Understanding the content, pedagogical and technological knowledge of beginning teachers using technology in relation to geometric constructions using dynamic software.

A thesis submitted for the Degree of Doctor of Philosophy at the

University of Nottingham, United Kingdom

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March 2024

ABSTRACT

This study explores the content, pedagogical, and technological knowledge (TPACK) of beginning teachers engaging in geometric constructions with dynamic geometry software. It aims to examine how carefully designed tasks can promote productive mathematical talk and dialogic learning among these teachers, improving their TPACK knowledge and deepening their understanding of the knowledge required to teach geometry in technology-rich environments.

To achieve this, the study involved designing effective geometric construction tasks as core components of a dynamic learning ecosystem, within an online platform's dynamic geometry software environment. These tasks incorporated principles such as scaffolding, collaborative paired learning (via platforms like Microsoft Teams with screen sharing), reflective practices, dynamic manipulation using GeoGebra, instrumental orchestration, Bruner's (1974) modes of representation, a balance of ostensive and non-ostensive objects, feedback mechanisms, meaningful goals, visible mathematics, and open-ended yet scaffolded task. Twelve beginning teachers from Mathematics postgraduate certificate in education (PGCE) programmes across four UK universities participated, with four teachers in phase one and eight in phase two. Data collection methods included video analysis, questionnaires, focus groups, and interviews. Analytical approaches employed interpretive video deductive coding, drawing on dialogic talk and learning principles, and the TPACK framework, with both inductive and deductive thematic analyses providing detailed insights into TPACK development for teaching geometric constructions.

The analysis revealed that certain design principles embedded in the tasks, as set out in the paragraph above, supported productive mathematical talk, dialogic learning and TPACK development among beginning teachers. In addition, participants identified specific features they felt best supported their TPACK knowledge-building, including task complexity, integration of multiple geometric concepts, structured sequencing with progression and scaffolding, alignment with relevant and authentic learning goals, engagement, motivation, and a balance of appropriate challenge and support. Findings also revealed that beginning teachers perceived effective geometry teaching in technology-integrated environments as manifold, requiring a solid understanding of geometric relationships, cross-curricular connections, classroom management in technological settings, software proficiency, and differentiated teaching strategies. This leads to the conclusion that for TPACK development for classroom practice, beginning teachers need a well-rounded skill set that integrates technological fluency, pedagogical strategies, and strong content knowledge.

Analysis of dialogue between beginning teachers in the study confirmed that the carefully designed tasks led to complex interactions of dialogic talk that in turn led to (dialogic and active) learning, critical thinking, and a deep understanding of geometric concepts. The learning ecosystem, within which the designed tasks played a key role, provides proof of concept and an example of how TPACK development for classroom practice of beginning teachers can be achieved.

The thesis concludes with practical recommendations advocating the integration of technology-rich tasks in mathematics teacher training to strengthen beginning

teachers' TPACK knowledge and preparedness. Recommendations for teacher training programmes, educational institutions, researchers, and educational technology developers include embedding technology in training, designing tasks that foster productive mathematics talk, promoting effective TPACK development, and recognising the diverse roles of teachers in technology-centred settings. This research makes a meaningful contribution to the discourse on technology integration in mathematics education, offering valuable insights for educators, researchers, and policymakers.

ACKNOWLEDGEMENTS

I am profoundly grateful to my esteemed supervisors, Professor Geoffrey Wake and Professor Andrew Noyes, whose unwavering support, prompt guidance, and generous mentorship have been instrumental throughout my doctoral study. Their inspiring supervision has shaped my academic journey and deepened my understanding of academic writing. The invaluable feedback they provided at each stage of this thesis significantly contributed to the depth of my analysis and interpretation of qualitative data. Despite the challenges faced during this winding journey, their constant encouragement served as a beacon of motivation. The relationship we have cultivated over the past four years feels more like a supportive family, and I am truly honoured.

I extend my sincere gratitude to Professor Mary Oliver, whose excellent advice as the internal assessor of my confirmation of status paper and as the annual reviewer of my progress has been invaluable. I deeply appreciate her commitment to ensuring the quality and rigour of my research.

Special thanks are due to Dr Marc North for his insightful suggestions on the initial design of my study tasks, and to Dr Marie Joubert for her thoughtful reading of my thesis and valuable suggestions. I am also indebted to Dr Nicola Bretscher for generously sharing insights from her own data analysis during her PhD, which influenced my analytical approach. I express heartfelt appreciation to Dr Ipek Saralar for her helpful suggestions and meticulous formatting of my confirmation of status paper, which laid the foundation for my thesis writing.

My gratitude also extends to all my colleagues at the School of Education, University of Nottingham, for providing the necessary facilities to support the successful completion of this study. I gratefully acknowledge the Initial Teacher Education directors, the PGCE course leaders, and the PGCE mathematics teams from Schools of Education in UK universities where my study participants were based. Their collaboration was vital to the success of this research.

I am indebted to the Ghana Scholarship Secretariat for its sponsorship, which significantly alleviated the financial burden of my doctoral journey.

To my family—my wife, Dorcas, and my children, Kwaku Awortwe and Ewura Bema—your unwavering support has been my anchor. I also extend my appreciation to my mother, Angelina Awortwe, and my siblings—Comfort, Dorcas, Hannah, Joana, Esther, and Emmanuel—for their constant encouragement. Special thanks go to all my cousins and the entire extended family for their enduring support and kind words. I am also thankful to my wife's family and in-laws for their steadfast encouragement throughout this journey. Your collective warmth has been a source of strength.

Finally, I wish to express my sincere appreciation to Professor Alison Clark-Wilson (External Examiner), Professor Mary Oliver (Internal Examiner), and Dr Eugenia Katartzi (Independent Chair) for their careful and thoughtful engagement during my viva voce examination. I am especially grateful to Professors Clark-Wilson and Oliver for their rigorous and constructive feedback, which prompted meaningful revisions that have strengthened the coherence, clarity, and academic rigour of this thesis. These revisions reflect my commitment to scholarly improvement and my gratitude for their insightful contributions to the final development of this work.

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CHAPTER 1: INTRODUCTION

This introductory chapter provides an overview of the thesis, starting with an explanation of the study's motivation. The contextual background of the research is then presented, focusing on the preparation of beginning or pre-service mathematics teachers in developing the required TPACK knowledge for teaching geometric constructions using dynamic geometry software. The chapter also outlines the purpose of the study and presents the research questions that guide the investigation. The theoretical framework that underpins the study is introduced briefly. This study's philosophical stances and research paradigms are explained and justified. Finally, the organisation and structure of the rest of the thesis are outlined.

1.1 The motivation for the study

The motivation for this study stems from my deep-rooted passion for modern digital technologies, particularly dynamic geometry software (DGS), and its transformative potential in mathematics education. My passion has been nurtured over years of personal experience and observation, from my early encounters with educational technology during secondary school to my continued exploration throughout tertiary education and professional endeavours. Throughout this journey, I have witnessed first-hand the profound impact that technology, such as DGS, can have on facilitating student learning in mathematics. From graphical calculators to sophisticated software like Autograph and GeoGebra, these tools can potentially engage students, enhance conceptual understanding, and unlock new pathways for exploration and discovery.

However, amidst the promise of these technologies, a glaring gap exists between their potential benefits and their limited integration into classroom instruction, particularly among beginning teachers. Despite the availability of tools like DGS, research indicates that many teachers struggle to effectively incorporate them into their teaching practices (Clark-Wilson, Robutti, & Sinclair, 2014; Ertmer, Ottenbreit-Leftwich, Sadik, Sendurur, & Sendurur, 2012; Galindo & Newton, 2017; Mouza, Sheridan, Lavigne, & Pollock, 2023). This discrepancy is often attributed to various factors, including a lack of expertise, limited access to training, and systemic barriers in educational institutions (Ruthven, 2018a).

As a result, the full potential of DGS and similar technologies remains largely untapped in mathematics education. This realisation has spurred my determination to explore this issue further, to understand the underlying challenges faced by beginning teachers in using DGS for teaching geometry. By conducting this study, I aim to shed light on these challenges and uncover insights that can inform the development of targeted interventions and support mechanisms for beginning teachers. Specifically, I seek to explore how beginning teachers acquire the technological pedagogical content knowledge (TPACK) necessary to effectively integrate DGS into their teaching. In addition, I aim to develop pedagogical strategies tailored to the unique needs and contexts of secondary school mathematics classrooms.

Ultimately, my goal is not only to bridge the gap between the potential of DGS and its implementation but also to contribute to the broader advancement of mathematics education. Empowering beginning teachers with the knowledge, skills, and confidence to harness the power of modern digital technologies like

DGS, I aspire to promote a learning environment centred on carefully designed tasks where all students can thrive and excel in mathematics.

1.2 Background and context of the study

The study was set within the context of professional development for pre-service teachers enrolled in postgraduate certificate in education (PGCE) programmes, with a focus on secondary mathematics teachers. Shulman's (1986) seminal work emphasised the critical role of teachers' knowledge and skills in effective mathematics teaching. This foundational concept underlines the significance of teachers possessing comprehensive content mastery and pedagogical expertise to facilitate meaningful learning experiences. This amalgamation of expertise can ensure that teachers grasp the content they teach and possess the instructional or pedagogical strategies to convey the content effectively.

This present study stresses the importance of preparing beginning teachers to navigate the complexities of content, pedagogy, and technology integration within this context. Beginning teachers must be equipped with a firm grasp of mathematical concepts and expert pedagogical techniques to cater for the different learning abilities of their pupils. To address these manifold demands, robust teacher education programmes are imperative. These programmes should provide beginning teachers with an understanding of mathematical content, empower them with pedagogical strategies, and foster their adaptability to meet the needs of pupils with different learning abilities.

The potential of technology integration in modern classrooms cannot be overstated. Technology can offer unprecedented avenues to engage students, foster active learning and provide a more profound understanding of complex mathematical concepts (Aliyu, Osman, Kumar, Talib, & Jambari, 2022; Asranqulova, 2023; Cardullo, Wilson, & Zygouris-Coe, 2018; Majeed, 2023; Serin, 2023). To harness these benefits, mathematics teachers must be well-versed in integrating technology into their teaching. This requires the development of their technological pedagogical content knowledge (TPACK) (Koehler & Mishra, 2009), enabling teachers to expertly employ specific tools and software to teach mathematical concepts with efficacy (Aliyu et al., 2022; Asranqulova, 2023; Cardullo et al., 2018; Majeed, 2023; Marbán & Sintema, 2021; Redmond-Sanogo, Stansberry, Thompson, & Vasinda, 2018; Serin, 2023; Wilson, Ritzhaupt, & Cheng, 2020). However, empirical evidence highlights a persistent hesitancy among mathematics teachers, including beginning teachers, to embrace technology fully in their teaching (Clark-Wilson et al., 2014; Ertmer et al., 2012; Galindo & Newton, 2017; Mouza, Yadav, & Ottenbreit-Leftwich, 2021; Zheng, Warschauer, Lin, & Chang, 2016). Various factors contribute to this reticence, including limited familiarity with the technology and software, inadequate training, time constraints, and concerns over classroom management. This reluctance hampers the realisation of technology's full potential in enhancing mathematics education and, hence, teaching and learning mathematics.

Among the avenues for technology integration, dynamic geometry software stands out for its capacity to revolutionise geometry teaching and other mathematics topics. The software can allow students to visualise and manipulate geometric figures, nurturing exploration and deep conceptual understanding. Nevertheless, effective integration of dynamic geometry software is fraught with challenges. Educators or teachers may lack the training and support required to fully exploit the software's capabilities, thus curbing its integration. Overcoming these hurdles necessitates tailored professional development and resources that equip teachers, especially beginning teachers, to seamlessly and effectively integrate dynamic geometry software into their teaching practices.

A crucial pivot towards addressing these challenges is ensuring experiences of using appropriate technology early, in Initial Teacher Education Programmes. Embedding experiences of using technology in pre-service education can support beginning teachers in developing the requisite skills, familiarity, and confidence to proficiently use technology tools in their future classrooms. Their experience must encompass theoretical grounding and hands-on application, empowering beginning teachers to explore various technologies, recognise their instructional potential, and develop pedagogical strategies that align with the use of technology.

Commencing such experiences of the use of technology from the outset of a teacher's education journey has the potential to ensure that teachers enter their careers primed to seamlessly blend technology into their mathematical instruction or teaching. This fusion of technology into their mathematical teaching can establish an enriched learning milieu with resources, tools, and interactive platforms that might kindle school pupils' engagement with mathematics in ways that can deepen their grasp of mathematical concepts. This fusion of technology, starting at the initial stages of beginning teachers' programmes of preparation for teaching, may develop a generation of mathematics teachers who are well-equipped to embrace technology as a powerful tool for enhancing student learning outcomes.

1.3 The problem of the study

This study addresses how we might tackle the problem of the limited integration of technology, particularly dynamic geometry software in mathematics teaching, in the context of beginning teachers. Many teachers hesitate to use technology tools owing to limited familiarity, lack of training, time constraints, and concerns about classroom management (Clark-Wilson et al., 2014; Ertmer et al., 2012; Galindo & Newton, 2017; Mouza et al., 2023). The central problem is the lack of comprehensive insights into the ways that carefully designed tasks can facilitate productive mathematical talk and dialogic learning among beginning teachers working with dynamic geometry software to develop their TPACK knowledge. While online platforms and DGS are increasingly adopted, there is insufficient evidence on designing tasks to maximise their effectiveness.

1.4 The aim/purpose/objective of the study

The study aims to explore and analyse ways in which carefully designed tasks can facilitate productive mathematical talk and dialogic learning among beginning teachers working with dynamic geometry software in ways that develop their TPACK knowledge. Moreover, the research seeks to understand how beginning teachers perceive, understand, and talk when exploring the knowledge needed to teach geometry in a technology-based environment. This investigation is intended to understand better the overall preparedness of beginning teachers in teaching geometry, particularly in conjunction with technological tools.

1.5 Research questions

The overall research question of the study is:

RQ0: What are the key features of efficacious geometric construction tasks within an online platform's dynamic geometry software environment that have the potential to support beginning teachers in their development of knowledge necessary to support their future teaching of geometric constructions using dynamic geometry software?

The following questions guide the study:

RQ1: In what ways can carefully designed tasks facilitate productive mathematical talk and dialogic learning among beginning teachers working with dynamic geometry software in ways that develop their TPACK knowledge?

RQ2: In what ways do beginning teachers perceive, understand and talk when exploring the knowledge needed for teaching geometry in a technology-based environment?

1.6 Significance of the study

This study greatly interests me as it addresses the crucial need to improve technology integration including dynamic geometry software (DGS) in mathematics education, especially among beginning teachers. The study aims to equip beginning teachers with early experiences of using dynamic geometry software to investigate a potential way of supporting the challenges that they often face in using technology tools effectively in their classrooms. This research has the potential to contribute valuable insights into improving teacher education programmes so that they might promote the seamless integration of technology in geometry teaching and provide a new generation of mathematics teachers proficient in using technology to potentially improve school pupils' learning outcomes.

1.7 Scope and delimitations

This research focuses on beginning teachers using dynamic geometry software in the context of geometric constructions. The study's limitations include the small sample size of beginning teachers, which may affect the generalisability of the findings to a larger population. The reliance on voluntary participation may introduce a self-selection bias, potentially affecting the representativeness of the sample.

1.8 Conceptual framework and theoretical background

The conceptual frameworks that underpin this study are the technological pedagogical content knowledge (TPACK), the zone of proximal development (ZPD), and dialogic learning. Dialogic learning is a foundational construct to support the analysis of the data collected.

1.8.1 Technological pedagogical content knowledge (TPACK)

The TPACK framework is central to this study, with the potential to offer a structured approach to integrating technology into pedagogy and content knowledge (Koehler & Mishra, 2009). It serves as a guiding paradigm to explore how beginning teachers can develop competencies across technology, pedagogy, and content domains, particularly in teaching geometric constructions using technology including dynamic geometry software.

1.8.2 The zone of proximal development (ZPD)

Rooted in Vygotsky's sociocultural theory, the zone of proximal development (ZPD) can be a crucial concept for understanding how learning and knowledge construction occur through social interactions (Daniels, Cole, & Wertsch, 2007; Vygotsky, 1978). The ZPD represents the gap between what learners can achieve independently and what they can achieve with guidance and support from others.

In this study, the ZPD theory (Vygotsky, 1978) informs the design, providing scaffolding and tailored support as beginning teachers develop their technological pedagogical content knowledge (TPACK). To operationalise the ZPD, I observed beginning teachers in postgraduate certificate in education (PGCE) classes. During these observations, I noted that some beginning teachers struggled with basic geometric constructions, such as constructing an equilateral triangle or a perpendicular line to a line segment. Their inability to perform these tasks highlighted gaps in their geometric construction knowledge. In addition, a pre-questionnaire was administered to participants, revealing that only four out of 12 had experience with dynamic geometry software in their university courses. Moreover, only one participant had used such software during their teaching practice. These findings provided insights into their current technological and content knowledge and informed the design of tasks aimed at improving their TPACK in the context of geometric constructions and dynamic geometry software.

The ZPD theory emphasises that learning can occur through interactions that bridge the gap between current knowledge and latent capabilities. To support beginning teachers' growth in TPACK concerning geometric constructions and dynamic geometry software, I promoted social interactions and employed geometric construction tasks alongside dynamic geometry software as scaffolding tools. This approach could enable beginning teachers to learn within their ZPD, gradually expanding their knowledge and skills through structured support (tasks) and peer collaboration.

1.8.3 Alignment of TPACK and ZPD

The combination of the TPACK framework and the ZPD theory forms a theoretical foundation for this study, aligning with its objectives. These frameworks offer inclusive perspectives on the interplay between technology, pedagogy, content knowledge, and collaborative discourse. Drawing on both these frameworks, this approach stresses the importance of dialogue and collaborative discussions in the co-construction of knowledge. It supports the study's aim of identifying how carefully designed tasks can facilitate productive mathematical talk and dialogic learning among beginning teachers working with dynamic geometry software. This approach can support their development of TPACK knowledge, increase their understanding of how they perceive, understand, and talk when exploring the knowledge needed to teach geometry in a technology-based environment.

1.8.4 Dialogic learning

Dialogic learning, an educational philosophy, prioritises interactive communication and purposeful discourse for knowledge construction (Mercer, 2002). In a dialogic learning environment, participants engage in collaborative inquiry, fostering critical thinking and social knowledge construction through questions, investigations, and shared exploration (Alexander, 2020). This approach views learning as a dynamic, collective process shaped by interactions (Flecha, 2000; Mercer, 2002).

A hallmark of dialogic learning can be its emphasis on respecting diverse perspectives. Participants can express thoughts openly, listen attentively, and educators (teachers) can guide an inclusive dialogue (Alexander, 2020). This may transcend traditional teaching, promoting co-construction of knowledge through collaborative efforts.

Reflective practice can be an integral part of dialogic learning, encouraging participants to introspect on thinking, dialogue, and the learning process (Schön, 2017). This intentional reflection has the potential to boost metacognition, applicable across diverse educational settings, thus promoting open dialogue, critical thinking, and the social construction of knowledge (Moon, 2013).

In this study, dialogic learning emerged naturally, becoming a foundational construct for the analysis of the data collected. The technological pedagogical content knowledge (TPACK) of beginning teachers was made visible through dialogue, including mathematical talk that emerged, enabling the analysis of how knowledge was co-constructed. As I observed and listened to the participants' conversations, it became clear that analysing their dialogue using dialogic talk and learning principles as analytical tools was crucial for understanding their knowledge construction process. This analysis of participants' conversations using dialogic talk and learning principles as tools is presented in Chapter 4 and Chapter 5A.

The study highlights dialogic learning's role in technology-enhanced learning for beginning teachers. The designed geometric construction tasks within an online platform's dynamic geometry software environment have the potential to facilitate diverse dialogic interactions, that can support the development of beginning teachers' knowledge for teaching geometric constructions. Emphasising the creation of spaces (tasks) that encourage open dialogue and collaboration, the study demonstrates how dialogic learning principles can naturally emerge in technology-enhanced environments, enriching collective knowledge co-construction.

1.9 Philosophical stances/research paradigm

In this section, I outline the philosophical underpinnings guiding my research, addressing epistemological, ontological, and axiological perspectives that shape the study's design. From an epistemological standpoint, I view knowledge as constructed socially and contextually, rather than objectively. My ontological position recognises that reality is relational, shaped by interactions and that learners interpret knowledge through cultural and historical contexts. Finally, the axiological foundation of this study values inclusivity, diverse perspectives, and collaborative learning. These perspectives provide a rationale for the social constructivist paradigm and qualitative methods chosen, ensuring coherence across the study's approach, design, and data interpretation.

1.9.1 Social constructivist approach

The social constructivist approach, central to my research, recognises that knowledge is actively constructed through social interactions and shared cultural contexts. This approach contrasts with traditional, deductive teaching methods, often criticised for encouraging rote learning (Tularam, 2018). Adopting a social constructivist lens, I focus on collaborative learning as a pathway for beginning teachers to co-construct knowledge in dynamic, technology-enhanced environments. Grounded in the theories of Vygotsky (1978) who emphasised the role of social interaction and the zone of proximal development (ZPD), and Piaget (1977), who highlighted active engagement in knowledge construction, this perspective shapes my study design. To facilitate this process, I encouraged beginning teachers to participate in collaborative tasks and discussions in an online collaborative environment, creating a social context in which they could construct their understanding of geometric constructions and the use of dynamic geometry software.

I contend that adopting a social constructivist approach can offer several benefits for knowledge construction. Through social interaction, beginning teachers can share ideas, perspectives, and interpretations, leading to an understanding of the subject matter. Collaborative learning environments have the potential to support critical thinking as individuals engage in dialogue, challenge assumptions, and consider alternative viewpoints (Thomas & Kell, 2021; Thompson, 2020). This approach may also encourage the exploration of multiple solutions, promoting creativity and innovation in problem-solving (Thomas & Kell, 2021; Thompson, 2020). Involving beginning teachers in such social interactions, I aimed to create an environment centred on carefully designed tasks that have the potential to stimulate active engagement, reflection, and deep learning.

In this study, I designed a socio-cultural learning environment that brought together carefully designed tasks that supported collaborative learning (in pairs), a dynamic geometry platform, all within an online communication system (in this particular instance Microsoft Teams). The collaborative tasks to be worked on with dynamic geometry software were specifically designed to encourage beginning teachers to engage in dialogue and problem-solving in pairs, advancing understanding and collaborative learning. These tasks aimed to facilitate social interactions that promote critical thinking, reflective questioning, and knowledge sharing, fostering a more profound understanding of geometric concepts. Through collaborative engagement, beginning teachers could share multiple perspectives, test ideas, and co-develop solutions (Thompson, 2020), thereby enriching their professional learning experience and knowledge of technology use in learning.

Furthermore, a social constructivist approach assumes that knowledge is inherently contextual and shaped by cultural influences (Thomas & Kell, 2021). Recognising the impact of cultural backgrounds, I aimed to create an inclusive learning environment that values diverse perspectives in mathematics education, with the goal of bridging understanding across educational contexts.

The constructivist paradigm, which can promote learner autonomy and ownership of learning, is embedded in the study's structure to encourage active learning and exploration. Through peer interactions in the ZPD, participants developed technical language and an understanding of geometric constructions. This stance contrasts with positivist views that see learning as objective and measurable; instead, I acknowledge learning as subjective and influenced by social interactions and contexts. The social constructivist approach which emphasises active engagement, problem-solving, and reasoning, can align well with the nature of mathematics education taught in schools (Thomas & Kell, 2021; Thompson, 2020). Mathematics is not merely knowledge to be memorised but a dynamic field that requires exploration, conjecture, and justification.

Through collaborative tasks within dynamic geometry software environments, beginning teachers could actively engage in mathematical investigations, developing a deep understanding of geometric concepts and techniques. This approach can promote the development of mathematical thinking and problem-solving skills, which I consider crucial for effective teaching and learning of geometry. Although direct instruction can have benefits in reducing misconceptions (Hattie, 2008; Kamii & Dominick, 1998; Kirschner, Sweller, & Clark, 2006), I argue that constructivist methods, especially with technology integration, can foster essential exploratory and critical skills for complex concepts.

1.9.2 Qualitative methods

Given my commitment to social constructivism, I selected a qualitative approach to investigate how beginning teachers develop TPACK for teaching geometry with dynamic geometry software. This method sits well with my epistemological belief in co-constructed, subjective knowledge. Qualitative research is well-suited to capture the complexity of participants' interactions and perspectives, providing rich, detailed insights beyond the quantitative scope. Using interviews, observations, video analysis, and open-ended questionnaires, I explored participants' experiences, revealing the depth of their learning and professional growth.

Selecting a small sample of beginning teachers resonates with qualitative principles prioritising depth over generalisability (Creswell & Poth, 2017). This choice allowed me to engage closely with each participant pair, facilitating rapport and in-depth discussion. Focusing on individual cases, I could examine each participant's unique experience, gaining insights into their challenges, needs, and evolving understanding of dynamic geometry software. This approach enabled a detailed interpretation of their learning process, making it possible to address diverse aspects of learning geometry with technology.

The qualitative approach also aligns with the study's goal to explore beginning teachers' subjective, context-specific experiences to capture the complexity of knowledge co-construction that would not be easily quantified. Quantitative methods, while useful for testing hypotheses, may limit the flexibility needed to explore participants' in-depth experiences and the processes underpinning their learning. Focusing on qualitative analysis, I aimed to investigate the contextual and relational factors influencing how beginning teachers develop TPACK knowledge and manage the challenges of technology-enhanced teaching.

Overall, the study's qualitative approach supports the constructivist goal of investigating the social and collaborative aspects of learning. This methodological choice reflects my epistemological stance that knowledge is socially constructed rather than objectively discovered, and my ontological position that reality is relational and shaped by interactions. Prioritising a qualitative framework, I embraced an approach that acknowledges the contextual, subjective, and co-constructed nature of knowledge development in technology-enhanced geometry teaching.

1.9.3 Two-phase study: pilot and main study

In this study, I employed a two-phase approach comprising a pilot study followed by the main study. This method allowed for the development and refinement of the interventions based on data collected from participants during each phase.

The pilot study (phase one) provided an initial framework for the intervention, enabling the identification of areas requiring improvement. Based on the insights gained from this phase, I made necessary modifications to the task designs and their implementation.

The main study (phase two) then incorporated these modifications to stand a chance of creating a more effective and comprehensive learning experience. This iterative process could ensure that the interventions were continually refined to better meet the needs of the participants.

Owing to the time limitations inherent in my PhD studies, conducting the research in two phases allowed for systematic data collection, analysis, and the implementation of improvements. This two-phase approach ensured that sufficient time was allocated to evaluate the effectiveness of the changes made after the pilot study, thus enhancing the overall quality of the intervention.

1.10 The structure of the thesis

This thesis comprises seven chapters, each contributing uniquely to the overarching goal of addressing the challenges faced by beginning teachers in integrating technology, particularly dynamic geometry software (DGS), into mathematics teaching.

Chapter 2 initiates my exploration with a review of relevant literature. It explores the role of technology in supporting teacher knowledge development, particularly for beginning teachers tasked with teaching geometric constructions. This chapter sets the stage for understanding the challenges surrounding technology integration in mathematics education and the necessity to empower beginning teachers with the skills to effectively use DGS.

Chapter 3 unfolds the carefully designed tasks within an online platform's DGS environment that can support beginning teachers' developing knowledge for teaching geometric constructions. Rooted in social constructivism, this chapter explains the foundational principles underpinning the creation of a conducive environment centred on carefully designed tasks to support beginning teachers to co-construct technological pedagogical content knowledge (TPACK) for geometry teaching using DGS.

Following this, Chapter 4 describes the research methods and design employed in this study. It delineates the strategies used to investigate the carefully designed tasks within an online platform's DGS environment in enhancing beginning teachers' TPACK knowledge for teaching geometric constructions.

Chapters 5A and 5B explore the analysis and findings of the study. Chapter 5A addresses RQ1, focusing on ways carefully designed tasks can facilitate productive mathematical talk and dialogic learning among beginning teachers working with DGS in ways that develop their TPACK knowledge. Chapter 5B tackles RQ2, exploring ways beginning teachers perceive, understand and talk when exploring the knowledge needed for teaching geometry in a technology-based environment.

In Chapter 6A, I engage in a discussion of the ways carefully designed tasks can facilitate productive mathematical talk and dialogic learning among beginning teachers working with dynamic geometry software in ways that develop their TPACK knowledge. This discussion illuminates the intricate interplay between dialogic learning (rich in different dialogic talk types guided by dialogic learning principles) and the development of TPACK knowledge.

Similarly, Chapter 6B discusses how beginning teachers perceive, understand and talk when exploring the knowledge needed for teaching geometry in a technology-based environment.

Finally, Chapter 7 draws together the threads of the study, encapsulating the key research findings and offering recommendations to address the challenges identified. It culminates with a call to action for the integration of technology into mathematics education for the empowerment of beginning teachers and the advancement of pedagogical practices.

1.11 The introductory chapter summary of the thesis

The introductory chapter of the thesis lays out the motivation, background, problem statement, aim, research questions, significance, scope, and theoretical framework of the study. It begins with the researcher's passion for modern digital technologies, particularly dynamic geometry software (DGS), and highlights the challenges beginning teachers face in integrating technology into teaching, especially in geometry. The study aims to address these challenges by providing effective training and strategies. Shulman's (1986) emphasis on teachers' knowledge stresses the importance of comprehensive content mastery and pedagogical expertise. The study aims to explore and analyse how carefully designed tasks can facilitate the development of beginning teachers' TPACK knowledge for teaching geometric constructions using dynamic geometry software. The research focuses on how these tasks promote productive mathematical talk and dialogic learning, as well as how beginning teachers perceive, understand, and discuss teaching in technology-focused contexts. The study's theoretical foundation includes the TPACK framework and the ZPD; it adopts a social constructivist perspective, qualitative methodology, and design-based research framework. The chapter concludes by outlining the thesis structure and providing a breakdown of the chapters and their content.

1.12 Closing remarks

In conclusion, this research holds significant potential in addressing a critical need within mathematics education: effective technology integration in teaching mathematics, specifically the teaching of geometric constructions, particularly among beginning teachers. Providing these teachers with early technology training, I aim to address the challenges they often face in using technology tools effectively in their classrooms. This endeavour carries implications for the future of education, with the potential to revolutionise teaching methods and elevate student learning outcomes.

The significance of this study extends beyond the immediate goal of equipping beginning teachers. It can contribute valuable insights to the broader field of teacher education. Identifying ways in which carefully designed tasks can facilitate productive mathematical talk and dialogic learning among beginning teachers working with DGS in ways that develop their TPACK knowledge, the study has the potential to inform the design and implementation of more effective teacher education programmes.

CHAPTER 2: LITERATURE REVIEW

2.1 Introduction

This literature review considers the essential role of technology in supporting teacher knowledge development, particularly its support for beginning teachers in teaching geometric constructions. It highlights the persistent challenge teachers face in integrating technology into mathematics teaching owing to factors including unfamiliarity and inadequate training. The primary aim is to equip beginning teachers with the necessary skills to effectively use dynamic geometry software (DGS) and other technological tools.

The review starts by contextualising teacher knowledge within the integration of technology into pedagogy, shifting from a focus on content knowledge to effective teaching methods. It discusses the emergence of technological knowledge as a new dimension of teacher knowledge, particularly pedagogical content knowledge (PCK), emphasising the integration of technological, pedagogical, and content knowledge as per the TPACK framework. The review further examines contextual factors surrounding technological pedagogical content knowledge (TPACK), particularly in supporting beginning mathematics teachers using DGS. It discusses the historical development and relevance of DGS, emphasising its interactive nature and pedagogical aspects.

The chapter also explores the intersection of dialogic pedagogy and digital technology, acknowledging the transformative potential despite challenges. It discusses the didactical principles in task design, advocating for student-centred learning with sociocultural approaches. The section on task design in DGS for beginning teacher education integrates and discusses principles from Bruner's

(1974) modes of representation, Rabardel's (1995) instrumental genesis, and Sfard's (2008) commognitive framework, demonstrating how these principles can support varied cognitive processes and accommodate different learning preferences.

2.2 Chronological outline of thinking about teacher knowledge

Throughout the twentieth century, thinking about the knowledge required for effective teaching has evolved significantly. Initially, the dominant belief was that teachers simply needed a thorough understanding of the content they were teaching (Grimmett & MacKinnon, 1992; Niess, 2008; Parkay & Stanford, 2008; Shulman, 1987). This perspective, however, lacked depth and failed to account for the complexities of teaching. In response, Shulman (1987), introduced a more detailed view, arguing that teaching demands more than just content expertise. He identified several types of knowledge essential for teachers, including content knowledge, general pedagogical knowledge, curriculum knowledge, pedagogical content knowledge (PCK), knowledge of learners, knowledge of educational contexts, and an understanding of educational ends, purposes and values (Shulman, 1987).

Shulman's (1986) introduction of pedagogical content knowledge was a critical moment in rethinking teacher knowledge. PCK, as he defined it, merges content knowledge with pedagogy and can be essential for effective teaching because it has the potential to address how subject matter is made understandable to students with diverse learning needs. While Shulman's (1986) framework initially focused on mathematics education, its implications reach across various subjects (Ball, Thames, & Phelps, 2008; Gess-Newsome, 2002; Grossman, 1990; Niess, 2005; Shulman, 1986). However, Shulman's (1986) framework left some gaps, particularly in its practical application. While the identification of PCK was a groundbreaking step, there remains ambiguity in how this knowledge should be applied across different subjects and educational settings. Much of the subsequent literature has sought to expand or clarify Shulman's (1986) categories, but these efforts have often complicated rather than simplified the practical realities of teacher knowledge.

Several institutions have sought to incorporate Shulman's (1986) ideas into teacher preparation programmes. For instance, Oregon State University restructured its programmes to focus on a comprehensive understanding of PCK, presenting it as the hub connecting various knowledge domains, including learners, pedagogy, curriculum, subject matter, and school (Niess, 2012; Suharwoto & Niess, 2001). While this approach acknowledges the complexity of teaching, it raises questions about whether such frameworks genuinely reflect the dynamic nature of teacher knowledge in practice.

A critical concern is that these models, while conceptually sound, risk over-compartmentalising (or over-categorising) teacher knowledge into distinct domains. In reality, teaching involves a continuous interplay of knowledge types. Rather than being neatly segmented, teacher knowledge must be adaptive, fluid, and responsive to diverse classroom contexts. This rigid categorisation can sometimes obscure the practical synthesis of knowledge that effective teaching requires.

Ball et al. (2008) further refined Shulman's (1986) ideas, arguing that understanding content for teaching required clearer distinctions between types of content knowledge. They introduced a model that divided content knowledge into common content knowledge (CCK), horizon content knowledge (HCK), and specialised content knowledge (SCK). Pedagogical content knowledge was similarly divided into knowledge of content and students (KCS), knowledge of content and teaching (KCT), and knowledge of content and curriculum (KCC). These domains are shown in Figure 2.1.

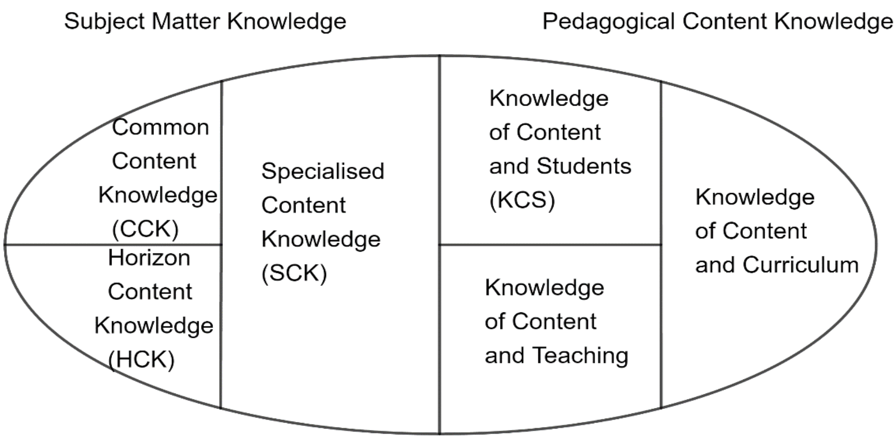


Figure 2.1 Domains of mathematical knowledge for teaching
(Ball et al., 2008)

Common content knowledge (CCK) was defined as the mathematical knowledge and skills that are used both within and outside of the teaching context in the same way. Specialised content knowledge (SCK) was defined as the mathematical knowledge and skills that are unique to teaching. This includes the ability to identify patterns in student errors and to use nonstandard methods that are effective in general. Knowledge of content and students (KCS) is the knowledge that teachers have about the specific students they teach. This requires an understanding of how students think and what their common conceptions and misconceptions are. Teachers must also be able to identify what interests and motivates students to learn, in order to tailor their teaching accordingly. Knowledge of content and teaching (KCT) is the amalgamation of knowledge about teaching and knowledge about mathematics. This requires teachers to understand how to sequence content for teaching and to evaluate the best instructional approaches and procedures for teaching specific content.

While this refinement provides more specificity, it also raises new challenges. The boundaries between CCK and SCK, for example, are not always clear in practice. Similarly, the placement of horizon content knowledge, which relates to an awareness of how topics are connected across the curriculum, remains ambiguous (Loewenberg Ball, Thames, & Phelps, 2008, p. 403). Is it part of subject matter knowledge or should it be seen as an aspect of PCK? These complexities make the model difficult to operationalise, as teaching is not a linear process where distinct categories of knowledge can be applied in isolation.

A major issue with the frameworks of Shulman (1986) and Ball, Thames and Phelps (2008) is their tendency to compartmentalise (or categorise) teacher knowledge. Teaching can be a dynamic, integrated practice where teachers must simultaneously draw on multiple forms of knowledge. While helpful for academic analysis, these models can be limiting when applied to real-world teaching, where adaptability and holistic understanding are critical. Teachers often need to shift between different knowledge domains in fluid ways, responding to the unique needs of their students and the content they are teaching.

In the context of this study, which aims to explore how beginning teachers engage with DGS centred on carefully designed tasks to develop TPACK, the limitations of these frameworks become particularly apparent. Both Shulman's

(1986) model and that of Ball, Thames and Phelps (2008) focus on content and pedagogy but do not adequately address the growing importance of technology in modern classrooms. This is a critical oversight in an era where digital tools seem to be increasingly shaping teaching and learning experiences.

The advent of digital technologies in the 1980s introduced a new dimension to teacher knowledge (American Association for the Advancement of Science, 1990) . Traditional frameworks did not account for the integration of technology into teaching. As computer-based technologies became more prevalent in education, the need for teachers to develop technological knowledge alongside pedagogical and content knowledge became clear (Parkay & Stanford, 2008; Roblyer, 2009). This gap led to the development of the TPACK framework, which highlights the need for teachers to integrate content, pedagogy, and technology effectively.

Despite the progress made in refining teacher knowledge models, the existing literature has not fully embraced the role of technology. This gap is especially relevant given this study's focus on how beginning teachers use dynamic geometry software to foster productive mathematical talk and dialogic learning in ways that can develop their TPACK knowledge. The introduction of technology into teacher knowledge frameworks like PCK or TPACK requires further development to reflect the complexities of teaching in technology-rich environments.

2.3 The emergence of technological aspect into teacher knowledge

Building upon the evolution of teacher knowledge frameworks, integrating technology into education seems to have become a central focus in the twenty-first century (Chander & Arora, 2021; Consoli, Désiron, & Cattaneo, 2023; Haleem, Javaid, Qadri, & Suman, 2022). The shift from traditional teaching methods to technology-enhanced learning environments presents both opportunities and challenges for educators. While technology can offer significant potential to deepen students' mathematical understanding, as demonstrated in studies (Noss et al., 2020; Radović, Marić, & Passey, 2019), its effective use requires teachers to develop new dimensions of knowledge beyond content and pedagogy. This necessity calls for a broader reconceptualisation of teacher knowledge, which traditional models, such as Shulman's (1987) pedagogical content knowledge (PCK), do not fully address.

As noted earlier, Shulman's (1987) introduction of PCK represented a crucial advancement in understanding the intersection of content and pedagogy. However, the framework might have been developed in a pre-digital age, leaving a significant gap in accounting for how technology mediates teaching and learning processes. Today, educational authorities worldwide encourage integrating technology into the classroom (Department for Education, 2020; Education Scotland, 2016; European Commission, 2020; UNESCO, 2020; Welsh Government, 2018), yet the challenge persists: it is not enough to merely introduce technology into teaching. Teachers must know how and when to use it effectively to promote meaningful learning.

This challenge is particularly relevant for beginning teachers. While many of them are comfortable using everyday digital tools like social media, they often lack the pedagogical knowledge required to use technology as a teaching and learning tool for educational purposes (Azad, 2023; Eslit, 2023). This gap stresses the urgency for teacher preparation programmes to address the integration of technology into pedagogical practice.

One promising and significant development in this area is the emergence of the technological pedagogical content knowledge (TPACK) framework (Koehler & Mishra, 2008; Mishra & Koehler, 2006) which builds on Shulman's (1987) original PCK model. TPACK reflects the interconnectedness of technology, pedagogy, and content knowledge, recognising that these knowledge domains must be integrated to create effective technology-enhanced learning environments. For mathematics teachers, particularly in geometry teaching, this means understanding how digital tools like DGS can promote conceptual learning and support mathematical reasoning.

However, the integration of technology into teacher knowledge, as described by the TPACK framework, is not without its difficulties. Combining these three domains of knowledge (content, pedagogy, and technology) is complex, and effective teaching in technology-rich environments requires an adaptive and flexible understanding of how these elements interact. It is not enough to treat each domain as a discrete entity, rather, teachers must synthesise them dynamically to respond to the needs of their students and the instructional goals of the lesson.

In the context of this study, which investigates how beginning teachers use dynamic geometry software to develop their TPACK knowledge, these challenges become particularly salient. While existing models such as PCK and

TPACK provide valuable conceptual insights, they also highlight the gaps that remain in fully operationalising the use of technology in real-world classrooms. This underlines the importance of exploring how carefully designed tasks, in combination with digital tools, can facilitate productive mathematical discussions and facilitate dialogic learning, ultimately enriching beginning teachers' technological pedagogical content knowledge. Critically reflecting on the gaps in traditional teacher knowledge frameworks and understanding how TPACK attempts to address these challenges, this study aims to contribute to an understanding of the complexities of teaching mathematics with technology. As digital tools become more integrated into classrooms, it becomes increasingly crucial to equip teachers with the knowledge and skills they need to use these tools and use them in ways that can improve learning.

2.4 The technological pedagogical content knowledge (TPACK) framework

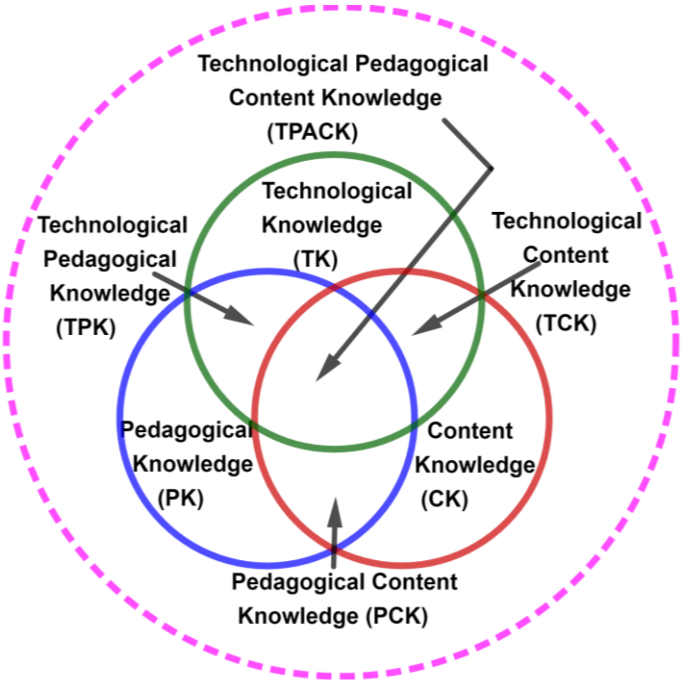


Figure 2.1: TPACK framework
(Koehler & Mishra, 2009)

This section discusses the various knowledge components essential for effective teaching drawing on the framework of technological pedagogical content knowledge (TPACK). The TPACK framework was developed to provide insight into the knowledge and skills required by teachers to integrate technology effectively into their teaching practices. The framework highlights the various knowledge components: content knowledge (CK), technology knowledge (TK), pedagogical knowledge (PK), pedagogical content knowledge (PCK), technological pedagogical knowledge (TPK), technological content knowledge (TCK), and technological pedagogical content knowledge (TPACK).

Content knowledge: CK is the knowledge of the subject matter that is being taught or learned in a particular field or level of education. It includes concepts, central facts, theories, ideas, organisational frameworks, knowledge of the rules of evidence and proof, and practical approaches to developing such knowledge (Shulman, 1986). Teachers must clearly understand the content they teach and the differences in knowledge across various fields or subjects to avoid misrepresenting information to their students (Schmidt *et al*., 2009).

Technology knowledge: TK is knowledge about the use and operation of traditional and digital technologies. This knowledge includes the skills required for using or operating the various types of technology, as well as how to adapt to changes in technology and its uses (Chai, Koh, & Tsai, 2011, 2013; Koehler & Mishra, 2009).

Pedagogical knowledge: PK refers to profound knowledge about the procedures, practices, and methods of teaching and learning. It includes understanding the nature of student learning, classroom standards and management, scripting lesson plans and their implementation, and lesson and student evaluation. Pedagogical knowledge also involves knowledge and understanding of cognitive, social, and developmental learning theories and how they apply to students in a classroom (Chai et al., 2011, 2013; Koehler & Mishra, 2009).

Pedagogical content knowledge: PCK is the knowledge of how to sequence content for teaching. It involves knowing the best techniques or methods that fit the content and arranging the content elements for effective teaching (Ball et al., 2008; Shulman, 1986). PCK also includes knowledge and understanding of student misconceptions and conceptions and the appropriate instructional techniques to address them for meaningful understanding (Ball et al., 2008).

Technological pedagogical knowledge: TPK refers to a teacher's understanding of the various technologies available and how to effectively use them in the classroom to facilitate learning activities, regardless of the subject matter (Chai et al., 2013). This involves selecting the most appropriate technological tool(s) to support the learning process.

Technological content knowledge: TCK is a teacher's knowledge of how to use technology to represent and research subject matter in diverse ways. TCK is not related to teaching but focuses on combining content and technology. Examples of TCK include familiarity with subject-specific websites, online dictionaries, and ICT tools such as GeoGebra, Autograph, and Geometer's Sketchpad (Chai et al., 2013).

Technological pedagogical content knowledge: TPACK is the intersection of content, pedagogy and technical knowledge in ways that allow for effective teaching. It also involves using technology to connect students' prior knowledge to new concepts and assisting students who face challenges in learning a certain subject matter (Koehler & Mishra, 2009). TPACK involves understanding and negotiating the relationship between technology, pedagogy, and content knowledge to achieve genuine technology integration in the classroom (Chai et al., 2013; Koehler & Mishra, 2009).

In conclusion, the TPACK framework emphasises the importance of teachers having a combination of content, technology, and pedagogical knowledge, as well as pedagogical content knowledge, technological content knowledge, and technological pedagogical knowledge, to be effective in their teaching practices. Teachers with this combined knowledge can be better equipped to integrate technology into their teaching practices and promote effective learning outcomes for their students.

2.5 The contextual factors surrounding technological pedagogical content knowledge (TPACK) for supporting beginning mathematics teachers in the context of dynamic geometry software.

The effective integration of technology, particularly DGS, in mathematics teaching requires careful consideration of the contextual factors surrounding TPACK. For beginning mathematics teachers, these contextual factors related to TPACK concerning dynamic geometry software may include the following:

Technology access and availability: While beginning mathematics teachers may have unlimited access to DGS such as GeoGebra, Autograph Maths, and others, challenges arise when using them for teaching in schools or districts. Factors like limited availability of devices, lack of technical support, or inadequate training can hinder their ability to effectively integrate technology into their teaching practices (Archambault & Crippen, 2009; Keengwe, Onchwari, & Onchwari, 2009; Mishra & Koehler, 2006; Niess et al., 2009).

Curriculum and standards alignment: To successfully integrate dynamic geometry software, beginning mathematics teachers, particularly those in the UK, must thoroughly understand the mathematics curriculum and standards in the context of the UK. They need to recognise how this technology can enhance student learning and achievement in alignment with these standards (Archambault & Crippen, 2009; Harris & Hofer, 2011; Manches & Price, 2011; Niess, 2007; Niess et al., 2009).

Teacher preparation programmes: In the UK, Initial Teacher Education (ITE) for beginning mathematics teachers should prioritise equipping them with the knowledge and skills to effectively use DGS in their teaching. Pre-service teacher preparation programmes must provide hands-on experience with various educational technologies, including DGS, to develop their TPACK and increase their ability to integrate technology effectively (Archambault & Crippen, 2009; Ertmer et al., 2012; Mishra & Koehler, 2006; Niess et al., 2009).

Professional development: In-service professional development opportunities can be vital for beginning mathematics teachers to stay updated with the latest technology tools and strategies, especially concerning the integration of DGS. Continuous professional development should encourage collaboration and reflection, fostering their TPACK development and promoting effective technology integration (Archambault & Barnett, 2010; Voogt, Fisser, Pareja Roblin, Tondeur, & van Braak, 2013).

School and district policies: School and district policies significantly impact the availability and use of technology, including DGS, in the mathematics classroom. Beginning teachers must be familiar with these policies and their implications for integrating technology effectively into their teaching practices (Ertmer et al., 2012; Foulger, Graziano, Schmidt-Crawford, & Slykhuis, 2017; Graziano, Foulger, Schmidt-Crawford, & Slykhuis, 2017; Shulman, 1987).

Considering these contextual factors, beginning mathematics teachers in the UK can navigate challenges and capitalise on opportunities for integrating DGS effectively, ultimately enhancing their TPACK and improving student learning experiences in mathematics education. However, it is essential to investigate whether schools, education districts, or the Department of Education in the UK have specific policies for the use of technology in teaching mathematics to address potential barriers and promote successful technology integration.

2.6 The historical development of dynamic geometry software (DGS) and its relevance in mathematics education

The historical development of DGS traces back to the emergence of computer-based technologies in mathematics education. The origins of DGS can be attributed to the pioneering work of Sutherland (1963), who developed the first interactive computer drawing programme called Sketchpad in the early 1960s (Sutherland, 1963). Sketchpad allowed users to create geometric shapes and explore mathematical properties through dynamic manipulation, laying the groundwork for the future development of DGS. Throughout the 1980s and 1990s, the field of mathematics education witnessed significant advancements in computer technology, leading to the introduction of more sophisticated DGS tools. Cabri Geometry, introduced in the late 1980s, was one of the early commercial DGS software programs that gained popularity among educators (Laborde, 1993). Cabri Geometry could enable users to construct and explore geometric figures dynamically, offering a powerful tool for developing geometric understanding and problem-solving. Following the success of Cabri Geometry, other DGS software like Geometer's Sketchpad and GeoGebra were developed, further expanding the capabilities and accessibility of DGS tools. Geometer's Sketchpad, released in the early 1990s, became widely used in classrooms owing to its user-friendly interface and extensive functionality (Roschelle et al., 1999). GeoGebra, introduced in the early 2000s, emerged as a versatile DGS tool that integrated dynamic geometry with algebraic capabilities, making it a valuable resource for mathematics educators at all levels (Hohenwarter & Preiner, 2007).

As technology continued to advance, DGS software evolved to become more intuitive, powerful, and compatible with various devices and operating systems. Cloud-based DGS platforms emerged, allowing for seamless collaboration and access to DGS tools from different devices, including computers, tablets, and smartphones (Hollebrands, Laborde, & Strässer, 2008; Olive, 2010). In recent years, the proliferation of DGS tools and their integration with other technologies, such as virtual reality and augmented reality, has opened up new possibilities for mathematics education. These developments have contributed to the expansion of DGS beyond traditional geometric constructions to encompass dynamic explorations of mathematical concepts in various branches of mathematics.

In the context of DGS and geometry, the term ‘dynamic’ refers to the ability of geometric figures and constructions to change, move, and respond in real time as users interact with them. Unlike traditional static representations of geometric shapes on paper, DGS can enable users to manipulate points, lines, and shapes dynamically, witnessing how they transform and adjust in response to various actions. In DGS, geometric objects are not fixed but rather flexible and malleable, allowing for immediate feedback as users modify the figures. For instance, users can drag points to change the shape of a triangle, resize a circle, or a point on a slide (a line segment) to explore different angles. As these modifications occur, other geometric properties, such as lengths, angles, and areas, are automatically updated, providing learners with a visual representation of the changes and their consequences. The dynamic nature of DGS fosters a more interactive and engaging learning experience, where students can actively explore geometric concepts and relationships. It promotes active discovery and empowers learners to experiment, make conjectures, and validate mathematical statements through direct manipulation. DGS is particularly beneficial for enhancing students’ geometric intuition and problem-solving skills. It allows them to observe how geometric properties are interconnected and how changes to one aspect affect the entire figure. This hands-on approach to learning geometry can help students develop a deep understanding of geometric concepts and encourages them to think critically and analytically about mathematical relationships. Therefore, the advantages and functionalities highlighted regarding DGS have spurred its advocates, researchers, teachers, and educators to recognise its potential in transforming mathematics teaching into a laboratory-based science. This shift aims to facilitate profound learning and improve students' understanding of mathematical concepts (Buchholtz, Kaiser, & Schwarz, 2023; Olive, 2000).

The historical development of DGS showcases its evolution, potentially revolutionising geometry education. From Sketchpad to modern cloud-based tools, DGS continually enhances students' understanding and problem-solving skills. Its versatility empowers educators globally to create engaging learning experiences. DGS literature highlights its potential to enhance geometric understanding and problem-solving skills; however, successful integration requires well-designed tasks and effective teacher training. Despite challenges, DGS has the potential to offer opportunities to improve mathematics education and support beginning teachers in acquiring the knowledge needed for teaching geometry effectively, making it valuable for technology integration in mathematics teaching.

2.7 Pedagogical aspects of using dynamic geometry software

The use of dynamic geometry software (DGS) in mathematics teaching has garnered significant attention from researchers and educators owing to its potential to transform the learning experience for students (Bretscher, 2023; Clark-Wilson et al., 2014; Ruthven, 2018b). Dynamic geometry software like GeoGebra software has the potential to integrate geometry, algebra, graphs, statistics, and calculus, in ways that are suitable for all educational levels (Bretscher, 2023; Drijvers et al., 2016). It is dynamic mathematics software that facilitates the visual exploration of mathematical relationships (Donevska-Todorova, 2018a, 2018b). Dynamic software can help teach underlying mathematical concepts, enhancing teacher-student interaction (Clark-Wilson, 2017). This aligns with research suggesting digital technology can support students' and teachers' learning (Hoyles, 2018). This section of the literature review focuses on the pedagogical aspects of using DGS in the classroom (Thomas & Palmer, 2014), emphasising effective teaching strategies that influence the capabilities of this technology to promote active learning and problem-solving among students.

One crucial aspect of using DGS is its ability to promote active learning through manipulation and exploration. Hohenwarter and Preiner (2007) highlight the significance of hands-on engagement, where students actively manipulate geometric figures and observe changes in real time. This approach can promote students' construction of knowledge, formulation of conjectures, and testing of mathematical statements, leading to a deep understanding of geometric concepts.

Dynamic geometry software (DGS) can significantly improve students' problem-solving skills. Banson and Arthur-Nyarko (2021) demonstrated that integrating computer-based geometry applications, including DGS, improved academic performance, revealing its potential for enhancing problem-solving abilities. Kuzle (2017) research emphasised DGS as a cognitive tool for creative problem-solving in non-routine geometry tasks. Similarly, the study by Granberg and Olsson (2015) illustrated how DGS, particularly GeoGebra, fostered collaborative creative reasoning during mathematical problem-solving, further affirming its role in nurturing problem-solving skills. These findings endorse DGS as a tool for developing students' problem-solving proficiency.

Another aspect is the potential to increase visualisation and conceptual understanding. DGS provides a powerful tool to enhance students' visualisation skills, as emphasised by Stylianou and Silver (2004). DGS's dynamic and interactive nature allows students to visualise geometric objects and concepts in real time, bridging the gap between abstract ideas and tangible representations (Laborde & Laborde, 2014). This visualisation can improve conceptual understanding and promote connections between various geometric properties (Donevska-Todorova, 2018a; Hohenwarter & Preiner, 2007).

DGS also supports collaborative learning through small-group activities (Drijvers et al., 2010; Geraniou, Baccaglini-Frank, Finesilver, & Mavrikis, 2023). Students collaboratively explore geometric concepts and solve problems, fostering peer interaction, communication, and argumentation. Collaborative activities can promote a better understanding of geometric principles and cultivate positive attitudes towards mathematics, enhancing motivation and interest in the subject (Resnick, 1993).

Moreover, DGS can empower teachers to provide individualised instruction based on students' learning needs and progress (Trouche & Drijvers, 2010). Monitoring individual interactions with DGS allows teachers to identify areas of difficulty and provide targeted interventions, catering to diverse learning styles and proficiency levels, leading to improved learning outcomes (Trgalová, Clark-Wilson, & Weigand, 2018).

Integrating DGS into mathematics teaching aligns with the broader goal of incorporating technology into education (Artigue, 2002; Niess, 2005). Incorporating DGS into the curriculum can enable teachers to create dynamic and interactive learning environments, enhancing traditional teaching methods and engaging students in meaningful learning experiences. However, effective

DGS integration requires teachers to possess the necessary knowledge and skills (Clark-Wilson & Hoyles, 2017). Trgalová et al. (2018) highlight the importance of providing adequate training during teacher education programmes. Beginning teachers who receive DGS training demonstrate increased confidence and competence in integrating technology into their geometry lessons. Ongoing professional development is also crucial to support teachers in staying updated with DGS developments and implementing best practices in their classrooms (Clark-Wilson & Hoyles, 2019).

Despite the potential benefits, DGS integration comes with challenges. Trouche and Drijvers (2010) highlight technical issues, steep learning curves for both teachers and students, and potential distractions owing to the interactive nature of the software. Adequate resources and technology infrastructure are also necessary for successful DGS implementation. Careful task design and alignment with the curriculum can be crucial to creating meaningful learning experiences (Clark-Wilson, Donevska-Todorova, Faggiano, Trgalová, & Weigand, 2021).

In conclusion, the literature review on the pedagogical aspects of using dynamic geometry software underlines the potential of DGS in promoting active learning, problem-solving, visualisation, and conceptual understanding in mathematics education. However, teachers must acquire appropriate pedagogical knowledge to fully harness the benefits of DGS. Combining technology with classroom pedagogy and creating the right learning environment can allow educators to improve students' mathematical understanding and problem-solving skills. For beginning teachers to effectively use DGS as a tool in teaching geometry, providing them with the necessary training and support is crucial, thereby bridging the gap between traditional and modern methods of geometry teaching.

2.8 Promoting critical thinking, effective communication, and emotional engagement in technology-enhanced dialogic learning

Dialogic learning, with its emphasis on interactive and collaborative discourse, can present an important opportunity to promote critical thinking, communication, and emotional engagement in educational settings. However, its effectiveness is not guaranteed, and several factors, including the context and the nature of the tasks, must be critically considered when applying dialogic learning principles, particularly in technology-enhanced environments like DGS.

Riel's (1992) research on collaborative problem-solving tasks illustrates the potential of dialogic learning to promote critical thinking. However, the extent to which such collaboration fosters deep reflection often depends on the quality of the dialogue itself and the participants' ability to engage critically. While Riel's (1992) work demonstrates positive outcomes, it leaves unexamined how variables such as classroom dynamics, teacher experience, and students' familiarity with both dialogue structures and technology might affect the effectiveness of this approach. The assumption that merely engaging in discussions leads to reflective thinking overlooks potential challenges, such as the need for scaffolding and guidance to help students move beyond surface-level conversations.

Similarly, Alexander's (2020) principles of dialogic teaching, which are collective, cumulative, and purposeful, can offer a framework for promoting critical thinking and effective communication. Yet, the transferability of these principles to digital learning environments, particularly with beginning teachers using dynamic geometry software, remains open to question. Although these principles can encourage reciprocal and supportive classroom interactions, the digital medium may impose limitations on spontaneity and the fluidity of conversations, making it more difficult to sustain meaningful dialogue. For beginning teachers, mastering both the content (geometry) and the technology (dynamic geometry software) might detract from their ability to effectively facilitate such productive discussions.

The notion of accountable talk, proposed by Resnick, Asterhan and Clarke (2018), advocates for structured dialogue as a means of developing critical thinking. While structured approaches can provide clear expectations for discourse, they may also restrict the natural flow of discussion, potentially limiting creative exploration of ideas. In a dynamic geometry context, where exploration and conjecture can be key to mathematical discovery, overly rigid dialogue structures could impede spontaneous mathematical reasoning. The balance between maintaining structure and fostering open-ended inquiry can be crucial, and the details of this balance may vary depending on the teacher’s comfort with both the technology and the dialogic approach.

The knowledge-building dialogue discussed by Scardamalia and Bereiter (2006, 2021) can provide insight into the iterative development of ideas in collaborative settings. Their work underlines the value of continuous dialogue in advancing understanding. However, the success of such knowledge-building relies on participants’ sustained engagement, which can be particularly challenging in technology-enhanced learning environments where distractions and technical issues may interrupt the flow of discourse. In addition, beginning teachers may struggle to facilitate this iterative dialogue effectively if they are simultaneously learning to use the technology themselves.

The emphasis by Paul and Elder (2006) and Elder and Paul (2020) on deliberative dialogue can offer a valuable lens for understanding how structured discourse facilitates critical thinking (Elder & Paul, 2020; Paul & Elder, 2006). However, their focus on structured conversation may oversimplify the complexity of dialogic interactions in real classrooms, where students' varying levels of participation, engagement, and prior knowledge complicate the dialogue process. In a technology-based learning environment, especially for beginning teachers, the added layer of managing both the software and the dialogue may hinder their ability to guide such structured conversations towards deep critical thinking.

The concept of disciplined dialogue, as explored by Engeström and Sannino (2010), advocates for purposeful, structured communication that drives collective understanding and problem-solving. While their model sides with the principles of deliberation and promotes critical thinking, its practical application in technology-enhanced settings, particularly with beginning teachers, is not straightforward. The disciplined nature of such dialogue requires both careful facilitation and a deep understanding of the subject matter and the tools being used. Beginning teachers, who are still developing their content knowledge and pedagogical skills, may find it difficult to manage these competing demands, leading to less effective dialogic interactions.

Moreover, while scholars such as Alexander (2018), Coultas and Booth (2019), Howe (2014), Howe (2023), Knight and Mercer (2015) and Palinscar and Brown (1984) highlight various forms of dialogic talk (transactional, expressive, exploratory, evaluative, deliberative, interrogatory, imaginative, and expository), the adaptability of these forms in technology-rich environments remains an open question. Technology can both enhance and constrain communication. While tools like dynamic geometry software can provide visual and interactive representations of mathematical ideas, they may also limit the richness of verbal interactions if not used carefully. The emotional engagement that dialogic learning aims to foster may be hindered if the technological interface creates a barrier rather than a bridge between participants.

In reflecting on the aims and research questions of this study, the critical analysis of dialogic learning in the context of beginning teachers and dynamic geometry software reveals important considerations. The study's aim, to explore how carefully designed tasks can facilitate productive mathematical talk and dialogic learning, assumes that such tasks will naturally lead to deeper engagement with both mathematics and technology. However, the literature suggests that the success of dialogic learning hinges on more than task design alone. Factors such as teacher expertise, familiarity with the software, and the ability to manage classroom discourse are equally important. The findings from (Alexander, 2020) and (Resnick et al., 2018), while promising, they also highlight potential pitfalls that must be considered when implementing these principles in practice.

The research questions, particularly those relating to how beginning teachers perceive and understand the knowledge required to teach geometry with technology, also warrant further exploration. The assumption that dialogic learning principles can seamlessly translate into a technology-rich environment may overlook the additional cognitive load placed on teachers. The need to simultaneously develop technological, pedagogical, and content knowledge (as outlined in the TPACK framework) may challenge beginning teachers' ability to foster the kind of deep, reflective discussions that dialogic learning requires.

While dialogic learning presents significant opportunities for promoting critical thinking, effective communication, and emotional engagement, its implementation in technology-enhanced environments requires careful consideration. Beginning teachers face the dual challenge of mastering both the pedagogical strategies of dialogic learning and the technological tools required to teach geometry. Without adequate support and scaffolding, these challenges may undermine the potential benefits of dialogic learning. Therefore, while the literature supports the use of dialogic principles in education, the complexity of integrating these principles with dynamic geometry software suggests that a more detailed approach is needed, one that considers the unique needs and challenges of beginning teachers.

2.9 Dialogic pedagogy and the integration of digital technology

The evolution of technology seems to be influencing modern education, especially in teaching and learning. As digital technologies are being integrated into classrooms and everyday life, educators must grasp how to synergise these advances with dialogic approaches (Knight, 2019). This aspect of the literature review explores the role of technology in dialogue and vice versa, uncovering the dynamic interplay that characterises the integration of dialogic pedagogy and digital technology (Major & Warwick, 2019). The symbiotic relationship between dialogic methods and digital tools sets the stage for a new paradigm in education.

The affordances of digital technologies within educational contexts are complex. Digital technologies can offer a unique platform for nurturing dialogic learning by making ideas visible, providing spaces for scrutinising concepts, and tracking the evolution of thoughts (Major & Warwick, 2019). This visibility not only has the potential to enrich the learning process but also facilitates the analysis and reflection of polyphonic characteristics within dialogue (Trausan-Matu, 2013, 2019). As a result, the integration of digital technology can catalyse the gathering and storage of data for more profound insights into the dynamics of dialogic learning.

Furthermore, digital tools can redefine the boundaries of dialogue and interaction within educational settings. The emergence of platforms like Knowledge Forum empowers educators to create environments conducive to knowledge-building (Chan, Tong, & van Aalst, 2019). Knowledge Forum is a platform developed by the Ontario Institute for Studies in Education (OISE) at the University of Toronto. This technology can bridge the formal and informal aspects of learning, facilitating shared understanding and coherent learning trajectories (Staarman & Ametller, 2019).

However, the integration of digital technologies brings forth both opportunities and challenges. Joint attention, a fundamental element in education, may be crucial for productive classroom dialogue (Tomasello, 1995). Yet, the digital landscape can complicate the establishment and maintenance of joint attention

(Dietz & Henrich, 2014). Ground rules, as highlighted by Mercer and colleagues, can play an essential role in facilitating productive classroom discussions (Mercer & Littleton, 2007). In the digital age, the need for adapted ground rules to harness the power of technology in shaping dialogue is evident (Lantz-Andersson, Vigmo, & Bowen, 2013).

The original ground rules, such as sharing ideas, giving reasons, questioning ideas, considering diverse perspectives, agreeing, involving everyone, and accepting responsibility (Mercer and Littleton, 2007), have been foundational in fostering effective classroom dialogue. However, as education embraces digital technology, the adaptation of these ground rules becomes paramount.

