Topics in half integral weight modular forms

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To my parents

Abstract

This thesis is divided in three independent chapters. In chapter 1, we prove some explicit results concerning images under Shimura's map, the methods are elementary. In chapter 2, we provide results on two kernels of half integral weight. Chapter 3 is an exposition to the results obtained as part of the collaboration with Wissam Raji, Larry Rolen and one of my supervisors Nikolaos Diamantis [1], where we study periods in the half integral weight setting.

Acknowledgements

Four people are of particular mention, and would like now to take the time to acknowledge them, for I doubt this thesis would have been finished were it not for their help.

I feel obliged to extend appreciations to both of my supervisors Fredrick and Nikos for their mathematical help in assisting my unending stream of silly questions, doubts and queries. Your constant optimism has been of great confidence.

Lastly, I am grateful my parents for their unending reassuring support. I would like to dedicate this thesis to both of you.

Contents

Introduction

This thesis is divided into three distinct and independent chapters. These can be described as 1) explicit computation of Shimura images, 2) analytic properties of the Cohen kernel $\mathcal{C}_{k/2}$ and of the double Eisenstein series $E_{k/2}$; and finally, 3) period polynomials for half integral weight modular forms.

In Chapter 1, we consider the product of an integral weight cusp form against a theta series and develop a method to compute its Shimura image by explicitly giving its coefficients. The culmination of this chapter is Theorem 1.2.2. The proof relies crucially on Lemma 1.1.15 which relies on an idea that can be traced back to Selberg. The methods of this chapter are completely elementary and do not use much beyond Möbius inversion. In addition to adding this novel tool to our understanding, this helps recontextualise results that were previously scattered in the literature. The author believes that the full power of this tool has not yet been realised. In this direction, I give 3 open problems in §1.4. We now give a brief idea of the main theorem. To each fundamental discriminant D, one can define the Shimura map σ_D. In fact for most purposes, it will suffice to look at the case σ_1 (see remark 1.1.9). Let ψ be a Dirichlet character modulo N_ψ of parity ν, that is, $\psi(-1) = (-1)^\nu$ with $\nu = 0$ or 1. Attached to ψ there is a theta series $h_\psi(\tau) = \sum_n \psi(n) n^\nu q^{n^2}$ which we multiply by a cusp form f of integral weight $\lambda - \nu$ (more precisely we multiply by a "scaled up" version of f, namely $f(N_\psi \tau)$). Then Theorem 1.2.2 gives the coefficients of the q-expansion of $\sigma_1(f(N_\psi\tau)h_\psi(\tau))$. In fact, when λ and ν have the same parity, we can determine this image exactly. We provide as well several examples as applications of our main result.

Chapter 2 is roughly split into two parts. The first, up to §2.5 contains the main theoretical content of the chapter, where the analytic properties of $C_{k/2}$ and $E_{k/2}$ are discussed. These two objects should be thought of as the integral weight analogs of the Cohen kernel and the double Eisenstein series that appeared in the papers [7] and [8]. We first give the definition of these two objects in Definition 2.2.1 and 2.1.1. The aims up to §2.5 are

(I) Prove absolute convergence

(II) Express the application of the kernel to a function f in terms of the L-functions of f

(III) Give a meromorphic continuation.

We tackle (I)-(III) for the Cohen kernel $\mathcal{C}_{k/2}$ first. This is because part (III) for $\mathcal{C}_{k/2}$ is used in the proof of part (II) for $E_{k/2}$. We first prove (I) for $\mathcal{C}_{k/2}$ on the vertical strip $\sigma \in \left(\frac{1}{4}, \frac{k}{2} - \frac{1}{4}\right)$. On this region, when σ is fixed, we show that the Cohen kernel is a cusp form (Corollary 2.2.9). Actually, the region of absolute convergence can be improved to a left plane $\sigma < \frac{k}{4} - 1$ by considering the convergence of the Hurwitz zeta function $\sum_n \frac{1}{(\tau+n)^s}$ (Proposition 2.2.10). By a typical unfolding argument, we show part (II) in Proposition 2.2.11 and thence deduce (III) in Corollary 2.2.12. Moving on to $E_{k/2}$, part (I) is proved for s lying in the vertical strip $(4, k-4)$ and w lying on a left plane. On the other hand, part (II) involves a rather technical use of lifts and epsilon factors, and as mentioned before, uses part (III) for $\mathcal{C}_{k/2}$. This culminates in Theorem 2.5.2 and part (III) follows from this. This concludes the theoretical part of Chapter 2.

The next part, is separate and independent of work up to §2.5. We present two supplementary results. The first result is inspired by work of Köhnen [15], we give a general construction of how to compute the Shintani lift of a general kernel in Lemma 2.6.4. Then we apply this to the integral weight double

Eisenstein series $E_{2\lambda}$ introduced in [8]. This is Theorem 2.6.1. The second result involves a half integral weight analogue of Zagier's appendix in [17]. To our knowledge, this appears to be new. With these two supplementary results, we conclude Chapter 2.

Chapter 3 is the culmination of joint work with W. Raji, L. Rolen and one of my supervisors N. Diamantis. For every half integral weight cusp form f on the group $\Gamma_0^*(4N)$, we define a polynomial $P_a(\tau)$ and show that (just as in the integral weight case) this polynomial encodes the values of the L-function of f within its coefficients. In §3.1 we use this polynomial to construct a parabolic cocycle $\pi_f \in H^1_{par}(\mathbf{PSL}_2(\mathbb{Z}), I_\lambda)$. The main theorem in this chapter is Theorem 3.1.4, where for each cusp form f of (half integral) weight $k/2$ on $\Gamma_0^*(4N)$, we show there exists a modular form g of weight λ on $\Gamma_0^*(4)$ satisfying a relation between the L-values of f and those of g. The map $f \mapsto g$ can be thought of as a "lift" and in section §3.2 we give an explicit way of expressing g using only the information given by f.

Background

We begin by encouraging the reader to refer to Appendix A of all the notation used in this thesis. Here we mention implicitly, always and throughout, $k = 2\lambda + 1$, where λ is a non-negative integer. We will often write $k/2 = \lambda + 1/2$ for the weight of a modular form in question. We give an overview of the properties and known facts that we will need for the next three chapters. We mostly follow the exposition in [24] and [26], especially §13 of [26] for the section on Hecke operators.

0.1 Preliminaries

Given a discrete subgroup Γ of $\mathbf{SL}_2(\mathbb{R})$ the projection of Γ onto $\mathbf{PSL}_2(\mathbb{R})$ acts discontinuously on $\mathfrak{H}$. In this thesis, we will only discuss the discrete congruence subgroups $\Gamma_0(4N)$, $\Gamma_0^*(4N)$ and Γ^ϑ. These groups are defined as follows.

Definition 0.1.1. (Congruence subgroup)

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{SL}_2(\mathbb{Z}) : \ c \equiv 0 \pmod{N} \right\}.$$

For a fixed integer $N \geq 1$ we denote

$$W_{4N} := \begin{pmatrix} 0 & -1/2\sqrt{N} \\ 2\sqrt{N} & 0 \end{pmatrix} \qquad T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Definition 0.1.2. $\Gamma_0^*(4N)$ is the subgroup of $\mathbf{SL}_2(\mathbb{R})$ generated by $\Gamma_0(4N)$ and W_{4N}.

To aid in exposition, we shall slightly abuse notation by writing γ for its image $\overline{\gamma}$ in $\mathbf{PSL}_2(\mathbb{R})$ with the understanding that $\gamma = -\gamma$. For $N = 1$, it is known (cf. [3] p26) that T and $W_4TW_4^{-1} = \left(\begin{smallmatrix} 1 & \\ -4 & 1 \end{smallmatrix}\right)$ generate the image of $\Gamma_0(4)$ in $\mathbf{PSL}_2(\mathbb{R})$. Given this, it follows that $\Gamma_0^*(4) = \langle T, W_4 \rangle$. For $N > 1$, the finite index subgroup $\langle T, W_{4N} \rangle$ need not generate all of $\Gamma_0^*(4)$. We mention that $\Gamma_0^*(4N)$ is the normaliser of $\Gamma_0(4N)$ in $\mathbf{PSL}_2(\mathbb{R})$. Explicitly, the action of W_{4N} is defined to be

$$W_{4N}f := (f|_{k/2}W_{4N})(z) := (-2i\sqrt{N}z)^{-k/2}f(-1/(4Nz)). \tag{0.1.1}$$

We define the theta group Γ^{ϑ} as the group generated by the matrices $T^2 = \left(\begin{smallmatrix} 1 & 2 \\ & 1 \end{smallmatrix}\right)$ and $S = \left(\begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix}\right)$.

Quadratic residue symbol

Fix $a \in \mathbb{Z}$ and an odd $b \in \mathbb{Z} \setminus \{0\}$. Then the Kronecker extension of the Legendre symbol $\left(\frac{a}{b}\right)$ enjoys the properties

1. $\gcd(a,b) \neq 1 \Rightarrow \left(\frac{a}{b}\right) = 0$.

2. If b is prime, then
$$\left(\frac{a}{b}\right) = \#\{x(\mathrm{mod} b) : x^2 \equiv a \mod b\} - 1.$$

3. If $b > 0$ then the character $\chi(a) = \left(\frac{a}{b}\right)$ has modulus b.

4. If $a \neq 0$ then the character $\psi(b) = \left(\frac{a}{b}\right)$ has modulus dividing $4a$ and conductor equal to the minimal r such that $\mathbb{Q}(\sqrt{a}) \subseteq \mathbb{Q}(\zeta_r)$.

5. $\left(\frac{a}{-1}\right) = \mathrm{sgn}(a), \quad a \neq 0$.

6. $\left(\frac{0}{\pm 1}\right) = 1$.

This is the notation found in [24] p442 and is not conventional. In addition, one has
$$\left(\frac{a}{b}\right) = (a,b)_\infty \left(\frac{a}{|b|}\right),$$
where $(a,b)_\infty$ is the Hilbert symbol (for a definition of this see A).

0.1.1 Multiplier systems

We are now going to define a function σ. To do this, fix for now an arbitrary $\tau \in \mathfrak{H}$. Let Γ be a Fuchsian group and consider the function $\sigma : \Gamma \times \Gamma \to \mathbb{C}$ defined by

$$\sigma(\gamma_1, \gamma_2) := \frac{j(\gamma_1, \gamma_2\tau)^{1/2} j(\gamma_2, \tau)^{1/2}}{j(\gamma_1\gamma_2, \tau)^{1/2}}. \tag{0.1.2}$$

On the right hand side we define $z^{1/2} = e^{\log(z)/2}$, where the implied logarithm is taken along its principal branch, so that $-\pi < \arg(z) \leq \pi$. It is well known that the value of $\sigma(\gamma_1, \gamma_2)$ is independent of the choice of τ. We denote by $\mathbb{T} = \{z \in \mathbb{C} : \ |z| = 1\}$.

Definition 0.1.3 (Multiplier system of weight 1/2)**.** A map $v : \Gamma \to \mathbb{T}$ satisfying

1. $v(\gamma_1\gamma_2) = \sigma(\gamma_1, \gamma_2) v(\gamma_1) v(\gamma_2)$
2. $v(-I) = -i$.

is called a *multiplier system of weight* 1/2.

Example 0.1.4. The theta multiplier

$$v_\theta(\gamma) := \varepsilon_d^{-1} \left(\frac{c}{d}\right)$$

is a multiplier system for $\Gamma_0(4)$ of weight 1/2. Here ε_d takes the value 1 or i depending on whether $d \equiv 1 \pmod 4$ or $d \equiv 3 \pmod 4$.

We remark that the square of v_θ is the non-trivial character $\chi_{1,4}$ modulo 4. This next example involves a multiplier whose image lies in μ_8 the cyclic subgroup of $\mathbb{T}$ generated by the eighth root of unity $\zeta_8 = e^{2\pi i/8}$.

Example 0.1.5. The map $v_\theta^* : \Gamma_0^*(4) \to \mu_8$ given by sending $T \mapsto 1$ and $W_4 \mapsto \zeta_8^{-1}$ is a multiplier system for $\Gamma_0^*(4)$ that agrees with v_θ on $\Gamma_0(4)$. This is a consequence of $\Gamma_0^*(4) = \langle T, W_4 \rangle$ and the transformation formulas for the Jacobi theta function

$$\theta(\tau) = \sum_{n \in \mathbb{Z}} q^{n^2} = 1 + 2q + 2q^4 + 2q^9 + 2q^{16} + O(q^{25}),$$

namely

$$\theta(\tau + 1) = \theta(\tau) \qquad \theta(-1/4\tau) = \sqrt{-2i\tau}\,\theta(\tau) = \zeta_8^{-1}\sqrt{2\tau}\,\theta(\tau).$$

In general, given a multiplier system v on Γ of weight 1/2, we can construct

$$\mathcal{J}_v(\gamma, \tau) := v(\gamma)(c_\gamma\tau + d_\gamma)^{1/2}, \qquad \gamma = \left(\begin{smallmatrix} * & * \\ c_\gamma & d_\gamma \end{smallmatrix}\right) \in \Gamma, \ \ \tau \in \mathfrak{H}.$$

Later on, in section 0.2.2 we will specify v to v_θ^* from the above example. We remind the reader that $k = 2\lambda + 1$ is always an odd integer. If v is a multiplier system of weight $1/2$, then for any $k \geq 1$ we call v^k a multiplier system of weight $k/2$.

Definition 0.1.6 (Slash action, half integral weight)**.** Let v be a multiplier system of weight $k/2$ for Γ. Given a function $f : \mathfrak{H} \to \mathbb{C}$ we set

$$f|_{k/2,v}\gamma := v(\gamma)^k (c\tau + d)^{-k/2} f(\gamma\tau), \qquad \gamma = \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \in \Gamma.$$

By $z^{k/2}$ we understand $e^{k\log(z)/2} = (z^{1/2})^k$. When $v = v_\theta^*$ of example 0.1.5, we will suppress the v and just write $f|_{k/2}\gamma$.

0.2 Covers and lifts

0.2.1 The quadruple cover

We follow the exposition of [24]. In general, given any group G and an abelian group C, one can consider C-extensions $\mathcal{G}$ of G; namely exact sequences

$$0 \to C \to \mathcal{G} \to G \to 0.$$

Such exact sequences are said to be central if the image of C in $\mathcal{G}$ lies entirely in the centre of $\mathcal{G}$. This will be used in the sequel in the case when $G = \mathbf{GL}_2^+(\mathbb{R})$ and $C = \mu_4$, because (informally) the correct object capturing the level structure of a half integral weight modular form will be a quadruple cover of a congruence subgroup (see §0.2.2 for a more precise statement). We denote $\mathrm{Hol}(\mathfrak{H})$ the set of holomorphic functions on the upper half plane. We define set $\mathcal{G}$ to be

$$\left\{(\gamma, \phi) \in G \times \mathrm{Hol}(\mathfrak{H}) : \ \phi(\tau)^2 = \epsilon(\det\gamma)^{-1/2}(c\tau + d), \, \forall \tau \in \mathfrak{H} \ \epsilon = \pm 1\right\}.$$

We equip $\mathcal{G}$ with a product defined by

$$(\gamma_1, \phi_1) \star (\gamma_2, \phi_2) := (\gamma_1\gamma_2, \phi_3),$$

where $\phi_3(\tau) = \phi_1(\gamma_2\tau)\phi_2(\tau)$. We refer to $\mathcal{G}$ as the *quadruple cover* of $\mathbf{GL}_2^+(\mathbb{R})$. The reason for the quadruple name is due to the fact that $\phi(\tau)(\det\gamma)^{1/4}/(c\tau + d)^{1/2}$ lies in μ_4, namely $\sqrt{\epsilon} \in \{1, i\}$ is a fourth root of unity. There is a projection map proj : $\mathcal{G} \to \mathbf{GL}_2^+(\mathbb{R})$ given by $\mathrm{proj}(\gamma, \phi) = \gamma$ and an exact sequence

$$0 \to \mu_4 \to \mathcal{G} \to \mathbf{GL}_2^+(\mathbb{R}) \to 0.$$

Definition 0.2.1. The slash action for an element $g = (\gamma, \phi) \in \mathcal{G}$ is defined to be

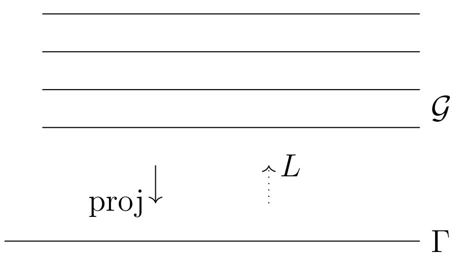
$$(f|_{k/2}g)(\tau) := \phi(\tau)^{-k} f(\gamma\tau).$$

For the rest of this chapter, we shall consider a subgroup H of $\{g \in \mathcal{G} |\det g = 1\}$. For $g \in \mathcal{G}$, by $\det g$ we mean $\det \mathrm{proj}(g)$. We shall say H is Fuchsian if it satisfies:

(i) $\mathrm{proj}(H)$ is a discrete subgroup of $\mathbf{SL}_2(\mathbb{R})$.

(ii) The projection map is a bijection of H onto its image and $\mathrm{proj}^{-1}(-I) = \{(-I, 1)\}$.

If $\Gamma = \mathrm{proj}(H)$, condition (ii) guarantees the existence of a section $L : \Gamma \to H$ satisfying $\mathrm{proj} \circ L = \mathrm{id}$ and $L(I) = (I, 1)$. We stress that L is not defined globally on $\mathbf{GL}_2^+(\mathbb{R})$ since L is not defined outside Γ. One should have the following picture in mind:



Because of (ii) we have

$$(\mathrm{proj} \circ L)(\alpha) = \alpha.$$

The action of Γ is determined by that of H, so that if one considers $\mathfrak{b}$ to be a cusp of Γ then the stabilizer of $\mathfrak{b}$ in H is

$$H_{\mathfrak{b}} = \{h \in H |\, h(\mathfrak{b}) = \mathfrak{b}\},$$

and there exists an element $R_{\mathfrak{b}}$ such that $H_{\mathfrak{b}}$ is either of the form $\langle R_{\mathfrak{b}} \rangle$ or of the form $\langle R_{\mathfrak{b}} \rangle \times \langle (-I, 1) \rangle$ when $-I \in \Gamma$. If we denote by $\gamma_{\mathfrak{b}}$ the matrix sending $\mathfrak{b} \mapsto \infty$ and we set $L(\gamma_{\mathfrak{b}}) = h_{\mathfrak{b}}$ then for some $w \in \mathbb{Z}$,

$$h_{\mathfrak{b}}^{-1} R_{\mathfrak{b}} h_{\mathfrak{b}} = \left(\pm \left(\begin{smallmatrix} 1 & w \\ & 1 \end{smallmatrix} \right), \epsilon \right), \qquad \epsilon \in \mu_4.$$

Without losing generality, $w > 0$ (otherwise swap $h_{\mathfrak{b}}$ by its inverse). Notice that ϵ only depends on $\gamma_{\mathfrak{b}}$ and not on the choice of $R_{\mathfrak{b}}$ nor the choice of $h_{\mathfrak{b}}$. For a function f on $\mathfrak{H}$ to be meromorphic at $\mathfrak{b}$ we mean that

$$f|_{k/2} h_{\mathfrak{b}} = \sum_{-\infty \ll n} c_n q^{(n+r)/w}$$

where $r \in \{0, 1/4, 1/2, 3/4\}$ is such that $\epsilon^k = e(r)$. We say that f is holomorphic at $\mathfrak{b}$ if for all $n < 0$, we have $c_n = 0$. We say that f vanishes at $\mathfrak{b}$ if it is holomorphic at $\mathfrak{b}$ and if $r = 0$ we have $c_0 = 0$. In the sequel we shall consider the case when $\mathrm{proj}(H) = \Gamma_0^*(4)$ where we can be more explicit, namely we can replace the condition of holomorphicity by imposing

$$f(\tau) = \sum_{n \geq 0} a_f(n) q^n \qquad f|_{k/2} L \left(\begin{smallmatrix} 1 & \\ -2 & 1 \end{smallmatrix} \right) = \sum_{n \geq 0} a_f^{(1/2)}(n) q^n$$

at each of the nonequivalent cusps $\{i\infty, 1/2\}$ of $\Gamma_0^*(4)$.

Definition 0.2.2 (Modular form weight $k/2$)**.** A function $f \in \mathrm{Hol}(\mathfrak{H})$ satisfying

- $f|_{k/2} h = f$ for all $h \in H$
- f is holomorphic at all cusps of $\Gamma = \mathrm{proj}(H)$

is said to be a modular form of weight $k/2$ on H.

This definition does not appear to depend on a choice of multiplier. The reason for this choice is that such a multiplier is determined provided we know the choice of lift L. In this way it appears implicitly. We say that f is a cusp form if it is modular and vanishes at all the cusps of Γ. We denote the space of modular forms of weight $k/2$ on H by $M_{k/2}(H)$ and the subspace of cusp forms by $S_{k/2}(H)$. For $\beta \in \mathbf{GL}_2^+(\mathbb{R})$, define

$$\Gamma_\beta := \beta^{-1} \Gamma \beta \qquad \text{and } \Gamma^\beta := \Gamma \cap \Gamma_\beta.$$

If $\beta \in \mathbf{GL}_2^+(\mathbb{R})$ satisfies $[\Gamma : \Gamma^\beta] < \infty$ and $[\Gamma_\beta : \Gamma^\beta] < \infty$ (i.e. Γ_β and Γ are commensurable) then the lift of an element of Γ^β gives rise to a multiplier v_β on Γ^β. More precisely, if $B \in \mathcal{G}$ satisfies $\mathrm{proj}(B) = \beta$, then

$$L(\beta\gamma\beta^{-1}) = BL(\gamma)B^{-1}(I, v_\beta(\gamma)) \qquad \gamma \in \Gamma^\beta. \tag{0.2.1}$$

We obtain a multiplier $v_\beta : \Gamma^\beta \to \mu\!\!\!\mu_4$. We set $K_{v_\beta} := \{\gamma \in \Gamma^\beta : \; v_\beta(\gamma) = 1\}$. Let $H_\beta = B^{-1}HB$ and

$$H^\beta := H \cap B^{-1}HB = H \cap H_\beta.$$

Definition 0.2.3 (Double coset action)**.** Let $\{\xi_\nu\}$ be a set of coset representatives for $H\backslash H\xi H$. Then

$$f|[H\xi H] := \det(\xi)^{k/4-1} \sum_\nu f|\xi_\nu.$$

Definition 0.2.4 (Commensurable)**.** We say H is commensurable with H' if and only if $H \cap H'$ is of finite index in H and also in H'.

Proposition 0.2.5 (Shimura, Prop.1.0 in [24])**.** *Suppose* $\beta \in \mathbf{GL}_2^+(\mathbb{R})$ *satisfies* $[\Gamma : \Gamma^\beta] < \infty$ *and* $[\Gamma_\beta : \Gamma^\beta] < \infty$*. If* $[\Gamma^\beta : K_{v_\beta}] < \infty$*, then*

(i) $H^\beta = L(K_{v_\beta})$*;*

(ii) H *is commensurable with* H_β*;*

(iii) if v_β^k *is non-trivial then* $f|HhH = 0$ *for all* $f \in M_{k/2}(H)$*.*

Proof (i) We have

$$\begin{aligned} h \in L(K_{v_\beta}) &\iff h = L(\gamma) \text{ with } v_\beta(\gamma) = 1 \\ &\iff BL(\gamma)B^{-1} = L(\beta\gamma\beta^{-1}) \in H \\ &\iff L(\gamma) = h \in H^\beta. \end{aligned}$$

(ii) Applying proj and using (i) we see

$$[H : H^\beta] = [\mathrm{proj}(H) : \mathrm{proj}(H^\beta)] = [\Gamma : K_{v_\beta}] \le [\Gamma : \Gamma^\beta][\Gamma^\beta : K_{v_\beta}] < \infty.$$

Similarly,

$$[H_\beta : H^\beta] = [\mathrm{proj}(H_\beta) : \mathrm{proj}(H^\beta)] = [\Gamma_\beta : K_{v_\beta}] \le [\Gamma_\beta : \Gamma^\beta][\Gamma^\beta : K_{v_\beta}] < \infty.$$

(iii) Let $\{\gamma_i\}$ and $\{\delta_j\}$ be a set of coset representatives for $\Gamma^\beta\backslash\Gamma$ and $K_{v_\beta}\backslash\Gamma^\beta$ respectively. Then, since L is a homomorphism on Γ,

$$\begin{aligned} H &= \coprod_i L(\Gamma^\beta)L(\gamma_i) \\ &= \coprod_{i,j} L(K_{v_\beta}\delta_j)L(\gamma_i) \\ &\overset{(i)}{=} \coprod_{i,j} H^\beta L(\delta_j)L(\gamma_i). \end{aligned} \tag{0.2.2}$$

Multiplying the last expression by HB,

$$\begin{aligned} HBH &= \coprod_{i,j} HB(H\cap H_\beta)L(\delta_j)L(\gamma_i) \\ &= \coprod_{i,j} HB(H_\beta)L(\delta_j)L(\gamma_i) \\ &= \coprod_{i,j} HB(B^{-1}HB)L(\delta_j)L(\gamma_i) \\ &= \coprod_{i.j} HBL(\delta_j)L(\gamma_i). \end{aligned}$$

Since

$$L(\beta\delta_j\beta^{-1})B(I,v(\delta_j)^{-1}) = BL(\delta_j),$$

and $L(\beta\delta_j\beta^{-1})$ lies in H and f is H invariant, we see

$$\begin{aligned} f|HBH &= \det B^{k/4-1}\sum_{i,j} f|BL(\delta_j)|L(\gamma_i) \\ &= \det(\beta)^{k/4-1}\sum_{i,j} f|L(\beta\delta_j\beta^{-1})B(I,v_\beta(\delta_j)^{-1})|L(\gamma_i) \\ &\overset{L(\beta\delta_j\beta^{-1})\in H}{=} \det(\beta)^{k/4-1}\sum_{i,j}(f|B)|(I,v_\beta(\delta_j))^{-1}|L(\gamma_i) \\ &= \det(\beta)^{k/4-1}\sum_{i,j} v_\beta(\delta_j)^k f|B|L(\gamma_i) \\ &= \det(\beta)^{k/4-1}\sum_i \underbrace{\left(\sum_j v_\beta(\delta_j)^k\right)}_{=:X} f|B|L(\gamma_i). \end{aligned}$$

We now claim that the value of the sum $X = 0$. By assumption, the multiplier system v_β^k is non-trivial, so $v_\beta(-I)^k = (-i)^k \neq 1$ as k is always odd. We claim that $\sigma(-I, \delta_j)$ is always equal to one for all j. This is because in general,

$$\sigma(\gamma_1, \gamma_2) = e^{is(\gamma_1,\gamma_2)/2}$$

where

$$s(\gamma_1, \gamma_2) = \arg j(\gamma_1, \gamma_2\tau) + \arg j(\gamma_2, \tau) - \arg j(\gamma_1\gamma_2, \tau).$$

Substituting $\gamma_1 = -I$ and $\gamma_2 = \beta$ gives

$$\begin{aligned} s(-I, \delta_j) &= \arg(-1) + (\arg j(\delta_j, \tau) - \arg j(-\delta_j, \tau)) \\ &= -\pi + \pi = 0. \end{aligned}$$

Thus, $\sigma(-I, \delta_j) = 1$ and therefore $v_\beta(-\delta_j) = v_\beta(-I)v_\beta(\delta_j)$. We have

$$v_\beta(-I)^k X = \sum_j v_\beta(-I)^k v_\beta(\delta_j)^k = \sum_j v_\beta(-\delta_j)^k = X.$$

This implies $X = 0$ as claimed. □

The conditions of this proposition are easy to satisfy. For instance, we have

Example 0.2.6. If β lies in the commensurator of $\Gamma_0^*(4)$ then the conditions of the proposition are met, since $[\Gamma_0^*(4) : K_{v_\beta}] \leq [\Gamma_0^*(4) : \Gamma_0^*(4) \cap \Gamma_1(4)] < \infty$.

Now let us specialize to the case when v_β is trivial on Γ^β. This motivates the following lemma.

Lemma 0.2.7. *If B is any element of $\mathcal{G}$ such that* $\mathrm{proj}(B) = \beta$, *then the following are equivalent*

(a) $L(\Gamma^\beta) = H^\beta$

(b) For all $\gamma \in \Gamma^\beta$, we have $L(\beta^{-1}\gamma\beta) = B^{-1}L(\gamma)B$

(c) v_β is trivial on Γ^β

(d) proj *maps HBH bijectively onto* $\Gamma\beta\Gamma$.

Moreover, if (a)-(d) hold then $HBH = \bigcup_i HL(\beta_i)$ is a disjoint union if and only if $\Gamma\beta\Gamma = \bigcup_i \Gamma\beta_i$ is a disjoint union.

Proof By (i) of the previous proposition, the RHS of (a) is $L(K_{v_\beta})$ and this shows $(a) \Leftrightarrow (b) \Leftrightarrow (c)$ because of equation (0.2.1). Now by (0.2.2) we have that $BL(\delta_j\gamma_i)$ is a set of representatives for $H\backslash HBH$. Since $\beta\gamma_i$ is a set of coset representatives for $\Gamma\backslash\Gamma\beta\Gamma$, it follows that $\text{proj}(BL(\delta_j\gamma_i)) = \beta\gamma_i$ and

$$\begin{aligned}(d) \quad &\Leftrightarrow \quad \text{proj}(L(\delta_i)L(\gamma_j)) = \gamma_j \\ &\Leftrightarrow \quad \text{proj}(L(\delta_i)) = 1 \quad \forall i \\ &\Leftrightarrow \quad \Gamma^\beta = K_{v_\beta} \\ &\Leftrightarrow \quad v_\beta = 1 \text{ on } \Gamma^\beta \Leftrightarrow (c)\end{aligned}$$

The remaining claim follows from this. □

0.2.2 The lift

We now consider the case $\Gamma = \Gamma_0^*(4N)$. To do this, we will choose a lift L. Refering back to Example 0.1.5, denote

$$v_\theta^*(\gamma) = \frac{\theta(\gamma\tau)}{(c_\gamma\tau + d_\gamma)^{1/2}\theta(\tau)}.$$

Set L to be the choice $\phi(\gamma,\tau) = \mathcal{J}(\gamma,\tau) = v_\theta^*(\gamma)(c\tau+d)^{1/2}$, namely

$$\begin{aligned}L : \Gamma &\to H \\ \gamma &\mapsto (\gamma, \mathcal{J})\end{aligned}$$

We point out as well that under $\gamma \mapsto L(\gamma)$,

$$f|_{k/2}L(\gamma) = f|_{k/2,v_\theta}\gamma := v_\theta(\gamma)^{-k}(c\tau+d)^{-k/2}f(\gamma\tau)$$

which in turn is the same as $\mathcal{J}(\gamma,\tau)^{-k}f(\gamma\tau)$. We denote $S_{k/2}(\Gamma) = S_{k/2}(H)$ the space of cusp forms of weight $k/2$ for Γ with multiplier system v_θ. For a character χ of Γ, we say that $f \in S_{k/2}(\Gamma,\chi)$ if the slash action in definition 0.2.2 can be replaced by

$$f|_{k/2}L(\gamma) = \chi(d)f, \quad \gamma = \left(\begin{smallmatrix} * & * \\ c & d\end{smallmatrix}\right) \in \Gamma.$$

We can even set $f|_{k/2,\chi}L(\gamma) := \chi(d)^{-1}f|_{k/2}L(\gamma)$ so that $f \in S_{k/2}(\Gamma,\chi)$ satisfies $f|_{k/2,\chi} = f$. Notice that when $\left(\begin{smallmatrix} * & * \\ c & d\end{smallmatrix}\right) = \gamma \in \Gamma_0(4)$, we have the explicit expression

$$v_\theta^*(\gamma) = v_\theta(\gamma) = \epsilon_d^{-1}\left(\frac{c}{d}\right)$$

as in example 0.1.4 so that by construction $L(\Gamma_0(4N))$ is a subgroup of H. We write $S_{k/2}(4N,\chi)$ to mean $S_{k/2}(L(\Gamma_0(4N)),\chi)$. Similarly, we will write $S_{k/2}(\Gamma_0^*(4N),\chi)$ to mean $S_{k/2}(H,\chi)$.

Extending the definition of $\mathcal{J}$

From now on, we shall loosely follow [26]. Recall that the definition of $\mathcal{J}(\gamma,z)$ is valid only for $\gamma\in\Gamma_0^*(4)$. In chapter 2 we shall need to extend the definition of $\mathcal{J}$ to cover cases such as diagonal matrices in $\mathbf{SL}_2(\mathbb{Q})$. To this end let P be the subgroup of upper triangular matrices in $\mathbf{SL}_2(\mathbb{Q})$. For $\beta\in P$ set

$$\tilde{\mathcal{J}}(\beta,\tau) := |d_\beta|^{1/2}.$$

Define $\tilde{\mathcal{J}}(\beta\gamma,\tau) = |d_\beta|^{1/2}\mathcal{J}(\gamma,\tau)$ for $\beta\in P$ and $\gamma\in\Gamma_0^*(4)$. The restriction of $\tilde{\mathcal{J}}$ to $\Gamma_0^*(4)$ equals $\mathcal{J}$. Moreover $\tilde{\mathcal{J}}$ satisfies

$$\tilde{\mathcal{J}}(\beta\gamma\gamma',\tau) = \tilde{\mathcal{J}}(\beta,\tau)\tilde{\mathcal{J}}(\gamma,\gamma'\tau)\mathcal{J}(\gamma',\tau), \quad \beta\in P,\ \gamma\in P\Gamma_0^*(4),\ \gamma'\in\Gamma_0^*(4).$$

The transfer map

The purpose of this subsection is to define and prove a bijection from $\mathcal{G}$ to a new subgroup $\mathcal{G}_k$. This is achieved in Lemma 0.2.9. Here $\mathcal{G}_k$ is constructed in a similar way to $\mathcal{G}$ but only for matrices with determinant one. The "tradeoff" is that we must keep track of the weight so that the action of $\mathcal{G}_k$ on modular forms of weight $k/2$ agrees with that of $\mathcal{G}$. We assume throughout that $\psi\in\mathrm{Hol}(\mathfrak{H})$. We define $\mathcal{G}_k$ to be the set of pairs $(\alpha,\psi)\in\mathbf{SL}_2(\mathbb{Q})\times\mathrm{Hol}(\mathfrak{H})$ such that

$$\psi(\tau)^2 = \epsilon\tilde{\mathcal{J}}(\alpha,\tau)^k$$

with $\epsilon\in\{\pm1\}$ and $\mathcal{J}$ as above. Then $\mathcal{G}_k$ has a multiplication law

$$(\alpha_1,\psi_1)\star(\alpha_2,\psi_2) = (\alpha_1\alpha_2,\psi_1(\alpha_2\tau)\psi_2(\tau))$$

and the projection map $\mathrm{proj}:\mathcal{G}_k\to\mathbf{SL}_2(\mathbb{Q})$ is the map $\mathrm{proj}(\alpha,\psi)=\alpha$.

