On the emergence of physics beyond the Standard Model from right-handed neutrinos

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'Gravity matters'

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Abstract

Sakharov's induced gravity is a semiclassical mechanism where classical spacetime dynamics emerge from quantised matter fields. In this thesis, we find the Lagrangian terms from Sakharov's induced gravity where the matter fields are exclusively right-handed neutrinos. The right-handed neutrino induced terms consist of two main types: gravitational and fermionic. Our key result from the gravitational piece is a Newton constant that is consistent with observation, whence it is possible for the Einstein-Hilbert action to be an emergent consequence of right-handed neutrinos alone. We also obtain a cosmological constant and gravitational couplings of curvature-squared order, but these are incompatible with existing bounds and can be discarded. The fermionic piece leads to a seesaw mechanism, which gives active (left-handed) and sterile (right-handed) neutrino masses. The active neutrino masses turn out to be consistent with experimental data, but this comes at the expense of naturalness and staying within the perturbative coupling domain. The sterile neutrino masses lead to a potential neutrino dark matter candidate. Hence, we observe that right-handed neutrino induced gravity accounts for the combination of realistic spacetime dynamics, neutrino masses and a dark matter candidate, which is unaccounted for by the Standard Model. The right-handed neutrino induced Einstein-Hilbert term in particular further motivates right-handed neutrino existence.

In the non-commutative formulation of geometry due to Connes, coordinates under multiplication do not commute. A simple matrix model within Connes' non-commutative geometry is considered. It is observed that the full action of this model is emergent in a manner akin to Sakharov's induced gravity. In the model, this induced gravity perspective provides further explanation for a non-commutative geometry principle which is a stronger version of diffeomorphism-invariance at the level of the action. The findings are restricted to our model and constitute work in progress. It is hoped that non-commutative geometry will gain more understanding due to the induced gravity perspective.

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Notation and conventions

These follow Barrett [1], which follows Misner-Thorne-Wheeler [2].

We assume natural units $\hbar = c = 1$.

Index notation for spacetime:

- Greek indices denote spacetime indices.
- Latin letters $a, b, c, \dots$ denote local tangent space indices.
- For simplicity, we use the same spacetime indices in both the Lorentzian and Euclidean. The same holds for the tangent indices.

Metric signatures:

- In the Lorentzian, we use $(-, +, +, +)$, i.e. $\eta_{ab} = diag(-1, 1, 1, 1)$.
- The Euclidean regime just has $(+, +, +, +)$, i.e. $\delta_{ab} = diag(1, 1, 1, 1)$.

The Euclidean Ricci scalar R_E on S^2 is assumed to be positive.

Contents

1 Introduction **8**

1.1 Background and motivation . . . 8

1.1.1 Context of Sakharov's induced gravity . . . 8

1.1.2 Previous developments in Sakharov's induced gravity . . . 9

1.2 Main contribution . . . 12

1.2.1 Outline . . . 12

1.2.2 Novelty and impact statement . . . 13

1.2.3 Limitations . . . 13

1.3 Literature survey . . . 15

1.4 Organisation . . . 19

2 Induced gravity and induced Standard Model **28**

2.1 Sakharov's induced gravity . . . 29

2.1.1 Main steps . . . 29

2.1.2 Technical details . . . 31

2.1.3 Regularisation toolkit . . . 37

2.1.4 Modifications . . . 38

2.2 Induced Standard Model . . . 39

3 Lorentzian-Euclidean transition **42**

3.1 Functional integrals . . . 43

3.2 Euclideanisation . . . 46

4 Beyond the Standard Model **48**

4.1 Right-handed neutrinos . . . 49

4.2 Connes' non-commutative geometry . . . 52

4.2.1 Non-commutative Standard Model 56
4.2.2 Finite spectral triples 59
4.3 Weinberg operator and beyond 60

5 Beyond Standard Model physics from right-handed neutrinos 62
5.1 Setting up the functional integral 63
5.2 Evaluating the functional integral 66
5.3 Total effective action 69
5.4 Bosonic action 70
5.4.1 Evaluating the Pfaffian 70
5.4.2 Bosonic action 72
5.4.3 Bosonic action under scalar symmetry breaking 79
5.4.4 Gravitational constants 82
5.5 Fermionic action 84
5.5.1 Induced neutrino masses 89

6 Further aspects of induced gravity models 92
6.1 Right-handed neutrino induced fermionic action: next-to-leading term 92
6.2 Integrating out other fields 96
6.3 Non-commutative induced gravity 97

7 Conclusions 101

A Lorentzian and Euclidean spinors in curved spacetime 107
A.1 Lorentzian spinors 108
A.2 Euclidean spinors 112

B Complex symmetric matrices 115

Chapter 1

Introduction

1.1 Background and motivation

1.1.1 Context of Sakharov's induced gravity

Our most experimentally successful understanding of the matter in our universe is encapsulated in the Standard Model (SM): this gives the fundamental matter fields, their interactions via gauge bosons, and the Higgs boson, in a unified description according to quantum field theory. However, the SM does not include gravity, which is currently best described by the classical theory of general relativity. Finding the correct quantum description for gravity remains a significant unsolved problem.

The problem of gravity has seen the development of several candidates for quantum gravity (see [3] for a history). These candidates all take spacetime as quantised, i.e. at the Planck length spacetime has a particular discrete structure (depending on the candidate). The quantum gravity candidates have different treatments of gravity and matter, but generally work as follows: prioritise the quantisation of gravity and afterwards incorporate matter (c.f loop quantum gravity [3] and causal dynamical triangulations [4]), or consider gravity and matter fields within a unified quantum framework (c.f string theory [5, 6] and causal fermion systems [7, 8]).

In this thesis, we focus on another approach to a quantum description of gravity: Sakharov's induced gravity. According to this concept, matter fields are considered fundamental, and the dynamics of classical gravity (but not the gravitational field itself) emerge from the matter fields upon their quantisation. These roles of matter and gravitation contrast with those in general quantum gravity candidates. For general matter fields, the induced gravitational dynamics is given by an Einstein-Hilbert action and cosmological constant term in addition to terms of curvature-squared order (the latter terms can be viewed as general relativity corrections arising from a classical limit of quantum gravity). Thus, there are several merits of Sakharov's induced gravity:

- The Einstein-Hilbert action gains a new layer of meaning: it is an emergent quantity coming from quantised matter, rather than being innate;
- Any curved spacetime quantum field theory admits gravitational dynamics;
- The origin of gravitational dynamics may be semiclassical and may not need quantum gravity.

Correspondingly, the advent of Sakharov's induced gravity spawned a line of research in this area, including extensions to the basic concept. We now outline some of the key developments in this research line.

1.1.2 Previous developments in Sakharov's induced gravity

We start with Sakharov's concise introduction of his induced gravity [9]. According to this original version, the source of spacetime dynamics is fluctuations of the quantum vacuum. This fact is an immediate consequence of Lagrangian quantum field theory as follows. One assumes the background spacetime $(\mathcal{M}, g)$ (in the Lorentzian regime, see appendix A) as well as a quantum vacuum. One lets $\mathcal{L}_{vac}$ be the Lagrangian of spacetime curvature due to the quantum vacuum fluctuations. The form of the Lagrangian is given by the curvature expansion

$$\mathcal{L}_{vac} = \tilde{c}_0 \int k^3 dk + \tilde{c}_1 \Big(\int k dk \Big) R + \mathcal{O}(R^2) \tag{1.1}$$

where $\tilde{c}_0, \tilde{c}_1$ are dimensionless constants of order unity and R is the Ricci scalar. Equation (1.1) comes about as follows: it is an expansion in all diffeomorphism-invariants given by g, with coefficients proportional to momentum integrals for virtual particles produced by quantum vacuum fluctuations, and dimensional analysis gives the integrand powers. Equation (1.1) gives a cosmological constant at leading order and an Einstein-Hilbert term immediately above leading order. The Einstein-Hilbert term leads to a Newton constant G_I given by (in our conventions)

$$\frac{1}{16\pi G_I} = \tilde{c}_1 \int k dk. \tag{1.2}$$

The leading term in equation (1.1) is (in our conventions) $\frac{-2\Lambda_I}{16\pi G_I}$ where Λ_I is the induced cosmological constant, hence using equation (1.2) gives

$$\Lambda_I = -\frac{\tilde{c}_0}{2\tilde{c}_1} \frac{\int k^3 dk}{\int k dk}. \tag{1.3}$$

The momentum integrals give rise to UV divergences, which make equations (1.2) and (1.3) ill-defined. These divergences are regularised by introducing a UV momentum cutoff Λ which is set at the Planck scale. This gives a non-zero Newton constant that fits observation. However, the cutoff also means the cosmological constant Λ_I has size of order Λ^2 and is thus much larger than experiment [10, 11]. Neglecting the cosmological constant, one is left with the Einstein-Hilbert term. As a corollary of his mechanism, Sakharov showed free particles source dynamics of spacetime, since applying the mechanism to free particles of masses Λ leads to a Newton constant $G = \Lambda^{-2}$.

Sakharov's original concept was given a reformulation by Visser [12]. This reformulation used the conventional language of quantum field theory, in particular the functional integral quantisation and the method of regularisation. One starts with a theory of quantum matter fields $\mathcal{F}$ in classical curved spacetime $(\mathcal{M}, g)$. Given action S_m for the matter fields, quantisation gives the functional integral

$$Z = \int_{F \in \mathcal{F}} e^{iS_m[F;g]} DF = e^{i\Gamma} \tag{1.4}$$

where Γ is the quantum 1-loop effective action [13] induced by the matter. The regularisation of Γ is given by the expression

$$\Gamma_{reg}[g] = \int_{\mathcal{M}} \Omega \Big(\frac{1}{16\pi G_I} \Big(-2\Lambda_I + R \Big) + \mathcal{O}(R^2) \Big) \tag{1.5}$$

giving an Einstein-Hilbert term with cosmological constant and curvature-squared terms. Visser showed that the mechanism works with scalars, spinors and other field types (including supermultiplets). He also identified several pathways for induced gravity, which propose differing sets of conditions that are enforced on the induced gravitational constants.

The effective action regularisation involves a technique known as the heat kernel expansion [14, 15], which expresses the solution of the heat equation on curved spacetime as an asymptotic expansion whose coefficients are diffeomorphism-invariants depending on spacetime curvature. Denardo and Spallucci were the first to employ the heat kernel expansion in induced gravity (see [16] and references therein). The heat kernel expansion was also used early in induced gravity within the Kaluza-Klein model of [17]. Induced gravity now has the heat kernel expansion as a standard component. We will discuss the heat kernel expansion in more detail in chapter 2.

The induced gravity mechanism has also been considered for a curved spacetime with non-zero torsion. This was done first for a Euclidean formalism [18] and later for a Lorentzian formalism [19]. In these cases, the induced regularised effective action also encodes the dynamics of torsion. We only mention torsion for completeness purposes and will not give explicit consideration to torsion in this work.

The mechanism has been extended by Broda and Szanecki [16]: in addition to spacetime dynamics corresponding to gravitational coupling constants, this extended mechanism gives rise to the dynamics of gauge fields. A special case of this is the SM, for which it is shown that matter fields comprising the SM fermions and the Higgs field furnish SM gauge coupling constants. Values for the induced gravitational and gauge couplings that are a good match with experiment are derived from the mechanism.

For the purposes of the historical background for the present thesis, an important extension to the basic induced gravity mechanism is the 'induced Standard Model' [20]. In this model, the matter fields are taken to be all fermions of the SM where the right-handed counterparts to the neutrinos are also added. These matter fields are integrated out giving an effective action that is purely bosonic: the action contains the gauge-Higgs Lagrangian of the SM coupled to curved spacetime in addition to cosmological, Einstein-Hilbert and curvature-squared terms. Hence, the induced Standard Model provides a mechanism for dynamical spacetime in the SM, thus proposing a solution to the lack of gravitational dynamics within the SM. The effective bosonic action has connections with the Connes-Chamseddine action of non-commutative geometry (which will be discussed in chapter 4) and is compatible with the state sum formalism of quantum gravity.

1.2 Main contribution

1.2.1 Outline

Our main contribution in this thesis[1] is finding the effective action terms induced by integrating out only right-handed neutrinos. In this induced gravity mechanism, the initial matter theory is taken as three right-handed neutrinos minimally coupled to the SM within curved spacetime. This theory is physically observable in the Lorentzian regime, so our mechanism starts here. However, in the Euclidean regime, the standard techniques of induced gravity, in particular the heat kernel expansion, are better suited than they are to the Lorentzian regime. Hence, a significant fraction of our calculations are done in the Euclidean regime. Here, we introduce a real singlet scalar needed to fix problems in the Higgs sector within a non-commutative geometry formulation of the SM (for non-commutative geometry, see section 4.2). It is in the Lorentzian regime where we give our results, and this is to ensure they can (in principle) be observed. The integration out of the right-handed neutrinos ultimately gives an effective action containing a gravitational part and a fermionic part. The gravitational part gives rise to cosmological, Newton and curvature-squared constants of which the Newton constant turns out to have a value which is consistent with observation (for the cosmological and curvature-squared constants, see subsection 1.2.3). The fermionic term leads to a real scalar modification of the standard (type-I) seesaw mechanism for neutrino masses (see section 4.1). The active (left-handed) neutrino masses from the seesaw turn out to be consistent with experimental results.

[1]The original work in this thesis has not yet been published. The author expects to prepare a paper on this work and submit the paper to an appropriate journal in due course.

The seesaw also gives rise to a sterile (right-handed) neutrino state that may be a candidate for dark matter.

1.2.2 Novelty and impact statement

In [20] the total bosonic effective action induced by all SM fermions and right-handed neutrinos was found, but decomposing the action into contributions from each species of fermion has not been done. One can in principle start to fill this gap by considering any fermion species. We choose the right-handed neutrinos since these give the case that is simplest and is valid in the largest energy range of all fermions in the induced Standard Model. To our knowledge, we are the first to consider the case of an induced gravity mechanism from right-handed neutrinos. In the gravitational sector, the key new result is that right-handed neutrinos furnish an observationally consistent Newton constant and thus realistic dynamics of spacetime. This result implies the Einstein-Hilbert action is possibly not innate but rather emergent from right-handed neutrinos. The main new result from the fermionic sector is the emergence of the scalar modified seesaw within Sakharov's induced gravity.

The right-handed neutrinos are already known to have implications for particle physics phenomenology, in particular explaining neutrino masses and dark matter (see section 4.1). Even though the explanations of neutrino masses and (potentially) dark matter in our right-handed neutrino induced gravity mechanism are not new, the explanation of the Einstein-Hilbert term is new, and thus the emergence of the combination of all three properties in our mechanism is a new result. This emergence of all three aspects in our mechanism potentially solves three deficiencies in the SM (see chapter 4). In particular, our result of the emergence of realistic spacetime dynamics from right-handed neutrinos provides an additional motivation for the existence of right-handed neutrinos.

1.2.3 Limitations

The main limitations of our right-handed neutrino mechanism are as follows:

- Due to the heuristic nature of functional integrals and some details of the treatment of the effective action, our right-handed neutrino mechanism is heuristic;

- Our mechanism uses a spacetime $\mathcal{M}$ with several assumptions (see appendix A). These assumptions give physical inconsistencies:
  - Assuming $\mathcal{M}$ is compact but boundaryless rules out $\mathbb{R}^n$ topology and a fully physical Lorentzian structure but makes the mathematics easier[2].
  - The manifold $\mathcal{M}$ admits unphysical (Euclidean) fields on the same footing as physical (Lorentzian) fields. Ghost fields are unphysical fields that can arise in gauge theories via gauge fixing, but the Euclidean fields are not ghosts.

  The first point means that the Lorentzian regime of our right-handed neutrino mechanism approximates Lorentzian physics. Nevertheless we use the Lorentzian regime since it is closer to observable physics than the Euclidean. Resolution of the above points is left to further work;

- The right-handed neutrino induced cosmological constant is much larger than the experimental value;

- The curvature-squared couplings give rise to undesirable tachyonic behaviour;

- The experimentally consistent active neutrino masses come at the expense of neutrino Yukawa couplings that are large enough to be in the unnatural and non-perturbative regime;

- We have omitted a full analysis to check the viability of the dark matter candidate due to our seesaw;

- Our seesaw gives sterile neutrino states of masses well above current accelerators;

- For the whole effective action due to the right-handed neutrinos, one should ideally conduct a comprehensive parameter space search to investigate

[2]The author thanks John W. Barrett for raising this point to the author.

which regions give experimentally consistent induced quantities. Instead of this, we simply provide one region of this parameter space. Further exploration of the parameter space is beyond our scope.

1.3 Literature survey

Our right-handed neutrino induced gravity mechanism is new as a whole, but particular aspects of it have antecedents in the literature. These particular aspects are surveyed in this section. We have already given a historical survey of Sakharov's induced gravity, but this section is more general.

Sakharov's induced gravity

We discuss some earlier concepts that are similar to models in Sakharov's induced gravity. There is the proposal that fermionic matter comprises the gauge bosons and graviton [21, 22, 23, 24], and this is similar to the induced Standard Model. In [25], the Einstein-Hilbert action with cosmological constant emerges from a flat-spacetime model of a Higgs field in six dimensions. Emergent spacetime dynamics due to gauge fields in addition to scalars and spinors was considered in [26].

Another earlier proposal [27, 28, 29, 30] is that spontaneous symmetry breaking furnishes the Einstein-Hilbert action. This is similar to our right-handed neutrino mechanism (see section 5.4.3), where gravitational corrections will arise upon spontaneous real scalar symmetry breaking.

The induced Standard Model was introduced in [20]. This reference is the only substantial work on the induced Standard Model (to the present author's knowledge).

Heat kernel expansion

The standard reference for the heat kernel expansion is the book of Gilkey [14]. Another notable reference is the review article of Vassilevich [15], which shows the universal nature of the heat kernel expansion by treating in detail many different cases. However, the heat kernel expansion is only well-defined in the

Euclidean regime (this is discussed in chapter 2 using the discussion from [15]). To the present author's knowledge, a Lorentzian formulation of the heat kernel expansion is not generally accepted. We remark that a candidate for this Lorentzian formulation is Hadamard expansions [31, 32], but these are not considered in this thesis as it is unclear how these can be applied to our right-handed neutrino mechanism. To the author's knowledge, the optimal definition of the heat kernel expansion is in the Euclidean.

Lorentzian-Euclidean transition

In chapter 3, we discuss a particular framework for transitioning between the Lorentzian and Euclidean regimes, and this framework (here called the Lorentzian-Euclidean transition) was introduced in [1] and has not received further work (to the present author's knowledge). We remark on connections the Lorentzian-Euclidean transition has to the literature:

- An antecedent for the Lorentzian-Euclidean transition is the Wick rotation. The Wick rotation (along with some of its previous references) is discussed in chapter 3.

- We state connections to the older framework of Euclidean quantum gravity. The Lorentzian-Euclidean translation uses the practice of Hawking [33] of using the first-order formalism for spacetime, i.e. tetrads and spin connections. Also, Gibbons-Hawking-Perry end up with a pathological Euclidean Einstein-Hilbert term [34], which is what one gets in the Lorentzian-Euclidean transition. Both Gibbons-Hawking-Perry in [35, 34] and the Lorentzian-Euclidean transition use contour rotations, but in different forms: the Gibbons-Hawking-Perry rotation was used to cure the Einstein-Hilbert problem in a purely Euclidean framework, while in the Lorentzian-Euclidean transition the rotation is a key step in going to the Euclidean theory from the Lorentzian.

- In the Lorentzian-Euclidean transition, the functional integrals of the Lorentzian and Euclidean theories are equated using the contour rotation. This resembles the situation in [36] and attempts along these lines in [37, 38, 39].

In this thesis, we use the Lorentzian-Euclidean transition since this turns out to fit naturally with our right-handed neutrino mechanism.

Connes' non-commutative geometry

We discuss this area in section 4.2 and give references later in this chapter. In the older non-commutative geometry references (e.g. [40, 41]), there is one type of dimension corresponding to the 'metric' dimension. Then in [42, 43], it was found that two dimension types ('metric' and 'KO') are needed to fix certain problems within a non-commutative formulation of the SM (with right-handed neutrinos). The two dimensions are discussed in section 4.2.

In references 3-7 in [44], there are considerations of coupling to the Higgs a (possibly complex) scalar field transforming as a singlet under the internal SM symmetry. This has two main effects. Firstly, it allows the selection of the experimental value of the Higgs mass. Secondly, it fixes the problem identified in [45] that at high energies the Higgs self-coupling runs negative and hence the Higgs vacuum becomes unstable. These ideas have been realised within the non-commutative geometry description of the SM and right-handed neutrinos. In [46], a real singlet scalar was introduced through the right-handed neutrinos and furnished the same Higgs couplings, though the scalar couplings were not considered to run. In the follow-up [44], the running scalar couplings were accounted for and this resulted in both of the effects on the Higgs sector. The non-commutative geometry real singlet scalar has couplings to the Higgs which emerge from the coupling of the scalar to right-handed neutrinos. Given the neutrino-scalar coupling, the scalar is naturally compatible with right-handed neutrino induced gravity. This form of scalar induced gravity has not been considered in the induced Standard Model [20] or (to the present author's knowledge) in the wider literature.

We remark that a similar complex scalar field is derived in [47]. From the view of non-commutative geometry, this complex field is more natural than the real field. However, here we just consider the real field, and leave the complex field to future work.

With respect to the induced Standard Model, we remark that [48] contains a related proposal where a fermionic action quantum property leads to a bosonic action within a non-commutative setting.

Weinberg operator and beyond

In the framework of Broncano-Gavela-Jenkins [49, 50], the right-handed neutrinos are integrated out resulting in non-renormalisable fermionic terms, one of which is the Weinberg operator [51]. We remark that the Weinberg operator also comes from the effective framework in [52, 53]. These articles are set in flat Minkowski spacetime. Thus, without gravity, the right-handed neutrinos still induce fermionic terms. Having gravitational terms along with fermionic terms emerge from right-handed neutrinos à la induced gravity has not been studied in the literature (to the author's knowledge). We consider such a simultaneous induction here, and use the Broncano-Gavela-Jenkins framework to compare fermionic terms. The non-renormalisable terms and the Broncano-Gavela-Jenkins framework are discussed in chapter 4.

Integration out

The integration out of the right-handed neutrinos typically means (e.g. [52, 49]) restricting to the locus of right-handed neutrino configurations satisfying the right-handed neutrino equations of motion. In terms of the functional integral view on this, the explicit evaluation of the functional integral over right-handed neutrinos has not been done in the literature (to the author's knowledge). The evaluation of the right-handed neutrino functional integral is a key problem in this work. This problem is non-trivial since functional integrals are generally intractable. Some hints for the integration out are found in the literature:

- In [54], a functional integral of a similar form to ours is worked out within the framework of Euclidean finite non-commutative geometries. The result of the integral in [54] is a Pfaffian, i.e. a function whose square is the determinant.

- A problem in the non-commutative SM and right-handed neutrinos is fermion doubling [55, 56, 57] (see also [58] for a particularly clear account).

This has a solution [43]: a functional determinant to the power of $1/4$ cures all fermion doubling. This precise expression was found for the Pfaffian in [54].

1.4 Organisation

Chapters 2 to 4 are review chapters that form the core background of this thesis, and these chapters contain no original material. Chapters 5 and 6 contain all the original content of this thesis: chapter 5 has the bulk of this content, and the remainder is in chapter 6. In all these chapters, our presentation is from the perspective of a physicist, i.e. we use heuristic definitions and omit some rigorous proofs. We now give a chapter summary.

Chapter 2: Induced gravity and induced Standard Model

Chapter 2 is a review of Sakharov's induced gravity, and in particular includes the induced Standard Model as a generalisation of the concept.

A large part of the review is the key steps and technical details of induced gravity. This mainly follows Visser's induced gravity reformulation [12], used since it contains a full exposition of the steps and details without torsion, the latter of which is not explicitly considered in this thesis. In [19] there is an induced gravity mechanism with torsion containing the steps and details, and we use these in our presentation (but torsion is neglected in our case). We remark that our presentation also uses effective field theory in the form of a clear lecture by Melville [59].

The induced gravity regularisation includes several standard techniques, which we discuss in subsection 2.1.3:

- One such technique is the Schwinger proper-time formalism. This is found in the more recent induced gravity articles (e.g. [12, 16, 19]) and is discussed in [60];

- We also discuss the heat kernel expansion, which in this subsection follows the Euclidean formulation in the review article of Vassilevich [15]. The relevant but more lengthy formulae in that review article are not reproduced in this subsection since there is little advantage to doing so. We choose the Vassilevich article [15] as it has the same Ricci scalar conventions as this work. Henceforth in this work, we assume the heat kernel expansion as set out in this subsection.

In subsection 2.1.4, we further discuss torsion and other modifications to the standard induced gravity mechanism, but these specific modifications are not considered in further detail in this thesis.

In section 2.2, we discuss the induced Standard Model. This section follows [20] since there are no other articles to use. In his induced Standard Model, Barrett assumes the Euclidean regime to find the SM fermion and right-handed neutrino contribution. In contrast, the present work will use a scheme to transition to the Euclidean from the Lorentzian, and we will review this scheme in the next chapter.

Chapter 3: Lorentzian-Euclidean transition

We clarify what is meant here by a Lorentzian geometry on spacetime manifold $\mathcal{M}$. A classical Lorentzian geometry is given by appendix A as either: a Lorentzian metric g which is a pseudo-Riemannian metric of signature $(-,+,+,+)$ (second-order formalism) or; or a tetrad l^a defined by equation (A.5) (first-order formalism), and both descriptions are equivalent. A quantum Lorentzian geometry can be defined as a particular discrete geometry emerging from quantum gravity (with quantised $\mathcal{M}$) e.g. casual dynamical triangulation [4], causal sets [61], and loop quantum gravity [3], that corresponds to a Lorentzian metric in the classical limit. Our notion of Lorentzian geometries includes both classical and quantum geometries. One can set matter on a Lorentzian geometry, which corresponds to classical or quantum fields. By a Lorentzian theory, we mean a set of matter fields in a Lorentzian geometry on $\mathcal{M}$. Since the universe is set in Lorentzian spacetime, Lorentzian theories are the experimentally observable theories. However, one encounters problems in Lorentzian theories, in particular (for this work) divergent functional integrals and ill-defined heat kernel expansions.

One has similar notions in the Euclidean regime, except with metric signature $(+,+,+,+)$. In contrast with Lorentzian theories, Euclidean theories are unphysical but allow for better functional integral convergence and greater compatibility with existing frameworks, in particular the heat kernel expansion. Thus in induced gravity which is centered around functional integrals and heat kernel expansions, the optimal choice is to start with the measurable Lorentzian physics and then perform the computations in the more tractable Euclidean regime.

Given the complimentary nature of Lorentzian and Euclidean theories, one wishes to transition between them. A standard procedure for this is known from flat-spacetime quantum field theory as the Wick rotation. In chapter 3 we discuss a different scheme herein called the Lorentzian-Euclidean transition, which is the transition procedure used in this work. The Lorentzian-Euclidean transition was recently introduced in [1], which is the reference this chapter follows (and the notion of geometry we use comes from this reference). The idea of the Lorentzian-Euclidean transition is to start from a Lorentzian theory and relate this via a series of steps to a similar Euclidean theory. A caveat is that the manifold does not rotate, but the geometries and matter on the manifold do, which is a different situation to the Wick rotation. One of the steps of the Lorentzian-Euclidean transition is 'Euclideanisation' [1], which is itself significant and thus gets its own discussion in section 3.2. In [1], the transition was explicitly considered in the direction of Lorentzian to Euclidean, but the reverse transition is in theory allowed. Correspondingly, the chapter concludes with remarks on the transition from the Euclidean to the Lorentzian. This section emphasises practical results with relatively few technical aspects, and we defer to [1] for the details.

A major reason why we use the spinor presentations [62, 1] is because these are used for the Lorentzian-Euclidean transition [1]. In addition, these frameworks use the same manifold $\mathcal{M}$, and we follow this practise. Thus, our use of these frameworks is consistent.

Chapter 4: Beyond the Standard Model

Chapter 4 concerns three main relevant areas of beyond-SM physics.

