Growing Classifications: Widths, Ehrhart Theory and Spherical Geometry

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Abstract

This thesis focuses on three classifications of convex polytopes, which are separate, but the methods of each influences those that follow. There are links to combinatorial algebraic geometry throughout, particularly to toric and spherical geometry. This is most explicit in the third project, which is additionally a classification of certain spherical varieties.

In the first project we introduce the multi-width of a polytope, which is an extension of its lattice width. We study the classification of lattice simplices by their multi-width in dimensions two and three. This is motivated by computational questions in toric geometry. We completely classify lattice triangles by their multi-width and also classify lattice tetrahedra of small multi-width.

The second project concerns the Ehrhart theory of rational polygons. The Ehrhart theory of lattice polygons is already well understood and here we make steps towards a similar understanding of denominator two polygons. We classify denominator two polygons containing up to four lattice points, including a description of infinite families of polygons with no interior points. Using this data, we completely classify the Ehrhart polynomials of denominator two polygons with zero interior points and find three sharp bounds on the coefficients when there are interior points.

In the final project we study spherical varieties, which generalise toric and flag varieties. We discuss isomorphisms between spherical varieties and describe a class of lattice automorphisms which are induced by isomorphisms of spherical varieties. We define a normal form for lattice polytopes up to this group of automorphisms. This normal form is vital to our classification of spherical canonical Fano four-folds. Like toric Fano varieties, spherical Fano varieties correspond to polytopes. Therefore, we can classify the varieties by classifying the corresponding polytopes.

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List of Papers

1. Classification of lattice triangles by their two smallest widths [Ham23a] [Chapter 3]
2. Classification of width 1 lattice tetrahedra by their multi-width [Ham24a] [Chapter 4]

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Chapter 1

Introduction

The central thread of this thesis is a trio of polytope classifications. The contexts and motivations for the classifications are different, but the methods of each influences the next. An algebraic geometry perspective is relevant throughout via toric geometry, and the third classification is a geometric one.

Toric varieties are an important class of algebraic variety, which have a beautiful description in terms of convex geometry. Spherical varieties are a generalisation of toric varieties which share a similar combinatorial description. Both toric and spherical Fano varieties are in correspondence with certain polytopes, and we can learn about the varieties by studying these polytopes. We recall the necessary basics of toric geometry in Chapter 2. For a more in-depth study of toric varieties see, for example, [Ful93] or [CLS11].

1.1 Classifying Simplices by Width

In this project we classify lattice simplices by their widths. In dimension 2 this consists of a classification of the lattice triangles which are a subset of any given lattice rectangle. We complete this in Chapter 3 and find that it shows unexpected regularity and links with geometry. In Chapter 4 we partially extend this classification to three dimensions, where it shows similar patterns to the two-dimensional case.

We can associate lattice polytopes to toric Fano varieties. Their vertices are points in the lattice of one-parameter subgroups of the torus and they contain the origin in their interior. Many important invariants of the variety can be computed from properties of this polytope. In dimension greater than one, there are infinitely many equivalent polytopes associated to the variety. Computing the desired invariants from any of these polytopes will give the same solution but may have a different compute time, since the vertices of the polytopes are different. This suggests the question: for which equivalent polytope can we compute an invariant fastest? A first guess may be to find a polytope whose vertices are 'as close to the origin as possible' so that the numbers we compute with are small. We formalise this idea by considering the widths of a polytope.

The lattice width of a polytope P with respect to a dual lattice point u counts the number of integral hyperplanes, with normal vector u, which intersect a polytope, and is defined as follows

$$\mathrm{width}_u(P) \coloneqq \max_{x \in P}\{u \cdot x\} - \min_{x \in P}\{u \cdot x\}.$$

The *width* of P is the minimum value which $\mathrm{width}_u(P)$ can take for all non-zero u. We extend this definition into the multi-width of a polytope. For a d-dimensional polytope the multi-width is

$$\mathrm{mwidth}(P) \coloneqq \min_{u_1,\ldots,u_d \in \mathbb{Z}^d}\{(\mathrm{width}_{u_1}(P), \ldots, \mathrm{width}_{u_d}(P))\}$$

where the collection of u_i are required to be linearly independent and minimum means with respect to lexicographic order. The multi-width describes the dimensions of the smallest parallelepiped containing a polytope.

We completely classify lattice triangles with multi-width (w_1, w_2) in Theorem 3.0.2. This can be adapted to count the lattice triangles which are a subset of a square, yielding the following surprising result.

Theorem. *Let t_n denote the number of lattice triangles, up to affine equivalence, which are a subset of $[0, n]^2$. Then $t_n = f(n)$ for $n \in \mathbb{Z}_{\geq 0}$ where f is the Hilbert function of a hypersurface of degree 8 in $\mathbb{P}(1, 1, 1, 2, 2, 2)$. Additionally $t_n = |nP \cap \mathbb{Z}^4|$ for $n \in \mathbb{Z}_{\geq 0}$ where*

$$P \coloneqq \operatorname{conv}\left(\left(\tfrac{1}{2}, 0, 0, 0\right), \left(0, \tfrac{1}{2}, 0, 0\right), \left(0, 0, \tfrac{1}{2}, 0\right), (0, 0, 0, 1), (-1, -1, -1, -1)\right).$$

This suggests some deeper structure in the classification which would be interesting to investigate further.

It is straightforward to extend this style of classification to one-dimensional simplices but extending it to higher dimensions is challenging. The method we use to classify triangles can be generalised to classify tetrahedra, however the number of cases which needs to be checked by hand increases dramatically. We extend the theoretical classification to tetrahedra with multi-width $(1, w_2, w_3)$ and algorithmically classify tetrahedra with multi-width $(2, w_2, w_3)$ for small w_2 and w_3. Excitingly, very similar patterns appear in these extensions.

1.2 Rational Polygons

The *denominator* of a polygon P is the smallest integer r such that rP is a lattice polytope. In Chapter 5 we classify rational polygons with small denominator, containing a small number of lattice points, up to affine equivalence. Using the results we study the Ehrhart theory of denominator 2 polygons.

There are infinitely many rational polygons with a fixed denominator containing a fixed number of lattice points. However, we show that all but finitely many denominator r polygons containing k lattice points are equivalent to a subset of $[0, 1] \times \mathbb{R}$. This allows us to make a meaningful classification despite the infinite number of polygons. The classification method is based on a growing algorithm. This means we determine a collection of minimal polygons, then successively grow them by adding points. In this way we classify denominator 2 polygons containing $0, 1, 2, 3$ and 4 lattice points and denominator 3 polygons containing 0 lattice points which are not equivalent to a subset of $[0, 1] \times \mathbb{R}$. They are enumerated in Table 5.1.

Ehrhart theory is concerned with counting lattice points in integral dilations of a polytope. A quasi-polynomial is a polynomial whose coefficients are periodic functions with integer period. The *Ehrhart polynomial* of a rational polytope P is the quasi-polynomial $\mathrm{ehr}_P(n)$ with the property that $\mathrm{ehr}_P(n) = |nP \cap \mathbb{Z}^d|$ for all integers n. Ehrhart proved that there is such a quasi-polynomial for all rational polytopes [Ehr62]. An interesting problem is to classify the Ehrhart polynomials for a given class of polytope. For lattice polygons the classification is known. The Ehrhart polynomial of a lattice polygon P is determined by the number of interior and boundary lattice points of P. Scott proved that if b is the number of boundary points of a lattice polygon and i is the number of interior points then $b \geq 3$ and either $i = 0$, $b \leq 2i + 6$ or $(b, i) = (9, 1)$ [Sco76]. Hasse and Schicho gave examples of polygons realising all such pairs (b, i) completing the classification of Ehrhart polynomials in this case [HS09].

We seek a similar classification in the case of denominator 2 polygons. As in the lattice polygons case, the Ehrhart polynomial of a denominator 2 polygon is completely determined by the number of interior and boundary points in P and $2P$. Our main result is the following. Its proof depends on the idea of multi-width, defined in the previous project.

Theorem. *Let P be a denominator 2 polygon and define b_1, i_1, b_2 and i_2 to be the number of boundary and interior points of P and $2P$ respectively. For all but finitely many such polygons, the integers b_1, i_1, b_2 and i_2 satisfy one of the following conditions:*

- $i_1 = 0$, $i_2 = 0$ *and* $b_2 \geq \max(3, 2b_1)$,
- $b_1 = 0$, $i_1 = 0$, $b_2 = 4$ *and* $i_2 > 0$,
- $i_1 = 0$, $i_2, b_1 > 0$, $\max(3, 2b_1) \leq b_2 \leq 2b_1 + 4$ *and* $b_2 \leq 2i_2 + 6$ *or*

- $i_1 > 0$, $b_2 \geq \max\{3, 2b_1\}$, $i_2 \geq b_1 + 2i_1 - 1$ *and* $b_2 + i_2 \leq 2b_1 + 6i_1 + 7$.

1.3 Canonical Spherical Fano Four-Folds

Classification of spherical varieties using their combinatorics requires a combinatorial understanding of isomorphisms of spherical varieties. In Chapter 6 we describe a class of lattice automorphisms which are induced by isomorphisms of spherical varieties. We also define a normal form for lattice polytopes up to this class of automorphisms. Using this, in Chapter 7 we describe a method to classify the non-toric spherical canonical Fano four-folds.

Our classification will include the non-toric spherical Gorenstein Fano four-folds, completing Kreuzer and Skarke's seminal classification of the toric case. It will also include the non-toric, locally factorial spherical Fano four-folds, so is an extension of Delcroix and Montagard's classification of the same varieties in rank at most 2 [DM23].

Our motivation comes from mirror symmetry, which is an active area of research in algebraic geometry with applications in string theory. In this setting, people care about finding pairs of Calabi-Yau varieties called mirror pairs. Batyrev [Bat94] gave a way to construct many such pairs as hypersurfaces in toric Gorenstein Fano varieties. There is a completely combinatorial description of the polytopes associated to these varieties opening the doors to classification. Kreuzer and Skarke [KS00] classified all four-dimensional toric Gorenstein Fano varieties. This is a dimension of particular interest in physics since it produces three-dimensional (complex) Calabi-Yau's which are the compactifications of the six extra (real) dimensions theorised in string theory.

The combinatorial description of toric Gorenstein Fano varieties has been generalised to the spherical case. The polytopes are called G/H-reflexive and are the objects we describe how to classify. Spherical Gorenstein Fano varieties are a subset of spherical canonical Fano varieties. We define G/H-canonical polytopes which are easier to work with, although there are many more of them. We give a method to classify the spherical canonical Fano four-folds and can then retrieve the Gorenstein classification as a subset.

The polytopes are found using modified versions of the growing algorithm used in Chapter 5 and the growing algorithm used in the classification of toric canonical Fano three-folds [Kas10]. For polytopes to represent spherical varieties we must also combine them with Luna data. In spherical geometry a Luna datum is a collection of objects which plays the role of the character lattice in toric geometry. The full classification is expected to take months to run, so it has not been practical to complete it before finishing this thesis. Therefore, we present the classification method here and not the classification.

Chapter 2

Background

In this chapter we recall the basics of convex geometry, toric varieties and Ehrhart theory, since these topics will be relevant throughout the remaining chapters. Convex geometry and lattice polytopes are central to the whole thesis, so we begin there.

A *lattice* is a finite dimensional $\mathbb{Z}$-module $M \cong \mathbb{Z}^d$ for some non-negative d. It is contained in a natural rational vector space $M_{\mathbb{Q}} := \mathbb{Q} \otimes_{\mathbb{Z}} M \cong \mathbb{Q}^d$ and real vector space $M_{\mathbb{R}} := \mathbb{R} \otimes_{\mathbb{Z}} M \cong \mathbb{R}^d$. There is a dual lattice $N = M^* := \mathrm{Hom}(L, \mathbb{Z})$ with a natural dual paring $M \times N \to \mathbb{Z}$. A *lattice polytope* is the convex hull of finitely many lattice points $v_1, \ldots, v_n \in M$. A *rational polytope* is the same except we allow rational points $v_1, \ldots, v_n \in M_{\mathbb{Q}}$. A polytope is called a *polygon* if it is two-dimensional and a *simplex* if its vertices are affinely independent. For example, line segments, triangles and tetrahedra are simplices.

We will usually consider polytopes up to some equivalence relation. Sometimes this is *unimodular equivalence*, where polytopes are defined up to the action of the general linear group $\mathrm{GL}_d(\mathbb{Z})$. At other times we use *affine equivalence*, where polytopes are defined up to integral translations and unimodular maps. Many properties of a polytope are preserved by these equivalences such as its volume and the number of lattice points in its boundary and interior. Another affine invariant of a polytope is its lattice width. Given $u \in N$ the *width of P with respect to u* is

$$\mathrm{width}_u(P) \coloneqq \max_{x\in P}\{u\cdot x\} - \min_{x\in P}\{u\cdot x\}.$$

The *width* of P is the smallest $\mathrm{width}_u(P)$ for all non-zero dual vectors u:

$$\mathrm{width}(P) = \min_{u\in N\setminus\{0\}}\{\mathrm{width}_u(P)\}.$$

A polytope P has *dual polytope* which is the set of points u in the dual space $N_{\mathbb{R}}$ such that $u \cdot v \leq 1$ for all points v in P. The dual of a lattice polytope is a rational polytope but need not be a lattice polytope.

A *(lattice) cone* is the collection of non-negative linear combinations of a finite collection of lattice points $v_1, \ldots, v_n \in M$:

$$\mathrm{cone}(v_1,\ldots,v_n) \coloneqq \left\{\sum_{i=1}^{n}\lambda_i v_i : \lambda_i \in \mathbb{Q}_{\geq 0}\right\}.$$

One-dimensional cones are called *rays*. An *affine cone* is the translation of a cone σ by some point $x \in M_{\mathbb{R}}$ written $x + \sigma$. The *dual cone* σ^* of a lattice cone σ is the set of points u in the dual space such that $u \cdot v \geq 0$ for all points $v \in \sigma$. A face of a lattice cone σ is a subset of σ 'cut out by' a linear form u in σ^*. That is, the set of points v in σ such that $u \cdot v = 0$. A cone is said to be *pointed* if it has the origin as a face. A *lattice fan* is a finite set $\mathcal{F}$ of pointed lattice cones such that

1. for all $\sigma, \tau \in \mathcal{F}$ the intersection $\sigma \cap \tau$ is also in $\mathcal{F}$ and
2. for all $\sigma \in \mathcal{F}$ and all faces τ of σ, τ is also in $\mathcal{F}$.

2.1 Toric Geometry

A *toric variety* is a normal irreducible variety X such that an algebraic torus $T = (\mathbb{C}^\times)^d$ is a Zariski open dense subset of X and the action of T on itself extends to an action of T on X. An algebraic torus has two, mutually dual, lattices associated to it: the *character lattice* $M \coloneqq \mathfrak{X}(T) = \mathrm{Hom}(T, \mathbb{C}^\times)$ and the *lattice of one-parameter* subgroups $N \coloneqq \mathrm{Hom}(\mathbb{C}^\times, T)$. To each toric variety X we can associate a lattice fan in N denoted by $\mathcal{F}_X$ and to each lattice fan $\mathcal{F}$ in N we can associate a toric variety $X_\mathcal{F}$.

Let X_1 and X_2 be toric varieties with algebraic tori $T_i \hookrightarrow X_i$ for $i = 1, 2$. A *toric morphism* is a morphism $\varphi : X_1 \to X_2$ such that $\varphi(T_1) \subseteq T_2$ and $\varphi|_{T_1} : T_1 \to T_2$ is a morphism of algebraic tori. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be lattice fans in N_1 and N_2 respectively, then a *morphism of lattice fans* is a lattice homomorphism $\phi : N_1 \to N_2$ such that for every cone $\sigma_1 \in \mathcal{F}_1$ there is a cone $\sigma_2 \in \mathcal{F}_2$ such that $\phi(\sigma_1) \subseteq \sigma_2$. There is a covariant equivalence of categories:

$$\{\text{toric varieties}\} \leftrightarrow \{\text{lattice fans}\}$$
$$X \mapsto \mathcal{F}_X$$
$$(\varphi : X_1 \to X_2) \mapsto (\varphi_* : N_1 \to N_2, \lambda \mapsto \varphi|_T \circ \lambda)$$

Theorem 2.1.1 (Orbit-cone correspondence)**.** *Let X be a toric variety, then there is a bijection between the cones in $\mathcal{F}_X$ and the set of torus orbits $T \cdot x$ for $x \in X$. The dimension of a cone $\sigma \in \mathcal{F}_X$ is equal to the codimension of its corresponding T-orbit in X and a cone $\tau \in \mathcal{F}_X$ is a face of σ if and only if the closure of the orbit corresponding to σ contains the orbit corresponding to τ.*

Proof. See for example [CLS11, Theorem 3.2.6]. □

A *Fano variety* is a normal projective variety X with restricted singularities such that the anticanonical divisor $-K_X$ is an ample $\mathbb{Q}$-Cartier divisor. Different authors place different restrictions on the singularities of a Fano variety. Here all Fano varieties have at worst log-terminal singularities (see Definition 2.1.4). To represent toric Fano varieties with lattice polytopes rather than lattice fans we need the following definitions. The *lattice length* of a lattice line segment is the number of lattice points it contains minus 1. A lattice point v is called *primitive* if the line segment $\mathrm{conv}(\mathbf{0}, v)$ has lattice length 1.

Let $\mathcal{F}$ be a fan and let $\{\rho_1, \ldots, \rho_n\}$ be the set of rays in $\mathcal{F}$. Each ray ρ_i is generated by a primitive lattice point v_i and $\mathcal{F}$ corresponds to a Fano toric variety if and only if the points v_i are the vertices of a convex lattice polytope. From this we get a correspondence between toric Fano varieties and lattice polytopes, motivating the following definitions.

Definition 2.1.2. A lattice polytope P with the origin in its interior is called *Fano* if its vertices are primitive lattice points. A Fano polytope is called

- *Canonical* if the only interior point in P is the origin,
- *Terminal* if the only lattice points in P are the origin and its vertices,
- *Reflexive* if its dual polytope is also a lattice polytope.

Each of these correspond to a type of toric variety.

Definition 2.1.3. A normal variety X is Gorenstein (resp. $\mathbb{Q}$-Gorenstein) if the anticanonical divisor $-K_X$ is Cartier (respectively $\mathbb{Q}$-Cartier).

Definition 2.1.4. Let X be a normal $\mathbb{Q}$-Gorenstein variety and let $f : V \to X$ be a resolution of X, that is f is birational and V is smooth. Then we have $K_V - f^*(K_X) = \sum_{i \in I} a_i E_i$ where $\{E_i : i \in I\}$ is the set of exceptional divisors of f. We say that X is

- *canonical* if, for all $i \in I$, $a_i \geq 0$,
- *terminal* if, for all $i \in I$, $a_i > 0$,
- *log-canonical* if, for all $i \in I$, $a_i \geq 1$ and
- *log-terminal* if, for all $i \in I$, $a_i > 1$.

Often this is referred to as X having at worst canonical, terminal, log-canonical or log-terminal singularities.

Theorem 2.1.5 ([Rei87, Bat94])**.** *Toric Fano (resp. canonical, terminal, Gorenstein) varieties up to toric morphisms are in correspondence with Fano (resp. canonical, terminal, reflexive) lattice polytopes up to unimodular equivalence.*

These descriptions naturally lead to classifications. Toric canonical and terminal varieties have been classified up to dimension 3 [Kas10, Kas06], and toric Gorenstein varieties up to dimension 4 [KS98, KS00].

2.2 Ehrhart Theory

Ehrhart theory is concerned with counting lattice points in integral dilations of rational polytopes. For example, the number of lattice points in the n-th dilation of the empty triangle $\operatorname{conv}((0,0),(0,1),(1,0))$ is the triangular number $\frac{(n+1)(n+2)}{2}$ which is a polynomial in n with rational coefficients. To describe how this appears in general we define quasi-polynomials

Definition 2.2.1. A *quasi-polynomial* is a polynomial whose coefficients are periodic functions with integral period, which map the integers into some field K. In particular, for some integer r, a quasi-polynomial f can be written:

$$f(n) = \begin{cases} f_0(n) & \text{if } n \equiv 0 \mod r \\ \vdots & \vdots \\ f_{r-1}(n) & \text{if } n \equiv r-1 \mod r \end{cases}$$

for some polynomials $f_0, \ldots, f_{r-1} \in K[n]$. The integer r is called the *period* of f. We will always assume that $K = \mathbb{Q}$.

The central result of Ehrhart theory is that we can always count the points in dilations of a rational polytope with some quasi-polynomial.

Theorem 2.2.2 ([Ehr62])**.** *Let P be a d-dimensional rational polytope, then there exists a quasi-polynomial* ehr_P *such that for all positive integers n*

$$\operatorname{ehr}_P(n) = |nP \cap \mathbb{Z}^d|.$$

This is called the Ehrhart polynomial *of P. The* quasi-period *of P is defined to be the minimum period of* ehr_n.

The Ehrhart polynomial of a polytope P has degree equal to the dimension of P, its leading coefficient is constant and equals the volume of P. Remarkably, it is also meaningful to consider the values taken by ehr_P at negative integers:

Theorem 2.2.3 (Ehrhart–Macdonald reciprocity [Mac71])**.** *Let $P \subseteq \mathbb{R}^d$ be a rational polytope and let n be a positive integer. Then,*

$$(-1)^{\dim(P)}\,\mathrm{ehr}_P(-n) = |nP^\circ \cap \mathbb{Z}^d|$$

where P° is the relative interior of P.

A denominator of a rational polytope P is an integer r such that rP is a lattice polytope. Some authors require r to be the minimum such integer but we do not include that condition. The quasi-period of a polytope P will always divide the denominator of P. If the quasi-period is less than all denominators of P then we say that P exhibits *quasi-period collapse*. For example the denominator 2 triangle $\mathrm{conv}((0,0),(2,0),(0,\frac{1}{2}))$ has quasi-polynomial $\frac{1}{2}n^2 + \frac{3}{2}n + 1$ and quasi-period 1.

It is sometimes useful to consider the series

$$\mathrm{Ehr}_P(t) \coloneqq 1 + \sum_{n=1}^{\infty} \mathrm{ehr}_P(n)t^n$$

called the *Ehrhart series* of P. This always sums to $\frac{h^*(t)}{(1-t)^{d+1}}$ where $h^*(t)$ is a polynomial of degree at most d and $h^*(1)$ is non-zero. Note that the dimension of P and the polynomial h^* completely determine the Ehrhart theory of P.