Definition 0.2.8. The slash action for $\xi = (\alpha, \psi) \in \mathcal{G}_k$ is

$$(f||\xi)(\tau) := \psi(\tau)^{-1} f(\alpha\tau).$$

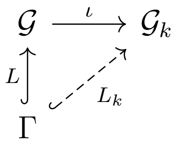
Lemma 0.2.9. *There exists an isomorphism of groups*

$$\begin{aligned} \iota : \mathcal{G} &\to \mathcal{G}_k \\ (\gamma, \phi) &\mapsto (\alpha, \psi) = \left(\begin{pmatrix} 1/\det\gamma & \\ & 1/\det\gamma \end{pmatrix} \gamma, \det(\gamma)^{k/2} \phi^k \right) \end{aligned}$$

that preserves actions, namely

$$f|_{k/2}(\gamma, \phi) = f||(\alpha, \psi).$$

The proof consists of unraveling the definitions. Indeed, $\gamma\tau = \alpha\tau$ is clear. After choosing the lift, $\phi(\tau) = \mathcal{J}(\alpha, \tau)/\sqrt{\det \alpha}$ and putting $\psi = \phi^k$ then we see $\phi(\tau)^{-k} f(\gamma\tau) = v(\alpha)^k (\det \alpha)^{k/2} \mathcal{J}(\alpha, \tau)^{-k} f(\alpha\tau) = \psi(\tau)^{-1} f(\alpha\tau)$. From this observation we see



hence we can obtain a new lift $L_k : \Gamma \to \mathcal{G}_k$. In particular because $\gamma \in \Gamma$ has $\det(\gamma) = 1$, if $L(\gamma) = (\gamma, \mathcal{J}(\gamma, \tau))$ then the diagram above gives $L_k(\gamma) = (\gamma, \mathcal{J}(\gamma, \tau)^k)$.

0.3 Hecke operators

0.3.1 The Hecke algebra

We summarize the Hecke theory for half integral weight modular forms. We follow the exposition in chapter IV §13 in [26]. We now fix H to be the lift of $\Gamma_0^*(4N)$ as in §0.2.2. Recall that the hyperbolic measure $d\mu(z) = \frac{dxdy}{y^2}$ on $\mathfrak{H}$ is invariant under the action of $\Gamma = \Gamma_0^*(4N)$.

Proposition 0.3.1. *For any $g \in \mathcal{G}$, assume gHg^{-1} is commensurable with H. Then*

$$|H \backslash HgH| = [H : H \cap g^{-1}Hg] = [H : H \cap gHg^{-1}] = |HgH/H|$$

and there exists a set of coset representatives $\{h_i\}$ such that

$$HgH = \coprod Hh_i = \coprod h_i H.$$

Proof Let $\Gamma = \mathrm{proj}(H)$ so that $H = L(\Gamma)$, and put $\gamma = \mathrm{proj}(g)$. Then, since μ is Γ-invariant, acting by γ gives

$$\begin{aligned} \mu(\Gamma\backslash\mathfrak{H})[\Gamma : \Gamma\cap\gamma^{-1}\Gamma\gamma] &= \mu(\Gamma\cap\gamma^{-1}\Gamma\gamma\backslash\mathfrak{H}) \\ &= \mu(\Gamma\cap\gamma\Gamma\gamma^{-1}\backslash\mathfrak{H}) \\ &= \mu(\Gamma\backslash\mathfrak{H})[\Gamma : \Gamma\cap\gamma\Gamma\gamma^{-1}]. \end{aligned}$$

Giving $[\Gamma : \Gamma\cap\gamma\Gamma\gamma^{-1}] = [\Gamma : \Gamma\cap\gamma^{-1}\Gamma\gamma]$ and therefore $[H : H\cap g^{-1}Hg] = [H : H\cap gHg^{-1}]$. This proves the middle inequality, for the other two, apply either the bijection $h_i \mapsto g^{-1}h_ig$ or $h_i \mapsto gh_ig^{-1}$. To prove the claim about the existence of $\{h_i\}$, suppose we have

$$HgH = \coprod H\eta_i' = \coprod \eta_i H.$$

Since $\eta_i' \in H\eta_iH$ we can find $\alpha_i, \beta_i \in H$ such that $\eta_i' = \alpha_i\eta_i\beta_i$. Then, setting $h_i = \alpha_i^{-1}\eta_i'$ we have

$$\coprod Hh_i = \coprod H\alpha_i^{-1}\eta_i' = \coprod H\eta_i' = \coprod \eta_i H = \coprod \eta_i\beta_i = \coprod h_i H.$$

□

We now fix a subset $\Xi \subset \mathcal{G}$ containing H. By Definition 0.2.3, each ξ defines a double coset $H\xi H$ and in turn an automorphism of $M_{k/2}(H)$ via $f \mapsto f|[H\xi H]$. We denote by $\mathfrak{R}(H, \Xi)$ the set of formal finite sums of the form

$$\sum_{\xi\in\Xi} c_\xi H\xi H \qquad c_\xi \in \mathbb{Q}.$$

For a subgroup $\Delta \subset \mathbf{GL}_2(\mathbb{Q})$, we can define in a similar way $\mathfrak{R}(\Gamma, \Delta)$ to be the set of formal finite sums $\sum_{\delta\in\Delta} c_\delta\Gamma\delta\Gamma$ with $c_\delta \in \mathbb{Q}$. One should think of elements in $\mathfrak{R}(\Gamma, \Delta)$ as behaving well under conjugation, for instance with our usual notation of $\Gamma_\alpha = \alpha^{-1}\Gamma\alpha$, we obtain

Proposition 0.3.2. *Let Γ be a congruence subgroup. Let $\alpha \in M_2(\mathbb{Z})$ have positive determinant. Then, the map sending $\mathfrak{R}(\Gamma, \Delta)$ to $\mathfrak{R}(\Gamma_\alpha, \alpha^{-1}\Delta\alpha)$ is an isomorphism.*

Proof If $\{\gamma_i\}$ is a set of coset representatives for $\Gamma^\delta\backslash\Gamma$ (here $\Gamma^\delta = \Gamma\cap\Gamma_\delta$), then $\{\delta\gamma_i\}$ is a set of coset representatives for $\Gamma\backslash\Gamma\delta\Gamma$. Under $\gamma\mapsto\alpha^{-1}\gamma\alpha$, Γ^δ has image

$$\alpha^{-1}(\delta^{-1}\Gamma\delta\cap\Gamma)\alpha = (\alpha^{-1}\delta\alpha)^{-1}\Gamma_\alpha(\alpha^{-1}\delta\alpha)\cap\Gamma_\alpha = (\Gamma_\alpha)^{\alpha^{-1}\delta\alpha}.$$

But this means that $\Gamma^\delta\backslash\Gamma \to (\Gamma_\alpha)^{\alpha^{-1}\delta\alpha}\backslash\Gamma_\alpha$ bijects and so does $\Gamma\backslash\Gamma\delta\Gamma \to \Gamma_\alpha\backslash\Gamma_\alpha(\alpha^{-1}\delta\alpha)\Gamma_\alpha$. The isomorphsim we seek is $\Gamma\delta\Gamma\mapsto\Gamma_\alpha(\alpha^{-1}\delta\alpha)\Gamma_\alpha$. □

If $\{\alpha_i\}$ and $\{\beta_j\}$ denote a set of representatives for $H\backslash H\alpha_0 H$ and $H\backslash H\beta_0 H$ respectively then

$$H\alpha_0 H\beta_0 H = \bigcup H\alpha_i\beta_j = \coprod H\xi H$$

where in the last equality, the finite union is chosen to be disjoint for some choice of representatives $\{\xi\}$. We define a multiplication on $\mathfrak{R}(H,\Xi)$ as

$$\begin{aligned} \mathfrak{R}(H,\Xi)\times\mathfrak{R}(H,\Xi) &\to \mathfrak{R}(H,\Xi) \\ (H\alpha_0 H, H\beta_0 H) &\mapsto \sum c_\xi H\xi H \end{aligned}$$

where c_ξ is the number of (i,j) such that $H\alpha_i\beta_j = H\xi$. It can be verified (see [26] p117) that this map is independent of the choice of representatives $\{\alpha_i\},\{\beta_j\}$ and $\{\xi\}$; so that $\mathfrak{R}(H,\Xi)$ now becomes an associative algebra. We call this algebra the *Hecke algebra*.

In general, $\mathfrak{R}(H,\Xi)$ acts on modular forms in the following way. Given an element $H\xi H\in\mathfrak{R}(H,\Xi)$, we let $\{\xi_\nu\}_\nu$ be a set of coset representatives of $H\backslash H\xi H$ so that by Lemma 0.2.7 part (d), if we set $\alpha_\nu = \mathrm{proj}(\xi_\nu)$ then $\{\alpha_\nu\}_\nu$ becomes a set of coset representatives of $\Gamma\backslash\Gamma\alpha\Gamma$. This gives us a well defined isomorphism

$$\begin{aligned} \mathfrak{R}(H,\Xi) &\to \mathfrak{R}(\Gamma,\mathrm{proj}(\Xi)) \\ H\xi H &\mapsto \Gamma\alpha\Gamma. \end{aligned}$$

We stress that this isomorphism depends on the choice of the lift L. We mention here that the slash action on the right-hand side of Def.0.2.3 is the one with respect to $\mathcal{G}$. Of course, if $\Xi' \subset \mathcal{G}_k$, one can analogously the algebra of formal finite rational sums of double cosets as before. To stress that we are working over $\mathcal{G}_k$, we shall denote $\mathfrak{R}_k(H, \Xi')$ to be this algebra, with the subscript reminding us that the slash action is with respect to $\mathcal{G}_k$. In this case we have

$$f||[H\xi' H] := (d_{\xi'})^{k/2-2} \sum_{\nu} f||\xi'_\nu.$$

Here $d_{\xi'}$ is the bottom right entry of $\alpha' = \mathrm{proj}(\xi')$, which is always an integer. We shall exclusively use $||$ to denote the action within $\mathcal{G}_k$ (as defined in §0.2.2). This gives a map $\mathfrak{R}_k(H, \Xi') \to \mathfrak{R}_k(\Gamma, \mathrm{proj}(\Xi'))$.

Using the isomorphism of Lemma 0.2.9, if Ξ' denotes the image of Ξ under $\iota : \mathcal{G} \to \mathcal{G}_k$, then

$$H\xi H \mapsto H\iota(\xi)H$$

extends to a map $\mathfrak{R}(H, \Xi) \to \mathfrak{R}_k(H, \Xi')$. All of this can be summarized in the commutative diagram below.

$$\begin{array}{ccc} \mathfrak{R}(H,\Xi) & \xrightarrow{\quad\iota\quad} & \mathfrak{R}_k(H,\Xi') \\ \downarrow & & \downarrow \\ \mathfrak{R}(\Gamma, \mathrm{proj}(\Xi)) & \longrightarrow & \mathfrak{R}_k(\Gamma, \mathrm{proj}(\Xi')) \end{array} \tag{0.3.1}$$

Let us now specialize to the case $\Gamma = \Gamma_0^*(4)$ with the choice of lift as in §0.2.2. Let P be the subgroup of $\mathbf{SL}_2(\mathbb{Q})$ of upper triangular matrices. Let P_n be the subgroup of P consisting of matrices of the form $\left(\begin{smallmatrix} n & * \\ 0 & 1/n \end{smallmatrix}\right)$ for some $n \in \mathbb{Z} \setminus \{0\}$. The set Ξ_{n^2} is the subset of $\mathcal{G}$ consisting of elements (γ, ϕ) with $\gamma \in M_2(\mathbb{Z})$ and $\det \gamma = n^2$.

Lemma 0.3.3. *The following hold:*

(i) $\mathbf{SL}_2(\mathbb{Q}) = P\Gamma^{\vartheta}$.

(ii) $\mathbf{SL}_2(\mathbb{Q}) = P\Gamma_0^*(4)$.

(iii) The map $\mathrm{proj}(\Xi_{n^2}) \to P_n\Gamma_0^*(4)$ *given by*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} n & t \\ 0 & 1/n \end{pmatrix} \begin{pmatrix} x & y \\ c & d \end{pmatrix}$$

is a bijection.

Proof In order to prove parts (i) and (ii) we first claim that $\mathbf{SL}_2(\mathbb{Q}) = P \cdot \mathbf{SL}_2(\mathbb{Z})$. Let $\beta \in \mathbf{SL}_2(\mathbb{Q})$ then (clearing denominators) the last row of β can be written as $q\mathbf{w}^T$ where $q \in \mathbb{Q}^\times$ and $\mathbf{w}$ is a primitive vector in $\mathbb{Z}^2$. Let $\gamma \in \mathbf{SL}_2(\mathbb{Z})$ be the unique matrix with bottom row $\mathbf{w}$. Then $\beta\gamma^{-1} = \left(\begin{smallmatrix} * & * \\ 0 & q \end{smallmatrix}\right) \in P$ showing that $\mathbf{SL}_2(\mathbb{Q}) = P \cdot \mathbf{SL}_2(\mathbb{Z})$ as claimed. With this in mind we can now prove parts (i) and (ii).

(i) Notice that the generators of Γ^ϑ are T^2 and S, but since $T^{-1} \in P$, $T = T^{-1}T^2$ lies in $P \cdot \Gamma^\vartheta$. Thus $P \cdot \Gamma^\vartheta = P \cdot \mathbf{SL}_2(\mathbb{Z}) = \mathbf{SL}_2(\mathbb{Q})$ as well.

(ii) Using the previous part and conjugating by $\begin{pmatrix} 1/2 & \\ & 1 \end{pmatrix}$ we see that

$$P\Gamma_0^*(4) = P \begin{pmatrix} 1/2 & \\ & 1 \end{pmatrix} \Gamma^\vartheta \begin{pmatrix} 2 & \\ & 1 \end{pmatrix} = P\Gamma^\vartheta \begin{pmatrix} 2 & \\ & 1 \end{pmatrix} = \mathbf{SL}_2(\mathbb{Q}).$$

(iii) If $\det \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = n^2$ then $\left(\begin{smallmatrix} a/n & b/n \\ c/n & d/n \end{smallmatrix}\right)$ lies in $\mathbf{SL}_2(\mathbb{Q})$. By parts (i) and (ii), this matrix also lies in $P\Gamma_0^*(4)$. For bottom rows to agree, it must be of the form

$$\begin{pmatrix} a/n & b/n \\ c/n & d/n \end{pmatrix} = \begin{pmatrix} n & t \\ 0 & 1/n \end{pmatrix} \begin{pmatrix} x & y \\ c & d \end{pmatrix},$$

hence lie in $P_n\Gamma_0^*(4)$. Here $x, y \in \mathbb{Z}$ solve $xd - cy = 1$ and t satisfies both $tc = a/n - nx$ and $td = b/n - ny$.

□

0.3.2 Definition of the Hecke operator

We have seen in the previous section that the map ι induces an isomorphism between the Hecke algebras $\mathfrak{R}(H, \Xi)$ and $\mathfrak{R}_k(H, \Xi')$. Fix a congruence subgroup Γ and a choice of lift $L : \Gamma \hookrightarrow H$. In this section, we first construct an operator $\mathfrak{T}_m$ inside $\mathfrak{R}(H, \Xi)$, find its description in $\mathfrak{R}_k(H, \Xi')$ under ι. For now we maintain generality for Γ and L (and thence also H); later, we specialise to the case when $\Gamma = \Gamma_0^*(4)$ and $L(\gamma) = (\gamma, v_\theta^*(\gamma)\mathcal{J}(\gamma,\tau))$. Recall $\Xi = \bigcup_{n\in\mathbb{N}} \Xi_{n^2}$ where Ξ_{n^2} is the subset of $\mathcal{G}$ consisting of integral matrices of determinant n^2.

Definition 0.3.4. Let $m \in \mathbb{N}$. In $\mathfrak{R}(H,\Xi)$, the Hecke operator is defined to be

$$\mathfrak{T}_m := \left[H\left(\begin{pmatrix}1 & \\ & m^2\end{pmatrix}, \sqrt{m}\right)H\right].$$

Now since $\det \operatorname{diag}(1,m^2)^{k/2} = (m^2)^{k/2}$,

$$\iota\left(\begin{pmatrix}1 & \\ & m^2\end{pmatrix}, \sqrt{m}\right) = \left(\begin{pmatrix}1/m & \\ & m\end{pmatrix}, (m^2)^{k/2}(\sqrt{m})^{-k}\right).$$

It follows that the image of $\mathfrak{T}_m$ in $\mathfrak{R}_k(H,\Xi')$ is

$$\left[H\left(\begin{pmatrix}1/m & \\ & m\end{pmatrix}, m^k\right)H\right].$$

Informally, this means that the action of the a single element of the Hecke action corresponding to $m\alpha_m$ in $\mathcal{G}$ gets passed to the Hecke action corresponding to α_m in $\mathcal{G}_k$ under ι. More precisely, for $m \in \mathbb{Z}$ let $\alpha_m = \left(\begin{smallmatrix}1/m & 0\\ 0 & m\end{smallmatrix}\right)$, and set

$$\xi_m = L(\alpha_m) = (\alpha_m, m^{k/2}).$$

Writing $H\xi_m H = \coprod H\xi_\nu$, we see that for $f \in M_{k/2}(4N,\chi)$,

$$\mathfrak{T}_m f = m^{k/2-2}\sum_\nu \chi(d_{\alpha_\nu})^{-1} f|_{k/2}\alpha_\nu.$$

On the other hand, one can consider the element

$$\mathfrak{T}_m := H\left(m\alpha_m, m^{1/2}\right)H = H\left(\begin{pmatrix}1 & \\ & m^2\end{pmatrix}, m^{1/2}\right)H$$

so that for $f \in M_{k/2}(4N,\chi)$,

$$\mathfrak{T}_m f := \det(m\alpha_m)^{k/4-1}\sum_\nu \chi(ma_{\alpha_\nu}) f|_{k/2}\xi_\nu, \quad \alpha_\nu = \operatorname{proj}(\xi_\nu).$$

By construction, the action of both of these is the same. Such a $\mathfrak{T}_m$ is written $T^N_{\kappa,\chi}(m^2)$ in [24], and we call it the *Hecke operator*.

Lemma 0.3.5. *Suppose either $(m,n)=1$ or every prime factor of m divides N. Then $\mathfrak{T}_n\mathfrak{T}_m = \mathfrak{T}_{nm}$.*

For a proof see Lemma 13.6 in [26]. We now specialise $\Gamma = \Gamma_0^*(4N)$ with lift $L(\gamma) = (\gamma, v_\theta^*(\gamma)\mathcal{J}(\gamma,\tau))$. As in the integral case, for a prime p, we can describe the effect $\mathfrak{T}_p$ has on the coefficients $a_f(n)$ of $f = \sum a_f(n)q^n$:

Theorem 0.3.6. *Let $f = \sum_{n\geq 0} a_f(n)q^n \in M_{k/2}(4N,\chi)$. Then $\mathfrak{T}_p f = \sum b(n)q^n$ where*

$$b(n) = a_f(p^2n) + \chi(p)\left(\frac{-1}{p}\right)^\lambda \left(\frac{n}{p}\right) p^{\lambda-1}a_f(n) + \chi(p^2)p^{k-2}a_f(n/p^2).$$

Recall the convention that $a_f(n)$ vanishes when n is not an integer. We include a sketch proof of this theorem so that we explicitly present the representatives of $H\backslash H\xi_p H$. For a detailed proof, see either Theorem 1.7 in [24] or Theorem 13.9 in [26].

Proof [Shimura] We lift for each $0 \leq \nu < p^2$ and each $0 < h < p$ the representatives

$$\beta_\nu = \left(\begin{smallmatrix} 1/p & \nu/p \\ & p\end{smallmatrix}\right), \qquad \gamma_h = \left(\begin{smallmatrix} 1 & h/p \\ & 1\end{smallmatrix}\right), \qquad \delta = \left(\begin{smallmatrix} p & \\ & 1/p\end{smallmatrix}\right)$$

of $\Gamma\backslash\Gamma\left(\begin{smallmatrix} 1/p & \\ & p\end{smallmatrix}\right)\Gamma$ to the representatives

$$L(\beta_\nu) = (\beta_\nu, p^{k/2}), \quad L(\gamma_h) = \left(\gamma_h, \epsilon_p^{-1}\left(\frac{-h}{p}\right)\right), \quad L(\delta) = (\delta, p^{-k/2}) \quad (0.3.2)$$

of $H\backslash H\xi_p H$. This gives

$$\begin{aligned}
\mathfrak{T}_p f &= p^{k/2-2}\left(\sum_\nu \chi(pa_{\beta_\nu})f|L(\beta_\nu) + \sum_h \chi(pa_{\gamma_h})f|L(\gamma_h) + \chi(pa_\delta)f|\delta\right)\\
&= p^{-2}\sum_\nu f\left(\frac{\tau+\nu}{p^2}\right) + p^{k/2-2}\epsilon_p^{-1}\chi(p)\sum_h \left(\frac{-h}{p}\right) f(\tau + h/p)\\
&+ p^{k/2-2}p^{k/2-2}\chi(p^2)p^{k/2}f(p^2\tau)\\
&= p^{-2}\sum_{n\geq 0} a_f(n)q^{n/p^2}\sum_{\nu=0}^{p^2-1} e(n\nu/p^2)\\
&+ p^{k/2-2}\epsilon_p^{-1}\chi(p)\sum_{n\geq 0}a_f(n)q^n \sum_{h=1}^{p-1}\left(\frac{-h}{p}\right)e(nh/p)\\
&+ p^{k-2}\chi(p^2)\sum_{n\geq 0}a_f(n)q^{np^2}
\end{aligned}$$

To conclude, use Gauss sums and equate coefficients. □

The following two corollaries are jointly stated in Corollary 1.8 of [24]:

Corollary 0.3.7. *We have that* $a_{\mathfrak{T}_2 f}(n) = a_f(4n)$. *Moreover, if* f *is an eigenfunction for* $\mathfrak{T}_2$ *with eigenvalue* ω_2*, then*

$$\omega_2 a_f(4^m n) = a_f(4^{m+1}n),\ m \geq 1.$$

Corollary 0.3.8. *Let* $D \in \mathbb{N}$ *and* p *be prime and assume* $p^2 \nmid D$. *If* $\mathfrak{T}_p f = \omega_p f$ *then*

1. $\omega_p a_f(D) = a_f(p^2 D) + \chi(p) \left(\frac{-1}{p}\right)^\lambda \left(\frac{D}{p}\right) p^{\lambda-1} a_f(p)$

2. $\omega_p a_f(p^{2m} D) = a_f(p^{2m+2} D) + \chi(p^2) \left(\frac{-1}{p}\right)^\lambda p^{k-2} a_f(p^{2m-2} D)$

and formally,

$$\sum_{n\geq 1} \frac{a_f(Dn^2)}{n^s} = \sum_{p\nmid n} \frac{a_f(Dn^2)}{n^s} \frac{1-\chi(p)\left(\frac{-1}{p}\right)^\lambda \left(\frac{D}{p}\right) p^{\lambda-1-s}}{1-\omega_p p^{-s} + \chi(p)^2 p^{2\lambda-1-2s}}.$$

For odd k and $f, g \in S_{k/2}(\Gamma_0^*(4N), \chi)$ we define the Petersson scalar product as

$$\langle f, g\rangle := \int_{\Gamma_0^*(4N)\backslash\mathfrak{H}} f(z)\overline{g(z)} y^{k/4} \frac{dxdy}{y^2}. \tag{0.3.3}$$

Lemma 0.3.9. *If* $f, g \in S_{k/2}(\Gamma_0^*(4N), \chi)$ *then for all primes* $p \nmid 4N$,

$$\langle \mathfrak{T}_p f, g\rangle = \chi(p)^2 \langle f, \mathfrak{T}_p g\rangle.$$

Proof Notice that if $\{\xi_\nu\}$ is a set of representatives for $H\xi_p H$ then $\{\xi_\nu^{-1}\}$ is a set of representatives for $H\xi_p^{-1}H$. The top left entries a_ν and a'_ν of $p\text{proj}(\xi_\nu)$ and $p\text{proj}(\xi_\nu^{-1})$ are related via $a_\nu a'_\nu \equiv p^2 \mod 4N$. So $\chi(a_\nu)\chi(a'_\nu) = \chi(p)^2$ and

$$\begin{aligned} p^{2-k/2}\langle \mathfrak{T}_p f, g\rangle &= \langle f|H\xi_p H, g\rangle = \sum_\nu \chi(a_\nu)\langle f|\xi_\nu, g\rangle \\ &= \chi(p)^2 \langle f, \sum \chi(a'_\nu) g|\xi_\nu^{-1}\rangle = \chi(p)^2 \langle f, g|H\xi_p^{-1}H\rangle \\ &= \chi(p)^2 p^{2-k/2} \langle f, \mathfrak{T}_p g\rangle. \end{aligned}$$

□

Of course, when we are working over $\Gamma_0^*(4)$ we have $\chi(p)^2 = 1$ for all characters so the above simplifies, when p is odd to

$$\langle \mathfrak{T}_p f, g\rangle = \langle f, \mathfrak{T}_p g\rangle.$$

In any case, as a consequence of the previous lemma, we see that $\overline{\chi}(p)\mathfrak{T}_p$ is a Hermitian operator for any odd prime p. This proves the existence an orthogonal Heigenbasis (a Heigenbasis is a Hecke eigenbasis) for all odd n.

0.4 The L function and its functional equation

In Chapter 3, we will need to extend the slash-action of Definition 0.1.6 to $\mathbb{C}[\Gamma_0^*(4N)]$ by linearity. This means that $\sum c_\gamma \gamma$ acts on f as $\sum c_\gamma (f|\gamma)$. We shall only be interested in the case $\Gamma = \Gamma_0^*(4N)$ for some N, we will denote the space of cusp forms of weight $k/2$ for a group by $S_{k/2}(\Gamma_0^*(4N))$. As standard, we let T, S and U be the following elements of $\mathbf{SL}_2(\mathbb{Z})$:

$$T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), \qquad S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right), \qquad U = TS = \left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right).$$

To a form

$$f(z) = \sum_{n\geq 1} a_f(n) e^{2\pi i n z} \in S_{k/2}(\Gamma)$$

we attach the L-*series*

$$L(f,s) := \sum_{n\geq 1} \frac{a_f(n)}{n^s}.$$

This is absolutely convergent for $\Re(s) \gg 1$ and can be analytically continued to the entire complex plane. Its "completed" L function is

$$L^*(f,s) = \frac{\Gamma(s)}{(2\pi)^s} \sum_{n\geq 1} \frac{a_f(n)}{n^s} = \int_0^\infty f(it) t^s \frac{dt}{t}.$$

It satisfies the functional equation

$$L^*(f,s) = N^{\frac{k}{4}-s} L^*\left(W_{4N} f, \frac{k}{4} - s\right). \tag{0.4.1}$$

We now state without proof a well-known proposition. Consider a sequence $\{a(n)\}_{n=1}^\infty \subset \mathbb{C}$ such that $|a(n)| \ll n^\alpha$ for some $\alpha \in \mathbb{R}$. Put $\mathcal{A}(\tau) = \sum_{n\geq 1} a(n) q^n \in$

$\mathrm{Hol}(\mathfrak{H})$ and put also

$$(-2i\tau)^{-k/2}\mathcal{A}(-1/4\tau) = \sum_{n\geq 1} \hat{a}(n)q^n.$$

We assume as well that $|\hat{a}(n)| \ll n^\alpha$. Then

Proposition 0.4.1. *With the setting as above, the series*

$$\frac{\Gamma(s)}{\pi^s}\sum_{n\geq 1}\frac{a(n)}{n^{k/2-s}} = 2^{s-k/2}\frac{\Gamma(s)}{\pi^s}\sum_{n\geq 1}\frac{\hat{a}(n)}{n^s}$$

is entire.

This is a very special case of Theorem 3.2 in [25].

0.5 Shimura's isomorphism

We present a simplified version of Shimura's result. For a general statement, we recomend the more modern exposition in §12 chapter IV in the book [26] rather than the original paper [24].

Theorem 0.5.1 (Shimura)**.** *Let $D \in \mathbb{Z}$ square-free. Set $\check{\chi}(m) := \chi(m)\left(\frac{D}{m}\right)$. Then there exists a map*

$$\sigma_D : S_{k/2}(4N, \chi) \to M_{2\lambda}(2N, \chi^2)$$

such that the image $\sigma_D f$ of a form $f = \sum a_f(n)q^n$ has its L-function equal to

$$L(\check{\chi}, s-\lambda+1)\sum_{n\geq 1}\frac{a_f(|D|n^2)}{n^s}. \tag{0.5.1}$$

Proof We only provide a sketch of the case for $N = 1$ and $D = 1$. We do this in two steps. Begin formally with the L-series

$$L(s) := L(\check{\chi}, s-\lambda+1)\sum_{n\geq 1}\frac{a_f(n^2)}{n^s} = \sum_{n\geq 1}\frac{A(n)}{n^s}$$

defined by (0.5.1) and check that $L(s)$ satisfies the conditions of Weil's converse theorem; namely that for any primitive character ψ modulo r_ψ, the twist of $L(s)$ by ψ has absolute convergence on a half-plane, is bounded on every vertical strip, has a holomorphic continuation and a functional equation. When all these conditions are met, this will show that $L(s)$ must be the L function of a modular form. In other words, $\sum A(n)q^n \in M_{2\lambda}$. The absolute convergence and boundedness on vertical strips are easy to verify; and the holomophic continuation will follow from the functional equation. Therefore it suffices to check that

$$\sum_{n\geq 1}\frac{\psi(n)A(n)}{n^s}=\sum_{n\geq 1}\frac{\check{\chi}(n)\psi(n)}{n^{s+1-\lambda}}\sum_{m\geq 1}\frac{a(m^2)\psi(m)}{m^s}$$

has a functional equation. This is already known for the first sum, because it is a Dirichlet series. Hence we must check that $\sum a(n^2)\psi(n)n^{-s}$ has functional equation. This is achieved by the Rankin-Selberg method. We consider unfolding $B(z,s)=f(z)\overline{h_\psi(z)}y^{s+1}$ where $h_\psi(z)=\sum_{n\geq 1}\bar{\psi}(n)n^\nu q^{n^2}$. We have

$$\begin{aligned}\frac{\Gamma(s)}{(4\pi)^s}\sum_{n\geq 1}\frac{\psi(n)a(n^2)}{n^{2s-\nu}} &= \int_{\Gamma_\infty\backslash\mathfrak{H}} B(z,s)d\mu(z)\\ &= \int_{\Gamma_0(4r_\psi^2)\backslash\mathfrak{H}} B(z,s)\sum_{\Gamma_\infty\backslash\Gamma_0(4r_\psi^2)}\mathcal{B}(\gamma,z,s)d\mu(z)\\ &= \int_{\Gamma_0(4r_\psi^2)\backslash\mathfrak{H}} B(z,s)\tilde{E}(z,s)d\mu(z).\end{aligned}$$

Here $\mathcal{B}(\gamma,z,s)=\frac{B(\gamma z,s)}{B(z,s)}$. It turns out that the Eisenstein series $\tilde{E}$ can be expressed as

$$\tilde{E}(z,s)=\frac{\Gamma(s)}{\pi^s}y^s\sum_{c,d}\frac{\psi\check{\chi}(d)(cz+d)^{\lambda-\nu}}{|cz+d|^{2s-2\nu+1}}$$

where c,d run through coprime integers such that $0<c\equiv 0(4r_\psi^2)$. We include $(c,d)=(0,1)$ in the sum as well. Such a series should be thought of as an Eisenstein series of weight $(\lambda-\nu)/2$. The strategy now to complete the proof of the functional equation is to show the functional equation for $\tilde{E}$. This is a deep and nontrivial result, but is well known in the literature. For a proof of the functional equation of $\tilde{E}$ see [24] Lemma 3.3. □

We end this section with a comment on what occurs when f is a Heigenform. A Heigenform is a normalised Hecke eigenform. To do this we state [26] Theorem 13.11,

Proposition 0.5.2 (Shimura). *Let $f\in S_{k/2}(4N,\chi)$ be a Heigenform for all primes. Then there exists a normalised integral weight Heigenform $g\in$*

$M_{2\lambda}(2N, \chi^2)$ *such that*

$$L(\check{\chi}, s+1-\lambda) \sum_{n \geq 1} \frac{a_f(|D|n^2)}{n^s} = a_f(|D|)L(g,s).$$

In particular if f is a Heigenform, there exists a normalised Heigenform g such that

$$\sigma_D f = a_f(|D|)g. \tag{0.5.2}$$

0.6 Conjugating the theta group

Recall that Γ^ϑ is the subgroup of $\mathbf{PSL}_2(\mathbb{Z}) = \Gamma_1$ generated by T^2 and S. Conjugating this group by $\left(\begin{smallmatrix} 2 & \\ & 1 \end{smallmatrix}\right)$ gives the group generated by T and W_4, namely $\Gamma_0^*(4)$. Denote by Δ° the set of matrices of positive odd determinant and integer entries. Write $\Delta^\vartheta \subset \Delta^\circ$ for the set of matrices with $a \equiv d \not\equiv c \equiv b$ modulo 2.

