On Fuzzy Analogues of Physical Spaces in Noncommutative Geometry

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Abstract

Since the development of noncommutative geometry, it has become common to construct fuzzy versions of well-known geometric spaces, such as the fuzzy sphere. This work follows efforts to construct approximations to physical spaces that are as consistent as possible with the definitions of a spectral triple. The first part looks at a way of constructing lattice-like spectral triples that are able to approximate simple geometries such as the line and the circle. The latter model is successfully interpreted in terms of a fermionic state-sum model with a U(1) connection. The second and main part of the work develops a model for a fuzzy complex projective plane within the formalism of matrix geometries, itself a formulation of real spectral triples in terms of matrices. An argument is presented that justifies the expectation that such fuzzy spaces with symmetry groups may by necessity approximate coadjoint orbits. A Hilbert space and Dirac operator that approximate the complex spinor geometry of the complex projective plane are then constructed, which are much simpler and more concrete than those obtained by earlier efforts in the literature.

[. . .] so in me I quote on when the panting stops scraps of that ancient voice on itself its errors and exactitudes on us millions on us three our couples journeys and abandons on me alone I quote on my imaginary journeys imaginary brothers in me when the panting stops that was without quaqua on all sides bits and scraps I murmur them [...]

–Samuel Beckett, *How It Is*

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Contents

Chapter 1

Introduction

Among the many research programmes developed along the way to obtain a theory of quantum gravity, many have converged on the need of obtaining a new structure of spacetime. Approaches such as loop quantum gravity, causal dynamical triangulation and causal set theory, among others, each possess their own framework of representing spacetime structure and quantising it[1]. Indeed, this focus can be seen to result from the conceptual underpinnings of General Relativity, where the gravitational field, causal structure, and geometry of space(time) are combined into one. Correspondingly, attempts to reconcile the theory with quantum dynamics thus proceed, depending on one's preferred outlook, by interpreting quantum gravity as a quantisation of a gauge theory mediated by the graviton; a theory of how classical notions of causality and even locality fit within the quantum formalism; a theory of how space and the usual associated notions of area, volume, etc. can be quantised; or even some combination of these views. Simultaneously, these ap-

[1]See for example [Ori09]

proaches need to display how in the classical limit the notions of general relativity emerge.

The approach taken by this work is that it is worthwhile to pursue the quantisation of spacetime structure on its own. The mathematical formalism of choice for the investigation is that Noncommutative Geometry[Con94; CM07; Van15].

The field of noncommutative geometry arose at first from the purely mathematical field of operator algebras. Influenced by deep mathematical theorems expressing dualities between algebraic and topological data, it was possible to develop the theory as a means of expressing unique and pathological spaces algebraically. Eventually, the field was developed to the point where it could represent algebraically manifolds endowed with metric geometry[Con13].

The language of operator algebras and Hilbert spaces is something that is also vital in many formulations of quantum mechanics. Subsequently, a description of the structure of spacetime is thus sought here in terms of this language. Whether this description will turn out to be fundamental is a matter for another work. Nonetheless, once one is committed to the development of physics in this algebraic language, two valuable points need to be addressed. These include (1) that geometries in the traditional sense are describable within the framework, and (2) that concepts are developed in the framework that make connection with the classical notions of geometry, and thus allows one to provide some meaningful interpretation to new geometries. Noncommutative geometry provides answers both of these concerns. Indeed, much of the work done during the early days of noncommutative geometry focused on the development of algebraic equivalents of analytic and geometric notions that corresponds exactly with the traditional notions whenever one inputs a commutative algebra to the formalism. The starting point of the theory is the well known duality due to Gelfand[Gel87] between commutative C^* algebras and (locally compact) topological spaces. Beyond that, for example, infinitesimal quantities are interpreted as compact operators, points as characters on an algebra, and the notion of the dimension of a space is interpreted in a purely spectral way. In what was ultimately a crucial development for physical applications, the Dirac operator took central stage when it came to defining the differential[Con85] and metrical[CL92] structures of noncommutative spaces. Thus the object of study representing a 'noncommutative space' was formalised through the notion of a *spectral triple*, a collection $(\mathcal{A}, \mathcal{H}, D)$, where $\mathcal{A}$ represents the algebra of functions, $\mathcal{H}$ the Hilbert space of square-integrable spinor fields, and D the Dirac operator commonly used in field theory to define the fermion propagator.

It is at this point that the development of the field became intricately tied with its application to physics. In particular, the formalism was used to express the standard model[CL91; Con96]. Along the way the concept of a *real* spectral triple arose, essentially combining the chirality and charge conjugation operators on fermions into the formalism[Con95]. It is precisely this structure that ended up being capable of fully encapsulating the Standard Model within the framework of noncommutative geometry[Bar07;

CCM07]. The model is a prime example of how spectral triples can encode both 'classical' systems of manifolds and their geometry, and 'quantum' systems of observables as operators on Hilbert spaces, unifying them into a single object. This uses the construct of an *almost-commutative* geometry, essentially a product structure between the spectral triples of a commutative manifold and a discrete internal space.

To complete the quantum mechanical picture of a noncommutative space, the dynamical evolution of a system is here taken to be based on the path-integral formalism. This has given rise to the spectral action principle[CC97], whereby the bosonic part of the action strictly depends on the spectrum of the Dirac operator, which includes gravitational degrees of freedom in addition to the gauge bosons and Higgs sector. Fluctuations of the Dirac operator about some background value can be written in terms of the unitary element of the algebra. Importantly, if the algebra and Hilbert space are taken to be finite dimensional, the Dirac operator has a finite spectrum, and the integration measure associated with one's preferred action becomes well defined. The system then has an intrinsic energy cutoff expressed by the maximal eigenvalue of the Dirac operator, which can be identified with the Planck scale. Equivalently, this can be interpreted in terms of the Plank length, whose existence can now be explained heuristically using the noncommutativity of the algebra expressing the coordinates on the noncommutative manifold, precluding the resolution of points beyond this length scale. The dynamical framework does suffer from the all-to-common issue of being formulated in Euclidean rather than Lorentzian signature, which is sufficient to describe traditional geometries, but of course not enough to directly model spacetime itself. The difficulties in defining a Lorentzian spectral triple are the ones that are typically encountered in such cases, such as the positivity of the inner product on state space, or the integration measure of the path integral not necessarily being well defined. These questions are important in the grand scheme of using spectral triples to model spacetime, but will not be investigated in this work. As such the geometries will be taken to be Euclidean.

The idea of using operators on finite dimensional Hilbert spaces, i.e. matrices, as the quantum correspondents of coordinate functions is precisely the starting point taken by many existing models, commonly known as *fuzzy spaces*. Such explicit models include the fuzzy sphere[Mad92; GP95], fuzzy complex projective spaces[Ale+01; GS99; Dol+08], and fuzzy flag manifolds[MS08], among others. These projects are influenced by noncommutative geometry to varying degrees.

One of the main aims of the current investigation is the development of such fuzzy spaces. Its starting point is a formal definition presented in [Bar15] that fits fully within the currently accepted axiomatic framework of real spectral triples[CM07]. This is unlike many of the aforementioned examples off fuzzy spaces. This insistence on constructing fuzzy spaces whose definition conforms strictly to the axioms of noncommutative geometry is a choice that can be justified conceptually. For example, the properties and objects of spectral triples mentioned above are oftentimes the only known generalisations of geometric notions to the realm of quantised geometry. Of course, the current formulation of the axioms of noncommutative geometry is somewhat contingent, something that is evident from the consideration of how the axioms have changed since their inception. As a striking example, what was once known as the orientability axiom[Con95], perfectly well-motivated from the point of view of the Riemanninan geometry of oriented manifolds, was left behind once it was realised that the spectral triple of the standard model does not satisfy it[Ste06]. Indeed, more recent formulations of spectral triples exist[BF14] that are candidate generalisations of the commonly accepted one. It is thus quite possible that the definition of a (real) spectral triple may change in the future. For the time being, the current choice of programme, viz. starting with the accepted axiomatic framework, is taken to be justified from a pragmatic point of view. One of the results of the work in [Bar15] is that, once a finite dimensional algebra and a finite dimensional Hilbert space are chosen, the axioms of a real spectral triple serve to constrain the form of the Dirac operator, to the extent that its solution space is directly computable. It turns out to be completely characterised in terms of a vector space of completely independent Hermitian and anti-Hermitian matrices. This means that it can be converted into a random matrix model. Consequently, the models become directly amenable to direct computation using numerical and even analytical methods. For example, using Monte-Carlo simulations, certain models were shown to undergo a phase transition where the coupling constants in the action serve as order parameters[BG16; KP21].

Parallel to calculations using these Dirac ensembles, it is important to keep developing individual fuzzy spaces within the matrix geometry formalism. These will allow one to connect with the classical realm of geometry, by displaying how such spaces can emerge from their fuzzy analogues. Moreover, recent examples suggest that fuzzy spaces, such as the fuzzy sphere, can be observed as vacuum solutions of certain Dirac matrix models[DAr22]. They may thus already have a very real use when one is working in the semi-classical regime of Dirac ensembles.

As it stands the matrix geometry formalism contains two examples of fuzzy spaces. These are the fuzzy sphere[Bar15] and the fuzzy torus[BG25]. It is the aim of this investigation to extend this catalogue of fuzzy spaces. Much like the original sphere example, heavy use will be made of group theoretical methods to construct such fuzzy spaces. This follows the age old process of describing spaces and their properties using their symmetries. The quintessential example pursued here will end up being the two-dimensional complex projective space $\mathbb{C}P(2)$, a symmetric space under the action of the symmetry group SU(3). This is a four-dimensional space and so may serve as a valuable model to use when modelling spacetime.

1.1 Outline of the Thesis

The majority of chapter 2 will focus on the necessary background from the point of view of noncommutative geometry. Sections 2.1–2.2 introduce sequentially the concepts that constitute a real spectral triple, providing along the way the minimal information necessary to understand their structure. Section 2.3 then introduces the formalism of matrix geometries within which fuzzy spaces are defined. Finally, section 2.4 provides a quick overview of the fuzzy sphere, the main example motivating the course of the subsequent investigation.

Before delving into fuzzy spaces proper, chapter 3 focuses on another effort to construct physical models using the axioms of a real spectral triple. This can be viewed as a complementary point of view to constructing fuzzy spaces, as its aim is ultimately to be able to construct lattice models within noncommutative geometry. Beyond the basic ingredients of the framework, the majority of the chapter contains original work investigating relatively geometrically simple models, and evaluating their usefulness as potential physical lattice models. Section 3.1 introduces the underlying framework and its assumptions, while the section 3.2 applies the formalism to specific models, including a line and a circle model. Connections are made between the circle model and a particular state-sum model.

Chapter 4 introduces and develops the main arguments of the fuzzy space investigation, and proceeds to recall the necessary background for the $\mathbb{C}P(2)$ model developed in the following chapter. All results in this section are already present in the literature, though the derivations are at places presented in a unique manner that appears important to the following discussion. The discussion starts in section 4.1 by explaining why the class of coadjoint orbits is taken as the class of homogeneous spaces to be approximated by a fuzzy construction. Focus is then narrowed down to the main geometry to be considered in the subsequent chapter, that of $\mathbb{C}P(2)$. Its usual continuous geometry, including its bundle of complex spinors and spectrum of the associated Dirac operator, will be derived in section 4.4, as much as possible, using group theoretical methods.

The original results of the thesis concerning the fuzzy construction of $\mathbb{C}P(2)$ are then presented in Chapter 5. The emphasis is on using Fock-space methods to construct the (fuzzy) spinor bundle, together with an explicit form for the Dirac operator. The resulting expressions are much simpler than other formulations of a fuzzy $\mathbb{C}P(2)$ found in the literature. The geometry is then shown to converge for large matrix size, though the full packaging of the data terms of a spectral triple is still found to be missing.

Finally, Chapter 6 is devoted to a discussion of what has been obtained using the commutative models of Chapter 3 and the fuzzy $\mathbb{C}P(2)$ model of Chapter 5, together with what is still left to be done in future investigations.

Chapter 2

Noncommutative Geometry

2.1 Complex Spectral Triples

2.1.1 Algebras and Topology

Definition 2.1.1. A (complex) *-algebra is a complex algebra $\mathcal{A}$ together with an anti-linear involution, denoted *, that satisfies the relations

$$(\alpha a)^* = \bar{\alpha} a^*, \tag{2.1}$$

$$(ab)^* = b^* a^*, \tag{2.2}$$

for all a, b in $\mathcal{A}$ and $\alpha \in \mathbb{C}$. If $\mathcal{A}$ is additionally a Banach algebra, it will be required that the involution is isometric, that is that

$$||a^*|| = ||a||, \tag{2.3}$$

for all a in $\mathcal{A}$.

Definition 2.1.2. A C^*-algebra is a Banach Algebra $\mathcal{A}$ together with a *-involution satisfying the conditions above together with the C^*-*Condition*:

$$||aa^*|| = ||a||^2, \tag{2.4}$$

for all a in $\mathcal{A}$.

In this work it will be assumed that, unless stated otherwise, all algebras discussed are unital.

Example 1 (Continuous functions)**.** Given a compact topological space X, the set of continuous, complex-valued, functions $C(X)$ with point-wise addition and products is a C^*-algebra. The norm is given by the supremum norm

$$||f|| := \sup_{x \in X} |f(x)|, \tag{2.5}$$

for any f in $C(X)$. Note additionally that this algebra is commutative since for any pair of functions f, g in $C(X)$ one has

$$f(x)g(x) = g(x)f(x), \tag{2.6}$$

for all points $x \in X$, and so also at the level of the algebra.

Example 2 (Differentiable functions)**.** Given the example above where now X additionally has a differentiable structure (e.g. when it is a differentiable manifold) the algebra of complex-valued smooth functions $C^\infty(X)$ is a subalgebra of $C(X)$, but as it is not complete it is only a *-algebra.

Definition 2.1.3. A **character** of a Banach algebra is a nonzero algebra homomorphism $L : \mathcal{A} \to \mathbb{C}$.

Let $X(\mathcal{A})$ denote the set of characters of $\mathcal{A}$, and endow it with the weak* topology. An important starting point for the field is the fact that whenever the algebra in question corresponds to an algebra of continuous functions on some topological space, $X(\mathcal{A})$ turns out to be homeomorphic to this space. In fact, so long as $\mathcal{A}$ is commutative, the algebra of functions on $X(\mathcal{A})$ is isomorphic to the starting algebra. This equivalence is facilitated by the *Gelfand transform*.

Definition 2.1.4. The *Gelfand Transform* of a commutative Banach algebra $\mathcal{A}$ is a map $\tau : \mathcal{A} \to C(X(\mathcal{A}))$ defined by

$$\tau(a)(\chi) = \chi(a), \tag{2.7}$$

for all χ in $X(\mathcal{A})$.

Theorem 2.1.1 (Gelfand). *The Gelfand transform of any commutative C^*-algebra is an isometric *-isomorphism onto $C(X(\mathcal{A}))$.*

Proof. See for example [Mur90]. □

One can relax the condition that the algebra in question be unital, with the result that the topological space X in question is no longer compact, but only locally compact. The algebra of functions is now replaced with $C_0(X)$, the algebra of continuous functions vanishing at infinity.[1]

[1] Concretely, this algebra is obtained using the one-point compactification of X, and taking the algebra of continuous functions that vanish on the adjoined point.

2.1.2 Modules and Bundles

The equivalence exhibited by the Gelfand transform is strict enough to retain the whole topological data of a space, or indeed even the differential structure of the space, using the associated commutative algebra of functions. But, when a space comes endowed with further geometric data, it may not be possible to exhibit that data using simply an algebra of functions. The most important example at this stage is that of a Riemannian manifold, which most simplistically requires the additional datum of a Riemannian metric tensor to be defined. Moreover, when it comes to physical theories, the presence of matter fields requires additional structure.

Traditionally, both of the examples above can be expressed through the use of vector bundles. This ubiquity of vector bundles is the motivation behind introducing the next object in what will eventually defined as a spectral triple, namely, that of a module over the algebra in question.

Let $E \to X$ be a complex vector bundle over the compact space X. Evidently, the space $\mathcal{E} \coloneqq \Gamma(E)$ of sections of the bundle is a module over the algebra $\mathcal{A} = C(X)$, using the pointwise action at $x \in X$ given by

$$(f \cdot s)(x) \coloneqq f(x) \cdot s(x),$$

where f is a function in $C(X)$, and s a section in $\Gamma(E)$. In fact, one finds that $\mathcal{E}$ is always finitely generated and projective.

Proposition 2.1.1. *The module $\mathcal{E}$ over $\mathcal{A}$ is finitely generated and projective*

To prove this it will be useful to refer to a basic and well-known result from topological K-theory:

Lemma 2.1.1. *Let X be a paracompact space. Then given any vector bundle $E \to X$, there exists another vector bundle $E' \to X$ that stably trivialises E, that is, $E \bigoplus E' \cong X \times \mathbb{C}^N$ for some positive integer N.*

Proof. See for example [AA18]. □

Proof of Proposition 2.1.1. Both properties are easily derived using a partitions of unity argument. If the partitions of unity are subordinate to neighbourhoods that locally trivialise E, any section can immediately be expanded in terms of a finite basis, so that $\mathcal{E}$ is indeed finitely generated. Let E' be a bundle that stably trivialises E, as is guaranteed to exist from Lemma 2.1.1. One can assume that each bundle is endowed with a Hermitian metric, as is guaranteed to exist using partitions of unity. Then one can readily use the metrics to decompose $\Gamma(E) \bigoplus \Gamma(E') \cong \Gamma(E \bigoplus E') \cong \mathcal{A}^N$, with the latter equivalence following by stabilisation condition. But, the statement $\mathcal{E} \bigoplus \mathcal{F} \cong \mathcal{A}^N$ for some positive integer N is equivalent to the condition that $\mathcal{E}$ is projective. □

Much like in the case of the Gelfand duality, it turns out that the association $(E \to X) \rightsquigarrow \Gamma(E)$, sending a vector bundle E over X to a finitely generated projective module over $C(X)$, gives rise to an equivalence

Theorem 2.1.2 (Serre-Swan)**.** *There is an equivalence between the collection of vector bundles over a compact topological space X and* *the collection of finitely generated projective modules over the commutative C^*-algebra $C(X)$.*

Proof. Associating to the vector bundle $E \to X$ the $C(X)$-module $\Gamma(E)$ provides one way of the equivalence map. A construction of the map going the other way, together with a proof that these indeed form an equivalence, can be found for example in [GVF01]. □

A few comments are in order. Firstly, just as in the case of the previous section, the condition on X being compact can be relaxed to the conditions that X is only locally compact *and* paracompact. Then the arguments above follow but with the module being $\Gamma_0(E)$, the collection of sections vanishing at infinity.

Secondly, by restricting to the submodule $L^2(E) \subseteq \Gamma(E)$ of square integrable sections, one can turn the module into a Hilbert space. This fact becomes also important when one is interested in discussing measure theoretic properties of noncommutative spaces, that make use of the theory of von Neumann algebras and where the representation spaces are naturally Hilbert spaces. Hence, the C^*-algebra modules are as a matter of definition taken to be Hilbert spaces.

2.1.3 Riemannian Geometry

It has already been mentioned that typically, a differential manifold M is given a Riemannian structure by endowing it with a metric tensor. This serves as a bilinear form $\Gamma(TM) \times \Gamma(TM) \to C^\infty(M)$ on the tangent bundle TM of M. This means that, using the language of projective modules from the previous section, additional information is still needed to fully encode such a geometry.

The route taken by noncommutative geometry uses the Dirac operator. For the purposes of physical applications, it is clear that the Dirac operator is a necessary ingredient in quantum theory since it allows one to write down the propagator for fermion fields. Using the Dirac operator to encode geometric data obviously requires manifolds that are at least spin$^{\mathbb{C}}$; again, this is anyhow desirable from a physical point of view.

That the metrical aspects of a Riemannian spin manifold M are recoverable from the Dirac operator D follows from the distance formula

$$d(x,y) = \sup_{f \in C(X)} \{|f(x) - f(y)|, \| [D,f] \| \leq 1\} \tag{2.8}$$

for any two points x, y in M. This turns out to be precisely the geodesic distance between the two points, using the Riemannian metric from which the Dirac operator was derived. Note that by interpreting the values on the right of the distance formula using the evaluation maps, one obtains a purely functional theoretic definition of Riemannian geometry, as now x and y are viewed as characters on the algebra.

2.1.4 Spin Geometry

Since Noncommutative geometry relies heavily on Dirac operators as a means to represent geometric data of Riemannian manifolds, it will be necessary to review several aspects of Clifford algebra and spinor theory that will be useful in the sequel. Standard references on Clifford algebras and Spinors include [ABS64; LM89].

Let (V, Q) be a quadratic space, consisting of vector space V together with quadratic form Q. It will be assumed throughout that the vector space is finite dimensional. One has a natural inner product defined using the polarisation identity

$$\langle v, u\rangle \coloneqq Q(v+u) - Q(u) - Q(v),$$

where v, u are in V. The *Clifford algebra* $Cl(V, Q)$ is defined as the quotient algebra of $T(V)$, the tensor algebra over V, by the ideal generated by elements of the form $vu + uv - 2\langle v, u\rangle$ for all v, u in V. Obviously one has an injection $j : V \to Cl(V, Q)$.

The Clifford algebra satisfies the following universal property.

Proposition 2.1.2. *Let (V, Q) be a quadratic space and $Cl(V, Q)$ its Clifford algebra. Then given any vector space homomorphism $\phi : V \to A$ between V and a unital associative algebra A, such that $\phi(u)\phi(v) + \phi(v)\phi(u) = 2\langle u, v\rangle 1_A$ for all u, v in V, there exists a unique algebra homomorphism $\Phi : Cl(V, Q) \to A$ such that the diagram*

$$\begin{array}{ccc} V & \xrightarrow{j} & Cl(V,Q) \\ & \overset{\phi}{\searrow} & \downarrow \Phi \\ & & A \end{array}$$

commutes.

Practically, the universal property of 2.1.2 implies that any finite dimensional Clifford algebra can be mapped, using a transformation that diagonalises Q, to a Clifford algebra generated by an orthogonal basis where the quadratic form corresponds to p basis vectors that square to $+1$ and q basis vectors that square to -1. This Clifford algebra will be labelled as $Cl_{p,q}$, with the understanding that it is an algebra over $\mathbb{R}$.

Let $Cl(V,Q)^{\times}$ be the group of units in $Cl(V,Q)$. The *Pin*-group is the group generated by the elements $v \in Cl(V,Q)^{\times} \cap V$.

The Clifford algebra has a natural $\mathbb{Z}_2$ grading using the tensor algebra grading defined using $deg(v) = +1$ for $v \in V$. One subsequently defines the *Spin* group using this grading by $\mathrm{Spin}(V,Q) := Cl(V,Q)^0 \cap \mathrm{Pin}(V,Q)$.

One may also define a Clifford algebras over $\mathbb{C}$ in a manner similar to the real case. In fact, such Clifford algebras may be viewed as the complexification of the real ones. The signature of the original Clifford algebra is immaterial, as a complex phase can be used transform from one signature to the other. Thus for any positive integer n there is a unique complex Clifford algebra over $\mathbb{C}^n$, and one has an isomorphism $\mathbb{C}l_n \cong \mathbb{C} \otimes Cl_{p,q}$ for all nonnegative p, q for which $n = p + q$.

Definition 2.1.5. A *Clifford module* is a representation of the Clifford algebra $Cl_{p,q}$.

Proposition 2.1.3. *Let $Cl_{p,q}$ be a real Clifford algebra of signature (p,q). If $p+q$ is odd, the algebra has two inequivalent irreducible algebra representations, while if $p+q$ is even, there is only one such irreducible representation.*

Proof. The classification of Clifford algebras[ABS64; LM89] demonstrates that any such algebra is isomorphic to a matrix algebra over a division algebra. This matrix algebra is either simple or a direct sum of two simple algebras. There then exists only one irreducible algebra representation associated to each simple algebra factor. A useful way to view this is to use the following construction. Consider $Cl_{p,q}$ as a Clifford algebra generated by the elements γ_i for $i = 1, 2, \ldots, p+q$. The element

$$\gamma \coloneqq \gamma_1 \gamma_2 \cdots \gamma_{p+q}$$

commutes with the rest of the γ_i's whenever $p+q$ is odd. This element then serves to decompose any representation of $Cl_{p,q}$ into two irreducible representations depending on whether γ is $+1$ or -1 when restricted to the relevant subspace. These end up corresponding the the two irreducible representations of the two simple algebra factors. □

2.1.5 Spectral Triples

It turns out that, as in the last couple of sections, going the other way is possible. This has been proven by Connes, most recently in [Con13], and is known as the *Manifold Reconstruction Theorem.* This theorem ultimately justifies trading the traditional notion of a space by an algebraic definitions based a commutative C^*-algebra $\mathcal{A}$, a Hilbert space $\mathcal{H}$ on which $\mathcal{A}$ acts, and a Dirac operator D.

The springboard into noncommutative geometry proper is achieved, as the name of the field suggests, by relaxing the requirement that an algebra $\mathcal{A}$ be commutative, thereby working with a more general notion of a 'space'. To discuss geometry on this weaker notion of space, one turns to the algebraic equivalents mentioned above to define the notion of a *spectral triple*.

Definition 2.1.6 (Spectral Triple)**.** A *Spectral Triple* consists of the data $(\mathcal{A}, \mathcal{H}, D)$ where:

- $\mathcal{A}$ is a complex *-algebra with a faithful *-representation $L : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ in terms of bounded operators on the Hilbert space $\mathcal{H}$.

- The operator D, acting on $\mathcal{H}$, is self-adjoint and with compact resolvent. It is called the Dirac operator in parallel with the commutative case.

- The commutator $[D, L(a)]$ is a bounded operator for all a in $\mathcal{A}$.

When it is clear from the context, the action of an element a of $\mathcal{A}$ on a vector v in $\mathcal{H}$ will be written with L suppressed, so that by writing $a \cdot v$ it will be implicit that the action is through L.

Example 3. The *Canonical Spectral Triple* of a Riemannian spin manifold (M, g) is given by the triple of data $(C(M), L^2(S), D)$, where $S \to M$ is the spinor bundle on M, and D is the Atiyah-Singer Dirac operator.

The Reconstruction theorem of Connes now takes the following form.

Theorem 2.1.3 (Manifold Reconstruction Theorem[Con13]). *Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple with a unital commutative algebra. Then there exists a unique compact Riemannian manifold for which $\mathcal{A} \cong C^0(X)$. In addition, $\mathcal{H} \cong L^2(E)$ for some vector bundle E over X, and D is a Dirac-type operator acting on that bundle.*

For a proof see for example Connes's paper[Con13]. In most cases $E = S$ corresponds to a spinor bundle and D is the usual Dirac operator, so that the spectral triple is interpreted as a canonical spectral triple as per example 3. The theorem in fact requires that the commutative spectral triple satisfy extra conditions, mostly analytic, in order to hold. For the purpose of this work, which concerns mostly finite dimensional algebras, their explicit form has been omitted. Indeed, the delicate analytical conditions that are required by the theorem make the possibility of an explicit reconstruction rather impractical. Nevertheless, the theorem points towards conditions that necessarily hold for a given (commutative) spectral triple in order to guarantee that it corresponds to the usual notion of a Riemannian manifold.

Oftentimes, a spectral triple may come with a $\mathbb{Z}_2$ grading on the Hilbert space $\mathcal{H}$, represented by the operator Γ satisfying $\Gamma^2 = 1$ and $\Gamma^* = \Gamma$, and subject to the further constraints

$$\Gamma a = a\Gamma$$

for all a in $\mathcal{A}$, and

$$\Gamma D = -D\Gamma.$$

This allows one to talk of left-handed and right-handed spinors, where Γ is then the *chirality operator*. A spectral triple together with such a chirality operator is known as an *even spectral triple*, while one without such operator is known as an *odd spectral triple*.

2.2 Real Spectral Triples

Up to now all algebras considered were complex algebras. This is the most natural arena in which to develop the spectral theory of operators on Hilbert spaces, but from the point of view of geometry one would prefer to work with the notion of a real manifold. Additionally, from the point of view of physics, the notion of charge conjugation of fermions is paramount. These considerations lead the way to the introduction of another ingredient in the data that form spectral triples, namely that of a *real structure*.