Chapter 3

Classification of lattice triangles by their two smallest widths

In this chapter we classify lattice triangles by the smallest rectangle they are a subset of. We describe this in terms of lattice widths. Recall that for a lattice polytope $P \subseteq \mathbb{R}^d$ and a primitive dual vector $u \in (\mathbb{Z}^d)^*$ the *width of P with respect to u* is $\text{width}_u(P) \coloneqq \max_{x \in P}\{u \cdot x\} - \min_{x \in P}\{u \cdot x\}$ and the *width* of P is the minimum width with respect to non-zero u.

Restricting the width of polytopes is a powerful tool towards classifying them. It was shown in [BHHS21] that in each dimension d there is some constant $w \in \mathbb{N}$ such that the number of lattice polytopes with width larger than w containing n lattice points is finite. When $d > 2$ there are infinitely many polytopes with small width containing n lattice points, so to classify them one must classify the finitely many exceptional polytopes with large width and the infinitely many polytopes with small width. This was done, for example, in the classification of empty four-dimensional lattice simplices [IVnS21]. Although there are finitely many polygons containing a given number of lattice points, there are polygon classifications which follow a similar pattern. For example, the classification of lattice polygons which contain a point from which all other lattice points are visible [MT21] and the classification of denominator r polygons containing k lattice points (Chapter 5) both consist of finitely many exceptional polygons and infinitely many polygons with small width.

Here we introduce the higher widths of a polytope as a way to break an infinite class of polytopes with a given width into finite pieces. In particular, the first and second widths of P are the two smallest widths of P in linearly independent directions, denoted by $\operatorname{width}^1(P)$ and $\operatorname{width}^2(P)$ respectively (see Definition 3.1.1). If P has dimension at least 2, and w_1 and w_2 are the first and second widths then we always have $0 < w_1 \leq w_2$, so we assume that w_1 and w_2 are such integers for the rest of this chapter unless otherwise stated.

We demonstrate the use of higher widths by describing the finite set

$$\mathcal{T}_{w_1,w_2} \coloneqq \{T = \operatorname{conv}(v_1, v_2, v_3) : v_i \in \mathbb{Z}^2, \operatorname{width}^1(T) = w_1, \operatorname{width}^2(T) = w_2\}/\sim$$

of lattice triangles whose first and second widths are w_1 and w_2, where $\sim$ denotes affine equivalence. Theorem 3.0.2 achieves this by establishing a bijection between $\mathcal{T}_{w_1,w_2}$ and the set $\mathcal{S}_{w_1,w_2}$ defined as follows.

Definition 3.0.1. Let $\mathcal{S}_{w_1,w_2}$ be the set of lattice triangles T such that

- If $w_1 < w_2$ then T is equal to
  - (A) $\operatorname{conv}((0,0),(w_1,y_1),(0,w_2))$ where $0 \leq y_1 \leq (w_2 - y_1 \mod w_1)$ or
  - (B) $\operatorname{conv}((0,0),(w_1,y_1),(x_2,w_2))$ where $0 < x_2 \leq \frac{w_1}{2}$ and $0 \leq y_1 \leq w_1 - x_2$ or
  - (C) $\operatorname{conv}((0,y_0),(w_1,0),(x_2,w_2))$ where $1 < x_2 < \frac{w_1}{2}$ and $0 < y_0 < x_2$.
- If $w_1 = w_2$ then T is equal to
  - (A) $\operatorname{conv}((0,0),(w_1,y_1),(0,w_1))$ where $0 \leq y_1 \leq (w_1 - y_1 \mod w_1)$ or
  - (B) $\operatorname{conv}((0,0),(w_1,y_1),(x_2,w_1))$ where $0 < x_2 \leq \frac{w_1}{2}$ and $x_2 \leq y_1 \leq w_1 - x_2$.

These triangles are each of one of the three types depicted in Figure 3.1. The triangles in $\mathcal{S}_{w_1,w_2}$ for $w_1 \leq w_2 \leq 4$ are depicted in Figure 3.2. For example, the triangles in $\mathcal{S}_{2,3}$ are $\mathrm{conv}((0,0),(2,0),(0,3))$, $\mathrm{conv}((0,0),(2,0),(1,3))$ and $\mathrm{conv}((0,0),(2,1),(1,3))$.

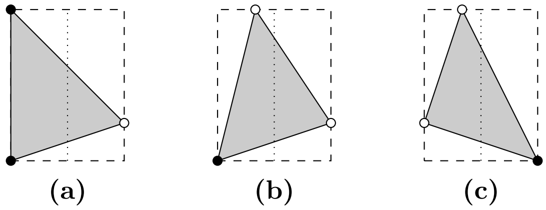


Figure 3.1: *The three types of triangle in the set $\mathcal{S}_{w_1,w_2}$ as in Definition 3.0.1. Black vertices are fixed for a given type while white vertices vary within fixed ranges along the boundary of the rectangle.*

Theorem 3.0.2. *There is a bijection from $\mathcal{S}_{w_1,w_2}$ to $\mathcal{T}_{w_1,w_2}$ given by the map taking T to its affine equivalence class. In particular,*

- *when $w_1 < w_2$ the cardinality of $\mathcal{T}_{w_1,w_2}$ is*
  - $\frac{w_1^2}{2} + 2$ *if w_1 and w_2 are even*
  - $\frac{w_1^2}{2} + 1$ *if w_1 is even and w_2 is odd*
  - $\frac{w_1^2}{2} + \frac{1}{2}$ *if w_1 is odd*
- *and when $w_1 = w_2$ the cardinality of $\mathcal{T}_{w_1,w_1}$ is*
  - $\frac{w_1^2}{4} + \frac{w_1}{2} + 1$ *if w_1 is even*
  - $\frac{w_1^2}{4} + \frac{w_1}{2} + \frac{1}{4}$ *if w_1 is odd*

	w_2										
w_1	0	1	2	3	4	5	6	7	8	9	10
0	1	1	2	2	3	3	4	4	5	5	6
1	0	1	1	1	1	1	1	1	1	1	1
2	0	0	3	3	4	3	4	3	4	3	4
3	0	0	0	4	5	5	5	5	5	5	5
4	0	0	0	0	7	9	10	9	10	9	10
5	0	0	0	0	0	9	13	13	13	13	13
6	0	0	0	0	0	0	13	19	20	19	20

Table 3.1: The number of lattice triangles with multi-width (w_1, w_2) up to affine equivalence for small w_1 and w_2.

The idea of the proof is to use the fact that a triangle with first two widths w_1 and w_2 is equivalent to a subset of a $w_1 \times w_2$ rectangle. It is straightforward to classify the collection of possible x-coordinates of vertices of a triangle in $[0, w_1] \times [0, w_2]$ up to affine equivalence. The y-coordinates of each vertex are then integers in the range $[0, w_2]$ and we use facts about the width of the triangles to bound these integers. By removing duplicates from the resulting list we obtain the set of triangles $\mathcal{S}_{w_1,w_2}$. It remains to check that these triangles have the desired widths and are distinct. This is made easier by the fact that when $w_1 < w_2$ we know that there is a unique (up to sign) direction in which the triangle has width w_1.

A consequence of Theorem 3.0.2 is that we can completely classify lattice triangles by their affine automorphism group. Another comes from the fact that a lattice polygon is always equivalent to a subset of a rectangle with dimensions given by its first two widths. Not only is this an integral part of the proof of Theorem 3.0.2 but it also allows us to classify lattice triangles by the smallest square they are a subset of. We can extend the classification to include degenerate triangles, that is multi-sets of three collinear lattice points. If we do so then, up to affine equivalence, the number of lattice triangles which are a subset of $[0, n]^2$ is equal to the cardinality of the set $nQ \cap \mathbb{Z}^4$ where Q is the four-dimensional simplex

$$Q := \tfrac{1}{2}\,\mathrm{conv}\left((1,0,0,0)\,, (0,1,0,0)\,, (0,0,1,0)\,, (0,0,0,2), (-2,-2,-2,-2)\right).$$

Additionally, the sequence counting the triangles in $[0, n]^2$ up to affine equivalence is given by a Hilbert function of a degree 8 hypersurface in $\mathbb{P}(1, 1, 1, 2, 2, 2)$.

In two dimensions the Ehrhart polynomial is completely determined by the number of boundary points b and the number of interior points i of a lattice polygon. The possible pairs (b, i) for lattice triangles can be plotted (see Figure 3.4), hinting at some beautiful combinatorial structures. Hofscheier–Nill–Öberg [HNO18] described infinitely many empty cones in this plot. They also observed that points in the strips between these cones appear to form periodic patterns. Using our classification of lattice triangles we will explain why these patterns appear and partially describe them.

In Section 3.1 we will formally define the first two widths. We will prove that any lattice polygon with widths w and w' in linearly independent directions is equivalent to a subset of a $w \times w'$ rectangle and as a result prove that $\mathcal{T}_{w_1,w_2}$ is equal to the set of triangles which are a subset of a $w_1 \times w_2$ rectangle and no smaller up to equivalence. In Section 3.2 we prove Theorem 3.0.2. Propositions 3.2.1, 3.2.2 and 3.2.4 show that the map taking a triangle to its affine equivalence class is a well-defined, bijective map from $\mathcal{S}_{w_1,w_2}$ to $\mathcal{T}_{w_1,w_2}$. In Section 3.3 we discuss corollaries of the main theorem. We classify lattice triangles by their affine automorphism group, examine the generating functions of sequences arising in the classification and identify some related sequences which appear in the On-Line Encyclopedia of Integer Sequences [OEI23]. Finally, in Section 3.4 we discuss the Ehrhart theory of lattice triangles. We do so by studying the plot of the number of boundary and interior points of lattice triangles. We provide an explanation for the periodic patterns which appear in strips of points in this plot and, by colouring the plot with the first and second widths of triangles realising each point, we provide an intuitive description of the triangles appearing in these strips.

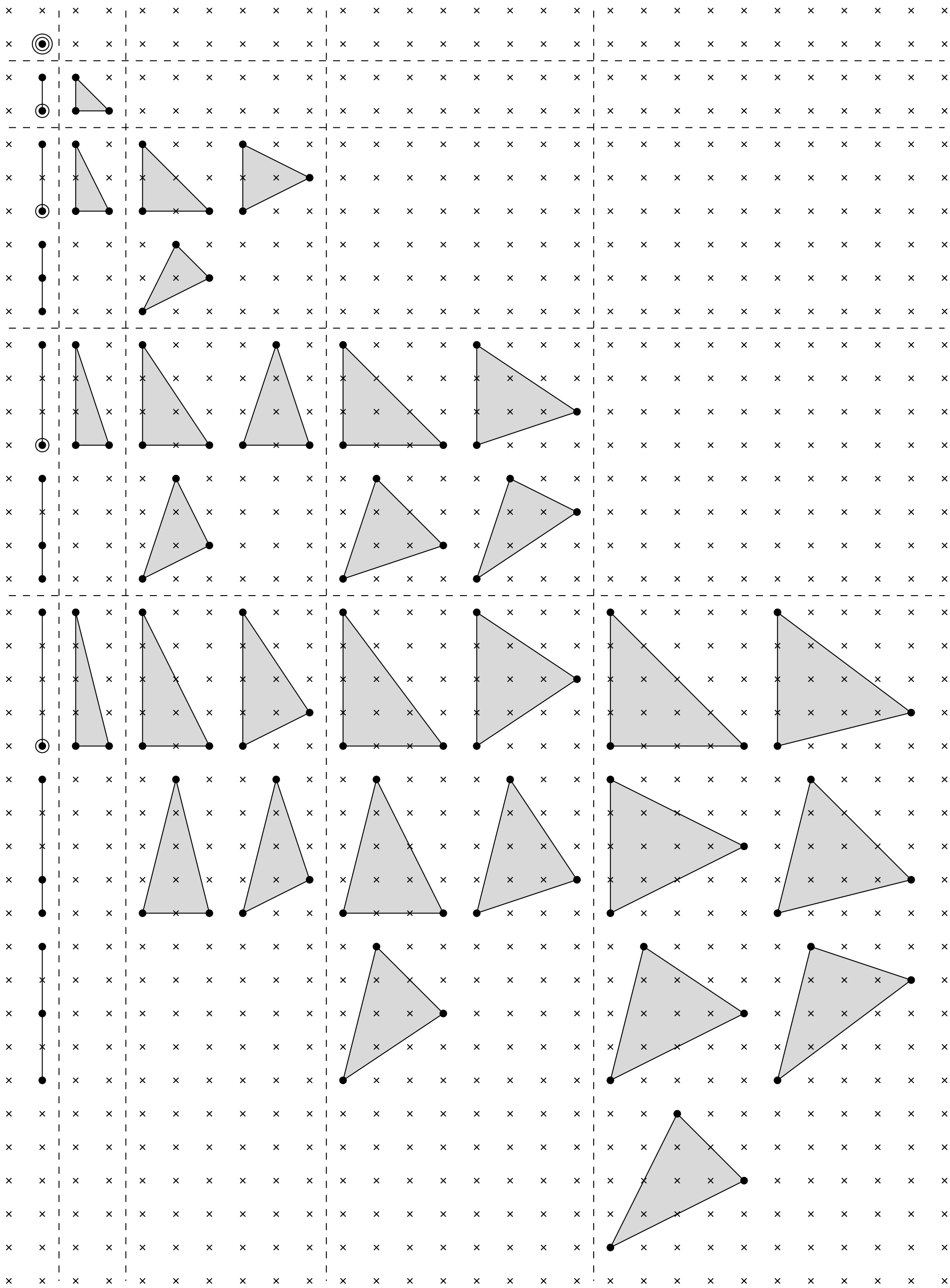


Figure 3.2: *The triangles $T \in \mathcal{S}_{w_1,w_2}$ where $0 \leq w_1 \leq w_2 \leq 4$. Columns (resp. rows) contain triangles with first (resp. second) width 0 to 4 from left to right (resp. top to bottom). When a width 0, degenerate triangle (see Section 3.3) has multiple identical vertices these are denoted by concentric circles.*

3.1 Width and Rectangles

Let $N \cong \mathbb{Z}^d$ be a lattice, $N^* := \mathrm{Hom}(N, \mathbb{Z}) \cong \mathbb{Z}^d$ its dual lattice and $N_{\mathbb{R}} := \mathbb{R} \otimes_{\mathbb{Z}} N \cong \mathbb{R}^d$ the real vector space containing N. For two tuples of integers $w = (w_1, \ldots, w_d)$ and $w' = (w'_1, \ldots, w'_d)$ we say that $w <_{lex} w'$ when there is some $1 \leq i \leq d$ such that $w_i < w'_i$ and $w_j = w'_j$ for all $j < i$. This defines the *lexicographic order* on $\mathbb{Z}^d$.

Definition 3.1.1. Let P be a lattice polytope and $u \in N^*$ a dual vector. We define the *width of P with respect to u* to be

$$\mathrm{width}_u(P) := \max_{x \in P}\{u \cdot x\} - \min_{x \in P}\{u \cdot x\}.$$

Since the widths are non-negative it is possible to define

$$\mathrm{mwidth}(P) := \min_{u_1, \ldots, u_d \in N^*}(\mathrm{width}_{u_1}(P), \ldots, \mathrm{width}_{u_d}(P))$$

where the minimum is taken with respect to lexicographic order and $u_1, \ldots, u_d$ are required to be linearly independent. We call this tuple the *multi-width* of P and call $\mathrm{width}_{u_i}(P)$ the *i-th width* of P written $\mathrm{width}^i(P)$.

We now define a polytope $\mathcal{W}_P := (P + (-P))^*$ which is the dual of the Minkowski sum of P and $-P$. Notice that $\mathcal{W}_P$ is a rational polytope but need not be a lattice polytope. This polytope encodes the widths of P in the following sense.

Lemma 3.1.2. *For $u \in N^*$, $\mathrm{width}_u(P) \leq w$ if and only if $u \in w\mathcal{W}_P$.*

Proof. By definition, $\mathcal{W}_P$ is the set of rational points u such that $u \cdot x \leq 1$ for all $x \in P + (-P)$. For a fixed lattice point u, $\mathrm{width}_u(P) \leq w$ if and only if $u \cdot (x_1 - x_2) \leq w$ for all pairs of points x_1 and x_2 in P. This is equivalent to saying that $\frac{1}{w}u \cdot x \leq 1$ for all points x of $P + (-P)$, or in other words $u \in w\mathcal{W}_P$. $\square$

Note that this is equivalent to saying that the i-th width of P is the i-th successive minimum of $\mathcal{W}_P$. For a definition of the successive minima of a polytope see [KL88, p. 581]. We use Lemma 3.1.2 to prove the following result.

Proposition 3.1.3. *Let $d \geq 2$ and $P \subset N_{\mathbb{R}}$ be a lattice polytope. If P has widths w_1 and w_2 with respect to two linearly independent, primitive dual vectors, then P is equivalent to a subset of $[0, w_1] \times [0, w_2] \times \mathbb{R}^{d-2}$.*

Proof. Relabelling if necessary we may assume $w_1 \leq w_2$. Pick linearly independent, primitive dual vectors u_1 and u_2 which realise the stated widths of P. By Lemma 3.1.2 we know that $u_1, u_2 \in w_2\mathcal{W}_P$. As real vectors, u_1 and u_2 generate a two-dimensional vector space containing a sublattice of N^*. The triangle $\mathrm{conv}(0, u_1, u_2) \subseteq w_2\mathcal{W}_P$ contains a lattice point u_2' such that $\{u_1, u_2'\}$ is a basis for this sublattice. Since $u_2' \in w_2\mathcal{W}_P$ we know that $\mathrm{width}_{u_2'}(P) \leq w_2$. After a change of basis, we may assume that u_1 and u_2' are the first two standard basis vectors. This change of basis and a translation are sufficient to map P to a subset of $[0, w_1] \times [0, w_2] \times \mathbb{R}^{d-2}$. □

Notice that the fact that the triangle $\mathrm{conv}(0, u_1, u_2)$ contains a basis of the rank 2 sublattice is an artefact of dimension 2. It depends on the fact that empty lattice triangles (i.e. those whose only lattice points are their vertices) are all affine equivalent. In dimensions 3 and higher we may no longer assume that all empty lattice simplices are equivalent.

We utilise this special case in dimension 2 to reformulate the classification. Given a lattice triangle T, there is a natural lattice rectangle with dimensions $\mathrm{width}_{(1,0)}(T) \times \mathrm{width}_{(0,1)}(T)$ which T is a subset of. Another triangle equivalent to T may have a different associated rectangle so it is interesting to consider what the "smallest" rectangle containing T is up to equivalence. One can make rigorous the idea of "smallest" using lexicographic order on rectangles or by requiring a triangle to be equivalent to no subset of a sub-rectangle. This is formalised as follows.

Definition 3.1.4. Let $\mathcal{T}^{lex}_{w_1,w_2}$ be the set of lattice triangles T, up to affine equivalence, which are a subset of $[0, w_1] \times [0, w_2]$ such that for all (w'_1, w'_2) lexicographically less than (w_1, w_2), T is not equivalent to a subset of the rectangle $[0, w'_1] \times [0, w'_2]$.

Let $\mathcal{T}^{sub}_{w_1,w_2}$ be the set of lattice triangles T, up to affine equivalence, which are a subset of $[0, w_1] \times [0, w_2]$ such that for all lattice points (w'_1, w'_2) in the rectangle $[0, w_1] \times [0, w_2]$ if T is equivalent to a subset of $[0, w'_1] \times [0, w'_2]$ then $(w'_1, w'_2) = (w_1, w_2)$.

For a lattice triangle T there is a unique pair of integers (w_1, w_2) such that the equivalence class of T is in $\mathcal{T}^{lex}_{w_1,w_2}$. However, while there is at least one pair (w_1, w_2) such that the equivalence class of T is in $\mathcal{T}^{sub}_{w_1,w_2}$, it is not immediate that there is only one such pair. In fact, if we allow $w_2 < w_1$ there is more than one for infinitely many triangles. The following proves that if we require $w_1 \leq w_2$ this pair is unique and that both of these sets are equal to $\mathcal{T}_{w_1,w_2}$.

Proposition 3.1.5. *When $w_1 \leq w_2$, the sets $\mathcal{T}_{w_1,w_2}$, $\mathcal{T}^{lex}_{w_1,w_2}$ and $\mathcal{T}^{sub}_{w_1,w_2}$ are equal.*

Proof. If the equivalence class of T is in $\mathcal{T}_{w_1,w_2}$ then there are linearly independent, primitive directions with respect to which it has widths w_1 and w_2. Therefore, by Proposition 3.1.3 T is equivalent to a subset of $[0, w_1] \times [0, w_2]$. If T were equivalent to a subset of a $w'_1 \times w'_2$ rectangle where $(w'_1, w'_2) <_{lex} (w_1, w_2)$ this would contradict the widths of T so $\mathcal{T}_{w_1,w_2} \subseteq \mathcal{T}^{lex}_{w_1,w_2}$. If T were equivalent to a subset of some $w'_1 \times w'_2$ rectangle where $(w'_1, w'_2) \in [0, w_1] \times [0, w_2] \cap \mathbb{Z}^2$ then its widths would force $(w'_1, w'_2) = (w_1, w_2)$ so $\mathcal{T}_{w_1,w_2} \subseteq \mathcal{T}^{sub}_{w_1,w_2}$.

If the equivalence class of T is in $\mathcal{T}^{lex}_{w_1,w_2}$ then its widths are lexicographically at most w_1 and w_2. If we had $(\mathrm{width}^1(T), \mathrm{width}^2(T)) <_{lex} (w_1, w_2)$ then by Proposition 3.1.3 T would be equivalent to a subset of $[0, \mathrm{width}^1(T)] \times [0, \mathrm{width}^2(T)]$ contradicting the definition of $\mathcal{T}^{lex}_{w_1,w_2}$. Therefore, T has widths w_1 and w_2 and $\mathcal{T}_{w_1,w_2} = \mathcal{T}^{lex}_{w_1,w_2}$.

If the equivalence class of T is in $\mathcal{T}^{sub}_{w_1,w_2}$ then its widths are lexicographically at most w_1 and w_2. We may assume that $T \subseteq [0, w_1] \times [0, w_2]$. If it had width w_2' in some direction linearly independent to $(1, 0)$ and $w_2' < w_2$ then by Proposition 3.1.3, T is equivalent to a subset of $[0, w_1] \times [0, w_2']$ which contradicts the definition of $\mathcal{T}^{sub}_{w_1,w_2}$. If it had width $w_1' < w_1$ with respect to $(1, 0)$ then, possibly after a translation, it would be a subset of $[0, w_1-1]\times[0, w_2]$ which also contradicts the definition of $\mathcal{T}^{sub}_{w_1,w_2}$. Therefore, T has widths w_1 and w_2 and $\mathcal{T}_{w_1,w_2} = \mathcal{T}^{sub}_{w_1,w_2}$. □

From now on we will freely use any of these definitions to describe $\mathcal{T}_{w_1,w_2}$.