The first section of this chapter discusses right-handed neutrinos (in the Lorentzian). The main element of this section for the next chapter is the SM and right-handed neutrinos, which is used as the origin for right-handed neutrino induced gravity. The main contribution is also supplemented by discussion in this section on neutrino mass ranges, seesaw mechanisms and a certain SM plus right-handed neutrino regime. This section approximately follows Drewes' review article [52] that proved to be a useful source of material and references. Drewes' article describes Majorana and Dirac neutrinos, but we do not explicitly consider this distinction in this work. This section also uses the review article [63] in covering other (non type-I) seesaw mechanisms.

The second section (section 4.2) reviews elements of Connes' non-commutative geometry, which we place under the label of beyond-SM physics since there is a non-commutative formulation of the SM and right-handed neutrinos. Whereas a Riemannian geometry is defined by a smooth manifold, a non-commutative geometry is defined by an algebra, a Hilbert space and a set of operators on the Hilbert space, and the algebra does not commute under multiplication of its elements. The presence of the ingredients for a non-commutative geometry ensures consistency with the cases of the Riemannian spacetime in isolation and as a background for the SM with right-handed neutrinos. Hence, despite non-commutative geometry having origins in pure mathematics, it has links to physics. There are non-commutative analogues of several concepts from Riemannian geometry including spacetime, metric, dimension, orientability and geodesic distance. In particular, the analogue of the action for SM bosons and spacetime is the Connes-Chamseddine action. In addition to Riemannian spaces that have in a sense infinite size, non-commutative geometry admits finite spaces.

Section 4.2 does not follow any one single reference. The standard reference for non-commutative geometry is the book of Connes [64]. The axioms for Connes' framework come from [40]. There is also a recent set of general non-commutative geometry lectures by Barrett [65], which the special case of the Riemannian manifold follows herein. The Euclidean SM and right-handed neutrino case is discussed in [42, 43, 46]. The first reference in particular discusses non-commutative analogues of gauge symmetries as well as two separate non-commutative dimensions, namely metric and KO (more in chapter 4), and the latter two references give a general overview of non-commutative geometry. We use the definition of the metric dimension from [66]. Non-commutative symmetries are automorphisms, and this is discussed in [67, 40], the latter of which in particular introduces the inner automorphisms and inner fluctuations and that these give rise to SM bosons. The spectral action principle and Connes-Chamseddine action come from [67, 41], the latter of which also gives a physics-friendly discussion of general non-commutative geometry aspects. The non-commutative fermionic action comes from [42, 43]. The real singlet scalar appeared in [46]. A clear and more explanatory account of the scalar and its properties is given in [68] and references therein, as well as [69]. The effects of the scalar interactions, in particular the experimental Higgs mass and vacuum stability, were fully captured and worked out in [44]. The section concludes with finite spectral triples, which are non-commutative geometries that are in a sense finite dimensional. The finite spectral triple discussion mainly uses [70], the article [71] containing special finite spectral triples, and the recent article of Barrett [54].

In the last section, we consider a specific class of Lagrangian terms which: are non-renormalisable (mass dimension > 4 in our units), are invariant under SM gauge symmetries and have two SM lepton doublets and feature the Higgs twice. One such term is the Weinberg operator, which we cover first. The Weinberg operator was introduced in [51], and useful discussions of the term are found in [52, 49]. We remark on other fermionic terms, but we stay below mass dimension 7 in this work. Lastly, we review the Broncano-Gavela-Jenkins framework [49, 50], which we compare our right-handed neutrino induced gravity mechanism to in the next chapter.

The first and last sections of this chapter are elsewhere typically in flat Minkowski spacetime. However, to ensure our discussion is consistent with induced gravity, for the first and last sections, we take the setting of the curved spacetime manifold $\mathcal{M}$ assuming the Lorentzian regime. The fields are assumed to be minimally coupled to $\mathcal{M}$, thus the first and last sections are effectively not changed compared to the literature.

Chapter 5: Beyond Standard Model physics from right-handed neutrinos

Chapter 5 contains the main original contribution of this thesis: the Lagrangian terms due to an induced gravity mechanism sourced by right-handed neutrinos. In this chapter, our mechanism is in a sense commutative, but we use non-commutative geometry as a conceptual and structural tool for some steps.

The opening section of this chapter starts with the Lorentzian regime since this is physical. In particular, we have the functional integral for quantised right-handed neutrinos and the classical SM set on classical $\mathcal{M}$. We then transition to the analogous Euclidean functional integral, which is more compatible with our main methods and techniques, in particular the heat kernel expansion.

In the next section, we conduct the evaluation of the Euclidean right-handed neutrino functional integral, which is the first significant stage of this chapter. The right-handed neutrino functional integral evaluation follows [54]. The key result is that the Euclidean functional integral gives a Pfaffian and an extra fermionic piece due to couplings between the right-handed neutrinos and SM leptons.

The next section sees the Euclidean functional integral defining an effective action containing bosonic and fermionic pieces, where the bosonic piece comes from the Pfaffian. For the schematic effective action, we transition back to the Lorentzian.

The rest of this chapter concerns the individual bosonic and fermionic terms of the effective action. The next section is a study of the bosonic piece, and the fermionic term is looked at in the section after. In both sections, we start in the Euclidean regime and perform the transition to the Lorentzian regime in more detail. In both regimes, the sections also give the explicit calculations and comparisons of results produced by the corresponding terms to frameworks in the literature.

The bosonic term is the subject of the next section (section 5.4), which begins in the Euclidean. We conduct the evaluation of the Pfaffian using results from [54], and we end up with the Pfaffian as a functional determinant 4th root (neglecting a trivial index term). This functional determinant gives the bosonic term. For this bosonic term, we use standard regularisation techniques from induced gravity, including the Schwinger proper-time formalism and the heat kernel expansion, though at this point we keep our formulae general. We remark that we are still in the Euclidean and thus the heat kernel expansion is well-defined. Now, we bring into our framework the non-commutative geometry real singlet scalar (slightly modified), which gives an additional internal $\mathbb{Z}_2$ symmetry. We now compute the specific details of the induced gravity regularisation. Putting together our results for the regularisation gives the bosonic action in terms of curvature and scalar terms, where the bosonic action is $\mathbb{Z}_2$-invariant. A provisional comparison of our bosonic action with the (Euclidean) Connes-Chamseddine action from non-commutative geometry gives good agreement between both actions. The real scalar $\mathbb{Z}_2$ symmetry is then spontaneously broken, which leads to a non-zero scalar vacuum expectation value that collapses the scalar terms to pure gravity terms. The resulting pure gravity action is transitioned to the Lorentzian. We then extract the gravitational constants from the pure gravity action. The constants depend on three right-handed neutrino mass parameters, and our framework is consistent for two large masses and one small mass. This mass regime turns out to give a Newton constant, but we also get a cosmological constant that is too large and tachyons coming from the curvature-squared terms. We remark that tachyons arise in string theory, and correspondingly our reference for tachyons is [5].

In the last section of this chapter, we start from the Euclidean effective fermionic action and the corresponding Lagrangian. Some of the steps for the fermionic action are identical to those for the bosonic action, and for these steps the order is unchanged. This ensures that the bosonic and fermionic terms are treated consistently in the full effective action. We put the (modified) non-commutative geometry scalar into the fermionic term. We then expand the fermionic Lagrangian in the right-handed neutrino mass parameters, and in this section restrict to leading order. After some manipulation of the fermionic term, we assume the $\mathbb{Z}_2$-breaking scalar vacuum expectation value. We now transition back to the Lorentzian. The resulting fermionic Lagrangian is a Weinberg operator, which upon further electroweak symmetry breaking gives physical masses for the left-handed neutrinos. Compared to the Lorentzian Broncano-Gavela-Jenkins framework [49, 50], our left-handed neutrino masses have an extra factor due to the scalar. For one small right-handed neutrino mass parameter and two large, one gets experimental active neutrino masses that come with the cost that the couplings between the SM and right-handed neutrinos are of large enough magnitude to be unnatural and non-perturbative. In addition, we get sterile neutrino masses and comment on their values.

Chapter 6: Further aspects of induced gravity models

In chapter 6 we consider a continuation of the right-handed neutrino induced gravity mechanism as well as other models of induced gravity, but this material is currently incomplete and hoped to be continued in further work.

In the first section, we consider the next-to-leading term in the expansion of the fermionic Lagrangian. The procedure for this term is similar to that for the leading term, but both procedures are not exactly identical since their details differ. In the Lorentzian, and after electroweak and real scalar $\mathbb{Z}_2$ breaking, we end up with a kinetic term for left-handed neutrinos. Again comparing to Broncano-Gavela-Jenkins [49, 50], our kinetic term differs by a scalar factor, which gives additional suppression in the right-handed neutrino mass parameter regime.

The next section gives some remarks on the integration out of fields other than right-handed neutrinos. We provide such remarks for SM fermions, in particular the top quark, the SM gauge fields and the metric. In addition, we discuss approaches and further work for the induced Standard Model.

For the last section, we attempt to realise the induced gravity mechanism in the framework of non-commutative geometry. This involves building upon previously obtained results for integration on finite non-commutative geometries. The background of this section is finite spectral triples, which have the function of being useful toy models for non-commutative geometry. More specifically, we restrict to a certain finite spectral triple model. The main element of this model is a functional integral over geometries and fermionic fields, which contains an action with geometry and matter terms. Within this setting, one can view the

Connes-Chamseddine action as being induced from the fermion integration. The geometry action (with a further constraint) may be also considered induced by a field resembling a Higgs field coupled to geometry.

Chapter 2

Induced gravity and induced Standard Model

This chapter is an introduction to the relevant background of induced gravity (without original material). Section 2.1 mostly concerns the basic induced gravity mechanism. In subsection 2.1.1, we outline the key steps of the mechanism. In subsection 2.1.2, we specialise to simple cases to illustrate how the mechanism works in more detail. The presentation in these two subsections follows the presentation of [12], which formulates Sakharov's idea in terms of the functional integral quantisation of field theory[1]. Our presentation also uses the steps and details as shown in [19] but we do not include torsion. Some important techniques in induced gravity are discussed in subsection 2.1.3, and we give the references for this in chapter 1. Particular ways of modifying the basic induced gravity mechanism are discussed in subsection 2.1.4, which is short since the precise modifications covered therein are outside the scope of the present thesis. This subsection serves as a natural preamble to the modification in section 2.2, which reviews an induced gravity structure in the SM (modified by right-handed neutrinos) named the induced Standard Model. For this section we follow [20], which is the only available reference on the induced Standard Model (to the present author's knowledge). We remark that the level of detail possible in section 2.2 is constrained by the fact that the discussion in [20] is largely qualitative

[1] The conventions of [12] are different to ours, and our presentation translates to our own conventions.

and no original content is in our section.

2.1 Sakharov's induced gravity

The main steps of induced gravity are given below. Later in this section, these steps will be precisely formulated in special cases.

2.1.1 Main steps

- We start with the spacetime manifold $\mathcal{M}$ (see appendix A) endowed with Lorentzian metric g and a space of matter fields $\mathcal{F}$. At this point, we assume an action containing only matter terms, i.e. $(\mathcal{M}, g)$ is non-dynamical.

- In the functional integral quantisation the matter fields are quantised but $(\mathcal{M}, g)$ remains classical throughout, i.e. one has the functional integral

$$Z = \int_{F \in \mathcal{F}} e^{iS_m[F;g]} DF \tag{2.1}$$

  where S_m is the action functional for matter fields $F \in \mathcal{F}$ and may also depend on the metric g (the action S_m will take specific forms later). Note that due to the still non-dynamical $(\mathcal{M}, g)$, one has a quantum field theory in curved spacetime.

- One has the following result from curved spacetime quantum field theory [13]:

$$Z = e^{i\Gamma[g]} \tag{2.2}$$

  where $\Gamma[g]$ is the 1-loop effective action[2] depending on the metric g. Hence, this step consists of integrating out the matter fields $\mathcal{F}$ to get the 1-loop effective action Γ.

- The 1-loop effective action is regularised, and we denote the result of this by Γ_{reg}. The regularisation procedure involves three key techniques: Schwinger's proper time formalism, cutoff regularisation and heat kernel

[2]In this formalism gravitons are fluctuations of an ambient metric, and any number of them can couple to the 1-loop Feynman diagrams, and the sum of all such diagrams gives Γ.

expansion (we will see how these work in more detail later). However, no renormalisation is performed. Generically, one ends up with

$$\Gamma_{reg} = \int_{\mathcal{M}} \Omega \Big(\frac{1}{16\pi G_I} \Big(-2\Lambda_I + R \Big) + \mathcal{O}(R^2) \Big) \tag{2.3}$$

where Ω is the spacetime volume form (see appendix A). From the perspective of effective field theory, equation (2.3) is the low-energy effective action that is induced from the matter theory present at high energies.

- The spacetime $(\mathcal{M}, g)$ now obtains dynamics described by the regularised 1-loop effective action as expressed in equation (2.3). Hence, one gets a cosmological constant Λ_I, a Newton constant G_I and curvature-squared coupling constants. These constants have cutoff dependence as well as dependence on the parameters of the matter fields.

We make some general remarks on induced gravity. Given the induced gravitational constants, one has the option to introduce additional constraints (e.g. Sakharov's 1-loop dominance and Pauli's 1-loop finiteness), none of which are physically privileged [12]. As a whole, the induced gravity mechanism is semiclassical since quantisation is only applied to matter while the spacetime $(\mathcal{M}, g)$ remains purely classical. A consequence of the mechanism is that quantum field theory in curved spacetime comes at once with induced gravity. Note that the matter only induces the dynamics of $(\mathcal{M}, g)$ but not $(\mathcal{M}, g)$. Hence, if quantum gravity is needed to account for $(\mathcal{M}, g)$, induced gravity should be viewed as a complimentary piece towards an understanding of all spacetime features.

We now go into detail for the workings of the induced gravity mechanism. To do this, we consider the standard cases of scalar and pure Dirac spinor fields set on $\mathcal{M}$. Our presentation is based on the presentation in [12]: we find the effective action contributions of the individual fields, then combine these contributions before regularisation.

2.1.2 Technical details

Scalar field

Let $\phi \in C^\infty(\mathcal{M})$ be a scalar field of mass m_{sc} with non-minimal coupling term $\propto -\xi R\phi^2$ where R is the Ricci scalar and ξ is a (real) dimensionless constant [72]. Using notation from section A.1, the matter action in this case is the action of ϕ:

$$S_{sc}[\phi; g] = \int_{\mathcal{M}} \Omega\Big(-\frac{1}{2}\eta^{ab}(\nabla_{l_a}\phi)(\nabla_{l_b}\phi) - \frac{1}{2}m_{sc}^2\phi^2 - \frac{1}{2}\xi R\phi^2\Big). \tag{2.4}$$

Equivalently, we can write

$$S_{sc}[\phi; g] = \int_{\mathcal{M}} \Omega\Big(\frac{1}{2}\phi(\Box - m_{sc}^2 - \xi R)\phi\Big) \tag{2.5}$$

where $\Box = \eta^{ab}\nabla_{l_a}\nabla_{l_b} = g^{\mu\nu}\nabla_\mu\nabla_\nu$ is the d'Alembertian[3], and we used the Leibniz rule and neglected a boundary term. We quantise the scalar only so that the functional integral is

$$Z_{sc} = \int e^{iS_{sc}[\phi;g]}D\phi \tag{2.6}$$

A standard fact is that the result of a complex Gaussian integral is proportional to the inverse square root of a determinant. Equations (2.5) and (2.6) constitute an infinite-dimensional generalisation of this case. Hence, we have

$$Z_{sc} = \mathrm{Det}(\Box - m_{sc}^2 - \xi R)^{-\frac{1}{2}} \tag{2.7}$$

where Det is the functional determinant and we neglected an infinite multiplicative constant that does not change the correlation functions. The 1-loop effective action is given by $Z_{sc} = e^{i\Gamma_{sc}}$, thus

$$\begin{aligned} \Gamma_{sc} &= \frac{i}{2}\ln\,\mathrm{Det}(\Box - m_{sc}^2 - \xi R) \\ &= \frac{i}{2}\mathrm{Tr}\,\mathrm{Ln}(\Box - m_{sc}^2 - \xi R) \end{aligned} \tag{2.8}$$

where in the second line we used the relation ln Det = Tr Ln involving the functional trace Tr and the functional logarithm Ln. We note both Det and Tr run over points on $\mathcal{M}$, but there may also be indices (e.g. spinor, internal) in which case all indices are run over as well.

[3]One has $\nabla_\mu := \nabla_{\partial_\mu}$ where ∂_μ is the standard partial derivative.

Dirac spinor field

A pure Dirac spinor field is understood in the usual sense as a section of a corresponding spinor bundle. We follow similar steps as those for the scalar. In the present spinor case (and referring to the definitions in section A.1), the matter action is the action for an anti-commuting pure Dirac spinor ψ of mass m:

$$S_{sp}[\psi; g] = \int_{\mathcal{M}} \Omega\Big(\overline{\psi}(\not{D} - m)\psi\Big). \tag{2.9}$$

The functional integral is taken to be

$$Z_{sp} = \int e^{iS_{sp}[\psi,\overline{\psi};g]} D\overline{\psi} D\psi \tag{2.10}$$

where we adopt the standard practice of integrating independently over ψ and its conjugate $\overline{\psi}$. The infinite-dimensional generalisation of the multi-variable Grassmann integral yields

$$Z_{sp} = \mathrm{Det}(i(\not{D} - m)). \tag{2.11}$$

Given $Z_{sp} = e^{i\Gamma_{sp}}$, we have

$$\begin{aligned}\Gamma_{sp} &= -i\ln\,\mathrm{Det}(i(\not{D} - m)) \\ &= -i\mathrm{Tr}\ \mathrm{Ln}(i(\not{D} - m))\end{aligned} \tag{2.12}$$

where we again used ln Det = Tr Ln. At this point, we have an issue: equation (2.12) has a first-order differential operator (see section A.1), but the heat kernel expansion (more about this later on) requires a second-order differential operator. This issue is resolved using the following method from [19]. The spinor chirality operator γ_M and its properties are given in appendix A.1. Since γ_M is involutive and anti-commutes with the Dirac operator, one has

$$\gamma_M \not{D} \gamma_M = -\not{D}. \tag{2.13}$$

Now, one defines

$$\begin{aligned}\mathcal{D} &= i(\not{D} - m), \\ \widetilde{\mathcal{D}} &= -i(\not{D} + m).\end{aligned} \tag{2.14}$$

Using this, we have

$$\gamma_M \mathcal{D} \gamma_M = \widetilde{\mathcal{D}} \tag{2.15}$$

where we used equation (2.13) and the fact that γ is involutive. Taking the functional logarithm of both sides of equation (2.15) gives

$$\mathrm{Ln}(\mathcal{D}) = \mathrm{Ln}(\widetilde{\mathcal{D}}) \tag{2.16}$$

where we again used the involutory property of γ_M. Given equation (2.12), we have

$$\begin{aligned} \Gamma_{sp} &= -i \mathrm{Tr}\ \mathrm{Ln}(\mathcal{D}) \\ &= -\frac{i}{2} \mathrm{Tr}\ \mathrm{Ln}(\mathcal{D}\widetilde{\mathcal{D}}) \end{aligned} \tag{2.17}$$

where in the first line we used equation (2.14), and to get the last line we used equation (2.16). The differential operator in the second line of (2.17) is second-order and thus compatible with the heat kernel expansion. However, we can do further manipulations. Using $\mathcal{D}\widetilde{\mathcal{D}} = \not{D}^2 - m^2$, we have

$$\Gamma_{sp} = -\frac{i}{2} \mathrm{Tr}\ \mathrm{Ln}(\not{D}^2 - m^2). \tag{2.18}$$

In [73], Lichnerowicz derived the formula

$$\not{D}^2 = \Box - \frac{1}{4} R. \tag{2.19}$$

One substitutes this into equation (2.18) to get

$$\Gamma_{sp} = -\frac{i}{2} \mathrm{Tr}\ \mathrm{Ln}\Big(\Box - m^2 - \frac{1}{4} R\Big). \tag{2.20}$$

This result has the same form as equation (2.8). Hence, we can now treat both the scalar and spinor fields on equal footing.

Combining scalars and spinors

Let us now have a collection of scalars and Dirac spinors where each field is labelled by an index κ. Supersymmetry [74, 75] provides the supertrace of an arbitrary operator $\mathcal{O}$ as

$$\mathrm{Str}(\mathcal{O}) := \sum_{\kappa} \mathrm{Tr}\Big((-1)^F \mathcal{O}\Big) \tag{2.21}$$

where for scalars $(-1)^F = 1$ and for spinors $(-1)^F = -1$. Hence, combining equations (2.8) and (2.20), we write the total scalar-spinor effective action as

$$\Gamma_{\text{tot}} = \frac{i}{2}\text{Str Ln}(\Box - m^2 - \iota R) \tag{2.22}$$

where we have assumed all fields have the same mass m (c.f [12]), and for scalars $\iota = \xi$ and for spinors $\iota = \frac{1}{4}$.

We now perform the regularisation. Using the Schwinger proper time formalism (more about this and the cutoff regularisation later in this section), which gives relative but not absolute results, we have

$$\begin{aligned}\Gamma_{\text{tot}}[g] = \Gamma_{\text{tot}}[g_0] - \frac{i}{2}\text{Str}\int_0^\infty \Big(&\exp[-(\Box[g] - m^2 - \iota R[g])\ t] \\ &- \exp[-(\Box[g_0] - m^2 - \iota R[g_0])\ t]\Big)\frac{dt}{t}\end{aligned} \tag{2.23}$$

where we defined an auxiliary metric g_0 on $\mathcal{M}$. The cutoff regularisation is implemented in equation (2.23) by introducing a UV cutoff Λ at the Planck scale:

$$\begin{aligned}\Gamma_{\text{tot}}[g] = \Gamma_{\text{tot}}[g_0] - \frac{i}{2}\text{Str}\int_{\Lambda^{-2}}^\infty \Big(&\exp[-(\Box[g] - m^2 - \iota R[g])\ t] \\ &- \exp[-(\Box[g_0] - m^2 - \iota R[g_0])\ t]\Big)\frac{dt}{t}.\end{aligned} \tag{2.24}$$

The operator exponential in equation (2.24) has the expansion ([12])

$$\exp[-(\Box[g] - m^2 - \iota R[g])\ t] \overset{t\to 0^+}{\sim} \frac{1}{(4\pi t)^2}\sum_{n=0}^{\infty} b_n[g]\ t^n \tag{2.25}$$

where b_n are combinations of local diffeomorphism invariants and encode the parameters m and ι. Equation (2.25) is the heat kernel expansion, and b_n are heat kernel coefficients.

We pause to discuss the heat kernel expansion. As discussed in [15], the expansion faces a potential problem of unsuppressed contributions from non-local geometry: in the Euclidean regime this does not arise so the expansion is well-defined, but in the Lorentzian regime the problem does occur so the expansion is ill-defined. In particular, equation (2.25) is Lorentzian and thus poorly defined. However, this issue will be ignored for now (i.e. we view equation (2.25) as a formal power series). After this subsection we will not use equation (2.25), but will only consider a Euclidean version which is discussed in more detail in subsection 2.1.3.

Substituting equation (2.25) into equation (2.24) gives

$$\Gamma_{\text{tot}}[g] \overset{t\to 0^+}{\sim} \Gamma_{\text{tot}}[g_0] - \frac{i}{2(4\pi)^2}\text{Str}\sum_{n=0}^{\infty}\Big(b_n[g] - b_n[g_0]\Big)\Big(\int_{\Lambda^{-2}}^{\infty} t^{n-2}\frac{dt}{t}\Big). \tag{2.26}$$

The first three integrals (with $n \in \{0, 1, 2\}$) are

$$\begin{aligned}
&\int_{\Lambda^{-2}}^{\infty} t^{-2}\frac{dt}{t} = \frac{1}{2}\Lambda^4,\\
&\int_{\Lambda^{-2}}^{\infty} t^{-1}\frac{dt}{t} = \Lambda^2,\\
&\int_{\Lambda^{-2}}^{\infty} \frac{dt}{t} = \ln(\Lambda^2) + \text{div}\\
&= \Big(\ln(\Lambda^2) - \ln(m^2)\Big) + \Big(\ln(m^2) + \text{div}\Big)\\
&= \ln\Big(\frac{\Lambda^2}{m^2}\Big) + \text{div}'
\end{aligned} \tag{2.27}$$

where div and div′ are divergences coming from the infrared regime. For now these infrared divergences will be neglected, but later we will return to them and discuss how to systematically treat them. One substitutes equation (2.27) in (2.26), which leads to

$$\begin{aligned}
\Gamma_{\text{tot}}[g] \overset{t\to 0^+}{\sim} \Gamma_{\text{tot}}[g_0] - \frac{i}{2(4\pi)^2}\text{Str}\Big(&\Big(b_0[g] - b_0[g_0]\Big)\frac{1}{2}\Lambda^4\\
&+ \Big(b_1[g] - b_1[g_0]\Big)\Lambda^2 + \Big(b_2[g] - b_2[g_0]\Big)\ln\Big(\frac{\Lambda^2}{m^2}\Big) + \mathcal{O}(\Lambda^{-2})\Big)
\end{aligned} \tag{2.28}$$

where the form of the $\mathcal{O}(\Lambda^{-2})$ terms was deduced from the next t integral. Equation (2.28) has the typical quantum field theory UV divergence configuration. The first three heat kernel coefficients b_0, b_1, b_2 are given by ([12])

$$\begin{aligned}
b_0 &= 1,\\
b_1 &= m^2 - k_1 R,\\
b_2 &= \frac{1}{2}m^4 - m^2 k_1 R + k_2 Weyl^2 + k_3 Ric^2 + k_4 R^2 - k_5 \Box R.
\end{aligned} \tag{2.29}$$

Here $Weyl$ and Ric are respectively the Weyl and Ricci tensors, and $k_1, \ldots, k_5$ are dimensionless numbers containing ι. The heat kernel coefficients given by equation (2.29) are sufficient since they correspond to the dominant contributions to the supertrace in equation (2.28), and thus we will neglect the $\mathcal{O}(\Lambda^{-2})$ terms. For an arbitrary operator $\mathcal{O}$, let

$$\text{Str}[\mathcal{O}] := \int_{\mathcal{M}} \Omega(\text{str}[\mathcal{O}]). \tag{2.30}$$

By using this and equation (2.29), one can re-cast b_2 using the Chern-Gauss-Bonnet and divergence theorems:

$$\int_{\mathcal{M}} \Omega(b_2) = \int_{\mathcal{M}} \Omega\Big(\frac{1}{2}m^4 - m^2 k_1 R + k_2' Weyl^2 + k_4' R^2\Big) \tag{2.31}$$

where k_2, k_3, k_4 are absorbed into new numbers k_2' and k_4'. The remaining k numbers are specified in Table 1 in [12]. Substituting equations (2.29), (2.30) and (2.31) into equation (2.28) gives

$$\begin{aligned}\Gamma_{\text{tot}}[g] \overset{t\to 0^+}{\sim} \Gamma_{\text{tot}}[g_0] \\ &- \frac{i}{2(4\pi)^2}\Big(\text{str}\Big[\frac{1}{2}\Lambda^4 + m^2\Lambda^2 + \frac{1}{2}m^4 \ln\Big(\frac{\Lambda^2}{m^2}\Big)\Big]\Big(\int_{\mathcal{M}} \Omega[g] - \Omega[g_0]\Big) \\ &+ \text{str}\Big[-k_1\Lambda^2 - m^2 k_1 \ln\Big(\frac{\Lambda^2}{m^2}\Big)\Big]\Big(\int_{\mathcal{M}} R[g]\ \Omega[g] - R[g_0]\ \Omega[g_0]\Big) \\ &+ \text{str}\Big[k_2' \ln\Big(\frac{\Lambda^2}{m^2}\Big)\Big]\Big(\int_{\mathcal{M}} Weyl[g]^2\ \Omega[g] - Weyl[g_0]^2\ \Omega[g_0]\Big) \\ &+ \text{str}\Big[k_4' \ln\Big(\frac{\Lambda^2}{m^2}\Big)\Big]\Big(\int_{\mathcal{M}} R[g]^2\ \Omega[g] - R[g_0]^2\ \Omega[g_0]\Big)\Big)\end{aligned} \tag{2.32}$$

where Ω depends on the metric g via the tetrad l (see section A.1). The induced gravitational constants are given in a model-independent form by

$$\Gamma_{\text{tot}} = \int_{\mathcal{M}} \Omega\Big(\frac{1}{16\pi G_{ind}}(-2\Lambda_{ind} + R) + K_{ind}^{Weyl} Weyl^2 + K_{ind}^{R^2} R^2\Big). \tag{2.33}$$

Matching this with equation (2.32) gives

$$\begin{aligned}\frac{2\Lambda_{ind}}{16\pi G_{ind}} &= \frac{i}{2(4\pi)^2}\text{str}\Big[\frac{1}{2}\Lambda^4 + m^2\Lambda^2 + \frac{1}{2}m^4 \ln\Big(\frac{\Lambda^2}{m^2}\Big)\Big], \\ \frac{1}{16\pi G_{ind}} &= -\frac{i}{2(4\pi)^2}\text{str}\Big[-k_1\Lambda^2 - m^2 k_1 \ln\Big(\frac{\Lambda^2}{m^2}\Big)\Big], \\ K_{ind}^{Weyl} &= -\frac{i}{2(4\pi)^2}\text{str}\Big[k_2' \ln\Big(\frac{\Lambda^2}{m^2}\Big)\Big], \\ K_{ind}^{R^2} &= -\frac{i}{2(4\pi)^2}\text{str}\Big[k_4' \ln\Big(\frac{\Lambda^2}{m^2}\Big)\Big].\end{aligned} \tag{2.34}$$

Equation (2.34) gives the induced cosmological, Newton and curvature-squared constants due to a collection of scalars and Dirac spinors. One has the problem that the last three gravitational constants in equation (2.34) are imaginary, and we attribute this to the ill-definedness of the Lorentzian heat kernel expansion. We will define the heat kernel expansion in a consistent manner later.