Definition 2.2.1. A *real structure* is an antilinear isometry $J : \mathcal{H} \to \mathcal{H}$ that satisfies either $J^2 = 1$ or $J^2 = -1$.

The interaction between the real structure, the Dirac operator, and the chirality operator (when it exists) is modelled after the situation in Clifford algebras. The main point is that the real structure either commutes or anti-commutes with the other two operators. One can characterise all possibilities by expressing these relations as

$$J^2 = \epsilon,$$

$$DJ = \epsilon' JD,$$

and

$$J\Gamma = \epsilon''\Gamma J,$$

where each of ϵ, ϵ' and ϵ'' is either 1 or -1. Of course, for an odd spectral triple the relation that holds between the real structure and the chirality operator does not exist.

Following from Clifford algebra theory, there turn out to be only eight such possible combinations, including both even and odd spectral triples. They are typically characteristed by the mod-8 parameter s, named the *KO-dimension* in honour of the Bott periodicity of Clifford algebras, since the periodicity exhibited by those algebras extends also to the relations that their respective real structures satisfy. The actual signs determined by the value of s are listed in Table (2.1).

s	0	1	2	3	4	5	6	7
ϵ	1	1	-1	-1	-1	-1	1	1
ϵ'	1	-1	1	1	1	-1	1	1
ϵ''	1		-1		1		-1	

Table 2.1: Signs for given KO-dimension s.

Definition 2.2.2. A *Real Spectral Triple* consists of the collection $(\mathcal{A}, \mathcal{H}, D, J)$, together with optionally an operator Γ, where

- $(\mathcal{A}, \mathcal{H}, D)$ is a spectral triple as in Definition 2.1.6.
- The operator $J : \mathcal{H} \to \mathcal{H}$ forms a real structure of *KO*-dimension s with respect to Dirac operator D and, when it exists, the chirality operator $\Gamma : \mathcal{H} \to \mathcal{H}$.

- In addition to the algebra left action L, there is a right action R of the algebra on $\mathcal{H}$ defined as $R(a) := JL(a)^* J^{-1}$, and it satisfies the *zeroth order condition* $[L(a), R(b)] = 0$ for all a, b in $\mathcal{A}$.

- The *first order condition* is satisfied:

$$[[D, L(a)], R(b)] = 0$$

  for all a, b in $\mathcal{A}$.

The first order condition, serves to enforce the condition that D is a first-order differential operator, as it very much is in the commutative case. In local coordinates:

$$[D, f] = \sum_i \gamma^i \partial_i f \tag{2.9}$$

for any function f which acts via multiplication on sections square integrable spinors. It immediately follows that

$$[[D, f], g] = 0. \tag{2.10}$$

Conversely, an order 1 differential operator mapping from one bundle to another can by definition be written locally in terms of order one differentials and suitable coefficient functions. To generalise this commutative definition, one needs to use the right action on the algebra and to see that that commutes with the left action of $[D, a]$, for all $a \in A$. This is precisely because A is no longer commutative, and so the condition 2.10 typically fails purely due to the nature of the algebra itself. So the order one condition is a necessary condition for the abstract operator D to be interpreted as a Dirac operator. The first order condition can also be understood as arising naturally if one uses a bimodule point of view. If $\mathcal{H}$ is a bimodule over $\mathcal{A}$ together with a Dirac operator D, then it simply has the form

$$D(x \cdot v \cdot y) = x \cdot D(v \cdot y) + D(x \cdot v) \cdot y - x \cdot D(v) \cdot y, \qquad (2.11)$$

where $x, y \in \mathcal{A}$, $v \in \mathcal{H}$ and the left and right actions are understood based on positioning.

2.3 Matrix Geometries

The main appeal of Noncommutative Geometry from the point of view of quantum gravity is that by by extending the definition of a geometric space, it provides a framework in which to formulate a quantised structure of space time. One sees that, by defining the notion of geometry using the language of operator algebras on Hilbert space, the data that models spacetime is already in a form that is amenable to quantum mechanical calculations.

Ultimately, the language of spectral triples provides a way to formalise the age-old approach to the quantisation of spacetime that demands that the coordinate functions should fail to commute in the quantum regime. This results in the inability of states to be simultaneous eigenvectors of all coordinate operators, and thus dissolving the concept of a 'point' in the space. This heuristic point of view is at the core of what has been termed 'fuzzy' space in literature. The amount of fuzziness is expressed by the Plank length, ℓ_p, which is the minimal length scale that can be resolved in a space. In terms of energy, this translates, as is usual in quantum mechanics, to a high energy cutoff. This is where the Dirac operator comes into play, since its spectrum precisely determines the energy modes of the system. So long as one is working with compact spaces, this means that the spectrum of the Dirac operator is discrete and in fact finite, so long as the Planck length is not zero. But as $\ell_p \to 0$, the spectrum of the Dirac operator must agree more and more with the spectrum of a classical space.

The above picture can be rephrased in the language of Noncommutative Geometry as follows. One forms a sequence of spectral triples $(\mathcal{A}_n, \mathcal{H}_n, D_n)$ labelled by the positive integers. As $n \to \infty$, one wants this sequence of spectral triples to converge in some sense to a commutative spectral triple $(\mathcal{A}, \mathcal{H}, D)$. The algebras $\mathcal{A}_n$ should converge to $\mathcal{A}$ in some norm, and so should the Hilbert spaces $\mathcal{H}_n$ to $\mathcal{H}$. The spectrum of each sequential Dirac operator should converge towards the spectrum of the limiting geometry. Additionally, if the spectral triples are real, than there should be a similar sense in which the real structures converge to a limiting one.

At least in the case of compact manifolds, where the Dirac spectrum is discrete, constructing a sequence of fuzzy spaces, each with an energy cutoff, entails using finite dimensional algebras and Hilbert spaces.

The majority of the following work will be based on the task of finding examples of such spectral triples, such that they approximate a classical space. As a first step, it shall be necessary to specify a working definition of what such an approximation of spaces ought to look like. This has already been done in [Bar15] and goes by the name of a *Matrix Geometry.*

Definition 2.3.1. A *Matrix Geometry* of type (p, q) is a finite real spectral triple $(\mathcal{A}, \mathcal{H}, D, J, \Gamma)$ satisfying the following conditions.

- Its KO-dimension is given by $s \equiv q - p (mod\ 8)$
- The algebra $\mathcal{A}$ is isomorphic to some finite matrix algebra.
- $\mathcal{H} \cong V \otimes \mathcal{H}_0$ where V is a Clifford module of type (p, q) and $\mathcal{H}_0$ is a representation space of $\mathcal{A}$. Thus $L(a)v \otimes m = v \otimes a \cdot m$.
- The Hilbert space inner product is inherited from the inner products of the factors, so that $\langle v \otimes m, v' \otimes m' \rangle = (v, v) \langle m, m' \rangle$.
- $\Gamma = \gamma \otimes 1$, whenever the Clifford module chirality operator γ exists.
- $J = C \otimes j_0$, where C is a charge conjugation operator on the Clifford module and j_0 is a real structure defined on the matrix algebra.

The core motivation behind the definition of a Matrix Geometry, beyond the necessity of the algebra of functions being a finite dimensional matrix algebra, is that just like in the classical case, much of the real structure and chirality behaviours exemplified by the KO-dimension are inherently tied to the behaviour of spinors on the spacetime. As a result, their action in the definition above factors in such a way that it acts on the space of spinors at each point.

Oftentimes, it is necessary to to work only with the collection of data $(\mathcal{A}, \mathcal{H}, J, \Gamma)$, i.e. of a matrix geometry without a Dirac operator. This shall be termed a *fermion space*. Fermion spaces are particularly common when one works in quantum field theory, since it may be desirable to vary the allowable geometries while keeping the fermionic degrees of freedom fixed, for example when one wishes to integrate over the geometries. A further simplification of the formalism is afforded by considering *fuzzy spaces*.

Definition 2.3.2. A *fuzzy space* is a Matrix Geometry $(\mathcal{A}, \mathcal{H}, D, J, \Gamma)$ that satisfies the following additional conditions.

- The algebra $\mathcal{A}$ is a simple matrix geometry, of the form $M_n(\mathbb{F})$ for some integer n and field $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$.
- The Hilbert space factor $\mathcal{H}_0$ is itself equal to the matrix algebra $M_n(\mathbb{C})$, and is endowed with the Hermitian inner product induced using the trace.
- The real structure is given by $J(v \otimes m) = Cv \otimes m^*$

The benefit of using the definition of a fuzzy space, instead of the broader definition of a matrix geometry, is that the set of allowable Dirac operators is easy to compute, and is in fact a vector space.

In shorthand, the action of the Dirac operator is

$$D(v \otimes m) = \sum_\alpha \omega^\alpha \cdot v \otimes \left(K_\alpha m + \epsilon' m K_\alpha^*\right), \tag{2.12}$$

where for each α one has some product of gamma matrices ω^α and a matrix $K_\alpha \in M_n(\mathbb{C})$. Further details are explained in [Bar15].

2.3.1 Transformations of Spectral Triples

The main focus in the second part of this work will be on fuzzy spaces that are related to classical spaces that are homogeneous spaces. Thus, it is important to consider how fuzzy spaces can be endowed with group actions of their own to replicate the symmetry transformations of the limiting spaces. This provides one benefit of finite spectral triple over usual lattice-based approximations, since the latter necessarily break some of the symmetries of a space at any finite order of the approximation. The fuzzy spaces, meanwhile, retain the full symmetry group throughout.

Definition 2.3.3. A *transformation* of a real spectral triple $(\mathcal{A}, \mathcal{H}, D, J, \Gamma)$ consists of a unitary transformation $U : \mathcal{H} \to \mathcal{H}$ that respects the fermion space structure of the spectral triple. Namely, one has the following conditions.

- There exists an isometric *-automorphism ϕ of the algebra $\mathcal{A}$ such that $UL(a)U^* = L(\phi(a))$ for all a in $\mathcal{A}$.
- $U\Gamma = \Gamma U$.
- $UJ = JU$.

Letting $D' = UDU^*$, one has a transformed spectral triple $(\mathcal{A}, \mathcal{H}, D', \Gamma, J)$.

A transformation is called a *symmetry* of the spectral triple if it satisfies $D' = D$.

2.4 The Fuzzy Sphere

The first noteworthy formulation of the fuzzy sphere was that given by Madore[Mad92]. It is based on the fact that the algebra $C(S^2)$ of continuous functions on the 2-sphere S^2 can be sequentially approximated by the algebra $M_n(\mathbb{C})$ of $n \times n$ complex matrices.

Firstly, let S^2 be embedded in $\mathbb{R}^3$ as the sphere of radius R. Using the global coordinates x^1, x^2, x^3, one can view the sphere as the locus of points with coordinates satisfying the relation

$$\left(x^1\right)^2 + \left(x^2\right)^2 + \left(x^3\right)^2 = R^2. \tag{2.13}$$

Indeed, the algebra of polynomial functions over the sphere is simply the algebra of polynomials $\mathbb{R}[x^1, x^2, x^3]$ modulo the ideal generated by the polynomial $(x^1)^2 + (x^2)^2 + (x^3)^2 - R^2$, enforcing the relation above. This allows one to speak of the algebra of polynomial functions on the sphere as simply having the form

$$f(x^1, x^2, x^3) = f_0 + f_i x^i + f_{ij} x^i x^j \cdots ,$$

where each $f_{i_1 i_2 \ldots i_k}$ is a symmetric tensor of order k. This algebra is in fact dense in $C(S^2)$ and so one can use its own separable basis as a basis for the algebra of continuous functions. At any polynomial order greater than one, all symmetric monomials that can be written are not linearly independent since relation (2.13) holds between the coordinates. In fact, at order k there are only $2k + 1$ linearly independent monomials that do not factor into a product of a lower order monomial with the above relation. Thus, if one truncates the algebra of polynomials up to order N, one has a vector space of dimension $\sum_{j=0}^{N-1}(2j+1) = N^2$.

Now consider some results from group theory. In terms of Lie algebras, one may identify the Lie algebras so(3) and su(2) and so speak of them interchangeably. Let the su(2) generators be labelled by J_{jk} for $1 \leq j < k \leq 3$. From the representation theory of su(2) one knows that finite dimensional irreducible representations are labelled by a nonnegative integer l, with the associated representation space V_l having dimension $N = 2l+1$. Fixing N, it is clear that the endomorphism algebra over V_l is simply $M_N(\mathbb{C})$, of dimension N^2. It is thus clear that the truncation at order N of the algebra polynomials on the sphere and that of the $n \times n$ matrices are isomorphic, at least as vector spaces. This on its own may not be very spectacular. But what is more important, is that one can write down a mapping from the one space to another that in the limit of large N becomes closer and closer to a C^*-algebra isometry.

From the point of view of noncommutative geometry, Madore's construction only touches upon the first part of a spectral triple, with no reference to the Hilbert space or Dirac operator. Those were constructed soon after by Grosse and Prešnajder[GP95]. Their construction applied to more general spin$^{\mathbb{C}}$ structures on the sphere, but for the purpose of the current discussion it is only necessary to discuss the case when one is dealing with the spin structure. Explicitly, they take for any positive integer n the Hilbert space $\mathcal{H}_n = \mathbb{C}^2 \otimes M_n(\mathbb{C})$. It is a module over $M_n(\mathbb{C})$ with the algebra action over itself via left action. The $\mathbb{C}^2$ factor has a natural su(2) action using the Pauli matrices σ_i, or, viewing it as a Clifford module, using the spin generators $S_{jk} = \frac{1}{2}\sigma_j\sigma_k$. Meanwhile, let J_{jk} be as above the matrices representing the su(2) generators on the space $V_n \cong \mathbb{C}^n$.Then using the vector space isomorphism $M_n(\mathbb{C}) \cong V_n \otimes (V_n)^*$, one can act on the matrix algebra factor using the adjoint action $[J_{jk}, m] = J_{jk} \cdot m - m \cdot J_{jk}$ for any m in $M_N(\mathbb{C})$.

Now, the Dirac operator on $\mathcal{H}_n$ is given the form

$$D_{GP} = \sum_{j<k} S_{jk} \otimes J_{jk} + 1. \tag{2.14}$$

This Dirac operator turns out to have a spectrum identical to that of the Classical Dirac operator on the sphere, up to some truncated value.

The triple $(M_n(\mathbb{C}), M_n(\mathbb{C}) \otimes \mathbb{C}^2, D_{GP})$ serves as a noncommutative finite spectral triple that still converges in the $n \to \infty$ limit to that of the sphere. But note that it constitutes only the basic definition of a spectral triple given by definition (2.1.6), while it is desirable that each such triple also be a real spectral triple just as that that the limiting classical geometry affords. Most concretely, one finds that the Grosse and Prešnajder construction does not allow one to introduce a chirality, such as the one that exists in the classical case. Thus one is working with a KO-dimension $s = 3$ instead of the $s = 2$ one required by the sphere.

Using the definition of fuzzy space introduced in the previous definition, which is inherently a real spectral triple, it is possible[Bar15] to define the fuzzy sphere as the fuzzy space $(M_n, M_n(\mathbb{C}) \otimes \mathbb{C}^4, J, \Gamma, D)$

for a given positive integer n. The Clifford module structure of the $\mathbb{C}^4$ factor is of type $(1,3)$, so that the Clifford algebra action is generated by the Hermitian matrix γ^0 satisfying $(\gamma^0)^2 = 1$ together with the original anti-Hermitian matrices γ^i for $i = 1,2,3$.

The Dirac operator has the form

$$D = \gamma^0 + \sum_{j<k=1}^{3} \gamma^0\gamma^j\gamma^k \otimes [L_{jk}, \cdot] \,,$$

where much like before the L_{jk}'s are the representation matrices of the Lie algebra generators of su(2) on V_n.

Note that the Clifford module dimension has been doubled.

Proposition 2.4.1. *The Dirac operator D on the fuzzy sphere has spectrum*

$$Spec(D) = \{\pm l | l = 1, 2, \ldots, n\} \,,$$

with multiplicity $2(2l+1)$ except for the values $\pm n$, that have only multiplicity $2n+1$.

The above spectrum can be compared with that of the spectrum of the usual Dirac operator on the sphere[Var06]. Indeed, as $n \to \infty$, the spectrum converges towards the spectrum of the classical Dirac operator.

Chapter 3

Finite Commutative Spectral Triples

In this chapter, a formalism for finite commutative real spectral triples will be developed. By virtue of the axioms for a spectral triple, the Dirac operator for such models turns out to be heavily constrained. This allows one to interpret the models as certain lattice models with a left and right action of the algebras. The formalism will then be applied to the simple models of the line and the circle. It will be shown that by using a fermionic kinetic term as an action, the latter model can be interpreted as a discretised topological state sum model with a $U(1)-$gauge symmetry. Everything up to the constraint of proposition (3.1.1) has essentially been known since Krajewski's work[Kra98], though the choice not to impose the equality between left and right action for a commutative real spectral triple is novel. All subsequent examples and constructions are thus new results.

3.1 The General Model

Attempts to apply noncommutative geometry to discrete spaces, by use of finite spectral triples, where all objects are finite dimensional, immediately followed the development of noncommutative geometry and its algebra[PS98; Kra98]. Indeed, such spaces serve the role of the internal space when expressing the Standard Model or its extensions in noncommutative geometry. But another natural question to ask is whether it is possible to construct physically relevant lattice models using such finite spectral triples. Much work has been done in this direction, mainly aiming to harness the full consequences of using noncommutative spaces[Lan97; Per24]. But another possible direction to take is to consider finite commutative spectral triples as a means of representation traditional lattice models from physics. This possibility was ruled out quite early in the development of the field, for example in [GS98]. But such approaches relied on on two assumptions, the first of which was the now-discarded orientability axiom. A more common assumption, which persists up to this day[Van15], is that any spectral triple with a commutative algebra must by definition contain only a single algebra action, since commutativity suggests the equality of the left and right actions. In fact, the current investigation will proceed by relaxing this condition, by constructing Hilbert spaces on which a commutative algebra acts on the left and on the right in a different manner.

Arguably, constructing a model with a commutative algebra with different left and right actions is no stranger than, say, taking the product of a commutative spectral triple and a noncommutative spectral triple to obtain an almost-commutative manifold. In fact, constructing an algebra associated with a lattice theory that possesses a differing left- and right-actions on the Hilbert space has been done before, for example by Dimakis and collaborators[DMS93]. But such models rely on a noncommutative extension of the original algebra, corresponding to the algebra of differential forms on the lattice and no longer simply the original algebra of functions. Such approaches are distinct from the current investigation, where the algebra in question is the commutative algebra of functions throughout; it does not appear that this setting has been investigated prior. From here, construction will follow on the assumption that a justification for the choice of differing actions may be justified by the results of the formalism.

3.1.1 Structure of Finite Dimensional C^*-Algebras

A finite dimensional complex C^* algebra A is isomorphic to a matrix algebra:

$$A \simeq \bigoplus_{i=1}^{N} M_{n_i}(\mathbb{C}). \tag{3.1}$$

It is well known[Van15] that for any complex matrix algebra, the only irreducible algebra representation is given by the algebra's defining representation. Thus the algebra $M_n(\mathbb{C})$ acts irreducibly on the vector space $\mathbb{C}^n$. It follows that the structure space of A, denoted $\hat{A}$, is given as the discrete N-point space with each point denoting $\underline{n_i}$, the irreducible algebra representation of the i-th algebra factor. Note that for each such irreducible representation, all other algebra factors act trivially, and so the irreducible algebra representations of A, in the cases when $2 \leq N$, are never faithful. In fact, the smallest faithful representation of A is given by $\underline{n_1} \oplus \underline{n_2} \oplus \cdots \oplus \underline{n_N}$.

In terms of the defining representation, the *-involution becomes the usual Hermitian conjugation, and as all other representations are constructed from the definition representation, this holds for the general representation spaces to be used here. Thus, for whatever Hilbert space chosen, the operation * will be understood as Hermitian conjugation for all operators, which evidently agrees with the involution on the algebra.

3.1.2 Commutative Finite Spectral Triples

Finite commutative algebras over the complex numbers are uniquely determined by their dimension. Such an algebra is of the form $A = \mathbb{C}^n$ for some integer n. The finite dimensional representations of the algebra are given by the unique irreducible representation, $\mathbb{C}$ itself, together with a potential multiplicity factor. For A as a whole one has a decomposition into representations of each factor, so that the most general finite dimensional representation of A is given by

$$\mathcal{H} \simeq \bigoplus_{i=1}^{n} \mathbb{C}^{m_i}, \tag{3.2}$$

with m_i the multiplicity of the representation corresponding to the i-th factor. The characters of A are easy to find, as they are simply the projections onto the individual components of an element $a \in A$.

Thus, the spaces represented by commutative finite spectral triples are simply collection of points, n for the algebra $A = \mathbb{C}^n$.

Reality Conditions

To handle real commutative finite spectral triples, treat J as the antiunitary operator intertwining the left and right actions of the commutative algebra A. The Hilbert space needs to be an algebra representation of the combined left and right actions, equivalent to being a representation of the algebra $A \otimes A^{op}$. Using the fact that $(\mathbb{C}^n)^{op} = \mathbb{C}^n$ in the commutative case, the algebra tensor product is simply

$$A \otimes A^{op} = (\mathbb{C}^n)^{op} \otimes \mathbb{C}^n \cong \mathbb{C}^{n^2}. \tag{3.3}$$

As above, such a Hilbert space is simply the direct sum of a representation of each factor of $\mathbb{C}$, each with its own multiplicity. To keep the left and right actions of the original algebra distinguished, each factor will be labelled by the pair (i, j) corresponding to an original basis of A. A representation is then simply

$$\mathcal{H} = \bigoplus_{i,j=1}^{n} \mathcal{H}_{ij} := \bigoplus_{i,j=1}^{n} \mathbb{C}^{m_{ij}}, \tag{3.4}$$

where again m_{ij} corresponds to the multiplicity of the factor. The left action of an element $a = (a_i) \in A$ is given by diagonal matrices containing a_i a total of m_{ij} times, namely

$$L(a) = \sum_i a_i \mathbb{I}_{m_{ij}}. \tag{3.5}$$

The right action meanwhile has a_j a total of m_{ij} times, or

$$R(a) = \sum_j a_j \mathbb{I}_{m_{ij}}. \tag{3.6}$$

This is not the whole story, since the antiunitary operator J must exist to guarantee that one is working with a real spectral triple. Such an operator will have to intertwine the operators $L(a)$ and $R(a^*)$ for all $a \in A$. It is easy to see that for this to be possible, one must have $m_{ij} = m_{ji}$ for all pairs i, j. One can then define the operator J block-wise by using complex-antilinear isomorphisms $J : \mathcal{H}_{ij} \to \mathcal{H}_{ji}$.

It is at this point that discussions concerning commutative algebras impose the requirement that $L(a) = R(a^*)$ for all a in A, essentially restricting the Hilbert space to be diagonal in the labels above. The point of departure of the current approach is that this is not assumed to hold.

Chirality

It is necessary to consider cases where the Hilbert space comes with a **chirality operator**. This requires an operator Γ, satisfying $\Gamma^* = \Gamma$ and $\Gamma^2 = 1$ which guarantee that its only eigenvalues are ± 1. In addition, it is necessary require that $\Gamma L(a) = L(a)\Gamma$ for all $a \in \mathcal{H}$, so that the decomposition of the total Hilbert space into right-handed and left-handed spaces, $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$, is also a decomposition of the space as an algebra representation space.

The situation above is the same in the case one has a real spectral triple, except that in addition compatibility is imposed between the chirality and real structure. Recall that this is encoded by the relation

$$J\Gamma = \epsilon'' \Gamma J \tag{3.7}$$

where $\epsilon'' = \pm 1$. In the positive case the total Hilbert space thus decomposes into mutual eigenspaces of both operators, while in the negative case the real structure maps between the spaces of definite chirality. Thus for the latter case, for example, the real structure J maps a right handed particle to a left-handed anti-particle.

Dirac Operator

Once provided with a finite commutative algebra and its representation as above, it is possible to explicitly determine what form the Dirac operator D must take.

As always, the Dirac operator must be self-adjoint $D = D^*$. Further constraints are obtained using the spectral triple axioms, and involve the real structure J:

- Relation with real structure:

$$DJ = \epsilon' JD \tag{3.8}$$

- First order condition:

$$[[D, L(a)], R(b)] = 0 \qquad \forall a, b \in A \tag{3.9}$$

- In the case of an even spectral triple one requires that

$$D\Gamma = -\Gamma D. \tag{3.10}$$

These heavily constrain the allowed form of the Dirac operator.

Proposition 3.1.1 (Krajewski [Kra98])**.** *Let $D_{ij;kl}$ label the matrix element of the Dirac operator that maps from the space $\mathcal{H}_{kl}$ to $\mathcal{H}_{ij}$. Then one can have $D_{ij;kl} \neq 0$ only if* ***at least one*** *of the conditions $i = k$ or $j = l$ holds.*

Proof. Recall that in the Hilbert space $\mathcal{H}_{ij}$ the action of an element a can be written using summation convention as

$$L(a)_{kl} = a_i \delta_{ik} = a_k, \tag{3.11}$$

and similarly one has $R(a)_{kl} = a_l$. Working with the element $D_{ij;kl}$ one has

$$[D_{ij;kl}, L(a)] = D_{ij;kl} a_k - a_i D_{ij;kl} = (a_k - a_i) D_{ij;kl} \tag{3.12}$$

and with a further commutator action one has

$$[[D_{ij;kl}, L(a)], R(b)] = (a_k - a_i)(b_l - b_j) D_{ij;kl} \tag{3.13}$$

for the first order condition. For this to vanish for all a and b in A, one evidently must have that at least one of the factors in front of $D_{ij;kl}$ vanishes, and the proposition follows. □

Example 4. For the three dimensional commutative algebra, consider the following representation where each factor is one dimensional:

$$\begin{aligned} A =&\mathbb{C}^3 \\ \mathcal{H} =&\mathcal{H}_{12} \oplus \mathcal{H}_{21} \oplus \mathcal{H}_{23} \oplus \mathcal{H}_{32} \coloneqq \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}. \end{aligned}$$

Construct a $KO-$dimension $s = 7$ triple by taking the real structure

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \cdot C, \tag{3.14}$$

where C stands for the complex conjugation operation. Then the constrained Dirac operator takes the form

$$D = \begin{pmatrix} d_1 & 0 & 0 & f \\ 0 & d_1 & \bar{f} & 0 \\ 0 & f & d_2 & 0 \\ \bar{f} & 0 & 0 & d_2 \end{pmatrix} \tag{3.15}$$

where $d_1, d_2 \in \mathbb{R}$ and $f \in \mathbb{C}$.

Example 5. The same algebra and Hilbert space data can be used to calculate the Dirac operator for all (odd) KO-dimensional algebra.

These can be constructed by taking real structure

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \epsilon & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \epsilon & 0 \end{pmatrix} \cdot C, \tag{3.16}$$

so that $J^2 = \epsilon$. This leads to a Dirac operator of the form:

$$D = \begin{pmatrix} d_1 & 0 & 0 & f \\ 0 & \epsilon' d_1 & \epsilon\epsilon' \bar{f} & 0 \\ 0 & \epsilon\epsilon' f & d_2 & 0 \\ \bar{f} & 0 & 0 & \epsilon' d_2 \end{pmatrix}. \tag{3.17}$$

Again, $d_1, d_2 \in \mathbb{R}$ and $f \in \mathbb{C}$. Note that as may be expected, for all KO-dimensions the Dirac operator has the same form and degrees of freedom, with the only difference given by signs determined from table 2.1.