3.2 Proof of Theorem 3.0.2

The following results form the proof of Theorem 3.0.2 by showing that the map taking a triangle to its affine equivalence class is a well-defined bijection from $\mathcal{S}_{w_1,w_2}$ to $\mathcal{T}_{w_1,w_2}$. First we prove surjectivity.

Proposition 3.2.1. *Let T be a lattice triangle with first and second width w_1 and w_2 respectively. Then there exists a triangle $T' \in \mathcal{S}_{w_1,w_2}$ which is affine equivalent to T.*

Proof. By Proposition 3.1.5 we may assume that T is a subset of $[0, w_1]\times[0, w_2]$. We may further assume that there is a vertex of T contained in each edge of the rectangle otherwise it would be a subset of a smaller rectangle. Consider the three-point set we obtain by projecting the vertices of T onto the first coordinate. This set is $\{0, x_2, w_1\}$ for some integer $x_2 \in [0, w_1]$. By a reflection in the line $x = w_1/2$ we may assume that $x_2 \leq w_1/2$.

Now, for some integers $y_0, y_1, y_2 \in [0, w_2]$

$$T = \mathrm{conv}((0, y_0), (w_1, y_1), (x_2, y_2)).$$

Since T has vertices in each edge of the rectangle one of these y-coordinates must be 0 and one must be w_2. Suppose, towards a contradiction, that $0 <$

$y_2 < w_2$. Then, possibly after a reflection in the line $y = w_2/2$, we can assume that $y_0 = 0$ and $y_1 = w_2$ so the affine map $(x, y) \mapsto (x, y - x)$ takes T to

$$\operatorname{conv}((0,0),(w_1,w_2-w_1),(x_2,y_2-x_2)).$$

However, this is a subset of a smaller rectangle which contradicts the widths of T, so we may assume that $y_2 = 0$ or w_2. By a reflection in the line $y = w_2/2$ we assume that $y_2 = w_2$.

Now, for integers $y_0, y_1 \in [0, w_2]$, one of which is zero, we have

$$T = \operatorname{conv}((0,y_0),(w_1,y_1),(x_2,w_2))$$

and we need to consider three different cases corresponding to the three different types of triangle in $\mathcal{S}_{w_1,w_2}$. We will prove each of the following facts:

(1) If $x_2 = 0$ then we may assume T is of type (A),

(2) If $x_2 > 0$ and $y_0 = 0$ then we may assume T is of type (B)

(3) If $x_2 > 0$ and $y_0 > 0$ then we may assume T is of type (C).

(1) Suppose $x_2 = 0$ and consider the image of the vertices of T under $(1, 1)$:

$$\{y_0, w_1 + y_1, w_2\}.$$

By the widths of T, this must not be a subset of $[1, w_2]$. We know $y_1 = 0$ or $y_0 = 0$. If $y_1 = 0$ then $w_1 + y_1$ and w_2 are both contained in $[1, w_2]$ so $y_0 = 0$ too. Therefore, we always have $y_0 = 0$. By a shear about the y-axis we may assume that $y_1 < w_1$. Define y_1' to be $(w_2 - y_1 \mod w_1)$. Notice that, since $0 \le y_1 < w_1$, we have $y_1 = (w_2 - y_1' \mod w_1)$. If $y_1 \le y_1'$ then T is of type (A). Otherwise, let k be the integer such that $y_1' = w_2 - y_1 + kw_1$ then the map $(x, y) \mapsto (x, w_2 - y + kx)$ takes T to the following triangle of the form (A):

$$T' = \operatorname{conv}((0,0),(w_1,y_1'),(0,w_2)).$$

(2) If $x_2 > 0$ and $y_0 = 0$ then consider the image of T under (-1,1):

$$\{0, y_1 - w_1, w_2 - x_2\}.$$

Due to the second width of T this must not be a subset of a line segment of length less than w_2 which means that $(w_2 - x_2) - (y_1 - w_1)$ must be at least w_2. We can rearrange this to show that $0 \le y_1 \le w_1 - x_2$. If additionally $w_1 = w_2$ then we have the following two triangles which are equivalent, under the map exchanging x- and y-coordinates.

$$\operatorname{conv}((0,0),(w_1,y_1),(x_2,w_1)), \quad \operatorname{conv}((0,0),(w_1,x_2),(y_1,w_1))$$

We choose the triangle with the smaller x-coordinates and so may assume that $y_1 \ge x_2$.

(3) If $x_2 > 0$ and $y_0 > 0$ then we must have $y_1 = 0$. If $x_2 = w_1/2$ then a reflection takes us to the previous case so we may assume $x_2 < w_1/2$. Consider the image of T under $(1,1)$:

$$\{y_0, w_1, w_2 + x_2\}.$$

We know that y_0 and w_1 are both less than $w_2 + x_2$ so to prevent this fitting in a line segment of length less than w_2 we must have either y_0 or w_1 must be less than or equal to x_2. It is fixed that $x_2 < w_1$ so we must have $y_0 \le x_2$ If $y_0 = x_2$ then, under the map $(x, y) \mapsto (x, y - x_2 + x)$, T is equivalent to $\operatorname{conv}((0,0),(w_1,w_1 - x_2),(x_2,w_2))$ which is included in case (2) so we may assume that $y_0 < x_2$. A consequence of this is that $x_2 > 1$. If $w_1 = w_2$ then the map $(x, y) \mapsto (y, w_1 - x)$ takes T to $\operatorname{conv}((0,0),(y_0,w_1),(w_1,w_1 - x_2))$. From our previous bounds we have $y_0 < x_2 < w_1/2$ so this image of T is of the form addressed in case (2). Therefore, we may assume case (3) only occurs when $w_1 < w_2$. This shows that T is equivalent to a triangle in $\mathcal{S}_{w_1,w_2}$. $\square$

We now show that the widths of triangles in $\mathcal{S}_{w_1,w_2}$ are w_1 and w_2 which we will use to show that the map taking a triangle to its equivalence class is a map from $\mathcal{S}_{w_1,w_2}$ to $\mathcal{T}_{w_1,w_2}$.

Proposition 3.2.2. *For $T \in \mathcal{S}_{w_1,w_2}$, the first and second widths of T are w_1 and w_2 respectively.*

Proof. The triangles in $\mathcal{S}_{w_1,w_2}$ fall into one of the following types regardless of whether $w_1 = w_2$ or $w_1 < w_2$. Therefore, it suffices to prove the result for each of the following triangles for all positive integers $w_1 \leq w_2$.

(A) $T = \operatorname{conv}((0,0),(w_1,y_1),(0,w_2))$ where $0 \leq y_1 \leq (w_2 - y_1 \mod w_1)$

(B) $T = \operatorname{conv}((0,0),(w_1,y_1),(x_2,w_2))$ where $0 < x_2 \leq \frac{w_1}{2}$ and $0 \leq y_1 \leq w_1 - x_2$

(C) $T = \operatorname{conv}((0,y_0),(w_1,0),(x_2,w_2))$ where $1 < x_2 < \frac{w_1}{2}$, $0 < y_0 < x_2$ and $w_1 < w_2$.

These triangles all have widths w_1 and w_2 with respect to $(1,0)$ and $(0,1)$ respectively so it suffices to show that they have width at least w_2 in all directions linearly independent to $(1,0)$. That is, with respect to all dual lattice vectors $u = (u_x, u_y)$ where u_y is non-zero. It suffices to consider when u_y is positive. We do this in each of the three cases.

(A) This case it is immediate since the width of the triangles with respect to u is at least $|u \cdot (0,w_2) - u \cdot (0,0)| = u_y w_2$ which is at least w_2.

(B) The image of the vertices of this case under u is

$$\{0, w_1u_x + y_1u_y, x_2u_x + w_2u_y\}.$$

Suppose for contradiction that this is a subset of a line segment with length less than w_2. Then we have $x_2u_x + w_2u_y < w_2$ so u_x is negative. We also need $x_2u_x + w_2u_y - w_1u_x - y_1u_y < w_2$ and so $\frac{w_2(u_y-1)-y_1u_y}{w_1-x_2} < u_x$. However, we know that $w_2 > w_1 - x_2$ and $y_1 \leq w_1 - x_2$ so this shows that u_x is greater than -1 which is a contradiction.

(C) The image of the vertices of this case under u is

$$\{y_0u_y, w_1u_x, x_2u_x + w_2u_y\}.$$

Suppose for contradiction that this is a subset of a line segment with length less than w_2. Then we have $x_2u_x + w_2u_y - w_1u_x < w_2$ so u_x is positive. We also need $x_2u_x + w_2u_y - y_0u_y < w_2$ and so $u_x < \frac{y_0u_y - w_2(u_y-1)}{x_2}$. However, we know that $w_2 > 2x_2$ and $y_0 < x_2$ so $u_x < 2 - u_y \leq 1$ which is a contradiction. $\square$

The next lemma will be used to prove distinctness of the triangles in $\mathcal{S}_{w_1,w_2}$.

Lemma 3.2.3. *Let* $T = \operatorname{conv}((0, y_0), (w_1, y_1), (x_2, w_2))$ *be a triangle in* $\mathcal{S}_{w_1,w_2}$. *If* u *is a dual vector such that the image of the vertices of* T *under* u *is equivalent to* $\{0, x_2', w_1\}$ *for some integer* $x_2' \in [0, w_1]$, *then* $x_2' \geq x_2$. *That is,* x_2 *is minimal. Moreover,* u *can only be one of the vectors* $(1, 0)$, $(0, 1)$ *and* $(-1, 1)$.

Proof. When $x_2 = 0$ this is immediate. When $w_1 < w_2$ it follows from the facts that $\operatorname{width}_u(T) = w_1$ for a unique (up to sign) choice of u and that $x_2 \leq w_1/2$. Therefore, we need only prove this for triangles of the form (B) when $w_1 = w_2$. That is, $T = \operatorname{conv}((0, 0), (w_1, y_1), (x_2, w_1))$ with $0 < x_2 \leq w_1/2$ and $x_2 \leq y_1 \leq w_1 - x_2$.

Let $u = (u_x, u_y)$ be a dual vector linearly independent to $(1, 0)$, we may assume that $u_y \geq 1$. The image of the vertices of T under u is

$$\{0, u_xw_1 + u_yy_1, u_xx_2 + u_yw_1\}.$$

Pick u such that this is equivalent to $\{0, x_2', w_1\}$ for some $x_2' \in [0, w_1]$. This places strong restrictions on u. The difference between each pair of elements in this set must be at most w_1. In particular $u_xx_2 + u_yw_1 \leq w_1$ and $u_xw_1 + u_yy_1 \geq -w_1$. These can be rearranged into

$$-(u_yy_1 + w_1)/w_1 \leq u_x \leq w_1(1 - u_y)/x_2$$

By the conditions on T we know that $w_1/x_2 \geq 2$ and $y_1/w_1 < 1$ so we can further show that

$$-1 - u_y < u_x \leq 2(1 - u_y). \tag{3.1}$$

In particular $u_y < 3$. Substituting $u_y = 1$ and $u_y = 2$ into (3.1) and considering the possible integers u_x in each case shows that u is equal to $(0, 1)$, $(-1, 1)$ or $(-2, 2)$. By definition of u the width of T with respect to u is w_1, so if $u = (-2, 2)$ then the width of T with respect to $(-1, 1)$ is $\frac{w_1}{2}$ which contradicts the widths of T. For this reason we can discard the case $u = (-2, 2)$.

The images of the vertices of T under $(0, 1)$ and $(-1, 1)$ are

$$\{0, y_1, w_1\}, \quad \text{and} \quad \{0, y_1 - w_1, w_1 - x_2\}$$

respectively. The properties of the coordinates of T allow us to order the elements of each of these sets: $0 < y_1 < w_1$ and $y_1 - w_1 < 0 < w_1 - x_2$. This allows us to identify which point in each set is sent to x_2' under the equivalence with $\{0, x_2', w_1\}$. We see that $x_2' = y_1$, $w_1 - y_1$ or $w_1 - x_2$. For any of these $x_2' \geq x_2$ as desired. □

Now we can prove affine distinctness of the triangles in $\mathcal{S}_{w_1,w_2}$.

Proposition 3.2.4. *The triangles in $\mathcal{S}_{w_1,w_2}$ are all distinct under affine maps.*

Proof. Let T and T' be equivalent triangles in $\mathcal{S}_{w_1,w_2}$. Let the variables associated to T and T' be denoted by y_0, y_1, x_2 and y_0', y_1', x_2' respectively. By Lemma 3.2.3 $x_2 = x_2'$. Therefore, both are of the form (A) or neither are. We will show in each of the following four facts.

1. If T and T' are both of type (A) then $T = T'$
2. If T and T' are both of type (B) then $T = T'$
3. If T and T' are both of type (C) then $T = T'$

4. If T is of type (B) and T' is of type (C) then we have a contradiction.

(1) Either $y_1 = y_1' = 0$ and $w_1 = w_2$ or they each have a unique edge of lattice length w_2. Therefore, either $T = T'$ or the affine map taking T to T' preserves the line segment from $(0,0)$ to $(0,w_2)$. The reduces us to maps of the form

$$(x, y) \mapsto (x, y + kx) \quad \text{and} \quad (x, y) \mapsto (x, w_2 - y + kx).$$

for integers k. Since $0 \leq y_1, y_1' < w_1$ if a map of the first from takes T to T' then $y_1 = y_1'$ and $T = T'$. If a map of the second form takes T to T' then $y_1 \leq (w_2 - y_1 \mod w_1) = y_1'$ and symmetrically $y_1' \leq y_1$ so again $T = T'$.

(2) The normalised volumes of T and T' are $w_1w_2 - y_1x_2$ and $w_1w_2 - y_1'x_2$ which must be equal so $y_1 = y_1'$ and $T = T'$.

(3) The normalised volumes of T and T' are $w_1w_2 - w_1y_0 + y_0x_2$ and $w_1w_2 - w_1y_0' + y_0'x_2$ which must be equal so $y_0 = y_0'$ and $T = T'$.

(4) Since T' is of type (C) we know that $1 < x_2 < w_1/2$ and $w_1 < w_2$. By $w_1 < w_2$ we see that the dual vector u such that $\mathrm{width}_u(T') = w_1$ is unique. We can use it to distinguish the three vertices of T and T' by their images under u. In this way we see there is only one order in which to map the vertices of T to the vertices of T' that is $(0,0)$, (w_1, y_1) and (x_2, w_2) map to $(0, y_0')$, $(w_1, 0)$ and (x_2, w_2) respectively. Thus the following matrix must be unimodular

$$\begin{pmatrix} w_1 & x_2 \\ -y_0' & w_2 - y_0' \end{pmatrix} \begin{pmatrix} w_1 & x_2 \\ y_1 & w_2 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ \frac{y_1y_0' - y_0'w_2 - y_1w_2}{w_1w_2 - x_2y_1} & 1 \end{pmatrix}.$$

Since $w_2 > y_1$ we must have $y_1y_0' - y_0'w_2 - y_1w_2 < 0$ so for the matrix to have integral entries we must have $y_1w_2 + y_0'w_2 - y_1y_0' \geq w_1w_2 - x_2y_1$. Since $y_1 \leq w_1 - x_2$ the right hand side is at least $w_2(x_2 + y_1) - x_2y_1$. Cancelling terms and dividing by $w_2 - y_1$ gives $y_0' \geq x_2$ which is the desired contradiction. □

We bring together the results of this section to prove the main theorem.

Proof of Theorem 3.0.2. Proposition 3.2.2 shows that the map taking a triangle to its affine equivalence class is a map from $\mathcal{S}_{w_1,w_2}$ to $\mathcal{T}_{w_1,w_2}$. By Propositions 3.2.1 and 3.2.4 this map is bijective.

To calculate the cardinality of $\mathcal{T}_{w_1,w_2}$ we need only count the triangles in $\mathcal{S}_{w_1,w_2}$. For triangles of type (A) we need to compute the number of integers $y_1 \in [0, w_1)$ such that $y_1 \leq (w_2 - y_1 \mod w_1)$. Pick $q, r \in \mathbb{Z}$ with $0 \leq r < w_1$ such that $w_2 = qw_1 + r$. Then $y_1 \leq (w_2 - y_1 \mod w_1)$ if and only if $0 \leq y_1 \leq \frac{r}{2}$ or $r < y_1 \leq \frac{r+w_1}{2}$ which can be seen by considering the plot of (y_1, y_1) and $(y_1, w_2 - y_1 \mod w_1)$ for $y_1 \in [0, w_1)$. For any integer a the number of points in $[0, \frac{a}{2}] \cap \mathbb{Z}$ is $\lceil \frac{a+1}{2} \rceil$ so the number of y_1 satisfying one of the above is

$$\left\lceil \frac{r+1}{2} \right\rceil + \left\lceil \frac{w_1 - r + 1}{2} \right\rceil - 1.$$

If we substitute in $r = w_2 - qw_1$ and consider cases for w_1, w_2 and q odd and even it can be shown that this is $\lceil \frac{w_1}{2} \rceil$ when w_2 is odd and $\lceil \frac{w_1+1}{2} \rceil$ when w_2 is even.

For triangles of type (B) and (C) we need only calculate

$$\sum_{x_2=1}^{\lfloor w_1/2 \rfloor} (w_1 - x_2 + 1) + \sum_{x_2=2}^{\lfloor (w_1-1)/2 \rfloor} (x_2 - 1) = \begin{cases} \frac{w_1^2}{2} - \frac{w_1}{2} + 1 & \text{if } w_1 \text{ even} \\ \frac{w_1^2}{2} - \frac{w_1}{2} & \text{if } w_1 \text{ odd} \end{cases}$$

when $w_1 < w_2$ and

$$\sum_{x_2=1}^{\lfloor w_1/2 \rfloor} (w_1 - 2x_2 + 1) = \begin{cases} \frac{w_1^2}{4} & \text{if } w_1 \text{ even} \\ \frac{w_1^2}{4} - \frac{1}{4} & \text{if } w_1 \text{ odd} \end{cases}$$

when $w_1 = w_2$. Combining these sums and separating odd and even cases gives the result. □

3.3 Corollaries

A consequence of Theorem 3.0.2 is that we have defined a normal form for lattice triangles from which their first two widths can be read. This normal form is compatible with scaling in the sense that for a positive integer λ and a triangle $T \in \mathcal{S}_{w_1,w_2}$ we have $\lambda T \in \mathcal{S}_{\lambda w_1,\lambda w_2}$. We can also read the affine automorphism group of a triangle from this normal form as shown in the following corollary. Let S_3 denote the group of permutations of $\{0,1,2\}$, written in cycle notation, and $\mathrm{Aut}(T)$ denote the group of affine maps which map T to itself.

Corollary 3.3.1. *Let $T \in \mathcal{S}_{w_1,w_2}$ then*

- $\mathrm{Aut}(T) \cong S_3$ *if and only if one of the following hold*
  - $w_1 = w_2$ *and* $T = ((0,0),(w_1,0),(0,w_1))$ *or*
  - $w_1 = w_2$ *and* $T = ((0,0),(w_1,\frac{w_1}{2}),(\frac{w_1}{2},w_1))$
- $\mathrm{Aut}(T) \cong \langle(012)\rangle$ *if and only if the following holds*
  - $w_1 = w_2$ *and* $T = \mathrm{conv}((0,0),(w_1,y_1),(w_1-y_1,w_1))$ *such that* $y_1 \neq \frac{w_1}{2}$
- $\mathrm{Aut}(T) \cong \langle(01)\rangle$ *if and only if one of the following hold*
  - $T = \mathrm{conv}((0,0),(w_1,y_1),(0,w_2))$ *such that* $y_1 \equiv (w_2 - y_1 \mod w_1)$ *and either* $y_1 > 0$ *or* $w_1 < w_2$,
  - $T = \mathrm{conv}((0,0),(w_1,0),(\frac{w_1}{2},w_2))$
  - $T = \mathrm{conv}((0,0),(w_1,\frac{w_1}{2}),(\frac{w_1}{2},w_2))$ *and* $w_1 < w_2$,
  - $T = \mathrm{conv}((0,0),(w_1,y_1),(y_1,w_1))$, $w_1 = w_2$ *and* $2y_1 < w_1$
- $\mathrm{Aut}(T) \cong \{\iota\}$ *otherwise.*

Proof. We consider each of the three types of triangle in $\mathcal{S}_{w_1,w_2}$ and find conditions for them to have each automorphism group.

Triangles T of type (A) have automorphism group S_3 when $w_1 = w_2$ and $y_1 = 0$ since this is just a dilation of the standard simplex $\operatorname{conv}((0,0),(1,0),(0,1))$. Otherwise any automorphism of T must preserve the edge from $(0,0)$ to $(0,w_2)$ so we are reduced to maps of the form $(x,y) \mapsto (x, y+kx)$ and $(x,y) \mapsto (x, w_2 - y + kx)$ for integers k. The first of these maps can only map T to itself if $k = 0$ which is just the identity map. The second can only map T to itself if $y_1 = (w_2 - y_1 \mod w_1)$. So T has automorphism group isomorphic to $\langle(12)\rangle$ in this case and trivial automorphism group otherwise.