In the digital age, sharing ideas can extend beyond verbal communication, encompassing digital platforms, collaborative tools, and multimedia expression. Giving reasons has the potential to gain strength through the integration of multimedia support and can allow for a more comprehensive presentation of evidence. Questioning ideas can become dynamic in online forums, where asynchronous discussions and various digital communication channels have the potential to enhance inquiry.

Consideration of diverse perspectives has the potential to be amplified in the digital landscape, where students can explore viewpoints presented through online articles, videos, and interactive content. Agreement can be facilitated by digital collaboration tools, enabling shared document editing, virtual meetings, and collaborative decision-making. Involving everyone has the potential to extend to virtual participation, ensuring accessibility and inclusivity across physical locations. Finally, in the digital realm, everybody accepting responsibility can encompass individual accountability and adherence to digital protocol and ethical considerations.

These adapted ground rules and digital technology can facilitate the co-construction of knowledge, a central aim of dialogic pedagogy (Mercier, Rattray, & Lavery, 2015). Platforms such as Talkwall reinforce these ground rules, promoting idea-building and collaborative creativity (Rasmussen & Hagen, 2015).

Creativity, once seen as an individual endeavour, is now recognised as a social process where individuals collaboratively generate novel solutions with communal value (Loveless, 2007). This evolution aligns with a contemporary view of creativity and can be greatly facilitated by dialogue that embodies characteristics such as open-ended discussions, open-mindedness, consideration of diverse perspectives, and collaborative creative strategies (Palmgren-Neuvonen, Korkeamäki, & Littleton, 2017; Vass, Littleton, Jones, & Miell, 2014; Wegerif et al., 2010). Interactive technologies provide vital affordances for supporting collaborative creativity by offering accessibility, connectivity, visibility, interactivity, and multi-modal representation (Hämäläinen & Vähäsantanen, 2011; Rogers & Lindley, 2004; Sakr, 2018).

To create situations for collaborative creativity, educators can often begin with real-world social challenges that engage students in open and authentic dialogue, promoting multi-voiced new ideas and expanding their understanding (Pifarré, 2019). These dialogues necessitate the establishment of ground rules for effective communication, which are recorded and revisited through technology (Mercer & Littleton, 2007; Sams & Dawes, 2004). Combining small-group and whole-class dialogue can enable students to share, discuss, and integrate various perspectives into the discussion (Pifarré, 2019).

The impact of materiality within educational settings is underlined in a study, emphasising the evolving role of materials in teaching and learning (Kumpulainen, Rajala, & Kajamaa, 2019). The concept of hybrid communication spaces is introduced, where formal and informal knowledge intersect, connecting educational activities with students' everyday lives and communities (Kumpulainen, Mikkola, & Rajala, 2018). Here, students can be encouraged to creatively engage with material objects, leading to transformations in materiality through student agency (Kumpulainen, Kajamaa, & Rajala, 2018).

The dialogic perspective of learning, which sees dialogue as more than the exchange of words, explores how digital technology reshapes dialogic practices in education (Barwell, 2016). This perspective leads to the idea of dialogic teaching, which emphasises active and sustained student participation in classroom dialogue, empowering students to voice their ideas without fear of providing incorrect answers (Alexander, 2008). The integration of digital technologies in the classroom can be used to open up dialogic space, connecting students' identities and voices with other voices and discourses, expanding the learning environment (Erstad, 2014; Kumpulainen & Rajala, 2017; Mercier et al., 2015).

Pedagogical link-making becomes a vital tool in understanding and fostering dialogic teaching, as it establishes connections between ideas in teaching and learning that support knowledge construction (Scott, Mortimer, & Ametller, 2011). Teachers can use digital technology to create these pedagogical links, modelling ways in which students can develop understanding by connecting new information and ideas across various contexts (Vygotsky, 1987).

In summary, the integration of digital technology and dialogic pedagogy has the potential to lead to a transformative shift in education. Digital tools offer unprecedented affordances for fostering dialogic learning and collaborative creativity, making the learning process more dynamic and visible. However, the implementation of technology in educational settings brings challenges such as maintaining joint attention and adapting ground rules to this evolving landscape. Through dialogue, technology can mediate and enrich the educational experience, creating opportunities for creativity, knowledge-building, and deep understanding.

2.10 Integrating didactical principles in task design for student-centred learning

The theory of didactical situations (TDS) (Brousseau, 1997) has significantly influenced mathematics education, especially within the French tradition, by promoting a student-centred, inquiry-based learning approach. Brousseau's (1997) TDS proposes a pedagogical method where the teacher designs the learning environment in which students can actively construct their knowledge. The teacher's role lies in the devolution and institutionalisation phases, whereas the students are active in the adidactical phases, encompassing action, formulation, and validation. This approach involves designing learning environments that encourage active engagement and foster effective learning (Brousseau, 1997).

While the TDS presents strengths in promoting student agency and constructive feedback, it may not fully cater to the needs of all learners, and its assessment framework could be improved. To address these limitations, integrating sociocultural approaches to learning is suggested (Mercer & Littleton, 2007; Rogoff, 1990; Vygotsky, 1978, 1987).

In this current study, although I did not explicitly use the Theory of Didactical Situations (TDS) in my task design, the geometric construction tasks I created implicitly adhered to its principles. As the teacher-researcher, I designed tasks that seemed to resonate with the devolution phase of TDS, enabling the beginning teachers, acting as learners, to take responsibility for the adidactical phases. This approach could support student agency and allow them to, independently of the teacher, construct knowledge and receive constructive feedback aligned with formative assessment principles (Black & Wiliam, 2018; Hattie & Timperley, 2007; Tondeur, 2020; Tondeur, Van Braak, Ertmer, & Ottenbreit-Leftwich, 2017; Wiliam, 2011).

Task design can play a key role in teaching as it can facilitate student learning and achievement. Well-designed tasks can enhance student engagement, motivation, critical thinking, and problem-solving skills (Biggs, Tang, & Kennedy, 2022; Hattie, 2012; Wiggins & McTighe, 2005). In addition, tasks serve as a means of assessing student learning outcomes, providing evidence of their understanding and achievement (AkbaŞ & Başaran, 2023; Biggs et al., 2022).

Several theoretical constructs can be important in informing task design. For example, we may draw on cognitive load theory, situated learning theory, and constructivism. Cognitive load theory advocates for matching tasks to students' cognitive abilities to avoid cognitive overload (Sweller, 2023; Timothy, Fischer, Watzka, Girwidz, & Stadler, 2023). While situated learning theory emphasises the importance of designing authentic, relevant tasks set in real-world contexts (Amundrud *et al*., 2021; Lippman, 2023), constructivism, conversely, highlights tasks that promote active learning, collaboration, and student-centredness (Dabbagh & Kitsantas, 2012; Jonassen & Rohrer-Murphy, 1999; Stacey & Gerbic, 2008). Task design principles may be vital in creating effective learning experiences. These principles include clarity, relevance, authenticity, complexity, and alignment (Mayer & Estrella, 2014; van Merriënboer & Kirschner, 2018). Clarity can ensure tasks are easily understood, while relevance makes them meaningful and exciting. Authenticity in task design reflects real-world contexts, and complexity challenges students to think critically. Finally, alignment ensures that tasks align with instructional goals and assessments.

Current trends in task design can encompass technology use, project-based learning, and performance-based assessments. Integrating technology can enhance task interactivity and engagement (Foster & Warwick, 2018; Hira & Anderson, 2021). Project-based learning focuses on developing critical thinking and collaboration skills (Hira & Anderson, 2021; Loyens, Van Meerten, Schaap, & Wijnia, 2023). Lastly, performance-based assessments require students to apply knowledge in real-world situations, providing authentic learning experiences (Darling-Hammond & Snyder, 2000; Herrington & Oliver, 2000; Spooner-Lane, Broadley, Curtis, & Grainger, 2023; Willis, Arnold, & DeLuca, 2023).

In conclusion, implicitly integrating the theory of didactical situations with task design appears to show promising potential in enhancing students' learning experiences and supporting the professional development of beginning teachers. The designed geometric construction tasks implicitly foster student agency and constructive feedback while promoting active engagement, motivation, critical thinking, and problem-solving skills. Considering sociocultural aspects in technology-focused contexts, this integrated approach can potentially create dynamic and interactive learning environments that cater to diverse student needs and foster comprehensive knowledge construction. Future research should continue exploring the potential of this integration to advance mathematics education and instructional design practices.

2.11 Task design in dynamic geometry software for beginning teacher education

Dynamic geometry software (DGS), such as GeoGebra and the Geometer's Sketchpad, can offer interactive and visual experiences crucial for exploring mathematical concepts. Its significance in mathematics education is emphasised by its ability to facilitate visualisation, exploration, and discovery, which are essential from a socio-constructive perspective for effective teaching and learning. Laborde (2002) notes that DGS allows dynamic manipulation of geometric figures, enabling users to investigate mathematical properties and relationships hands-on. Patsiomitou (2018) emphasises the interactivity of DGS, which has the potential to facilitate a more intuitive grasp of abstract concepts.

Two primary frameworks guide the design of technology-based tasks in mathematics education: the dynamic geometry task analysis (DGTA) framework and the instrumental orchestration (IO) framework. The DGTA framework, developed by Trocki and Hollebrands (2018), can provide a structured approach for designing and analysing tasks by focusing on mathematical depth and technological actions. This framework categorises tasks based on cognitive demand, from recalling facts to generalising concepts, and assesses the technological actions required, such as drawing or measuring, to ensure they effectively support mathematical exploration (Trocki, 2015). In contrast, the IO framework, introduced by Trouche (2004), focuses on integrating technological tools with teaching strategies. This framework emphasises the interplay between tools and instructional practices, detailing configurations including the arrangement of tools and classroom settings; and exploitation modes encompassing various ways of using technology, including demonstrations, discussions, and student-led presentations (Drijvers *et al.*, 2010). The IO framework can be useful for managing and using technology in instructional settings, providing a strategy for integrating DGS into teaching practices.

The practical implementation of these frameworks in teacher education has been examined in various studies. Bozkurt and Koyunkaya (2022) demonstrate the effectiveness of incorporating the DGTA and IO frameworks into a restructured practicum course for prospective mathematics teachers. Their approach involved integrating seminars on digital technologies and task design using DGS, coupled with opportunities for micro-teaching and classroom implementation. The study found that applying these frameworks improved task design and teaching practices by offering structured methods for integrating technology into teaching. This approach might allow teachers to develop more effective and engaging technology-based tasks, enhancing their pedagogical skills. Micro-teaching sessions can play a critical role in refining technology-based tasks.

These controlled practice environments can enable prospective teachers to experiment with and adjust their use of technology. Bozkurt and Koyunkaya (2022) highlight that reflective practices during and after micro-teaching sessions are essential for evaluating and improving task design and implementation skills. Reflective analysis can allow teachers to critically assess their use of DGS, identify areas for improvement, and refine their instructional practices accordingly.

Collaborative learning, particularly in the form of paired working, can enhance effective task design in DGS environments. Bauersfeld (2012) and Vygotsky (1978) emphasise the importance of social interaction in cognitive development, asserting that collaborative tasks can foster understanding through peer dialogue and shared reasoning. Johnson, Johnson and Smith (2014) further support this, highlighting that collaboration can promote problem-solving skills and social competence. In the context of beginning teacher education, paired working can facilitate cognitive development and also prepare beginning teachers to encourage collaborative learning in their future classrooms. The principle of scaffolded instructions, as articulated by Wood, Bruner and Ross (1976), involves breaking down tasks into manageable steps to progressively support learners. This approach can be essential in reducing cognitive overload and aiding learners in managing complex geometric constructions. Scaffolded instructions can provide the necessary support for beginning teachers to gradually develop their skills and confidence in using DGS, hoping to effectively guide their future students through similar learning processes.

The use of DGS in task design can be justified by its capacity to provide an interactive and engaging learning environment. Hershkowitz (2020) and Leung and Lee (2013) highlight that DGS can improve student engagement and understanding through hands-on manipulation and immediate visual feedback. These interactive elements can allow learners to explore and discover geometric relationships dynamically, fostering a growing and more intuitive grasp of geometric concepts. Incorporating Bruner's (1974) modes of representation, enactive, iconic, and symbolic, into task design can support varied cognitive processes and learning styles. Providing multiple ways for students to engage with geometric content through tasks can cater to different abilities of learners, ensuring that abstract concepts are accessible through concrete actions and visual representations. This approach can be particularly beneficial for beginning teachers, who draw on these varied representations to support their students' learning.

Rabardel's (1995) theory of instrumental genesis stresses the transformation of tools into cognitive instruments through usage. Designing tasks that encourage epistemic interactions among beginning teachers can support them in developing effective cognitive schemes and becoming adept at using DGS as both a practical and theoretical tool. This dual emphasis on practical efficiency and theoretical understanding can be necessary for preparing beginning teachers to integrate technology meaningfully into their teaching. Artigue (2002), Brousseau (1997) and Duval (2006) emphasise the importance of didactical variables and the use of ostensive and non-ostensive objects in task design. These elements can promote active engagement and facilitate the transition from concrete examples to abstract concepts. For beginning teachers, understanding and applying these didactical considerations can be vital for creating effective learning experiences that bridge tangible activities with abstract mathematical thinking.

Sfard’s (2008) commognitive framework highlights the role of mathematical discourse in learning. Tasks designed to encourage discursive practices, including using keywords, visual mediators, and routines, can improve communication and peer learning. Morgan, Mariotti and Maffei (2009) support this by demonstrating how meaningful discussions about mathematical tools and processes contribute to the development of mathematical thinking. For beginning teachers, engaging in and facilitating mathematical discourse seems to be necessary for promoting understanding and critical thinking. In connection with this, Leung and Bolite-Frant (2015) and Sinclair (2013) discuss the mediation of tools between physical actions and conceptual understanding. Connecting personal meanings to mathematical meanings via tool use can allow tasks to bridge hands-on experiences with abstract concepts. This mediation process can be decisive for beginning teachers, as it helps scaffold their personal experiences with mathematical concepts, making abstract ideas more accessible and meaningful.

Trouche (2004) emphasises the importance of feedback mechanisms in task design. Providing immediate feedback through DGS can allow learners to refine their understanding in real time to improve the learning processes and outcomes. For beginning teachers, this immediate feedback can be invaluable in developing their skills and understanding of geometric constructions and enable them to provide timely and effective feedback to their future students. Clark-Wilson and Timotheus (2013) advocate for designing tasks with meaningful goals and visible mathematics. Making abstract concepts relatable and tangible can allow tasks to increase student motivation and engagement. This principle can be particularly important for beginning teachers, who need to create engaging and relevant learning experiences that resonate with their students. Practical aspects of task design, as discussed by Wood et al. (1976), can ensure that tasks are manageable and effective in teaching complex concepts. For beginning teachers, focusing on pragmatic or practical considerations helps in designing tasks that are achievable and impactful in facilitating an understanding of geometric principles.

The integration of DGS into beginning teacher education requires thoughtful task design that incorporates collaborative learning, scaffolded instructions, and interactive environments. Applying principles from Bruner's (1974) modes of representation, Rabardel's (1995) instrumental genesis, and Sfard's (2008) commognitive framework, can enable tasks to support varied cognitive processes and learning styles (different learning preferences). Didactical and discursive considerations, along with effective feedback mechanisms and engaging contexts, can enhance the learning experience. Pragmatic considerations can ensure tasks are manageable and impactful. The use of rubrics can ensure that task design is both effective and evaluative, providing a framework for preparing beginning teachers to integrate DGS into their future classrooms effectively.

Drawing on theoretical and empirical literature on task design in dynamic geometry environments, this study was guided by iterative and cyclical models of task development, emphasising continual refinement through feedback, reflection, and learner interaction. Such cycles of task design, as articulated by Bozkurt and Koyunkaya (2022), Trocki and Hollebrands (2018), and Trouche (2004), stress the importance of designing, implementing, evaluating, and revising tasks based on observed learning outcomes and dialogic engagement. These iterative processes integrate scaffolding (Wood, Bruner & Ross, 1976), collaborative problem-solving (Johnson et al., 2014; Vygotsky, 1978), and reflective practice as essential mechanisms for improvement. Instrumental orchestration (Trouche, 2004) and the theory of instrumental genesis (Rabardel, 1995) further illuminate how teachers transform digital tools into cognitive instruments through recursive engagement. Additionally, the integration of Bruner's (1974) representational modes and Brousseau's (1997) didactical variables ensures tasks are designed with cognitive accessibility and conceptual progression in mind. In particular, Brousseau's didactical variables underpin decisions balancing learner engagement with both ostensive objects—concrete, visible geometric elements manipulable by learners—and non-ostensive objects—the abstract geometric relationships and principles requiring interpretation beyond direct observation (a distinction drawn from Duval, 2006). This calibrated balance supports the gradual transition from hands-on interaction to theoretical understanding, enabling learners to connect physical actions with abstract reasoning. These cyclical, theory-informed design processes enable continual tuning of tasks to foster visible mathematics, meaningful goals (Clark-Wilson & Timotheus, 2013), and discursive learning (Sfard, 2008). As a result, beginning teachers are afforded opportunities to experience, reflect on, and internalise pedagogical strategies they may later enact. This recursive approach to task design was central to structuring the dynamic geometric construction tasks in this research, cultivating dialogic spaces for rich mathematical talk and the development of TPACK among beginning teachers.

2.12 TPACK in the context of enhancing mathematical talk and dialogic learning with dynamic geometry software

The technological pedagogical content knowledge (TPACK) framework can be fundamental for understanding how technology can be effectively integrated into teaching practices. This framework is central to my research, which explores how carefully designed tasks can facilitate mathematical talk and dialogic learning among beginning teachers using dynamic geometry software. A review of the literature reveals insights into how TPACK is operationalised and highlights areas where further research can be needed.

Hernawati and Jailani (2019) applied the TPACK framework to mobile learning in mathematics, emphasising the integration of technology with pedagogy and content to improve student engagement and achievement. Their study stresses the importance of supporting technology use with TPACK principles, showing that mobile devices can improve educational outcomes. However, their focus on mobile learning rather than dynamic geometry software limits the direct applicability of their findings to my research. While their insights into technology integration are valuable, they do not address the specific dynamics of using dynamic geometry software to facilitate mathematical talk and dialogic learning.

In contrast, Young (2016) provides a broader perspective by reviewing various technologies' impacts on mathematics education through the TPACK framework. Examining tools such as calculators, computer-assisted instruction, and mathematics-specific software, the study emphasises the importance of connecting technology integration with TPACK components. Although this review gives a general understanding of technology's role in mathematics education, it does not focus specifically on dynamic geometry software or the facilitation of mathematical talk, which are central to my research.

Stapf and Martin (2019) examined how TPACK is used in elementary mathematics teacher preparation. Their review highlights the benefits of integrating TPACK through coursework and practical experiences, emphasising modelling, collaboration, and reflection. This resonates with my research's focus on how beginning teachers can develop their TPACK knowledge. However, their emphasis on elementary education and general teacher preparation practices may not fully capture the specific details of dynamic geometry software or advanced mathematical discourse.

Rakes *et al*. (2022) investigated how secondary mathematics teacher candidates incorporated technology into their teaching during the COVID-19 pandemic, using the TPACK framework. Their study's use of the TPACK levels rubric to assess technology integration is particularly relevant, as it provides insights into how technology use can be measured and improved. The study highlights the need for targeted professional development and explicit strategies for technology integration, which resonates with my research's focus on developing beginning teachers' TPACK through structured tasks involving dynamic geometry software. However, the study's findings reveal that improvements in teaching practices did not always correlate with enhanced TPACK, emphasising the necessity of intentional professional development.

Li, Vale, Tan and Blannin (2024) offer a systematic review of TPACK research in primary mathematics education, focusing on lesson design, teacher evaluation, and professional development. Their emphasis on designing technology-integrated maths lessons and evaluating teachers' TPACK knowledge is highly relevant to my research. They highlight the importance of effective lesson design and professional development programmes in improving TPACK. Nonetheless, their focus on primary education and general TPACK applications leaves a gap in understanding how dynamic geometry software specifically contributes to mathematical talk and dialogic learning.

Kartal and Çınar (2022) provide insights into how pre-service mathematics teachers develop TPACK through teaching polygons with GeoGebra. Their study's focus on structured interventions, such as workshops and micro-teaching, is relevant to my research as it shows how TPACK levels can be advanced through reflective practices and hands-on experiences. Although their specific focus on polygons and GeoGebra may limit the generalisability of the findings, their insights into the progression of TPACK levels and the role of feedback and confidence are applicable to enhancing TPACK in dynamic geometry contexts.

Overall, the literature reveals a common theme: effective technology integration requires alignment with TPACK components and benefits from structured interventions, professional development, and reflective practices. Studies such as those by Rakes *et al*. (2022) and Kartal and Çınar (2022) provide practical insights into operationalising TPACK, while reviews by Stapf and Martin (2019) and Li *et al*. (2024) emphasise the importance of integrating TPACK into teacher preparation and professional development. However, there remains a notable gap in research specifically addressing tasks designed within dynamic geometry software environments and their role in facilitating mathematical talk, dialogic learning and development of TPACK. This study aims to fill that gap by focusing on how carefully designed tasks within a dynamic geometry environment can improve beginning teachers' TPACK, particularly in the context of teaching geometry. Insights gained from this research can inform effective technology use in mathematics education and support the development of beginning teachers' TPACK in increasingly technology-rich classrooms. Ultimately, this work seeks to contribute to understanding how targeted, technology-enhanced interventions can prepare beginning teachers for the demands of modern mathematics teaching.

2.13 Beginning teachers' perceptions, attitudes, and understanding of the knowledge needed for teaching mathematics with technology

Technology integration into mathematics teaching seems to significantly impact beginning teachers' perceptions, attitudes, and understanding of the requisite knowledge for effective teaching. This complex landscape can present opportunities and challenges, particularly in the context of teaching geometry using technological tools. The current research focuses on exploring how beginning teachers specifically perceive, understand, and talk about the knowledge necessary for teaching geometry within a technology-based environment, revealing detailed insights into their preparedness and instructional approaches.

Research by Naidoo and Govender (2019) highlights a dual impact of technology on beginning teachers' confidence and hesitancy. While dynamic geometry software like GeoGebra can boost teachers' ability to understand and demonstrate complex geometric concepts, it also introduces challenges. Some teachers express concerns about the universal applicability of technology and face initial difficulties, which can undermine their confidence. Over time, however, many pre-service teachers (PTs) develop increased independence and problem-solving skills, suggesting a transition from initial uncertainty to greater proficiency. This transition can be crucial for understanding how PTs perceive and adjust to the technological demands of teaching geometry.

Zambak (2014) stresses that PTs' beliefs about technology can be focal in their development of content knowledge. PTs with positive attitudes towards technology demonstrate greater improvements in diagnosing and addressing student errors. This indicates that PTs' perceptions of technology can significantly influence their ability to integrate it into their teaching practices, including their approach to teaching geometry. Similarly, Belbase (2015) demonstrates that instructional guidance can facilitate the evolution of teachers' beliefs, enabling them to better use technology for geometric transformations.

Despite these advancements, significant challenges remain. Batane and Ngwako (2017) document a disparity between PTs' enthusiasm for technology and its actual application in the classroom. Barriers such as limited access, inadequate knowledge, and financial constraints persist, indicating a need for better training and support. This is echoed by Pierce and Ball (2009), who identify time constraints, cost, and technological issues as major obstacles. Gender disparities in the display of confidence, with male teachers generally exhibiting greater confidence than their female counterparts, further highlight the need for tailored professional development to address these gaps in the context of teaching geometry.

Akkaya (2016) reveals that while training programmes can improve perceptions of technology's benefits, challenges such as inadequate infrastructure and increased workload continue to hinder effective technology integration. This emphasises the need for better resources and ongoing support, specifically targeting the integration of technology in geometry teaching.

The technological pedagogical content knowledge (TPACK) framework is crucial for understanding how technology can be effectively integrated into teaching practices. Harris, Mishra and Koehler (2009) stress that successful technology integration relies on the confluence of teacher perceptions, technological knowledge, and pedagogical strategies. Research shows that training aligned with the TPACK model can significantly improve PTs' ability to integrate technology into their teaching (Agyei & Voogt, 2016). This highlights the importance of combining technology, pedagogy, and content knowledge to improve instructional practices in geometry.

However, concerns about the adequacy of current pre-service training programmes remain. Yaylak (2019) notes that the complexities of technology integration, including teacher competencies and technological infrastructure, are not always sufficiently addressed in pre-service training. This inadequacy contributes to ineffective technology use in classrooms (Burns, 2023; Ertmer, 1999; Jerald & Orlofsky, 1999; Rahman & Sandra, 2024). Effective training should focus on technology use and its integration into specific teaching practices, particularly in geometry (da Silva Bueno & Niess, 2023; Dockendorff & Solar, 2018).

The integration of technology into mathematics teaching necessitates addressing several key areas: increasing beginning teachers' confidence, overcoming barriers to technology use, and improving training programmes. The current research specifically explores how beginning teachers perceive, understand, and articulate the knowledge needed for teaching geometry in a technology-based environment. Focusing on the development of TPACK and providing targeted professional development can enable beginning teachers to be better prepared to use technology in their geometry teaching. This tailored approach aims to bridge the gap between theoretical understanding and practical application, ultimately improving their teaching effectiveness and benefiting their students.

CHAPTER 3: THE CAREFULLY DESIGNED TASKS WITHIN AN ONLINE PLATFORM'S DYNAMIC GEOMETRY SOFTWARE ENVIRONMENT

3.1 Introduction

This chapter describes how I carefully designed geometric construction tasks within an online platform's dynamic geometry software environment for beginning teachers to engage with. As I explained in Chapter One, my own philosophical beliefs and stances influence this design. The core of this research lies in adopting a social constructivist approach to support beginning teachers in the development of their TPACK knowledge, specifically geared towards using dynamic geometry software (DGS) for teaching geometric constructions to students. To facilitate this development, I incorporated several theoretical principles, including the dynamic geometry task analysis (DGTA) and instrumental orchestration (IO) frameworks, along with theories related to reflective practices, collaborative learning, and scaffolded instruction. These principles, detailed in Table 3.1, guided the creation of tasks designed to challenge and engage participants, eventually boosting their TPACK through the intersection of technology, pedagogy, and content.

The ontological social constructivist approach I adopted, places the learners, in this case, beginning teachers, at the centre of their learning, emphasising active, collaborative knowledge construction rather than passive information absorption (Piaget, 1977; Vygotsky, 1978). The research was conducted online owing to the COVID-19 pandemic, providing an unexpected opportunity to explore the effectiveness of online learning environments, a method I had considered for years. The online learning environment, hosted on Microsoft Teams used GeoGebra, the dynamic geometry software, as a learning tool, allowing collaboration and collective knowledge-building among beginning teachers. This chapter also presents and describes the tasks I designed for a two-phase study, with refinements made in the second phase based on the first phase's outcomes. An example task is provided to illustrate the step-by-step design process, followed by samples that demonstrate how participants can use the instructions to create initial dynamic geometric constructions for onward explorations, showcasing the tasks' practical application and potential in a digital environment.

3.2 Operationalisation of task design principles in geometric construction tasks using Task 1 as an example

In designing geometric construction tasks for beginning teachers to engage with using dynamic geometry software, I applied key task design principles to develop learning tasks that have the potential to support their knowledge development. Scaffolding was used to gradually build complexity, helping participants develop confidence and understanding step by step. Collaborative learning was used to potentially foster peer interaction and promote shared insights and deeper discussions. Reflective practices were also incorporated into the tasks to encourage critical thinking about geometric relationships, hoping to support self-regulation and conceptual growth. The dynamic manipulation in GeoGebra provided real-time feedback to strengthen understanding, while feedback mechanisms, both from the software and peers, allowed for continuous adjustment and improvement. These principles aimed to support participants' knowledge development of dynamic geometry software, GeoGebra, conceptual mastery in geometric constructions and potential pedagogical strategies and approaches for teaching.

In Task 1, I operationalised the following task design principles to create an effective and engaging learning experience for beginning teachers. The principle of **scaffolding** was implemented through a structured progression from basic to more complex tasks. Participants initially constructed simple geometric elements, including line segments, a slider to control the circles' radii and circles, and then advanced to more sophisticated tasks, including constructing intersecting lines and investigating geometric relationships. This scaffolded approach enabled participants to build their skills incrementally, increasing their confidence and understanding. For example, after learning how to construct line segments and circles, participants progressed to determining the intersection of these circles and constructing perpendicular lines, providing them with manageable steps towards mastering more complex geometric concepts.

Collaborative learning was embedded throughout the various phases of Task 1, including its final stages. Participants were encouraged to work through the construction steps together and engage in joint reasoning, critical dialogue, and shared decision-making. These collaborative structures were designed to support peer interaction, mutual learning, and the development of mathematical communication skills. In the initial sketching phase, participants constructed key geometric elements including line segments, circles, and intersection points. Collaboration was encouraged as they created line segment *AB* and constructed circles at points *A* and *B* using dynamic radii (*CE* and *CF*). During these steps, participants worked in pairs to interpret and execute the instructions, often clarifying tool usage and construction techniques with each other. As they progressed to constructing the *line 'h'* through points *G and H*, collaboration became more conceptually focused. Participants were prompted to discuss their observations about the nature of this line, especially in relation to line segment *AB*. They were encouraged to explore when *line h* functioned as *a perpendicular bisector, a tangent*, or when it lost its geometric significance (i.e, just about to disappear). These discussions required them to justify their reasoning, consider alternative viewpoints, and articulate geometric relationships, key features of dialogic learning.

The Investigate phase further embedded collaborative learning by asking participants to manipulate the construction dynamically and interpret their observations together. Questions such as '*What do you observe*?' and '*Why are you observing what you see happening?*' promoted shared sense-making. Partners often took turns dragging points or made joint predictions, encouraging real-time feedback and comparison of interpretations. This collaborative inquiry allowed participants to test and refine hypotheses through discussion, increasing their conceptual understanding. In the Conjecture phase, participants were asked to generalise their findings and articulate construction methods for perpendicular bisectors. This phase deepened collaboration as they were expected to justify their procedures and reflect on why their constructions worked, a process that benefited from peer critique and support. Finally, the Present Your Findings stage explicitly promoted collaboration by requiring participants to compare and discuss their constructions with a partner or group. This created an opportunity for consolidation, critical reflection, and exposure to different strategies or interpretations. Participants refined their understanding by receiving peer feedback, defending their approaches, or reconsidering them in light of others' perspectives.

Reflective practices were integrated throughout the task. After constructing the geometric figures, participants were prompted to investigate and reflect on the relationships between different elements, including the intersections of circles and lines. The task design prompted them to think critically about their construction methods and findings. For example, after constructing *line h* through points *G* and *H* (the intersection points of the two circles), participants were asked to reflect on the geometric relationship between *line h* and the original *line segment AB*. This continuous cycle of reflection and inquiry was essential for promoting a growing understanding and developing self-regulation in their learning.

The task could also balance **didactical variables** by engaging participants with both **ostensive** (concrete geometric figures) and **non-ostensive** (abstract geometric relationships) objects. While participants interacted with tangible objects like points, line segments, lines and circles, they were also required to analyse the underlying geometric principles through observation and manipulation. For example, they manipulated points *E* and *F* on line segment *CD (serving as a slider to circles' radii)*, observing how the geometric relationships changed in real time, which might help them connect physical actions with theoretical understanding.

GeoGebra's dynamic environment played a role in Task 1, where participants constructed and manipulated geometric figures using tools, including the compass and move tools. While real-time visual feedback is an inherent feature of GeoGebra, the *design principle* embedded in the task was the intentional use of this dynamic feedback to support exploratory learning and deepen geometric understanding. Participants were encouraged to actively experiment with constructions, for instance, by altering the positions of points *E* and *F* to observe how the intersection points of the circles and the resulting line *h* changed. These manipulations provided immediate visual cues, allowing participants to test conjectures, identify patterns, and refine their understanding of geometric relationships. For example, as *line h* nearly disappeared or the intersection *point I* shifted, participants were prompted to re-evaluate their reasoning and engage in reflective discussion. This design approach promoted active learning through continuous interaction and feedback, reinforcing conceptual understanding through visual verification and dynamic exploration.

Instrumental orchestration was another design element in Task 1. As participants navigated GeoGebra, they learned to use its tools strategically to manage the complexity of the geometric tasks. For instance, as they adapted their methods for constructing perpendicular lines or bisectors, they had to modify their strategies in real time, developing their problem-solving skills in the process. This dynamic interaction between the learners, in this case beginning teachers, and the software could facilitate a growing grasp of geometric principles and the tools available for their exploration.

Task 1 also incorporated **Bruner's (1974) modes of representation**, supporting diverse learning preferences. The task combined enactive (hands-on interaction with GeoGebra), iconic (visualisation of geometric figures on the computer screen), and symbolic (abstract reasoning about geometric properties) modes of representation. Participants dynamically manipulated points and lines, observed the visual relationships between elements and applied geometric rules to reason abstractly about their constructions. For example, when constructing the perpendicular bisector of a line segment, participants physically constructed the bisector, visualised its relationship with the original segment, and reasoned symbolically about why their method worked.

Feedback mechanisms, inherent to DGS like GeoGebra, were strategically incorporated as a design feature, providing real-time visual feedback and fostering peer interactions. Participants could instantly see the effects of their actions in GeoGebra; for instance, dragging points *E* and *F* together showed how the *line h* and *point I* responded. This immediate feedback, coupled with peer input, enabled participants to make informed adjustments to their constructions and deepen their understanding of geometric concepts.

Finally, Task 1 was designed with **meaningful goals** and **visible mathematics** to ensure that participants engaged with relevant geometric concepts in a practical context. The task focused on constructing and analysing geometric figures, making the learning objectives clear and directly linked to observable outcomes. For example, participants were tasked with constructing a perpendicular bisector and analysing how it related to other elements in the figure, providing a clear connection between their practical activity and the underlying theoretical principles.

Task 1, like all the tasks, successfully integrated the task design principles to create an effective learning experience for beginning teachers. The structured scaffolding, collaborative and reflective practices, balanced engagement with tangible and abstract elements, and dynamic use of GeoGebra all contributed to participants' understanding of geometric concepts. Operationalising these principles in Task 1 facilitated an effective learning experience for participants to develop both the skills and conceptual knowledge necessary to succeed in future teaching of geometric constructions.

Table 3.1 summarises how the task design principles were operationalised in designing geometric construction tasks using GeoGebra to help beginning teachers develop technical skills in dynamic geometry software, geometric construction techniques and pedagogical strategies for teaching.

Table 3.1 Operationalisation of task design principles in geometric construction tasks

Task Design Principle	Application in the Task	Justification
Scaffolding	The task progressed from constructing basic geometric elements (for example, line segments and circles) to more complex tasks (including intersecting circle points and constructing perpendicular lines).	This structured progression helped participants build skills incrementally, boosting confidence and supporting conceptual understanding of geometric constructions through manageable steps.
Collaborative learning	Participants worked in pairs throughout the task, constructing and discussing elements including *line segment AB, circles, and line h*. Joint reasoning and discussion were encouraged during all phases, including investigation, conjecture, and presentation.	Collaborative learning promoted mutual support, critical dialogue, shared decision-making, and development of mathematical communication skills, enriching learning through peer interaction.
Reflective practices	Participants were prompted to investigate and reflect on relationships between constructions, such as *line h and line segment AB*. They were also asked to reflect on the	Reflection supported critical thinking, self-regulation, and deeper understanding of geometric relationships and construction strategies.

Task Design Principle	Application in the Task	Justification
	methods used and why their constructions worked.	
GeoGebra's dynamic environment	GeoGebra tools were used to construct and manipulate figures. For example, participants changed the positions of *points E and F* to explore how intersections and the resulting *line h* were affected.	Dynamic manipulation allowed for real-time exploration and testing of ideas, reinforcing conceptual understanding through visual feedback and exploratory learning.
Feedback mechanisms	GeoGebra provided immediate visual feedback (for instance, as line h shifted or disappeared). Peer interaction during collaborative phases also served as a source of feedback.	Immediate software feedback and peer input enabled continuous refinement of constructions and understanding, encouraging active and responsive learning.
Instrumental Orchestration	Participants used GeoGebra's compass, move, and intersection tools strategically (instrumental genesis), learning to adapt and manage construction methods (for example, constructing perpendicular bisectors).	Strategic tool use encouraged problem-solving skills and supported the development of both technical fluency with GeoGebra and pedagogical strategies relevant to teaching geometry.
Bruner's (1974) modes of representation	The task engaged participants through enactive (hands-on manipulation), iconic (visual representation on screen), and symbolic (abstract reasoning about geometric principles) modes.	These multiple modes supported diverse learning preferences and encouraged deeper understanding by integrating action, imagery, and symbolic thought.
Ostensive and non-ostensive objects	Participants interacted with visible geometric objects (including points, lines, circles) while analysing abstract geometric relationships (for example, perpendicular bisectors, tangents).	This dual engagement helped participants connect concrete constructions with theoretical principles, promoting a more integrated understanding of geometry.
Meaningful goals and visible mathematics	The task focused on constructing and analysing meaningful geometric figures (for instance, a perpendicular bisector), making the mathematics explicit and observable throughout the process.	Clear objectives linked practical activity with theoretical understanding, helping participants see the relevance of the mathematics and engage purposefully with the content.

3.3 The incidental benefits of conducting the research online.

The global COVID-19 pandemic has increased, by necessity, the popularity and importance of online learning environments. Before the pandemic, there was a question of whether online elements could replace some aspects of physical classroom time while providing greater flexibility and opportunities for quality educational performance (Owston & York, 2018). As universities and schools began to adopt online platforms for teaching and learning during the pandemic, many institutions considered moving some or all of their classroom teaching and learning to an online environment in the short and long term (Peters et al., 2022; Zhu & Bonk, 2022).

The incidental benefits of conducting research (teaching/learning) online are considerable. Conducting the study (teaching/learning) online can provide the beginning teachers with an opportunity to learn at their own pace, manage their time efficiently, improve their communication and technical language skills, and refine their critical thinking and problem-solving skills while collaborating with other students (Swan & Shih, 2005; Zhu & Bonk, 2022). Online learning can create a relational environment that reduces barriers to learning, supports the sharing of ideas, increases curiosity and inquisitiveness, and builds interpersonal relationships in learning (Palloff & Pratt, 2003; Picciano, 2018; Swan & Shih, 2005; Zhu & Bonk, 2022). Pairing students to learn online can encourage genuine participation, reduce isolation, increase willingness to cooperate, and establish a shared sense of purpose (Palloff & Pratt, 2003; Picciano, 2018; Swan & Shih, 2005).

Conducting the research online allowed the beginner teachers to learn and share their collaborative work from the comfort of their geographical zones on the computer screen through Teams, much like sitting in the front seat of a traditional classroom and viewing the screen as a whiteboard or interactive board. This online setting facilitated the recording of all onscreen activities of geometric construction and the beginning teachers' actions, along with automatic transcription obtained in Teams. The following section discusses why Microsoft Teams was chosen for this research.

3.4 Microsoft Teams as an online learning platform

Because of data protection and security requirements, the University of Nottingham mandated the use of Microsoft Teams for all research and teaching activities during the pandemic, leading to its selection as the online platform for the virtual classroom. Microsoft Teams is a chat-based collaboration platform that enables users to share documents, hold online meetings, and access various communication features. SharePoint online, which includes a default document library folder, automatically saves all files shared in discussions, ensuring the secure storage of sensitive information (Teams, 2021).

Microsoft Teams supports up to 10 000 participants in online meetings, offering scheduling tools, note-taking features, file uploads, and in-meeting chats. It also includes functionalities such as video calling, screen sharing, and audio/video conferencing, enabling face-to-face virtual interaction with or without an internet connection (Teams, 2021). These features made Teams ideal for creating an interactive, collaborative learning environment for the study. The following subheading discusses the designed interactive setting of the research and how it was implemented in the Microsoft Teams online environment.

3.4.1 Designing collaborative and interactive learning with Microsoft Teams

The study purposefully used Microsoft Teams to create a collaborative and interactive learning environment that supported beginning teachers' engagement with dynamic geometry tasks. Features including video calls, screen sharing, chat functions, and shared file access enabled participants to work together remotely while completing technology-based construction tasks using GeoGebra. Paired beginning teachers engaged in both synchronous and asynchronous sessions, using Teams' Groups and Channels to coordinate their work and share insights.

During synchronous sessions, participants met virtually via Teams' video and screen-sharing tools, allowing them to collaboratively construct and manipulate geometric figures in GeoGebra. Control of the shared screen could be passed between partners, encouraging active participation and joint problem-solving.

In asynchronous sessions, participants accessed tasks through the 'Files' tab and continued discussions via the Teams chat function. This integration of real-time and self-paced interaction advanced a flexible, dialogic learning environment that encouraged reflection, collaboration, and deeper conceptual engagement with geometry.

3.4.2 Task communication and access to GeoGebra

The geometric tasks were situated within the Teams environment using dedicated Teams Channels for each task. Instructions, links, and task resources were posted within the 'Files' section of the Channel, providing participants with easy access. GeoGebra, the dynamic geometry software used in the study, was accessible via a direct link within Teams, potentially ensuring seamless integration with the learning environment.

Participants were instructed to save their task progress in the designated 'Files' tab within their Team's Channel. This ensured that all participants had easy access to their saved work, which could be revisited during subsequent sessions. The collaborative features of Teams, combined with the dynamic capabilities of GeoGebra, helped facilitate an interactive, technology-enhanced learning experience.

3.4.3 Dynamic geometry software, GeoGebra integration in Teams

GeoGebra was chosen for its ability to visually represent and manipulate geometric constructions, making it an essential tool for the study. As a free, web-based platform, GeoGebra is easily accessible across a range of devices, allowing participants to engage with it from their laptops or tablets without requiring specialised software installation. Participants accessed GeoGebra directly from links within Teams, and they could save their work either locally on their devices or within Teams' 'Files' section.

GeoGebra's integration with the overall learning design enabled beginning teachers to explore geometric concepts dynamically, share their constructions with their partners, and discuss their findings in real time. This combination of Teams and GeoGebra provided a structured environment for beginning teachers to engage with technology-based learning, ensuring that they could collaborate effectively, share resources, and reflect on their learning. Organising the synchronous and asynchronous elements within Teams created a rich, flexible environment that facilitated both productive mathematical talk and the development of TPACK knowledge necessary for teaching geometry with technology.

3.5 The aims of the tasks designed for the beginning teachers

The tasks I designed aimed to support beginning teachers in acquiring three key areas of knowledge and developing their technological pedagogical content knowledge (TPACK) specifically related to using dynamic geometry software, GeoGebra, for teaching geometry and geometric constructions. The tasks had the following main objectives:

3.5.1 Developing proficiency in using GeoGebra

One of the primary aims was to equip beginning teachers with the skills to effectively use GeoGebra for dynamic geometric constructions. This was facilitated through step-by-step instructions that guided participants through the essential tools of the software, allowing them to engage with basic dynamic constructions. These tasks were accessible even to those with limited or no prior experience, ensuring that participants could follow along, learn how to use the software's tools, and build confidence. In addition, participants were provided with ongoing support, as I, the researcher, was available to offer advice and troubleshoot any difficulties encountered with the software.

3.5.2 Enhancing understanding of geometric construction techniques

The second aim was to deepen participants' knowledge of geometric constructions by engaging them in tasks that incorporated scaffolding principles.

The tasks were structured to progressively build their understanding of geometry by allowing them to actively explore, investigate, and discover core geometric concepts and truths. Each task was designed to encourage participants to work within their zone of proximal development (ZPD), guiding them to construct geometric figures, manipulate elements including points, line segments and circles, and investigate relationships between these elements.

For example, in one of the tasks, participants were instructed to perform various constructions, including creating line segments, using sliders to control the radii of circles, and altering point positions to observe geometric relationships. As they manipulated these constructions, they were encouraged to explore, drag points, and observe how changes affected the geometric relationships, fostering a growing understanding. This phase of the task culminated in formulating conjectures about properties like the perpendicular bisector of a line segment, encouraging critical thinking and reinforcing their conceptual understanding. Collaborative discussion and comparison of findings further reinforced these insights.

3.5.3 Fostering collaborative pedagogical reflection and strategy development

The final aim was to help beginning teachers reflect on their pedagogical practices and strategies through collaborative investigation. Participants worked together on dynamic constructions, including creating sliders to manipulate geometric figures. The tasks encouraged them to discuss and share their developing insights into both geometric constructions (geometry) and their pedagogy. These collaborative discussions enabled participants to consider how such tools and techniques could be integrated into their future teaching, promoting the development of pedagogical approaches that integrate technology meaningfully in the classroom.

3.6 The tasks for the two-phase study and modifications made.

This section outlines and describes all the tasks used in the study, which consisted of four main tasks in each phase. For phase two, some modifications were made to certain tasks originally used in phase one. Each pair of participants was assigned the same four tasks, with each individual taking turns to lead two of these tasks. During their assigned tasks, the designated leader was required to share their screen with their partner. Active participation and the exchange of suggestions from both participants were expected for each task, which lasted between 35 and 45 minutes. Therefore, a maximum of 1.5 hours was allotted for completing two tasks, or 3 hours for all four tasks. To offer flexibility, participants, in collaboration with me (researcher), could choose to complete all four tasks in a single 3-hour session or opt for a 2-day format, dedicating 1.5 hours each day to complete two tasks. I will begin by briefly describing the tasks in Table 3.2, followed by a detailed description and presentation of each task.

Table 3.2 Tasks for two phases

Task	Phase One	Phase Two
Task 1	Identifying a perpendicular line to a line segment and understanding its properties. Knowing when a perpendicular line becomes tangent to two circles and the associated properties as well as when it becomes a perpendicular bisector and its related properties. Constructing a tangent to a circle, a perpendicular line to a line segment, a perpendicular bisector of a line segment, and a kite.	Same Task 1 in phase one
Task 2	Exploring and constructing a perpendicular line through a point on a line segment, a line, or at the end of a line segment.	Task 2 was a progressive task, building on Task 1. Understanding the relationship between a kite, a rhombus, and a square and their properties. Constructing a kite, a rhombus, and a square.
Tasks 3a and 3b	Exploring and constructing angle bisectors, examples, and non-examples of an angle bisector.	Same Tasks 3a and 3b.
Task 4	Exploring and constructing a perpendicular to a line segment from or through a point above or below the line segment and discussing its associated properties.	4a. Exploring and constructing an angle bisector, kites, congruent triangles, similar triangles, isosceles triangles, isosceles trapeziums, and arrowheads with a straightedge only. 4b. Exploring and constructing a perpendicular line at the endpoint of a line segment with a pair of compasses and a straightedge without extending the line segment.

3.6.1 Tasks in phase one

Task 1 in Phase One

The first task centred on exploring, identifying, and constructing various geometric elements, including a perpendicular line to a line segment, a perpendicular bisector of a line segment, and a tangent to two circles, along with understanding their related properties. Participants were guided to discern when a perpendicular line acts as a tangent to two circles and when it serves as a perpendicular bisector of a line segment. In addition, they were provided with the opportunity to learn how to construct a tangent to a circle, a perpendicular to a line segment, a perpendicular bisector of a line segment, and a kite.

Task 1: Identifying a perpendicular line to a line segment and understanding its properties.

Sketch

1. Create a line segment *AB*.
2. Create a line segment *CD*.
3. Create two points *E* and *F* on the line segment *CD*.
4. Construct a circle at *A* with a radius *CE* – use the Compasses tool.
5. Construct another circle at B with a radius of *CF.*
6. Use the move tool to alter the position of points *E* and *F* so that the circles intersect.
7. Use the point tool intersect to construct points *G* and *H* where the two circles intersect.
8. Construct a line ‘h’ through points *G* and *H*.
9. Change the colour of the line ‘h’, so it stands out.
10. Construct the point of intersection of line h with line segment *AB*, ‘*I’*.

Investigate

1. What geometrical relationship does the line *GH* have to line segment *AB*? Why?
2. Drag point *E* or point *F.*
3. What do you observe? What about when the line 'h' is just about to disappear?
4. Drag point *A* or point *B*. What do you observe? Why are you observing what you see happening?
5. Select points *E* and *F* together (by holding the control key). Move points *E* and *F* together towards point *C* or point *D*. What happens to the line '*h*'? What happens to the point '*I*'?
6. Select points *E* and *F* together (by holding the control key). Move point F to point *D*. Now move point *E* to point *D.*
7. Drag the two points E and F together towards point *C*. What happens to the line '*h*'? What happens to the point '*I*'? Why?

Conjecture

1. How would you construct the perpendicular bisector of a line segment?
2. Why does your method work?

Present your findings

Compare and discuss your construction with your partner or group.

Task 2 in Phase One

The second task focused on constructing a perpendicular line through a point on a line segment, a line, or at the end of a line segment. This task also gave participants the chance to talk about the geometrical relationship that exists between the constructed perpendicular lines and why the methods used in their construction work.

Task 2: Exploring and constructing a perpendicular line through a point on a line segment, a line, or at the end of a line segment.

1. Create a line *AB* using the line tool.
2. Construct a circle at *A* with a radius of 2 units using the centre and radius tool.
3. Use the intersect tool to construct points *C* and *D*, where circle *A* and line *AB* intersect.
4. Change the colour of circle *A*.
5. Create a line segment *EF*.
6. Construct a circle at *C* with a radius *EF* using the compass tool.
7. Construct a circle at *D* with a radius *EF* using the compass tool.
8. Construct Points *G* and *H*, where the circle *C* and *D* intersect.
9. Construct a line through *GH* or *AH* or *AG*.
10. Change the colour of the line *GH.*
11. What geometrical relationship does the line *GH* have to line segment *AB*? Why?

Investigate

1. Drag point *E* or *F*
2. What do you observe?
3. Is the line *GH* still passing through point *A*?
4. Why are you observing what you are seeing?
5. Drag point *A*
6. What do you observe?

7. Is the line *GH* still passing through point *A*?
8. Why are you observing what you are seeing?

Conjecture

1. How would you construct a perpendicular line through a point on a line segment?
2. How would you construct a perpendicular line through a point on a line?
3. How would you construct a perpendicular line through a point at the end of a line segment?
4. Why does your method work?

Present your findings

Compare and discuss your construction with your partner or group

Task 3 in Phase One

The third task comprised two subtasks; the first specifically focused on the concept of an angle bisector. The second subtask dealt with examples and non-examples of an angle bisector, including their associated properties.

Task 3a: Exploring an angle bisector and its associated properties

Sketch

1. Construct the line segments *AB* and *BC*.
2. Construct a circle at *B* using the centre and radius tool. Use 2 units as the radius.
3. Use the point tool intersect to construct points *D* and *E*, where the line segments *AB* and *BC* intersect Circle *B*.
4. Create a line segment *FG*.
5. Construct a circle at *D* with a radius *FG* using the compass tool.

6. Construct another circle at *E* with a radius *FG* using the compass tool.
7. Use the point tool intersect to construct points *H* and *I*, where the circles *D* and *E* intersect.
8. Construct a line through points *H* and *I* using the line tool.
9. Change the colour of the line *HI* so that it stands out.

Investigate

1. What geometrical relationship does the line that passes through points *H* and *I* have to the angle *ABC*?
2. Drag point *A* or *C*
3. What do you observe?
4. Why are you observing what you are seeing?
5. Drag point *B*.
6. What do you observe?
7. Why are you observing what you are seeing?
8. Drag point *F* or *G*
9. What do you observe?

Why are you observing what you are seeing?

Task 3b: Exploring examples and non-examples of an angle bisector, and their associated properties

Sketch

1. Construct line segments *AB* and *BC*.
2. Construct a circle at *B* using the centre and radius tool. Use 2 units as the radius.

3. Use the point tool intersect to construct points *D* and *E*, where the line segments *AB* and *BC* intersect Circle *B*.
4. Change the colour of circle *B*.
5. Create a line segment *FG*.
6. Create two points *H* and *I* on the line segment *FG*.
7. Construct a circle centred at *D* with a radius of *FH* using the compass tool.
8. Construct another circle centred at *E* with a radius of *FI* using the compass tool.
9. Construct points *J* and *K*, where the two circles *D* and *E* intersect.
10. Construct a line through *J* and *K* using the line tool.
11. Construct another line through *J* and *B* using the line tool.
12. Give different colours to the lines *JK* and *JB* to stand out.

Investigate

1. What geometrical relationship does the line that passes through points J and *K* have to the angle *ABC*? And why?
2. What geometrical relationship does the line that passes through points J and *B* have to the angle *ABC*? And why?
3. Drag point *A* or *C*
4. What do you observe?
5. Why are you observing what you are seeing?
6. Drag point *B*.
7. What do you observe?
8. Why are you observing what you are seeing?
9. Drag point *I* or *H*
10. What do you observe?

11. What happens to the two lines, when points *I* and *H* are closer or coming closer to each other?
12. Why are you observing what you are seeing?
13. Select Points *I* and *H* together (by holding the control key). Move point *H* to point *G*. Now move point *I* to point *G*.
14. What do you observe now? And why?
15. Drag the two points *I* and *H* together towards point *F*. What is still happening? And why?

Conjecture

1. How would you construct the angle bisector of an angle?
2. Why does your method work?

Present Your Findings

1. Compare and discuss your construction with your partner or group.

Task 4 in phase one: Exploring and constructing a perpendicular line to a line segment from or through a point on either side of the line segment

The fourth and final task of Phase One involved two subtasks focused on constructing a perpendicular line to a line segment from or through a point on either side of the line segment. This task provided participants with the opportunity to understand the properties of the constructed perpendicular lines and their relationship to the original line segment.

Task 4a: Constructing a special line from or through a point to a line segment

Sketch

1. Create a line segment *AB*.

2. Create a point *C* above or below the line segment *AB*.
3. Construct a circle at *A* with a radius AC – use the compass tool.
4. Construct another circle at *B* with a radius of *BC*.
5. Use the point tool to construct point *D*, where the two circles intersect.
6. Construct a line through *C* and *D* using the line tool.
7. Change the colour of the line *CD* to stand out.
8. Can you predict the geometrical relationship the line *CD* has to the line segment *AB*? And why?
9. Use the point tool intersect to construct point *E*, where the line *CD* and the line segment *AB* intersect.
10. Use the angle tool on the measurement tools to measure angle *BEC* by clicking on the points *B*, *E*, and *C* in that order.

Investigate

1. Drag point *A* or *B*
2. What do you observe?
3. Why are you observing what you are seeing?
4. Drag point *C*.
5. What do you observe?

Why are you observing what you are seeing?

Task 4b: Exploring and constructing a perpendicular line to a line segment from or through a point on either side of the line segment

Sketch

1. Create a line segment *AB*.
2. Create point *C* above or below line segment *AB*.

3. Construct a circle at *A* with a radius *AC* using the compass tool.
4. Create a circle with radius *BC* and centre *D* and put the circle above the line segment *AB* but passing through *C*.
5. Construct a point *E*, where the two circles intersect.
6. Use the line tool to construct a line through points *C* and *E*.
7. Change the colour of the line *CE* to stand out.
8. Use the point tool intersect to construct point *F*, where the line *CE* and the line segment *AB* intersect.
9. Use the angle tool on the measurement tools to measure angle *BFC* by clicking on the points *B*, *F*, and *C* in that order.

Investigate

1. Drag point *A*
2. What do you observe?
3. Why are you observing what you are seeing?
4. Drag point *B*.
5. What do you observe?
6. Why are you observing what you are seeing?
7. Drag point *C*.
8. What do you observe?
9. Why are you observing what you are seeing?
10. Move point *D* to point *B*.
11. What do you observe now? What reason can you give to this?
12. Now drag point *A* again. What do you observe? And Why?
13. Select points *B* and *D* together by holding the control key. Drag points *B* and *D* together. What do you observe?

Conjecture

1. How would you construct a perpendicular line from or through a point to a line segment?
2. Why does your method work?

Present your findings

Compare and discuss your construction with your partner or group.

3.6.2 Tasks in phase two

Task 1 in phase two: Exploring perpendicular lines and other geometric concepts

This first phase two task is the same as the first task in phase one, centred on exploring, identifying, and constructing various geometric elements, including a perpendicular line to a line segment, a perpendicular bisector of a line segment, and a tangent to two circles, along with understanding their related properties. Participants were guided to discern when a perpendicular line acts as a tangent to two circles and when it serves as a perpendicular bisector of a line segment. In addition, it also served as a scaffold for the beginning teachers to construct tangents to circles, perpendiculars to line segments, perpendicular bisectors of line segments, kites, and other geometric figures.

Sketch

1. Create a line segment *AB*
2. Create a line segment *CD*
3. Create two points *E* and *F* on the line segment *CD*.
4. Construct a circle at *A* with a radius of *CE* – use the *Compasses tool*
5. Construct another circle at *B* with a radius of *CF*.

6. Use the *move tool* to alter the position of points *E* and *F* so that the circles intersect.
7. Use the *point tool intersect* to construct points *G* and *H* where the two circles intersect
8. Construct a line 'h' through points *G* and *H*
9. Change the colour of the line '*h*', so it stands out.
10. Construct the point of intersection of line *h* with line segment *AB*, '*I*'

Investigate

1. What geometrical relationship does the line *GH* have to line segment *AB*? Why?
2. Drag point *E* or point *F*.
3. What do you observe? What about when the line '*h*' is just about to disappear?
4. Drag point *A* or Point *B*. What do you observe? Why are you observing what you see happening?
5. Select Points *E* and *F* together (by holding the control key). Move points *E* and *F* together towards point *C* or point *D*. What happens to the line '*h*'? What happens to the point '*I*'?
6. Select Points *E* and *F* together (by holding the control key). Move point *F* to point *D*. Now move point *E* to point *D*
7. Drag the two points *E* and *F* together towards point *C*. What happens to the line '*h*'? What happens to the point '*I*'? Why?

Conjecture

1. How would you construct the perpendicular bisector of a line segment?

Task 2 in phase two: Exploring the relationship between a kite, a rhombus, and a square and constructing them

Task 2 was a progressive task building on the first task of phase two, where the focus was on the same perpendicular line to a line segment as task one. This task aimed to prompt investigation of the relationship between a kite, a rhombus, and a square and the properties associated with each of them. The goal was for beginning teachers to learn how to construct these geometric figures on their own, without external guidance.

Sketch

1. Create a line segment *AB*.
2. Create a line segment *CD*.
3. Create two points *E* and *F* on the line segment *CD*.
4. Construct a circle at *A* with a radius *CE* – use the *Compasses tool*.
5. Construct another circle at B with a radius *CF*.
6. Use the *move tool* to alter the position of points *E* and *F* so that the circles intersect.
7. Use the *tool* to construct points *G* and *H* where the two circles intersect.
8. Construct a line '*h*' through points *G* and *H*.
9. Change the colour of the line '*h*', so it stands out.
10. Construct the point of intersection of line *h* with line segment *AB*, '*I*'.
11. Construct line segments *AG*, *GB*, *BH*, and *HA*.
12. What quadrilateral is *AGBH*? And why?
13. What are the properties of this quadrilateral?
14. What will you say about points?
    i) *G* and *H*

ii) *A* and *B*

15. How will you construct this quadrilateral with

    iii) The software?

    iv) A pair of compasses and a straightedge?

16. Construct a circle at point '*I'* with radius *AI* and change its colour.
17. Construct a circle at point '*I*' with radius *BI* and change its colour.
18. Construct a circle at point '*I*' with radius *GI* or *HI* and change its colour.
19. What type of circles are they?
20. Highlight the circle centred at *A*, go to the top right corner of the GeoGebra screen, click on and then click on *set line style*, reduce the size to one (1) and choose *the dotted line style*.
21. Highlight the circle centred at *B*, go to the top right corner of the GeoGebra screen, click on and then click on *set line style*, reduce the size to one (1) and choose *the dotted line style*.

Investigate

1. Move point *E* or *F* along the line segment *CD*
2. What do you observe in terms of the quadrilateral?
3. What do you observe in terms of the circles?
4. What about putting the points *E* and *F* on top of each other? What happens to the quadrilateral and the circles?
5. What are the properties of this quadrilateral?
6. What will you say about points

   i) *G* and *H*

   ii) *A* and *B*

7. How will you construct this quadrilateral with

iii) The software?

iv) A pair of compasses and a straightedge?

8. Select points *E* and *F* together by holding the control key. Drag points *E* and *F* together to point *D*, and then back toward *C*. What do you observe? Leave them about three-quarters of the length of the line segment *CD* from *C* or a quarter from *D*.
9. Drag *A* or *B* until the three circles coincide or move *E* and *F* together until the three circles coincide.
10. What type of quadrilateral is this?
11. What are the properties of this quadrilateral?
12. How will you construct this quadrilateral with

    v) The software?

    vi) A pair of compasses and a straightedge?
13. What is the relationship between the first, the second and the third quadrilaterals? And why?
14. What pedagogical knowledge have you gained from these investigations or tasks?
15. What pedagogical strategies have you gained from these investigations or tasks?
16. What content knowledge have you gained from these investigations or tasks?
17. What technical knowledge have you gained from these investigations or tasks?

Task 3 in phase two

As in phase one, task 3 was a two-in-one task. The first subtask focused specifically on the concept of an angle bisector, while the second subtask centred on identifying examples and non-examples of an angle bisector, as well as their associated properties.

Task 3a: Exploring an angle bisector and its associated properties

Sketch

1. Construct the line segments *AB* and *BC*.
2. Construct a circle at *B* using the centre and radius tool. Use 2 units as the radius.
3. Use the intersect tool to construct points *D* and *E*, where the line segments *AB* and *BC* intersect Circle *B*.
4. Create a line segment *FG*.
5. Construct a circle at *D* with radius *FG* using the compasses tool.
6. Construct another circle at *E* with a radius *FG* using the compass tool.
7. Use the intersect tool to construct points *H* and *I*, where the Circles *D* and *E* intersect.
8. Construct a line through points *H* and *I* using the line tool.
9. Change the colour of the line *HI* so that it stands out.

Investigate

1. What geometrical relationship does the line that passes through points *H* and *I* have to the angle *ABC*?
2. Drag point *A* or *C*
3. What do you observe?
4. Why are you observing what you are seeing?

5. Drag point *B*.
6. What do you observe?
7. Why are you observing what you are seeing?
8. Drag point *F* or *G*
9. What do you observe?

Why are you observing what you are seeing?

Task 3b: Exploring examples and non-examples of an angle bisector, and their associated properties

Sketch

1. Construct the line segments *AB* and *BC*.
2. Construct a circle at *B* using the centre and radius tool. Use 2 units as a radius.
3. Use the intersect tool to construct points *D* and *E*, where the line segments *AB* and *BC* intersect Circle *B*.
4. Change the colour of circle *B*.
5. Create a line segment *FG*.
6. Create two points *H* and *I* on the line segment *FG*.
7. Construct a circle at *D* with a radius of *FH* using the compasses tool.
8. Construct another circle at *E* with a radius *FI* using the compasses tool .
9. Construct points *J* and *K*, where the two circles *D* and *E* intersect.
10. Construct a line through *J* and *K* using the line tool.
11. Construct another line through *J* and *B* using the line tool.
12. Give different colours to the lines *JK* and *JB* to stand out.

Investigate

1. What geometrical relationship does the line that passes through points *J* and *K* have to the angle *ABC*? And why?
2. What geometrical relationship does the line that passes through points *J* and *B* have to the angle *ABC*? And why?
3. Drag point *A* or *C*
4. What do you observe?
5. Why are you observing what you are seeing?
6. Drag point *B*.
7. What do you observe?
8. Why are you observing what you are seeing?
9. Drag point *I* or *H*
10. What do you observe?
11. What happens to the two lines, when points *I* and *H* are closer or coming closer to each other?
12. Why are you observing what you are seeing?
13. Select Points *I* and *H* together (by holding the control key). Move point *H* to point *G*. Now move point *I* to point *G*. What do you observe now? And why?
14. Drag the two points *I* and *H* together towards point *F*. What is still happening? And why?

Conjecture

1. How would you construct the angle bisector of an angle?
2. Why does your method work?

Present your findings

Compare and discuss your construction with your partner or group.

Task 4 in phase two

Task 4, the last task of phase two, was also two-in-one. The first subtask involved exploring and constructing an angle bisector and other geometric figures with a straightedge only. This task aimed to allow beginning teachers to explore new ways of constructing geometric figures, including kites, congruent triangles, similar triangles, isosceles triangles, isosceles trapeziums, and arrowheads. The second subtask involved constructing a perpendicular line at the endpoint of a line segment with a pair of compasses and a straightedge without extending the line segment.

Task 4a in phase two: Exploring and constructing an angle bisector and other geometric figures with a straightedge only

Sketch

1. Create the line segment *AB*.
2. Construct line segment *BC*
3. Create the line segment *DE*.
4. Construct two points *F* and *G* on the line segment *DE*
5. Construct a circle at *B* with a radius *DF* using the compasses tool.
6. Use the intersect tool to construct points *H* and *I*, where the line segments *AB* and *BC* intersect the circle centred at *B*.
7. Construct another circle at *B* with radius *DG* using the compasses tool

.

8. Use the intersect tool to construct points *J* and *K*, where the line segments *AB* and *BC* intersect the second circle centred at *B*.
9. Construct line segment *HK*
10. Construct line segment *IJ*
11. Name the point of intersection '*L*' of the line segments *HK* and *IJ* using the point tool or intersect tool
12. Construct a line through points *B* and *L*

Investigate

1. What geometrical relationship does the line have with angle *ABC*? And why?
2. Drag point *A* or *C*
3. What do you observe?
4. Why are you observing what you are seeing?
5. Drag point *B*.
6. What do you observe?
7. Why are you observing what you are seeing?
8. Drag point *F* or *G*
9. What do you observe?
10. Why are you observing what you are seeing?
11. Drag point *D* or *E*
12. What do you observe?
13. Why are you observing what you are seeing?

How will you construct an angle bisector with a markable ruler ONLY?

Task 4b: Exploring and constructing a perpendicular line at the endpoint of a line segment with a pair of compasses and a straightedge without extending the line segment.

Sketch

1. Create a line segment *AB*.
2. Create another line segment *CD*.
3. Use the compasses tool to construct a circle centred at *A* with a radius *CD*.
4. Construct point *E* at the point of intersection of the circle and the line segment *AB*.
5. Construct another circle centred at point *E* with the same radius *CD*.
6. Construct point *F* at the point of intersection of the two circles above the line segment *AB*.
7. Construct a ray from point *E* through point *F*.
8. Construct another circle centred at point *F* to intersect the ray.
9. Name the point of intersection '*G*' of the ray and circle centred at *F*.
10. Construct a ray from point *A* through point *G* and change its colour.

Investigate

1. What geometrical relationship does the ray from point A through G have with line segment *AB*? And why?
2. Drag point *A* or *B*
3. What do you observe?
4. Why are you observing what you are seeing?
5. Drag point *C* or *D*

6. What do you observe?
7. Why are you observing what you are seeing?

Conjecture

1. How will you construct a perpendicular line at the endpoint of a line segment with a pair of compasses and a straightedge without extending the line segment?