Example 6. The even KO-dimensional cases of the above setting can be easily tackled by firstly construction a chirality operator consistent with the given real structure, and then imposing relation (3.10) on the Dirac operator from the previous example. Using the commutativity of the chirality with the algebra left-action, together with the condition imposed using real structure from the previous example, one finds that

$$\Gamma = \begin{pmatrix} c_1 & 0 & 0 & 0 \\ 0 & \epsilon'' c_1 & 0 & 0 \\ 0 & 0 & c_2 & 0 \\ 0 & 0 & 0 & \epsilon'' c_2 \end{pmatrix} \tag{3.18}$$

where $c_1, c_2 = \pm 1$. Without loss of generality set $c_1 = 1$, and rewrite the chirality operator as

$$\Gamma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \epsilon'' & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \alpha\epsilon'' \end{pmatrix}, \tag{3.19}$$

where similarly to before, $\alpha = \pm 1$. Using this expression for Γ, the constraints on the Dirac operator are then

$$\begin{aligned} d_1 &= 0 \\ (\alpha\epsilon'' + 1) f &= 0 \\ d_2 &= 0. \end{aligned}$$

Immediately one sees that all diagonal Dirac elements vanish. Whether the parameter f remains depends on the KO-dimension together with how Γ was set up. So to make sure the Dirac operator does not vanish, one fixes $\alpha = -\epsilon''$ so that the chosen Chirality operator is

$$\Gamma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \epsilon'' & 0 & 0 \\ 0 & 0 & -\epsilon'' & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{3.20}$$

Geometric Interpretation

Using the above allowable Dirac operators one can develop a geometric picture of the resulting space. Working with a finite dimensional algebra, one always obtains a discrete space. The fundamental points arise as characters of the algebra. In the case of the commutative algebra $\mathcal{A} = \mathbb{C}^n$, the characters are simply the n coordinate functions μ_i

$$\mu_i(a) = a_i \qquad i = 1, 2, ..., n \tag{3.21}$$

for all $a \in \mathcal{A}$, so one simply has an n-point space. Recall that in the formalism of noncommutative geometry, the distance between two points is given in terms of the distance formula (2.8), using the characters associated to those points. If the points are labelled 1 to n, the main constraint on the Dirac operator above implies that two points may only have a finite distance between them if they are both connected to a third point as a nearest neighbour. Note that even in this picture, it is not possible in all cases to localise fermions to points consistently since they correspond to edges in the original diagram. In fact, it might be beneficial to simply talk about the relative distance between two fermions without strict reference to points, thus obtaining a purely relational model. Nonetheless, the following two models to be investigated, the line and the circle, will each separately exhibit the characteristic that the representation diagram has the same shape as the geometric diagram obtained by drawing the geometric distances from the respective Dirac operator. The only anomaly in the line case is that, if one starts with a representation diagram that segments a line using n points, one obtains only $n-2$ distance parameters, which would correspond to the line being segmented using $n-1$ vertices rather than the starting n.

It should be emphasised that evaluating distances using the distance formula (2.8) guarantees that, for any chosen lattice, the resulting distances satisfy the triangle inequality so that the lattice is indeed physical. To see this explicitly, consider for example a sequence of nodes i, j, k, l that are connected linearly via Hilbert spaces, i.e. node i is connected to node j, that is in turn connected also to node k, and so on. Than the analysis above shows that the Dirac operator has two unconstrained complex parameters, say d_1 and d_2, that arise from the fact that the pair of nodes i and k, and the pair of nodes j and l, constitute respectively connected next-nearest neighbours. Then the distance formula evaluated for the pair of nodes i and k yields a distance $d_{ik} = \frac{1}{|d_1|}$, while the same formula evaluated for the pair of nodes j and l yields a distance $d_{jl} = \frac{1}{|d_2|}$. Now considering the pair of nodes i and l, that are connected only as next-next-nearest neighbours, one finds that the distance between them is constrained to be $d_{il} \leq |\frac{1}{d_1} + \frac{1}{d_2}| \leq d_{ik} + d_{jl}$, showing that the triangle inequality is satisfied. This argument naturally generalises to larger lattices, so that one can talk of these lattices without worrying that distances between nodes might be infinite or ill-defined.

3.2 Specific Constructions

3.2.1 Line Model

As a step towards calculating the action for the discretised circle, it would be beneficial to evaluate the Dirac operator and its determinant for the line. It was shown above that geometric distances are calculated between the edges representing the relevant Hilbert space, rather than nodes in the representation diagram. Hence, a system set up with a row of nodes, each pair of neighbours connected via Hilbert space edges, represents a discretised line geometrically as well.

General Dirac Operator

For each dimension n the algebra $\mathcal{A}_n = \mathbb{C}^n$ is fixed with a Hilbert space of the form $\mathcal{H}_n = \mathcal{H}_{12} \oplus \mathcal{H}_{21} \oplus \mathcal{H}_{23} \oplus \mathcal{H}_{32} \oplus \cdots \oplus \mathcal{H}_{n-1n} \oplus \mathcal{H}_{nn-1}$. For now, each factor will be chosen to be one dimensional, so that in total $\mathcal{H}_n \cong \mathbb{C}^{2n-2}$. Then, using a standard basis with respect to the real structure J_n one obtains the matrix

$$J_n = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & & & \cdots & 0 & 0 \\ \epsilon & 0 & 0 & 0 & \cdots & & & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & & & \vdots & \vdots \\ 0 & 0 & \epsilon & 0 & 0 & 0 & & & & \\ \vdots & \vdots & 0 & 0 & 0 & 1 & & & & \\ & & 0 & 0 & \epsilon & 0 & & & \vdots & \vdots \\ & & & & & & \ddots & & 0 & 0 \\ \vdots & \vdots & & & & & & \ddots & 0 & 0 \\ 0 & 0 & \cdots & & & \cdots & 0 & 0 & 0 & 1 \\ 0 & 0 & \cdots & & & \cdots & 0 & 0 & \epsilon & 0 \end{pmatrix} \cdot C, \tag{3.22}$$

where C denotes the operation of complex conjugation.

Proposition 3.2.1. *Given the fermion space associated to the line model given by $(\mathcal{A}_n, \mathcal{H}_n, J_n)$, the Dirac operator D is determined by $n-1$ real parameters d_i and $n-2$ complex parameters f_j. The square root of its determinant can be evaluated as follows.*

Let $\underline{s} = (s_1, s_2, \cdots s_r)$ be a sequence of length r where $s_i = 1, 2$, such that $\sum_{i=1}^{r} s_i = n-1$. Label by m the number of times the number 2 appears in the sequence, and the partial sum up to s_j by $\sigma_j := \sum_{i=1}^{j} s_j$. Then the square root of the determinant of the Dirac operator is given by

$$\sqrt{\det(D_n)} = \left(\sqrt{\epsilon'}\right)^{\frac{1+(-1)^n}{2}} \sum_{\underline{s} \in S} (-\epsilon')^m g_1 g_2 \cdots g_r, \tag{3.23}$$

where S is the set of all sequences s, and g is defined as the function

$$g_j = \begin{cases} |f_{\sigma_j}|^2, & \text{if } s_j = 2 \\ d_{\sigma_j}, & \text{if } s_j = 1. \end{cases} \tag{3.24}$$

Proof. The Dirac operator is constrained much like before. In fact, since a nonvanishing matrix element only exists between two Hilbert spaces that share a node in the representation diagram, one easily finds that the Dirac operator must be block diagonal, extending example 5. Explicitly one has

$$D_n = \begin{pmatrix} d_1 & 0 & 0 & f_1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \epsilon' d_1 & \epsilon\epsilon' \bar{f}_1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \epsilon\epsilon' f_1 & d_2 & 0 & 0 & f_2 & & & \vdots & \vdots \\ \bar{f}_1 & 0 & 0 & \epsilon' d_2 & \epsilon\epsilon' \bar{f}_2 & 0 & \ddots & & & \\ 0 & 0 & 0 & \epsilon\epsilon' f_2 & d_3 & 0 & & & & \\ 0 & 0 & \bar{f}_2 & 0 & 0 & \epsilon' d_3 & & & \vdots & \vdots \\ 0 & 0 & & \ddots & & & \ddots & & 0 & f_{n-2} \\ \vdots & \vdots & & & & & & \ddots & \epsilon\epsilon' \bar{f}_{n-2} & 0 \\ 0 & 0 & \cdots & & & \cdots & 0 & \epsilon\epsilon' f_{n-2} & d_{n-1} & 0 \\ 0 & 0 & \cdots & & & \cdots & \bar{f}_{n-2} & 0 & 0 & \epsilon' d_{n-1} \end{pmatrix}, \tag{3.25}$$

where d_i for $i = 1, 2, ..., n-1$ are real numbers, and f_j for $j = 1, 2, ..., n-2$ are complex. Using induction it is straightforward to calculate the value of the determinant for $n \geq 3$. Setting $X_n :=$

$\sqrt{\det(D_n)}$ one obtains the recurrence relations

$$X_{2k} = \sqrt{\epsilon'} d_{2k-1} X_{2k-1} - \epsilon' |f_{2k-2}|^2 X_{2k-2} \tag{3.26}$$

$$X_{2k+1} = -\sqrt{\epsilon'} d_{2k} X_{2k} + |f_{2k-1}|^2 X_{2k-1}, \tag{3.27}$$

together with the initial values

$$\begin{aligned} X_1 &= 1 \\ X_2 &= \sqrt{\epsilon'} d_1. \end{aligned} \tag{3.28}$$

Note that then

$$det(D_3) = X_3^2 = (|f_1|^2 - \epsilon' d_1 d_2)^2, \tag{3.29}$$

which is what one obtains by explicitly calculating the determinant in example 5 above.

Using these recurrence relations, it is possible to obtain a more or less closed expression for the determinant. Let $Z_{2k} = \frac{1}{\sqrt{\epsilon'}} X_{2k}$. The relations are now

$$\begin{aligned} Z_{2k} &= d_{2k-1} X_{2k-1} - \epsilon' |f_{2k-2}|^2 Z_{2k-2} \\ X_{2k+1} &= -\epsilon' d_{2k} Z_{2k} + |f_{2k-1}|^2 X_{2k-1}. \end{aligned}$$

Considering the determinant function labelled by n (i.e. either Z_{2k} or X_{2k+1}), we see that its dependence on the subsequent functions, those labelled by $n-1$ and $n-2$, is obtained through multiplying by a factor with the same label, d_{n-1} and $|f_{n-2}|^2$ respectively. Moreover, these factors have the same weight in terms of the variables

$\{d_j, |f_k|\}$ as the change in the label itself. Since the same pattern is satisfied by the subsequent functions, it is possible, using hops of 1 or 2 in the label, to follow expansion of the lower functions all the way to an initial value in (3.28). For example, by tracing the sequence of integers $(n, n-1, n-3, n-4, ..., 2, 1)$, where each integer is followed by another that is either 1 or 2 less than it, one finds in the expansion above the monomial $d_{n-1}|f_{n-3}|^2 d_{n-4}...d_2 d_1$ that must exist in the expansion. In fact the determinant functions are exclusively evaluated by taking all such allowable sequences, and associating the corresponding polynomial to each of them by the rule

$$\text{Difference of } 1 \implies d_l$$
$$\text{Difference of } 2 \implies |f_{l-1}|^2.$$

For simplicity one can equivalently track the sequence used as an example above via the sequence of differences, i.e taking $\underline{s} = (1, 1, 2, 1, \ldots, 1)$. Combining all such sequences together with the corresponding factors gives precisely the desired result. □

Chiral Dirac Operator

Much like the real structure matrix and the Dirac operators, extending the chirality operator to a line simply involves repeating the operator from example 6, which corresponds to Γ_3 now, in a block diagonal manner. Recall that in the Γ of (3.20), there was a freedom to fix the chirality of the first pair of Hilbert spaces, while if one wanted a nontrivial Dirac operator the chirality of the second pair was fixed by a factor of $\alpha = -\epsilon''$ with respect to that of the first. Repeating the argument for all consecutive Hilbert space pairs in $\mathcal{H}_n$, one sees that even in this case it is only the first pair that is free to be fixed and the chirality of all the subsequent Hilbert spaces is fixed with respect to it:

$$\Gamma_n = \begin{pmatrix} 1 & 0 & 0 & 0 & & & \\ 0 & \epsilon'' & 0 & 0 & & & \\ 0 & 0 & -\epsilon'' & 0 & 0 & 0 & \\ 0 & 0 & 0 & -1 & 0 & 0 & \\ & & 0 & 0 & 1 & 0 & \ddots \\ & & 0 & 0 & 0 & \epsilon'' & \\ & & & \ddots & & & \ddots \end{pmatrix}, \tag{3.30}$$

where again the first Hilbert space factor was set to be right handed. So long as there is a nontrivial chirality operator for the system, the diagonal elements in of the Dirac operator (3.25) must vanish. With the Γ_n above, this is all that vanishes and the f_i's remain free.

Corollary 3.2.1. *The determinant of the Dirac operator for the chiral line model is given by*

$$det(D_n) = \begin{cases} |f_1|^4 |f_3|^4 \dots |f_{n-2}|^4 & n \text{ is odd.} \\ 0 & n \text{ is even.} \end{cases} \tag{3.31}$$

Proof. The result follows by taking proposition 3.2.1 setting all $d_i =$

0, and squaring the remaining term. □

3.2.2 Circle Model

The circle model can be viewed as an extension of the line model, if one realises that the process of gluing the end nodes of the line are obtained by simply adding an extra pair of Hilbert space factors representing the additional edge.

General Dirac Operator

As before n labels the algebra dimension $\mathcal{A}_n = \mathbb{C}^n$. The Hilbert space now takes the form $\mathcal{H}_n = \mathcal{H}_{12} \oplus \mathcal{H}_{21} \oplus \mathcal{H}_{23} \oplus \mathcal{H}_{32} \oplus \cdots \oplus \mathcal{H}_{n-1n} \oplus \mathcal{H}_{nn-1} \oplus \mathcal{H}_{n1} \oplus \mathcal{H}_{1n}$ which now satisfies $\mathcal{H}_n \cong \mathbb{C}^{2n}$. The real structure has the same form (3.22) with the addition of an extra 2×2 block. The Dirac operator, meanwhile, not only extends diagonally by an additional block, but also by non-vanishing parameters at the two 2×2 blocks at the top right and lower left corners of the matrix. This gives

$$
D_n = \begin{pmatrix}
d_1 & 0 & 0 & f_1 & 0 & 0 & \cdots & 0 & 0 & \epsilon\epsilon' f_n \\
0 & \epsilon' d_1 & \epsilon\epsilon' \bar{f}_1 & 0 & 0 & 0 & \cdots & 0 & \bar{f}_n & 0 \\
0 & \epsilon\epsilon' f_1 & d_2 & 0 & 0 & f_2 & & & 0 & 0 \\
\bar{f}_1 & 0 & 0 & \epsilon' d_2 & \epsilon\epsilon' \bar{f}_2 & 0 & \ddots & & \vdots & \vdots \\
0 & 0 & 0 & \epsilon\epsilon' f_2 & d_3 & 0 & & & & \\
0 & 0 & \bar{f}_2 & 0 & 0 & \epsilon' d_3 & & & \vdots & \vdots \\
\vdots & \vdots & & \ddots & & & \ddots & & 0 & f_{n-1} \\
0 & 0 & & & & & & \ddots & \epsilon\epsilon' \bar{f}_{n-1} & 0 \\
0 & f_n & 0 & \cdots & & \cdots & 0 & \epsilon\epsilon' f_{n-1} & d_n & 0 \\
\epsilon\epsilon' \bar{f}_n & 0 & 0 & \cdots & & \cdots & \bar{f}_{n-1} & 0 & 0 & \epsilon' d_n
\end{pmatrix}. \tag{3.32}
$$

Note that in this form, the matrix is invariant under a cyclic translation along the diagonal together with an associated relabelling of the coefficients, as would be expected.

Chiral Dirac Operator

The considerations arising when introducing a chiral structure are the same as in the line case (with correct number of blocks), aside from one important fact. The chirality of the Hilbert spaces now has to wrap back to the original Hilbert space. For $KO-$dimensions $s = 2, 6$, when $\epsilon'' = -1$, this poses no issues since all Hilbert space pairs have the same relation between chirality and particle/anti-particle labelling. But for $s = 0, 4$, when $\epsilon'' = 1$, consecutive Hilbert space pairs must have opposite chirality. This means that for odd n, when the number of edges is correspondingly odd, this cannot be done consistently along the circle. This implies that so long as one is working with nonvanishing Dirac operators, a system with $KO-$ dimensions $s = 0, 4$ cannot be assigned when n is odd.

Regardless of the above observation, one knows that when a chirality operator and nonvanishing Dirac operator consistently exist, the form of the Dirac operator of (3.32) is constrained only via $d_i = 0$ for all $i = 1, 2, \dots, n$. Since this form does not depend on ϵ'', a calculation of its determinant and possible Pfaffian can be made independent of whether the $KO-$dimension actually prohibits the system when fixing the dimension later on.

Proposition 3.2.2. *The determinant Dirac operator for the Chiral model is*

$$det(D_n) = \begin{cases} (1+\epsilon\epsilon')^2 |f_1 f_2 \dots f_n|^2 & n \text{ is odd.} \\ |f_1 f_3 \dots f_{n-1} - (-\epsilon\epsilon')^{\frac{n}{2}} f_2 f_4 \dots f_n|^4 & n \text{ is even.} \end{cases} \tag{3.33}$$

Proof. This follows a repeated use of the following elementary matrix identity. If the matrix M is decomposed into blocks as

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where A is an invertible matrix, then

$$det(M) = det(A) \cdot det(D - CA^{-1}B).$$

This relation is then repeatedly applied to the case where one takes A to be the top-left 4×4 block of the Dirac operator. □

3.2.3 Fermion Model on the Circle

With the form of the Dirac operator constrained as above for the circle model, one can write down the fermion part of the action as the kinetic term

$$S[D, \phi, \bar{\phi}] = \frac{1}{2}\langle \bar{\phi}, D\phi \rangle, \tag{3.34}$$

using the obvious inner product over the fermion Hilbert space. As is usual in such systems, the field and anti-field degrees of freedom are independent of each other.

For a chiral theory, one is able to halve the degrees of freedom that appear in the action by using the real structure J to map the set of field degree of freedom to the anti-field Hilbert space, giving an action

$$S_{chiral}[D, \phi] = \langle J\phi, D\phi \rangle. \tag{3.35}$$

Since one is now working with the same set of fermionic variables, the action needs to be antisymmetric under the exchange of both sides of the inner product, as happens when one applies the real structure. Recall that one of the conditions of J being a real structure on a (bosonic) Hilbert space $\mathcal{H}$ is that, with respect to the inner product, $\langle a, b \rangle = \langle Jb, Ja \rangle$ for all $a, b \in \mathcal{H}$. Since one now has a fermionic space, the exchange requires that

$$\langle J\phi, D\phi \rangle = -\langle JD\phi, J^2\phi \rangle = -\epsilon\epsilon' \langle DJ\phi, \phi \rangle = -\epsilon\epsilon' \langle J\phi, D\phi \rangle, \tag{3.36}$$

implying that a chiral fermionic theory can only be written in a $KO-$dimension for which $\epsilon\epsilon' = -1$. This is the case only for $s = 2, 4$. Assuming from now on that such a $KO-$dimension has been picked, the action functional S_{chiral}, which is bilinear in ϕ, can be written as an antisymmetric matrix. Explicitly, the total Hilbert space can be divided into field and antifield variables $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$, for example $\mathcal{H}^+ = \mathcal{H}_{12} \oplus \mathcal{H}_{23} \oplus \cdots \oplus \mathcal{H}_{n1}$ and $\mathcal{H}^- = \mathcal{H}_{21} \oplus \mathcal{H}_{32} \oplus \cdots \oplus \mathcal{H}_{1n}$. If $\{e_i\}$ is a basis for $\mathcal{H}^+$, then $\{Je_i\}$ serves as a basis for $\mathcal{H}^-$. Most importantly, the bilinear in S_{Chiral} can be written as

$$\langle J\phi, D\phi \rangle = \phi^T \cdot M \cdot \phi, \qquad (3.37)$$

where M is the antisymmetric matrix defined by $M_{ij} = \langle Je_i, De_j \rangle$. It is basis dependent, but the action is basis independent. Indeed, using the above action one can integrate out the Fermionic degrees of freedom to obtain

$$\int D\phi e^{-S_{Chiral}[\phi, D]} = pf(-M), \qquad (3.38)$$

since one is carrying out Berezin integration using a Euclidean path integral. The Pfaffian can then be integrated with respect to the Dirac degrees of freedom to obtain the partition function of the system.

To evaluate the Pfaffian explicitly, label the real structure operator of 3.22 as $J_n = N \cdot C$, where again C is the complex conjugation operation and N is the matrix factor in front of it in the formula. Explicitly C is defined with respect to the given basis above, acting trivially on the basis and extended by complex anti-linearity to give $C\alpha e_j = \bar{\alpha} e_j$. The matrix N in this chosen basis is the complex linear part of the transformation. Note that with respect to Hermitian conjugation $N^* = \epsilon N$ so that N is Hermitian or anti-Hermitian depending on the value of ϵ. This allows the matrix M to be written in this basis as $M = \epsilon N D$. For the $n-$dimensional algebra one then obtains the matrix

$$M_n = \begin{pmatrix} 0 & 0 & -\epsilon f_1 & 0 & 0 & 0 & \cdots & 0 & \epsilon f_n & 0 \\ 0 & 0 & 0 & \bar{f}_1 & 0 & 0 & \cdots & 0 & 0 & -\bar{f}_n \\ \epsilon f_1 & 0 & 0 & 0 & -\epsilon f_2 & 0 & & & 0 & 0 \\ 0 & -\bar{f}_1 & 0 & 0 & 0 & \bar{f}_2 & & & \vdots & \vdots \\ 0 & 0 & \epsilon f_2 & 0 & 0 & 0 & \ddots & & & \\ 0 & 0 & 0 & -\bar{f}_2 & 0 & 0 & & \ddots & \vdots & \vdots \\ \vdots & \vdots & & & \ddots & & \ddots & & -\epsilon f_{n-1} & 0 \\ 0 & 0 & & & & \ddots & & 0 & 0 & \bar{f}_{n-1} \\ -\epsilon f_n & 0 & 0 & \cdots & & \cdots & \epsilon f_{n-1} & 0 & 0 & 0 \\ 0 & \bar{f}_n & 0 & \cdots & & \cdots & 0 & -\bar{f}_{n-1} & 0 & 0 \end{pmatrix}, \tag{3.39}$$

which is indeed antisymmetric. Calculating its Pfaffian gives

$$pf(M_n) = \begin{cases} 0 & n \text{ is odd.} \\ \epsilon^{\frac{n}{2}} |f_1 f_3 \ldots f_{n-1} - f_2 f_4 \ldots f_n|^2 & n \text{ is even.} \end{cases} \tag{3.40}$$

Note that in accordance with the determinant calculations above $pf(M_n)^2 = det(D_n)$. The Dirac path integral itself then gets a factor of $pf(-M_n) = (-1)^n pf(M_n)$ in the integrand, with a similar phase factor of i^n if one is evaluating a Lorentzian path integral.

3.2.4 $U(1)$ Gauge Model

It is possible to interpret the discretised circle as a lattice $U(1)$-gauge model. In such a setting, the Dirac operator corresponds to the gauge potential coupled to the Fermion fields. To see how this works, restrict the variables of D to the case where for each i, $|f_i| = 1$, so that one can think of each step taken clockwise along the circle contributing a phase factor of $f_j = e^{i\theta_j}$ for field variables and a factor $-\bar{f}_j = -e^{-i\theta_j}$ for anti-field variables. Returning to the starting point along the circle one obtains a holonomy factor $e^{i\Theta} := (-1)^{n/2} e^{\sum_{j=1}^{\frac{n}{2}}(\theta_{2j}-\theta_{2j-1})} = (-1)^{\frac{n}{2}} f_1\bar{f}_2 f_3\bar{f}_4 \dots f_{n-1}\bar{f}_n$. Thus, when working with $KO-$dimension $s = 2, 4$, and considering even n models, one finds that

$$pf(-M_n) = |1 - f_1\bar{f}_2 f_3\bar{f}_4 \dots \bar{f}_n|^2 = (1-e^{i\Theta})(1-e^{-i\Theta}) = 2(1-\cos\Theta). \tag{3.41}$$

This can be compared with the result in [BKL13], where a single set of one dimensional fermions localised at vertices along a discretised circled contributed a factor of $(1 - e^{i\Theta})$, where again Θ is the holonomy around the circle of a $U(1)$ gauge connection. The doubling in (3.41) may be attributed that this model has a doubling of the fermionic fields, each original having an anti-field partner.

Chapter 4

Commutative Constructions

This chapter sets down the necessary geometric background needed to construct the geometry that will be approximated by the fuzzy complex projective space $\mathbb{C}P(2)$. As such all results are essentially already known in the literature. Firstly, in section 4.1 it will be explained why in general it is expected that fuzzy spaces with a Lie symmetry group may only approximate coadjoint orbits. The space $\mathbb{C}P(2)$ is indeed the next simplest example of such a space, after the 2-sphere, although it is not spin. Luckily, section 4.2 will present several properties of coadjoint orbits, the important of which for the discussion of spinors is that they are Kähler. This guarantees that coadjoint orbits, including $\mathbb{C}P(2)$ itself, are at least spin$^{\mathbb{C}}$. Section 4.3 will then develop explicitly the necessary geometry of $\mathbb{C}P(2)$ and its complex spinor bundles in terms of representation theory. This includes writing down the spectrum of the Dirac operator. For the benefit of the reader, the end of the chapter contains an appendix where the necessary group-theoretic terminology is recalled.

4.1 Motivation

The definition of a fuzzy space presented in chapter 2 was developed hand in hand with its use in formalising the structure of the fuzzy sphere. In an effort to extend the usage of the formalism to other geometries it is natural to focus on spaces with higher symmetry, since their properties are usually easier to compute, and subsequently extremely useful for physical applications. In fact, the geometry of the sphere itself, including most importantly the spectrum of the Dirac operator, can be most easily computed using representation theory. And the actual calculations of the fuzzy sphere, including the spectrum of its respective Dirac operator, depended at their core on the ability to utilise representation theory. It thus seems natural to search for other examples of fuzzy spaces that correspond to homogeneous spaces.

Symmetries of real spectral triples were mentioned in chapter 2, where they were defined in terms of transformations of real spectral triples that keep the Dirac operator invariant. Such transformations consist of an automorphism of the algebra, either inner or outer, together with a unitary transformation on the Hilbert space. In the case of a fuzzy space, where the algebra is taken to be a simple matrix algebra, the Skolem-Noether theorem implies any transformation on the matrix algebra is inner. Thus, if one wishes to construct a fuzzy space $(\mathcal{A}, \mathcal{H}, D, J, \Gamma)$ together with a group of symmetries G, the group action on the algebra $\mathcal{A} = M_n(\mathbb{C})$ will rely on a representation $U : G \to GL_n(\mathbb{C})$ to define the action $\rho : G \to Aut(\mathcal{A})$

via the action by conjugation

$$\rho(g)(a) = U(g) \cdot a \cdot U(g^{-1})$$

for all g in G and a in $\mathcal{A}$. Recall that both the norm on the algebra $\mathcal{A}$ and the inner product on the Hilbert space $\mathcal{H}$ are obtained from the inner product $\langle a, b\rangle = Tr[a^*b]$ on the algebra. For the group action to be an isometry, its representation must obviously then be unitary. Moreover, since our aim is to approximate commutative spaces that are closed differential manifolds with a Lie group of transformations, the group G will usually be taken to be a real and compact Lie group.

Fixing a Lie group G, let V be the carrier space of a unitary representation U as introduced above, and let V^* be its dual space. V^* is acted upon by G using the dual representation $\bar{U}$. Since $M_n(\mathbb{C}) \cong V \otimes V^*$ as vector spaces, the action of G defined on the matrix algebra is evidently given by the representation $U \otimes \bar{U}$. This action can now be factored as

$$G \xrightarrow{\Delta} G \times G \xrightarrow{U \otimes \bar{U}} End(\mathcal{A}), \tag{4.1}$$

where Δ is the diagonal mapping sending g in G to (g, g). Thus the action of G on the matrix algebra may be viewed as the representation $U \otimes \bar{U}$ of $G \times G$ restricted along the diagonal. This straightforward change of view will in fact be useful in showing that the natural homogeneous spaces that lend themselves to be approximated matrix geometries are coadjoint orbits.