For a triangle T of type (B) let φ be an affine map taking T to itself. It is defined by multiplication by a unimodular matrix $U \in \mathrm{GL}_2(\mathbb{Z})$ followed by a translation $t \in \mathbb{Z}^2$. Let $v_0 = (0,0)$, $v_1 = (w_1, y_1)$ and $v_2 = (x_2, w_2)$ be the vertices of T then the set of numbers $\{(1,0)\cdot\varphi(v_i) : i = 0,1,2\}$ is equal to $\{0, x_2, w_1\}$. This means that the set of numbers $\{(1,0)\cdot Uv_i^T : i = 0,1,2\}$ is equivalent to $\{0, x_2, w_1\}$ and so T has width w_1 with respect to the vector $(1,0)U$. If $w_1 < w_2$ this forces the first row of U to be $(\pm1, 0)$. Otherwise, by Lemma 3.2.3 the first row of U must be $(\pm1,0)$, $(0,\pm1)$ or $(\pm1,\mp1)$. Therefore, for some integer k, U is one of the following

$$\begin{pmatrix} 1 & 0 \\ k & \pm1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ k & \pm1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ \pm1 & k \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ \pm1 & k \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ \pm1-k & k \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ \pm1-k & k \end{pmatrix}.$$

The image of T under each of these maps, in the same order, is

$$T_1 = \operatorname{conv}((0,0),(w_1, kw_1 \pm y_1),(x_2, kx_2 \pm w_2))$$

$$T_2 = \operatorname{conv}((0,0),(-w_1, kw_1 \pm y_1),(-x_2, kx_2 \pm w_2))$$

$$T_3 = \operatorname{conv}((0,0),(y_1, \pm w_1 + ky_1),(w_2, \pm x_2 + kw_2))$$

$$T_4 = \operatorname{conv}((0,0),(-y_1, \pm w_1 + ky_1),(-w_2, \pm x_2 + kw_2))$$

$$T_5 = \operatorname{conv}((0,0),(w_1 - y_1, (\pm1-k)w_1 + ky_1),(x_2 - w_2, (\pm1-k)x_2 + kw_2))$$

$$T_6 = \operatorname{conv}((0,0),(y_1 - w_1, (\pm1-k)w_1 + ky_1),(w_2 - x_2, (\pm1-k)x_2 + kw_2)).$$

These must be translations of T. In each case, sorting the vertices by the size of their x-coordinates indicates the permutation of the vertices of T they correspond to. This allows us to find the translation vector t. The sign of the permutation indicates weather the determinant of U is positive or negative. This is enough to explicitly compute k and obtain a collection of restrictions on the variables. In the following we execute this procedure in each case:

If T_1 is a translation of T it comes from the identity permutation, so $t = (0, 0)$ and the determinant of U is 1. By equating the vertices of $T_1 + t$ and T we can show that $k = 0$ so φ is the identity map.

If T_2 is a translation of T it comes from the permutation (01) of the vertices, so $t = (w_1, y_1)$ and the determinant of U is -1. By equating the vertices of $T_2 + t$ and T we can show that $2x_2 = w_1$ and $k = -2y_1/w_1$ which is only an integer if $y_1 = 0$ or $y_1 = w_1/2$. Then $k = 0$ or -1 each of which lead to valid automorphisms.

If T_3 is a translation of T it comes from the permutation (12) of the vertices, so $t = (0, 0)$ and the determinant of U is -1. By equating the vertices of T_3 and T we can show that $y_1 = x_2$, $w_1 = w_2$ and $k = 0$ which leads to a valid automorphism of T.

If T_4 is a translation of T it comes from the permutation (012) of the vertices, so $t = (w_1, y_1)$ and the determinant of U is 1. By equating the vertices of T_4+t and T we can show that $w_1 = w_2 = x_2 + y_1$ and $k = -1$ which leads to a valid automorphism of T.

If T_5 is a translation of T it comes from the permutation (02) of the vertices, so $t = (x_2, w_2)$ and the sign of U is -1. By equating the vertices of $T_5 + t$ and T we can show that $w_1 = w_2 = 2x_2 = 2y_1$ and $k = -1$ which gives a valid automorphism of T.

If T_6 is a translation of T it comes from the permutation (021) of the vertices, so $t = (x_2, w_2)$ and the determinant of U is 1. By equating the vertices of T_6+t and T we can show that $w_1 = w_2 = x_2 + y_1$ and $k = 0$ which gives a valid automorphism of T.

To describe the remaining triangles in the corollary with non-trivial automorphism group we combine the above types of triangle. Note that each triangle defines a permutation and conditions under which $T \in \mathcal{S}_{w_1,w_2}$ has this permutation in its automorphism group. If $y_1 = w_1/2$ and $w_1 = w_2$ then $\operatorname{Aut}(T_2)$ is S_3. Otherwise, T_2 satisfies none of the other conditions so $\operatorname{Aut}(T_2)$ is $\langle(01)\rangle$. If $y_1 = w_1/2$ then $\operatorname{Aut}(T_3)$ is S_3. Otherwise, T_3 satisfies none of the other conditions so $\operatorname{Aut}(T_3)$ is $\langle(12)\rangle$. If $y_1 = w_1/2$ then $\operatorname{Aut}(T_4)$ and $\operatorname{Aut}(T_6)$ are both S_3. Otherwise, $\operatorname{Aut}(T_4)$ and $\operatorname{Aut}(T_6)$ are both $\langle(012)\rangle$. Finally, T_5 satisfies the necessary conditions for all the other permutations so $\operatorname{Aut}(T_5)$ is S_3.

Triangles of type (C) all have trivial automorphism group. We show this using a very similar method to the type (B) triangles above. For a triangle T of type (C) let φ be an affine map taking $T-(0,y_0)$ to itself (where we subtract $(0,y_0)$ so we may assume T has a vertex at the origin). This is defined by a unimodular matrix U and translation vector t. Since $w_1 < w_2$ the first row of U must be $(\pm 1, 0)$ so, for some integer k, U is one of the following

$$\begin{pmatrix} 1 & 0 \\ k & \pm 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 \\ k & \pm 1 \end{pmatrix}.$$

The image of $T-(0,y_0)$ under each of these maps is

$$T_7 = \operatorname{conv}((0,0),(w_1, kw_1 \mp y_0),(x_2, kx_2 \pm (w_2 - y_0))) \text{ or}$$

$$T_8 = \operatorname{conv}((0,0),(-w_1, kw_1 \mp y_0),(-x_2, kx_2 \pm (w_2 - y_0))).$$

If T_7 is a translation of $T-(0,y_0)$ it comes from the identity permutation so $t = (0,0)$ and the determinant of U is 1. By equating the vertices of $T_7 + t$ and $T-(0,y_0)$ we see that $k = 0$ so this is the identity map.

If T_8 is a translation of T it comes from the permutation (01) so $t = (w_1, -y_0)$ and the determinant of U is -1. By equating the x-coordinates of $T_8 + t$ and $T-(0,y_0)$ we see that $w_1 - x_2 = x_2$ so $x_2 = w_1/2$ which contradicts the definition of type (C) triangles. $\square$

So far we have restricted ourselves to non-degenerate triangles, that is triangles with non-zero volume. However, we can extend our definitions and proofs to multi-sets containing three collinear points and call these points the vertices of a degenerate triangle. Let the widths of such a set be the widths of its convex hull. It has first and second widths 0 and l where l is its lattice length. In this sense we can see that there are $\lceil (w_2+1)/2 \rceil$ triangles with first width 0 and second width w_2 up to affine equivalence for all $w_2 \geq 0$. They can be assumed to have vertices $(0,0)$, $(0,y_1)$ and $(0,w_2)$ for integers $y_1 \in [0, w_2/2]$.

Corollary 3.3.2. *The number of (degenerate and non-degenerate) triangles, up to affine equivalence, which are a subset of* $[0,n]^2$ *is equal to* $|nQ \cap \mathbb{Z}^4|$ *where* Q *is the four-dimensional rational simplex*

$$Q := \operatorname{conv}\left(\left(\frac{1}{2},0,0,0\right),\left(0,\frac{1}{2},0,0\right),\left(0,0,\frac{1}{2},0\right),(0,0,0,1),(-1,-1,-1,-1)\right).$$

Furthermore, the generating function of this sequence is the Hilbert series of a degree 8 hypersurface in $\mathbb{P}(1,1,1,2,2,2)$.

Proof. First notice that we can rewrite the cardinality of $|\mathcal{T}_{w_1,w_2}|$ described in Theorem 3.0.2 using the terms $(-1)^w_1$ and $(-1)^w_2$ instead of cases as follows

$$|\mathcal{T}_{w_1,w_2}| = \begin{cases} \frac{w_1^2}{4} + \frac{w_1}{2} + \frac{5}{8} + \frac{3}{8}(-1)^{w_1} & \text{when } w_1 = w_2 \\ \frac{w_1^2}{2} + 1 + \frac{1}{2}(-1)^{w_1} + \frac{1}{4}(-1)^{w_2} + \frac{1}{4}(-1)^{w_1+w_2} & \text{when } w_1 < w_2. \end{cases}$$

Thus, for a fixed w_1, the generating function $\sum_{w_2=w_1}^{\infty} s^{w_2}|\mathcal{T}_{w_1,w_2}|$ is

$$\frac{w_1^2}{4} + \frac{w_1}{2} + \frac{5}{8} + \frac{3}{8}(-1)^{w_1} + \sum_{w_2=w_1+1}^{\infty} s^{w_2}\left(\frac{w_1^2}{2} + 1 + \frac{1}{2}(-1)^{w_1} + \frac{1}{4}(-1)^{w_2} + \frac{1}{4}(-1)^{w_1+w_2}\right).$$

This sum can be simplified by applying the polylogarithm equations

$$\sum_{k=1}^{\infty} s^k = \frac{s}{1-s}, \quad \sum_{k=1}^{\infty} ks^k = \frac{s}{(1-s)^2}, \quad \sum_{k=1}^{\infty} k^2 s^k = \frac{s(1+s)}{(1-s)^3}$$

after which we obtain the following result (when $w_1 > 0$)

$$\sum_{w_2=w_1}^{\infty} s^{w_2}|\mathcal{T}_{w_1,w_2}| = s^{w_1}\left[\frac{w_1^2(1+s)}{4(1-s)} + \frac{w_1}{2} + \frac{5+6s+5s^2}{8(1-s^2)} + (-1)^{w_1}\frac{3+2s+3s^2}{8(1-s^2)}\right].$$

When $w_1 = 0$ we can rewrite the cardinality $|\mathcal{T}_{0,w_2}|$ as $\frac{w_2}{2} + \frac{3}{4} + \frac{1}{4}(-1)^{w_2}$ and following the same procedure as above we show that

$$\sum_{w_2=w_1}^{\infty} s^{w_2}|\mathcal{T}_{w_1,w_2}| = \frac{1}{(1-s^2)(1-s)}.$$

Consider the two generating functions we have found as polynomials in variables w_1 and $(-1)^{w_1}$. We can substitute these polynomials into the generating function $\sum_{w_1=0}^{\infty}\sum_{w_2=w_1}^{\infty} t^{w_1}s^{w_2}|\mathcal{T}_{w_1,w_2}|$ and use the polylogarithm equations to simplify and obtain the following expression for the generating function

$$\sum_{w_1=0}^{\infty}\sum_{w_2=w_1}^{\infty} t^{w_1}s^{w_2}|\mathcal{T}_{w_1,w_2}| = \frac{-s^7t^4 + s^6t^3 - s^5t^2 + s^4t^3 - s^3t + s^2t^2 - st + 1}{(1-s)^2(1+s)(1-st)^3(1+st)}. \tag{3.2}$$

Let T be a triangle with first and second width w_1 and w_2 respectively. To be equivalent to a subset of a square of side length w, T must have width at most w in two linearly independent directions. Therefore, the smallest square which T is equivalent to a subset of has side length w_2 and the number of triangles which are equivalent to a subset of $[0, w_2]^2$ and no smaller square is $\sum_{w_1=0}^{w_2} |\mathcal{T}_{w_1,w_2}|$. Reordering the sums, the generating function (3.2) is also equal to $\sum_{w_2=0}^{\infty}\sum_{w_1=0}^{w_2} t^{w_1}s^{w_2}|\mathcal{T}_{w_1,w_2}|$. Setting $t = 1$ in this function gives us the generating function for the sequence counting triangles which are a subset of $[0, w_2]^2$ and no smaller square. The result is

$$\frac{1-s^8}{(1-s^2)^3(1-s)^2}.$$

To compute instead the generating function of the number of triangles which are a subset of $[0, w_2]^2$ (and possibly smaller squares) we divide this by $(1-s)$.

The result is the Hilbert series of a degree 8 hypersurface in the weighted projective space $\mathbb{P}(1,1,1,2,2,2)$ and also the Ehrhart series of Q, that is

$$\mathrm{Ehr}_Q(t) \coloneqq \sum_{n=0}^{\infty} |nQ \cap \mathbb{Z}^4| s^n = \frac{1-s^8}{(1-s^2)^3(1-s)^3}.$$

The Ehrhart series of Q is found using computational algebra. By definition of the Ehrhart series this proves the result. These two descriptions of the generating function are related via mirror symmetry in the sense of [Prz13]. Let f be a Laurent polynomial which is a mirror partner for a degree 8 hypersurface in $\mathbb{P}(1,1,1,2,2,2)$, then Q is the dual of the Newton polytope of f. □

Finally, we note some sub-sequences which appear in the On-Line Encyclopedia of Integer Sequences (OEIS) [OEI23]. We refer to these sequences by their OEIS reference number. The sequence counting triangles which have an edge of lattice length n and are a subset of $[0,n]^2$ is A140144. The sequence counting these same triangles but excluding those with zero volume is A135276. Let $(a_n)_{n\geq 0}$ be the sequence counting triangles which are a subset of an $n \times n$ square and no smaller and which also have no edge of lattice length n. Then the sequence $(a_n - a_{n-1})_{n\geq 1}$ is the rounded up staircase diagonal on the natural numbers, shown in Figure 3.3, which is A080827. We distinguish between triangles with an edge of lattice length n and not since these are exactly the triangles of type (A) and those with the longest possible edge contained in a square. It is not obvious why any of these sequences should coincide.

16	23	31	40	50	61
11	17	24	32	41	51
7	12	18	25	33	42
4	8	13	19	26	34
2	5	9	14	20	27
1	3	6	10	15	21

Figure 3.3: *The rounded up staircase diagonal on the natural numbers.*

3.4 Ehrhart Theory of Lattice Triangles

In this section we discuss the pairs (b, i) where b and i are the number of boundary and interior points of a non-degenerate lattice triangle. For a polygon P, let $b(P)$ be the number of boundary lattice points and $i(P)$ the number of interior lattice points. The Ehrhart polynomial of a lattice polygon is

$$\operatorname{ehr}_P(n) = \left(i(P) + \frac{b(P)}{2} - 1\right) n^2 + \frac{b(P)}{2} n + 1.$$

Therefore, studying pairs $(b(P), i(P))$ is equivalent to studying the Ehrhart theory of lattice polygons. In [HNO18] it was proven that for integers $c \geq 1$ the cones

$$\sigma_c^\circ := \left\{ (b, i) \in \mathbb{R}^2_{\geq 0} : \frac{c-1}{2} b - (c-1) < i < \frac{c}{2} b - c(c+2) \right\}$$

contain no points $(b(T), i(T))$ where T is a lattice triangle. The boundaries of consecutive cones are parallel lines and describe strips of the plane where points $(b(T), i(T))$ can fall. Hofscheier–Nill–Öberg observed periodic patterns in the points in these strips which can be observed in Figure 3.4 and 3.5.

In Figure 3.4 we plot points $(b(T), i(T))$ and denote which are realised by a triangle with or without an edge of lattice length $\operatorname{width}^2(T)$. The periodic strips seem to be made up of triangles with a long edge, suggesting that the patterns observed relate somehow to a triangle having one relatively long edge. Additionally, Figure 3.5 shows the same plot only now it is coloured by the smallest and largest first and second width of lattice triangles realising each point. In all of these plots, clear patterns emerge in the strips away from the i-axis. When coloured by first width, each strip has its own consistent colour. When coloured by second width, each strip has an even gradient of colour. This suggests that the points in a given strip are realised mainly by triangles with fixed w_1 and evenly increasing w_2. We make all of this precise in the following result.

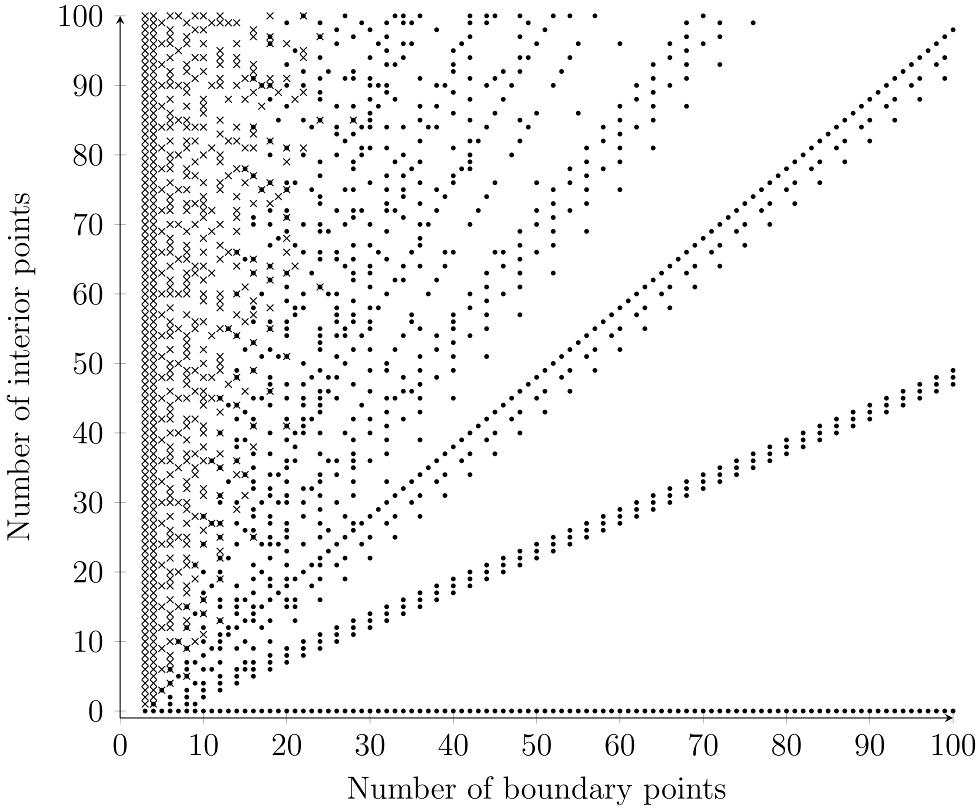


Figure 3.4: *Number of boundary and interior points of lattice triangles with multi-width (w_1, w_2) where $0 < w_1 \leq w_2 \leq 100$ and where the number of interior or boundary points does not exceed 100. Dots (resp. crosses) denote points which are realised by triangles with (resp. without) an edge of length w_2. Some points are realised by both.*

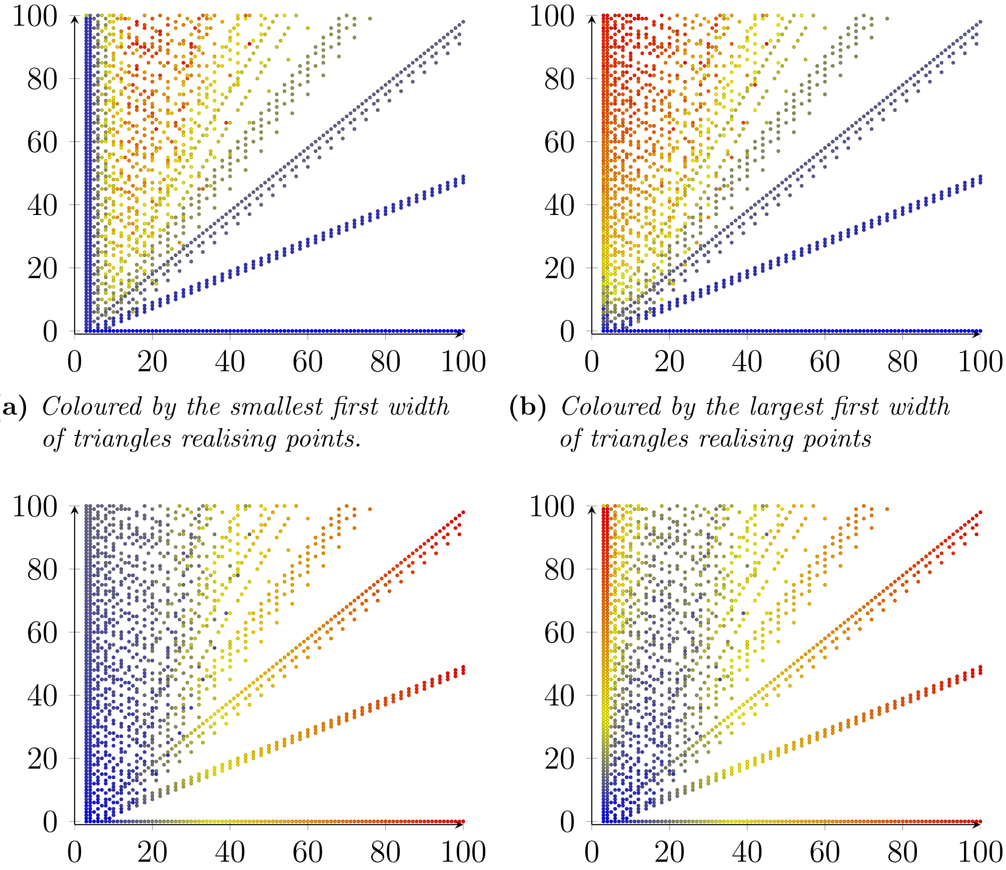
Proposition 3.4.1. *Let S_w be the set of points $(b(T), i(T))$ where T has width w with respect to the normal to one of its edges. Then we have*

$$S_w \subseteq \left\{ (b, i) \in \mathbb{Z}^2 : 1 - w^2 \leq i - \frac{w-1}{2} b \leq 1 - w \right\}$$

and

$$S_w + \left(w, \frac{w(w-1)}{2} \right) \subseteq S_w$$

Moreover, there are finitely many triangles $T_1, \ldots, T_r$, which have width w with respect to a normal to an edge, such that all points of S_w can be written $(b(T_j), i(T_j)) + k(w, w(w-1)/2)$ for some positive integer k. In other words, the points of S_w form a periodic pattern in a strip of the plane, generated by a finite collection of triangles. The set S of points $(b(T), i(T))$ where T is a lattice triangle is the union of these S_w for $w \geq 1$.



(a) *Coloured by the smallest first width of triangles realising points.*

(b) *Coloured by the largest first width of triangles realising points*

(c) *Coloured by the smallest second width of triangles realising points.*

(d) *Coloured by the largest second width of triangles realising points.*

Figure 3.5: *The plot shown in Figure 3.4, now coloured according to the widths of triangles which realise each point.*

Proof. It is immediate that S is the union of the sets S_w for $w \geq 1$ so it remains to prove the properties of S_w. Let T be a lattice triangle and say that u is a normal to one of its edges such that $\text{width}_u(T) = w$. There is an affine map which takes one of the end points of this edge to the origin, the other to the positive x-axis and the third vertex above the x-axis. We now have a triangle of the form $\text{conv}((0,0),(l,0),(a,w))$ for some positive integer l.