According to induced gravity, one takes Λ_{ind} as the induced value for the cosmological constant, despite the lack of renormalisation. From equation (2.34), we see that (the modulus of) Λ_{ind} is of order Λ^2. Given the experimental cosmological constant value $\Lambda^{exp} \sim 10^{-122}\Lambda^2$ [10, 11], one has $\Lambda_{ind} >> \Lambda^{exp}$, which is the cosmological constant problem [76][4]. We remark that the cosmological constant problem commonly arises in induced gravity models.

2.1.3 Regularisation toolkit

We cover in more details some techniques used in the regularisation of the 1-loop effective action.

Schwinger proper time formalism

This comes in the form of the integral

$$\mathrm{Ln}\left[\frac{B}{A}\right] = -\int_0^\infty \left[e^{-Bt} - e^{-At}\right]\frac{dt}{t} \tag{2.35}$$

where A and B are positive operators. In induced gravity, the operators A and B have mass dimension 2, hence the dummy variable t has mass dimension -2, i.e. is a genuine proper time parameter. Equation (2.35) gives a UV divergence coming from $1/t$ as $t \to 0^+$. The UV cutoff Λ is needed for the UV regularisation, which is done in the t-integral in equation (2.35) by letting $t = 0 \to t = \Lambda^{-2}$ which preserves the dimensions of proper time. The cutoff Λ is taken to be at the Planck scale (c.f Sakharov [9]). The curvature-squared terms give another divergence (see earlier and [16, 19]), this one due to the limit $t = \infty$ in equation (2.35) and thus the infrared regime. This infrared divergence gets regularised by a separate IR cutoff ϵ.

Heat kernel expansion

Here, we introduce the heat kernel expansion used in this work, and this is done in the Euclidean regime on $\mathcal{M}$ (see appendix A). Given that $\mathcal{M}$ is a 4-dimensional

[4] For completeness, we mention a more precise formulation of the cosmological constant problem in [77]. However, we keep with the formulation in [76] since it is simpler to use (the cosmological constant problem does not play a huge role in the present thesis).

compact boundary-less manifold, the heat kernel expansion takes the form

$$\mathrm{Tr}\Big[e^{-tP}\Big] \overset{t\to 0^+}{\sim} \sum_{n\geq 0} t^{\frac{n-4}{2}} a_n(1,P) \tag{2.36}$$

where P is a second-order differential operator and $a_n(1,P)$ are the heat kernel coefficients. Equation (2.36) is given by equation (2.21) in [15] for the special case of $\dim(\mathcal{M}) = 4$ and $f = 1$, which we consider in this work. Locally, the operator P is a generalised Laplacian of the form

$$P = -g_E^{\mu\nu}\nabla_\mu\nabla_\nu \otimes I_3 - E \tag{2.37}$$

where $\nabla_\mu = \nabla_{\partial_\mu}$ and E is a function from $\mathcal{M}$ to matrices. All odd-labelled heat kernel coefficients a_n vanish and the even-labelled a_n are expressed exclusively in terms of local coordinate invariants: the first four even coefficients are given by (4.26 - 4.29) in [15]. In particular, a_4 corresponds to curvature-squared terms. The coefficients beyond this correspond to terms of order greater than curvature-squared and are thus suppressed compared to the terms at curvature-squared order and below. Given this, in the present work we only consider heat kernel coefficients not exceeding a_4, and this allows us to describe the important semiclassical gravitational contributions. We remark that the heat kernel expansion is universal, i.e. lots of different situations have the same a_n.

2.1.4 Modifications

The gravity-gauge extension of the original induced gravity mechanism [16] also produces kinetic terms in arbitrary gauge fields (but the gauge fields themselves do not get induced), and specialises to where SM gauge couplings are induced from the SM fermions and Higgs. We remark that one may view their formalism as anticipating the induced Standard Model (section 2.2).

In the models involving spacetime with torsion [18, 19] one splits the torsion into several components, and the induced 1-loop contributions are terms in curvature and terms in the components of the torsion. The mechanism of [19] also gives rise to curvature-torsion interactions.

One can consider a spacetime boundary $\partial\mathcal{M}$. The (Euclidean) heat kernel expansion for the case with a boundary [15] has coefficients with additional boundary terms which involve invariants of $\partial\mathcal{M}$. The boundary modification may be worth consideration of others as it would remove an unphysical property of the Lorentzian regime corresponding to $\mathcal{M}$ (see subsection 1.2.3).

2.2 Induced Standard Model

This is set in the manifold $\mathcal{M}$, and one first considers the Lorentzian regime (see appendix A). For a precise formulation of the induced Standard Model, we follow a Lorentzian non-commutative geometry model of spacetime and fermions [1] (the Euclidean SM non-commutative geometry will be discussed in section 4.2). The induced Standard Model starts with fundamental matter consisting of all SM fermions in addition to right-handed neutrinos (for more details about the latter, see section 4.1). Correspondingly, the space of matter fields is

$$\mathcal{S} := C^\infty(\mathcal{M}, \hat{S}) \otimes \mathcal{S}_F \tag{2.38}$$

where $C^\infty(\mathcal{M}, \hat{S})$ is the space of Dirac spinors corresponding to spinor bundle $\hat{S}$ and $\mathcal{S}_F \simeq \mathbb{C}^{96}$ with basis corresponding to SM fermions and right-handed neutrinos (assuming three of the latter). Any element of $\mathcal{S}$ is given by $\Psi = \psi \otimes \mathfrak{f}$ where $\psi \in C^\infty(\mathcal{M}, \hat{S})$ and $\mathfrak{f} \in \mathcal{S}_F$ and has a conjugate $\overline{\Psi} \in \mathcal{S}$ given by analogy with section A.1. Given this, the Dirac inner product on $\mathcal{S}$ is given by

$$\langle \Psi, \Psi' \rangle_{ISM} = \langle \psi, \psi' \rangle \, (\mathfrak{f} \cdot_F \mathfrak{f}') \tag{2.39}$$

where $\langle \cdot, \cdot \rangle$ is the standard spinor Dirac product (see section A.1) and $\cdot_F$ is the dot product on $\mathcal{S}_F$. The gravitational and bosonic fields are encoded in a Dirac operator[5] on $\mathcal{S}$ of the form

$$\mathfrak{D} = \not{D} \otimes 1 + (i\gamma_M) \otimes D_{ferm} + \mathfrak{B} \tag{2.40}$$

where D_{ferm} is a Dirac operator encoding all right-handed neutrino Majorana masses (more about these in section 4.1) and Yukawa couplings, and $\mathfrak{B}$ is an operator on $\mathcal{S}$ that encodes the SM gauge and Higgs bosons. One defines the matter action as

$$S_{ISM} = \int_{\mathcal{M}} \Omega \Big(\langle \Psi, \mathfrak{D}\Psi \rangle_{ISM} - 2\Lambda_0 \Big) \tag{2.41}$$

[5]For further discussion on the notion of a Dirac operator, see sections 4.2 and A.1.

where Λ_0 is a cosmological constant counter-term introduced in anticipation that the cosmological constant problem will arise. Note equation (2.41) does not have curvature or bosonic terms, and this is in line with the standard induced gravity mechanism. Only the SM fermions and right-handed neutrinos are quantised, leading to the functional integral

$$Z_{ISM} = \int e^{iS_{ISM}} D\overline{\Psi} D\Psi Dg \tag{2.42}$$

where the integrations over $\Psi, \overline{\Psi} \in \mathcal{S}$ are independent. The metric functional integral is only approachable with quantum gravity (as is done in [20]) and thus outside our semiclassical scope, but the integration out of the fermions is a standard quantum field theory result giving

$$Z_{ISM} = \int \mathrm{Det}(i\mathfrak{D})\; e^{i\int_{\mathcal{M}} \Omega \left(-2\Lambda_0\right)} Dg. \tag{2.43}$$

In this expression, the most important part is the determinant which gives the effective action contributions, and thus one may drop everything else.

The mechanism now goes over to the Euclidean regime. The rules for doing this will be given in chapter 3, but for now we note that for the spacetime Dirac operator $\not{D} = i\not{D}_E$ and D_{ferm} does not change ([1]). Thus $\mathfrak{D} = i\mathfrak{D}_E$ where

$$\mathfrak{D}_E = \not{D}_E \otimes 1 + \gamma_M \otimes D_{ferm} + \mathfrak{B}_E \tag{2.44}$$

where we assumed $\mathfrak{B} = i\mathfrak{B}_E$. Hence, we get

$$\mathrm{Det}(i\mathfrak{D}) = \mathrm{Det}\ \mathfrak{D}_E \tag{2.45}$$

where a multiplicative constant with no effect on the induced dynamics was neglected. The effective action Γ_B induced by the fermions is defined by

$$e^{-\Gamma_B} := \mathrm{Det}\ \mathfrak{D}_E. \tag{2.46}$$

Hence

$$\begin{aligned} \Gamma_B &= -\ln \mathrm{Det}\ \mathfrak{D}_E \\ &= -\mathrm{Tr}\ \mathrm{Ln}\ \mathfrak{D}_E \\ &= -\frac{1}{2}\mathrm{Tr}\ \mathrm{Ln}\ \mathfrak{D}_E^2 \end{aligned} \tag{2.47}$$

where the second line used ln Det = Tr Ln. The IR regularisation in this case occurs by modifying (2.47) as

$$\Gamma_B = -\frac{1}{2}\text{Tr Ln } \mathfrak{D}_E^2 - \Big(-\frac{1}{2}\text{Tr Ln } \mathfrak{D}_{E0}^2 \Big) \tag{2.48}$$

where $\mathfrak{D}_{E0}$ is another Dirac operator, and the regularisation happens via the elimination of divergent contributions due to the logarithmic pole. The UV regularisation is then given as standard by the UV cutoff Λ. The regularised effective action is

$$\Gamma_B = -\frac{1}{2}\text{Tr Ln}\Big(\frac{\mathfrak{D}_E^2}{\Lambda^2}\Big) - \Big(-\frac{1}{2}\text{Tr Ln}\Big(\frac{\mathfrak{D}_{E0}^2}{\Lambda^2}\Big)\Big). \tag{2.49}$$

This can be approximated as

$$\Gamma_B = \frac{1}{2}\text{Tr Ei}\Big(\frac{\mathfrak{D}_E^2}{\Lambda^2}\Big) - \frac{1}{2}\text{Tr Ei}\Big(\frac{\mathfrak{D}_{E0}^2}{\Lambda^2}\Big) \tag{2.50}$$

where Ei is the exponential integral. It turns out [20] that equation (2.50) has an expansion giving curvature terms as well as SM gauge and Higgs terms. With respect to the Connes-Chamseddine action of non-commutative geometry (see section 4.2), equation (2.50) turns out [20] to be approximately equal to the Connes-Chamseddine action on the condition of a fourth SM fermion generation. Hence, the fact that the fermionic integration out gives a result depending only on Dirac operator eigenvalues implies that the same holds for the Connes-Chamseddine action, i.e. the spectral action principle (discussed in section 4.2).

Chapter 3

Lorentzian-Euclidean transition

The need for going between Lorentzian and Euclidean theories is motivated as follows. One ought to start with the Lorentzian physics since this is what can be probed in experiments. On the other hand, the Euclidean theories are unphysical in the sense that they cannot be measured, but are mathematically well-defined and compatible with existing techniques (as will be seen later in this chapter). This suggests that the Lorentzian physics should be the starting point, but it is more mathematically convenient to perform the required calculations in the Euclidean regime. This necessitates having a method to go between Lorentzian and Euclidean theories. This chapter is a review of the Lorentzian-Euclidean transition from [1]. The material covered here is not standard since it was introduced recently (in [1]), hence why we dedicate a chapter to reviewing the material. This chapter has no original material, but the material herein will be used as part of our original work in chapter 5.

A well-known procedure for mapping between Lorentzian and Euclidean theories is the Wick rotation. This procedure is conventionally formulated in Minkowski spacetime by making the time coordinate imaginary. For curved spacetime, the Wick rotation has a second-order formulation as a complexification of the spacetime metric [78] or a first-order formulation as a complexification of the tetrad [79, 58]. As will be seen later, the Wick rotation is inequivalent to the Lorentzian-Euclidean transition, but both have aspects in common. This leaves the question of why one may wish to forgo the Wick rotation in favour of the Lorentzian-Euclidean transition. To answer this, we compare the two schemes as follows:

- The Wick rotation acts at the level of the metric, and maps between a Lorentzian spacetime and an inequivalent Euclidean spacetime. Hence, the Lorentzian and Euclidean theories have different physical properties.

- Fundamentally, the Lorentzian-Euclidean transition is at the functional integral level. However, the functional integrals of both theories are not just related, but identical (more below). This gives an immediate bridge between the Lorentzian and Euclidean theories.

Thus, compared to the Wick rotation, the Lorentzian-Euclidean transition is more practical for the functional integral quantisation (and this will apply to chapter 5) and connects the Lorentzian and Euclidean theories in a more physically robust manner. These reasons make it physically advantageous to use the Lorentzian-Euclidean transition over the Wick rotation, as we do in chapter 5.

3.1 Functional integrals

The Lorentzian-Euclidean transition is centered around a pair of Lorentzian and Euclidean functional integrals. We define these first.

One considers the Lorentzian functional integral

$$Z = \int_{\Gamma \in \mathcal{L}} \int_{F \in \mathcal{F}} e^{iS[\Gamma, F]} DFD\Gamma \tag{3.1}$$

where $\mathcal{L}$ is a real vector space of Lorentzian geometries, $\mathcal{F}$ is a space of matter fields and S is the action. Equation (3.1) corresponds to a theory of matter fields $\mathcal{F}$ set in geometries $\mathcal{L}$ where both are quantised. Such a theory could be ordinary field theory in Lorentzian spacetime, but is an abstraction of this since the geometries in $\mathcal{L}$ are allowed to be classical or quantum. In particular, the theory can be the matter phase of induced gravity (neglecting the $\mathcal{L}$ integral) as in chapter 2. Since the Lorentzian theory is typically known from experiment, the corresponding functional integral (3.1) gives the physical amplitudes and observables and is thus sought after. On the other hand, since equation (3.1) is an extrapolation of quantum field theory to the Planck scale, the equation is not expected to exactly describe all details of physics down to that scale. In addition, since the complex exponential integrand in equation (3.1) is highly oscillatory, the functional integral will not converge.

A Euclidean version of the functional integral is given by

$$Z_E = \int_{\mathcal{E}} \int_{\mathcal{F}_E} e^{-S_E[\Gamma_E, F_E]} DF_E D\Gamma_E \tag{3.2}$$

with similar Euclidean definitions. This also gives an abstract Euclidean theory, which may in particular be non-commutative geometry (see chapter 4). The integral (3.2) does converge if

$$\begin{aligned} S_E \to \infty \text{ as } \Gamma_E \to \infty \\ \min S_E > -\infty \end{aligned} \tag{3.3}$$

where the first condition applies in some sense and requires that the rate of increase of S_E is not too small[1]. This is because (3.3) means the integrand of (3.2) is upper bounded and goes to zero as needed. Note (3.3) is not applicable to fermions since these are required to be anticommuting variables. In addition to having better convergence, the Euclidean functional integral is more amenable to mathematical physics tools, e.g. the heat kernel expansion (chapter 2), compared to the Lorentzian integral. However, the Euclidean integral is not directly relevant to observable physics.

The key proposal of the Lorentzian-Euclidean transition is

$$Z = Z_E. \tag{3.4}$$

We have chosen not to label this result as a theorem since a rigorous proof is unavailable. Indeed, the intractability of functional integrals is expected to be a barrier to getting a proof. Instead, equation (3.4) should currently be understood as a conjecture.

A motivation of (3.4), which may be viewed as an informal proof for the special case of Lorentzian and Euclidean structures on $\mathcal{M}$, was given by Barrett [1]. This has three main stages, which we cover in the following. We give the main results of each stage and relegate any further technical details to [1].

One starts with the Lorentzian functional integral (3.1).

[1] The author acknowledges comments by John W. Barrett.

- *Complexification*

  The complexification of $\mathcal{L}$ is given by the space of geometries $\mathcal{C}$. Into this, one takes the real subspace $\mathcal{L} \subset \mathcal{C}$ and the analytic continuation of S.

- *Rotation*

  One considers another subspace $\mathcal{L}' \subset \mathcal{C}$ defined by real and imaginary tetrads and spin connection coefficients. The subspace $\mathcal{L}$ is rotated to $\mathcal{L}'$ while keeping in $\mathcal{C}$. The result from this step is

  $$\int_{\mathcal{L}} e^{iS} D\Gamma = \int_{\mathcal{L}'} e^{iS} D\Gamma. \tag{3.5}$$

  This requires the vanishing of the integral over a contour $\mathcal{C}$.

- *Euclideanisation*

  As vector spaces,

  $$\mathcal{L}' \simeq \mathcal{E}, \tag{3.6}$$

  i.e., there is a correspondence between imaginary Lorentzian geometric data and real Euclidean data. Concretely, this applies to frames and metrics. A general result of Euclideanisation is the relation

  $$S = iS_E \tag{3.7}$$

  on the imaginary contour $\mathcal{L}'$. Equations (3.6) and (3.7) give

  $$\int_{\mathcal{L}'} e^{iS} D\Gamma = \int_{\mathcal{E}} e^{-S_E} D\Gamma_E. \tag{3.8}$$

  This (with the right continuation of matter fields) gives the right-hand side of (3.4).

In this method, the Lorentzian functional integral is used to define the Euclidean functional integral via the relations (3.4) and (3.7). In other words, the Euclidean theory is determined by the Lorentzian theory and thus the observable physics. Using (3.4) and (3.7), one can transition from the Lorentzian theory to the Euclidean theory.

We remark that the Euclideanisation stage gives similar results to the Wick rotations defined in [79, 58]. However, $\mathcal{E}$ directly relates to $\mathcal{L}'$ but not to $\mathcal{L}$. Hence, the method in this section is distinct from (but resembles) the Wick rotation.

3.2 Euclideanisation

Euclideanisation [1] is perhaps the most significant and well-formulated stage of the method. This stage has given practical results for theories without and with matter. Several of these results are reviewed below and are taken from [1]. Note all relations from Euclideanisation are assumed on $\mathcal{L}'$ as only here can one compare imaginary Lorentzian and real Euclidean geometries.

- *Spacetime only*

  For $N = \delta_{a0} + \delta_{b0} + \delta_{c0} - \delta_{f0}$, the Lorentzian and Euclidean curvature tensors are related by

  $$^E R^f_{cab} = (-i)^N R^f_{cab}. \tag{3.9}$$

  The Lorentzian and Euclidean Ricci scalars match:

  $$R = R_E. \tag{3.10}$$

  The Lorentzian and Euclidean volume forms obey

  $$\Omega = -i\Omega_E. \tag{3.11}$$

  The Einstein-Hilbert action (with cosmological constant)

  $$S_{Grav} = \frac{1}{16\pi G}\int_{\mathcal{M}}(R - 2\Lambda)\Omega \tag{3.12}$$

  satisfies

  $$S_{Grav} = i^E S_{Grav} \tag{3.13}$$

  where

  $$^E S_{Grav} = -\frac{1}{16\pi G}\int_{\mathcal{M}}(R_E - 2\Lambda)\Omega_E \tag{3.14}$$

  is the corresponding Euclidean action. Note the sign differences of the Lorentzian and Euclidean gravitational actions. It was pointed out in [34] that the Euclidean action (3.14) violates the second condition in (3.3) as the action admits contributions beyond the Planck scale. In this work, we assume the Planck scale is sufficient to restore the convergence condition for any theory containing the action (3.14) (c.f [1]). This is natural to impose in induced gravity (2) in which the effective action is regularised by a cutoff and consists of the action (3.14) with additional curvature-squared terms.

- *Scalar fields*

  For a scalar field Φ, the Euclideanisation of the action (3.7) is satisfied given

$$\Phi = \Phi_E. \tag{3.15}$$

- *Dirac spinor fields*

  The relation (3.7) holds (including Majorana mass terms) if, in going from Lorentzian to Euclidean, the spinor ψ is unchanged and one replaces the conjugate $\overline{\psi}$ (see section A.1) by an *independent* spinor $\widetilde{\psi}$. Given chiral ψ, the independent Euclidean counterpart spinor $\widetilde{\psi}$ has opposite chirality, and thus this property is preserved from the Lorentzian.

  One also has the Dirac operator Euclideanisation

$$\not{D} = i\not{D}_E. \tag{3.16}$$

  We assume Euclideanisation leaves mass and coupling parameters invariant.

In this whole section, we discussed only the transition from the Lorentzian to Euclidean theory. However, we will require the reverse transition, i.e. Euclidean to Lorentzian. We assume it is sufficient to just use (3.4) and (3.7) and the rest of the Euclideanisation relations. In other words, we use only the Euclideanisation stage and leave out the complexification and rotation stages. Any rigorous formulation of the transition (either way) is beyond our present scope.

Chapter 4

Beyond the Standard Model

In particle physics, the SM is the theory that gives the most comprehensive account of observed phenomena and has the most experimental verification. However, one can determine that the SM is not complete as:

- Observations of solar neutrinos suggest left-handed neutrinos exhibit flavour oscillations, which occur if left-handed neutrinos have masses $m_{\nu p} \neq 0$. However, the SM has $m_{\nu p} = 0$.
- The SM particle spectrum has no dark matter candidates.
- The SM has not explained the fact that the universe has a baryon asymmetry, i.e. a larger number of baryons than anti-baryons.
- Gravity is absent from the SM.

This serves as a motivation to consider physics beyond the SM, which modifies the SM or constructs new theories with the SM as a special case. Physics beyond the SM is a broad area, so we will focus on certain aspects of this here.

This chapter is organised as follows. Section 4.1 is a review of known material concerning right-handed neutrinos, and this is set in the Lorentzian regime. The SM with three right-handed neutrinos, as well as the type-I seesaw, are standard, but the alternative seesaws and Neutrino Minimal Standard Model are not (hence why this section is not an appendix). All this will be needed for our original work in chapter 5. Section 4.2 is a review of non-commutative geometry. This has the SM with right-handed neutrinos as a special case and therefore appears in this chapter. In section 4.3, we review the Weinberg operator and higher-dimensional invariants with two lepton doublets and two Higgs, as well as the work of Broncano-Gavela-Jenkins [49, 50] which includes such terms. The material in sections 4.2 and 4.3 is not new, but is placed into this chapter since the material is non-standard. Sections 4.2 and 4.3 will be important for our original work later on.

4.1 Right-handed neutrinos

Our point of view is that right-handed neutrinos are outside the SM particle spectrum[1]. This does not necessarily rule out the existence of right-handed neutrinos. Assuming that right-handed neutrinos do exist, continuing the SM particle classification, we see that they would be right-handed fermions that are SM gauge singlets. Hence, right-handed neutrinos interact very little with baryonic matter. As such, the existence of right-handed neutrinos has not been unambiguously verified by experiments.

Right-handed neutrinos may be added to the SM particle spectrum, and one can add any number of them. In this work, we consider the special case of the SM with three right-handed neutrinos (which naturally fits into grand unified theories of $SO(10)$ [81, 82] and Pati-Salam [83]). Hence, we consider one extra right-handed neutrino per SM generation but with no other changes to the SM particle spectrum and keeping the same SM gauge group (i.e. minimal modification). The three right-handed neutrinos are right-handed Dirac spinors with Majorana mass terms (see appendix A) and the right-handed neutrinos couple to the SM only via Yukawa couplings to the left-handed neutrinos. We denote the right-handed neutrinos by ν_R^p where the generational indices $p, q, \ldots$ are valued in $\{1, 2, 3\}$. Using notation from appendix A, the three extra right-handed neutrinos contribute the following terms to the SM Lagrangian:

$$\mathcal{L}_N := \overline{\nu_{Rp}} \not{D} \nu_R^p + \mathcal{L}_Y + \mathcal{L}_M \tag{4.1}$$

where $\overline{\nu_{Rp}} = C\nu_R^p$ and the last two terms will be discussed. The indices $p, q, \ldots$

[1] For completeness, we acknowledge two different views: the one we adopt (c.f [52]) and that where right-handed neutrinos are SM particles (c.f [80]).

can be arbitrarily raised and lowered since this is done by the flat metric δ_{pq}.

Equation (4.1) has the Yukawa term

$$\mathcal{L}_Y = Y_{pq}(\overline{l_L^p} \cdot_{\mathbb{C}^2} \phi)\nu_R^q + c.c. \tag{4.2}$$

where the Yukawa matrix Y is a 3×3 complex matrix, $l_L^p = (e_L\ \nu_L)^{Tp}$ are the $SU(2)$ left-handed lepton doublets, ϕ is the Higgs doublet and $\cdot_{\mathbb{C}^2}$ is the dot product on $\mathbb{C}^2$ with respect to the internal $SU(2)$ indices. The electroweak symmetry is spontaneously broken by the Higgs vacuum $\phi_v = (0\ \frac{v}{\sqrt{2}})^T$ where $v \sim 246$GeV. This reduces the Yukawa term (4.2) to a Dirac mass term of the form (A.18):

$$[\mathcal{L}_Y]_{\phi=\phi_v} = m_{Dpq}\overline{\nu_L^p}\nu_R^q + c.c. \tag{4.3}$$

where the mass parameter is now a 3×3 mass matrix $m_D = \frac{v}{\sqrt{2}}Y$. The right-handed neutrinos ν_R have no other SM couplings in the electroweak broken regime.

The last term in (4.1) is a Majorana mass term (A.24):

$$\mathcal{L}_M = \frac{1}{2}M_{pq}\nu_R^p\nu_R^q + c.c. \tag{4.4}$$

with a 3×3 complex symmetric matrix M as the Majorana mass for three generations. The diagonalisation of M is given in appendix B and results in the Majorana mass eigenvalues.

We consider the magnitudes of Y and M to be respectively given by the elements of Y and eigenvalues of M. Existing constraints on the Y and M magnitudes appear to be weak (see [52, 84, 53]). Several bounds on M have been motivated by phenomenology and grand unified theory [52]: $M = 0$ gives pure Dirac right-handed neutrinos whose Y must be unnaturally smaller than those of SM fermions so that m_D matches experimental (left-handed) neutrino masses; M of order keV gives right-handed neutrinos that are candidates for stable dark matter (see also [85]); M of grand unification order ($\mathfrak{M} \sim 10^{15}$GeV) is the type-I seesaw regime (more below). One can have theories with multiple bounds at once, and examples of these are later considered.