Proposition 0.6.1 ([23] Prop 3.30)**.** *There exists a homomorphism*

$$\mathfrak{R}(\Gamma^\vartheta, \Delta^\vartheta) \to \mathfrak{R}(\Gamma_1, \Delta^\circ).$$

Fix $\delta \in \Delta^\circ$ and find $\gamma \in \Gamma_1$ such that $\gamma \equiv \delta \mod 2$. This can always be done since $\det \delta \equiv 1(2)$ so $\delta \mod 2$ can be viewed as an element of $\mathbf{SL}_2(\mathbb{Z}/2\mathbb{Z})$ and Γ_1 surjects onto $\mathbf{SL}_2(\mathbb{Z}/2\mathbb{Z})$. Therefore $\gamma^{-1}\delta \in \Delta^\vartheta$ so $\Gamma^\vartheta\gamma^{-1}\delta\Gamma^\vartheta$ maps to $\Gamma_1\gamma^{-1}\delta\Gamma_1 = \Gamma_1\delta\Gamma_1$ showing the map is surjective. Moreover, if $\Gamma_1\delta_1\Gamma_1 = \Gamma_1\delta_2\Gamma_1$ then either $\delta_1 \equiv \delta_2 \mod 2$ or $\delta_1 \equiv S\delta_2 \mod 2$. But since $S \in \Gamma^\vartheta$, we have

$$\Gamma^\vartheta\delta_1\Gamma^\vartheta = \begin{cases} \Gamma^\vartheta\delta_2\Gamma^\vartheta \text{ if } \delta_1 \equiv \delta_2 \\ \\ \Gamma^\vartheta S\delta_2\Gamma^\vartheta \text{ if } \delta_1 \equiv S\delta_2 \end{cases} = \Gamma^\vartheta\delta_2\Gamma^\vartheta.$$

For clarification, in either case, $\Gamma^\vartheta\delta_1\Gamma^\vartheta = \Gamma^\vartheta\delta_2\Gamma^\vartheta$. This shows that the homomorphism above is in fact a bijection and therefore we have proved

Proposition 0.6.2. *The map* $\mathfrak{R}(\Gamma^\vartheta, \Delta^\vartheta) \to \mathfrak{R}(\Gamma_1, \Delta^\circ)$ *sending* $\Gamma^\vartheta\alpha\Gamma^\vartheta \mapsto \Gamma_1\alpha\Gamma_1$ *is an isomorphism.*

We point out that this is a different result to Proposition 3.31 of [25], because

Γ^ϑ is not of the form considered in equation (3.3.1') of that book.

Corollary 0.6.3. *Let $\Delta_0^*(4)$ be the image of Δ^ϑ under conjugation by $\left(\begin{smallmatrix} 2 & \\ & 1 \end{smallmatrix}\right)$. Then $\mathfrak{R}(\Gamma^\vartheta, \Delta^\vartheta) \to \mathfrak{R}(\Gamma_0^*(4), \Delta_0^*(4))$ is an isomorphism.*

Proof Apply Proposition 0.3.1 with $\Gamma = \Gamma^\vartheta$ and $\alpha = \left(\begin{smallmatrix} 2 & \\ & 1 \end{smallmatrix}\right)$ and combine this with the Proposition 0.6.2. □

Explicitly, $\Delta_0^*(4)$ can be expressed as

$$\left\{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) :\ a, d \in \mathbb{Z},\ c \in 2\mathbb{Z},\ b \in \frac{1}{2}\mathbb{Z},\ a \equiv d(2),\ c/2 \equiv 2b(2),\ 0 < ad - bc \equiv 1(2)\right\}$$

Now fix the choice $\Gamma = \Gamma_0^*(4)$ and the choice of lift as in §0.2.2. Then, because of Lemma 0.2.7(d), since the image of $\Delta_0^*(4)$ is $\Xi_0^*(4)$ we see that the map

$$\Xi_0^*(4) \to \Delta_0^*(4) \tag{0.6.1}$$

is a bijection. In particular, $\mathfrak{R}(H, \Xi_0^*(4)) \to \mathfrak{R}(\Gamma_0^*(4), \Delta_0^*(4))$ is also a bijection. The commutative diagram in (0.3.1) allows us to push forward this bijection using ι. If Δ' and Ξ' are (respectively) the images of $\Delta_0^*(4)$ and $\Xi_0^*(4)$ under ι, then

$$\mathfrak{R}_k(H, \Xi') \to \mathfrak{R}_k(\Gamma_0^*(4), \Delta')$$

is a bijection. Given a pair of matrices $(\left(\begin{smallmatrix} * & * \\ a & b \end{smallmatrix}\right) \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right)) \in \Gamma^\vartheta \times \Gamma^\vartheta$ then provided that $a \equiv d(2)$ and $ad - bc > 0$ we see $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Delta^\vartheta$. The converse need not be true; for instance, $3S \in \Delta^\vartheta$ is not of this form. This is because the image will always have $\gcd(a, b) = \gcd(c, d) = 1$. Recall that for an integer $u \geq 1$, P_u is the subset of $\mathbf{SL}_2(\mathbb{Q})$ consisting of matrices of the form $\left(\begin{smallmatrix} 1/u & * \\ 0 & u \end{smallmatrix}\right)$. In Chapter 2, we shall need to sum over elements of Δ^ϑ. In order to do this we shall exploit the following bijection. Let

$$\Delta_{u,v}^\vartheta := \{\delta \in \Delta^\vartheta :\ \gcd(a_\delta, b_\delta) = u,\ \gcd(c_\delta, d_\delta) = v\}.$$

Then it's clear that $\Delta^\vartheta = \bigcup_{u,v \geq 1} \Delta_{u,v}^\vartheta$ and in fact

$$\bigcup_{u \geq 1} P_u \Gamma^\vartheta \times \bigcup_{v \geq 1} P_v \Gamma^\vartheta \to \Delta^\vartheta \tag{0.6.2}$$

is one-to-one and onto. This bijection is allowed to pass under the map ι in (0.3.1). We obtain

$$\bigcup_{u\geq 1} P_u\Gamma_0^*(4) \times \bigcup_{v\geq 1} P_v\Gamma_0^*(4) \to \Delta_0^*(4) \tag{0.6.3}$$

In Chapter 2 §2.5 we shall need to take a sum over the action of matrices in $\Delta_0^*(4)_{u,v} := \{\delta \in \Delta_0^*(4) : \ \gcd(a_\delta, b_\delta) = u, \ \gcd(c_\delta, d_\delta) = v\}$. We claim that summing over matrices in $\Delta_0(4)_{u,v}$ is the same as summing over matrices in $\Delta_0(4)_{u^2,v^2}$. Indeed those matrices $A_{uv} \notin \Delta_0(4)_{u^2,v^2}$ with $A_{uv} \in \Delta_0(4)_{u,v}$ will have non-trivial $v^k_{A_{uv}}$, so by Proposition 0.2.5(iii), the action of A_{uv} will not contribute to the sum. We will come back to this in chapter 2 §2.5 and give another argument there. Similar arguments as these can be seen for instance in [13], Chapter IV §3 Proposition 12 p204.

0.7 Definition of cohomolgy groups

In chapter 3 we will fix an integer $\ell \geq 0$ and work with one of three choices for Γ, namely either $\mathbf{PSL}_2(\mathbb{Z})$, Γ^ϑ or $\Gamma_0^*(4)$. We denote by $\mathbb{C}_\ell[\tau]$ the space of polynomials with coefficients in $\mathbb{C}$ of degree $\leq \ell$. We equip $\mathbb{C}_\ell[\tau]$ with the following action

$$(P|_{-\ell}\gamma)(\tau) = (c_\gamma\tau + d_\gamma)^\ell P\left(\frac{a\tau+b}{c\tau+d}\right) \quad P \in \mathbb{C}_\ell[\tau] \ \gamma \in \Gamma. \tag{0.7.1}$$

We say that $\pi : \Gamma \to \mathbb{C}_\ell[\tau]$ is a cocycle if $\pi(\gamma_1\gamma_2) = \pi(\gamma_1)|_{-\ell}\gamma_2 + \pi(\gamma_2)$ for any $\gamma_1, \gamma_2 \in \Gamma$. We say that a cocycle π is a coboundary if there exists a polynomial $P \in \mathbb{C}_\ell[\tau]$ such that $\pi(\gamma) = P|_{-\ell}(\gamma - I)$. The space of cocycles modulo coboundaries is call the cohomology group of Γ with values in $\mathbb{C}_\ell[\tau]$ and is denoted $H^1(\Gamma, \mathbb{C}_\ell[\tau])$. Parabolic cocycles are those $\pi \in H^1(\Gamma, \mathbb{C}_\ell[\tau])$ satisfying $\deg(\pi(T^w)) < \ell$ where $w = 2$ in the case $\Gamma = \Gamma^\vartheta$ and $w = 1$ for $\Gamma = \mathbf{PSL}_2(\mathbb{Z})$ or $\Gamma_0^*(4)$. The space of parabolic cocycles is denoted $H^1_{par}(\Gamma, \mathbb{C}_\ell[\tau])$.

Chapter 1

Explicit images of Shimura's map

1.1 Introduction

The aim of this chapter is to prove explicit examples of the Shimura lift σ_D using only elementary techniques. As in the previous chapter, let λ be a positive integer and let $\nu \in \{0,1\}$ and suppose that $f \in S_{\lambda-\nu}(N_f, \chi_f)$ is a holomorphic cusp form of weight $\lambda - \nu$. To a primitive Dirichlet character ψ of conductor r_ψ satisfying $\psi(-1) = (-1)^\nu$, we shall consider the theta series $h_\psi \in S_{1/2+\nu}(4r_\psi^2, \psi\chi_{-4}^\nu)$ given by

$$h_\psi(\tau) = \sum_{m\in\mathbb{Z}} \psi(m) m^\nu q^{m^2}.$$

An example of this is the choice $\psi = (12/\cdot)$, $\nu = 0$ which gives rise to the well-known eta product

$$\sum_{n\geq 1} \left(\frac{12}{n}\right) q^{n^2} = \eta(24\tau) := q \prod_{n\geq 1} (1 - q^{24n})$$

of level 576 and weight 1/2. Thus informally our goal will be to describe explicitly $\sigma(fh_\psi)$ in terms of f and the characters χ_f and ψ. We maintain the convention $k = 2\lambda + 1$. By a Heigenform we will mean a normalized Hecke newform. We shall denote η_a^b the function $\eta(a\tau)^b$ defined above e.g. $h_{(12/\cdot)} = \eta_{24}$. For a formal power series $f = \sum_{n\geq 1} a_f(n) q^n$ define the operators

V_d and U_d as follows

$$V_d f := \sum_{n\geq 1} a_f(n)q^{dn} \qquad U_d f := \sum_{n\geq 1} a_f(dn)q^n.$$

It is known that if $f \in S_{k/2}(N, \chi_f)$,

$$V_{|D|} f \in S_{k/2}\left(|D|N_f, \left(\frac{4|D|}{\cdot}\right)\chi_f\right),$$

and if $|D|$ divides N_f,

$$U_{|D|} f \in S_{k/2}\left(N_f, \left(\frac{4|D|}{\cdot}\right)\chi_f\right).$$

See for instance either Proposition 1.3 and Proposition 1.5 in [24] or Proposition 3.7 in [20]. Define the twist of $f = \sum a_f(n)q^n$ by a character ψ to be the power series $\sum a_f(n)\psi(n)q^n$ and denote it $f \otimes \psi$. Throughout this section we denote by $\chi_{0,4}$ and $\chi_{1,4}$ the trivial and nontrivial characters modulo four. We also denote $\chi_{0,N}$ the trivial character of modulus N. Given a half integral weight form $f \in S_{k/2}(4N, \chi)$, and a fundamental discriminant D, we define the checked character

$$\check{\chi}_D(d) := \left(\frac{4(-1)^\lambda |D|}{d}\right)\chi(d).$$

When $D = 1$ we simply write $\check{\chi} = \chi_{1,4}^\lambda \chi$.

Definition 1.1.1 (Shimura image)**.** Let $\lambda \in \mathbb{N}$ and D be a fundamental discriminant. For $f = \sum_{n\geq 1} a_f(n)q^n \in S_{k/2}(4N, \chi)$ set

$$\sigma_D f := \sum_{n\geq 1} A_D(n) q^n$$

where

$$A_D(n) = \sum_{d|n} \check{\chi}_D(d) d^{\lambda-1} a_f\left(\frac{|D|n^2}{d^2}\right).$$

Remark 1.1.2. If we wish to extend this definition to cover the case when f is a modular form (not necessarily a cusp form) we add the constant term $\frac{a_f(0)}{2} L((D/\cdot), 1-\lambda)$ to our definition of $A_D(n)$. In this chapter we will not discuss this case.

Remark 1.1.3. Definition 1.1.1 is taken so as to agree with the L-function relation of Theorem 0.5.1, namely

$$\sum_{n\geq 1}\frac{A_D(n)}{n^s} = L(\check{\chi}_D, s-\lambda+1)\sum_{n\geq 1}\frac{a_f(|D|n^2)}{n^s}.$$

We begin with a simple result that to our knowledge has not appeared in the literature. It says that Shimura images of h_ψ are twists of Eisenstein series. More precisely we have

Lemma 1.1.4. *Let $\nu = 1$, let ψ be a non-trivial character and let E_2 be the weight 2 Eisenstein series of level 2. Then*

$$\sigma_1 h_\psi = E_2 \otimes \psi.$$

Proof Since $a_{h_\psi}(n^2) = \psi(n)n$, we see that the coefficients of $\sigma_1 h_\psi$ are

$$\begin{aligned} A(n) &= \sum_{d|n}\psi(d)a_{h_\psi}\left(\frac{n^2}{d^2}\right) = \sum_{d|n}\psi(d)\psi(n/d)\cdot(n/d) \\ &= \psi(n)\sum_{\substack{d|n \\ (d,r_\psi)=1}}\frac{n}{d}. \end{aligned}$$

On the other hand, $a_{E_2}(n) = \sum_{d|n} d$. The result now follows. □

Example 1.1.5. Let ψ be the non-trivial character modulo 4. Then $h_\psi = \eta_8^3$ so the lemma gives

$$\sigma_1\eta_8^3 = \sigma_1 h_\psi = E_2 \otimes \psi = \frac{\eta_2^4\eta_8^4}{\eta_4^4}.$$

The last equality follows form checking enough coefficents.

Definition 1.1.6. We say $f \in M_{2\lambda}(N, \chi_f)$ is a CM form if there exists a non-trivial character χ such that $f \otimes \chi = f$.

From this simple lemma we deduce

Corollary 1.1.7. *No CM form is the Shimura image of some h_ψ under σ_1.*

Proof By the previous lemma, it suffices to show $E_2 \otimes \psi$ is never CM. Assume not, then there exists a non-trivial character χ such that

$$E_2 \otimes (\psi\chi) = (E_2 \otimes \psi) \otimes \chi = E_2 \otimes \psi.$$

This would imply χ is trivial modulo r_χ, a contradiction. □

We now prove that the operators U and V behave nicely under σ:

Lemma 1.1.8. *Assume that $|D|$ divides N. Then on the space $S_{k/2}(4N, \chi)$ we have*

(i) $\sigma_D V_{|D|} f = \sigma_1 f$

(ii) $\sigma_1 U_{|D|} f = \sigma_D f$

(iii) $\sigma_D U_{|D|} = U_{|D|} \sigma_1$

(iv) $U_{|D|} \sigma_1 V_{|D|} = \sigma_D$

(v) $V_{|D|} \sigma_1 = V_{|D|} U_{|D|} \sigma_1 V_{|D|^2}$

Proof Fix $f \in S_{k/2}(4N, \chi_f)$ and note that $a_{V_{|D|}f}(n) = a_f(n/|D|)$ and $a_{U_{|D|}f}(n) = a_f(|D|n)$. Also $U_{|D|}V_{|D|}$ is the identity but $V_{|D|}U_{|D|}$ is not.

(i) The checked character of $V_{|D|}f$ with respect to D is $((D/\cdot)\chi_f)^\vee_D = \check{\chi}$, where $\check{\chi}$ is $\check{\chi}_f$ considered modulo $\mathrm{lcm}(4N, |D|)$. Hence

$$\begin{aligned} a_{\sigma_D V_{|D|} f}(n) &= \sum_{d|n} ((D/\cdot)\chi_f)^\vee_D(d) d^{\lambda-1} a_{V_{|D|}f}\left(\frac{|D|n^2}{d^2}\right) \\ &= \sum_{d|\ |D|n} \check{\chi}(d) d^{\lambda-1} a_f\left(\frac{n^2}{d^2}\right) = a_{\sigma_1 f}(n). \end{aligned}$$

In the penultimate equality we used the fact that if $d|\ |D|$ then $\check{\chi}(d) = 0$ does not contribute to the sum.

(ii) As before, the checked character of $U_{|D|}f$ with respect to D is $((D/\cdot)\chi_f)^\vee_D = \check{\chi}$ considered again modulo $\mathrm{lcm}(4N, |D|)$. In a similar way,

$$\begin{aligned} a_{\sigma_1 U_{|D|} f}(n) &= \sum_{d|n} \check{\chi}(d) d^{\lambda-1} a_{U_{|D|}f}\left(\frac{n^2}{d^2}\right) \\ &= \sum_{d|n} \check{\chi}(d) d^{\lambda-1} a_f\left(\frac{|D|n^2}{d^2}\right) = a_{\sigma_D f}(n). \end{aligned}$$

We remark that (i) and (ii) are equivalent, because $\sigma_1 = \sigma_1 U_{|D|} V_{|D|} = \sigma_D V_{|D|}$.

(iii) Indeed,

$$\begin{aligned} a_{\sigma_D U_{|D|} f}(n) &= \sum_{d|n} ((D/\cdot)\chi_f)^\vee_D(d) d^{\lambda-1} a_{U_{|D|}f}\left(\frac{|D|n^2}{d^2}\right) \\ &= \sum_{d|n} \check{\chi}(d) d^{\lambda-1} a_f\left(\frac{(|D|n)^2}{d^2}\right) \\ &= a_{\sigma_1 f}(|D|n) = a_{U_{|D|}\sigma_1 f}(n). \end{aligned}$$

(iv) By part (iii), we have $(U_{|D|}\sigma_1)V_{|D|} = \sigma_D U_{|D|}V_{|D|} = \sigma_D$.

(v) By part (i) and (iv), we have $V_{|D|}\sigma_1 = V_{|D|}\sigma_D V_{|D|} = V_{|D|}(U_{|D|}\sigma_1 V_{|D|})V_{|D|}$.

□

Remark 1.1.9. Part (ii) of this lemma allows us to compute all images of σ_D provided we know all images of σ_1. In essence, (i) and (ii) say that finding σ_D is as hard as finding σ_1. The proof of *(ii)* was already known by Brown in his thesis (cf [2] Proposition 2.5).

Remark 1.1.10. Having proved this lemma, it is tempting and quite natural to conjecture that $V_{|D|}\sigma_D = \sigma_1 V_{|D|}$. However this is false and is explained by the fact that although $U_{|D|}V_{|D|}$ is the identity map, $V_{|D|}U_{|D|}$ is not.

We can remove the condition that $|D|$ divides N_f by use of L-functions. Indeed, by Theorem 0.5.1, on the region of absolute convergence we have

$$\begin{aligned}\frac{L(\sigma_D V_{|D|}f,s)}{L(\sigma_1 f,s)} &= \frac{L(((D/\cdot)\chi_f)_D^\vee, s-\lambda+1)}{L(\check{\chi}, s-\lambda+1)}\\ &= \frac{\prod_{p\nmid \mathrm{lcm}(4N,|D|)}\left(1-\check{\chi}(p)p^{\lambda-1-s}\right)^{-1}}{\prod_{p\nmid N}\left(1-\check{\chi}(p)p^{\lambda-1-s}\right)^{-1}}\\ &= \prod_{p\mid\, |D|/(N,|D|)}\left(1-\check{\chi}(p)p^{\lambda-1-s}\right).\end{aligned}$$

This proves the following formal equality of maps

$$\sigma_D \circ V_{|D|} = \prod_{p \mid \frac{D}{(N,D)}} (1-\check{\chi}(p)p^{\lambda-1}V_p)\circ\sigma_1 \tag{1.1.1}$$

It's easy to see now that 1.1.8(i) is a special case of this when $|D|/(N,|D|) = 1$.

With our notation, the Sturm bound[1] of the space $M_{2\lambda}(2N,\chi^2)$ is at most

$$L = \left\lfloor \frac{\lambda N}{2} \prod_{\substack{p\mid N\\ p \text{ odd}}} \left(1+\frac{1}{p}\right)\right\rfloor.$$

This of course can be much smaller if we work on the space of cusp forms $S_{2\lambda}(2N,\chi^2)$. If f and g are in correspondence $\sigma_D f = g$ then we should expect the Sturm bound on g to have an effect on the coefficients of f. Indeed, we have

[1]the smallest integer L such that if $f \neq g$ then there exists $1 \leq n \leq L$ with $a_f(n) \neq a_g(n)$.

Lemma 1.1.11. *Fix $f \in S_{k/2}(4N, \chi)$ and let L be the Sturm bound of $S_{2\lambda}(2N, \chi^2)$. Then, the following are equivalent:*

1. $\sigma_D f = 0$
2. $a_f(|D|n^2) = 0$ *for all* $n \geq 1$
3. $a_f(|D|n^2) = 0$ *for* $1 \leq n \leq L$
4. $a_f(|D|) = 0$

Proof Clearly (2) $\Rightarrow$ (1) $\Rightarrow$ (4) since $a_f(|D|) = A_D(1) = 0$. Also clear is the direction (2) $\Rightarrow$ (3) $\Rightarrow$ (4). Möbius inversion proves (1) $\Rightarrow$ (2), since

$$a_f(|D|n^2) = \sum_{d|n} \mu(d)\check{\chi}(d)d^{\lambda-1} \underbrace{A_D\left(\frac{n}{d}\right)}_{=0}$$

To show (4) $\Rightarrow$ (1) it suffices to show it for a Heigenbasis. This follows immediately from equation (0.5.2), since there exists a normalised Heigenform g such that $\sigma_D f = a_f(|D|)g = 0$. □

An alternative way of showing (4) $\Rightarrow$ (1) is to exploit Möbius inversion

$$a_f(|D|n^2) = a_f(|D|) \sum_{d|n} \mu(d)\check{\chi}(d)d^{\lambda-1} A_D\left(\frac{n}{d}\right),$$

but this also relies on the proof of [26] Theorem 13.11. Let $D \in \mathbb{N}$ be a square-free integer. Immediately we see that $f \in \mathrm{Ker}\sigma_D$ if and only if the coefficients of f on the square class $D \in \mathbb{Q}^\times/(\mathbb{Q}^\times)^2$ vanish. In particular, as D runs through square-free integers,

$$\bigcap_D \mathrm{Ker}\sigma_D = \{0\}.$$

Example 1.1.12. Since $S_2(30)$ has Sturm bound $L = 3$, any $f \in S_{3/2}(60)$ with $a_f(1) = a_f(4) = a_f(9) = 0$ will have $\sigma_1 f = 0$.

Definition 1.1.13. Given a Dirichlet character χ of modulus N, we define its Möbius inverse by

$$c_\chi(n) = \sum_{d|n} \mu(d)\chi\left(\frac{n}{d}\right).$$

We of course interpret $\chi(n/d) = 0$ when $(n/d, N) = 1$. We easily obtain:

Lemma 1.1.14. *The Möbius inverse of the trivial character $\chi_{0,4}$ modulo 4 is*

$$c_{\chi_{0,4}}(n) = \begin{cases} 1 & n = 1 \\ -1 & n = 2 \\ 0 & \textit{otherwise.} \end{cases}$$

Proof The first two cases are easily verified. It suffices to check prime powers so either $n = p^\alpha$ with $p \neq 2$ or $n = 2^\alpha$ with $\alpha \geq 2$. We have $c_{\chi_{0,4}}(p^\alpha) = \chi_{0,4}(p)^{\alpha-1}(\chi_{0,4}(p) - 1)$ for all p. If p is odd, $\chi_{0,4}(p) = 1$ and if $p = 2$, $\chi_{0,4}(2)^{\alpha-1} = 0$ as $\alpha \geq 2$. Either way, $c_{\chi_{0,4}}(p^\alpha)$ vanishes. □

1.1.1 Historic background

We shall need the following known (cf 2.2 in [4]) elementary fact, which we will refer as Selberg inversion.

Lemma 1.1.15 (Selberg Inversion)**.** *Let $f \in S_\lambda(N, \chi)$ be a normalised Heigenform. Then*

$$a_f(m_1 m_2) = \sum_{e|(m_1,m_2)} \mu(e)\chi(e)e^{\lambda-1} a_f\left(\frac{m_1}{e}\right) a_f\left(\frac{m_2}{e}\right).$$

Proof Set $D = ed$ and note that the right hand side is

$$\sum_{e|(m_1,m_2)} \mu(e)\chi(e)e^{\lambda-1} \sum_{d|(m_1/e,m_2/e)} \chi(d)d^{\lambda-1} a_f\left(\frac{m_1 m_2}{d^2 e^2}\right)$$
$$= \sum_{D|(m_1,m_2)} \left(\sum_{e|D} \mu(e)\right) \chi(D)D^{\lambda-1} a_f\left(\frac{m_1 m_2}{D^2}\right) = a_f(m_1 m_2).$$

□

We now give a brief historic survey of all previously known explicit Shimura lifts. The first is credited to an unpublished result of Selberg, which first appeared in Cipra's article [4]. We give an exposition of this statement in part (1) of the next proposition. Actually, as Cipra suggests, Selberg did not consider the case with characters. With similar ideas, we give a proof involving characters. In other words, whereas part (1) was previously known[2], part (2) of the next proposition appears to be new.

[2] see the main theorem in [4], although the version Cipra presents is more general, our formulation does not use Dirichlet series but rather careful use of the modulus of the character in Möbius inversion.

Proposition 1.1.16 (Selberg). *Let $f \in S_\lambda(N, \chi_f)$ be a normalised Heigenform. Consider*

$$f(4\tau)\theta(\tau) \in S_{k/2}(4N, \chi_f\chi_{0,4}).$$

1. *If λ is even and $N = 1$, we have*

$$\sigma_1(f(4\tau)\theta(\tau)) = f(\tau)^2 - 2^{\lambda-1}f(2\tau)^2$$

   lies in $S_{2\lambda}(2N, \chi_f^2)$. Otherwise, if $N \neq 1$, we have $\sigma_1(f(4\tau)\theta(\tau)) = f(\tau)^2$.

2. *If λ is odd then the coefficents of $\sigma_1(f(4\tau)\theta(\tau))$ are given by*

$$\sum_{s|n} \chi_f(n/s)(n/s)^{\lambda-1}c_{\chi_{1,4}}(n/s)a_{f^2}(s),$$

   where $c_{\chi_{1,4}}(n) = \sum_{e|n} \mu(e)\chi_{1,4}(n/e)$ is the Möbius inverse of the character $\chi_{1,4}$.

Proof (cf [4]) Since $f(4\tau)\theta(\tau)$ has coefficients

$$\sum_{m\in\mathbb{Z}} a_f\left(\frac{n-m^2}{4}\right) =: b(n)$$

and $f(\tau)^2$ has coefficients

$$\sum_{m\in\mathbb{Z}} a_f(m)a_f(n-m),$$

we have,

$$b(n^2) = \sum_m a_f\left(\frac{(n-m)(n+m)}{4}\right) = \sum_m a_f\left(\frac{m(2n-m)}{4}\right) = \sum_m a_f(m(n-m)).$$

The middle equality follows from the substitution $m \mapsto m - n$ and the second by noticing that m must be even so $m \mapsto 2m$ is allowed. By Selberg inversion (1.1.15),

$$\begin{aligned} b(n^2) &= \sum_m \sum_{e|(m,n)} \mu(e)\chi_f(e)e^{\lambda-1}a_f\left(\frac{m}{e}\right)a_f\left(\frac{n-m}{e}\right) \\ &= \sum_{e|n} \mu(e)\chi_f(e)e^{\lambda-1}\sum_m a_f(m)a_f(n/e-m) \\ &= \sum_{e|n} \mu(e)\chi_f(e)e^{\lambda-1}a_{f^2}(n/e) \end{aligned}$$

the last line is just for easier substitution later. The checked character associated to $f(4\tau)\theta(\tau)$ is $\chi\chi_{1,4}^{\lambda}$. Finally performing the Shimura lift gives

$$\begin{aligned} B(n) &= \sum_{d|n} \chi_f \chi_{1,4}^{\lambda}(d) d^{\lambda-1} \sum_{e|\frac{n}{d}} \mu(e)\chi_f(e) e^{\lambda-1} a_{f^2}\left(\frac{n}{ed}\right) \\ &= \sum_{d|n} \sum_{e|\frac{n}{d}} \chi_f \chi_{1,4}^{\lambda}(d) d^{\lambda-1} \mu(e)\chi_f(e) e^{\lambda-1} a_{f^2}\left(\frac{n}{ed}\right) \\ &= \sum_{d|n} \sum_{e|\frac{n}{d}} \chi_f(ed)(ed)^{\lambda-1} \mu(e) a_{f^2}\left(\frac{n}{ed}\right) \chi_{1,4}^{\lambda}(d) \end{aligned}$$

upon substituting $n = eds$,

$$\begin{aligned} B(n) &= \sum_{s|n} (n/s)^{\lambda-1} \chi_f(n/s) a_{f^2}(s) \sum_{e|\frac{n}{s}} \mu(e) \chi_{1,4}^{\lambda}(n/es) \\ &= \sum_{s|n} (n/s)^{\lambda-1} \chi_f(n/s) a_{f^2}(s) c_{\chi_{1,4}^{\lambda}}(n/s). \end{aligned}$$

This proves the case when λ is odd. By Lemma 1.1.14, when λ is even the last term $c_{\chi_{1,4}^{\lambda}}(n/s) = c_{\chi_{0,4}}(n/s)$ only the takes values $+1$ at $n = s$ and -1 at $n = 2s$ and vanishing otherwise, therefore

$$B(n) = a_{f^2}(n) - 2^{\lambda-1}\chi_f(2) a_{f^2}(n/2) = a_{f^2}(n) - 2^{\lambda-1}\chi_f(2) V_2 a_{f^2}(n),$$

which combined with the fact that $(V_2 f)^2 = V_2(f^2)$ the result follows. Note that when $N = 1$, $\chi_f(2) = 1$ and when $N \neq 1$ then χ_f considered modulo $2N$ so $\chi_f(2) = 0$. □

Example 1.1.17. Let $f = \eta^2\eta_{11}^2$ of level $N = 11 \neq 1$ and even weight $\lambda = 2$. By part (1) of Proposition 1.1.16, we have

$$\sigma(\eta_4^2 \eta_{44}^2 \theta) = \eta^4 \eta_{11}^4.$$

This can be verified for instance with the `pari-gp` script:

```
f = mffrometaquo([1,2;11,2]);
F = mfmul(mfbd(f,4),mfTheta());
mf = mfinit(F);
sF = mfshimura(mf,F)[2];
mfisequal(sF,mffrometaquo([1,4;11,4]))
%5 = 1
```

Let us now consider a variation of this problem. Instead of θ we consider the weight 3/2 theta function

$$\theta_1(\tau) = \sum_{n\in\mathbb{Z}} (-1)^n q^{n^2}.$$

Proposition 1.1.18. *Let $\lambda \in \mathbb{N}$ be even and assume $f \in S_\lambda(N,\chi)$ is a Heigenform. Then the coefficients of $\sigma(f(4\tau)\theta_1(\tau))$ are*

$$\begin{cases} (-1)^n a_{f^2}(n) & \text{when } N \neq 1, \\ (-1)^n \left[a_{f^2}(n) - 2^{\lambda-1} a_{f^2}(n/2)\right] & \text{when } N{=}1. \end{cases}$$

Proof We follow the same proof strategy as the proof of Proposition 1.1.16. The coefficients of $f(4\tau)\theta_1(\tau)$ are

$$b(n) = \sum_{m\in\mathbb{Z}} (-1)^m a_f\left(\frac{n-m^2}{4}\right),$$

so making the substitution $m \mapsto 2m - n$,

$$b(n^2) = \sum_m (-1)^{m-n} a_f\left(\frac{2m(2n-2m)}{4}\right) = (-1)^n \sum_m a_f(m(n-m)).$$

By Selberg inversion (Lemma 1.1.15),

$$b(n^2) = (-1)^n \sum_{e|n} \mu(e)\chi_f(e)e^{\lambda-1} a_{f^2}(n/e).$$

Since the checked character in this case is $\chi_f\chi_{0,4}$ (as λ is even) it follows that the Shimura lift we seek has coefficients

$$\begin{aligned} B(n) &= \sum_{d|n}\sum_{e|\frac{n}{d}} (ed)^{\lambda-1}\chi_f(ed)\mu(e)(-1)^{n/d} a_{f^2}\left(\frac{n}{ed}\right)\chi_{0,4}(d) \\ &= \sum_{s|n} (n/s)^{\lambda-1}\chi(n/s)a_{f^2}(s)\sum_{e|\frac{n}{s}} \mu(e)\chi_{4,0}\left(\frac{n}{es}\right)(-1)^{es}. \end{aligned}$$

We note that $\chi_{4,0}(n/es)$ only takes values when n/es is odd, so $(-1)^{es} = (-1)^n$. To conclude, pull the factor of $(-1)^n$ outside and apply Lemma 1.1.14. □

Proposition 1.1.18 appears to be a new result. We give a brief application in the next example.

Example 1.1.19. Let $f = \eta\eta_2\eta_7\eta_{14} \in S_2(14)$. It is easy to verify that $f(\tau)^2$ has q-expansion at ∞

$$q^2 - 2q^3 - 3q^4 + 6q^5 + 2q^6 - q^8 - 12q^9 + 4q^{10} + 4q^{11} - 2q^{12} + 2q^{13} - 7q^{14} + 16q^{15} + O(q^{16})$$

By Proposition 1.1.18, it follows that $\sigma_1(f(4\tau)\theta_1(\tau))$ has the q-expansion

$$q^2+2q^3-3q^4-6q^5+2q^6-q^8+12q^9+4q^{10}-4q^{11}-2q^{12}-2q^{13}-7q^{14}-16q^{15}+O(q^{16}),$$

obtained by changing the odd coefficients by -1. We have checked this agrees with `pari-gp`.

In Cipra's article [4], a generalisation of Selberg's result is presented by considering the problem when θ is replaced by the theta function h_ψ with respect to a primitive Dirichlet character ψ. Let $f \in S_{\lambda-\nu}(N,\chi)$ be a Heigenform and let ψ be a primitive Dirichlet character of conductor r. Assume r is a power of a prime. Set

$$g(\tau) = \begin{cases} f(\tau)f(r\tau) & \text{for } \nu = 0, \\ \frac{1}{2\pi i}\left(f'(\tau)f(r\tau) - rf(\tau)f'(r\tau)\right) & \text{for } \nu = 1. \end{cases}$$

Proposition 1.1.20 (Cipra, 1987)**.** *With f, ψ, g as above,*

$$\sigma_1(f(4r\tau)h_\psi(\tau)) = g \otimes \psi(\tau) - 2^{\lambda-1}\chi(2)\psi(2)g \otimes \psi(2\tau)$$

lies in $S_{2\lambda}(2rN_r, \chi^2\psi^2)$ where $N_r := \mathrm{lcm}(N, r)$.

As far as the author knows, this result was the first in the literature to give an explicit description of a Shimura image. Notwithstanding, this only works for *primitive* characters. The non-primitive case was answered by Hansen and Naqvi [9]. In order to state the theorem, we introduce the notation $\ell || r$ to mean ℓ is a divisor of r and $\gcd(\ell, r/\ell) = 1$. We caution the reader that although the conclusion is the same as that of Cipra's result, the g in this next theorem is not the same as Cipra's, but rather involves taking a sum over $\ell || r_\psi$, which takes into account the non-primitivity of ψ.