Pick's Theorem tells us that the normalised volume of a lattice polygon P is $2i(P)+b(P)-2$. The normalised volume of T is lw and its number of boundary points is at most $l+2w$ and at least $l+2$. Notice that $i(T)-(w-1)b(T)/2$ is equal to $(2i(T)+b(T))/2 - wb(T)/2$ and, using Pick's Theorem, this is equal to $(lw+2)/2 - wb(T)/2$. Therefore, the bounds on $b(T)$ give

$$1 - w^2 \leq i(T) - \frac{w-1}{2}b(T) \leq 1 - w.$$

Now consider the triangle $T' = \operatorname{conv}((0,0),(l+w,0),(a,w)$. Its volume is $(l+w)w$ and it has $b(T)+w$ boundary points. From Pick's Theorem we can compute that it has $i(T)+w(w-1)/2$ interior points. This shows that $S_w + (w, w(w-1)/2)$ is a subset of S_w.

To show that there is a finite collection of triangles generating S_w, consider the triangle $\operatorname{conv}((0,0),(l-kw,0),(a,w))$, where k is the integer such that $l-kw$ is the smallest positive integer possible. This has volume $(l-kw)w$, $b(T)-kw$ boundary points and $i(T)-kw(w-1)/2$ interior points. Therefore, if $k>0$ the point $(b(T),i(T))$ can be obtained from another point in S_w by adding $(w,w(w-1)/2)$. If $k=0$ then $l \in (0,w]$. We may assume by a shear that $a \in [0,w)$. Therefore, there are finitely many choices for T. $\square$

Notice that the lower and upper boundaries of S_w are in the same hyperplane as the upper boundary of σ_{w-1} and lower boundary of σ_w respectively. However, this result does not reproduce the result of [HNO18] since, as subsets of the real plane, the strips containing the sets S_w and the cones σ_c intersect non-trivially. This intersection could potentially contain points (b,i) if it were not for the proof that the cones are empty.

Chapter 4

Classification of width 1 lattice tetrahedra by their multi-width

In this chapter we classify lattice tetrahedra with width 1 by their multi-width, as an extension of the previous chapter to three dimensions. We also partially classify lattice tetrahedra with width 2 by their multi-width algorithmically.

Lattice simplices are recurring objects of study with multiple applications. Via toric geometry they are relevant to algebraic geometry and are closely related to toric $\mathbb{Q}$-factorial singularities. The toric Fano three-folds with at most terminal singularities were classified by finding all the three-dimensional lattice polytopes whose only lattice points were the origin and their vertices [Kas06]. A key step towards this was classifying the subset of tetrahedra among those polytopes. Simplices whose only lattice points are their vertices can give terminal quotient singularities by placing one vertex at the origin and considering the cone they generate. These are called *empty simplices* and were classified in dimension 3 and 4 in [Whi64] and [IVnS21] respectively. There are also applications of lattice simplices in mixed-integer and integer optimisation, see for example [AWW11] and [AKN20]. Width appears in the proofs of both [IVnS21] and [AWW11] so there is reason to investigate lattice tetrahedra of given width.

Recall from Section 3.1 that the multi-width of a d-dimensional polytope is a tuple of widths $(w_1, \ldots, w_d)$. Among other properties, this satisfies that $w_i \leq w_{i+1}$ for $i = 1, \ldots, d-1$ so for the rest of this chapter we assume, unless stated otherwise, that w_1, w_2 and w_3 are positive integers with $w_1 \leq w_2 \leq w_3$.

Ideally, we would describe the finite sets $\mathcal{T}_{w_1,w_2,w_3}$ defined

$$\mathcal{T}_{w_1,w_2,w_3} \coloneqq \{T = \text{conv}(v_1, v_2, v_3, v_4) : v_i \in \mathbb{Z}^3, \text{mwidth}(T) = (w_1, w_2, w_3)\}/\sim .$$

where $\sim$ denotes affine equivalence. Theorem 4.0.2 achieves this when the first width is 1 by establishing a bijection between $\mathcal{T}_{1,w_2,w_3}$ and a set of tetrahedra $\mathcal{S}_{1,w_2,w_3}$. To define $\mathcal{S}_{1,w_2,w_3}$ we define the four types of tetrahedron which our new classification will include. Let $\mathcal{S}_{w_1,w_2}$ be as in Definition 3.0.1.

Definition 4.0.1. The four types of tetrahedron which appear in $\mathcal{S}_{1,w_2,w_3}$ are

1. $\text{conv}(\{0\} \times t, (1, 0, 0))$ where $t \in \mathcal{S}_{w_2,w_3}$,
2. $\text{conv}((0, 0, 0), (0, w_2, z_1), (1, 0, 0), (1, 0, w_3))$ where $0 \leq z_1 \leq \frac{w_2}{2}$,
3. $\text{conv}((0, 0, 0), (0, w_2, z_1), (1, 0, w_3), (1, y_1, 0))$ where $0 < y_1 \leq w_2$ and $w_3 - w_2 \leq z_1 \leq w_3$,
4. $\text{conv}((0, 0, 0), (0, w_2, w_3), (1, 0, w_3), (1, y_1, z_1))$ where $0 < z_1 < y_1 < w_2$.

For examples of these see Figure 4.1.

If $w_3 > w_2 > 1$ let $\mathcal{S}_{1,w_2,w_3}$ be the set containing all tetrahedra of type 1-4.

If $w_3 = w_2 > 1$ then $\mathcal{S}_{1,w_2,w_2}$ is the set of all type 1 and 2 tetrahedra as well as type 3 tetrahedra satisfying $y_1 \leq z_1$ and type 4 tetrahedra satisfying $z_1 \leq w_2 - y_1$.

If $w_3 > w_2 = 1$ then

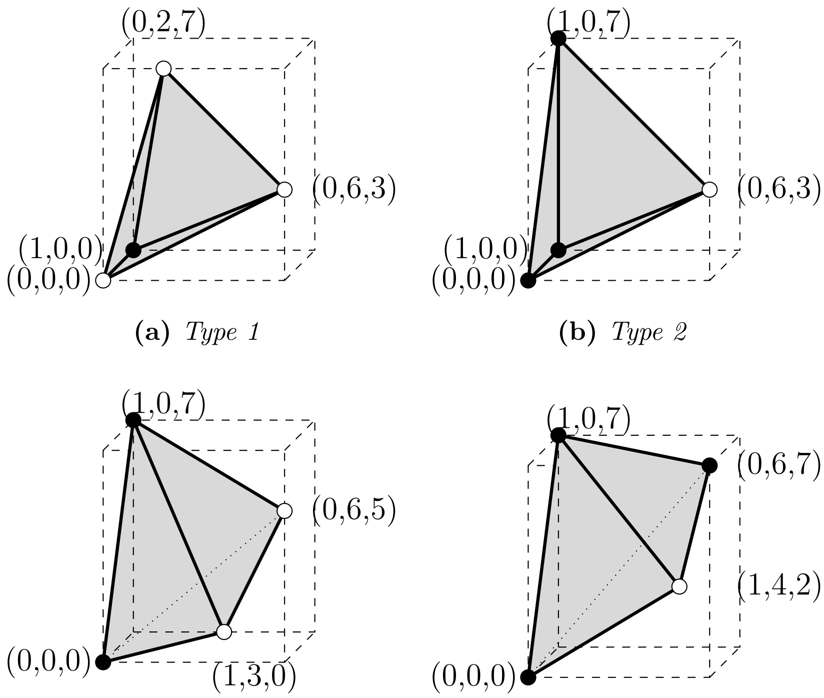
$$\begin{aligned}\mathcal{S}_{1,1,w_3} \coloneqq \{\, &\text{conv}((0, 0, 0), (0, 1, 0), (0, 0, w_3), (1, 0, 0)),\\ &\text{conv}((0, 0, 0), (0, 1, w_3 - 1), (1, 0, w_3), (1, 1, 0)),\\ &\text{conv}((0, 0, 0), (0, 1, w_3), (1, 0, w_3), (1, 1, 0))\}.\end{aligned}$$

If $w_3 = w_2 = 1$ then

$$\mathcal{S}_{1,1,1} \coloneqq \{\, \mathrm{conv}((0,0,0),(0,1,0),(0,0,1),(1,0,0)),$$
$$\mathrm{conv}((0,0,0),(0,1,1),(1,0,1),(1,1,0))\}.$$

These last two cases have elements of type 1 and 3.

The reason not all tetrahedra of type (1)-(4) necessarily belong to $\mathcal{S}_{1,w_2,w_3}$ is that in the special cases $w_2 = 1$, $w_2 = w_3$ and $w_2 = w_3 = 1$ some of these tetrahedra are affine equivalent to one-another, so we exclude duplicates. The following is the main classification result of this chapter.



(a) *Type 1* **(b)** *Type 2* **(c)** *Type 3* **(d)** *Type 4*

Figure 4.1: *Examples of tetrahedra of type 1-4 when $w_2 = 6$ and $w_3 = 7$. Black vertices are fixed for a given type while white vertices are variable.*

Theorem 4.0.2. *There is a bijection from $\mathcal{S}_{1,w_2,w_3}$ to $\mathcal{T}_{1,w_2,w_3}$ given by the map taking a tetrahedron to its affine equivalence class. In particular,*

- *when $w_3 > w_2 > 1$ the cardinality of $\mathcal{T}_{1,w_2,w_3}$ is*
  - $2w_2^2 + 4$ *if w_2 and w_3 even*

- $2w_2^2 + 3$ *if* w_2 *even and* w_3 *odd*
- $2w_2^2 + 2$ *if* w_2 *odd*

- *when* $w_2 > 1$ *the cardinality of* $\mathcal{T}_{1,w_2,w_2}$ *is*
  - $w_2^2 + w_2 + 2$ *if* w_2 *even*
  - $w_2^2 + w_2 + 1$ *if* w_2 *odd*
- *when* $w_3 > 1$ *the cardinality of* $\mathcal{T}_{1,1,w_3}$ *is* 3
- *and the cardinality of* $\mathcal{T}_{1,1,1}$ *is* 2.

	w_3											
w_2	1	2	3	4	5	6	7	8	9	10	11	12
1	2	3	3	3	3	3	3	3	3	3	3	3
2	0	8	11	12	11	12	11	12	11	12	11	12
3	0	0	13	20	20	20	20	20	20	20	20	20
4	0	0	0	22	35	36	35	36	35	36	35	36
5	0	0	0	0	31	52	52	52	52	52	52	52
6	0	0	0	0	0	44	75	76	75	76	75	76

Table 4.1: The number of lattice tetrahedra with multi-width $(1, w_2, w_3)$ up to affine equivalence for small w_2 and w_3.

Table 4.1 gives the cardinality of $\mathcal{T}_{1,w_2,w_3}$ when $w_2 \leq 6$ and $w_3 \leq 12$ described by Theorem 4.0.2. Surprisingly this follows the same pattern as the triangles case in the previous chapter: right of the diagonal, odd rows are constant and even rows alternate between two values. The idea of the proof is to first show that a tetrahedron with multi-width $(1, w_2, w_3)$ is equivalent to a subset of $[0, 1] \times [0, w_2] \times [0, w_3]$. We then successively classify the possible x-, y- and z-coordinates of the vertices of the tetrahedra. At each step we remove cases which have too small a width in some direction or are equivalent to other cases.

In earlier steps of the classification we consider multi-sets of points. In an abuse of notation we write $\{v_1, \ldots, v_n\}$ for the n-point multi-set containing lattice points $v_i \in \mathbb{Z}^d$ even when the v_i are not distinct. We define widths of these sets by saying the width of a set is the width of its convex hull.

We can completely classify the four-point sets in $\mathbb{Z}$ with width w_1. These can represent the possible x-coordinates of all four-point sets in $\mathbb{Z}^2$ with multi-width (w_1, w_2). The second width gives bounds on their possible y-coordinates and we can completely classify the four-point sets in the plane with multi-width $(1, w_2)$. Similarly, these represent the possible first two coordinates of the vertices of tetrahedra of multi-width $(1, w_2, w_3)$. By considering the possible z-coordinates we can assign to each point we obtain the classification above.

When $w_1 > 1$ the number of cases which needs to be checked for this proof style increases dramatically. However, the method can be used to create an algorithm which classifies the tetrahedra of a given multi-width. In this way we begin classifying the width 2 case for small multi-width. The number of tetrahedra classified can be found in Table 4.2. We take this classification only far enough to obtain and test Conjecture 4.4.3 on the number of multi-width $(2, w_2, w_3)$ tetrahedra. The extent to which we can extend the two-dimensional results to the three-dimensional case remains open, but the similarities in the results we have found so far seem hopeful.

	w_3										
w_2	2	3	4	5	6	7	8	9	10	11	12
2	17	45	47	45	47	45	47	45	47	45	47
3	0	87	178	175	178	175	178	175	178	175	178
4	0	0	161	320	325	320	325	320	325	320	325
5	0	0	0	244	493	490	493	490	493	490	493
6	0	0	0	0	358	716	721	716	721	716	721
7	0	0	0	0	0	482	970	967	970	967	970
8	0	0	0	0	0	0	636	1274	1279	1274	1279
9	0	0	0	0	0	0	0	801	1609	1606	1609
10	0	0	0	0	0	0	0	0	995	1994	1999

Table 4.2: The number of lattice tetrahedra with multi-width $(2, w_2, w_3)$ up to affine equivalence for small w_2 and w_3.

In Section 4.1 we prove some facts about a polytope of given multi-width. In particular a 3-dimensional lattice polytope with multi-width (w_1, w_2, w_3) is equivalent to a subset of $[0, w_1] \times [0, w_2] \times [0, w]$ where w is the smallest out of $w_1 + w_3 - 1$ and $\max\{w_1 + w_2, w_3\}$. In Section 4.2 we classify the four-point sets in $\mathbb{Z}$ with multi-width w_1 and the four-point sets in $\mathbb{Z}^2$ with multi-width (w_1, w_2) which have x-coordinates $\{0, 0, 0, w_1\}$ or $\{0, 0, w_1, w_1\}$. A corollary of this is the classification of multi-width $(1, w_2)$ four-point sets. In Section 4.3 we prove Theorem 4.0.2. Propositions 4.3.2, 4.3.3 and 4.3.5 show that the map taking a lattice tetrahedron to its equivalence class is a well-defined bijection from $\mathcal{S}_{1,w_2,w_3}$ to $\mathcal{T}_{1,w_2,w_3}$. In Section 4.4 we describe the computational extension of this classification. We classify the multi-width (w_1, w_2) four-point sets in the plane and the multi-width $(2, w_2, w_3)$ tetrahedra for small w_1, w_2 and w_3. Based on these classifications we make conjectures about the functions counting such sets and tetrahedra in general.

4.1 Width and Parallepipeds

We carry forward all definitions of width and multi-width from Section 3.1. Since we are no longer in the special case of dimension 2 multi-width is no longer equivalent to the dimensions of a minimal box containing a polytope. However, we can still bound the size of a box a polytope is equivalent to a subset of using its widths as follows.

Proposition 4.1.1. *Let Q be a 3-dimensional lattice polytope with widths w_1, w_2 and w_3 with respect to three linearly independent dual vectors. Assume that $0 < w_1 \leq w_2 \leq w_3$, then Q is equivalent to a subset of $[0, w_1] \times [0, w_2] \times [0, w_1 + w_3 - 1]$. Furthermore, if (w_1, w_2, w_3) is the multi-width of Q then Q is equivalent to a subset of $[0, w_1] \times [0, w_2] \times [0, \max\{w_1 + w_2, w_3\}]$.*

Proof. By Proposition 3.1.3 we may assume that Q is a subset of the parallelepiped

$$P \coloneqq \{v \in \mathbb{R}^3 : (1,0,0) \cdot v \in [0, w_1],\ (0,1,0) \cdot v \in [0, w_2],\ u \cdot v \in [a, a + w_3]\}$$

for some integer a and some dual vector u linearly independent to $(1,0,0)$ and $(0,1,0)$. Say $u = (u_x, u_y, u_z)$ then $u_z \neq 0$. In fact we may assume $u_z > 0$, otherwise replace u with $-u$ and adjust a so this does not change P. Therefore, we may pick integers k_x and k_y such that $0 \leq k_i u_z - u_i < u_z$. Now let φ be the shear described by

$$(x, y, z) \mapsto (x, y, k_x x + k_y y + z).$$

The coordinates of the vertices of $\varphi(P)$ are

$$(0, 0, \tfrac{a}{u_z}), \quad (0, 0, \tfrac{a+w_3}{u_z}), \quad (w_1, 0, \tfrac{a+w_1(k_x u_z - u_x)}{u_z}), \quad (w_1, 0, \tfrac{a+w_3+w_1(k_x u_z - u_x)}{u_z}),$$
$$(0, w_2, \tfrac{a+w_2(k_y u_z - u_y)}{u_z}), \quad (0, w_2, \tfrac{a+w_3+w_2(k_y u_z - u_y)}{u_z}),$$
$$(w_1, w_2, \tfrac{a+w_1(k_x u_z - u_x)+w_2(k_y u_z - u_y)}{u_z}) \text{ and } (w_1, w_2, \tfrac{a+w_3+w_1(k_x u_z - u_x)+w_2(k_y u_z - u_y)}{u_z}).$$

By inspecting the z-coordinates of the vertices of $\varphi(P)$, each of which is a sum of a and some non-negative terms all divided by u_z, we can see that

$$\begin{aligned}\text{width}_{(0,0,1)}(\varphi(Q)) \leq & \frac{w_1(k_x u_z - u_x) + w_2(k_y u_z - u_y) + w_3}{u_z} \\ \leq & w_1\left(1 - \frac{1}{u_z}\right) + w_2\left(1 - \frac{1}{u_z}\right) + w_3\frac{1}{u_z}\end{aligned}$$

Since $w_2 \leq w_3$ this is less that $w_1 + w_3$. After a translation this shows that Q is equivalent to a subset of $[0, w_1] \times [0, w_2] \times [0, w_1 + w_3 - 1]$. This uses the fact that Q is a lattice polytope so has integral widths.

Now suppose the multi-width of Q is (w_1, w_2, w_3) then we show that Q is equivalent to a subset of $[0, w_1] \times [0, w_2] \times [0, \max\{w_1 + w_2, w_3\}]$. In the above inequalities if $u_z = 1$ then $\text{width}_{(0,0,1)}(\varphi(Q)) \leq w_3$ so we are done. If $u_z \geq 2$ consider the fact that $\text{width}_{(0,0,1)}(\varphi(Q)) \leq w_1 + w_2 + \frac{w_3 - w_1 - w_2}{u_z}$. If $w_3 > w_1 + w_2$ then $w_1 + w_2 + \frac{w_3 - w_1 - w_2}{u_z}$ is at most $\frac{w_1 + w_2 + w_3}{2}$ which is less than w_3. If $w_3 \leq w_1 + w_2$ then $w_1 + w_2 + \frac{w_3 - w_2 - w_1}{u_z}$ is at most $w_1 + w_2$. Since the third width of $\varphi(Q)$ is w_3 and its first two widths are realised by $(1, 0, 0)$ and $(0, 1, 0)$ it cannot have width less that w_3 with respect to $(0, 0, 1)$. This eliminates the case $u_z \geq 2$ and $w_3 > w_1 + w_2$ and so, after a translation, $\varphi(Q)$ is a subset of the desired box. $\square$

This shows that any 3-dimensional lattice polytope with multi-width (w_1, w_2, w_3) is equivalent to a subset of $[0, w_1] \times [0, w_2] \times [0, w]$ where

$$w \coloneqq \min\{w_1 + w_3 - 1, \max\{w_1 + w_2, w_3\}\}.$$

This bound may not be sharp in general.

4.2 Four-point Sets in the Plane

The main aim of this section is to classify the four-point sets in the plane with first width 1 up to affine equivalence. The four-point sets with multi-width

$(1,1)$ are just

$$\{(0,0),(1,0),(0,1),(1,1)\} \quad \text{and} \quad \{(0,0),(0,0),(1,0),(0,1)\},$$

which we deduce by considering four-point subsets of $\{(0,0),(1,0),(0,1),(1,1)\}$. When $w_2 > 1$ the four-point sets with multi-width $(1, w_2)$ are

$$\{(0,0),(0,w_2),(0,y_0),(1,0)\} \quad \text{where} \quad y_0 \in [0, \tfrac{w_2}{2}]$$

and

$$\{(0,0),(0,w_2),(1,0),(1,y_1)\} \quad \text{where} \quad y_1 \in [0, w_2]$$

(for example, see Figure 4.2). This can be proven directly but here we will prove a more general result. We will classify four-point sets S in the plane with multi-width (w_1, w_2) where $w_2 > w_1$ with the additional condition that if $\text{width}_{u_1}(S) = w_1$ then all points of S are contained in the two hyperplanes with normal vector u_1 bounding S. This is sufficient to classify the width 1 four-point sets in the plane while being the most general classification which is practical to obtain with this method. We do this because it may be useful towards a future extension of the tetrahedron classification.

First we classify all four-point sets in $\mathbb{Z}$ of width w_1.

Proposition 4.2.1. *There is a bijection from the collection of lattice points in the triangle* $Q_{w_1} := \text{conv}((0,0),(0,w_1),(\frac{w_1}{2},\frac{w_1}{2}))$ *to the set of the equivalence classes of the four-point sets in* $\mathbb{Z}$ *with width* w_1*. It is given by the map*

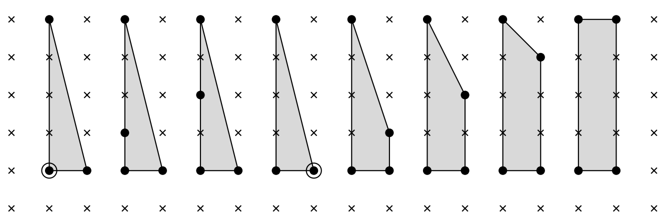


Figure 4.2: *The 4-point sets in* $\mathbb{Z}^2$ *with multi-width* $(1,4)$ *up to affine equivalence and their convex hulls. Two points with the same coordinates are denoted by a circled dot.*

taking (x_1, x_2) *to* $\{0, x_1, x_2, w_1\}$. *In particular, the number of such sets up to equivalence is*

$$\begin{cases} \frac{w_1^2}{4} + w_1 + 1 & \textit{if } w_1 \textit{ is even} \\ \frac{w_1^2}{4} + w_1 + \frac{3}{4} & \textit{if } w_1 \textit{ is odd.} \end{cases}$$

Proof. The map $(x_1, x_2) \mapsto \{0, x_1, x_2, w_1\}$ is a well-defined map taking a lattice point of Q_{w_1} to a four-point set of width w_1. For surjectivity notice that the convex hull of any four-point set of width w_1 is equivalent to $\operatorname{conv}(0, w_1)$. Therefore, we may assume that 0 and w_1 are points in such a set and that $x_1, x_2 \in [0, w_1]$ are the two remaining points. By relabeling of the x_i we may assume that $x_1 \leq x_2$. A reflection takes $\{0, x_1, x_2, w_1\}$ to $\{0, w_1 - x_2, w_1 - x_1, w_1\}$ so we may assume that $x_1 \leq w_1 - x_2$. This shows that $(x_1, x_2) \in Q_{w_1}$.