Theorem 4.1.1 ([Bar22]). *Let $\{M_{n_i}(\mathbb{C})\}_{i\in\mathbb{N}}$ be a sequence of matrix algebras, where each algebra is acted upon by a semisimple Lie group G via matrix conjugation. Moreover, assume that the sequence converges to the algebra $C_0(X)$ of continuous functions on some homogeneous G-space X. Then X is a coadjoint orbit.*

The proof relies on results obtained by Dooley and Rice[DR85] on the contraction of semisimple Lie groups. These will now briefly be described.

Let G be a semisimple Lie group, and let K be a subgroup such that G/K is reductive. Additionally it will be required that the pair (G, K) be a Riemannian symmetric pair[1]. Then the Lie algebra $\mathfrak{g}$ can be decomposed into a direct sum $\mathfrak{g} = \mathfrak{k} \oplus V$, where $\mathfrak{k}$ is of course the Lie algebra of subgroup K. The adjoint action of K on subspace V turns the latter into a representation space of K, since by definition $Ad(K)(V) \subseteq V$. Using this fact one can form the semidirect product group $V \rtimes K$, which can be viewed as the group generated by elements k of K and v of V subject to the relations $kvk^{-1} = Ad(k) \cdot v$. The theory of irreducible representations of this group can be understood using Mackey's theory[Mac75] to be determined using the following prescription. Let $\mathfrak{a} \subseteq V$ be a maximal abelian subalgebra of V, namely, a subspace of V that also closes as an abelian subalgebra that cannot be extended inside V itself. Its irreducible representations are determined by one forms $\psi \in \mathfrak{a}^*$. Globally, on the Lie group $A = \exp(\mathfrak{a})$ this determines a character $e^{i\psi}$. By setting ψ to be zero on the complement subspace to $\mathfrak{a}$, it

[1] Unless otherwise stated, all symmetric pairs are assumed to be of compact type.

can be extended to the entirety of V. Meanwhile, the group K can be defined to act on linear functions on V as $(k \cdot \psi)(v) \equiv \psi(k^{-1}vk)$. Now let $K_\psi \subseteq K$ be the stabiliser group of ψ under this action. Letting η be any irreducible representation of K_ψ on space H_η, the representation $e^{i\psi} \otimes \eta$, defined on $V \rtimes K_\psi$ as

$$(e^{i\psi} \otimes \eta)(vk) = e^{i\psi(v)}\eta(k), \tag{4.2}$$

is in fact well defined and irreducible. Since $V \rtimes K_\psi \subseteq V \rtimes K$, one may form the induced representation $\rho_{\psi,\eta}$ on the latter group and in fact this representation is also irreducible.

Consider now the irreducible representations of G. One may find a Cartan subalgebra $\mathfrak{t}$ compatible with the Lie subalgebra $\mathfrak{a}$ introduced above, in the sense that it can be written as $\mathfrak{t} = \mathfrak{t}_1 \oplus \mathfrak{a}$. Here $\mathfrak{t}_1$ is a Cartan subalgebra of the centraliser $\mathfrak{m}$ of $\mathfrak{a}$ in $\mathfrak{k}$. Let as above ψ define an irreducible representation of $\mathfrak{a}$, and let η denote (by abuse of notation) both an irreducible representation of $\mathfrak{m}$ and of the associated Lie group $M = \exp(\mathfrak{m})$. Then one can form the representation $e^{i\psi} \otimes \eta$ of the Lie group $MA \subseteq G$. Yet again, constructing the induced representation on G yields an irreducible representation, at least whenever the result is nonzero. The condition for the representation not to vanish is that ψ be a dominant integral weight in V^*. Denote such an induced representation by $\sigma_{\psi,\eta}$. Note that regardless of the chosen ψ, one has that $M \subseteq K_\psi$, and so by restriction one can view η as a representation of M and use it to define the representation $\sigma_{\psi,\eta}$ of G.

Theorem 4.1.2 (Dooley and Rice [DR85]). *Let ψ be an element of $\mathfrak{a}^*$ that is dominant and integral, and fix a representation η of K_ψ such that the irreducible representation $\rho_{\psi,\eta}$ exists and is well defined. Fix η, and define a sequence $\{\sigma_{\phi_n,\eta}\}_{n\in\mathbb{N}}$ of irreducible representations of G, such that (1) the sequence is unbounded in $\mathfrak{a}^*$ and (2) it converges asymptotically in the large n limit to the (positive side of the) line spanned by ψ. Then*

1. *The representation space of $\sigma_{\phi_n,\eta}$ converges to that of $\rho_{\psi,\eta}$.*
2. *The action of G via the representation $\sigma_{\phi_n,\eta}$, composed with the sequence of maps defining the contraction from G to $V \rtimes K$, converges to the action of $V \rtimes K$ via the representation $\rho_{\psi,\eta}$ on the same space.*

Note that, with the first statement, both representation spaces are almost always infinite dimensional separable Hilbert spaces, so long as they don't vanish. Hence the statement essentially implies that the limiting behaviour of the sequence of representations $\sigma_{\phi_n,\eta}$ is such that its representation space indeed does not vanish above a certain value of n.

Proof of Theorem 4.1.1. Consider the product Lie group $G' = G \times G$. It is also semisimple with Lie algebra isomorphic to $\mathfrak{g}' = \mathfrak{g} \oplus \mathfrak{g}$. The group-level Diagonal map introduced above comes with a similar diagonal map acting on the Lie algebra-level as

$$\begin{aligned} \Delta : \quad & \mathfrak{g} \to \mathfrak{g} \oplus \mathfrak{g} \\ & X \mapsto (X, X), \end{aligned}$$

together with a complementary anti-diagonal map

$$\begin{aligned}\nabla: \quad \mathfrak{g} &\to \mathfrak{g} \oplus \mathfrak{g} \\ X &\mapsto (X, -X).\end{aligned}$$

Let $\mathfrak{d}$ and $\mathfrak{a}$ respectively label the images of the diagonal and the anti-diagonal maps. Clearly there is a vector space decomposition $\mathfrak{g}' \cong \mathfrak{d} \oplus \mathfrak{v}$. But only $\mathfrak{d}$ is a Lie-subalgebra of $\mathfrak{g}'$, generating the subgroup $D = \Delta(G)$ that is the image the diagonal mapping on the group-level. In fact, one has

$$\begin{aligned}&[\mathfrak{d}, \mathfrak{d}] \subseteq \mathfrak{d}, \qquad [\mathfrak{d}, \mathfrak{v}] \subseteq \mathfrak{v}, \\ &[\mathfrak{v}, \mathfrak{v}] \subseteq \mathfrak{d},\end{aligned} \tag{4.3}$$

showing that $(G \times G, D)$ is a Riemannian symmetric pair.

First note that if $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$, then $\mathfrak{a} := \nabla(\mathfrak{t})$ is clearly a maximal commuting subalgebra of $\mathfrak{v}$ as well. The linear forms $\psi \in \mathfrak{a}^*$ then serve also as weights for $\mathfrak{g}$. Together with a representation η of the stabiliser group $\mathrm{Stab}(\psi)$, the induced representation $\rho_{\psi,\eta}$ consists explicitly of functions $f \in L^2(G, H_\eta)$ satisfying

$$f(xvk) = e^{-i\psi(v)}\eta(k^{-1}) \cdot f(x), \tag{4.4}$$

for all x in $\mathfrak{v} \rtimes G$ and vk in $\mathfrak{v} \rtimes K_\psi$. In fact, since in general $x = ug$ for some u in $\mathfrak{v}$ and g in G, one may use the semidirect product relation to write

$$f(x) = f(ug) = f(gAd(g^{-1})u) = e^{-i\psi(Ad(g^{-1})u)}f(g), \tag{4.5}$$

showing that any f satisfying (4.4) is fully determined by its restriction $\hat{f}$ to G, that satisfies

$$\hat{f}(gk) = \eta(k^{-1}) \cdot \hat{f}(g) \tag{4.6}$$

for all g in G and k in $\mathrm{Stab}(\psi)$. Such functions are precisely sections of the vector bundle over the coadjoint orbit $\mathcal{C}_\psi$ induced using the representation η. The important case where η is chosen to be the trivial representation 0 yields, upon completion of the representation space, the space of L^2-functions on $\mathcal{C}_\psi$. Next consider the representations $\sigma_{\psi_i,0}$. It was noted that for the representations to not vanish, ψ_i must be a dominant integral weight, which, since $\mathfrak{a}$ corresponds to a Cartan subalgebra of G, means that ψ must be dominant and integral with respect to the original Lie algebra $\mathfrak{g}$ and so it corresponds to a unitary irreducible representation $(\rho_{\psi_i}, V_{\psi_i})$ of G. Now, the way the anti-diagonal mapping sits inside $\mathfrak{g}'$ implies that $\sigma_{\psi,0}$ acts as the map $X \mapsto (\rho_{\psi_i}(X), -\rho_{\psi_i}(X))$. Since ρ_ψ is unitary, one has $-\rho_\psi(X) = \bar{\rho}_{\psi_i}(X)$ giving the dual representation $\left(\bar{\rho}_\psi, V^*_{\psi_i}\right)$. The matrix representation of $G \times G$ is now obtained by exponentiation, and is given by $\left((\rho_{\psi_i}, \bar{\rho}_{\psi_i}), V_{\psi_i} \otimes V^*_{\psi_i}\right)$. Once this is pulled back using the diagonal map to G, one gets the tensor product representation $\rho_{\psi_i} \otimes \bar{\rho}_{\psi_i}$ acting on the space $V_{\psi_i} \otimes V^*_{\psi_i} \cong End(V_{\psi_i})$, namely, the space of matrices of dimension $n = dim(V_{\psi_i})$ acted upon by conjugation.

Finally, with the representations identified, one can apply Theorem 4.1.2 to immediately prove the theorem. □

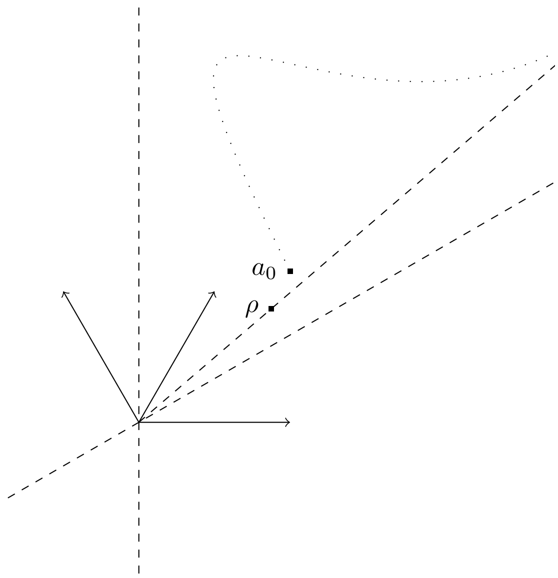

Figure 4.1: The setting of the convergence of Theorem 4.1.1 for the group SU(3). The sequence of representations $\{a_i\}$ converges asymptotically to the line spanned by the positive weight ρ. The theorem guarantees that the sequence of representations $\{a_i \otimes a_i^*\}$ converges to the function space $L^2(\mathcal{C}_\rho)$.

4.2 Geometric Ingredients

4.2.1 Coadjoint Orbits as Homogeneous Spaces

Let G be a semisimple Lie group with Lie algebra $\mathfrak{g}$, and fix a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$. Recall that a coadjoint orbit, as the name suggests, is a subspace $\mathcal{C} \subset \mathfrak{g}^*$ that is generated by the coadjoint group action. The latter group action $Ad^* : G \to \mathrm{End}(\mathfrak{g}^*)$ is defined using the representation dual to the adjoint action, namely, if $\mu \in \mathfrak{g}^*$ then the coadjoint action is defined as the linear transformation

$$\langle Ad^*(\mu), X\rangle \equiv \langle \mu, Ad_{g^{-1}}(X)\rangle, \tag{4.7}$$

for all X in $\mathfrak{g}$ and g in G. Coadjoint orbits are evidently homogeneous spaces of G, and they can be parameterised as follows. A well known result from the theory[Hel79] is that any coadjoint orbit intersects the positive Weyl chamber at one point, say at λ. Denote the resulting coadjoint orbit by $\mathcal{C}_\lambda$. Then evidently $\mathcal{C}_\lambda \cong G/Stab(\lambda)$, where $Stab(\lambda) = \left\{g \in G | Ad^*_g(\lambda) = 0\right\}$ is the stabiliser subgroup of λ in G.

Since G is taken to be semisimple, the nondegeneracy of the Killing form allows one to identify the coadjoint action with the adjoint one on $\mathfrak{g}$. This gives the following useful way to characterise the stabiliser of a weight $\lambda \in \mathfrak{g}^*$ using its dual element H_μ in $\mathfrak{g}$, obtained using the Killing form.

Proposition 4.2.1. *For a semisimple Lie group G, the stabiliser of the coadjoint orbit $\mathcal{C}_\mu$ is generated by the Lie algebra elements that* *are orthogonal to H_μ.*

Proof. Since G is semisimple, its Killing form can be used to identify $\mathfrak{h} \cong \mathfrak{h}^*$. This allows one to speak for each weight ω of the associated element $H_\omega \in \mathfrak{h}$. A previous result has shown that if $Z_\mathfrak{g}(H_\omega)$ is the centraliser of H_ω in $\mathfrak{g}$ with respect to the Lie bracket, then $Stab(\omega) = exp\,(Z_\mathfrak{h}(H_\omega))$. This for one implies that if $\mathcal{T}$ is the torus subgroup generated by $\mathfrak{h}$, then in general $\mathcal{T} \subset Stab(\omega)$. Now, consider the decomposition of the Lie algebra into root spaces $\mathfrak{g} = \mathfrak{h} \bigoplus_{\alpha\in\Delta} \mathbb{C}E_\alpha$, where for any root $\alpha \in \mathfrak{h}^*$ the associated element $E_\alpha \in \mathfrak{g}$ satisfies $[H, E_\alpha] = \alpha(H)E_\alpha$ for all $H \in \mathfrak{h}$. In particular to the case above, $E_\alpha \in Z_\mathfrak{g}(H_\omega)$ if and only if $\langle\alpha, \omega\rangle = \alpha(H_\omega) = 0$. Now the fundamental weights, which serve as the basis for the labelling of weights introduced above, form a basis in $\mathfrak{h}^*$ dual to the basis $\{\frac{2\alpha_i}{\langle\alpha_i,\alpha_i\rangle}\}$ of coroots with $i = 1, 2, \ldots, r$. This is with respect to the inner product induced by the restriction of the Killing form. Thus, if a weight ω has component $p_i = 0$, then $\langle\alpha_i, \omega\rangle = 0$. □

Let G be a Lie group, and let H be a Lie subgroup of G. There are two important procedures to obtain a representation of one of these groups from a representation of the other.

Given a representation μ of G, it is straightforward to obtain a representation of H by considering the group action restricted to H. This representation is the *restricted representation* of μ, and it will be denoted by $\mathrm{Res}^G_H(\mu)$. Note that in general, even if μ denotes an irreducible representation of G, the resulting restricted representation is not necessarily an irreducible representation of H.

The process of going the other way, obtaining a representation of

G from a representation λ of H, is known as *induction.* Explicitly, let $\rho_\lambda : H \to \mathrm{End}(V_\lambda)$ be the representation map of H on vector space V_λ. Then the induced representation obtained from $(\rho_\lambda, V_\lambda)$ is defined as the collection of functions $f : G \to V_\lambda$ satisfying the condition

$$f(gh^{-1}) = \rho_\lambda(h) \cdot f(g), \tag{4.8}$$

where g is an element of G and h an element of H. As for the general case of functions over G, the group action is defined by the left-regular representation, restricted to the class of functions satisfying the above condition. The resultant induced representation will be denoted by $\mathrm{Ind}_H^G(\lambda)$.

Let Rep_G be the category of representations of the group G. The two operations Ind_H^G and Res_H^G are functors between the categories Rep_G and Rep_H. There in fact exists an adjunction between them, with Ind_H^G being left-adjoint to Res_H^G. This is known as the *Frobenius Reciprocity.*

Theorem 4.2.1 (Frobenius Reciprocity)**.** *Let μ be a representation of G, and λ be a representation of H. Then there exists an adjunction between the functors Ind_H^G and Res_H^G. In terms of maps between group representations, one has the isomorphism*

$$\mathrm{Hom}_G(\mathrm{Ind}_H^G(\lambda), \mu) \cong \mathrm{Hom}_H(\lambda, \mathrm{Res}_H^G(\mu)). \tag{4.9}$$

For a proof of Frobenius reciprocity, see for example [Kna96]. In practical terms the reciprocity is useful when one wants to decompose a representation obtained using one of the two functors into irreducible representations. For example, if λ is a representation of H and one wishes to decompose $\alpha = \mathrm{Ind}_H^G(\lambda)$ in terms of irreducible representations of G, then by definition the multiplicity of an irreducible representation ν in α is given by $|\mathrm{Hom}_G(\alpha, \nu)| = |\mathrm{Hom}_G(\nu, \alpha)|$. One can then use the reciprocity to translate the problem into an equivalent one, in this case finding what irreducible representations ν of G contain λ once restricted to a representation of H, and with what multiplicities.

In the following, when it is clear from the context what the groups are, they may be dropped from the notation of the induction and restriction functors, which will thus be denoted Ind and Res, respectively.

4.2.2 spin$^{\mathbb{C}}$ structures

A nice property of coadjoint orbits is that they are always spin$^{\mathbb{C}}$, meaning that one can construct spinor fields compatible with their metric structure. This is related to the fact that they are Kähler.

Definition 4.2.1. A **Kähler Manifold** (M, g, J) is a Riemannian manifold together with an integrable almost-complex structure given by the operator $J : TM \to TM$, that thus turns M into a complex manifold. As always, $J^2 = -1$ and the operator J is additionally taken to be compatible with the Riemannian metric g, in the sense that

$$g(JX, JY) = g(X, Y), \tag{4.10}$$

for all vector fields X, Y.

Associated to any Kähler manifold is a closed, nondegenerate 2-form Ω defined by

$$\Omega(X,Y) \coloneqq g(X,JY), \tag{4.11}$$

which thus turns M into a symplectic manifold as well.

An important fact about the geometry of coadjoint orbits that will be used extensively later on is the following.

Proposition 4.2.2. *Coadjoint orbits are Kähler manifolds.*

Proof. If $\mathcal{C}$ is a coadjoint orbit of the Lie algebra $\mathfrak{g}$, it has a natural Riemannian structure obtained by restricting the inner product on $\mathfrak{g}^*$ induced by the Killing form. A well known fact from the theory is that $\mathcal{C}$ also comes with a natural symplectic form compatible with the Riemannian metric, known as the *Kirillov-Kostant-Souriau form*. These guarantee $\mathcal{C}$ is a Kähler manifold. For more details see [Kir04]. □

Recall now the definition of the group $\mathrm{spin}^{\mathbb{C}}(n)$. There are two common ways in which it is defined[ABS64]. The first one uses the theory of complex Clifford algebras to realise it as a subgroup of the Clifford group. This is the original motivation for the group, that can be seen as a complex version of the usual $spin(n)$ group.

A more direct definition consists of defining $spin^{\mathbb{C}}(n)$ as the quotient group

$$Spin^{\mathbb{C}}(n) \coloneqq (spin(n) \times U(1))/\mathbb{Z}_2, \tag{4.12}$$

using the identification $(\mathbb{I}_n, 1) \sim (-\mathbb{I}_n, -1)$.

As a consequence of the first definition, one can use the algebra representations of $\mathbb{C}l(n)$ to restrict to a *spin*$^{\mathbb{C}}(n)$ group representation. This gives a representation of $\mathrm{spin}^{\mathbb{C}}(n)$ of complex dimension $2^{\lfloor \frac{n}{2} \rfloor}$, the same dimension as the real dimension of the irreducible *spin* group representations. It will be termed the *complex spinor* representation. Such spinors are commonly known Dirac spinors.

It is evident from the second definition that, much like the group $\mathrm{spin}(n)$ is the double cover of the group $\mathrm{SO}(n)$ via a covering map $\xi : \mathrm{spin}(n) \to \mathrm{SO}(n)$, the group $\mathrm{spin}^{\mathbb{C}}$ is the double cover of the group $\mathrm{SO}(n) \times \mathrm{U}(1)$ via the map

$$\begin{aligned} \lambda : \quad & \mathrm{spin}^{\mathbb{C}}(n) \to \mathrm{SO}(n) \times U(1) \\ & [m, \alpha] \mapsto (\xi(m), \alpha^2). \end{aligned} \tag{4.13}$$

When discussing complex spinors on a Riemannian manifold of dimension n, the group $\mathrm{spin}^{\mathbb{C}}(n)$ takes up a role similar to the one the group $\mathrm{spin}(n)$ takes up in the case of (real) spinors. This relies on the definition of a $\mathrm{spin}^{\mathbb{C}}$-structure. Let M be an oriented Riemannian manifold of dimension n, and let F be the principal $\mathrm{SO}(n)$-bundle consisting of the set of frames at each point that are orthogonal with respect to the metric (and compatible with the orientation). This bundle is commonly known as the (oriented) *orthonormal frame bundle*. Then a $\mathrm{spin}^{\mathbb{C}}$ structure is defined as follows.

Definition 4.2.2. A *spin*$^{\mathbb{C}}$ *structure* over a Riemannian manifold M is a principal $\mathrm{spin}^{\mathbb{C}}(n)$-bundle P over M together with a principal $\mathrm{U}(1)$-bundle Q, also over M, and a bundle map $\Lambda : P \to F \times Q$ that over each point x of M agrees with the covering group map λ. It is thus a double-covering map, and it is assumed to be compatible with the right-group actions, meaning that the diagram

$$\begin{array}{ccc} P & \xrightarrow{\times \mathrm{spin}^{\mathbb{C}}(n)} & P \\ \downarrow{\scriptstyle \Lambda} & & \downarrow{\scriptstyle \Lambda} \\ F \times Q & \xrightarrow{\times (\mathrm{SO}(n) \times \mathrm{U}(1))} & F \times Q \end{array}$$

commutes.

While the existence of a spin$^{\mathbb{C}}$-structure on a manifold M is a weaker condition than the existence of a spin-structure, the definition is still quite restrictive. Not all manifolds possess a spin$^{\mathbb{C}}$-structure; a manifold that does is said to be spin$^{\mathbb{C}}$.

Example 7. Any spin-manifold is automatically spin$^{\mathbb{C}}$. To see this consider a spin-manifold M. Then any spin-structure P can be canonically extended to a spin$^{\mathbb{C}}$-structure by taking the trivial U(1)-bundle Q on M and forming the fibre-product bundle $\tilde{P} := P \times_{\mathbb{Z}_2} Q$.

Example 8. Any almost-complex orientable Riemannian manifold M is spin$^{\mathbb{C}}$. This is because the existence of an almost-complex structure automatically guarantees the vanishing of the Dixmier-Douady class. If J is an almost complex structure compatible with the metric structure of M, one can associate a spin$^{\mathbb{C}}$ structure to it as follows. The existence of J allows one to reduce the orthogonal frame bundle F to the principal $\mathrm{U}(n)$-bundle of frames compatible with J. Using the determinant map $\det : \mathrm{U}(n) \to \mathrm{U}(1)$, one obtains a principal $U(1)$-bundle Q. Choosing this bundle to define the spin$^{\mathbb{C}}$-structure, the resulting bundle is known as the **canonical spin$^{\mathbb{C}}$ structure**. Note that the complex line bundle over M that is associated to Q is the (complex) determinant line bundle over M.

Explicitly, the spin$^{\mathbb{C}}$-structure is defined locally using the representation map of $\mathrm{U}(n)$. The vanishing of the Dixmier-Douady class precisely guarantees that this bundle globalises.

If one were to use the inverse determinant map $\det^{-1} : \mathrm{U}(n) \to \mathrm{U}(1)$ to construct a spin$^{\mathbb{C}}$-structure as above, one would obtain the **anti-canonical spin$^{\mathbb{C}}$-structure**.

Kähler manifolds are by construction almost-complex manifolds. Thus, as a consequence of Proposition 4.2.2 all coadjoint orbits are spin$^{\mathbb{C}}$-manifolds. In particular, the canonical spin$^{\mathbb{C}}$-structure construction of Example 8 will be extremely important in the following sections.

From the comments above about the representation theory of the group spin$^{\mathbb{C}}(n)$, there is for a fixed spin$^{\mathbb{C}}$-structure a representation of complex spinors obtained from the irreducible representation of the Clifford algebra in which spin$^{\mathbb{C}}$ sits. The vector bundle associated to the spin$^{\mathbb{C}}$-structure using this representation will be denoted the *bundle of complex spinors* on the manifold. In the case of even-dimensional manifolds, which includes the case of Kähler manifolds, such a bundle can be decomposed into two subbundles, corresponding to complex spinor bundles of definite chirality, also known as bundles of half-spinors, or Weyl spinors. One thus has a direct sum $S = S^+ \oplus S^-$. A chirality operator Γ is then defined to be equal to $+1$ when acting on the subbundle of right-handed spinors, S^+, and to -1 when acting on the bundle of left-handed spinors, S^-.

Recall that in the case of a Riemannian manifold with a spin-structure, the Levi-Civita connection ∇ can be lifted to a connection on the spinor bundle. Let ω denote the connection form of the Levi-Civita connection, so that for vector fields X, Y one has

$$\nabla_X(Y) = X(Y) + \omega(X) \cdot Y, \tag{4.14}$$

where $X(Y)$ is defined locally in a coordinate basis using the component-wise derivative, taken with respect to the vector field $X = X^i \partial_i$. Then the Levi-Civita connection $\tilde{\nabla}$ on some spinor bundle S is defined in such a way that for any vector field X and spinor field Ψ one has

$$\tilde{\nabla}_X(\Psi) = X(\Psi) + \frac{1}{2}\tilde{\omega}(X) \cdot \Psi, \tag{4.15}$$

where $X(\Psi)$ is again defined in terms of the component-wise vector derivative. Note that $\frac{1}{2}\tilde{\omega} := A \circ \omega$ is the the composition of the connection form with the usual Lie algebra isomorphism $A : \mathfrak{so}(n) \xrightarrow{\cong} \mathfrak{spin}(n)$, and so uses the Clifford algebra action by γ-matrices.

A connection on a bundle of *complex* spinors can be defined as a similar lift, only this time one needs an extra ingredient in the form of a connection σ defined on the principal U(1)-bundle Q used to define the spin$^{\mathbb{C}}$-structure. Fix such a connection. Then lifting one gets

$$\bar{\nabla}_X(\Psi) := X(\Psi) + \frac{1}{2}\omega(X) \cdot \Psi + \frac{1}{2}\sigma(X) \cdot \Psi \tag{4.16}$$

Let e^i be an orthonormal frame field, potentially only locally defined. Then the Dirac operator is defined as

$$D = \gamma^i \bar{\nabla}_i \tag{4.17}$$

The definition in fact combines to a global differential operator. Let Σ be the curvature form associated with the U(1) connection σ. Using this notation, the Dirac operator D satisfies a Lichnerowicz-type relation.

Theorem 4.2.2. *Lichnerowicz Theorem (spin$^{\mathbb{C}}$ case)*

$$D^2 = \Delta + \frac{1}{4}R + \frac{i}{2}\Sigma, \tag{4.18}$$

where R is the scalar curvature of the Levi-Civita connection ∇, and the 2-form Σ acts on spinors via Clifford multiplication.

Proof. See [Fri00]. □

Let M be a Kähler manifold of real dimension $n = 2k$. Fix its canonical spin$^{\mathbb{C}}$-structure and consider its (full) bundle of complex spinors S. Using the Clifford action one can represent the Kähler form Ω as a linear operator on the bundle S. One can show that the spinor bundle decomposes[Fri00] as $S = \bigoplus_{p=0}^{k} S_p$, where S_p is the subbundle of spinor fields Ψ that satisfy

$$\Omega \cdot \Psi = i(k - 2p)\Psi. \tag{4.19}$$

Using the representation of $\mathbb{C}l(2k)$ in terms of differential forms, it additionally follows that $S_p \cong \Lambda^{0,p} \otimes S_k$, where $\Lambda^{0,p}(M)$ is the bundle of anti-holomorphic p-forms on M. For the canonical spin$^{\mathbb{C}}$-structure only S_k is isomorphic to the trivial complex line bundle.