3.6.3 Modifications in tasks for the phase two study

In line with design-based research principles, the second phase of the study involved modifications to the tasks used in phase one. Specifically, Task 2 from phase one was replaced by Task 4b in phase two. The former task involved constructing a perpendicular line through a point on a line segment, while the latter task required participants to construct a perpendicular line at the endpoint of a line segment using only a pair of compasses and a straightedge without extending the line segment. This modification was made to ensure that the construction would work for all situations, regardless of the location of the point on the line segment.

Task 2 in phase two was designed as a progressive task, building upon the findings from Task 1 in the phase one study. Analysis of the data from task one in phase one revealed that the participants did not explore other quadrilaterals, such as rhombuses and squares, beyond kites around perpendicular lines and perpendicular bisectors. As such, Task 2 in phase two aimed to supplement Task 1 and encourage the discovery of the relationships and associated properties of kites, rhombuses, and squares through dynamic construction.

Innovative and connective to Tasks 1, 2 and 3 in both phases, Task 4a in phase two aimed to improve participants' knowledge and introduce new approaches to constructing geometric figures. This task involved constructing various geometric figures with a straightedge only, including kites, congruent triangles, similar triangles, isosceles triangles, isosceles trapeziums, and arrowheads.

Modifying the tasks in phase two was carried out so as to enhance and broaden participants' knowledge and encourage new construction approaches. These changes align with design-based research principles, emphasising the iterative evaluation and development process to improve the study's effectiveness.

3.7 An example of the design of technology-based geometric construction tasks.

Below is one of the tasks which exemplifies the design of the tasks and provides insights into some of the ways the beginning teachers performed and investigated it.

Table 3.3 Example of the tasks

Task 1
Sketch

1. Create a line segment AB.
2. Create a line segment CD.
3. Create two points E and F on the line segment CD.
4. Construct a circle at A with a radius CE – use the *Compasses tool*
5. Construct another circle at B with a radius of CF.
6. Use the *move tool* to alter the position of points E and F so that the circles intersect.
7. Use the *point tool intersect* to construct points G and H where the two circles intersect
8. Construct a line 'h' through points G and H
9. Change the colour of the line 'h', so it stands out.
10. Construct the point of intersection of line h with line segment AB, 'I'

Investigate
What geometrical relationship does the line GH have to line segment AB? Why?

Drag point E or point F.

What do you observe? What about when the line 'h' is just about to disappear?

Drag point A or Point B. What do you observe? Why are you observing what you see happening?

Select Points E and F together (by holding the control key). Move points E and F together towards point C or point D. What happens to the line 'h'? What happens to the point 'I'?

Select Points E and F together (by holding the control key). Move point F to point D. Now move point E to point D.

Drag the two points E and F together towards point C. What happens to the line 'h'? What happens to the point 'I'? Why?

Conjecture
How would you construct the perpendicular bisector of a line segment?

Why does your method work?

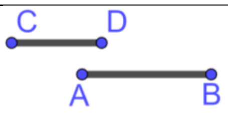
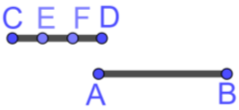
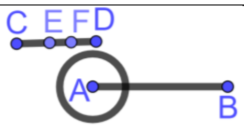
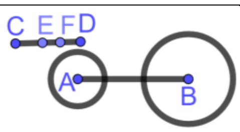
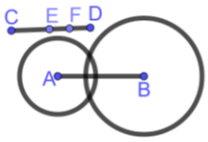
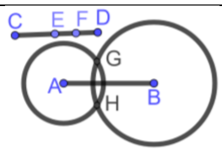
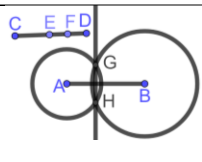
Present Your Findings
Compare and discuss your construction with your partner or group.

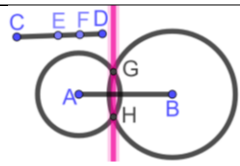
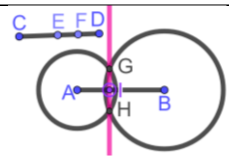
The section shows how to follow the tasks' instructions to construct the initial dynamic construction for onward investigation.

3.8 Sketching the initial dynamic geometric construction

This sketch section is where the step-by-step instructions provide the scaffolding for the beginning teachers to construct the initial dynamic construction using the software for onward exploration of it.

Table 3.4 The steps, tasks and associated responses

Step	Task	Expected Response
1	Create a line segment AB	
2	Create a line segment CD	
3	Create two points, E and F, on the line segment CD	
4	Construct a circle at A with a radius CE – use the ***Compasses tool***	
5	Construct another circle at B with the radius of CF	
6	Use the ***move tool*** to alter the position of points E and F so that the circles intersect	
7	Use the ***point tool intersect*** to construct points G and H where the two circles intersect	
8	Construct a line 'h' through points G and H	

Step	Task	Expected Response
9	Change the colour of the line ‘h’ so it stands out	
10	Construct the point of intersection of line h with line segment AB, I	

In the following section, I consider the potential exploration of the initial dynamic construction in Figure 3.1 .

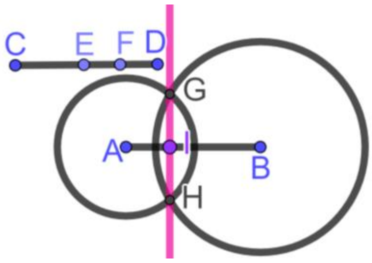


Figure 3. 1: The initial dynamic construction setup for onward exploration

3.9 Considering the potential exploration of the initial dynamic construction in Figure 3.1

The second component of each task encompasses an investigative phase focusing on the initial dynamic construction created by the beginning teachers. This aspect is presented in conjunction with the final dynamic figure developed in Task 1, as illustrated in Figure 3.1.

During the investigation stage, the beginning teachers are tasked to respond to questions regarding the geometrical relationships present in their initial dynamic construction. For instance, typical questions include:

What geometrical relationship does line GH have with line segment AB?

Why does that particular geometrical relationship exist?

If the beginning teachers already know the answer, they should be able to articulate that the line passes through points *G* and *H* and is perpendicular to line segment *AB*. If they do not know the answer, they start investigating whether they can get the answer. The subsequent question to address pertains to the rationale behind the existence of this geometrical relationship.

As part of the third step, the beginning teachers are instructed to drag point *E* or point *F*. This prompts them to make observations, such as:

What do you observe?

How does this change when *line h* is on the verge of disappearing?

This exploration encourages the teachers to engage actively with the dynamic elements of their constructions and draw meaningful insights from their observations. The goal is to observe the effects of this action, particularly when *line h* is nearly disappearing.

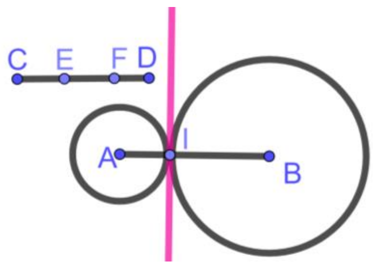
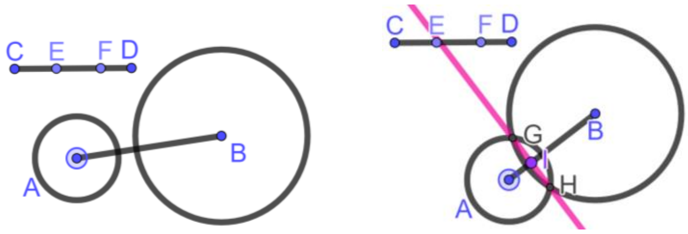


Figure 3. 2: Exploration of a perpendicular line tangent to two circles

As shown in Figure 3.2, the beginning teachers should be able to explain that the perpendicular line becomes tangent to two circles at the point when the intersection of point I of the line and line segment *AB* is at the point of tangency. In addition, it can be observed that the sum of the radii of these circles is equal to the length of line segment *AB*.

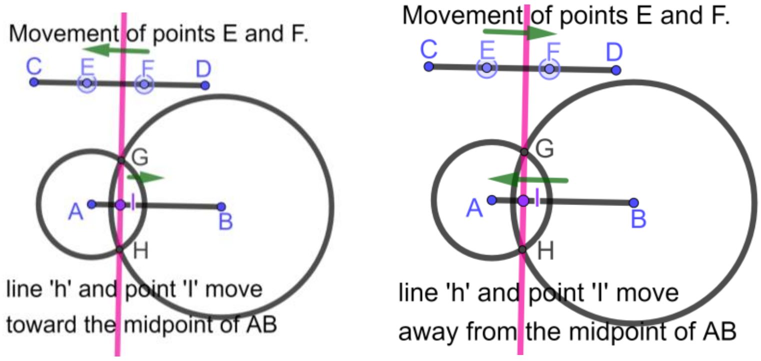
The beginning teachers are then instructed to drag point *A* or point *B* and provide explanations for their observations. In the process of dragging point *A* or point *B,* the following situations depicted in Figures 3.3a and 3.3b can potentially occur.



Figures 3.3a and 3.3b: Dragging point A

A series of further tasks follow, as outlined below:

1. The teachers are directed to select points *E* and *F* together (by holding the control key) and move them towards points *C* or *D*. They are prompted to observe the changes in line ‘*h*’ and point ‘*I*.’ The purpose of moving points *E* and F towards *C* or *D* is for participants to observe the moving of *line h* and *point I* in relation to the midpoint of *AB* when the radii of the two circles are decreasing or increasing by the same length.



Figures 3.4a and 3.4b: Exploring geometric relationship by dragging points *E* and *F to* change the two circles' radii

2. A subsequent action is to move point *F* to point *D* and then move point *E* to point *D*. At this stage if they have not already understood ideas relating to the perpendicular bisector of a line segment, they are expected to explore and discuss it here. This is the stage where the radii of two circles become equal, paving the way for them to know when a perpendicular line becomes a perpendicular bisector of a line segment, as shown in Figure 3.5.

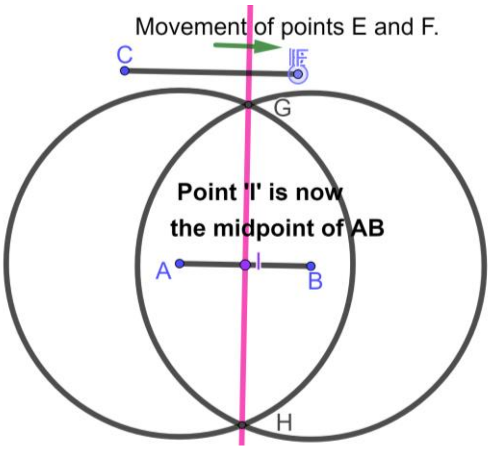


Figure 3.5: Exploration of a perpendicular bisector

3. In this subtask, the teachers are prompted by the task to drag points *E* and *F* together towards point *C* (see Figure 3.6). They are required to note the impact on line '*h*' and point '*I*' and provide explanations for their

observations. Moving points *E* and *F* towards point *C* reduces the equal radii of the two circles by the same length. The purpose of this stage is to let them understand that as long as the radii of the circles are equal and half the length of the line segment *AB*, line h will remain the perpendicular bisector at the midpoint of *AB*.

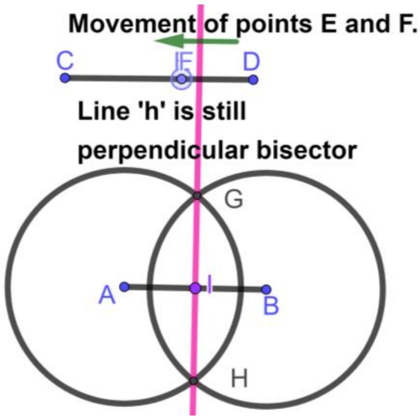
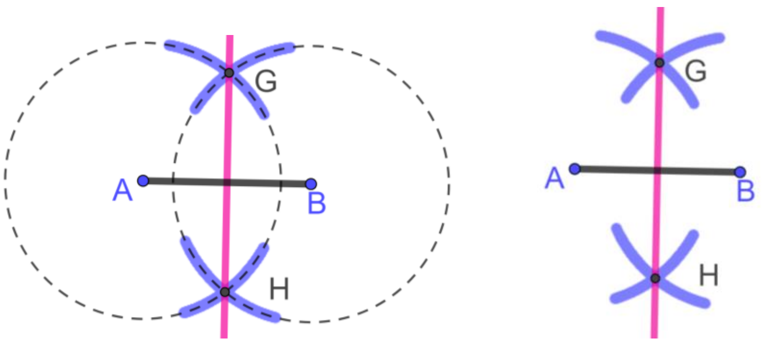


Figure 3.6: Exploration of a perpendicular bisector

4. **Conjecture**: The teachers are prompted to contemplate how they would construct the perpendicular bisector of a line segment. They are asked to explain the methodology behind their construction and why it works. Figures 3.7a and 3.7b show what some of them did with software.



Figures 3.7a and 3.7b: Construction of perpendicular bisector

5. **Present your findings**: The section concludes with the instruction for the teachers to compare and discuss their construction with a partner.

Throughout this series of tasks, the emphasis is on active exploration, observation, and collaborative work, where interactions and critical thinking play a central role in constructing understanding.

3.10 Conclusion

In conclusion, the carefully designed geometric construction tasks within dynamic geometry software, GeoGebra, in the online learning environment form the backbone of this study. These tasks were developed based on certain theoretical principles, including the Dynamic Geometry Task Analysis (DGTA), Instrumental Orchestration (IO) frameworks and other theories related to reflective practices, collaborative learning, and scaffolded instruction. These investigative collaborative learning tasks within the dynamic geometry software in this environment provided a platform for beginning teachers to engage in discussions and interactive exploration, intending to develop TPACK knowledge for their future teaching of geometric constructions and geometry using the software.

CHAPTER 4: RESEARCH METHODS AND DESIGN

4.1 Introduction

This methods chapter aims to outline the research design and the methods employed to investigate the research questions on developing beginning teachers' knowledge of TPACK for teaching geometric constructions with dynamic geometry software. Detailed explanations of the research design, ethical considerations, and data collection and analysis methods are explored. Furthermore, the chapter discusses issues relating to the trustworthiness of the findings and concludes with a chapter summary.

4.2 My positionality as a researcher

My background in mathematics education profoundly shapes my position as a researcher, along with my teaching experiences, and my commitment to integrating technology into teaching practices. With over 20 years of diverse teaching roles across Ghana and the UK, I have developed an understanding of mathematics pedagogy and classroom dynamics. My academic credentials include a Master of Philosophy in Mathematics Education from the University of Cape Coast, Ghana, and ongoing doctoral research at the University of Nottingham, alongside international and UK teaching certifications. This background equips me with insight into the challenges and opportunities within mathematics education, particularly regarding the integration of modern digital technologies.

My passion for technology in education, especially dynamic geometry software (DGS), significantly influences my research focus. Having observed first-hand the transformative potential of tools like GeoGebra and Autograph, I am motivated to explore how these technologies can address gaps in teaching practices. While this enthusiasm drives my research towards innovative solutions, it can also introduce a bias towards emphasising the positive impacts of technology, potentially underestimating the challenges involved in its implementation. This alignment between my professional experiences and my philosophical stance on education naturally led to my adoption of a social constructivist paradigm, which emphasises collaborative learning and knowledge co-construction, principles that resonate deeply with my educational philosophy.

In addition, my experience working with beginning teachers through various educational roles and professional development settings shapes my understanding of their needs and challenges. This experience can enable me to empathise with their difficulties and tailor my research to address their specific needs. However, it can also carry the risk of assuming that my viewpoint on these challenges is universally applicable, which may obscure other valuable perspectives or experiences. My adoption of a social constructivist paradigm reinforces the importance of acknowledging these diverse viewpoints, as it stresses collaborative learning and the co-construction of knowledge.

To address these biases and ensure the credibility of the study, I employ strategies such as peer debriefing. Collaborating with supervisors and incorporating diverse perspectives, I tried to provide a balanced and inclusive analysis of the research findings. Overall, my positionality as a researcher is characterised by a blend of professional experience, technological enthusiasm, and a commitment to understanding and supporting the needs of beginning teachers, with a conscious effort to maintain a balanced and comprehensive approach throughout the research process.

4.3 Philosophical stances and research paradigm

As explained and justified in Chapter One, I carefully considered factors in line with the research objectives and questions when designing this study. The study's design is grounded in specific philosophical foundations, including ontological, epistemological, axiological, and methodological aspects, that are linked to these objectives and questions. These foundations shape how knowledge can be understood, constructed, and valued throughout the research process, warranting that the research framework is consistent and coherent.

My chosen ontological stance is social constructivism, which views reality not as an external objective fact but as co-created through social interactions and shared cultural understandings. This ontological perspective can be grounded in the work of Vygotsky (1978) and Bruner (1974), who argue that reality and knowledge are shaped through dialogue and social engagement. In the context of teaching and learning geometry with dynamic software, this ontological stance may help to understand the learning process as a collaborative and socially mediated one. In my research, the reality of developing the necessary knowledge for teaching geometric concepts with dynamic software can be understood as a process where beginning teachers co-construct their understanding through interaction. Given my personal experiences and commitment to collaborative learning, the social constructivist paradigm naturally resonates with my research focus and provides a theoretical basis for exploring the participants' learning experiences.

Epistemology concerns how we come to know and what constitutes knowledge. Epistemologically, this study adopts a social constructivist view, which sees knowledge as actively constructed, not passively received. Knowledge emerges through social processes, interactions, and shared experiences. This perspective echoes Piaget (1977) and Vygotsky (1978), who emphasise that individuals construct knowledge through interaction with their environment and peers. In my study, this epistemological stance is operationalised through collaborative tasks and discussions among beginning teachers. These interactions can enable participants to construct their understanding of technological, pedagogical and geometric concepts through active engagement and social negotiation. The process of sharing diverse perspectives, engaging in critical dialogue, and collaboratively solving problems can encourage cognitive development, which may be crucial in a technology-enhanced environment.

Axiology can involve the values and ethics that underpin research. Axiologically, this research values inclusivity, collaboration, and empowerment. The study prioritises creating an inclusive learning environment that respects diverse perspectives and promotes mutual respect among participants. Ethical considerations are central to the research design, safeguarding participants' voluntary involvement, maintaining confidentiality, and fostering a safe space or environment where they can freely share their ideas and experiences. These values align with the constructivist approach, which emphasises the importance of participants' voices and the co-construction of knowledge. The significance of this study lies in its potential to support the development of beginning teachers' TPACK (technological pedagogical content knowledge), which can help them integrate technology effectively into their future teaching practices, potentially contributing to improved educational outcomes.

The research paradigm guiding this study is constructivism, which posits that knowledge is constructed through experience rather than discovered as an objective truth. This paradigm significantly influences or shapes my methodological choices, which are grounded in qualitative research methods, including interviews, observations, video recordings, and open-ended questionnaires. These methods are particularly suited for exploring participants' lived experiences, perspectives and understanding in-depth, in line with the constructivist emphasis on subjective and contextual understandings of knowledge. Employing qualitative methods, I can capture the details of participants' learning processes, casting light on how they develop their TPACK for teaching geometry with dynamic geometry software.

The qualitative methodology can enable a detailed exploration of how beginning teachers construct their understanding through active engagement with the software and one another. In connection with the constructivist framework, this approach prioritises depth over breadth, focusing on the meanings that participants assign to their experiences. In addition, reflexivity is an important aspect of this research. I remain conscious of my own biases, especially my enthusiasm for technology, and continually reflect on how these might influence the research process. This self-awareness can be crucial for maintaining the integrity and credibility of the study, as it may ensure a transparent and thoughtful approach to data collection and interpretation.

In summary, this study adopts a social constructivist paradigm, which supports the ontological view that reality is socially constructed and the epistemological stance that knowledge is actively co-created. Axiologically, the study emphasises inclusivity and empowerment, warranting that the voices and perspectives of participants are central to the inquiry. The qualitative methodology can allow for an in-depth exploration of beginning teachers' experiences with the tasks within dynamic geometry software in a collaborative environment. This coherent and justified approach can provide a foundation for examining the development of beginning teachers' TPACK in teaching geometry with technology, while ensuring the research is both ethically sound and methodologically rigorous.

4.4 Research design

In Chapter Three, the study outlines the carefully designed geometric construction tasks I designed for beginning teachers, serving as the intervention in this current research. In addition, Table 4.1 represents the research design of this study, detailing how each research question was addressed through a combination of data sources, participant engagement, and analytical methods. The design integrates interpretive video analysis, drawing on dialogic learning talk/principles, the TPACK framework, alongside inductive thematic analysis, to explore beginning teachers' knowledge construction. In addition, both inductive and deductive thematic analyses are applied to examine beginning teachers' perceptions and understanding of the knowledge needed to teach geometry in technology-enhanced environments. Each methodological approach is justified in relation to its role in revealing insights into beginning teachers' interactions, task reflections, and evolving competencies (the development and improvement of their technological, pedagogical, and content knowledge) with dynamic geometry software. The following research questions apply:

RQ1: In what ways can carefully designed tasks facilitate productive mathematical talk and dialogic learning among beginning teachers working with dynamic geometry software in ways that develop their TPACK knowledge?

RQ2: In what ways do beginning teachers perceive, understand and talk when exploring the knowledge needed for teaching geometry in a technology-based environment?

Table 4.1 Overview of research design addressing research questions

Research question	Data	Participant	Analytical method	Justification/ application	Findings
RQ1	Video recording of paired work (two pairs) Focus group discussion (in phase one all 4 participants and in pairs of 8 participants) Individual questionnaire and interview (all 12 participants)	4 Participants in phase one and 8 in phase two	Interpretive video deductive coding analysis drawing on first, dialogic talk/learning principles (dialogic learning analysis), and second, TPACK framework as analytical tools Inductive thematic analysis	Interpretive video analysis, combined with dialogic learning analysis, captured the dynamic interactions among participants by categorising different types of dialogue (for example, transactional, exploratory) and dialogic learning principles (for example, collective, deliberative). This analysis highlighted how these interactions facilitated the construction of TPACK knowledge. In addition, the TPACK framework provided valuable insights into participants' technological, pedagogical, and content knowledge as they engaged in collaborative tasks. Inductive thematic analysis captured emergent task design features (for example, complexity, progression, scaffolding) that supported TPACK development from participants' reflections on their task engagement	Chapter 5A

Research question	Data	Participant	Analytical method	Justification/ application	Findings
RQ2	Focus group discussion (in phase one all 4 participants and in pairs of 8 participants) Individual questionnaire and interview (all 12 participants)	12 Participants	Deductive Thematic Analysis (TPACK Framework), Inductive Thematic Analysis	The TPACK framework enabled structured analysis of participants' perceptions of technological, pedagogical, and content knowledge. Inductive thematic analysis enabled unanticipated themes on essential teaching competencies to naturally emerge from participants' reflections on their experiences with dynamic geometry software, providing insights into their perceptions and understanding of the knowledge needed for teaching geometry in a technology-based environment.	Chapter 5B
Addressing the overall research question (RO0)	The overall research question (RO0) covers research questions 1 and 2.				

4.5 Ethical considerations

In accordance with the British Educational Research Association (BERA) guidelines for educational research (BERA 2018), ethical considerations were taken into account throughout the research process. The guidelines cover important areas such as informed consent, confidentiality, and the participant's right to withdraw from the study at any time.

Before commencing the study, ethical approval was sought from the ethical review committee of the School of Education at the University of Nottingham, and this is documented (Appendix A). The research process was discussed with various parties, including the ITE directors, the PGCE course leaders, and the PGCE mathematics teams of the Schools of Education of the participants in both phases from the four UK universities involved in the research. Participants, who were PGCE students, were provided with participant information sheets and consent forms to ensure they were fully informed of the research aims and procedures and could make an informed decision about whether to participate.

Participants were informed that their onscreen activities would be video recorded and that all interviews would be audio recorded. They were also made aware of their right to withdraw from the study at any time without any consequences. By ensuring the ethical guidelines were followed throughout the study, I, the researcher, was able to uphold the rights and welfare of the participants and maintain the integrity and validity of the research findings.

4.6 Sampling of participants (beginning teachers) for the study

The study involved the participation of 12 individuals, all of whom were beginning teachers pursuing a Mathematics PGCE at four UK universities. Phase one of the study consisted of four participants, while phase two comprised eight. The PGCE students recruited for the study had either a first degree in mathematics or engineering, with some holding master's degrees in computer science or other subjects. Beginning teachers about to finish a one-year postgraduate university-based secondary course were specifically and purposefully chosen as participants. This was because it was at a stage during this one-year course where they had overcome some of the initial issues of first experiences of teaching, had some experiences of different pedagogies, and had some experiences of using technology both in school and university. Moreover, there is a scarcity of existing literature examining the collaborative use of dynamic software by beginning teachers to explore geometric construction tasks remotely, as well as the ease of their recruitment for such studies.

4.7 Research instruments/methods

The data collection process employed a range of instruments, including screen video and audio recordings of the participants' investigation of tasks using GeoGebra software. I also conducted interviews and focus group discussions, made observations, and kept detailed notes in notebooks I maintained. In addition, questionnaires were administered to the participants.

Using a variety of data collection methods, the study sought to capture multiple dimensions of the participants' experiences with the tasks and the software. The screen video and audio recordings offered a detailed view of the participants' interactions with the software, while interviews and focus group discussions provided more profound insight into their thought processes and reflections on the tasks. Observations and the researcher's notebooks helped contextualise the data. Lastly, questionnaires allowed for qualitative data collection, such as participants' pre- and post-experiences with technology, especially with the software.

Overall, the use of multiple data collection instruments ensured that the study was able to capture a rich and triangulated dataset, providing a detailed view of the participants' experiences and perceptions. This approach highlights the importance of employing a range of data collection methods in educational research to understand complex phenomena.

4.7.1 Video recordings

The study used video recordings to capture the tasks and activities undertaken by beginning teachers who were the participants. The recorded videos provided insight into some of the content and technological knowledge gained by the participants, particularly concerning the use of dynamic software for making conjectures. Throughout the inquiry, the participants engaged in discussions while performing the tasks, making it simpler to connect their actions with their thoughts and understanding of geometric constructions and associated tasks. The use of Microsoft Teams for video recording and automated transcription proved beneficial in this regard.

4.7.2 Observations

The study involved observing six pairs of beginning teachers as they investigated geometric construction tasks using the dynamic mathematics software GeoGebra. As the researcher, I observed their interactions and made notes, occasionally asking questions for clarification. During these observations, the focus was on the type of dialogic talk used by the participants in their conversations and discussions, how they used the software to investigate the tasks and construct geometric figures, and the progress of their knowledge development in the learning process.

As a participant-observer, my goal was to document key moments in the participants' actions, including their use of the software, discussions about the tasks, and responses to questions. These observations were conducted remotely via Teams, as the participants were learning in an online environment

4.7.3 Focus group discussion

In the first phase of the study, all four participants came together after investigating the geometric construction tasks to reflect on the knowledge they had gained and discuss how they could use the software in teaching geometry, particularly geometric constructions, to their students. This discussion provided an opportunity for the participants to share their perspectives and ideas, building on their collective experiences with the software.

In the study's second phase, post-investigation discussions involving eight participants were held in pairs according to the university teams. Here, the pair from each of the four universities took part in the post-investigation discussions. These discussions were also recorded using Microsoft Teams, and automatic transcripts were generated to facilitate analysis. In addition, participants sent notes they had taken while investigating the tasks with the software via email to me, the researcher. The inclusion of participants' notes further enriched the dataset, providing additional insights into their thought processes and reflections on the investigations.

4.7.4 Interview/post-investigation questionnaire

Post-investigation questionnaires and individual interviews were conducted to gather information about participants' experiences using the software to investigate and perform geometric construction tasks. The questionnaires and interviews were divided into four main sections covering appropriate geometry content knowledge gained, technical knowledge of using the GeoGebra software, potential pedagogical strategies developed to teach geometric constructions and geometry with the software, as well as general knowledge concerning the research questions and objectives. While the items in the questionnaire and the interview questions were constructed differently, they addressed the same purpose (see Appendices D and E).

The mathematics knowledge section of the questionnaire/interview asked how exploratory tasks helped participants learn appropriate geometry content, whether the inductive approach helped them learn about geometric constructions, and whether the tasks were carefully designed to follow relevant learning content. The pedagogical knowledge section focused on the pedagogical strategies developed or learned from investigating geometric exploratory tasks using the GeoGebra software and whether the pedagogical approach of working in pairs with the software was relevant. The technological knowledge section asked about the technical expertise gained through the exploratory tasks, the technical knowledge needed to teach geometric constructions with dynamic geometry software, and the nature of teacher knowledge for teaching geometry in technology-focused contexts.

The questionnaire and individual interviews assessed participants' learning and provided evidence of the knowledge gained. The questionnaire/interview questions also examined the knowledge teachers need to teach geometry using technology, the benefits of using dynamic software in the mathematics classroom, and whether participants believe they can design/create geometric construction tasks for students to complete using dynamic software. In addition, the interview captured their knowledge about the mathematical links they observed or noticed between what they investigated and other mathematical concepts. Participants were given the post-questionnaire to respond to first, followed by the individual interview. The interview was intended to check the consistency of responses and gather more detailed responses from them.

4.8 Data analysis methods

To analyse the data collected in the study, I employed a combination of interpretive video deductive coding analysis, deductive thematic analysis, and inductive thematic analysis to examine the development of knowledge for teaching geometry among beginning teachers using dynamic geometry software (GeoGebra). These complementary analytical approaches were strategically used to address the research questions and to explore different dimensions of the dataset (Farinola, 2023; Heath, Hindmarsh, & Luff, 2010; Schutz, 1976).

The interpretive video deductive coding analysis was applied to video recordings of collaborative task-based activities. This analysis focused on identifying and interpreting dialogic interactions and the development of TPACK components, using pre-defined frameworks.

1. The deductive thematic analysis was used to examine transcribed focus group discussions, questionnaires and interviews, guided by the TPACK framework (Koehler & Mishra, 2009).

2. The inductive thematic analysis sought to uncover emergent themes from participants' engagement with geometric construction tasks and technology-centred learning environments.

To consolidate and clarify the analytical procedures employed in the study, Table 4.2 presents a summary of the data analysis methods. It aligns each research question with the analytical approach applied, the rationale for its use, and illustrative examples drawn from the dataset. Complementing this, Figure 4.1 provides a visual overview of how the study's analytical approaches and methods were interconnected with the research questions to generate comprehensive insights into beginning teachers' knowledge development.

Table 4.2 Summary of data analysis methods for examining TPACK development in technology-enhanced geometry teaching

Research question	Analytical methods	Analytical methods justification	Examples of data categorised
RQ 1	Interpretive video deductive coding analysis draws on two key analytical tools: first, dialogic talk and learning	1. Interpretive video analysis was essential for examining interactions and dialogue, focusing on key moments by segmenting the video data into 30-second intervals. This analysis was grounded in the use of dialogic talk	Dialogue such as, '*Yeah, that's a perpendicular line. What happens if you move point A*?' was categorised as: - Transactional - Exploratory - Deliberative In addition, data related to participants' development of: - Technological knowledge using GeoGebra, Participant M stated,

Research question	Analytical methods	Analytical methods justification	Examples of data categorised
	principles, and second, the TPACK framework.	and learning principles as analytical tools. 2. Dialogic learning talk analysis specifically categorised participants' dialogue into types of talk and learning principles, using deductive coding based on Alexander's (2018) framework. 3. In addition, deductive coding was applied to identify data related to participants' development of technological, content, and pedagogical knowledge, informed by the TPACK framework.	'*I became familiar with tools like compasses, line segments, and drawing various shapes. More importantly, how to create a slider to control the radii of circles*.' - Geometric construction knowledge, Participant B, said, '*And I guess you could also think about it, like obviously all these things, it depends on what you've already proved. We've got a kite AGBH, and we know that's a kite because their radius AG and AH are the radii of a circle, and so are BG and BH, and we know that the diagonals of a kite cross at right angles, so, if we knew that already, that would tell us that these were perpendicular.*' Another participant stated, *'The tasks prompted me to transfer my understanding back to pen and paper. This shift from the digital environment to traditional tools allowed me to reflect on how the constructions could be replicated using different mediums.'* - Content knowledge in geometry, Participant A, stated, '*I mean, yes. You do. The fact is that the chord is a vertical property where the tangent to a circle is always at right angles to its radius. And if you've got a chord that's parallel to that tangent as we have here, then it's always going to be 90 degrees to the radius that is*

Research question	Analytical methods	Analytical methods justification	Examples of data categorised
			constructed.' [See Figure 5A.4c] - Pedagogical ideas, the participants emphasised the important using the software to teach '*how'* and '*why'* the method of constructing geometric figures works.
	Inductive thematic analysis	4. Inductive thematic analysis was used to identify the features of the geometric construction tasks that participants perceived and described as engaging with dynamic geometry software.	For example, Sequencing and progression of tasks were appreciated by the participants. Participants commended the tasks for their clear and logical sequencing, expressing their appreciation for the well-structured instructions. One participant affirmed, *'The instructions were well-structured, making it easy to follow along and understand what should be appearing on the screen at each stage of the construction.'*
RQ 2	Inductive thematic analysis Deductive thematic analysis, drawing on the TPACK framework	1. Inductive thematic analysis was employed to identify recurring themes from the data about the knowledge participants perceived, understood, and discussed as necessary for teaching in a technological-centred environment. 2. The TPACK framework informed deductive thematic analysis to identify aspects of transcribed data related to the	Themes identified included participants' views on the role of technology in learning geometry and aspects of TPACK such as technological knowledge, content knowledge, and pedagogical strategies. For instance, Participant J reiterated that, '*Teachers should be comfortable with the technology they are using, proficient in managing student behaviour during technology-based lessons,'* and skilful at providing '*clear instructions and asking effective questions.'* Participant A outlined the needed knowledge components, which

Research question	Analytical methods	Analytical methods justification	Examples of data categorised
	analytical tool	three main TPACK categories that participants perceived and understood as the knowledge they developed for teaching geometry in a technological-centred environment.	include, *'Sound grounding in geometric relationships, ability to spot cross-curricular links, classroom management in computer rooms, ability to use the software and clearly explain use to others, and the ability to understand the advantages and limitations of working with software when it comes to differentiation.'*

Figure 4.1 provides a visual overview of the analytical approaches and methods used in the study. It illustrates how these methods were interconnected with the research questions to generate comprehensive insights into beginning teachers' knowledge development.

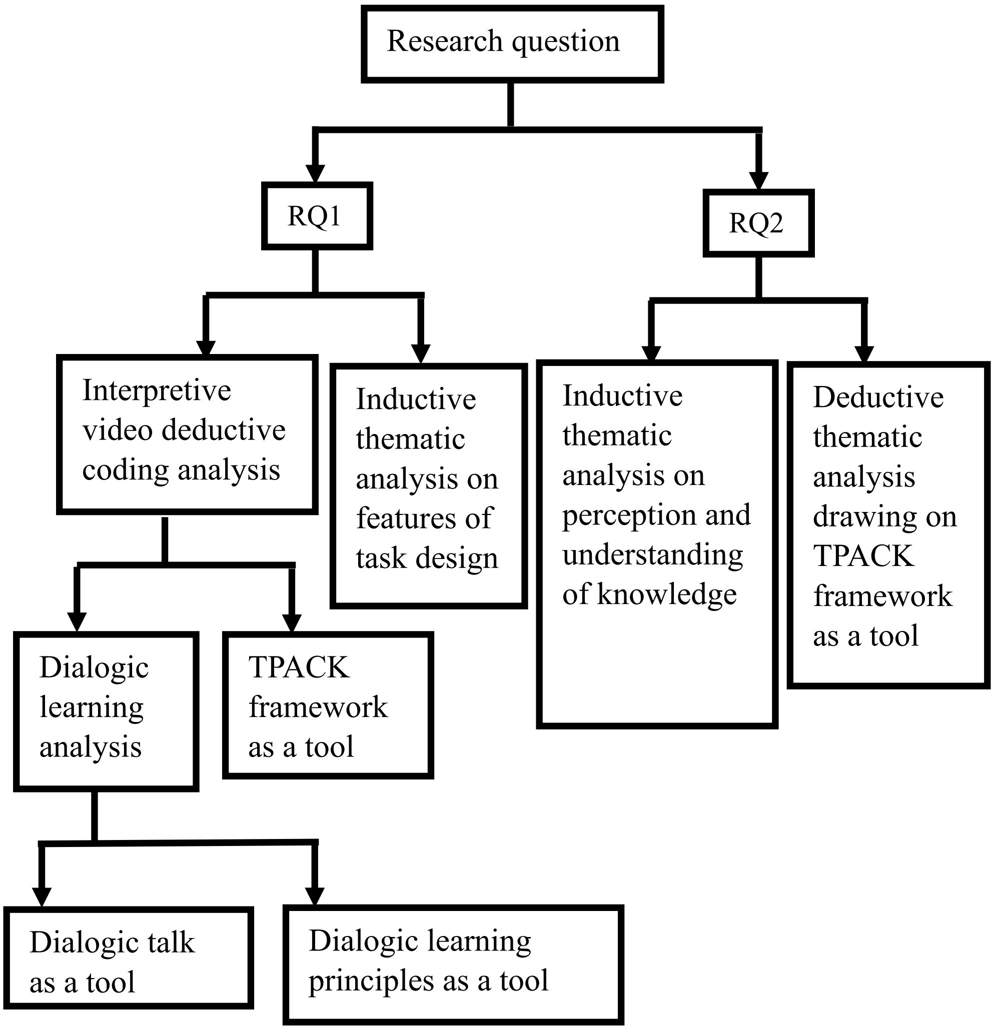


Figure 4.1: Analytical approaches and methods in addressing the research questions

Each of these analytical approaches is elaborated in the sub-sections that follow. The next section focuses specifically on the interpretive video deductive coding analysis.

4.8.1 Interpretive video deductive coding analysis drawing on dialogic talk/principles and TPACK framework as analytical tools

Building on the overview provided in Section 4.8, this section details the interpretive video deductive coding analysis I used to analyse beginning teachers' collaborative interactions during task-based activities. This analysis drew upon two key analytical lenses:

1. Dialogic talk and dialogic learning principles (Alexander, 2018)
2. The TPACK framework (Koehler & Mishra, 2009)

The rationale for using a dual-framework approach was to trace how participants' dialogue reflected knowledge development across technological, pedagogical, and content domains, and to characterise the types of talk that facilitated this development.

To facilitate systematic analysis, I segmented the video data into 30-second intervals. Each segment was independently coded using both dialogic talk and learning principles (for example, transactional, expressive, exploratory, evaluative, deliberative, imaginative, expository, and interrogatory talk) and TPACK dimensions (for example, TK, PK, CK, TPK, TCK, PCK, TPACK). This dual-coding strategy enabled a detailed interpretation of how technological collaborative tasks within a dynamic mathematical software environment supported knowledge growth.

The sub-sections that follow elaborate on the operationalisation of each analytical lens. Sub-section 4.8.1.1 focuses on dialogic talk and learning principles, followed by a discussion of the TPACK framework in Sub-section 4.8.1.2.

4.8.1.1 Interpretive video deductive coding analysis drawing on dialogic talk and principles as analytical tools

Through the lens of Vygotsky's sociocultural theory (1978), dialogic talk is seen as a means of communication and a fundamental process through which knowledge, particularly TPACK, is socially constructed. In this context, 'dialogic talk' refers to interactive communication between participants, characterised by open exchanges of ideas, active listening, deliberating, evaluating, questioning and shared exploration. Specifically, in learning geometric construction, this can be termed 'mathematical talk' or 'dialogic mathematical talk'. It involves a dynamic, collaborative process where individuals engage in dialogue to co-construct meaning, ask questions, and collectively develop an understanding of geometry or mathematics. To complement the qualitative analysis, I conducted a frequency analysis of the different types of dialogic talk observed in participants' conversations (see Chapter 5A). This helped identify patterns of dominant and underused talk types within and across participant pairs.

Building on the methodological approach outlined in Sections 4.8 and 4.8.1, this section illustrates how I applied interpretive video deductive coding analysis using Alexander's (2018) typology of dialogic talk and dialogic learning principles. These theoretical lenses served as analytical tools to explore how different types of talk emerged as participants engaged in collaborative task-based work using dynamic geometry software.

The analysis focused on participants' various interactions of spoken dialogue, gestures, and software manipulation, coded within 30-second video segments. The typology of dialogic talk, including transactional, expressive, exploratory, evaluative, deliberative, interrogatory, imaginative, and expository, was used to classify participant utterances and interactions according to pre-defined characteristics. These are summarised in Table 4.3 below. For example, exploratory talk was characterised by features such as genuine questioning, reasoning aloud, and testing ideas through collaborative manipulation of dynamic constructions.

Table 4.3 Typology-based classification of participant utterances using Alexander's framework of dialogic talk

Talk type	**Purpose**	**Key characteristics**	**Examples from data**
Transactional	Procedural, organisational	1. Giving/receiving instructions 2. Managing task steps 3. Confirming understanding 4. Limited conceptual engagement 5. Task-focused communication 6. Procedural and operational language 7. Used to regulate, coordinate, and sequence activities 8. Often short, closed, or confirmatory exchanges 9. Minimal elaboration or reasoning 10. May include tool-related instructions (for example, software navigation)	*'Click here.'/ 'Now drag point D.' 'Okay, I see it.' 'Let's construct a circle at point A.'*
Expressive	Expressing personal	1. Emotional or affective language 2. Reveals motivation, uncertainty or confidence	*'I feel like that might bisect angle ABC…'*

Talk type	Purpose	Key characteristics	Examples from data
	feelings or reactions	3. Commentary on personal experiences or opinions 4. Often spontaneous 5. Can be off-task or tangential, but reflects a genuine response 6. May include laughter, frustration, or delight 7. Sometimes figurative or informal	*'This is actually fun!'* *'I have never seen this before.'*
Exploratory	Joint reasoning and inquiry	1. Hypothesising and speculating 2. Tentative language (for example, 'maybe,' 'I think') 3. Critical engagement with peers 4. Testing ideas collaboratively 5. Conceptual meaning-making 6. Thinking aloud and joint reasoning 7. Willingness to challenge or build on each other's thinking	*'What happens if you move point A?'* *'I think it's an angle bisector.'* *'Let's test it by dragging the point and seeing what changes.'*
Evaluative	Judging or reflecting on progress	1. Reflecting on methods or results 2. Offering feedback or self-assessment 3. Judging/assessing the quality or correctness of an idea, method, or result 4. Reflection on strategy or process	*'That's not right—it doesn't bisect the angle.'* *'So, to summarise, it changes from a kite to a*

Talk type	Purpose	Key characteristics	Examples from data
		5. Often meta-cognitive or summative 6. May affirm or critique peer responses 7. Includes verbal assessments	*square.'* *'That method worked well.'* *'That's better,'* *'We missed a step'*
Deliberative	Considering options and making collective decisions	1. Weighing alternatives 2. Comparing strategies 3. Negotiating group choices 4. Seeking consensus 5. Reasoned choice-making 6. Encourages compromise and justification of choices	*'Should we use the circle tool or try to construct it manually?'* *'I think using a radius might be more accurate—what do you think?'*
Interrogatory	Probing or seeking clarification	1. Asking direct questions 2. Prompting explanations 3. Drives dialogue through inquiry 4. Probing for clarification, elaboration, or justification 5. Can be open or closed questions 6. Includes 'why' and 'how' questions 7. Central to scaffolding understanding	*'Why did you choose point C?'* *'What does this construction show us?'* *'Can you explain that step again?'*

Talk type	Purpose	Key characteristics	Examples from data
Imaginative	Creative or metaphorical thinking	1. Using playful or figurative language 2. Engaging in creative 'what if' scenarios 3. Stimulates flexible thinking 4. Use of metaphor, analogy, or creative language 5. Hypothetical reasoning (for example, 'what if' scenarios) 6. Playful manipulation of ideas or representations 7. Often speculative, beyond the immediate task 8. May use visualisation or storytelling to increase understanding	*'Imagine this line as a mirror...'* *'It's like folding the angle in half.'* *'What if the circles do not intersect?'*
Expository	Explaining or demonstrating understanding	1. Providing explanations 2. Describing factual procedures or concepts 3. Summarising reasoning 4. Teacher-like or informative tone 5. Can be monologic (one-way talk) 6. Intended to inform or clarify for others	*'You create the circle by selecting a centre and defining the radius.'* *'This line bisects the angle because it divides it into two equal parts.'*

Each video segment was analysed iteratively. To capture detailed shifts in participants' understanding and reasoning through the interpretive video deductive analysis, I developed analytic memos that documented the interplay between observed talk, emerging meanings, and evolving knowledge in geometry, pedagogy, and technology use. These memos fell into several categories, each serving a distinct analytical purpose and enabling the capture of the dynamic interplay between different dimensions of TPACK development as they unfolded through collaborative activity. Content-knowledge memos focused on participants' developing understanding of geometric construction techniques and geometry concepts, while technological-knowledge memos traced their growing fluency with GeoGebra's tools and affordances. Pedagogical-knowledge memos highlighted moments where participants shifted from task execution to reasoning about teaching strategies. Dialogic function memos captured how the nature of talk, whether exploratory, expository, or deliberative, enabled or constrained collaborative meaning-making.

Affective or engagement memos recorded expressions of enjoyment, curiosity, or frustration, offering insight into participants' motivational states and their openness to learning through experimentation. Integrated TPACK memos synthesised moments where content, pedagogy, and technology intersected meaningfully in participants' talk and actions. Finally, reflective shift memos identified points at which participants moved beyond procedural execution to articulate conceptual insights or instructional implications. These analytic memos, developed alongside the coding process, played a role in interpreting the significance of dialogic interactions and tracing the co-construction of knowledge across cognitive, technical, and pedagogical domains. Table 4.4 outlines the various categories of analytic memos used during the coding process.

Table 4.4 Types of analytic memos used during interpretive video deductive coding analysis

Memo Type	**Purpose**	**Illustrative Example**
Content-knowledge memos	Capture how talk reveals understanding/misunderstanding of geometry concepts	*'The line should cut it exactly in half because the radii of two circles are equal and it's perpendicular'* – shows emerging understanding of perpendicular bisectors via exploratory talk.
Technological-knowledge memos	Track participants' use of dynamic geometry software (GeoGebra)	*'Click here—use the compasses tool'* – transactional and evaluative talk reveal growing procedural fluency with tool selection and application.
Pedagogical-knowledge memos	Note when participants reason about how to teach concepts	*'If we let students try this first, they might notice the circle symmetry themselves'* – deliberative talk shows pedagogical planning and sequencing.
Dialogic function memos	Reflect on how types of talk facilitate or limit meaning-making	*'This is how it works...'* – dominant expository talk limits group dialogue, showing a missed opportunity for co-construction.
Affective/engagement memos	Identify expressions of motivation, curiosity, or frustration	*'This is actually fun!'* – expressive and imaginative talk signals affective engagement and playful experimentation.

Memo Type	Purpose	Illustrative Example
Integrated TPACK memos	Synthesise concurrent development of content, pedagogy, and technology	*Participants question symmetry, use dynamic geometry tools, and discuss student misconceptions* – an example of integrated TPACK development.
Reflective shift memos	Highlight transitions from doing to understanding or teaching	*'Now I get why we needed to use the move first, followed by the midpoint tool'* – evaluative talk reveals conceptual realisation and reflective learning.

In instances where participants read instructions aloud, these were also included in the analysis as they frequently initiated or steered the direction of the dialogue. This deductive lens allowed me to examine the content of participants' utterances and the nature and function of their talk within the collaborative meaning-making process. The findings offered insights into how different forms of dialogic interaction shaped the co-construction of knowledge and supported the development of participants' TPACK.

The next section provides concrete examples of these types of dialogic talk drawn from participants' conversations, demonstrating how these talk types played out during task engagement and contributed to knowledge development.

Exploring types of dialogic talk: examples from participant conversations

Exploratory talk: Exploratory talk was particularly prominent in participants' dialogic engagement, serving as a mechanism for collaborative meaning-making. Identified through established indicators such as tentative language (example, '*I think...*', '*I feel like...*'), dialogic questioning, hypothesis generation, and joint revision of ideas, this talk type embodied the core bases of dialogic learning. Participants engaged in open-ended inquiry, challenged each other's interpretations, and used GeoGebra's dynamic features to test and refine their geometric reasoning.

A salient example of exploratory talk emerged in Pair 1's discussion of the geometric relationship between a constructed line and angle *ABC*. Initially, the participants hypothesised that the line represented a perpendicular bisector. However, through manipulation and observation within GeoGebra, they revised this to an angle bisector. Their exchange, featuring utterances such as *'I think it's right'* and *'I feel like that is right'*, reflected a readiness to revise assumptions, collaboratively construct understanding, and test geometric relationships dynamically. The conversation from Pair 1 is as follows:

M: *'OK, what geometrical relationship does the line that passes through points H and I have to the angle ABC? A perpendicular bisector.'* [see Figure 5A.1a]

N: *Yeah, that's a perpendicular line. What happens if you move point A? All right, just move point A about there.'*

M: *'No, that's not it. It's not a perpendicular bisector.'* [As M moves the point A and translates to Figure 5A.1b]

N: *'Make it like ABC right triangle. OK. So, you see, it cuts the angle in half by bisecting the angle.'* [see Figure 5A.1b]

M: *'Yeah, that is right. Yeah. That's sure.'*

N: *'So, what geometrical relationship is that?'*

M: *'It is an angle bisector. OK. Is that right? I think it's right.'*

N: *'Yeah, I feel like that is right. So, when you move either of those points, you are just changing the angle, but the line still bisects the angle. It is good to learn concepts like this.'*

This episode revealed the core features of exploratory talk: a readiness to be wrong, collaborative negotiation of meaning, and the testing of geometric relationships using GeoGebra. Language such as '*I think*,' '*I feel like*,' and the open-ended question '*What happens if...*?' were taken as reliable indicators of exploratory engagement in the coding process.

Additional examples from this and other pairs affirmed this pattern. In Pair 1, Participant M speculated:

> *'I feel like if that line [blue line JK] is in a different position, it would bisect the angle, but I might be totally wrong. Can you try dragging point B around?'*

This prompts further manipulation and shared reflection, with Participant N saying,

> *'Yes, let me drag point B now. I don't know; it is still not bisecting the angle. I feel like that might bisect angle ABC if it was in the right location.'*

Similarly, N responds with, *'Mmm, I have not seen this before…'*—an admission that highlights the openness of exploratory talk to encountering and working through unfamiliar concepts.

In Pair 3, the interaction shows a similarly investigative stance:

A: *'Okay, now let's investigate the geometrical relationship between line GH and line segment AB. What do you think it is? So, GH to AB, perpendicular bisector.'*

B: *'Yep, but that is not always a bisector. At the moment, we got a perpendicular line, but it can be at any point along a line segment.*

A: *'If you could construct a tangent that also is perpendicular to AB, which will be at the side of the circle centred at B on AB.'*

These examples demonstrate that exploratory talk is not merely about talk itself, but represents an iterative process of idea generation, validation, and refinement, a process intimately linked with cognitive engagement and dialogic learning.

Transactional talk: While exploratory talk drove conceptual development, transactional talk provided essential structural support. Transactional talk was task-focused and directive, aiding the organisation of actions, procedural clarification, and coordination of roles. Though it typically lacked conceptual depth, its function was vital in facilitating the smooth execution of tasks within the dynamic GeoGebra environment.

Transactional exchanges were coded based on indicators including issuing instructions '*Click here*', confirming progress '*Done*', clarifying steps '*Let's* *create a line segment AB*', and managing roles. These utterances were often lacking in conceptual speculation but were crucial for orchestrating participants' engagement with the digital tasks.

For example, in Pair 2, Participants G and H, in some instances, worked procedurally to determine whether a quadrilateral was a square. Their talk was methodical and progression-focused:

G: '*So, if we want to prove that the quadrilateral is a square...*'

H: '*Right, so we can conclude that angle G is a right angle. Now, what about the other angles?*'

G: '*Well, at points A, B, and H, we can draw tangents...*'

H: '*So, to summarise, we've observed that the quadrilateral changes between a kite and a square...*'

G: '*Exactly. It's been a fascinating investigation...*'

Though reflective in parts, this dialogue remained procedural in nature, with coordination and summarisation dominating the interaction. It was coded as transactional talk due to its structural function in facilitating the execution of shared tasks.

In Pair 3, transactional talk was evident in a coordinated sequence of commands and clarifications:

A: '*Hey, let's create a line segment AB.'*

B: '*Yeah, got it.'*

A: ‘*Perfect! Looks like we used A as the centre, which is correct since the instruction says to create a circle at A with a radius of CE. Good job! Ok, I don’t know how you managed to construct the circle at A.’*

B: ‘*The first thing I did was to select the two points that define the radius CE, and then once you have done that, you then click where you want it to be centred.’*

A: ‘*Okay, now let’s investigate the geometrical relationship between line GH and line segment AB.’*

This interaction reflects clear role negotiation, technical explanation, and feedback exchange, supporting the joint construction process without yet involving conceptual interrogation. The transition at the end, ‘*now let’s investigate...*’, signals a shift toward exploratory talk.

Although often overlooked in conceptual analyses, transactional talk was essential in this study’s context. It scaffolded collaboration by ensuring clarity, synchronising actions, and managing the flow of engagement with GeoGebra. It also provided entry points into exploratory and evaluative modes of dialogue, suggesting that transactional talk, while less conceptually rich, served as a dialogic bridge toward deeper reasoning.

Expressive talk: Expressive talk allowed participants to voice their feelings, uncertainties, and reflections, often surfacing during moments of doubt, excitement, or realisation. This form of talk enriched the dialogic process by encouraging openness and trust, which contributed to collaborative exploration and co-construction of knowledge.

For instance, in Pair 1's engagement, expressive talk was evident when Participant M verbalised uncertainty and a tentative hypothesis:

M: *'I feel like if that line is in a different position, it would bisect the angle, but I might be totally wrong.'*

N: *'I don't know, it is still not bisecting the angle. I feel like that might bisect angle ABC if it was in the right location.'*

N: *'Mmm, I have not seen this before…'* [Participant N admits unfamiliarity with the situation, suggesting a new and potentially challenging concept of examples and non-examples of an angle bisector].

M: *'I see; it is true. Let's try to make it an angle bisector.'*

N: *'Yes.'*

Here, M's candid expression of doubt ('*...but I might be totally wrong*') did more than reflect uncertainty, it created an open space where exploration was safe and encouraged. This moment functioned as an invitation to collaborative sense-making, prompting a productive exchange where both participants acknowledged gaps in understanding and proposed a joint investigative effort. The willingness to embrace not-knowing facilitated deeper inquiry and exemplified the affective and epistemic value of expressive talk in sustaining a dialogic learning environment.

Similarly, expressive talk was also used to reflect on learning experiences and signal emotional engagement. In Pair 2, during a reflective moment at the conclusion of their task, one participant remarked:

'Exactly. It's been a fascinating investigation...'

This reflective utterance, though brief, served to affirm the process of exploration and recognise the intellectual satisfaction derived from collaborative problem-solving. Such expressions reinforced the social and emotional dimensions of learning, acknowledging effort, celebrating insight, and validating joint participation.

Evaluative talk: Evaluative talk involved critical assessment, comparison of strategies, and reflection on outcomes. It marked a shift from observation to analytical reasoning, as participants tested conjectures, challenged assumptions, and clarified geometric relationships. This talk supported deeper understanding and informed conclusions, facilitated by dynamic geometry software (DGS).

For example, in Pair 3, evaluative talk emerged as participants analysed the relationship between a constructed line and a line segment:

A: *'Okay. So, what geometrical relationship does it [the line] have to a line segment [AB]? Definitely perpendicular. Why is it perpendicular? Because you've got two sorts of right-angle triangles.'*

B: *'Yeah, so like, I guess we could use the circle theorem for that.'*

A: *'Yes, you are right! It's like the kite shape we described between AGBH, which rotates around the midpoint of AB, making them perpendicular.'* [Participant A confirms Participant B's observation, highlighting the connection between the kite shape and the perpendicular relationship].

Here, participants drew on theoretical knowledge (circle theorems and symmetry properties) and evaluated the connection between visual features and geometric principles. Evaluative talk thus provided a framework for participants to test conjectures, consider alternative strategies, and justify conclusions.

A similar evaluative process was observed in Pair 1, where participants reconsidered their construction of an angle bisector. Participant M stated:

'I feel like if that line is in a different position, it would bisect the angle, but I might be totally wrong.'

With Participant N, responding, saying, after further manipulation,

'I don't know; it is still not bisecting the angle. I feel like that might bisect angle ABC if it was in the right location.'

These utterances reflect ongoing critical assessment, leading to a collective refinement of understanding. When Participant M responded with:

'I see; it is true. Let's try to make it an angle bisector.'

N: *'Yes.'* (in agreement with M's reasoning)

The participants were engaging in evaluative processes, testing conjectures, revisiting visual evidence, and confirming emerging hypotheses.

In Pair 2, evaluative talk was used to classify a quadrilateral formed:

> *'So, if we want to prove that the quadrilateral is a square, we need to show that all the angles at A, B, H, and G are right angles.'*

Participant H confirmed and elaborated:

> *'That's correct. So, all the angles at A, B, H, and G are right angles since they become midpoints of the new square formed by the tangents... And when the circles are not of the same size, the quadrilateral becomes a kite, but it's not necessarily a rhombus unless the circles are equal.'*

These analytical reflections illustrate how participants reasoned through geometric classifications, integrating prior knowledge with visual representations. The talk was purposeful, evaluative, and grounded in geometric reasoning.

Collectively, evaluative talk strengthened the dialogic learning environment by encouraging rigorous reasoning and reflective decision-making. It complemented exploratory and expressive talk, contributing to an approach to understanding geometry in technology-enhanced settings.

Deliberative talk: Deliberative talk was characterised by participants weighing alternatives, proposing multiple solution paths, and collaboratively considering the best course of action. This form of talk frequently surfaced as participants encountered decision points in their constructions and needed to evaluate potential strategies.

In the exchange between Participants M and N (Pair 1), deliberative talk is evident as they explored how to correctly position a line to act as an angle bisector:

M: *'I feel like if that line is in a different position, it would bisect the angle, but I might be totally wrong.'*

N: *'When you drag the points around, I can see that the slope of the line JK always stays the same as the angle bisector, just that it is at the wrong place.'*

This dialogue shows a deliberate effort to hypothesise and refine a construction through strategic manipulation of the figure. Participant N's consideration of how dragging points affects alignment suggests active planning and analysis of dynamic geometry tools to achieve a specific goal. This kind of deliberation exemplifies dialogic learning, as participants do not settle for surface-level observation but instead collaboratively determine geometric validity through reasoning and testing.

Deliberative talk also featured in Pair 2's efforts to classify a quadrilateral formed by intersecting circles:

G: *'So, if we want to prove that the quadrilateral [Figure 5.3c] is a square, we need to show that all the angles at A, B, H, and G are right angles.'*

H: *'That's correct. So, all the angles at A, B, H, and G are right angles since they become midpoints of the new square formed by the tangents...'*

Here, the participants engaged in forward planning, identifying key mathematical properties that would validate their claim and developing a logical strategy to support their proof. The deliberative nature of this exchange is reflected in the focus on criteria selection and theoretical grounding, as they tied observable properties to geometric principles.

These instances highlight how deliberative talk supported shared reasoning, critical thinking, and effective use of technological tools. Working through various options and co-constructing strategies, participants demonstrated a deepening engagement with both geometric concepts and collaborative inquiry.

Imaginative talk: Imaginative talk enabled participants to speculate, visualise, and mentally simulate hypothetical geometric configurations. This mode of communication encouraged learners to think beyond the constraints of the immediate task and explore dynamic relationships creatively.

In Pair 3, imaginative talk surfaced as participants considered alternative geometric configurations that were not explicitly required by the task:

A: '*Imagine if we reflected this circle over the line AB, what kind of symmetry would we get?*'

B: '*That's interesting… it would be like a mirror image, so you'd have two tangents touching at opposite sides, probably still perpendicular, right?*'

A: ‘*Yes, like a butterfly shape. I wonder if we could make the two tangents intersect and form a kite.*’

B: ‘*Or even make a rhombus if we adjust the radius so that the circles are equal. That might change everything*.’

This dialogue exemplifies how imaginative talk enabled learners to creatively manipulate mental models and extend their thinking beyond the literal construction, considering symmetry, transformation, and shape classification in innovative ways.

In Pair 1, Participant N contemplated:

> *‘What if the line went the other way, like a horizontal line? Would that still bisect the angle or not?’*

Participant M dragged *point A* to make the line look horizontal, which then made Participant N respond, ‘*Yes, it does, doubting Thomas’*.

Such utterances, though brief, reflect imaginative reasoning, inviting speculation and prompting deeper exploration.

Pair 2 also demonstrated imaginative talk in their discussion on tangents and quadrilateral formation:

G: *‘Since the line from I to G is the radius of the circle centred at I, any tangent to that circle from point G would form a right angle.’*

H: *'You can imagine from what we are exploring that when this happens, all the intersection points of the tangents would form another square.'*

Here, Participant H uses visual and imaginative reasoning to predict an emergent geometric figure. This use of imaginative talk links abstract theoretical understanding (properties of tangents and radii) with a creative hypothesis about a new configuration, enabling a richer appreciation of the relationships at play.

Across all groups, imaginative talk enriched the dialogic process by allowing participants to project, speculate, and visualise, thereby improving their mathematical creativity and enabling a more flexible, exploratory approach to geometry in dynamic environments.

Interrogatory talk: Interrogatory talk was marked by participants posing open-ended or clarifying questions, often to initiate inquiry, test assumptions, or stimulate deeper reasoning. Such questioning served as a key mechanism for collaborative exploration and the co-construction of understanding in a dynamic geometry environment.

In Pair 1, Participants M and N demonstrated interrogatory talk while investigating the relationship between a line and an angle bisector:

N: *'Can you try dragging point B around?'*

M: *'Do you mean the gradient of this line is the same as the angle bisector?'*

This brief exchange reveals how questioning was used to prompt actions (for example, dynamic dragging) and clarify conceptual connections (for example, slope equivalence). Participant N's prompt invites exploratory manipulation, while Participant M's query tests a hypothesised geometric relationship, signalling a shared effort to interrogate the mathematical structure of the construction.

Similarly, in Pair 4, interrogatory talk emerged when C and D reflected on their angle bisector construction:

C: '*Wait, is it still an angle bisector when I move point A?*'

D: '*Good question. Why would it stop being one? Doesn't it always divide the angle into two equal parts?*'

C: '*But how do we know for sure? Should we measure the angles?*'

This flow of probing questions propelled the participants to seek verification through GeoGebra's measurement tools, exemplifying dialogic learning grounded in inquiry.

Across pairs, interrogatory talk played a central role in sustaining collaborative dialogue, prompting experimentation, and inviting clarification. It acted as a gateway to deeper investigation, encouraging participants to test hypotheses, probe definitions, and refine their understanding of geometric phenomena.

Expository talk: Expository talk occurred when participants assumed an explanatory role, verbalising geometric principles, describing construction steps, or articulating observed patterns. This form of talk helped structure the collaborative inquiry by clarifying processes and consolidating shared knowledge.

In the following excerpt from Pair 2, expository talk emerged as participants discussed how the classification of a quadrilateral varied with changes to circle size and alignment:

H: *'So, to summarise, we've observed that the quadrilateral changes between a kite and a square depending on the size of the circles. When all the circles coincide, it becomes a square, and when they are different sizes, it's a kite.'*

G: *'Exactly! So, at the point where the orange and blue circles coincide, we have a square, and as we move E and F along the line segment CD, the quadrilateral remains a square until the circles change sizes again.'*

Participant H provides a summary of key geometric observations, while Participant G elaborates, describing how dragging points affects shape properties. Their clear articulation and logical progression of ideas demonstrate how expository talk supports collective reasoning and affirms understanding within the pair.

Another example from Pair 3 showcases expository talk through explanation of construction steps and geometric theorems:

A: *'So, A is built upon C to E, and B is built upon C to F, so if E and F overlay, then it will be a perpendicular bisector.'*

B: *‘Because we’ve got a radius and chord.’*

A: *‘The fact is that the chord is a vertical property where the tangent to a circle is always at right angles to its radius.’*

Participant A outlines the procedural relationship between construction elements, while Participant B briefly references the geometric reasoning behind it. Participant A then reinforces the explanation by citing a known theorem. This exchange exemplifies how participants shared and extended ideas using precise language and conceptual reasoning.

In some cases, expository talk served to consolidate findings:

M: ‘*So, when you move either of those points, you are just changing the angle, but the line still bisects the angle. It is good to learn concepts like this.*’

Expository talk thus complemented exploratory and interrogatory modes by providing stability and clarification in the meaning-making process. It allowed participants to reflect on what they had discovered and articulate it for shared understanding.

Finally, expository talk also facilitated connections between current tasks and previous learning. In a dialogue from Pair 1, Participant N draws on a past activity to explain the current configuration:

N: *‘When you drag the points around, I can see that the slope of the line JK always stays the same as the angle bisector, just that it is at the wrong place.’*

M: *'Do you mean the gradient of this line is the same as the angle bisector?'*

N: '*You see, when you put point I on top of point H, it is now an angle bisector, the same as the previous task we did yesterday. So, the radii of the two circles must be the same.'*

Here, Participant N interprets visual behaviour and links it to geometric properties, further contextualised by recalling a similar previous task. The ability to draw from past experiences and articulate connections illustrates the cumulative nature of expository talk in enhancing mathematical understanding.

Through the application of these dialogic talk types as analytical tools, the dialogue and communication interactions among participants were analysed, shedding light on the richness and complexity of their collaborative learning experiences within the dynamic geometry software environment. This analytical approach further facilitated the exploration of knowledge co-construction among beginning teachers, thereby contributing to an understanding of their learning processes. Typical examples of how I represented this dialogic talk analysis are shown in Chapter 5A, the next chapter.

Exploring types of dialogic learning principles analysis

Building on the previously identified dialogic forms of talk, I extended the analysis by applying Alexander's (2008; 2018) dialogic learning principles to gain deeper insight into how participants collaboratively constructed knowledge during the geometric construction tasks. This theoretical framework enabled a more detailed interpretation of their interactions, highlighting the underlying structures of dialogic engagement as defined by Alexander's (2008; 2018) six principles, including collective, reciprocal, supportive, cumulative, purposeful, and deliberative. After systematically coding participants' discussions, actions, and exchanges, I organised these codes according to Alexander's (2008, 2018) dialogic principles. A colour-coded system was then used to map their engagement with each principle, offering a structured lens through which to explore how they approached the tasks and addressed the research questions guiding this study. (See Chapter 5A, for the colour-coded system).

The collective principle guided the categorisation of collaborative problem-solving, peer learning through discussion, shared exploration, and negotiated ideas. This principle illuminated how participants managed challenges and shared insights in a cohesive learning process. For example, participants' dialogue demonstrated this principle as they addressed a geometric construction task together: *'So A is built upon C to E and B is built upon C to F, so if E and F overlay, then it will be a perpendicular bisector,'* Participant A explained, with Participant B agreeing, *'Yeah, whatever E and F are the same... [as I am doing and showing on the screen].'* This exchange exemplifies how collective knowledge was co-constructed through discussion and shared visualisation.