For injectivity let (x_1, x_2) and (x_1', x_2') be lattice points in Q_{w_1} such that $\{0, x_1, x_2, w_1\}$ is equivalent to $\{0, x_1', x_2', w_1\}$. The only non-trivial affine automorphism of a line segment in $\mathbb{Z}$ is the reflection about its midpoint so either $(x_1, x_2) = (x_1', x_2')$ or $(x_1, x_2) = (w_1 - x_2', w_1 - x_1')$. In the first case we are done. In the second case notice that $x_1 = w_1 - x_2' \geq x_1'$ and $x_1' = w_1 - x_2 \geq x_1$ so $x_1 = x_1'$. Similarly $x_2 = x_2'$ which proves the result.

The counting can be seen by counting points in vertical lines of lattice points in Q_{w_1}. Thus there are $(w_1 + 1) + (w_1 - 1) + \cdots + 1$ points in total if w_1 is even and $(w_1 + 1) + (w_1 - 1) + \cdots + 2$ if w_1 is odd. These simplify to the given formulas. $\square$

We now move on to four-point sets in $\mathbb{Z}^2$ with multi-width (w_1, w_2). If a multi-width $(w_1, \ldots, w_d)$ polytope is a subset of a $w_1 \times \cdots \times w_d$ box it must have a vertex in each facet of this box otherwise it would have smaller multi-width. Therefore, a multi-width (w_1, w_2) four-point set which is a subset of $[0, w_1] \times [0, w_2]$ has x-coordinates equivalent to one of the above classified sets. We restrict to the case where the corresponding point of Q_{w_1} is either $(0, 0)$ or $(0, w_1)$ since this is sufficient to classify all multi-width $(1, w_2)$ four-point sets in $\mathbb{Z}^2$. We additionally assume that $w_1 < w_2$, since the four-point sets with multi-width $(1,1)$ are easy to identify and for $w_1 > 1$ classifying the multi-width (w_1, w_1) four-point sets adds unnecessary complexity to the proofs.

Proposition 4.2.2. *Let S be a four-point set in the plane with multi-width (w_1, w_2) where $0 < w_1 < w_2$. There is a dual vector u_1 such that* $\mathrm{width}_{u_1}(S) = w_1$ *and $u_1 \cdot S$ is equivalent to $\{0, 0, 0, w_1\}$ if and only if S is equivalent to exactly one of the following four-point sets:*

- $\{(0,0), (0,w_2), (0,y_0), (w_1,y_1)\}$ *where* $0 \le y_0 < \frac{w_2}{2}$ *and* $0 \le y_1 < w_1$*,*
- $\{(0,0), (0,w_2), (0,\frac{w_2}{2}), (w_1,y_1)\}$ *where* $0 \le y_1 \le (w_2 - y_1 \mod w_1)$ *and* w_2 *is even.*

Proof. First we show that the listed four-point sets have multi-width (w_1, w_2). It is enough to notice that in either case if $u = (u_x, u_y)$ is a dual vector with $u_y \neq 0$, then

$$\mathrm{width}_u(S) \ge |u \cdot (0, w_2) - u \cdot (0,0)| = |u_y w_2| \ge w_2.$$

The image of these sets under $u_1 = (1,0)$ is $\{0,0,0,w_1\}$ which proves the implication in one direction.

Next we show that all four-point sets S with multi-width (w_1, w_2) and a dual vector u_1 such that $u_1 \cdot S$ is equivalent to $\{0,0,0,w_1\}$ are equivalent to one of the two given cases. Let S be such a set, then by Proposition 3.1.3 we may assume it is a subset of $[0, w_1] \times [0, w_2]$. Since $w_1 < w_2$ the direction in which S has width w_1 is unique up to sign so $u_1 = \pm(1,0)$. Under u_1 points of S are mapped to (possibly -1 times) their x-coordinates so the only way for these to map to something equivalent to $\{0,0,0,w_1\}$ is to have three points of S on one vertical edge of the rectangle and the fourth point on the other vertical edge. Therefore, possibly after the reflection $(x,y) \mapsto (w_1 - x, y)$, we may assume that S contains three points with x-coordinate 0 and one with x-coordinate w_1. Also, S must contain $(0,0)$ and $(0,w_2)$ otherwise, by a shear $(x,y) \mapsto (x, y - kx)$ for some integer k, S is equivalent a subset of a smaller rectangle which contradicts the widths. Therefore, we assume that $S = \{(0,0),(0,w_2),(0,y_0),(w_1,y_1)\}$.

By the reflection $(x,y) \mapsto (x, w_2 - y)$ we may assume that $0 \leq y_0 \leq \frac{w_2}{2}$. By a shear $(x,y) \mapsto (x, y - kx)$ for some integer k we may assume that $0 \leq y_1 < w_1$. If $y_0 = \frac{w_2}{2}$ and $y_1 > (w_2 - y_1 \mod w_1)$ then we can make the y-coordinate of the vertex on $x = w_1$ smaller by a reflection in the line $y = \frac{w_2}{2}$ followed by a shear. In more precise terms, pick k such that $w_2 - y_1 - kx = (w_2 - y_1 \mod w_1)$ then the reflection and shear $(x,y) \mapsto (x, w_2 - y - kx)$ takes S to one of the given four-point sets. This proves that S is equivalent to a set of one of the given forms.

Finally we show that the four-points sets in the two cases are unique. Suppose

$$S = \{(0,0),(0,w_2),(0,y_0),(w_1,y_1)\} \sim \{(0,0),(0,w_2),(0,y_0'),(w_1,y_1')\} = S'$$

where S and S' are each of either of the forms from the proposition. We will show that S and S' are equal. We can think of these sets as their convex hulls, which are triangles, with a marked point. Since $w_2 > w_1$, considering the lattice length of line segments (i.e. the number of lattice points they contain minus 1) in $[0,w_1] \times [0,w_2]$, we see that the edge from $(0,0)$ to $(0,w_2)$ is the only edge of each triangle with lattice length w_2. Therefore, an affine map taking S to S' must map this edge back to itself. This reduces us to shears $(x,y) \mapsto (x, y - kx)$ and the reflection followed by a shear $(x,y) \mapsto (x, w_2 - y - kx)$ where k is an integer. The images of S under such maps are

$$\begin{gathered}\{(0,0),(0,w_2),(0,y_0),(w_1,y_1 - kw_1)\}, \quad \text{and} \\ \{(0,0),(0,w_2),(0,w_2 - y_0),(w_1,w_2 - y_1 - kw_1)\}\end{gathered}$$

Since $0 \leq y_0, y_0' \leq \frac{w_2}{2}$ this shows that $y_0 = y_0'$. The y-coordinate of the fourth points of these images must equal y_1' so, since $0 \leq y_1' < w_1$, we must always choose k so that this y-coordinate is reduced modulo w_1. Since $0 \leq y_1 < w_1$ if the shear takes S to S' this means that $y_1 = y_1'$ and $S = S'$. If instead the reflection followed by a shear takes S to S' then $y_0' = w_2 - y_0$ and since $y_0 = y_0'$ we have $y_0 = \frac{w_2}{2}$. This means that $y_1 \leq (w_2 - y_1 \mod w_1) = y_1'$ and by the symmetric argument exchanging S and S' we have $y_1' \leq (w_2 - y_1' \mod w_1) = y_1$ so $y_1 = y_1'$ and $S = S'$. □

Proposition 4.2.3. *Let S be a four-point set in the plane with multi-width (w_1, w_2) where $0 < w_1 < w_2$. There is a dual vector u_1 such that* $\text{width}_{u_1}(S) = w_1$ *and $u_1 \cdot S$ is equivalent to $\{0, 0, w_1, w_1\}$ if and only if S is equivalent to exactly one of the following four-point sets:*

- $\{(0,0),(0,w_2),(w_1,y_1),(w_1,y_2)\}$ *where* $0 \leq y_1 \leq y_2 \leq w_2$ *and* $y_1 \leq (w_2 - y_2 \mod w_1)$
- $\{(0,0),(0,y_0),(w_1,y_1),(w_1,w_2)\}$ *where* $\max\{w_2 - y_1, w_2 - (w_1 - y_1)\} \leq y_0 < w_2$

Proof. First we show that the listed four-point sets have multi-width (w_1, w_2). The first case follows by the same proof as that in Proposition 4.2.2 since it still contains $(0, 0)$ and $(0, w_2)$. In the second case, it suffices to show that for any dual vector $u = (u_x, u_y)$ with $u_y > 0$, $\text{width}_u(S) \geq w_2$. The image of S under u is

$$u \cdot S = \{0, u_y y_0, u_x w_1 + u_y y_1, u_x w_1 + u_y w_2\}.$$

Suppose for contradiction that the width of S with respect to u is less than w_2. This means that the difference of any two elements in $u \cdot S$ must be less than w_2 so then $u_x w_1 + u_y w_2 < w_2$ which implies $u_x < 0$. Also, $u_y y_0 - u_x w_1 - u_y y_1 < w_2$ which we rearrange to show

$$u_x > (u_y y_0 - w_2 - u_y y_1)/w_1. \tag{4.1}$$

By the conditions on y_0 and y_1 we have $y_0 - y_1 \geq w_2 - w_1$ so we know that $u_y(y_0 - y_1) \geq w_2 - w_1$. Combining this with (4.1) shows that $u_x > -1$ which is the desired contradiction. Under $(1,0)$ these sets are taken to $\{0,0,w_1,w_1\}$ which shows the implication in one direction.

Let S be a four-point set with multi-width (w_1,w_2) with a dual vector u_1 such that $u_1 \cdot S$ is equivalent to $\{0,0,w_1,w_1\}$. We will show that S is equivalent to one of the sets listed in the proposition. By Proposition 3.1.3 we may assume this is a subset of $[0,w_1] \times [0,w_2]$. Since $w_1 < w_2$ the direction in which the first width is realised is unique up to sign so $u_1 = \pm(1,0)$. This shows that S contains two points with x-coordinate 0 and two with x-coordinate w_1. Also, S must contain points with y-coordinates 0 and w_2 or it would have smaller multi-width. This means $(0,0)$ or $(w_1,0)$ is in S and $(0,w_2)$ or (w_1,w_2) is in S. Possibly after applying the reflection $(x,y) \mapsto (w_1-x,y)$ we may assume that $(0,0) \in S$. We may assume one of the following two possibilities:

A. $S = \{(0,0),(0,w_2),(w_1,y_1),(w_1,y_2)\}$ where $0 \leq y_1 \leq y_2 \leq w_2$,

B. $S = \{(0,0),(0,y_0),(w_1,y_1),(w_1,w_2)\}$ where $0 \leq y_0 < w_2$ and $0 < y_1 \leq w_2$

where the additional inequalities come from relabeling the y_i and removing overlap between the cases.

In A we aim to minimise the y-coordinates of vertices on the line $x = w_1$ and can do so either by a shear about the y-axis or by a reflection in the line $y = \frac{w_2}{2}$ followed by such a shear. We may assume by a shear that $0 \leq y_1 < w_1$. Consider the map given by $(x,y) \mapsto (x, w_2 - y - kx)$ for the integer k such that $w_2 - y_2 - kw_1 = (w_2 - y_2 \mod w_1)$. This map is self inverse and takes S to $\{(0,0),(0,w_2),(w_1,y_1'),(w_1,y_2')\}$ which is also of the form A and $y_1' < w_1$. Either $y_1 \leq (w_2 - y_2 \mod w_1) = y_1'$ or $y_1' \leq (w_2 - y_2' \mod w_1) = y_1$. In either case S is equivalent to one of the sets listed in the proposition.

In case B we can get a set of the same form by applying the affine map $(x,y) \mapsto (w_1 - x, w_2 - y)$, which replaces y_0 and y_1 with $w_2 - y_1$ and $w_2 - y_0$ respectively. Therefore, we may assume that $y_0 \geq w_2 - y_1$. Now consider the image $(-1,1) \cdot S = \{0, y_0, y_1 - w_1, w_2 - w_1\}$. To prevent the width of S

with respect to $(-1,1)$ being less that w_2 the difference between some pair of elements of $(-1,1)\cdot S$ must be at least w_2. However, checking these differences case by case only $y_0-(w_2-w_1)$ and $y_0-(y_1-w_1)$ can be at least w_2. If $y_0-(w_2-w_1)\geq w_2$ then definitely $y_0-(y_1-w_1)\geq w_2$ so we may assume the latter holds. Therefore, S is equivalent to one of the sets listed in the proposition.

Now we show that the sets listed in the proposition are distinct up to equivalence. The two cases are distinct since the convex hull of the first case has an edge of lattice length w_2 and the second does not. Let S and S' be of the first form and suppose

$$S=\{(0,0),(0,w_2),(w_1,y_1),(w_1,y_2)\}\sim\{(0,0),(0,w_2),(w_1,y_1'),(w_1,y_2')\}=S'.$$

Either these are both equal to the vertices of $[0,w_1]\times[0,w_2]$ or their convex hulls each have exactly one edge of lattice length w_2. If they are equal we are done, otherwise the map taking S to S' must preserve the line segment from $(0,0)$ to $(0,w_2)$. This reduces us to shears $(x,y)\mapsto(x,y-kx)$ and the reflection followed by a shear $(x,y)\mapsto(x,w_2-y-kx)$ for integers k. Since $0\leq y_1,y_1'<w_1$ if a shear maps S to S' then $S=S'$. If the reflection followed by a shear maps S to S' then $y_1'=(w_2-y_2 \mod w_1)\geq y_1$ and symmetrically $y_1\geq y_1'$ so $y_1=y_1'$. Since the volume of the convex hulls of S and S' must be equal this shows that $y_2=y_2'$ and $S=S'$.

Now let S and S' be of the second form listed in the proposition and suppose

$$S=\{(0,0),(0,y_0),(w_1,y_1),(w_1,w_2)\}\sim\{(0,0),(0,y_0'),(w_1,y_1'),(w_1,w_2)\}=S'.$$

By the multi-width of these we know that $\pm(1,0)$ are the only dual vectors under which they have width w_1. Therefore, a map taking the convex hull of S to the convex hull of S' must take edges with normal $(1,0)$ to edges with normal $(1,0)$. Thus it suffices to consider the lengths of the vertical edges of the convex hulls of S and S'. By the conditions on S and S' their left-most vertical edge is at least as long as their right most vertical edge so $y_0 = y_0'$, $y_1 = y_1'$ and $S = S'$. $\square$

We use these results to classify the four-point sets of multi-width $(1, w_2)$:

Corollary 4.2.4. *Let S be a four-point set in $\mathbb{Z}^2$ with multi-width $(1, w_2)$ then if $w_2 > 1$ either S is equivalent to*

- $\{(0,0),(0,w_2),(0,y_0),(1,0)\}$ *with* $y_0 \in [0, \frac{w_2}{2}]$ *or*
- $\{(0,0),(0,w_2),(1,0),(1,y_1)\}$ *with* $y_1 \in [0, w_2]$.

Counting the possible integers y_0 and y_1 shows that these are counted by

$$\begin{cases} \frac{3w_2}{2} + 2 & \textit{if } w_2 \textit{ even} \\ \frac{3w_2}{2} + \frac{3}{2} & \textit{if } w_2 \textit{ odd.} \end{cases}$$

If instead $w_2 = 1$ then S is equivalent to

- $\{(0,0),(0,1),(1,0),(1,1)\}$ *or*
- $\{(0,0),(0,0),(1,0),(0,1)\}$.

4.3 Proof of Theorem 4.0.2

In this section we prove Theorem 4.0.2, that is we show that the map taking a tetrahedron to its affine equivalence class defines a bijection from the set of tetrahedra $\mathcal{S}_{1,w_2,w_3}$ to the set $\mathcal{T}_{1,w_2,w_3}$ of tetrahedra of multi-width $(1, w_2, w_3)$ up to affine equivalence.

We begin with a preparatory lemma towards surjectivity.

Lemma 4.3.1. *Let T be a lattice tetrahedron with multi-width $(1, w_2, w_3)$. Then there exists a tetrahedron T' of type 1, 2, 3 or 4, as in Definition 4.0.1, which is equivalent to T.*

Proof. By Proposition 4.1.1 we may assume that T is a subset of $[0,1]\times[0,w_2]\times[0,w_3]$. The vertices of T can only be in the planes $x=0$ and $x=1$ so, possibly after a reflection, we may assume the x-coordinates of the vertices of T are either $\{0,0,0,1\}$ or $\{0,0,1,1\}$. We will show that

- If the x-coordinates of T are $\{0,0,0,1\}$ then T is equivalent to a type 1 tetrahedron,
- If the x-coordinates of T are $\{0,0,1,1\}$ then T is equivalent to a type 2, 3 or 4 tetrahedron.

If the x-coordinates are $\{0,0,0,1\}$ then T is the convex hull of a triangle embedded in the plane $x=0$ and a point with x-coordinate 1. For integers k_1 and k_2, the shears $(x,y,z)\mapsto(x,y+k_1x,z+k_2x)$, can take the vertex with x-coordinate 1 to any lattice point with x-coordinate 1 without changing the triangle in the plane $x=0$. Therefore, we may assume that, under the projection onto the last two coordinates, T is mapped to the triangle. The triangle is a subset of a $w_2\times w_3$ rectangle so its multi-width is at most (w_2,w_3). If it had multi-width lexicographically smaller than (w_2,w_3) there would be dual vectors u_2' and u_3' linearly independent from $u_1=(1,0,0)$ such that $(1,\mathrm{width}_{u_2'}(T),\mathrm{width}_{u_3'}(T)) <_{lex} (1,w_2,w_3)$ which is a contradiction. Therefore, this triangle has multi-width (w_2,w_3). By Theorem 3.0.2 we can assume the triangle is in $\mathcal{S}_{w_2,w_3}$. Then by another shear of the same form we can move the fourth vertex to $(1,0,0)$ which proves that T is equivalent to a tetrahedron of the form 1.

If the x-coordinates are $\{0,0,1,1\}$ we need to consider the four-point set we get by projecting the vertices of T onto the first two coordinates. This must have multi-width $(1,w_2)$ otherwise T has smaller multi-width. By Corollary 4.2.4 the set is equivalent to one of the sets $\{(0,0),(0,w_2),(1,0),(1,y_1)\}$ for an integer $y_1\in[0,w_2]$ and by an affine map on T we may assume they are equal. It remains to determine the z-coordinates of the vertices of T. These are integers in $[0,w_3]$. At least one of them must equal 0 and one w_3 otherwise T

has width less than w_3 with respect to $(0, 0, 1)$ contradicting the multi-width.

There are 12 ways to assign 0 and w_3 to two of the vertices. See Figure 4.3 for the full list. By a reflection in the plane $z = \frac{w_3}{2}$, as depicted in Figure 4.4, we may swap which vertices are assigned 0 and w_3. In this way we see that (g)-(l) are equivalent to (a)-(f) so we disregard the last 6 cases.

We are left with cases (a)-(f). By a reflection in the plane $y = \frac{w_2}{2}$ followed by the shear $(x, y, z) \mapsto (x, y - (w_2 - y_1)x, z)$, as depicted in Figure 4.4, we can swap the z-coordinates assigned to the lower two vertices with those assigned to the upper two vertices. In this way we see that (d) and (e) are equivalent to (c) and (b) respectively so we may disregard (d) and (e) also.

We are left with cases (a), (b), (c) and (f). In case (a), after a shear about the plane $x = 0$, we may assume that $z_1 = w_3$ or $z_2 = w_3$. Therefore, (a) is included in case (b) or (c) and we may disregard (a). Similarly, in case (f), after a shear about the plane $x = 1$ we may assume that $z_1 = 0$ or $z_2 = 0$. Therefore, (f) is included in case (c) or case (e) and thus (b). We disregard (f) as a result.

We are left with only cases (b) and (c). We will show that we can also disregard case (c) by showing that its width is incorrect unless it is also equivalent to a type (b) tetrahedron. In case (c), if $y_1 = 0$, $z_1 = 0$ or $z_2 = w_3$ then these tetrahedra would be included in case (b) (or (e) and thus (b)) so we assume these three equalities are false. The images of the vertices of (c) under $(-z_2, 0, 1)$, $(0, -1, 1)$ and $(w_3 - y_1, 1, -1)$ are

$$\{0, z_1, z_2, w_3 - z_2\}, \quad \{0, z_1 - w_2, z_2, w_3 - y_1\} \quad \text{and} \quad \{0, w_2 - z_1, w_3 - y_1 - z_2, 0\}$$

respectively. If any of these is a subset of $[0, w_3)$ then T would have width less than w_3 in some direction linearly independent to $(1, 0, 0)$ and $(0, 1, 0)$ which is a contradiction. To prevent this all three of the following points must be true.

- $z_1 = w_3$ or $z_2 = w_3$ or $z_2 = 0$

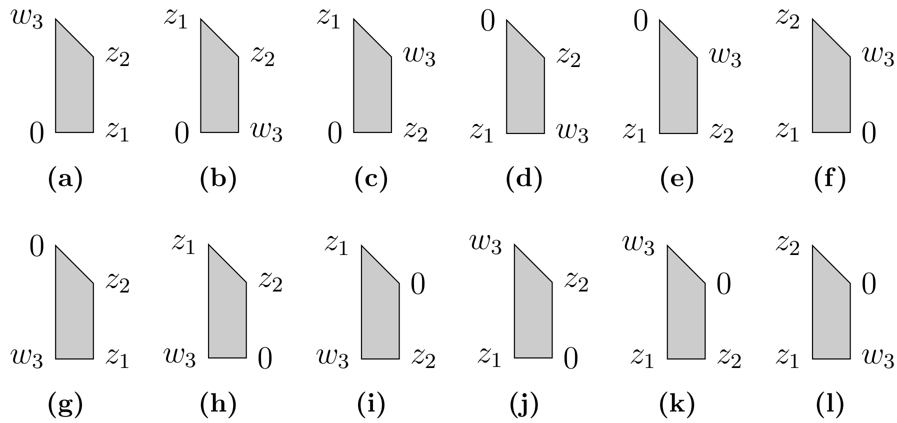


Figure 4.3: *Image* $\mathrm{conv}((0,0),(0,w_2),(1,0),(1,y_1))$ *of a tetrahedron under projection onto the first two coordinates. Labels denote the* z*-coordinates at each vertex. Numbers* z_1 *and* z_2 *are integers in the range* $[0, w_3]$ *and these twelve cases include all affine equivalence classes of tetrahedra with multi-width* $(1, w_2, w_3)$ *and* x*-coordinates* $\{0, 0, 1, 1\}$ *in* $[0, 1] \times [0, w_2] \times [0, w_3]$.