This makes it possible to identify

$$S \cong \bigoplus_{p=0}^{k} \Lambda^{0,p}(M), \tag{4.20}$$

at least at the level of vector bundles. Moreover, the Dirac operator (4.17) can also be represented in the language of differential forms. Let ∂ be the holomorphic exterior differential and δ the respective codifferential on the bundle of differential forms on M. The Dirac operator D is now identified with the operator $\partial + \delta$, in the sense that the diagram

$$\begin{array}{ccc} S & \xrightarrow{\cong} & \sum_i \Lambda^{0,i} \\ {\scriptstyle D}\downarrow & & \downarrow{\scriptstyle \partial+\delta} \\ S & \xrightarrow{\cong} & \sum_i \Lambda^{0,i} \end{array}$$

commutes. The operator $\partial + \delta$ is commonly known as the *Dirac-Dolbeault* operator. In addition, whenever M is also a homogeneous space, the decomposition above respects the group action.

Recall that a spin$^{\mathbb{C}}$-structure, when it exists, is fixed by a choice of principal U(1)-bundle Q_0, which it covers as in definition 4.2.2. It is possible to change from one spin$^{\mathbb{C}}$-structure to another by taking the tensor product of the bundle with another principal U(1) bundle Q. This statement can be translated to the language of vector bundles by using the bundle of complex spinors. Thus, given such a bundle S_0 associated to a spin$^{\mathbb{C}}$-structure, a bundle of complex spinors S_Q associated with another spin$^{\mathbb{C}}$-structure may be obtained by taking the tensor product

$$S_Q = S_0 \otimes L_Q, \tag{4.21}$$

where L_Q is the natural complex line-bundle associated with the U(1) bundle Q. The new spin$^{\mathbb{C}}$-structure is precisely the one obtained by taking the product of the original one with Q. This point of view will be especially useful in later sections, where all spin$^{\mathbb{C}}$-structures of $\mathbb{C}P(2)$ will be obtained by twisting the complex spinor bundle of the canonical spin$^{\mathbb{C}}$-structure.

4.3 The Geometry of CP2

It is now time to develop the geometry of $\mathbb{C}P(2)$ using the group theoretic machinery developed earlier in the section.

In general the complex projective space $\mathbb{C}P(n)$ is realisable as the symmetric space $\mathbb{C}P(n) \cong \mathrm{U}(n+1)/\left(\mathrm{U}(n) \times U(1)\right) \cong \mathrm{SU}(n+1)/S\left(U(n) \times U(1)\right)$. For the purposes of this investigation it is also important that it is a coadjoint orbit.

Proposition 4.3.1. *The space $\mathbb{C}P(n)$ is a coadjoint orbit of the Lie group $\mathrm{SU}(n+1)$.*

Proof. Let $G = SU(n+1)$ with Lie algebra $\mathfrak{g}$. By Proposition 4.2.1, it is enough to find a linear form $\mu \in \mathfrak{g}^*$ whose stabiliser subgroup is isomorphic to $K = S(\mathrm{U}(n) \times \mathrm{U}(1))$. In fact, a weight with this property can easily be found. Choose a Cartan subalgebra $\mathfrak{h}$ together with a root system Δ, and work with the lattice of integral weights in $\mathfrak{h}^*$. Let r be the rank of the Lie group, which in the case of $SU(n+1)$ is $r = n$. As always, dominant integral weights are of the form $(p_1, p_2, \ldots p_r)$ with $p_i \in \mathbb{Z}_{\geq 0}$. In particular, note that the weight $\mu = (p, 0, 0, \ldots, 0)$, with $p \neq 0$, is orthogonal to r simple roots. Consequently it is also orthogonal to all roots obtained as sums of these roots. Root vectors associated to this set of roots, together with an appropriate r-dimensional subspace of $\mathfrak{h}$, generate a Lie subalgebra isomorophic to $\mathfrak{su}(n)$, since the only set of roots not included in this set contain multiples of $\pm\alpha_1$. The remaining $1-$dimensional subspace of $\mathfrak{h}$ naturally generates a U(1) subgroup. Thus, once restricting to the compact real form of the stabiliser subalgebra one finds the stabiliser subgroup $\mathrm{Stab}(\mu) = S\left(\mathrm{U}(n) \times \mathrm{U}(1)\right)$. A similar result is obtained using the integral weight $\mu = (0, 0, \ldots, 0, p)$. □

4.3.1 CP2 Geometry from Representation Theory

From now on the discussion will focus on the case of $\mathbb{C}P(2)$. Fix $G = \mathrm{SU}(3)$, $K = S(\mathrm{U}(2) \times \mathrm{U}(1))$, and consider a Cartan-Weyl basis of the Lie algebra $\mathfrak{g}$ of G. Since one is dealing with a rank two group, it has two simple roots, α and β, and two elements H_α and H_β identified as the basis of the chosen Cartan subalgebra. The associated Lie algebra elements $X_{\pm\alpha}$, $X_{\pm\beta}$ are labelled by the roots that correspond to their behaviour under the adjoint action of H_α and H_β. Specialising further to the Chevalley basis, this action is specified by the Cartan matrix of A_2:

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}. \tag{4.22}$$

As a consequence of this choice of basis, the triple $\{X_{\pm\alpha}, H_\alpha\}$ closes as a copy of $SU(2)$, and satisfies the relations

$$[H_\alpha, X_{\pm\alpha}] = \pm 2X_\alpha, \tag{4.23}$$

and

$$[X_\alpha, X_{-\alpha}] = H_\alpha. \tag{4.24}$$

Identical relations hold for the triple $\{X_{\pm\beta}, H_\beta\}$. Using again the Cartan matrix one also has

$$\begin{aligned}[H_\alpha, X_\beta] &= -X_\beta \\ [H_\beta, X_\alpha] &= -X_\alpha.\end{aligned} \tag{4.25}$$

There are also two more Lie algebra elements $\{X_{\pm(\alpha+\beta)}\}$ whose behaviour under the action of the Cartan subalgebra is given by the roots $\pm(\alpha + \beta)$. They are derived from the simple roots via the relations

$$\begin{aligned}[X_\alpha, X_\beta] =& X_{\alpha+\beta} \\ [X_{-\alpha}, X_{-\beta}] =& - X_{-(\alpha+\beta)}.\end{aligned} \tag{4.26}$$

All other relations are derivable from the above relations together with the Jacobi identity.

The negative of the Killing form B can be used to find the metric on the space of weights. Its restriction to the Cartan subalgebra in

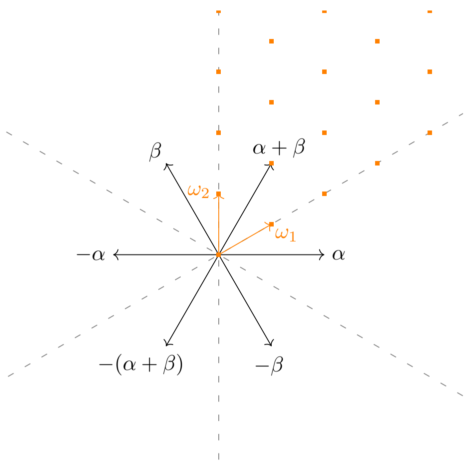

Figure 4.2: Elements in the weight lattice of SU(3). The fundamental weights ω_i span the weight lattice, of which only those elements in the positive Weyl chamber are indicated. The root system spanned by the simple roots α and β is also present in the diagram.

the basis $\{H_\alpha, H_\beta\}$ is

$$B|_{\mathcal{L}_\mathcal{T}} = 6 \cdot \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}. \tag{4.27}$$

Unsurprisingly, the Killing form is proportional to the symmetrised Cartan matrix. In fact, the factor of 6 is precisely the second Dynkin index of the adjoint representation[FS03]. The above basis thus provides a metric on $\mathfrak{h}^*$ given by the quadratic form matrix

$$G = \frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \tag{4.28}$$

written out in terms of a basis $\{\omega_1, \omega_2\}$ of fundamental weights. See figure 4.2 for the resulting weight lattice configuration that will be used throughout.

Using this metric one has the following

Proposition 4.3.2. *The Quadratic Casimir* $C(p,q)$ *of* SU(3) *evaluated on the irreducible representation space labelled by* (p,q) *is given by*

$$C(p,q) = \frac{2}{3}\left(p^2 + q^2 + pq + 3p + 3q\right). \tag{4.29}$$

Proof. It is well known[Kna96] that the quadratic Casimir for the irreducible representation λ takes the form

$$C(\lambda) = (\rho + \lambda, \rho + \lambda) - (\rho, \rho), \tag{4.30}$$

where ρ is the Weyl vector. In the case where $G = SU(3)$, and where the Dynkin labels are obtained using the Chevalley basis above, one has $\rho = (1,1)$, and the inner product on the space of weights is fixed by the Euclidean metric given in Equation 4.28. Evaluating the Casimir element using $\lambda = (p,q)$ then gives the desired result. □

When dealing with calculations, it will be convenient to fix the matrix form of H_α and H_β in some Chevalley basis. These will be chosen as

$$H_\alpha = \begin{pmatrix} i & & \\ & -i & \\ & & 0 \end{pmatrix}, \qquad H_\beta = \begin{pmatrix} 0 & & \\ & i & \\ & & -i \end{pmatrix}. \tag{4.31}$$

From here on, the construction of $\mathbb{CP}(2)$ as a coadjoint orbit will be obtained by choosing a weight $\mu = k\omega_2$ where $k \in \mathbb{R}^{\geq 0}$ The compact real form of the stabiliser group K is generated by the real

Lie subalgebra

$$\mathcal{K} = span_{\mathbb{R}}(\{iH_\alpha, iH_\beta, X_\alpha - X_{-\alpha}, i(X_\alpha + X_\alpha)\}), \tag{4.32}$$

so that $K = \exp \mathcal{K}$. In fact, $K \cong (U(1) \times SU(2))/\mathbb{Z}_2$, as any element $A \in K$ can be written as a matrix

$$A = \begin{pmatrix} h \cdot \lambda & \\ & \lambda^{-2} \end{pmatrix} \tag{4.33}$$

where $\lambda \in \mathrm{U}(1)$ and $h \in \mathrm{SU}(2)$. Restricting to the upper left-hand block of A one obtains the matrix $M = h \cdot \lambda \in \mathrm{U}(2)$, and this process in fact provides an isomorphism $K \cong \mathrm{U}(2)$ with the obvious inverse. This mapping has the benefit of identifying the Cartan subalgebras of the respective Lie algebras. To take the U(2) as a start, note that the λ term corresponds to the U(1) factor, the global phase factor generated by the Lie algebra element $T_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$. Together with the generator $T_2 = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$ one has a basis for the Cartan subalgebra. The matrices T_1 and T_2 are mapped by the isomorphism to the $\mathcal{K}$ elements

$$X = \begin{pmatrix} i & & \\ & -i & \\ & & 0 \end{pmatrix}, \tag{4.34}$$

and

$$Y = \begin{pmatrix} i & & \\ & i & \\ & & -2i \end{pmatrix}, \tag{4.35}$$

respectively, which themselves form a basis for a Cartan subalgebra of $\mathcal{K}$. It will be convenient to use this basis to define the representation labels of K. The label (n, ϕ) will thus correspond to the highest weight of irreducible representation K defined on the Lie algebra via $X \cdot v = nv$ and $Y \cdot v = \phi v$. Note that n is constrained to be a nonnegative integer, while ϕ can be any real number. When performing restriction from G to K, or the induction in the other way, it is important to know how the Cartan subalgebras and their chosen bases are related. Evidently $X = H_\alpha$ while $Y = H_\alpha + 2H_\beta$. This means that, formally, a vector in a K-module with weight (a, b) with respect to Cartan basis (X, Y) has weight $\big(a, \frac{1}{2}(b-a)\big)$ with respect to the Cartan basis (H_α, H_β). It is with respect to the latter basis that restriction and induction can be carried.

The tangent space of $\mathbb{C}P(2)$ at the basepoint $o = [e]$ can be identified with the subspace of $\mathfrak{g}$ complement to $\mathcal{K}$ with respect to B. In the picture of $\mathbb{C}P(2)$ as a (co-)adjoint orbit of SU(3), this subspace can be used as a model for local coordinates of $\mathbb{C}P(2)$. Consider explicitly the subspace

$$\mathrm{m} := \left\{ \begin{pmatrix} & & z_1 \\ & & z_2 \\ -\overline{z}_1 & -\overline{z}_2 & \end{pmatrix} : z_1, z_2 \in \mathbb{C} \right\} \tag{4.36}$$

of $\mathfrak{g}$. Evidently, one has the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. The adjoint action of X and Y on a generic $Z \in \mathfrak{m}$ becomes

$$[X, Z] = \begin{pmatrix} & & iz_1 \\ & & -iz_2 \\ -\overline{iz_1} & -\overline{-iz_2} & \end{pmatrix}. \tag{4.37}$$

and

$$[Y, Z] = \begin{pmatrix} & & 3iz_1 \\ & & 3iz_2 \\ -\overline{3iz_1} & -\overline{3iz_2} & \end{pmatrix} \tag{4.38}$$

Using $\vec{z} = (z_1, z_2)$ as complex coordinates, one sees that the adjoint action is equivalent to the U(2) action on $\mathbb{C}^2$ with highest weight $(1, 3)$. By the isomorphism above used to identify U(2) and K, this is also the highest weight of the adjoint action of K on $\mathfrak{m}$. Converting to the basis compatible with the weights of $\mathfrak{g}$, one obtains the highest weight $(1, 1)$. This is the isotropy representation of K acting on the tangent space at the point o. Hence, the tangent bundle of $\mathbb{C}P(2)$ is constructed as the associated vector bundle

$$T\mathbb{C}P(2) \cong G \times_{(1,1)} \mathbb{C}^2. \tag{4.39}$$

Note that this makes use of $\mathbb{C}P(2)$ as a complex manifold of dimension two.

In a similar vein other important bundles over $\mathbb{C}P(2)$ can be characterised. Firstly note that the dual of the irreducible representation (a, b) of K is isomorphic to the irreducible representation

$(a, -b)$. Hence, the space of one-forms at o transform under K as the irreducible representation $(1, -1)$. This allows one to identify the bundle of (holomorphic) one forms on $\mathbb{C}P(2)$ as

$$T^*\mathbb{C}P(2) \cong G \times_{(1,-1)} \mathbb{C}^2. \tag{4.40}$$

To evaluate tensor products of representations of K, use the original Cartan subalgebra basis, for which the basis elements are orthogonal with respect to the restriction of the Killing metric of SO(4) to $K \subset \mathrm{SO}(4)$. The embedding is of course given by the adjoint action on $\mathfrak{m}$ viewed as a real space, and so respects the orthogonal decomposition of vectors with respect to the Fubini-Study metric on $\mathbb{C}P(2)$. The tensor product decomposition $(1, -3) \otimes (1, -3) \cong (0, -6) \oplus (2, -6)$, allows one to identify the space of holomorphic 2-forms at o with irreducible representation $(0, -6)$. Converting now to the representation labels compatible with H_α and H_β, one gets the representation label $(0, -3)$. This allows one to identify the bundle of holomorphic $2-$forms on the entirety of $\mathbb{C}P(2)$ as

$$\Lambda^{2,0}(\mathbb{C}P(2)) \cong G \times_{(0,-3)} \mathbb{C}. \tag{4.41}$$

Typically, the bundle of holomorphic differential forms provides the more common choice in which to describe complex geometry. But for the purposes of the specific complex spinor geometry of $\mathbb{C}P(2)$ used here, the identification provided by equation 4.20 models the canonical bundle of complex spinors in terms of the bundle of *anti-holomorphic* differential forms. Luckily, since both bundles are dual to each other, this relationship carries forward to the associated bundle construction, so that one simply needs to use the conjugate representations of K to construct the dual bundles. In this case one has

$$\begin{aligned} \Lambda^{0,1}(\mathbb{C}P(2)) &\cong G \times_{(1,1)} \mathbb{C}, \\ \Lambda^{0,2}(\mathbb{C}P(2)) &\cong G \times_{(0,3)} \mathbb{C}, \end{aligned} \tag{4.42}$$

and it is these representations that need now to be decomposed into irreducible SU(3) representations.

4.3.2 Decomposition of Homogeneous Bundles over CP2

As in the previous sections, let $G = \mathrm{SU}(3)$ and $K = S(\mathrm{U}(2) \times \mathrm{U}(1))$, so that $\mathbb{C}P(2) \cong G/K$. With the geometric structures of $\mathbb{C}P(2)$ fixed in terms of the group structure of G, the next task is to determine for the required G-equivariant vector bundles $E \to \mathbb{C}P(2)$ how the Hilbert space $\mathcal{H} = L^2(E)$ of square integrable sections decomposes into irreducible representations of G. As always, this decomposition into irreducible will form a dense subspace whose completion is $L^2(E)$ itself. The obvious starting point is to determine the decomposition of the function space $L^2(\mathbb{C}P(2))$.

Theorem 4.3.1. *The space of square integrable functions on* $\mathbb{C}P(2)$*, with the group* SU(3) *acting on it via the right-regular representa-*

tion, decomposes as

$$L^2(\mathbb{C}P(2)) = \bigoplus_{j=0}^{\infty}(j, j). \tag{4.43}$$

Proof. By Frobenius reciprocity it is enough to consider the multiplicity of the trivial irreducible representation $(0, 0)$ of K in the restriction of the representation (p, q) of G to K. Choosing the set $\{H_\alpha, H_\beta, X_{\pm\alpha}\}$ to generate $\mathfrak{k}$, the restriction of (p, q) to representations of K in terms of the conventional picture allows one to read off the K representations as the horizontal lines that are a part of the weight diagram of (p, q). The multiplicity of weight $(0, 0)$ is directly obtained from the Kostant partition function, giving

$$\text{mult}(0, 0) = \min(p, q) + 1, \tag{4.44}$$

which can be understood by subtracting from (p, q) the root α_i corresponding to the larger Dynkin label Λ_i until one reaches line in the positive Weyl chamber spanned by α_3 at the point (r, r), where now $r = \min(p, q)$, i.e. the other Dynkin label. Since $\alpha_3 = \alpha_1 + \alpha_2 = (1, 1)$, the number of ways to algebraically write down the path from this point to the origin consists of associating to each of the r segments either α_3 or $\alpha_1 + \alpha_2$ in an unordered fashion. This number is clearly $r + 1$.

The multiplicity $m_{(0,0)}$ of $(0, 0)$ as a *representation*, though, depends on the number of times the weight $(0, 0)$ appears in the diagram that that do not belong to any other root strings. This is simply a question of finding on the one hand how far along the horizontal axis lies the outermost weight of the representation (p, q) (when it exists), and on the other hand how deep along the horizontal axis from this extremal weight one reaches maximum multiplicity, which by the Kostant multiplicity formula is always given by the value on the right-hand side of equation (4.44). Purely geometric calculations give the depth along the horizontal axis to be $\max\left(\frac{2p+q}{3}, \frac{p+2q}{3}\right)$, which is always greater than or equal to the maximal multiplicity. It also restricts to cases where $p \equiv q \pmod 3$. Thus, the only instance when a weight $(0,0)$ does not belong to a nontrivial horizontal root string is when $p = q = j$ for some nonnegative integer j. In all such cases the multiplicity is one. In summary one has

$$m_{(0,0)} = \begin{cases} 1 & \text{if } (p,q) = (j,j); \\ 0 & \text{otherwise.} \end{cases}$$

This then also gives the multiplicity of (p, q) in $L^2(\mathbb{C}P(2))$. □

In the previous section, the tangent bundle and the bundles of anti-holomorphic differential forms over $\mathbb{C}P(2)$ were expressed in equations (4.39) and (4.42) as vector bundles associated to the principal K-bundle $G \to \mathbb{C}P(2) \cong G/K$. In fact, all G-equivariant vector bundles over $\mathbb{C}P(2)$ can be viewed (up to isomorphism) as such associated vector bundles. This stems from the following general fact. Let X be a homogeneous space of the group H, with stabiliser subgroup $L \subset H$ so that $X \cong H/L$. Given any (H-equivariant) vector bundle $E \to H/L$, fix a point $o \in X$. The left-action of the stabiliser subgroup L must then map the fibre E_o of E over o linearly into itself, fixing the fibre as a representation space of L under some representation μ over the carrier space E_o. One can then show that necessarily $E \cong G \times_\mu V_o$ as H-bundles over X; see for example Chapter 5 of [Wal18]. This guarantees that all H-bundles are equivalent up to isomorphism to vector bundles associated to the principal L-bundle $H \to X$. Now, fix a unitary irreducible representation (μ, V_μ) of L. The function space of the H-equivariant vector bundle $E_\mu \coloneqq G \times_\mu V_\mu$ inherits an H-action, through the obvious regular representation. In fact, by the very definition of the induction functor, one has the isomorphism of H-representations

$$L^2(E_\mu) \cong \mathrm{Ind}(\mu). \tag{4.45}$$

Thus, in therms of the groups relevant to $\mathbb{C}P(2)$, the following proposition concerning the decomposition of induced representations from K to G in terms of irreducible representations of G is paramount.

Proposition 4.3.3. *Let μ be the highest weight of the irreducible representation V_μ of K.*

1. *If $\mu = (m, n)$, where $0 \leq m, n$, then*

$$\mathrm{Ind}_K^{SU(3)}(m,n) \cong \bigoplus_{k=0}^{m} \big[\bigoplus_{j=0}^{\infty} (j+m-k, j+n+2k) \big]. \tag{4.46}$$

2. *If $\mu = (m, -n)$, where $0 \leq m \leq n$, then*

$$\mathrm{Ind}_K^{SU(3)}(m,-n) \cong \bigoplus_{k=0}^{m} \big[\bigoplus_{j=0}^{\infty} (j+n-m+2k, j+m-k) \big]. \tag{4.47}$$

3. If $\mu = (m, -n)$ *where* $0 \leq n \leq m$ *then*

$$\begin{aligned}\mathrm{Ind}_K^{SU(3)}(m,-n) \cong & \bigoplus_{k=0}^{m-n}\Big[\bigoplus_{j=0}^{\infty}(j+m-n-k, j+n+2k)\Big] \\ & \oplus \bigoplus_{k=1}^{n}\Big[\bigoplus_{j=0}^{\infty}(j+m-n+2k, j+n-k)\Big]\end{aligned} \tag{4.48}$$

Proof. The proof relies on the use of the Frobenius reciprocity in a manner similar to that used in Theorem 4.3.1.

1. The basic case is when $0 \leq m, n$. One is now looking for weight diagrams of irreducible representations (p, q) of SU(3) containing a horizontal root string that goes from weight (m, n) to weight $(-m, m+n)$ using the negative root $-\alpha = (-2, 1)$. Since the multiplicity of the weight (m, n) is less than or equal to weights contained in the string, to count the number of times the representation (m, n) of K appears it is enough to determine how many copies of the weight $\lambda = (m, n)$ do not contribute to larger horizontal root strings. As (m, n) is no longer restricted to be at the apex of the positive Weyl chamber, there are a few more cases that need to be covered, but this can be done simply using an inductive argument.

It is necessary that (p, q) be a weight higher than λ, namely, that it can be expanded as $(p, q) = (m, n) + s\alpha + t\beta$ for some nonnegative integers s, t, and moreover one requires $p - q \equiv m - n \pmod 3$ to guarantee that the weights are connected by roots. Now consider the boundary of the allowable region where this holds. The irreducible representation labelled by highest weight $(p, q) = (m, n)$ itself contains the desired root string, with multiplicity one that obviously does not contribute to any larger root string. Taking the irreducible representation $(p, q) = (m, n) + \alpha = (m + 2, n - 1)$, one sees that (m, n) is still a boundary weight with multiplicity one, but now it belongs to the horizontal root string that starts at (p, q), so the multiplicity of the irreducible representation is now zero. The situation is the same further on, when considering $(p, q) = (m, n) + s\alpha$ for larger values of s, until one hits the axis spanned by ω_1 at $\sigma = (m + 2n, 0)$. Going further along this axis, the multiplicity of all weights remains one, so unless one started on this axis (i.e. unless n=0), the multiplicity of the irreducible representation (m, n) is always zero here. Next, the irreducible representation $(p, q) = (m, n) + \beta$ also has (m, n) as a weight on the boundary of the diagram, but without a larger root string containing it. This continues further on along the line $(p, q) = (m, n) + t\beta$, and so (m, n) exists as an irreducible representation of K along this entire line until one hits the ω_2 (vertical) axis at $\tau = (0, m + 2n)$. Finally, continuing along the vertical axis, the weight (m, n) still has multiplicity one, but it is no longer on the boundary of the weight diagram and so it is taken up by a larger horizontal root string.

To complete all cases, it can be noted that whatever multiplicity the irreducible representation (m, n) of K has when restricting (p, q), the same multiplicity exists when restricting the irreducible representation $(p, q) + \alpha + \beta = (p + 1, q + 1)$. This is because, on the one hand, the multiplicity of (m, n) as a weight has increased by one, but on the other hand, the new weight diagram is geometrically similar to the original one, having been extended by one additional

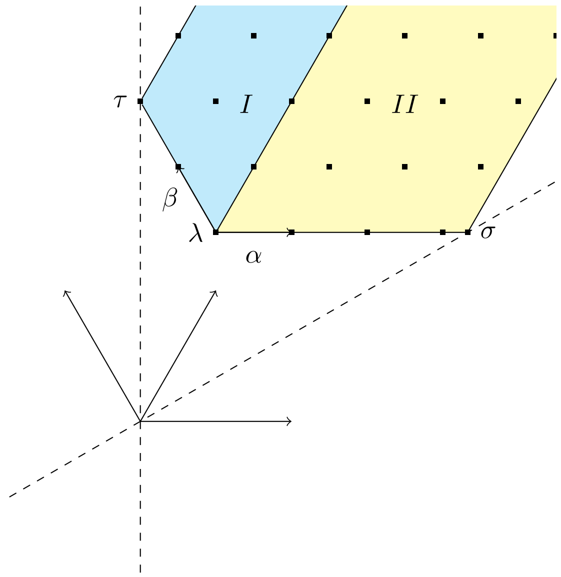

Figure 4.3: The regions contributing to the multiplicities in all cases, with the coordinates of λ depending on the case being considered. (1) Only region I contributes. (2) Only region II contributes. (3) Both regions contribute.

layer of weights at the boundary. So there is simultaneously a single extra (m, n) weight and a larger root string containing it, resulting in the same overall multiplicity of the irreducible representation (m, n). Now, all allowable irreducible representations (p, q) can be written as $(p_0, q_0) + k(1, 1)$ for (p_0, q_0) on the boundary of the region, and nonnegative integer k. This determines the multiplicities completely.

$$m_{(m,n)} = \begin{cases} 1 & \text{if } (p,q) = (m-k+j, n+2k+j) \text{ for } 0 \le k \le m \text{ and } 0 \le j; \\ 0 & \text{otherwise.} \end{cases}$$

2. The case of irreducible representation $(m, -n)$ when $0 \le m \le n$ can be solved in a manner nearly identical to the first case. This

time, the weight (m,n) is in the fifth Weyl chamber anticlockwise starting from the positive Weyl chamber. Given that the weight diagram of any irreducible representation (p,q) is invariant under Weyl reflection, one can use the unique reflection that maps $(m,-n)$ to the positive Weyl chamber to solve the equivalent multiplicity problem that gives the same answer. This particular reflection maps the weight to the positive weight $\lambda = (n-m,m)$. It also maps the root α to the root β. So the solution is obtained by taking the solution in the first case with weight $(n-m,m)$ and swapping $\alpha \leftrightarrow \beta$. Explicitly one has

$$m_{(m,-n)} = \begin{cases} 1 & \text{if } (p,q) = (n-m+2k+j, m-k+j) \text{ for } 0 \leq k \leq m \text{ and } 0 \leq j; \\ 0 & \text{otherwise.} \end{cases}$$

3. In the final case of irreducible representation $(m,-n)$ where $0 \leq n \leq m$, the weight is now in the sixth Weyl chamber that neighbours the positive one. The relevant Weyl reflection maps this weight to the positive weight $\lambda = (m-n,n)$, and the root α to the root $\alpha+\beta$. This time both the line segment from λ to $\tau = (0,2m-n)$ and the line segment from λ to $\sigma = (2m-n)$ contribute multiplicity one. Thus, one obtain

$$m_{(m,-n)} \begin{cases} 1 & \text{if } (p,q) = (m-n+2k+j, n-k+j) \text{ for integers } 0 \leq k \leq m \text{ and } 0 \leq j; \\ 1 & \text{if } (p,q) = (m-n-k+j, n+2k+j) \text{ for integers } 0 \leq k \leq m \text{ and } 0 \leq j; \\ 0 & \text{otherwise.} \end{cases}$$

$\square$

Using Proposition 4.3.3 on the bundles introduces at the end of the last section one has the following decompositions.