Theorem 1.1.21 (Hansen, Naqvi, 2007 [9])**.** *Let ψ be a character modulo r (not necessarily primitive) of parity ν. Suppose that $f \in M^{new}_{\lambda-\nu}(N,\chi)$. Then*

$$\sigma_1(f(4r\tau)h_\psi(\tau)) = g \otimes \psi(\tau) - 2^{\lambda-1}\psi(2)\chi(2)(g \otimes \psi)(2\tau)$$

lies in $M_{2\lambda}(2rN_r, \chi^2\psi^2)$.

In more precise terms, if ψ is a (not necessarily primitive) Dirichlet character modulo r_ψ, then we can factor

$$\psi = \prod_\ell \psi_\ell = \prod_i \psi_{p_i^{\alpha_i}} \text{ where } r_\psi = \prod_i p_i^{\alpha_i}.$$

Then the g in the theorem of Hansen and Naqvi [9] is given by

$$g(\tau) = \begin{cases} \sum_{l||r_\psi} \psi_l(-1) f(l\tau) f\left(\frac{r\tau}{l}\right) & \text{if } \psi \text{ is even,} \\ \frac{1}{i\pi} \sum_{l||r_\psi} \psi_l(-1) l f'(l\tau) f\left(\frac{r\tau}{l}\right) & \text{if } \psi \text{ is odd.} \end{cases} \tag{1.1.2}$$

If we write

$$g_l(\tau) = \begin{cases} f(l\tau) f\left(\frac{r\tau}{l}\right) & \text{if } \psi \text{ is even,} \\ \frac{l}{i\pi} f'(l\tau) f\left(\frac{r\tau}{l}\right) & \text{if } \psi \text{ is odd;} \end{cases}$$

then we can express $g = \sum_{l||r_\psi} \psi_l(-1) g_l$. We give a new proof of this using the same Dirichlet convolution method as before:

Proof (We illustrate only the case when ψ is even). First note that

$$a_{g_l}(n) = \sum_m a_f\left(\frac{n - lm}{r/l}\right) a_f(m).$$

Also note that the coefficients of $f(4r\tau)h_\psi(\tau)$ are given by

$$b(n) = \sum_m \psi(m) a_f\left(\frac{n - m^2}{4r}\right).$$

If $n + m$ is odd then so is $n^2 - m^2 = (n+m)(n-m)$ meaning $(n^2 - m^2)/4r$ is not an integer so doesn't contribute to the sum. This means we are allowed the substitution $n + m = 2k_1 l$ and $n - m = 2k_2 r/l$ for each $l||r$ for some integers k_1, k_2. We have

$$\begin{aligned} b(n^2) &= \sum_m \psi(m) a_f\left(\frac{n^2 - m^2}{4r}\right) \\ &= \sum_{l||r} \sum_{k_1, k_2} \psi(k_1 l - k_2 r/l) a_f(k_1 k_2) \end{aligned}$$

where the second sum is taken over integer k_1, k_2 satisfying $k_1 l + k_2 r/l = n$. If $k_1 = m$ then $k_2 = (n - lm)/(r/l)$. Using $\psi(2ml - n) = \psi(-1)\psi(n) =$

$\psi_l(-1)\psi(n)$ we can apply Selberg inversion to obtain

$$\begin{aligned} b(n^2) &= \sum_{l||r}\sum_{m}\psi(2ml-n)a_f\left(m\cdot\frac{n-lm}{r/l}\right) \\ &= \psi(n)\sum_{l||r}\psi_l(-1)\sum_{m}\sum_{e|\left(m,\frac{n-\ell m}{r/\ell}\right)}\mu(e)\chi(e)e^{\lambda-1}a_f\left(\frac{m}{e}\right)a_f\left(\frac{n-lm}{re/l}\right). \end{aligned}$$

If e divides r/ℓ then $\chi_f(e)$ vanishes and does not contribute to the sum. Therefore $e\nmid r/\ell$ and $e|\left(m,\frac{n-\ell m}{r/\ell}\right)=\left(m,\frac{n}{r/\ell}\right)$. Thus we can sum over $e|(m,n)$ and it follows that

$$\begin{aligned} b(n^2) &= \psi(n)\sum_{l||r}\psi_l(-1)\sum_{e|n}\mu(e)\chi(e)e^{\lambda-1}\sum_{m}a_f(m)a_f\left(\frac{n/e-lm}{r/l}\right) \\ &= \psi(n)\sum_{l||r}\psi_l(-1)\sum_{e|n}\mu(e)\chi(e)e^{\lambda-1}a_{g_l}(n/e) \\ &= \psi(n)\sum_{e|n}\mu(e)\chi(e)e^{\lambda-1}a_g(n/e). \end{aligned}$$

Therefore

$$\begin{aligned} B(n) &= \sum_{d|n}\chi(d)\psi(d)\chi_{0,4}(d)d^{\lambda-1}b\left(\frac{n^2}{d^2}\right) \\ &= \sum_{d|n}\chi(d)\psi(d)\chi_{0,4}(d)d^{\lambda-1}\psi(n/d)\sum_{e|\frac{n}{d}}\mu(e)\chi(e)e^{\lambda-1}a_g(n/de) \\ &= \sum_{d|n}\sum_{e|\frac{n}{d}}(de)^{\lambda-1}\chi\psi(de)\psi(n/de)a_g(n/de)\mu(e)\chi_{0,4}(d) \\ &= \sum_{s|n}\left(\frac{n}{s}\right)^{\lambda-1}\chi\psi\left(\frac{n}{s}\right)\psi(s)a_g(s)\sum_{e|\frac{n}{s}}\mu(e)\chi_{0,4}\left(\frac{n}{se}\right) \\ &= \left[a_{g\otimes\psi}(n)-2^{\lambda-1}\chi(2)\psi(2)a_{g\otimes\psi}(n/2)\right]. \end{aligned}$$

These are precisely the coefficients we claimed. □

We note that in all of these theorems so far, one takes a product of an old form $f(4r\tau)$ with a theta function h_ψ. We mention another result in a slightly different direction, due to Brown [2]:

Theorem 1.1.22 (Brown, 2013, [2] Theroem 2.4)**.** *We have*

(a) Let $f\in M_\lambda(N,\chi)$ be a Heigenform. Then

$$\sigma(f(24\tau)\eta_{24}(\tau))=(f(\tau)f(6\tau)-f(2\tau)f(3\tau))\otimes\left(\frac{12}{\cdot}\right)$$

lies in $S_{2\lambda}(144N,\chi^2)$.

(b) Let $f\in M_{\lambda-1}(N,\chi)$ *be a Heigenform. Then*

$$\sigma(f(8\tau)\eta_8^3(\tau)) = \frac{1}{2\pi i}\cdot(f(2\tau)f'(\tau)-f(\tau)f'(2\tau))\otimes\left(\frac{-4}{\cdot}\right)$$

lies in $S_{2\lambda}(16N,\chi^2)$.

We shall be able to recover the proof of (a) as an application of our main theorem. All of our Shimura images so far have involved the h_ψ. One can also ask what happens if we replace h_ψ by $V_D h_\psi$. This was answered recently by

Theorem 1.1.23 (Pandey, Ramakrishnan, 2022 [21])**.** *Let* $0<D|N$ *be square-free and assume* $(D,N/D)=1$*. Let* $f\in S_{\lambda-\nu}(N,\chi)$ *be a normalised Hecke eigenform. Fix* ψ *primitive mod* $r=r_\psi$ *with* $(r_\psi,D)=1$*. Set*

$$g_D = a_f(D)g$$

where g *is as in (1.1.2). Then*

$$\sigma_D(f(4r\tau)h_\psi(D\tau)) = g_D\otimes\psi(\tau) - 2^{\lambda-1}\chi(2)\psi(2)g_D\otimes\psi(2\tau)$$

lies in $S_{2\lambda}(2rN_r,\chi^2\psi^2)$

In the same paper, Pandey and Ramakrishnan also prove

Theorem 1.1.24 (Pandey, Ramakrishnan, 2022 [21])**.** *Let* $f\in S_{\lambda-\nu}(N,\chi)$ *be a normalised Heigenform and let* ψ *be primitive mod* $r=r_\psi$*. If* $D>0$ *is square-free then*

$$\sigma_D(f(4rD\tau)h_\psi(D\tau)) = \prod_{p|2D}(1-\chi(p)\psi(p)p^{\lambda-1}V_p)(g\otimes\psi)$$

lies in $S_{2\lambda}(2rN_r,\chi^2\psi^2)$.

1.2 The main theorem

In this chapter we fix $\xi\in\{1,2,4\}$ and define

$$(\xi',\xi'') := \begin{cases}(1,1) \text{ if } \xi=1\\ (2,1) \text{ if } \xi=2\\ (2,2) \text{ if } \xi=4.\end{cases}$$

Lemma 1.2.1. *Suppose* $\nu \in \{0,1\}$ *and* $\xi \in \{1,2,4\}$. *Let* $f(\tau) = \sum_{n\geq 1} a_f(n)q^n \in S_{\lambda-\nu}(N,\chi_f)$ *be a Heigenform and let* ψ *be a Dirichlet character modulo* N_ψ. *Let*

$$g_\ell = \begin{cases} V_\ell f \cdot U_{\xi'/\xi''} V_{N_\psi/\ell} f & \nu = 0, \\ (V_{\xi'\ell} f) \cdot q\frac{d}{dq}\left(V_{\xi'' N_\psi/\ell} U_2 f\right) & \nu = 1. \end{cases}$$

and let $\xi^* = 2$ *exactly when* $\nu = 0$, $\xi' = 1$ *and* $\xi^* = 1$ *otherwise. If* $f(\xi N_\psi \tau) h_\psi(\tau)$ *has coefficients* $b(n)$ *in the* q*-expansion, then*

$$b(n^2) = \psi(n) \sum_{\ell || N_\psi} \psi_{N_\psi/\ell}(-1) \sum_{e | \frac{2n}{\xi'' N_\psi/\ell}} \mu(e)\chi_f(e) e^{\lambda-1} a_{g_l}\left(\frac{\xi^* n}{e}\right).$$

We remind the reader that the notation $\ell || N_\psi$ means that ℓ is a divisor of N_ψ and that $\gcd(\ell, N_\psi/\ell) = 1$.

Proof A similar computation to those done previously shows

$$b(n^2) = \sum_{m\in\mathbb{Z}} a_f\left(\frac{n^2 - m^2}{\xi N_\psi}\right)\psi(m)m^\nu.$$

When $\frac{n^2-m^2}{\xi N_\psi}$ is not an integer, $a_f\left(\frac{n^2-m^2}{\xi N_\psi}\right) = 0$. Now let $\ell = \gcd\left(\frac{n-m}{\xi'}, N_\psi\right)$. We always have

$$n \equiv m \pmod{\xi'\ell} \quad \text{and} \quad n \equiv -m \pmod{\xi'' N_\psi/\ell}.$$

We claim that if $\ell' = (\ell, N_\psi/\ell) > 1$ then there is no contribution to the sum. Indeed, if $\ell' \neq 1$, by the above congruences we would have $m \equiv n \equiv -n \equiv 0$ mod $\xi'\ell'$ meaning $\psi(m) = 0$, hence no contribution. Therefore, without loss of generality we can assume that $\ell || N_\psi$. We have

$$\psi(m) = \psi_\ell(m)\psi_{N_\psi/\ell}(m) = \psi_\ell(n)\psi_{N_\psi/\ell}(-n) = \psi_{N_\psi/\ell}(-1)\psi(n).$$

Now we change the sum over m to a sum over m' via the substitution $n = m + \xi'\ell m'$. Making the change of variables $m \leftarrow m'$, we obtain

$$b(n^2) = \psi(n) \sum_{\ell || N_\psi} \psi_{N_\psi/\ell}(-1) \sum_{m\in\mathbb{Z}} a_f\left(\frac{m(2n - \xi'\ell m)}{\xi'' N_\psi/\ell}\right)(n - \xi'\ell m)^\nu.$$

Apply Selberg inversion (Lemma 1.1.15) the inner sum becomes

$$\sum_{m\in\mathbb{Z}} \sum_{e|(m,\frac{2n-\xi'\ell m}{\xi'' N_\psi/\ell})} \mu(e)\chi_f(e)e^{\lambda-\nu-1} a_f\left(\frac{m}{e}\right) a_f\left(\frac{2n-\xi'\ell m}{e\xi'' N_\psi/\ell}\right)(n-\xi'\ell m)^\nu$$

$$= \sum_{e|\frac{2n}{\xi'' N_\psi/\ell}} \mu(e)\chi_f(e)e^{\lambda-1} \sum_{m\in\mathbb{Z}} a_f(m)\, a_f\left(\frac{2n}{e\xi'' N_\psi/\ell} - \frac{\xi'\ell m}{\xi'' N_\psi/\ell}\right)\left(\frac{n}{e}-\xi'\ell m\right)^\nu.$$

When $\nu = 0$, we observe that the function

$$g_\ell = V_\ell f \cdot U_{\xi'/\xi''} V_{N_\psi/\ell} f$$

depends on ξ and has coefficients

$$a_{g_\ell}(n) = \sum_{m\in\mathbb{Z}} a_f(m) a_f\left(\frac{\xi'}{\xi''}\cdot\frac{n-\ell m}{N_\psi/\ell}\right).$$

Therefore,

$$b(n^2) = \psi(n)\sum_{\ell||N_\psi} \psi_{N_\psi/\ell}(-1) \sum_{e|\frac{2n}{\xi'' N_\psi/\ell}} \mu(e)\chi_f(e)e^{\lambda-1} a_{g_l}\left(\frac{\xi^* n}{e}\right).$$

When $\nu = 1$,

$$g_\ell = (V_{\xi'\ell} f)\cdot q\frac{d}{dq}\left(V_{\xi'' N_\psi/\ell} U_2 f\right)$$

has coefficients

$$a_{g_\ell}(n) = \sum_{m\in\mathbb{Z}} a_f(m) a_f\left(\frac{2n-\xi'\ell m}{\xi'' N_\psi/\ell}\right)(n-\xi'\ell m)$$

meaning

$$b(n^2) = \psi(n)\sum_{\ell||N_\psi} \psi_{N_\psi/\ell}(-1) \sum_{e|\frac{2n}{\xi'' N_\psi/\ell}} \mu(e)\chi_f(e)e^{\lambda-1} a_{g_l}\left(\frac{n}{e}\right).$$

□

With the aid of this lemma, we are now in a position to state and prove the main theorem. The lemma allows us to obtain explicitly, the coefficients of $f(\xi N_\psi \tau)h_\psi(\tau)$ under σ_1.

Theorem 1.2.2. *Suppose $\nu \in \{0,1\}$ and $\xi \in \{1,2,4\}$. Let ξ', ξ'', ξ^* be as in Lemma 1.2.1. Let $f \in S_{\lambda-\nu}(N,\chi_f)$ be a Heigenform and let ψ be a Dirichlet character modulo N. The image of $f(\xi N_\psi\tau)h_\psi(\tau)$ under the Shimura map σ_1*

has coefficients $B(n)$ equal to

$$\psi(n)\sum_{\ell||N_\psi}\psi_{N_\psi/\ell}(-1)\sum_{s|\frac{2n\ell}{\xi''N_\psi}}\chi_f\left(\frac{2n\ell}{s\xi''N_\psi}\right)\left(\frac{2n\ell}{s\xi''N_\psi}\right)^{\lambda-1}a_{g_\ell}\left(\frac{s\xi^*\xi''N_\psi}{2\ell}\right)c_{\chi_{1,4}^{\lambda+\nu}}\left(\frac{2n\ell}{s\xi''N_\psi}\right).$$

In particular, when $\lambda+\nu$ is even, if we set

$$g=\sum_{\ell||N_\psi}\psi_{N_\psi/\ell}(-1)g_\ell$$

with g_ℓ as in Lemma 1.2.1, then we have

$$\sigma_1(f(\xi N_\psi\tau)h_\psi(\tau))=\left[U_{\xi^*}-\chi_f(2)2^{\lambda-1}U_{\xi^*}V_2\right]g\otimes\psi.$$

Proof We use the fact that the coefficients of the image are

$$B(n)=\sum_{d|n}\chi_f\psi\chi_{1,4}^{\lambda+\nu}(d)d^{\lambda-1}b\left(\frac{n^2}{d^2}\right)$$

Indeed, the character of $f(\xi N_\psi\tau)h_\psi(\tau)$ is $\chi_f\psi\chi_{-4}^\nu$ whose checked character is $(\chi_f\psi\chi_{1,4}^\nu)\chi_{1,4}^\lambda=\chi_f\psi\chi_{1,4}^{\lambda+\nu}$. Applying Lemma 1.2.1, setting $2n\ell=eds\xi''N_\psi$ and changing the order of the sum,

$$\begin{aligned}B(n)&=\sum_{d|n}\chi_f(d)\psi(d)\chi_{1,4}^{\lambda+\nu}(d)d^{\lambda-1}\psi(n/d)\\&\quad\cdot\sum_{\ell||N_\psi}\psi_{N_\psi/\ell}(-1)\sum_{e|\frac{2n\ell}{d\xi''N_\psi}}\mu(e)\chi_f(e)e^{\lambda-1}a_{g_\ell}\left(\frac{\xi^*n}{de}\right)\\&=\psi(n)\sum_{\ell||N_\psi}\psi_{N_\psi/\ell}(-1)\sum_{d|n}\sum_{e|\frac{2n\ell}{d\xi''N_\psi}}(ed)^{\lambda-1}\chi_f(de)\mu(e)\chi_{1,4}^{\lambda+\nu}(d)a_{g_\ell}(\xi^*n/ed).\end{aligned}$$

This simplifies to

$$B(n)=\psi(n)\sum_{\ell||N_\psi}\psi_{N_\psi/\ell}(-1)\sum_{s|\frac{2n\ell}{\xi''N_\psi}}\left(\frac{2n\ell}{s\xi''N_\psi}\right)^{\lambda-1}\chi_f\left(\frac{2n\ell}{s\xi''N_\psi}\right)a_{g_\ell}\left(\frac{s\xi^*\xi''N_\psi}{2\ell}\right)$$
$$\cdot\sum_{e|\frac{2n\ell}{s\xi''N_\psi}}\mu(e)\chi_{1,4}^{\lambda+\nu}\left(\frac{2n\ell}{es\xi''N_\psi}\right).$$

Using the definition of the Möbius inverse proves the claim about $B(n)$. Now if $\lambda+\nu$ is even, we can apply Lemma 1.1.14,

$$c_{\chi_{0,4}}\left(\frac{2n\ell}{s\xi''N_\psi}\right)=\begin{cases}1 & 2n\ell=s\xi''N_\psi\\ -1 & 2n\ell=2s\xi''N_\psi\\ 0 & \text{otherwise.}\end{cases}$$

As such, when $g=\sum_{\ell||N_\psi}\psi_{N_\psi/\ell}(-1)g_\ell$,

$$B(n)=\psi(n)\left[a_g(\xi^*n)-\chi_f(2)2^{\lambda-1}a_g(\xi^*n/2)\right].$$

Reading off the coefficients yields the result. □

1.3 Consequences

In this section we look at special cases of the main theorem.

1.3.1 Trivial theta character

We can apply Theroem 1.2.2 when ψ is trivial. Let us first look at the case $N_\psi=1$ and $\nu=0$. By reading off the coefficients we obtain

Corollary 1.3.1. *Let $f\in S_\lambda(N,\chi_f)$ be a Heigenform. Then*

$$\sigma_1(f(\tau)\theta(\tau))=U_2(f(\tau)^2)-2^{\lambda-1}\chi_f(2)f(\tau)^2$$

and

$$\sigma_1(f(2\tau)\theta(\tau))=\left[I-\chi_f(2)2^{\lambda-1}V_2\right](f\cdot U_2f).$$

Example 1.3.2. When $\xi=4$ we immediately recover Selberg's Proposition 1.1.16, namely

$$\sigma_1(f(4\tau)\theta(\tau))=f(\tau)^2-2^{\lambda-1}\chi_f(2)V_2f(\tau)^2.$$

We make the choice $\psi=(12/\cdot)$ and $\xi=2$. This means $\xi'=2=\xi^*$ and $\xi''=1$. As discussed in the introduction, $h_\psi=\eta_{24}$. If f is a Heigenform of even level, $\chi_f(2)=0$. We factor $\psi=\psi_3\chi_{1,4}$ where ψ_3 is the non-trivial character modulo 3. Combining all of this with our main theorem gives

$$\begin{aligned}\sigma_1(f(24\tau)\eta_{24}(\tau)) &= (U_2V_2-\chi_f(2)2^{\lambda-1}V_2)g\otimes\psi=g\otimes\psi\\ &= \sum_{\ell||12}\psi_{12/\ell}(-1)V_\ell f\cdot U_2V_{12/\ell}f\otimes\psi.\end{aligned}$$

The four terms in the sum are

$$\begin{aligned}\sum_{\ell||12}\psi_{12/\ell}(-1)V_\ell f\cdot U_2V_{12/\ell}f &= \left(\frac{12}{-1}\right)f\cdot U_2V_{12}f+\chi_{1,4}(-1)V_3f\cdot U_2V_4f\\ &+ \psi_3(-1)V_4f\cdot U_2V_3f+V_{12}fU_2f\\ &= f(\tau)f(6\tau)-f(3\tau)f(2\tau)\\ &- f(4\tau)U_2f(3\tau)+f(12\tau)U_2f(\tau).\end{aligned}$$

We now use the fact that since $2|N$, U_2 acts as the Hecke operator $U_2f = T_2f = a_f(2)f$ which commutes with V_2, so

$$\begin{aligned}V_4f\cdot U_2V_3f = V_4f\cdot V_3U_2f &= V_4fV_3T_2f = a_f(2)V_4f\cdot V_3f\\ &= V_4U_2f\cdot V_3f = U_2V_4f\cdot V_3f = V_2f\cdot V_3f.\end{aligned}$$

A similar argument shows $V_{12}f\cdot U_2f = f\cdot V_6f$. Therefore we obtain

$$\sigma_1(f(24\tau)\eta(24\tau)) = 2\left[f(\tau)f(6\tau)-f(3\tau)f(2\tau)\right]\otimes\left(\frac{12}{\cdot}\right).$$

Up to a multiple of 2 (we used a different re-normalisation), this proves Theroem 1.1.22(a).

We now give an example for the case $\xi = 1$:

Example 1.3.3. Let $f \in S_9^{new}(4, \chi_{-4})$ be given by

$$f(z) = q + 16q^2 + 256q^4 - 1054q^5 + O(q^6)$$

then $f\theta \in S_{19/2}^{new}(4)$ and

$$f(z)^2 = q^2 + 32q^3 + 256q^4 + 512q^5 + 6048q^6 - 33728q^7 + 65536q^8 + O(q^9).$$

We check that the first four coefficients (Sturm bound) are

$$\sigma(f\theta) = q + 256q^2 + 6048q^3 + 65536q^4 + O(q^5)$$

which are precisely the even coefficients of f^2, hence $U_2f^2 = \sigma(f\theta)$ and we do indeed get the conclusion of the theorem.

Theorem 1.3.4. *Suppose λ is even and let $f \in S_\lambda^{new}(N_f, \chi_f)$ be a normalized newform and let ψ be an even character of conductor dividing N_f. Set*

$$g := (f\otimes\psi)\cdot f.$$

Then

$$\sigma_1(fh_\psi) = U_2g - 2^{\lambda-1}\chi_f(2)\psi(2)g.$$

Proof As usual we put $\chi = \chi_f \psi$. The coefficients of fh_ψ are

$$b(n) = \sum_m a_f(n-m^2)\psi(m)m^\nu$$

Since ψ is even this corresponds to the case when $\nu = 0$. Selberg inversion Lemma 1.1.15 allows

$$\begin{aligned} b(n^2) &= \sum_m a_f\left((n+m)(n-m)\right)\psi(m)m^\nu \\ &= \sum_{e|n} \mu(e)\chi_f(e)e^{\lambda-1} \sum_m a_f(m) a_f\left(\frac{2n}{e}-m\right)\psi\left(\frac{2n}{e}-m\right) \\ &= \sum_{e|n} \mu(e)\chi(e)e^{\lambda-1} a_g\left(\frac{2n}{e}\right). \end{aligned}$$

A similar argument to that of Proposition 1.1.16 gives

$$\begin{aligned} B(n) &= \sum_{d|n} \check{\chi}(d)d^{\lambda-1} \sum_{e|\frac{n}{d}} \mu(e)\chi(e)e^{\lambda-1} a_g\left(\frac{2n}{ed}\right) \\ &= \sum_{s|n} \chi(n/s)(n/s)^{\lambda-1} c_{\chi_{0,4}}(n/s) a_g(2s) \\ &= a_g(2n) - 2^{\lambda-1}\chi_f(2)\psi(2)a_g(n), \end{aligned}$$

which are precisely the coefficients of the claimed form. □

Example 1.3.5. Apply the previous theorem with $f = \Delta \in S_{12}^{new}(1)$, so $\lambda = 12$, $N_f = 1$ and χ_f is trivial modulo 1. Then with ψ trivial modulo 1,

$$\sigma_1(\Delta\theta) = U_2\Delta^2 - 2^{11}\Delta^2.$$

1.4 Future directions

The purpose of this section is to briefly speculate and provide some questions for future work. The author feels that the elementary methods of this chapter can be pushed even further. We provide three problems where it is hoped that the methods of this chapter could aid in their solution.

Problem 1: In [14], it is shown that there is an isomorphism $M_\lambda \oplus M_{\lambda-2} \to M^+_{k/2}(4)$ mapping two forms $(f,g) \mapsto f(4\tau)\theta(\tau) + g(4\tau)H_{5/2}(\tau)$. Here $H_{5/2}$ denotes the Cohen-Eisenstein series as in §3 of [5]. We have already shown in Proposition 1.1.16 how to treat $\sigma_1(f(4\tau)\theta(\tau))$. To give a complete explicit answer in the level four case, it is desirable to obtain, if possible, an expression for $\sigma_1(g(4\tau)H_{5/2}(\tau))$. Can this be done?

Problem 2: Let Λ be a unimodular lattice and let the map $Q : \Lambda \to \mathbb{Z}$ given by $\mathbf{x} \mapsto Q[\mathbf{x}]$ be an integral quadratic form with respect to Λ of level N_Q. This means that Q restricted to Λ is integer-valued. Let $\theta_Q(\tau) = \sum_{\mathbf{x}\in\Lambda} q^{Q[\mathbf{x}]}$. By factoring $n^2 - Q[\mathbf{x}]$ can one apply Lemma 1.1.15 to find $\sigma_1(f(N_Q\tau)\theta_Q(\tau))$?

Problem 3: In a beautiful paper, Tunnel [29] computed for a specific weight one modular form f, the image $\sigma_1(f\theta)$ and matched it to the corresponding elliptic curve E to obtain an expression between $a_f(n)$ and $a_E(n)$. Can one use the main theorem to find an expression in general for $\sigma_1(f\theta)$ for any form f of weight one?

Chapter 2

Linear reproducing kernels

2.1 Introduction

The aim of this chapter is to study the properties of two linear reproducing kernels of half integral weight, the first being the Cohen kernel $\mathcal{C}_{k/2}(\tau, s; p/q)$ and the second being the double Eisenstein series $E_{k/2}(\tau; s, w)$. We shall define these two objects and prove their absolute convergence and functional equations.

2.1.1 Motivation

We introduce some of the ideas by considering first the integral weight setting. To maintain generality for now, let Γ be a Fuchsian group and $\mathfrak{a}$ a cusp of Γ. Consider first a general class of linear reproducing kernels of the form

$$\mathcal{K}_{\mathfrak{a}}(\tau) = \sum_{\gamma \in \Gamma_{\mathfrak{a}} \backslash \Gamma} (h|_{2\lambda}\gamma(\tau). \tag{2.1.1}$$

Here h is a nice enough function. The freedom in the choice of h allows, in practice, many examples occur in this way. These are of interest since (after a standard Rankin-Selberg argument) they enjoy the property that

$$\langle f, \mathcal{K}_{\mathfrak{a}} \rangle_{\Gamma} = \langle f, h \rangle_{\Gamma_{\mathfrak{a}}} \tag{2.1.2}$$

whenever f is a cusp form of weight 2λ. In the setting of (2.1.2), we say $\mathcal{K}_{\mathfrak{a}}$ is a reproducing kernel for h. When $h = h(\tau, z)$ is a function of two variables, the right hand side of (2.1.2) can be interpreted as an integral operator:

$$(Kf)(z) = \langle f, h(\cdot, z)\rangle_{\Gamma_{\mathfrak{a}}}$$

whose trace is found by integrating h along a "diagonal"

$$\mathrm{Tr}(K) = \int_{\Gamma\backslash\mathfrak{H}} \overline{h(\tau, -\overline{\tau})} y^{2\lambda} d\mu(\tau).$$

Standard examples of kernels include $h = 1$ (Eisenstein series) and $h(\tau) = e^{2\pi i n\tau}$ (Poincare series). Another example that has recently been of interest (see [7][8]), is the following choice:

$$h(z) = \sum_{\substack{\delta\in\Gamma_\infty\backslash\Gamma \\ c_\delta>0}} c_\delta^{w-1} j(\delta, z)^s.$$

Here $\Gamma = \mathbf{PSL}_2(\mathbb{Z})$ and c_δ denotes the bottom-left entry of the matrix representative $\delta \in \Gamma_\infty\backslash\Gamma$. Upon substituting this into (2.1.1), we obtain

$$E_{2\lambda}(\tau; s, w) := \sum_{\substack{\gamma,\delta\in\Gamma_\infty\backslash\Gamma \\ c_{\delta\gamma^{-1}}>0}} c_{\delta\gamma^{-1}}^{w-1} \left(\frac{j(\gamma, \tau)}{j(\delta, \tau)}\right)^{-s} j(\gamma, \tau)^{-2\lambda} \tag{2.1.3}$$

we stress that since the lower left entry c_γ of a matrix $\gamma \in \mathbf{PSL}_2(\mathbb{Z})$ is only defined up to ± 1, we take c_γ to be the unique representative such that $c_\gamma > 0$. This ensures as well that

$$\Im\left(\frac{j(\gamma, \tau)}{j(\delta, \tau)}\right) = c_{\delta\gamma^{-1}} \Im(\tau) > 0,$$

so that (2.1.3) is well defined. We mention as well that the construction in (2.1.1) allows us, provided that h decays sufficiently fast, to average over the whole group Γ not just the quotient $\Gamma_{\mathfrak{a}}\backslash\Gamma$:

$$\mathcal{K}(\tau) = \sum_{\gamma\in\Gamma} h|_{2\lambda}\gamma.$$

With less control on the decay of h, in general we should expect that this limits the range of convergence of the corresponding $\mathcal{K}$. For instance, if $R_{h,\epsilon}$ is the region of points $z \in \mathfrak{H}$ such that

$$|h(z)| \ll_\epsilon y^{-1-\epsilon} \qquad \text{as } y \to \infty,$$

then $\mathcal{K}$ is absoultely convergent, uniformly on compact subsets on $R_{h,\epsilon}$. To see this, observe

$$\begin{aligned} |\mathcal{K}(z)| &\leq \sum_{\gamma} |j(\gamma, z)|^{-2\lambda} |h(\gamma z)| \\ &\ll \sum_{(c,d)=1} ((cx+d)^2 + c^2y^2)^{-\lambda} \Im(\gamma z)^{-1-\epsilon} \\ &= \sum_{(c,d)=1} ((cx+d)^2 + c^2y^2)^{-\lambda+1+\epsilon} y^{-1-\epsilon} \\ &\ll y^{-1-\epsilon} \sum_{c \neq 0} c^{-2(\lambda-1)} \\ &\ll y^{-1-\epsilon}. \end{aligned}$$

Let us illustrate this with an example. Consider $h(\tau, s) = (\tau + p/q)^{-s}$ where p, q are integers. Then h has corresponding kernel

$$C_{2\lambda}(\tau, s; p, q) := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_1} j(\gamma, \tau)^{-2\lambda} (\gamma\tau + p/q)^{-s}, \tag{2.1.4}$$

the so called Cohen kernel. Diamantis and O'Sullivan ([7] Prop.5.1.(ii)) showed that $C_{2\lambda}(\tau, s)$ is absolutely convergent on the strip $0 < \Re(s) < \lambda$. Our aim will be to define (2.1.4) and (2.1.3) in the half integral weight setting. Analogous to the classical kernels above, for a Fuschsian subgroup H of $\mathcal{G}$ as in §0.2.2, we consider

$$\mathcal{K}_{\mathfrak{a}} = \sum_{\gamma \in H_{\mathfrak{a}} \backslash H} h|_{k/2} L(\gamma).$$

The choice

$$h(\tau) = \sum_{\substack{\delta \in \Gamma_\infty \backslash \Gamma \\ c_\delta > 0}} c_\delta^{w-1} \mathcal{J}(\delta, \tau)^s, \quad w,\, s \in \mathbb{C}$$

leads us to consider

Definition 2.1.1 (Double Eisenstein)**.**

$$E_{k/2}(\tau; s, w) := \sum_{\substack{\gamma, \delta \in L(B) \backslash H \\ c_{\gamma\delta^{-1}} > 0}} c_{\gamma\delta^{-1}}^{w-1} \left(\frac{\mathcal{J}(\gamma, \tau)}{\mathcal{J}(\delta, \tau)} \right)^{-s} \mathcal{J}(\delta, \tau)^{-k}.$$

Here B denotes the upper triangular matrices in $\Gamma = \mathrm{proj}(H)$. It is useful as well to define the completed double Eisenstein series

$$E^*_{k/2}(\tau; s, w) := \zeta(2 - 2w + s)\zeta(2 - 2w + k - s)E_{k/2}(\tau; s, w). \tag{2.1.5}$$

2.2 Meromorphic continuation of the Cohen kernel

As discussed in the introduction of this chapter, the integral weight kernel (2.1.4) was shown by Diamantis and O'Sullivan ([7]) to be absolutely convergent and have a meromorphic continuation. Our aim in this section will be to introduce the analogous object in the half integral weight setting. We begin with the natural definition:

Definition 2.2.1 (Cohen Kernel)**.** For $p/q \in \mathbb{Q}$ a cusp representative of a Fuchsian group H, we set

$$\mathcal{C}_{k/2}(\tau, s; p/q) := \sum_{\gamma\in H} (\gamma\tau + p/q)^{-s} \mathcal{J}(\gamma, \tau)^{-k}.$$

Implicitly, by $\mathcal{C}_{k/2}(\tau, s)$ we shall mean $\mathcal{C}_{k/2}(\tau, s; 0)$. Let us henceforth consider a cusp $\mathfrak{a}$ of H and set

$$c_H := \max\left\{|c|^{-2} : c \neq 0, \left(\begin{smallmatrix} * & * \\ c & * \end{smallmatrix}\right) \in H\gamma_{\mathfrak{a}}\right\}.$$

Then the following lemma is useful

Lemma 2.2.2 (Jorgenson, O'sullivan[12] Lem.A1)**.** *For any $\tau \in \mathfrak{H}$ and any cusp $\mathfrak{a}$,*

$$\max_{\gamma\in H} \Im(\gamma\gamma_{\mathfrak{a}}\tau) \leq (c_H + c_H^{-1})(y + y^{-1}).$$

Proof Let Y be the right hand side of the inequality. Assume (for the sake of contradiction) that there exists $\gamma \in H$ such that $\Im(\gamma\gamma_{\mathfrak{a}}\tau) > Y$. Then

$$|j(\gamma\gamma_{\mathfrak{a}}, \tau)|^2 = \frac{\Im\tau}{\Im(\gamma\gamma_{\mathfrak{a}}\tau)} < \frac{y}{Y}.$$

If $c_{\gamma\gamma_{\mathfrak{a}}} = 0$ this implies $y = \Im(\gamma\gamma_{\mathfrak{a}}\tau) > Y$, so the integer $|d_{\gamma\gamma_{\mathfrak{a}}}| = |j(\gamma\gamma_{\mathfrak{a}}, \tau)| < 1$ must vanish, a contradiction. If $c_{\gamma\gamma_{\mathfrak{a}}} \neq 0$, then $|c_{\gamma\gamma_{\mathfrak{a}}} y|^2 < y/Y$ so $Y < c_H/y$ but this implies $c_H \leq (c_H + c_H^{-1})(y^2 + 1) = Yy < c_H$, a contradiction. □

Lemma 2.2.3. *Fix $\tau \in \mathfrak{H}$ and let $B(\tau)$ be the ball of radius $1/2$ centered at τ. Set*

$$B := \left\{ w \in \mathfrak{H} : \Im(w) < \frac{3e}{2}(y + y^{-1}) \right\}.$$

Then

$$\bigcup_{\gamma \in H} B(\gamma\gamma_{\mathfrak{a}}\tau) \subseteq B,$$

where each point is counted with multiplicity $\ll y + y^{-1}$.