There is a pair of (left-handed) neutrino mass orderings given by $(m_1^L)^2 < (m_2^L)^2 < (m_3^L)^2$ and $(m_2^L)^2 > (m_1^L)^2 > (m_3^L)^2$, which respectively are the normal and inverted hierarchies. These come about from a sign ambiguity in a squared mass difference. However, the hierarchies will not be important here as we only consider orders of neutrino masses.

We discuss in more detail the emergence of neutrino masses for $M \sim \mathfrak{M}$. This occurs via the type-I seesaw mechanism [86, 87, 88, 89] (our presentation is based on [52]) as follows. After electroweak breaking,

$$\mathcal{L}_Y + \mathcal{L}_M = \frac{1}{2}\Big[\begin{pmatrix} C\nu_L & \nu_R \end{pmatrix}, \begin{pmatrix} 0 & m_D \\ m_D^T & M \end{pmatrix}\begin{pmatrix} C\nu_L \\ \nu_R \end{pmatrix}\Big] \tag{4.5}$$

where we used (A.20) and the Majorana product symmetry. In the above, the matrix consisting of Dirac and Majorana blocks is complex symmetric and is hence diagonalisable (appendix B). The full diagonalisation (see [52]) results in three left-handed active neutrinos and three right-handed sterile ones[2] with respective masses $m_{\nu p} \sim Y^2 M^{-1}$ and $M_{\nu p} \sim M$. The active and sterile masses scale in opposite fashion as M increases, which gives the seesaw. In the seesaw regime $M \sim \mathfrak{M}$ with $Y \sim 1$, one gets small active masses $m_{\nu p}$ of experimental order and large sterile masses $M_{\nu p} \sim \mathfrak{M}$ whose significance is only felt around grand unification. A Yukawa coupling $Y \sim 1$ is natural relative to the SM[3], which is an improvement on $M = 0$ right-handed neutrinos.

We comment on other seesaw mechanisms that are reviewed in [63]:

- In general, the seesaws share the type-I properties of seesaw mass behaviour and small active neutrino masses but replace the right-handed neutrinos.
- The type-I seesaw can be modified by a complex scalar field due to an additional $U(1)_X$ gauge symmetry.
- Type II has a scalar $SU(2)_L$ triplet and type III uses several leptonic $SU(2)_L$ triplets.

[2]Precisely, these eigenstates are linear combinations of ν_L and ν_R and their conjugates. However, the terms of opposite chirality are suppressed by powers of M^{-1}, which is small in the seesaw regime $M \sim \mathfrak{M}$.

[3]The largest SM Yukawa coupling, which corresponds to the top quark, is of order $Y_t \sim 1$ approximately [90].

- Seesaws of the form I+II and I+III give active neutrino masses by adding pieces from the individual seesaws.

Also, the type I $\times$ II seesaw of Wong and Chen [91] multiplies the active mass pieces. The authors order types I-III and their model by the seesaw scale and find theirs is the lowest entry.

A special case of the earlier SM with three right-handed neutrinos is the Neutrino Minimal Standard Model (νMSM) (see [52] and references therein) where M does not exceed the electroweak scale. The theory gives a sterile neutrino that is keV dark matter and two electroweak-GeV sterile neutrinos that give rise to the type-I seesaw and baryon asymmetry from the mechanism in [92].[4] The heavy sterile neutrinos are approximately degenerate.

In this review section, we have seen that three right-handed neutrinos as a minimal modification to the SM account for the neutrino masses, dark matter and baryon asymmetry, and all three of these phenomena cannot be explained by the SM without right-handed neutrinos. In our original work in chapter 5, we will see that the right-handed neutrinos also source gravitational dynamics, thus accounting for another beyond-SM phenomenon.

The right-handed neutrino Majorana mass terms can give a scalar modification to the SM and right-handed neutrinos. This will arise in the next section.

4.2 Connes' non-commutative geometry

In the SM the leptons and quarks as well as the gauge and Higgs bosons form separate sectors. This has motivated attempts to find a structure underpinning the SM, e.g. grand unified theories [94, 81] and supersymmetry [74, 95]. One such attempt is Connes' non-commutative geometry, which is a distinct approach to geometry based on non-commuting coordinates and spectra of operators. Connes' framework gives Euclidean-signature spacetime and the SM fields (plus right-handed neutrinos) a unified geometrical description, which provides accurate predictions for the top quark and Higgs masses [46, 44]. The non-commutative description of the SM also leads to further insights: gravity is

[4] A different mechanism for baryon asymmetry is found in [93].

included along with realistic dynamics [43], and the action for SM bosons coupled to gravity only depends on the eigenvalues of a certain differential operator (the spectral action principle) [67, 41]. Non-commutative geometry also naturally admits beyond-SM physics [68, 69] and grand unification [96]. Lastly, Connes' framework gives rise to matrix models for quantum gravity without matter [97] and with matter [54, 98, 99]. Hence, the physics connections of non-commutative geometry make it a subject of significant interest.

A non-commutative geometry is specified by a spectral triple $(\mathcal{A}, \mathcal{H}, D)$. The first entry $\mathcal{A}$ is a *-algebra, which is an associative algebra with an anti-linear map $^* : \mathcal{A} \to \mathcal{A}$ such that $\forall\ a, b \in \mathcal{A}$

$$\begin{aligned} (a^*)^* &= a, \\ (ab)^* &= b^* a^*. \end{aligned} \tag{4.6}$$

The second entry $\mathcal{H}$ is a Hilbert space. The third entry is a *Dirac operator*, which is an operator $D : \mathcal{H} \to \mathcal{H}$ which is self-adjoint ($D^* = D$) with compact resolvent $(D - wI)^{-1}$ where $w \in \mathbb{C}$ here[5]. We also have (see [71]) a faithful representation $\pi : \mathcal{A} \to \mathrm{End}(\mathcal{H})$ that commutes with *, thus π represents $\mathcal{A}$ by bounded operators in the Hilbert space[6]. There are two distinct notions of dimension in non-commutative geometry. The first is the metric dimension, which is a non-negative integer d_m where the eigenvalues $\lambda_k(D^{-1})$ of the inverse D^{-1} satisfy

$$\lambda_k(D^{-1}) \sim k^{-\frac{1}{d_m}}. \tag{4.7}$$

The second is the KO dimension, which is an integer $s \in \frac{\mathbb{Z}}{8}$ specified by the axioms of non-commutative geometry (as will be seen later).

The non-commutative geometry data are subject to a set of axioms [40]:

- *Orientability*:

  An even spectral triple has even s, and in this case there is a chirality

[5] The compact resolvent property means that D has a real discrete spectrum (see Theorem 4.28 in [100]).

[6] We will abuse notation by not explicitly showing the representation, hence the algebra elements and operators on $\mathcal{H}$ are denoted the same.

operator (or $\mathbb{Z}/2$-grading) $\gamma : \mathcal{H} \to \mathcal{H}$ satisfying

$$\begin{aligned} \gamma^* &= \gamma, \\ \gamma^2 &= 1, \\ \{\gamma, D\} &= 0, \\ [\gamma, a] &= 0 \ \ \forall a \in \mathcal{A}. \end{aligned} \tag{4.8}$$

The first and second entries implies γ has the real eigenvalues 1 and -1. The corresponding eigenspaces are respectively $\mathcal{H}_+$ and $\mathcal{H}_-$, and these decompose the Hilbert space as $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$. The third entry means D maps $\mathcal{H}_+$ to $\mathcal{H}_-$ and vice versa. The fourth entry is non-trivial for non-commutative geometry but trivial for commutative geometry (such as Riemannian geometry).

For odd s, we have an odd spectral triple where the chirality operator is replaced with the identity 1.

- *Reality*:

  A real spectral triple is equipped with a real structure [101], which is an anti-linear isometry J on $\mathcal{H}$ satisfying

$$\begin{aligned} J^2 &= \epsilon, \\ JD &= \epsilon' DJ, \\ J\gamma &= \epsilon'' \gamma J, \\ [a, Jb^* J^{-1}] &= 0 \ \ \forall a, b \in \mathcal{A} \end{aligned} \tag{4.9}$$

  where ϵ, ϵ', ϵ'' are signs depending on s as shown in Table 4.1. For commutative geometry, the fourth relation is replaced with $JaJ^{-1} = a^* \ \forall a \in \mathcal{A}$.

  An even real spectral triple is a spectral triple that is both even and real.

- *1st order condition*:

$$[[D, a], Jb^* J^{-1}] = 0 \ \ \forall a, b \in \mathcal{A}. \tag{4.10}$$

  This is the statement that D is local in the sense that the derivation $[D, \cdot\,]$ on $\mathcal{A}$ acts locally.

s	ϵ	ϵ'	ϵ''
0	+	+	+
1	+	−	
2	−	+	−
3	−	+	
4	−	+	+
5	−	−	
6	+	+	−
7	+	+	

Table 4.1: Signs ϵ, ϵ', ϵ'' as functions of s, reproduced from [40]. The sign ϵ'' (such that $J\gamma = \epsilon''\gamma J$) only exists if a chirality operator exists, i.e. for even s, which is why there are gaps in the ϵ'' column for odd s.

- *Smooth coordinates, finiteness and Poincaré duality*:

  These are not important for this work. Furthermore, Poincaré duality does not always hold and is not actually necessary[7].

A concrete example of a Connes-like structure is given by the spacetime manifold $\mathcal{M}$ in the Euclidean signature (see appendix A). This is given by an even real spectral triple with the algebra $C^\infty(\mathcal{M})$ of smooth functions on $\mathcal{M}$, the Hilbert space $L^2(\mathcal{M}, S)$ of square-integrable spinor fields on $\mathcal{M}$, the Dirac operator in (A.31), the chirality operator on spinors (A.3) and the real structure equal to the Euclidean charge conjugation operator C_E. The manifold $\mathcal{M}$ corresponds to a *commutative geometry*. Connes' article [40] contains the axioms for such a geometry, some of which coincide with the non-commutative axioms stated earlier, but we do not state all commutative axioms here. In the manifold case, the commutative axioms are satisfied with metric dimension 4 and KO dimension 4 mod 8. The manifold commutative geometry is subject to the reconstruction theorem due to Connes [102], which states that given the even real spectral triple, one can reconstruct the manifold $(\mathcal{M}, g_E)$. This means the Riemannian and spectral triple descriptions of the manifold are equivalent. For

[7]The author acknowledges correspondence with John W. Barrett on this point.

example, in the spectral triple description, the geodesic distance is given by

$$d(x,y) = sup\{|a(x) - a(y)| : a \in C^\infty(\mathcal{M}), ||[\not{D}_E, a]|| \leq 1\}, \tag{4.11}$$

which reproduces the standard Riemannian geodesic distance [40]. Hence, the Dirac operator D is the non-commutative analogue of the Riemannian metric. There is also a non-commutative notion of a point in $\mathcal{M}$ as a complex-valued homomorphism of $C^\infty(\mathcal{M})$.

4.2.1 Non-commutative Standard Model

As a key example of Connes' framework, the SM and right-handed neutrinos set in $\mathcal{M}$ in the Euclidean regime can be embedded into a *bona fide* non-commutative geometry[8]. The key postulate is that spacetime is the manifold $\mathcal{M}$ times a finite non-commutative space F, i.e. spacetime takes the form $\mathcal{M} \times F$. This resembles the form of spacetime under compactification of dimensions as in Kaluza-Klein theories [104, 105] and string theory [5, 6]. The space F is given by a finite even real spectral triple (more on finite spectral triples in section 4.2.2) of metric dimension 0 and where 6 *mod* 8 is the KO dimension[9]. The metric and KO dimensions are additive over Cartesian products, hence $\mathcal{M} \times F$ has metric dimension $4 + 0 = 4$ and KO dim $4 + 6 = 10 = 2$ mod 8. Denoting the data of the finite spectral triple with a subscript F, the even real spectral triple of $\mathcal{M} \times F$ is given by the following product non-commutative geometry:

$$\mathcal{A}_P = C^\infty(\mathcal{M}) \otimes \mathcal{A}_F \tag{4.12}$$

where $\mathcal{A}_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ is the direct sum of the *-algebras of complex numbers, quaternions and 3×3 complex matrices;

$$\mathcal{H}_P = L^2(\mathcal{M}, S) \otimes \mathcal{H}_F \tag{4.13}$$

where $\mathcal{H}_F$ is a 96-dimensional Hilbert space with basis enumerated by the SM fermions and right-handed neutrinos;

$$D_P = \not{D}_E \otimes 1 + \gamma_M \otimes D_F \tag{4.14}$$

[8]More precisely, this is an *almost-commutative geometry* [103] because it has commutative and non-commutative components.

[9]Connes showed [42] that $KO(F) = 6$ *mod* 8 eliminates several problems, in particular fermion doubling. Before this (and thus distinct dimensions), F had dimension 0 *mod* 8.

where D_F is a 96×96 matrix encoding the SM parameters and right-handed neutrino Majorana mass matrix;

$$\gamma_P = \gamma_M \otimes \gamma_F; \tag{4.15}$$

$$\mathcal{J}_E = C_E \otimes J_F. \tag{4.16}$$

The SM gauge group is given by the special unitary group for the finite space:

$$SU(\mathcal{A}_F) = \{u \in \mathcal{A}_F : u^*u = uu^* = 1, det(u) = 1\} \tag{4.17}$$

whose adjoint action on $\mathcal{H}_F$ gives the gauge group action on SM fermions.

The symmetries of the modified SM in question here are spacetime diffeomorphisms and internal gauge symmetries. In the non-commutative structure, all symmetries are unified together in one group: the automorphism group $Aut(\mathcal{A}_P)$.

A special class of automorphisms of $\mathcal{A}_P$ are the *inner automorphisms*, i.e. the automorphisms given by conjugation of elements of $\mathcal{A}_P$ with unitary elements. The group of inner automorphisms is a normal subgroup of $Aut(\mathcal{A}_P)$, and assumes the role of the gauge symmetries. The inner automorphisms give rise to an equivalence class generated by a metric as follows: given the product Dirac operator D_P, the *inner fluctuations* are defined as $D_P \to D_P^A$ where

$$D_P^A = D_P + A + \mathcal{J}_E A \mathcal{J}_E^{-1} \tag{4.18}$$

with the self-adjoint operator $A = \sum_i a_i[D_P, b_i]$ where $a_i, b_i \in \mathcal{A}_P$. The inner fluctuations (4.18) furnish the SM gauge and Higgs bosons given the unimodularity condition

$$\mathrm{Tr}(A) = 0, \tag{4.19}$$

which gives the $SU(3)$ gauge fields and removes $U(3)$ fields. The real singlet scalar σ also appears, but not from inner fluctuations. This is because the Dirac operators (4.18) also satisfy the 1st order condition (4.10) which cancels the σ part (see [69, 68]). The σ field instead comes in through the right-handed neutrino Majorana mass terms as

$$M \to \tilde{M}\sigma. \tag{4.20}$$

The σ field gives an additional $\mathbb{Z}_2$ internal symmetry leading to $\sigma \to -\sigma$ and invariance for all other fields. We remark on, but do not consider further, options to get σ as an inner fluctuation by modifying the finite algebra [69] or throwing the first-order condition (4.10) out [96]. This latter point raises the question of whether the first-order condition is a required axiom.

The action in the spectral triple description is given by the 'spectral action principle' [67, 41]: the action is invariant under $Aut(\mathcal{A}_P)$, and hence is a function exclusively of the Dirac operator spectrum. This principle is satisfied by

$$S = S_{CC} + S_{CCM} \tag{4.21}$$

where both terms will be discussed in the following. To our knowledge, the presence of the spectral action principle has not been explained.

The first term is the *Connes-Chamseddine action*:

$$S_{CC} = \mathrm{Tr}\; f\left(\frac{D_P^A}{\Lambda}\right) \tag{4.22}$$

where Λ is a UV cut-off and $f(x)$ is a cut-off Fermi-Dirac function that cuts off smoothly around $x = 1$. One assumes the inner fluctuations (4.18) satisfy the unimodularity condition (4.19) and the σ field has entered via (4.20). The Connes-Chamseddine action admits a heat kernel expansion in Sakharov terms, SM bosonic terms, Higgs couplings to the Ricci scalar and σ and terms in σ (the full formula is (5.49) in [46]). The running of the resulting gravity and gauge couplings gives rise to several predictions [43, 46]: these include consistent Newton and curvature-squared couplings, grand unification gauge coupling and weak mixing angle relations as well as an experimentally consistent top quark mass. Initially, by taking σ constant and treating it as a spectator during the running, the resulting Higgs mass was too large to fit with experiment [46]. The subsequent incorporation of the running σ couplings [44] led to the admission of the experimental Higgs mass in addition to a stable Higgs vacuum at high energies.

We remark on an interpretation of the Connes-Chamseddine action in terms of induced gravity. Recall that in chapter 2 we covered a link between the Connes-Chamseddine action and the 1-loop effective action from quantised SM fermions and right-handed neutrinos. Barrett [20] has justified the spectral action principle using the link. The justification should be unchanged by the σ field.

The other piece in (4.21) does not play a key role in this work, but for completeness we briefly discuss it. The term is the Connes-Chamseddine-Marcolli action (see [42, 43])

$$S_{CCM} = \frac{1}{2} \langle \mathcal{J}_E \Psi, D_P^A \Psi \rangle_P \tag{4.23}$$

where $\langle \cdot, \cdot \rangle_P$ is the inner product on $\mathcal{H}_P$ and we make the same assumptions of unimodularity condition and σ. Equation (4.23) gives the action for the SM fermions and right-handed neutrinos coupled to curved spacetime [43, 46].

4.2.2 Finite spectral triples

The spectral data is now a matrix algebra $\mathcal{A}$, a finite-dimensional Hilbert space $\mathcal{H}$, a matrix Dirac operator D on $\mathcal{H}$ and (for the even and real cases) matrix operators γ and J. Specialising the non-commutative axioms to the finite case [70] preserves the algebraic ones and makes the rest trivial (excluding Poincaré duality).

Examples of finite spectral triples are provided by *matrix geometries* and *fuzzy spaces*, which were expounded by Barrett in [71]. The matrix geometries were constructed from an initial space by taking products with spinor spaces. Each matrix geometry is a set of spectral triple data with KO dimension (signature) $s := q - p \bmod 8$ where the *type* is two non-negative integers (p, q). Matrix geometries are of physical interest since they are effectively Riemannian finite spectral triples. The *fuzzy spaces* arise from constraining the matrix geometries. The specifics of these particular spaces are not required in this work and are given in [71].

A finite spectral triple admits quantum geometry in the sense of promoting D to an integral over a vector space $\mathcal{G}$ of matrices [71]. This gives the partition function

$$Z = \int_{\mathcal{G}} e^{-S} dD. \tag{4.24}$$

Integrals of this form have been numerically computed in [97, 106].

We review recent results from defining fermionic Grassmann integration for finite spectral triples found by Barrett [54] (which is in the Euclidean regime). We assume D is fixed. Given $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ is the inner (Dirac) product on $\mathcal{H}$, the complex integral over $\psi \in \mathcal{H}$ and $\overline{\psi} \in \mathcal{H}$ of the form

$$F[D] = \int_{\mathcal{H}} \int_{\mathcal{H}} e^{iS_f} d\overline{\psi} d\psi \tag{4.25}$$

where the action is

$$S_f = \langle \psi, D\psi \rangle_{\mathcal{H}} \tag{4.26}$$

gives a determinant of D. A Majorana product on $\mathcal{H}$ is defined by analogy with (A.20):

$$[\psi_1, \psi_2]_{\mathcal{H}} = \langle J\psi_1, \psi_2 \rangle_{\mathcal{H}} . \tag{4.27}$$

The real integral over only $\psi \in \mathcal{H}$ given by

$$F[D] = \int_{\mathcal{H}} e^{iS_f} d\psi \tag{4.28}$$

where

$$S_f = \frac{1}{2} [\psi, D\psi]_{\mathcal{H}} \tag{4.29}$$

produces a Pfaffian whose modulus is basis independent. The integration over chiral fermions works similarly to the real case. However, an important example of this (”Example 2” in [54]) occurs for a $s = 4$ spectral triple. In this case, two copies of the triple combine to give a new $s = 2$ spectral triple and a basis-independent Pfaffian in terms of the 4th root of a functional determinant of D. We remark that similar results to [54] are given in [98].

4.3 Weinberg operator and beyond

Here, we refer to the terms specified in the introduction to this chapter. The Weinberg operator (see below) is a source of neutrino masses that only requires SM fields. Furthermore, the higher-dimensional terms have effects on phenomenology involving left-handed leptons. Hence, the Weinberg and higher-dimensional terms are of phenomenological interest. We restrict to these terms in our discussion in this section, which is in the Lorentzian regime.

For mass dimension 5, one can only write the Weinberg operator [51]

$$\mathcal{O}_W = \frac{1}{2} f_{pq} (l_L^p \cdot_{\mathbb{C}^2} \phi^*)(l_L^q \cdot_{\mathbb{C}^2} \phi^*) + c.c. \tag{4.30}$$

where f_{pq} is a 3×3 coefficient matrix. In the electroweak broken regime, equation (4.30) becomes a ν_L Majorana mass term

$$[\mathcal{O}_W]_{\phi=\phi_v} = \frac{1}{2} m_{pq} \nu_L^p \nu_L^q + c.c. \tag{4.31}$$

where $m_{pq} = \frac{v^2}{2} f_{pq}$ is the active neutrino mass matrix.

We now consider mass dimension 6. Several operators of this form have been found to give corrections to interactions involving left-handed leptons and SM bosons [107, 108]. Hence, more than one dimension 6 operator is allowed.

In the SM coupled to right-handed neutrinos (section 4.1), integrating the right-handed neutrinos out (in the informal sense) is known to give an effective Lagrangian containing the Weinberg term and higher dimensional operators (see [52, 49]). This happens in the framework in [49, 50] of Broncano-Gavela-Jenkins[10], which we will now discuss. The effective Lagrangian the authors find has the following expansion (using their notation) in terms of dimension d:

$$\mathcal{L}_{eff} = \delta\mathcal{L}^{d=5} + \delta\mathcal{L}^{d=6} + \dots \tag{4.32}$$

where the $d = 5$ term is of order $1/M$, the $d = 6$ term is of order $1/M^2$ etc.. The $d = 5$ term is the Weinberg operator with coefficient matrix $|c^{d=5}| \sim |Y^2 M^{-1}|$. We remark that the full coefficient expression (14) in [49] has a matrix η of phases used to rotate the right-handed neutrino basis to set up the integration out. The $d = 5$ coefficient matrix gives the standard type-I seesaw active masses (section 4.1). The $d = 6$ term is the operator (see [49, 50] for notation)

$$\delta\mathcal{L}^{d=6} = (\overline{l_L}\widetilde{\phi}) i\partial\!\!\!/ (c^{d=6} \widetilde{\phi}^\dagger l_L) \tag{4.33}$$

where the Hermitian matrix $|c^{d=6}| \sim |Y^2 M^{-2}|$. This operator is composed of other $d = 6$ operators and furnishes corrections to (left-handed) neutrinos in their gauge couplings and oscillations.

[10]Some of the conventions used by the authors for the right-handed neutrino terms differ from ours. We will factor this in later.

Chapter 5

Beyond Standard Model physics from right-handed neutrinos

The modern scope of Sakharov's induced gravity is such that matter fields give rise to spacetime dynamics in addition to the dynamics of gauge fields. This is encapsulated by the induced Standard Model (see section 2.2) in which the SM fermions including right-handed neutrinos furnish the purely bosonic action for the spacetime metric and all SM bosons, and this is a potential solution to the problem of coupling the SM to dynamical spacetime. What is missing from the induced Standard Model is the contributions to the induced action from each individual fermion species. In order to start filling this gap, we specialise to the right-handed neutrinos for two reasons. Firstly, right-handed neutrinos are the only fermions leaving the gauge sector unaffected and thus constitute the simplest case of induced spacetime dynamics in the induced Standard Model. Secondly, considering all fermions, and assuming a GUT seesaw scale, the effective theory due to right-handed neutrinos is valid in the largest energy range, and thus the right-handed neutrino induced action is applicable and testable at the most scales. Hence, we make it the goal of this chapter to find the terms induced by right-handed neutrinos with GUT seesaw scale.

This chapter is organised as follows. In section 5.1, we prepare the right-handed neutrino functional integral, which contains the induced terms we seek. The functional integral is evaluated in section 5.2. This step is necessarily heuristic since functional integrals are too. A real fermion integral involving Majorana mass terms was evaluated in [54], and this case is built upon in section 5.2 by including Yukawa terms and three fermion generations due to right-handed neutrinos. Our functional integral evaluation gives a bosonic action and a fermionic action, the latter of which does not arise in the induced Standard Model [20]. Section 5.3 gives the total effective action. In section 5.4, we show that the bosonic piece gives a Newton constant as well as cosmological and curvature-squared constants, and we discuss the values of the constants. In section 5.5, the fermionic action receives a $1/M$ expansion since it is of the right form for this, and we consider only the leading term since this is the dominant contribution. The leading term results in an active neutrino mass term that agrees with [49, 50] up to a new real scalar factor. We show in subsection 5.5.1 this new factor leads to a seesaw mechanism that gives experimentally consistent active neutrino masses at the price of unnaturally large neutrino Yukawa couplings.

5.1 Setting up the functional integral

We consider the minimal modification of the SM with three right-handed neutrinos in curved spacetime $\mathcal{M}$ (section 4.1). The inclusion of right-handed neutrinos accounts for particle physics phenomena left unexplained by the SM alone, namely neutrino mass, dark matter and the baryon asymmetry of the universe [52]. In particular, we are interested in the extent to which the right-handed neutrinos give rise to realistic spacetime dynamics. In this chapter, our original contribution is to show that right-handed neutrinos simultaneously induce spacetime curvature terms along with a new real scalar modification to the type-I seesaw. This contribution starts with quantised right-handed neutrinos and all other fields being classical, which means we consider the functional integral over right-handed neutrinos only. The functional integral is central to the goal of this chapter since the resulting 1-loop effective action contains all the right-handed neutrino induced terms.

Lorentzian

We begin with the Lorentzian physics since this is already known or interpolated from experiments. Our functional integral is given by

$$Z_N = \int e^{iS_N} D\overline{\nu_R} D\nu_R \tag{5.1}$$

and the action consists of only right-handed neutrino terms;

$$S_N = \int_{\mathcal{M}} \Omega \Big(\overline{\nu_{Rp}} \not{D} \nu_R^p + \Big[Y_{pq} (\overline{l_L^p} \cdot_{\mathbb{C}^2} \phi) \nu_R^q + \frac{1}{2} M_{pq} \nu_R^p \nu_R^q + \text{c.c.} \Big] \Big) \tag{5.2}$$

where +c.c. applies to the terms in the square brackets. Note equation (5.1) integrates over ν_R and $\overline{\nu_R}$ independently. In contrast to the induced Standard Model ([20], also see section 2.2), we omitted a cosmological constant counter-term and an integration over geometries since both are spectators in the following. We assume a type-I seesaw scale $\mathfrak{M}$ (see section 4.1) at the grand unification scale.

We now write equation (5.2) in full. Using the properties of the Dirac and Majorana products, we write

$$\text{c.c.} = Y_{pq}^{\dagger} \overline{\nu_R^p} (l_L^q \cdot_{\mathbb{C}^2} \phi^*) + \frac{1}{2} M_{pq}^* \overline{\nu_R^p \nu_R^q} \tag{5.3}$$

where * denotes complex conjugation. Hence, equation (5.2) becomes

$$\begin{aligned} S_N = \int_{\mathcal{M}} \Omega \Big(& \overline{\nu_{Rp}} \not{D} \nu_R^p + Y_{pq} (\overline{l_L^p} \cdot_{\mathbb{C}^2} \phi) \nu_R^q + Y_{pq}^{\dagger} \overline{\nu_R^p} (l_L^q \cdot_{\mathbb{C}^2} \phi^*) \\ & + \frac{1}{2} M_{pq} \nu_R^p \nu_R^q + \frac{1}{2} M_{pq}^* \overline{\nu_R^p \nu_R^q} \Big). \end{aligned} \tag{5.4}$$

Lorentzian → Euclidean

We want to integrate out the right-handed neutrinos from equations (5.1) and (5.4). This relies on a choice of basis for the right-handed neutrinos, and crucially the result of the integration out must be basis-independent. This requirement has been met in a finite dimensional NCG model for $Y = 0$ and one fermion generation with KO dimension corresponding to the Euclidean signature [54], and thus an infinite dimensional extension of this result is possible. However, an analogous result for the Lorentzian regime (which has a different KO dimension) has not been found. In order to guarantee ourselves a chance of our integration out being basis-independent, we now go over to the Euclidean regime[1], which

[1] A basis-independent integration out in the Lorentzian regime is beyond our scope.

will be important when we consider the heat kernel expansion (section 5.4.2). We use the Lorentzian-Euclidean transition scheme reviewed in chapter 3, and this scheme fits most naturally to our right-handed neutrino model since both originate at the functional integral level.