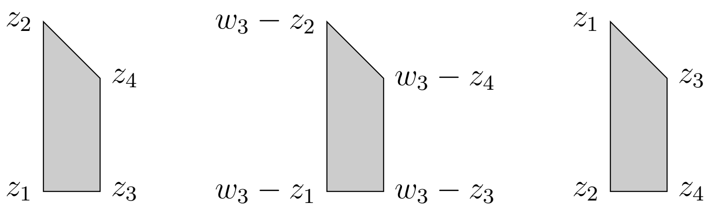


Figure 4.4: *Image of* $T = \mathrm{conv}((0,0,z_1),(0,w_2,z_2),(1,0,z_3),(1,y_1,z_4))$ *and two equivalent tetrahedra under projection onto the first two coordinates. Labels denote the* z*-coordinates at each vertex. The second tetrahedron is obtained from the first by a reflection in the plane* $z = \frac{w_3}{2}$*. The third is obtained from the first by a reflection in the plane* $y = \frac{w_2}{2}$ *followed by the shear* $(x, y, z) \mapsto (x, y - (w_2 - y_1)x, z)$.

- $z_1 < w_2$ or $z_2 = w_3$ or $y_1 = 0$

- $(w_2 = w_3$ and $z_1 = 0)$ or $z_1 > w_2$ or $y_1 = z_2 = 0$ or $y_1 + z_2 > w_3$

Eliminating options we have assumed are not true we see that it is impossible to satisfy all of these conditions at once. Therefore, we may discard (c) entirely and assume that when the x-coordinates of our tetrahedron are $\{0, 0, 1, 1\}$ then it is equivalent to

$$T = \mathrm{conv}((0,0,0),(0,w_2,z_1),(1,0,w_3),(1,y_1,z_2))$$

for some integers z_1 and z_2 in $[0, w_3]$. We will now refine this general tetrahedron into a type 2, 3 or 4 tetrahedron considering the cases $y_1 = 0$ and $y_1 > 0$ separately.

If $y_1 = 0$ then the image of the vertices of T under $(-z_2, 0, 1)$ and $(-z_2, -1, 1)$ are

$$\{0, z_1, w_3 - z_2, 0\}, \quad \text{and} \quad \{0, z_1 - w_2, w_3 - z_2, 0\}$$

respectively. Neither of these can be a subset of $[0, w_3)$ as this would contradict the widths of T. Therefore, $z_2 = 0$ since otherwise we would need both $z_1 = w_3$ and $z_1 < w_2$ which is false. By a shear $(x, y, z) \mapsto (x, y, z - ky)$ and possibly the reflection in the plane $y = \frac{w_2}{2}$ we may assume $z_1 \leq (-z_1 \mod w_2)$ and so $0 \leq z_1 \leq \frac{w_2}{2}$. This shows that T is of the form 2.

If instead $y_1 > 0$ consider the images of the vertices of T under $(-z_2, 0, 1)$, $(-1, 1, 1)$ and $(-1, -1, 1)$ which are

$$\{0, z_1, w_3 - z_2, 0\}, \quad \{0, w_2 + z_1, w_3 - 1, y_1 + z_2 - 1\}, \quad \text{and} \quad \{0, z_1 - w_2, w_3 - 1, z_2 - y_1 - 1\}$$

respectively. Again, none of these can be a subset of $[0, w_3)$ so all three of the following points must be true. For each set we identify the sign of each entry and thus define a bound it would satisfy if it was not an element of $[0, w_3)$.

- $z_1 = w_3$ or $z_2 = 0$

- $w_2 + z_1 \geq w_3$ or $y_1 + z_2 - 1 \geq w_3$
- $z_1 < w_2$ or $z_2 < y_1 + 1$.

To satisfy these we must have either $z_1 = w_3$ and $z_2 \leq y_1$ or $z_2 = 0$ and $z_1 \geq w_3 - w_2$. The tetrahedron when $z_1 = w_3$ and $z_2 = y_1$ is equivalent but not equal to that when $z_2 = 0$ and $z_1 = w_3 - w_2$ by the shear $(x, y, z) \mapsto (x, y, z-y)$. Therefore, we may also assume that $z_2 < y_1$ to avoid duplicates. This proves that T is of the form 3 or 4. □

The following shows that the map taking a tetrahedron to its affine equivalence class gives a surjective map from $\mathcal{S}_{1,w_2,w_3}$ to $\mathcal{T}_{1,w_2,w_3}$.

Proposition 4.3.2. *Let T be a lattice tetrahedron with multi-width $(1, w_2, w_3)$. Then there exists some $T' \in \mathcal{S}_{1,w_2,w_3}$ which is equivalent to T.*

Proof. By Lemma 4.3.1 and the definition of $\mathcal{S}_{1,w_2,w_3}$, if $1 < w_2 < w_3$ then we are done.

Now we consider the special cases. If $w_2 = w_3$ and T is a tetrahedron of the form 3 then the image of T under the map $(x, y, z) \mapsto (1 - x, z_1 - z + x(w_2 - z_1), y)$ is

$$\operatorname{conv}((0, 0, 0), (0, w_2, y_1), (1, 0, w_2), (1, z_1, 0)).$$

If $y_1 > z_1$ this is also of the form 3 but with the roles of y_1 and z_1 swapped. Therefore, to remove duplicates we may assume that $y_1 \leq z_1$. If $w_2 = w_3$ and T is a tetrahedron of the form 4 then the image of T under the map $(x, y, z) \mapsto (x, w_2 - z, w_2 - y)$ is

$$\operatorname{conv}((0, 0, 0), (0, w_2, w_2), (1, 0, w_2), (1, w_2 - z_1, w_2 - y_1)).$$

This is also of the form 4 and the map is self inverse so unless T is equal to its image we need to eliminate one of these tetrahedra to remove duplicates. Our T is equal to its image only when $z_1 + y_1 = w_2$ so we may assume that $z_1 \leq w_2 - y_1$.

When $w_2 = 1$ substituting into tetrahedra 1-4 and simplifying reduces us to the following cases:

- $\operatorname{conv}((0,0,0),(0,0,w_3),(0,1,0),(1,0,0))$
- $\operatorname{conv}((0,0,0),(0,1,0),(1,0,0),(1,0,w_3))$
- $\operatorname{conv}((0,0,0),(0,1,z_1),(1,0,w_3),(1,1,0))$ where $z_1 = w_3 - 1$ or $z_1 = w_3$.

Under $(x,y,z) \mapsto (1-x-y,y,z)$ the second of these maps to the first so we are left with the three tetrahedra appearing in $\mathcal{S}_{1,1,w_3}$. Finally, when $w_3 = 1$ two of these are equivalent to the convex hull of the empty triangle embedded in the plane $x = 0$ and the point $(1,0,0)$ so it reduces to the two cases in $\mathcal{S}_{1,1,1}$. $\square$

The following proves that the map taking a tetrahedron to its affine equivalence class gives a map from $\mathcal{S}_{1,w_2,w_3}$ to $\mathcal{T}_{1,w_2,w_3}$.

Proposition 4.3.3. *Let $T \in \mathcal{S}_{1,w_2,w_3}$, then the multi-width of T is $(1,w_2,w_3)$.*

Proof. By definition, the tetrahedra in $\mathcal{S}_{1,w_2,w_3}$ are always of one of the forms 1-4. Therefore, it suffices to show that a tetrahedron satisfying one of these conditions has multi-width $(1,w_2,w_3)$ for any $w_3 \geq w_2 \geq 1$. Let T be in one of these forms. Lattice polytopes have integral widths and only have width zero in some direction if their dimension is less than that of the space they are in. Since $\operatorname{width}_{(1,0,0)}(T) = 1$ and T has non-zero volume its first width is 1. Furthermore, T has widths w_2 and w_3 realised by the dual vectors $(0,1,0)$ and $(0,0,1)$ respectively.

There are two ways in which the remaining two widths can fail. Either there is a dual vector u linearly independent to $(1,0,0)$ such that $\operatorname{width}_u(T) < w_2$ or there is a dual vector u linearly independent to $\{(1,0,0),(0,1,0)\}$ such that $\operatorname{width}_u(T) < w_3$. To prove these do not occur it suffices to show that for all $u = (u_x,u_y,0)$ with $u_y \neq 0$ $\operatorname{width}_u(T) \geq w_2$ and for all $u = (u_x,u_y,u_z)$ with $u_z \neq 0$ $\operatorname{width}_u(T) \geq w_3$.

Let π be the projection onto the first two coordinates then

$$\text{width}_{(u_x,u_y,0)}(T) = \text{width}_{(u_x,u_y)}(\pi(T)).$$

However, the four-point set which is the image of the vertices of T under π has multi-width $(1, w_2)$ by Corollary 4.2.4. The first width of this set is realised by $(1, 0)$ therefore, for all $u = (u_x, u_y, 0)$ with $u_y \neq 0$, we have $\text{width}_u(T) = \text{width}_{(u_x,u_y)}(\pi(T)) \geq w_2$.

Now suppose for contradiction there exists dual vector $u = (u_x, u_y, u_z)$ with $u_z \neq 0$ is such that $\text{width}_u(T) < w_3$. Without loss of generality we may assume that $u_z > 0$. Then by the proof of Proposition 4.1.1 a map of the form $(x, y, z) \mapsto (x, y, k_1x + k_2y + z)$ takes T to a subset of a $1 \times w_2 \times \text{width}_u(T)$ box for some integers k_1 and k_2. The image of T under this map is one of the following corresponding to the form of T.

1. $\text{conv}(\{0\} \times \left(\begin{smallmatrix} 1 & 0 \\ k_2 & 1 \end{smallmatrix}\right) t, (1, 0, k_1))$ where $t \in \mathcal{S}_{w_2,w_3}$
2. $\text{conv}((0, 0, 0), (0, w_2, z_1 + k_2w_2), (1, 0, k_1), (1, 0, w_3 + k_1))$ where $0 \leq z_1 \leq \frac{w_2}{2}$
3. $\text{conv}((0, 0, 0), (0, w_2, z_1 + k_2w_2), (1, 0, w_3 + k_1), (1, y_1, k_1 + k_2y_1))$ where $0 < y_1 \leq w_2$ and $w_3 - w_2 \leq z_1 \leq w_3$,
4. $\text{conv}((0, 0, 0), (0, w_2, w_3+k_2w_2), (1, 0, w_3+k_1), (1, y_1, z_1+k_1+k_2y_1))$ where $0 < y_1 < w_2$ and $0 < z_1 < y_1$.

We will show that it is impossible for any of these to have width less than w_3 with respect to $(0, 0, 1)$.

1. Let π be the projection onto the last two coordinates then for a polytope P we have

$$\text{width}_{(0,0,1)}(P) = \text{width}_{(0,1)}(\pi(P)).$$

This means that $\text{conv}(\{0\} \times \left(\begin{smallmatrix} 1 & 0 \\ k_2 & 1 \end{smallmatrix}\right) t, (1, 0, k_1))$ where $t \in \mathcal{S}_{w_2,w_3}$ never has width less than w_3 with respect to $(0, 0, 1)$ thanks to the widths of t.

2. The tetrahedron of the form 2 has width at least w_3 with respect to $(0,0,1)$ due to the vertices $(1,0,k_1)$ and $(1,0,w_3+k_1)$.

3. In a tetrahedron of the form 3 if the width with respect to $(0,0,1)$ was less than w_3 the difference between every pair of z-coordinates must be less than w_3. In particular we would need to have $w_3+k_1-k_1-k_2y_1<w_3$ and $z_1+k_2w_2<w_3$. The first of these implies that $k_2>0$ which combines with the second to show that $z_1+w_2<w_3$. However, $z_1 \geq w_3-w_2$ which is a contradiction.

4. Similarly, in a tetrahedron of the form 4 we would need $w_3+k_2w_2<w_3$ and $w_3+k_1-z_1-k_1-k_2y_1<w_3$. The first of these implies that $k_2<0$ and the second implies that $k_2>-z_1/y_1>-1$ which is a contradiction. □

In Proposition 4.3.5 we will show that if any two tetrahedra in $\mathcal{S}_{1,w_2,w_3}$ are affine equivalent then they are equal. To prove this we will need the following extra result about the tetrahedra whose x-coordinates are $\{0,0,1,1\}$.

Lemma 4.3.4. *Let $1<w_2\leq w_3$ and let T be a tetrahedron in $\mathcal{S}_{1,w_2,w_3}$ whose x-coordinates are $\{0,0,1,1\}$. Suppose the projection of T onto the first two coordinates is $\mathrm{conv}((0,0),(0,w_2),(1,0),(1,y_1))$. Then, for any surjective lattice homomorphism $\pi:\mathbb{Z}^3\to\mathbb{Z}^2$, if $\pi(T)$ is equivalent to $\mathrm{conv}((0,0),(0,w_2),(1,0),(1,y_1'))$ for some integer $y_1'\in[0,w_2]$ then $y_1'\geq y_1$. In other words, y_1 is minimal.*

Proof. For tetrahedra of the form 2 this is immediate since $y_1=0$.

Let T be of the form 3 or 4. Let $\pi:\mathbb{Z}^3\to\mathbb{Z}^2$ be a surjective lattice homomorphism such that $\pi(T)$ is equivalent to $\mathrm{conv}((0,0),(0,w_2),(1,0),(1,y_1'))$ for an integer $y_1'\in[0,w_2]$. Let $P=(p_{ij})$ be the 2×3 integral matrix defining π. By Proposition 3.1.3 there are dual vectors u_1 and $u_2\in(\mathbb{Z}^2)^*$ which form a basis of $(\mathbb{Z}^2)^*$ and which realise the first two widths of $\pi(T)$. This means that $\mathrm{width}_{u_i}(\pi(T))=\mathrm{width}_{u_i\circ\pi}(T)=w_i$ for $i=1$ and 2. Since $w_2>1$ the direction in which P has width 1 is unique so, possibly after changing the sign of u_1, we have $u_1P=(1,0,0)$. Let U be the matrix with rows u_1 and u_2 then replace P with UP and change π accordingly. In this way we can assume that the first row of P is $(1,0,0)$. This does not alter our previous assumptions about π since this operation is a unimodular map in $\mathbb{Z}^2$.

If $w_2 < w_3$ a vector realising the second width of P must be of the form $(u_x, u_y, 0)$ so the final entry of $u_2 P$ is 0. This allows us to assume $p_{23} = 0$. Now we have image

$$\pi(T) = \operatorname{conv}((0,0),(0,p_{22}w_2),(1,p_{21}),(1,p_{21}+p_{22}y_1)).$$

This has an edge of lattice length $|p_{22}w_2|$ so p_{22} must be 0 or ± 1. If $p_{22} = 0$ then $\pi(T)$ is just a line segment, contradicting our assumptions on π. If $p_{22} = \pm 1$ then $y_1' = |p_{22}y_1 + p_{21} - p_{21}| = y_1$ so $y_1' \geq y_1$ as desired.

It remains to consider the case $w_3 = w_2 > 1$. By replacing P with $\left(\begin{smallmatrix} 1 & 0 \\ -p_{21} & 1 \end{smallmatrix}\right) P$ we may assume $p_{21} = 0$. This leaves us with the following two cases corresponding to 3 and 4

$$\pi(T) = \operatorname{conv}((0,0),(0,p_{22}w_2 + p_{23}z_1),(1,p_{23}w_2),(1,p_{22}y_1) \quad \text{or}$$

$$\pi(T) = \operatorname{conv}((0,0),(0,p_{22}w_2 + p_{23}w_2),(1,p_{23}w_2),(1,p_{22}y_1 + p_{23}z_1)).$$

The dual vector $(1,0)$ is the unique dual vector realising the first width of both $\pi(T)$ and $\operatorname{conv}((0,0),(0,w_2),(1,0),(1,y_1'))$. These quadrilaterals each have (at most) two facets with normal vector $(1,0)$ so these facets must be equivalent. This means that one of the vertical edges of $\pi(T)$ must have length w_2 and the other must have length in $[0, w_2]$. Therefore, to complete the proof it suffices to show that for each choice of vertical edge to be length w_2 the other vertical edge must have length at least y_1.

First consider the quadrilateral associated to case 3. If $|p_{22}w_2 + p_{23}z_1| = w_2$ then, after a possible change of sign of the second line of P, we may assume that $p_{23} = w_2(1-p_{22})/z_1$. Then $|p_{23}w_2 - p_{22}y_1| = |w_2^2/z_1 - p_{22}(w_2^2/z_1 + y_1)|$. This is the modulus of a linear function in p_{22} so to find the smallest values it takes we notice that it is zero when $p_{22} = w_2^2/(w_2^2 + y_1 z_1) \in [0,1]$. Since p_{22} is an integer the length is smallest when either $p_{22} = 0$ or 1 which gives values w_2^2/z_1 and y_1 respectively. Both of these are at least y_1 given the conditions on a tetrahedron of the form 3 when $w_2 = w_3$.

On the other hand, if $|p_{23}w_2 - p_{22}y_1| = w_2$ then as above we may assume that $p_{22} = w_2(p_{23} - 1)/y_1$. Then $|p_{22}w_2 + p_{23}z_1| = |-w_2^2/y_1 + p_{23}(w_2^2/y_1 + z_1)|$ which is zero when $p_{23} = w_2^2/(w_2^2 + z_1y_1) \in [0, 1]$. Since p_{23} is an integer it is actually smallest at either w_2^2/z_1 or z_1 both of which are at least y_1 given our assumptions.

Now for the quadrilateral associated to case 4. If $|p_{22}w_2 + p_{23}w_2| = w_2$ then as above we may assume that $p_{22} + p_{23} = 1$. Then $|p_{23}(w_3 - z_1) - p_{22}y_1| = |p_{23}(w_2 - z_1 + y_1) - y_1|$ which is zero when $p_{23} = y_1/(w_2 - z_1 + y_1) \in [0, 1]$. Since p_{23} is an integer it is actually smallest at either $w_2 - z_1$ or y_1 both of which are at least y_1 given our assumptions.

On the other hand, if $|p_{23}(w_2 - z_1) - p_{22}y_1| = w_2$ then notice that $|p_{22}w_2 + p_{23}w_2| = |p_{22} + p_{23}|w_2$. Since these are all integers this is at least y_1 unless $p_{23} = -p_{22}$. However, then we would have $w_2 = |p_{23}|(w_2 - z_1 + y_1)$. We can't let both p_{22} and p_{23} be zero so then $w_2 \geq w_2 - z_1 + y_1 > w_2$ which is a contradiction. $\square$

The following shows injectivity of the map taking a tetrahedron to its affine equivalence class from $\mathcal{S}_{1,w_2,w_3}$ to $\mathcal{T}_{1,w_2,w_3}$.

Proposition 4.3.5. *Tetrahedra in $\mathcal{S}_{1,w_2,w_3}$ are distinct under affine maps.*

Proof. The two tetrahedra in $\mathcal{S}_{1,1,1}$ have normalised volumes 1 and 2. Since volume is an affine invariant they must be distinct. If $w_3 > 1$ the three tetrahedra in $\mathcal{S}_{1,1,w_3}$ have normalised volumes w_3, $2w_3 - 1$ and $2w_3$ so must also be distinct.

In the remaining cases our goal is to show that if T and T' are equivalent tetrahedra in $\mathcal{S}_{1,w_2,w_3}$ then either $T = T'$ or this leads to a contradiction. We have $w_2 > 1$ so up to sign $u_1 = (1, 0, 0)$ is the unique vector such that each tetrahedron has width 1 with respect to u_1. Let T and T' be equivalent tetrahedra in $\mathcal{S}_{1,w_2,w_3}$ then the image of the vertices of T and T' under u_1 must be equivalent. In other words, the set of x-coordinates of two equivalent tetrahedra in $\mathcal{S}_{1,w_2,w_3}$ is also equivalent. Therefore, tetrahedra of the form 1 are always distinct from the others. Furthermore, if T and T' are equivalent tetrahedra of the form 1 then they each have a unique facet with normal u_1, so these facets must be equivalent too. These facets were triangles in $\mathcal{S}_{w_2,w_3}$ so by Theorem 3.0.2 this means $T = T'$.

Now let T and T' be equivalent tetrahedra in $\mathcal{S}_{1,w_2,w_3}$ of the forms 2, 3 or 4. By Lemma 4.3.4 if we project the sets of vertices of T and T' onto the first two coordinates we get the same set. That is the x- and y-coordinates of T and T' are the same. This immediately tells us that tetrahedra of the form 2 are distinct from those of the forms 3 and 4. For the rest we will show the following four facts:

A. If both T and T' are of the form 2 then $T = T'$

B. If both T and T' are of the form 3 then $T = T'$

C. If both T and T' are of the form 4 then $T = T'$

D. If T is of the form 3 and T' is of the form 4 then we get a contradiction.

A. Unless $w_2 = w_3$ and $z_1 = z_1' = 0$ the only edge of T or T' with lattice length w_3 is the one from $(1, 0, 0)$ to $(1, 0, w_3)$. Therefore, either $T = T'$ or the affine map taking T to T' preserves this edge. There are only four ways to map vertices of T to T' satisfying this. Define the map θ by $(x, y, z) \mapsto (x, y, z, 1)$ and let $\pi : \mathbb{Z}^4 \to \mathbb{Z}^3$ be the projection onto the first three coordinates. Affine maps in $\mathbb{Z}^3$ are exactly the maps $(x, y, z) \mapsto \pi(\theta(x, y, z)U)$ where U is a unimodular matrix with last column $(0, 0, 0, 1)^T$. Let M and M' be the matrices whose rows are the vertices of $\theta(T)$ and $\theta(T')$ respectively. Let σM denote the matrix obtained by permuting the rows of M according to σ. At least one of $M^{-1}M'$, $((12)M)^{-1}M'$, $((34)M)^{-1}M'$ or $((12)(34)M)^{-1}M'$ is unimodular.