Remark 2.2.4. Since $c_H \geq 1$ for any Fuschsian group H, the sharpest bound in Lemma 2.2.2 is $\frac{3}{2}(y + y^{-1})$.

Proof If $w \in B(\tau)$ then $\Im(w) < ey$ so if $w \in B(\gamma\gamma_{\mathfrak{a}}\tau)$, by Lemma 2.2.2 $\Im(w) < e\Im(\gamma\gamma_{\mathfrak{a}}\tau) \leq \frac{3e}{2}(y + 1/y)$. □

In fact, Lemma 2.2.2 (see [12] Lem.A.1) remains valid if we change γ to $\sigma_{p,q}^{-1}\gamma$ where $\sigma_{p,q}^{-1}$ is the inverse of the matrix in $\mathbf{SL}_2(\mathbb{Z})$ that sends $i\infty$ to $-p/q$. In this case we have

$$\frac{\Im(\gamma\gamma_{\mathfrak{a}}\tau)}{|q|^2|\gamma\gamma_{\mathfrak{a}}\tau + p/q|^2} = \Im(\sigma_{p,q}^{-1}\gamma\gamma_{\mathfrak{a}}\tau) \ll_H y + y^{-1}.$$

Multiplying by $|q|^2$ we obtain

Lemma 2.2.5. *Let $\gamma \in H$ and $\tau \in \mathfrak{H}$. Then*

$$\frac{\Im(\gamma\gamma_{\mathfrak{a}}\tau)}{|\gamma\gamma_{\mathfrak{a}}\tau + p/q|^2} \ll_{q,H} y + y^{-1}.$$

Proposition 2.2.6 (Absolute Convergence of Cohen)**.** *On the region*

$$1 < 4\sigma < 2k - 1,$$

the series $\mathcal{C}_{k/2}(\tau, s; p/q)$ converges absolutely and uniformly on compact subsets.

Proof We use the integral bound of Imamoglu-O'Sullivan, valid for $1 \leq k \in \mathbb{R}$,

$$\frac{\Im(\tau)^k}{|\tau + p/q|^{4\sigma}} \ll \iint_{B(\tau)} \frac{\Im(w)^k}{|w + p/q|^{4\sigma}} d\mu w. \tag{2.2.1}$$

Here $B(\tau)$ is a ball centered at τ of radius 1/2. We have

$$\begin{aligned}
|\mathcal{J}(\gamma_{\mathfrak{a}},\tau)^{-k}\mathcal{C}_{k/2}(\gamma_{\mathfrak{a}}\tau,s)|^4 &\leq \sum_{\gamma\in H}|\gamma\gamma_{\mathfrak{a}}\tau+p/q|^{-4\sigma}|\mathcal{J}(\gamma\gamma_{\mathfrak{a}}\tau)|^{-4k}\\
&\ll y^{-k}\sum_{\gamma\in H}\frac{\Im(\gamma\gamma_{\mathfrak{a}}\tau)^k}{|\gamma\gamma_{\mathfrak{a}}\tau+p/q|^{4\sigma}}\\
&\overset{(2.2.1)}{\ll} y^{-k}\sum_{\gamma\in H}\iint_{B(\gamma\gamma_{\mathfrak{a}}\tau)}\frac{\Im(w)^k}{|w+p/q|^{4\sigma}}d\mu w.
\end{aligned}$$

Let $w=u+iv$. Now Lemma 2.2.5 implies $|w+p/q|^{-2}\ll v^{-1}(y+y^{-1})$ provided that $w\in B(\gamma\gamma_{\mathfrak{a}}\tau)$ and this also holds for $w\in\cup_\gamma B(\gamma\gamma_{\mathfrak{a}}\tau)\subseteq B$ by Lemma 2.2.3. For $1<r<4\sigma$, we have

$$\frac{1}{|w+p/q|^{4\sigma-r}}\ll_q v^{r/2-2\sigma}(y+y^{-1})^{2\sigma-r/2}. \tag{2.2.2}$$

We bound

$$\begin{aligned}
\iint_B\frac{\Im(w)^k}{|w+p/q|^{4\sigma}}d\mu w &\leq \iint_B\frac{v^k}{|w+p/q|^r|w+p/q|^{4\sigma-r}}\frac{dudv}{v^2}\\
&\overset{(2.2.2)}{\ll} \int_0^{\frac{3e}{2}(y+y^{-1})}\int_{-\infty}^{\infty}\frac{v^{k+r/2-2\sigma}(y+y^{-1})^{2\sigma-r/2}}{((u+p/q)^2+(v^2))^{r/2}}\frac{dudv}{v^2}\\
&\ll (y+y^{-1})^{2\sigma-r/2}\int_0^{\frac{3e}{2}(y+y^{-1})}v^{k-r/2-2\sigma-2}dv\\
&\ll (y+y^{-1})^{2\sigma-r/2}(y+y^{-1})^{k-2\sigma-r/2-1}\\
&= (y+y^{-1})^{k-r-1}.
\end{aligned} \tag{2.2.3}$$

We see that for $k-r/2>2\sigma$,

$$\begin{aligned}
|\mathcal{J}(\gamma_{\mathfrak{a}},\tau)^{-k}\mathcal{C}_{k/2}(\gamma_{\mathfrak{a}}\tau,s)|^4 &\ll y^{-k}\sum_{\gamma\in H}\iint_{B(\gamma\gamma_{\mathfrak{a}}\tau)}\frac{\Im(w)^k}{|w+p/q|^{4\sigma}}d\mu w\\
&= y^{-k}(y+y^{-1})\iint_B\frac{\Im(w)^k}{|w+p/q|^{4\sigma}}d\mu w\\
&\overset{(2.2.3)}{\ll} y^{-k}(y+y^{-1})(y+y^{-1})^{k-r-1}\\
&\ll y^{-r}+y^{r-2k}.
\end{aligned}$$

On the region $1<r<4\sigma<2k-r<2k-1$, absolute convergence and uniform convergence on compact subsets now follows. □

Remark 2.2.7. As discussed in Diamantis-O'Sullivan (cf Prop 5.2(iii) in [8]) one finds that the domain of absolute convergence of the integral weight Cohen kernel $\mathcal{C}_k$ is $1 < \sigma < k-1$ which is in some sense "double the length" of our domain of convergence.

Remark 2.2.8. The domain of convergence of $\mathcal{C}_{k/2}$ permits values on the line $\sigma = 1$ to converge absolutely.

Corollary 2.2.9. *On the region $1 < 4\sigma < 2k-1$, $\mathcal{C}_{k/2}(\cdot, s; p/q)$ lies in $S_{k/2}(\Gamma)$.*

Proof Prop.2.2.6 proves uniform convergence in this region. Therefore $\mathcal{C}_{k/2}$ is holomorphic and the value of the coefficient $a_{\mathcal{C},\mathfrak{a}}(m)$ of the q expansion of $\mathcal{C}_{k/2}$ at the cusp $\mathfrak{a}$ is

$$\lim_{Y\to\infty}\int_0^1 \mathcal{J}(\gamma_{\mathfrak{a}}, x+iY)^{-k}\mathcal{C}_{k/2}\big(\gamma_{\mathfrak{a}}(x+iY), s; p/q\big)e(-mx)e^{2\pi mY}\,dx$$
$$= \lim_{Y\to\infty}(Y^{-r}+Y^{r-2k})e^{2\pi mY}.$$

When $m \leq 0$ this vanishes. □

In the special case when $k \geq 5$ is odd, we can exploit the symmetrized Hurwitz zeta function $\zeta_{\mathbb{Z}}(\tau, s) = \sum_{n\in\mathbb{Z}}(\tau+n)^{-s}$. Well known (cf [7](5.20)) is the estimate

$$\zeta_{\mathbb{Z}}(\tau, s) \ll_s \begin{cases} e^{-2\pi y}(1+y^{-\sigma}) & \sigma \neq 0 \\ e^{-2\pi y}(1+|\log y|) & \sigma = 0. \end{cases} \tag{2.2.4}$$

We can now use this bound to obtain a left plane of absolute convergence:

Proposition 2.2.10. *Suppose $k \geq 5$. Then $\mathcal{C}_{k/2}(\tau, s; p/q)$ is absolutely convergent on the left-plane $\sigma < \frac{k-4}{4}$.*

Proof Fix $\tau \in \mathfrak{H}$. The substitution $\gamma \mapsto \gamma\begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix}$ gives, after summing $n \in \mathbb{Z}$,

$$\begin{aligned}
|\mathcal{C}_{k/2}(\tau, s; p/q)| &= \left|\sum_{\gamma\in L(B)\backslash H}\sum_{n\in\mathbb{Z}}(\gamma\tau+n+p/q)^{-s}\mathcal{J}(\gamma,\tau)^{-k}\right| \\
&= \left|\sum_{\gamma\in L(B)\backslash H}\frac{\zeta_{\mathbb{Z}}(\gamma\tau+p/q, s)}{\mathcal{J}(\gamma,\tau)^k}\right| \\
&\overset{(2.2.4)}{\ll} y^{-k/4}\sum_{\gamma\in L(B)\backslash H}\Im(\gamma\tau)^{k/4-\sigma}+\Im(\gamma\tau)^{k/4} \\
&= y^{-k/4}\left[E(\tau, k/4-\sigma)+E(\tau, k/4)\right]
\end{aligned}$$

proving that $\sigma < k/4 - 1$ is sufficient for absolute convergence, since we assumed $k \geq 5$. □

Now fix p, q as before and consider the character $\chi_{p,q}$ that sends $m \mapsto e(mp/q)$ whenever $(m, q) = 1$ and zero otherwise. We are now in a position to prove the analog of Proposition 5.4 of [7]. By the L-function of $\bar{f} \otimes \chi_{p,q}$ we will mean the Dirichlet series given by

$$L^*(\bar{f} \otimes \chi_{p,q}, s) := \pi^{-s}\Gamma(s) \sum_{n \geq 1} \frac{\overline{a_f(n)}\chi_{p,q}(n)}{n^s}.$$

Observe that since p/q is a representative of a cusp for Γ,

$$L^*(\bar{f} \otimes \chi_{p,q}, s) = \int_0^\infty \bar{f}\left(iy + \frac{p}{q}\right) y^s \frac{dy}{y}$$

is analytic for all $s \in \mathbb{C}$.

Proposition 2.2.11. *For $k \geq 5$, $\sigma < k/4 - 1$ we have*

$$\langle \mathcal{C}_{k/2}(\cdot, s; p/q), f \rangle = \frac{\pi\Gamma(k/2-1)}{2^{k/2-2}e^{si\pi/2}\Gamma(s)\Gamma(k/2-s)} L^*(\bar{f} \otimes \chi_{p,q}, k/2 - s).$$

Proof We make use of

$$\mathcal{J}(\gamma,\tau)^{-k}\overline{\mathcal{J}(\gamma,\tau)^{-k}}(\Im\tau)^{k/2} = (\Im\gamma\tau)^{k/2}$$

and unfold

$$\begin{aligned}
\langle \mathcal{C}_{k/2}(\cdot, s; p/q), f \rangle &= \int_{H\backslash\mathfrak{H}} \sum_{\gamma \in L(B)\backslash H} \zeta_{\mathbb{Z}}(\gamma\tau + p/q, s)\overline{f(\gamma\tau)}\mathcal{J}^{-k}\overline{\mathcal{J}^{-k}}(\Im\tau)^{k/2} d\mu(\tau) \\
&= \int_{H\backslash\mathfrak{H}} \sum_{\gamma \in L(B)\backslash H} \zeta_{\mathbb{Z}}(\gamma\tau + p/q, s)\overline{f(\gamma\tau)}(\Im\gamma\tau)^{k/2} d\mu(\gamma\tau) \\
&= \int_{L(B)\backslash\mathfrak{H}} \zeta_{\mathbb{Z}}(\tau + p/q, s)\overline{f(\tau)} y^{k/2} d\mu(\tau) \\
&= \int_0^\infty \int_0^1 \frac{(2\pi)^s}{e^{si\pi/2}\Gamma(s)} \sum_{m,n} m^{s-1} e(m\tau - n\bar{\tau})\overline{a_f(n)} e(pm/q) y^{k/2} \frac{dxdy}{y^2} \\
&= \frac{(2\pi)^s}{e^{si\pi/2}\Gamma(s)} \int_0^\infty \sum_m m^{s-1} e^{-4\pi my}\overline{a_f(m)}\chi_{p,q}(m) y^{k/2-1} \frac{dy}{y} \\
&= \frac{(2\pi)^s}{e^{si\pi/2}\Gamma(s)} \frac{\Gamma(k/2-1)}{(4\pi)^{k/2-1}} \sum_{m \geq 1} \frac{\overline{a_f(m)}\chi_{p,q}(m)}{m^{k/2-s}} \\
&= \frac{\pi\Gamma(k/2-1)}{2^{k/2-2}e^{si\pi/2}\Gamma(s)\Gamma(k/2-s)} L^*(\bar{f} \otimes \chi_{p,q}, k/2 - s).
\end{aligned}$$

The last line follows by definition. □

Corollary 2.2.12. *For $k \geq 5$, $\mathcal{C}_{k/2}(\tau, s; p/q)$ has a meromorphic continuation to all of $s \in \mathbb{C}$.*

Proof Take a normalised[1] eigenbasis $\{f_j\}$ of the Hecke operators $\mathfrak{T}_p$ on the space of weight $k/2$ level 4 cusp forms for all primes away from 2. Then we inherit meromorphic continuation from that of $L^*(\bar{f} \otimes \chi_{p/q}, s)$:

$$\begin{aligned} \mathcal{C}_{k/2}(\tau, s; p/q) &= \sum_j \langle \mathcal{C}_{k/2}, f_j \rangle f_j \qquad (2.2.5) \\ &= \frac{\pi\Gamma(k/2-1)}{2^{k/2-2} e^{si\pi/2}\Gamma(s)\Gamma(k/2-s)} \sum_j L^*(\bar{f}_j \otimes \chi_{p,q}, k/2-s) f_j. \end{aligned}$$

As a consequence, if $k \geq 5$ the left hand side of (2.2.5) gives a meromorphic continuation of $\mathcal{C}_{k/2}$ to all of $\mathbb{C}$. □

2.3 Absolute convergence of the double Eisenstein series

We remind ourselves that in Def.2.1.1 we defined the double Eisenstein series of weight k,

$$E_{k/2}(\tau; s, w) := \sum_{\substack{\gamma,\delta \in L(B)\backslash H \\ c_{\gamma\delta^{-1}} > 0}} c_{\gamma\delta^{-1}}^{w-1} \left(\frac{\mathcal{J}(\gamma, \tau)}{\mathcal{J}(\delta, \tau)} \right)^{-s} \mathcal{J}(\delta, \tau)^{-k}.$$

To each cusp $\mathfrak{a}$ we associate also

Definition 2.3.1 (Double Eisenstein with cusp $\mathfrak{a}$)**.** Fix a cusp $\mathfrak{a}$ and let $\gamma_{\mathfrak{a}}$ be the usual element that maps $i\infty$ to $\mathfrak{a}$. Define

$$E_{k/2,\mathfrak{a}}(\tau; s, w) := \mathcal{J}(\gamma_{\mathfrak{a}}, \tau)^k \sum_{\gamma,\delta \in L(B)\backslash H_{\mathfrak{a}}} c_{\gamma\delta^{-1}}^{w-1} \left(\frac{\mathcal{J}(\gamma, \tau)}{\mathcal{J}(\delta, \tau)} \right)^{-s} \mathcal{J}(\delta, \tau)^{-k}.$$

Here $H_{\mathfrak{a}} = \gamma_{\mathfrak{a}}^{-1} H \gamma_{\mathfrak{a}}$.

We next prove the absolute convergence of this kernel.

Proposition 2.3.2 (Absolute Convergence of Double Eisenstein)**.** *Fix $\tau \in \mathfrak{H}$ and $k \geq 9$ an odd integer. Let $r = \Re w$ and $\sigma = \Re s$. If $s, w \in \mathbb{C}$ are such that*

$$\begin{gathered} 4 < \sigma < k-4, \\ r < \min\left\{ \frac{k-\sigma}{2} - 1, \frac{\sigma}{2} - 1 \right\}, \end{gathered}$$

[1]By normalised we mean $\langle f_j, f_j \rangle = 1$ here.

then $E_{k/2,\mathfrak{a}}(\tau; s, w)$ converges absolutely and uniformly on compact sets. Moreover, for fixed s, w the form $E_{k/2,\mathfrak{a}}(\tau; s, w)$ lies in $S_{k/2}(\Gamma_0(4))$.

Proof Let $\max\{r, 1\} := r'$. We use the bound (cf [8] lemma 4.1)

$$c_{\gamma\delta^{-1}}^{r-1} \ll (\Im\gamma\tau)^{\frac{1-r'}{2}} (\Im\delta\tau)^{\frac{1-r'}{2}}.$$

Valid for $\gamma\delta^{-1} \notin L(B)$. To see this, note

$$\begin{aligned} c_{\gamma\delta^{-1}} &\leq j(\delta,\tau)\Im j(\gamma,\tau) - j(\gamma,\tau)\Im j(\delta,\tau) \\ &= \left(\frac{\overline{j(\delta,\tau)}}{j(\delta,\tau)} - \frac{\overline{j(\gamma,\tau)}}{j(\gamma,\tau)}\right) \frac{j(\gamma,\tau)j(\delta,\tau)}{2iy} \end{aligned}$$

since the first factor of the right hand side has absolute value ≤ 2, we see that

$$|c_{\gamma\delta^{-1}}| = \frac{|j(\gamma,\tau)||j(\delta,\tau)|}{y} = \Im\gamma\tau^{-1/2}\Im\delta\tau^{-1/2}.$$

Raising to $r - 1$ powers and ensuring that $\gamma\delta^{-1} \notin L(B)$, we see the claim. With this, since $|\mathcal{J}| = |j|^{1/2}$,

$$\begin{aligned} |\mathcal{J}(\gamma_\mathfrak{a},\tau)^{-k}E_{k/2,\mathfrak{a}}(\tau; w, s)| &\leq \sum_{\substack{\gamma,\delta\in L(B)\backslash H_\mathfrak{a} \\ \gamma\delta^{-1}\notin L(B)}} |c_{\gamma\delta^{-1}}^{w-1}| \left|\frac{\mathcal{J}(\gamma,\tau)}{\mathcal{J}(\delta,\tau)}\right|^{-\sigma} |\mathcal{J}(\delta,\tau)^{-k}| \\ &\ll y^{-k/4} \sum_{\substack{\gamma,\delta\in L(B)\backslash H_\mathfrak{a} \\ \gamma\delta^{-1}\notin L(B)}} c_{\gamma\delta^{-1}}^{r-1} (\Im\gamma\tau)^{\frac{\sigma}{4}} (\Im\delta\tau)^{\frac{k-\sigma}{4}} \\ &\ll y^{-k/4} \left(\sum_{\gamma,\delta\in L(B)\backslash H_\mathfrak{a}} - \sum_{\substack{\gamma,\delta\in L(B)\backslash H_\mathfrak{a} \\ \gamma\delta^{-1}\in L(B)}} \right) (\Im\gamma z)^{\frac{2-2r'+\sigma}{4}} (\Im\delta z)^{\frac{2-2r'+k-\sigma}{4}} \\ &= y^{-k/4} \left[E_\mathfrak{a}\left(\gamma_\mathfrak{a}\tau, \frac{2-2r'+\sigma}{4}\right) E_\mathfrak{a}\left(\gamma_\mathfrak{a}\tau, \frac{2-2r'+k-\sigma}{4}\right) \right. \\ &\quad \left. - E_\mathfrak{a}\left(\gamma_\mathfrak{a}\tau, 1 - r' + \frac{k}{4}\right) \right] \end{aligned}$$

In the last line we used that $\gamma\delta^{-1} \in L(B)$ means that $\Im\gamma\tau = \Im\delta\tau$. In particular, when $\sigma \in (4, k-4)$, the standard fact that $E_\mathfrak{a}(\tau, s)$ is absolutely convergent for $\Re s > 1$ proves the uniform convergence of $E_{k/2,\mathfrak{a}}$ in the desired region. A standard argument shows modularity, so it remains to prove that $E_{k/2,\mathfrak{a}}$ is cuspidal in τ provided we fix $s, w \in \mathbb{C}$. To do this, we want to show that the expansion at a different cusp is also bounded. Indeed, by choosing a different cusp $\gamma_{\mathfrak{b}}$ we see $\mathcal{J}(\gamma_{\mathfrak{b}}, \tau)^{-k} E_{k/2,\mathfrak{a}}(\gamma_{\mathfrak{b}}\tau; s, w)$ is bounded by

$$y^{-k/4}\left[E_{\mathfrak{a}}\left(\gamma_{\mathfrak{b}}\tau, \frac{2-2r'+\sigma}{4}\right)E_{\mathfrak{a}}\left(\gamma_{\mathfrak{b}}\tau, \frac{2-2r'+k-\sigma}{4}\right)\right.$$
$$\left. - E_{\mathfrak{a}}\left(\gamma_{\mathfrak{b}}\tau, 1-r'+\frac{k}{4}\right)\right] \ll T.$$

□

2.4 Lemmas on epsilon factors

The purpose of this section is to introduce two factors ε and $\tilde{\varepsilon}$ and prove some properties of these. This section can be ommited on first reading. These properties will be needed in the proof of the analytic continuation of the double Eisenstein series in section §2.5. Throughout this section we set $\gamma = \left(\begin{smallmatrix} * & * \\ a & b \end{smallmatrix}\right) \quad \delta = \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right)$ to be two distinct elements of $\Gamma_0^*(4)$. For a fixed integer $u \geq 1$ we always have

$$u^{-s/2}\left[v_\theta^*(\gamma)(a\tau+b)^{1/2}\right]^{-s} = \left(v_\theta^*(\gamma)(au\tau+bu)^{1/2}\right)^{-s}.$$

Set

$$\beta_u = \begin{pmatrix} 1/u & \\ & u \end{pmatrix}$$

and consider $\gamma_u = \beta_u^{-1}\gamma\beta_u$. Likewise for $v \geq 1$, consider $\delta_v = \beta_v^{-1}\delta\beta_v$. We define a new multiplier system on $\Gamma_u = \beta_u^{-1}\Gamma_0^*(4)\beta_u$ via

$$\tilde{v}(\gamma_u) := v_\theta^*(\gamma)\frac{\sigma(\gamma, \beta_u)}{\sigma(\beta_u, \gamma_u)} \tag{2.4.1}$$

where σ is as in (0.1.2), namely

$$\sigma(\gamma_1, \gamma_2) = \frac{j(\gamma_1, \gamma_2\tau)^{1/2} j(\gamma_2, \tau)^{1/2}}{j(\gamma_1\gamma_2, \tau)^{1/2}}.$$

Lemma 2.4.1. *For $\gamma \in \Gamma_0^*(4)$ and notation as above, we have*

(i) $\sigma(\gamma, \beta_u) = 1$

(ii) $\sigma(\beta_u, \gamma_u) = 1$

(iii) $\sigma(\gamma_u\delta_v^{-1},\delta_v) = \sigma(\gamma\delta^{-1},\delta) = \begin{cases} -1 & \text{if } a<0,\ c\geq 0 \\ 1 & \text{otherwise.} \end{cases}$

Proof (i) We are going to use Theorem 4.1 of [27], namely that

$$\sigma(\gamma,\beta_u) = (\sigma_\gamma\sigma_{\gamma\beta_u}, \sigma_{\beta_u}\sigma_{\gamma\beta_u})_\infty s(\gamma)s(\beta_u)s(\gamma\beta_u)^{-1}. \tag{2.4.2}$$

As a reminder, if $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ we denote by

$$\sigma_\gamma = \begin{cases} c & c\neq 0, \\ d & c=0 \end{cases} \qquad s(\gamma) = \begin{cases} 1 & c\neq 0 \\ sgn(d) & c=0. \end{cases}$$

The Hilbert symbol $(x,y)_\infty$ is -1 if both entries are negative, $+1$ otherwise. A quick calculation shows

$$s(\gamma) = \begin{cases} 1 \quad c\neq 0, \\ sgn(d) \quad c=0. \end{cases} = \begin{cases} 1 \quad c/u\neq 0, \\ sgn(du) \quad c/u=0. \end{cases} = s(\gamma\beta_u).$$

Since $s(\beta_u)=1$, substituting this into (2.4.2), we have

$$\sigma(\gamma,\beta_u) = (\sigma_\gamma\sigma_{\gamma\beta_u}, \sigma_{\beta_u}\sigma_{\gamma\beta_u})_\infty = \begin{cases} (c^2/u, c)_\infty & c\neq 0 \\ (d^2u, du^2)_\infty & c=0 \end{cases} = 1.$$

(ii) As in part (i), bearing in mind that $s(\beta_u)=1$ and $s(\gamma_u)=s(\gamma\beta_u)$

$$\begin{aligned} \sigma(\beta_u,\gamma_u) &= (\sigma_{\beta_u}\sigma_{\gamma\beta_u}, \sigma_{\gamma_u}\sigma_{\gamma\beta_u})_\infty s(\gamma_u)s(\gamma\beta_u)^{-1} \\ &= (\sigma_\gamma\sigma_{\gamma\beta_u}, \sigma_{\gamma_u}\sigma_{\gamma\beta_u})_\infty \\ &= \begin{cases} (c^2/u, c/u^3)_\infty & c\neq 0 \\ (du^2, d^2u)_\infty & c=0 \end{cases} = 1. \end{aligned}$$

(iii) Here we use Theorem 16 in [18]. A computation gives

$$4w(\gamma_u\delta_v^{-1},\delta_v) = \begin{cases} (1+\text{sgn}(c))(1-\text{sgn}(a)) & ac\neq 0 \\ 0 \quad a=0, & c\neq 0 \\ 2(1-\text{sgn}(a_{\delta_v})) \quad a\neq 0, & c=0. \end{cases}$$

Upon noticing that $\operatorname{sgn}(a_{\delta_v}) = \operatorname{sgn}(a_\delta)$ we see that the left hand side does not depend on u nor v. As such,

$$w(\gamma_u \delta_v^{-1}, \delta_v) = w(\gamma\delta^{-1}, \delta)$$

giving

$$\sigma(\gamma_u \delta_v^{-1}, \delta_v) = e^{i\pi w(\gamma_u \delta_v^{-1}, \delta_v)} = \sigma(\gamma\delta^{-1}, \delta).$$

For the final equality observe that by theorem 4.1 of [27][2] we have, for $ac \neq 0$,

$$\sigma(\gamma\delta^{-1}, \delta) = (na, ca)_\infty = (a, ca)_\infty = (a, -c)_\infty$$

since $n > 0$. Moreover, for $a = 0 \neq c$,

$$\sigma(\gamma\delta^{-1}, \delta) = (nb, cb)_\infty \operatorname{sgn}(b) = (nb, -n)_\infty \operatorname{sgn}(b) = 1.$$

Furthermore, for $a \neq 0 = c$,

$$\sigma(\gamma\delta^{-1}, \delta) = (na, da)_\infty \operatorname{sgn}(d) = (na, n)_\infty \operatorname{sgn}(d) = \operatorname{sgn}(d) = \operatorname{sgn}(a).$$

This concludes the proof. □

An immediate application of this lemma gives us that

$$\left(\frac{au^2\tau + bu^2}{cv^2\tau + dv^2}\right)^{1/2} = \sigma(\gamma\delta^{-1}, \delta)\frac{(au^2\tau + bu^2)^{1/2}}{(cv^2\tau + dv^2)^{1/2}}. \tag{2.4.3}$$

Proof Substitute $\gamma_1 = \gamma_{u^2}\delta_{v^2}^{-1}$, $\gamma_2 = \delta_{v^2}$ into (0.1.2). Then since

$$j(\gamma_{u^2}\delta_{v^2}^{-1}, \delta_{v^2}\tau) = \frac{\det(\delta_{v^2})(au^2\tau + bu^2)}{cv^2\tau + dv^2},$$

and $\det(\delta_{v^2}) = 1$, by (iii) of the previous lemma, we have $\sigma(\gamma_{u^2}\delta_{v^2}^{-1}, \delta_{v^2}) = \sigma(\gamma\delta^{-1}, \delta)$ so the claim follows. □

Definition 2.4.2. Set $\varepsilon : \Gamma_0^*(4) \times \Gamma_0^*(4) \to \{\pm 1, \pm i\}$ where

$$\varepsilon(a, b, c, d) = \varepsilon(\gamma, \delta) := \frac{v_\theta^*(\gamma)}{v_\theta^*(\delta)}\sigma(\gamma\delta^{-1}, \delta)^{-1}, \qquad \gamma, \delta \in \Gamma_0^*(4).$$

[2] One can just as well use theorem 16 in [18].