We denote the Euclidean functional integral by ${}^{E}Z_N$ and the Euclidean action by ${}^{E}S_N$. Using the Lorentzian-Euclidean transition rules (see chapter 3), we find the action Euclideanises in the expected manner as $S_N \to i({}^{E}S_N)$ from which we find an explicit expression for ${}^{E}S_N$. The functional integral Euclideanises as ${}^{E}Z_N = Z_N$, which gives ${}^{E}Z_N$ upon substituting in the action Euclideanisation and the replacement of $\overline{\nu_R}$ with the independent spinor $\widetilde{\nu_R}$.

Euclidean

The resulting explicit expressions for the functional integral and action are

$$
{}^{E}Z_N = \int e^{-({}^{E}S_N)} D\widetilde{\nu_R} D\nu_R \tag{5.5}
$$

and

$$
\begin{aligned}
{}^{E}S_N = -\int_{\mathcal{M}} \Omega_E \Big(& \widetilde{\nu_{Rp}}(i\not{D}_E)\nu_R^p + Y_{pq}(\widetilde{l}_L^p \cdot_{\mathbb{C}^2} \phi)\nu_R^q + Y_{pq}^{\dagger}\widetilde{\nu_R}^p(l_L^q \cdot_{\mathbb{C}^2} \phi^*) \\
& + \frac{1}{2}M_{pq}\nu_R^p\nu_R^q + \frac{1}{2}M_{pq}^*\widetilde{\nu_R}^p\widetilde{\nu_R}^q \Big)
\end{aligned} \tag{5.6}
$$

where $\widetilde{l}_L$ is the independent spinor counterpart to l_L.

Before we move on, we do some book-keeping for the action (5.6). In most of the following, the lepton doublets and Higgs are spectators. Hence, for the Yukawa terms, we define

$$
\psi_{Lp} := Y_{pq}^{\dagger}(l_L^q \cdot_{\mathbb{C}^2} \phi^*) \tag{5.7}
$$

which is a left-handed spinor with a free generation index, and the counterpart spinor

$$
\widetilde{\psi_{Lp}} := (\widetilde{l}_L^q \cdot_{\mathbb{C}^2} \phi)Y_{qp}. \tag{5.8}
$$

Substituting these definitions into equation (5.6) gives

$$
\begin{aligned}
{}^{E}S_N = -\int_{\mathcal{M}} \Omega_E \Big(& \widetilde{\nu_{Rp}}(i\not{D}_E)\nu_R^p + \widetilde{\nu_R}^p\psi_{Lp} + \widetilde{\psi_{Lp}}\nu_R^p \\
& + \frac{1}{2}M_{pq}\nu_R^p\nu_R^q + \frac{1}{2}M_{pq}^*\widetilde{\nu_R}^p\widetilde{\nu_R}^q \Big).
\end{aligned} \tag{5.9}
$$

5.2 Evaluating the functional integral

First, we re-cast each class of terms in equation (5.9) but omit any $-$ signs for now. We find

$$\int_{\mathcal{M}} \Omega_E\Big(\widetilde{\nu_{Rp}}(i\not{D}_E)\nu_R^p\Big) = \int_{\mathcal{M}} \Omega_E\Big(\nu_R^p(i\not{D}_E)\widetilde{\nu_{Rp}}\Big) \tag{5.10}$$

where we used the Majorana product symmetry on fermions, the property

$$[\varphi,\zeta] = \langle C_E\varphi,\zeta\rangle_E\,, \tag{5.11}$$

the relation $\not{D}_E C_E = C_E\not{D}_E$, and the self-adjointness of $\not{D}_E$ with respect to $\int_{\mathcal{M}}\Omega_E(\langle\cdot,\cdot\rangle)$ since we assume no torsion (for these properties, see appendix A). Thus, in equation (5.9), the kinetic term becomes

$$\begin{aligned}\int_{\mathcal{M}} \Omega_E\Big(\widetilde{\nu_{Rp}}(i\not{D}_E)\nu_R^p\Big) &= \int_{\mathcal{M}} \Omega_E\Big(\frac{1}{2}\widetilde{\nu_{Rp}}(i\not{D}_E)\nu_R^p + \frac{1}{2}\nu_R^p(i\not{D}_E)\widetilde{\nu_{Rp}}\Big)\\ &= \int_{\mathcal{M}} \Omega_E\Big(\frac{1}{2}\left\langle C_E\widetilde{\nu_{Rp}},(i\not{D}_E)\nu_R^p\right\rangle_E + \frac{1}{2}\left\langle C_E\nu_R^p,(i\not{D}_E)\widetilde{\nu_{Rp}}\right\rangle_E\Big)\end{aligned} \tag{5.12}$$

where we used equation (5.11) in line two. Using equation (5.11) and the fact that the spinor chirality operator acts on chiral spinors as $\gamma_M\zeta_L = \zeta_L$ and $\gamma_M\zeta_R = -\zeta_R$, we write the Majorana mass terms in equation (5.9) as

$$\begin{aligned}&\int_{\mathcal{M}} \Omega_E\Big(\frac{1}{2}M_{pq}\nu_R^p\nu_R^q + \frac{1}{2}M_{pq}^*\widetilde{\nu_R}^p\widetilde{\nu_R}^q\Big)\\ =&\int_{\mathcal{M}} \Omega_E\Big(-\frac{1}{2}\left\langle C_E\nu_R^p, M_{pq}\gamma_M\nu_R^q\right\rangle_E + \frac{1}{2}\left\langle C_E\widetilde{\nu_R}^p, M_{pq}^*\gamma_M\widetilde{\nu_R}^q\right\rangle_E\Big).\end{aligned} \tag{5.13}$$

After combining equations (5.12) and (5.13), the sum of the kinetic and Majorana mass terms becomes

$$\begin{aligned}\int_{\mathcal{M}} \Omega_E\Big(&\frac{1}{2}\left\langle C_E\widetilde{\nu_{Rp}},(i\not{D}_E)\nu_R^p\right\rangle_E + \frac{1}{2}\left\langle C_E\nu_R^p,(i\not{D}_E)\widetilde{\nu_{Rp}}\right\rangle_E\\ &-\frac{1}{2}\left\langle C_E\nu_R^p, M_{pq}\gamma_M\nu_R^q\right\rangle_E + \frac{1}{2}\left\langle C_E\widetilde{\nu_R}^p, M_{pq}^*\gamma_M\widetilde{\nu_R}^q\right\rangle_E\Big).\end{aligned} \tag{5.14}$$

Lastly, using equation (5.11) and the symmetry of $[\cdot,\cdot]$ on fermions, we write the Yukawa terms in equation (5.9) as

$$\int_{\mathcal{M}} \Omega_E\Big(\widetilde{\nu_R}^p\psi_{Lp} + \widetilde{\psi_{Lp}}\nu_R^p\Big) = \int_{\mathcal{M}} \Omega_E\Big(\left\langle C_E\psi_{Lp},\widetilde{\nu_R}^p\right\rangle_E + \left\langle C_E\widetilde{\psi_{Lp}},\nu_R^p\right\rangle_E\Big). \tag{5.15}$$

We now consider the Hilbert space $\mathcal{H}^f = L^2(\mathcal{M},S)\otimes\mathbb{C}^3$. The Dirac product on $\mathcal{H}^f$ is denoted by $\langle\cdot,\cdot\rangle_E^\circ$ and is a simultaneous $\langle\cdot,\cdot\rangle_E$ and dot product on $\mathbb{C}^3$.

Given $\zeta_L, \zeta_R \in \mathcal{H}^f$ and the basis $e_p \in \mathcal{H}^f$, we have

$$\begin{aligned} \langle \zeta_L, \zeta_R \rangle_E^\circ &= \langle \zeta_L^p e_p, \zeta_R^q e_q \rangle_E^\circ \\ &= \langle \zeta_L^p, \zeta_R^q \rangle_E \left(e_p \cdot_{\mathbb{C}^3} e_q \right) \\ &= \langle \zeta_L^p, \zeta_R^p \rangle_E \end{aligned} \tag{5.16}$$

where in the last line we used $e_p \cdot_{\mathbb{C}^3} e_q = \delta_{pq}$. In terms of elements of $\mathcal{H}^f$, we write equation (5.14) as

$$\begin{aligned} \int_{\mathcal{M}} \Omega_E \Big(\frac{1}{2} \langle C_E \widetilde{\nu_R}, (i \not{D}_E) \nu_R \rangle_E^\circ + \frac{1}{2} \langle C_E \nu_R, (i \not{D}_E) \widetilde{\nu_R} \rangle_E^\circ \\ - \frac{1}{2} \langle C_E \nu_R, M \gamma_M \nu_R \rangle_E^\circ + \frac{1}{2} \langle C_E \widetilde{\nu_R}, M^* \gamma_M \widetilde{\nu_R} \rangle_E^\circ \Big) \end{aligned} \tag{5.17}$$

where $\nu_R = \nu_R^p e_p$ with similar expressions for other fermions and we used the result of (5.16). By similar means, equation (5.15) becomes

$$\int_{\mathcal{M}} \Omega_E \Big(\langle C_E \psi_L, \widetilde{\nu_R} \rangle_E^\circ + \langle C_E \widetilde{\psi_L}, \nu_R \rangle_E^\circ \Big). \tag{5.18}$$

We now consider the 'doubled' Hilbert space $\mathcal{H}_2^f = \mathcal{H}^f \oplus \mathcal{H}^f$, which has the Dirac product $\langle\langle \cdot, \cdot \rangle\rangle_E$ that combines $\langle \cdot, \cdot \rangle_E^\circ$ and $\cdot_{\mathbb{C}^2}$. Equation (5.17) can now be written as

$$\int_{\mathcal{M}} \Omega_E \Big(\frac{1}{2} \langle\langle \mathcal{C}_E N_R, i \mathcal{D}_E N_R \rangle\rangle_E \Big) \tag{5.19}$$

where we define

$$N_R := \begin{pmatrix} \widetilde{\nu_R} \\ \nu_R \end{pmatrix} \in \mathcal{H}_2^f, \tag{5.20}$$

$$\mathcal{C}_E := \begin{pmatrix} & C_E \\ C_E & \end{pmatrix} \tag{5.21}$$

where empty entries are block zeroes, and

$$i \mathcal{D}_E = \begin{pmatrix} i \not{D}_E & -M \gamma_M \\ M^* \gamma_M & i \not{D}_E \end{pmatrix} \tag{5.22}$$

and the last two equations define operators on $\mathcal{H}_2^f$. Similarly, equation (5.18) becomes

$$\int_{\mathcal{M}} \Omega_E \Big(\langle\langle \mathcal{C}_E \Psi_L, N_R \rangle\rangle_E \Big) \tag{5.23}$$

where

$$\Psi_L := \begin{pmatrix} \widetilde{\psi_L} \\ \psi_L \end{pmatrix} \in \mathcal{H}_2^f. \tag{5.24}$$

Compared to Example 2 in [54], we have 3×3 complex matrices in place of complex numbers, which does not affect the KO dimensions. For the Hilbert space $\mathcal{H}_2^f$, we define the Majorana product

$$[[\Psi, \Phi]] = \langle\langle \mathcal{C}_E \Psi, \Phi \rangle\rangle_E\,, \quad \Psi, \Phi \in \mathcal{H}_2^f. \tag{5.25}$$

Using this, we express equation (5.19) as

$$\int_{\mathcal{M}} \Omega_E \Big(\frac{1}{2} [[N_R, i\mathcal{D}_E N_R]] \Big) \tag{5.26}$$

and equation (5.23) as

$$\int_{\mathcal{M}} \Omega_E \Big([[\Psi_L, N_R]] \Big). \tag{5.27}$$

In equation (5.9), we replace the kinetic plus Majorana terms by (5.26) and the Yukawa terms by (5.27), and doing this gives

$$^E S_N = - \int_{\mathcal{M}} \Omega_E \Big(\frac{1}{2} [[N_R, i\mathcal{D}_E N_R]] + [[\Psi_L, N_R]] \Big). \tag{5.28}$$

Hence, the Euclidean functional integral in equation (5.5) becomes

$$^E Z_N = \int \exp \Big[\int_{\mathcal{M}} \Omega_E \Big(\frac{1}{2} [[N_R, i\mathcal{D}_E N_R]] + [[\Psi_L, N_R]] \Big) \Big] DN_R. \tag{5.29}$$

We now consider the substitution

$$N_R \to N_R - (i\mathcal{D}_E)^{-1} \Psi_L. \tag{5.30}$$

This gives

$$\frac{1}{2} [[N_R, (i\mathcal{D}_E) N_R]] + [[\Psi_L, N_R]] \to \frac{1}{2} [[N_R, (i\mathcal{D}_E) N_R]] - \frac{1}{2} [[\Psi_L, (i\mathcal{D}_E)^{-1} \Psi_L]], \tag{5.31}$$

i.e. the term linear in N_R is eliminated. Thus, making the substitution in equation (5.29) gives

$$\begin{aligned} ^E Z_N &= \int \exp \Big[\int_{\mathcal{M}} \Omega_E \Big(\frac{1}{2} [[N_R, (i\mathcal{D}_E) N_R]] - \frac{1}{2} [[\Psi_L, (i\mathcal{D}_E)^{-1} \Psi_L]] \Big) \Big] DN_R \\ &= \Big(\int \exp \Big[\int_{\mathcal{M}} \Omega_E \Big(\frac{1}{2} [[N_R, (i\mathcal{D}_E) N_R]] \Big) \Big] DN_R \Big) \\ &\quad \times \exp \Big[\int_{\mathcal{M}} \Omega_E \Big(- \frac{1}{2} [[\Psi_L, (i\mathcal{D}_E)^{-1} \Psi_L]] \Big) \Big] \\ &= \mathrm{Pf}[iM_D] \exp \Big[- \int_{\mathcal{M}} \Omega_E \Big(\frac{1}{2} [[\Psi_L, (i\mathcal{D}_E)^{-1} \Psi_L]] \Big) \Big] \end{aligned} \tag{5.32}$$

where $\mathrm{Pf}[iM_D]$ is the Pfaffian [1, 54] and M_D is the matrix of elements of the Dirac operator $\mathcal{D}_E$ in a chosen basis. The Pfaffian is well-defined since M_D

is anti-symmetric for our case where $s = 2$ [54]. In the last line of equation (5.32), the first factor (Pfaffian) is purely bosonic and the second factor is purely fermionic.

We remark on comparisons with integrals in [54] (also see section 4.2.2). The Euclidean functional integral (5.5) is complex since it integrates over the right-handed neutrinos ν_R and their 'conjugates' $\widetilde{\nu_R}$ independently. Equation (5.5) was then expressed as a real integral (5.29) with operators and fermions that are twice as large (in terms of matrices). Hence, our functional integral gives a Pfaffian, rather than the determinant due to a genuine complex integral. Overall, the current section is essentially a generalisation of Example 2 in [54] where we have three right-handed neutrinos and additional Yukawa terms.

5.3 Total effective action

In the last line of equation (5.32), the Pfaffian encodes a 1-loop effective bosonic action denoted by ${}^E S_B$, and this action is defined as

$$\mathrm{Pf}[iM_D] := e^{-{}^E S_B}. \tag{5.33}$$

In the second factor, there is a 1-loop effective fermionic action

$${}^E S_F := \int_{\mathcal{M}} \frac{1}{2}[[\Psi_L, (i\mathcal{D}_E)^{-1}\Psi_L]]\Omega_E. \tag{5.34}$$

Note ${}^E S_B$ and ${}^E S_F$ are respectively bosonic and fermionic as their labels suggest, but both originate from integrating out right-handed neutrinos. We can now write the last line of (5.32) as

$${}^E Z_N = e^{-{}^E \Gamma_N} \tag{5.35}$$

where the full 1-loop effective action is

$${}^E \Gamma_N := {}^E S_B + {}^E S_F. \tag{5.36}$$

Equation (5.36) applies below the type-I seesaw scale $\mathfrak{M}$.

Euclidean → Lorentzian

We transition back to the Lorentzian regime to obtain the physics that can be measured (whereas the Euclidean 'physics' cannot be). To return to the

Lorentzian regime, we use the Lorentzian-Euclidean transition (chapter 3) but in reverse. We assume the Euclidean-to-Lorentzian transition works similarly to the Lorentzian-to-Euclidean transition.

For completeness, we formally apply the Euclidean-to-Lorentzian transition to the full effective action. Under Euclideanisation, the Euclidean and Lorentzian versions of each piece in (5.36) are related by $S_E = -iS$. Hence, Euclideanisation gives

$$^{E}\Gamma_N = -i(S_B + S_F) \tag{5.37}$$

where we have the Lorentzian bosonic and fermionic 1-loop effective actions on the right-hand side. Substituting equations (5.37) and (5.35) into $Z_N = {}^{E}Z_N$ gives

$$Z_N = e^{i\Gamma_N} \tag{5.38}$$

where the Lorentzian 1-loop total effective action is

$$\Gamma_N := S_B + S_F. \tag{5.39}$$

For the bosonic and fermionic terms on the right-hand side of (5.39), we must now perform an explicit Euclidean-to-Lorentzian transition as well as explicitly conduct the necessary analysis. This occupies the rest of the present chapter. In the next section we consider the bosonic action, and in section 5.5 we consider the fermionic action.

5.4 Bosonic action

5.4.1 Evaluating the Pfaffian

Euclidean

We start by recalling that we derived the Pfaffian $\text{Pf}[iM_D]$ in the Euclidean regime, where we remain for the time being. The Pfaffian is over spacetime and fermion generations. To deal with the generations, we assume the Pfaffian decomposes into a product of Pfaffians that each correspond to a generalisation of a finite spectral triple contribution from [54]. Thus, we write

$$\text{Pf}[iM_D] = \Pi_p \text{Pf}_p[iM_D] \tag{5.40}$$

where $\mathrm{Pf}_p[iM_D]$ is a spinor Pfaffian of iM_D for a given p.

We consider the second-order differential operator

$$\not{D}_E^2 + |M|^2 \tag{5.41}$$

where $|M|^2 = M^*M$. In equation (5.41) some Kronecker products are suppressed, and writing these in full makes the equation read

$$\not{D}_E^2 \otimes 1 + 1 \otimes |M|^2, \tag{5.42}$$

which is a Kronecker sum [109]. A generalised definition of $|\cdot|^2$ for any square matrix R is

$$|R|^2 := R^\dagger R. \tag{5.43}$$

A result due to appendix B is that

$$|M|^2 = U_M^* \mathrm{diag}\Big(\{M_p^2\}\Big) U_M^T \tag{5.44}$$

where U_M is a unitary matrix and the eigenvalues $M_p^2 > 0$. Using this, we write (5.42) as

$$\not{D}_E^2 \otimes U_M^* U_M^T + 1 \otimes U_M^* \mathrm{diag}(\{M_p^2\}) U_M^T \tag{5.45}$$

where we used $U_M^* U_M^T = 1$. Equation (5.45) factorises as

$$(1 \otimes U_M^*)(\not{D}_E^2 \otimes 1 + 1 \otimes \mathrm{diag}(\{M_p^2\}))(1 \otimes U_M^T). \tag{5.46}$$

The middle factor of this is

$$\not{D}_E^2 \otimes 1 + 1 \otimes \mathrm{diag}(\{M_p^2\}) = \mathrm{diag}(\not{D}_E^2 + M_p^2). \tag{5.47}$$

Thus, we have the block diagonalisation

$$\not{D}_E^2 \otimes 1 + 1 \otimes |M|^2 = (1 \otimes U_M^*) \mathrm{diag}(\not{D}_E^2 + M_p^2)(1 \otimes U_M^T). \tag{5.48}$$

Given this, applying the functional determinant over $\mathcal{H}_f$ to both sides gives

$$\mathrm{Det}[\not{D}_E^2 + |M|^2] = \Pi_p \mathrm{Det}_{L^2(\mathcal{M},\mathcal{S})}[\not{D}_E^2 + M_p^2] \tag{5.49}$$

where $\mathrm{Det}_{L^2(\mathcal{M},\mathcal{S})}$ is the determinant over spinors on $\mathcal{M}$. This step used the fact that multiplying block determinants gives the determinant of a block diagonal matrix and that $1 \otimes U_M$ is unitary. We take the fourth root of equation (5.49), which must be the positive choice since $\not{D}_E^2 + |M|^2$ is positive:

$$\text{Det}[\not{D}_E^2 + |M|^2]^{\frac{1}{4}} = \Pi_p \text{Det}_{L^2(\mathcal{M},\mathcal{S})}[\not{D}_E^2 + M_p^2]^{\frac{1}{4}}. \tag{5.50}$$

Example 2 in [54] gives

$$\text{Det}_{L^2(\mathcal{M},\mathcal{S})}[\not{D}_E^2 + M_p^2]^{\frac{1}{4}} = \text{Pf}_p[iM_D]e^{-i\theta\frac{I_D}{2}} \tag{5.51}$$

where I_D is the Dirac operator index and for us $\theta = \frac{\pi}{2}$ as we have real M_p. Putting (5.51) into (5.50) leads to

$$\text{Det}[\not{D}_E^2 + |M|^2]^{\frac{1}{4}} = \Big(\Pi_p \text{Pf}_p[iM_D]\Big)\Big(\Pi_p e^{-i\theta\frac{I_D}{2}}\Big). \tag{5.52}$$

The first factor is $\text{Pf}[iM_D]$ by definition (equation (5.40)). There is also an extra factor of $\exp[-i\frac{3\pi}{2}\frac{I_D}{2}]$. Hence, (5.52) gives

$$\text{Pf}[iM_D] = \text{Det}[\not{D}_E^2 + |M|^2]^{\frac{1}{4}} \exp\Big[i\frac{3\pi}{2}\frac{I_D}{2}\Big]. \tag{5.53}$$

In equation (5.53), the index I_D is an invariant [71] and the determinant is a spectral function, and thus the Pfaffian is independent of the basis expressing the matrix M_D. Also, equation (5.53) ensures the cancellation of fermion doubling as in [43]. Compared to the result of Example 2 in [54], our result (5.53) is an extension to 3×3 internal mass matrices. This is expected since both results emerge from similar methods.

The exponential in (5.53) gives an action term $-i\frac{3\pi}{2}\frac{I_D}{2}$. This is an index term (c.f [54]) which is purely topological. Since the physics is not changed by the index term, we now drop it.

5.4.2 Bosonic action

Substituting equation (5.53) (with no index part) into (5.33) gives

$$e^{-{}^E S_B} = \text{Det}[\not{D}_E^2 + |M|^2]^{\frac{1}{4}}. \tag{5.54}$$

We let D_N be a Dirac operator whose square is

$$D_N^2 = \not{D}_E^2 + |M|^2 \tag{5.55}$$

where D_N itself is to be found. Equation (5.55) is the functional determinant argument in equation (5.54), i.e.

$$e^{-{}^E S_B} = \text{Det}[D_N^2]^{\frac{1}{4}}. \tag{5.56}$$

Taking the logarithm gives

$${}^E S_B = -\frac{1}{4}\text{Tr Ln}[D_N^2] \tag{5.57}$$

where Tr is the functional trace and we used Ln Det = Tr Ln. Equation (5.57) is now approached with standard induced gravity methods.

Induced gravity methods

For this we follow the regularisation procedure of Visser [12] (also see section 2) but work in the Euclidean regime. This sub-subsection is independent of the precise form of D_N. Viewing the bosonic action (5.57) as a function of D_N, we introduce a reference Dirac operator D_{N0} so that

$${}^E S_B[D_N] = {}^E S_B[D_{N0}] - \frac{1}{4}\text{Tr}\Big[\text{Ln}[D_N^2] - \text{Ln}[D_{N0}^2]\Big]. \tag{5.58}$$

This permits the use of the Schwinger parameterisation identity

$$\ln\Big(\frac{b}{a}\Big) = -\int_0^\infty \Big[e^{-bt} - e^{-at}\Big]\frac{dt}{t}, \tag{5.59}$$

which gives

$${}^E S_B[D_N] = {}^E S_B[D_{N0}] + \frac{1}{4}\int_0^\infty \Big[\text{Tr } e^{-tD_N^2} - \text{Tr } e^{-tD_{N0}^2}\Big]\frac{dt}{t}. \tag{5.60}$$

We introduce a UV cutoff Λ on the order of the Planck mass and an IR cutoff ϵ where we assume $\epsilon << \Lambda$ (c.f [16, 19]). In this case, given the suggestion in section 3.2, the cutoff Λ plays the role of fixing the conformal factor-induced divergence in the path integral. The Schwinger parameter t has mass dimension -2, hence keeping track of dimensions and sizes dictates that we must replace the integration limits as

$${}^E S_B[D_N] \sim {}^E S_B[D_{N0}] + \frac{1}{4}\int_{\Lambda^{-2}}^{\epsilon^{-2}} \Big[\text{Tr } e^{-tD_N^2} - \text{Tr } e^{-tD_{N0}^2}\Big]\frac{dt}{t}. \tag{5.61}$$

For now we keep our formulae general, but it will be shown that the second-order differential operator D_N^2 is of the Laplace form (2.37), thus permitting us to use the heat kernel expansion. The heat kernel expansion is well-defined in the Euclidean regime but not in the Lorentzian regime (see subsection 2.1.2), hence our use of the Euclidean regime. The heat kernel expansion in this case is given by (2.36) with $P = D_N^2$:

$$\mathrm{Tr}\ e^{-tD_N^2} \overset{t\to 0^+}{\sim} \sum_{n\geq 0} t^{\frac{n-4}{2}} a_n(1, D_N^2). \tag{5.62}$$

Given that $a_n = 0$ for odd n, and for the remaining (even) terms we put $n = 2m$, we have

$$\mathrm{Tr}\ e^{-tD_N^2} \overset{t\to 0^+}{\sim} \sum_{m\geq 0} t^{m-2} a_{2m}(1, D_N^2). \tag{5.63}$$

We may substitute this into (5.61), which upon re-arranging yields

$${}^{E}S_B[D_N] \sim \frac{1}{4} \sum_{m\geq 0} \left(\int_{\Lambda^{-2}}^{\epsilon^{-2}} t^{m-2} \frac{dt}{t} \right) a_{2m}(1, D_N^2) \tag{5.64}$$

where we neglected the reference action which has the role of a spectator[2].

Introducing the real singlet scalar

We now specialise the heat kernel coefficients to our Dirac operator (5.55). We use the real singlet scalar from non-commutative geometry (see section 4.2) since this is needed (in the non-commutative SM) to ensure an experimentally consistent Higgs mass and a stable Higgs self-coupling [44].

We first need an explicit formula for the Dirac operator D_N, or the square root of equation (5.55). This is given by

$$D_N = \not{D}_E \otimes I_3 + \gamma_M \otimes |M| \tag{5.65}$$

where $|M| = \sqrt{M^* M}$ is the positive matrix square root since $M_p^2 > 0$. A check shows that (5.65) squares to (5.55) and hence is a good square root. In terms of the non-commutative SM (section 4.2.1), equation (5.65) can be interpreted as the vacuum SM Dirac operator (4.14) with the internal space F restricted so that $D_F = |M|$.

Our real singlet scalar is the field σ from the non-commutative SM (see section 4.2.1). This scalar enters via

$$M \to \widetilde{M} \sigma(x) \tag{5.66}$$

[2]This neglect plays out via the reasoning in [19], which applies since $\mathcal{M}$ is compact.

where $\widetilde{M}$ is a constant non-zero 3×3 symmetric matrix. We invoke the additional assumption $\sigma \neq 0$, which will be used later. In our mechanism, the σ field and the metric g_E are the only fields coupling to right-handed neutrinos. We choose to put the mass dimension 1 into $\widetilde{M}$, so that σ is dimensionless.