Cumulative knowledge-building was visible in instances where participants' understanding evolved through incremental discussions that built on prior knowledge and connected concepts over time. This principle enabled me to track how their learning evolved and deepened. One excerpt illustrates this cumulative learning: *'You see, when I put point I on top of point H, it is now an angle bisector, the same as the previous task we did yesterday. So, the radii of the two circles must be the same,'* to which another participant replied, *'Yes, you are* *right, you are right.'* This interaction highlights how participants revisited and applied prior knowledge to approach more complex tasks.

The deliberative principle became evident in episodes where participants critically considered different strategies, evaluated outcomes, and reflected on next steps. This thoughtful dialogue added depth to their engagement with the tasks. For example, one participant noted: *'So if we want to prove that the quadrilateral is a square, we need to show that all the angles at A, B, H, and G are right angles.'* This careful assessment underlines their intent to validate their understanding through analytical thinking and collaborative reasoning.

In line with the purposeful principle, many discussions were directed toward specific learning objectives and strategic problem-solving. Participants often focused their talk on achieving task goals and exploring relevant features of the dynamic geometry software. For instance, when working towards proving the properties of a quadrilateral, one participant remarked: *'If the circles are equal, then the quadrilateral is both a kite and a rhombus,'* demonstrating a purposeful link between their geometric reasoning and the intended learning outcome.

The reciprocal principle was demonstrated through mutual assistance, feedback, and shared construction of meaning. One participant encouraged another with: *'Make it like ABC right triangle. OK. So, you see, it cuts the angle in half by bisecting the angle.'* This interaction illustrates reciprocal learning, where participants collaboratively clarified geometric relationships and provided real-time support.

Lastly, the supportive principle was reflected in the encouragement and practical help participants gave each other, helping to build a caring and responsive learning environment. For example, Participant A acknowledged Participant B's success in constructing a circle: *'Perfect! Looks like we used A as the centre, which is correct since the instruction says to create a circle at A with a radius of CE. Good job!'* When Participant A later expressed confusion, Participant B responded with guidance: *'The first thing I did was to select the two points that define the radius CE, and then once you've done that, you then click where you want it to be centred.'* This exchange provided constructive support and reinforced a positive learning atmosphere.

Applying these dialogic principles, I gained insight into the collective, cumulative, deliberative, purposeful, reciprocal, and supportive ways participants interacted and co-constructed knowledge. This analysis revealed how structured dialogue and a responsive learning environment meaningfully shaped their engagement with the geometric construction tasks. In doing so, it addressed the study's research focus on how dialogic learning and task design facilitate beginning teachers' development of technological, pedagogical, and content knowledge.

4.8.1.2 Interpretive video deductive coding analysis drawing on the TPACK framework as an analytical tool

Building on the interpretive video deductive coding approach outlined earlier, this section details how the TPACK framework was applied as an analytical lens to examine beginning teachers' knowledge development during collaborative geometric construction tasks using dynamic geometry software (GeoGebra).

The TPACK framework supported a layered analysis of participants' engagement with technology (technological knowledge, TK), their mathematical reasoning (content knowledge, CK), and their emerging pedagogical thinking (pedagogical knowledge, PK). Importantly, the framework also enabled interpretation of how these domains intersected, highlighting moments of technological pedagogical knowledge (TPK), technological content knowledge (TCK), pedagogical content knowledge (PCK), and integrated technological pedagogical content knowledge (TPACK).

To facilitate fine-grained analysis, video data were segmented into 30-second intervals. Within each segment, verbal and non-verbal interactions were coded according to relevant TPACK components. For example, using GeoGebra's move tool to manipulate geometric objects was coded as TK, while articulating geometric properties or relationships reflected CK. Instances where technology was strategically employed to support teaching and learning were coded as PK, showcasing participants' emerging pedagogical reasoning.

An illustrative episode involved participants A and B investigating the geometric relationship between a line formed by the intersection points of two circles and the line segment connecting the circles' centres (AB). Participant A hypothesised that the intersecting line was a perpendicular bisector of AB. Participant B initially agreed but then dynamically manipulated the circles to test this assumption. This exploration revealed that the relationship varied depending on point positions and radii, prompting further reasoning and refinement. Their dialogue and interactions demonstrated distinct and overlapping elements of CK, TK, and PK as they collaboratively explored, challenged, and refined their understanding. Their discussion unfolded as follows:

A: *'Okay, now let's investigate the geometrical relationship between line GH and line segment AB. What do you think it is? So, GH to AB, perpendicular bisector.'* [The geometrical relationship at this moment, as it shows on their computer screen, was perpendicular].

B: *'Yep,' [meaning Participant B agrees that the relationship is a perpendicular bisector. However, Participant B drags a point to change the radius of one of the two circles and observes that] 'but that is not always a bisector. At the moment, we got a perpendicular line, but it can be at any point along a line segment.' [This challenges the initial assumption and fosters a more detailed understanding.]*

A: '*So A is built upon C to E and B is built upon C to F, so if E and F overlay, then it will be a perpendicular bisector.'*

B: *Yeah, whatever E and F are the same...*

A: *But they will intersect only if C to E and C to F are over half of the length AB.*

B: *Yeah, yeah.*

I coded this episode using the TPACK framework, exemplifying the interplay among its components:

1. **Content knowledge (CK):** Participants discussed the nature of the geometric relationship, identifying and questioning whether the line was a perpendicular bisector.

2. **Technological knowledge (TK):** Participants used GeoGebra's *move tool* to dynamically adjust the radii of the circles, observing how changes affected the relationship, thus demonstrating their use of the software's capabilities.

3. **Pedagogical knowledge (PK):** Through these manipulations, participants reasoned about how such dynamic visualisations could aid learners in understanding geometric properties, reflecting pedagogical thinking.

Analytic memos accompanied the coding process, documenting the presence and integration of individual TPACK components as participants collaboratively engaged in meaning-making. These memos provided insight into how technology, pedagogy, and content knowledge evolved and intersected during the tasks.

In addition to the TPACK codes, metacognitive talk (MT) was included as a supplementary code to capture participants' reflective thinking and regulation of their cognitive processes. This dimension was particularly important for interpreting how beginning teachers negotiated uncertainty, tested hypotheses, and revised their understanding, processes that are critical for developing integrated TPACK and for learning to teach effectively in dynamic, technology-rich environments (Flavell, 1979; Mercer, 2002).

Table 4.5 presents excerpts from this episode alongside screenshots from the video, corresponding TPACK codes, interpretive commentary and rationale for the code. This triangulation of visual, verbal, and analytical data provides evidence of how beginning teachers negotiated and developed their TPACK in action. The colours correspond to:

CK (Content Knowledge) – Blue

PK (Pedagogical Knowledge) – Green

TK (Technological Knowledge) – Purple

PCK (Pedagogical Content Knowledge) – Yellow

TCK (Technological Content Knowledge) – Red

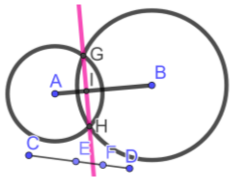
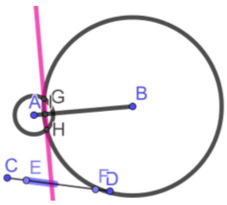
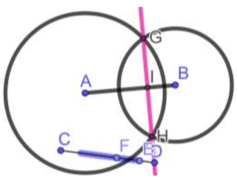
TPK (Technological Pedagogical Knowledge) – Orange

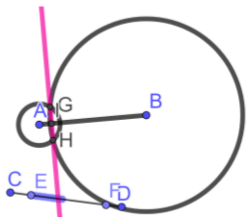
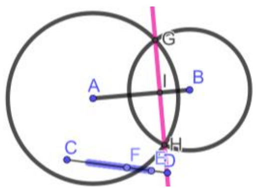
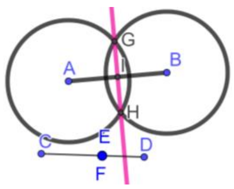
MT (Metacognitive Talk) – Brown

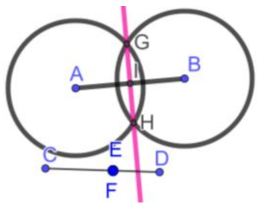
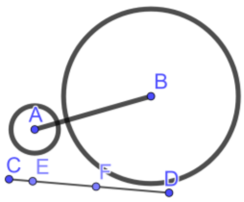
The **metacognitive talk (MT)** code is used to capture moments when participants reflect on their own thinking, evaluate their reasoning, or adjust their understanding or strategy in response to new observations. It resonates with the notion of '*thinking about thinking*' (Flavell, 1979), a critical process in learning and conceptual development, particularly in dialogic and exploratory learning environments. Mercer (2002) similarly highlights that exploratory and metacognitive talk in peer dialogue can significantly improve understanding. In this episode, participants tested geometric relationships and evaluated and refined their reasoning, demonstrating clear instances of metacognitive engagement.

Table 4.5 Screenshots of geometric figures, excerpts, colour-coded TPACK codes, interpretations and rationales.

Screenshot	Excerpt	Code(s)	Interpretation	Rationale for Code(s)

	A: *'So, GH to AB, perpendicular bisector.'* B: *'Yep [means perpendicular bisector]'*	**CK**	Participants hypothesise a geometric relationship based on visual observation, though a misconception exists—it is a perpendicular, not a bisector.	Demonstrates initial geometric reasoning about perpendicular bisectors.
	B: *'But that is not always a bisector. At the moment, we got a perpendicular line, but it can be at any point along a line segment.'*	**CK,** **MT**	However, Participant B drags *point E* to change the radius of the circle centred at *point A* to revise their initial assumption, recognising that perpendicularity does not imply bisecting.	CK: Refines content understanding. Metacognitive: B reflects on and corrects a misconception.
	B: *[Participant B drags points E or F to adjust the*	**TK,** **TCK** **MT**	Use of the *move tool* reflects growing fluency with GeoGebra's dynamic	TK: Demonstrates skill in using the software. TCK: Applies technology to

	radii of the circles]		features and informed content-based adjustments.	manipulate content. 🟤Metacognitive: Reflects on the purpose of manipulation.
 	B: *[Participant B tests the revised geometric relationship through manipulation]*	🟢 **PK**, 🟡**PCK**, 🟠 **TPK**	Strategic testing of hypotheses using technology reflects an emerging pedagogical idea about using dynamic tools for learning.	🟢PK: Uses manipulation to test a hypothesis. 🟡PCK: Pedagogically considers geometry. 🟠TPK: Technology used purposefully for exploration.
	A: *'So A is built upon C to E and B is built upon C to F, so if E and F overlay, then it will be a perpendicular bisector.'*	🔵 **CK**, 🔴**TCK**	Participant A expresses understanding through verbal reasoning supported by the visual construction.	🔵CK: Understands geometric principles of perpendicular bisectors. 🔴TCK: Integrates construction steps with visual confirmation.

	B: *'Yeah, whatever E and F are the same...'*	**PK**, **TPK**, **MT**	Participant B adjusts radii to match, demonstrating understanding of geometric conditions and use of technology to test a hypothesis.	PK: Pedagogical aim to confirm the hypothesis. TPK: Uses GeoGebra to illustrate and confirm. Metacognitive: Reflects on actions.
	A: *'But they will intersect only if C to E and C to F are over half of the length AB.' B: 'Yeah, yeah.'*	**CK**, **TCK**	Clarifies when circle intersection occurs, strengthening understanding of conditions for accurate geometric construction.	CK: Clarifies geometric condition. TCK: Uses a tool to explore specific mathematical dependency.

These interactions illustrate the interplay between TPACK components during collaborative problem-solving. Participant A's initial assertion and subsequent reasoning reflect the development of CK, while Participant B's manipulation of the dynamic geometry environment reveals TK and PK. The dialogue demonstrates how participants used GeoGebra not only to explore and validate geometric properties but also to think pedagogically about how such tools can support mathematical understanding.

This analysis, supported by analytic memos and visual data, underscores how technology-mediated tasks can foster the integration of pedagogical, content, and technological knowledge. Through collaborative dialogue and hands-on exploration, beginning teachers made meaningful connections that reflect the complexity of learning to teach geometry in technology-enhanced environments.

4.8.2 Conducting inductive thematic analysis to reveal emergent features of efficacious geometric construction tasks

In my research, I used inductive thematic analysis to explore and identify the emergent features of efficacious geometric construction tasks, intending to understand how these tasks facilitated participants' learning experiences. The process followed the traditional stages of inductive analysis, including data familiarisation, coding, theme development, and reviewing, allowing themes to emerge from the raw data itself, rather than imposing predefined categories. This approach was guided by the six-phase framework of thematic analysis developed by Braun and Clarke (2006) and further informed by their later work on reflexive thematic analysis, which emphasises the researcher's active role in theme development and interpretive engagement with the data (Braun & Clarke, 2021).

Step 1: Familiarisation with the data

The first step involved immersing myself in the raw data, which primarily consisted of participants' reflections, focus group discussions, and interviews. This process was essential in gaining an understanding of the participants' experiences with the designed geometric construction tasks. Data familiarisation was particularly important, as it allowed me to profoundly engage with the details in their responses, identifying recurring thoughts, concerns, and reflections on their engagement with the tasks.

For instance, Participant A reflected: '*The task helped me see the relationships between different geometric shapes in a way I hadn't before*.' Such reflections seem to signal the importance of exploring underlying features that made the tasks efficacious. In this stage, I also repeatedly reviewed video recordings, field notes, and transcripts to capture the details of their experiences.

Step 2: Generating initial codes

After thoroughly familiarising myself with the data, I proceeded to systematically generate initial codes. Using a bottom-up approach, I assigned codes to specific excerpts that appeared significant or insightful, without predetermining which features might emerge. The codes focused on identifying both explicit mentions of task features (for example, clarity, engagement, motivation) and implicit experiences (for example, frustration or discovery during task completion). This open coding was an essential component of the inductive process, as it could allow the analysis to be data-driven.

For example, when Participant B mentioned, '*The tasks were challenging but not impossible; they helped me build confidence over time*,' I coded this reflection under '*challenge level*' and '*confidence building*.' Similarly, when participants discussed understanding the connections between geometric concepts, such statements were coded under '*conceptual connections*' and '*task structure*.'

Step 3: Searching for themes

Once I had a comprehensive list of codes, I began identifying broader themes by examining the relationships among the codes. This stage involved clustering the codes into coherent patterns to form potential themes that addressed key aspects of participants' experiences. As the data was rich with feedback on various aspects of the tasks, patterns regarding task complexity, clarity, engagement, and relevance started to emerge.

For instance, multiple participants referred to how breaking down the tasks into manageable steps helped them feel less overwhelmed. From this recurring pattern, a theme surrounding '*scaffolding and progression'* emerged. Similarly, participants frequently highlighted their engagement and motivation when using dynamic geometry software, leading to the development of the theme '*active exploration and engagement*.'

Step 4: Reviewing and refining themes

At this stage, I refined the themes to ensure that they accurately represented the data. This involved reviewing the coded data for each theme and checking if the themes were distinct, internally consistent, and relevant to the research aims. I re-examined the themes in the context of the entire data set to ensure they captured the core features of the geometric construction tasks that were identified as efficacious by participants.

For instance, the theme of '*scaffolding and progression*' was refined to include sub-themes related to *sequencing, task difficulty, and gradual conceptual development.* This brought the refined final theme to be ***sequencing, progression, and varied challenges with scaffolding***. Participants' feedback, such as *'The instructions were well-structured, making it easy to follow along* *and understand what should be appearing on the screen at each stage of the construction.'*

'The building up of the model step-by-step and the progressive questioning were both useful.'

'They struck a good balance between providing guidance and allowing for independent exploration,' solidified the relevance of this theme to understanding how task design impacted learning.

Step 5: Defining and naming themes

At this stage, I finalised the themes and named them in a way that accurately represented the underlying features. For each theme, I defined the scope and boundaries to clarify how they contributed to understanding the effectiveness of the geometric construction tasks.

For example, the theme '*active exploration and engagement'* was defined as participants' ability to independently explore geometric relationships using dynamic geometry software, which facilitated motivation and active learning. Participant M's reflection, '*The software let me explore relationships between geometric elements, which kept me engaged throughout the task,'* served as a key piece of evidence in defining this theme.

Step 6: Writing up the results

The final stage involved synthesising the themes into a coherent narrative that explained how the emergent features contributed to the effectiveness of the geometric construction tasks. Each theme was illustrated with excerpts from the data, providing concrete examples of participants' experiences and insights. (see Chapter 5 for the write-up of the results).

4.8.3 Deductive thematic analysis using the TPACK framework for revealing beginning teachers' perceptions of knowledge requirements for geometry teaching in technology-integrated contexts

The analysis of beginning teachers' perceptions of the knowledge requirements for teaching geometry in technology-integrated contexts was conducted through a deductive thematic analysis grounded in the technological pedagogical content knowledge (TPACK) framework. Developed by Koehler and Mishra (2009), the TPACK framework can provide a structured lens to explore the interconnectedness of technological, pedagogical, and content knowledge in educational practice.

Data preparation

The analysis commenced with the preparation of data gathered from interviews and reflective discussions among beginning teachers. Audio recordings were transcribed to create a rich qualitative dataset, capturing participants' insights and experiences in using dynamic geometry software (DGS), such as GeoGebra, for geometry instruction.

Coding framework development

Building upon the TPACK framework, I established a coding framework encompassing three primary components:

1. **Technological knowledge (TK)**: Understanding the technology and its educational applications.

2. **Pedagogical knowledge (PK)**: Familiarity with teaching methodologies and instructional strategies.
3. **Content knowledge (CK)**: Understanding of the subject matter, particularly geometric concepts.

These categories served as predetermined themes for the analysis, enabling systematic data categorisation.

Deductive coding process

During the review of transcripts, I applied the established codes to participants' comments, focusing on their reflections about integrating technology into geometry instruction. Each statement was analysed to determine which TPACK category it best aligned with. For instance:

Technological knowledge (TK): Comments that illustrated participants' experiences with GeoGebra's features or their comfort level in using the software were categorised as TK. An example included a participant stating, '*I felt more confident using GeoGebra after this task*,' indicating their growing proficiency.

Pedagogical knowledge (PK): Statements reflecting instructional strategies or teaching methods were classified under this theme. For example, when a participant discussed the use of technology to create engaging learning experiences, it was categorised as PK, demonstrating how technology could enhance their pedagogical approaches.

Content knowledge (CK): Remarks showing an understanding of geometric concepts or relationships were assigned to this category. For instance, a participant noted, '*I now understand the connections between different* *geometric constructions better*,' highlighting an improved understanding of the subject matter.

Theme identification and interpretation

After coding the data, I synthesised the categorised statements to identify overarching themes that emerged from the participants' reflections. This step involved reviewing the coded data to form comprehensive insights regarding beginning teachers' perceptions of the knowledge requirements for teaching geometry in technology-integrated contexts.

One prominent theme that emerged was **technical support and usability issues**, reflecting participants' expressions of needing robust support and guidance. This theme emphasised the technological challenges they faced and highlighted their evolving perceptions of the knowledge necessary for effective technology-enhanced teaching.

In conclusion, the deductive thematic analysis framed by the TPACK model systematically revealed beginning teachers' perceptions of the knowledge requirements for geometry teaching in technology-integrated environments. Applying predetermined codes aligned with the TPACK framework, I critically assessed participants' insights and illuminated the complex interplay between technological, pedagogical, and content knowledge in their learning practices. This analysis provided valuable insights into how beginning teachers can improve their instructional approaches by effectively integrating technology into geometry education.

4.9 Establishing the trustworthiness of the findings of the study

In this section, I discuss the criteria for establishing the trustworthiness of qualitative research, focusing on Lincoln and Guba's model (Lincoln & Guba, 1985). Many authors believe that the criteria determining the trustworthiness of quantitative research cannot be used to assess the trustworthiness of qualitative research (Braun & Clarke, 2019; Clarke & Braun, 2013; Lincoln & Guba, 1985). The use of Lincoln and Guba's model to establish the trustworthiness of qualitative research is not new in the literature (Braun & Clarke, 2019; Houghton, Casey, Shaw, & Murphy, 2013; Krefting, 1991). To ensure the trustworthiness of qualitative research, four criteria must be met: credibility, transferability, dependability, and confirmability. In this section, I first focus on credibility, which refers to confidence in the truth of research findings.

To establish credibility in my study, I followed the recommended strategies of Lincoln and Guba (1985) , such as prolonged field engagement. Collaborating with supervisors, ITE directors, PGCE course leaders, and mathematics teams from Schools of Education, I aimed to build rapport with participants, fostering their willingness to provide accurate information. Lincoln and Guba (1985) noted that prolonged engagement helps mitigate researcher biases and allows for a deep understanding of informants' perspectives. In this case, my extensive teaching experience with the software and discussions with key figures in ITE significantly enhanced the research's credibility.

The credibility of qualitative research is determined by the truth of the findings based on the participants' original data and their actual views (Braun & Clarke, 2019; Clarke & Braun, 2013; Lincoln & Guba, 1985; Noble & Smith, 2015). The strategies proposed by Lincoln and Guba to establish credibility, such as peer debriefing and member checking, have been shown to increase the trustworthiness of qualitative research (Lincoln & Guba, 1985). In my study, the involvement of knowledgeable and experienced professionals in qualitative research, such as supervisors, was critical to ensuring the credibility of research findings (Braun & Clarke, 2019; Clarke & Braun, 2013; Krefting, 1991; Noble & Smith, 2015). My contact with my two supervisors, who had rich experience in qualitative research, helped to polish the interpretation of the results reported in the study through their insightful suggestions and comments. Furthermore, my academic and research background, skills, and training increased the study's credibility (Krefting, 1991; Lincoln & Guba, 1985; Noble & Smith, 2015).

Another crucial criterion is the enhancement of transferability to establish the trustworthiness of qualitative research. Transferability refers to the extent to which the findings of a study can be applied to other contexts or settings beyond the original research context. Unlike generalisability, transferability does not seek universal claims but acknowledges the contextual nature of qualitative research (Houghton *et al*., 2013). To bolster transferability, I provided details about the research context, participants, and process. For instance, in the study, I engaged participants from four different universities across varied geographical locations and employed diverse mathematics curricula, amplifying the potential application of results to comparable contexts. The research unfolded in two distinct phases, each occurring at different times and years.

The deliberate inclusion of participants from different universities seems to extend the applicability of findings beyond a specific educational context and contribute to the study's external validity. Notably, temporal variation is interwoven into the research design, introducing depth and indicating that the conclusions of the findings transcend time-based constraints.

A crucial element contributing to transferability is the careful documentation of each phase. Appendix F provides detailed tables outlining tasks, participants, and learning environments, furnishing a transparent roadmap for comprehending the study's details. This transparency facilitates both replication and the potential application of similar methodologies in different educational settings.

Dependability concerns the stability and consistency of the study's findings over time. To ensure dependability, I used strategies such as an audit trail, code-recode, stepwise replication, and peer examination. An audit trail provides a record of the research process, which external auditors can review to ensure dependability. Code-recode involves coding the same data several times to check for consistency in the coding process. Stepwise replication involves repeating the study with different participants or in different contexts to test the findings' generalisability. Peer examination involves other researchers reviewing the study's findings to ensure their dependability. In my study, I used code-recode and engaged two supervisors to provide critical insights into the research process.

Confirmability concerns the degree to which other researchers can confirm or corroborate the findings of a qualitative study. To achieve confirmability, I use strategies such as an audit trail, triangulation, and reflexive practice. Triangulation involves using multiple methods or data sources to confirm the findings, while reflexive practice involves reflecting on my role and biases in the research process.

Moreover, in the research, I employed a diverse array of data collection methods, including video data, individual interviews, focus group discussions, and questionnaires. This methodological diversity captures a holistic view of participants' perspectives, enriching the data and fortifying confirmability and transferability. The strategic amalgamation of qualitative data collection techniques ensures a nuanced understanding of the researched phenomenon.

In summary, I used Lincoln and Guba's (1985) model to establish the credibility of my qualitative research, which included prolonged engagement in the field, peer debriefing, member checking, collaboration with supervisors and knowledgeable professionals, and my academic and research background, skills, and training (Krefting, 1991; Lincoln & Guba, 1985).

4.10 Conclusion

This chapter explained the context and philosophical standpoint of the study, research design, ethical considerations, research methods or instruments, how the data was analysed, and the trustworthiness of the findings in connection with the study's purpose and research questions.

Drawing on the principles of qualitative research methodology, design-based research was adapted to understand how inquiry tasks within a dynamic geometry software environment can support beginning teachers' development of technical knowledge, appropriate content, pedagogical and technological knowledge, and understanding of knowledge for teaching in technology-focused contexts with a focus on geometry. The findings of the study are reported in detail in the following chapters.

4.11 The next chapters: Chapters 5A and 5B

In Chapters 5A and 5B, the findings of the study are examined concerning the research questions. Chapter 5A focuses on how carefully designed tasks within a dynamic geometry software (DGS) environment fostered productive dialogic mathematical talk among beginning teachers, contributing to the development of their technological pedagogical content knowledge (TPACK). The chapter explores how task design principles promoted meaningful interactions that supported collaborative knowledge construction and pedagogical reflections, essential for TPACK development. Chapter 5B addresses how beginning teachers perceive, understand, and communicate the knowledge needed for teaching geometry in a technology-based environment. Through interviews and discussions, this chapter provides insights into their experiences and the challenges of integrating technology into their teaching practices.

CHAPTER 5A: TASK DESIGN PRINCIPLES, PRODUCTIVE DIALOGIC MATHEMATICAL TALK, AND TPACK DEVELOPMENT

5A.1 Introduction

This chapter examines how carefully designed tasks within a dynamic geometry software (DGS) environment fostered productive mathematical talk and dialogic learning among beginning teachers and, in turn, facilitated the development of their technological pedagogical content knowledge (TPACK). The research question guiding this analysis is: *In what ways can carefully designed tasks facilitate productive mathematical talk and dialogic learning among beginning teachers working with dynamic geometry software in ways that develop their TPACK knowledge?*

The chapter is organised into four main sections. The first section briefly revisits the task design principles used in designing the geometric construction tasks. The second section presents findings on how these design principles might have promoted productive mathematical talk and dialogic learning among beginning teachers in an online environment. These principles likely encouraged the beginning teachers to actively share ideas, ask questions, and engage in collaborative problem-solving, key aspects of effective dialogic mathematical talk.

In the third section, the focus shifts to how these well-constructed tasks supported the development of the participants' TPACK, particularly in relation to their use of dynamic geometry software like GeoGebra, their understanding of geometric constructions, and how they might apply these insights in their future teaching. Lastly, the chapter provides additional analysis and findings focused on specific features of the geometric construction tasks that participants identified as particularly impactful. Through this integrated analysis, I demonstrate that the combination of carefully designed tasks and a supportive online environment was essential in facilitating the dialogic mathematical talk necessary for TPACK development among beginning teachers.

5A.2 Examining how and why these task design principles were incorporated into the geometric construction tasks designed for beginning teachers and their theoretical background

Before discussing the findings related to how task design principles facilitated productive mathematical talk and dialogic learning, it is important to revisit how and why these principles were incorporated into the geometric construction tasks. This section explores the rationale for selecting and integrating key design principles, setting the stage for understanding the outcomes observed in this study.

I used the principle of scaffolding in the task design, providing a structured progression from basic to more complex tasks. For instance, in Task 1, beginning teachers were introduced to fundamental constructions, including creating line segments, sliders, and circles, before progressing to more advanced activities like investigating geometric relationships and constructing intersecting lines. This gradual increase in complexity seems to have enabled beginning teachers to build their skills incrementally, fostering an understanding of geometric concepts. The inclusion of scaffolding was based on Vygotsky's theory of the zone of proximal development (ZPD), which emphasises that learners can reach higher levels of understanding with appropriate support (Vygotsky, 1978). The scaffolding of the tasks ensured that participants were neither overwhelmed by the complexity nor under-challenged, facilitating a learning environment that promoted confidence and understanding, both crucial for developing their TPACK (Mishra & Koehler, 2006).

Collaborative problem-solving and reflective practices were also embedded within the tasks. Throughout Task 1, participants were encouraged to compare and discuss their geometric constructions with peers, promoting mutual learning and the exchange of ideas. In addition, participants were prompted to engage in reflective practices by observing, manipulating, and analysing geometric relationships. Reflection extended to the conjecture and presentation stages, where participants critically assessed their own methods and findings. These practices were grounded in Vygotsky's social constructivism, which emphasises the role of social interaction in learning (Vygotsky, 1978), and Schön's reflective practitioner model, which highlights the importance of reflection in professional growth (Schön, 2017). The integration of collaborative and reflective practices aimed to foster productive mathematical talk by encouraging dialogue, peer feedback, and active engagement, all of which were key to the development of TPACK (Mishra & Koehler, 2006; Shulman, 1987).

The use of dynamic geometry software (GeoGebra) played a pivotal role in the design of these tasks. GeoGebra allowed participants to interactively manipulate geometric objects, offering a dynamic, visual environment to explore and experiment with mathematical concepts. In Task 1, for example, and likewise all the tasks, participants used GeoGebra's compass and move tools to investigate geometric relationships, receiving immediate feedback from the software. The decision to incorporate GeoGebra was grounded in research on the use of technology in mathematics education (Laborde, Kynigos, Hollebrands, & Strässer, 2006), particularly about instrumental orchestration (Trouche, 2004), where learners actively engage with digital tools to deepen their understanding. This dynamic interaction between learners, in this case beginning teachers and the software, potentially helped facilitate the development of their technological knowledge, a core component of TPACK, while fostering dialogic learning as participants explored, justified, and critiqued different solutions.

Instrumental orchestration further enhanced the integration of GeoGebra, as participants learned to use/navigate the functionalities/complexities of the software while solving geometric problems. This process seems to have encouraged the development of problem-solving strategies and adaptation in real time, as participants refined their understanding of both the tool and the mathematical concepts. The principle of instrumental orchestration is grounded in theories of technology-mediated learning (Trouche, 2004), ensuring that participants engaged meaningfully with the software, thus contributing to the growth of their TPACK.

In designing these tasks, Bruner's (1974) modes of representation, enactive (hands-on manipulation), iconic (visual representation), and symbolic (abstract reasoning), were incorporated to cater to different learning styles or preferences. In Task 1 and likewise, all the tasks, participants engaged with geometric figures through physical manipulation via GeoGebra, visualised their constructions dynamically and reasoned abstractly about geometric relationships. This multi-modal approach, rooted in Bruner's cognitive theory (Bruner, 1974), was selected to ensure that participants could connect physical actions with visual feedback and abstract reasoning, thus supporting an understanding of geometric concepts.

The principle of open-endedness was also important to task design. Tasks were structured to allow multiple approaches and solutions, encouraging participants to explore different strategies and perspectives. For example, in Task 1, participants were free to investigate various geometric constructions, which promoted dialogue and critical thinking. This flexibility, grounded in constructivist theories (Stein, Smith, Henningsen, & Silver, 2009), encouraged a richer form of mathematical exploration and reflection. Participants were required to justify their approaches and engage in collaborative problem-solving, all of which are central to productive mathematical talk.

The tasks also struck a balance between engagement with tangible geometric objects (ostensive) and abstract geometric relationships (non-ostensive). Participants manipulated concrete geometric figures using GeoGebra while simultaneously analysing the underlying abstract principles governing these figures. This balance, supported by Bruner's theories of representation (Bruner, 1974) and dual coding theory (Paivio, 1990), allowed participants to connect physical manipulation with theoretical understanding, promoting a holistic approach to learning geometric concepts.

Finally, feedback mechanisms and the focus on meaningful goals and visible mathematics were integral to task design. GeoGebra's real-time feedback provided immediate visual updates, while collaborative discussions encouraged peer feedback. These mechanisms reinforced understanding and allowed participants to make adjustments to their constructions based on constructive feedback. The tasks were also designed with clear and meaningful goals, ensuring that participants engaged with relevant geometric concepts in a practical context. The visibility of mathematical ideas through GeoGebra further supported participants in linking practical activities to theoretical learning, enhancing their overall grasp of geometric principles (Hiebert, 1997).

These task design principles were deliberately chosen and integrated to support the development of TPACK, foster productive mathematical talk, and encourage dialogic learning among beginning teachers. Scaffolding tasks, promoting collaboration and reflection, using dynamic geometry software, and incorporating feedback mechanisms seem to have created a learning environment that promoted good exploration of mathematical concepts and enhanced pedagogical understanding. These principles laid the foundation for the productive dialogue and TPACK development that emerged as key findings in this study.

5A.3 Using task design principles in facilitating productive mathematical talk and dialogic learning among beginning teachers

5A.3.1 Facilitation of productive mathematical talk through task design principles (or carefully designed tasks)

In designing these geometric construction tasks (outlined in Chapter 3), I carefully aligned them with key task design principles that could foster productive mathematical talk. This alignment could ensure that participants engage profoundly with mathematical concepts and articulate their ideas effectively.

Scaffolding played a role in facilitating mathematical talk. Structuring tasks progressively, from simpler constructions to more complex ones, the tasks could allow participants to build their understanding incrementally. For instance, participants began with basic line segments and circles before moving on to more complex tasks like constructing intersecting lines and exploring geometric relationships. This gradual increase in complexity encouraged participants to explain their reasoning at each step, building their confidence and ensuring they could articulate their thought processes clearly. For example, when paired participants A and B were constructing the initial dynamic construction for onward exploration, Participant A said, '*Looks, like we used A as the centre, which is correct since the instruction says to create a circle at A with a radius of CE,'* after Participant B had constructed a circle at *A* with a radius of *CE.* The scaffolding in the task instructions led Participant A to engage in this mathematical talk by providing clear, incremental steps that built their understanding. As they followed the instructions, Participant A could connect their actions to the task requirements, prompting them to verbalise their reasoning. This structured approach encouraged active engagement, leading to Participant A's accurate and confident articulation of the process.

Collaborative learning was another principle embedded in the task design. Tasks were crafted to encourage paired or group work, which could foster discussions among participants. For example, in the task involving the construction of perpendicular bisectors, participants were prompted to compare their methods and discuss any discrepancies in their results. This exchange of perspectives could help clarify misunderstandings and enrich participants' understanding of geometric concepts by exposing them to different approaches.

While working on Task 1, which involved exploring the geometric relationship between the line passing through the intersection points of the circles centred at points *A* and *B* and the line segment *AB*, Participant A initially stated that the relationship was a '*perpendicular bisector,*' to which Participant B agreed by saying, '*yes.*' However, after collaborating to manipulate, drag, and change the radius of one circle, Participant B observed that, '*it is not always a bisector; at the moment, we have a perpendicular line, but it [the perpendicular line] can be at any point along a line segment.*' Based on this, Participant A hypothesised, '*So A is built upon C to E and B is built upon C to F, so if E and F overlay, then it will be a perpendicular bisector.*' This indicates that when the radii of the two circles, centred at the endpoints of segment AB, are equal and intersect, the line connecting their points of intersection forms the perpendicular bisector of segment AB. The collaborative learning process facilitated productive mathematical talk by encouraging Participants A and B to test and refine their understanding together. Participant B's observation that the line isn't always a bisector prompted Participant A to reconsider and clarify the conditions under which the line becomes a perpendicular bisector, deepening their conceptual grasp.

The **reflective practices** integrated into the tasks played a role in facilitating productive mathematical talk. Encouraging participants to review and discuss their geometric constructions and methods, these practices could facilitate a better engagement with the concepts. Reflection prompts guided participants to consider what they did and why they did it, thereby increasing their ability to articulate their reasoning. For example, when participants M and N explored examples and non-examples of angle bisector and their relationship to line JK, reflective dialogue significantly advanced their understanding. Participant N observed, '*When you drag the points around, I can see that the slope of the line JK always stays the same as the angle bisector, just that it is at the wrong place.'* This insight was further examined through Participant M's probing question, '*Do you mean the gradient of this line is the same as the angle bisector?'* Participant N then linked this observation to prior knowledge, concluding, '*When you put I on top of H, it is now an angle bisector. bisector, the same as the previous task we did yesterday. So, the radii of the two circles must be the same.'* These reflective discussions could clarify the participants' observations and connect new insights with previous tasks, leading to a more profound and communicative understanding of geometric principles.

The **balance between ostensive and non-ostensive objects** within the tasks significantly supported productive mathematical talk. Engaging participants with tangible geometric figures (ostensive) while also exploring abstract geometric principles (non-ostensive), the tasks could facilitate a connection between practical construction activities and theoretical concepts. This integration encouraged richer discussions and deeper understanding. For instance, when Participants A and B explored and manipulated geometric objects to investigate relationships among different kites, such as general kites, rhombuses, and squares, they handled physical models and examined their underlying properties. During their discussion, Participant A remarked, '*I think what I've taken from this is that I often thought of quadrilaterals being defined by almost the external lines. I don't think I've necessarily understood until now how the construction lines, with similarities and differences.'* This reflection highlights a shift from viewing shapes by their external features to understanding them through their construction.

Furthermore, Participant A noted the importance of technology in this learning process, stating, '*Yes, the software allows you to grab vertices and see how moving them around affects the model according to the constraints that have been created.'* This comment underlines how dynamic geometry software facilitated the exploration of geometric concepts, linking physical manipulation with abstract understanding and enhancing the participants' ability to articulate and discuss their findings effectively.

The use of **dynamic geometry software** (GeoGebra) as a mediator tool could enhance productive mathematical talk by providing a shared interactive environment where participants could explore and discuss geometric concepts more effectively. GeoGebra allowed participants to manipulate geometric figures in real time, which could offer immediate visual feedback that made abstract concepts more tangible and easier to discuss.

For example, when Participants M and N were exploring angle bisectors, they used GeoGebra to examine the relationship between a line h and angle $\angle ABC$. This interactive tool allowed them to manipulate the geometric elements dynamically to advance a growing understanding through collaborative exploration. Participant N remarked, '*Yeah, that's a perpendicular line. What happens if you move point A? All right, just move point A about there.'* Responding to this prompt, Participant M dragged point A and observed, '*No, that's not it. It's not a perpendicular bisector.'* As Participant M moved point A, they were able to see the effect of this manipulation on the geometric construction, which led to a visual and conceptual clarification when Participant N further explained, '*Make it like a right triangle ABC. OK. So, you see, it cuts the angle in half by bisecting the angle.'* This exchange, supported by the real-time visual feedback from GeoGebra, helped solidify their understanding of angle bisectors.

In another instance, Participants G and H engaged in a discussion about the relationship between quadrilaterals, specifically focusing on kites, rhombuses, and squares. GeoGebra facilitated their exploration by allowing them to manipulate points on the quadrilateral and observe changes in its shape. Participant G observed, '*Exactly! So, at the point where the orange and blue circles coincide, we have a square, and as we move E and F along the line segment CD, the quadrilateral remains a square until the circles change sizes again.'* This interaction demonstrated how the software allowed them to visualise and discuss the dynamic nature of the quadrilateral's transformation. Participant H concluded the exploration by summarising, '*So, to summarise, we have observed that the quadrilateral changes between a kite and a square depending on the size of the circles. When all the circles coincide, it becomes a square, and when they are different sizes, it's a general kite.'* This summary highlighted the connection between the geometric elements and their corresponding properties, which was made possible through the use of GeoGebra as a dynamic and interactive mediator.

Instrumental orchestration played a role in supporting productive mathematical talk during the tasks by facilitating an environment where participants could explore and articulate their understanding of geometric concepts through dynamic interaction with GeoGebra's tools. The tasks were designed to require participants to manage the complexity of geometric constructions in real time, which led to discussions about the underlying mathematical principles and problem-solving techniques.

For example, participants G and H were exploring how to construct angle bisectors using a straightedge only, Participant G asked, '*What geometrical relationship does the line have with the angle ABC and why?*'; it prompted H to explain that '*It bisects it.*' This initial exchange set the stage for a deeper exploration of why the bisector relationship held true under different manipulations. As they dragged points *A* and *C*, Participant H observed, '*It's still bisecting it,*' which led to further inquiry about the consistency of this geometric property.

The instrumental orchestration became evident as the participants used the tools to dynamically adjust the figures and immediately see the impact of their actions. For instance, when Participant G prompted Participant H to, '*drag point B,*' and observed that the bisector remained consistent, Participant H explained, '*H remains in between J and B on a straight line, and I remains in between B and K on a straight line. So, they're just moving in proportion, which is why it works.*' This shows how the participants were using the software to manipulate the figures and engaging in profound mathematical reasoning to understand the relationships between the elements.

The orchestration of tasks also encouraged participants to explore different scenarios, such as changing the radius of the circles by dragging points *F* and *G*. Participant H noted that this did not alter the bisector's property because it had, '*all to do with the fact that they share the same centre point.'* This real-time adjustment and observation allowed participants to solidify their understanding by connecting the geometric manipulations to the underlying concepts.

Moreover, when Participants G and H shifted their focus to constructing an angle bisector using only a ruler, they engaged in a detailed step-by-step process. Participant G guided the conversation, stating, '*So basically, we don't need to draw any circles,'* leading Participant H to articulate a method for doing so. Participant H outlined the approach, saying, '*I've got my ruler. I'm going to draw my angle [MNO], and I want to draw a bisector of it.'* They then described how they would create line segments of specific lengths and use these to determine the bisector, culminating in H's statement, '*And then I just need to draw a line which passes through [point N and the] intersection T. And then we have done it.'* G affirmed this approach, replying, '*We have done it. Yeah, exactly how I would have done it.'*

This detailed interaction exemplifies how instrumental orchestration, through the use of both digital tools like GeoGebra and traditional tools like a ruler, supported productive mathematical talk. The participants could explore and manipulate geometric concepts and articulate their understanding and reasoning, thereby deepening their collective comprehension of the task at hand.

Bruner's (1974) modes of representation, enactive, iconic, and symbolic, could facilitate productive mathematical talk, enabling participants to explore and articulate geometric concepts more profoundly. Integrating these modes into the tasks allowed participants to engage with the material in a manner that supported diverse learning styles (preferences) and enriched their understanding through multi-modal expression. To illustrate how these modes of representation facilitated productive mathematical talk, let's first examine the conversation between Participants A and B as they explored the relationships between kites, rhombuses, and squares.

A: *Construct a circle at point I with radius BI and change the colour. Wow, too many circles. What properties can we find here? Finally, please construct a circle at point I with a radius GI or HI and change the colour. What kind of circles are they?*

B: *Concentric because, well they have the same centre by construction.*

A: *Yeah, so what do you think about this? So, our green circle was IB, wasn't it?*

B: *Our purple circle is IA and our orange circle is IG.*

A: *Yeah, so if AI and IB were the same length, we expect the two circles to overlap, so the purple and green should overlap, it's a rhombus.*

B: *With AI, so if we made these the same... Rhombus, yeah, and so we can make them all overlap, which is exactly when we're going to get a square.*

A: *Yeah, it makes sense…. And what happens to the quadrilateral? So when the green and purple line up, we get a rhombus, isn't it?*

B: *Rhombus, yeah.*

A: *But then if you increase them both together, then you're going to a point where all three circles overlap, you get a square.*

B: *Yeah.*

A: *And then, if you keep on going, you end up with a rhombus in the other direction.*

B: *Yes, if I select E and F, and then I make them bigger, then we get everything overlapping, and we get a square. And that's because, in a square, you have perpendicularly bisecting diagonals. Yeah, a rhombus, anything with bisecting diagonals. Is that true? Or anything with bisecting diagonals is a rhombus.* [Here, B was using the construction on their computer screen to confirm her statement]

A: *A rectangle has bisecting diagonals, doesn't it? Not perpendicular, but they are bisecting.*

B: *Oh yeah, oh sorry, I mean that each one of the diagonals is bisected. A rectangle has to have the same length. So a parallelogram has... umm...*

A: *So a parallelogram will bisect, but they won't necessarily be perpendicular.*

B: *Yeah.*

A: *But in the event of a square, they will be perpendicular bisectors.*

B: *Yeah.*

A: *In the event of a rhombus...*

B: *We don't have these lengths the same, so that's what our circles are telling us.*

A: *Yeah.*

B: *They're all perpendicular bisecting each other, but AI is not equal to GI.*

A: *Yeah, correct.*

B: *And that's why when we then move them so the circles are the same size* [B moved the two coinciding points on the slider until the circles coincided or were the same], *that then tells us that AI is the same as GI, and that's why we get a square.*

A: *So, what are the properties of the quadrilateral? What would you say about the points G and H, and A and B?*

B: *They are equidistant from point I. In fact, GI, HI, AI, and BI are congruent because, again, it's a square.*

With enactive representation, participants interacted physically with the mathematical concepts using GeoGebra, which allowed them to manipulate geometric shapes directly. For example, when Participant A instructed their partner to, '*construct a circle at point I with radius BI and change the colour'*, this hands-on activity (enactive mode) helped the participants engage with the concept of concentric circles, facilitating a profounder understanding of their properties.

In the iconic representation, the visualisation provided by GeoGebra allowed participants to see the geometric relationships more clearly. When Participant B observed, '*Concentric because, well they have the same centre by construction,'* they were using the iconic mode to recognise and describe the visual properties of the circles. This visualisation was further enhanced when they noticed how, '*if AI and IB were the same length, we expect the two circles to overlap, so the purple and green should overlap,'* leading to a discussion about the relationship between the shapes and the formation of a rhombus.

Finally, within the symbolic representation, the conversation also included abstract reasoning and symbolic understanding, particularly when discussing the implications of their constructions. For example, when Participant B stated, '*In a square, you have perpendicularly bisecting diagonals. Yeah, a rhombus, anything with bisecting diagonals. Is that true?'* They were engaging in symbolic reasoning, connecting their visual and physical interactions with the abstract properties of these geometric figures.

Through these representations, participants were able to transition from manipulating and visualising the shapes to discussing and understanding their properties in a symbolic and abstract manner. This multi-modal approach seems to have supported their individual learning styles and facilitated a richer, more productive mathematical discourse, as seen when they concluded that, '*GI, HI, AI, and BI are congruent because, again, it's a square.'* This progression from enactive to iconic to symbolic representation highlights the effectiveness of Bruner's (1974) modes in enhancing mathematical understanding and communication.

Feedback mechanisms were a crucial component in the task design, significantly supporting productive mathematical talk by fostering continuous dialogue among participants. The integration of dynamic geometry software like GeoGebra provided immediate visual feedback, which reinforced understanding and prompted further exploration. In addition, peer feedback during collaborative activities offered new insights, creating a dual feedback loop that encouraged participants to discuss their constructions, make adjustments, and refine their ideas based on both visual and peer input.

For instance, in the conversation between Participants M and N during their exploration of an angle bisector in Task 3a, GeoGebra's feedback allowed them to correct their initial misconceptions. M asked, *'What geometrical relationship does the line that passes through points H and I have to the angle ABC?'* and initially thought it was a perpendicular bisector. However, upon moving *point A* and receiving visual feedback, M quickly realised this was incorrect, stating, *'No, that's not it. It's not a perpendicular bisector.'* This immediate correction, facilitated by the software, helped M and N arrive at the correct conclusion that the line was not actually an angle bisector.

In Task 3b, where M and N explored examples and non-examples of an angle bisector, the feedback mechanism again played a pivotal role. As N suggested dragging point B around, they observed that the slope of *line JK* remained the same as the imaginary angle bisector but wasn't correctly positioned. This insight, prompted by visual feedback from GeoGebra, led M to realise, *'Let's try to make it [an] angle bisector,'* eventually achieving the correct construction when they aligned points *I* and *H*.

Similarly, in the conversation between Participants A and B, as they explored the relationships between kites, rhombuses, and squares, feedback mechanisms played a crucial role. When Participant A asked Participant B to construct circles with different radii, B used GeoGebra to visualise the concentric circles: '*Concentric because, well they have the same centre by construction.'* This visual feedback led to a discussion on the properties of the shapes they were constructing, with Participant A noting, '*Yeah, so if AI and IB were the same length, we expect the two circles to overlap, so the purple and green should overlap, it's a rhombus.'* The feedback from GeoGebra could enable the participants to test their conjectures in real time, facilitating a richer dialogue and a clearer understanding of the geometric relationships.

The integration of feedback mechanisms, both from GeoGebra and peer interaction, was instrumental in facilitating productive mathematical talk. These mechanisms allowed participants to immediately see the consequences of their actions, discuss their observations, and collaboratively refine their understanding, leading to more profound insights and clearer articulation of geometric concepts.

Meaningful goals and visible mathematics within the tasks supported productive mathematical talk by ensuring participants were engaged in relevant and practical geometric explorations. Focusing on tasks with clear, observable outcomes, such as constructing and analysing geometric figures, participants directly connect their actions to theoretical learning to facilitate more focused and substantive discussions grounded in visible results.

For instance, in the conversation between Participants M and N while exploring an angle bisector using GeoGebra, the task was designed to establish a meaningful goal, identifying the geometric relationship between a line and an angle. Initially, Participant M incorrectly identified the line as a perpendicular bisector, saying, '*A perpendicular bisector.'* However, as they continued to manipulate the figure and engage in dialogue, Participant N guided the exploration further by asking, '*What happens if you move point A?'* This question prompted Participant M to reconsider, leading to the realisation that the line was an angle bisector. M then corrected themselves, stating, '*No, that's not it. It's not a perpendicular bisector.'* The visible mathematics in this task, combined with a clear goal, helped M and N refine their understanding through a productive exchange rooted in observable geometric properties.

In another example, M and N worked on a task to explore examples and non-examples of an angle bisector. The goal of distinguishing between correct (examples) and incorrect (non-examples) cases of an angle bisector made the mathematical concepts more concrete. Participant N expressed uncertainty by saying, '*I feel like if that line [blue line JK] is in a different position, it would bisect the angle, but I might be totally wrong.'* This meaningful goal pushed the participants to experiment further, leading Participant M to conclude, '*Let me drag the other points around to see…,'* and eventually, they correctly identified the geometric relationship by using visible feedback from GeoGebra to verify their hypotheses. This exchange shows how the combination of a clear goal and visible mathematics facilitated an understanding and effective mathematical discourse.

Similarly, the conversation between Participants A and B as they explored the relationships between kites, rhombuses, and squares illustrates the power of meaningful goals and visible mathematics. The task required them to construct and compare geometric shapes, with the visible outcome of their constructions directly tied to the theoretical concepts they were discussing. As Participant A instructed, Participant B to, '*Construct a circle at point I with radius BI and change the colour,'* they both observed and discussed the properties of the resulting figures. Participant B's response, '*Concentric because, well they have the same centre by construction,'* and their subsequent dialogue about overlapping circles and the formation of a square could demonstrate how the visible mathematics made abstract geometric properties tangible. The task's meaningful goal, understanding the relationships between different quadrilaterals, anchored their discussion, making it easier to articulate and explore concepts such as perpendicular bisectors and congruence in squares.

Integrating these task design principles, scaffolding, collaborative learning, reflective practices, balancing ostensive and non-ostensive objects, dynamic use of GeoGebra, instrumental orchestration, Bruner's (1974) modes of representation, feedback mechanisms, and meaningful goals, the tasks were designed to allow participants to learn geometric concepts and foster an environment where productive mathematical talk could thrive. This careful design could ensure that participants were consistently engaged in meaningful, reflective, and collaborative discussions, leading to an understanding of geometric principles and the ability to communicate them effectively.

5A.3.2 Facilitation of dialogic learning through task design principles (or carefully designed tasks)

To answer RQ1 further, which seeks to identify how the carefully designed tasks facilitated dialogic learning that supports the co-construction of knowledge among beginning teachers, it is imperative to demonstrate that these design principles indeed led to dialogic learning interactions and, ultimately, co-construction of knowledge.

Dialogic learning talk and principles: This section elaborates on the eight distinct types of dialogic talk and the six dialogic learning principles proposed by Alexander (2008; 2018), which were used as analytical tools to examine participants' discourse and communicative interactions, as described in Chapter 4, subsection 4.8.1.1. Table 5A.1 presents descriptions of the eight types of dialogic talk, while Table 5A.2 outlines the six dialogic learning principles, each illustrated with examples from the study. These tools were employed to provide empirical evidence of how dialogic talk and learning principles emerged within the collaborative learning environment. They enabled a detailed analysis of the conversational complexity and highlighted the dialogic learning principles in action.

Table 5A.1 Dialogic talk types and examples in geometric exploration

Talk Type	Brief description	In the study…
Transactional	Efficient exchanges conveying information or completing tasks, characterised by clarity and practical engagement in sharing experiences and problem-solving.	Participants define, correct, and confirm, as seen in proving a quadrilateral is a square, showcasing a structured problem-solving approach.
Exploratory	Open-ended discussions that stimulate curiosity and intellectual exploration, involving questioning, idea generation, and diverse perspective exploration for deeper understanding.	Participants embraced exploratory talk, engaging in open-ended discussions, questioning, speculation, and collaborative inquiry, fostering hands-on learning and an understanding of dynamic geometric relationships.
Deliberative	Focused on thoughtful consideration and decision-making, participants weigh options, deliberate choices, and collaboratively reach decisions.	Participants focus on deliberative talk, identifying bisectors and exploring line behaviour, proving a quadrilateral to be a square, and discussing geometrical relationships.

Talk Type	Brief description	In the study…
Interrogatory	Features questioning and inquiry, where participants pose questions to seek clarification, gather information, provoke thought, and request feedback.	Participants engaged in interrogatory talk by asking questions, seeking insights, and requesting feedback, fostering collaborative exploration and shared understanding of geometric concepts.
Evaluative	Involves critical assessment, constructive critique, and informed opinions. Participants evaluate strategies, reflect on outcomes, and compare approaches in decision-making processes.	Participants critically assess geometric relationships, evident in quadrilateral discussions, emphasising collaborative refinement and agreement in evaluative talk.
Expressive	Participants articulate thoughts, emotions, and viewpoints, fostering self-expression, sharing of feelings and personal experiences within a conversation.	Participants freely share insights on quadrilateral properties, employing expressive talk in collaborative construction and discussions on geometric relationships.
Expository	Involves clear, concise explanations and sharing factual information to convey knowledge and understanding.	Participants employed expository talk to explain geometric properties, describe processes, and share factual information, enhancing collective understanding through clear communication and summarisation.

Talk Type	Brief description	In the study…
Imaginative	Encourages creative thinking, envisioning hypothetical scenarios, and generating innovative ideas to foster creativity and imagination within the conversation.	Participants employ imaginative talk to explore angles dynamically, express uncertainty, propose solutions, and envision shape formation, fostering creative collaborative exploration.

The six dialogic learning principles offer a conceptual framework for understanding how meaningful dialogue and learning unfold within this context:

Table 5A.2 Dialogic learning principles in geometric exploration

Dialogic learning principle	Brief description	In the study…
Collective	Collaborative learning involves shared efforts and contributions towards common goals, promoting a sense of community.	Participants collectively collaborated, engaging in joint problem-solving to discern geometrical relationships and explore specific tasks.
Cumulative	Building knowledge progressively through continuous contributions, accumulating insights, and deepening understanding over time.	Cumulative learning unfolded as participants progressively refined and deepened their understanding through ongoing discussion and exploration.

Dialogic learning principle	Brief description	In the study…
Deliberative	Thoughtful and careful consideration of ideas, encouraging thorough exploration and critical analysis within the learning community.	Deliberative learning unfolded through reflective questioning and thoughtful exploration, fostering critical thinking and contributing to participants' growing understanding of geometric relationships and concepts.
Purposeful	Learning activities are designed with clear objectives and intentional goals, providing direction and focus for meaningful collaborative engagement.	Purposeful learning persisted as participants actively engaged in dialogue, specifically aiming to understand geometric relationships and achieve clarity on the study's objectives.
Reciprocal	Participants actively share, respond to, and enhance each other's ideas, creating an interactive and collaborative learning experience.	Reciprocal learning was evident as participants engaged in a dynamic exchange of ideas, explanations, and questions, fostering mutual interaction and collaborative understanding.
Supportive	Creating a nurturing atmosphere where participants encourage, assist, and value each other's contributions, promoting a positive and inclusive learning experience.	A supportive learning environment was fostered as participants collaborated, providing feedback and guidance, highlighting the significance of support in effective learning.

In essence, these eight types of dialogic talk and six dialogic learning principles can be invaluable tools for analysing and understanding the dynamics of participants' interactions within the collaborative learning environment. They have the potential to offer a structured lens through which the complex nature of dialogue and its role in knowledge co-construction can be viewed.

In the subsequent sections, I showcase the richness and diversity of dialogic exchanges that the learning environment stimulated and how this led to the co-construction of knowledge among beginning teachers. As I continue, I will provide two sample analyses on dialogic talk, one sample analysis on dialogic learning principles, and a frequency distribution analysis.

5A.3.3 Examples of productive dialogic mathematical talk facilitated by carefully designed tasks

This section provides examples of productive dialogic mathematical talk facilitated by carefully designed tasks for beginning teachers and outlines the steps taken to illustrate these examples. As mentioned in Chapter 4, some of the task instructions were included in the analysis because, during the task performance and investigation, one participant in each pair read the instructions aloud as part of the collaborative process. This reading was integral to the conversation, guiding the initial geometric constructions and subsequent exploration. Therefore, it is justified to treat the reading of task instructions as part of the dialogic talk, as it contributed directly to the dialogue and the collaborative nature of the learning experience.

Firstly, I converted some participants' verbal exchanges into geometric figures to offer visual representations of their discussions. These visual aids help create an intuitive understanding of the findings. I will begin by presenting these geometric illustrations, which act as a bridge between the participants' dialogue and the subsequent analysis.

Next, I will explore the transcripts of the participants' conversations. This direct analysis of their dialogue can provide insights into their thought processes and collaborative interactions. Following this, I will detail the coding process, where I categorised and highlighted aspects of the transcripts according to different types of dialogic talk. This coding analysis can offer a detailed view of participants' engagement.

The final part of this section will present the findings from each pair's conversation, showcasing the various types of dialogic talk observed. I will also provide a summary of all analyses, connecting the findings to draw meaningful conclusions about how dialogue contributes to the co-construction of knowledge in collaborative learning environments.

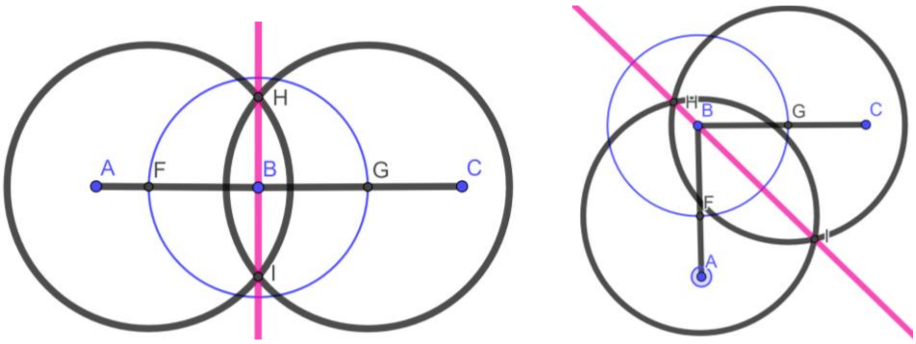
This structured approach aims to offer a clear and comprehensive understanding of the analysis and findings, ensuring that each component contributes to a narrative of the participants' collaborative learning experience. It is important to view the diagrams and tables of the analysis in colour.

5A.3.3.1 First sample of analysis and findings

The first part of this analysis is the conversation between Participants M and N during their exploration of an angle bisector (on Task 3a) using the dynamic geometry software GeoGebra.

5A.3.3.2 Analysis of the conversation

The following geometric figures (Figures 5A.1a and 5A.1b) depict some of the concepts in their dialogue. These figures illustrate snapshots of the pair's work on Task 3a, outlined in Chapter 3. Task 3a involves exploring and constructing an angle bisector, emphasising associated properties. The objective is to support beginning teachers in learning geometric construction techniques, fostering an understanding of teaching angle bisectors using dynamic geometry software (DGS) in secondary schools.



Figures 5A.1a and 5A.1b: Exploration of angle bisector

Now, let us consider their dialogue or conversation below with respect to Figures 5A.1a and 5A.1b.

M: *'OK, what geometrical relationship does the line that passes through points H and I have to the angle ABC?A perpendicular bisector.'* [see Figure 5A.1a]

N: *'Yeah, that's a perpendicular line. What happens if you move point A? All right, just move point A about there.'*

M: *'No, that's not it. It's not a perpendicular bisector.'* [As M moves the point A and translated to Figure 5A.1b]

N: *'Make it like ABC right triangle. OK. So, you see, it cuts the angle in half by bisecting the angle.'* [see Figure 5A.1b]

M: *'Yeah, that is right. Yeah. That's sure.'*

N: *'So, what geometrical relationship is that?'*

M: *'It is an angle bisector. OK. Is that right? I think it's right.'*

N: *'Yeah, I feel like that is right. So, when you move either of those points, you are just changing the angle, but the line still bisects the angle. It is good to learn concepts like this.'*

Table 5A.3 shows how I analysed the dialogue above and categorised them into dialogic mathematical talk using Alexander's (2018) types of learning talk as an analytical tool.

Table 5A.3 Analysis of dialogic talk sample 1

Talk	Text showing coding of interplay of dialogic talk types	Talk
Transactional	*M:* *OK, what geometrical relationship does the line that passes through points H and I have to the angle ABC?* *A perpendicular bisector.* [see Figure 5A.1a]	Deliberative
Expressive	*N:* *Yeah, that's a perpendicular line. What happens if you move point A? All right, just move point A about there.* *M:* *No, that's not it. It's not a perpendicular bisector.*	Expository
Exploratory	*N:* *Make it like ABC right triangle. OK. So, you see, it cuts the angle in half by bisecting the angle.* [see Figure 5A.1b] *M:* *Yeah, that is right. Yeah. That's sure.* *N:* *So, what geometrical relationship is that?*	Interrogatory
Evaluative	*M:* *It is an angle bisector. OK. Is that right? I think it's right.* *N:* *Yeah, I feel like that is right. So, when you move either of those points, you are just changing the angle, but the line still bisects the angle. It is good to learn concepts like this.*	Imaginative

Text exemplifying dialogic talk types	Code
N: Yeah, that's a perpendicular line. What happens if you move point A? M: No, that's not it. It's not a perpendicular bisector.	Transactional Talk
M: No, that's not it. It's not a perpendicular bisector. M: Yeah, that is right. Yeah. That's sure. N: Yeah, I feel like that is right.	Expressive Talk
N: What happens if you move point A? All right, just move point A about there. M: No, that's not it. It's not a perpendicular bisector. N: Make it like ABC right triangle. OK. So, you see, it cuts the angle in half by bisecting the angle. N: So, what geometrical relationship is that? M: It is an angle bisector. OK. Is that right? I think it's right.' N: Yeah, I feel like that is right. So, when you move either of those points, you are just changing the angle, but the line still bisects the angle.'	Exploratory Talk
M: Yeah, that is right. Yeah. That's sure. N: So, what geometrical relationship is that? M: It is an angle bisector. OK. Is that right? I think it's right. N: It is good to learn concepts like this.	Evaluative Talk
N: What happens if you move point A? N: So, what geometrical relationship is that? M: It is an angle bisector. OK. Is that right? I think it's right.	Deliberative Talk
N: So, you see, it cuts the angle in half by bisecting the angle. N: So, when you move either of those points, you are just changing the angle, but the line still bisects the angle. It is good to learn concepts like this.	Expository Talk
N: What happens if you move point A? All right, just move point A … N: So, what geometrical relationship is that? M: It is an angle bisector. OK. Is that right? I think it's right.	Interrogatory Talk

5A.3.3.3 Findings from the analysis of participants M and N's conversation

The analysis of the Pair M and N's conversation provided insights into a dynamic interplay of various types of dialogic talk, each playing a distinct role in facilitating a productive and collaborative learning experience centred around the exploration of geometric concepts.

Transactional talk laid the foundation for their conversation, providing a structured approach to understanding the geometric relationships at hand. In this phase, participants initiated their exploration with questions and logical guidance. For example, M raised the question, '*OK, what geometrical relationship does the line that passes through points H and I have to the angle ABC?*' This structured approach formed the basis of their investigation, as they deliberated on the geometrical relationship of a line passing through points *H* and I in relation to angle *ABC*.

As the conversation unfolded, *expressive talk* emerged. Participants expressed agreement, certainty, and confidence in their responses, creating a positive and supportive learning atmosphere. For instance, M and N reassured each other by stating, '*Yeah, that's a perpendicular line. What happens if you move point A?*' This type of talk reinforced mutual assurance in their understanding.

Exploratory talk took centre stage as participants engaged in collaborative exploration of geometric concepts. They jointly investigated various scenarios and deepened their understanding through hypothesis and experimentation. For instance, N asked, '*What happens if you move point A? All right, just move point A about there.*' This type of talk fostered a sense of shared discovery.

Evaluative talk served to reinforce the participants' understanding. They confirmed the correctness of their answers and expressed confidence in their responses. For instance, M affirmed, '*Yeah, that is right. Yeah. That's sure.*' This evaluation affirmed their learning and supported their self and mutual assurance in tackling geometric concepts.

Deliberative talk acted as a guiding force, directing discussions towards possible scenarios and outcomes. Participants considered the implications of moving points, guiding their exploratory efforts with thoughtful deliberation. For example, N inquired, '*What happens if you move point A?*' This type of talk prompted thoughtful exploration.

Expository talk emerged when participants provided explanations and clarified critical concepts. Their descriptions elucidated how a line bisects an angle and highlighted the significance of mastering such fundamental principles. For instance, N explained, '*So, you see, it cuts the angle in half by bisecting the angle.*' This type of talk contributed to a growing understanding of the subject matter.

Throughout the conversation, *interrogatory talk* propelled the discourse forward. Participants posed questions aimed at prompting further exploration and understanding, driving critical thinking. For example, M initiated, '*OK, what geometrical relationship does the line that passes through points H and I have to the angle ABC?*' This inquiry-driven approach encouraged the participants to analyse the geometric relationships under scrutiny.

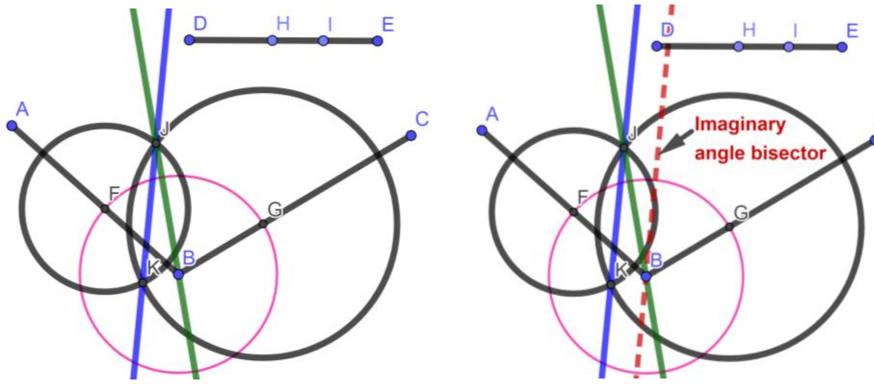
In summary, the diverse types of dialogic talk combine to create a rich and dynamic learning environment. The participants' interaction was marked by structured reasoning, mutual agreement, exploratory discourse, evaluation of findings, thoughtful deliberation, explanatory insights, and probing inquiries. Together, these facets enhance their grasp of geometric concepts and emphasise the importance of varied dialogic talk types in fostering collaborative learning and meaningful exploration in educational settings.