Corollary 4.3.1. *Let* $\Lambda^{0,1}(\mathbb{C}P(2))$ *and* $\Lambda^{0,2}(\mathbb{C}P(2))$ *be the bundles of anti-holomorphic* 1*-forms and* 2*-forms, respectively. Then the spaces of their square integrable sections decompose as irreducible representations of* SU(3) *as*

$$L^2(\Lambda^{0,1}(\mathbb{C}P(2))) \cong \bigoplus_{j=0}^{\infty} [(j+1, j+1) \oplus (j, j+3)] \tag{4.49}$$

$$L^2(\Lambda^{0,2}(\mathbb{C}P(2))) \cong \bigoplus_{j=0}^{\infty} (j, j+3) \tag{4.50}$$

Proof. This is a straightforward application of Proposition 4.3.3 to $\mathrm{Ind}_K^{\mathrm{SU}(3)}(1,1)$ and $\mathrm{Ind}_K^{\mathrm{SU}(3)}(0,3)$. □

4.3.3 Dirac Operator and Spectrum

It is now possible to characterise the canonical spinor bundle over $\mathbb{C}P(2)$ and calculate the spectrum of the Dirac operator. At the end of section 4.2.2 it was explained how the fundamental spinor bundle $S = S^+ \oplus S^-$ obtained from the canonical spin$^{\mathbb{C}}$-structure on $\mathbb{C}P(2)$ is isomorphic to the bundle of holomorphic differential forms. Thus one has the following.

Lemma 4.3.1. *The fundamental spinor bundle on* $\mathbb{C}P(2)$ *decom-*

poses in terms of SU(3) *representations as*

$$L^2(S^+) \cong \bigoplus_{j=0}^{\infty} [(j,j) \oplus (j,j+3)]\,, \tag{4.51}$$

$$L^2(S^-) \cong \bigoplus_{j=0}^{\infty} [(j+1,j+1) \oplus (j,j+3)]\,. \tag{4.52}$$

Proof. Since $\mathbb{C}P(2)$ is Kähler, the end of section 4.2.2 implies that upon choosing the canonical spin$^{\mathbb{C}}$-structure one has

$$S^+ \cong \sum_{j=0}^{1} \Lambda^{0,2j},$$

and

$$S^- \cong \Lambda^{0,1}.$$

The lemma now follows directly from theorem 4.3.1, together with equations 4.49, and 4.50. □

With the spinor bundle S completely decomposed into irreducible representations of SU(3), the spectrum of the Dirac operator can be calculated using the fact that the Dirac operator is equivalent to the (Dirac)-Dolbeault operator.

Theorem 4.3.2. *The nonzero part of the spectrum of the Dirac operator D on the canonical spinor bundle S is given by the two sequences*

1. $\pm\lambda_r = \pm\sqrt{(r+1)(r+3)}, \qquad r = 0,1,2,3\ldots.$

   The eigenvalues $\pm\lambda_r$ have multiplicity $(r+2)^3$.

2. $\pm\rho_r = \pm\sqrt{(r+2)(r+3)}, \qquad r = 0,1,2,3\ldots.$

   The eigenvalues $\pm\rho_r$ have multiplicity $\frac{1}{2}(r+1)(r+4)(2r+5)$.

Finally, the Dirac operator has one zero mode, giving an index $\mathrm{Index}(D) = 1$.

Proof. Recall that the Lie algebra elements $\{X_{\pm\beta}, X_{\pm(\alpha+\beta)}\}$ span the tangent space to the basepoint $o = [e]$. The push-forward of this basis using the group action of SU(3) forms a local frame field that is orthogonal with respect to the metric. Denote these vector fields by the the same symbol as the Lie algebra element, and using the identification above decompose S in terms of the bundles of anti-holomorphic differential forms as $S \cong \Lambda^{0,2} \oplus \Lambda^{0,0} \oplus \Lambda^{0,1}$. Let $\tilde{Z}$ denote the one-form dual to the vector field Z, and fix the ordered basis of S to be $\left(\tilde{X}_\beta \wedge \tilde{X}_{\alpha+\beta}, 1, \tilde{X}_\beta, \tilde{X}_{\alpha+\beta}\right)$. Converting the Dirac operator to the equivalent representations in terms of differential forms, it can be now identified with the Dirac-Dolbeault operator, and in the basis above it takes the form

$$D = \begin{pmatrix} 0 & 0 & X_{\alpha+\beta} & X_\beta \\ 0 & 0 & -X_{-\beta} & X_{-(\alpha+\beta)} \\ X_{-(\alpha+\beta)} & -X_\beta & 0 & 0 \\ X_{-\beta} & X_{\alpha+\beta} & 0 & 0 \end{pmatrix}, \tag{4.53}$$

where the vector fields act by differentiation. This is simply the expression of the action of the (co-)differential operators on the space of differential forms. The definition agrees up to a factor of $\sqrt{2}$ with the Dirac operator in [GS99]. One now finds that

$$D^2 = \frac{1}{2}\Delta_{\mathbb{C}P(2)} - \frac{1}{2}H_\alpha - H_\beta, \tag{4.54}$$

where the connection Laplacian is expressed in terms of the left-invariant vector fields as

$$\Delta_{\mathbb{C}P(2)} = (X_\beta X_{-\beta} + X_{-\beta} X_\beta + X_{\alpha+\beta} X_{-(\alpha+\beta)} + X_{-(\alpha+\beta)} X_{\alpha+\beta}) \cdot \mathbb{I}_4. \tag{4.55}$$

Note that the left-invariant vector fields H_α and H_β vanish when acting by differentiation on sections of the spinor bundles. This follows from the interpretation of the induced representations as vector-valued functions on the group G that are right-invariant under the stabiliser subgroup K. Thus, the square of the Dirac operator is equal in this case and normalisation to half of the Laplacian, which is well known[IT78]. It also agrees with the quadratic Casimir, whose value on each irreducible representation was given in (4.29). Thus using lemma 4.3.1 one immediately writes down the spectrum. The multiplicity of each value is the dimension of the irreducible representation that corresponds to it. □

It is in fact straightforward to calculate the spectrum of the Dirac operator for other spin$^{\mathbb{C}}$-structures on $\mathbb{C}P(2)$. Firstly, recall that the remaining spin$^{\mathbb{C}}$-structures on $\mathbb{C}P(2)$ are obtained by twisting by the different principal U(1)-bundles. In terms of the complex spinor bundle, this is achieved by twisting by the associated complex line bundles. These are parameterised uniquely on $\mathbb{C}P(2)$ by their integer Chern number m, and so the resulting spin$^{\mathbb{C}}$-structures may be labelled by m as well. Denote the bundle of complex spinors associated to this spin$^{\mathbb{C}}$-structure by S_m. It is first necessary to decompose these bundles into irreducible representations of SU(3).

Proposition 4.3.4. *The spinor bundle* $S_m = S_m^+ \oplus S_m^-$ *over* $\mathbb{C}P(2)$ *decomposes in terms of irreduicble representations of* SU(3) *as*

$$\begin{aligned} S_m^+ &:= \left(S_m^-\right)_1 \oplus \left(S_m^+\right)_2 = \mathrm{Ind}(0,m) \oplus \mathrm{Ind}(0,m+3) \\ S_m^- &= \mathrm{Ind}(1,1+m) \end{aligned} \tag{4.56}$$

Proof. A line bundle over $\mathbb{C}P(2)$ whose Chern number is given by m transforms locally under the isotropy representation of K labelled by $(0,m)$. This representation is thus tensored with the isotropy representation of the complex spinor bundle associated with the canonical spin$^{\mathbb{C}}$ structure to give S_m. Note that for K one has $(a,b) \otimes (0,m) \cong (a, b+m)$. Thus locally one finds

$$\begin{aligned} S_m^+\big|_o &\cong (0,m) \oplus (0,m+3) \\ S_m^+\big|_o &\cong (1,1+m), \end{aligned} \tag{4.57}$$

and constructing global bundles using the induction functor gives the desired expression. □

Note that when $m = 0$ one indeed gets the canonical spin$^{\mathbb{C}}$-structure.

Corollary 4.3.2. *Let* S_m *be the complex spinor bundle associated with the m-th spin*$^{\mathbb{C}}$ *structure on* $\mathbb{C}P(2)$. *Then the spectrum of its Dirac operator is given, in terms of the series of Theorem 4.3.2, by the two sequences*

1. $\pm\lambda_r^m = \pm\sqrt{r(r+2) + |m|(r+1) - m}$, $\qquad r = 0,1,2,3\ldots.$
   The eigenvalues $\pm\lambda_r$ *have multiplicity* $\frac{1}{2}(r+2)(r+|m|+2)(2r+|m|+4)$.

2. $\pm\rho_r^m = \pm\sqrt{r(r+2)+|m+3|(r+1)+m+3}, \qquad r = 0,1,2,3\ldots.$ *The eigenvalues* $\pm\rho_r$ *have multiplicity* $\frac{1}{2}(r+1)(r+|m+3|+1)(2r+|m+3|+2)$.

The Dirac operator has index $\mathrm{Index}(D) = \frac{(m+1)(m+2)}{2}$.

Proof. The sections of S_m are obtained from the sections of S by a twisting by the line bundle whose isotropy representation under K is given by $(0,m)$. Now the left-invariant vector fields of K, which vanished in 4.3.2, transform under this irreducible representation. Let C_K contain the terms in the quadratic Casimir C that span in the subgroup K. In the chosen Chevalley basis, where the Lie algebra element orthogonal to H_α with respect to the Killing form is $H_\perp := \frac{1}{\sqrt{3}}(H_\alpha + 2H_\beta)$, this is given by

$$C_K = X_\alpha X_{-\alpha} + X_{-\alpha}X_\alpha + \frac{1}{2}H_\alpha^2 + \frac{1}{2}H_\perp^2. \tag{4.58}$$

There is a Dirac operator D_m defined on S_m in the canonical way, namely, by using the connection on the spinor bundle that is obtained via pull-back. In this case, the Dirac operator's square gives

$$D^2 = \frac{1}{2}\Delta_{\mathbb{C}P(2)} - \frac{1}{2}H_\alpha - H_\beta = \frac{1}{2}\left(C - C_K - H_\alpha - 2H_\beta\right), \tag{4.59}$$

where now the extra terms do not necessarily vanish. The quadratic Casimir C acts on the total representation space as a scalar whose value can be calculated, as before, by decomposing the bundle S_m into irreducible representations of G and using (4.29). The extra terms, as mentioned, act on sections of this bundle via the $(0,-m)$

irreducible representation of K. Letting v be a vector in this representation, that is one dimensional, one gets

$$[H_\alpha + 2H_\beta + C_K] \cdot v = \left[H_\alpha + 2H_\beta + \frac{2}{3}\left(H_\alpha^2 + H_\alpha H_\beta + H_\beta^2\right)\right] \cdot v = (2m + \frac{2}{3}m^2)v. \quad (4.60)$$

Hence, the square of the Dirac operator for the spin$^{\mathbb{C}}$ structure labelled by m is given by

$$D^2 = \frac{1}{2}C - m - \frac{1}{3}m^2. \quad (4.61)$$

Each sequence can now be considered separately. First note from the different cases of proposition 4.3.3 that $(S_m^+)_1 \cong \mathrm{Ind}(0, m)$ is constructed from a sequence of irreducible representations $(j, j + m)$ or $(j + m, j)$, where j is a nonnegative integer, depending on whether $0 \leq m$ or $m \leq 0$, respectively. The spectrum of C and the multiplicities are thus invariant under a change of sign of m, so it will be enough to consider the case $0 \leq m$ and substituting $|m|$ in those functions. From equation (4.61) one obtains the sequence

$$\pm\lambda_r^m = \pm\sqrt{r(r+2) + m(r+1) - m}, \quad (4.62)$$

for $r = 0, 1, 2, \ldots$. The multiplicity of $\pm\lambda_r^m$ is given by the dimension of the irreducible representation $(r + 1, r + m + 1)$, which gives precisely the desired multiplicity. Note that the label r has been shifted by one; there is an additional irreducible representation $(0, m)$ at the start of the sequence that, only when m is positive, corresponds to zero modes of D_m. Now, only the m in the second term inside the square root needs to be replaced by $|m|$, since the last one comes from equation (4.61).

The second sequence corresponds to $(S_m^+)_2 \cong \mathrm{Ind}(0, m+3)$. This decomposes into irreducible representations of the form $(j, j+m+3)$ or $(j-m-3, j)$ depending on whether $-3 \leq m$ or $m \leq -3$, respectively. As in the previous sequence, consider just the case $-3 \leq m$. This gives a spectrum

$$\pm\rho_r^m = \pm\sqrt{r(r+2) + (m+3)(r+1) + m + 3}, \tag{4.63}$$

for $r = 0, 1, 2, \ldots$. The multiplicity of $\pm\rho_r^m$ is given by the dimension of the irreducible representation $(r, r+m+3)$. Yet again, only the $(m+3)$ factor in the second term needs to be replaced by its absolute value. Note that this time there is no shift in the variable r, since whenever $-3 \leq m$ there are no zero modes in this sequence. These arise only when $m \leq -3$, for the irreducible representation $(-m-3, 0)$. This has dimension equal to $\frac{(m+1)(m+2)}{2}$, which is exactly the same polynomial in m as before. Thus, whatever the value of m is, the dimension of the space contributing to the zero modes of D_m is given by this same amount. The bundle S_m^-, meanwhile, has no zero modes and this determines the index completely.

□

4.A Review of Lie Algebra Theoretic Definitions

In this section, the basic terms of Lie algebra representation theory will be briskly reviewed. This shall be done more so to make sure that the definition are fixed than to explain the basics of the theory. Such expositions are plentiful; see, for example, [BR86; FS03; Kna96].

The Lie algebras considered throughout will almost always be semisimple, if not entirely simple. They will be defined over $\mathbb{R}$. Let $\mathfrak{g}$ be a semisimple Lie algebra. Fix a Cartan subalgebra $\mathfrak{h}$, namely, a maximal abelian subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$. The rank r of the Lie algebra $\mathfrak{g}$ is defined as the dimension of its Cartan subalgebras, which is unique since all such choices of Cartan subalgebra are isomorphic.

The action of the Lie algebra on itself using the Lie bracket is known is as the **adjoint representation** $ad : \mathfrak{g} \to End(\mathfrak{g})$. One can use this adjoint action to define the symmetric bilinear form

$$B(X,Y) \coloneqq Tr\left[ad(X) \circ ad(Y)\right], \tag{4.64}$$

for any pair X, Y in $\mathfrak{g}$. A well known fact is that in the case of (semi-)simple Lie algebras, this is a negative-definite inner product, known as the **Killing form**.

Using the adjoint representation, one can find a basis for the complexified Lie algebra $\mathfrak{g}^{\mathbb{C}}$ such that the action of $\mathfrak{h}$ on the whole Lie algebra is diagonal. One then has the vector space decomposition

$$\mathfrak{g} \cong \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathbb{C} \cdot v_\alpha, \tag{4.65}$$

where v_α is the basis vector in $\mathfrak{g}$ for which the adjoint action of $\mathfrak{h}$ is given by the dual vector $\alpha : \mathfrak{h} \to \mathbb{C}$, and $R \subset \mathfrak{h}^*$ is the set of all such dual vectors. These are known as **roots**. For any $\alpha \in R$, one has $-\alpha \in R$ as well. The roots generate a lattice of dimension r in $\mathfrak{h}^*$. Fix a set of **positive roots** R^+ using a hyperplane in $\mathfrak{h}^*$ passing through the origin. The set of positive roots is in general linearly dependent, and one can find a set $\Delta \subseteq R^+$ of r linearly independent positive roots that generate all other positive roots and thus the whole root-lattice. These are known as the **simple roots**.

A **weight** w is defined as an irreducible representation $w : \mathfrak{h} \to \mathbb{C}$. Let $\langle \cdot, \cdot \rangle$ be the inner product on $\mathfrak{h}^*$ induced by the Killing form. Using it, one can fix a distinguished set of weights by taking the basis in $\mathfrak{h}^*$ dual to the basis of simple roots. These r weights ω_i are known as the **fundamental weights**. As is well known, necessary and sufficient conditions for a general weight w to correspond to an irreducible representation of $\mathfrak{g}$ as a whole is that

- The weight is **dominant**: For all simple roots α one has $0 \leq \langle \alpha, w \rangle$.

- The weight is **integral**: For all fundamental weights ω_i one has $\langle \omega_i, w \rangle \in \mathbb{Z}$.

Assuming a weight w is dominant and integral, the irreducible representation of $\mathfrak{g}$ can be viewed as the representation of $\mathfrak{g}$ induced from the representation w of $\mathfrak{h}$. For compact Lie algebras, this exhausts the collection of irreducible representations. Such an irreducible representation has the starting w as highest weight, and will frequently be labelled by its carrier space V_w.

Recall that the weight lattice is invariant under reflections about the hyperplanes orthogonal to the roots in the root system R. These reflections generate the **Weyl group** W, that acts as the group of outer automorphisms of the Lie algebra. The Weyl reflections serve to partition the space $\mathfrak{h}^*$ into equivalent regions, known as the **Weyl chambers**. Given a set of simple roots Δ as above, there is one Weyl chamber that contains all the dominant weights. It is known as the **positive Weyl chamber**. All integral weights in this chamber are dominant, and in terms of the basis of fundamental weights are expressible in as vectors $w = (a_1, a_2, \ldots a_r)$, where the components a_i are integers. These integers are known as the Dynkin labels of the irreducible representation associated to w.

An important element of $\mathfrak{h}^*$ that is ubiquitous in representation theory is the **Weyl Vector** ρ, which is defined as half the sum of positive roots. It is not necessarily an integral weight, though for the special unitary group $\mathrm{SU}(n)$ it turns out to be so, with Dynkin labels given by $\rho = (1, 1, 1, \ldots 1)$.

Recall that a simple Lie algebras can be completely reconstructed from its Cartan matrix A_{ij}. A Lie algebra basis commonly used that is based on the Cartan matrix is the *Chevalley* basis.

$$\begin{aligned}
&[H_j, H_k] = 0 \\
&\left[H_j, X_k^{\pm}\right] = \pm A^{kj} X_k^{\pm} \\
&\left[X_j^{+}, X_k^{-}\right] = \delta_{jk} H_j \\
&\left(ad_{X_j^{\pm}}\right)^{1-A_{kj}} \left(E_k^{\pm}\right) = 0.
\end{aligned} \tag{4.66}$$

The Racah-Speiser algorithm[FS03] is used to decompose the tensor product of representations $V_\lambda \otimes V_\mu$ into irreducible representation. It is essentially an expression of the Kostant multiplicity formula in terms of a linear procedure that can be carried out easily. Its steps are as follows.

1. Fix either of the two highest weights, say λ. Form the weight diagram of the representation with highest weight μ, and shift it by λ in the weight lattice so that it is centred about λ.

2. Shift the whole weight diagram again by the Weyl vector ρ.

3. Taking note of the multiplicities of each weight, use Weyl reflections to fold each part of the resulting diagram onto the fundamental Weyl chamber. Consider a weight ν in the *interior* of the fundamental Weyl chamber. If another weight τ is mapped onto it by the procedure above, then there exists a Weyl reflection w_τ that maps $w_\tau(\tau) = \nu$. take the set S of all such weights τ that map onto ν, and note that it includes ν itself for which $w_\nu = id$. Now define the integer

$$m_\nu = \sum_{\sigma \in S} (-1)^{|w_\sigma|} \text{mult}(\sigma), \tag{4.67}$$

where mult(σ) is the multiplicity of the weight σ in shifted diagram, and $|w_\sigma|$ is the sign of the corresponding reflection.

4. Shift the folded diagram back by the value of the Weyl vector, i.e. by the vector $-\rho$.

5. The multiplicity of the irreducible representation V_ν for each dominant weight ν is now given by the number $V_{\nu+\rho}$. One explicitly has

$$V_\lambda \otimes V_\mu \cong \bigoplus_\nu m_{\nu+\rho} V_\nu, \tag{4.68}$$

where the sum is over all dominant integral weights ν.

Chapter 5

Fuzzy Constructions

This chapter presents a construction of a fuzzy $\mathbb{C}P(2)$ model. Explicitly this will consist of constructing a sequence of Hilbert spaces whose representation content converges to that of the spinor bundle in the traditional geometry, for whatever spin$^{\mathbb{C}}$ structure one chooses. Additionally, a sequence of Dirac operators will be constructed and their spectrum will be shown to converge to that of the traditional Dirac operator. The construction of the Dirac operator will use Fock-space methods. Section 5.1 motivates this by reviewing the construction of a Dirac operator on the fuzzy sphere, in terms of this operator formalism. Then in section 5.2, the $\mathbb{C}P(2)$ construction will be presented.

5.1 Prelude: The Fuzzy Sphere with Spinc structures

The fuzzy sphere as presented in Chapter 2 used Dirac spinors in its formulation, for the explicit aim of obtaining the desired real spectral triple with KO-dimension $s = 2$. Indeed the reason that the fuzzy space could be defined consistently follows from fact that for S^2, the bundle of Dirac spinors is trivial. The chosen Hilbert space for the fuzzy sphere $\mathcal{H}_n = \mathbb{C}^4 \otimes M_n(\mathbb{C})$ then precisely expresses the structure of the bundle as a product between the coordinate algebra and the spinor vector space. Chiral spinors, meanwhile, do not form a trivial bundle. This fact is crucial in the case of $\mathbb{C}P(2)$, where the left-handed and right-handed spinors transform as different representations under the isotropy subgroup $S(\mathrm{U}(2)\times\mathrm{U}(1))$, and so must be treated separately. This is clear if only because the Dirac operator on the total space of spinors has a nonzero index for a generic spin$^{\mathbb{C}}$-structure.[Var06].

The formalism that will be chosen in the sequel to define fuzzy $\mathbb{C}P(2)$ shall now be applied to the sphere. It uses a two-particle Fock space to represent the Hilbert space in terms of raising and lowering operators. This was essentially done already in [GKP96], but will be presented here in a simpler manner that still provides the necessary results.

Let $|0\rangle$ be the vacuum state in Fock space, and define a_i and a_i^*, for $i = 1, 2$, as the annihilation and creation operators, respectively.

The usual relations are taken to hold, namely

$$[a_i, a_j^*] = \delta_{ij}, \tag{5.1}$$

$$[a_i, a_j] = [a_i^*, a_j^*] = 0. \tag{5.2}$$

A generic Fock state is then formed out of a sum of states of the form

$$|l, m\rangle := \frac{1}{\sqrt{l!m!}} (a_1^*)^l (a_2^*)^m |0\rangle. \tag{5.3}$$

The operator $N := \sum_{i=1}^{2} a_i^* a_i$ serves as the usual number operator, and evaluates to multiplication by $k = l + m$ on vectors of the form (5.3). Clearly, the space of k particle states is of dimension $k + 1$, and has a basis consisting of vectors of the form above. The Lie algebra $\mathfrak{su}(2)$ has a natural action on the whole Fock space using the defining representation. Let $X_{ij} = -\bar{X}_{ji}$ be a Lie algebra element in this representation. Then one can map

$$X_{ij} \mapsto a_i^* X_{ij} a_j, \tag{5.4}$$

which then acts on the Fock space. In fact, a simple application of the commutation relations shows that the operator N commutes with the representation matrices, so that the subspace of k-particle states is itself a representation of $\mathfrak{su}(2)$. Let $\tilde{\sigma}_j$, $i = 1, 2, 3$, be the usual (Hermitian) Pauli matrices. Then a basis of $\mathfrak{su}(2)$ can be fixed using the matrices $\sigma_i = \frac{i}{2}\tilde{\sigma}_i$. With respect to this basis, the operator

$$\sigma_3 = \frac{i}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{5.5}$$

is diagonal on the space of n-particle states with spectrum $m = \frac{n}{2}, \frac{n}{2} - 1, \ldots, -\frac{n}{2}$, clearly identifying the space with the irreducible $\mathfrak{su}(2)$ representation of spin $\frac{n}{2}$. Label this space by V_n. The fuzzy analogues of the chiral spinors are now defined as

$$S_n^+ \coloneqq V_n \otimes V_{n+1} \tag{5.6}$$

for the right-handed spinors, and

$$S_n^- = V_{n+1} \otimes V_n \tag{5.7}$$

for the left-handed spinors. The total bundle of (Dirac) spinors is then $S_n = S_n^+ \oplus S_n^-$. A straightforward application of the Clebsch-Gordan decomposition shows that the bundles have the same decomposition into irreducible $\mathfrak{su}(2)$ as the corresponding bundle of spinors on S^2, up to the cut-off value of $2n+1$. Within this formalism the chirality operator Γ is defined from the start to be $+1$ on S_n^+ and -1 on S_n^-. Consider the operator $d_n : S_n^+ \to S_n^-$ defined by

$$d_n = a_i^* \otimes a_i, \tag{5.8}$$

where summation convention implied. The Dirac operator on the spinor bundle is defined to be

$$D_n = \begin{pmatrix} 0 & d_n^* \\ d_n & 0 \end{pmatrix}. \tag{5.9}$$

Proposition 5.1.1. *The spectrum of D_n is given by*

$$\lambda_r = \pm (r+1), \tag{5.10}$$

where $r = 0, 1, \ldots n$. It thus agrees with the spectrum of the Dirac operator for Dirac spinors on S^2 up to truncation at n.

Proof. Note that

$$D_n^2 = \begin{pmatrix} d_n^* d_n & 0 \\ 0 & d_n d_n^* \end{pmatrix} = \begin{pmatrix} c + \frac{1}{4} & 0 \\ 0 & c + \frac{1}{4} \end{pmatrix}, \tag{5.11}$$

where c is the quadratic Casimir on the whole tensor product space, which using the normalisation above acts on the irreducible representation V_m via the scalar $\frac{m}{2}\left(\frac{m}{2}+1\right)$. The spectrum now follows from the Clebsch-Gordan decomposition into irreducible representations. □

There exists a real structure j that acts on Fock space by mapping a_1 to a_2 and a_2 to $-a_1$, in combination with complex conjugation. This can be extended to a real structure J on S_n that additionally swaps the left and right factors in the tensor products. This J satisfies $J^2 = -1$, $JD_n = D_n J$ and $J\Gamma = \Gamma J$, showing that the resulting construction as $KO-$dimension $s = 2$ as desired.

The above construction can be easily extended to other spin$^{\mathbb{C}}$ structures on S^2.The fundamental spinor bundles of such structures are obtained from the real spinor bundle of S^2 using a twist by a respective line bundle. At the level of the fuzzy version one simply has to choose the bundles

$$S_n^{m+} := V_n \otimes V_{n+1+m} \tag{5.12}$$

and

$$S_n^{m-} := V_{n+1} \otimes V_{n+m}, \tag{5.13}$$

and choosing the Hilbert space module $S_n = S_n^{m+} \oplus S_n^{m-}$. This amounts precisely to tensoring the original spinor bundle with the line bundle of Chern number m. Note that in this case the real structure is no longer well defined, which is an inevitable consequence of working with a nontrivial spin$^{\mathbb{C}}$-structure. The definition of the Dirac operator is the same as the spin-case, with the only difference that the spectrum of D_n^2 is now perturbed by Chern number m. This is in fact immediately clear from the Clebsch-Gordan decomposition of the new bundles.