Making the scaling (5.66) into the Dirac operator (5.65) gives

$$D_N = \not{D}_E \otimes I_3 + \gamma_M \otimes |\widetilde{M}||\sigma|. \tag{5.67}$$

This replaces the Dirac operator (5.65).

Computing the integrals and heat kernel coefficients

The heat kernel coefficients depend on the coefficients of the Laplace form (2.37), which we must find for our case $P = D_N^2$. Explicitly this operator is given by

$$D_N^2 = \not{D}_E^2 \otimes I_3 + \gamma_E^\mu \gamma_M (\partial_\mu |\sigma|) \otimes |\widetilde{M}| + I \otimes |\widetilde{M}|^2 \sigma^2. \tag{5.68}$$

The Lichnerowicz formula [73] states that the squared Dirac operator $\not{D}_E^2$ is the Laplace-Beltrami operator up to a scalar curvature term:

$$\not{D}_E^2 = -g_E^{\mu\nu} \nabla_\mu \nabla_\nu + \frac{1}{4} R_E \tag{5.69}$$

where $g_E^{\mu\nu}$ are the components of the inverse Euclidean metric, ∇_μ are the components of the covariant derivative (which will be specified later), and R_E is the Euclidean Ricci scalar[3]. Substituting equation (5.69) into equation (5.68), we get the Laplace form (2.37) for $P = D_N^2$ where

$$-E := \frac{1}{4} R_E \otimes I_3 + \left(\gamma_E^\mu \gamma_M \otimes |\widetilde{M}|\right) \partial_\mu |\sigma| + I \otimes |\widetilde{M}|^2 \sigma^2. \tag{5.70}$$

We remark on why we introduced the real singlet scalar σ where we did (in the previous sub-subsection). We followed the non-commutative SM [46] in the sense of introducing the scalar at the level of the internal space in the (undoubled) Dirac operator. Furthermore, if we introduced σ at the $|M|^2$ level as in the Dirac operator squared (5.55), in equation (5.70) the term on the right-hand side involving $\partial_\mu |\sigma|$ would be absent. This missing term will turn out to correspond

[3]The Euclidean equation (5.69) is related to the Lorentzian equation (2.19) via the Lorentzian-Euclidean transition in chapter 3.

to a scalar kinetic term in the bosonic action (c.f equation (5.89)), and this term is consistent with the heat kernel expansion of the Connes-Chamseddine action [46]. Since the heat kernel coefficients in equation (5.64) are at the D_N^2 level, introducing σ in the heat kernel coefficients would lead to the missing scalar kinetic term. Hence, our choice of location to introduce σ ultimately ensures our mechanism is consistent with non-commutative geometry, and we will check this later. The consistency is important since it functions as a verification of our mechanism.

For our case, we now compute the integrals over t and heat kernel coefficients up to $m = 2$ inclusive.

$\underline{m = 0}$:

The integral over t is

$$\int_{\Lambda^{-2}}^{\epsilon^{-2}} \frac{dt}{t} t^{-2} \sim \frac{1}{2}\Lambda^4 \tag{5.71}$$

where we have dropped terms below leading order in the cutoffs. The lowest heat kernel coefficient is of the form

$$a_0(1, D_N^2) = (4\pi)^{-2} \int_{\mathcal{M}} dV \Big(\text{tr } 1\Big) \tag{5.72}$$

where tr is the trace over spinor and generational indices and $dV = \sqrt{\det(g_E)}d^4x$. The trace evaluates to $\text{tr}(1) = 12$ due to 4 spinor index values times 3 generation index values, thus giving

$$a_0(1, D_N^2) = (4\pi)^{-2} 12 \int_{\mathcal{M}} dV. \tag{5.73}$$

$\underline{m = 1}$:

The integral to leading order is

$$\int_{\Lambda^{-2}}^{\epsilon^{-2}} \frac{dt}{t} t^{-1} \sim \Lambda^2. \tag{5.74}$$

The heat kernel coefficient is

$$a_2(1, D_N^2) = (4\pi)^{-2} 6^{-1} \int_{\mathcal{M}} dV \Big(\text{tr}[6E + R_E]\Big). \tag{5.75}$$

Substituting in (5.70) and evaluating the traces gives

$$a_2(1, D_N^2) = -(4\pi)^{-2} \int_{\mathcal{M}} dV \Big(R_E + 4c\sigma^2\Big) \tag{5.76}$$

where we have defined $c := \text{tr } |\widetilde{M}|^2$ (c.f non-commutative geometry e.g. [43, 46]). To get this, we used the trace identity

$$\text{tr}(\gamma_E^\mu \gamma_M) = 0 \tag{5.77}$$

due to the trace containing an odd number of gamma matrices.

$\underline{m = 2}$:

We have

$$\int_{\Lambda^{-2}}^{\epsilon^{-2}} \frac{dt}{t} t^0 = \ln\Big(\frac{\Lambda^2}{\epsilon^2}\Big) \tag{5.78}$$

and the heat kernel coefficient

$$\begin{aligned} a_4(1, D_N^2) =& (4\pi)^{-2} 360^{-1} \int_{\mathcal{M}} dV \Big(\text{tr}\Big[60 E_{;\ \mu}^{\ \mu} + 60 R_E E + 180 E^2 + 12 R_{E;\ \mu}^{\ \ \mu} + 5 R_E^2 \\ & - 2 Ric_E^2 + 2 Riem_E^2 + 30 \mathfrak{F}^2 \Big] \Big) \end{aligned} \tag{5.79}$$

where the subscript ; denotes a covariant derivative; Ric_E and $Riem_E$ are respectively the Euclidean Ricci and Riemann tensors and; the 2-form $\mathfrak{F}$ is the curvature of a connection 1-form $\mathfrak{f}$ defined by

$$\nabla_\mu = \partial_\mu + \mathfrak{f}_\mu. \tag{5.80}$$

The Euclidean spin connection action (A.29) gives

$$\mathfrak{f}_\mu = -\omega_{\mu ab} S_E^{ab}. \tag{5.81}$$

Hence, reading off from known results [15, 46] gives

$$\mathfrak{F}_{\mu\nu} = -R_{E\mu\nu ab} S_E^{ab}. \tag{5.82}$$

By substituting (5.70, 5.82) into (5.79) and evaluating the traces, we find

$$\begin{aligned} a_4(1, D_N^2) = (4\pi)^{-2} \int_{\mathcal{M}} dV \Big(& \frac{1}{24} R_E^2 - \frac{1}{15} Ric_E^2 - \frac{29}{15} Riem_E^2 \\ & + \frac{1}{3} c R_E \sigma^2 + 2c (\nabla_e |\sigma|)^2 + 2d\sigma^4 \Big) \end{aligned} \tag{5.83}$$

where $d := \text{tr } |\widetilde{M}|^4$ (again c.f e.g. [43, 46]). In deriving this, we used the trace identities

$$\begin{aligned} \text{tr}(\gamma_E^\mu \gamma_M \gamma_E^\nu \gamma_M) &= 4 g_E^{\mu\nu}, \\ \text{tr}[S_E^{ab} S_E^{cd}] &= 4(\delta^{ad}\delta^{bc} - \delta^{bd}\delta^{ac}) \end{aligned} \tag{5.84}$$

and dropped all total derivative terms of the form $\cdot_{;\ \mu}^{\ \mu}$. Some results for the parameters c and d are given in appendix B, in particular $c, d > 0$.

Explicitly deriving the bosonic action

We return to the bosonic action form (5.64), for which we only consider the $m \leq 2$ terms. Substituting in the expressions for the integrals and heat kernel coefficients gives

$$\begin{aligned}
{}^{E}S_B \sim \frac{1}{4}(4\pi)^{-2}\Big[& 6\Lambda^4 \int_{\mathcal{M}} \Omega_E \\
& - \Lambda^2 \int_{\mathcal{M}} \Omega_E \Big(R_E + 4c\sigma^2 \Big) \\
& + \ln\Big(\frac{\Lambda^2}{\epsilon^2}\Big) \int_{\mathcal{M}} \Omega_E \Big(\frac{1}{24} R_E^2 - \frac{1}{15} Ric_E^2 - \frac{29}{15} Riem_E^2 \\
& + 2d\sigma^4 + \frac{1}{3} c R_E \sigma^2 + 2c(\nabla_e |\sigma|)^2 \Big) \\
& + \mathcal{O}(\epsilon^{-2} a_6) \Big]
\end{aligned} \tag{5.85}$$

where the form of the remainder $\mathcal{O}(\epsilon^{-2}a_6)$ is determined from writing out the $m = 3$ term in equation (5.64). For the above result, we assumed $dV = \Omega_E$, which holds if the Lorentzian-Euclidean transition (chapter 3) has a suitably performed rotation [1]. The curvature-squared terms can be reduced further due to the result that a particular curvature-squared combination is a topological term (i.e. Chern-Gauss-Bonnet theorem):

$$Riem_E^2 - 4Ric_E^2 + R_E^2 = Top \tag{5.86}$$

where the right-hand side integrates to (a multiple of) the Euler characteristic of $\mathcal{M}$. Hence, eliminating $Riem_E^2$ gives

$$\frac{1}{24} R_E^2 - \frac{1}{15} Ric_E^2 - \frac{29}{15} Riem_E^2 = \frac{79}{40} R_E^2 - \frac{39}{5} Ric_E^2 - \frac{29}{15} Top. \tag{5.87}$$

Moreover from $\nabla_{e_a}|\sigma| = (\text{sgn } \sigma)\nabla_{e_a}\sigma$ we have

$$(\nabla_e |\sigma|)^2 = (\nabla_e \sigma)^2. \tag{5.88}$$

In this working one gets a factor of $(\text{sgn } \sigma)^2 = 1$. If we did not impose $\sigma \neq 0$, the factor $(\text{sgn } \sigma)^2$ would give a non-physical discontinuity at $\sigma = 0$. Substituting (5.87) and (5.88) into the bosonic action gives

$$\begin{aligned}
{}^{E}S_B \sim & \frac{1}{4(4\pi)^2} \int_{\mathcal{M}} \Omega_E \Big(6\Lambda^4 - \Lambda^2 [R_E + 4c\sigma^2] \\
& + \ln\left(\frac{\Lambda^2}{\epsilon^2}\right) \Big[\frac{79}{40} R_E^2 - \frac{39}{5} Ric_E^2 \\
& + \frac{1}{3} c R_E \sigma^2 + 2d\sigma^4 + 2c(\nabla_e \sigma)^2 \Big] + \mathcal{O}(\epsilon^{-2} a_6') \Big)
\end{aligned} \tag{5.89}$$

where we omitted a non-dynamical topological term and put the remainder terms involving the non-integrated heat kernel coefficient a_6' into the Lagrangian. This result contains curvature terms, the σ^4 Lagrangian and a scalar-curvature coupling through the term $\propto R_E\sigma^2$ in a combination possessing scalar $\mathbb{Z}_2$ symmetry. There are no SM bosons since the right-handed neutrinos couple only to g_E and σ in the high-energy fermionic theory. The cosmological constant and Einstein-Hilbert term have the signs of Euclidean signature (see equation (3.14)).

We can perform a sanity check by making a rough comparison between equation (5.89) and the g_E-σ part of the Connes-Chamseddine action (5.49 in [46]): both have the same signs and invariants, but the numerical coefficients are not equal. The latter observation can be attributed to the differences in the high-energy geometry and matter actions. The comparison excludes the curvature-squared terms due to differing invariant choices, and we do not pursue this further. A more rigorous comparison between the bosonic and Connes-Chamseddine actions is deferred to further work.

5.4.3 Bosonic action under scalar symmetry breaking

Euclidean

We wish to find the σ potential for symmetry breaking. The kinetic σ term is normalised by defining $\widetilde{\sigma}$ such that

$$\sigma = \left(\frac{(4\pi)^2}{c\,\ln(\Lambda^2/\epsilon^2)}\right)^{\frac{1}{2}}\widetilde{\sigma}. \tag{5.90}$$

Note $\sigma \neq 0$ means $\widetilde{\sigma} \neq 0$. We now define

$$\begin{aligned}\mu_\sigma^2 &:= \frac{2\Lambda^2}{\ln(\Lambda^2/\epsilon^2)},\\ \lambda_\sigma &:= \frac{2(4\pi)^2}{r\,\ln(\Lambda^2/\epsilon^2)}\end{aligned} \tag{5.91}$$

where $r := \frac{c^2}{d}$ in the above[4]. Given the range of r in appendix B, both expressions in (5.91) are positive. The normalisation with the parameters gives

$$
\begin{aligned}
{}^{E}S_B \sim \int_{\mathcal{M}} \Omega_E \Big(& \frac{3}{2(4\pi)^2}\Lambda^4 - \frac{\Lambda^2}{4(4\pi)^2} R_E \\
& + \frac{\ln(\Lambda^2/\epsilon^2)}{4(4\pi)^2}\Big[\frac{79}{40}R_E^2 - \frac{39}{5}Ric_E^2\Big] \\
& + \frac{1}{12}R_E\widetilde{\sigma}^2 + \frac{1}{2}(\nabla_e\widetilde{\sigma})^2 + V[\widetilde{\sigma}] + \mathcal{O}(\epsilon^{-2}a_6')\Big)
\end{aligned}
\tag{5.92}
$$

with the scalar potential

$$
V[\widetilde{\sigma}] := -\frac{1}{2}\mu_\sigma^2\widetilde{\sigma}^2 + \frac{1}{4}\lambda_\sigma\widetilde{\sigma}^4. \tag{5.93}
$$

The pure σ terms in (5.92) and (5.93) have suitable symmetry breaking forms (given $m^2 \to -\mu^2$) (c.f [110]).

In what follows, we explicitly break $\mathbb{Z}_2$ and find symmetry broken quantities. The global minima of (5.93) are given by $\widetilde{\sigma} = \pm\left(\frac{\mu_\sigma^2}{\lambda_\sigma}\right)^{\frac{1}{2}}$, which are non-zero as $\mu_\sigma^2, \lambda_\sigma > 0$ and are $\mathbb{Z}_2$ equivalent. The scalar $\mathbb{Z}_2$ breaking is done by the choice of $\widetilde{\sigma}$ vacuum $\langle\widetilde{\sigma}\rangle := w$ where $w = \left(\frac{\mu_\sigma^2}{\lambda_\sigma}\right)^{\frac{1}{2}}$ is the vacuum expectation value (VEV). Fluctuations around the vacuum give the usual singlet scalar mass

$$
m_\sigma = \sqrt{2}\mu_\sigma = \frac{2\Lambda}{\ln(\Lambda^2/\epsilon^2)^{\frac{1}{2}}} \tag{5.94}
$$

where we used (5.91). The above expression gives $m_\sigma \sim 2 \times 10^{18}$ GeV taking $\epsilon \sim m_\nu^{\text{exp}}$ where $m_\nu^{\text{exp}} \sim 10^{-11}$ GeV is the active neutrino mass scale datum[5]. Since the scale of $\widetilde{\sigma}$ fluctuations $m_\sigma >> \mathfrak{M}$, we may neglect them in the effective action (including the fermionic piece). Thus, fixing $\widetilde{\sigma} = w$ in the induced action (5.92) gives the curvature expansion

$$
\begin{aligned}
{}^{E}S_B \sim \int_{\mathcal{M}} \Omega_E \Big(& \frac{3}{2(4\pi)^2}\Lambda^4 - \frac{m_\sigma^4}{16\lambda_\sigma} \\
& - \frac{1}{4}\Big(\frac{\Lambda^2}{(4\pi)^2} - \frac{m_\sigma^2}{6\lambda_\sigma}\Big)R_E \\
& + \frac{1}{(4\pi)^2}\Big(\frac{\Lambda}{m_\sigma}\Big)^2\Big[\frac{79}{40}R_E^2 - \frac{39}{5}Ric_E^2\Big] \\
& + \mathcal{O}(\epsilon^{-2}a_6')\Big).
\end{aligned}
\tag{5.95}
$$

[4]Note that subscript labels σ apply to the *re-scaled* scalar $\widetilde{\sigma}$.

[5]Our estimating ϵ follows [19] except with different active mass, but this is insignificant as we also only see ϵ inside the logarithm (as for the fermionic action later). Constraints reported in [52] imply a total active mass $\sim$ 0.1eV, and we estimate m_ν^{exp} as a third of this, i.e. $m_\nu^{\text{exp}} \sim 0.01\text{eV} = 10^{-11}\text{GeV}$.

To get this, we used the following results: equation (5.94), which gives

$$w^2 = \frac{\mu_\sigma^2}{\lambda_\sigma} = \frac{m_\sigma^2}{2\lambda_\sigma}, \tag{5.96}$$

and putting this into (5.93) gives

$$V[w] = -\frac{\mu_\sigma^2 w^2}{4} = -\frac{m_\sigma^4}{16\lambda_\sigma}. \tag{5.97}$$

Further, note w goes through ∇_e, thus the scalar kinetic term disappears.

Euclidean $\rightarrow$ Lorentzian

The transition to the Lorentzian theory follows chapter 3. Equation (5.95) shows that we need the Euclideanisation of the square of the Ricci tensor. From the pure gravity relations of section 3.2, we have the Riemann tensor Euclideanisation

$${}^E R^f_{cab} = (-i)^N R^f_{cab} \tag{5.98}$$

where $N = \delta_{a0} + \delta_{b0} + \delta_{c0} - \delta_{f0}$. In other words, for (contractions of) the Riemann tensor, each upper tangent space index gives a factor i and each lower index gives $-i$. These factors for the squared Ricci tensor cancel, giving in MTW conventions

$$Ric_E^2 = Ric^2. \tag{5.99}$$

Now, we can get the Lorentzian action. Using the Euclideanisation relations (including the above result) on (5.95) gives ${}^E S_B = -iS_B$ where

$$\begin{aligned} S_B \sim \int_{\mathcal{M}} \Omega \Big(&- \Big(\frac{3\Lambda^4}{2(4\pi)^2} - \frac{m_\sigma^4}{16\lambda_\sigma} \Big) \\ &+ \frac{1}{4} \Big(\frac{\Lambda^2}{(4\pi^2)} - \frac{m_\sigma^2}{6\lambda_\sigma} \Big) R \\ &+ \frac{1}{(4\pi)^2} \Big(\frac{\Lambda}{m_\sigma} \Big)^2 \Big[-\frac{79}{40} R^2 + \frac{39}{5} Ric^2 \Big] + \mathcal{O}(\epsilon^{-2} b_6') \Big) \end{aligned} \tag{5.100}$$

where b_6' is the result of transitioning the Euclidean a_6' to the Lorentzian regime. We interpret (5.100) as the Lorentzian heat kernel expansion of the action in our case.

Lorentzian

Equation (5.100) can be alternatively written as

$$\begin{aligned} S_B \sim \int_{\mathcal{M}} \Omega\Big(&- \frac{1}{2(4\pi)^2}\Big(3\Lambda^2 - \frac{r}{4}m_\sigma^2\Big)\Lambda^2 \\ &+ \frac{\Lambda^2}{4(4\pi)^2}\Big(1 - \frac{r}{3}\Big)R \\ &+ \frac{1}{(4\pi)^2}\Big(\frac{\Lambda}{m_\sigma}\Big)^2\Big[- \frac{79}{40}R^2 + \frac{39}{5}Ric^2\Big] + \mathcal{O}(\epsilon^{-2}b_6')\Big) \end{aligned} \tag{5.101}$$

where in the first two terms we used the λ_σ expression in (5.91) and (5.94) to eliminate a factor of m_σ^2.

5.4.4 Gravitational constants

From here, we neglect the remainder terms $\mathcal{O}(\epsilon^{-2}b_6')$ (which are highly suppressed). We define the right-handed neutrino induced gravitational constants implicitly by

$$S_B = \int_{\mathcal{M}} \Omega\Big(\frac{1}{16\pi G_N}\Big(- 2\Lambda_N + R\Big) + a_N R^2 + b_N Ric^2\Big). \tag{5.102}$$

We remark that this is the higher-derivative gravity action [111, 112], which is mathematically well-defined. Equation (5.101) is equated to (5.102). By matching terms, we can read off the gravitational constants. The Einstein-Hilbert term gives a Newton constant such that

$$\frac{1}{16\pi G_N} = \frac{\Lambda^2}{4(4\pi)^2}\Big(1 - \frac{r}{3}\Big). \tag{5.103}$$

The cosmological constant from the cosmological term can be cast using equation (5.103) into the form

$$\Lambda_N = \frac{3\Lambda^2 - \frac{r}{4}m_\sigma^2}{1 - \frac{r}{3}}. \tag{5.104}$$

This and G_N in (5.103) are singular at the value $r = 3$, which we therefore exclude from the r bound in appendix B so that now $1 < r < 3$. A separate result is $m_\sigma < \Lambda$ as we later show. Both bounds give $\frac{1}{16\pi G_N}, \Lambda_N > 0$, which ensures Lorentzian signs for these terms. From the curvature-squared terms, we find

$$\begin{aligned} a_N &= \frac{1}{(4\pi)^2}\Big(\frac{\Lambda}{m_\sigma}\Big)^2\Big(- \frac{79}{40}\Big), \\ b_N &= \frac{1}{(4\pi)^2}\Big(\frac{\Lambda}{m_\sigma}\Big)^2\Big(\frac{39}{5}\Big). \end{aligned} \tag{5.105}$$

The scalar had no couplings to curvature squared, and hence the curvature-squared couplings did not receive corrections depending on r when $\mathbb{Z}_2$ was broken. This has resulted in (5.105) having no dependence on the Majorana mass regime. Together, the induced constants are

$$\begin{aligned}\Lambda_N &= \frac{3\Lambda^2 - \frac{r}{4}m_\sigma^2}{1-\frac{r}{3}},\\ \frac{1}{16\pi G_N} &= \frac{\Lambda^2}{4(4\pi)^2}\Big(1-\frac{r}{3}\Big),\\ a_N &= \frac{1}{(4\pi)^2}\Big(\frac{\Lambda}{m_\sigma}\Big)^2\Big(-\frac{79}{40}\Big),\\ b_N &= \frac{1}{(4\pi)^2}\Big(\frac{\Lambda}{m_\sigma}\Big)^2\Big(\frac{39}{5}\Big)\end{aligned} \tag{5.106}$$

where $1 < r < 3$.

For the spectrum of $\widetilde{M}$ (see appendix B), we assume two eigenvalues $\widetilde{M}_p \sim \mathfrak{M}$ and one $\widetilde{M}_p << \mathfrak{M}$. To motivate this regime, we consider a slightly more general one with k heavy eigenvalues (for non-negative $k \leq 3$). From (B.3) we have in general

$$r = \frac{\Big(\sum_p \widetilde{M}_p^2\Big)^2}{\sum_p \widetilde{M}_p^4} \tag{5.107}$$

whence for k heavy eigenvalues $r \sim (k\mathfrak{M})^2/(k\mathfrak{M}^4) = k$. The r bound prefers a k value away from the limits, with particular disfavour of the degenerate regime $k = 3$ near the singularity of (5.106). The optimal choice is $k = 2$, i.e. our chosen regime. This somewhat relates our theory to the νMSM (section 4).

For one eigenvalue $\widetilde{M}_p << \mathfrak{M}$ and the other two $\widetilde{M}_p \sim \mathfrak{M}$ (so $r = 2$), we find, using $m_\sigma \sim 2 \times 10^{18}$GeV,

$$\begin{aligned}\Lambda_N &\sim 9\Lambda^2,\\ \frac{1}{16\pi G_N} &= \frac{\Lambda^2}{12(4\pi)^2},\\ a_N &\sim -0.4,\\ b_N &\sim 2.\end{aligned} \tag{5.108}$$

We now discuss these values.

The induced cosmological constant is positive, as is the case for fermionic spinors (c.f [19]). The cosmological constant from experiment [10, 11] is $\Lambda^{exp} \sim 10^{-122}\Lambda^2$. Thus, we have the cosmological constant problem (section 2). We expect that also integrating out SM fermions would make this worse since these would give additional cosmological contributions.

Equation (5.108) gives $G_N = 12\pi\Lambda^{-2}$. The classical Newton constant $G = \Lambda^{-2}$ is smaller. The sign ambiguity mentioned in [19] is ruled out since $G_N > 0$.

Only very weak bounds on the curvature-squared couplings appear to exist via experiment: [113] shows

$$|a|, |b| \leq 10^{60} \tag{5.109}$$

and other bounds therein are only weaker. The bounds (5.109) have sufficiently large tolerance to admit (5.108) and possibly additional SM contributions. The curvature-squared couplings also have the tachyon-free constraints [111, 112] (also see [113])

$$\begin{aligned} &b < 0, \\ &3a + b > 0. \end{aligned} \tag{5.110}$$

The first bound is violated by (5.108) (but the second is satisfied). Hence, our effective action is tachyonic (has particles with $-m^2$), i.e. the vacuum state of the effective action is uncertain (if it exists). Attempting to find the genuine vacuum state is left as an open problem.

5.5 Fermionic action

Euclidean

We recall the fermionic part of the total effective action:

$${}^{E}S_F = \int_{\mathcal{M}} \Omega_E\left(\frac{1}{2}[[\Psi_L, (i\mathcal{D}_E)^{-1}\Psi_L]]\right) \tag{5.111}$$

where

$$\Psi_L = \begin{pmatrix} \widetilde{\psi_L} \\ \psi_L \end{pmatrix} \tag{5.112}$$

and

$$i\mathcal{D}_E = \begin{pmatrix} i\not{D}_E & -M\gamma_M \\ M^*\gamma_M & i\not{D}_E \end{pmatrix}. \tag{5.113}$$

The Lagrangian term is given by ${}^{E}S_F = \int_{\mathcal{M}} \Omega_E({}^{E}\mathcal{L}_F)$ where

$${}^{E}\mathcal{L}_F = \frac{1}{2}[[\Psi_L, (i\mathcal{D}_E)^{-1}\Psi_L]]. \tag{5.114}$$

We now impose the same σ modification as for the bosonic action:

$$M \to \widetilde{M}\sigma, \tag{5.115}$$

which gives

$$i\mathcal{D}_E \to \begin{pmatrix} i\not{\partial}_E & -\widetilde{M}\sigma\gamma_M \\ (\widetilde{M}\sigma)^*\gamma_M & i\not{\partial}_E \end{pmatrix}. \tag{5.116}$$

The above makes the action (5.111) violate the scalar $\mathbb{Z}_2$ symmetry. This will later be realised in full.

We now take a $1/M$ expansion. To prepare, we make a Dirac-Majorana split of (5.116) such that

$$\mathcal{D}_E = \mathcal{D}_D + \mathcal{D}_M \tag{5.117}$$

where

$$\begin{aligned} \mathcal{D}_D &:= [\mathcal{D}_E]_{\sigma=0}, \\ \mathcal{D}_M &:= [\mathcal{D}_E]_{\not{\partial}_E=0} \end{aligned} \tag{5.118}$$

and the evaluations on zero are only used here to keep the notation economical. Inverting (5.117) gives the Maclaurin series in M^{-1}

$$\mathcal{D}_E^{-1} = \mathcal{D}_M^{-1} - \mathcal{D}_M^{-1}\mathcal{D}_D\mathcal{D}_M^{-1} + \dots \tag{5.119}$$

and we assume the convergence condition $\rho(\mathcal{D}_D\mathcal{D}_M^{-1}) < 1$. Substituting this into (5.114) gives

$$\begin{aligned} {}^E\mathcal{L}_F &= \frac{1}{2}[[\Psi_L, (i\mathcal{D}_M)^{-1}\Psi_L]] + \frac{1}{2}[[\Psi_L, \mathcal{D}_M^{-1}(i\mathcal{D}_D)\mathcal{D}_M^{-1}\Psi_L]] + \dots \\ &\sim \frac{1}{2}[[\Psi_L, (i\mathcal{D}_M)^{-1}\Psi_L]] \end{aligned} \tag{5.120}$$

where for now we only preserve the leading order term in $\widetilde{M}^{-1}$, which is most significant.