It follows from (2.4.3) that

$$\frac{v^*_\theta(\gamma)(au^2\tau+bu^2)^{1/2}}{v^*_\theta(\delta)(cv^2\tau+dv^2)^{1/2}} = \varepsilon(a,b,c,d)\left(\frac{au^2\tau+bu^2}{cv^2\tau+dv^2}\right)^{1/2}. \tag{2.4.4}$$

Let us now set

$$\mu_\gamma = \frac{v^*_\theta(-\gamma)}{v^*_\theta(\gamma)}.$$

Note that

$$\mu_\gamma = \begin{cases} -i & \text{if } a>0 \text{ or } a=0,\ b=-1, \\ i & \text{if } a<0 \text{ or } a=0,\ b=1 \end{cases} = \begin{cases} -\operatorname{sgn}(a)i & \text{if } a\neq 0, \\ \operatorname{sgn}(b)i & \text{if } a=0. \end{cases}$$

Lemma 2.4.3. *We have*

1. $\varepsilon(-a,-b,-c,-d)=\varepsilon(a,b,c,d)$
2. $\varepsilon(-a,-b,c,d)=\varepsilon(a,b,-c,-d)$
3. $\varepsilon(2b,-a/2,2d,-c/2)=\varepsilon(a,b,c,d)$
4. $\varepsilon(a,b-a,c,d-c)=\varepsilon(a,b,c,d)$

Proof 1. We have

$$\frac{\mu_\gamma}{\mu_\delta} = \begin{cases} \operatorname{sgn}(ac), & a\neq 0\neq c \\ -1 & a=0\neq c \\ 1 & a\neq 0=c \end{cases} = \frac{\sigma(\gamma\delta^{-1},\delta)}{\sigma(\gamma\delta^{-1},-\delta)}.$$

Hence

$$\frac{\varepsilon(-a,-b,-c,-d)}{\varepsilon(a,b,c,d)} = \frac{\mu_\gamma}{\mu_\delta}\frac{\sigma(\gamma\delta^{-1},\delta)}{\sigma(\gamma\delta^{-1},-\delta)} = \left(\frac{\mu_\gamma}{\mu_\delta}\right)^2 = 1.$$

2. We can multiply μ_γ/μ_δ by $\mu_\delta^2=-1$ to see

$$\mu_\gamma\mu_\delta = \begin{cases} -\operatorname{sgn}(ac) & ac\neq 0 \\ 1 \quad a=0\neq c \\ -1 \quad a\neq 0=c \end{cases} = \frac{\sigma(-\gamma\delta^{-1},-\delta)}{\sigma(-\gamma\delta^{-1},\delta)}.$$

Hence

$$\frac{\varepsilon(-a,-b,c,d)}{\varepsilon(a,b,-c,-d)} = \mu_\gamma\mu_\delta\frac{\sigma(-\gamma\delta^{-1},-\delta)}{\sigma(-\gamma\delta^{-1},\delta)} = (\mu_\gamma\mu_\delta)^2 = 1.$$

3. Recall that

$$\sigma(\gamma_1,\gamma_2\gamma_3)\sigma(\gamma_2,\gamma_3)=\sigma(\gamma_1\gamma_2,\gamma_3)\sigma(\gamma_1,\gamma_2)$$

and observe

$$\gamma W_4^{-1}=\begin{pmatrix} * & * \\ 2b & -a/2 \end{pmatrix}.$$

Set $\gamma_1=\gamma\delta^{-1},\gamma_2=\delta$ and $\gamma_3=W_4^{-1}$ into the above to see that

$$\begin{aligned}\varepsilon(2b,-a/2,2d,-c/2) &= \frac{v_\theta^*(\gamma W_4^{-1})}{v_\theta^*(\delta W_4^{-1})}\sigma(\gamma W_4^{-1}(\delta W_4^{-1})^{-1},\delta W_4^{-1})^{-1} \\ &= \frac{v_\theta^*(\gamma)}{v_\theta^*(\delta)}\frac{\sigma(\gamma,W_4^{-1})}{\sigma(\delta,W_4^{-1})}\sigma(\gamma\delta^{-1},\delta W_4^{-1})^{-1} \\ &= \frac{v_\theta^*(\gamma)}{v_\theta^*(\delta)}\sigma(\gamma\delta^{-1},\delta)^{-1}=\varepsilon(a,b,c,d)\end{aligned}$$

4. Similar to 3 but with T in place of W_4.

□

We need one final lemma on ε. Set

$$\tilde\varepsilon(au,bu,cv,dv):=\frac{\tilde v(\gamma_u)}{\tilde v(\delta_v)}\sigma(\gamma_u\delta_v^{-1},\delta_v).$$

Lemma 2.4.4. *For $u,v\geq 1$ we have*

$$\tilde\varepsilon(au,bu,cv,dv)=\varepsilon(a,b,c,d).$$

Proof This is just an application of Lemma 2.4.1. Indeed, substituting (2.4.1), by definition

$$\tilde\varepsilon(au,bu,cv,dv)=\frac{\frac{\sigma(\gamma,\beta_u)}{\sigma(\beta_u,\gamma_u)}}{\frac{\sigma(\delta,\beta_v))}{\sigma(\beta_v,\delta_v)}}\cdot\frac{v_\theta^*(\gamma)}{v_\theta^*(\delta)}\sigma(\gamma_u\delta_v^{-1},\delta_v),$$

and by parts (i),(ii) of Lemma 2.4.1 the factor on the left of v_θ^* is one. By part (iii), the factor on the right is $\sigma(\gamma\delta^{-1},\delta)$. □

This means that we can rewrite (2.4.4) as

$$\frac{v_\theta^*(\gamma)(au^2\tau+bu^2)^{1/2}}{v_\theta^*(\delta)(cv^2\tau+dv^2)^{1/2}}=\tilde\varepsilon(au^2,bu^2,cv^2,dv^2)\left(\frac{au^2\tau+bu^2}{cv^2\tau+dv^2}\right)^{1/2}. \qquad (2.4.5)$$

Now for $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ let $\varepsilon(A) = \varepsilon(a,b,c,d)$. By Lem2.4.3 part (1) we see $\varepsilon(-A) = \varepsilon(A)$. Likewise since S, T generate $\mathbf{SL}_2(\mathbb{Z})$, combining Lem2.4.3 parts (3) and (4) gives

$$\varepsilon(A\gamma^{-1}) = \varepsilon(A) \qquad \gamma \in \mathbf{SL}_2(\mathbb{Z}).$$

In particular $\tilde{\varepsilon}(Bh) = \tilde{\varepsilon}(B)$. In the sequel, $A \in \mathbf{GL}_2(\mathbb{Q})$ and so if $\tau \in \mathfrak{H}$, $(A\tau)^{1/2}$ will have argument in $(0, \pi/2)$. Since ε only takes values at fourth roots of unity, we see that $\varepsilon(A)(A\tau)^{1/2}$ never crosses the branch cut, so

$$(\varepsilon(A)(A\tau)^{1/2})^{-s} = \varepsilon(A)^{-s}(A\tau)^{-s/2}.$$

If we now set $A_{uv} = \left(\begin{smallmatrix} au^2 & bu^2 \\ cv^2 & dv^2 \end{smallmatrix}\right)$, lemma 2.4.4 says that $\tilde{\varepsilon}(A_{uv}) = \varepsilon(A)$. By the same argument,

$$\left(\tilde{\varepsilon}(A_{uv})(A_{uv}\tau)^{1/2}\right)^{-s} = \tilde{\varepsilon}(A_{uv})^{-s}(A_{uv}\tau)^{-s/2}. \tag{2.4.6}$$

Recall that Ξ is the subset of $\mathcal{G}$ consisting of elements $(\gamma, \phi) \in \mathcal{G}$ with $\gamma \in M_2(\mathbb{Z})$. We write Ξ_{n^2} for those elements in Ξ of determinant n^2 A consequence of Lemma 0.3.3(iii) we obtain the commutative diagram

$$\begin{array}{ccc} \mathcal{G} & \xrightarrow{\iota} & \mathcal{G}_k \\ {\scriptstyle \text{proj}}\downarrow & & \downarrow{\scriptstyle \text{proj}} \\ \mathbf{GL}_2^+(\mathbb{Q}) & \xrightarrow[\alpha \mapsto \frac{\alpha}{\sqrt{\det \alpha}}]{} & \mathbf{SL}_2(\mathbb{Q}) \end{array}$$

which after restricting to Ξ_{n^2} gives

$$\begin{array}{ccc} \Xi_{n^2} & \xrightarrow{\iota} & \Xi'_{n^2} \\ {\scriptstyle \text{proj}}\downarrow & & \downarrow{\scriptstyle \text{proj}} \\ nP_n\Gamma_0^*(4) & \longrightarrow & P_n\Gamma_0^*(4) \end{array}$$

As such by extension, we obtain an isomorphism of Hecke algebras

$$\mathfrak{R}(H, \Xi_{n^2}) \to \mathfrak{R}(\Gamma_0^*(4), P_n)$$

The usefulness of the bijection (iii) above is that it can now be pushed forward onto $\tilde{\varepsilon}$ to obtain a map $\mathbb{N} \times P\Gamma_0^*(4) \to \{\pm 1, \pm i\}$.

2.5 Analytic continuation of the double Eisenstein series

We now come back to the main object of interest, namely the completion of the Eisenstein series $E^*_{k/2}(\tau; s, w)$ stated in (2.1.5):

$$E^*_{k/2}(\tau; s, w) := \zeta(2 - 2w + s)\zeta(2 - 2w + k - s)E_{k/2}(\tau; s, w), \tag{2.5.1}$$

which expressed differently is

$$\sum_{u,v\geq 1} u^{2w-2-s}v^{2w-2+s-k} \sum_{\left(\begin{smallmatrix} * & * \\ a & b \end{smallmatrix}\right), \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \in \Gamma_0^*(4)} (ad-bc)^{w-1} \left(\frac{v_\theta^*(\gamma)(a\tau+b)^{1/2}}{v_\theta^*(\delta)(c\tau+d)^{1/2}}\right)^{-s} \mathcal{J}(\delta,\tau)^{-k}$$

which equals

$$\sum_{u,v\geq 1}\sum_{a,b,c,d} (au^2dv^2-bu^2cv^2)^{w-1} \left(\frac{v_\theta^*(\gamma)(au^2\tau+bu^2)^{1/2}}{v_\theta^*(\delta)(cv^2\tau+dv^2)^{1/2}}\right)^{-s} (v_\theta^*(\delta)(cv^2\tau+dv^2)^{1/2})^{-k}.$$

In this section we shall frequently reference the previous section §2.4 and refer to the notation therein. By (2.4.5), if we set $A_{uv} = \left(\begin{smallmatrix} au^2 & bu^2 \\ cv^2 & dv^2 \end{smallmatrix}\right)$ we have

$$\begin{aligned} \left(\frac{v_\theta^*(\gamma)(au^2\tau+bu^2)^{1/2}}{v_\theta^*(\delta)(cv^2\tau+dv^2)^{1/2}}\right)^{-s} &= \left(\tilde{\varepsilon}\begin{pmatrix} au^2 & bu^2 \\ cv^2 & dv^2 \end{pmatrix}\left(\frac{au^2\tau+bu^2}{cv^2\tau+dv^2}\right)^{1/2}\right)^{-s} \\ &= \tilde{\varepsilon}(A_{uv})^{-s}(A_{uv}\tau)^{-s/2}. \end{aligned}$$

The last equality is justified by the following. Suppose $\det A_{u,v} \neq 0$. Since we are assuming $-I \in \Gamma_0^*(4)$, we can assume after a change of variables $(au^2, bu^2, cu^2, du^2) \mapsto (-au^2, -bu^2, cu^2, du^2)$ if necessary, without loss of generality, that $A_{uv} \in \mathbf{GL}_2^+(\mathbb{Q})$ holds. Therefore if $\tau \in \mathfrak{H}$ then so does $A_{uv}\tau \in \mathfrak{H}$ so $(A_{uv}\tau)^{1/2}$ has argument lying in $(0, \pi/2)$. Since $\tilde{\varepsilon}$ only takes values at fourth roots of unity, using the fact that $\tilde{\varepsilon}$ remains unchanged under $(au^2, bu^2, cu^2, du^2) \mapsto (-au^2, -bu^2, cu^2, du^2)$ (for this see Lemma 2.4.3) $A_{uv}\tau$ never crosses the branch cut. This justifies the last equality.

Recall §0.6 we considered the map

$$P_{u^2}\Gamma_0^*(4) \times P_{v^2}\Gamma_0^*(4) \to \Delta_0^*(4)_{u^2,v^2}$$
$$\begin{pmatrix} * & * \\ au^2 & bu^2 \end{pmatrix}\begin{pmatrix} * & * \\ cv^2 & dv^2 \end{pmatrix} \mapsto \begin{pmatrix} au^2 & bu^2 \\ cv^2 & dv^2 \end{pmatrix} = A_{uv}$$

Recall that in (0.6.3) we obtained the bijection

$$\bigcup_{u\geq 1} P_u\Gamma_0^*(4) \times \bigcup_{v\geq 1} P_v\Gamma_0^*(4) \to \Delta_0^*(4)$$

Recall also the bijection (0.6.1)

$$\Delta_0^*(4) \to \Xi$$
$$A_{uv} \mapsto A$$

We now abuse notation slightly and also call $\tilde{\varepsilon}$ on Ξ the map defined by $\tilde{\varepsilon}$ on $\Delta_0^*(4)$ under the above bijection. We can now exploit both of these bijections. Indeed combining all this together we see

$$\begin{aligned} E_{k/2}^*(\tau; s, w) &= \sum_{u,v\geq 1} \sum_{A_{uv}\in\Delta_0^*(4)_{u^2v^2}} \det(A_{uv})^{w-1}\tilde{\varepsilon}(A_{uv})^{-s}(A_{uv}\tau)^{-s/2}\tilde{\mathcal{J}}(A_{uv},\tau)^{-k} \\ &= \sum_{\substack{A\in\Xi \\ \det A=\square}} \det(A)^{w-1}\tilde{\varepsilon}(A)^{-s}(A\tau)^{-s/2}\tilde{\mathcal{J}}(A,\tau)^{-k}. \end{aligned}$$

If we let $\xi_n = \left(\left(\begin{smallmatrix} 1 & \\ & n^2 \end{smallmatrix}\right), \sqrt{n}\right)$ then every $A \in \Xi$ of determinant n^2 in the sum above must appear in some coset $A \in H\xi_\nu$ of $H\xi_n H$ for some $\xi_\nu \in H\backslash H\xi_n H$. This means

$$\begin{aligned} E_{k/2}^*(\tau; s, w) &= \sum_{n\geq 1} n^{2w-2} \sum_{h\in H\xi_n H} \tilde{\varepsilon}(h)^{-s}\,(h\tau)^{-s/2}\,\tilde{\mathcal{J}}(h,\tau)^{-k} \\ &= \sum_{n\geq 1} n^{2w-2} \sum_{\xi_\nu\in H\backslash H\xi_n H} \left(\sum_{h\in H} \tilde{\varepsilon}(h)^{-s}(h\tau)^{-s}\mathcal{J}(h,\tau)^{-k}\right)|\xi_\nu. \end{aligned}$$

Now by Lemma 2.4.4 we see that $\tilde{\varepsilon}(h) = \varepsilon(W_4 h, h) = v_\theta^*(W_4) = \zeta_8^{-1}$ is independent of h. As such, recalling the definition of $\mathcal{C}_{k/2}$, we see

$$E_{k/2}^*(\tau; s, w) = \zeta_8^s \sum_{n\geq 1} n^{2w-2} \sum_{\xi_\nu\in H\backslash H\xi_n H} \mathcal{C}_{k/2}(\tau; s/2)|\xi_\nu$$

which can be expressed as

$$\zeta_8^s \sum_{n\geq 1} \frac{\mathfrak{T}_n \mathcal{C}_{k/2}(\tau; s/2)}{n^{k/2-2w}}.$$

By Corollary 2.2.12, we know that $\mathcal{C}_{k/2}(\cdot; s)$ is a cusp form for all $s \in \mathbb{C}$. Since $\mathfrak{T}_n$ preserves cusp forms, we see that $\langle \mathfrak{T}_n \mathcal{C}_{k/2}, f\rangle$ is meaningful provided f is a cusp form.

Remark 2.5.1. The reason for using u^2 and v^2 instead of the more natural u and v is the following. Had we chosen the latter, we would have obtained the expansion

$$E^*_{k/2}(\tau; s, w) = \zeta_8^s \sum_{n\geq 1} n^{w-1} \left(\mathcal{C}_{k/2}(\tau; s/2) | \left[H \left(\begin{smallmatrix} 1 & 0 \\ & n \end{smallmatrix} \right) H\right]\right).$$

Now observe that the multiplier $v^k_{\left(\begin{smallmatrix} 1 & \\ & n \end{smallmatrix} \right)}$ in Proposition 0.2.5(iii) is trivial if and only if n is a perfect square, hence there is no contribution for those terms.

We are finally in a position to prove the meromorphic contintuation of $E^*_{k/2}$

Theorem 2.5.2. *Let $k \geq 5$ be odd. If $\tau \in \mathfrak{H}$ is fixed, then $E^*_{k/2}(\tau; s, w)$ has a meromorphic continuation with respect to $(s, w) \in \mathbb{C}^2$.*

Proof Let $\{f_j\}$ be a Heigenbasis for $\Gamma_0^*(4)$ of weight $k/2$. By Lemma 0.3.9,

$$\begin{aligned} E^*_{k/2}(\tau; s, w) &= \sum_j \frac{\langle E^*_{k/2}, f_j\rangle}{\langle f_j, f_j\rangle} f_j = \sum_{n\geq 1}\sum_j \frac{\langle \mathfrak{T}_n \mathcal{C}_{k/2}(; s/2), f_j\rangle}{n^{k/2-2w}\langle f_j, f_j\rangle} f_j \\ &= \zeta_8^s \sum_{n\geq 1}\sum_j \frac{\langle \mathcal{C}_{k/2}(; s/2), \mathfrak{T}_n f_j\rangle}{n^{k/2-2w}\langle f_j, f_j\rangle} f_j. \end{aligned}$$

Now Thm.0.3.6 shows that $\mathfrak{T}_n f_j = b_j(n) f_j$ with

$$b_j(n) = \prod_{p|n} \left(a_j(p^2) + \chi(p) p^{\lambda-1} a_j(p)\right)$$

where $a_j(n)$ is the n-th coefficient of f_j in its q-expansion at ∞. We can bound this by using $|a_j(n)| \ll n^{k/4}$ to see

$$|b_j(n)| \ll n^{\max\{k/2, \lambda-1+k/4\}}.$$

This implies that $\sum \frac{b_j(n)}{n^s}$ is absolutely convergent for $s \gg 1$. Apply now

Prop.0.4.1 for the sequence $b_j(n)$ to see that

$$\sum_{n\geq 1}\frac{b_j(n)}{n^{2w}} = 2^{2w-k/2}\sum_{n\geq 1}\frac{\hat{b}_j(n)}{n^{k/2-2w}},$$

where $\hat{b}_j(n)$ denotes the coefficients of the image of W_4 on $\sum_{n\geq 1} b_j(n)q^n$. By Cor.2.2.12,

$$E^*_{k/2}(\tau; s, w) = \frac{4\pi\Gamma\left(k/2-1\right)\zeta_8^s}{2^{k/2}e^{si\pi/4}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{k-s}{2}\right)}\sum_j L^*(\bar{f}_j, k/2-s)\left(\sum_{n\geq 1}\frac{b_j(n)}{n^{k/2-2w}}\right)\frac{f_j}{\langle f_j, f_j\rangle}.$$

Finally, combining the functional equation (0.4.1) for f_j with the above, we see that

$$E^*_{k/2}(\tau; s, w) = \frac{4^{s+w+1}\pi\Gamma\left(k/2-1\right)}{2^k\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{k-s}{2}\right)}\sum_j L^*(\bar{f}_j, s)\left(\sum_{n\geq 1}\frac{\hat{b}_j(n)}{n^{2w}}\right)\frac{f_j}{\langle f_j, f_j\rangle}.$$

The right hand side now gives the desired meromorphic continuation. □

From the proof of this theorem, we see immediately the following corollary:

Corollary 2.5.3. *Denote*

$$\mathbf{E}^*_{k/2}(\tau; s, w) := \frac{2^k i^s\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{k-s}{2}\right)}{4^{s+w+1}\pi\Gamma\left(k/2-1\right)}E^*_{k/2}(\tau; s, w).$$

Let $f \in S_{k/2}(\Gamma_0^*(4), \chi)$ be a Heigenform with Heigenvalues $b(n)$, namely $\mathfrak{T}_n f = b(n)f$. Set

$$\left(\sum_{n\geq 1} b(n)q^n\right)\Bigg| W_4 = \sum_{n\geq 1}\hat{b}(n)q^n$$

Then for all $(s, w) \in \mathbb{C}^2$,

$$\langle \mathbf{E}^*_{k/2}(\,\cdot\,; s, w), f\rangle = L^*(\bar{f}, s)\sum_{n\geq 1}\frac{\hat{b}(n)}{n^{2w}}.$$

Remark 2.5.4. Although we have shown the invariance under $s \mapsto k/2 - s$ and $w \mapsto k/4 - w$, it is unclear if there are more "intertwining" functional relations between s and w, say of the form $(s, w) \mapsto (w, s)$ or $(s, w) \mapsto (2w, s)$. This establishes at least a $\mu_2 \times \mu_2$ symmetry as an analog of the D_8 symmetry equations (2-14) and (2-15) in [7]. We expect that no larger symmetry group exists, but this remains an open question.

2.6 The Shintani lift of the double Eisenstein series and its kernel

In this section we shall only deal with the special case of level one. Our ultimate aim will be to give an explicit expression in terms of L-functions of the Petersson inner product of an arbitrary half integral weight modular form f against the Shintani lift of $E_{2\lambda}(\tau; s, w)$ defined in (2.1.3). We set

$$A(\lambda, s) := \frac{1}{3 \cdot 2^{\lambda-1}} \frac{(-2\pi i)^s}{\Gamma(s)} \frac{\Gamma(2\lambda-1)}{(4\pi)^{4\lambda-1}},$$

and

$$A^*(\lambda, s, w) = \frac{(2\pi)^{2\lambda-s}(2\pi)^{2\lambda-w}}{\Gamma(2\lambda-s)\Gamma(2\lambda-w)} \cdot A(\lambda, s), \qquad w \in \mathbb{C}.$$

Recall that the plus space $S^+_{k/2}(4)$ is the subspace of $S_{k/2}(\Gamma_0(4))$ consisting of forms whose n-th Fourier coefficient vanishes whenever $\check{n} := (-1)^\lambda n \equiv 2, 3 \pmod 4$. This is the notation adopted in Kohnen's thesis [14] and different to that adopted in [15]. Given a fundamental discriminant D, for a level one cusp form ϕ of weight 2λ, the Shintani lift of ϕ with respect to D is defined to be

$$\sigma^*_D \phi := \sum_{\substack{m \geq 1 \\ \check{m} \equiv 0,1 \pmod 4}} r(\phi; D, \check{m}) q^m,$$

where

$$r_\lambda(\phi; D, \check{m}) := \sum_{Q \in \mathcal{Q}_{|D|m}} \omega_D(Q) \int_{C(Q)} \phi(z) d_Q z.$$

The sum above runs through reduced binary quadratic forms Q of discriminant $|D|m$ and when $Q = [A, B, C]$, the integrand is taken over a cycle $C(Q)$ defined by the image of $A|z|^2 + B\Re(z) + C = 0$ in $\mathbf{SL}_2(\mathbb{Z})\backslash\mathfrak{H}$. We take this path to be anticlockwise. Finally, $d_Q z = Q(z, 1)^\lambda dz$. This will be clearer next subsection.

Our main result in this section is the following:

Theorem 2.6.1. *Let $f \in S^+_{k/2}(4)$ and suppose $\phi_D \in S_{2\lambda}(1)$ is such that $\sigma_D f = \phi_D$ under Shimura's map. Then $\sigma^*_D E_{2\lambda}(\cdot; s, w)$ is the kernel for the product of four L functions, $\langle f, \sigma^*_D E^*_{2\lambda}(\cdot; s, w)\rangle$ has a meromorphic continuation with respect to $(s, w) \in \mathbb{C}^2$ and*

$$\langle f, \sigma^*_D E_{2\lambda}(\cdot; s, w)\rangle = A^*(\lambda, s, w) \frac{L^*(\phi_D, s) L^*(\phi_D, w)}{\zeta(2\lambda - s - w)\zeta(1 + s - w)}.$$

Moreover, when $\Re(s) > \lambda + 1$, $\langle f, \sigma_D^* E_{2\lambda}(\,\cdot\,; s, w)\rangle$ *equals*

$$A(\lambda, s)\frac{L(\check{\chi}, s-\lambda+1)L(\check{\chi}, w-\lambda+1)}{\zeta(2\lambda-s-w)\zeta(1+s-w)}\left(\sum_{n\geq 1}\frac{a_f(|D|n^2)}{n^s}\right)\left(\sum_{n\geq 1}\frac{a_f(|D|n^2)}{n^w}\right).$$

The main tool used is a result of Kohnen ([15] Thm 2) which we will review in the next section.

2.6.1 Background on Kohnen's theorem

Let $f \in S^+_{k/2}(4)$ be a cusp form and fix a fundamental discriminants D and $\check{m} = (-1)^\lambda m$. Denote by $\mathcal{Q}_D$ the set of binary quadratic forms of discriminant D. We define a map $\omega_D : \mathcal{Q}_{|D|m} \to \mathbb{C}$ via

$$Q = [A, B, C] \mapsto \begin{cases} 0 & \text{if } (A, B, C, |D|m) = 1 \\ \left(\frac{|D|m}{n}\right) & \text{if } (n, |D|m) = 1 \text{ and } n \text{ is represented by } Q. \end{cases}$$

This can be shown to be independent of the representative n and thus well defined. We set

$$f_\lambda(z; D, \check{m}) = \sum_{Q\in\mathcal{Q}_{|D|\check{m}}}\frac{\omega_D(Q)}{Q(z,1)^\lambda}.$$

Consider the Kohnen kernel defined by the following expansion with respect to $\tau \in \mathfrak{H}$,

$$\Omega_{\lambda,D}(z,\tau) := c_{\lambda,D}^{-1}\sum_{\substack{m\geq 1\\ \check{m}\equiv 0,1(4)}} m^{\lambda-1/2} f_\lambda(z; D, \check{m})e(m\tau), \qquad z \in \mathfrak{H}$$

for some constant $c_{\lambda,D}$. It turns out that one can express the Fourier series in z of $\Omega_{\lambda,D}(z,\tau)$ with coefficients involving half integral weight Poincaire series in the variable τ. The precise statement we mean is given in equation (2.6.3) which we will give shortly, so let us now give more details on this. First, let us define

Definition 2.6.2 (Poincare series, half integral weight)**.**

$$P^m_{k/2}(\tau) := \sum_{\gamma\in L(B)\backslash H}\mathcal{J}(\gamma,\tau)^{-k}e^{2\pi i m\gamma\tau}.$$

It is known (see [10](3.19)), that the q-expansion of $P^m_{k/2}$ is

$$\sum_{n\geq 1}\left[(n/m)^{(k-2)/4}\left(\delta_{nm}+2\pi i^{-k/2}\sum_{c>0}\frac{S_\theta(m,n;c)}{c}J_{k/2-1}\left(\frac{4\pi\sqrt{mn}}{c}\right)\right)\right]q^n,$$

where

$$S_\theta(m,n;c)=\sum_{\left(\begin{smallmatrix}a&*\\c&d\end{smallmatrix}\right)\in\Gamma_\infty\backslash\Gamma_0^*(4)/\Gamma_\infty}v_\theta^*(\gamma)e\left(\frac{ma+nd}{c}\right),\tag{2.6.1}$$

and

$$J_{k/2-1}\left(\frac{4\pi\sqrt{mn}}{c}\right)=\sum_{j\geq 0}\frac{(-1)^j(2\pi\sqrt{mn}/c)^{2j+k/2-1}}{j!\Gamma(j+k/2)}.$$

Moreover, if $\check{m}\equiv 0,1(4)$ then for any $f\in S_{k/2}(4)$ one also has[3]

$$\langle f,P^m_{k/2}\rangle=\frac{\Gamma(k/2-1)}{6(4\pi m)^{k/2-1}}a_f(m),\qquad m\geq 1.\tag{2.6.2}$$

Kohnen has established (Thm 1 of [15]) the expansion of $\Omega_{\lambda,D}(z,\tau)$ with respect to $z\in\mathfrak{H}$, namely

$$\Omega_{\lambda,D}(z,\tau)=B(\lambda,D)\sum_{n\geq 1}n^{2\lambda-1}\omega_n(-\bar{\tau})e(nz),\tag{2.6.3}$$

where

$$\omega_n(-\bar{\tau})=\sum_{d|n}\left(\frac{D}{d}\right)d^{-\lambda}P^{n^2|D|/d^2}_{k/2}(\tau).$$

Here $B(\lambda,D)=\frac{2^{2\lambda}(2\pi)^{\lambda-1}|D|^{\lambda-1/2}(\lambda-1)!}{(2\lambda-2)!}$. We can now state the result we shall need in the sequel.

Theorem 2.6.3 (Kohnen, [15] Thm.2.Eq.10))**.** *Let $\phi\in S_{2\lambda}(1)$. Then for every fixed $\tau\in\mathfrak{H}$, the form $\Omega_{\lambda,D}(\cdot,-\bar{\tau})$ lies in $S_{2\lambda}(1)$ and*

$$\langle\phi,\Omega_{\lambda,D}(\cdot,-\bar{\tau})\rangle=\sigma_D^*\phi.$$

2.6.2 A general lemma

Our aim in this next section will be to apply Thm.2.6.3 with the choice $\phi=\mathcal{K}$. The following lemma is of independent interest since it can be applied to any kernel $\mathcal{K}$ provided that (a) it is a cusp form and (b) its reproducing kernel h is "nice" in the sense described below.

[3] Compare with [15] Eq.4 where Kohnen's k is our λ and $i_{4N}^{-1}=i_4^{-1}=1/6$. See also [10] Thm.3.3.

Lemma 2.6.4. *Let $\lambda \geq 2$ and let $\mathcal{K} \in S_{2\lambda}$ be a kernel of the form (2.1.1) where*

$$h(z) = \sum_{m\geq 1} a_m(y) e^{2\pi i m x},$$

with $|e^{2\pi m y} a_m(y)| \ll 1$ uniformly in y. Then, the D-th Shintani lift $\sigma_D^* \mathcal{K}$ of $\mathcal{K}$ is

$$B(\lambda, D) \sum_{n\geq 1} n^{2\lambda-1} \omega_n(-\bar{\tau}) \int_0^\infty a_n(y) e^{-2\pi n y} y^{2\lambda-1} \frac{dy}{y}.$$

Here $B(\lambda, D) = \frac{2^{2\lambda}(2\pi)^{\lambda-1}|D|^{\lambda-1/2}(\lambda-1)!}{(2\lambda-2)!}$ and

$$\omega_n(-\bar{\tau}) = \left(\sum_{d|n} \left(\frac{D}{d}\right) d^{-\lambda} P_\lambda^{n^2|D|/d^2}(-\bar{\tau}) \right).$$

Proof By Thm.2.6.3, we unfold $\langle \mathcal{K}, \Omega_{\lambda,D} \rangle$, that is

$$\begin{aligned}
&\int_{\Gamma\backslash\mathfrak{H}} \sum_{\Gamma_\infty\backslash\Gamma} h(\gamma z) j(\gamma, z)^{-2\lambda} \overline{\Omega_{\lambda,D}(z, -\bar{\tau})} y^{2\lambda} d\mu(z) \\
&= \int_{\Gamma\backslash\mathfrak{H}} \sum_{\Gamma_\infty\backslash\Gamma} h(\gamma z) j(\gamma, z)^{-2\lambda} \overline{j(\gamma, z)^{-2\lambda} \Omega_{\lambda,D}(\gamma z, -\bar{\tau})} y^{2\lambda} d\mu(z) \\
&= \int_{\Gamma\backslash\mathfrak{H}} \sum_{\Gamma_\infty\backslash\Gamma} h(\gamma z) \overline{\Omega_{\lambda,D}(\gamma z, -\bar{\tau})} (\Im \gamma z)^{2\lambda} d\mu(\gamma z) \\
&= \int_{\Gamma_\infty\backslash\mathfrak{H}} h(z) \overline{\Omega_{\lambda,D}(z, -\bar{\tau})} y^{2\lambda} \frac{dx dy}{y^2}.
\end{aligned}$$

Now substitute the expansion of $\Omega_{\lambda,D}$ and the assumed expansion of h. By orthogonality we see that

$$\begin{aligned}
&B(\lambda, D) \int_0^\infty \sum_{m,n} n^{2\lambda-1} \omega_n(-\bar{\tau}) \left(\int_0^1 e(mx) e(-nx) dx \right) a_m(y) e^{-2\pi n y} y^{2\lambda-1} \frac{dy}{y} \\
&= B(\lambda, D) \sum_{n\geq 1} n^{2\lambda-1} \omega_n(-\bar{\tau}) \int_0^\infty a_n(y) e^{-2\pi n y} y^{2\lambda-1} \frac{dy}{y},
\end{aligned}$$

and the lemma follows. □

We point out that this method does not extend to kernels lying in $M_{2\lambda}$, due to convergence issues with the integral.

2.6.3 Proof of the theorem

Given $s, w \in \mathbb{C}$ fixed we will now find the Shintani lift of $E_{2\lambda}(z; s, w)$. We want to use Lemma 2.6.4 for $\mathcal{K}(z) = E_{2\lambda}(z; s, w)$. We need to find what the $a_n(y)$

are in this case, so we require the q-expansion of the sum

$$h(z) = \sum_{c_\gamma > 0} c_\gamma^{w-1} \sum_{(c,d)=1} (cz+d)^{-s},$$

as this will allow us to find $a_n(y)$, thence by Lemma 2.6.4 obtain the desired lift. First let $\chi_{0,c}(n)$ be the trivial character modulo c, that is $\chi_{0,c}(n) = 1$ if and only if $(n,c) = 1$ and zero otherwise. Then by the Lipschitz formula (see [6]§3.5),

$$\begin{aligned} \sum_{d\in\mathbb{Z}} \frac{\chi_{0,c}(d)}{(c\tau+d)^s} &= \frac{1}{c^s}\sum_{l=0}^{c-1}\chi_{0,c}(l)\sum_{d\in\mathbb{Z}}(\tau+l/c+d)^{-s} \\ &= \frac{(-2\pi i)^s}{c^s\Gamma(s)}\sum_{n\geq 0}\left(\sum_{l=0}^{c-1}\chi_{0,c}(l)e(ln/c)\right)n^{s-1}q^n, \qquad \Re(s) > 1. \end{aligned}$$

The inner sum can be expressed as a Ramanujan sum (see [11] Eq.3.1)

$$S(0,n;c) := \sum_{\ell\in(\mathbb{Z}/c\mathbb{Z})^\times} e\left(\frac{\ell n}{c}\right).$$

It is a well known fact (see [28](1.5.4)) that

$$Z_n(s) := \sum_{c\geq 1}\frac{S(0,n;c)}{c^s} = \frac{\sigma_{1-s}(n)}{\zeta(s)}, \qquad \Re(s) > 1.$$

We can read off the coefficient $a_n(y)$ of h to be

$$a_n(y) = \frac{(-2\pi i)^s}{\Gamma(s)} Z_n(s-w+1)n^{s-1}e^{-2\pi n y},$$

defined on the region $\Re(s) > \max\{1, \Re(w)\}$. In this same region, the conditions of Lemma 2.6.4 are satisfied. We deduce that up to a factor of $B(\lambda, D)$, $\sigma_D^* E_{2\lambda}(\cdot; s, w)$ equals

$$\begin{aligned} &\sum_n n^{2\lambda-1}\omega_n(-\bar\tau)\int_0^\infty \frac{(-2\pi i)^s}{\Gamma(s)}Z_n(s-w+1)n^{s-1}e^{-4\pi n y}y^{2\lambda-1}\frac{dy}{y} \\ = &\frac{(-2\pi i)^s}{\Gamma(s)}\sum_n n^{2\lambda-1}\omega_n(-\bar\tau)Z_n(s-w+1)n^{s-1}\int_0^\infty e^{-4\pi n y}y^{2\lambda-1}\frac{dy}{y} \\ = &\frac{(-2\pi i)^s}{\Gamma(s)}\sum_n n^{2\lambda-1}\omega_n(-\bar\tau)Z_n(s-w+1)n^{s-1}\frac{\Gamma(2\lambda-1)}{(4\pi n)^{2\lambda-1}} \\ = &\frac{(-2\pi i)^s\Gamma(2\lambda-1)}{(4\pi)^{2\lambda-1}\Gamma(s)}\sum_n n^{s-1}Z_n(s-w+1)\omega_n(-\bar\tau). \end{aligned}$$

Multiplying both sides by $B(\lambda, D)$ gives

$$\sigma_D^* E_{2\lambda}(\cdot; s, w) = \frac{2|D|^{\lambda-1/2}\Gamma(\lambda)(-2\pi i)^s}{(2\pi)^\lambda \Gamma(s)} \sum_{n\geq 1} n^{s-1} Z_n(s-w+1)\omega_n(-\bar{\tau}) \quad (2.6.4)$$

Proof [of Theorem 2.6.1] We have the relation $\sigma_D f = \phi_D$, which in terms of coefficients is

$$a_{\phi_D}(n) = \sum_{d|n} \left(\frac{D}{n}\right) d^{\lambda-1} a_f\left(\frac{n^2|D|}{d^2}\right).$$

Using (2.6.2), we see

$$\begin{aligned} \langle f, \omega_n(-\bar{\tau})\rangle &= \sum_{d|n} \left(\frac{D}{d}\right) d^{-\lambda} \langle f, P_\lambda^{n^2|D|/d^2}(-\bar{\tau})\rangle \\ &= \frac{\Gamma(k/2-1)}{6(4\pi)^{k/2-1}|D|^{k/2-1}} n^{2-k} \sum_{d|n} \left(\frac{D}{n}\right) d^{\lambda-1} a_f\left(\frac{n^2|D|}{d^2}\right) \\ &= \frac{\Gamma(k/2-1)}{6(4\pi|D|)^{k/2-1}} n^{1-2\lambda} a_{\phi_D}(n). \end{aligned} \quad (2.6.5)$$

Let us set

$$A(\lambda, s) = \frac{1}{3\cdot 2^{\lambda-1}} \cdot \frac{(-2\pi i)^s}{\Gamma(s)} \cdot \frac{\Gamma(2\lambda-1)}{(4\pi)^{2\lambda-1}}.$$

Then by (2.6.4) and (2.6.5),

$$\begin{aligned} \langle f, \sigma_D^* E_{2\lambda}(\cdot; s, w)\rangle &= \frac{3(-2\pi i)^s \Gamma(2\lambda-1)}{(2\pi)^\lambda \Gamma(s)\Gamma(\lambda)} \sum_{n\geq 1} n^{s-1} Z(s-w+1)\langle f, \omega_n(-\bar{\tau})\rangle \\ &= A(\lambda, s) \sum_{n\geq 1} \frac{Z_n(s-w+1)}{n^{2\lambda-s}} a_{\phi_D}(n), \end{aligned}$$

which simplifies to

$$A(\lambda, s) \sum_{n\geq 1} n^{s-2\lambda} a_{\phi_D}(n) \sum_{c\geq 1} c^{w-s-1} S(0, n; c) = \frac{A(\lambda, s)}{\zeta(s-w+1)} \sum_{n\geq 1} \frac{\sigma_{w-s}(n) a_{\phi_D}(n)}{n^{2\lambda-s}}.$$

This latter sum is known to converge to (see for instance corollary 10.8.2 in [6]),

$$\sum_{n\geq 1} \frac{\sigma_{w-s}(n) a_{\phi_D}(n)}{n^{2\lambda-s}} = \frac{L(\phi_D, 2\lambda-s) L(\phi_D, 2\lambda-w)}{\zeta(2\lambda-s-w)},$$

valid on $\lambda + 1 - \delta > \Re(s)$ for some $\delta > 0$. Our original region $\Re(s) > \max\{0, \Re(w)\}$ of absolute convergence for $\langle f, \sigma_D^* E_{2\lambda}\rangle$ has non-empty intersection with the region $\lambda + 1 - \delta > \Re(s)$ on the right hand side. Since the right hand side admits also a meromorphic continuation, we deduce that $\langle f, \sigma_D^* E_{2\lambda}\rangle$ does also. As such,

$$\langle f, \sigma_D^* E_{2\lambda}(\cdot; s, w)\rangle = A(\lambda, s)\frac{L(\phi_D, 2\lambda - s)L(\phi_D, 2\lambda - w)}{\zeta(2\lambda - s - w)\zeta(1 + s - w)}.$$

By the functional equation, we see

$$\langle f, \sigma_D^* E_{2\lambda}(\cdot; s, w)\rangle = A^*(\lambda, s)\frac{L^*(\phi_D, s)L^*(\phi_D, w)}{\zeta(2\lambda - s - w)\zeta(1 + s - w)}.$$

Finally, combine the above expression with equation (0.5.1) of Theorem 0.5.1; namely that we can express the L function of ϕ as an L-function of f as follows:

$$L(\phi_D, s) = L(\check{\chi}, s - \lambda + 1)\sum_{n\geq 1}\frac{a_f(|D|n^2)}{n^s}, \qquad \Re(s) > \lambda + 1.$$

Rearranging terms concludes the proof. □

2.7 A kernel for the Hecke operator

This section is independent of the other sections in this chapter. Our aim will be to find a reproducing kernel for the map $f \mapsto \mathfrak{T}_m f$. This is the analog of Thm.1 of Zagier's Appendix in [17] for the half integral weight setting.