5A.3.3.4 Second sample of analysis and findings

This section is another analysis of the conversation between the same participants M and N during their exploration of examples and non-examples of an angle bisector (Task 3b) using the dynamic geometry software GeoGebra.

5A.3.3.5 Analysis of the conversation

The following geometric Figures 5A.2a, 5A.2b, and 5A.2c depict some of the geometric construction concepts in their dialogue. These figures depict three snapshots of the pair's work on Task 3b, which is detailed in Chapter 3. Task 3b focuses on the exploration and construction of examples and non-examples related to an angle bisector, along with its associated properties. The objective of this task is to assist beginning teachers in acquiring geometric construction skills, specifically in constructing examples and non-examples of an angle bisector. The ultimate goal is to improve their understanding and proficiency in teaching the concepts of angle bisector examples and non-examples to secondary school students using dynamic geometry software.



Figures 5A.2a and 5A.2b: Exploring non-examples of an angle bisector

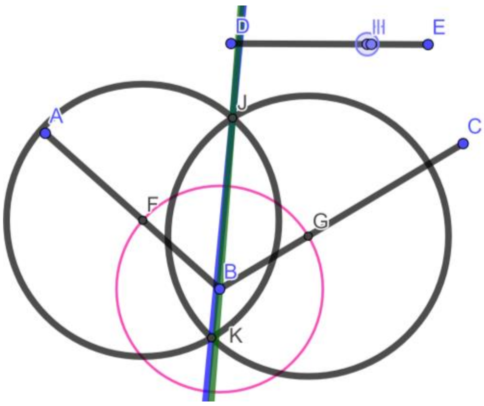


Figure 5A.2c: Exploring examples and non-examples of an angle bisector

The conversation below is considered alongside Figures 5A.2a, 5A.2b and 5A.2c:

N: *'I feel like if that line [blue line JK] is in a different position, it would bisect the angle, but I might be totally wrong. Can you try dragging point B around?* [see Figure 5A.2a and 5A.2c]*'*

M '*Yes, let me drag point B now. I don't know, it is still not bisecting the angle. I feel like that might bisect angle ABC if it was in the right location.'*

N *'Mmm, I have not seen this before…'*

M: *'Let me drag the other points around to see…'*

N: *'When you drag the points around, I can see that the slope of the line JK always stays the same as the [imaginary] angle bisector, just that it is at the wrong place.'*

M: *'Do you mean the gradient of this line is the same as the [imaginary] angel bisector?'* [see Figure 5A.2b]

N: *'Yes'.*

M: '*I see, it is true. Let's try to make it angle bisector.'*

N: *'How do we do that?... Oh okay. Can you put I and H on top of each other?'*

M: *'Yes, point I on top H, …'* [see Figure 5A.2c]

N: *'You see when put I on top of H, it is now an angle bisector, the same as the previous task we did yesterday. So, the radii of the two circles must be the same.'*

M: *'Yes, you are right, you are right.'*

Table 5A.4 shows the analysis and categorisation of the kinds of dialogic mathematical talk (Alexander, 2018) present in participants M and N's dialogue above.

Table 5A.4 Analysis of dialogic talk sample 2

Talk	Text showing coding of interplay of dialogic talk types	Talk
Transactional	N: *I feel like if that line is in a different position, it would bisect the angle, but I might be totally wrong. Can you try dragging point B around?* [see Figure 5A.2a and 5A.2c]	Deliberative
Expressive	M *Yes, let me drag point B now. I don't know, it is still not bisecting the angle. I feel like that might bisect angle ABC if it was in the right location.* N *Mmm, I have not seen this before…*	Expository
Exploratory	M: *Let me drag the other points around to see…* N *when you drag the points around, I can see that the slope of the line JK always stays the same as the angle bisector, just that it is at the wrong place.* M: *Do you mean the gradient of this line is the same as the angel bisector?* [see Figure 5A.2b]	Interrogatory
Evaluative	N: *Yes.* M: *I see, it is true. Let's try to make it angle bisector.* N: *How do we do that?... Oh okay. Can you put I and H on top of each other?* M: *Yes, point I on top H, …* [see Figure 5A.2c] N: *You see, when put I on top of H, it is now an angle bisector, the same as the previous task we did yesterday. So, the radii of the two circles must be the same.* M: *Yes, you are right, you are right.*	Imaginative

Text exemplifying dialogic talk types	Code
N: Can you try dragging point B around? M: Yes, let me drag point B now. M: Let me drag the other points around to see. M: Can you put I and H on top of each other? M: Yes, point I on top H...	Transactional Talk
N: I feel like if that line is in a different position, it would bisect the angle... M: I don't know, it is still not bisecting the angle. I feel like that might bisect angle ABC if it was in the right location. N: Mmm, I have not seen this before...	Expressive Talk
N: I feel like if that line is in a different position, it would bisect the angle... N: Can you try dragging point B around? M: Let me drag the other points around to see... N: When you drag the points around, I can see that the slope of the line JK always stays the same as the angle bisector, just that it is at the wrong… M: Do you mean the gradient of this line is the same as the angel bisector? N: How do we do that?... Oh okay. Can you put I and H on top of each… M: I don't know, it is still not bisecting the angle. N: You see, when we put I on top of H, it is now an angle bisector, …	Exploratory Talk
M: I feel like that might bisect angle ABC if it was in the right location.' N: Mmm, I have not seen this before… M: I see, it is true. Let's try to make it an angle bisector. N: You see, when put I on top of H, it is now an angle bisector, the same as the previous task we did yesterday. So, the radii of the two circles must.. M: Yes, you are right, you are right.	Evaluative Talk
N: Can you try dragging point B around? M: How do we do that?... Oh okay. Can you put I and H on top of each …	Deliberative Talk

N: Can you try dragging point B around? M: Do you mean the gradient of this line is the same as the angel bisector? M: How do we do that?... Oh okay. Can you put I and H on top of each…	Interrogatory Talk

5A.3.3.6 Findings from the analysis of participants M and N's conversation

The analysis of the Pair M and N's conversation offers a view of the diverse types of dialogic talk employed during their discussion, shedding light on the collaborative exploration of geometric concepts.

Transactional talk formed the foundation of their interaction, providing clear guidance for their exploration. Participants used this type of talk to instruct each other, setting the stage for manipulating points and elements within the geometric scenario. Instructions such as moving point *B*, repositioning points *I* and *H*, or experimenting with other points were all manifestations of transactional talk, ensuring a structured approach to their investigation.

Expressive talk emerged naturally as participants shared their thoughts and emotions regarding the geometric scenario. It was a channel for them to express uncertainty, curiosity, and novelty in their observations. Expressive talk became evident when one participant admitted, *'I feel like if that line is in a different position, it would bisect the angle...'* while the other responded, *'I don't know, it is still not bisecting the angle. I feel like that might bisect angle ABC if it was in the right location.'* They expressed feelings such as, *'Mmm, I have not seen this before...'* fostering an atmosphere of open expression and shared learning experiences.

Exploratory talk took centre stage in the conversation, serving as a driving force for collaborative exploration. Participants actively engaged in hypothesising, questioning, and testing their ideas. They pondered how moving points may affect the geometric relationships, discussing whether a line's position would result in angle bisecting. The dialogue was marked by their curiosity. When N said, *'I feel like if that line is in a different position, it would bisect the angle, but I might be totally wrong. Can you try dragging point B around?',* M responded with, *'Yes, let me drag point B now. I don't know, it is still not bisecting the angle. I feel like that might bisect angle ABC if it was in the right location.'* They jointly examined various scenarios, as N prompted, *'When you drag the points around, I can see that the slope of the line JK always stays the same as the angle bisector, just that it is at the wrong place.'* M reacted '*Do you mean the gradient of this line is the same as the angle bisector?'* Furthermore, M pressed, '*I see, it is true. Let's try to make it angle bisector.'* M questioned, *'How do we do that?... Oh okay. Can you put I and H on top of each other?'* Their exploratory discourse fuelled curiosity and facilitated a deepening understanding of the subject matter.

As the participants engaged in dialogue, *evaluative talk* played a role in their learning process. This form of discourse involved reflection on their findings and hypotheses, aimed at assessing the validity of their observations and strengthening their grasp of geometric concepts. Participants used *evaluative talk* when discussing whether a line bisected an angle, confirming their conclusions, or drawing connections between the current scenario and previous tasks. For instance, M shared, *'I feel like that might bisect angle ABC if it was in the right location'* to which N responded, *'Mmm, I have not seen this before…'* Subsequently, they affirmed, *'You see, when we put I on top of H, it is now an angle bisector, the same as the previous task we did yesterday. So, the radii of the two circles must be the same,'* and *'Yes, you are right, you are right.'* This phase signified a stage of self-assessment and the consolidation of knowledge, reinforcing the significance of *evaluative talk* in the collaborative learning process.

Deliberative talk served as a practical tool for coordinating actions and planning manoeuvres within the geometric scenario. Participants discussed how to move points, align elements, and execute specific actions to achieve their learning objectives. Deliberative talk ensured a clear and coordinated approach to their exploration, facilitating effective experimentation. They engaged in dialogue like, *'Can you try dragging point B around?'* and *'How do we do that?... Oh okay. Can you put I and H on top of each other?'* These conversations not only helped them navigate the complexities of the geometric problem but also sharpened their problem-solving skills. Deliberative talk played a role in creating a productive and engaging learning environment for participants.

Interrogatory talk played a role in prompting further inquiry and clarification. Participants used questions to seek understanding, gather information, and drive the conversation forward. When they inquired about the gradient of a line or about the process for aligning points, interrogatory talk fuelled their collaborative learning journey. For example, N inquired, *'Can you try dragging point B around?'* and M asked, *'Do you mean the gradient of this line is the same as the angle bisector?'* They sought clarity with questions like, *'How do we do that?... Oh okay. Can you put I and H on top of each other?'* These questions facilitated the exchange of ideas and promoted a deep understanding of geometric concepts.

In summary, the pair's conversation illustrated a dynamic interplay of various types of dialogic talk. Transactional guidance provided structure, expressive sharing fostered open expression, exploratory discourse fuelled curiosity, evaluative reflection consolidated knowledge, deliberative planning ensured coordination, and interrogatory inquiry drove further exploration. Together, these facets contributed to a productive and collaborative learning environment, enhancing the exploration and comprehension of geometric concepts.

5A.3.3.7 A summary of the findings for all pairs of participants

The analysis of the pairs of participants' conversations provides an overview of the diverse types of dialogic talk employed in their discussions, showcasing the role each type plays in facilitating productive and collaborative learning experiences in the context of exploring geometric concepts to co-construct knowledge for teaching.

Transactional talk serves as the initial building block for all pairs, providing a structured foundation for their investigations. This type of talk is evident as participants exchange instructions and logical guidance to guide their exploration. It ensures a systematic approach to understanding geometric relationships and solving problems. For example, as participants, G and H, engaged in the conversation, they broke down the task into steps. G stated, *'So if we want to prove that the quadrilateral is a square, we need to show that all the angles at A, B, H, and G are right angles. Let's start with point G. Since the line from I to G is the radius of the circle centred at I, any tangent to that circle from point G would form a right angle,'* to which H responded, *'Right, so we can conclude that angle G is a right angle.'* This exchange reflected their efforts in guiding the discussion logically and ensuring a clear direction. Participant G further elaborated, *'Well, at points A, B, and H, we can draw tangents to the* *circles centred at I. These tangents will also form right angles since they are perpendicular to the radii.'* This illustrated how transactional talk helped establish the foundational steps necessary for their geometric exploration.

In another instance of participants A and B's conversation, they exchanged instructions, with Participant A initiating by saying, *'Hey, let's create a line segment AB.'* Participant B responded, *'Yeah, got it,'* and Participant A acknowledged when Participant B constructed a circle at point A, saying *'Perfect! Looks like we used A as the centre, which is correct since the instruction says to create a circle at A with a radius of CE. Good job! Ok, I don't know how you managed to construct the circle at A.'* Participant B explained, *'The first thing I did was to select the two points that define the radius CE, and then once you've done that, you then click where you want it to be centred.'* This initial phase ensured both participants were aligned in their approach to the task.

In the instances of participants M and N, where I presented in Sample 1 of Task 3a, M raised the question, '*OK, what geometrical relationship does the line that passes through points H and I have to the angle ABC?*' This structured approach formed the basis of their investigation, as they deliberated on the geometrical relationship of a line passing through points *H* and *I* in relation to angle *ABC*.

Expressive talk emerges as a recurring theme in the conversations, reflecting participants' emotions and enthusiasm for the learning process. They express agreement, curiosity, and novelty in their observations, fostering a positive and supportive atmosphere for mutual learning experiences. In the conversation between participants A and B, Participant A acknowledged B's actions with expressions like, *'Great! Now let's create another line segment CD. Lovely!'*

In another pair, which was G's and H's investigation, Participant G enthusiastically stated, *'Exactly! So, at the point where the orange and blue circles coincide, we have a square...'* H later responded with agreement to a different statement by G, *'Agreed. I've learnt a lot from this activity,'* and G concurred, *'A lot!'* These expressions of agreement and enthusiasm demonstrated how expressive talk fostered a positive learning environment by allowing participants to express their satisfaction and acknowledge the value of the learning activity.

Exploratory talk takes centre stage across all pairs, serving as a driving force for collaborative exploration. Participants actively engage in hypothesising, questioning, and testing their ideas. Through joint exploration of scenarios and shared discovery, they deepen their understanding of geometric concepts. For instance, in G's and H's conversation, Participant G encouraged this exploration by saying, *'You can imagine from what we are exploring that when this happens, all the intersection points of the tangents would form another square.'* H added, *'That's correct. So, all the angles at A, B, H, and G are right angles since they become midpoints of the new square formed by the tangents...'* These quotes showcased how exploratory talk actively engaged participants in discussing various aspects of the problem and encouraged them to envision geometric concepts.

Participants A and B used exploratory talk in their discussions about the relationships between geometric elements, hypothesised about potential outcomes, and explored different scenarios. Participant B explained, *'Yep, but that is not always a bisector. At the moment, we got a perpendicular line, but it* *can be at any point along a line segment. Yeah, so I can make the circle smaller.'* Participant A responded, *'So, A is built upon C to E, and B is built upon C to F, so if E and F overlay, then it will be a perpendicular bisector.'* This type of talk underpinned their joint effort to deepen their understanding of the geometric problem at hand.

Evaluative talk plays a critical role in all conversations as participants assess the validity of their observations and express confidence in their conclusions. This type of talk reinforces their learning, supports self-assurance, and signifies a vital stage of self-assessment and knowledge consolidation. For example, Participant G commented, *'That's right. If the circles are equal, then the quadrilateral is both a kite and a rhombus. Otherwise, it's just a kite.'* Participant H summarised, *'So, to summarise, we've observed that the quadrilateral changes between a kite and a square depending on the size of the circles,'* and Participant G agreed that they have learned, *'A lot!'* These quotes highlighted how evaluative talk reinforced motivation to learn by evaluating progress and recognising the success of the investigation.

In another instance, Participant B responded, *'Yep, but that is not always a bisector,'* and Participant A agreed, *'Definitely perpendicular.'* Participant A further reinforced, *'Yeah, absolutely,'* and Participant B questioned, *'So, what tangent are you looking at?'* when they were imagining a tangent to a radius of a circle and parallel to a line perpendicular to a line segment. These instances of evaluative talk helped the participants refine their understanding and arrive at more robust conclusions.

Deliberative talk serves as a practical tool for planning actions and coordinating efforts within geometric scenarios. Participants discuss how to move points, align elements, and execute specific actions to achieve their learning objectives. Deliberative talk ensures a clear and coordinated approach to their exploration. Participant G initiated deliberation by saying, *'So, if we want to prove that the quadrilateral is a square, we need to show that all the angles at A, B, H, and G are right angles. Let's start with point G...'* H participated actively, asking, *'Right, so we can conclude that angle G is a right angle. Now, what about the other angles?'* These exchanges demonstrated how deliberative talk encouraged participants to strategically discuss their next steps in the analysis, leading to an understanding of the problem. In addition, the utterance, *'Now, what about the other angles'* by Participant H also reflects elements of interrogatory talk. This type of talk prompted a further inquiry into specific angles, fostering a more comprehensive exploration.

Deliberative talk was also employed as participants A and B deliberated on the logic behind their conclusions and reasoned through the steps taken to solve the problem. Participant A stated, *'OK, I don't know how you managed to construct the circle at A.'* Participant B responded, providing a detailed explanation, *'The first thing I did was to select the two points that defined the radius CE, and then once you've done that, you then click where you want it to be centred.'* Participant A added, *'So, A is built upon C to E, and B is built upon C to F, so if E and F overlay, then it will be a perpendicular bisector.'* This exchange demonstrated how deliberative talk encouraged participants to strategically discuss their next steps in the analysis, leading to a deep understanding of the problem.

Interrogatory talk is a key element in maintaining a productive flow of conversation, ensuring understanding, and prompting further inquiry and discussion. Participants use questions to seek clarification and drive critical thinking, encouraging deep exploration of geometric relationships. For instance, Participant B asked, *'Is that true?'* when they were trying to explain why a line is perpendicular to a line segment using a chord and radius concept, and Participant A responded, saying '*I mean, yes. You do. The fact is that the chord is a vertical property where the tangent to a circle is always at right angles to its radius. And if you've got a chord that's parallel to that tangent as we have here, then it's always going to be 90 degrees to the radius that is constructed.'* On another occasion, Participant B enquired, *'So, what tangent are you looking at?'* These questions played a crucial role in maintaining a productive flow of conversation and ensuring that both participants were on the same pace.

Imaginative talk stimulates creative thinking, inviting participants to envision hypothetical scenarios and possibilities related to the geometric concepts under exploration. This type of talk encourages a deep engagement with the subject matter. **Imaginative talk** emerged as the participants considered hypothetical scenarios and 'imaginary lines' to explore the geometric relationships. Participant B wondered, *'Oh, I see. The line AB? So, are we thinking about the imaginary line here?'* This encouraged creative problem-solving and outside-the-box thinking.

Expository talk is prevalent in these conversations. Participants actively engage in collaborative exploration and discussion while also conveying established knowledge and providing explanations. This reflects their dynamic approach to constructing understanding through shared problem-solving and the exchange of informative insights. For instance, Participant B detailed the process of constructing a circle, *'The first thing I did was to select the two points that define the radius CE, and then once you've done that, you then click where you want it to be centred.'* In addition, Participant A elucidated the geometric relationship between points A, B, C, E, and F, stating, *'So, A is built upon C to E, and B is built upon C to F, so if E and F overlay, then it will be a perpendicular bisector'* [See Figure 5A.4b]. Furthermore, discussions on circle properties, chords, and tangents contributed to the expository nature of the conversation, enriching the participants' understanding of geometric principles.

In summary, these diverse types of dialogic talk collectively create a dynamic and engaging learning environment for the participants. Their interactions are marked by transactional talk, expressive talk, exploratory talk, evaluative talk, deliberative talk, imaginative talk, expository talk, and interrogatory talk. Together, these facets enhance their grasp of geometric concepts and underscore the importance of varied dialogic talk types in fostering collaborative learning and meaningful exploration in educational settings.

5A.3.3.8 Sample analysis and findings of dialogic learning principles

The shift towards analysing dialogic learning principles is motivated by the realisation that, despite the intentional design of the tasks in the online learning environment with specific design principles listed earlier in the introduction of this chapter and also in Chapter 3, there was an unexpected emergence of dialogic talk types among the beginning teachers. This unplanned occurrence suggests a natural integration of dialogic elements within the learning environment.

With this, I will be transitioning the analysis to dialogic learning principles to broaden the comprehension of the unintentional dynamics that contributed to the emergence of dialogic talk types. Dialogic learning principles can provide a macro-level framework that helps interpret the overarching strategies shaping the learning environment. This analytical shift aims to explore how intentional design, as described in the components (see Chapter 3 for detail), interacted with the natural development of dialogic interactions among beginning teachers.

In essence, the shift acknowledges the importance of examining the deliberate components of the tasks in the learning environment and the emergent principles that played an unforeseen yet meaningful role in shaping the collaborative online learning experience. Understanding these dialogic learning principles can allow for a wide-ranging exploration of how intentional planning and unintentional dynamics collectively contributed to the effectiveness of the learning environment in promoting open dialogue, critical thinking, and collaborative knowledge construction among beginning teachers.

5A.3.3.9 Analysis of the conversation using dialogic learning principles

Table 5A.5 Analysis of dialogic learning principles sample 5

Principle	Text showing coding of interplay of dialogic principles	Principle
Collective	M: *OK, what geometrical relationship does the line that passes through points H and I have to the angle ABC?* *A perpendicular bisector.* [see Figure 5A.1a] N: *Yeah, that's a perpendicular line. What happens if you move point A? All right, just move point A about there.*	Purposeful
Cumulative	M: *No, that's not it. It's not a perpendicular bisector* N: *Make it like an ABC right triangle. OK. So, you see, it cuts the angle in half by bisecting the angle.* [see Figure 5A.1b] M: *Yeah, that is right. Yeah. That's sure.*	Reciprocal
Deliberative	N: *So, what geometrical relationship is that?* M: *It is an angle bisector. OK. Is that right? I think it's right.* N: *Yeah, I feel like that is right. So, when you move either of those points, you are just changing the angle, but the line still bisects the angle. It is good to learn concepts like this.*	Supportive

	Colour coding of interplay of dialogic learning principles every 30 seconds								
Principle	30	30	30	30	30	30	30	30	30
Collective									
Cumulative									
Deliberative									
Purposeful									
Reciprocal									
Supportive									

5A.3.3.10 Findings from the analysis of participants' conversation

The analysis of the conversation between participants M and N revealed the application of dialogic learning principles in the context of geometric constructions. These dialogic learning principles encompassed collective, cumulative, deliberative, purposeful, reciprocal, and supportive aspects of the learning process that influenced their understanding and knowledge acquisition.

One salient finding was the embodiment of the **collective** principle. Throughout the conversation, M and N engaged collaboratively to discern the geometrical relationship between the line passing through points *H* and *I* and angle *ABC*. Their joint exploration demonstrated the power of collective learning, as the collaborative exchange of ideas contributed to an understanding of the concept.

Furthermore, the dialogue exemplified the **cumulative** principle of dialogic learning. Initially, M's response suggested a misunderstanding of the concept of *'perpendicular bisector'*. However, as the conversation unfolded, their understanding developed, Participant M recognised that the line under investigation is '*... not a perpendicular bisector'* of line segment AB but '*a perpendicular*' illustrating how knowledge and understanding can accumulate and mature during thoughtful discussion. In another instance where participants M and N were performing and exploring examples and non-examples of an angle bisector, they were able to link this task to the previous task with Participant N, saying, '*You see when put I on top of H, it is now an angle bisector, the same as the previous task we did yesterday. So, the radii of the two circles must be the same,'* showing how knowledge and understanding can accumulate and link during thoughtful discussion.

The **deliberative** principle was another key finding. N's probing question, *'What happens if you move point A?'* encouraged reflective discussion and prompted M to think critically about the relationship. This element of critical thinking and deliberation played a role in deepening their understanding.

Moreover, the conversation maintained a clear learning purpose, adhering to the **purposeful** principle of dialogic learning. Both participants actively engaged in the discussion with the specific goal of understanding the geometrical relationship, ensuring that the conversation remained focused and productive.

Reciprocity was evident as well, as both participants actively contributed ideas, explanations, and questions throughout the conversation. This mutual interaction embodied the **reciprocal** principle of dialogic learning, fostering active engagement and the exchange of diverse perspectives.

Finally, the conversation created a **supportive** learning environment, aligning with the **supportive** principle of dialogic learning. N provided feedback, encouragement, and guidance to M, facilitating M's journey towards the correct understanding of the concept. In another instance, where a different pair, participants A and B, explored Task 1, Participant A asked a question, '*I don't know how you managed to construct the circle at A,'* and Participant B, responded, '*The first thing I did, was to select the two points that define the radius CE, and then once you've done that, you then click where you want it to be centred.'* This shows a supportive principle of dialogic learning at play. This supportive atmosphere fostered effective learning and highlighted the importance of a supporting learning environment.

In conclusion, the conversation analysis between participants M and N highlighted the effective application of dialogic learning principles in developing the understanding and knowledge acquisition process. These principles contributed significantly to the depth of understanding of the geometrical concept under discussion, emphasising the invaluable role of dialogic learning in educational settings.

5A.3.3.11 Insights into the complex nature of dialogic talk types in collaborative online learning environments

The study's findings shed light on the spontaneous emergence of dialogic learning principles and diverse dialogic talk among beginning teachers within collaborative online learning environments. Despite the absence of a deliberate framework for dialogic learning, participants seemed to have actively engaged in dialogic exchanges, which appears to showcase their adaptability and proficiency in dialogue facilitation. Notably, dialogic learning principles, such as fostering open communication and collaborative knowledge construction, seem to have organically surfaced during interactions, indicating learners' appreciation for dialogue's pedagogical value.

A noteworthy aspect of the findings was the complex nature of dialogic talk types evident in participants' dialogue, which appeared instrumental in facilitating knowledge co-construction. Interestingly, the tasks in the collaborative online learning setting, initially lacking explicit incorporation of dialogic learning principles, inadvertently facilitated the emergence of diverse dialogic talk types among participants. These observations seem to highlight learners' adaptability in recognising the significance of dialogue and varied talk types in their learning processes.

While the explicit integration of dialogic learning principles could improve collaborative learning experiences, educators and instructional designers can design learning environments that are conducive to fostering open discourse.

Encouraging participants to employ diverse talk types for knowledge co-construction and deep understanding can be essential for maximising the potential of collaborative online learning environments.

5A.3.3.12 Frequency distribution of types of dialogic talk within and across paired participants

In this section, the analytical focus shifts towards exploring the frequency distribution of various dialogic talk types among beginning teachers engaged in collaborative online learning. The decision to explore the quantitative aspect of dialogic interactions is driven by the quest to unravel patterns and insights that might be concealed within the sheer volume of conversational data. Multiple coding of utterances, initially employed to categorise and highlight dialogic talk types qualitatively, is now complemented by a quantitative lens. This shift is rooted in the acknowledgement that while qualitative analysis shows the details of dialogic interactions, a quantitative approach allows for a broader understanding of prevalence, trends, and variations in dialogic talk types. Comparing qualitative and quantitative analyses can show how dialogic learning unfolds among beginning teachers to shed light on the qualitative richness and the quantitative distribution of dialogic talk within their collaborative online learning experiences. Table 5A.6 shows the frequency distribution of types of dialogic talk within and across paired participants.

Table 5A.6 Frequency distribution of types of dialogic talk

Talk Type/Pair	Pair 1	Pair 2	Pair 3	Pair 4	Pair 5	Pair 6	Total	%
Transactional	429	459	864	711	657	594	**3714**	15
Exploratory	735	671	935	1111	1034	935	**5421**	22
Deliberative	606	552	780	792	744	684	**4158**	17
Interrogatory	231	429	676	607	676	693	**3312**	13
Evaluative	222	360	624	588	516	504	**2814**	11
Expressive	561	341	385	341	308	275	**2211**	9
Expository	352	336	492	444	408	348	**2380**	10
Imaginative	132	189	207	126	99	117	**870**	3
Total	**3268**	**3337**	**4963**	**4730**	**4442**	**4150**	**24880**	100

Table 5A.7 Percentage frequency distribution of dialogic talk types within each pair and all the pairs

Talk Type/Pair	Pair 1 (%)	Pair 2 (%)	Pair 3 (%)	Pair 4 (%)	Pair 5 (%)	Pair 6 (%)	All Pairs (%)
Transactional	13	14	17	15	15	14	**15**
Exploratory	22	20	19	23	23	23	**22**
Deliberative	19	17	16	17	17	17	**17**
Interrogatory	7	13	14	13	15	16	**13**
Evaluative	7	11	13	12	12	12	**11**
Expressive	17	10	8	7	7	7	**9**
Expository	11	10	10	9	9	8	**10**
Imaginative	4	6	4	3	2	3	**3**
Total	**100**	**100**	**100**	**100**	**100**	**100**	**100**

Table 5A.8 Percentage frequency distribution of each dialogic talk type across all pairs

Talk Type/Pair	Pair 1 (%)	Pair 2 (%)	Pair 3 (%)	Pair 4 (%)	Pair 5 (%)	Pair 6 (%)	Total (%)	All Pairs (%)
Transactional	12	12	23	19	18	16	**100**	**15**
Exploratory	14	12	17	20	19	17	**100**	**22**
Deliberative	15	13	19	19	18	16	**100**	**17**
Interrogatory	7	13	20	19	20	20	**100**	**13**
Evaluative	8	13	22	21	18	18	**100**	**11**
Expressive	25	15	17	15	14	12	**100**	**9**
Expository	15	14	21	19	17	15	**100**	**10**
Imaginative	15	22	24	14	11	13	**100**	**3**

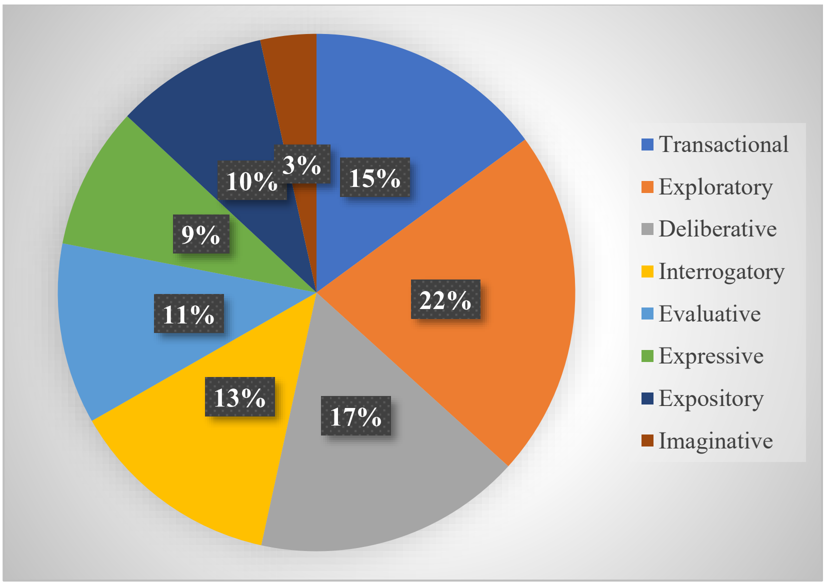


Figure 5A.2 Pie chart of dialogic talk types

The frequency distribution of dialogic talk types in Table 5A.6 and the accompanying pie chart can offer valuable insights into the complexities of communication dynamics within collaborative online learning environments through task facilitation. This analysis seems to provide an additional understanding of how participants engage in dialogue and interact with each other. Exploratory talk emerges as the dominant form, constituting 22% of total instances, indicating collaborative exploration and knowledge co-construction. The 22% of total instances account for 22% of the sum of each total number of dialogic talk types tallied from participants' conversations. Assuming that all conversation was dialogic talk, this percentage falls between 22% exclusively and 100% inclusively. This suggests an overlap of dialogic talk types during their conversation, where the same dialogic talk could be categorised as more than one type of talk. The deliberative talk was closely followed by 17%, showcasing participants' commitment to in-depth analysis. However, despite being less frequent, imaginative talk at 3% demonstrates participants' creativity and hypothetical thinking, injecting innovation and exploration into interactions.

Reflecting on the distribution reveals distinct patterns across pairs, suggesting varying emphasis on different forms of dialogue. For instance, Pair Four exhibits a prevalence of exploratory talk, which seems to suggest a collaborative effort towards idea generation and exploration. In contrast, Pair Three demonstrates a propensity for evaluative talk, which appears to imply a focus on critically evaluating ideas and concepts. These discrepancies likely stem from the unique dynamics, backgrounds, and communication styles of participants, as well as how they looked at the complexity of the tasks undertaken.

Further exploration into specific instances of differences between pairs seems to reveal deeper insights into the factors that may have influenced talk patterns. For example, Pair Six displays a higher frequency of interrogatory talk, indicating a participant's inclination towards seeking clarification through questioning. This disparity may arise from individual preferences or disparities in subject matter familiarity. Similarly, Pair Four emphasises expository talk, possibly owing to one participant's instructional role or task requirements necessitating a didactic approach.

Reflecting on the reasons behind these patterns underlines the influence of external factors on dialogue dynamics. The prevalence of transactional talk across all pairs seems to give an impression of a common need for coordination and task management. Task complexity, familiarity with the subject matter, and group dynamics can be fundamental in shaping talk distributions. The dominance of exploratory talk in Pair Three appears to foster a collaborative and explorative learning environment, while the abundance of evaluative talk in Pair Five can cultivate critical thinking and reflection.

From Table 5A.7, it is evident that the proportion of transactional talk within Pair Three exceeds the collective proportion of transactional talk across all six pairs. Similarly, in Pair One, the percentage of deliberative talk surpasses that of any other pair or the aggregate across all pairs. However, it is crucial to note that the percentage of deliberative talk in Pair One does not necessarily imply a higher share of deliberative talk out of the total deliberative talk across all pairs, as demonstrated in Table 5A.8 For instance, Table 5A.7 indicates that 19% of dialogic talk within Pair One is deliberative, exceeding Pair Three's 16% and the 17% in each of the other pairs as well as across all pairs; Table 5A.8 reveals that Pair One contributes only 15% of the total deliberative talk across all pairs, which is less than Pair Three's and Pair Four's 19%, Pair Five's 18%, and Pair Six's 16%.

Exploring deeper questions surrounding observed talk patterns facilitates critical reflection on their broader implications. For instance, understanding variations in expressive talk may illuminate the role of affective expression in collaborative learning. Considerations of task design and instructional strategies are imperative for optimising dialogue dynamics and enhancing learning outcomes.

In summary, while the frequency distribution of talk types seems to provide additional insights into understanding the quality of discourse in collaborative learning contexts, its application necessitates careful consideration. While it provides valuable insights into communication dynamics, particularly in identifying trends and patterns, it may oversimplify the complexity inherent in dialogic interactions. The limitations of frequency distribution in capturing contextual details and non-verbal cues can feature the need for a comprehensive understanding of dialogic talk types. Integrating qualitative analysis to supplement the quantitative findings, as I did in earlier sections, can offer a more detailed interpretation, enriching the understanding of dialogic communication dynamics and their impact on collaborative learning outcomes. This holistic approach can ensure a robust analysis, fostering advancements in collaborative online learning research.

5A.4 Beginning teachers developing TPACK knowledge through exploring geometric construction tasks

In this section, I present how the tasks within the dynamic geometry software environment supported the development of the participants' (beginning teachers') TPACK through their performance, investigation, and discussion of these tasks on the online platform.

5A.4.1 The development of participants' technological knowledge (TK) of dynamic geometry software through carefully designed tasks

The study's tasks significantly contributed to the development of participants' technological knowledge (TK) of dynamic geometry software, particularly GeoGebra. Engaging directly with the software allowed participants to deepen their understanding of geometric concepts while simultaneously enhancing their ability to navigate and use GeoGebra's features effectively.

Task 1 served as an introduction, where participants A and B explored the basic functionalities of GeoGebra by constructing and manipulating geometric figures. This task allowed them to experiment with the software's dynamic capabilities, such as moving points and observing real-time changes in geometric relationships. Initially, participants grappled with misconceptions, like incorrectly identifying the perpendicular bisector of a segment. However, through active manipulation and observation, they corrected these errors, demonstrating the iterative nature of learning with dynamic geometry tools. For example, Participant B stated: '*Yep, but that is not always a bisector. At the moment, we got a perpendicular line, but it can be at any point along a line segment. Yeah, so I can make the circle smaller.'*

The excerpt shows developing technological knowledge in GeoGebra because the speaker differentiates between a perpendicular line and a bisector, demonstrates the ability to manipulate objects by resizing a circle, and engages dynamically with the software to explore geometric concepts, indicating growing proficiency in using the tool effectively for geometric tasks.

Similarly, participants G and H encountered difficulties with GeoGebra's compass tool, but through persistent exploration and collaboration, they gained confidence and proficiency in using the software to construct and understand geometric properties. For instance, let us consider the excerpt below:

G: *'Why wouldn't the software let me construct the circle? What is happening here?'*

H: *'I think, we need to undo the point you have selected first.'*

G: *'Right, so here it says select a segment or two points for radius then centre point. OK so want to select CE and then bang [it on point A]. OK, that wasn't too hard. Now I know how to do it. [H] doesn't know how to do it, so I'm going to do that again. Watch [H]. So, [H], did you see that [how I constructed the circle centred at point A]?'*

H: *'Yeah, I did. I did see it. I missed, the exact button you clicked before you had to select the segment.'*

The excerpt shows developing technological knowledge of GeoGebra because participants were troubleshooting software use, interpreting tool instructions (selecting a segment and centre point for a circle), and demonstrating mastery by teaching themselves, which could reflect growing understanding and competence in using the software's features for geometric construction.

Building on this foundation, **Task 2** deepened participants' technological knowledge by challenging them to explore the properties of quadrilaterals dynamically. Participants A and B, for example, investigated how varying the positions of points E and F along a line could transform a quadrilateral into different specific shapes like a rhombus or square. This task highlighted the power of GeoGebra in visualising and understanding the relationship between side lengths and shape classification. As participants manipulated points and observed the resulting transformations, they gained a more detailed understanding of how the software could be used to explore geometric concepts interactively. For example, Participant A stated: *'But then if you increase them [the circles] both together [as I am doing], then, you're going to a point where all three circles overlap, and you get a square.'*

The excerpt shows developing technological knowledge of GeoGebra as Participant B demonstrates an understanding of how manipulating dynamic objects (increasing the size of circles) affects geometric relationships, predicting the outcome (overlap of circles forming a square), which reflects growing proficiency in using the software to explore geometric constructions. The hands-on experience in Task 2 reinforced their geometric knowledge and enhanced their ability to use GeoGebra as a tool for mathematical exploration.

In **Task 3a**, participants focused on constructing and analysing angle bisectors using GeoGebra. This task provided them with the opportunity to observe consistent geometric properties, such as how the line HI bisected the angle ABC regardless of the points' positions. Through experimenting with different configurations, participants deepened their understanding of the software's potential to maintain geometric properties dynamically. This task further developed their technological knowledge as they explored the interconnectedness of geometric elements within the GeoGebra environment. For instance, consider Participants A and B's excerpt below:

A: *'It changes the location of the centre of the circle. Isn't it? Yeah.'*

B: *'It changes the point of intersection with the circle centred at B, so changes where the circle centred at D is. But our line HI keeps on bisecting.'*

A: *'It is interesting that lines AB and BC intersect in such a way that the points stay H and I are always on that line. I didn't actually necessarily think that would be the case, be with replacement line HI.'*

B: *'Yeah, which obviously we need if we're going to bisect that angle yeah, it's not immediately apparent from how we constructed it why that would be the case.'*

The excerpt shows developing technological knowledge of GeoGebra as participants analysed how manipulating geometric elements (for example, changing circle centres and line intersections) affects the overall construction. Their discussion reveals an evolving understanding of the software's dynamic features and geometric relationships, as they reflect on unexpected outcomes and make sense of the underlying constructions.

Task 3b introduced participants to the concept of non-examples, emphasising the importance of symmetry and equal circle radii in establishing angle bisectors. As participants like A and B engaged with the software to test different scenarios, they developed a more sophisticated understanding of how GeoGebra can be used to explore and confirm geometric principles. The task highlighted the participants' growing ability to leverage the software's features to distinguish between valid and invalid geometric constructions, emphasising their advancement in technological knowledge. For example,

A: *'Okay, right then, question one what geometrical relationship does the line that passes through JandK have to the angle ABC? And why?*

If we grab either AB, B or BC and wave it around a bit, we will hopefully see it moving and get a feel for what's happening on here.'

B: *'So, we're focusing on JK with angle ABC.'*

A: *'Correct!'*

B: *'OK so JK remains the same. No, it's not that's not true. So JK is changing, it's getting bigger as our angle gets smaller and then it's getting smaller and smaller and smaller and smaller.'*

A: *'If you have I and H overlapping presumably point K or line JK would sit or overload JB, wouldn't it?'*

B: *'Yeah, they are. So, at the point where our circles are the same size, we get the same result as in the case before.'*

A: *'It's a bisector yeah.'*

The excerpt shows developing technological knowledge of GeoGebra as participants explore how manipulating dynamic geometric elements (lines and angles) affects the relationship between them. Their discussion, which includes testing geometric behaviour by moving objects and observing changes in line and angle properties, reflects growing proficiency in using the software to understand and explain geometric relationships, such as bisectors.

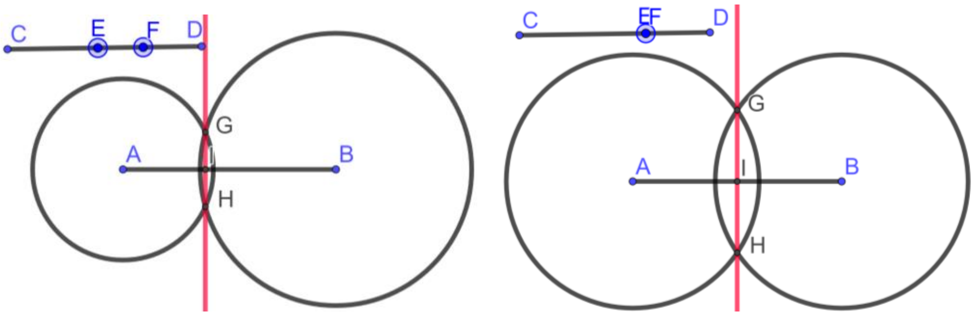
Task 4a and **Task 4b** further challenged participants to apply their growing technological knowledge to more complex geometric relationships. In Task 4a, participants like A and B constructed and analysed isosceles trapeziums, using GeoGebra to identify congruent angles and symmetrical configurations. This task demonstrated their increasing proficiency in using the software to visualise and analyse complex geometric constructions. Similarly, in Task 4b, participants explored the properties of perpendicular lines and angles, further solidifying their understanding of how GeoGebra can be used to generalise geometric principles and apply them to dynamic scenarios.

Throughout these tasks, participants' interaction with GeoGebra significantly enhanced their Technological Knowledge (TK) of dynamic geometry software. By actively engaging with the software, experimenting with its features, and overcoming challenges, they developed a deeper understanding of how to use GeoGebra as a powerful tool for exploring and understanding geometric concepts. This process of exploration and discovery not only improved their proficiency with the software but also reinforced the value of dynamic geometry environments in facilitating meaningful mathematical learning.

5A.4.2 The development of participants' geometric content knowledge (CK) through carefully designed tasks

In Task 1, participants demonstrated an understanding of geometric principles related to perpendicular lines, bisectors, and symmetry through their explorations with GeoGebra. For instance, Participants A and B identified that for *GH* to act as a perpendicular bisector, the radii of the circles centred at points *A* and *B* must be equal, an important insight supported by Figures 5A.4a and 5A.4b. They also explored the properties of symmetry and mirror lines, discovering that line *AB* serves as a mirror line causing symmetric reflections of points *G* and *H*. This reflects their ability to translate theoretical knowledge into practical constructions using dynamic geometry tools. In addition, Participants

G and H’s investigation into tangents and perpendicular radii revealed an understanding of the geometric relationship between tangents and radii, as illustrated in Figure 5A.5.



Figures 5A.3a and 5A.4b: Exploring perpendicular line and perpendicular bisector of line segment AB

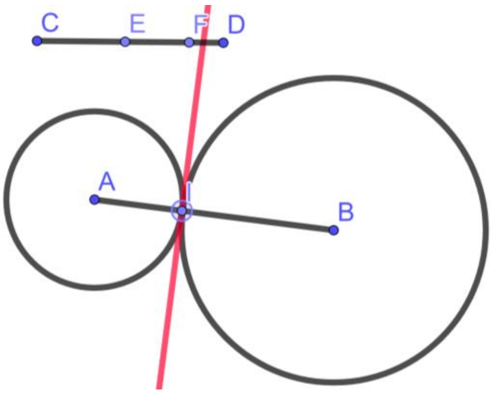


Figure 5A.4: Exploration of a perpendicular line tangent to two circles

In Figure 5A.6, Participants E and F engaged in exploring the concept of image and object points using GeoGebra. Their discussion centred on the role of line segment *AB* as a mirror line, with particular emphasis on how this mirror line affects the reflection of points *G* and *H*. They observed that points *G* and *H* were symmetrical with respect to line *AB*, meaning that each point’s image was a mirror reflection of its object point across *AB*. This understanding was crucial for their exploration of geometric transformations involving reflections. Participants E and F discussed how changing the position of *AB* as the mirror line would alter the location of the reflected points while maintaining symmetry. They concluded that the key property of reflections is that distances from the mirror line are preserved, ensuring congruence between the original and reflected images. Their ability to articulate these reflections and the associated symmetry demonstrated a robust grasp of geometric transformations and the properties of reflections.

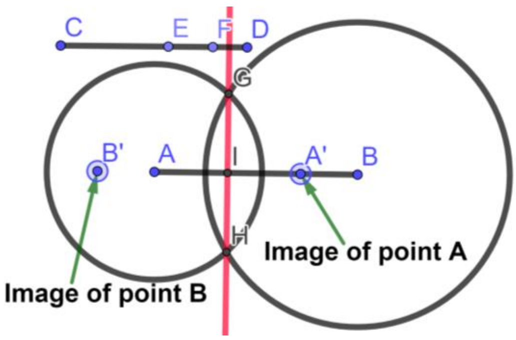
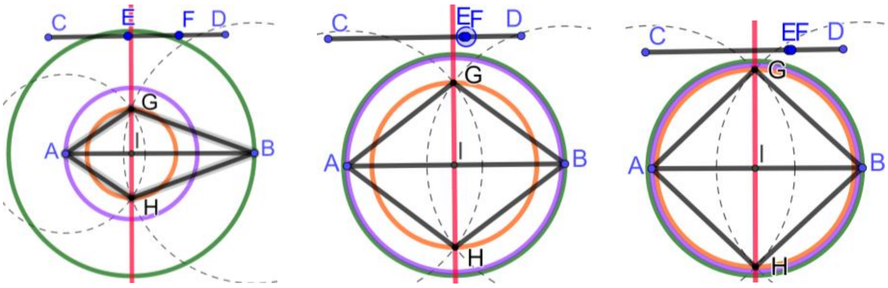


Figure 5A.5: Exploration of the concept of image and object points

Participants C and D's work with the perpendicular bisector of line segment *AB*, forming a rhombus *AGBH* (Figure 5A.7b), demonstrated their ability to identify geometric properties and apply them to dynamic constructions. Their insights into the relationships among quadrilateral properties, such as those forming kites and squares, were illustrated in Figures 5A.7a-c. They recognised how the geometric configuration changes with varying distances between points *E* and *F*, which highlights their understanding of shape classification and transformations.



Figures 5A.6a, 5A.7b and 5A.7c: Exploratory of a general kite, rhombus and square

In Figure 5A.8, Participants G and H discussed the use of tangent *AH* to explain the right angles in a square. They observed that when the circles centred at points A and B coincided, the quadrilateral *AGBH* transformed into a square. A significant insight was the role of the tangent line *AH*, which intersected the circle centred at *B* at point *H*. They recognised that the tangent line *AH* was perpendicular to the radius *HB* of the circle, forming a right angle at *H*. This observation was linked to the square's internal right angles. Participants noted that since the tangents to a circle at a point of tangency are perpendicular to the radius, this property helped in verifying the right angles at each vertex of the square. They discussed how the perpendicularity of tangents and the properties of the radii ensured that the quadrilateral's internal angles were right angles, validating the square's geometric properties.

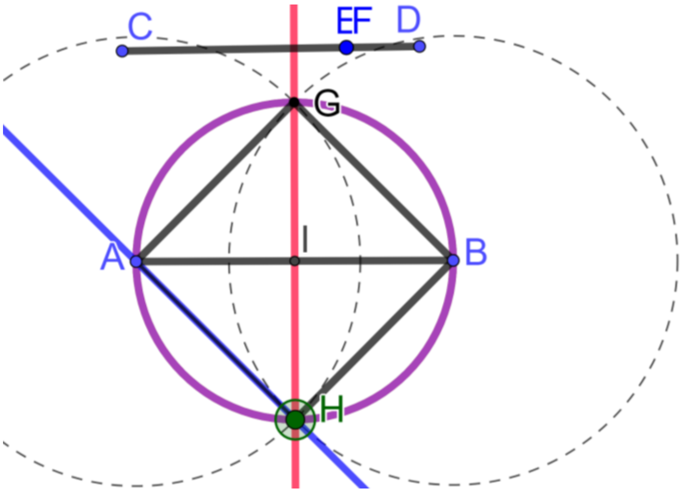
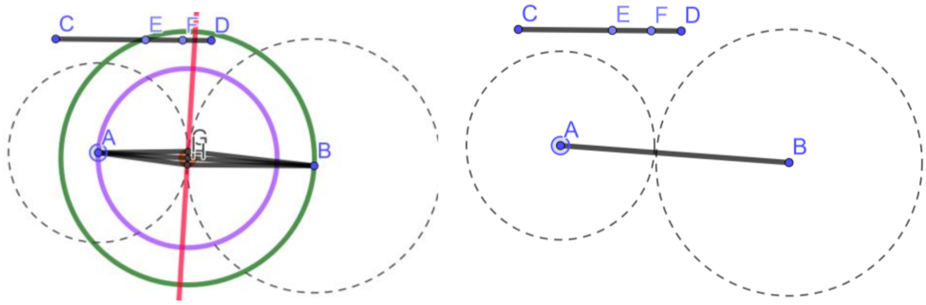


Figure 5A.7: Using tangent *AH* in explaining a square's right angles

In Task 2, participants extended their knowledge by analysing how the distances between points *E* and *F* altered quadrilaterals from kites to rhombuses and squares, as shown in Figures 5A.7a-c. They examined properties such as equal sides and perpendicular diagonals, leading to a growing understanding of geometric element interactions and classifications. Participants C and D's discovery that a general kite becomes a rhombus when points *E* and *F* merge (Figures 5A.7a, 5A.7b, 5A.9a and 5A.9b) exemplifies their advanced understanding of geometric transformations.



Figures 5A.8a and 5A.9b: Exploring when kite AGBH disappears by dragging point A

In Figure 5A.10, Participants E and F explored the transformation of a kite into an arrowhead diagram. They began by constructing a kite, focusing on its distinct properties such as two pairs of adjacent sides being equal and one pair of opposite angles being equal. As they manipulated the positions of points *E* and *F*, they observed how the kite's shape evolved. They discovered that when the points *E* and *F* were adjusted in specific ways, the kite's shape progressively changed into an arrowhead. This transformation occurred when the quadrilateral's diagonals began to align differently, altering the kite's original structure. Participants E and F discussed the geometric implications of this transformation, noting how the arrowhead's angles and side lengths differed from those of the kite. Their exploration highlighted an understanding of how altering geometric configurations can lead to significant shape changes, enriching their grasp of geometric transformations and properties.

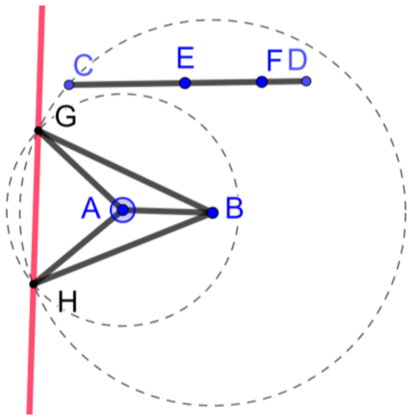
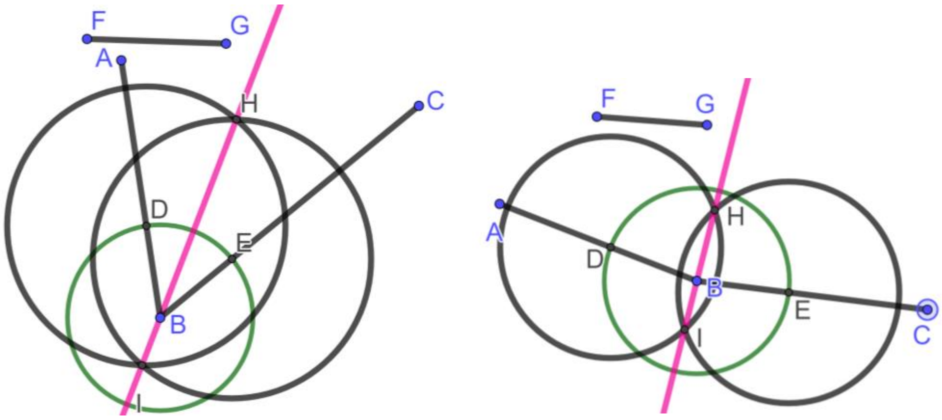


Figure 5A.9: Exploring when a kite becomes an arrowhead diagram

Task 3a involved participants manipulating dynamic constructions of angle bisectors, consistently observing that line *HI* bisected angle *ABC* regardless of point positions (Figures 5A.11a and 5A.11b). This revealed their grasp of invariant properties related to angle bisectors and how circle radii impact these properties. They noted that deviations in radii affect the angle bisector's functionality, as seen in Figure 5A.12.



Figures 5A.10a and 5A.11b: Exploring invariant angle bisector

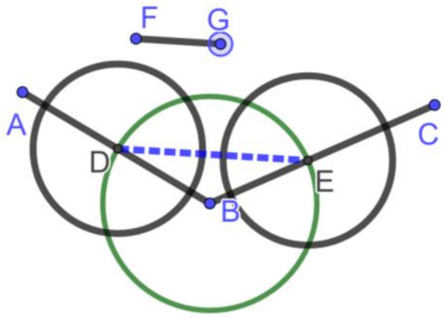
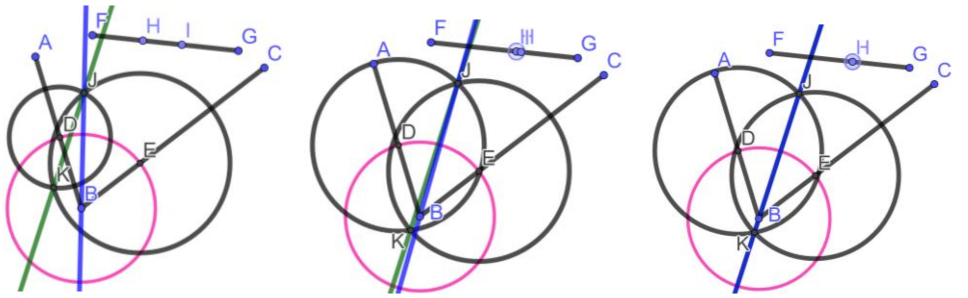


Figure 5A.11: Exploring when invariant angle bisector disappears

Similarly, in Task 3b, participants confirmed that equal circle radii are necessary for constructing accurate angle bisectors, as evidenced by Figures 5A.13a-c. They explored both successful and unsuccessful examples, highlighting the critical role of radii in maintaining angle bisector properties.



Figures 5A.12a, 5A.13b and 5A.23c: Exploration of examples and non-examples of angle bisector

Further analysis in Task 3b revealed that when points *I* and *H* were manipulated to move apart, causing the radii to differ, neither line *JK* nor line *JB* functioned as an angle bisector. An interesting observation emerged regarding line *JK*, which consistently maintained the same gradient as the angle bisector, irrespective of whether the radii of the two circles were uniform or not, as demonstrated in Figure 5A.14. Consequently, line *JK* was determined to be parallel to the angle bisector.

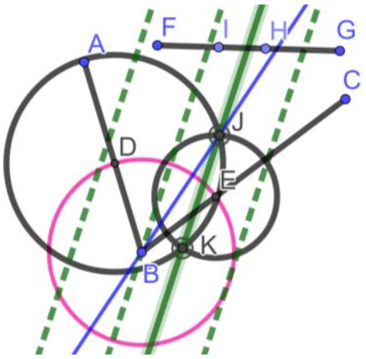


Figure 5A.13: Exploring non-examples of angle bisector parallel to the angle bisector

In Task 4a, participants examined angle bisectors and geometric relationships through various methods, including using a straightedge for construction (Figures 5A.15a-c). They identified congruent angles and symmetrical properties in isosceles trapeziums and kites, demonstrating an understanding of geometric principles. Participants G and H's investigation into proportionality and alignment further reflected their geometric reasoning skills, while Participants E and F's consistent observations regarding segment *BL* and its role as an angle bisector illustrated their proficiency with dynamic geometry software.

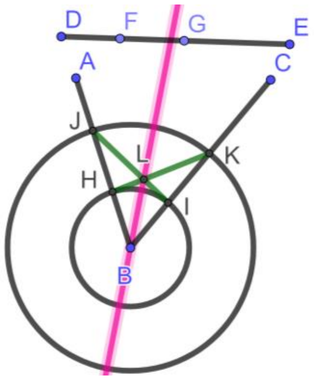


Figure 5A.14a: Exploration of invariant angle bisector

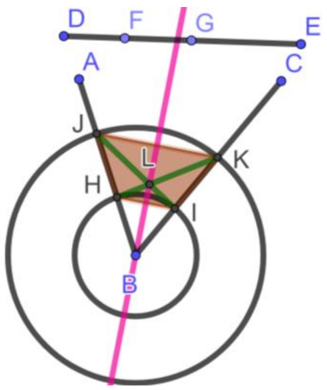


Figure 5A.15b: Exploration of an isosceles trapezium and an angle bisector

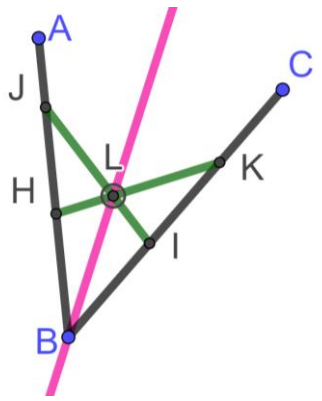
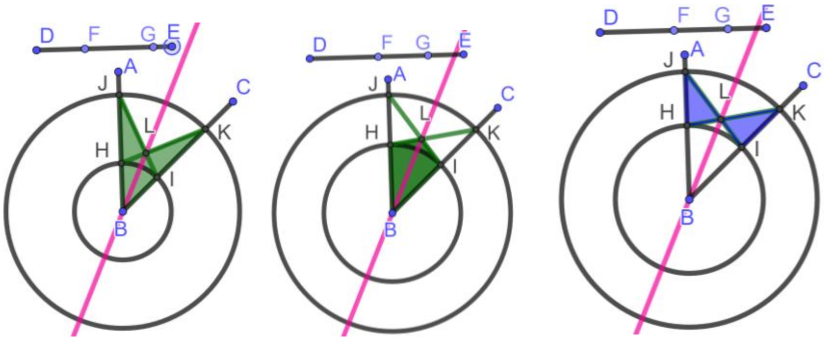


Figure 5A.15c: Exploring and constructing an angle bisector with a straightedge only

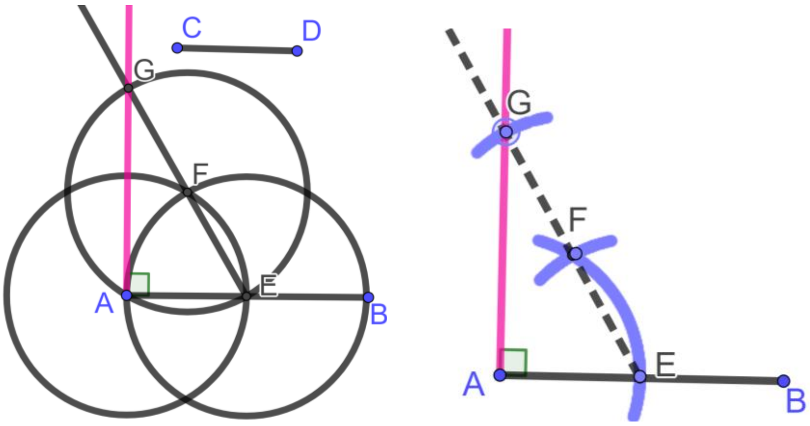
In Figures 5A.16a, 5A.16b, and 5A.16c, Participants C and D engaged in an exploration of geometric shapes to construct using only a straightedge. Their investigation involved with the construction of an arrowhead, as depicted in

Figure 5B.16a. The participants concentrated on the specific angles and side lengths of the arrowhead, using the straightedge to construct its two non-parallel lines. They carefully aligned the straightedge to ensure precision, discussing how this approach was necessary for accurately capturing the geometric relationships inherent in the shape. Following this, in Figure 5A.16b, the focus shifted to constructing a kite. The participants highlighted the kite's distinctive symmetry and the congruence of its adjacent sides. They employed the straightedge to draw the kite's diagonals, ensuring these intersected at right angles, a fundamental property of kites. Their discussion featured the importance of precision in maintaining the kite's defining characteristics, reflecting their understanding of the geometric properties associated with this shape. Finally, in Figure 5A.16c, Participants C and D tackled the construction of congruent triangles. Their work emphasised the critical requirement for equal side lengths and angles to establish congruence. Carefully using the straightedge, they verified that the constructed triangles were congruent by ensuring that corresponding sides and angles matched exactly. This step demonstrated their proficiency in applying geometric principles to achieve precise constructions and their ability to use a basic tool effectively for geometric verification.



Figures 5A.16a, 5A.16b and 5A.16c: Exploring and constructing arrowheads, kites and congruent triangles using a straightedge only

Finally, Task 4b revealed participants' proficiency in constructing perpendicular lines and applying circle theorems. For instance, Participants A and B showed how a line through point *A* creates a right angle with segment *AB*, confirmed by Figures 5A.17a and 5A.17b. Their discussions on circle theorems and perpendicularity seem to demonstrate their ability to generalise geometric principles into practical applications, while Participants C and D's exploration affirmed the reliability of their findings and the effectiveness of dynamic geometry tools.



Figures 5A.17a and 5A.17b: Exploring and constructing a perpendicular at A without extending the line segment AB

5A.4.3 The development of participants' pedagogical knowledge (PK) through carefully designed tasks

Participants in the study developed pedagogical insights through their engagement with carefully designed tasks using dynamic geometry software, notably GeoGebra. Although they did not teach during the study, their experiences with the tasks enabled them to formulate strategies they could apply in future geometry teaching.

One prominent strategy was using GeoGebra to promote conceptual understanding. Participants frequently discussed how technology could move students beyond procedural steps to grasp underlying geometric principles. For instance, Participant G illustrated how GeoGebra could visually demonstrate the essential elements of a perpendicular bisector, such as intersecting arcs, without needing to draw entire circles. In a reflective exchange, G and H discussed their realisations:

G: '*Do you think this would be a good way to introduce construction? … They kind of constructions, even if they [students] can get the procedure, they have no idea why it works or why they're doing it*.'

H: '*Absolutely, yeah. I mean, when I have learnt perpendicular bisectors…, I kind of just learnt the method for it and not with the understanding of why it works*.'

G: '*So, if you think about it in GeoGebra, it probably seems intuitive, though maybe not for foundation Year 10 kids. Essentially, when we're doing one of these constructions, we're only using the necessary parts of the circles…*'

H: '*Yeah, we just need to know that the circles are equal, there are identical circles*.'

G: '*…which is what a compass does, but we do not need to draw the whole circle*.'

These insights resonate with G and H's earlier observation in Task 1: *'With GeoGebra, they [students] are not just doing steps, they can see why things work. That's more powerful.'* This exchange highlights the pedagogical emphasis on conceptual learning, encouraging students to understand the reasoning behind constructions rather than simply executing a sequence of steps.

Another key pedagogical strategy involved using dynamic geometry software to identify and correct misconceptions in real time. In Task 3a, Participants A and B initially mistook a perpendicular line for a bisector. Through manipulation in GeoGebra, they challenged their assumptions and revised their understanding:

A: '*Okay, now let's investigate the geometrical relationship between line GH and line segment AB. What do you think it is? So, GH to AB, perpendicular bisector.*'

B: '*Yep… [Participant B drags a point] …it is not always a bisector. At the moment, we got a perpendicular line, but it can be at any point along a line segment*.'

This dynamic interaction highlights the software's ability to support exploratory and corrective learning. Their evolving understanding mirrors their later insight in Task 1: *'The circles kind of show that GH has to be symmetrical to AB so that students can see that connection more easily.'* Such moments demonstrate how learners can test, revise, and reinforce their geometric understanding through guided exploration.

Collaboration and peer interaction also emerged as significant strategies in the development of pedagogical knowledge. The participants often worked together in pairs or small groups, which allowed them to share ideas, discuss complex problems, and build on each other's understanding. For instance, Participants G and H encountered difficulties using GeoGebra's compass tool, but through collaboration, they were able to overcome these challenges and successfully explore perpendicular bisectors. This experience highlights the value of collaborative learning, where students can benefit from each other's insights and support, leading to a more growing understanding of geometric concepts. This sentiment was echoed by Participant A, expressing, '*There were times during the exercise when I was able to discuss fragments of ideas with my partner and vice versa. These insights have inspired me to consider implementing the strategy of pair work in teaching geometry with GeoGebra*.'

This was repeated in Task 3a, where A and B described their task as a '*mini-investigation*,' promoting shared hypothesis testing and discovery: *'We think this point is the centre of rotation… but we need to test that more. It's like a mini-investigation.'*

Scaffolding and the use of critical thinking prompts were other recurring strategies. In Task 2, Participant A noted:

'*If we ask, 'Is this always true?' it makes them really think about why the lines behave that way*.'

These types of reflective questions encouraged learners to interrogate assumptions, promoting deeper reasoning and generalisation. This scaffolding was evident in other tasks as well, such as Task 1, when Participant A recognised how a step-by-step breakdown clarified the purpose of each action:

‘*OK, so, I understood that when we were doing that we were creating it using A as the centre, the instruction says a circle at A with a radius CE, oh I see. … These kinds of instruments work for me.*’

Such structured prompts and stepwise guidance can improve conceptual clarity and offer a pedagogical model for building independence in learners.

Participants also frequently emphasised the importance of visual manipulation and iterative construction. Participant C remarked in Task 2:

‘*I moved the points to see if it still stayed an angle bisector, that is something I will get my students to try*.’

Similarly, Participant E shared:

‘*Doing it manually helped me think about each step… I want to use that to teach students the construction*.’

These insights reveal how hands-on manipulation can lead to internalisation of geometric principles and promote active learning strategies.

Iterative learning and revisiting concepts also emerged as pedagogical themes. In Task 3a, Participants E and F drew on prior learning:

‘*We used a similar method in the last one.*’

This cyclical approach of experimenting, reflecting, and reapplying facilitated insight and a cumulative understanding of geometric constructions. Encouraging such iteration in classrooms may similarly support the development of mathematical reasoning over time.

Another noteworthy pedagogical move was the blending of traditional and digital tools. In Task 4a, Participants E and F explored constructing an angle bisector using only a straightedge. Their reflections indicated that combining traditional hands-on tools with dynamic software could increase conceptual understanding, as Participant E stated:

'Exploring it in GeoGebra first helps to confirm if the construction works, and then going back to the straightedge really makes you think about what's necessary for the construction.'

This dual approach of merging physical and digital representations offers students diverse ways to engage with abstract concepts. A similar sentiment was expressed by Participant A, who noted:

'Using only a straightedge makes you think differently, which could be a nice challenge after they have used GeoGebra.'