We find an explicit formula for $\mathcal{D}_M^{-1}$. By the definitions in (5.118)

$$\mathcal{D}_M = \begin{pmatrix} & i\widetilde{M}\sigma\gamma_M \\ -i(\widetilde{M}\sigma)^*\gamma_M & \end{pmatrix}. \tag{5.121}$$

We write

$$\mathcal{D}_M^{-1} = \begin{pmatrix} \alpha & \beta \\ \gamma' & \delta \end{pmatrix} \tag{5.122}$$

where $\alpha, \beta, \gamma', \delta$ are block matrices to be found. Imposing $\mathcal{D}_M^{-1}\mathcal{D}_M = 1$ and solving gives

$$\begin{aligned} &\alpha = \delta = 0, \\ &\beta = i\gamma_M(\widetilde{M}\sigma)^{*-1}, \\ &\gamma' = -i\gamma_M(\widetilde{M}\sigma)^{-1}. \end{aligned} \tag{5.123}$$

We now express the Lagrangian (5.120) via formulae from the high-energy theory (section 5.2). We have

$$^E\mathcal{L}_F \sim \frac{1}{2}\langle\langle \mathcal{C}_E\Psi_L, (i\mathcal{D}_M)^{-1}\Psi_L\rangle\rangle_E \tag{5.124}$$

where $[[\Psi, \Phi]] = \langle\langle \mathcal{C}_E\Psi, \Phi\rangle\rangle_E$. The inverse of $\mathcal{D}_M^{-1}$ can be given as

$$(i\mathcal{D}_M)^{-1} = \begin{pmatrix} & \gamma_M(\widetilde{M}\sigma)^{*-1} \\ -\gamma_M(\widetilde{M}\sigma)^{-1} & \end{pmatrix}. \tag{5.125}$$

Taking the product of this with (5.112) leads to

$$(i\mathcal{D}_M)^{-1}\Psi_L = \begin{pmatrix} (\widetilde{M}\sigma)^{*-1}\psi_L \\ (\widetilde{M}\sigma)^{-1}\widetilde{\psi_L} \end{pmatrix}. \tag{5.126}$$

In addition, using

$$\mathcal{C}_E = \begin{pmatrix} & C_E \\ C_E & \end{pmatrix}, \tag{5.127}$$

we have

$$\mathcal{C}_E\Psi_L = \begin{pmatrix} C_E\psi_L \\ C_E\widetilde{\psi_L} \end{pmatrix}. \tag{5.128}$$

Substituting equations (5.128) and (5.126) into equation (5.124) gives

$$^E\mathcal{L}_F \sim \frac{1}{2}\langle C_E\psi_L, (\widetilde{M}\sigma)^{*-1}\psi_L\rangle_E^\circ + \frac{1}{2}\langle C_E\widetilde{\psi_L}, (\widetilde{M}\sigma)^{-1}\widetilde{\psi_L}\rangle_E^\circ \tag{5.129}$$

where we used the decomposition of the doubled Dirac product $\langle\langle\cdot,\cdot\rangle\rangle_E$ into the dot product on $\mathbb{C}^2$ and the Dirac product $\langle\cdot,\cdot\rangle_E^\circ$ on generational triplets. Equation (5.129) in terms of generational components reads

$$^E\mathcal{L}_F \sim \frac{1}{2}(\widetilde{M}\sigma)^{*-1}_{pq}\langle C_E\psi_L^p, \psi_L^q\rangle_E + \frac{1}{2}(\widetilde{M}\sigma)^{-1}_{pq}\langle C_E\widetilde{\psi_L}^p\widetilde{\psi_L}^q\rangle_E \tag{5.130}$$

where we used (5.16). Restoring the Majorana product through $[\varphi, \zeta] = \langle C_E\varphi, \zeta\rangle_E$ yields

$$^E\mathcal{L}_F \sim \frac{1}{2}(\widetilde{M}\sigma)^{*-1}_{pq}\psi_L^p\psi_L^q + \frac{1}{2}(\widetilde{M}\sigma)^{-1}_{pq}\widetilde{\psi_L}^p\widetilde{\psi_L}^q. \tag{5.131}$$

This gives the action

$$ {}^{E}S_F \sim \int_{\mathcal{M}} \Omega_E \Big(\frac{1}{2} (\widetilde{M}\sigma)^{*-1}_{pq} \psi_L^p \psi_L^q + \frac{1}{2} (\widetilde{M}\sigma)^{-1}_{pq} \widetilde{\psi_L}^p \widetilde{\psi_L}^q \Big). \tag{5.132} $$

We exchange the book-keeping field and its conjugate for the original lepton and Higgs fields via (5.7) and (5.8):

$$ {}^{E}S_F \sim \int_{\mathcal{M}} \Omega_E \Big(\frac{1}{2} f_{rs} \sigma^{-1} (l_L^r \cdot_{\mathbb{C}^2} \phi^*)(l_L^s \cdot_{\mathbb{C}^2} \phi^*) + \frac{1}{2} f^*_{rs} \sigma^{-1} (\widetilde{l}_L^r \cdot_{\mathbb{C}^2} \phi)(\widetilde{l}_L^s \cdot_{\mathbb{C}^2} \phi) \Big) \tag{5.133} $$

where

$$ f = Y^* \widetilde{M}^{*-1} Y^\dagger. \tag{5.134} $$

The results of the real scalar symmetry breaking were shown in section 5.4.3 and are unaffected by the fermionic action. From this point until the end of the section, we impose symmetry breaking, i.e. we take

$$ \sigma(\widetilde{\sigma}) = \Big(\frac{(4\pi)^2}{c \ln(\Lambda^2/\epsilon^2)} \Big)^{\frac{1}{2}} \widetilde{\sigma} \tag{5.135} $$

and evaluate this on the $\widetilde{\sigma}$ VEV to get

$$ \sigma(w) = \Big(\frac{\Lambda^2 r}{c \ln(\Lambda^2/\epsilon^2)} \Big)^{\frac{1}{2}}, \tag{5.136} $$

which is positive.

Euclidean $\rightarrow$ Lorentzian

By making the transition to the Lorentzian theory using the rules in chapter 3, we have ${}^{E}S_F \rightarrow -iS_F = -i \int_{\mathcal{M}} \Omega(\mathcal{L}_F)$ where

$$ \mathcal{L}_F \sim -\frac{1}{2} f_{rs} \sigma(w)^{-1} (l_L^r \cdot_{\mathbb{C}^2} \phi^*)(l_L^s \cdot_{\mathbb{C}^2} \phi^*) + \text{c.c.} \tag{5.137} $$

with coefficient matrix (5.137). This corresponds precisely to the Weinberg operator (section 4.3).

We remark that the potential fermion doubling in the Euclidean from taking spinors and their charge conjugates as independent fields was restricted to the Euclidean, and this fermion doubling disappears in going back to the Lorentzian.

Lorentzian

The electroweak breaking from $\phi = \phi_v$ leaves unchanged the induced bosonic action due to the absence of Higgs terms, and thus transforms our Weinberg operator (5.137) into the Majorana mass term

$$\mathcal{L}_F \sim -\frac{1}{2} m_{rs} \nu_L^r \nu_L^s + \text{c.c.} \tag{5.138}$$

where

$$m_\nu = \frac{v^2}{2} f \sigma(w)^{-1}. \tag{5.139}$$

The mass matrix contains the piece from the type-I seesaw (section 4):

$$m_\nu^I = \frac{v^2}{2} Y^* \widetilde{M}^{*-1} Y^\dagger. \tag{5.140}$$

However, our mechanism is not strictly type-I due to the σ field. Indeed, the mass matrix (5.139) factorises as

$$m_\nu = m_\nu^I \sigma(w)^{-1}. \tag{5.141}$$

This makes our mechanism a form of multiplicative seesaw [91], in our case Type I $\times$ σ.

We compare our results to those of [49, 50]. To change sign and notation conventions between the former and the latter, we use

$$Y \to -Y_\nu,\ \widetilde{M} \to -M,\ \phi \to \widetilde{\phi},\ \not{D} \to i\not{D}. \tag{5.142}$$

In the special case where M is real and diagonal, this gives

$$f \to -c^{d=5}. \tag{5.143}$$

To ensure this we have the constraint $\eta = 1$, which is a natural correspondence to our formalism where we assume positive Majorana mass eigenvalues. Given the invariance of the Majorana mass traces and cutoffs, we simply have

$$\sigma(w) \to \sigma(w). \tag{5.144}$$

Putting the transformations together gives

$$\mathcal{L}_F \to \mathcal{L}'_F \sim \mathcal{L}^{d=5} \sigma(w)^{-1}. \tag{5.145}$$

Our Lagrangian has an additional factor of $\sigma(w)^{-1}$, which arises through (5.66) with the σ fluctuation terms neglected. We can succinctly express this deviation in the form

$$m \to m^{d=5}\sigma(w)^{-1}. \tag{5.146}$$

Hence, it is sufficient to consider the impact of the σ factor on the active neutrino masses and their agreement with experiment. This is the subject of the next subsection.

5.5.1 Induced neutrino masses

We found the masses to be

$$m_\nu = m_\nu^I \sigma(w)^{-1}. \tag{5.147}$$

The mass matrix (5.147) is complex symmetric and thus diagonalisable by a unitary matrix (see appendix B) since (5.140) is exactly likewise. Thus, the relation (5.147) also holds in eigenvalue form:

$$m_{\nu p} = m_p^I \sigma(w)^{-1}. \tag{5.148}$$

Consider the regime where one $\widetilde{M}_p << \mathfrak{M}$ and the other two $\widetilde{M}_p \sim \mathfrak{M}$. Of the $\widetilde{M}_p$ eigenvalue contributions to active masses, we assume those from the light $\widetilde{M}_p$ are negligible (in a similar fashion to the νMSM [52]). Hence, in (5.148), the first factor is the standard type-I expression

$$m_p^I \sim \frac{v^2|Y_{pq}|^2}{2\mathfrak{M}} \tag{5.149}$$

and we cast the second factor using (5.136) as

$$\sigma(w) \sim \frac{m_\sigma}{2\mathfrak{M}} \tag{5.150}$$

where we used $m_\sigma = 2\Lambda/\ln(\Lambda^2/\epsilon^2)^{\frac{1}{2}} \sim 10^{18}$GeV. Making the substitutions gives

$$m_{\nu p} \sim \frac{v^2|Y_{pq}|^2}{m_\sigma}. \tag{5.151}$$

Setting this of order $m_\nu^{\text{exp}} \sim 10^{-11}$GeV, we solve for the Yukawa couplings to get $|Y_{pq}| \sim 20$. The SM Yukawa couplings are about an order of magnitude below this, which leads us to view our Yukawa scale as unnatural[6]. Furthermore, our Yukawa couplings lie in the strong coupling range, i.e. outside the perturbative range. Given that fermions with real scalar Yukawa coupling exhibit fermionic confinement at large Yukawa coupling magnitude [114], we speculate that a similar effect occurs for the right-handed neutrinos and thus one would get a $\nu_R\overline{\nu_R}$ bound state (assuming ν_R and $\overline{\nu_R}$ are not Majorana spinors). Compared to toponium [115], our proposed bound state is expected to be much heavier since the right-handed neutrino mass scale is significantly greater than the top mass.

The sterile neutrino masses are given in our Type I $\times\, \sigma$ seesaw by

$$M_{\nu p} \sim \widetilde{M}_p \sigma(w). \tag{5.152}$$

Using (5.150) results in

$$M_{\nu p} \sim \frac{m_\sigma}{2\mathfrak{M}} \widetilde{M}_p. \tag{5.153}$$

Hence, in our chosen regime, we have one $M_{\nu p} << m_\sigma$ and two $M_{\nu p} \sim m_\sigma$. Relative to the νMSM [52], our spectrum admits a keV neutrino but a far greater heavy scale.

We remark on our Type $1\times\sigma$ seesaw. This is different to type-I, but originates in the manner of type-I, i.e. from right-handed neutrinos coupling to the SM. The seesaw scale is the singlet scalar mass m_σ due to the σ factor trading this for the type-I seesaw scale $\mathfrak{M} \sim 10^{15}$GeV. In contrast to type-I, our scale is *bosonic* instead of fermionic. Further $m_\sigma >> \mathfrak{M}$, placing our seesaw higher than type I (and type 3) in the seesaw ordering by new scales in [91], which was the cause of our unnatural Y. Our seesaw is similar in form to the type-I mechanism in a modified SM with further gauge symmetry $U(1)_X$ [63].

Our last remark is on equation (5.151). This equation determines the active neutrino masses. However, we already assumed their experimental value $m_\nu^{\text{exp}} \sim 10^{-11}$GeV in order to compute m_σ. This appears to be a circular argument, but we now explain why it is not. The experimental value for the active masses was taken as part of the input data for equation (5.151), along with $v \sim 246$GeV and the cutoffs $\epsilon \sim m_\nu^{\text{exp}}$ and Λ of Planck order. This leaves $|Y_{pq}|$ as the only unknown

[6] Another reason for our view is that our Yukawa coupling exceeds the natural type-I Yukawa coupling $Y_I \sim 1$ by a factor ~ 20.

in equation (5.151), and this unknown was subsequently computed. We now see that there is no over-determination of any parameters in equation (5.151), and hence there is no circular argument.

Chapter 6

Further aspects of induced gravity models

In the previous chapter, we derived the right-handed neutrino induced fermionic action, and we considered the leading $\mathcal{O}(\widetilde{M}^{-1})$ term only. The term immediately above leading order is considered in this chapter. The present chapter also presents results for additional induced gravity models, which are motivated by a need to account for SM fields and problems in non-commutative geometry. All projects in this chapter are currently work in progress, and we hope further work on them will give important insights and applications corresponding to the induced gravity models in question.

This chapter is organised as follows. In section 6.1, we briefly investigate the next-to-leading order term from our right-handed neutrino induced fermionic action. Section 6.2 concerns qualitative discussions of integrating out SM fields and the metric. Lastly, section 6.3 gives induced gravity interpretations to a simple random non-commutative geometry with fermions.

6.1 Right-handed neutrino induced fermionic action: next-to-leading term

Recall that the $1/M$ expansion of the right-handed neutrino induced fermionic action in the Euclidean regime is given by equation (5.120) (in section 5.5). Our interest in the next-to-leading term is twofold. Firstly, given our methodology in section 5.5, it will be fairly straightforward to compute the next-to-leading term. Secondly, Broncano-Gavela-Jenkins [49, 50] have found a dimension-6 term along similar lines to ours, so our next-to-leading term provides another test of our right-handed neutrino mechanism.

Euclidean

Equation (5.120) gives the above leading order (i.e. $\mathcal{O}(\widetilde{M}^{-2})$) term as

$$^{E}\delta\mathcal{L}_F = \frac{1}{2}[[\Psi_L, \mathcal{D}_M^{-1}(i\mathcal{D}_D)\mathcal{D}_M^{-1}\Psi_L]]. \tag{6.1}$$

This is written in terms of the doubled Dirac product via $[[\Psi,\Phi]] = \langle\langle \mathcal{C}_E\Psi, \Phi\rangle\rangle_E$ so that

$$^{E}\delta\mathcal{L}_F = \frac{1}{2}\,\langle\langle \mathcal{C}_E\Psi_L, \mathcal{D}_M^{-1}(i\mathcal{D}_D)\mathcal{D}_M^{-1}\Psi_L\rangle\rangle_E\,. \tag{6.2}$$

The anti-linear entry is given by the previous chapter (equation (5.128)):

$$\mathcal{C}_E\Psi_L = \begin{pmatrix} C_E\psi_L \\ C_E\widetilde{\psi_L}\end{pmatrix}. \tag{6.3}$$

We now explicitly find the linear entry. From the last chapter,

$$\mathcal{D}_M^{-1} = \begin{pmatrix} & i\gamma_M(\widetilde{M}\sigma)^{*-1} \\ -i\gamma_M(\widetilde{M}\sigma)^{-1} & \end{pmatrix} \tag{6.4}$$

and

$$\mathcal{D}_D = \begin{pmatrix} \not{D}_E & \\ & \not{D}_E \end{pmatrix} \tag{6.5}$$

where we suppress Kronecker products with identities. A matrix product gives

$$\mathcal{D}_M^{-1}(i\mathcal{D}_D)\mathcal{D}_M^{-1} = i\begin{pmatrix} \gamma_M(\widetilde{M}\sigma)^{*-1}\not{D}_E\gamma_M(\widetilde{M}\sigma)^{-1} & \\ & \gamma_M(\widetilde{M}\sigma)^{-1}\not{D}_E\gamma_M(\widetilde{M}\sigma)^{*-1}\end{pmatrix}. \tag{6.6}$$

We exploit the commuting of the Majorana matrices with the data on $\mathcal{M}$. This gives factors of $(\widetilde{M}\sigma)^{-1}(\widetilde{M}\sigma)^{*-1} = |\widetilde{M}\sigma|^{-2}$ and the transpose by $\widetilde{M}$, and

$$\gamma_M\not{D}_E\gamma_M = -\not{D}_E \tag{6.7}$$

due to $\gamma_M^2 = 1$ and $\{\gamma_M, \not{D}_E\} = 0$. Thus, equation (6.6) becomes

$$\mathcal{D}_M^{-1}(i\mathcal{D}_D)\mathcal{D}_M^{-1} = -i\begin{pmatrix} \not{D}_E|\widetilde{M}\sigma|^{-2^T} & \\ & \not{D}_E|\widetilde{M}\sigma|^{-2}\end{pmatrix}. \tag{6.8}$$

This acts on $\Psi_L = (\widetilde{\psi_L}\ \psi_L)^T$ as

$$(\mathcal{D}_M^{-1}(i\mathcal{D}_D)\mathcal{D}_M^{-1})\Psi_L = -i\begin{pmatrix}|\widetilde{M}\sigma|^{-2^T}\not{D}_E\widetilde{\psi_L}\\ |\widetilde{M}\sigma|^{-2}\not{D}_E\psi_L\end{pmatrix}. \tag{6.9}$$

For equation (6.2), we substitute in (6.3) and (6.9) and then decompose the doubled Dirac product to get

$${}^E\delta\mathcal{L}_F = -\frac{i}{2}\Big[\langle C_E\psi_L, |\widetilde{M}\sigma|^{-2^T}\not{D}_E\widetilde{\psi_L}\rangle_E^{\circ} + \langle C_E\widetilde{\psi_L}, |\widetilde{M}\sigma|^{-2}\not{D}_E\psi_L\rangle_E^{\circ}\Big]. \tag{6.10}$$

In component form, this is (c.f equation (5.16))

$${}^E\delta\mathcal{L}_F = -\frac{i}{2}\Big[|\widetilde{M}\sigma|^{-2}_{qp}\langle C_E\psi_L^p, \not{D}_E\widetilde{\psi_L}^q\rangle_E + |\widetilde{M}\sigma|^{-2}_{pq}\langle C_E\widetilde{\psi_L}^p, \not{D}_E\psi_L^q\rangle_E\Big]. \tag{6.11}$$

The first term in the square brackets gives upon integration over $\mathcal{M}$

$$\int_{\mathcal{M}}\Omega_E\Big(|\widetilde{M}\sigma|^{-2}_{qp}\langle C_E\psi_L^p, \not{D}_E\widetilde{\psi_L}^q\rangle_E\Big) = \int_{\mathcal{M}}\Omega_E\Big(|\widetilde{M}\sigma|^{-2}_{pq}\widetilde{\psi_L}^p\not{D}_E\psi_L^q\Big) \tag{6.12}$$

where we used the self-adjointness of $\not{D}_E$ (since torsion vanishes), the relation $\not{D}_E C_E = C_E\not{D}_E$ and the properties of $[\cdot,\cdot]$. Similarly, from the second term

$$\int_{\mathcal{M}}\Omega_E\Big(|\widetilde{M}\sigma|^{-2}_{pq}\langle C_E\widetilde{\psi_L}^p, \not{D}_E\psi_L^q\rangle_E\Big) = \int_{\mathcal{M}}\Omega_E\Big(|\widetilde{M}\sigma|^{-2}_{pq}\widetilde{\psi_L}^p\not{D}_E\psi_L^q\Big). \tag{6.13}$$

We see the equality of (6.12) and (6.13), which is analogous to the computation in [49]. Hence, defining ${}^E\delta S_F = \int_{\mathcal{M}}\Omega_E({}^E\delta\mathcal{L}_F)$, we have

$${}^E\delta S_F = \int_{\mathcal{M}}\Omega_E\Big(-|\widetilde{M}\sigma|^{-2}_{pq}\widetilde{\psi_L}^p(i\not{D}_E)\psi_L^q\Big). \tag{6.14}$$

Equations (5.7) and (5.8) give the original fields:

$${}^E\delta S_F = \int_{\mathcal{M}}\Omega_E\Big(-g_{rs}|\sigma|^{-2}(\widetilde{l_L}^r\cdot_{\mathbb{C}^2}\phi)(i\not{D}_E)(l_L^s\cdot_{\mathbb{C}^2}\phi^*)\Big) \tag{6.15}$$

where

$$g = Y|\widetilde{M}|^{-2}Y^\dagger. \tag{6.16}$$

For the normalised scalar $\widetilde{\sigma}$ given by

$$\sigma(\widetilde{\sigma}) = \Big(\frac{(4\pi)^2}{c\ \ln(\Lambda^2/\epsilon^2)}\Big)^{\frac{1}{2}}\ \widetilde{\sigma}, \tag{6.17}$$

we impose the VEV $\widetilde{\sigma} = w$.

Euclidean → Lorentzian

Using the transition rules from chapter 3, we have ${}^{E}\delta S_F \to -i\delta S_F$ with the Lorentzian action $\delta S_F = \int_{\mathcal{M}} \Omega(\delta\mathcal{L}_F)$ where

$$\delta\mathcal{L}_F = g_{rs}\sigma(w)^{-2}(\overline{l_L^r} \cdot_{\mathbb{C}^2} \phi)\not{D}(l_L^s \cdot_{\mathbb{C}^2} \phi^*). \tag{6.18}$$

Lorentzian

Equation (6.18) under electroweak breaking $\phi = \phi_v$ becomes

$$\delta\mathcal{L}_F = \frac{v^2}{2} g_{rs}\sigma(w)^{-2}\overline{\nu_L^r}\not{D}\nu_L^s. \tag{6.19}$$

This contributes to the SM ν_L kinetic term.

The term (6.19) is compared to the $d = 6$ operator in [49, 50]. This follows the comparison done for the leading term in the last chapter. For Majorana mass real and diagonal, both terms deviate as

$$\delta\mathcal{L}_F \to \delta\mathcal{L}^{d=6}\sigma(w)^{-2}, \tag{6.20}$$

which is natural at order $\widetilde{M}^{-2}$. The term $\delta\mathcal{L}^{d=6}$ involves a Hermitian matrix λ ((48) in [50]). Our case sees this replaced by λ', which is related to λ as

$$\lambda' = \lambda\sigma(w)^{-2}. \tag{6.21}$$

This is equivalent to (6.20). In other words, with respect to Broncano-Gavela-Jenkins [49, 50], our next-to-leading term is in agreement up to a real scalar factor that does not significantly alter the agreement since it amounts to a rescaling of the parameters.

Analysis of the effects of the $\mathcal{O}(\widetilde{M}^{-2})$ term is conducted in [49, 50]. Our only desire here is to estimate the contributions of the σ factor. Using appendix B, we get the eigenvalue form of (6.21):

$$\frac{\lambda'_p}{\lambda_p} = \sigma(w)^{-2}. \tag{6.22}$$

For one $\widetilde{M}_p << \mathfrak{M}$ and two $\widetilde{M}_p \sim \mathfrak{M}$, the above is

$$\sigma(w)^{-2} \sim 10^{-6} \tag{6.23}$$

where we used (5.150) with $m_\sigma \sim 10^{18}$GeV. Hence, the $\mathcal{O}(\widetilde{M}^{-2})$ effects in our case are suppressed by an extra factor given by (6.23).

6.2 Integrating out other fields

We have seen that right-handed neutrinos do not induce the pure gauge or Higgs terms of the SM. However, we know from the induced Standard Model [20] that these pure bosonic terms do arise from the set of all SM fermions and right-handed neutrinos. This suggests it is important to consider integrating out the SM fermions in order to account for the pure bosonic terms. The SM fermion integration out is considered in this section. Induced gravity from SM fermions has been considered before in [16], but our discussion of induced fermionic terms is new to our knowledge. For completeness, we also consider integrating out gauge bosons and (very briefly) the metric. Lastly, we suggest some approaches to further develop the induced Standard Model beyond its current early stages. This is new to our knowledge since [20] constitutes the only work on the induced Standard Model. The discussion in this section is currently qualitative, and a quantitative formulation is left to future work.

For an SM fermion f, the (Lorentzian) functional integral is similar to the right-handed neutrino case:

$$Z_f = \int e^{iS_f} D\overline{f} Df \tag{6.24}$$

where S_f is the relevant action. In the integration out, the fermion f would have extra gauge couplings that should add difficulty, but however S_f has no gauge-violating Majorana mass terms initially present. Compared to the right-handed neutrino case, the effective action Γ_f works at smaller scales because f has mass $m_f << \mathfrak{M}$. A rule of thumb is that Γ_f contains terms for fields that couple to f in the high-energy matter theory. Given the results from chapter 5, we expect that Γ_f will contain two parts: a heat kernel expansion in curvature and SM bosonic terms and a series of higher-dimensional fermionic operators due to Yukawa couplings.

The SM fermion falling immediately below the right-handed neutrinos (via mass scales) is the top quark $f = t$ with mass $m_t \sim 173$GeV. The effects of the effective action Γ_t are significant below m_t and thus in the range of present experiments. Hence, the special case of just integrating out $f = t$ is a natural possibility for further work which gives testable induced gravity.

One can also consider the integration out of the SM gauge bosons. This is inconsistent with the induced Standard Model since the latter theory only integrates out fermions. Despite this, results have been found for gauge bosons in more general induced gravity models [12], and this suggests integrating out SM gauge bosons is allowed within general induced gravity. In the case of the SM gauge bosons, one has to handle the gauge kinetic terms and gauge-fermion interactions. In addition, to integrate out the gauge bosons, one must fix a gauge and thus introduce additional ghost fields.

The metric integration in the induced Standard Model (see chapter 2) needs quantum gravity, e.g. causal dynamical triangulations [4] or spin foams [3].

The methods seen in chapters 3 and 5 as well as the σ field from chapter 4 may be used in future work to fill in more details of the induced Standard Model as presented by Barrett [20]. A test for the induced Standard Model is to do comparisons with the induced Lorentzian curvature and SM bosonic terms, because Barrett [20] used the Euclidean. Assuming the Lorentzian physics works, the induced Standard Model would be a natural compliment to the SM with three right-handed neutrinos, which already has wide physical breadth (see section 4.1 and [52]).

An alternative approach to the induced Standard Model of Barrett [20] is to perform the integration out stepwise for each fermion species. In this case, the order of integration should be important. For example, starting with ν_R and integrating this out before l_L gives higher-dimensional terms in l_L (see sections 5.5 and 6.1) which are absent if one starts with l_L and then ν_R. The higher-dimensional fermionic operators from SM fermions and right-handed neutrinos were not found by Barrett [20] and may complicate the integration out.

6.3 Non-commutative induced gravity

In this section, we make a first attempt at interpreting pure Connes' non-commutative geometry in terms of Sakharov's induced gravity. Previously, the induced Standard Model has explained the spectral action principle for the case of the non-commutative SM [20] (also see section 2.2). However, to our knowledge the spectral action principle justification in the strictly non-commutative case has not been done. This gap is the motivation behind the present section, where we consider an induced gravity perspective of pure non-commutative geometry in an attempt to fill the gap. A caveat is that we restrict to a simple finite toy geometry containing additional fermionic fields, and defer the general case to future work.

We are concerned with (Euclidean) finite spectral triples from section 4.2.2. Our focus is the type $(0, 1)$ functional integral [54, 98]

$$Z = \int e^{-S[D,\psi]} d\psi dD \tag{6.25}$$

where

$$S[D, \psi] = \mathrm{tr}[g_2 D^2 + g_4 D^4] + \langle \psi, D\psi \rangle \,. \tag{6.26}$$

We assume all D eigenvalues are non-zero and the restrictions $g_2 > 0$ and $g_4 \geq 0$.