Lemma 2.7.1. *Let $f \in S_{k/2}(\Gamma)$. Then*

$$\int_{-\infty}^{\infty}\frac{f(x+iy)}{(x - iy - w)^{k/2}}dx = \frac{2\pi}{\Gamma(k/2)}\sum_{n\geq 1} a_f(n)n^{k/2-1}e^{-4\pi ny}e(nw).$$

Proof Let C_R be the semicircle along $[-R, R]$. Then,

$$\left|\int_{C_R}\frac{e(nx)}{(x - iy - w)^{k/2}}dx\right| = o(e^{-R}R^{-k/2}).$$

Therefore we have

$$\begin{aligned}\int_{-\infty}^{\infty}\frac{e(nx)}{(x - iy - w)^{k/2}}dx &= 2\pi i \mathrm{Res}_{x=iy+w}\frac{e(nx)}{(x - iy - w)^{k/2}} \\ &\quad - \lim_{R\to\infty}\int_{C_R}\frac{e(nx)}{(x - iy - w)^{k/2}}dx \\ &= 2\pi i\frac{e(niy + nw)}{i\Gamma(k/2)}n^{k/2-1} - \lim_{R\to\infty} o(e^{-R}R^{-k/2}) \\ &= \frac{2\pi n^{k/2-1}}{\Gamma(k/2)}e^{-2\pi ny}e(nw).\end{aligned}$$

Then

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{f(x+iy)}{(x-iy-w)^{k/2}} dx &= \sum_{n\geq 1} a_f(n) e^{-2\pi n y} \int_{-\infty}^{\infty} \frac{e(nx)}{(x-iy-w)^{k/2}} dx \\ &= \frac{2\pi}{\Gamma(k/2)} \sum_{n\geq 1} a_f(n) n^{k/2-1} e^{-4\pi n y} e(nw),\end{aligned}$$

as desired. □

We introduce the following kernel

Definition 2.7.2. For $m \in \mathbb{N}$, $z, w \in \mathfrak{H}$, set

$$\mathcal{K}_m(z,w) := \sum_{\xi \in \Xi_{m^2}} (w+z)^{-k/2}|_{k/2}\xi,$$

the action here is with respect to the first variable.

As per our discussion in §2.1.1, for fixed w, $|w+z|^{-k/2} \ll |y|^{-k/2}$. It follows that for $k \geq 3$, $\mathcal{K}_m(\cdot, w)$ is holomorphic. We are now going to fix $\Gamma = \Gamma_0^*(4)$ and the choice of lift H as in §0.2.2. Then for $L(\gamma) \in H$, since $\Xi_{m^2} H = \Xi_{m^2}$,

$$\mathcal{K}_m(\cdot, w)|L(\gamma) = \sum_{\xi \in \Xi_{m^2}} (z+w)^{-k/2}|\xi L(\gamma) = \mathcal{K}_m(\cdot, w)$$

shows modularity in the first variable. This calculation remains valid for the two transition matrices $\gamma_\infty = T \in \Gamma_0^*(4)$ and $\gamma_{1/2} = \left(\begin{smallmatrix} 1 & \\ -2 & 1\end{smallmatrix}\right)$ and gives a holomorphic right hand side. This shows that $\mathcal{K}_m(\cdot, w)$ is a cusp form as well.

Proposition 2.7.3. *For all $f \in S_{k/2}(4, \chi)$,*

$$\langle f, \mathcal{K}_m \rangle = \frac{\chi(m)^2 (4\pi k)}{(4\pi m)^{k/2-1}} \mathfrak{T}_m f.$$

Proof We first prove it for $m = 1$. Since $\Xi_1 = H$,

$$\begin{aligned}\langle f, \mathcal{K}_1 \rangle &= \int_{\Gamma\backslash\mathfrak{H}} f(z) \overline{\mathcal{K}_1(z, -\bar{w})} y^{k/2} d\mu(z) \\ &= \int_{\Gamma\backslash\mathfrak{H}} f(z) \sum_{h\in H} (-w+\bar{z})^{-k/2}|h \; y^{k/2} d\mu(z) \\ &= \int_{\Gamma\backslash\mathfrak{H}} f(\gamma z) \sum_{\gamma\in\Gamma} (-w+\gamma\bar{z})^{-k/2} \Im(\gamma z)^{k/2} d\mu(\gamma z) \\ &= \int_{\mathfrak{H}} (-w+\bar{z})^{-k/2} f(z) y^{k/2} d\mu(z) \\ &= \int_0^\infty \int_{-\infty}^{\infty} \frac{f(x+iy)}{(x-iy-w)^{k/2}} dx y^{k/2-1} \frac{dy}{y}.\end{aligned}$$

Now the previous lemma tells us that

$$\begin{aligned}\langle f, \mathcal{K}_1\rangle &= \frac{2\pi}{\Gamma(k/2)}\sum_{n\geq 1} a_f(n)\int_0^\infty e^{-4\pi ny}(ny)^{k/2-1}\frac{dy}{y}e(nw)\\ &= k(4\pi)^{2-k/2}f(w).\end{aligned}$$

Replacing f by $\mathfrak{T}_m f$ shows $\langle \mathfrak{T}_m f, \mathcal{K}_1\rangle = k(4\pi)^{2-k/2}\mathfrak{T}_m f(w)$. Therefore

$$\begin{aligned}\langle f, \mathcal{K}_m\rangle &= \frac{\langle f, \mathfrak{T}_m \mathcal{K}_1\rangle}{m^{k/2-1}} = \chi(m)^2\frac{\langle \mathfrak{T}_m f, \mathcal{K}_1\rangle}{m^{k/2-1}}\\ &= \frac{\chi(m)^2(4\pi k)}{(4\pi m)^{k/2-1}}\mathfrak{T}_m f.\end{aligned}$$

□

Consider a Heigenbasis $\{f_j\}$ of $S_{k/2}(4,\chi)$. Then expanding $\mathcal{K}_m(\cdot, w)$ in this basis, we may write

$$\mathcal{K}_m(z,w) = \sum_j \frac{f_j(z)}{\langle f_j, f_j\rangle}g(w)$$

for some holomorphic function g. If f_i has Heigenvalue $\omega_{i,m}$, subsituting $f = f_i$ into the previous proposition shows

$$\omega_{i,,m}\frac{\chi(m)^2(4\pi k)}{(4\pi m)^{k/2-1}}f_i(w) = g(w)$$

therefore $g(w)$ is also a cusp form and

$$\mathcal{K}_m(z,w) = \sum_i \frac{\omega_{j,m}}{\langle f_i, f_i\rangle}\frac{\chi(m)(4\pi k)}{(4\pi m)^{k/2-1}}f_j(z)f_j(w).$$

Another interesting consequence of the proposition is the fact that it allows an explicit computation of the trace of $\mathfrak{T}_m$. Indeed,

$$\mathrm{Tr}(\mathfrak{T}_m) = \frac{(4\pi m)^{k/2-1}}{\chi(m)^2(4\pi k)}\int_{\Gamma_0^*(4)\backslash\mathfrak{H}} \mathcal{K}_m(z,-\bar{z})y^{k/2}\frac{dxdy}{y^2}. \qquad (2.7.1)$$

We end this section with the following open question. Can one exploit some kind of bijection (analogous to the integral weight case), such that matrices with fixed trace in Ξ_{m^2} get mapped to quadratic forms? If so, what is the arithmetic significance of the right hand side of (2.7.1) and can one obtain a similar result to Theorem 2 in Zagier's Appendix of Lang's book [17]?

Chapter 3

Period polynomials

This chapter is based on the joint work [1] coauthored with N. Diamantis, W. Raji and L. Rolen. After proving the key Lemma 3.0.2 on the family of transformation of Φ_a for each a, we construct the analogue of the period polynomial P_a for half integral weight cusp forms. We maintain the usual convention $k = 2\lambda + 1$, $\lambda > 2$. In this chapter it will be convenient to set $\ell = \lambda - 2$ and $\kappa = \ell + 1/2 = k/2 - 2$. This will mean that $\frac{k-5}{2} = \ell$.

3.0.1 The definition of P_a

Consider $f \in S_{k/2}(\Gamma_0^*(4N))$ with expansion

$$f = \sum_{n \geq 1} a_f(n) q^{n/w}$$

where w is the width of the cusp $i\infty$. Then, the usual bound on the coefficients $a_f(n)$ (we raise to the power of half the weight, i.e. $|a_f(n)| \ll n^{\frac{1}{2}\left(\frac{k}{2}\right)} = n^{k/4}$), gives

$$L^*(f, s) := \int_0^{i\infty} f(z) z^s \frac{dz}{z} = \frac{\Gamma(s) w^s}{(2\pi i)^s} \sum_{n \geq 1} \frac{a_f(n)}{n^s}, \qquad \Re(s) > k/4 + 1,$$

which has functional equation

$$L^*(f, s) = i^{k/2} (4N)^{k/4 - s} L^*(f|_{W_{4N}}, k/2 - s).$$

When m, n are half integral, we set

$$\binom{n}{m} := \frac{\Gamma(n+1)}{\Gamma(m+1)\Gamma(n-m+1)}.$$

It will be implicitly assumed unless otherwise specified that $a \in [0, \kappa+\ell]$. With this in mind we set, for fixed $a \in [0, \kappa+\ell]$

$$\Phi_a(\tau, z) := \sum_{n=0}^{\ell} \left[\binom{\kappa}{n} (4iN\tau)^n z^{a-n} + \frac{i^{k/2}}{(4N)^{1/4}} \binom{\kappa}{n+1/2} (i\tau)^n (-4Nz)^{\kappa+\ell-a-n} \right].$$

The analogue of the period polynomial is

Definition 3.0.1 (Period polynomial)**.** The period polynomial is defined to be

$$P_a(\tau) := \int_0^{i\infty} f(z)\Phi_a(\tau, z)dz.$$

We introduce as well

$$F_a(\tau) := \int_\tau^{i\infty} f(z)\Phi_a(\tau, z)dz.$$

Lemma 3.0.2. *Let* $\lambda > 2$, $a \in [0, \kappa+\ell]$. *Then*

1. $-(2i\sqrt{N}\tau)^\ell(-2i\sqrt{N}z)^\kappa (i\tau)^{n-\ell}(-4Nz)^{l-n-a} = \frac{i^{k/2}}{(4N)^{1/4}}(i\tau)^n(-4Nz)^{\kappa+\ell-a-n}$

2. $-(2i\sqrt{N}\tau)^\ell(-2i\sqrt{N}z)^\kappa \frac{i^{k/2}}{(4N)^{1/4}}(4iN\tau)^{n-\ell}z^{a-n-\kappa} = (4iN\tau)^n z^{a-n}$

3. $-(2i\sqrt{N}\tau)^\ell(-2i\sqrt{N}z)^\kappa \Phi_a(W_{4N}\tau, W_{4N}z) = \Phi_a(\tau, z)$.

Proof Observe first that $(4N)^{\kappa/2-\ell/2} = (4N)^{1/4}$.

1. If we use $-(-iz)^\kappa = i^{k/2}(-z)^\kappa$ then

$$-(4N)^{\ell/2+\kappa/2}(-iz)^\kappa = -\frac{1}{(4N)^{1/4}}(-4iNz)^\kappa = \frac{i^{k/2}}{(4N)^{1/4}}(-4Nz)^\kappa$$

   and (1) follows.

2. Use instead $-i^{k/2}(-iz)^\kappa = z^\kappa$ so

$$-(4N)^{\kappa/2-\ell/2}\frac{i^{k/2}}{(4N)^{1/4}}(-iz)^\kappa = -i^{k/2}(-iz)^\kappa = z^\kappa$$

   and (2) follows.

3. We make the change of variables $n \mapsto \ell - n$ so that $\binom{\kappa}{n}$ changes to

$\binom{\kappa}{\ell-n} = \binom{\kappa}{n+1/2}$. With this, we have that the left hand side of (3) is

$$\begin{aligned}
&-(2i\sqrt{N}\tau)^{\ell}(-2i\sqrt{N}z)^{\kappa}\sum_{n=0}^{\ell}\Bigg[\binom{\kappa}{n}(i\tau)^{-n}(-4Nz)^{n-a} \\
&\qquad + \frac{i^{k/2}}{(4N)^{1/4}}\binom{\kappa}{n+1/2}(4iN\tau)^{-n}z^{a+n-\ell-\kappa}\Bigg] \\
&\stackrel{n\mapsto\ell-n}{=} -(2i\sqrt{N}\tau)^{\ell}(-2i\sqrt{N}z)^{\kappa}\sum_{n=0}^{\ell}\Bigg[\binom{\kappa}{n+1/2}(i\tau)^{n-\ell}(-4Nz)^{\ell-n-a} \\
&\qquad + \frac{i^{k/2}}{(4N)^{1/4}}\binom{\kappa}{n}(4iN\tau)^{n-\ell}z^{a-n-k}\Bigg].
\end{aligned}$$

Apply parts (1) and (2), to conclude. □

By a character χ of $\Gamma_0^*(4N)$ we mean a character χ of $\Gamma_0(4N)$ with $\chi(W_{4N}) \in \{\pm 1\}$. Since $\Gamma_0^*(4N)$ is generated by $\Gamma_0(4N)$ and W_{4N} this is enough to completely determine the character.

Proposition 3.0.3. *Let $\lambda > 2$, $f \in S_{k/2}(\Gamma_0^*(4N), \chi)$, $a \in [0, \kappa + \ell]$. Then*

$$F_a|_{-\ell,\chi}(I - i^{\ell}W_{4N}) = P_a.$$

Proof By part (3) of Lemma 3.0.2,

$$\begin{aligned}
i^{\ell}F_a|_{-\ell,\chi}W_{4N} &= \chi(W_{4N})^{-1}(2i\sqrt{N}\tau)^{\ell}\int_{W_{4N}\tau}^{\infty} f(z)\Phi_a(W_{4N}\tau, z)dz \\
&= \chi(W_{4N})^{-1}(2i\sqrt{N}\tau)^{\ell}\int_{\tau}^{0} f(W_{4N}z)\Phi_a(W_{4N}\tau, W_{4N}z)d(W_{4N}z) \\
&= -(2i\sqrt{N}\tau)^{\ell}\int_{\tau}^{0} f(z)(-2i\sqrt{N}z)^{\kappa}\Phi_a(W_{4N}\tau, W_{4N}z)dz \\
&= \int_{\tau}^{0} f(z)\Phi_a(\tau, z)dz \\
&= \left(-\int_{0}^{i\infty} + \int_{\tau}^{i\infty}\right) f(z)\Phi_a(\tau, z)dz \\
&= -P_a(\tau) + F_a(\tau).
\end{aligned}$$

Rearranging, we obtain the claim. □

Corollary 3.0.4. *If ℓ is even, then $P_a|_{-\ell}(I + i^{\ell}\chi(W_{4N})W_{4N}) = 0$.*

Proof First write

$$F_a|_{-\ell,\chi}(I - i^{\ell}W_{4N})) = F_a|_{-\ell}(I - i^{\ell}\chi(W_{4N})W_{4N}).$$

Since ℓ is even and $\chi(W_{4N})^2 = 1$, by Proposition 3.0.3 it follows that

$$\begin{aligned} P_a|_{-\ell}(I + i^\ell \chi(W_{4N})W_{4N}) &= F_a|_{-\ell}(I - i^\ell \chi(W_{4N})W_{4N})(I + i^\ell \chi(W_{4N})W_{4N}) \\ &= F_a|_{-\ell}(I - W_{4N}^2) = 0. \end{aligned}$$

□

Remark 3.0.5. If ℓ is even and $\chi(W_{4N}) = \mp i^\ell$ then $P_a|_{-\ell}W_{4N} = \pm P_a$.

Remark 3.0.6. We emphasise that in contrast to [1], we do not assume that ℓ is a multiple of four.

Our aim now will be to explicitly compute the polynomial $P_a(\tau)$. We will do this by using the definition of $\Phi_a(\tau, z)$ and swapping the sum and the integral. In the next proposition it will be useful to consider the norm 1 complex number

$$\eta(n, k, a) := e\left(\frac{1}{2}\left[\frac{3n - k + 9}{2} + a\right]\right). \tag{3.0.1}$$

Proposition 3.0.7. *The period polynomial $P_a(\tau)$ satisfies*

$$\begin{aligned} P_a(\tau) = \sum_{n=0}^{\ell} \Bigg[&\binom{\kappa}{n} (4iN)^n L^*(f, a - n + 1) \\ &+ \binom{\kappa}{n + 1/2} (4N)^{\ell/2} \eta(n, k, a) L^*(f, a + n + 1 - \ell) \Bigg] \tau^n. \end{aligned} \tag{3.0.2}$$

Here η is as in (3.0.1).

Proof Substituting definition (3.0.1) of Φ_a into P_a we see

$$\begin{aligned} P_a(\tau) = \sum_{n=0}^{\ell} &\binom{\kappa}{n} (4iN\tau)^n \int_0^{i\infty} f(z) z^{a-n} dz \\ &+ \frac{i^{k/2}}{(4N)^{1/4}} \binom{\kappa}{n + 1/2} (i\tau)^n \int_0^{i\infty} f(z)(-4Nz)^{\kappa+\ell-a-n} dz \end{aligned}$$

which can be expressed as

$$\begin{aligned} P_a(\tau) &= \sum_{n=0}^{\ell} \binom{\kappa}{n} (4iN\tau)^n L^*(f, a - n + 1) \\ &+ \frac{i^{k/2}}{(4N)^{1/4}} \binom{\kappa}{n + 1/2} (i\tau)^n (4N)^{\kappa+\ell-a-n} e^{-\pi i(\kappa+\ell-a-n)} L^*(f, \kappa + \ell - a - n + 1). \end{aligned}$$

In the second integral we have used the fact that $(-z)^s = e^{-\pi i s} z^s$ for $z \in \mathfrak{H}$

and $s \in \mathbb{R}$. The functional equation then allows us to write

$$L^*(f, \kappa + \ell - a - n + 1) = i^{k/2}(4N)^{k/4-(\kappa+\ell-a-n+1)}L^*(f, a + n + 1 - \ell).$$

Substituting and simplifying we obtain the result. □

We shall term the polynomial $P_a(\tau)$ in (3.0.2) the *period polynomial* attached to f. Observe that P_a encodes within its coefficients the L values of the $L^*(f, s)$ at $s = a+1-\ell, a+2-\ell, \dots, a, a+1$. In the sequel, since it will be useful for us to just consider the τ^n coefficient, we introduce some notation. Given a polynomial $f(\tau) = \sum_{j=0}^{J} a_j\tau^j$ we denote $[\tau^j]f(\tau) = a_j$ the coefficient of τ^j. This way, we may write

$$\begin{aligned} [\tau^n]P_a(\tau) &= \binom{\kappa}{n}(4iN)^n L^*(f, a-n+1) \\ &+ \binom{\kappa}{n+1/2}(4N)^{\ell/2}\eta(n,k,a)L^*(f, a+n+1-\ell), \end{aligned} \quad (3.0.3)$$

where $\eta(n, k, a)$ is given by (3.0.1).

Remark 3.0.8. Under the mild assumption that ℓ is even, we can deduce that the coefficient of $\tau^{\ell/2}$ is

$$\left[\binom{\kappa}{\ell/2}i^{\ell/2} + \eta(n,k,a)\binom{\kappa}{(\ell+1)/2}\right](4N)^{\ell/2}L^*(f, a+1-\ell/2).$$

We also mention that we have allowed certain flexibility in a. We will next show that the choice $a = \kappa$ allows an interpretation in terms of the Eichler cocycle map

$$\begin{aligned} \psi_f : \Gamma_0^*(4N) &\to \mathrm{Hol}(\overline{\mathfrak{H}^-}) \\ \gamma &\mapsto \int_{\gamma^{-1}i\infty}^{i\infty} f(z)(z-\tau)^\kappa dz. \end{aligned}$$

Here the image is contained in the set of holomorphic functions on the closure of the lower half plane $\mathfrak{H}^- = \{\tau \in \mathbb{C} : \Im(\tau) < 0\}$. In fact, we will show that the evaluation $\psi_f^* = \psi_f(W_{4N})$ of the Eichler cocycle ψ_f at W_{4N} is (up to an error bound) the period polynomial at $a = \kappa$ at the point iy. More precisely, we have

Proposition 3.0.9. *For $y > 1$,*

$$P_\kappa(iy) = \psi_f^*(4Ny) + (-1)^{\lambda+1}(2\sqrt{N}y)^\ell \psi_f^*(1/y) + O(y^{\ell+1}).$$

Proof Since $\ell + 1 > \kappa$, we can bound the error term in the Taylor expansion

$$(1 + 4Niy/t)^\kappa - \sum_{n=0}^{\ell} \binom{\kappa}{n} (4Niy/t)^n$$
$$\ll_\ell \int_0^1 (1-\xi)^\ell (1 + 4Niy\xi/t)^{\kappa-\ell-1} (4Niy/t)^{\ell+1} d\xi \ll_\ell (y/t)^{\ell+1}.$$

Similarly,

$$\left[(1 + i/ty)^\kappa - \sum_{n=0}^{\ell} \binom{\kappa}{n} (i/ty)^n \right] \ll (yt)^{-\ell-1}.$$

Combining these two error terms with the assumption that $y > 1$, we see that in both cases we can bound by $O_\ell(y^{\ell+1})$. We now substitute the Taylor expansion with this error term and use the substitution $n \mapsto \ell - n$ in the second sum to see

$$\begin{aligned}
& \psi_f^*(4Ny) + (-1)^{\lambda+1}(2\sqrt{N}y)^\ell \psi_f^*(1/y) \\
= \quad & i^{\kappa+1} \int_0^\infty f(it) \left[(1 + 4Niy/t)^\kappa + (-1)^{\lambda+1}(2\sqrt{N}y)^\ell (1 + i/yt)^\kappa \right] t^\kappa dt \\
= \quad & \sum_{n=0}^{\ell} \binom{\kappa}{n} \left[(4Niy)^n i^n - i^{2\lambda+n}(2\sqrt{N}y)^\ell (i/y)^n \right] L^*(f, \kappa+1-n) + O_\ell(y^{\ell+1}) \\
\overset{n\mapsto\ell-n}{=} \quad & \sum_{n=0}^{\ell} \Bigg[\binom{\kappa}{n} (4Ni)^n L^*(f, \kappa+1-n) \\
- \quad & i^{2(\lambda+\ell)-3n} \binom{\kappa}{n+1/2} (4N)^{\ell/2} L^*(f, \kappa+1-\ell+n) \Bigg] (iy)^n + O_\ell(y^{\ell+1}) \\
\overset{(3.0.2)}{=} \quad & P_\kappa(iy) + O_\ell(y^{\ell+1}).
\end{aligned}$$

In the last step, to check that the powers of i agree with those of the period polynomial (3.0.2), it is easy to check that $-i^{2\lambda+2\ell-3n} = i^{3n+k} e^{-\pi i(k-9/2-\kappa)}$.
$\square$

A general theme in studying the period polynomials of modular forms is that there exists a construction of these polynomials using the Eichler integral. In the half integral weight setting, such an object was studied by Kohnen and

Raji [16], namely given a cusp form $f \in S_{k/2}(\Gamma_0^*(4N))$ they considered

$$\mathcal{E}_f(\tau) := \frac{1}{\sqrt{\pi}} \sum_{n\geq 1} \frac{a_f(n)}{n^{k/2-1}} \left(e(-n\tau)\Gamma(1/2, -2\pi n z) - (2\pi n \tau)^{-1/2} \right)$$

and showed ([16] Thm2 p19) that

$$\mathcal{E}_f|_{-\kappa}(I - W_{4N}) = \sum_{n=0}^{\ell} \left(\frac{L(f, \kappa+1-n)}{\Gamma(n+1)} + \frac{L(f, \lambda - n)}{\Gamma(n+\frac{1}{2})} \left(\frac{2\pi\tau}{i}\right)^{-\frac{1}{2}} \right) \left(\frac{2\pi\tau}{i}\right)^{n}.$$

In general, the right hand side of this expression does not lie in a finite dimensional vector space. In joint work, [1], it was shown that the main contribution of $\mathcal{E}_f$ is an integral also.

Proposition 3.0.10. *For* $\tau \in \mathfrak{H}$,

$$\mathcal{E}_f(\tau) = \alpha_k(-i\tau)^{\frac{1}{2}} \int_\tau^{i\infty} F_f(z)(z-\tau)^\ell dz - \frac{L(f, \lambda)}{\sqrt{-2\pi^2 i\tau}}$$

where

$$F_f(z) := \int_0^\infty \frac{f(tz)}{\sqrt{t+1}} t^\lambda \frac{dt}{t}$$

and

$$\alpha_k = \frac{(-2\pi i)^{\kappa+1}}{\pi^{\frac{1}{4}} \ell!}.$$

Proof Upon using

$$\Gamma(1/2, w) = w^{1/2} \int_1^\infty e^{-wt} \frac{dt}{\sqrt{t}}, \qquad \Re(w) > 0$$

we see that

$$\begin{aligned}
\mathcal{E}_f &= \frac{1}{\sqrt{\pi}} \sum_{n\geq 1} \frac{a_f(n)}{n^{k/2-1}} \left(e(-n\tau)\Gamma(1/2, -2\pi i n\tau) - (-2\pi n\tau)^{-1/2} \right) \\
&= \frac{1}{\sqrt{\pi}} \sum_{n\geq 1} \frac{a_f(n)}{n^{k/2-1}} e(-n\tau)\sqrt{-2\pi i n\tau} \int_1^\infty e(nt\tau) \frac{dt}{\sqrt{t}} - \frac{(-2\tau)^{-1/2}}{\pi} \sum_{n\geq 1} \frac{a_f(n)}{n^{(k-1)/2}} \\
&= \sqrt{-2i\tau} \int_1^\infty \sum_{n\geq 1} \frac{a_f(n)}{n^\kappa} e(n(t-1)\tau) \frac{dt}{\tau} - \frac{L(f,\lambda)}{\pi(-2\tau)^{1/2}} \\
&= (-2i\tau)^{1/2} \int_1^\infty \frac{(-2\pi i)^{\ell+1}}{\ell!} \int_{(t-1)\tau}^\infty f(w)(w - \tau(t-1))^\ell dw dt - \frac{L(f,\lambda)}{\pi(-2\tau)^{1/2}} \\
&= \alpha_k \sqrt{\tau} \int_0^\infty \frac{t^{\lambda-1}}{(t+1)^{1/2}} \int_\tau^{i\infty} f(tz)(z-\tau)^\ell dz dt - \frac{L(f,\lambda)}{\pi(-2\tau)^{1/2}}.
\end{aligned}$$

On the last line we used the change of variables $t \mapsto t-1$ and $z \mapsto w/(t-1)$. Now change the order of integration to conclude. □

3.1 The main theorem

The aim of this section will be to give an explicit expression between the L-values of a half integral weight cusp form on one side and the L-values of an integral weight modular form on the other. The exact expression is given in Theorem 3.1.4. This method exploits the isomorphism between $\Gamma_0^*(4)$ and Γ^ϑ. We construct and describe fully a parabolic cocycle $\pi_f \in H^1_{par}(\mathbf{PSL}_2(\mathbb{Z}), I_\lambda)$ with values in a *finite* dimensional space I_λ. With π_f we apply a result of Pasol and Popa [22] and again compare coefficients. The construction of π_f is done in several steps. From the period polynomial P_a of the previous section, we construct a cocycle π_f^ϑ on Γ^ϑ. From π_f^ϑ we induce a cocycle $\tilde{\pi}_f$. Sadly, $\tilde{\pi}_f$ is not necessarily parabolic, but this can be fixed, we shall prove that a small adjustment π_f of $\tilde{\pi}_f$ *is* parabolic.

A possible set of coset representatives for $\Gamma^\vartheta\backslash\mathbf{PSL}_2(\mathbb{Z})$ is $\{I, T, U\}$. We shall fix these representatives. If we denote by $\psi(x)$ the element of $\{I, T, U\}$ that corresponds to the coset $\Gamma^\vartheta x$. For example, it is easy to check that $\psi(T^{-1}) = T$. The relation $T^2SU = UT^{-1}$ shows that

$$\psi(UT^{-1}) = \psi(T^2SU) = \psi(U) = U. \tag{3.1.1}$$

The map

$$\begin{aligned} \alpha : \Gamma^\vartheta\backslash\mathbf{PSL}_2(\mathbb{Z}) \times \mathbf{PSL}_2(\mathbb{Z}) &\to \mathbf{PSL}_2(\mathbb{Z}) \\ (\Gamma^\vartheta x, g) &\mapsto \psi(x)g\psi(xg)^{-1} \end{aligned}$$

is well defined independent of the choice of the coset representative x (see property (i) below). In addition, this map also enjoys the properties

(i) $\Gamma^\vartheta x = \Gamma^\vartheta x' \Rightarrow \psi(x) = \psi(x')$, $\psi(xg) = \psi(x'g) \Rightarrow \alpha(x, g) = \alpha(x', g)$.

(ii) $x \in \Gamma^\vartheta \Rightarrow \alpha(x, g) = g\psi(g)^{-1}$ (put $x' = 1$ into (i))

(iii) $\alpha(x, g_1g_2) = \alpha(x, g_1)\alpha(xg_1, g_2)$.

This ensures that α is well defined. We now define

$$\mathcal{I} := \{v : \Gamma^\vartheta \backslash \mathbf{PSL}_2(\mathbb{Z}) \to \mathrm{Hol}(\mathfrak{H})\}.$$

With the setting $4|\ell$ and λ as before, we also consider the subset $I_\lambda \subset \mathcal{I}$ defined by

$$I_\lambda := \{v : \Gamma^\vartheta \backslash \mathbf{PSL}_2(\mathbb{Z}) \to \tau^{-1}\mathbb{C}_\lambda[\tau]\}.$$

Since $g \in \mathbf{PSL}_2(\mathbb{Z})$ acts on $\mathrm{Hol}(\mathfrak{H})$ via $\cdot|_{-\ell}g$, we can use this to define an action on $\mathcal{I}$:

$$(v||g)(x) := v(\psi(xg^{-1}))|_{-\ell}g \qquad x \in \Gamma^\vartheta \backslash \mathbf{PSL}_2(\mathbb{Z}) \ g \in \mathbf{PSL}_2(\mathbb{Z}).$$

Notice that in general, $\cdot||g$ does not preserve the space I_λ as $\tau^{-1}\mathbb{C}_\lambda[\tau]$ is not preserved under $\cdot|_{-\ell}g$. However, on the subspace

$$W_\lambda = \{v \in I_\lambda : \ v||(S+I) = v||(U^2+U+I) = 0\}$$

the $||$ action *is* preserved. From now on we impose the condition that $4|\ell$ so that $\lambda > 2$ is always even. Analogously to the classical setting, if $\varepsilon = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, then W_λ decomposes into $W_\lambda^+ \oplus W_\lambda^-$. We put

$$v^\pm = \frac{v \pm v||\varepsilon}{2} \in W_\lambda^\pm.$$

The period polynomial for Γ^ϑ of weight λ

In this subsection we give an exposition of a result of Pasol and Popa ([22]) which is a generalisation of the fact that the period polynomial of a modular form f is essentially given by $\tilde{f}|(I-S)$, the Eichler integral slashed by $I-S$. Our setting here will be for weight λ and on the theta group Γ^ϑ. Suppose that $f \in M_\lambda(\Gamma^\vartheta)$ has q-expansion $f = \sum_{n\geq 0} a_f(n)q^n$ and denote by f_0

$$f_0(\tau) = f(\tau) - a_f(0).$$

This makes f_0 vanish at the ∞ cusp. Since we still maintain the notation $\ell = \lambda - 2$, the Eichler integral is the holomorphic function

$$\tilde{f}(\gamma)(\tau) := \int_{\tau}^{i\infty} (f|\gamma)_0(w)(w-\tau)^{\ell} dw, \qquad \gamma \in \mathbf{PSL}_2(\mathbb{Z}),\ \tau \in \mathfrak{H}.$$

The slash action $f|\gamma = f|_{\lambda}\gamma$ is understood to be of weight λ here and will be ommited henceforth. Observe this defines an element $\tilde{f} \in \mathcal{I}$. We denote by $r_f(\gamma) := \tilde{f}(\gamma)||(I-S)$.