Further, participants explored the value of working with both examples and non-examples to surface geometric conditions. For instance, in Task 3b, Participants G and H tested cases where angle bisectors did not behave as expected due to unequal radii:

'*It's useful to let them change it and see what breaks… that's how they will understand what matters*.'

This exploration of edge cases promoted metacognitive awareness and helped clarify the underlying requirements for geometric validity.

Finally, participants discussed strategies to differentiate instruction and stimulate critical thinking. Tasks involving prompts such as '*always, sometimes, never*' supported such differentiation. Participant A described these as:

'*A way to get them thinking more deeply about properties, like with quadrilaterals, it helps to work through the possibilities*.'

Participant L also expressed a preference for inductive learning:

'*I like an inductive approach a lot… it builds problem-solving skills.*'

These comments underline how exploratory, open-ended strategies can accommodate different learners while encouraging critical thinking.

In conclusion, participants developed a rich repertoire of pedagogical strategies through carefully designed tasks involving dynamic geometry and collaborative reasoning. These strategies include emphasising conceptual understanding, iterative exploration, real-time error correction, blended tool use, structured scaffolding, and promoting metacognitive awareness through examples and non-examples. Their engagement with GeoGebra enhanced their understanding and helped them envision classroom practices that support active, reflective, and differentiated learning. These insights suggest valuable directions for future teaching, offering pathways to promote critical thinking and deeper engagement in geometry learning.

5A.5 Emergent features of efficacious geometric construction tasks

This section builds on the earlier discussion of how task design principles facilitated productive dialogic mathematical talk and the development of TPACK among participants. Here, I present an additional analysis focusing on the specific features of effective geometric construction tasks, as identified by the participants themselves, in the dynamic geometry software environment. This report is essential because it highlights the particular aspects of task design that further enhanced the participants' ability to develop their TPACK knowledge and their potential to teach geometric constructions effectively. I categorise these findings under the following sub-headings:

1. Complexity of construction tasks;
2. Integration of multiple geometric concepts;
3. Sequencing, progression, and varied challenges with scaffolding;
4. Relevance, alignment, and authenticity;
5. Engagement and motivation;
6. Difficulty level appropriate to learners;
7. Aligning tasks with prior knowledge; and
8. Relevance and learning continuity.

5A.5.1 The complexity of construction tasks

Participants appreciated the inductive approach to construction tasks as it enabled them to engage in the process and offered chances for investigation and learning. One Participant, R, stated, *'Despite these challenges, the inductive approach allowed me to actively participate in the construction process and provided opportunities for exploration and discovery.'*

Explorations of various constructions, such as constructing circles with the same radius and creating intersecting points, helped participants understand concepts such as constructing a perpendicular bisector. Participant M expressed, *'Through* *these explorations, I was able to understand concepts like bisecting line segments. It solidified my understanding of the topic.'*

Building their own dynamic geometric constructions for onward exploration by themselves allowed participants to understand how such models fit together. They felt that they might have been overwhelmed if they were presented with complete models and asked to explore relationships. As Participant A explained, *'By building the models ourselves, we were able to see how the model fitted together. Had we been presented with a complete model and been asked to play with it and look at relationships, I think that I would have been overwhelmed by the relationships.'*

Participants were able to understand, for instance, how diagonals interact and define distinct quadrilaterals by concentrating on the construction process rather than the exterior edges of the geometric shapes, such as quadrilaterals. Participant A further shared, *'The focus on the "skeleton" of the model - i.e., the way in which it was constructed - rather than the external edges of the quadrilateral gave me a greater appreciation of the interaction of the diagonals in quadrilaterals and how these can be used to define different quadrilaterals.'*

In addition to broadening the participants' understanding of the subject matter, construction tasks allowed them to provide more concise and innovative justifications for the underlying assumptions of some theorems related to constructions and circle geometry. Participant H stated, *'It helped me with my own subject knowledge, which I believe would help me be able to give clearer and alternate explanations as to why certain theorems within constructions and circle geometry are always true.'*

While many participants acknowledged the well-designed nature of the tasks, some expressed the necessity for an instructor (researcher) to pose the question of whether they have finished the tasks owing to the occasionally open-ended nature of the tasks, leading to uncertainty about task completion. Participant B mentioned, *'The tasks were very well designed, but I found having the instructor on hand to ask questions was essential as sometimes the tasks felt so open-ended that we were not sure if we had finished a task.'*

The tasks were perceived by participants as instrumental in fostering a deeper learning of constructions, illustrating how constructions involving full circles and lines are linked to the broader topic of constructions in the classroom. Participant J noted, *'The tasks helped me to form a better understanding of constructions by showing the way that constructions can be done with full circles and lines and how these fit into the topic of constructions in the classroom.'*

The clarity and helpfulness of the instructions provided, for creating the initial dynamic constructions for further exploration, were widely recognised, enabling participants to quickly grasping the software. A participant, G, mentioned, *'The instructions for making constructions were clear and helped me get to grips with the software quickly.'*

In summary, the participants' feedback indicates that the detailed nature of the construction tasks within the online platform's dynamic geometry software environment was beneficial for their learning. The tasks encouraged exploration, discovery, and a growing understanding of geometric constructions. Involving multiple aspects and adopting an exploratory approach, the tasks actively engaged participants in the construction process, enabling them to uncover and realise the relationships between geometric elements. The emphasis on the construction process itself, rather than solely on the final result, fostered a greater appreciation for the complexities and interactions within geometric constructions, ultimately leading to a more profound understanding.

5A.5.2 Integration of multiple geometric concepts

The feature of integrating multiple geometric concepts within the tasks emerged as an element in participants' perceptions. This feature encapsulated the essence of tasks that spanned across diverse geometric principles and their interconnections, contributing to an enriched understanding of geometry.

The participants demonstrated an awareness of the integration of multiple geometric concepts within the tasks. They applied some geometric principles such as parallel lines, equal angles, and equal lengths of line segments to fundamentally, demonstrate circle theorems. *'By recognising parallel lines, equal angles, and equal lengths of line segments, we were essentially applying circle theorems,'* emphasised one participant. Specifically, during investigations and discussions, one participant said, *'…it feels like it's kind of like the converse to the perpendicular bisector theorem. A perpendicular bisector theorem says that a perpendicular bisector of a chord passes through the centre, and this is kind of saying that we've got a line that we know passes through the centre, so it's got to perpendicularly bisect that chord, which is kind of like the converse theorem.'* Moreover, these show that they highlighted the in-depth nature of these geometric explorations and discussions, drawing parallels with established theorems, demonstrating that the tasks were conducive to the exploration of converse theorems.

The construction tasks themselves required participants to engage with various geometrical concepts, such as constructing circles with the same radius and creating intersecting points. These constructions encouraged participants to apply their existing knowledge and explore the relationships between these concepts. *'We explored various constructions, like drawing circles with the same radius and creating intersecting points,'* expressed one participant. These tasks served as a bridge, fostering connections across various areas of geometry, enhancing the participants' understanding.

Moreover, participants acknowledged the interconnectedness of constructions with broader facets of geometry, recognising the tasks as a valuable opportunity to interconnect and apply their knowledge across diverse geometric aspects. *'Constructions link to wider geometry topics (e.g., similarity, circle theorems, properties of 2D shapes, etc.) also to general ideas of mathematical thinking (e.g., proof, invariants), also links to the history of maths via Euclid,'* underscored one participant. This approach allowed participants to consolidate their understanding and see geometry as an interconnected and holistic discipline.

Participants voiced their recognition of the tasks' potential to facilitate the transfer of understanding from the digital environment to traditional tools, such as a pair of compasses and a straightedge. This transition enabled them to contemplate how the constructions could be replicated using different mediums. As one participant stated, *'The tasks prompted me to transfer my understanding back to pen and paper. This shift from the digital environment to traditional tools allowed me to reflect on how the constructions could be replicated using different* *mediums.'* This aspect added depth to their learning experiences and encouraged reflection on the versatility of their geometric knowledge.

In conclusion, participants embraced the feature of integrating multiple geometric concepts as a key to their learning journey. The tasks effectively encouraged them to establish elaborate connections across various facets of geometry, propelling a growing understanding of the subject. The participants linking principles such as circle theorems, angle facts, and properties of 2D shapes, show that the tasks provided a learning experience that transcended isolated geometric concepts. The interdisciplinary nature of geometric constructions, extending into algebraic concepts, equations, ratios, and topics like Pythagoras' theorem, further enriched participants' mathematical horizons. This holistic perspective facilitated a transfer of understanding from digital to traditional tools, amplifying the impact of the tasks on the participants' geometric cognition.

5A.5.3 Sequencing, progression, and varied challenges with scaffolding

The participants appreciated the sequencing and progression of tasks. Participants commended the tasks for their clear and logical sequencing, expressing their appreciation for the well-structured instructions. *'The instructions were well-structured, making it easy to follow along and understand what should be appearing on the screen at each stage of the construction,'* affirmed one participant. The structured questions, alongside the layout of the tasks, were lauded for their clarity and organisation. These clear directives rendered the constructions accessible to all participants while guiding their explorations.

In addition, the step-by-step construction of models and the progressive questioning were highlighted. *'The building up of the model step-by-step and the progressive questioning were both useful,'* emphasised one participant. This incremental approach facilitated the learning process and allowed participants to grasp each concept thoroughly. Collaborative learning, enabled through working in pairs, was perceived as a beneficial aspect, as it encouraged idea-sharing and provided support where individual struggles might have arisen.

The participants also found value in the concept of using knowledge gained from the tasks to describe manual constructions with a compass and a ruler. This step fostered a growing understanding and an appreciation of how to apply digital knowledge to manual techniques.

Varied levels of challenge and scaffolding were also recognised and acknowledged by participants**.** In their exploration of the tasks, participants commended the artful balance struck between guidance and independent exploration. *'They struck a good balance between providing guidance and allowing for independent exploration,'* commented one participant. This balance facilitated a conducive environment for exploration and discoveries while ensuring essential support was readily available.

The advantages of working in pairs became apparent as participants highlighted the value of collaborative learning. *'I particularly enjoyed working in pairs because we could support and help each other. If we had done it individually, we might not have discovered the same things,'* was an observation made by one participant. This collaborative approach enriched the learning experience and encouraged the discovery and development of ideas, exemplifying the tasks' efficacy.

Participants also recognised the importance of using clear mathematical language, particularly during discussions in pairs, as it enhanced their understanding of the mathematical concepts embedded in the tasks. A participant mentioned*, 'Working in pairs also highlighted the importance of using clear mathematical language.'*

In addition, participants identified the tasks as a means to foster a profound understanding and the exploration of mathematical content. The ability to manipulate vertices and witness their impact within the software environment, along with the use of specific tools, contributed to the development of this understanding. One participant mentioned, as embedded in the tasks, *'The software allows you to grab vertices and see how moving them around affects the model according to the constraints that have been created.'*

However, one participant suggested that tasks could be further improved by minimising repetition and offering more general instructions for point manipulation, thereby enabling students to discover the properties relevant to specific constructions independently.

User-friendly technology was emphasised as an essential element in the learning process. Participants highlighted the need for intuitive software, ensuring that the lesson's core focus remained on construction and exploration rather than software navigation.

Working in pairs was identified as the optimal approach for this type of task, as it encouraged discussion, idea-sharing, and hypothesis testing. *'I think that working in pairs is optimum for this type of task as it encourages discussion through sharing ideas and testing different hypotheses against one another,'* remarked a participant.

Overall, participants appreciated the systematic sequencing and progression of tasks, along with the structured guidance they provided, while also valuing collaborative learning and the practical translation of digital knowledge into manual skills, which enhanced their overall learning experience. This reflects their recognition of the tasks' ability to strike a balance between challenge and scaffolding. They highlighted the effective use of collaborative learning, clear mathematical language, and software tool exploration in enhancing the learning experience and deepening their understanding of geometric constructions. These strategies were considered valuable in accommodating participants with varying levels of expertise.

5A.5.4 The relevance, alignment, and authenticity

The relevance, alignment, and authenticity feature points out the role of geometric construction tasks that are authentic, relevant and closely aligned with foundational concepts and learning objectives. This feature has the potential to promote tasks that mirror real-world scenarios, striking a balance between accessibility and intellectual stimulation while maintaining continuity in the learning process.

Participants emphasised the significance of tasks closely aligned with real-life geometric constructions. One participant offered insights into the gradual nature of understanding fostered by these tasks, reflecting that, *'The tasks were quite helpful in getting me to think about the relationships between angles and lines. It gradually led me to understand the connection between them.'* Participant M underscored the practical skills gained through these tasks, emphasising the tangible learning outcomes, stating, *'I learnt how to construct perpendicular lines to another line and the significance of using circles or arcs to construct such lines.'*

Collaborative learning was illuminated by Participant R, who found working in pairs to be effective, emphasising the value of discussion and diverse viewpoints, noting, *'I found working in pairs helpful because it allowed for discussion and bouncing ideas off one another.'* In addition, Participant J shared a gradual appreciation for the use of technology in real-world examples, highlighting, *'At first, it didn't seem like it would help much, but the tasks did help me appreciate the use of GeoGebra and its application in real-world examples.'*

Within this overarching feature of task authenticity, a subtheme emerged, underlining the importance of tasks that reflect real-world scenarios and practical applications, as noted by Participant M who was identifying, *'the value of tasks mirroring practical situations, enabling the transfer of skills to real-life contexts.'* Moreover, this alignment with manual construction methods enhanced the authenticity of these tasks. An additional subtheme emphasised the efficacy of an inductive approach to learning, wherein participants engage in tasks with uncertain outcomes, as expressed by Participant R, *'I found the exploratory task to be quite beneficial. During our exploration, I did experience some confusion, especially in the initial task. Interestingly, this confusion actually turned out to* *be helpful, as it allowed me to gain a deeper insight into the potential confusion that students might face while working on a task.'*

The participants' feedback revealed a consensus on the importance of well-designed tasks that align with learning objectives. They emphasised clear instructions, explicit expectations, and the encouragement of independent thinking and problem-solving abilities. As one participant, G, mentioned, *'The tasks were well thought through with regard to the ordering and wording of the questions, it followed logical steps to come to the skills and information targeted.'* These well-structured tasks not only facilitate task completion but also encourage exploration and a growing understanding of essential concepts, promoting relational understanding, as expressed by participants.

Furthermore, the participants acknowledged that effective teaching involves comprehensive knowledge, encompassing content, technical, and pedagogical knowledge, to design tasks that not only align with foundational concepts but also use technology for effective learning. They recognised the value of tasks that delved into fundamental concepts, helping them to develop a profound understanding of essential geometric principles. As one participant highlighted, *'The tasks were well-designed with clear and easy-to-follow steps. What stood out to me was the presence of clear steps. As we followed these steps, we ended up with a model that we could explore further.'*

In conclusion, this comprehensive exploration of the relevance, alignment, and authenticity theme highlights the various aspects of task design that contribute to engaging and effective learning experiences. It emphasises the importance of tasks that are authentic, relevant, and well-aligned with learning objectives and foundational concepts while maintaining a delicate balance between accessibility and intellectual stimulation. These findings underscore the role of tasks that have the potential to foster real-world connections and gradual understanding, encourage students to think independently, and promote the development of deep, relational understanding.

5A.5.5 Engagement and motivation

The exploration into engagement and motivation within the context of geometric construction tasks revealed a many-sided perspective from participants, highlighting the paramount importance of designing tasks that have the potential to captivate and sustain learners' interest and motivation throughout the learning process.

Participant M's reflection explained the gradual yet impactful nature of engagement: *'The tasks were quite helpful in getting me to think about the relationships between angles and lines. It gradually led me to understand the connection between them. Having an explanation [about answers or what we are looking for] from the start would have made the tasks more meaningful, though.'* Participant M's words underlined the role of tasks as a pathway to curiosity, revealing that a fine balance between exploration and initial direction can create meaningful engagement.

Participant M also highlighted the potential of dynamic geometry software for captivating learners, *'I never thought about using GeoGebra to teach math before, but now I see its potential. It lets students explore relationships and investigate on their own. It's a great tool for visualising geometric concepts.'* This viewpoint emphasised the nature of interactive exploration, where learners are not passive recipients but active participants, driving their own learning process.

Furthermore, Participant M's perspective on collaborative learning added depth to the feature, *'I found working in pairs helpful because it allowed for discussion and bouncing ideas off one another. It leads to more learning than doing it alone. Working in groups might be too much input for this task, so using pairs is the best strategy.'* This insight echoed the sentiment that peer interaction has the potential to enhance engagement and introduces diverse viewpoints, fostering collaborative problem-solving.

The subtheme of curiosity and active engagement became evident as participants shared their perspectives. Participant R's experience revealed the transformative potential of exploratory tasks, *'In our investigation, one main thing that stood out was the importance of having a structured approach to instruction... engaging in exploratory tasks significantly enhanced my learning experience.'* This highlighted how a well-crafted task can not only impart knowledge but also inspire learners to actively explore geometric relationships.

Participant R's view on the exploratory task's benefits reinforced the significance of appropriate challenges, *'Absolutely, I found the exploratory task to be quite beneficial... This experience helped me understand the criteria for constructing the "diagram" correctly and also gave me an insight into the areas where students could potentially make mistakes.'* This recognition of potential pitfalls reiterated that tasks need to be challenging enough to provoke thought yet manageable enough to avoid frustration.

Moreover, Participant R's words reflected on the interactive nature of the dynamic geometry software, *'The tasks were well-designed with clear and easy-to-follow steps... They were structured through step-by-step instructions while also allowing room for investigation.'* This perspective highlighted the software's capacity to maintain engagement, as learners could explore geometric properties dynamically, witnessing the outcomes first-hand.

Participant L's perspective further enriched the feature, emphasising the guided learning process, *'It was good because it guided you through the content without revealing the outcome at the beginning... I think the tasks were really good because, for learning content specifically, if you don't find out what the answer is at the beginning, which we do too much, it makes it less exciting.'* Participant L's viewpoint echoed the idea that tasks that withhold outcomes initially and gradually lead learners to discoveries maintain curiosity and engagement.

Participant L also shared insights about the benefits of an inductive approach, *'We learned how to construct a perpendicular bisector of a line... I like an inductive approach a lot... it builds problem-solving skills.'* This aligned with the feature's emphasis on problem-solving and engagement, suggesting that allowing students to discover concepts on their own fosters active involvement.

In addition, Participant L's thoughts on task design highlighted the significance of clear instructions, *'So, I thought they were designed really well... it's sort of like a funnel that guides you to the content that we were trying to learn at the end.'* This perspective resonated with the importance of providing structured guidance to maintain learners' focus and interest.

Participant J's contributions were invaluable, weaving a fresh perspective into the narrative. His exploration of GeoGebra's potential and the revelation of geometric relationships through dynamic interactions highlighted the allure of active exploration, *'At first, it didn't seem like it would help much, but the tasks did help me appreciate the use of GeoGebra and its application in real-world examples.'* This perspective attested to the transformation of engagement from scepticism to appreciation, mirroring the feature's exploration.

Furthermore, Participant J's recognition of the potential of the inductive approach and its connection to problem-solving aligned seamlessly with the established sub-themes, *'I found the instructions very detailed and easy to follow. … It was enjoyable once everything came together, but it would have been more useful if I had understood what I was looking at from the beginning. I had trouble describing the relationship between the different components until the task made it clear.'* Participant J's perspective reflected the importance of gradual comprehension in maintaining engagement.

In conclusion, the engagement and motivation feature combined the voices of participants, revealing the interplay of curiosity, active engagement, collaborative learning, and the role of dynamic software. This synthesis showcased the profound impact of well-designed tasks, indicating that fostering sustained motivation is not merely about the end result but the journey itself, rich with exploration, collaboration, and interactive discovery.

5A.5.6 Difficulty level appropriate to learners

Within the domain of geometric construction tasks, the feature of balancing the difficulty level for learners emerged as another key consideration. Participants highlighted the imperative of tasks that offer challenges within the learners' grasp, fostering progress and skill development without causing frustration or disengagement.

Participant M's insights exemplified the essence of balancing challenges, *'The tasks were well-designed. The steps were clear and easy to follow, progressing from one concept to another. The sequence was effective, leading to an understanding of perpendicular lines and more.'* Participant M's words accentuated the importance of a structured progression that gradually builds complexity while ensuring participants remain engaged and invested in their learning journey.

Moreover, Participant M's experience resonated with alignment to prior knowledge, *'I learnt how to use the dynamic geometry software GeoGebra confidently. I became familiar with tools like compasses, line segments, and drawing various shapes. More importantly, how to create a slider to control the radii of circles.'* This reflection underscored the significance of incorporating familiar tools and concepts, creating a bridge that aids learners in comprehending new concepts while refining their existing skills.

Participant R's perspective further deepened the feature, *'If you learn in a rigid way, looking to find the answer from the beginning, then it poses challenges further down the road. That's my opinion on that.'* Participant R's insights delved into the intricacies of learning approaches, underscoring the importance of not just task difficulty but the process of learning itself. This echoed the sentiment that tasks should not just provide answers but foster understanding through exploration.

The view of task effectiveness and learner engagement was articulated by Participant L, *'Tasks that were too easy or too difficult wouldn't have worked... So, I think if it's an appropriate level of challenge, then the students would be interested and would want to solve it.'* Participant L's perspective validated that the sweet spot of challenge is vital for engagement. It highlighted the notion that tasks should provoke curiosity, compelling learners to explore problem-solving without feeling overwhelmed or disheartened.

The subtheme of dynamic software's role in engagement was evident in Participant L's perspective, *'The advantages of incorporating dynamic software in the mathematics classroom are evident in the time saved at the end of a lesson... The potential benefits are many when the software is used effectively, allowing for the exploration of open-ended questions.'* This viewpoint accentuated the value of dynamic tools in presenting challenges and enabling learners to experiment and explore, leading to meaningful insights.

In conclusion, the feature of difficulty level appropriateness harmonised the voices of participants, revealing the symbiotic relationship between challenge, engagement, and the learning process itself. This amalgamation stressed the intricacies of task design, where challenges tailored to learners' abilities, coupled with thoughtful progression and interactive tools, can create a rich learning experience that fosters skill development, curiosity, and active exploration.

5A.5.7 Aligning tasks with prior knowledge

The feature of aligning tasks with learners' prior knowledge and foundational concepts revealed insights into effective pedagogical strategies. Participant L's perspective on learning from the tasks highlighted the value of familiarity,*' I* *learnt how to construct a perpendicular bisector of a line, and that one was really good because you could experiment with the conditions required for that to work... it made a lot of sense.'* Participant L's experience indicated that tasks that connect with learners' existing knowledge bridge the gap between known concepts and new skills.

Participant L's thoughts on task design for teaching brought forth the practical dimension, *'I believe I will create tasks similar to these if I have well-behaved students and the time to do it is justified... If time is limited, I could demonstrate it on the interactive whiteboard and lead a class discussion about it.'* This perspective acknowledged the alignment of tasks with learners' prior knowledge and the need for effective strategies within the constraints of classroom dynamics.

Furthermore, Participant L's experience emphasised the role of proficiency in task execution, *'This proficiency allows me to follow instructions and concentrate on what's happening mathematically rather than getting caught up in figuring out how to perform basic actions, such as creating a line segment or using the compass tool.'* This insight highlighted that learners' familiarity with software tools contributes to effective task completion, enabling them to focus on mathematical concepts.

5A.5.8 Relevance and learning continuity

The integration of technology like GeoGebra into teaching geometry prompted reflections on relevance and learning continuity. Participant J's perspective on technology adoption spoke to the need for structured approaches, *'Boosted my confidence in learning to use it [GeoGebra] for teaching... should have* *continued with that approach... have a whole module on using technology like GeoGebra for teaching.'* This viewpoint underscored the potential benefits of dedicated modules at universities on the use of technology for teaching, which could enhance educators' comfort and proficiency in using technology.

Participant R's perspective on technology integration offered a balanced perspective, *'If we were learning geometry and using GeoGebra as part of that process... wouldn't be the main focus, but we'd still gain knowledge that's useful going forward.'* This recognition highlighted that technology, while not the primary focus, could be an integral part of the learning journey at the universities, contributing to broader knowledge acquisition.

In addition, Participant J's reflection on technology's role in illustrating principles reinforced its impact: *'Great for illustrating fundamental principles... encouraging students to figure out... activities involving examples and non-examples of angle bisectors.'* This insight emphasised the potential role of technology in enhancing visualisation and exploration, thereby contributing to understanding.

In summary, these findings collectively underscore the significance of clear connections to foundational concepts, the balance between accessibility and intellectual stimulation, alignment with prior knowledge, and the relevance of technology. The synthesis of participants' perspectives showcased the intricate interplay of these elements, revealing effective strategies for engaging learners and promoting meaningful learning outcomes.

5A.5.9 Criticism or shortcomings of the tasks

Participants engaged with geometric construction tasks, fostering exploration and discovery, yet some shortcomings emerged. Occasional open-endedness led to uncertainty about completion, necessitating clearer task endpoints. Despite perceived relevance, initial scepticism about practical use was noted, revealing a potential gap in learning objectives communication. While collaborative learning was valued, promoting individual exploration for independent problem-solving skills was suggested. Although tasks integrated multiple concepts effectively, minimising repetition could maintain engagement. Facilitating a seamless transition from digital to manual tools could enhance learning further. Participants praised gradual curiosity development but desired a clearer initial direction. Dynamic geometry software's role in captivating learners was acknowledged, alongside the benefit of aligning tasks with prior knowledge. Suggestions included offering dedicated technology integration modules in universities. Overall, while the tasks demonstrated effectiveness in engaging learners, addressing concerns about initial clarity, task difficulty, and technology integration could further enhance their educational impact.

5A.6 Conclusion

This chapter's findings illustrate how carefully designed tasks within dynamic geometry software on an online platform facilitated productive mathematical talk and dialogic learning, enriched by diverse dialogic talk types and guided by dialogic learning principles. The participants' interactions, marked by productive mathematical talk and dialogic learning, appeared to foster the co-construction of TPACK, emphasising the dynamic nature of collaborative learning and the vital role of dialogue and discourse in the learning process.

This potential can be realised when tasks within the learning environment are designed with principles that include scaffolding, collaborative paired learning using platforms like Microsoft Teams with screen sharing, reflective practices, dynamic manipulation with GeoGebra, instrumental orchestration, Bruner's (1974) modes of representation, a balance of ostensive and non-ostensive objects, feedback mechanisms, meaningful goals with visible mathematics, and inquiry and hypothesis testing. These insights can offer valuable considerations for future task design and instructional practices in educational technology, underlining the importance of creating tasks within spaces that promote open dialogue, collaborative exploration, and the natural emergence of dialogic learning principles.

CHAPTER 5B: EXPLORING BEGINNING TEACHERS' PERSPECTIVES ON GEOMETRY TEACHING IN TECHNOLOGY-ENHANCED LEARNING ENVIRONMENTS

5B.1 Introduction

This chapter addresses Research Question 2 (RQ2): *In what ways do beginning teachers perceive, understand and talk when exploring the knowledge needed for teaching geometry in a technology-based environment?*

To answer this question, both inductive and deductive thematic analyses were employed to extract insights from participants' interactions with dynamic geometry software (DGS). The inductive thematic analysis uncovers emergent themes, shedding light on the diverse perceptions and understandings that beginning teachers develop regarding the integration of content, pedagogy, and technology in geometry teaching. This analysis highlights the interplay between pedagogical practices and technological tools, offering a detailed perspective on the knowledge required for effective teaching in technology-enhanced contexts.

In parallel, the deductive thematic analysis uses the TPACK framework to explore how beginning teachers evolve their technological, pedagogical, and content knowledge through their experiences with DGS, particularly GeoGebra. This approach identifies key themes related to pedagogical strategies, the facilitation of conceptual understanding, and the challenges faced in integrating technology into classroom practice. Together, these analyses provide a comprehensive view of how beginning teachers talk and navigate their perception of geometry teaching in a technology-infused environment, setting the foundation for the implications discussed in later sections.

5B.2 Inductive thematic analysis: emergent themes on perceptions and understanding of geometry teaching in technology-integrated contexts

This section focuses on the results from the inductive thematic analysis, revealing emergent themes related to how beginning teachers perceive and understand the knowledge required for teaching geometry in a technology-enhanced environment. It highlights the spontaneous insights and reflections gathered from participants' experiences with dynamic geometry software, showcasing the diverse perspectives that contribute to a nuanced understanding of technology integration in mathematics teaching.

Beginning teachers' perceptions of knowledge requirements for geometry teaching in technology-focused contexts are manifold. Participants articulated essential components necessary for effective pedagogy, including sound grounding in geometric relationships, the ability to spot cross-curricular links, classroom management in computer rooms, software proficiency, and an awareness of the advantages and limitations of working with technology for differentiation.

Participant A outlined the components, which include, *'Sound grounding in geometric relationships, ability to spot cross-curricular links, classroom management in computer rooms, ability to use the software and clearly explain use to others, and the ability to understand the advantages and limitations of working with software when it comes to differentiation.'*

Participant A's delineation of essential components encompasses an understanding of technology integration in geometry teaching. Emphasising a grasp of geometric relationships, cross-curricular connections, classroom management in technology-infused environments, software proficiency, and awareness of software's capabilities and constraints for differentiation, Participant A demonstrated the density of effective technology integration in pedagogical practice. This insight highlights the importance of teachers possessing a diverse skill set to optimise instructional outcomes and adapt to evolving educational landscapes.

Participant B concurred, stating that, '*Teachers should possess knowledge of both geometry and the technology they are using.'* Participant B affirmed the necessity for teachers to be proficient in geometry and the technology they use, showing the importance of a complete skill set in technology-focused teaching contexts.

Moreover, Participant M reinforced the idea that, '*Teachers need a combination of content knowledge, teaching techniques, and technical skills'* to use '*technology effectively in teaching'*. Participant M emphasised the requirements for effective technology integration in teaching, highlighting the need for teachers to possess a blend of content knowledge, pedagogical strategies, and technical proficiency. This accentuates the requirements necessary for the successful implementation of technology in educational settings.

Participant J reiterated that, '*Teachers should be comfortable with the technology they are using, proficient in managing student behaviour during technology-based lessons',* and adept at providing '*clear instructions and asking effective questions.'*

Participant J emphasised the importance of teacher competence in technology integration, stressing proficiency in both technical skills and classroom management during technology-based lessons. In addition, clarity in instructions and effective questioning techniques were highlighted as essential components for successful technology-enabled teaching and learning experiences.

Participant R stressed the necessity of *'Teachers knowing the content, curriculum, software, and how to use it.'* Participant R emphasised the crucial requirement for teachers to possess complete knowledge of the subject content, curriculum, and the software used in teaching. This includes understanding both the functionalities of the software and how to effectively integrate it into instructional practices.

Beginning teachers' perceptions and understanding of the benefits of using dynamic software in the mathematics classroom are manifold. Participant A highlighted the power of technology in enabling students to, *'See relationships, form expectations, ask themselves questions, and systematically work through an understanding of a problem'* without needing traditional tools. Participant A elucidated the potential of technology, illuminating its capacity to facilitate students' visualisation of relationships, formulation of hypotheses, self-directed inquiry, and systematic problem-solving. This highlights the efficacy of technology in fostering independent learning and critical thinking skills, transcending the constraints of conventional instructional tools.

Participant B expanded on this, emphasising the ability of dynamic software, *'To reveal aspects of diagrams that remain invariant'*. In addition, it '*allows for rapid and accurate experimentation',* enabling teachers '*to present a variety of examples to students.'*

Participant B elaborated on the capabilities of dynamic software, accentuating its capacity to unveil invariant elements within diagrams. Moreover, it facilitates swift and precise experimentation, empowering teachers to furnish students with diverse exemplars effectively. This seems to highlight the instrumental role of dynamic software in enhancing pedagogical strategies and enriching students' learning experiences through interactive and adaptive visualisations.

Participant G pointed out the, '*Rapid responsiveness'* of dynamic software, which '*allows students to visualise the impact of small variations in geometric parameters'*. Participant G highlighted the rapid responsiveness of dynamic software, enabling students to visually understand the effects of minor alterations in geometric parameters. This functionality can enhance students' understanding by providing immediate feedback, fostering deeper engagement and facilitating exploration of mathematical concepts. Such insights underscore the pivotal role of dynamic software in promoting interactive and experiential learning environments, ultimately enhancing students' proficiency in geometry.

This dynamic exploration has the potential to foster a deep grasp of geometry. Participant M highlighted, '*The element of excitement and engagement'* that technology '*brings to the classroom*, *making mathematics more exciting and encouraging independent exploration'*.

Participant M described the value of technology in injecting excitement and fostering engagement within the classroom, particularly in the field of mathematics. Through the use of technology, teachers can create dynamic learning environments that encourage independent exploration and discovery among students. This heightened level of engagement has the potential to make mathematics more appealing and cultivate a sense of curiosity and empowerment, thereby enhancing students' overall learning experiences and outcomes.

Participant L articulated, *'The time-saving aspect of dynamic software*, *particularly when compared to manual methods involving traditional tools*.'

This observation seems to place emphasis on technology's impact on learning, streamlining tasks and enhancing productivity. By using dynamic software, teachers can optimise instructional time, allowing for focused engagement with content. This recognition of technology's capacity to expedite processes highlights its value as a tool for efficiency and effectiveness in learning. It signifies a shift towards more streamlined and productive educational practices, fostering deeper learning experiences for students. This time efficiency can allow for a more profound exploration of open-ended questions.

Participant J considered the practical advantages of dynamic software, especially in terms of *'Manipulating geometric constructions and instantly visualising dynamic effects'*.

This observation underscores the use of such software in facilitating hands-on exploration and immediate feedback, which can enhance learners' understanding of geometric concepts. By enabling users to interactively manipulate constructions and observe real-time changes, dynamic software can foster a more engaging and insightful learning experience, aligning with modern pedagogical approaches emphasising active participation and visualisation in mathematics teaching.

Furthermore, it aligns with the trend of technological integration in mathematical learning, preparing students for future educational and professional landscapes. In supporting this statement, L stated, '*Another advantage is the alignment with the trend in higher education towards technological integration in mathematical learning. As education progresses, students are increasingly expected to use technology to solve mathematical problems, making early exposure to software crucial. Introducing this software at an earlier stage helps prevent students from becoming disappointed with mathematics…'*

Participant L highlighted the alignment between dynamic software integration and the evolving landscape of higher education, emphasising the importance of technological integration in mathematical learning. This acknowledgement seems to reflect the contemporary trend towards using technology as a tool for problem-solving in mathematics, underscoring the significance of early exposure to such software. By introducing dynamic software at an earlier stage, educators can mitigate potential disappointment with mathematics among students, which can potentially foster a smoother transition towards advanced mathematical concepts.

In conclusion, in technology-focused contexts, beginning teachers perceive the knowledge necessary for teaching geometry as manifold and essential for effective pedagogy. Participants articulated insights highlighting the broad skill set required for successful technology integration in geometry teaching. They emphasised the importance of understanding geometric relationships, cross-curricular connections, software proficiency, and the advantages and limitations of technology for differentiation. In addition, they stressed the need for teachers to possess knowledge of both geometry and technology, as well as a blend of content knowledge, pedagogical strategies, and technical skills. Moreover, participants stressed the significance of teacher competence in technology use, classroom management during technology-based lessons, and effective instructional practices. Insights into the benefits of dynamic software in mathematics teaching, including its capacity to facilitate independent learning, visualise geometric concepts, reveal invariant elements, and provide immediate feedback, highlighted the pivotal role of technology in modern geometry teaching. In addition, the time-saving aspect of dynamic software aligns with the trend of technological integration in mathematical learning, which can enhance students' overall learning experiences and outcomes. Overall, these insights highlight the critical importance of technology in modern geometry lessons and the diverse skill set required for its effective integration into teaching practice.

5B.3 Deductive thematic analysis: exploring knowledge development through the TPACK framework

In this section, the deductive analysis findings are presented, using the TPACK framework to explore how beginning teachers develop their technological, pedagogical, and content knowledge. The insights gained reveal the framework's influence on their understanding of technology integration in geometry teaching.

5B.3.1 Pedagogical insights among beginning teachers

The insights gathered from participants' engagement with exploratory tasks in GeoGebra contribute to ongoing discussions on effective pedagogical strategies in geometry teaching. Many participants emphasised the innovative use of GeoGebra's tools in explaining geometric concepts, recognising its potential to facilitate conceptual connections among learners. This shift towards technology integration reflects the growing pedagogical trend of using dynamic modern tools to create interactive and engaging learning experiences.

A prominent theme was the value of hands-on learning facilitated by exploratory tasks. Participants noted that constructing dynamic geometric models (constructions) deepened their understanding of geometric relationships, encouraged critical thinking, and catered to diverse learning methods. As Participant A explained: *'By building the models [constructions] ourselves, we were able to see how the model fits together.'* This active approach promotes an appreciation of geometric concepts and reveals the connections between geometric elements, providing a deeper understanding of both the '*how*' and the '*why*' behind geometric constructions.

Participant G echoed this view, remarking on the shift from traditional tools to dynamic geometry platforms: *'Constructions are usually taught with pencils and compasses, but this approach using circles and lines highlighted the connections between them. With these insights, I can replicate this approach with my students.'* This reflection underlines the pedagogical benefits of using dynamic software like GeoGebra, suggesting that such tools can support students in exploring conceptual relationships more effectively.

Participants also emphasised the importance of differentiation in instruction. Participant A noted the need for tasks that accommodate varied student abilities: *'I would design written investigative geometric tasks for higher-attaining students to explore relationships between rhombuses, kites, squares, rectangles, while lower-attaining students have time to get through to the basic initial understanding.'* This insight reflects an approach to differentiated learning that warrants both higher- and lower-attaining students are appropriately supported and challenged.

Collaborative learning emerged as another key pedagogical strategy. Several participants emphasised the value of pair work in developing understanding. As Participant A shared: *'There were times during the exercise when I was able to discuss fragments of ideas with my partner, and vice versa.'* This mutual exchange of ideas improved participants' proficiency with GeoGebra and demonstrated the potential of collaborative learning to facilitate meaningful dialogue and shared insights.

Participant H reinforced this, stating that pair work *'encourages discussion through sharing ideas and testing different hypotheses against one another, a strategy I will consider in my teaching.'* Such reflections highlight the pedagogical benefits of collaborative work, where knowledge-sharing and hypothesis testing can stimulate critical thinking and active engagement among students.

The effectiveness of scaffolding and progressive questioning was also a recurring theme. Participant A commented on the impact of these strategies, noting: *'The step-by-step approach and progressive questioning employed in the tasks were highly effective.'* This structured guidance supported participants in acquiring new knowledge, emphasising the value of scaffolded learning experiences and clearly defined objectives in promoting conceptual understanding.

Participants also reflected on the balance between teacher-led and student-led learning. Participant R observed: *'It felt more like a teacher-led scenario as the teacher had designed the task and outlined all the steps. However, the students were the ones carrying out the task… It seemed like there was potential for insights for exploration, but the process for initial constructions itself wasn't truly inductive due to its well-defined nature.'* This reflection suggests the need to strike a balance between providing guidance and allowing room for student exploration, particularly in the context of exploratory learning tasks.

Many participants endorsed an inductive approach to learning geometric constructions. Participant G affirmed that this was *'A good way to learn geometric constructions,'* highlighting the benefits of hands-on experimentation and inquiry-based learning. Similarly, Participant A explained how GeoGebra supports this process: *'The software allows you to grab vertices and see how moving them around affects the model according to the constraints that have been created.'* This interactive engagement promotes an understanding of geometric relationships, illustrating the pedagogical value of technology-enhanced learning.

In conclusion, the pedagogical insights shared by beginning teachers highlight the efficacy of hands-on, exploratory learning in geometry education. Their reflections emphasise the importance of differentiated instruction, scaffolding, and collaborative learning in creating inclusive, engaging learning environments. Moreover, participants recognised the role of dynamic geometry software such as GeoGebra in promoting conceptual understanding and student-centred inquiry. These insights may contribute to a framework for developing effective pedagogical practices aimed at advancing meaningful learning experiences in the teaching of geometry.

5B.3.2 Content knowledge insights among beginning teachers

This section examines the content knowledge gained by beginning teachers through interactive tasks and technology integration. The exploratory tasks using dynamic geometry software (DGS) had a significant impact on developing their mathematical content knowledge, particularly in geometry.

Participant G explained how the tasks within GeoGebra supported their visualisation and understanding of geometric concepts, stating, *'Dynamic software visually links concepts and helps students understand why certain* *actions are taken. It clarifies relationships between angles, lengths, and more, making abstract ideas more tangible and memorable.'*

This reflection highlights how dynamic software aids in connecting geometric principles, but its effectiveness depends on instructional design and student diversity. Effective integration requires skilled facilitation and sound pedagogy to optimise learning outcomes.

Participants also noted the broader relevance of DGS to other mathematical concepts. Participant H shared, *'The exploratory tasks linked to circle theorems, properties of quadrilaterals, and even some triangle knowledge. It helped me understand various aspects of geometry and how different concepts interrelate.'*

This suggests that exploratory tasks can foster a growing understanding of geometry, though further research is needed to evaluate the approach's efficacy across different learner groups.

The potential for DGS to foster mathematical creativity was also recognised. Participant M noted, *'DGS can inspire mathematical creativity by allowing students to experiment and discover new geometric relationships. It encourages them to think outside the box and explore mathematical ideas in a dynamic way.'*

This highlights DGS's ability to promote creativity and critical thinking through exploratory learning, encouraging students to transcend conventional problem-solving approaches.

Participant A reflected on a shift in their understanding of quadrilaterals, stating, *'I often thought of quadrilaterals being defined by almost the external lines. I don't think I've necessarily understood until now how the construction lines, with similarities and differences.'*

This reflection demonstrates a grasp of the role of construction lines in defining geometric shapes. Participant A's insights accentuate the value of understanding geometric 'skeletons' or constructions to appreciate their internal structure.

The hands-on experimentation afforded by DGS was acknowledged by Participant A, who commented, *'The software allows you to grab vertices and see how moving them around affects the model according to the constraints that have been created.'*

This illustrates how DGS can support interactive learning to boost conceptual understanding and problem-solving skills.

In addition, Participant G remarked on the inclusivity of DGS for beginners, stating, *'Even if you are learning constructions for the first time, this way is definitely beneficial to developing a deeper understanding after a basic knowledge of constructions.'*

This reflects DGS's potential as a scaffolding tool, helping learners build on foundational knowledge to grasp more complex concepts.

Participant H further noted the inductive approach used in tasks, explaining, *'For me, the inductive approach was useful because I was able to find out truths for myself, which helped me develop my 'mathematical' approach to problem-solving.'*

This emphasises the pedagogical benefits of inductive learning, allowing learners to uncover mathematical truths autonomously, thereby increasing problem-solving skills and critical thinking.

The practical learning aspects of DGS were echoed by Participants M, J, and R, who detailed their acquisition of specific geometric construction techniques. Participant R, for example, shared, *'I learnt how to construct a line perpendicular to another line, the importance of using different circles or arcs, constructing an angle bisector of a line segment, and making a line perpendicular to a point on the line.'*

These insights reflect the participants' improved content knowledge and understanding of geometric relationships, gained through active engagement with DGS.

In conclusion, the reflections of beginning teachers on their use of DGS stress its transformative impact on content knowledge acquisition in geometry. From fostering creativity to deepening relational understanding, the dynamic tasks helped participants gain critical insights into geometric principles. The practical experimentation and inductive approaches embedded in DGS-supported tasks enriched their content knowledge, equipping them with the skills to apply these concepts in teaching geometry effectively.

5B.3.3 Unravelling technological insights among beginning teachers

This section explores the technological insights gleaned from participants' experiences, highlighting the knowledge acquired in navigating dynamic geometry software (DGS). A prominent insight is the developed comfort and proficiency with DGS, particularly GeoGebra, which participants reported as crucial for integrating technology into mathematics teaching.

Participant H remarked, *'I've gained a better understanding of using GeoGebra, especially its features related to points, lines, movement, tracing, and circles. It's* *allowed me to experiment and explore different applications, improving my technical skills.'*

This statement reflects significant growth in GeoGebra proficiency. However, further investigation is needed to understand how these technical skills translate into effective instructional strategies in secondary classrooms.

Participants recognised technology's potential to make geometry lessons more interactive and engaging. Participant M stated, *'We brainstormed ways to use the software to create interactive demonstrations and visualisations. It was exciting to see how technology could make geometry lessons more interactive and engaging for students.'*

Participant M's reflection stresses the collaborative exploration of technology's educational potential. Yet, the effectiveness of these strategies in promoting meaningful learning experiences and conceptual understanding in geometry warrants further scrutiny.

Participant R expressed, *'I think I understand better how to use the basic tools of the dynamic geometry software GeoGebra to teach geometric constructions. So, using this software to teach, you can show a lot of stuff in class. For instance, you can use it to show examples and non-examples of some concepts, like an angle bisector. I really like to use this because, something that you can't easily do on paper or on a PowerPoint or any other way than with software. It is actually quite exciting, now I know more about the use of this software.'*

Participant R's enthusiasm highlights GeoGebra's capacity to facilitate interactive learning experiences, although more research is needed to evaluate its impact on student learning outcomes.

Participants gained proficiency with DGS through hands-on tasks that boosted their familiarity and confidence. For example, Participant L noted, *'I spent time exploring different features of the software, like creating constructions and manipulating objects. It helped me become more comfortable with the software interface and its functionalities.'*

This hands-on exploration led to increased comfort and understanding of the software's functionalities.

Participant R further articulated their learning journey, *'During these activities, I learnt how to draw circles using GeoGebra's dynamic geometry software to find the perpendicular line to a given line. I also learnt to use circles and lines to create various shapes, such as a triangle or kite. These tools can be useful in teaching children how to draw perpendicular lines and understand different shapes. I learnt how to use this software to teach geometric constructions through various tasks. I liked how this software could be used to teach about constructions, such as how a kite is constructed and how its diagonals interact. It helped me understand how to use the dynamic geometry software GeoGebra to teach different mathematical concepts. Although I still need to work on using all of the features, I have the basic building blocks of thinking about how they can be used in the classroom.'*

Participant R emphasised their learning journey with GeoGebra's dynamic geometry software, focusing on drawing circles or arcs to find perpendicular lines and creating shapes like triangles and kites. They appreciated the software's use in teaching geometric constructions and understanding the properties of shapes. While acknowledging the need for further exploration, R recognised the software's potential to improve classroom teaching and expressed confidence in using its features effectively.

Participant N also highlighted their growing fluency with GeoGebra's tools*, 'I became more fluent in using GeoGebra's general tools like point tool, intersect tool, circle tool, and others through activities we just did. More importantly, how to change the colours of different components of a figure to avoid confusion.'*

Participant N reflected on their developed proficiency in using GeoGebra's fundamental tools, attained through practical engagement. In addition, they highlighted the significance of mastering colour alterations within figures to increase clarity and mitigate confusion, showcasing a detailed understanding of software functionalities for instructional efficacy.

Participant M added, *'Through the exploratory tasks, I better understood how dynamic geometric software could work and how it can be used to show non-examples. I also learnt to use GeoGebra better and feel more confident using the different tools and troubleshooting errors. … I learnt the steps to create a model, whether it is on paper or software. I gained insight into how detailed and how many steps are necessary to create a geometric object, which is useful knowledge to have.'*

Participant M's reflections demonstrated a grasp of the functionality of the dynamic geometric software, particularly in illustrating non-examples, exemplifying its versatility in pedagogical settings. Moreover, they highlighted an enhanced proficiency in GeoGebra manipulation, alongside a newfound confidence in tool utilisation and error resolution. In addition, M's acknowledgement of the process involved in model creation, both analogously and digitally, signifies an appreciation for the steps essential for geometric representation, thus developing their technological knowledge.

The participants' reflections revealed a commitment to improving their technological knowledge. Participant M stated, *'After using the software for a while, I would reflect on my experiences and think about what worked well and what could be improved. This reflection helped me identify areas where I needed to focus and learn more.'*

This iterative learning approach promotes personal growth and an understanding of effective software use.

While some participants initially expressed scepticism about GeoGebra's use in teaching, ongoing engagement shifted their perspectives. Participant R noted, *'At first, I didn't see the benefits of using GeoGebra, but I now recognise its potential. The software is user-friendly and offers various functions that traditional methods cannot.'*

This transformation reflects the value of experiential learning in overcoming initial reservations.

Participant J cautioned, *'While GeoGebra is powerful, some students might encounter technical issues or struggle with the interface.'*

This insight highlights the importance of addressing potential barriers to ensure equitable access and optimal learning experiences.

In summary, the integration of technology into mathematics teaching is a dynamic journey for beginning teachers, marked by significant advancements in their technological knowledge. As they navigate DGS, participants demonstrate enhanced proficiency, collaborative exploration, and reflective practices, all of which contribute to their growing confidence in using these tools. While challenges remain, particularly regarding usability and technical issues, participants' experiences emphasise the transformative potential of technology in improving geometry teaching. Ongoing reflection, exploration, and pedagogical adaptation are essential for maximising technology's impact in mathematics education.

5B.4 Comparison and synthesis of inductive and deductive findings

This section synthesises the findings from inductive and deductive analyses, comparing emergent themes and structured insights to form a cohesive understanding of beginning teachers' knowledge development and dialogue in the technology-based learning environment.

Inductive thematic analysis: emergent themes on geometry teaching in technology-integrated contexts

The inductive analysis revealed multiple themes regarding beginning teachers' perceptions and understanding of the knowledge required for teaching geometry in a technology-enhanced environment. Participants articulated various essential components necessary for effective pedagogy, including:

1. **Geometric relationships**: A solid grounding in geometric concepts was deemed vital for teaching effectively.
2. **Cross-curricular connections**: The ability to integrate concepts from different subjects was highlighted as beneficial for enriched learning.

3. **Classroom management in technology-infused environments**: Proficiency in managing student behaviour during technology-based lessons was considered crucial.

4. **Software proficiency**: Understanding how to effectively use dynamic geometry software (DGS) was seen as fundamental to teaching geometry.

5. **Awareness of technology limitations**: Teachers acknowledged the importance of understanding the advantages and constraints of technology to differentiate instruction effectively.

Participants also expressed appreciation for the benefits of dynamic software in encouraging engagement, enabling students to visualise relationships, and promoting independent exploration. The feedback highlighted technology's role in increasing students' critical thinking, offering rapid experimentation, and facilitating collaborative learning experiences.

Deductive thematic analysis: exploring knowledge development through the TPACK framework

The deductive analysis using the TPACK framework illustrated how beginning teachers developed their technological, pedagogical, and content knowledge. Key insights included:

1. **Pedagogical Insights**:

    i. **Innovative use of technology**: Participants highlighted GeoGebra's effective integration in teaching geometric concepts, reinforcing connections among learners.

ii. **Hands-on learning**: The construction of dynamic models was seen as instrumental in deepening understanding.

iii. **Differentiation**: Participants recognised the necessity for tailored tasks to accommodate diverse learning needs.

iv. **Collaborative learning**: Pair work was valued for developing understanding through dialogue.

v. **Scaffolding**: Clear objectives and progressive questioning were emphasised to promote conceptual understanding.

2. **Content Knowledge Insights**:

   i. **Interactive tasks**: Exploratory tasks were identified as crucial for increasing the understanding of geometric concepts.

   ii. **Connections between concepts**: Tasks revealed interrelationships within geometry, demonstrating DGS's potential for deeper insights.

   iii. **Fostering mathematical creativity**: DGS was acknowledged for encouraging experimentation and discovery in geometric relationships.

3. **Technological insights**:

   i. **Growth in comfort with DGS**: Participants reported increased proficiency and comfort with GeoGebra, facilitating its integration into teaching.

ii. **Collaborative exploration**: Engaging in brainstorming sessions highlighted the software's potential to boost geometry lessons.

iii. **Iterative learning**: Continuous reflection fostered personal growth in technological knowledge.

iv. **Addressing initial scepticism**: Initial hesitations about DGS were overcome through hands-on experience.

Synthesis of findings

Both analyses converge on the necessity of a well-rounded skill set for beginning teachers in technology-integrated geometry teaching. While the inductive analysis illuminated participants' perceptions of required knowledge, emphasising their many-sided understanding of geometry teaching and technology, the deductive analysis framed this knowledge within the TPACK model, highlighting the interconnectedness of technological, pedagogical, and content knowledge.

1. **Complementary skill sets**: Both analyses stress the critical importance of integrating content knowledge, pedagogical strategies, and technological proficiency. This integration can enable beginning teachers to optimise instructional outcomes and adapt effectively to evolving educational contexts.
2. **Collaborative and reflective practices**: The findings emphasise the role of collaborative learning and reflection as central to knowledge development. Participants recognised the value of working together, exchanging ideas, and continuously reflecting on their practices to augment their teaching effectiveness.

3. **Engagement and independence**: Both analyses reveal the potential of dynamic software to facilitate engagement and independent exploration among students. Beginning teachers perceived technology as a tool that can improve critical thinking, facilitate visualisation, and promote a more interactive and experiential learning environment.
4. **Challenges and growth**: Acknowledging potential challenges in using technology, including usability issues, participants expressed a commitment to developing their skills and addressing barriers to ensure equitable access to technology for all students.

The synthesis of findings from both the inductive and deductive analyses provides a detailed understanding of beginning teachers' knowledge development and dialogue in technology-enhanced learning environments. The insights gained underline the complex nature of knowledge required for effective geometry teaching, the role of dynamic geometry software in nurturing meaningful learning experiences, and the importance of collaborative and reflective practices in supporting teachers' growth in technology integration.

5B.5 Conclusion

The findings from this chapter elucidate the complex view of beginning teachers' perceptions, understanding, and talk regarding geometry teaching in technology-enhanced environments. Synthesising insights from both inductive and deductive thematic analyses we can see a clear convergence on the necessity for an involved skill set that encompasses technological proficiency, pedagogical strategies, and content knowledge. The inductive analysis reveals a rich needlepoint of participants' reflections on the essential components of effective geometry teaching, while the deductive analysis, framed within the TPACK model, highlights the interconnectedness of these elements in supporting meaningful learning experiences.

The implications of this study underline the importance of encouraging collaborative and reflective practices among beginning teachers, as these approaches can improve their professional growth and contribute to the creation of engaging and inclusive learning environments. The integration of dynamic geometry software emerges as a key factor in facilitating student engagement, critical thinking, and independent exploration. Furthermore, while challenges in technology use persist, beginning teachers demonstrate a commitment to overcoming these barriers through ongoing skill development and targeted professional development opportunities.

Ultimately, this chapter's findings can contribute to a growing understanding of the knowledge landscape required for effective geometry teaching in technology-rich contexts, laying the groundwork for future research and the enhancement of teacher education programmes. Highlighting the critical role of collaborative and reflective practices and the practical applications of dynamic geometry software, this study advocates for ongoing support and training to equip beginning teachers with the skills necessary for technology integration in mathematics teaching.

CHAPTER 6A: DISCUSSION OF TASK DESIGN PRINCIPLES IN FACILITATING PRODUCTIVE DIALOGIC MATHEMATICAL TALK, AND TPACK DEVELOPMENT OF BEGINNING TEACHERS CONCERNING RESEARCH QUESTION ONE (RQ1)

6A.1 Introduction

The literature review in Chapter Two highlights the global significance of integrating technology into the educational landscape. Various scholarly sources (Chander & Arora, 2021; Consoli, Désiron, & Cattaneo, 2023; Haleem, Javaid, Qadri, & Suman, 2022) have documented the emergence of technology as an important issue worldwide. In response to this, educational authorities and ministries (Department for Education, 2020; Education Scotland, 2016; European Commission, 2020; UNESCO, 2020; Welsh Government, 2018) have recommended technology's incorporation as a teaching and learning tool in schools, recognising its vast potential to augment students' mathematical proficiency, understanding, and overall learning experience (Noss et al., 2020; Radović et al., 2019). However, the literature review also uncovered the challenges teachers face when integrating technology into their classroom lessons. The hesitation among many teachers is driven by a range of factors, including limited familiarity with technology, inadequate training, time constraints, and concerns about classroom management (Clark-Wilson et al., 2014; Ertmer et al., 2012; Galindo & Newton, 2017; Mouza et al., 2023). It is crucial to note that these challenges extend beyond mere technology integration, encompassing the fundamental questions of *when* and *how* to effectively use technology. A vital issue identified during the literature review is the deficiency in skills and knowledge, particularly among beginning teachers, related to the effective integration of technology, including dynamic geometry software, into their classroom teaching (Azad, 2023; Eslit, 2023). This substantial knowledge gap paved the way for formulating two essential research questions to understand how we may address this pressing issue.

The first research question (see page 7, subsection 1.5) explored how carefully designed tasks can facilitate productive mathematical talk and dialogic learning among beginning teachers using dynamic geometry software, thereby supporting the development of their TPACK knowledge. This chapter builds on the findings from Chapter 5A, focusing on how task design supported the co-construction of knowledge among beginning teachers working with dynamic geometry software. The discussion is organised into three sections: the first examines how task design principles fostered dialogic learning and productive mathematical talk; the second explores how these principles contributed to the development of TPACK; and the third reflects on beginning teachers' perceptions of the design features in geometric construction tasks that supported their TPACK growth or development.

6A.2 Task design principles in facilitating dialogic learning and productive mathematical talk

This section focuses specifically on how the implemented task design principles facilitated productive dialogic learning and mathematical talk among beginning teachers working with dynamic geometry software. While RQ1 broadly explores how such tasks support dialogic learning and TPACK development, the emphasis here is on how the tasks promote collaborative reasoning, exploratory dialogue, and the co-construction of mathematical meaning. The integration of carefully considered design features, including scaffolding, collaborative problem-solving, reflective practices, instrumental orchestration, representational modes, open-ended structures, tangible geometric objects (ostensive) and abstract geometric relationships (non-ostensive), feedback mechanisms, meaningful goals, and visible mathematics, helped to create dialogic spaces in which rich mathematical discourse could emerge. These task design principles played a crucial role in fostering interactive environments that supported collaboration, reflection, and inquiry, thereby improving participants' engagement with geometric concepts. This aligns with Barwell's (2016) emphasis on sustained participation and open-ended dialogue, in which learners are encouraged to freely explore mathematical ideas and engage in conversations that deepen conceptual understanding. Illustrative references or examples are provided in this section to substantiate these claims, with further detailed evidence available in Chapter 5A.

6A.2.1 Scaffolding in facilitating dialogic learning and productive mathematical talk

Scaffolding emerged as a principle in structuring the tasks to support participants' incremental reasoning and articulation. Beginning teachers first engaged in constructing basic geometric shapes, gradually progressing to more complex tasks that required them to explore relationships among those shapes. This progressive structure encouraged verbalisation at each stage, boosting confidence and understanding.

For instance, Participant A affirmed Participant B’s action during a task: ‘*Perfect! Looks like we used A as the centre, which is correct since the instruction says to create a circle at A with a radius of CE. Good job! Ok, I don’t know how you managed to construct the circle at A.*’ In response, Participant B explained, ‘*The first thing I did was to select the two points that define the radius CE, and then once you’ve done that, you then click where you want it to be centred.*’ This exchange reflects how task scaffolding supported mutual understanding and joint problem-solving.

Such interactions align with Wood, Bruner, and Ross’s (1976) concept of scaffolded instruction, which emphasises the role of structured support in promoting learner independence. Additionally, the dialogue demonstrates Bruner’s (1974) modes of representation—verbal, visual, and symbolic—as participants articulated their actions while manipulating geometric constructions. Furthermore, this scaffolded progression illustrates Scott, Mortimer, and Ametller’s (2011) notion of pedagogical link-making by connecting practical constructions to underlying geometric concepts, thus facilitating a conceptual understanding.

6A.2.2 Collaborative problem-solving in fostering dialogic learning and productive mathematical talk

The emphasis on collaborative problem-solving and reflection further cultivated a culture of open dialogue among participants. Engaging in mutual discussions and reflective practices enabled them to share insights and challenge one another’s ideas. This resonates with Alexander’s (2018) principles of dialogic teaching, which stress the importance of exploratory dialogue/talk and reducing the fear of making mistakes. Participants engaged in reflective dialogue that often led to deliberative talk, as they critically examined their approaches and considered the implications of their findings (see Section 5A.3 for illustrations that back this discussion). This collaborative atmosphere could reflect Resnick, Asterhan and Clarke's (2018), concept of accountable talk, which often promotes structured dialogue that improves critical thinking. The mutual engagement promoted by collaborative problem-solving illustrates how participants could evaluate and reflect on their methods, thereby supporting Howe's (2014, 2023) and Alexander's (2018) exploration of various forms of dialogic talk. This supportive environment contributed to participants' confidence and willingness to engage deeply with complex geometric concepts.

6A.2.3 Reflective practices in supporting dialogic learning and productive mathematical talk

Integrating reflective practices prompted participants to engage in dialogue about their processes and outcomes. Reflection prompts encouraged them to connect new insights with prior knowledge, resulting in richer discussions about geometric concepts. This focus on reflection sides with Vygotsky's (1978) sociocultural theory, which emphasises the role of social interaction in cognitive development. As participants compared methods and outcomes, their dialogue could deepen their knowledge of geometric relationships. This aspect of the findings is noteworthy as it highlights how reflective dialogue led to a collaborative learning community, echoing Palmgren-Neuvonen, Korkeamäki and Littleton's (2017) emphasis on open-ended discussions and diverse perspectives in collaborative learning. The emphasis on reflection also connects to Jonassen and Rohrer-Murphy's (1999) constructivist approach, which stresses the importance of active, student-centred learning through scaffolded tasks.

6A.2.4 Instrumental orchestration in enabling dialogic learning and productive mathematical talk

Instrumental orchestration with GeoGebra supported an immediate feedback loop that enriched participants' engagement with geometry tasks. The software's real-time visual responses allowed learners to revise their constructions and thinking in situ. As Major and Warwick (2019) observed, digital technologies can improve dialogic learning by making learners' reasoning visible, an effect mirrored in this study as participants verbalised their actions, interpretations, and revisions collaboratively. This dynamic interaction facilitated productive mathematical talk, helping participants explore and consolidate their understanding of geometric properties and relationships. These findings resonate with Laborde (2002) and Patsiomitou (2018), who noted how dynamic geometry software promotes exploratory dialogue and reflective thinking.

Although traditionally applied to analyse activity in teacher-directed settings, Trouche's (2004) concept of instrumental orchestration can also be extended to thoughtfully designed learning environments. In this study, the task structure, use of GeoGebra, and collaborative pair work functioned together as orchestrational elements, enabling real-time feedback and dialogic interaction that supported meaning-making in geometry.

6A.2.5 Representational modes in enhancing dialogic learning and productive mathematical talk

Bruner's (1974) representational modes of enactive, iconic, and symbolic were embedded into the task design to support cognitive development and foster dialogic engagement around geometric ideas. These modes scaffolded the progression from *doing*, to *seeing*, to *saying*, aligning with the principles of dialogic learning and productive mathematical talk.

Enactive representation, rooted in physical manipulation, was activated through dynamic interaction with GeoGebra. In Task 2, Participant A's prompt, *'Construct a circle at point I with radius BI and change the colour. Wow, too many circles. What properties can we find here?',* initiated hands-on exploration. This embodied engagement externalised thought and opened a dialogic space for inquiry, consistent with Wood, Bruner, and Ross's (1976) scaffolding model. The absence of explicit answers encouraged exploratory talk, hypothesis testing, and sense-making through direct manipulation, reflecting Laborde's (2002) emphasis on DGS as a medium for conjecture and discovery.

Transitioning into iconic representation, participants interpreted geometric properties visually. GeoGebra's dynamic traces and colour-coded objects made abstract relationships visible, prompting participants to articulate their interpretations. Participant B's observation, *'So I can see the green and purple circles are exactly overlapping now. I think it becomes a rhombus here?',* illustrates how visual cues sparked interpretive reasoning and shared dialogue. These images became semiotic anchors for collective thinking, enabling what Sfard (2008) terms *discursive objectification*. Visual structures thus enhanced understanding and served as conversational artefacts around which collaborative reasoning unfolded.

Dialogic interactions often transitioned to symbolic representation, where participants employed formal mathematical language. For instance:

> B: *'That's because, in a square, you have perpendicularly bisecting diagonals...'*
>
> A: *'A rectangle has bisecting diagonals, doesn't it? Not perpendicular, but they are bisecting.'*

Here, participants moved from intuitive observations to formal argumentation, refining definitions collaboratively. This shift illustrates inter-thinking (Mercer, 2002), where participants jointly constructed and negotiated meaning. It also reflects the movement from empirical to deductive reasoning (Jones, 2000), a hallmark of geometric conceptual growth.

The strength of this task design lies in how it scaffolds fluid movement across representational modes. As participants transitioned from manipulating constructions to interpreting visuals and articulating properties, their talk evolved—from interrogatory ('*What properties can we find?*'), to exploratory ('*Is that true?*'), to expository ('*That's because, in a square*…'). These multimodal engagements raised what Resnick, Asterhan and Clarke's (2018) terms accountable talk, grounded, reasoned, and collaborative discourse.

Ultimately, these representational shifts enabled beginning teachers to develop mathematical fluency and practise the discursive teaching practices of explaining, questioning, reasoning, and clarifying in technology-rich environments.

This study embedded Bruner's (1974) representational modes into task design, creating powerful scaffolds for dialogic learning. The progression from enactive manipulation to iconic interpretation and symbolic reasoning supported the co-construction of geometric knowledge and dialogic competence. Such tasks helped participants learn *to think, to talk,* and potentially *to teach* geometry meaningfully in dynamic, technology-enhanced environments.

6A.2.6 Open-ended yet scaffolded tasks in promoting dialogic learning and mathematical talk

Although the tasks in this study were open-ended in nature, they were deliberately scaffolded to support productive engagement and dialogic learning. The open-endedness emerged through opportunities for participants to explore multiple strategies, construct conjectures, and justify their reasoning. At the same time, scaffolding was embedded through carefully sequenced steps, targeted use of GeoGebra tools, and structured prompts that supported exploration without prescribing a single correct approach.

Task 1 in Phase Two, for instance, exemplified this duality. Participants were guided step-by-step to construct a geometric figure involving perpendicular lines, perpendicular bisectors, and tangents to circles, using tools such as the compasses, point, and line tools in GeoGebra. These initial instructions served as a scaffold, ensuring participants could access the geometric scenario and become familiar with manipulating dynamic objects. However, once the figure was constructed, the *'Investigate'* and *'Conjecture'* sections opened the task up for critical exploration. Participants were prompted to manipulate points, observe dynamic changes, and articulate the underlying geometric relationships, including whether the constructed line was a perpendicular bisector or a tangent, and why this depended on the positioning of certain elements. The prompts asked open-ended questions like: *'What do you observe?'*, *'Why are you observing what you see?'*, and *'What happens when…?'*, encouraging participants to formulate explanations, test conjectures, and revise their understanding based on visual feedback.