5.2 A Fuzzy Version of CP2

5.2.1 CP2 bundle representations

The first identities regarding tensor products of SU(3) representations will be stated in more generality in terms of SU(n), since the generalisation to the higher dimensional case is immediate. The

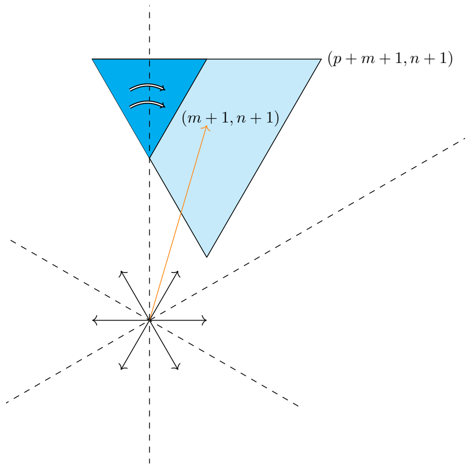

Figure 5.1: An example of the application of the Racach-Speiser algorithm to the tensor product representation $(p, 0) \otimes (m, n)$ of SU(3). Up to relabelling this agrees exactly with case 1 of Proposition 5.2.1. The weight diagram of $(p, 0)$ is shifted so that it is centred about $(m+1, n+1)$. The weights of interest are those in the large triangle in the diagram reachable from highest weight $(p+m+1, n+1)$ using roots. The top-left region is folded into the positive Weyl chamber using a single Weyl reflection, and as all weights have multiplicity one, the overlapping regions cancel each other out. The remaining weights that are present in the lightly shaded strip are then shifted by the vector $(-1, -1)$. The resulting weights that are still positive after the shift contribute a highest weight representation to the tensor product decomposition, with multiplicity equal to the multiplicity of the weight.

proof itself anyway relies on the SU(3) case.

Theorem 5.2.1. *For any nonnegative integer n one has*

$$(n, 0, \ldots, 0) \otimes (0, 0, \ldots 0, n) = \bigoplus_{j=0}^{n} (j, 0, \ldots, 0, j).$$

Proof. It will be sufficient to prove the theorem for SU(3), since the geometry of the weight lattice for higher groups in the A_n series allows one to restrict the calculation to the 2-dimensional face of the positive Weyl chamber that is spanned by fundamental weights

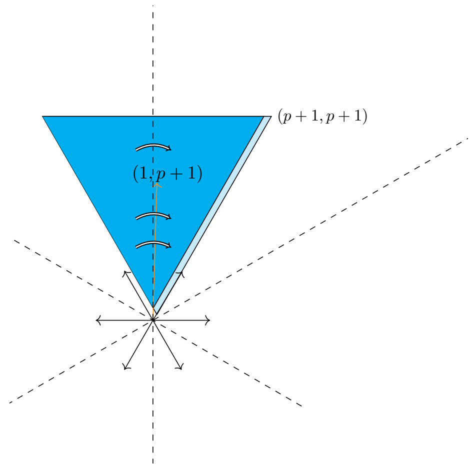

Figure 5.2: The application of the Racah-Speiser algorithm to the tensor product representation $(p, 0) \otimes (0, p)$ of SU(3). Here, the only set of weights in the diagram that are not cancelled by the Weyl reflection lie on the edge of the diagram spanned by the vector $(1, 1)$.

w_1 and w_{n-1}.

The proof is essentially a straightforward application of the Racah-Speiser algorithm. See figure 5.2.1 for a simple visualisation of the general process, and 5.2 for the exact process as it applies to the current theorem.

Geometrically, one starts by taking the weight diagram of $(0, p)$, an upside-down equilateral triangle, and translating it by the vector $(p + 1, 1)$ in the weight lattice. At this point the only part of the diagram that does not lie inside the positive Weyl chamber is a triangle that is then mapped into the positive Weyl chamber using a Weyl reflection. The multiplicities of all the weights in the diagram are equal to one, and since the fold is enacted by a Weyl reflection of sign -1, the overlapping areas cancel each other. Particular to case at hand, the edge of the Weyl chamber that served as the line of reflection was displaced by the root β from the apex of the original upside-down triangle. As a result, the folding cancels all weights in this diagram *except* the edge that connects the apex to the highest weight $(p+1, p+1)$. Shifting back by the Weyl vector ρ, this is diagonal segment connecting $(0,0)$ with (p,p), and so the integral weights that lie on this segment are the only ones whose corresponding irreducible representations contribute to the tensor product, with multiplicity one. □

The benefit of using the Racah-Speiser algorithm over other methods of computing the tensor product becomes apparent in the proof of the following proposition, which is an extension of the previous result.

Proposition 5.2.1. *Fix a nonnegative integer p, together with any irreducible representation with highest weight (m,n) of* SU(3)*.*
(1) Whenever $m \leq p$, one has

$$(p,0) \otimes (m, p+n) = \bigoplus_{k=0}^{m} \Big[\bigoplus_{j=0}^{p-k} (j+m-k, j+n+2k) \Big]. \tag{5.14}$$

(2) The cases when $p < m$ all give the same result as when $m = p$.

Proof. The proof is similar to that of Theorem 5.2.1. The only difference is that the weight diagram of $(p,0)$, that was originally shifted by $(0,p)+\rho$, is now shifted by an additional vector (n,m), where of course m,n are nonnegative. This can only have the effect of shifting more of the region that was in the Weyl chamber neighbouring the positive Weyl chamber into the latter one. Note that translation by $(0,n)$ shifts the weight diagram vertically, and thus does note change the axis of reflection. So it is only m that shifts the axis of reflection, by the same amount of units along the lower-left side of the diagram. This adds an additional m segments that do not cancel to the one that contributed originally, starting on the lower-left side and ending on the top side. This proves (1). Of course, whenever $m = p$, the whole weight diagram is in the positive Weyl chamber and thus contributes to the decomposition. Increasing m further does not serve to change this, and so the formula remains identical in the case where $p < m$ to the $m = p$ case. □

Similar relations hold for negative values of n. For subsequent applications it will be enough to state them for when m is at most one.

Proposition 5.2.2. *Fix a nonnegative integer p, and let $m, n \leq p$ be integers such that $m = 0, 1$. Then in terms of representations of* SU(3)*, one has the tensor product decomposition*

$$(p, 0) \otimes (m, p-n) = \bigoplus_{k=0}^{m} \Big[\bigoplus_{j=0}^{p-k} (j + n - m + 2k, j + m - k) \Big]. \quad (5.15)$$

Proof. The proof is essentially the same as that of Proposition 5.2.1. The main difference is with the regions of the weight diagram that cancel out during the application. There are several edge cases to consider, namely the cases where $(m, -n) = (0, -1), (1, -1), (1, -2)$, since each introduces a distinct intersection pattern with all the other Weyl chambers. But in all such cases, for any additional region that is mapped onto the positive Weyl chamber using an even product of Weyl reflections there corresponds an identical region in some other Weyl chamber that is mapped onto the positive one using an odd product of Weyl reflections, and the additional regions cancel each other. The only contributions that change the multiplicities in the positive Weyl chamber arise from the two Weyl chambers that neighbour the positive one, and the weights that exist in the region which remains after cancellation precisely correspond to the the ones in the proposition. □

The key results that will be used in construction follow immediately from Propositions 5.2.1 and 5.2.2.

Theorem 5.2.2. *Let (m, n) be an irreducible representation of the subgroup $K = S(\mathrm{U}(2) \times \mathrm{U}(1))$ of $\mathrm{SU}(3)$, where either (1) $0 \leq m \leq n$, or (2) $m = 0, 1$ and n is any negative integer. Then in terms of representations of $\mathrm{SU}(3)$, one has*

$$\lim_{N \to \infty} (N, 0) \otimes (m, N + n) \cong \mathrm{Ind}(m, n). \tag{5.16}$$

Proof. Evidently, it is enough to consider integers N that are large enough, in the sense that $m, |n| < N$.

(1) Note that in Proposition 5.2.1 and in part 1 of Proposition 4.3.3 it was shown that both the tensor product $(N, 0) \otimes (m, N + n)$ and $\mathrm{Ind}(m, n)$ decompose into the direct sum of $m + 1$ series of irreducible representations, with the only difference being that for the former the series terminate at an irreducible representation of order $O(N)$. It is clear then that as $N \to \infty$, the former decomposition converges into the latter.

(2) This case arises in the same way, by comparing Proposition 5.2.2 and part 2 of Proposition 4.3.3. □

Corollary 5.2.1. *Let* $T^N_{p,q} = (N,0) \otimes (p, N+q)$*, and define*

$$\begin{aligned} T^{N+}_m &= T^N_{(0,m)} \oplus T^N_{(0,3+m)} \\ T^{N-}_m &= T^N_{(1,1+m)}. \end{aligned} \tag{5.17}$$

Then

$$\lim_{N\to\infty} T^{N+}_m = S^+_m, \tag{5.18}$$

and

$$\lim_{N\to\infty} T^{N-}_m = S^-_m. \tag{5.19}$$

Proof. This is a direct consequence of Theorem 5.2.2, since the representations that the right hand side of (5.17) converge precisely to the induced representations that correspond to the m-th spin$^{\mathbb{C}}$-structure on $\mathbb{C}P(2)$. □

Thus, the module of SU(3) representations given by $T^N_m := T^{N+}_m \oplus T^{N-}_m$ converges in the large N limit to the bundle of total spinors obtained from the m-th spin$^{\mathbb{C}}$ structure. Note that convergence here is understood in terms of the decomposition of the modules into into irreducible representations of G. This is analogous to the case of spherical harmonics on the sphere, that can be truncated at some total angular momentum value l. As this value is increased, one recovers more and more of the complete orthonormal set of spherical harmonics, the total set of which serves as a separable basis of the Hilbert space of smooth functions on the sphere. Here, a similar procedure occurs in the sense that, as $N \to \infty$, an orthonormal basis of the finite dimensional module T^N_m converges to a separable basis in $\Gamma^\infty(S_m)$, whose completion is precisely $L^2(S_m)$.

Hence, the module T_m^N serves as a promising candidate for a Hilbert space $\mathcal{H}_N$ to be used in defining a fuzzy $\mathbb{C}P(2)$. Indeed, this seems to agree with previous efforts[GS99] to define such a fuzzy space.

To continue with the current investigation, though, it will be necessary to switch to a different formulation of the module. The reason for this is as follows. Once the module H_N is fixed as a direct sum of tensor product representations, say T_m^N itself, the next important step is to determine the Dirac operator. For it to be SU(3)-equivariant, it must be constructed using intertwiners between the right handed and left handed spinors, which in the case above are simply the T_m^{N+} and T_m^{N-} terms. But since all tensor product terms now have the same irreducible representation $(N, 0)$ on the left hand side of the tensor product, no such linear operators can be constructed, since they would necessarily have to replace one irreducible representation on the right hand side of the product with another one. For example, one would need to find an intertwiner between $(N, 0) \otimes (0, N+m)$ and $(N, 0) \otimes (1, N+1+m)$, which clearly requires an invariant operator from $(0, N + m)$ to $(1, N + 1 + m)$. As is well known from Schur's lemma, this cannot be done.

The solution is to instead use the following definition for the module of spinors.

Definition 5.2.1. Define the total module of spinors for fuzzy $\mathbb{C}P(2)$ with its m-th spin$^{\mathbb{C}}$structure as the module $S_m^N := S_m^{N+} \oplus S_m^{N-}$, where

$$\begin{aligned} S_m^{N+} &= \left(S_m^+\right)_1 \oplus \left(S_m^{N+}\right)_2 = \\ &\qquad (N-m,0)\otimes(0,N)\oplus(N-m-1,0)\otimes(0,N+2), \\ S_m^{N-} &= (N-m-1,1)\otimes(0,N+1). \end{aligned} \tag{5.20}$$

It is first necessary to see that in the large N limit this agrees with the module S_m.

Proposition 5.2.3. *The bundles agree in the limit of $N \to \infty$ with the total spinor bundle associated with the mth spin$^{\mathbb{C}}$ structure.*

Proof. Consider first the module $\left(S_m^{N+}\right)_1 = (N-m,0)\otimes(0,N)$. If $0 \leq m$, it can be expanded as

$$(N-m,0)\otimes(0,N) = (N-m,0)\otimes(0,(N-m)+m), \tag{5.21}$$

and since the limit $N \to \infty$ agrees with the limit $N-m \to \infty$ for fixed m, and one evidently recovers by 5.2.2 the module $\mathrm{Ind}(0,m)$. If $m \leq 0$, expand

$$(N-m,0)\otimes(0,N) = (N+|m|,0)\otimes(0,N) = \overline{(N,0)\otimes(0,N+|m|)}, \tag{5.22}$$

where the relation between an irreducible representation and its dual $\overline{(p,q)} = (q,p)$ has been used. Expanding using 5.2.1 one gets

$$(N-m,0)\otimes(0,N) = \bigoplus_{j=0}^{N} \overline{(j,j+|m|)} = \bigoplus_{j=0}^{N} (j+|m|,j). \tag{5.23}$$

Note that Theorem 5.2.2 implies that in the large N limit this converges to $\mathrm{Ind}(0,-|m|) = \mathrm{Ind}(0,m)$.

The module $\left(S_m^{N+}\right)_2 = (N-m-1,0)\otimes(0,N+2)$ can be expanded when $-3 \leq m$ as

$$(N-m-1,0)\otimes(0,N+2) = (N-m-1,0)\otimes(0,(N-m-1)+(m+3)), \tag{5.24}$$

and when $m \leq -3$ as

$$(N-m-1,0)\otimes(0,N+2) = \overline{(N+2,0)\otimes(0,N+2-(m+3))}. \tag{5.25}$$

A similar application of the propositions as before shows that for all values of m, this converges in the large N limit to $\mathrm{Ind}(0,3+m)$. Hence, S_m^{N+} converges to $\mathrm{Ind}(0,m)\oplus\mathrm{Ind}(0,3+m) \cong S_m^+$. Similarly, one finds that S_m^{N-} converges to S_m^-. □

With the modules $\mathcal{H}_N := S_m^N$ now determined for all positive N, the next step is to find a Dirac operator D_N for each such module.

5.2.2 The Dirac operator

Inspired by the new fuzzy sphere, the Dirac operator will be constructed using a Fock-space representation. The first step will be to understand how irreducible representations of SU(3) are constructed concretely in this space.

Let a_i, where $i = 1, 2, 3$, transform as a basis of the defining representation $(1, 0)$ of SU(3). Similarly, let b_i transform in the complex conjugate of the defining representation, $(0, 1)$. Construct the Fock space using a_i, b_j as lowering operators with respective raising operators a_i^* and b_j^*, subject to the relations

$$\begin{aligned}[a_i, a_j^*] =& \delta_{ij}, \\ [b_i, b_j^*] =& \delta_{ij},\end{aligned} \tag{5.26}$$

with all other commutators vanishing. Note that within the Fock space representation, the operators

$$N_a = \sum_{j=1}^{3} a_j^* a_j \tag{5.27}$$

and

$$N_b = \sum_{j=1}^{3} b_j^* b_j \tag{5.28}$$

are the number operators for the a particles and the b particles respectively. Then a vector of the form

$$Q_{i_1 i_2 \ldots i_p j_1 j_2 \ldots j_q} a_{i_1}^* \cdots a_{i_p}^* b_{j_1}^* \cdots b_{j_q}^* |0\rangle, \tag{5.29}$$

where summation convention is implied, contains p particles of species a and q particles of species b.

The Fock space above has a natural $SU(3)$ action defined explicitly as follows. Focus will be on the Lie-algebra level with the Lie group obtained by exponentiation. Let T^σ, where $\sigma = 1, 2, \ldots, 8$ be an orthonormal basis of the Lie algebra $\mathfrak{su}(3)$ in its defining representation, normalised so that

$$Tr[T^{\sigma *} T^\tau] = \delta_{\sigma\tau}. \tag{5.30}$$

These matrices can be mapped into operators on the Fock space as

$$M(T) \coloneqq a_i^* T_{ij} a_j + b_i^* \bar{T}_{ij} b_j, \tag{5.31}$$

where the complex conjugation on the second factor is due to the transformation properties of the b operators. Using the commutation relations 5.26 it follows easily that

$$[M(T), M(S)] = a_i^* \left[T, S\right]_{ij} a_j + b_i^* \left[\bar{T}, \bar{S}\right]_{ij} b_j = M([T, S]), \tag{5.32}$$

turning the Fock space into a representation space of $\mathfrak{su}(3)$. In fact, one sees that the tensors N_a and N_b are SU(3) invariant under this group and so the subspaces of constant particle number are SU(3) representation spaces. Consider the subspace spanned by vectors with p particles of type a and none of type b, namely those of the form

$$Q_{i_1 i_2 \ldots i_p} a_{i_1}^* \cdots a_{i_p}^* |0\rangle. \tag{5.33}$$

There are clearly $\binom{p+2}{2}$ linearly-independent monomials in the a's that span this subspace. This is precisely the dimension of the $\mathfrak{su}(3)$ representation space $(p, 0)$, identifying them as representation spaces. A similar thing happens within the subspace spanned by q particles of type b, which thus transform as the $(0, q)$ irreducible representation. Furthermore, taking the subspace with p particles of type a and q of type b result in the representation space $(p, 0) \otimes (0, q)$.

A simple consequence of Proposition 5.2.1 is that

$$(p,0)\otimes(0,q)\cong\bigoplus_{r=0}^{min(p,q)}(p-r,q-r), \tag{5.34}$$

so in the case of SU(3) representations the Fock space of a and b particles contains all the irreducible finite dimensional representations of SU(3).

In order to obtain a generic (p,q) representation, it is necessary to restrict to the relevant invariant subspace of the Fock subspace $(p,0)\otimes(0,q)$. The condition on tensors in $(p,0)\otimes(0,q)$ is that they be traceless under contractions between an a-index and a b-index. In the Fock space representation, this translates to the requirement that any state $|Q\rangle$ in $(p,0)\otimes(0,q)$ satisfies the relation $a_j b_j\cdot|Q\rangle=0$. Denote the relevant element by

$$\Pi=a_j b_j. \tag{5.35}$$

One finds that at the level of operators

$$[\Pi,\Pi^*]=N_a+N_b+3, \tag{5.36}$$

a relation which will be used freely below.

Definition 5.2.2. The Dirac operator is defined in terms of the following two operators.

$$\begin{aligned} d_1:(N-m-1,1)\otimes(0,N+1)\to(N-m,0)\otimes(0,N) \\ d_1=a_i^* b_j\otimes b_k\epsilon^{ijk}, \end{aligned} \tag{5.37}$$

where ϵ^{ijk} is the usual totally anti-symmetric (Levi-Civita) tensor in three dimensions, and

$$\begin{aligned} d_2 : (N-m-1,1) \otimes (0,N+1) &\to (N-m-1,0) \otimes (0,N+2) \\ d_2 = \sqrt{N-m+1} \cdot b_j \otimes b_j^* &- \frac{1}{\sqrt{N-m+1}} a_k^* a_j b_j \otimes b_k^*. \end{aligned} \tag{5.38}$$

Note that the condition $N > m+1$ on N is implicit at the level of the representation space. Choose a basis of the total spinor bundle using the decomposition $S_m = (S_m^+)_1 \oplus S_m^- \oplus (S_m^+)_2$. Then the Dirac operator in this basis has the form

$$D_m = \begin{pmatrix} 0 & d_1 & 0 \\ d_1^* & 0 & d_2^* \\ 0 & d_2 & 0 \end{pmatrix}. \tag{5.39}$$

A priori, either of the operators d_1 and d_2 may have an adjoint with image existing in the larger space $(N-m-1,0) \otimes (0,1) \otimes (0,N+1)$. In other words, applying the Dirac operator on any vector χ in, say $(N-m-1,0) \otimes (0,N+2)$, may result, once applying the Dirac operator, in the vector $d_2^*(\chi)$ that is in the space $(N-m-1,0) \otimes (0,1) \otimes (0,N+1)$, rather than in the domain of d_2 in S_m. That this is not the case will follow from the following.

Proposition 5.2.4. *The Dirac operator 5.39 is well defined on the module S_m and is* SU(3)*-equivariant.*

Proof. The states χ in the relevant space $(N-m-1,1) \otimes (0,N+1) \subset (N-m-1,0) \otimes (0,1) \otimes (0,N+1)$ are those states that satisfy the condition $(\Pi \otimes 1)(\chi) = 0$. So for d_1 and d_2 to be compatible with the structure of the module S_m, they must identically satisfy $(\Pi \otimes 1) \circ d_1^* = 0$ and $(\Pi \otimes 1) \circ d_2^* = 0$.

Note first that for the d_1 part of the Dirac operator, any operator of the form

$$d_1' = \alpha a_i^* b_j \otimes b_k \epsilon^{ijk} \tag{5.40}$$

for some complex number α satisfies

$$(\Pi \otimes 1) \circ d_1'^* = \bar{\alpha}(a_l b_l \otimes 1)(b_j^* a_i \otimes b_k \epsilon^{ijk}) = \bar{\alpha}(b_j^* b_l + \delta_{lj}) a_l a_i \otimes b_k \epsilon^{ijk}, \tag{5.41}$$

where the commutation relations have been used throughout. The first term on the right vanishes automatically, since the factor b_l can be commuted to act on vectors in the representation $(N-m, 0)$ to annihilate them, while the second term is the product of a term symmetric in the indices i, j with the ϵ tensor that is antisymmetric. Hence the product of these operators vanishes identically and is well defined on S_m for all (nonzero) α. The normalisation $\alpha = 1$ will be shown later on to give the desired spectrum.

The evaluation of d_2 involves a few more steps. The lowest order terms in the a and b operators that map from S_m^- to $(S_m^+)_2$ are given by

$$f = b_j \otimes b_j^* \tag{5.42}$$

$$e = a_k^* a_j b_j \otimes b_k^* = (a_k^* \otimes b_k^*)(\Pi \otimes 1), \tag{5.43}$$

though they do not satisfy the condition $(\Pi \otimes 1) \circ d_2^* = 0$ independently. It is necessary to solve for the coefficients in the following expression

$$d_2' = Af + Be, \tag{5.44}$$

which for simplicity will be taken to be real. Note that using the commutation relations one sees that

$$\Pi \otimes 1 \cdot e^* = \Pi\Pi^*\Pi \otimes 1 \cdot d_2^* = (N - m + 1)\Pi \otimes 1 \cdot d_2^*, \tag{5.45}$$

so that

$$(\Pi \otimes 1)d_2'^* = [A + (N - m + 1)B]\,(\Pi \otimes 1)f^*, \tag{5.46}$$

which evidently gives the condition

$$A = -(N - m + 1)B \tag{5.47}$$

on the coefficients. Again, the global factor chosen to normalise d_2 will be shown to yield the desired spectrum later on. In fact, it is the only normalisation that can be chosen to make the terms in the spectrum of d_2 independent of N up to the cutoff value.

The equivariance of d_1 and d_2 under SU(3) transformations is a direct consequence of the way they have been constructed, using the tensors δ_{ij} and ϵ^{ijk} that are invariant under the action in the defining representation of SU(3). Indeed, let $L(T) := M(T) \otimes 1$ be the action of $\mathfrak{su}(3)$ element T using the representation on the left-hand side factor of the tensor product, and $R(T) := 1 \otimes M(T)$ its action on the right. Then a straightforward calculation shows that

$$[d_1, L(T) + R(T)] = 0, \tag{5.48}$$

and

$$[d_2, L(T) + R(T)] = 0, \tag{5.49}$$

for all matrices T in $\mathfrak{su}(3)$. This automatically extends to the adjoint operators d_1^* and d_2^*, since T^* is also in the Lie algebra, and so the Dirac operator as a whole satisfies this invariance relation. □

Remark 1. The terms above are not the only ones that result in a Dirac operator with the desired properties. For example, the next highest-order terms that may contribute to d_2 are the terms $f\Pi\Pi^*$ and $f\Pi^*\Pi$. One can then solve for the operator

$$d_2' = \tilde{A}f + \ Be + \tilde{C}f\Pi^*\Pi + Df\Pi\Pi^*, \tag{5.50}$$

which can in fact always be recast as

$$F = Af + Be + Df\Pi^*\Pi, \tag{5.51}$$

using $A = \tilde{A} + D(N - m + 3)$, and $C = \tilde{C} + D$. Using the commutation relations, the condition that the image of $d_2'^*$ lies in the correct subspace yields the condition

$$A = -(N - m + 1)(B + D) \tag{5.52}$$

on the remaining coefficients. Evidently this gives a family of solutions with the one chosen for d_2 included. As it stands, the choice made throughout this investigation of $C = D = 0$ is motivated by simplicity.

The remainder of this section will be devoted to the determination of the spectrum of D_m, and the demonstration that it agrees with the typical geometry in the large N limit. The spectrum is given by the following

Theorem 5.2.3. *(1) For a given N the spectrum of d_1 and d_2 is as follows. The operator d_1 has spectrum*

$$\lambda_r = \sqrt{r(r+2) + |m|(r+1) - m}, \tag{5.53}$$

for $r = 0, 1, \ldots, \min(N-m, N)$, with value λ_r having multiplicity $\mathrm{mult}(\lambda_r) = \frac{1}{2}(r+1)(r+|m|+1)(2r+|m|+2)$. The operator d_2 has spectrum

$$\rho_r = \sqrt{r(r+2) + |m+3|(r+1) + m + 3} \tag{5.54}$$

for $r = 0, 1, \ldots, \min(N-m-1, N+2)$, with value ρ_r having multiplicity $\mathrm{mult}(\rho_r) = \frac{1}{2}(r+1)(r+|m+3|+1)(2r+|m+3|+2)$.
(2) The spectrum of the Dirac operator D_m is given by combining the two sequences together with their negatives into $\{\pm\lambda_r\}$ and $\{\pm\rho_r\}$, each index ranging as above and with the multiplicities of the negative values equal to the multiplicities of the respective positive ones. Moreover, the Dirac operator has index

$$\mathrm{Index}(D_m) = \frac{(m+1)(m+2)}{2}. \tag{5.55}$$

Proof. (1) Note first that

$$d_2 d_1^* = \left(\sqrt{N-m+1}\cdot a_i b_l b_j^* \otimes b_l^* b_k^* \right.$$
$$\left. - \frac{1}{\sqrt{N-m+1}} a_l^* a_m a_i b_m b_j^* \otimes b_l^* b_k^* \right) \epsilon^{ijk} =$$
$$\left(\sqrt{N-m+1}\cdot a_i \otimes b_j^* b_k^* - \frac{1}{\sqrt{N-m+1}} a_l^* a_j a_i \otimes b_l^* b_k^* \right) \epsilon^{ijk} = 0, \tag{5.56}$$

which follows from the commutation relations together with the fact that the b_j operators vanish on states in $(N-m, 0)$. The final equality follows since both terms are a product of even terms by the odd ϵ tensor. Thus, the pair of operators d_1 and d_2 decomposes S_m^- into two orthogonal parts, each mapped isomorphically onto by one of the two operators. Hence, spectrum can be evaluated separately for each operator.