Proposition 3.1.1 (Pasol, Popa [22]). *We have the following*

(1) The cocycle r_f lies in W_λ and

$$\begin{aligned} r_f(\gamma)(\tau) &= \int_{\tau_0}^{i\infty} (f|\gamma)_0(w)(w-\tau)^{\ell} dw - a_{f|\gamma}(0) \int_0^{\tau_0} (w-\tau)^{\ell} dw \\ &- \left[\int_{S\tau_0}^{i\infty} (f|\gamma S)_0(w)(w-\tau)^{\ell} dw - a_{f|\gamma S}(0) \int_0^{S\tau_0} (w-\tau)^{\ell} dw \right] |_{-\ell} S \\ &+ \frac{a_{f|\gamma(0)}}{\lambda - 1} \tau^{\lambda-1} + \frac{a_{f|\gamma S}(0)}{\lambda-1} \tau^{-1} \end{aligned}$$

for any $\gamma \in \mathbf{PSL}_2(\mathbb{Z})$ and any $\tau_0, \tau \in \mathfrak{H}$. Moreover, $r_f(\gamma)$ is independent of τ_0.

(2) The map

$$\begin{aligned} r^- : M_\lambda(\Gamma^\vartheta) &\to W_\lambda^- \\ f &\mapsto r_f^- \end{aligned}$$

is an isomorphism.

Proof (1) See [22] Prop.8.1

(2) Use the fact that Γ^ϑ is $\mathbf{PSL}_2(\mathbb{Z})$ conjugate to $\Gamma(2)$ which is of the form prescribed in [22] Prop.4.4. This allows us to apply [22]Prop.8.4(b).

□

We are now going to define a 1-cocycle $\hat{\pi}_f$ on $\Gamma_0^*(4)$ with values in $\mathbb{C}_\ell[\tau]$, the space of polynomials of degree $\leq \ell$ with coefficients in $\mathbb{C}$. Let $f \in$

$S_{k/2}(\Gamma_0^*(4N), \chi)$ and set

$$\hat{\pi}_f : \Gamma_0^*(4N) \to \mathbb{C}_\ell[\tau]$$
$$W_4 \mapsto P_a(\tau/\sqrt{N}) =: \hat{P}_a(\tau)$$

By imposing that $\hat{\pi}_f(T) = 0$, that ℓ is even and that $\chi(W_{4N}) \in \{\pm 1\}$, the map $\hat{\pi}_f$ is well-defined since the cocycle action gives

$$\begin{aligned}\hat{\pi}_f(W_{4N}^2) &= \hat{\pi}_f(W_{4N}) + \hat{\pi}_f(W_{4N})|_{-\ell,\chi} W_{4N} = \hat{P}_a|_{-\ell,\chi}(I + W_{4N}) \\ &= \hat{P}_a|_{-\ell}(I \pm W_{4N}) = 0.\end{aligned}$$

Here the action $(Q|_{-\ell,\chi}\gamma)(\tau)$ means $\chi(\gamma)j(\gamma,\tau)^\ell Q(\gamma\tau)$. From §0.6 we know that $\Gamma_0^*(4)$ and Γ^ϑ are conjugate. The homomorphism sending $T \mapsto T^2$ and $W_4 \mapsto S$ is a group isomorphism from $\Gamma_0^*(4)$ onto Γ^ϑ. This allows us to transfer $\hat{\pi}_f$ to Γ^ϑ:

$$\pi_f^\vartheta(S)(\tau) = \hat{P}_a(\tau/2) \qquad \pi_f^\vartheta(T^2) = 0.$$

We have obtained a non-trivial class in $H^1_{par}(\Gamma^\vartheta, \mathbb{C}_\ell[\tau])$ with action defined by $|_{-\ell}$.

Following the construction of Pasol and Popa [22], we can induce a 1-cocycle $\tilde{\pi}_f$ on $\mathbf{PSL}_2(\mathbb{Z})$ with values in $\mathcal{I}$ from π_f^ϑ as follows

$$\tilde{\pi}_f(g)(x) = \pi_f^\vartheta(\alpha(x, g^{-1})^{-1})|_{-\ell}\psi(x), \qquad \Gamma^\vartheta\backslash\mathbf{PSL}_2(\mathbb{Z}).$$

We now claim that $\tilde{\pi}_f$ is indeed a 1-cocycle. To show this we make use of two facts, the first

$$\alpha(x, g_2^{-1})^{-1}\psi(x) = \psi(xg_2^{-1})g_2 \tag{3.1.2}$$

follows from the definition of α. The second, uses property (iii) of §3.2, namely

$$\alpha(x, g_2^{-1}g_1^{-1})^{-1} = \alpha(xg_2^{-1}, g_1^{-1})^{-1}\alpha(x, g_2^{-1})^{-1}. \tag{3.1.3}$$

We have

$$\begin{aligned}
(\tilde{\pi}_f(g_1)||g_2)(x) + \tilde{\pi}_f(g_2)(x) &= \tilde{\pi}_f(g_1)(xg_2^{-1})|g_2 + \pi_f^{\vartheta}(\alpha(x,g_2^{-1})^{-1})|\psi(x) \\
&= \pi_f^{\vartheta}(\alpha(xg_2^{-1},g_1^{-1})^{-1})|\psi(xg_2^{-1})g_2 \\
&\quad +\pi_f^{\vartheta}(\alpha(x,g_2^{-1})^{-1})|\psi(x) \\
&\overset{(3.1.2)}{=} \pi_f^{\vartheta}(\alpha(xg_2^{-1},g_1^{-1})^{-1})|\alpha(x,g_2^{-1})^{-1}\psi(x) \\
&\quad +\pi_f^{\vartheta}(\alpha(x,g_2^{-1})^{-1})|\psi(x) \\
&= \left[\pi_f^{\vartheta}(\alpha(xg_2^{-1},g_1^{-1})^{-1})|\alpha(x,g_2^{-1})^{-1}\right. \\
&\quad \left.+\,\pi_f^{\vartheta}(\alpha(x,g_2^{-1})^{-1})\right]|\psi(x) \\
&= \left[\pi_f^{\vartheta}(\alpha(xg_2^{-1},g_1^{-1})^{-1}\alpha(x,g_2^{-1})^{-1})\right]|\psi(x) \\
&\overset{(3.1.3)}{=} \pi_f^{\vartheta}(\alpha(x,g_2^{-1}g_1^{-1})^{-1})|\psi(x) \\
&= \tilde{\pi}_f(g_1g_2)(x).
\end{aligned}$$

This proves our claim that $\tilde{\pi}_f$ is a cocycle. Alternatively, we could have shown that $\tilde{\pi}_f$ is a 1-cocycle by invoking Shapiro's lemma (cf [19] p59).

The next proposition we will prove is rather surprising, although we cannot claim that $\tilde{\pi}_f$ is parabolic[1], a small adjustment will be and, even better, the coefficients of this adjusted cocycle will have values in I_λ. Precisely, we have

Proposition 3.1.2. *There exists* $v : \Gamma^{\vartheta}\backslash\mathbf{PSL}_2(\mathbb{Z}) \to \mathbb{C}_{\lambda-1}[\tau]$ *such that*

$$\pi_f(g) := \tilde{\pi}_f(g) - v||(g-1) \quad \forall g \in \mathbf{PSL}_2(\mathbb{Z})$$

is a parabolic cocycle with coefficients in I_λ. *Furthermore,* $\pi_f(S) \in W_\lambda$.

Proof We evaluate

$$\tilde{\pi}_f(T)(x) = \pi_f^{\vartheta}(\psi(xT^{-1})T\psi(x)^{-1})|\psi(x)$$

at $x = I, T, U$ respectively. One finds

$$\tilde{\pi}_f(T)(I) = \pi_f^{\vartheta}(\psi(T^{-1}T)) = \pi_f^{\vartheta}(T^2) = 0,$$

$$\tilde{\pi}_f(T)(T) = \pi_f^{\vartheta}(T\psi(T)^{-1})|\psi(T) = \pi_f^{\vartheta}(I)|T = 0.$$

[1] we will prove $\tilde{\pi}_f(T)$ is not always vanishing

Moreover, since $UTU^{-1} = S^{-1}T^{-2}$ and $\psi(UT^{-1}) = U$,

$$\begin{aligned}\tilde{\pi}_f(T)(U) &= \pi_f^{\vartheta}(\psi(UT^{-1})T\psi(U)^{-1})|\psi(U) \\ &= \pi_f^{\vartheta}(UTU^{-1})|U = \pi_f^{\vartheta}(S^{-1}T^{-2})|U \\ &= \pi_f^{\vartheta}(S)|T^2U + \pi_f^{\vartheta}(T^2)|U = \pi_f^{\vartheta}(S)|T^2U \\ &= \hat{P}_a(\tau/2)|_{-\ell}T^2U.\end{aligned}$$

We now consider a polynomial $Q_f(\tau) \in \mathbb{C}_{\lambda+1}[\tau]$ with the property that

$$Q_f(\tau)|(T-I) = Q_f(\tau+1) - Q_f(\tau) = \hat{P}_a(\tau/2)|_{-\ell}T^2U. \tag{3.1.4}$$

This exists since $\deg Q_f = \deg \hat{P}_a + 1$ (for a proof of this see [1] Lem.4.3). Alternatively, we can find Q_f in the following way. Set $[\tau^{\ell+1}]Q_f = \frac{[\tau^\ell]\hat{P}_a(\tau/2)|_{-\ell}T^2U}{\ell+1}$ and use the recursive formula

$$\sum_{j=n+1}^{\ell+1} \binom{j}{n} \left([\tau^j]Q_f\right) = [\tau^n]\left(\hat{P}_a(\tau/2)|_{-\ell}T^2U\right), \quad 0 \le n \le \ell - 1.$$

Therefore, we can construct v as follows, we impose

$$v(I) = v(T) = 0 \qquad v(U) = Q_f.$$

With v shown to exist, we are now in a position to define π_f to be the difference

$$\pi_f(g) := \tilde{\pi}_f(g) - v||(g-1).$$

Subtracting the coboundary $v||(g-I)$ from the cocycle $\tilde{\pi}_f(g)$ implies π_f is indeed a cocycle. To prove that π_f is parabolic, it suffices to show that $\pi_f(T)(x)$ vanishes for all $x \in \Gamma^{\vartheta}\backslash\mathbf{PSL}_2(\mathbb{Z})$. Unraveling the definitions, we have for $g \in \mathbf{PSL}_2(\mathbb{Z})$, $x \in \Gamma^{\vartheta}\backslash\mathbf{PSL}_2(\mathbb{Z})$

$$\pi_f(g)(x) = \pi_f^{\vartheta}(\psi(xg^{-1})g\psi(x)^{-1})|\psi(x) - v(\psi(xg^{-1}))|g + v(x). \tag{3.1.5}$$

Indeed, substituting $g = T$ and $x = I$ above,

$$\pi_f(T)(I) = \tilde{\pi}_f(T)(I) - (v||(T-I))(I) = 0 - (v(T{-}1)|T - v(I)) = 0,$$

and

$$\pi_f(T)(T) = \tilde{\pi}_f(T)(T) - (v||(T-I))(T) = 0 - (v(I)|T - v(T)) = 0.$$

Finally,

$$\begin{aligned} \pi_f(T)(U) &= \tilde{\pi}_f(T)(U) - (v||(T-I))(U) \\ &= Q_f|(T-I) - \big(v(UT^{-1})|T - v(U)\big) \\ &= v(U)|(T-I) - v(U)|(T-I) = 0. \end{aligned}$$

Moreover, since v takes values in $\mathbb{C}_{\lambda-1}[\tau]$ we see that

$$\pi_f(g)(x) = \underbrace{\tilde{\pi}_f(g)(x)}_{\in\mathbb{C}_\ell[\tau]} - \left[\underbrace{v(xg^{-1})|_{-\ell}g}_{\in\tau^{-1}\mathbb{C}_\ell[\tau]} - \underbrace{v(x)}_{\in\mathbb{C}_{\lambda-1}[\tau]}\right],$$

meaning that $\pi_f(g)(x)$ takes values in (at worst) $\tau^{-1}\mathbb{C}_{\lambda-1}[\tau]$. Therefore, $\pi_f(g)$ takes values in I_λ. To conclude the proof it remains to show that $\pi_f(S)$ lies in W_λ, but this follows from the cocycle condition, as $\pi_f(S) = \pi_f(U)$ and

$$\pi_f(S)||(S+I) = \pi_f(S)||(U^2+U+I) = 0.$$

□

Remark 3.1.3. It is possible to explicitly construct Q_f from (3.1.4).

In fact, the existence part of the proposition allows us to compute explicitly, the value of $\pi_f(S)$. For $x = I$,

$$\pi_f(S)(I) = \pi_f^\vartheta(\psi(S^{-1}S))|\psi(S^{-1}) + \underbrace{v(I)}_{=0} - \underbrace{v(I)|S}_{=0} = \pi_f^\vartheta(S) = \hat{P}_a\left(\frac{\cdot}{2}\right). \quad (3.1.6)$$

We found that $\tilde{\pi}_f(S)(x)$ vanishes for $x = T, U$ in which case only the contribution of $\pi_f(S)$ comes from v, namely

$$\pi_f(S)(x) = -(v||(S-I))(x) = v(x) - v(xS^{-1})|S \qquad x = T, U.$$

For $x = T$,

$$\pi_f(S)(T) = v(T) - v(TS^{-1})|S = -v(U)|S = -Q_f|_{-\ell}S.$$

For $x = U$,

$$\pi_f(S)(U) = v(U) - v(US^{-1})|S = v(U) - v(T)|S = Q_f.$$

Combining this with the previous remark gives a full description of π_f.

Theorem 3.1.4. *Suppose $\ell > 0$ is a multiple of four. Fix $f \in S_{k/2}(\Gamma_0^*(4N))$ and $a \in [0, \kappa + \ell]$. There exists $g \in M_\lambda(\Gamma_0^*(4))$ such that for all odd $n \in \{1, \cdots, \ell - 1\}$,*

$$\begin{aligned}
&(4N)^{-n/2}\left[\binom{\kappa}{n}(4iN)^{n/2}L^*(f, a+1-n)\right.\\
+\quad &\left.\binom{\kappa}{n+1/2}(4N)^{\ell/2}\eta(n,k,a)L^*(f, \kappa+\ell-1-a-n)\right]\\
=\quad &-2^{\ell+1-n}\binom{\ell}{n}L^*(g, \ell+1-n).
\end{aligned}$$

Proof By Proposition 3.1.2 we know $\pi_f(S) \in W_\lambda$. It follows that the element

$$\frac{\pi_f(S) - \pi_f(S)||\varepsilon}{2} = \pi_f(S)^-$$

lies in W_λ^-. We can now apply the isomorphism r^- of Proposition 3.1.1(2) to obtain the existence of a form g_1 in $M_\lambda(\Gamma^\vartheta)$ such that

$$\pi_f(S)^- = r_{g_1}^-. \tag{3.1.7}$$

Since $\psi(\varepsilon T \varepsilon) = T$ and $\psi(\varepsilon U \varepsilon) = U$, we see that $\pi_f(S)^-$ and $r_{g_1}^-$ are the odd parts of $\pi_f(S)$ and r_{g_1} respectively. In particular the polynomials $\pi_f(S)(I)(\tau)^-$ and $r_{g_1}(I)(\tau)^-$ agree, and we are led to compare the coefficients between these two polynomials. From (3.1.6) we have

$$\pi_f(S)(I)(\tau) = \hat{P}_a(\tau/2) = P_a(\tau/2\sqrt{N}).$$

This gives us the coefficients on the left hand side. For the right hand side use Proposition 3.1.1 part (1) and a binomial expansion to see that the n-th term is

$$\begin{aligned}
r_{g_1}(I)(\tau) = \sum_{n=0}^{\ell}\binom{\ell}{n}&\left[\int_{\tau_0}^{i\infty} g_1(w)_0 w^{l-n}dw - \frac{a_{g_1}(0)\tau_0^{\ell-1+n}}{\ell-1+n}\right.\\
-(-1)^n\int_{S\tau_0}^{i\infty}&\left.(g_1|S)_0(w)w^n dw - \frac{a_{g_1|S}(0)(S\tau_0)^{n+1}}{n+1}\right](-\tau)^n.
\end{aligned}$$

Set $\tau_0 = i$. Using $g_1|S = g_1$ and using the integral expansion

$$L^*(g_1, s) = \int_0^{i\infty} g_1(w)_0 w^s \frac{dw}{w},$$

we see

$$r_{g_1}(I)(\tau) = \sum_{n=0}^{\ell} \binom{\ell}{n} L^*(g_1, \ell + 1 - n)(-\tau)^n + \frac{a_{g_1}(0)}{\ell + 1}\tau^{\ell+1} + \frac{a_{g_1}(0)}{\ell + 1}\tau^{-1}.$$

For odd $0 \leq n \leq \ell$, the n-th coefficient of $r_{g_1}(I)$ is

$$[\tau^n]r_{g_1}(I) = -\binom{\ell}{n} L^*(g_1, \ell + 1 - n).$$

Finally the substitution $g(\tau) = g_1(2\tau) \in M_\lambda(\Gamma_0^*(4))$ gives $2^{-s}L^*(g, s) = L^*(g_1, s)$, so the desired coefficients on the right hand side. □

3.2 Explicit description of the lift

The purpose of this section is to put into context the result of Theorem 3.1.4, namely we want to explicitly describe the map

$$S_{k/2}(\Gamma_0^*(4N)) \to M_\lambda(\Gamma_0^*(4))$$
$$f \mapsto g$$

This will be achieved by writing the image g as a linear combination of cuspidal and Eisenstein components. To describe this we will need another result of Pasol and Popa [22].

We introduce some notation to this end. From the normalised weight λ Eisenstein series

$$E_\lambda(\tau) = 1 - \frac{2\lambda}{B_\lambda} \sum_{n \geq 1} \sigma_{\lambda-1}(n) q^n,$$

we set

$$\mathcal{E}_1(2\tau) := E_\lambda(2\tau), \qquad \mathcal{E}_2(2\tau) := 2^\lambda E_\lambda(4\tau) + E_\lambda(\tau)$$

to be the two forms that form a basis for the Eisenstein subspace in $M_\lambda(\Gamma_0^*(4))$. Given any $g \in M_\lambda(\Gamma_0^*(4))$ there exists some cusp form $\mathring{g} \in S_\lambda(\Gamma_0^*(4))$ such that

$$g(\tau) = \mathring{g}(\tau) + \alpha_1\mathcal{E}_1(2\tau) + \alpha_2\mathcal{E}_2(2\tau) \qquad \text{for some } \alpha_1, \alpha_2 \in \mathbb{C}.$$

The change of variable $\tau \mapsto \tau/2$ ensures $\mathcal{E}_1(\tau), \mathcal{E}_2(\tau) \in M_\lambda(\Gamma^\vartheta)$ and $\mathring{g}_1(\tau) \in S_\lambda(\Gamma^\vartheta)$. The Eisenstein series have constant terms

$$a_{\mathcal{E}_1}(0) = 1 \quad \text{and} \quad a_{\mathcal{E}_2}(0) = 2^\lambda + 1$$

respectively. Therefore, by (3.1.7),

$$\pi_f(S)^- = r_{g_1}^- = r_{\mathring{g}_1}^- + \alpha_1 r_{\mathcal{E}_1}^- + \alpha_2 r_{\mathcal{E}_2}^-.$$

Claim: Let $b_{\ell+1}$ denote the coefficient of $\tau^{\ell+1}$ of the polynomial Q_f discussed in (3.1.4). Then we have

$$\alpha_1 = \frac{2^\lambda + 1}{2^\lambda - 1}(\ell+1)b_{\ell+1} \quad \text{and} \quad \alpha_2 = \frac{-1}{2^\lambda - 1}(\ell+1)b_{\ell+1}.$$

Proof Since $\mathring{g}$ is cuspidal, $[\tau^{-1}]r_{\mathring{g}}^- = 0$. Therefore

$$[\tau^{-1}]\pi_f(S)(x)^- = \alpha_1[\tau^{-1}]r_{\mathcal{E}_1}(x)^- + \alpha_2[\tau^{-1}]r_{\mathcal{E}_2}(x)^-.$$

Since

$$[\tau^{-1}]\pi_f(S)(I) = 0 \quad \text{and} \quad [\tau^{-1}]\pi_f(S)(U) = [\tau^{-1}]Q_f = b_{\ell+1},$$

we have

$$\begin{aligned} 0 = [\tau^{-1}]\pi_f(S)(I)^- &= \alpha_1 \cdot [\tau^{-1}]r_{\mathcal{E}_1}(I)^- + \alpha_2 \cdot [\tau^{-1}]r_{\mathcal{E}_2}(I)^- \\ &= \frac{\alpha_1 a_{\mathcal{E}_1}(0) + \alpha_2 a_{\mathcal{E}_2}(0)}{\ell+1} \\ &= \frac{\alpha_1 + \alpha_2(2^\lambda+1)}{\ell+1}. \end{aligned}$$

It follows that $\alpha_1 = -(2^\lambda + 1)\alpha_2$. Let's compute $a_{\mathcal{E}_2|U}(0)$, observe $\frac{1}{2}(U\tau) = ST^{-2}S \cdot \left(\frac{\tau-1}{2}\right)$ so

$$\begin{aligned} \mathcal{E}_2|_\lambda U &= E_\lambda(\tau/2)|_\lambda U + 2^\lambda E_\lambda(2\tau)|_\lambda \left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right) \\ &= \tau^{-\lambda} E_\lambda\left(\frac{1}{2}(U\tau)\right) + 2^\lambda \tau^{-\lambda} E_\lambda\left(2\left(\frac{\tau-1}{\tau}\right)\right) \\ &= \tau^{-\lambda} E_\lambda\left(ST^{-2}S \cdot \left(\frac{\tau-1}{2}\right)\right) + j(S, \tau/2)^{-\lambda} E_\lambda(S(\tau/2)) \\ &= E_\lambda\left(\frac{\tau-1}{2}\right) + E_\lambda(\tau/2). \end{aligned}$$

Hence $a_{\mathcal{E}_2|U}(0) = a_{E_\lambda}(0) + a_{E_\lambda}(0) = 2$. Moreover, $a_{\mathcal{E}_1|U}(0) = a_{\mathcal{E}_1}(0) = 1$. Therefore,

$$\begin{aligned} b_{\ell+1} &= [\tau^{-1}]\pi_f(S)(U)^- = \alpha_1 \cdot [\tau^{-1}]r_{\mathcal{E}_1}(U)^- + \alpha_2 \cdot [\tau^{-1}]r_{\mathcal{E}_2}(U)^- \\ &= \frac{\alpha_1 a_{\mathcal{E}_1|U}(0) + \alpha_2 a_{\mathcal{E}_2|U}(0)}{\ell+1} \\ &= \frac{\alpha_1 + 2\alpha_2}{\ell+1}. \end{aligned}$$

With these two relations, solving for α_1, α_2 gives the claim. □

We have completely determined the Eisenstein component. To determine the cuspidal component, we seek a description of $\mathring{g}_1$. Define the set $\mathfrak{I}_\ell$ to be the set of $f \in I_\lambda$ with the property that $f(x) \in \mathbb{C}_\ell[\tau]$ for all $x \in \Gamma^\vartheta \backslash \mathbf{PSL}_2(\mathbb{Z})$. We define an inner product $\langle\!\langle \cdot, \cdot \rangle\!\rangle : \mathfrak{I}_\ell \times \mathfrak{I}_\ell \to \mathbb{C}$

$$\langle\!\langle f, g \rangle\!\rangle := \sum_{x \in \Gamma^\vartheta \backslash \mathbf{PSL}_2(\mathbb{Z})} \langle f(x), g(x) \rangle,$$

where the inner product $\langle \cdot, \cdot \rangle : \mathbb{C}_\ell[\tau] \times \mathbb{C}_\ell[\tau] \to \mathbb{C}$ is defined by

$$\left\langle \sum_{n=0}^{\ell} a_n \tau^n, \sum_{n=0}^{\ell} b_n \tau^n \right\rangle := \sum_{n=0}^{\ell} (-1)^n \binom{\ell}{n}^{-1} a_n b_{\ell-n}.$$

In order to not cause confusion, we shall denote (g, h) for the Petersson inner product in $S_\lambda(\Gamma^\vartheta)$. If $g \in M_\lambda(\Gamma^\vartheta)$ but $h \in S_\lambda(\Gamma^\vartheta)$ impose $(g, h) = (\mathring{g}, h)$. By [22] Prop.8.1, we can decompose $r \in W_\lambda$ as

$$r = \mathring{r} + \tilde{r}||(I - S)$$

for some $\mathring{r} \in \mathfrak{I}_\ell$ and some $\tilde{r}(x)$ of the form $c(x)\tau^{\ell+1}$ with $x \in \Gamma^\vartheta \backslash \mathbf{PSL}_2(\mathbb{Z})$. With this decomposition, we can define the inner product $\{\cdot, \cdot\} : W_\lambda \times W_\lambda \to \mathbb{C}$

$$\{r_1, r_2\} := \langle\!\langle \mathring{r}_1||(T - T^{-1}), \mathring{r}_2 \rangle\!\rangle + \langle\!\langle 2\tilde{r}_1||(T - T^{-1}), \mathring{r}_2 \rangle\!\rangle + \langle\!\langle \mathring{r}_1, 2\tilde{r}_2||(T^{-1} - T) \rangle\!\rangle.$$

Theorem 3.2.1 (Pasol, Popa [22] Thm.8.6(c))**.** *Let $h \in S_\lambda(\Gamma^\vartheta)$ and $g_1 \in M_\lambda(\Gamma^\vartheta)$. Then*

$$-3(2i)^{\ell+1}(\mathring{g}_1, h) = \{r_{g_1}^-, \overline{r_h^+}\}.$$

Actually the theorem proved in [22] is much more general, but for our purposes, this will suffice. Since $r_h = r_{\mathring{h}} = \mathring{r}_h$, we conclude from Theorem 3.2.1 that

$$-3(2i)^{\ell+1}(\mathring{g}_1, h) = \langle\!\langle \mathring{r}_{g_1}^- || (T - T^{-1}), \overline{r_h^+} \rangle\!\rangle + \langle\!\langle 2\tilde{r}_{g_1} || (T - T^{-1}), \overline{r_h^+} \rangle\!\rangle.$$

Claim: We have that $\tilde{r}_{g_1}(x) = \tilde{r}_{g_1}(x)^- = c(x)\tau^{\ell+1}$ with $c(I) = c(T) = 0$ and $c(U) = b_{\ell+1}$.

Proof Observe

$$\tilde{r}_{g_1} || (I - S)(x) = c(x)\tau^{\ell+1} + \frac{c(xS)}{\tau}.$$

Since $c(S) = c(I)$, substituting $x = I$ gives $0 = c(I)(\tau^{\ell+1} + 1/\tau)$ so $c(I) = 0$. Substituting $x = T$ gives $\frac{b_{\ell+1}}{\tau} = c(T)\tau^{\ell+1} + c(U)/\tau$ so $c(T) = 0$ and $c(U) = b_{\ell+1}$.

□

We have

$$\mathring{r}_{g_1}^- || (T - T^{-1})(x) = \mathring{r}_{g_1}^-(xT)|_{-\ell}(T - T^{-1}) = \sum_{\substack{j=0 \\ j \text{ even}}}^{\ell} s_j(xT)\tau^j, \tag{3.2.1}$$

where

$$s_j(x) = 2\sum_{i=0}^{\ell} \binom{i}{j} [\tau^i]\pi_f(S)(x).$$

Also

$$\tilde{r}_{g_1}^- || (T - T^{-1})(x) = c(xT)[(\tau+1)^{\ell+1} - (\tau-1)^{\ell+1}] = \sum_{\substack{j=0 \\ j \text{ even}}}^{\ell} 2c(xT)\binom{\ell+1}{j}\tau^j. \tag{3.2.2}$$

Theorem 3.2.2. *Let ℓ be even and $\chi(W_4) = \pm i^{-\ell}$. For $f \in S_{k/2}(\Gamma_0^*(4N), \chi)$ and $a \in [0, \kappa + \ell]$, the form $g \in M_\lambda(\Gamma_0^*(4))$ in Theorem 3.1.4 satisfies*

$$\begin{aligned} g(\tau) &= \frac{-(2i)^{-1-\ell}}{3} \sum_{x \in \Gamma^\vartheta \backslash \mathbf{PSL}_2(\mathbb{Z})} \sum_{\substack{j=0 \\ j \text{ even}}}^{\ell} \left[s_j(xT) + 2c(xT)\binom{\ell+1}{j} \right] R_{x,j}^+(2\tau) \\ &+ \frac{2^\lambda + 1}{2^\lambda - 1}(\ell+1)b_{\ell+1}\mathcal{E}_1(2\tau) + \frac{-1}{2^\lambda - 1}(\ell+1)b_{\ell+1}\mathcal{E}_2(2\tau). \end{aligned}$$

Proof We work over Γ^ϑ first. For any $h \in S_\lambda(\Gamma^\vartheta)$, we have $-3(2i)^{\ell+1}(g, h) =$

$-3(2i)^{\ell+1}(\mathring{g}_1, h)$ which equals

$$\begin{aligned}
&= \langle\!\langle \mathring{r}^{-}_{g_1} || (T - T^{-1}), \overline{r^{+}_{h}} \rangle\!\rangle + \langle\!\langle 2\tilde{r}_g || (T - T^{-1}), \overline{r^{+}_{h}} \rangle\!\rangle \\
&= \sum_{x \in \Gamma^{\vartheta} \backslash \mathbf{PSL}_2(\mathbb{Z})} \langle \mathring{r}^{-}_{g_1}(xT)|_{-\ell}(T - T^{-1}) + 2\tilde{r}_g || (T - T^{-1})(x), \overline{r^{+}_{h}} \rangle \\
&= \sum_{x \in \Gamma^{\vartheta} \backslash \mathbf{PSL}_2(\mathbb{Z})} \left\langle \sum_{\substack{j=0 \\ j \text{ even}}}^{\ell} \left(s_j(xT) + 2c(xT) \binom{\ell+1}{j} \right) \tau^j, \sum_{j=0}^{\ell} \binom{\ell}{j} \rho^{+}_{h}(x, j) \tau^{\ell - j} \right\rangle \\
&= \sum_{x \in \Gamma^{\vartheta} \backslash \mathbf{PSL}_2(\mathbb{Z})} \sum_{\substack{j=0 \\ j \text{ even}}}^{\ell} \left[s_j(xT) + 2c(xT) \binom{\ell+1}{j} \right] \overline{\rho^{+}_{h}(x, n-j)} \\
&= \sum_{x \in \Gamma^{\vartheta} \backslash \mathbf{PSL}_2(\mathbb{Z})} \sum_{\substack{j=0 \\ j \text{ even}}}^{\ell} \left[s_j(xT) + 2c(xT) \binom{\ell+1}{j} \right] (R^{+}_{x,j}, h).
\end{aligned}$$

It follows that

$$\mathring{g}_1(\tau) = \frac{-(2i)^{1-\ell}}{3} \sum_{x \in \Gamma^{\vartheta} \backslash \mathbf{PSL}_2(\mathbb{Z})} \sum_{\substack{j=0 \\ j \text{even}}}^{\ell} \left[s_j(xT) + 2c(xT) \binom{\ell+1}{j} \right] R^{+}_{x,j}.$$

To switch back to $\Gamma_0^*(4)$ perform the change of variable $\tau \mapsto 2\tau$. □

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Appendix A

List of Notation

$\mathbb{N}$	$\{1, 2, 3, \cdots\}$
$\mathfrak{H}$	Upper half plane $\{\tau \in \mathbb{C} : \Im(\tau) > 0\}$
$a(c)$	a modulo c (when context is clear)
$\left(\frac{c}{d}\right)$	See section 0.1
ε_d	1 if $d \equiv 1(4)$ and i if $d \equiv 3(4)$
I	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
T	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$
S	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
U	$\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} = ST$
W_N	$\begin{pmatrix} 0 & -1/\sqrt{N} \\ \sqrt{N} & 0 \end{pmatrix}$
$\Gamma_0(N)$	See Definition 0.1.1
$\Gamma_0^*(4N)$	$\langle W_{4N}, \Gamma_0(4N)\rangle$, the group generated by $\Gamma_0(4N)$ and W_{4N}
$\mathbb{T}$	The set of complex numbers of modulus one
H	The lift of $\Gamma_0^*(4)$ (see section 0.2.2)
$j_\gamma(z) = j(\gamma, z)$	$cz + d$ if $\gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix}$
$\mathcal{J}_v$	
$\mathbf{SL}_2(R)$	Determinant one 2×2 matrices with coefficients in R
$\mathbf{PSL}_2(R)$	$\mathbf{SL}_2(R)/\{\pm I\}$
v_θ	See example 0.1.5
q	$e^{2\pi i\tau}$
$(a, b)_\infty$	Hilbert symbol: -1 if $a < 0$ and $b < 0$; $+1$ otherwise
μ_n	The cyclic subgroup of $\mathbb{T}$ of order n
ζ_n	$e(1/n) = e^{2\pi i/n}$
$\mathrm{Hol}(\mathfrak{H})$	The set of holomorphic functions on $\mathfrak{H}$
Heigenform	Hecke eigenform (shorthand)
Heigenbasis	Hecke eigenbasis (shorthand)
$\chi_{1,4}$	the non-trivial character modulo 4
$\mathrm{diag}(a, b)$	$\begin{pmatrix} a & \\ & b \end{pmatrix}$