This combination of structure and openness reflects Pifarré's (2019) emphasis on encouraging critical and creative thinking through dialogue, as well as Mercer and Littleton's (2007) dialogic principles, particularly the importance of sharing ideas, giving reasons, and engaging in collaborative evaluation. The task encouraged participants to collaboratively develop mathematical ideas and fostered a space for evaluative and exploratory talk, which were frequently observed in their interactions (see Section 5A.3 in Chapter 5A for illustrations of the dialogic mathematical talk types). Eventually, this task design approach, *structured enough to support learning, open enough to invite dialogue and discovery*, was instrumental in promoting productive mathematical talk and deepening participants' understanding of geometric relationships within a dynamic, technology-enhanced environment.

6A.2.7 Ostensive and non-ostensive objects in facilitating dialogic learning and mathematical talk

The integration of ostensive (visible, manipulable) and non-ostensive (abstract, conceptual) elements within the task design played a role in advancing productive mathematical talk among beginning teachers. This balance enabled participants to engage with tangible geometric representations through dynamic tools like GeoGebra while simultaneously exploring underlying mathematical properties and relationships. These dual representations acted as semiotic bridges, supporting transitions from visual exploration to conceptual generalisation, consistent with Duval's (2006) theory of semiotic registers and Brousseau's (1997) and Artigue's (2002) distinction between ostensive and non-ostensive elements.

The findings show that such designs provoked reflective, exploratory talk, participants frequently questioned, hypothesised, and negotiated meaning, embodying what Mercer (2002) and Alexander (2008) characterise as dialogic discourse. For instance, when Participants A and B explored and manipulated geometric objects to investigate relationships among different kites, including general kites, rhombuses, and squares, they transitioned from perceiving geometric shapes in static, superficial ways to reasoning about their constructional and transformational properties. This conceptual shift is articulated by Participant A, who reflected:

'I think what I have taken from this is that I often thought of quadrilaterals being defined by almost the external lines. I don't think I have necessarily understood until now how the construction lines, with similarities and differences.'

This remark reveals a movement from ostensive surface features to deeper non-ostensive geometric understanding, aligning with Radford's (2003) notion of knowledge transformation through visual and discursive interaction, and Arzarello et al.'s (2002) work on semiotic mediation in technological environments.

Moreover, Participant A further noted:

'Yes, the software allows you to grab vertices and see how moving them around affects the model according to the constraints that have been created.'

This comment stresses the role of GeoGebra as a shared cognitive artefact (Morgan et al., 2009), providing immediate visual feedback that supports experimentation, joint attention (Tomasello, 1995), and collaborative construction of meaning (Vygotsky, 1978).

Similarly, when Participants M and N explored angle bisectors using GeoGebra, their dialogue illustrated how ostensive manipulation led to non-ostensive understanding. Participant N prompted:

'Yeah, that's a perpendicular line. What happens if you move point A? All right, just move point A about there.'

Participant M responded through interaction:

'No, that's not it. It's not a perpendicular bisector.'

Through this joint manipulation, the pair interrogated the accuracy of their assumptions. Eventually, Participant N concluded:

'Make it like a right triangle ABC. OK. So, you see, it cuts the angle in half by bisecting the angle.'

This exchange illustrates how real-time feedback from the DGS encouraged hypothesis testing and promoted conceptual clarification, showcasing the exploratory talk that Mercer (1995) associates with dialogic engagement and critical thinking.

In another instance, Participants G and H collaboratively examined the transformations of quadrilaterals. Participant G noted:

‘Exactly! So, at the point where the orange and blue circles coincide, we have a square, and as we move E and F along the line segment CD, the quadrilateral remains a square until the circles change sizes again.’

This demonstrated how visual cues (ostensive) were linked to abstract geometric constraints (non-ostensive), enabling precise reasoning. Participant H summarised the conceptual insight gained:

‘So, to summarise, we have observed that the quadrilateral changes between a kite and a square depending on the size of the circles. When all the circles coincide, it becomes a square, and when they are different sizes, it’s a general kite.’

This reflection shows how DGS-enabled manipulation facilitates conceptual generalisation, a shift from recognising isolated shapes to understanding families of geometric figures based on invariant properties, supporting Trocki and Hollebrands’ (2018) assertion that dynamic environments facilitate abstraction through guided interaction.

These examples also exemplify instrumental genesis (Rabardel, 1995), where tools like GeoGebra transition from instruments of manipulation to instruments of mathematical reasoning. As participants used the software to visualise transformations, constraints, and symmetries, they co-constructed knowledge, contributing to their TPACK development (Mishra & Koehler, 2006; Niess, 2011).

In sum, the coordinated use of ostensive and non-ostensive elements within carefully designed DGS tasks scaffolded mathematical understanding and cultivated rich dialogic spaces for meaning making. This highlights the value of semiotically diverse, interactive, and collaborative environments in beginning teachers' mathematical and pedagogical development. As such, the findings reinforce Sinclair's (2013) idea of polyphonic spaces—where learners co-narrate, test, and revise their understandings in dynamic, technology-enhanced settings.

6A.2.8 Feedback mechanisms supporting dialogic learning and productive mathematical talk

The task design included feedback mechanisms central to promoting collaborative dialogue and increasing mathematical understanding. Feedback from the dynamic geometry software (GeoGebra) provided immediate visual cues, which helped participants correct misconceptions and refine their geometric reasoning in real-time. This aligns with Laborde's (2002) work on how DGS encourages real-time exploration, allowing users to modify constructions based on instant visual outcomes.

For example, in Task 3a, when *Participant M* initially mistook an angle bisector for a perpendicular bisector, the immediate visual feedback from GeoGebra allowed them to adjust their construction and correct their error, strengthening their understanding. Similarly, peer feedback played an essential role in refining participants' ideas, exemplified in Task 3b, where the collaborative exploration of examples and non-examples of an angle bisector led to deeper insights, further supported by GeoGebra's feedback. This two-way feedback loop, between software and peers, mirrors Vygotsky's (1978) Zone of Proximal Development (ZPD), where learners benefit from social interaction in understanding and refining concepts.

The feedback mechanisms used in this study also align with the theories of semiotic mediation (Patsiomitou, 2018) and instrumental orchestration (Trouche, 2004), where feedback loops from both GeoGebra and peer interactions supported the development of mathematical concepts through dialogic learning.

6A.2.9 Meaningful goals and visible mathematics facilitating dialogic learning

Meaningful, goal-oriented tasks were central to the task design, promoting engagement with visible mathematics in concrete geometric constructions and abstract relationships. Focusing on tasks with clear goals, including distinguishing between correct and incorrect angle bisectors, participants were encouraged to deepen their understanding through experimentation and dialogue. In Task 3b, participants used GeoGebra's real-time feedback to adjust their constructions, refining their hypotheses and facilitating more focused discussions. This iterative process of trial and error, supported by software feedback, resonates with Clark-Wilson and Timotheus' (2013) assertion that tasks should allow learners to interact with mathematical concepts in visible, meaningful ways, which aligns with Hershkowitz's (2020) emphasis on '*thinking in pictures.*'

The principle of visible mathematics, grounded in Bruner's (1974) modes of representation, enabled participants to manipulate geometric objects, making abstract concepts tangible. GeoGebra's interactive features allowed participants to test and explore geometric relationships, supporting the findings of Laborde (2002) and Leung & Lee (2013), who emphasise the importance of making abstract geometric properties visible to facilitate mathematical dialogue.

In Task 3, for example, participants explored the relationships between kites, rhombuses, and squares. The visible outcomes of their constructions led to richer dialogue. Participant A noted, '*If AI and IB were the same length, we expect the two circles to overlap, so the purple and green should overlap, it's a rhombus.*' This observation illustrates how meaningful goals and visible mathematics, as advocated by Clark-Wilson and Timotheus (2013), promote productive mathematical exchanges, enabling learners to engage deeply with both the content and each other

In conclusion, the task design principles explored in this section demonstrate how carefully crafted tasks within a dynamic geometry software environment promote productive dialogic learning and mathematical talk among beginning teachers. Integrating scaffolding, collaborative problem-solving, reflective practices, and instrumental orchestration, alongside representational modes, open-ended structures, feedback mechanisms, and meaningful goals, these tasks facilitated rich, interactive dialogue that deepened participants' engagement with geometric concepts. Involving ostensive (tangible geometric objects) and non-ostensive (abstract geometric relationships) elements further boosted conceptual understanding, allowing teachers to co-construct mathematical knowledge collaboratively. The emergence of various types of dialogic talk, including transactional, exploratory, and evaluative, highlighted how thoughtfully designed tasks developed a collaborative environment conducive to meaningful mathematical discourse. Eventually, these principles supported the exploration of geometric concepts and the articulation of reasoning and reflection, critical elements in developing pedagogical insights among beginning teachers. The study offers insights into the effectiveness of task design principles in promoting dialogic learning through dynamic geometry software, thus contributing to the broader discourse on effective teaching practices in technology-enhanced mathematics education.

6A.3 Facilitating the development of TPACK knowledge through task design principles

This section explores how the design principles embedded in the geometric construction tasks contributed to the development of the participants' TPACK knowledge. The following three subheadings focus on the key TPACK components that emerged from the participants' engagement with the tasks. For illustrative examples of participants' excerpts supporting this discussion, refer to Section 5A.4.

6A.3.1 Facilitating the development of technological knowledge through task design principles

The findings from this study indicate that the deliberate application of task design principles significantly facilitated the development of participants' technological knowledge within a structured, collaborative, and reflective learning environment. Through sustained engagement with GeoGebra, participants were able to improve both their technical proficiency and their understanding of how technology can be meaningfully integrated into future teaching practices.

This supports the assertions of Laborde (2002) and Sinclair (2003, 2013), who emphasise the capacity of dynamic geometry software to enable interactive and visual exploration of mathematical concepts, an essential affordance for effective geometry instruction. The current study extends these insights by demonstrating how participants' exploration of GeoGebra enabled them to make sense of complex geometric relationships and to recognise the pedagogical value of DGS in promoting mathematical inquiry.

Participants' growing proficiency with GeoGebra aligns with the Dynamic Geometry Task Analysis (DGTA) framework proposed by Trocki and Hollebrands (2018), which advocates for task design that balances cognitive demands with technological actions. As participants moved from basic constructions to more complex explorations, their trajectory exemplified the kind of progression envisioned by the DGTA model, reinforcing the importance of task sequencing in technology-enhanced mathematics learning.

Participants' ability to articulate the connections between their actions and geometric properties, including manipulating points and lines, provided further evidence of how hands-on interaction supported conceptual understanding. This reflects Hohenwarter and Preiner's (2007) findings on the importance of real-time manipulation in constructing mathematical knowledge.

Moreover, the integration of digital and traditional tools represented a significant pedagogical development. Participants transferred insights gained through GeoGebra to manual constructions, developing precision and conceptual clarity.

This reflects Trouche's (2004) Instrumental Orchestration framework, which highlights the pedagogical potential of coordinating digital tools with teaching strategies to enrich mathematical understanding.

The task design also incorporated principles of scaffolding, allowing participants to progress incrementally from foundational constructions to more complex geometric relationships. This echoes the classic scaffolding model proposed by Wood, Bruner and Ross (1976), and is supported by recent research (Kartal and Çınar, 2022; Naidoo and Govender, 2019) highlighting the importance of structured support in building confidence and technical competence with DGS.

The structured tasks further allowed for real-time feedback and iterative learning, which improved participants' ability to refine their understanding through experimentation and validation. These findings resonate with Trouche's (2004) emphasis on immediacy in feedback and with research by Harris and Hofer (2011) and Voogt et al. (2013), who highlight the role of formative feedback in promoting active learning in technology-rich environments.

Collaboration also played a role in participants' learning. The tasks encouraged peer dialogue and knowledge sharing, contributing to a community of practice in which participants co-constructed understanding. This aligns with sociocultural perspectives on learning, particularly those of Vygotsky (1978) and Bauersfeld (2012), who emphasise the centrality of social interaction in cognitive development. Mercer and Littleton's (2007) work further stresses the importance of exploratory talk in advancing learning outcomes.

As participants progressed, they increasingly articulated the value of using DGS to make abstract geometric concepts more accessible and tangible, an insight that reflects an important shift in their developing technological pedagogical knowledge. These findings support Radović, Marić and Passey's (2019) and Noss et al.'s (2020) arguments for extending teacher knowledge to include technological fluency as a core component of effective instructional practice.

In summary, the structured and collaborative design of the tasks enabled participants to develop their technological knowledge through scaffolded exploration, real-time feedback, and peer dialogue. These experiences improved their competence in using GeoGebra and contributed to their broader TPACK development, equipping them with the skills and confidence to integrate technology into mathematics teaching. The findings offer insights into how targeted task design can bridge the gap between theoretical knowledge and practical application in technology-centred learning.

6A.3.2 Facilitating the development of geometric content knowledge through task design principles

This study's structured task design principles supported participants' development of geometric content knowledge by enabling deeper engagement with fundamental concepts and relationships. This reflects Shulman's (1986, 1987) emphasis on the integration of content and pedagogy and aligns with Ball, Thames and Phelps's (2008) articulation of content knowledge for teaching, which stresses conceptual understanding over procedural recall.

Participants engaged in constructing and analysing geometric figures, including perpendicular lines, perpendicular bisectors and quadrilaterals, within scaffolded tasks, allowing them to uncover relationships and generalise properties. These activities, enriched through hands-on interaction and guided inquiry, encouraged movement towards higher-order reasoning. Dialogue around the classification and properties of kites, rhombuses, and squares exemplified this progression and echoed Clark-Wilson et al.'s (2014) findings on how dynamic geometry environments encourage collaborative problem-solving and meaningful mathematical discussion.

GeoGebra facilitated the dynamic exploration of geometric transformations, enabling participants to manipulate and reflect upon configurations involving reflections and rotations. These experiences strengthened the connection between theoretical knowledge and practical application, expanding upon earlier work by Hohenwarter and Preiner (2007) and supporting Brousseau's (1997) Theory of Didactical Situations, which advocates for inquiry-driven, student-centred learning. Participants' ability to articulate the implications of different transformations demonstrated growing content fluency and spatial reasoning.

The exploration of examples and non-examples further deepened conceptual understanding by prompting critical reflection on the necessary conditions for geometric properties to hold. This investigative approach encouraged analytical thinking and aligned with Trocki and Hollebrands' (2018) advocacy for maintaining cognitive demand in technology-centred tasks. The ability to construct, deconstruct, and reason through geometric relationships echoes Granberg and Olsson's (2015) emphasis on creative, collaborative reasoning within DGS environments.

Visualisation through GeoGebra played a role in mediating abstract concepts, including congruence and symmetry. The software's dynamic features allowed participants to bridge theoretical properties with visual representations, supporting Laborde's (2001) argument that DGS promotes active conceptual construction. Importantly, reflective dialogue arising from these visualisations revealed detailed understandings not commonly reported in existing studies, highlighting the underexplored value of dialogue in reinforcing geometric content knowledge.

Additionally, automatically embedded feedback mechanisms within GeoGebra provided immediate insight, facilitating iterative learning. Combined with scaffolded progression and opportunities for reflection, the tasks supported participants' growing confidence and precision, reinforcing Bruner's (1974) and Wood, Bruner and Ross's (1976) models of guided discovery and support. For instance, participant discussions about angle bisectors clarified key conceptual distinctions and reinforced the application of geometric reasoning in context. These findings align with Naidoo and Govender's (2019) work on the dual benefits of technology in supporting both competence and confidence among beginning teachers.

The gradual transition from practical manipulation to abstract generalisation marked a critical shift in participants' engagement with geometry. This synthesis of experience and theory enriched their geometric content knowledge and highlighted the importance of TPACK-aligned training (Agyei and Voogt, 2016). Eventually, this study contributes to the literature by illustrating how task design, when grounded in scaffolded inquiry, dynamic technology, and collaborative dialogue, can effectively improve beginning teachers' geometric knowledge and prepare them for teaching in technology-rich classrooms.

6A.3.3 Facilitating the development of pedagogical strategies and ideas through task design principles

The development of participants' pedagogical knowledge in this study was shaped by their engagement with carefully designed, exploratory tasks using dynamic geometry software, especially GeoGebra. These tasks provided opportunities for participants to reflect on how learners might experience and make sense of geometric ideas. Importantly, even in the absence of direct teaching, the participants began to articulate and internalise pedagogical strategies grounded in conceptual understanding, diagnostic reasoning, differentiation, and collaborative inquiry, all key indicators of pedagogical growth.

A prominent theme in the findings was the shift from procedural to conceptual understanding in geometry teaching. Participants such as G and H emphasised how GeoGebra allowed them, and potentially students, to grasp *why* constructions like perpendicular bisectors work, not just *how* to execute them. This transition mirrors what Hohenwarter and Preiner (2007) describe as a core affordance of DGS, the visualisation and manipulation of mathematical properties, which enables learners to build strong conceptual foundations. The realisation by G and H that students could observe the equality of radii or the necessity of equal intersecting arcs without drawing entire circles exemplifies the kind of conceptual engagement central to meaningful mathematics learning (Granberg & Olsson, 2015).

These conceptual engagements were not limited to static understandings but extended to dynamic, corrective learning. For instance, A and B's discovery that a perpendicular line was not always a bisector, revealed through dragging and manipulation in GeoGebra, reflects the type of self-corrective learning process described in Brousseau's (1997) *Theory of Didactical Situations*. Tasks became didactical tools through which participants could simulate student misconceptions, explore alternative outcomes, and, in doing so, develop formative, adaptive pedagogical strategies.

Collaboration played another role in amplifying these insights. Many participants explicitly credited peer discussion for helping them make sense of complex constructions and uncover conceptual insights. This resonates with Vygotsky's (1987) sociocultural theory and Alexander's (2020) conception of dialogic teaching, both of which stress the centrality of talk in learning and teaching. The 'mini-investigation' essence of tasks, as referenced by Participants A and B, facilitated joint problem solving and mirrored the kinds of dialogic learning environments participants aspired to replicate in their future classrooms.

Participants also came to value scaffolding as a deliberate instructional tool. Their comments around breaking down tasks and prompting critical questions such as '*Is this always true?*' show a growing awareness of how task design can support cognitive load management (Sweller, 2023) and promote generalisation. For example, Participant A's reflection on step-by-step instructions suggests a sensitivity to sequencing and learner autonomy. This pedagogical shift from mere execution to orchestration of learning experiences is echoed in Laborde's (2001) assertion that DGS tasks, when scaffolded effectively, help learners internalise mathematical structures through progressive abstraction.

Further, participants began to express pedagogical intentionality around differentiation. They moved beyond abstract notions of learner variability to consider concrete strategies, including using '*always, sometimes, never*' prompts or layering tool use (for example, straightedge and GeoGebra), to accommodate diverse needs. This adaptive thinking aligns with Clark-Wilson and Hoyles' (2019) findings that DGS environments afford customisation and responsive teaching pathways, allowing beginning teachers to simulate and plan for differentiated teaching. For instance, Participant L's endorsement of inductive learning reflects a pedagogical preference for open-ended inquiry, a powerful strategy for engaging a broad range of learners in critical thinking (Dabbagh & Kitsantas, 2012).

Another pedagogical move supported by task design was the exploration of examples and non-examples. Tasks that allowed constructions to fail or behave differently under certain conditions enabled participants to surface and interrogate the boundaries of geometric validity. This practice of working through contrasts and '*broken*' configurations echoes Sfard's (2008) *commognitive framework*, which posits that learning occurs when students confront and revise their discourse about mathematical ideas. Participants' desire to replicate these experiences in their teaching suggests a commitment to promoting metacognitive awareness in their students, a sophisticated pedagogical aim.

The blending of traditional and digital tools, such as using a straightedge alongside GeoGebra, reflected an emerging pedagogical flexibility. Participants recognised that this dual-modality approach deepens conceptual understanding and caters to different learner preferences. As highlighted in Bozkurt and Koyunkaya (2022), such blending facilitates the development of both procedural fluency and conceptual reasoning. Participants' comments about how using GeoGebra first confirmed conceptual understanding, while physical tools later reinforced reasoning, suggest an appreciation for the complementarity of representations in geometry teaching.

In sum, participants' development of pedagogical knowledge was significantly mediated by the design of the tasks they engaged with. These tasks did more than teach geometry; they *modelled pedagogy*. Through these structured, exploratory, and collaborative engagements, participants internalised strategies including:

1. Prioritising conceptual understanding over procedures,
2. Using visual and interactive representations to analyse and correct misconceptions,
3. Employing scaffolding to guide reasoning and promote autonomy,
4. Designing for dialogue and collaboration to enrich understanding,
5. Facilitating differentiation through layered supports and task variation,
6. Encouraging critical thinking via open-ended, inductive prompts, and
7. Fostering metacognition through the examination of both examples and non-example cases.

Together, these strategies point to a growing *pedagogical intentionality* among participants, a shift from doing mathematics to *teaching* it thoughtfully. These findings reinforce the literature's claim (for example, Clark-Wilson et al., 2014; Resnick et al., 2018) that task-based professional learning using DGS develops mathematical insight and acts as a fertile ground for developing pedagogical expertise.

6A. 4 Beginning teachers' perceptions of design features in geometric construction tasks that supported their TPACK development

Building on the findings in Section 5A.5 of Chapter 5A, this discussion explores beginning teachers' perceptions regarding the design features of geometric construction tasks that they believed aided their development of TPACK knowledge. The findings reveal how thoughtfully structured tasks can improve understanding and facilitate effective teaching practices in technology-centred learning environments. Participants highlighted several key design features as instrumental in their TPACK development, including task complexity, integration of multiple geometric concepts, task sequencing and progression, relevance and authenticity, collaboration, balance between exploration and independent learning, user-friendliness of technology, engagement, motivation, and alignment with prior knowledge. Supporting illustrations are provided through participants' excerpts in Section 5A.5.

One prominent finding was the value participants placed on task complexity, structured to promote engagement and exploration. The gradual increase in task difficulty encouraged active learning, echoing the hands-on interaction emphasised in dynamic geometry software contexts, as noted by Laborde (2001). This approach sides with constructivist theories by Vygotsky (1978), where learners actively construct knowledge through exploration. Moreover, participants' experiences resonated with Shulman's (1987) concept of Pedagogical Content Knowledge (PCK), as they developed deeper content knowledge alongside pedagogical insights. The need for scaffolding to balance autonomy and guidance reflects the principles of scaffolded instruction proposed by Wood, Bruner and Ross (1976), ensuring tasks challenge learners and provide the necessary support. The TPACK framework by Mishra and Koehler (2006) further highlights how well-designed tasks that integrate technological, pedagogical, and content knowledge can enhance teaching efficacy in geometry.

The integration of multiple geometric concepts within tasks influenced a perception of geometry as a cohesive discipline among participants. This approach aligns with Niess *et al*. (2009), who stress the importance of connecting DGS with curricula to increase teachers' content knowledge. Tasks bridged digital and traditional tools, promoting technological proficiency through GeoGebra exploration, thereby supporting the TPACK framework (Mishra and Koehler, 2006). The careful sequencing and scaffolding of tasks reflected Clark-Wilson and Timotheus's (2013) emphasis on meaningful, goal-oriented tasks that have the potential to enhance pedagogical knowledge.

Participants also emphasised the importance of sequencing and progression in task design, which allowed for a gradual mastery of complex geometric concepts. This finding is consistent with Trocki and Hollebrands' (2018) categorisation of tasks by cognitive demand, confirming that learners engage step-by-step, reinforcing content understanding. Scaffolding provided necessary support while encouraging independent exploration, supporting Clark-Wilson and Timotheus (2013), who advocated for tasks with clear goals and visible mathematics. Furthermore, Niess (2008) highlighted the role of technology in improving PCK, while Vygotsky (1978) emphasised the importance of social interaction in learning, reinforcing the significance of collaborative problem-solving.

The relevance and authenticity of tasks were critical in bridging abstract geometric concepts with practical applications, developing participants' content knowledge and pedagogical content knowledge. As noted by Clark-Wilson and Timotheus (2013), tasks with meaningful goals and visible mathematics can facilitate better understanding, aligning with participants' perceptions. Authentic tasks that mirror real-world scenarios supported the integration of technological pedagogical knowledge, illustrating how digital and manual tools can complement each other. This perspective resonates with Laborde's (2001) emphasis on DGS for hands-on learning and Vygotsky's (1978) sociocultural theory, which highlights the role of context-based learning.

Collaboration and communication emerged as essential design features that supported the development of pedagogical knowledge in technology-enhanced geometry learning. Vygotsky's (1978) sociocultural theory highlights the importance of social interaction in learning, mirroring participants' experiences of working in pairs to foster peer-to-peer learning, exchange ideas, and facilitate clearer mathematical communication. This echoes the findings of Granberg and Olsson (2015) and Geraniou *et al*. (2023), who emphasise collaborative reasoning in DGS environments. Peer collaboration allowed participants to test hypotheses, refine their understanding of geometric concepts, and build confidence in using technology, reinforcing their content knowledge and pedagogical strategies (Mishra and Koehler, 2006). Moreover, the role of productive mathematical talk resonates with Mercer and Littleton's (2007) emphasis on dialogue, facilitating exploratory problem-solving and supporting broader TPACK development (Clark-Wilson and Timotheus, 2013).

Participants valued a balance between guided instruction and opportunities for independent exploration, which supported their critical thinking and problem-solving skills. This perspective resonates with constructivist theories articulated by Dabbagh and Kitsantas (2012), who stress the importance of active learning in promoting meaningful engagement. Independent use of GeoGebra allowed participants to develop their content knowledge and technological knowledge, contributing to overall TPACK development (Mishra and Koehler, 2006). The tasks' flexibility, accommodating various skill levels, links with Laborde's (2001) emphasis on DGS as a tool for hands-on learning, improving conceptual understanding. Kuzle (2017) further highlights DGS as a cognitive tool that promotes creative problem-solving, encouraging independent learning and conceptual insights in technology-centred geometry teaching.

The user-friendly nature of GeoGebra, as highlighted by participants, aligns with research on technology integration that emphasises reducing the cognitive load to develop PCK. Mishra and Koehler (2006) and Koehler and Mishra (2008) indicate that ease of technology use can facilitate the development of TPACK. Harris and Hofer (2011) stress the importance of technology use in improving learning outcomes. In addition, the balance between task difficulty and accessibility reflects Vygotsky's (1978) concept of the zone of proximal development (ZPD), where appropriate scaffolding supports learning progression. Clark-Wilson and Timotheus (2013) advocate for tasks featuring meaningful goals and visible mathematics, reinforcing how thoughtful task design and sequencing foster PCK and engagement in technology-enhanced learning.

Engagement and motivation emerged as another aspect of task design, consistent with Radović, Marić and Passey's (2019) findings, which emphasise technology's role in deepening mathematical understanding. Engaging tasks stimulate curiosity and active participation, enhancing TPACK development (Mishra and Koehler, 2006). These tasks promote critical thinking and independent learning, essential for future classroom practice. Balancing task difficulty, as noted by participants, is crucial for maintaining motivation, reflecting the cognitive demands discussed by Trocki and Hollebrands (2018) in DGS environments. This balance sides with the need for scaffolded instruction (Wood, Bruner and Ross, 1976) and Vygotsky's (1978) ZPD, where learners thrive on appropriately challenging tasks. These insights contribute to PCK (Shulman, 1987) by demonstrating how dynamic tasks can facilitate progression and engagement while accommodating diverse learning needs (Clark-Wilson and Timotheus, 2013).

A critical finding of this study is that tasks aligned with participants' prior knowledge facilitated smoother learning experiences. This supports Niess (2008), who advocates for integrating technology into PCK, enabling teachers to build lessons that connect students' existing knowledge with advanced geometric concepts. Mishra and Koehler's (2006) TPACK framework highlights the synergy between technological, pedagogical, and content knowledge, crucial for effective lesson design. Laborde (2001) reinforces the value of DGS in promoting hands-on learning, while the socio-cultural theory of Vygotsky (1978) emphasises social interaction's role in knowledge construction, reflected in collaborative learning experiences. The principles of scaffolded instruction proposed by Wood, Bruner and Ross (1976) resonate with participants' insights, emphasising sequencing and support in task design. These findings align with contemporary research advocating for tasks with meaningful goals (Clark-Wilson and Timotheus, 2013) and the constructive use of DGS to deepen understanding (Radović, Marić and Passey, 2019), highlighting the interplay of prior knowledge, technology, and pedagogical strategies in enhancing TPACK development.

The continuity and relevance of task design emerged as pivotal for enhancing participants' skills in using GeoGebra, fostering both technological knowledge (TK) and technological pedagogical knowledge (TPK). Continuous exposure to DGS enabled participants to visualise and manipulate geometric relationships effectively, reinforcing the practical application of technology in classroom instruction. This aligns with Mishra and Koehler's (2006) TPACK framework, which underscores the integration of technological, pedagogical, and content knowledge essential for effective teaching. Laborde (2001) supports this by highlighting how DGS facilitates hands-on manipulation of geometric figures, enhancing understanding.

Furthermore, the relevance of tasks to real-world scenarios was critical in solidifying learning continuity, as participants recognised the significance of authentic, meaningful tasks, echoing Clark-Wilson and Timotheus (2013).

Shulman's (1987) PCK concept reinforces the importance of contextually relevant task design in developing TPACK. The principles of scaffolded instruction proposed by Wood, Bruner and Ross (1976) resonate with participants' insights, advocating for a structured approach to task sequencing and support. Ongoing engagement with GeoGebra not only deepened participants' understanding of geometric concepts but also enhanced their pedagogical strategies, reflecting the collaborative learning aspects emphasised by Granberg and Olsson (2015) and Dabbagh and Kitsantas (2012). Ultimately, this comprehensive approach to task design, grounded in the principles of sociocultural learning (Vygotsky, 1978), enables beginning teachers to develop the competencies necessary for effective geometry teaching in technology-enhanced environments.

6A. 5 Conclusion

This chapter explores task design principles that facilitate productive dialogic mathematical talk and the TPACK development of beginning teachers, focusing on their perceptions of design features in geometric construction tasks. It highlights key principles including scaffolding, collaboration, reflection, and instrumental orchestration, which engage beginning teachers using dynamic geometry software (GeoGebra). These principles enable meaningful dialogue and mathematical discussion through incrementally designed tasks that encourage reasoning and critical reflection. The open-ended nature of the tasks promotes diverse approaches and exploration of geometric concepts, improving participants' understanding and encouraging productive classroom dialogue.

The study demonstrates how these task design principles support the development of TPACK knowledge, particularly technological skills with GeoGebra. Structured tasks and real-time feedback allow participants to progressively improve their proficiency, transitioning from basic to complex geometric concepts. Integrating digital tools with traditional construction methods allows participants to explore geometric relationships in innovative ways, reinforcing technology's role in enhancing mathematical understanding and preparing teachers for diverse educational settings.

In addition to developing technological proficiency, the task design improves geometric content knowledge. Participants engage in critical discussions about geometric relationships, manipulate objects, and explore transformations like reflections and rotations, deepening their conceptual understanding. Tasks involving examples (edge cases) and non-examples encourage reflective problem-solving, while the iterative design, coupled with real-time feedback, allows participants to refine their geometric reasoning.

Finally, these task design principles significantly boost participants' pedagogical strategies by promoting collaborative learning, differentiation, and scaffolding. The study highlights how dynamic tools like GeoGebra help engage students, foster critical thinking, and create interactive learning environments. Dialogic learning and reflective practices are vital for developing exploratory learning and problem-solving strategies, aligning with Vygotsky's social constructivism. Overall, the study provides valuable insights into how carefully structured tasks can support beginning teachers' development in technology-enhanced geometry education, emphasising the importance of thoughtfully designed tasks in enhancing their technological, pedagogical, and content knowledge.

CHAPTER 6B: DISCUSSION OF BEGINNING TEACHERS' PERSPECTIVES AND KNOWLEDGE DEVELOPMENT ON GEOMETRY TEACHING IN TECHNOLOGY-ENHANCED ENVIRONMENTS

6B.1 Introduction

This chapter discusses the findings related to Research Question 2 (RQ2): *In what ways do beginning teachers perceive, understand, and talk when exploring the knowledge needed for teaching geometry in a technology-based environment?* The discussion synthesises insights from both inductive and deductive analyses to engage critically with existing literature and address the study's broader aims. The inductive analysis revealed diverse perceptions regarding the integration of content, pedagogy, and technology in geometry teaching. In contrast, the deductive analysis applied the Technological Pedagogical Content Knowledge (TPACK) framework to examine how beginning teachers developed their knowledge through experiences with dynamic geometry software (DGS), particularly GeoGebra. This chapter highlights essential knowledge for effective geometry teaching, the role of DGS in developing that knowledge, and implications for teacher education and professional development. In doing so, this chapter affirms and extends existing research on beginning teachers' engagement with technology in mathematics education.

6B.2 Synthesis of knowledge development in geometry teaching using technology

This section synthesises findings regarding beginning teachers' perspectives on the knowledge required to teach geometry effectively in technology-centred environments. It draws on inductive and deductive thematic analyses to examine how teachers perceive, understand, and articulate the integration of content, pedagogical, and technological knowledge. The discussion explores their developing awareness of knowledge requirements for teaching geometry with technology, the pedagogical strategies they found effective, and their experiences with dynamic geometry software. It also considers how teachers approached fostering student engagement, promoting critical thinking, and encouraging independent exploration through the use of technology-based tasks and tools. Supporting examples and participant excerpts can be found in Chapter 5B.

6B.2.1 Beginning teachers' perceptions of knowledge requirements for teaching geometry with technology

The inductive thematic analysis revealed that beginning teachers perceive the knowledge required for effective geometry teaching in technology-integrated environments as multidimensional and complex. Their reflections emphasise the importance of content, pedagogical, and technological knowledge and the interplay between these domains, reinforcing the relevance of the TPACK framework (Mishra & Koehler, 2006; Harris, Mishra & Koehler, 2009). This emphasis on the need for integrated knowledge development is consistent with Agyei and Voogt's (2016) findings, which highlight how pre-service or beginning teachers’ TPACK growth through technology-enhanced tasks supports their capacity to design and implement effective, technology-rich instruction. While their perspectives affirm existing literature, they also contribute new insights that deepen our understanding of early-career teachers' knowledge development in dynamic learning contexts.

A key theme emerging from participants’ reflections was the need for deep *content knowledge*, especially an understanding of geometric properties, relationships, and the ability to generalise patterns emerging from dynamic manipulation. They emphasised the importance of relational thinking, recognising interdependencies among properties and figures, which is foundational for facilitating meaningful learning. This aligns with Ball, Thames, and Phelps’s (2008) concept of specialised content knowledge necessary for mathematics teaching.

Participants also identified the need to develop *technological proficiency* with tools such as GeoGebra, including fluency with dragging, measurement, construction, and transformation features. They acknowledged that such fluency supports exploratory and investigative learning, enabling learners to visualise and interact with abstract mathematical ideas. This reflects the technological content knowledge (TCK) dimension of the TPACK framework (Mishra & Koehler, 2006), where technology reshapes how content is accessed and represented. These findings echo Zambak’s (2014) observation that positive beliefs about technology correlate with improved content integration and enhanced instructional design.

Consistent with the pedagogical content knowledge (PCK) framework (Shulman, 1986, 1987), participants recognised the importance of pedagogical reasoning. They emphasised the need to anticipate students' misconceptions, to scaffold tasks appropriately, and to promote student autonomy through open-ended exploration. These reflections are in line with Thomas and Palmer's (2014) TPACK applications in mathematics education, advocating for pedagogical strategies that use dynamic tools to deepen conceptual understanding.

Participants also grappled with the *classroom management challenges* posed by technology-rich environments, especially in settings including computer labs. They stressed the need for strategies that maintain student engagement and support collaborative learning. These concerns resonate with Radović, Marić and Passey (2019), who emphasise that effective technology integration involves technical and pedagogical skills and classroom orchestration. Participant J's commentary on these difficulties aligns with Bretscher (2023) and Ruthven (2018), both of whom highlight the pedagogical orchestration required for DGS to support collaborative and interactive learning.

A distinctive contribution of this study lies in the emphasis participants placed on the *interdisciplinary and cross-curricular potential* of geometry, particularly its connections with art, architecture, and data analysis. This interdisciplinary vision reflects an emerging form of *curricular knowledge* (Ball et al., 2008) and suggests that DGS can support broader educational aims beyond mathematics. Additionally, participants recognised the role of GeoGebra in bridging concepts across mathematical domains, including algebra and geometry, and improving conceptual cohesion. These insights extend the current literature by suggesting that DGS can support the integration of visual, spatial, and algebraic reasoning in innovative ways.

Moreover, participants reported that using DGS prompted reflection on the *epistemological nature of mathematical knowledge*, which they began to perceive as dynamic, visual, and open to exploration. This observation makes an important contribution to the TPACK literature by illustrating how technology transforms the way content is taught and reshapes how knowledge itself is conceptualised and understood by beginning teachers.

Another unique insight was the *motivational and efficiency benefits* of DGS. Participants valued GeoGebra's capacity to streamline geometric construction, increase student engagement, and support independent inquiry from the teacher. These findings support Laborde's (2001) assertion of the benefits of hands-on digital manipulation for geometric learning. In particular, the depth of participants' comments on how students can explore variations and experiment with problems using DGS provides a new lens on how such tools can promote critical thinking and exploratory learning.

Finally, participants expressed a strong belief in the importance of *early exposure to technology*, noting its value in preparing students for higher education. Participant L specifically emphasised how familiarity with GeoGebra in secondary education supports transition to university-level mathematical studies. This observation supports Kartal and Çınar's (2022) call for greater alignment between secondary and tertiary mathematics education and highlights a gap in existing research that warrants further exploration.

In summary, beginning teachers in this study recognised the importance of integrating *content, pedagogy, and technology* in meaningful ways when teaching geometry with DGS. Their reflections resonate with theoretical frameworks including PCK (Shulman, 1986, 1987), TPACK (Mishra & Koehler, 2006; Harris et al., 2009), and specialised content knowledge (Ball et al., 2008), while also offering fresh contributions. These include attention to interdisciplinary applications, epistemological awareness, classroom orchestration, and educational transitions. The findings suggest that carefully designed professional development should focus on technical fluency and on helping beginning teachers build integrated, inquiry-driven practices that prepare students for dynamic, technology-rich mathematical learning environments. Building on these perceived knowledge requirements for teaching geometry in technology-integrated contexts, the next section examines how beginning teachers envisioned and developed pedagogical strategies to implement such knowledge effectively within dynamic and interactive learning environments.

6B.2.2 Pedagogical strategies and insights for geometry teaching in technology-enhanced contexts

The findings revealed a range of pedagogical insights developed by beginning teachers as they engaged in geometry tasks using dynamic geometry software (DGS), particularly GeoGebra. These insights reflected a shift in their pedagogical thinking, with participants recognising how technology integration could support more exploratory, collaborative, and student-centred approaches to teaching geometry. The data suggest that DGS-enabled tasks prompted beginning teachers to reflect on how geometry can be taught not merely as procedural knowledge but as conceptual understanding rooted in visualisation, manipulation, and justification.

A central finding was the strategic use of GeoGebra to facilitate conceptual understanding, an insight that is strongly aligned with the work of Radović, Marić, and Passey (2019), Noss et al. (2020), and Hohenwarter and Preiner (2007). Participants' recognition of GeoGebra's ability to render geometric relationships visible in real-time confirms these authors' assertions that dynamic geometry software (DGS) supports conceptual clarity through visual manipulation and immediate feedback. Participant G's shift from static tools like compass and straightedge to digital circles and lines echoes Noss et al.'s (2020) claim that DGS helps learners perceive emergent patterns and underlying relationships or properties, thereby enriching their understanding.

This also mirrors Bruner's (1974) theory of multiple modes of representation, which posits that knowledge is best acquired through a progression of enactive, iconic, and symbolic forms. Participant A's reflection on building constructions and '*seeing how the model fits together*' exemplifies this enactive and iconic engagement with mathematical ideas, illustrating how DGS bridges representations in ways that support understanding.

Participants' emphasis on hands-on, learner-centred learning also reinforces the findings of Chander and Arora (2021) and Dabbagh and Kitsantas (2012), who highlight the role of interactive technologies in fostering inquiry, metacognition, and self-regulation. Participants' descriptions of manipulating vertices and testing hypotheses independently suggest the development of learner autonomy, critical thinking, and inductive reasoning—attributes that are widely acknowledged in the literature as essential to effective technology-enhanced and constructivist learning environments (Tondeur et al., 2017; Voogt et al., 2013). These findings indicate a pedagogical shift among beginning teachers away from passive, didactic instruction toward more active, exploration-based approaches enabled by digital tools.

The theme of task design, particularly open-ended, sequenced, and structured to support dialogue and reflection, aligns with Alexander's (2008) notion of dialogic learning and Consoli et al.'s (2023) work on mathematical talk. The collaborative exchanges described by Participant A and H, where ideas were 'fragmented,' shared, and developed jointly, exemplify this principle. Through these dialogic encounters, participants improved their technological fluency and engaged in collective sense-making, highlighting the social dimension of learning that Alexander (2018) and Mercer and Littleton (2007) advocate.

This social constructivist orientation is further reinforced by Vygotsky's (1978) theory, which places social interaction at the heart of cognitive development. Pair work, as reflected in participants' narratives, provided a Zone of Proximal Development (ZPD) where peers acted as more capable partners, facilitating mutual growth. The findings confirm the enduring relevance of Johnson et al.'s (2014) work on cooperative learning, where collaborative environments promote deeper learning outcomes and emotional engagement.

In regard to scaffolding, Participant A's endorsement of step-by-step guidance and progressive questioning strongly supports the foundational work of Wood, Bruner and Ross (1976), who introduced the concept as a means to bridge students' current and potential levels of competence. These methods allowed participants to build understanding incrementally, mirroring the scaffolded instructional practices that are critical in developing complex geometric reasoning. This scaffolding resonates with Trouche's (2004) notion of instrumental orchestration, where teachers purposefully design and adapt technological tools and tasks to shape learning. Although participants were not in teaching roles, their collaborative discussions and conceptualisations of how to structure scaffolded learning sequences illustrate an emerging competence in orchestrating mathematical discourse with technology.

The emphasis on differentiated instruction, expressed through the design of tasks to support students at varying levels, reflects key principles from Trocki and Hollebrands (2018) and the Universal Design for Learning (UDL) framework. Participant A's intent to tailor tasks to students' attainment levels demonstrates an awareness of how technology can be employed to personalise instruction. These practices echo Brousseau's (1997) didactic situations and Artigue's (2002) emphasis on adapting teaching to students' cognitive needs, marking a shift from uniform instruction to responsive, inclusive pedagogy.

Furthermore, the reflective commentary on teacher-led versus student-led instruction, as seen in Participant R's observation, suggests a detailed understanding of pedagogical balance. While tasks were initially structured with clear steps, participants recognised that true inquiry lies in offering students opportunities for autonomous exploration. This reflection hints at the development of pedagogical imagination, as articulated by Lampert (2001), the capacity to anticipate, interpret, and adjust instruction based on learners' needs and responses.

Although Pierce and Ball (2009) caution that technology integration is often hindered by confidence, time, and technical challenges, the findings in this study suggest otherwise. Participants embraced DGS and identified its pedagogical affordances, pointing to a generational shift in teacher readiness and mindset. These findings highlight the evolving digital fluency of early-career teachers and challenge deficit-oriented narratives about teachers' technological uptake.

Finally, the interconnectedness of pedagogical strategies of combining scaffolding, collaboration, differentiation, and student-centred inquiry underlines participants' emerging TPACK (Mishra & Koehler, 2006). Their integrated approach illustrates how beginning teachers are not merely learning to use technology but are conceptualising pedagogy, content, and digital tools as mutually reinforcing elements of effective instruction.

The findings under 5B.3.1, when viewed through the lens of relevant literature, reveal a rich, evolving pedagogical consciousness among beginning teachers. Their insights affirm the value of dynamic technology, collaborative learning, and scaffolded instruction, while also extending the literature by highlighting how beginning teachers are reimagining the pedagogical function of technology in geometry teaching. These findings confirm the transformative potential of DGS, such as GeoGebra, and illustrate how early-career educators are developing adaptive, student-centred, and reflective pedagogical approaches, indicative of a shift toward more nuanced and integrated models of mathematics instruction in technology-enhanced learning environments.

Having considered the pedagogical strategies envisioned by beginning teachers for integrating dynamic geometry into instruction, the next section shifts focus to their technological proficiency. It examines how participants interacted with the affordances and limitations of GeoGebra, highlighting both the progress achieved and the challenges encountered in using dynamic geometry software effectively.

6B.2.3 Technological proficiency and challenges with dynamic geometry software

The findings from this study reveal a marked development in participants' technological proficiency and confidence when using GeoGebra, highlighting the vital role of technological skills in contemporary mathematics education. This parallels the work of Niess (2008), who emphasises the need to integrate technology into pedagogical content knowledge through the TPACK framework. Participants' growing competence with dynamic geometry software reflects the importance of equipping teachers with the technical expertise necessary for effective technology integration, a view also supported by Roblyer (2009), who highlights the significance of teacher education in facilitating such integration. In parallel, Radović, Marić and Passey (2019) argue that effective integration requires teachers to develop knowledge that transcends content and pedagogy alone, an observation reflected in the technological knowledge demonstrated by participants in this study.

Despite their growing competence, participants also reported specific challenges in using GeoGebra, particularly concerning technical issues and interface usability. These concerns echo those identified in prior studies by Trouche and

Drijvers (2010) and Clark-Wilson et al. (2021), which feature how such obstacles can hinder effective teaching and learning. These findings reinforce the view that technology integration is not inherently seamless and requires adequate support systems and continuous professional development (Voogt et al., 2013). Participants' reflections suggest that initial worries about using GeoGebra were gradually replaced by more positive evaluations following sustained interaction with the tool, demonstrating a learning trajectory marked by resistance and eventual engagement. This process resonates with the findings of Naidoo and Govender (2019), who describe the complex interplay between technological confidence and initial scepticism among beginning teachers.

A particularly unique insight from this study is the transformative journey that participants underwent, from initial scepticism to confident advocacy for GeoGebra. This evolution supports Zambak's (2014) argument that pre-service or beginning teachers' beliefs significantly influence their professional development, especially in relation to content knowledge and teaching practices. The data suggest that structured, hands-on experience with technology facilitated this shift, consistent with the conclusions drawn by Kartal and Çınar (2022), who stress the impact of targeted interventions in developing TPACK. Furthermore, these findings are in line with Harris, Mishra and Koehler (2009), who argue that successful technology integration requires technical knowledge and the alignment of teacher beliefs, pedagogical practices, and contextual understanding.

In summary, the findings support and extend the literature by illustrating the dual nature of beginning teachers' experiences with dynamic geometry software, characterised by opportunities for growth and significant initial barriers. This study's progression from hesitation to advocacy points out the need for teacher education programmes to prioritise continuous, hands-on engagement with digital tools. It also reinforces the role of sustained professional development in building technological proficiency and resilience in the face of instructional challenges. These insights contribute to an understanding of how beginning teachers can develop confidence and competence in technology-centred learning environments, particularly within the domain of geometry learning.

While grappling with GeoGebra's functionalities presented opportunities and challenges for beginning teachers, these experiences also informed their thinking about student engagement. The next section explores how participants envisioned using dynamic geometry software to teach concepts, stimulate curiosity, promote independent exploration, and foster deeper conceptual understanding among students.

6B.2.4 Teachers' perspectives on fostering engagement and independent exploration through DGS

The Findings from this study revealed that beginning teachers viewed dynamic geometry software (DGS) as a catalyst for promoting student engagement, autonomy, and critical thinking in geometry learning. Participants recognised that DGS, particularly GeoGebra, enabled exploratory and interactive learning environments where students could visualise geometric relationships dynamically, manipulate constructions, and receive immediate feedback. These features were seen as essential in supporting students' ability to form conjectures, test hypotheses, and construct understanding through active engagement, aligning with research by Hohenwarter and Preiner (2007), Niess (2008), and Noss et al. (2020), who emphasise the epistemic value of dynamic tools in mathematical learning.

Beginning teachers perceived DGS as a functional aid and a cognitive and metacognitive tool that facilitates inquiry, supports learner autonomy, and scaffolds independent and collaborative reasoning. This perspective echoes Dabbagh and Kitsantas's (2012) findings on the role of technology in supporting self-regulation and goal setting. Participants emphasised that open-ended, well-structured tasks within DGS environments encouraged deeper engagement and problem-solving, reinforcing principles of constructivist and inquiry-based pedagogy (Consoli et al., 2023). The dialogic potential of such tasks, where students could articulate, justify, and challenge mathematical ideas, was also highlighted, resonating with Alexander (2008) and Mercer and Littleton (2007) on the importance of meaningful mathematical discourse.

Crucially, the participants highlighted the role of thoughtful task design in stimulating exploration and engagement. Tasks that were overly procedural or closed in nature were seen as limiting, while those that incorporated guiding questions, multiple entry points, and space for prediction encouraged sustained inquiry. This resonates with Clark-Wilson and Timotheus's (2013) advocacy for tasks with meaningful goals and the need to balance structure with student freedom, especially important for beginning teachers developing confidence in orchestrating technology-enhanced learning.

Moreover, this study extends prior literature by highlighting the versatility of DGS in deepening conceptual understanding, accommodating diverse learning styles and promoting visual and verbal communication. The findings reinforce Kuzle's (2017) view of DGS as a cognitive partner in creative problem-solving and align with Bretscher (2023) and Drijvers et al. (2016) on the adaptive affordances of such tools. Despite not teaching in real classrooms, participants critically reflected on how DGS tasks could promote ownership of learning, enhance student motivation, and bridge abstract content with tangible exploration.

In summary, beginning teachers recognised DGS as a powerful medium for designing technology-rich learning experiences that foster independent exploration, conceptual engagement, and dialogic interaction. Their perspectives underscore the need for teacher education programmes to emphasise the pedagogical use of DGS and to support the development of tasks that promote critical thinking, creativity, and meaningful student engagement in mathematics learning.

6B.3 Implications for teacher education and professional development

The findings highlight the need for teacher education programmes and professional development initiatives to go beyond basic technology training, focusing instead on the integrated development of technological, pedagogical, and content knowledge (TPACK). Beginning teachers in this study viewed effective geometry teaching with technology as complex, requiring deep subject knowledge, strategic pedagogical thinking, and technological fluency. This reflects the TPACK framework developed by Harris, Mishra, and Koehler (2009), which emphasises the dynamic interplay among content, pedagogy, and technology in effective teaching practices. Thus, teacher preparation should adopt holistic approaches that explicitly address these intersections.

Firstly, teacher education must provide structured opportunities to develop specialised content knowledge, particularly geometry, and its representation through dynamic geometry software. The emphasis participants placed on relational thinking and geometric generalisation suggests the importance of supporting beginning teachers to explore geometry conceptually, not just procedurally, using tools like GeoGebra. This aligns with Zambak's (2014) findings, which indicate that positive beliefs about technology contribute to improved content knowledge and its integration into pedagogy.

Secondly, professional development should support teachers in building technological fluency and pedagogical imagination. This includes learning to orchestrate DGS tools effectively, designing open-ended, scaffolded, and differentiated tasks, and anticipating student thinking and misconceptions. These practices require simulated teaching experiences and task-based collaboration, enabling beginning teachers to rehearse responsive teaching in technology-rich settings. As highlighted by Naidoo and Govender (2019), teachers' ability to navigate the dual impact of technology hinges on targeted support in integrating tools meaningfully into teaching practices.

Furthermore, the findings stress the value of developing collaborative and dialogic learning environments. Teachers must be trained to facilitate productive mathematical talk, using DGS's visual and interactive capabilities to prompt exploration, justification, and reflection. Embedding dialogic principles in teacher education can empower beginning teachers to cultivate classrooms where mathematical ideas are co-constructed. This emphasis on dialogic interaction supports the work of Radović, Marić, and Passey (2019), who argue that such practices improve engagement and learning in technology-enhanced settings.

Participants' insights also point to the need for professional development to address classroom orchestration challenges, such as managing technology use, maintaining student focus, and encouraging independent inquiry. Equipping teachers with strategies for technical troubleshooting and pedagogical pacing in computer-based environments is essential. As Trocki and Hollebrands (2018) have shown, differentiating tasks according to cognitive demand is critical in supporting all learners in such settings.

In addition, teacher education should encourage interdisciplinary thinking and the exploration of geometry's connections with other domains (e.g., art, architecture, and data). This can broaden curricular vision and enrich mathematical understanding and engagement. Hohenwarter and Preiner (2007) also affirm the value of DGS tools in enabling interactive and manipulative experiences that foster deeper conceptual understanding.

Lastly, the findings signal the importance of continuity between secondary and tertiary education in mathematics. Programmes should promote early and consistent exposure to DGS, helping future teachers and students transition smoothly into higher-level mathematical thinking and tool use. The participants' journey from initial scepticism to advocacy for GeoGebra demonstrates the transformative potential of sustained, hands-on engagement with technology in teacher education.

6B.4 Conclusion

This chapter examined beginning teachers' perceptions and knowledge development in relation to teaching geometry within technology-enhanced environments. Drawing on inductive and deductive analyses, the findings illustrate the interplay between content, pedagogy, and technology in teaching with dynamic geometry software such as GeoGebra. Participants consistently emphasised the importance of integrated technological, pedagogical, and content knowledge, reinforcing the relevance of the TPACK framework.

Key insights included the value of strong content knowledge, growing technological fluency, and the use of pedagogical strategies such as scaffolding, collaborative learning, and hands-on tasks. While these findings resonate with existing literature, they also offer fresh perspectives, particularly regarding the interdisciplinary potential of geometry and the epistemological shifts brought about by the use of DGS.

Despite encountering challenges related to technology integration, participants demonstrated increasing competence and began to view technology as a means of improving pedagogy rather than as an external tool. These developments hold important implications for teacher education programmes, highlighting the need for professional development opportunities that encourage integrated, reflective, and inquiry-driven practices in technology-rich mathematics classrooms.

CHAPTER 7: CONCLUSION AND RECOMMENDATIONS

7.1 Introduction

This chapter concludes the study by summarising the key research findings relating to the two research questions and offering recommendations based on the research outcomes. The study overall aimed to explore efficacious geometric construction tasks within an online platform's DGS environment that have the potential to support beginning teachers in their development of knowledge necessary to support their future teaching of geometric constructions using DGS. The specific research questions focused on exploring how carefully designed tasks facilitate productive mathematical talk and dialogic learning among beginning teachers working with dynamic geometry software in ways that develop their TPACK knowledge, and understanding how beginning teachers perceive, understand and talk when exploring the knowledge needed for teaching geometry in a technology-based environment. After qualitatively analysing participants' learning and interactions with the tasks within the online platform's dynamic geometry software environment, it can be concluded that such an approach has the potential to assist beginning teachers in co-constructing the necessary knowledge for their future teaching of geometric constructions using dynamic geometry software. This potential can be realised when tasks within the environment are designed with principles that include scaffolding, collaborative paired learning using platforms such as Microsoft Teams with screen sharing, reflective practices, dynamic manipulation with GeoGebra, modes of representation (Bruner's (1974) theory), balancing ostensive and non-ostensive objects, feedback mechanisms, meaningful goals and visible mathematics, and inquiry and hypothesis testing.

7.2 Summary of key findings

7.2.1 Research question 1: In what ways can carefully designed tasks facilitate productive mathematical talk and dialogic learning among beginning teachers working with dynamic geometry software in ways that develop their TPACK knowledge?

To investigate this, I analysed participants' learning interactions and communication patterns using interpretive video deductive coding, guided first by dialogic talk and learning principles, and second by the TPACK framework as analytical tools. In addition, I conducted an inductive thematic analysis to identify and categorise the features that participants highlighted as important in the tasks I created for them.

The findings indicated that certain design principles embedded in the tasks, including scaffolding, reflective practices, dynamic manipulation with GeoGebra, modes of representation (Bruner's (1974) theory), balancing ostensive and non-ostensive objects, incorporating feedback mechanisms, meaningful goals and visible mathematics, and open-ended yet scaffolded, as described in the previous section supported productive mathematical talk, dialogic learning and TPACK development among beginning teachers. Also, collaborative paired learning using platforms such as Microsoft Teams with screen sharing was used in the implementation of the tasks, further supporting these learning processes. In addition, participants identified specific features they felt best supported their TPACK knowledge-building. These included task complexity, integration of multiple geometric concepts, structured sequencing with progression and scaffolding, alignment with relevant and authentic learning goals, engagement, motivation, and a balance of appropriate challenge and support (see Chapter 5A, Section 5A.5, Subsections 5A.5.1–5A.5.6, pp. 261–277). Tasks that aligned with participants' prior knowledge and used technology effectively were especially valued for their role in knowledge construction (see Chapter 5A, Section 5A.5, Subsections 5A.5.7–5A.5.8, pp. 277–279). From this, it was concluded that effective geometric construction tasks must incorporate a thoughtful blend of these design features to best support the development of TPACK knowledge among beginning teachers.

A noteworthy aspect of the findings was the complex nature of dialogic talk types evident in participants' discourse, which appeared instrumental in facilitating knowledge co-construction. These included transactional, expressive, exploratory, evaluative, deliberative, interrogatory, imaginative, and expository talk, demonstrating participants' engagement in logical reasoning, shared enthusiasm, reflective evaluation, inquiry, and creative thinking. Dialogic learning principles, such as collective, cumulative, deliberative, purposeful, reciprocal, and supportive learning, were also evident. The power of collective learning was demonstrated as beginning teachers engaged collaboratively, shared insights, and deepened their understanding. In addition, the cumulative dialogic learning principle was evident as participants' understanding evolved throughout their discussions. The deliberative, purposeful, and reciprocal dialogic learning principles facilitated critical thinking and mutual interaction. The online platform Microsoft Teams provided a supportive learning environment in ways that support effective learning. These task design principles, collectively integrated into the tasks used in the study guided participants' interactions and contributed to their co-construction of knowledge.

7.2.2 Research question 2: In what ways do beginning teachers perceive, understand and talk when exploring the knowledge needed for teaching geometry in a technology-based environment?

This study investigated beginning teachers' perceptions, understanding, and discussions about the knowledge required for teaching geometry in technology-enhanced settings. Their verbal interactions, including peer discussions during task engagement and reflections in interviews and focus groups, provided insights into how they articulated their evolving understanding of teaching with technology. To address this, I applied both inductive and deductive thematic analyses, using the TPACK framework as an analytical lens for the deductive portion.

Findings revealed that beginning teachers viewed effective geometry teaching in technology-integrated environments as requiring diverse knowledge. They highlighted core requirements, including a solid grasp of geometric relationships, the ability to make cross-curricular connections, managing classroom dynamics in computer settings, software proficiency, and understanding how technology can support differentiated learning. Together, the analyses showed a consistent need for a well-rounded skill set that integrates technological fluency, pedagogical strategies, and strong content knowledge.

Pedagogically, beginning teachers expressed a preference for hands-on learning and an inductive approach, facilitated by tools like GeoGebra (see Chapter 5B, Section 5B.3.1, pp. 289–292). During paired discussions, they engaged in exploratory talk, questioning geometric relationships, verbalising reasoning while manipulating dynamic objects, and negotiating strategies for constructing figures. Their dialogue revealed how they moved from procedural descriptions to deeper conceptual explanations, supporting their pedagogical growth. This approach, which involved collaboratively working in pairs, following written task instructions with open exploration, using DGS to construct and manipulate dynamic geometric objects, and observing the effects on their geometric properties, supports differentiated teaching and collaboration. It can help establish a foundation for a teaching framework that improves geometry teaching. Recognising the importance of integrating innovative technology, beginning teachers observed that tools such as GeoGebra can facilitate critical thinking and cater to diverse learning preferences, aligning with contemporary educational practices. This emphasis on hands-on learning echoes constructivist theories, while differentiated methods, including written and non-written designed tasks for higher-attaining and lower-attaining students, reinforce inclusivity. In addition, collaborative and inductive approaches featured the value of student-centred teaching, as evidenced in the nature of participants' discussions, where they built on each other's ideas and refined their reasoning through dialogue. Integrating dynamic geometry software (DGS) can boost engagement and encourage interactive learning through shared exploration.

Engagement with DGS deepened teachers' mathematical content knowledge, allowing them to explore the connections between geometric concepts and promote a genuine interest in the subject (see Chapter 5B, Section 5B.3.2, pp. 292–295). Their spoken reflections highlighted a growing awareness of how dynamic manipulation supports conceptual understanding, as they articulated discoveries about properties, dependencies, and transformations in geometry. Participants also recognised how DGS tasks supported critical thinking and mathematical creativity through active, hands-on learning (see Chapter 5B, Section 5B.3.2, pp. 293–295). The technology's role in developing an understanding of geometric constructions and visualising dynamic models further highlights its instructional value. Beyond traditional teaching methods, DGS was seen as a versatile tool that supports both creative and conceptual learning in geometry.

Through collaborative exploration, beginning teachers demonstrated growth in essential technological skills, which informed their understanding and supported innovative teaching practices (see Chapter 5B, Section 5B.3.3, pp. 295–300). Their spoken interactions revealed initial hesitancy in using GeoGebra, shifting toward more confident discussions of its pedagogical potential. They used interrogatory talk to ask each other about functions and features, and evaluative talk to assess the effectiveness of different approaches in constructing geometric figures. DGS facilitated hands-on concept exploration and structured learning experiences, although it also presented technical challenges that highlighted the importance of effective task design. Reflective practice emerged as a component of technological pedagogy, allowing teachers to adjust and enhance their instructional techniques.

In discussions and interviews, beginning teachers emphasised the importance of a diverse skill set of knowledge needed to teach geometry effectively in technology-integrated settings (see Chapter 5B, pp. 281–305). Their dialogue revealed evolving perspectives, as they shifted from concerns about technical challenges to a focus on pedagogical strategies. They employed reflective talk to articulate the benefits and limitations of dynamic geometry software, comparing its affordances to those of traditional static diagrams and debating its role in promoting student engagement. Acknowledging the positive effects of dynamic software on student engagement and critical thinking, they advocated for its continued integration into teaching. These insights are consistent with prior research, affirming that a diverse skill set is essential for successful geometry teaching in technology-rich environments.

7.3 Task design principles for developing geometric knowledge, pedagogical skills, and technological proficiency in dynamic learning environments

The task design principles applied in the study established a structured and supportive learning environment that aimed to facilitate beginning teachers' development of technological knowledge, geometric content knowledge and pedagogical strategies (see Section 5A.3.1). This was particularly relevant in the context of dynamic geometry software like GeoGebra. The design of these tasks provided participants with opportunities to build, test, and refine their understanding of geometric constructions while co-constructing knowledge in collaboration with their peers. Below is an explanation of how each principle was applied in the task design and how these principles contributed to participants' learning, with Task 4a serving as a focal example.

Scaffolding: Scaffolding in Task 4a was embedded through a structured sequence of tasks that gradually increased in complexity, allowing participants to develop their technological pedagogical content knowledge (TPACK) in a supportive manner. The task began with fundamental geometric constructions, such as creating line segments and points, before progressively introducing more complex tasks like constructing circles, intersections, and bisectors. This incremental approach enabled participants to systematically build on their prior knowledge, reducing cognitive overload and ensuring a smooth transition from basic to advanced concepts. The structured nature of the tasks acted as scaffolding, guiding participants through the integration of technological knowledge (manipulating GeoGebra tools), pedagogical knowledge (understanding how to sequence and support learning for students), and content knowledge (deepening their understanding of geometric relationships). Providing a clear learning pathway, the scaffolding raised participants' confidence in using GeoGebra effectively, allowing them to experiment, make conjectures, and refine their strategies while strengthening their overall understanding of teaching geometry in technology-enhanced environments (see page 188).

Collaborative learning: Collaborative learning in this study was both a designed feature and an emergent result of the online environment. The task structure intentionally required participants to work in pairs or small groups, engaging in peer-to-peer interactions to explore geometric relationships using GeoGebra. This collaboration, reinforced by real-time manipulation of figures, promoted dynamic discussions and collective problem-solving. Task 4a, for example, encouraged participants to compare findings and justify their methods for constructing an angle bisector, promoting dialogic learning through exploratory, evaluative, and transactional talk. Guided by dialogic principles (for example, cumulative, reciprocal, supportive), these interactions enabled participants to co-construct knowledge, refining their understanding through shared insights and reasoning. Ultimately, the collaborative nature of the tasks deepened both mathematical content knowledge and pedagogical strategies, as participants collectively developed approaches to using GeoGebra effectively (see Section 5A.4, pp. 238-260, and Section 5B.3.1-3, pp. 289-300, for participants' reflections on collaborative learning and pedagogical insights).

Reflective practices: Reflective practices were embedded in Task 4a through strategically designed prompts that encouraged participants to critically analyse their constructions and reasoning. Questions such as 'What do you observe?' and 'Why are you observing what you are seeing?' prompted participants to reflect on the mathematical properties they were exploring, encouraging metacognitive awareness and deep engagement with geometric concepts. These reflective moments encouraged participants to connect their technological actions in GeoGebra with conceptual understanding, reinforcing key geometric principles (for example, the properties of an angle bisector). Additionally, reflection facilitated pedagogical awareness, as participants considered how such prompts could be used to guide students' thinking in their future teaching. Articulating their observations and reasoning, participants engaged in critical thinking, refined their understanding, and developed strategies for supporting students in making sense of geometric relationships in a technology-enhanced learning environment (see page 189).

Dynamic manipulation with GeoGebra and real-time feedback: The dynamic manipulation capabilities of GeoGebra allowed participants to interact with their constructions and receive immediate visual feedback as they adjusted points and observed changes in geometric relationships. This real-time feedback strengthened participants' understanding of geometric constructions, as they could experiment with different configurations and immediately see the results of their actions. This hands-on approach provided an inquiry-based learning experience, where participants explored and validated their constructions, ultimately improving their conceptual understanding.

Instrumental orchestration: Instrumental orchestration refers to the deliberate structuring of digital tools used by the teacher or task designer to guide students' mathematical learning (Trouche, 2004). In this study, instrumental orchestration was embedded in Task 4a, where GeoGebra's tools (for example, compass, straightedge, intersection, and point tools) were strategically structured to support participants in constructing geometric figures using only a straightedge. The orchestration involved instrumentalisation (how participants adapted the tools to explore geometric relationships) and instrumental genesis (how they developed fluency in using the tools to construct and analyse geometric properties). This structured integration of tools helped participants manage the complexity of the tasks while maintaining a focus on key geometric concepts, encouraging both technological proficiency and conceptual understanding. Engaging in these orchestrated tasks, participants developed technological pedagogical content knowledge (TPACK), ensuring they could effectively and potentially use GeoGebra to facilitate dynamic, inquiry-based geometry teaching (see page 192 and Section 5A.4, pp. 238-260).