The integral over fermions ψ is known from [54, 98]:

$$F[D] = \int e^{-S_f} d\psi = \det(D) \tag{6.27}$$

where $S_f[D, \psi] = \langle \psi, D\psi \rangle$. We assume this is regularised as in [98], i.e. via an internal modification to the Dirac operator. Equation (6.27) gives

$$Z = \int e^{-S_G} F[D] dD \tag{6.28}$$

with the pure geometry term

$$S_G[D] = \mathrm{tr}[g_2 D^2 + g_4 D^4]. \tag{6.29}$$

We discuss aspects of interpreting the fermionic action[1]. The fermionic integral is positive[2] and so can be written as

$$F[D] = e^{-S_I[D]} \tag{6.30}$$

where

$$S_I[D] = -\frac{1}{2} \mathrm{tr}\ \ln(D^2), \tag{6.31}$$

which comes from ln det = tr ln. Convergence of the integral over geometries requires $S_G \neq 0$, in which case (6.31) is the action the fermions induce.

[1]The author attributes this to private communications with John W. Barrett.

[2]Also see [98].

We now consider fixed D with eigenvalues λ_i. Equation (6.31) corresponds to the Connes-Chamseddine action with, for $x \neq 0$,

$$f(x) = \begin{cases} -\frac{1}{2}\ln(x^2) & x^2 \leq \lambda_{max}^2, \\ 0 & x^2 > \lambda_{max}^2 \end{cases} \tag{6.32}$$

where λ_{max} is the maximum eigenvalue. This cutoff function includes all λ_i and is free from singularities as D^2 is positive. Thus, one has a well-defined induced Connes-Chamseddine action assuming $S_G \neq 0$. This explains the spectral action principle along similar lines to [20] but in a purely non-commutative setting in our case. We remark that a similar result was found in [48], but this was for the non-commutative SM and had a key role played by anomalies.

Moreover, we may also consider the pure geometry action where we set $g_4 = 0$. This case of the pure geometry action has been considered[3], and we wish to see if the geometry action also has an induced gravity interpretation. Equation (6.29) for $g_4 = 0$ is

$$S_G[D] = \sum_i g_2 \lambda_i^2. \tag{6.33}$$

Substituting this into (6.28) gives

$$e^{-S_G[D]} = \prod_i e^{-g_2\lambda_i^2}. \tag{6.34}$$

The Hubbard-Stratonovich transformation [116, 117] gives each factor on the right-hand side as

$$e^{-g_2\lambda_i^2} = \frac{1}{(\pi g_2)^{\frac{1}{2}}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{g_2} - 2i\lambda_i x} dx \tag{6.35}$$

and hence (up to an overall constant)

$$e^{-S_G[D]} = \int e^{-\mathrm{tr}\left[\frac{1}{g_2}X^2 + 2iDX\right]} dX \tag{6.36}$$

where the dummy matrix X has real eigenvalues. Hence, we can interpret S_G as being induced by a self-adjoint field X with action

$$S_{SA}[X; D] = \mathrm{tr}\left[\frac{1}{g_2}X^2 + 2iDX\right]. \tag{6.37}$$

Referring to the non-commutative matrix model in [99], we see that X is a Higgs field (without self-coupling) with a coupling to D and a quadratic coefficient

[3]The author again acknowledges John W. Barrett for this.

satisfying

$$a_2 = \frac{1}{g_2}. \tag{6.38}$$

Equation (6.38) is characteristic of induced gravity since it relates parameters corresponding to pure geometry (left-hand side) and pure matter (right-hand side). Hence, one can regard the geometry coupling as being induced by the Higgs-like field X, and one has a similar viewpoint for the pure geometry action as a whole. Thus, in this model we have shown that all geometry terms arise via an induced gravity mechanism from fundamental bosonic and fermionic fields.

Chapter 7

Conclusions

A major outstanding problem is to explain gravity in terms of quantum theory. One mechanism for this is the semiclassical concept of Sakharov's induced gravity, which proposes the emergence of the Einstein-Hilbert action plus cosmological constant and curvature-squared terms from 1-loop quantised matter. Since Sakharov's induced gravity was introduced, it has been shown to work for bosonic and fermionic matter, give rise to gravitational as well as gauge boson terms, and work for spacetimes without or with torsion. In particular, the action for gravity coupled to SM bosons has been shown to emerge from SM fermions and right-handed neutrinos, and this is the induced Standard Model [20]. However, the induced Standard Model does not resolve the total action into individual fermion contributions. In this work, we found the terms induced only by right-handed neutrinos, which compared to SM fermions give a model that is simpler and valid for a larger range of energies.

Our mechanism started with three quantised right-handed neutrinos minimally coupled to the classical SM within classical curved spacetime. This was set in the regime of the measurable Lorentzian physics. However, our calculational techniques, including the integration out and heat kernel expansion, were used in the Euclidean regime where they are well grounded in the literature (unlike in the Lorentzian regime). Our final results were given in the Lorentzian regime. We transferred between the two regimes using the Lorentzian-Euclidean transition (chapter 3).

In the Euclidean regime, we found that the effective action was the sum of a purely bosonic part, which was given by a Pfaffian, and a fermionic part involving the SM leptons and Higgs. This constitutes the first appearance in Sakharov's induced gravity (to the author's knowledge) of a 1-loop fermionic action and therefore of 1-loop bosonic and fermionic actions simultaneously. The total effective action was induced from the integration out of the right-handed neutrinos. This mirrored a case of finite spectral triple fermion integrals [54], but we also had three fermion generations and Yukawa couplings. The generations were included simply by enlarging the spinor Dirac product. More substantially, the Yukawa couplings gave rise to the fermionic term, which was absent from the finite spectral triple case.

Using the Lorentzian-Euclidean transition, we found a similar result for the effective action in the Lorentzian. Then, starting in the Euclidean, we explicitly transitioned and analysed each part of the effective action.

First, in order to get the bosonic action, we evaluated the Pfaffian. A key assumption we made was the split of the Pfaffian into generational contributions which each behave like the Pfaffian for a finite spectral triple [54]. The evaluation gave an expression for the Pfaffian in terms of a functional determinant 4th root, and thus the Pfaffian is basis-independent. We remark that ideally one would have more rigorous procedures for the functional integral and Pfaffian, which are left to future work. Applying standard induced gravity methods to the functional determinant gave an expansion of the bosonic action in curvature terms coupled to a real singlet scalar σ which came from the Majorana mass according to non-commutative geometry. Our bosonic action had good agreement with the relevant sector of the Connes-Chamseddine action, though this finding was provisional.

We then implemented the real scalar $\mathbb{Z}_2$ symmetry breaking. This sent the scalar terms in the induced bosonic action to cosmological and Einstein-Hilbert corrections without fundamentally affecting the curvature-squared sector. We then transitioned the resulting pure gravitational action to the Lorentzian.

The pure gravitational action gave cosmological, Newton and curvature-squared constants. These were computed in the regime where one Majorana mass eigenvalue is much below the type-I seesaw scale $\mathfrak{M}$ and two Majorana eigenvalues are on the order of $\mathfrak{M}$. We chose this regime because it was the only one (of the ones we considered) that is allowed by the model and gives no unphysical divergences. Observations of the induced gravitational constants are as follows:

- The induced cosmological constant far exceeds the experimental value, i.e. we have the cosmological constant problem. This can be resolved simply by cancelling the induced cosmological constant by restoring the cosmological counter-term we initially neglected (c.f [20], also see section 5.1). A more sophisticated option, which furnishes a small value of the cosmological constant, is sequestration of the vacuum energy [118, 119, 120, 121], which is discussed in the context of induced gravity in [19]. We leave a solution to our cosmological constant problem to future work.
- The induced Newton constant $G_N = 12\pi\Lambda^{-2} > G$ (where G is the classical Newton constant). Given that an induced Newton constant has a tolerance of within two orders of magnitude [16], this suggests our induced Newton constant is consistent with observation.
- The induced curvature-squared constants give rise to tachyonic behaviour, which is physically problematic since it means the effective action does not necessarily have a true vacuum state.

Invoking the Sakharov version of induced gravity [12], we can discard the inadmissible cosmological and curvature-squared constants, leaving us with the admissible Newton constant. Thus, the main result of our bosonic action is that the right-handed neutrinos induce a viable Newton constant, which is a new result (to the author's knowledge). Our result has the implication for induced gravity that only the right-handed neutrinos are needed to obtain realistic spacetime dynamics. Furthermore, our result leads to the possibility that the Einstein-Hilbert action is not fundamental, but is an emergent property of right-handed neutrinos alone.

The observations that the induced Newton constant $G_N > G$, as well as that the effective theory favours our chosen Majorana mass regime and disfavours degeneracy, remain fundamentally unexplained.

We now start from the Euclidean fermionic action. This had a $1/M$ expansion where the Majorana mass was modified by the real singlet scalar σ. Initially, we kept and worked out only the dominant leading term. Upon scalar $\mathbb{Z}_2$ breaking and transitioning to the Lorentzian, we found the fermionic action (leading term) was the Weinberg operator. After electroweak breaking, the operator furnished active neutrino masses corresponding to a Type $\mathrm{I} \times \sigma$ seesaw. To our knowledge, this is the first time such a seesaw has emerged from Sakharov's induced gravity. Compared to the standard type-I seesaw, our active masses deviated only by a factor involving σ, and we have a larger seesaw scale given by the mass m_σ of the σ field[1]. Our seesaw also gave sterile neutrino masses, and we computed both active and sterile masses in the Majorana mass regime used earlier. From this, we have the following observations:

- The active masses matched experiment, but the large size of the seesaw scale m_σ led to a Yukawa coupling in the unnaturally large and non-perturbative regime. A prospect for future work is to restore a natural Yukawa coupling, and this may happen if additional factors due to other fields are included in the mechanism.

- We found that one sterile mass is small enough to be keV. Hence, the corresponding neutrino state is a dark matter candidate (according to section 4.1). Checking the stability of the dark matter candidate is left as an open problem. Furthermore, we found two large sterile masses that were of order m_σ, thus the corresponding neutrino states are above the type-I seesaw scale and so even further beyond current accelerators.

Hence, we have solved the problem in the induced Standard Model of finding the dominant contributions due to right-handed neutrinos. We have shown for the first time that integrating out right-handed neutrinos alone has the three consequences of an Einstein-Hilbert action, active neutrino masses and a potential dark matter candidate. This means the integration out of the right-handed neutrinos has the potential to solve three problems within the SM. Given right-handed neutrino phenomenology [52], the new contribution to this from our

[1] The mass m_σ is also one of the scales governing the induced gravitational constants.

findings is realistic gravitational dynamics, which gives another motivation for right-handed neutrino existence. We have assumed that one can cope with the caveats of a large Yukawa coupling and heavy sterile neutrino states (the latter of which already appear in the type-I seesaw), and we defer further consideration of these caveats to further work.

Starting in the Euclidean, we briefly considered the next-to-leading fermionic term. Via a similar procedure as for the leading term, we found the next-to-leading term gave an active neutrino kinetic term in the Lorentzian regime under electroweak and $\mathbb{Z}_2$ symmetry breaking. Our result agrees with the literature up to a new σ factor that merely rescales parameters. In the earlier Majorana mass regime, the σ factor significantly suppresses the next-to-leading order effects. We deduce that computations of next-to-leading effects in our mechanism must account for the suppression due to σ. This is deferred to future work.

Moreover, we discussed integration out of individual fields from the induced Standard Model. The highlight from this was the hint towards a formulation of SM fermion induced gravity, which is expected to give new fermionic terms. It is hoped that this formulation will allow for fermionic beyond-SM physics to be incorporated into the induced gravity models. Furthermore, we discussed advancing the induced Standard Model, which is a new development. This came in the form of gaps to fill and alternative perspectives, and it is hoped that this will contribute towards the considerable explanatory power of the induced Standard Model. We hope that our qualitative discussion will be quantitatively developed and thus lead to novel contributions to induced gravity in future work.

Finally, we set down the first case (to our knowledge) of providing pure non-commutative geometry with an induced gravity interpretation. We considered a particular simple finite spectral triple with fermions where there is an action consisting of geometry and fermionic terms. In this case, we found the first explanation of the purely non-commutative spectral action principle, which was given by the fermionic integration out. Another original finding was that the geometry term in this model could also be viewed as induced, in this case by a Higgs-like field coupling to geometry. The key merit of the induced gravity interpretation is that the non-commutative spectral action principle would be mysterious without it, hence one has a conceptual use for induced gravity. Of course, our findings here only hold for our specific toy model. Future work could be to see if the induced gravity interpretation, including the spectral action principle justification, continues to hold similarly for more complicated finite geometries. In the long term, our hope is for the induced gravity interpretation to give a general explanation for other mysterious aspects of non-commutative geometry, e.g. the presence of the first-order condition.

Appendix A

Lorentzian and Euclidean spinors in curved spacetime

In this appendix, we review Lorentzian and Euclidean spinors in curved spacetime. None of this is original, but it underpins a large part of the present thesis, which concerns fermions on curved spacetime. Our review corresponds to standard well-known material, but uses a more modern presentation.

We start with a brief discussion of the spacetime manifold we use.

Lorentzian spinors are reviewed in section A.1. For this, we closely follow the recent presentations in [62, 1], the first of which is set in Minkowski spacetime but readily generalises to curved spacetime via the principle of general covariance. Standard spinor presentations exist for Minkowski spacetime [110, 122] and for curved spacetime [13]. Compared to these standard presentations, the ones used here are notationally distinct but physically identical. The presentations followed herein use a certain charge conjugate definition and introduce the Majorana product, both of which are not typically seen in standard presentations from the experience of the author.

The presentation of Euclidean spinors in section A.2 closely follows [1]. A more standard Euclidean presentation is in [123], but we have similar comparisons that we made for the Lorentzian case. For Euclidean spinors, as well as for Lorentzian spinors, we use the presentations in [62, 1] since they simplify spinor calculations and are compatible with the Lorentzian-Euclidean transition (chapter 3), which is used in this work.

Spacetime manifold

In this work, we let $\mathcal{M}$ be a spacetime manifold that is 4-dimensional, oriented, compact, smooth and without boundary or torsion. We assume $\mathcal{M}$ admits geometric data and bundles (for scalars, spinors, gauge fields, etc.) for Lorentzian theories, and that the analogous Euclidean structures *are also set on* $\mathcal{M}$. The properties of $\mathcal{M}$ are assumed throughout this work. These assumptions are motivated by the need to have consistent notation throughout the sections of this thesis and ensure all frameworks seen in this thesis are compatible.

A.1 Lorentzian spinors

The Clifford algebra for Lorentzian Dirac spinor fields is

$$\{\gamma^a, \gamma^b\} = -2\eta^{ab} \tag{A.1}$$

where the generators are the gamma matrices γ^a. We assume the Hermitian conjugation relations

$$\begin{aligned} (\gamma^0)^\dagger &= \gamma^0, \\ (\gamma^j)^\dagger &= -\gamma^j \end{aligned} \tag{A.2}$$

where $j \in \{1, 2, 3\}$. The gamma matrices determine a *chirality operator* on spinors given by

$$\gamma_M = i\gamma^0\gamma^1\gamma^2\gamma^3, \tag{A.3}$$

which is also called the fifth gamma matrix γ^5. This chirality operator satisfies

$$\begin{aligned} \gamma_M^\dagger &= \gamma_M, \\ \gamma_M^2 &= 1. \end{aligned} \tag{A.4}$$

These properties restrict the distinct eigenvalues of the chirality operator to the real numbers 1 and -1. The eigenspinors with these respective eigenvalues are the purely left-handed spinor ψ_L and the purely right-handed spinor ψ_R, which decompose a general Dirac spinor as $\psi = \psi_L + \psi_R$.

In this spinor formalism, the fundamental geometric structure on $\mathcal{M}$ is a frame, which is a set of four vector fields $l_a = l_a^\mu \partial_\mu$. The dual of the frame is a tetrad, which is a set of four one-forms $l^a = l^a_\mu dx^\mu$ on $\mathcal{M}$. The tetrad and frame respectively determine a Lorentzian metric and its inverse by

$$\begin{aligned} g &= \eta_{ab} l^a \otimes l^b, \\ g^{-1} &= \eta^{ab} l_a \otimes l_b. \end{aligned} \tag{A.5}$$

Since $\mathcal{M}$ is oriented, it is equipped with a non-zero volume form, which is determined by the tetrad as

$$\Omega = l^0 \wedge l^1 \wedge l^2 \wedge l^3. \tag{A.6}$$

In this work, we choose the positive sign for this volume form, which corresponds to us choosing the positive orientation.

We equip $\mathcal{M}$ with a spin connection ∇. Given a vector field X on $\mathcal{M}$, the spin connection defines a map $X \to \nabla_X$ acting on a Dirac spinor ψ as

$$\nabla_X \psi = X\psi + \sigma_X{}^b_c S^c_b \psi \tag{A.7}$$

where the matrix of Lorentzian 1-forms σ^a_b gives the linear map $X \to \sigma_X$ whose components are the numbers $\sigma_X{}^a_b$; and the generators of the Lorentzian spinor representation are the matrices

$$S^b_a = \frac{1}{4} \eta_{ac} [\gamma^b, \gamma^c]. \tag{A.8}$$

The spin connection action (A.7) defines a first-order differential operator on Dirac spinors called the *Dirac operator*:

$$D\!\!\!/ = i\gamma^a \nabla_{l_a}. \tag{A.9}$$

The Dirac and chirality operators satisfy

$$\gamma_M D\!\!\!/ = -D\!\!\!/ \gamma_M. \tag{A.10}$$

This means the Dirac operator maps a left-handed spinor to a right-handed one and vice versa.

Charge conjugation of Lorentzian spinors is implemented by an operator C which is anti-linear and satisfies

$$\begin{aligned} C^2 &= 1, \\ C\gamma_M &= -\gamma_M C, \\ C D\!\!\!/ &= D\!\!\!/ C. \end{aligned} \tag{A.11}$$

In particular, the second entry implies C reverses chirality of chiral spinors in the same manner as $\not{D}$. The other entries will come in later. Equation (A.11) defines C as the Lorentzian charge conjugation operator.

For a spinor ψ, one has the charge conjugate spinor [62, 1]

$$\overline{\psi} := C\psi \tag{A.12}$$

and the first entry of (A.11) gives

$$C\overline{\psi} = \psi, \tag{A.13}$$

where we assume that the conjugate of ψ is defined by (A.12). This replaces the Dirac adjoint $\psi^D = \psi^\dagger \gamma^0$ in standard spinor notation (e.g. [110]). The conjugate (A.12) equals the standard charge conjugate $\psi^c = M_C(\psi^D)^T$ where M_C is a charge conjugation matrix [62]. The second relation in (A.11) gives the following: for a left-handed spinor ψ_L, the spinor $\overline{\psi_L}$ is *right-handed* and for a right-handed spinor ψ_R, the spinor $\overline{\psi_R}$ is *left-handed*.

The spinors ψ_L and ψ_R are Weyl spinors and have 4 components. In other places (e.g. [110, 122]), the Weyl spinor name is applied to 2-component chiral spinors given by χ_L and χ_R. Both Weyl spinor definitions are related via

$$\psi_L = \begin{pmatrix} \chi_L \\ 0 \end{pmatrix}, \quad \psi_R = \begin{pmatrix} 0 \\ \chi_R \end{pmatrix}. \tag{A.14}$$

In this work, we will only consider 4-component spinors. Though we will not use Majorana spinors, these are present in the formalism as spinors ψ_m such that $\overline{\psi_m} = \psi_m$.

Given Dirac spinors ψ and χ, their Dirac inner product is defined as

$$\langle \psi, \chi \rangle := \overline{\psi}\chi. \tag{A.15}$$

Note this definition uses the conjugate (A.12). The Dirac product is sesquilinear, i.e. antilinear in the first argument and linear in the second. Complex conjugation sends (A.15) to

$$\langle \psi, \chi \rangle^* = \langle \chi, \psi \rangle. \tag{A.16}$$

Thus, for $m \in \mathbb{R}$, the Dirac mass term $m\overline{\psi}\psi$ is real. Assuming $\mathcal{M}$ has no torsion, the Dirac operator (A.9) is self-adjoint in that

$$\int_{\mathcal{M}} \langle \psi, \not{D}\chi \rangle \, \Omega = \int_{\mathcal{M}} \langle \not{D}\psi, \chi \rangle \, \Omega. \tag{A.17}$$

The Dirac operator gives the kinetic term $\overline{\psi}\not{D}\psi$. Thus, the Lagrangian for a pure Dirac spinor coupled to $\mathcal{M}$ is

$$\langle \psi, (\not{D} - m)\psi \rangle = \overline{\psi}\not{D}\psi - m\overline{\psi}\psi. \tag{A.18}$$

The corresponding equation of motion

$$(\not{D} - m)\psi = 0 \tag{A.19}$$

holds for ψ and $\overline{\psi}$ by the third property of (A.11) (as required by C symmetry).

We also define the Majorana inner product[1] [62, 1]

$$[\psi, \chi] := \langle C\psi, \chi \rangle \, . \tag{A.20}$$

Since $C^2 = 1$, we may write the Majorana product as

$$[\psi, \chi] = \psi\chi. \tag{A.21}$$

Note this is still a pairing on spinors with particular properties. The Majorana product is bilinear (linear in both arguments) since $\langle \cdot, \cdot \rangle$ and C are anti-linear. We assign $+$ to anti-commuting spinors and $-$ to commuting spinors. Then

$$[\psi, \chi] = \pm[\chi, \psi]. \tag{A.22}$$

Upon complex conjugation,

$$[\chi, \psi]^* = [C\psi, C\chi]. \tag{A.23}$$

This is independent of the sign and hence the commuting properties of the spinors. The Dirac Lagrangian (A.18) may be equivalently written in terms of the Majorana product. In addition, the Majorana product gives a new term

$$\frac{1}{2}M\psi\psi + \text{c.c.} \tag{A.24}$$

[1] Similar notation appears in the book of Srednicki [122] and supersymmetry (see [74, 95]).

where M is a mass parameter (independent from the Dirac mass m). This term breaks $U(1)$ due to the bilinearity of $[\cdot, \cdot]$. The expression $\psi\psi$ is not real due to (A.23) so the complex conjugate +c.c. is needed to give a real term. Equation (A.24) is the Majorana mass term, which has consequences for beyond-SM neutrino physics as we see in chapter 4. Note spinors with Majorana mass terms ($M \neq 0$) are generally distinct from Majorana spinors ($\overline{\psi} = \psi$).

A.2 Euclidean spinors

The manifold $\mathcal{M}$ remains unchanged. In this formalism, several data are defined either by analogy with or directly from the Lorentzian case. For example, the Euclidean versions of the gamma matrices are the anti-Hermitian matrices

$$\begin{aligned} \gamma_E^0 &= i\gamma^0, \\ \gamma_E^j &= \gamma^j \end{aligned} \tag{A.25}$$

where again $j \in \{1, 2, 3\}$. These matrices γ_E^a generate the Clifford algebra

$$\{\gamma_E^a, \gamma_E^b\} = -2\delta^{ab}. \tag{A.26}$$

The Euclidean chirality operator is γ_M from (A.3). This gives similar decompositions of Euclidean spinors in terms of chiral spinors as in the Lorentzian.

There is a fundamental Euclidean frame $e_a = e_a^\mu \partial_\mu$ and a dual tetrad $e^a = e_\mu^a dx^\mu$ which determine a Euclidean metric and its inverse as

$$\begin{aligned} g_E &= \delta_{ab} e^a \otimes e^b, \\ g_E^{-1} &= \delta^{ab} e_a \otimes e_b. \end{aligned} \tag{A.27}$$

The tetrad gives the Euclidean volume form as

$$\Omega_E = e^0 \wedge e^1 \wedge e^2 \wedge e^3 \tag{A.28}$$

where we again fix the positive orientation.

One has the same spin connection ∇ but a different action of it on a Euclidean Dirac spinor φ: for a vector field X,

$$\nabla_X \varphi = X\varphi + \omega_X{}^b{}_c S_E{}^c{}_b \varphi \tag{A.29}$$

where there is an analogous definition of $\omega_X{}^a_b$ in terms of a matrix of Euclidean 1-forms ω^a_b; and the generators for the Euclidean spinor representation are given as

$$S_E{}^b_a = \frac{1}{4}\delta_{ac}[\gamma^b_E, \gamma^c_E]. \tag{A.30}$$

The Euclidean Dirac operator is defined in terms of (A.29) as

$$\not{D}_E = \gamma^a_E \nabla_{e_a}, \tag{A.31}$$

which anti-commutes with γ_M.

The charge conjugation operator C_E in the Euclidean is anti-linear and satisfies

$$\begin{aligned} C_E^2 &= -1, \\ C_E\gamma_M &= \gamma_M C_E, \\ C_E\not{D}_E &= \not{D}_E C_E. \end{aligned} \tag{A.32}$$

These are similar to the Lorentzian identities (A.11) but have some signs flipped. The positive sign in the second equation of (A.32) implies spinor chirality does not change under the action of C_E, which is different to the Lorentzian case. The third equation of (A.32) corresponds to the Lorentzian version so that C_E implements the Euclidean charge conjugation symmetry.

Given Euclidean Dirac spinors φ and ζ, the Euclidean Dirac inner product is

$$\langle \varphi, \zeta \rangle_E = \varphi^\dagger \cdot_{\mathbb{C}^4} \zeta. \tag{A.33}$$

As with the Lorentzian Dirac product, this is sesquilinear. The self-adjoint property of the Dirac operator holds similarly in the Euclidean case in that (neglecting torsion):

$$\int_{\mathcal{M}} \langle \varphi, \not{D}_E\zeta \rangle_E \, \Omega_E = \int_{\mathcal{M}} \langle \not{D}_E\varphi, \zeta \rangle_E \, \Omega_E. \tag{A.34}$$

We remark that the operators $\not{D}_E$, γ_M and C_E as well as the inner product $\int_{\mathcal{M}} \langle \cdot, \cdot \rangle_E \, \Omega_E$ are indicative of a structure in Connes' framework. This applies to the whole manifold $\mathcal{M}$ given the Euclidean structure in this sub-section, and more detail on this will be given in chapter 4.

The Euclidean Majorana product [1] is (A.20) given also that

$$[\varphi, \zeta] = \langle C_E\varphi, \zeta \rangle_E . \tag{A.35}$$

The Lagrangian terms for Euclidean spinors are similar to those in the Lorentzian.

Appendix B

Complex symmetric matrices

The first topic is diagonalisation. For this, we follow [124]. As in the main text, all matrices are 3×3 matrices (i.e. 3 generations). The diagonalisation of a complex symmetric matrix M is

$$M = U_M \mathrm{diag}(\{M_p\}) U_M^T \tag{B.1}$$

for unitary U_M and $M_p > 0$. The M_p are not eigenvalues since U_M is not an eigenbasis. For the Hermitian matrix $|M|^2 := M^* M$, the above result gives

$$|M|^2 = U_M^* \mathrm{diag}\Big(\{M_p^2\}\Big) U_M^T. \tag{B.2}$$

In this case, the M_p^2 are positive and do correspond to a genuine eigenbasis U_M. Thus, we will anyway apply the name eigenvalue to M_p (as elsewhere e.g. [52]).

The next results are used in the right-handed neutrino induced effective theory.

Lemma B.0.1. *Let $c := tr|M|^2$ and $d := tr|M|^4$. Then*

$$\begin{aligned} c &= \sum_p M_p^2, \\ d &= \sum_p M_p^4. \end{aligned} \tag{B.3}$$

Proof. Both of these follow quickly from cyclic permutation of the trace and the unitarity of U_M. □

A corollary of the above lemma is that $c, d > 0$.

Lemma B.0.2. *Let* $r := \frac{c^2}{d}$*. Then*

$$1 < r \leq 3. \tag{B.4}$$

Proof. The three-variable HM-GM-AM-QM inequality (see [125]) immediately gives

$$c^2 \leq 3d. \tag{B.5}$$

Also, we can write $c^2 = \sum_{p,q}(M_p M_q)^2$. We observe that

$$\sum_{p,q}(M_p M_q)^2 > \sum_p M_p^4 \tag{B.6}$$

as the non-diagonal remainder terms are positive since $M_p > 0$. Thus

$$c^2 > d. \tag{B.7}$$

Joining (B.5) with (B.7) and eliminating d gives the claim. □

Remark. *The upper limit* $r = 3$ *corresponds to the degenerate case* $M_1 = M_2 = M_3$.

Remark. *Lemma B.0.2 is also given by results in [43], except here (B.7) gives a strict lower bound.*

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