To calculate the spectrum of d_1, note that

$$d_1 d_1^* = \epsilon^{ijk} \epsilon^{lmn} \left(a_i^* b_j^* a_m \otimes b_k b_n^* \right). \tag{5.57}$$

Using manipulations similar to the ones above one finds that

$$d_1 d_1^* = N_a \otimes (N_b + 2) - a_k^* a_i \otimes b_k^* b_i, \tag{5.58}$$

viewed as an operator from the space $(N-m, 0) \otimes (0, N)$ to itself. Letting $\Phi \coloneqq a_k^* a_i \otimes b_k^* b_i$ one has

$$d_1 d_1^* = (N-m)(N+2)(1 \otimes 1) - \Phi \tag{5.59}$$

In a parallel fashion, one finds for d_2 that

$$d_2 d_2^* = (N+2)(N-m+1)(1 \otimes 1) - \Phi, \tag{5.60}$$

viewed as an operator from the space $(N-m-1,0) \otimes (0, N+2)$ to itself. Thus, the spectra of both operators are determined by evaluating Φ on each irreducible representation. To evaluate this term, one can use the completeness relation

$$\sum_\sigma T^\sigma_{ij} \bar{T}^\sigma_{kl} + \frac{1}{3}\delta_{ij}\delta_{kl} = \delta_{ik}\delta_{jl}, \tag{5.61}$$

that holds for the $\mathfrak{su}(3)$ basis chosen above. Now one can use the fact that

$$\Phi = a_j^* a_k \otimes b_j^* b_k = \delta_{jl}\delta_{km} a_j^* a_k \otimes b_l^* b_m, \tag{5.62}$$

to split it as

$$\Phi = \sum_\sigma a_j^* T^\sigma_{jk} a_k \otimes b_l^* \bar{T}^\sigma_{lm} b_m + \frac{1}{3} N_a \otimes N_b. \tag{5.63}$$

Recall that the a-particles transform under the fundamental representation of $\mathfrak{su}(3)$ while the b-particles transform under the complex conjugate of that representation. Thus, labelling individually the matrices of the former representation by A^σ and the latter by B^σ, the first term in (5.63) is precisely $\sum_\sigma A^\sigma \otimes B^\sigma$. Now one can use the relation arising from the tensor product of representations to write this term as

$$
\begin{aligned}
2\sum_{\sigma} A^{\sigma} \otimes B^{\sigma} = \\
&\sum_{\sigma} (A^{\sigma} \otimes 1 + 1 \otimes B^{\sigma}) \cdot (A^{\sigma} \otimes 1 + 1 \otimes B^{\sigma}) - \\
&\qquad \sum_{\sigma} A^{\sigma} A^{\sigma} \otimes 1 - \sum_{\sigma} 1 \otimes B^{\sigma} B^{\sigma}. \quad (5.64)
\end{aligned}
$$

Now, with respect to the Hermitian and orthogonal basis, the quadratic Casimir of $\mathfrak{su}(3)$ is simply

$$
c = -\sum_{\sigma} T^{\sigma} T^{\sigma}. \quad (5.65)
$$

An important fact of the representation space $(N - m - 1, 0) \otimes (0, N + 2)$ is that each tensor factor transforms purely as one type of particle, i.e the representation on the left is given by A while that on the right by B. This allows one to identify the last two terms on the right hand side of (5.64) with the quadratic Casimirs c_1 and c_2 of the left side and the right side of the tensor product, respectively. Meanwhile, the first term of (5.64) is precisely the quadratic Casimir c of the total representation space. Therefore,

$$
\Phi = \frac{1}{2}(c_1 + c_2 - c) + \frac{1}{3} N_a \otimes N_b. \quad (5.66)
$$

In terms of the representation labels, the quadratic Casimir has the same form as in the previous chapter, given by equations (4.29).

Using the value of Φ, one now has

$$d_1 d_1^* = \frac{1}{2}c - m(\frac{1}{3}m + 1). \tag{5.67}$$

Now, the explicit decomposition of $(S_m^+)_1$ depends on the value of m. There are two cases to consider.

Case I, $0 \leq m$:

In this case the decomposition into irreducible representations is given by

$$(N-m, 0) \otimes (0, N) \cong \bigoplus_{r=0}^{N-m} (r, r+m). \tag{5.68}$$

Restricting to the irreducible representation factor $(r, r+m)$ the operator $d_1 d_1^*$ acts as multiplication by the scalar

$$r(r+2) + mr. \tag{5.69}$$

Case II, $m \leq 0$:

In this case one has the decomposition

$$(N-m, 0) \otimes (0, N) \cong \bigoplus_{r=0}^{N} (r+|m|, r), \tag{5.70}$$

where now $d_1 d_1^*$ acts on $(r+|m|, r)$ as the scalar

$$r(r+2) + |m|(r+1) - m. \tag{5.71}$$

Note that in both cases, the scalar values are nonnegative. Using the spectrum of $d_1 d_1^*$, together with the knowledge that it is a positive semidefinite operator, one knows that the eigenvalues of d_1 can be obtained by taking the square root of each eigenvalue. The spectrum of d_1 is thus given by

$$\lambda_r = \pm\sqrt{r(r+2)+|m|(r+1)-m}, \tag{5.72}$$

where $r = 1, 2, \ldots \min(N-m, N)$, in the representation space $(r, r+m)$ or $r+|m|, r)$ depending on whether $m \geq 0$ or $m \leq 0$, respectively. In fact, the multiplicity for all cases of m is

$$\mathrm{mult}(\lambda_r) = \frac{1}{2}(r+1)(r+|m|+1)(2r+|m|+2). \tag{5.73}$$

The spectrum of d_2 is obtained in a similar fashion. Using the expression for Φ one has

$$d_2 d_2^* = \frac{1}{2}c - m(\frac{1}{3}m+1), \tag{5.74}$$

As before, there are two cases to evaluate based on the sign and magnitude of the integer m.

<u>Case I, $-3 \leq m$:</u>

In this case the tensor product decomposes as

$$(N-m-1, 0) \otimes (0, N+2) \cong \bigoplus_{r=0}^{N-m-1} (r, r+m+3). \tag{5.75}$$

The operator $d_2 d_2^*$ acts as a scalar on each irreducible representation factor. For a given $r = 0, 1, \ldots, N-m-1$ one finds that the scalar is

$$\frac{1}{2}c - \frac{1}{3}m^2 - m = r^2 + r(m+5) + 2(m+3). \qquad (5.76)$$

<u>Case II, $m \leq -3$:</u>

In this case the main difference is that the tensor product decomposition is the sum

$$(N-m-1,0) \otimes (0,N+2) \cong \bigoplus_{r=0}^{N+2}(r+|m|-3,r). \qquad (5.77)$$

Again, for a particular value of r the operator $d_2 d_2^*$ now acts as multiplication by the scalar

$$c - \frac{1}{3}m^2 + |m| = r^2 + r(|m|-1). \qquad (5.78)$$

Taking the square root of this nonnegative spectrum, one finds that the spectrum of d_2 for all values of m is given by

$$\rho_r = \pm\sqrt{r(r+2) + |m+3|(r+1) + (m+3)} \qquad (5.79)$$

in the representation space $(r, r+m+3)$ or $(r+|m|-3, r)$ depending on whether $m \geq -3$ or $m \leq -3$. Yet again, there is a symmetry between the dimension of the representation in the two cases, with the multiplicity of ρ_r is given by

$$\mathrm{mult}(\rho_r) = \frac{1}{2}(r+1)(r+|m+3|+1)(2r+|m+3|+2). \qquad (5.80)$$

(2) The Spectrum of D_m immediately follows from (1). That both the positive and negative values of each eigenvalue appear in equal

multiplicities is evident as always from the condition between the chirality operator Γ and the Dirac operator, $\Gamma D = -D\Gamma$, which guarantees the symmetry of the spectrum about zero. The only part of the spectrum that remains to be determined is that of the zero modes. The module decomposition of subsection 5.2.1 shows that the zero modes only arise from right handed spinors, i.e. the submodule S_m^+, can only arise from irreducible representation of factor $(0, m)$ or $(|m|, 0)$, whenever they exist. This is evident from the spectrum above: When $m \geq 0$, only d_1 has eigenvalue zero, on the irreducible representation $(0, m)$. When $m \leq -3$, only d_2 has eigenvalue zero, on the irreducible representation $(|m|, 0)$. When $m = -1, -2$, neither operator has zero modes. In all cases this multiplicity results in an index of

$$\text{Index}(D_m) = \frac{(m+1)(m+2)}{2} \tag{5.81}$$

□

Corollary 5.2.2. *In the limit $N \to \infty$ the spectrum of D_m on the module S_m agrees with the spectrum of the Dirac operator determined by the m-th spin$^{\mathbb{C}}$-structure on $\mathbb{C}P(2)$.*

Proof. This is immediate from the spectrum of D_m, since it agrees with that of the m-th spin$^{\mathbb{C}}$ structure up to the N-dependent cut-off. As $N \to \infty$ this discrepancy between the spectrum of the fuzzy $\mathbb{C}P(2)$ and the spectrum obtained from the geometry of $\mathbb{C}P(2)$ vanishes. □

Chapter 6

Discussion

The work presented above was split into two parts, the common theme to which is the approximation of spaces in a pragmatic way using the language of noncommutative geometry. The results of each project will be discussed separately.

6.1 Commutative Lattice Models

With the minimal development of the formalism of finite commutative spectral triples in Chapter 3, it was possible already to reconstruct a simple fermionic state-sum model on the circle. This application opens up the field to further investigations, since many choices were made along the way to obtain the circle model. A traditional approach following on from this model may aim to extend it, by either simply increasing the number of Fermions, or, more interestingly, by seeing whether more complicated gauge fields can be incorporated into the model. It may be hoped that the formalism of almost-commutative manifolds may be applied here, with role of the (commutative) manifold geometry taken up by the discrete finite commutative spectral geometry.

More important to the original aim of this investigation, it will be vital to find further lattice geometries that can be obtained using the formalism. The geometric models solved here relied heavily on the structure of the spaces considered, be it the symmetry of the spaces or their inductive definition. For the formalism to stand on its own, it will be necessary to investigate whether a larger class of lattices may be obtained by its use. Natural systems to investigate include higher dimensional lattices with periodic boundary conditions, and higher dimensional simplices. These are especially important if one wishes to find whether higher-dimensional objects may be approximated within the formalism. Further results have been obtained in the recent weeks beyond those presented in Chapter 3, and the plan is currently to produce a preprint leading up to publication in the near future. Addressing the aforementioned problems will serve as a starting point in a programme developing commutative lattice models, vindicating the power of the power of noncommutative geometry with regards to the age-old belief that it has nothing to say about lattice field theories.

6.2 Fuzzy Coadjoint Orbits

The steps taken here to describe the fuzzy version of $\mathbb{C}P(2)$ have not yet yielded a spectral triple consistent with the chosen definition of a fuzzy space. The primary mathematical object required to complete the the definition is that of the algebra acting on the Hilbert space, but with the Dirac operator being fixed, the algebra must satisfy the property encoded by the first order condition. In fact, a 'spectral' triple is easily obtained by taking the direct sum of the matrix algebras acting on the left factors of the $\mathbb{C}P(2)$ spinor bundle. This is the approach that most fuzzy constructions have taken up to now, but it involves completely forgoing the first order condition as a part of the definition of a spectral triple. The easiest way forward would be to abandon it altogether. But, the condition is relevant in capturing the continuum geometry of a space, for the Dirac operator is by definition first order differential, and so necessarily satisfies this condition. Indeed, even formal approaches that eschew the first order condition in the primary definition of a spectral triple maintain that it must be recovered by the standard model at observable energy scale[CCV13]. The mechanism that is expected to impose it may be some sort of symmetry breaking, but no model of such a process has ever been explicitly proposed.

The absence of a first order condition was already present in redefined nontrivial fuzzy sphere of section 5.1, where the only ingredient not present in the construction was the algebra. The Hilbert space S_n^m has an obvious left-action of the algebra $A = M_n(\mathbb{C}) \oplus M_{n+1}(\mathbb{C})$, but this algebra does not satisfy the first order condition. In fact, the condition cannot be satisfied by any subalgebra of A that remains invariant under the unitary action of SU(2), excluding trivial cases such as when the subalgebra algebra is commutative.

A hint toward the solution in the case of the sphere is provided by the following fact.

Proposition 6.2.1 ([Bar25])**.** *Let S_n be the Hilbert space associated to the (real) spinor bundle, and let $\bar{S}_{n-1}$ be the same bundle of one dimension less and with its chirality swapped. Then there is a unitary equivalence*

$$S_n \oplus \bar{S}_{n-1} \cong M_n(\mathbb{C}) \otimes \mathbb{C}^4, \tag{6.1}$$

which is equivariant under the group action of $\mathrm{SU}(2)$*. Moreover, let $U : S_n \oplus \bar{S}_{n-1} \to M_n(\mathbb{C}) \otimes \mathbb{C}^4$ be the unitary equivalence map. Then for the Dirac operators one has*

$$U \begin{pmatrix} D_n & 0 \\ 0 & -D_{n-1} \end{pmatrix} U^{-1} = \begin{pmatrix} 0 & d_{GP} \\ d_{GP} & 0 \end{pmatrix}, \tag{6.2}$$

where d_{GP} is the Dirac operator of Grosse and Prešnajder[GP95].

The Hilbert space on the right hand side of the equivalence is evidently the Hilbert space of the fuzzy sphere construction of section 2.4. Similarly, the operator on the right of (6.2) is the Dirac operator in the same construction. It was seen that these do fit into a consistent spectral triple definition, with the algebra compatible with the first order condition. This suggests that the relation of a fuzzy geometry to the first order condition may be more subtle. Indeed, while the nontrivial formulation of fuzzy sphere itself may not have an algebra that satisfies the condition directly, its doubling does satisfy the condition. This preliminary result suggests that the first order condition, as heretofore conceived, may be valid only when one talks of 'trivial' bundles. This may not be surprising when one considers the representation theory of the matrix algebras considered here, which are Morita equivalent to copies of the fields over which they are defined. Thus, their algebraic K-theory is trivial. This formally expresses the fact that modules over these matrix algebras can only express trivial, viz. product, modules. In developing nontrivial bundle structures on matrix geometries, the saving grace of spectral triples may perhaps be the presence of the Dirac operator and its relation to the algebra via some algebraic condition. Where the first order condition was, a new one may need to be found.

To support this view it is necessary to find further examples of the trivialisation of fuzzy spaces. In fact, preliminary results suggest that the fuzzy $\mathbb{C}P(2)$ construction itself may be trivialised. If this is the case it would remain to show that an algebra on the total trivialised bundle satisfies the first order condition, and that it corresponds to the algebra of functions on $\mathbb{C}P(2)$ in the continuum limit.

Other fuzzy spaces need to be developed. For symmetric spaces, the apparatus developed here allows for direct application to other symmetric spaces, be they the higher dimensional complex projective spaces $\mathbb{C}P(n)$ or the coadjoint orbits of other simple Lie groups that are at the same time symmetric spaces. Interestingly, the trivialisations of the fuzzy spaces mentioned above seem to be related to the fact that the coadjoint orbits can be immersed in the dual Lie algebra of the symmetry group, with the trivial bundle corresponding to the restriction of the spinor space of the dual Lie algebra to the coadjoint orbit. This construction may serve as a unifying mechanism to trivialise the complex or real spinor bundles on all coadjoint orbits. Whence one may investigate further signatures of the first order condition for so called 'nontrivial' spinor bundles on fuzzy spaces.

Finally, the construction of fuzzy spaces that correspond to coadjoint orbits that are not symmetric spaces may require further tools. The most natural family of connections that are associated with the group structure is not Levi-Civita, with the corresponding Dirac operator having cubic terms[Agr03]. This still guarantees that it is self-adjoint, and so can still be used in the definition of a spectral triple. But it requires further understanding of role of torsion within the noncommutative framework.

Bibliography

[AA18] M. F. Atiyah and D. Anderson. *K-Theory*. CRC Press, Mar. 2018. ISBN: 9780429493546. DOI: 10.1201/9780429493546. URL: https://www.taylorfrancis.com/books/9780429962097.

[ABS64] M. F. Atiyah, R. Bott, and A. Shapiro. "Clifford modules". In: *Topology* 3.SUPPL. 1 (1964), pp. 3–38. ISSN: 00409383. DOI: 10.1016/0040-9383(64)90003-5.

[Agr03] I. Agricola. "Connections on naturally reductive spaces, their Dirac operator and homogeneous models in string theory". In: *Communications in Mathematical Physics* 232.3 (2003), pp. 535–563. ISSN: 00103616. DOI: 10.1007/S00220-002-0743-Y/METRICS. URL: https://link.springer.com/article/10.1007/s00220-002-0743-y.

[Ale+01] G. Alexanian et al. "Fuzzy CP2". In: *Journal of Geometry and Physics* 42.1-2 (Mar. 2001), pp. 28–53. DOI: 10.1016/S0393-0440(01)00070-5. arXiv: 0103023v2 [hep-th]. URL: http://arxiv.org/abs/hep-th/0103023%20http://dx.doi.org/10.1016/S0393-0440(01)00070-5.

[Bar07] J. W. Barrett. "Lorentzian version of the noncommutative geometry of the standard model of particle physics". In: *Journal of Mathematical Physics* 48.1 (2007), p. 12303. ISSN: 00222488. DOI: 10.1063/1.2408400. arXiv: 0608221 [hep-th]. URL: https://doi.org/10.1063/1.2408400.

[Bar15] J. W. Barrett. "Matrix geometries and fuzzy spaces as finite spectral triples". In: *Journal of Mathematical Physics* 56.8 (Feb. 2015), p. 82301. ISSN: 00222488. DOI: 10.1063/1.4927224. arXiv: 1502.05383. URL: https://doi.org/10.1063/1.4927224%20http://arxiv.org/abs/1502.05383%20http://dx.doi.org/10.1063/1.4927224.

[Bar22] J. W. Barrett. "Introduction to Non-commutative Geometry". Lectures given at the Erwin Schrödinger Institute, Vienna. 2022. URL: https://johnwbarrett.wordpress.com/ncgnotes/.

[Bar25] J. W. Barrett. "Quantum Gravity from Non-Commutative Geometry". Talk given at conference: Applications of Noncommutative Geometry to Gauge Theories, Field Theories and Quantum Space-Time. CIRM, Marseille. 2025. URL: https://www.cirm-math.fr/Schedule/screen_display.php?id_renc=3196.

[BF14] L. Boyle and S. Farnsworth. "Non-commutative geometry, non-associative geometry and the standard model of particle physics". In: *New Journal of Physics* 16.12 (2014), p. 123027.

[BG16] J. W. Barrett and L. Glaser. "Monte Carlo simulations of random non-commutative geometries". In: *Journal of Physics A: Mathematical and Theoretical* 49 (24 May 2016). ISSN: 17518121. DOI: 10.1088/1751-8113/49/24/245001.

[BG25] J. W. Barrett and J. Gaunt. "Finite spectral triples for the fuzzy torus". In: *Journal of Geometry and Physics* 207 (Jan. 2025), p. 105345. ISSN: 0393-0440. DOI: 10.1016/J.GEOMPHYS.2024.105345. arXiv: 1908.06796. URL: http://arxiv.org/abs/1908.06796.

[BKL13] J. W. Barrett, S. Kerr, and J. Louko. "A topological state sum model for fermions on the circle". In: *Journal of Physics A: Mathematical and Theoretical* 46 (18 2013). ISSN: 17518113. DOI: 10.1088/1751-8113/46/18/185201.

[BR86] A. Barut and R. Raczka. *Theory of Group Representations and Applications*. World Scientific, Nov. 1986. ISBN: 978-9971-5-0216-4. DOI: 10.1142/0352. URL: https://www.worldscientific.com/worldscibooks/10.1142/0352.

[CC97] A. H. Chamseddine and A. Connes. "The spectral action principle". In: *Communications in Mathematical Physics* 186.3 (1997), pp. 731–750. ISSN: 00103616. DOI: 10.1007/s002200050126. arXiv: 9606001 [hep-th].

[CCM07] A. H. Chamseddine, A. Connes, and M. Marcolli. "Gravity and the standard model with neutrino mixing". In: *Advances in Theoretical and Mathematical Physics* 11.6 (2007), pp. 991–1089. ISSN: 10950753. DOI: 10.4310/ATMP.2007.v11.n6.a3. arXiv: 0610241 [hep-th].

[CCV13] A. H. Chamseddine, A. Connes, and W. D. Van Suijlekom. "Beyond the spectral standard model: Emergence of Pati-Salam unification". In: *Journal of High Energy Physics* 2013.11 (Nov. 2013), pp. 1–36. ISSN: 10298479. DOI: 10.1007/JHEP11(2013)132/METRICS. arXiv: 1304.8050. URL: https://link.springer.com/article/10.1007/JHEP11(2013)132.

[CL91] A. Connes and J. Lott. "Particle models and noncommutative geometry". In: *Nuclear Physics B (Proceedings Supplements)* 18.2 (1991), pp. 29–47. ISSN: 09205632. DOI: 10.1016/0920-5632(91)90120-4.

[CL92] A. Connes and J. Lott. "The Metric Aspect of Noncommutative Geometry". In: *New symmetry principles in quantum field theory.* 1992, pp. 53–93. DOI: 10.1007/978-1-4615-3472-3_3.

[CM07] A. Connes and M. Marcolli. *Noncommutative geometry, quantum fields and motives.* Vol. 55. American Mathematical Soc., 2007.

[Con13] A. Connes. "On the spectral characterization of manifolds". In: *Journal of Noncommutative Geometry* 7 (1 2013), pp. 1–82. ISSN: 16616952. DOI: 10.4171/JNCG/108.

[Con85] A. Connes. "Non-commutative differential geometry". In: *Publications Mathématiques de L'Institut des Hautes Scientifiques* 62.1 (Dec. 1985), pp. 41–144. ISSN: 00738301. DOI: 10.1007/BF02698807. URL: http://link.springer.com/10.1007/BF02698807.

[Con94] A. Connes. *Non-commutative Geometry.* New York: Academic Press, 1994. DOI: 978-0121858605.

[Con95] A. Connes. "Noncommutative geometry and reality". In: *Journal of Mathematical Physics* 36.11 (Nov. 1995), pp. 6194–6231. ISSN: 0022-2488. DOI: 10.1063/1.531241. URL: /aip/jmp/article/36/11/6194/393935/Noncommutative-geometry-and-reality.

[Con96] A. Connes. "Gravity coupled with matter and the foundation of non-commutative geometry". In: *Communications in Mathematical Physics* 182.1 (Dec. 1996), pp. 155–176. ISSN: 0010-3616. DOI: 10.1007/BF02506388. URL: http://link.springer.com/10.1007/BF02506388.

[DAr22] M. D'Arcangelo. "Numerical simulation of random Dirac operators". PhD thesis. University of Nottingham, 2022. URL: https://eprints.nottingham.ac.uk/id/eprint/71106.

[DMS93] A. Dimakis, F. Muller-Hoissen, and T. Striker. "Noncommutative differential calculus and lattice gauge theory". In: *Journal of Physics A: Mathematical and General* 26.8 (1993), p. 1927.

[Dol+08] B. P. Dolan et al. "A universal Dirac operator and noncommutative spin bundles over fuzzy complex projective spaces". In: *Journal of High Energy Physics* 2008.3 (2008). ISSN: 11266708. DOI: 10.1088/1126-6708/2008/03/029.

[DR85] A. H. Dooley and J. W. Rice. "On Contractions of Semisimple Lie Groups". In: *Transactions of the American Mathematical Society* 289 (1 1985), p. 185. ISSN: 00029947. DOI: 10.2307/1999695.

[Fri00] T. Friedrich. *Dirac operators in Riemannian geometry.* American Mathematical Society, 2000. ISBN: 978-0-8218-2055-1.

[FS03] J. Fuchs and C. Schweigert. *Symmetries, Lie Algebras and Representations.* Cambridge University Press, 2003. ISBN: 9780521541190.

[Gel87] I. M. Gelfand. "Normierte Ringe". In: *Collected Papers.* Vol. 51. 1. 1987, pp. 181–201. DOI: 10.1007/978-3-642-61705-8_15.

[GKP96] H. Grosse, C. Klimčík, and P. Prešnajder. "Topologically nontrivial field configurations in noncommutative geometry". In: *Communications in Mathematical Physics* 178.2 (1996), pp. 507–526. ISSN: 00103616. DOI: 10.1007/BF02099460/METRICS. URL: https://link.springer.com/article/10.1007/BF02099460.

[GP95] H. Grosse and P. Prešnajder. "The dirac operator on the fuzzy sphere". In: *Letters in Mathematical Physics* 33.2 (Feb. 1995), pp. 171–181. ISSN: 0377-9017. DOI: 10.1007/BF00739805. URL: http://link.springer.com/10.1007/BF00739805.

[GS98] M. Göckeler and T. Schücker. "Does noncommutative geometry encompass lattice gauge theory?" In: *Physics Letters B* 434.1-2 (Aug. 1998), pp. 80–82. ISSN: 0370-2693. DOI: 10.1016/S0370-2693(98)00734-5. URL: https://www.sciencedirect.com/science/article/pii/S0370269398007345.

[GS99] H. Grosse and A. Strohmaier. "Towards a Nonperturbative Covariant Regularization in 4D Quantum Field Theory". In: (Feb. 1999). URL: https://arxiv.org/abs/hep-th/9902138v1.

[GVF01] J. M. Gracia-Bondía, J. C. Várilly, and H. Figueroa. *Elements of Noncommutative Geometry*. Boston, MA: Birkhäuser Boston, 2001. ISBN: 978-1-4612-6569-6. DOI: 10.1007/978-1-4612-0005-5.

[Hel79] S. Helgason. *Differential geometry, Lie groups, and symmetric spaces*. Vol. 80. Academic press, 1979.

[IT78] A. Ikeda and Y. Taniguchi. "Spectra and eigenforms of the laplacian on sn and pn(c)". In: *Osaka Journal of Mathematics* 15.3 (1978), pp. 515–546. ISSN: 00306126.

[Kir04] A. A. Kirillov. *Lectures on the Orbit Method*. American Mathematical Society, 2004.

[Kna96] A. W. Knapp. *Lie Groups Beyond an Introduction*. Birkhäuser Boston, 1996. ISBN: 978-1-4757-2455-4. DOI: 10.1007/978-1-4757-2453-0.

[KP21] M. Khalkhali and N. Pagliaroli. "Phase transition in random noncommutative geometries". In: *Journal of Physics A: Mathematical and Theoretical* 54 (3 Jan. 2021). ISSN: 17518121. DOI: 10.1088/1751-8121/abd190.

[Kra98] T. Krajewski. "Classification of finite spectral triples". In: *Journal of Geometry and Physics* 28.1-2 (Nov. 1998), pp. 1–30. ISSN: 0393-0440. DOI: 10.1016/S0393-0440(97)00068-5. arXiv: 9701081 [hep-th]. URL: https://www.sciencedirect.com/science/article/pii/S0393044097000685?via%7B%5C%%7D3Dihub.

[Lan97] G. Landi. *An Introduction to Noncommutative Spaces and Their Geometries*. Springer Verlag, 1997. ISBN: 978-3-540-14949-1.

[LM89] H. B. Lawson and M.-L. Michelsohn. *Spin Geometry*. (PMS-38). Princeton University Press, Apr. 1989. ISBN: 9780691085425. URL: http://www.jstor.org/stable/j.ctt1bpmb28.

[Mac75] G. W. Mackey. "On the Analogy between Semisimple Lie Groups and Certain Related Semi-Direct Product groups". In: *Lie Groups and Their Representations (Summer School of the Bolyai János Mathematical Society*. Ed. by I. Gelfand. Halsted Press, 1975, pp. 339–363.

[Mad92] J. Madore. "The fuzzy sphere". In: *Classical and Quantum Gravity* 9.1 (Jan. 1992), p. 69. ISSN: 0264-9381. DOI: 10.1088/0264-9381/9/1/008. URL: https://iopscience.iop.org/article/10.1088/0264-9381/9/1/008%20https://iopscience.iop.org/article/10.1088/0264-9381/9/1/008/meta.

[MS08] S. Murray and C. Sämann. "Quantization of flag manifolds and their supersymmetric extensions". In: *Advances in Theoretical and Mathematical Physics* 12.3 (2008), pp. 641–710. ISSN: 10950753. DOI: 10.4310/ATMP.2008.v12.n3.a5.

[Mur90] G. J. Murphy. *C*-algebras and operator theory*. Academic press, 1990.

[Ori09] D. Oriti, ed. *Approaches to Quantum Gravity*. Cambridge University Press, Mar. 2009. ISBN: 9780521860451. DOI: 10.1017/CBO9780511575549. URL: https://www.cambridge.org/core/product/identifier/9780511575549/type/book.

[Per24] C. I. Perez-Sanchez. "The loop equations for noncommutative geometries on quivers". In: (Sept. 2024). arXiv: 2409.03705. URL: https://arxiv.org/pdf/2409.03705.

[PS98] M. Paschke and A. Sitarz. "Discrete spectral triples and their symmetries". In: *Journal of Mathematical Physics* 39.11 (Nov. 1998), pp. 6191–6205. ISSN: 0022-2488. DOI: 10.1063/1.532623. arXiv: 9612029 [q-alg]. URL: /aip/jmp/article/39/11/6191/230904/Discrete-spectral-triples-and-their-symmetries.

[Ste06] C. A. Stephan. "Almost-commutative geometry, massive neutrinos and the orientability axiom in KO-dimension 6". In: *arXiv preprint hep-th/0610097* (2006).

[Van15] W. D. Van Suijlekom. *Noncommutative geometry and particle physics*. Springer Nature, 2015.

[Var06] J. C. Varilly. *An introduction to noncommutative geometry*. Vol. 4. European Mathematical Society, 2006.

[Wal18] N. R. Wallach. *Harmonic analysis on homogeneous spaces*. Second edition. Dover Publications, 2018.