Modes of representation (Bruner's (1974) theory): Task 4a applied Bruner's (1974) modes of representation of enactive, iconic, and symbolic, to support participants' understanding of geometric constructions. The enactive mode was embedded through hands-on manipulation in GeoGebra, where participants physically constructed geometric figures using digital tools (for example, compass, straightedge, and intersection tools). The iconic mode emerged through visual representations on the computer screen, as dynamic figures allowed participants to observe geometric properties by dragging points and analysing relationships. The symbolic mode was facilitated through verbal and written discussions, where participants articulated abstract geometric relationships (for example, reasoning about the properties of an angle bisector) and justified their constructions (see pp. 194-198). The design of Task 4a ensured that participants engaged with all three modes, allowing them to internalise geometric concepts through multiple representations. This multimodal approach helped bridge procedural fluency with conceptual understanding, promoting flexible thinking about geometric relationships. Additionally, the investigative prompts embedded in the task (for example, 'What do you observe?' and 'Why are you observing what you are seeing?') encouraged participants to translate between modes—linking their physical interactions with visual and symbolic reasoning. This process enhanced their knowledge of geometric constructions and developed their pedagogical awareness, as they reflected on how different representations could be used to support students' learning in technology-enhanced classrooms. Engaging with enactive, iconic, and symbolic representations, participants refined their TPACK knowledge, gaining insights into how digital tools can facilitate deeper mathematical understanding.

Balancing ostensive and non-ostensive objects: Balancing ostensive and non-ostensive objects refers to the relationship between visible, tangible mathematical representations (ostensive objects) and abstract, conceptual reasoning (non-ostensive objects) in learning geometry. Ostensive objects in Task 4a included the dynamic constructions made using GeoGebra's tools (for example, lines, line segments, circles, and bisectors), while non-ostensive objects encompassed the underlying geometric principles and reasoning that justified these constructions. The task structure facilitated a balance between these elements by requiring participants to construct geometric figures and articulate and justify their reasoning behind these constructions (see page 190 and Section 5A.4.2 pp 118-128). This balance promoted both procedural fluency and conceptual understanding, as participants engaged in hands-on exploration while deepening their knowledge of geometric properties and relationships.

Feedback mechanisms: Task 4a embedded multiple feedback mechanisms that supported participants' learning through real-time digital feedback, peer interactions, and self-correction opportunities (see page 198). GeoGebra's dynamic nature provided immediate visual feedback, allowing participants to manipulate geometric figures, drag points, and observe changes in real-time. This enabled them to test conjectures, identify errors, and refine their constructions iteratively. Additionally, the structured peer discussions embedded in the task design encouraged participants to compare strategies, question each other's reasoning, and collectively resolve misconceptions. The investigative prompts (for example, 'What do you observe?' and 'Why are you observing what you are seeing?') further reinforced feedback mechanisms by prompting participants to critically reflect on their actions and geometric reasoning. This iterative cycle of construction, feedback, and refinement supported participants in deepening their conceptual understanding of geometric relationships while also enhancing their technological and pedagogical awareness.

Meaningful goals and visible mathematics: Task 4a provided participants with a clear and purposeful mathematical goal—constructing an angle bisector—while also encouraging them to explore additional geometric structures, such as kites, congruent triangles, and isosceles trapeziums (see page 200). The exploratory nature of the task ensured that participants were not simply following procedural steps but were actively engaged in uncovering mathematical relationships. Engaging in hands-on constructions using dynamic tools such as GeoGebra’s compass and intersection tools, participants were able to visualise the evolving mathematical structures in real-time, reinforcing the conceptual underpinnings of their constructions. The investigative prompts (for example, ‘What do you observe?’ and ‘Why are you observing what you are seeing?’) further encouraged them to articulate and reason about the mathematical properties they were uncovering, making abstract geometric principles more explicit and accessible. Furthermore, the task promoted meaningful engagement by connecting participants’ explorations to real-world teaching applications. Constructing figures without being given the intended outcome, participants experienced a form of guided discovery, similar to how they might design tasks for their own students. This process deepened their geometric understanding and improved their pedagogical awareness, demonstrating how well-structured, investigative tasks can help students engage meaningfully with mathematical ideas in a technology-enhanced learning environment (see Section 5A.4).

In summary, the task design in this study, particularly Task 4a, exemplified the deliberate integration of multiple task design principles to achieve a careful balance between structure and openness. Scaffolding was embedded through sequenced instructions, tool constraints, and guided prompts that supported access to geometric constructions, while open-endedness was preserved through inquiry-driven investigations, opportunities for conjecture, and collaborative reasoning. The automatic inclusion of dynamic feedback, collaborative dialogue, and multiple representations enabled participants to engage in reflective and evaluative talk, encouraging deep conceptual understanding. Operationalising principles, including instrumental orchestration, Bruner's (1974) representational modes, and the interplay between ostensive and non-ostensive objects, the tasks promoted visible mathematics and meaningful learning goals. This purposeful combination of open-ended exploration and structured support was instrumental in promoting dialogic learning, enabling beginning teachers to develop technological fluency, pedagogical insight, and rich geometric content knowledge, thus potentially advancing their integrated TPACK in technology-enhanced environments.

Table 7.1 encapsulates the task design principles applied in Task 4a and outlines their respective contributions to the learning outcomes achieved by participants. It also provides examples of dialogic talk that occurred during the task, illustrating how these principles cultivated productive mathematical discussions and collaborative learning experiences. This synthesis clarifies how structured task design can facilitate pedagogical skills and technological proficiency in dynamic learning environments.

Table 7. 1 Task design principles for facilitating learning progression, productive mathematical talk, dialogic learning, and TPACK development in dynamic geometry software.

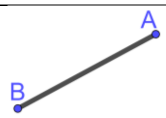
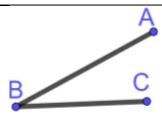
Task design principle	Exemplar of task progression	Learning outcome	Example of dialogic talk
Scaffolding	Progressing from basic constructions (line, point) to complex (circle, bisector)	Gradual knowledge building, avoiding cognitive overload	*'We can create a line segment slider first and then use it as a ruler*

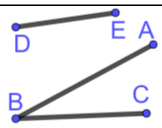
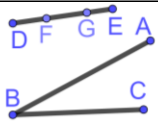
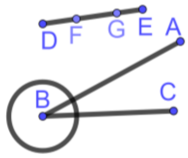
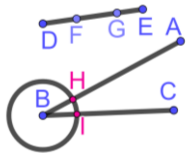
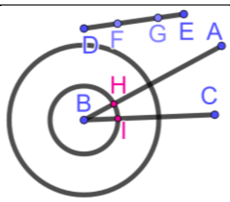
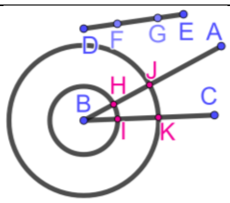
Task design principle	Exemplar of task progression	Learning outcome	Example of dialogic talk
			for constructing an angle bisector'
Collaborative learning	Discussion during the construction of an angle bisector	Peer-to-peer knowledge exchange	*'How did you manage to construct the circle? I did see that.'*
Reflective practices	Prompting reflection during the dragging phase (for example, 'What do you observe?')	Encourages critical thinking and deeper engagement	*'Why do you think this line is still bisecting the angle?'*
Dynamic manipulation with GeoGebra	Dragging points to explore changes in geometric figures	Immediate visual feedback, hypothesis testing	*'When I move this point, the bisector remains. Why is that so?'*
Instrumental orchestration	Using compass, point, and line tools in a guided manner	Enhancing both technical proficiency and geometric understanding	*'The compass tool helped me construct the bisector faster.'*
Modes of representation by Bruner (1974)	Enactive (constructing), iconic (visual), symbolic (reasoning)	Catering to multiple learning styles	*'I can see it visually, but how do we prove this algebraically?'*
Balancing ostensive and non-ostensive	Connecting visible constructions (segments, circles) with abstract reasoning	Understanding both practical and theoretical aspects	*'The line is there, but why does it bisect the angle exactly?'*
Feedback mechanisms	Real-time manipulation feedback + peer feedback in discussions	Supports iterative learning, allows for self-correction	*'I didn't get it right at first, but after dragging the points, I see why.'*
Meaningful goals and visible mathematics	A clear goal of constructing an angle bisector and exploring properties	Focus on the purpose of actions, making mathematics visible	*'We are not just constructing lines; we are seeing the bisector in action.'*
Inquiry and hypothesis testing	Testing by dragging points to explore angle bisector relationships	Encourages exploration, critical thinking, and validation	*'If I move the point, does it still bisect? Let's test that.'*

Task design principle	Exemplar of task progression	Learning outcome	Example of dialogic talk
Open-ended yet scaffolded tasks	Step-by-step initial dynamic construction (for example, building initial dynamic angle bisector for onward exploration), followed by open investigation prompts (for example, 'What do you observe? Why?')	Provides structured access to complex geometry while promoting conjecture, exploration, and reasoning	*'Is this an angle bisector here, and why does that depend on this point's position?' We can use these ideas to construct an angle bisector, a kite, or a trapezium with a ruler only'*

Table 7.2 details the step-by-step process for constructing the initial dynamic geometry task (Task 4a) along with expected responses and examples of mathematical dialogic talk that occurred among participants. This illustration highlights how the structured approach to task design encouraged engagement and collaborative learning, enabling participants to articulate their understanding of geometric constructions while actively testing their hypotheses.

Table 7. 2 shows the steps in constructing the initial dynamic construction of Task 4a, the expected responses and examples of mathematical dialogic talk

Step	Task	Expected response	Mathematical dialogic talk
1	Create the line segment AB.		G: *Create a line segment AB.* H: *Got it.*
2	Construct line segment BC		G: *Carry on from that with BC.* H: *Does it need to be straight?* G: *No, just construct a line segment BC.*

Step	Task	Expected response	Mathematical dialogic talk
3	Create the line segment DE.		H: *So, an angle. Done.* G: *Now, separately create a line segment DE.* H: *Alright.* G: *Construct two points, F and G, on the line segment DE.* H: *They're not quite on it.* G: *No, better use the point-on-an-object [tool], but as long as they're on the object, that's fine.* H: *Yeah.* G: *Actually, try moving F and G. Does it move them along the line?* H: *Yeah, it does. I can't move it up and down.* G: *Construct a circle at B with a radius DF using the compass tool.* H: *Compass tool at B, right?* G: *Yes.* H: *OK, so if I do that and then move point F, it changes the size of the circle constructed.* G: *Yeah, now move B around.* H: *Done. Moving B also works.* G: *Yeah, it's fine. I just wasn't sure if it was technically centred at B, if that makes sense.* H: *Yeah.* G: ... H: ...
4	Construct two points F and G on the line segment DE		
5	Construct a circle at B with a radius DF using the compasses tool.		
6	Use the intersect tool to construct points H and I, where the line segments AB and BC intersect the circle centred at B.		
7	Construct another circle at B with radius DG using the compasses tool.		
8	Use the intersect tool to construct points J and K, where the		

<table>
<tr><th>Step</th><th>Task</th><th>Expected response</th><th>Mathematical dialogic talk</th></tr>
<tr><td></td><td>line segments AB and BC intersect the second circle centred at B.</td><td></td><td rowspan="7">G: Using the intersection tool to construct and name the point of intersection L. Then construct a line, not a line segment, a full line through points B and L.

H: Yep.

G: OK. So, this seems like a long flipping way to do an angle bisector.

H: Are we at the investigation?

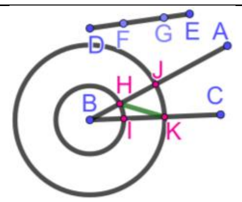
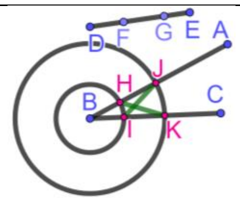
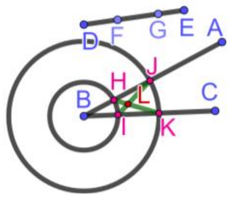
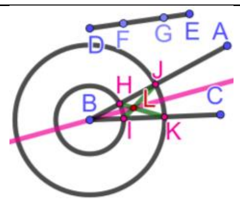
G: Yes. What's the relationship between the line and angle ABC?

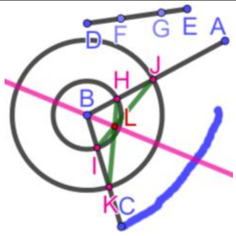
H: It bisects it.

G: Yeah, it does. Drag point A or C.

H: It's still bisecting it.

G: Yeah, it still bisects it. And why does it still bisect it?

H: So, if you look, let's call the angle between B and I, and the angle between B and K, zero. The angle between B, I, and K remains the same. B and I, and I and K, always remain the same. So, you're always the same distance around the circle, if that makes sense. They're in proportion to each other. I can't find the right words for it, but when you move C, I and K remain in line with each other; they remain in line with B. So, point I is always in the middle – not necessarily in the middle, but between B and K, and the angles are the same.</td></tr>
<tr><td>9</td><td>Construct line segment HK</td><td></td></tr>
<tr><td>10</td><td>Construct line segment IJ</td><td></td></tr>
<tr><td>11</td><td>Name the point of intersection 'L' of the line segments HK and IJ using the point tool or intersect tool</td><td></td></tr>
<tr><td>12</td><td>Construct a line through points B and L</td><td></td></tr>
<tr><td></td><td>Investigate</td><td></td></tr>
<tr><td>13</td><td>What geometrical relationship does the line have with angle ABC? And why?</td><td>An angle bisector. This is because point L is always equidistant from points H and I, and J and K</td></tr>
</table>

<table>
<tr><th>Step</th><th>Task</th><th>Expected response</th><th>Mathematical dialogic talk</th></tr>
<tr><td>14</td><td>Drag point A or C
What do you observe?
Why are you observing what you are seeing?</td><td></td><td rowspan="5">Does that make sense? I'm sure I haven't worded that very well.

G: Yeah, no, I get you.

H: The same thing happens when you do it this way [moving point A]. So, the fact that you always have that relationship means you always have the bisector. It will always bisect the angle.

G: Yeah, exactly. OK, so that's by dragging point C. Drag point B.

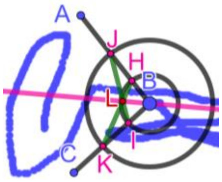
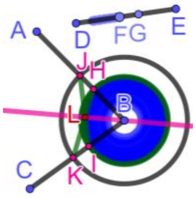
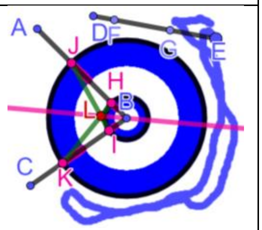
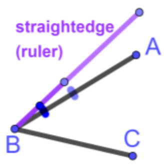
H: Yeah.

G: It's still always going to be an angle bisector. The intersections are there.

H: Yeah. So, at what point do we lose L?

G: We lose it when it goes over C because it no longer intersects the line HK; there's no KH.

H: Yeah, oh I see. Yeah, it still appears to be bisecting when I move point B around. All this does is, when we move C, H and J remain fixed; when we move A, I and K remain fixed. When we move B, H and J are moving, but they stay in proportion. H remains between J and B on a straight line, and I remains between B and K on a straight line. So, they're just moving in proportion, which is why it works.

G: OK, and then drag points F or G to change the radius of the two circles. As long as the circles aren't too big to get rid of the intersection points, it'll be the same.</td></tr>
<tr><td>15</td><td>Drag point B.
What do you observe?
Why are you observing what you are seeing?</td><td></td></tr>
<tr><td>16</td><td>Drag point F or G
What do you observe?
Why are you observing what you are seeing?</td><td></td></tr>
<tr><td>17</td><td>Drag point D or E
What do you observe?
Why are you observing what you are seeing?</td><td></td></tr>
<tr><td>18</td><td>How will you construct an angle bisector with a markable ruler ONLY?</td><td>First step


Second step</td></tr>
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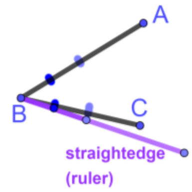
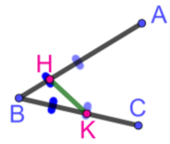
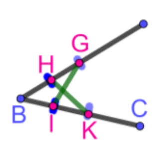
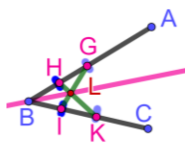
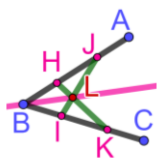
Step	Task	Expected response	Mathematical dialogic talk
		 Third step  Fourth step  Last step  	H: *Yeah, and that doesn't change anything when we do that [move point F to alter the radius of one of the circles] either, obviously. It's all based on the fact that everything rotates around point B.* G: *Mmm hmm.* H: *OK.* G: *Yeah, so this is all leading to the question of how you would construct an angle bisector with only a markable ruler.* H: *With a markable ruler only.* G: *So, basically, we don't need to draw any circles, but we can make a ruler and mark two centimetres on each side. So, if you make another angle…* H: *Yeah, got you, so here, let me make a ruler. I have got my ruler. I will draw my angle MNO and want to draw a bisector of it. So, I will do a segment of a given length, say 3cm along the line segment NM and then another line segment of 3cm on segment NO. I will repeat the same procedure of a given length…* G: *OK, yeah, make it smaller, make it 1cm.* H: *OK, yeah, alright. I will make 1cm and mark 1cm each from point N on these line segments of the angle I am trying to bisect. I will create segments which…* G: *Line segments SQ and PR.*

Step	Task	Expected response	Mathematical dialogic talk
			H: *Then I will draw a line passing through point N and the intersection T. We are done.* G: *We are done. Exactly, how I would have done it.* H: *Cool, I like that one. No compasses are involved.*

7.4 Developing TPACK through technology-enhanced geometric construction task designs for beginning teachers

The task designs provided in this study could offer valuable opportunities for beginning teachers to develop several elements of technological pedagogical content knowledge (TPACK). A key development area is **technological knowledge**, where participants become proficient in using GeoGebra's basic tools, including point, move, line, circle, and tracking tools. These tools enable them to construct and manipulate various geometric figures, enhancing their ability to integrate technology into geometry teaching (see Section 5A.4.1, pp 63). For example, in Task 4, participants learned to use these tools effectively, improving their technological fluency with GeoGebra and boosting their confidence in using digital resources to explore geometric relationships.

In addition to technological proficiency, these tasks strengthen **content knowledge** (see Section 5A.4.2, pp 77). Through tasks that require the construction of geometric elements, including angle bisectors, kites, congruent triangles, similar triangles, isosceles triangles, isosceles trapeziums, and arrowheads, beginning teachers gain insights into fundamental geometric principles. Task 4, in particular, helped participants improve their understanding of geometric construction techniques, such as using a straightedge to construct geometric shapes. Mastering these tasks allows participants to deepen their understanding of essential content, which can be crucial for effective geometry teaching.

The tasks also play a significant role in developing **pedagogical knowledge** (see Section 5A.4.3, pp 81). They are designed to guide beginning teachers through progressively complex geometric constructions, offering insights into how to scaffold lessons effectively. This scaffolding can ensure that tasks support students' learning progression and avoid overwhelming them. Task 4, for instance, helped participants develop pedagogical strategies, such as using sliders to control the radii of circles and employing a straightedge for constructing figures. This pedagogical insight can enable beginning teachers to design tasks that promote critical thinking and active student engagement in geometry lessons.

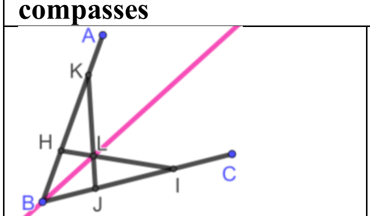
Through these experiences, beginning teachers enhance their **pedagogical content knowledge** by learning how to explain geometric concepts in a clear and accessible manner. Discussions surrounding geometric relationships and construction methods give teachers the tools to effectively communicate complex ideas to students. This blend of content and pedagogy can ensure that beginning teachers are equipped to teach geometric concepts in ways that foster student understanding.

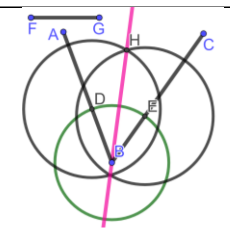
The integration of technology within these tasks also facilitates the development of **technological pedagogical knowledge**. Beginning teachers learn how to use dynamic geometry software for geometric construction and as a tool for promoting inquiry-based learning and active student participation. This can enable them to engage students in meaningful discussions about geometric relationships, helping to cultivate critical thinking skills. In Task 4, participants developed strategies for using technology including GeoGebra to enrich the learning experience and enhance the understanding of geometric content.

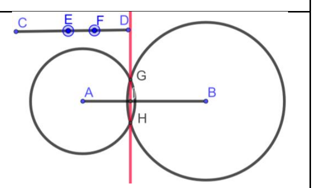
Ultimately, these task designs can facilitate the development of **technological pedagogical content knowledge (TPACK),** the core of the framework. Combining their understanding of geometric content with technological tools and effective pedagogical approaches empowers beginning teachers to create dynamic, technology-enhanced learning experiences. They developed ideas on designing tasks that convey geometric knowledge and encourage student engagement and inquiry, preparing them to teach geometry interactively and innovatively.

Table 7.3 illustrates various geometric constructions discussed by participants during their engagement with the tasks. These examples showcase the diverse methods beginning teachers used to explore, discuss, and construct geometric figures, employing either a straightedge alone or in conjunction with compasses.

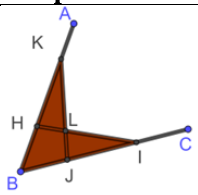
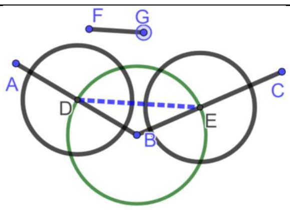
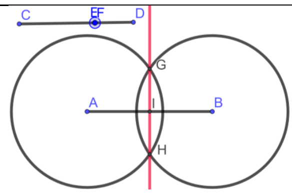
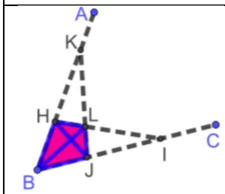
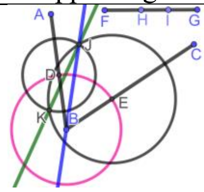
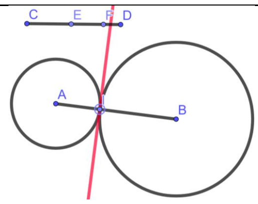
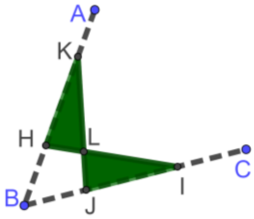
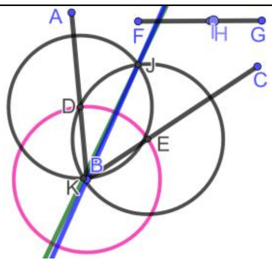
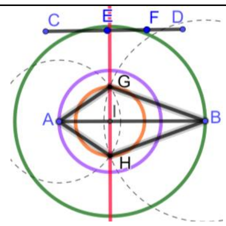
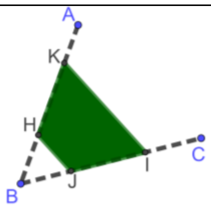
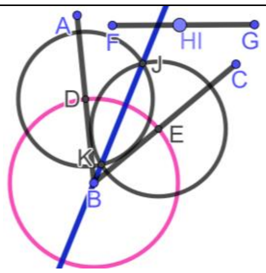
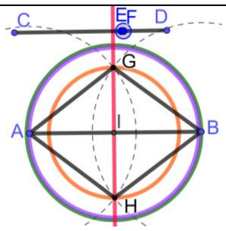
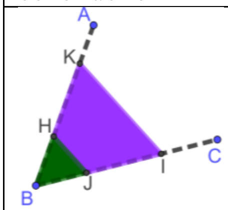
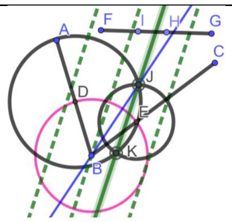
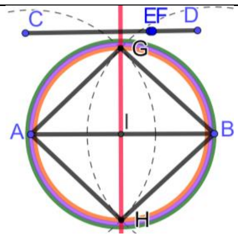
Table 7. 3 Examples of geometric constructions explored by participants using GeoGebra.

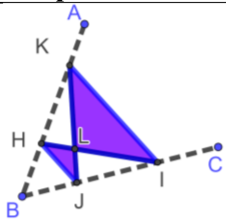
<table>
<tr><th colspan="3">Geometric figure construction with straightedge or straightedge and compasses</th></tr>
<tr>
<td>

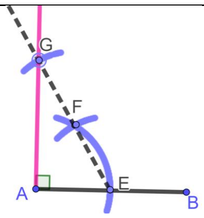
A straightedge-only angle bisector construction</td>
<td>

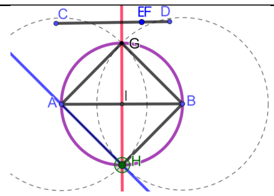
An angle bisector</td>
<td>

A perpendicular line</td>
</tr>
</table>

Geometric figure construction with straightedge or straightedge and compasses		
 A straightedge-only arrowhead construction	 An angle bisector disappearing	 A perpendicular sector
 A straightedge-only kite construction	 Non-examples of an angle bisector	 A tangent to two circles
 A straightedge-only congruent triangle construction	 Non-examples to examples of an angle bisector	 A general kite
 A straightedge-only isosceles trapezium construction	 An angle bisector example	 A rhombus
 A straightedge-only similar triangle construction	 Non-examples of angle bisector parallel to the angle bisector	 A square

<table>
<tr><th colspan="3">Geometric figure construction with straightedge or straightedge and compasses</th></tr>
<tr>
<td>

A straightedge-only similar triangle construction</td>
<td>

A perpendicular ray/line constructed without extending line segment AB</td>
<td>

A square and a tangent</td>
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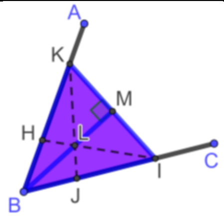
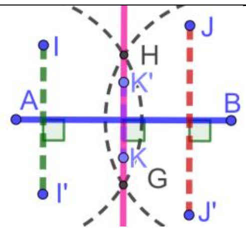
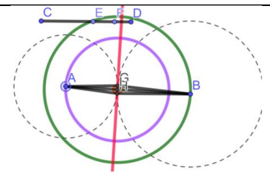
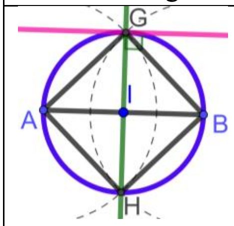
<tr>
<td>

An isosceles, congruent and scalene triangles</td>
<td>

Reflection and perpendicular</td>
<td>

A kite disappearing</td>
</tr>
<tr>
<td>

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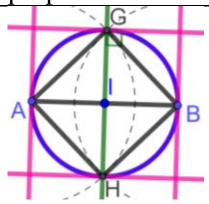
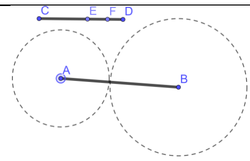
<td>

Midpoints, parallel lines and squares.</td>
<td>

A kite disappeared</td>
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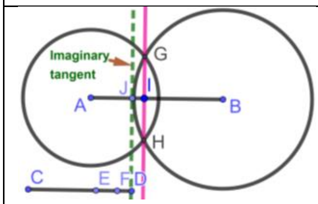
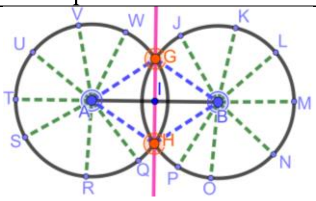
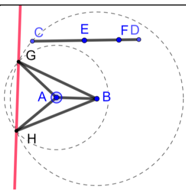
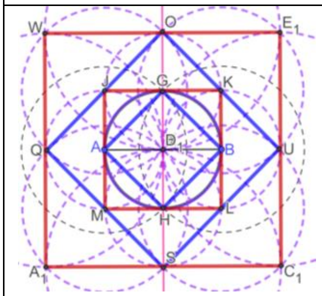
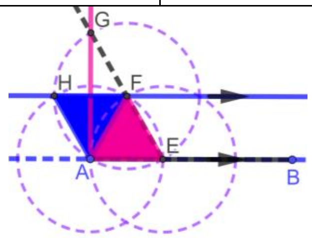
<td>

An imaginary tangent parallel to the perpendicular line</td>
<td>

Two equidistant points (G and H) from A and B, and an angle bisector.</td>
<td>

An arrowhead</td>
</tr>
<tr>
<td colspan="2">

Concepts of midpoints and parallel lines in constructing many squares.</td>
<td>

Equilateral triangles, parallelograms, kites, rhombus, perpendicular lines and parallel lines</td>
</tr>
</table>

7.5 Contribution to existing literature

My contributions to knowledge are both substantial and impactful. I designed a series of geometric construction tasks within an online dynamic geometry software platform, specifically aimed at supporting beginning teachers in learning and co-constructing knowledge for teaching geometric constructions. These tasks, embedded within a technology-rich environment, facilitated productive mathematical talk and dialogic learning, thereby supporting the development of beginning teachers' TPACK. The tasks I developed have the potential to be adapted or replicated in other learning environments to support similar goals. Moreover, the online dynamic mathematical software learning environment used in this study functions as a versatile learning ecosystem and could be effectively applied to tasks in other areas of mathematics, including the learning of functions and graphs.

Drawing on interpretive video deductive coding, guided by two analytical tools—first, the dialogic talk and learning principles established by Alexander (2008, 2020), and second, the TPACK framework developed by Koehler and Mishra (2009)—I analysed how beginning teachers co-construct knowledge within this environment. My study builds on and extends the work of Clark-Wilson (2017), Ruthven (2018a; 2018b), and Koehler and Mishra (2009), demonstrating how thoughtfully designed tasks can support the development of beginning teachers' knowledge for teaching geometry.

The analysis revealed that these tasks fostered complex interactions of dialogic talk, which in turn led to active learning, critical thinking, and a deep understanding of geometric concepts. This dialogic engagement proved to be a rich environment for knowledge co-construction, highlighting the complex nature of the discourse and its role in advancing TPACK development. Importantly, the learning ecosystem within which these tasks were embedded provides proof of concept for achieving TPACK development in classroom practice, illustrating how tasks and technology can collaborate to support teacher preparation.

Overall, my contribution to knowledge provides valuable insights into how we can design tasks within dynamic technology environments that promote the learning and co-construction of TPACK knowledge and foster meaningful dialogue that improves understanding. These findings contribute to improving teacher education programmes by suggesting effective strategies for supporting the integration of technology into geometry teaching, ultimately equipping the next generation of mathematics teachers with the necessary skills and knowledge to teach geometry in secondary schools effectively.

7.6 Limitations of the study

While this research provides valuable insights into the integration of technology in mathematics education for beginning teachers, it is important to acknowledge its limitations:

1. **Limited generalisability:** The findings of this study are based on a small number of specific groups of beginning teachers in a particular designed educational context. Generalising the results to a broader population of teachers or diverse educational settings should be done cautiously. Different contexts may yield different outcomes.

2. **Sample size:** The study's sample size was relatively small, limiting the diversity and representativeness of the participants. A larger and more

diverse sample could provide a broader perspective on the issues under investigation.

3. **Short-term focus:** The study primarily examined the short-term impact of technology integration on beginning teachers. The long-term effects and sustainability of the observed changes were not explored. A longitudinal study would be necessary to assess the durability of the findings.

4. **Technology platform specific:** The study focused on a specific dynamic geometry software (DGS) platform used in a particular online environment. The findings may not be directly applicable to other educational technology tools or platforms. Future research should explore the generalisability of the results to a broader range of technology applications.

5. **Teaching context:** The study's findings are dependent on the specific context of geometric construction tasks in mathematics education. The transferability of these findings to other areas of the geometry curriculum, other mathematical topics or subject areas remains an open question.

6. **Limited exploration of student outcomes:** The study primarily focused on beginning teachers' experiences and knowledge development. The impact of technology integration on student learning outcomes was not within the scope of this research and warrants further investigation.

7. **Sociocultural and contextual factors:** The research did not extensively investigate the influence of sociocultural and contextual factors on beginning teachers' experiences with technology integration. These factors may have a significant impact on the results.

Recognising these limitations, future research should aim to address these gaps and provide a more comprehensive understanding of the integration of technology in mathematics education, particularly for beginning teachers.

7.7 Recommendations

7.7.1 For teacher training programmes and educational institutions

The findings of this research offer valuable insights for educational practice and policy in the pursuit of teacher learning towards teaching geometry in the secondary curriculum. Based on the research outcomes, the following recommendations are suggested. They are organised by first addressing issues relating to the design of learning environments 1, 2, 3, secondly addressing issues relating to task design 4, 5, 6, and finally considering future directions for research in the field – section 7.7.2.

1. **Design of learning environments:** When designing technology-enhanced learning environments, designers and educators should prioritise open dialogue and collaboration in ways that encourage dialogue that promotes varied dialogic talk types, aligning with dialogic learning principles. Further, educators and designers should ensure that these designs have the potential for dialogic learning to naturally emerge and provide opportunities for structured reasoning, exploratory discourse, evaluative reflection, and inquiry-driven discussions.

2. **Designers of learning environments for use by beginning teachers should ensure that they:**

   a. Promote the use of exploratory tasks that connect the visual and interactive nature of technology and encourage collaborative pair work to promote dialogue and co-creation of knowledge. These strategies can help beginning teachers develop a versatile pedagogical toolbox and a deep understanding of geometric concepts.

   b. Encourage interactive and exploratory tasks that use software such as dynamic geometry software to deepen the mathematical content knowledge of beginning teachers. These tasks should help bridge the gap between theoretical understanding and practical application, accommodating diverse learning backgrounds and recognising the complex nature of teachers' roles in technology-focused mathematics education.

   c. Promote proficiency not only in the use of technological tools but also in content knowledge and pedagogical expertise, all of which are essential for bridging the gap between mathematical concepts and technological application

3. **Address challenges and provide support**: Designers of courses for beginning teachers should recognise and address challenges associated with integrating dynamic geometry software, such as technical issues and the initial learning curve. Offering proactive support and resources that have potential for immediate use in classrooms can ensure optimal technology integration in geometry lessons, thereby enhancing the learning experience for beginning teachers.

4. **Facilitate transition between digital and manual tools**: Courses of beginning teachers should provide support to participants (beginning teachers) to ensure a smooth transition between dynamic geometry software and traditional manual tools or vice versa. Guiding individuals with carefully designed tasks can lead to insight and understanding ensuring adaptability in applying geometric concepts across different modalities.

5. **Maintain engagement and motivation**: Tasks should be designed in ways that can stimulate curiosity, sustain motivation, and promote active engagement throughout the beginning teachers' learning process. Using dynamic geometry software can lead to interactive exploration and hands-on learning experiences, while collaborative learning approaches can encourage engagement through discussion, idea-sharing, and problem-solving.

6. **Task Design:** To support the development of TPACK knowledge among beginning teachers, task designers (teacher educators) should design tasks that allow them to effectively engage them in learning to use dynamic geometry software simultaneously with geometric construction techniques, and engage in discussions about their developing understanding and potential pedagogical approaches and strategies. This potential can be maximised by incorporating task design principles and features as set out in the previous section above.

7.7.2 Researchers and educational technology developers should:

1. Research the intentional design of learning environments that foster dialogic learning and support the co-construction of knowledge among beginning teachers.
2. Explore how different geometric construction tasks and their design features can be optimised to enhance beginning teachers' development of knowledge for teaching geometry in technology-rich environments.
3. Explore the long-term impact of proficiency with dynamic geometry software (DGS) on beginning teachers' technological, pedagogical, and content knowledge development and investigate how this proficiency can be used for more effective mathematics lessons.
4. Develop and test professional development programmes that explicitly focus on technology integration in mathematics education, targeting beginning teachers. They should evaluate the impact of such programmes on the development of teachers' TPACK knowledge and their instructional practices.

7.8 Closing remarks

This study has demonstrated the potential of efficacious geometric construction tasks when working with dynamic geometry software in an online platform learning ecosystem that fosters productive mathematical talk and dialogic learning among beginning teachers. These tasks encourage various dialogic talk types, grounded in dialogic learning principles, and emphasise the importance of collaboration and dialogue in developing TPACK knowledge. The findings illustrate how beginning teachers can be supported to perceive, understand, talk and co-construct the knowledge required for teaching geometry in technology-based environments.

In conclusion, this research contributes to ongoing discussions surrounding the integration of technology in mathematics education. It offers empirical insights into how carefully designed technology-supported tasks can enrich the professional preparation of beginning teachers. While the focus was on geometric constructions, the design principles and digital tools employed in this study offer broader applicability. The task design approach and the use of dynamic software demonstrated in this research can be extended to support learning in other mathematical domains, including functions, by advancing similar forms of conceptual exploration and dialogic engagement.

Addressing the recommendations identified in this study provides opportunities for teacher education programmes, educational institutions, researchers, policymakers, and technology developers to collaboratively improve mathematics education. Intentional integration of digital technologies into teacher preparation can contribute to the development of confident, pedagogically informed beginning teachers who are equipped to create meaningful and engaging learning experiences. While challenges remain, this research lays a foundation for future inquiry and stresses the transformative potential of well-designed tasks in dynamic, technology-rich learning environments.

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APPENDIX A: ETHICAL APPROVAL

School of Education
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13/05/2020

Our Ref: 2020/10

Dear Peter Awortwe

Thank you for your research ethics application for your project:

Understanding the content, pedagogical and technological knowledge of Beginning Teachers using technology in relation to teaching geometric constructions using dynamic software

Our Ethics Committee has looked at your submission and has the following comments.

□ Thank you for addressing the issues raised by the reviewers.

Based on the above assessment, it is deemed your research is:

□ Approved

This research is approved provided it is completed by August 2023

If your research overruns this date, please contact the Ethics Team to arrange an extension and update on any additions/changes to your work.

We wish you very good luck with your research study.

Professor Mary Oliver

Ethics Committee

APPENDIX B: CONSENT FORM FOR THE PARTICIPANTS

Participant Consent Form

Title of Study: Understanding the content, pedagogical and technological knowledge of Beginning Teachers using technology in relation to Geometric Construction using Dynamic Software

Name of Researcher: Peter Kwamina Awortwe (PhD student, University of Nottingham)

I would like to invite you to take part in my research study. Before you agree I would like you to understand why the research is being done and what it would involve for you.

What is the purpose of the study?

This study aims to understand how to improve teaching that focuses on the content, pedagogical and technological knowledge of beginning teachers when teaching geometric construction using dynamic software. The study will investigate ways in which beginning teachers use the GeoGebra dynamic mathematical software, as well as elicit their views about its use. The research will involve screen video and audio recording of participants using the software as well as interviewing them about the experience. The beginning Teachers will be PGCE Students in Secondary Mathematics Education in some UK universities. The research will be conducted in late June and July 2021. The research can be carried out with participants working remotely from the researcher. The participants (PGCE students) are assumed to already know how to use the online Microsoft Teams system or similar system(s) and it is envisaged that this online research will not prove problematic for them.

Why have I been invited?

You are being invited to take part because the use of technologies like GeoGebra software is being recommended in teaching and learning Mathematics. The study will provide insight into how you might best use the alternative methods of teaching and learning geometrical construction that dynamic GeoGebra software affords.

Do I have to take part?

It is up to you to decide whether or not to take part. If you do decide to take part, you will be given this information sheet to keep and be asked to sign a consent form. If you decide to take part, you are still free to withdraw at any time and without giving a reason. If you wish to withdraw you may contact the researcher using the details at the bottom of this sheet. This would not affect your legal rights.

What will happen to me if I take part?

Screen video and audio recordings of your investigatory use of GeoGebra software will be made. You will work as part of a pair with another PGCE student. You will be interviewed before and after the intervention concerning your knowledge of content, technology and pedagogy. The pre-interview (or questionnaire) will take 10-15minutes and post-interview will take 15-20 minutes. Each group of pairs will have two or four tasks to perform, and each participant will lead one or two tasks. The one leading will share his or her screen with their partner. Each participant will be expected to contribute and make suggestions for each task. Each task will take 35 to 45 minutes. (Maximum of 1.5 hours for 2 tasks or 3 hours for 4 tasks). In collaboration with you, we would either use one day with 3 hours to complete all 4 tasks or two days with 1.5 hours to complete 2 tasks each day. The post-interview will be on your experiences of using the software. These interviews will also gather some background data that provides some basic details in relation to your preparation for, and experience of, teaching mathematics at KS3 and 4.

Recordings will be transcribed. If you wish to see the transcriptions these will be provided.

What are the possible benefits of taking part?

A benefit to you of participating in the research will be learning how to use dynamic geometry software to teach geometry in secondary schools in a way that will bring a deeper understanding of geometry concepts. This modern way of using dynamic software to teach geometric constructions centrally involves inductive approaches and pedagogies that support a deeper understanding of geometry. This modern way of using dynamic mathematical software to teach supports the purpose for which the software was invented, such that it helps to turn mathematics into laboratory-based science. As a mathematics teacher, in this technological era, you will have opportunities to improve your technological pedagogical content knowledge for teaching geometric construction.

What if there is a problem?

If you have a concern about any aspect of this study, you should speak to the researcher who will do his best to answer your questions. If you remain unhappy and wish to complain formally, you can do this by contacting the School Research Ethics Officer. All contact details are given at the end of this information sheet.

Will my taking part in the study be kept confidential?

I will follow ethical and legal practice and all information about you will be handled in confidence. Participation will remain confidential and your data will be anonymised. All information which is collected about you during the course of the research will be kept strictly confidential. The data will be stored in the university's Microsoft Teams system, and this is encrypted within the University of Nottingham MS account.

Your personal data (contact details) will be kept for up to seven years after the end of the study so that I can contact you about the findings of the study (unless you advise me that you do not wish to be contacted). After this time your data will be disposed of securely. During this time all precautions will be taken to maintain your confidentiality, only the researcher will have full access to the data. Extracts of the data may be discussed with the researcher's supervisors. Only audio recordings and onscreen working with the dynamic geometry software will be shared with the researcher's supervisor ensuring that this research is kept totally separate from any identification in relation to any individual's academic ability.

Although what you say in the interview is confidential, should you disclose anything to me which I feel puts you or anyone else at any risk, I may feel it necessary to report this to the appropriate persons.

What will happen if I don't want to carry on with the study?

Your participation is voluntary, and you are free to withdraw at any time, without giving any reason, and without your legal rights being affected. You may withdraw by contacting the researcher using the details below. If you withdraw after one year it may not be possible to extract and erase some of your data from the project's analysis and write-up.

What will happen to the results of the research study?

The results of the study will be written up within my PhD thesis. I expect to present findings from my research at conferences and in academic and professional journals.

Who is organising and funding the research?

My research is being organised by the University of Nottingham, there is no external funding involved.

Who has reviewed the study?

All research at the University of Nottingham is looked at by a group of people, called a Research Ethics Committee, to protect your interests. This study has been reviewed and approved by this committee.

Further information and contact details:

Researcher: Peter Kwamina Awortwe

Email: Peter.Awortwe@nottingham.ac.uk

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Email: Geoffrey.Wake@nottingham.ac.uk

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School of Education Research Ethics Coordinator:

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APPENDIX C: INFORMATION SHEET

Participant Information Sheet

(Beginning Teachers)

Project title: Understanding the content, pedagogical and technological knowledge of Beginning Teachers using technology in relation to geometric construction using Dynamic Software

Researcher's name: Peter Kwamina Awortwe

Supervisor's name: Pofessor Geoff Wake

Supervisor's name: Professor Andrew Noyes

- I have read the Participant Information Sheet and the nature and purpose of the research project have been explained to me. I understand and agree to take part.
- I understand the purpose of the research project and my involvement in it.
- I understand that I may withdraw from the research project at any stage and that this will not affect my status now or in the future.
- I understand that while information gained during the study may be published, I will not be identified and my personal results will remain confidential.

- I understand that I will be audio/video recorded during my use of the dynamic geometry software GeoGebra and I will be interviewed concerning these.

- I understand the researcher will make transcripts of the recorded online work sessions and that I will have the right to see transcripts and check for accuracy.

- I understand that the data will be stored in the university's Microsoft Teams system, and this is encrypted within the University of Nottingham MS account and that only the researcher will have access to this material in full, and that extracts may be discussed between the researcher and supervisor during supervisions**.** Participants will be anonymised by the researcher and only audio recordings and onscreen working with the dynamic geometry software will be shared with the researcher's supervisor ensuring that this research is kept totally separate from any identification in relation to any individual's academic ability.

- I understand that I may contact the researcher or supervisor if I require further information about the research and that I may contact the Research Ethics Coordinator of the School of Education, University of Nottingham if I wish to make a complaint relating to my involvement in the research.

Signed ……………………………………………………………………

(research participant)

Print name ……………………………………………………………………

Date …………………………………

Contact details

Researcher: Peter Kwamina Awortwe

Email: Peter.Awortwe@nottingham.ac.uk

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Supervisor: Professor Andrew Noyes

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School of Education Research Ethics Coordinator:

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APPENDIX D: POST-INDIVIDUAL INTERVIEW QUESTIONS

The information gathered in this interview is intended for reference purposes only. All responses will be treated with the utmost confidentiality and anonymity. Please provide candid responses to each of the questions.

Understanding the Content, Pedagogical, and Technological Knowledge of Beginning Teachers Using Technology in relation to Geometric Constructions with Dynamic Software

Maths Knowledge

1. In what manner did the exploratory tasks assist you in acquiring relevant geometry content?

2. Did you find the inductive approach beneficial for learning about geometric constructions? If so, in what ways was it helpful?

3. Were the tasks carefully designed to facilitate learning of appropriate content? If yes, in what way?

Pedagogy

4. What pedagogical strategies have you acquired from these exploratory tasks using GeoGebra?

5. How pertinent do you find the pedagogical strategy of working in pairs with the software?

6. Do you plan to employ these pedagogical approaches in teaching geometric constructions?

Technological Knowledge

7. What technological knowledge have you gained from working through the exploratory tasks?

8. What technical knowledge have you acquired for teaching geometric constructions with dynamic geometry software?

9. What should be the nature of teacher knowledge for teaching geometry in technology-focused contexts?

General

10. What knowledge do teachers require to teach geometry using technology?

11. What do you perceive as the benefits of using dynamic software in the mathematics classroom?

12. Would you design or create geometric construction tasks for students to complete using dynamic software?

13. What mathematical connections do the concepts you worked with have with other mathematical concepts?

APPENDIX E: POST QUESTIONNAIRES

The information provided in this questionnaire will be used for reference only. All information will be treated with high confidentiality and anonymity. Please respond to each of the questions. Please be as candid as you can in your responses.

Understanding the content, pedagogical, and technological knowledge of Beginning Teachers using technology in relation to geometric constructions using dynamic software

Maths knowledge

1. In what way did the exploratory tasks help you to learn appropriate geometry content?
2. Did you find the inductive approach helpful in learning about geometric constructions?

   If yes, in what ways was it helpful?
3. Did you find the tasks carefully designed to follow to learn appropriate content?

   If yes, in what way?

Pedagogy Knowledge

4. What pedagogical strategies have you learned from these exploratory tasks using GeoGebra?
5. How relevant is the pedagogical strategy of working in pairs with the software?

6. Will you use these pedagogical approaches to teach geometric constructions?

Technological Knowledge

7. What technological knowledge have you learned from working through the exploratory tasks?

8. What technical knowledge have you learned to teach geometric constructions with dynamic geometry software?

9. What should be the nature of teacher knowledge for teaching geometry in technology-focused contexts?

General

10. What knowledge do teachers need to teach geometry using technology?

11. What do you think are the benefits of using dynamic software in the mathematics classroom?

12. Would you design/create geometric construction tasks for students to complete using dynamic software?

13. What mathematical links do the concepts you worked with have with other mathematical concepts?

APPENDIX F: TIMETABLE FOR THE STUDY

Appendix F shows the actual time all the events took place.

Phase One (Pilot Study)				
Day/Date	Time	Task/Activity	Participant	Learning environment
Day 1 29/06/2020	10:00-11:30 11:40-13:10	Tasks 1 and 2 Tasks 1 and 2	Pair 1 Pair 2	Online Teams
Day 2 30/06/2020	10:00-11:30 11:40-13:10	Tasks 3 and 4 Tasks 3 and 4	Pair 1 Pair 2	Online Teams
Day 3 01/07/2020	13:00-14:00	Focus Group Discussion	Four participants	Online Teams
Day 3 01/07/2020	14:30	Questionnaire see Appendix E	Four participants	Email
Day 4 02/07/2020 06/07/2020 08/07/2020 08/07/2020	9:15-09:45 10:00-10:30 12:00-12:30 15:30-16:00	Individual Interview see Appendix D	Participant 1 Participant 2 Participant 3 Participant 4	Online Teams

Phase Two (Main Study)				
Day/Date	Time	Task/Activity	Participant	Learning environment
Day 1 16/06/2021	10:00-11:30 10:00-11:30 12:00-13:30 13:00-14:30	Tasks 1 and 2 Tasks 1 and 2 Tasks 3 and 4 Tasks 3 and 4	Pair 3 Pair 4 Pair 3 Pair 4	Online Teams
Day 1 16/06/2021	13:30-14:00 14:30-15:00	Focus Group Discussion	Pair 3 Pair 4	Online Teams
29/07/21 30/07/21 Days 2/3	10:00-11:30 10:00-11:30 11:30-12:00	Tasks 1 and 2 Tasks 3 and 4 Focus Group Discussion	Pair 5 Pair 5 Pair 5	Online Teams
Day 3	10:00-11:30 10:30-12:00 12:00-12:30	Tasks 1 and 2 Tasks 3 and 4 Focus Group Discussion	Pair 6 Pair 6 Pair 6	Online Teams
Day 4		Questionnaire	6 participants	Email
Day 5 20/06/2021 24/06/2021 01/07/2021 04/08/2021 16/08/2021 25/08/2021 25/08/2021	12:00-12:30 13:00-13:30 16:00-16:30 11:00-11:30 9:30-10:00 9:00-9:30 11:00-11:30 12:00-12:30	Individual Interview see Appendix D	Participant 1 Participant 2 Participant 3 Participant 4 Participant 5 Participant 6 Participant 7 Participant 8	Online Teams

APPENDIX G: CODING AND ANALYSING VIDEO DATA WITH THE TPACK FRAMEWORK

This appendix presents a coded analysis of beginning teachers' engagement in a geometric construction task using GeoGebra. Episodes from the video data were transcribed and aligned with screenshots to identify evidence of Technological Knowledge (TK), Pedagogical Knowledge (PK), Content Knowledge (CK), and their intersections (TPK, TCK, PCK, TPACK). Each excerpt was coded and interpreted to show how participants explored geometric concepts, addressed misconceptions, and used technology to support reasoning. Appendix J combines visual, verbal, and analytical data to illustrate how TPACK developed through collaborative, technology-enhanced inquiry

Key for Colour-Coded Excerpts:

CK (Content Knowledge) – 🔵 Blue

PK (Pedagogical Knowledge) – 🟢 Green

TK (Technological Knowledge) – 🟣 Purple

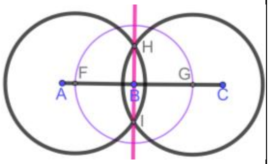
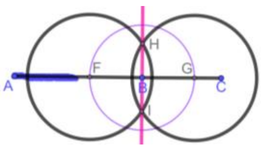
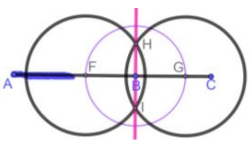
PCK (Pedagogical Content Knowledge) – 🟡 Yellow

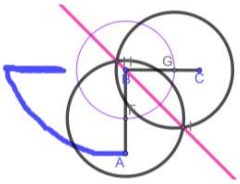
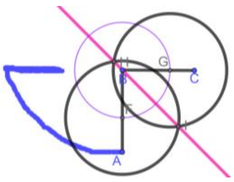
TCK (Technological Content Knowledge) – 🔴 Red

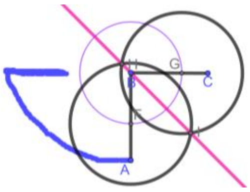
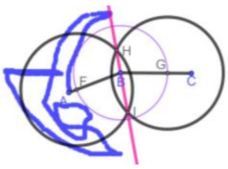
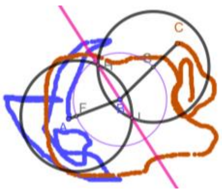
TPK (Technological Pedagogical Knowledge) – 🟠 Orange

Metacognitive Talk – 🟤 Brown

TPACK (Integrated Technological Pedagogical Content Knowledge) - ⚫ black

Screenshot	Excerpt	Codes	Interpretation	Rationale
	M: *'OK, what geometrical relationship does the line that passes through points H and I have to the angle ABC?'* M: *'A perpendicular bisector.'*	PCK CK CK	M initiates geometric reasoning and hypothesises a relationship, prompting conceptual exploration.	**PCK**: Uses questioning to engage peer thinking. **CK**: Understanding of geometric relationships.
	N: *'Yeah, that's a perpendicular line.* *What happens if you move point A?* *All right, just move point A about there.'* M: *[Drags point A almost horizontally.]*	TK, TPK, PCK	N encourages dynamic manipulation to test the hypothesis, while M uses GeoGebra for exploratory verification.	**TK**: Manipulation using GeoGebra. **TPK**: Tech-supported inquiry. **PCK**: Encourages hypothesis testing and peer dialogue.
	M: *'No, that's not it. It's not a perpendicular bisector.'*	CK, PCK	M corrects a conceptual misconception after testing it dynamically.	**CK**: Distinguishes incorrect geometric properties. **PCK**: Demonstrates

Screenshot	Excerpt	Codes	Interpretation	Rationale
				reflection and responsive pedagogy.
	N: *'Make it like ABC right triangle.'* M: *[Drags point A to form a right-angled triangle.]*	CK, PCK, TPK, TK	N suggests a construction, M follows with dynamic manipulation, refining conceptual understanding.	**CK**: Suggests task adaptation to uncover relationships. **TK**: Uses GeoGebra to implement idea. **TPK/PCK**: Combines strategy with conceptual goal.
	N: *'OK. So, you see, it cuts the angle in half by bisecting the angle.'* M: *'Yeah, that is right. Yeah. That's sure.'*	TPACK, CK, PCK	N and M consolidate the understanding that the line is an angle bisector through collaborative construction and reasoning.	**TPACK**: Seamless integration of tech, pedagogy, and geometry. **CK**: Recognition of angle bisector. **PCK**: Use of explanation to

Screenshot	Excerpt	Codes	Interpretation	Rationale
				consolidate learning.
	N: *'So, what geometrical relationship is that?'* M: *'It is an angle bisector. OK. Is that right?'*	PCK, CK	Encourages verbalisation of the concept to affirm understanding.	**CK**: Uses appropriate terminology. **PCK**: Scaffolds peer reflection and articulation.
 	N: *'Yeah, I feel like that is right. So, when you move either of those points, you are just changing the angle, but the line still bisects the angle. It is good to learn concepts like this.'*	TPACK, TPK, TK, CK	N reflects on how dynamic tools enhance concept generalisation and learning benefits.	**TPACK**: Fully integrated reflection. **TPK/TK**: Demonstrates tech's role in observing invariant properties. **CK**: Insight into conceptual consistency.

APPENDIX H: ANALYSIS OF TECHNOLOGICAL, CONTENT, AND PEDAGOGICAL KNOWLEDGE IN PAIRED PARTICIPANTS' DISCUSSIONS AND TASK PERFORMANCES

Table L.1: Indicates technological knowledge in the paired participants' discussions and performance of the four tasks.

Pair/Task	Technological Knowledge
A and B Task 1	1. Discussed using GeoGebra tools for constructing line segments, circles, points, and lines. 2. Observed how line *h* changed dynamically when moving points *E* and *F*.
C and D Task 1	3. Explored construction of circles and lines using GeoGebra.
E and F Task 1	4. Expressed initial difficulty using the GeoGebra compass tool. 5. Collaboratively addressed software-related challenges and learned to use GeoGebra tools.
G and H Task 1	6. Encountered struggles using the compass tool to draw circles. 7. Successfully constructed intersecting points *G* and *H* with guided support on using GeoGebra.
A and B Task 2	8. Investigated interactive capabilities of GeoGebra. 9. Engaged in hands-on manipulation (dragging points) to visualise quadrilateral transformations.
C and D Task 2	10. Cooperatively constructed circles with varied radii and altered line styles in GeoGebra for clarity. 11. Explored dynamic aspects of quadrilaterals and circles.
E and F Task 2	13. Used the *Move tool* to observe transformations by dragging points. 14. Demonstrated proficiency in the *Intersect tool* use and line customisation.

Pair/Task	Technological Knowledge
G and H Task 2	15. Manipulated circles and segments to examine quadrilateral properties. 16. Used visual illustrations with different line styles and constructions.
A and B Task 3a	17. Created line segments, circles, and manipulated elements dynamically. 18. Identified angle-bisecting properties through GeoGebra's dynamic feedback.
C and D Task 3a	19. Used the *Intersect tool* effectively to identify key points. 20. Engaged in dynamic exploration through constructed segments and circles.
E and F Task 3a	21. Systematically used tools like *Intersect* and *Centre and Radius* for construction. 22. Observed dynamic changes by dragging points.
G and H Task 3a	23. Explored angle bisectors and circles using GeoGebra tools. 24. Created multiple constructions involving segments and circles.
A and B Task 3b	25. Explored examples/non-examples of angle bisectors dynamically. 26. Engaged in detailed construction and manipulation using GeoGebra tools.
C and D Task 3b	27. Explored systematically, including changing the colour of circle B. 28. Discussed geometrical relationships as they dragged points.
E and F Task 3b	30. Noted construction changes through dynamic dragging.
G and H Task 3b	31. Investigated angle bisectors in relation to circles. 32. Manipulated points and lines to examine the impact on bisectors.
A and B Task 4a	33. Constructed circles and lines; explored dynamic changes. 34. Referenced traditional methods with ruler and proposed alternatives.

Pair/Task	Technological Knowledge
C and D Task 4a	35. Engaged in collaborative digital construction. 36. Discussed effects of point manipulation and radius alteration.
E and F Task 4a	38. Made keen observations and logical inferences based on digital interaction.
G and H Task 4a	39. Collaboratively constructed and manipulated geometric elements. 40. Critically examined how dynamic actions with GeoGebra affected the construction.
A and B Task 4b	41. Explored properties of perpendiculars and angles in semicircles. 42. Referenced tools like *circles, line segments, and rays*.
C and D Task 4b	43. Continued use of GeoGebra for geometric constructions. 44. Described the use of the *compass tool* and circle construction.
E and F Task 4b	45. Used the *compass and Intersect tools* proficiently. 46. Validated observations through dynamic dragging.
G and H Task 4b	47. Systematically used tools for constructing intersections and segments. 48. Discussed efficiency of constructing full circles vs. segments and software's optimisation features.

Table L.2: Indicates content knowledge in the paired participants' discussions and performance of the four tasks.

Pair/Task	Content Knowledge (Participants' Excerpts Expressions)
A and B Task 1	1. *'Wait, this line GH isn't just random—it's actually linked to the midpoint of AB, right?'* 2. *'Oh! But it's not exactly a perpendicular bisector unless the circles are equal.'* 3. *'When we made the circles smaller, the line moved differently. That's interesting!'* 4. *'To draw a perpendicular bisector manually, we need equal circles crossing over—then draw a line through their intersections.'*
C and D Task 1	5. *'Look, line GH is straight up and down between A and B. It must be perpendicular because of how the circles intersect.'* 6. *'As we drag point A or B, see how line h stays vertical? That symmetry's doing something.'* 7. *'It's like the software is forcing GH to become the perpendicular bisector when the circles are equal.'*
E and F Task 1	8. *'This looks like a kite—those sides are equal lengths.'* 9. *'Points G and H are equidistant from A and B, so this has to be a reflection.'* 10. *'If we play with the radii, the shape changes but still reflects balance between the points.'*
G and H Task 1	11. *'So, GH isn't always perpendicular—it depends on the circle size.'* 12. *'But if we make both circles the same, GH becomes the bisector of AB.'*
A and B Task 2	13. *'Quadrilateral AGBH is kind of like a [general] kite now, but when I move E here, it looks like a rhombus.'* 14. *'Changing E on CD changes the whole shape. See how the sides align differently?'* 15. *'We can tell it's a square when all sides look equal and the angles stay at 90 degrees.'*

Pair/Task	Content Knowledge (Participants' Excerpts Expressions)
C and D Task 2	16. *'This shape here—it's first a kite, then it turns into a rhombus and eventually a square!'* 17. *'We have got perpendicular diagonals here—that's one of the properties, right?'* 18. *'Let's add another circle and see what relationships it creates.'*
E and F Task 2	19. *'If E and F are too close, the shape squashes, but when spread evenly, AGBH looks like a kite.'* 20. *'These circles look like they share a centre—concentric?'* 21. *'Point G is reflecting over in the line segment AB, getting point H.'*
G and H Task 2	22. *'The shape looks like a rhombus when these sides are equal.'* 23. *'But when the radii change, it becomes more like a square.'* 24. *'There's something about the intersection points that changes the angle.'*
A and B Task 3a	25. *'Look, line HI cuts the angle perfectly—it's an angle bisector!'* 26. *'Even when I move A or B, HI still cuts evenly—that's interesting.'* 27. *'So, the radii must help in stabilising where the bisector goes.'*
C and D Task 3a	28. *'This line through H and I always seems to split angle ABC in half.'* *'Maybe it's because the radii are equal and so the points are equidistant.'*
E and F Task 3a	29. *'No matter how we change the circle or the angle, HI still bisects it.'* 30. *'The angle changes, but that central line stays stable.'*
G and H Task 3a	*'When both circles have equal radii, HI always bisects the angle—it's like a rule.'*
A and B Task 3b	31. *'The radii here make a big difference—if they're not equal, it's not a proper bisector.'* 32. *'It's symmetrical only when circles are the same size.'* 33. *'JB isn't a bisector here—but JK is when the distances match.'*
C and D Task 3b	*'The radii from D and E affect how the line between J and K behaves—it bends differently if one's larger.'*
E and F Task 3b	34. *'Unless the radii match, we can't say it's an angle bisector.'* 35. *'When the circles are equal, JK becomes the angle bisector.'*
G and H Task 3b	36. *'Equal radii are necessary for a true angle bisector.'* *'If we were using paper, this would be hard to replicate accurately.'*

Pair/Task	Content Knowledge (Participants' Excerpts Expressions)
A and B Task 4a	*37. 'We have got an isosceles trapezium here—and the diagonals are acting like bisectors.'* *38. 'Looks like a kite again—see how the diagonals intersect?'*
C and D Task 4a	*39. 'Does this line always bisect the angle at B? Let's move D and E.'* *'This shared centre means all the points are equidistant—must be why it stays stable [angle bisector].'*
E and F Task 4a	*40. 'When D and E move, the construction still works—it's consistent.'* *'The distances stay the same even when the radii change—that's interesting.'*
G and H Task 4a	*41. 'These angles, JBL and HBL, look equal. Same with BHI and BJK.'* *'When I move point H, the angle adjusts, but the relationships hold.'*
A and B Task 4b	*42. 'Ray AG is always perpendicular to AB. That has to do with the semicircle.'* *43. 'It fits with the Circle Theorem—angle in a semicircle is 90 degrees.'*
C and D Task 4b	*44. 'This looks like a right angle here between the ray and AB.'* *'There's a tangent line at play here—we didn't draw it, but it's implied.'*
E and F Task 4b	*45. 'That ray cuts AB at 90 degrees—has to be perpendicular.'* *'Let's test it with the circle theorem again.'*
G and H Task 4b	*46. 'The diameter controls the direction of the perpendicular line.'* *47. 'These intersections form predictable right angles depending on how the circles overlap.'*

Table L.3: Indicates pedagogical knowledge in the paired participants' discussions and performance of the four tasks.

Pair/Task	Pedagogical Knowledge	Participant Excerpts
A and B – Task 1	Recognition of symmetry's role in learning geometry	1. *'The circles kind of show that GH has to be symmetrical to AB so that students can see that connection more easily.'*
C and D – Task 1	Manual construction and significance of radii	2. *'It's all about where the radii intersect—students will realise the radii have to be the same length to get that perpendicular bisector.'*
E and F – Task 1	Value of hands-on, exploratory learning with DGS	3. *'When I used GeoGebra to play around with it, I understood it better… so I think kids need that time to experiment too.'*
G and H – Task 1	Discovering the 'why' through technology	4. *'With GeoGebra, they [pupils/students] are not just doing steps [following procedures] – they can see why things work. That's more powerful.'*
A and B – Task 2	Scaffolding and using critical thinking prompts	5. *'If we ask, 'Is this always true?' it makes them really think about why the lines [diagonals] behave that way.'*
C and D – Task 2	Improving clarity through visual manipulation	6. *'I moved the points to see if it still stayed a parallelogram—that is something I will get my students to try.'*
E and F – Task 2	Importance of step-by-step construction	7. *'Doing it manually helped me think about each step… I want to use that to teach students the construction.'*
G and H – Task 2	Investigating dynamic geometry in teaching	8. *'It's useful to let them change it and see what breaks, where the circles do not intersect… that's how they will understand what matters.'*

Pair/Task	Pedagogical Knowledge	Participant Excerpts
A and B – Task 3a	Collaborative discovery and hypothesising	9. *'We think this point is the centre of rotation… but we need to test that more. It's like a mini-investigation.'*
C and D – Task 3a	Connecting patterns to teaching complexity	10. *'Angle bisectors always seem to meet here… It's something I would want students to discover by themselves.'*
E and F – Task 3a	Reflecting on transfer of learning	11. *'We used a similar method in the last one, so maybe that's why it feels familiar. That could help students link ideas.'*
G and H – Task 3a	Use of DGS for angle bisector exploration	12. *'Dragging this changes the angle… see, it still works! That's the power of dynamic tools.'*
A and B – Task 3b	Using contrastive examples	13. *'JB and JK aren't the same—look at the radii. That's what changes whether it's an angle bisector or not.'*
C and D – Task 3b	Highlighting critical properties	14. *'If they see the radii lengths, they will know why this doesn't count as an angle bisector—it is not obvious.'*
E and F – Task 3b	Reinforcing key features across tasks	15. *'Again, it's the radii… just like before. Students can learn that pattern if we guide them.'*
G and H – Task 3b	Connecting theory to classroom practice	16. *'It's hard to do this without dynamic technology—students might not see what makes it work without dynamic feedback.'*
A and B – Task 4a	Balancing technological and traditional methods	17. *'Using only a straightedge makes you think differently—could be a nice challenge after they have used GeoGebra.'*
C and D – Task 4a	Exploring related concepts through discussion	18. *'It is like a kite but also like a trapezium—those overlaps help explain different properties to students.'*

Pair/Task	Pedagogical Knowledge	Participant Excerpts
E and F – Task 4a	Problem-solving with constraints	19. *'How can we do it without a compass? That's what made us think more deeply—and students could benefit from that.'*
G and H – Task 4a	Identifying constancy amid variation	20. *'Even though it changes, the angle proportions stay the same—that's a cool pattern for students to find.'*
A and B – Task 4b	Applying and synthesising knowledge	21. *'We're using everything—angle bisectors, perpendiculars, properties… it all comes together here.'*
C and D – Task 4b	Maintaining geometric properties dynamically	22. *'Even when I move it, the right angle stays. Students can test that themselves, not just take it from a diagram.'*
E and F – Task 4b	Investigating circle theorems collaboratively	23. *'This circle bit reminds me of that theorem with angles—if we could guide students to see that, it [understanding the circle theorems] will stick better.'*
G and H – Task 4b	Contextualising constructions	24. *'This could work in Year 9 but not lower years—it needs them to already know some basics.'*