These matrices are

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & (z_1' - z_1)/w_2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ -w_2 & -1 & 0 & w_2 \\ -z_1' & -(z_1 + z_1')/w_2 & 1 & z_1' \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ w_3 & (z_1 + z_1')/w_2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ -w_2 & -1 & 0 & w_2 \\ w_3 - z_1' & (z_1 - z_1')/w_2 & -1 & z_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Therefore, either $(z_1 - z_1')/w_2$ or $(z_1 + z_1')/w_2$ is an integer. In either case, the fact that $0 \leq z_1, z_1' \leq \frac{w_2}{2}$ forces $T = T'$.

B. The normalised volume of T and T' is $w_2w_3 + z_1y_1 = w_2w_3 + z_1'y_1$, therefore $z_1 = z_1'$ and $T = T'$.

C. The normalised volume of T and T' is $w_2w_3 - w_2z_1 + y_1w_3 = w_2w_3 - w_2z_1' + y_1w_3$, therefore $z_1 = z_1'$ and $T = T'$.

D. Let P and P' be the parallelograms obtained by intersecting $2T$ and $2T'$ with the plane $x = 1$. These are:

$$P = \operatorname{conv}((0, w_3), (y_1, 0), (w_2, w_3 + z_1), (w_2 + y_1, z_1))$$

$$P' = \operatorname{conv}((0, w_3), (y_1, z_1'), (w_2, 2w_3), (w_2 + y_1, w_3 + z_1'))$$

Since T and T' are equivalent so are P and P'. We will show that this leads to a contradiction. By the symmetries of a parallelogram, $P - (0, w_3)$ must be unimodular equivalent to either $P' - (0, w_3)$ or $P' - (y_1, z_1')$. Consider three matrices M, M_1 and M_2 whose rows are the following vertices of $P - (0, w_3)$, $P' - (0, w_3)$ and $P' - (y_1, z_1')$ adjacent to the origin:

$$M = \begin{pmatrix} y_1 & -w_3 \\ w_2 & z_1 \end{pmatrix}, \quad M_1 = \begin{pmatrix} y_1 & z_1' - w_3 \\ w_2 & w_3 \end{pmatrix}, \quad M_2 = \begin{pmatrix} -y_1 & w_3 - z_1' \\ w_2 & w_3 \end{pmatrix}.$$

One of $M_1^{-1}M, ((12)M_1)^{-1}M, M_2^{-1}M$ and $((12)M_2)^{-1}M$ must be a unimodular matrix. The necessary inverses are

$$M_1^{-1} = \frac{1}{V}\begin{pmatrix} w_3 & w_3 - z_1' \\ -w_2 & y_1 \end{pmatrix}, \quad M_2^{-1} = \frac{1}{V}\begin{pmatrix} -w_3 & w_3 - z_1' \\ w_2 & y_1 \end{pmatrix}$$
$$((12)M_1)^{-1} = \frac{1}{V}\begin{pmatrix} w_3 - z_1' & w_3 \\ y_1 & -w_2 \end{pmatrix}, \quad ((12)M_2)^{-1} = \frac{1}{V}\begin{pmatrix} w_3 - z_1' & -w_3 \\ y_1 & w_2 \end{pmatrix}$$

where $V = w_2w_3 - w_2z_1' + w_3y_1$. Using these we know that either

$$U = M_1^{-1}M = \frac{1}{V}\begin{pmatrix} w_2w_3 - w_2z_1' + w_3y_1 & -w_3^2 + w_3z_1 - z_1z_1' \\ 0 & w_2w_3 + z_1y_1 \end{pmatrix}$$

is a unimodular matrix or one of the following is an integer

$$\frac{w_2w_3 - y_1z_1' + w_3y_1}{V}, \quad \frac{z_1y_1 - w_2w_3}{V}, \quad \frac{-w_2w_3 + w_3y_1 - z_1'y_1}{V}.$$

These are entries $(1,1)$, $(2,2)$ and $(1,1)$ of $((12)M_1)^{-1}M, M_2^{-1}M$ and $((12)M_2)^{-1}M$ respectively. Since the diagonal entries of U are positive they must both be 1 for U to be unimodular. From this notice that

$$\begin{pmatrix} y_1 & z_1' - w_3 \\ w_2 & w_3 \end{pmatrix}\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} = M_1U = M = \begin{pmatrix} y_1 & -w_3 \\ w_2 & z_1 \end{pmatrix}$$

for some integer a. From this we show that $z_1' - w_3 - ay_1 = -w_3$ and $w_3 - aw_2 = z_1$. Since $w_3 - w_2 \leq z_1 \leq w_3$ either $a = 0$ or 1. Therefore, $z_1' = 0$ or y_1 both of which are contradictory so U cannot be unimodular.

Notice that $-z_1'y_1 > -w_2z_1'$ so the first of the fractions is at least 2. From this we show that $2w_2z_1' \geq w_2w_3 + w_3y_1 + z_1'y_1$ which is a contradiction since w_2w_3 and w_3y_1 are both greater than w_2z_1' and $z_1'y_1$ is non-negative. Since $y_1 < w_2$ and $z_1 \leq w_3$ the second fraction is at most -1 from which we show $w_2z_1' \geq z_1y_1 + w_3y_1$. This is contradictory since $w_3 \geq w_2$ and $y_1 > z_1'$. Finally, since $y_1 < w_2$ the third fraction is also at most -1 so $z_1'(w_2 + y_1) \geq 2w_3y_1$. This is a contradiction since $z_1' < y_1$ and $w_2 + y_1 < 2w_2 \leq 2w_3$. □

We now bring together the above results to prove Theorem 4.0.2.

Proof of Theorem 4.0.2. Proposition 4.3.3 shows that the map taking a tetrahedron to its equivalence class is a well-defined map from $\mathcal{S}_{1,w_2,w_3}$ to $\mathcal{T}_{1,w_2,w_3}$. Propositions 4.3.2 and 4.3.5 show that it is bijective. It remains to find the cardinality of $\mathcal{S}_{1,w_2,w_3}$. When $w_2 = 1$ it is immediate. When $w_2 > 1$ we combine the triangles classification with the new tetrahedra to get the desired counts. □

The generating function of the sequence counting lattice triangles with second width w_2 was the Hilbert series of a hypersurface in a weighted projective space so we investigate the generating function of $|\mathcal{T}_{1,w_2,w_3}|$.

Corollary 4.3.6. *The generating function of* $|\mathcal{T}_{1,w_2,w_3}|$ *is*

$$\sum_{w_2=1}^{\infty}\sum_{w_3=w_2}^{\infty} t^{w_2}s^{w_3}|\mathcal{T}_{1,w_2,w_3}| = \frac{f(s,t)}{2(1-s^2)(1-ts)^3(1+ts)}$$

where $f(s,t)$ *is the polynomial*

$$t^5s^7 + 5t^5s^6 + 4t^5s^5 - 2t^4s^7 - 5t^4s^6 - 9t^4s^5 - 4t^4s^4 + 4t^3s^6 + 13t^3s^5 + 7t^3s^4$$
$$- 6t^3s^3 - 5t^2s^4 - 3t^2s^3 + 4t^2s^2 - 4ts^4 - 10ts^3 - 4ts^2 + 2ts + 2s^3 + 14s^2 + 20s + 8.$$

Sketch of proof. We use Theorem 4.0.2. First note that $\sum_{w_3=w_2}^{\infty} s^{w_3}|\mathcal{T}_{1,w_2,w_3}|$ is

$$\frac{s(s+2)}{1-s}$$

when $w_2 = 1$ and

$$s^{w_2}\frac{w_2^2(s+1)^2 + w_2(1-s^2) + (\frac{3}{2}s^2 + \frac{5}{2}s + \frac{3}{2}) + \frac{1}{2}(-1)^{w_2}(s^2+s+1)}{1-s^2}.$$

otherwise. Combining these we get the desired result. This can all be done by hand using facts about the generating functions of polynomials or with the assistance of computer algebra. □

To instead count lattice tetrahedra with first width 1 and third width w_3 we let $t = 1$ in the above generating function resulting in

$$\frac{-s^7 + 4s^6 + 8s^5 - 6s^4 - 3s^3 + 22s + 8}{2(1-s)^4(1+s)^2}.$$

Neither of these generating functions share any of the properties of the one from triangles. However, this does not prevent a function counting lattice tetrahedra of a given multi-width in general from doing so.

4.4 Computational and Conjectural Results

The above classification method can be extended into a computer algorithm which classifies four-point sets and tetrahedra of a given multi-width. We implement all algorithms using MAGMA V2.27. The code and data produced can be found at [Ham24b] and [Ham23b] respectively. Algorithm 1 classifies four-point sets of multi-width (w_1, w_2). It takes the list of four-point sets in the line of width w_1 and assigns a y-coordinate in the range $[0, w_2]$ to each point of each set in every possible way. The resulting sets in the plane include all four-point sets of multi-width (w_1, w_2). We eliminate any which do not have the correct widths. Let P be the convex hull of such a set then we use the polytope $\mathcal{W}_P$ to check its multi-width. The i-th width of P is w if and only if the dimension of $\mathrm{conv}((w-1)\mathcal{W}_P \cap \mathbb{Z}^d)$ is less than i and the dimension of $\mathrm{conv}(w\mathcal{W}_P \cap \mathbb{Z}^d)$ is at least i. This allows us to check if the multi-width of a polytope is equal to (w_1, w_2) without necessarily calculating its multi-width. We also discard repeated sets using an affine unimodular normal form. Kreuzer and Skarke introduced a unimodular normal form for lattice polytopes in their PALP software [KS04]. This can be extended to an affine normal form by translating each vertex of a polytope to the origin in turn and finding the minimum unimodular normal form among these possibilities. If the convex hull of a four-point set is a quadrilateral we can use this normal form without adjustment. If the convex hull is a triangle we find the normal form of this triangle then consider the possible places the fourth point can be mapped to in this normal form. We choose the minimum such point and call the set of vertices of the triangle and this point the normal form of the four-point set denoted $\mathrm{NF}(S)$. Note that we keep the normal forms of each set as well as the set itself as we need the four-point sets written as a subset of $[0, w_1] \times [0, w_2]$ for the tetrahedra classification.

Algorithm 1: Classifying the four-point sets in $\mathbb{Z}^2$ with multi-width (w_1, w_2).

Data: The set $\mathcal{P}$ of all lattice points in $Q_{w_1} = \mathrm{conv}((0,0), (0, w_1), (\frac{w_1}{2}, \frac{w_1}{2})$.

Result: The set $\mathcal{A}$ containing all four-point sets in the plane with multi-width (w_1, w_2) written as a subset of $[0, w_1] \times [0, w_2]$.

$\mathcal{A} \longleftarrow \emptyset$
NormalForms $\longleftarrow \emptyset$
for $(x_1, x_2) \in \mathcal{P}$ **do**
  for $h_1, h_2, h_3, h_4 \in [0, w_2] \cap \mathbb{Z}$ *such that* $h_i = 0$ *and* $h_j = w_2$ *for some* $i < j$ **do**
    $S \longleftarrow \{(0, h_1), (x_2, h_2), (x_3, h_3), (w_1, h_4)\}$
    if $\mathrm{mwidth}(S) = (w_1, w_2)$ *and* $\mathrm{NF}(S) \notin$ NormalForms **then**
      $\mathcal{A} \longleftarrow \mathcal{A} \cup \{S\}$
      NormalForms $\longleftarrow$ NormalForms $\cup \{\mathrm{NF}(S)\}$

Running Algorithm 1 for small widths produces Table 4.3. We use this data to give estimates for a function counting the width (w_1, w_2) four-point sets in the plane. To decide how much data to compute we use the following.

Proposition 4.4.1. *There are at most* $(\frac{w_1^2}{4} + w_1 + c)(6w_2^2 + 1)$ *four-point sets in the plane with multi-width* (w_1, w_2) *where* $c = 1$ *if* w_1 *is even and* $c = \frac{3}{4}$ *if* w_1 *is odd.*

Proof. By Proposition 4.2.1 we know that there are

$$\begin{cases} \frac{w_1^2}{4} + w_1 + 1 & \text{if } w_1 \text{ even} \\ \frac{w_1^2}{4} + w_1 + \frac{3}{4} & \text{if } w_1 \text{ odd} \end{cases}$$

four-point sets in the line with width w_1. Let $(y_1, \ldots, y_4) \in [0, w_2]^4$ be a lattice point representing the y-coordinates we give to each point. We know that there exist indices i_0 and i_1 such that $y_{i_0} = 0$ and $y_{i_1} = w_2$. By a reflection we may assume that $i_0 < i_1$. We also assume these are as small as possible. Counting the possibilities in each of the six cases we show that there are at most $6w_2^2 + 1$ ways to assign y-coordinates to a four-point set in the line. □

	w_2											
w_1	1	2	3	4	5	6	7	8	9	10	11	12
1	2	5	6	8	9	11	12	14	15	17	18	20
2	0	13	31	42	49	60	67	78	85	96	103	114
3	0	0	39	101	123	148	170	195	217	242	264	289
4	0	0	0	114	282	342	394	454	506	566	618	678
5	0	0	0	0	254	624	727	835	938	1046	1149	1257
6	0	0	0	0	0	520	1239	1428	1605	1794	1971	2160
7	0	0	0	0	0	0	937	2206	2490	2781	3065	3356
8	0	0	0	0	0	0	0	1595	3682	4120	4542	4980
9	0	0	0	0	0	0	0	0	2527	5775	6380	6994
10	0	0	0	0	0	0	0	0	0	3851	8687	9534
11	0	0	0	0	0	0	0	0	0	0	5610	12555
12	0	0	0	0	0	0	0	0	0	0	0	7949

Table 4.3: The number of four-point sets with multi-width (w_1, w_2) up to affine equivalence.

A *quasi-polynomial* is a polynomial whose coefficients are periodic functions with integral period. By Theorems 3.0.2 and 4.0.2 the functions counting lattice triangles and width 1 lattice tetrahedra are piecewise quasi-polynomials whose coefficients have period 2 so we may expect a function counting four-point sets in the plane to be similar. By Proposition 4.4.1, if there is a quasi-polynomial counting four-point sets of multi-width (w_1, w_2) we expect it to be at most quadratic in w_2. We expect the case when $w_1 = w_2$ to be distinct due to the increased symmetry. Also we expect the cases when w_2 is odd and even to be distinct so consider them separately. By fitting a quadratic to the results for $(w_1, w_1+1), \ldots, (w_1, w_1+5)$ and $(w_1, w_1+2), \ldots, (w_1, w_1+6)$ we obtain the following conjecture which agrees with the entries of Table 4.3.

Conjecture 4.4.2. *The number of four-point sets of multi-width (w_1, w_2) if $w_1 < w_2$ is*

$$\begin{cases} 9w_2 + 6 & \text{if } w_2 \text{ even} \\ 9w_2 + 4 & \text{if } w_2 \text{ odd} \end{cases}$$

when $w_1 = 2$,

$$\begin{cases} \frac{47}{2}w_2 + 7 & \text{if } w_2 \text{ even} \\ \frac{47}{2}w_2 + \frac{11}{2} & \text{if } w_2 \text{ odd} \end{cases}$$

when $w_1 = 3$,

$$\begin{cases} 56w_2 + 6 & \text{if } w_2 \text{ even} \\ 56w_2 + 2 & \text{if } w_2 \text{ odd} \end{cases}$$

when $w_1 = 4$,

$$\begin{cases} \frac{211}{2}w_2 - 9 & \text{if } w_2 \text{ even} \\ \frac{211}{2}w_2 - \frac{23}{2} & \text{if } w_2 \text{ odd} \end{cases}$$

when $w_1 = 5$

$$\begin{cases} 183w_2 - 36 & \text{if } w_2 \text{ even} \\ 183w_2 - 42 & \text{if } w_2 \text{ odd} \end{cases}$$

when $w_1 = 6$

$$\begin{cases} \frac{575}{2}w_2 - 94 & \text{if } w_2 \text{ even} \\ \frac{575}{2}w_2 - \frac{195}{2} & \text{if } w_2 \text{ odd} \end{cases}$$

when $w_1 = 7$ and

$$\begin{cases} 430w_2 - 180 & \text{if } w_2 \text{ even} \\ 430w_2 - 188 & \text{if } w_2 \text{ odd} \end{cases}$$

when $w_1 = 8$.

It is tempting to fit cubics in w_1 to the coefficients of these polynomials to get a quasi-polynomial counting four-point sets whenever $w_1 < w_2$. However, the result is

$$\begin{cases} (\frac{5}{6}w_1^3 + \frac{1}{6}w_1 + 2)w_2 - \frac{5}{4}w_1^3 + \frac{39}{4}w_1^2 - \frac{47}{2}w_1 + 24 & \text{if } w_1, w_2 \text{ even} \\ (\frac{5}{6}w_1^3 + \frac{1}{6}w_1 + 2)w_2 - \frac{5}{4}w_1^3 + \frac{39}{4}w_1^2 - \frac{49}{2}w_1 + 24 & \text{if } w_1 \text{ even and } w_2 \text{ odd} \\ (\frac{5}{6}w_1^3 + \frac{1}{6}w_1 + \frac{1}{2})w_2 - w_1^3 + \frac{51}{8}w_1^2 - 10w_1 + \frac{53}{8} & \text{if } w_1 \text{ odd and } w_2 \text{ even} \\ (\frac{5}{6}w_1^3 + \frac{1}{6}w_1 + \frac{1}{2})w_2 - w_1^3 + \frac{51}{8}w_1^2 - \frac{21}{2}w_1 + \frac{53}{8} & \text{if } w_1, w_2 \text{ odd} \end{cases}$$

which disagrees with Table 4.3 whenever $w_1 \geq 9$. This suggests that either there is no such quasi-polynomial or that for small values of w_1 we have special cases and so cannot predict it from this data. Taking successive differences of a sequence can help to identify when it is given by a quasi-polynomial since higher order terms cancel making the pattern more obvious. Considering successive differences (and successive differences of these differences etc.) of the sequence counting multi-width $(w_1, w_1 + 1)$ four-point sets, it seems that if such a quasi-polynomial exists we would need significantly more data-points to estimate it. Therefore, we do not attempt to classify enough four-point sets to make such a conjecture.

Using the classification of four-point sets, we move on to classify tetrahedra. This uses a similar algorithm to the four-point set case (see Algorithm 2) with two main differences. We may no longer assume that all the tetrahedra we want to classify are contained in a $w_1 \times w_2 \times w_3$ box so must allow more z-coordinates to be assigned to each point. Also, since we are not extending this classification to a higher dimension, we need only store the normal form of each tetrahedron in order to count them.

Based on Theorem 4.0.2 we may hope that the tetrahedra of multi-width $(2, w_2, w_3)$ are counted by some quadratic functions in w_2. Since we can fit a quadratic to any three points we would like at least 4 points in each subsequence of $|\mathcal{T}_{2,w_2,w_3}|$ to make a reasonable conjecture. Including the case

Algorithm 2: Classifying the tetrahedra with multi-width (w_1, w_2, w_3).

Data: The set $\mathcal{A}$ containing all four-point sets in the plane with multi-width (w_1, w_2) written as a subset of $[0, w_1] \times [0, w_2]$.
Result: The set $\mathcal{T}$ containing all tetrahedra with multi-width (w_1, w_2, w_3).

```
T ⟵ ∅
for {v_1, v_2, v_3, v_4} ∈ A do
    for h_1, h_2, h_3, h_4 ∈ [0, max{w_1 + w_2, w_3}] ∩ Z such that h_i = 0 and
      h_j ≥ w_3 for some i < j do
        T ⟵ conv(v_i × {h_i} : i = 1, 2, 3, 4)
        if mwidth(T) = (w_1, w_2, w_3) then
            T ⟵ T ∪ {NF(T)}
```

$w_2 = 2$ makes the resulting polynomials higher degree so we need to classify at least multi-width $(2, w_2, w_2)$, $(2, w_2, w_2 + 1)$ and $(2, w_2, w_2 + 2)$ tetrahedra for $w_2 = 3, \ldots, 10$ to get enough data points. The classification of lattice tetrahedron with width 2, second width up to 10 and third width up to 12 can be found in the database [Ham23b] and they are counted in Table 4.2. Fitting polynomials to the sequences displayed in Table 4.2 produces the following.

Conjecture 4.4.3.

- *When $w_3 > w_2 > 2$ the cardinality of $\mathcal{T}_{2,w_2,w_3}$ is*
  - $\frac{1}{4}(81w_2^2 - 18w_2 + 76)$ *if w_2 and w_3 even*
  - $\frac{1}{4}(81w_2^2 - 18w_2 + 56)$ *if w_2 even and w_3 odd*
  - $\frac{1}{4}(81w_2^2 - 18w_2 + 37)$ *if w_2 odd and w_3 even*
  - $\frac{1}{4}(81w_2^2 - 18w_2 + 25)$ *if w_2 and w_3 odd*
- *when $w_2 > 2$ the cardinality of $\mathcal{T}_{2,w_2,w_2}$ is*
  - $\frac{1}{8}(81w_2^2 - 22w_2 + 80)$ *if w_2 is even*
  - $\frac{1}{8}(81w_2^2 - 20w_2 + 27)$ *if w_2 is odd*
- *when $w_3 > 2$ the cardinality of $\mathcal{T}_{2,2,w_3}$ is*
  - 47 *if w_3 is even*
  - 45 *if w_3 is odd*

- *and the cardinality of* $\mathcal{T}_{2,2,2}$ *is* 17.

More generally we may also guess that the following pattern will continue to hold.

Conjecture 4.4.4. *There is a piecewise quasi-polynomial with 4 components counting lattice tetrahedra of multi-width* (w_1, w_2, w_3). *There is a component for each combination of equalities in* $w_3 \geq w_2 \geq w_1 > 0$. *The leading coefficient in the case* $w_3 > w_2 > w_1 > 0$ *is double the leading coefficient in the case* $w_3 = w_2 > w_1 > 0$. *For fixed* w_1 *and* w_2 *there are at most three values which* $|\mathcal{T}_{w_1,w_2,w_3}|$ *can take depending of whether* w_3 *is odd, even or equal to* w_2.

Chapter 5

Half-Integral Polygons With Small Size

We call the number of lattice points contained in a rational polygon its *size*. In this chapter we classify polygons with small size and denominator up to affine equivalence and use the resulting classification as a starting point to investigate the Ehrhart theory of denominator 2 polygons.

Denominator 2 polygons are also of interest outside of Ehrhart theory and have applications in algebraic geometry. For example, in [BH20] denominator 2 polygons whose only lattice points are vertices are classified to understand canonical three-fold singularities with a complexity one torus action. They also appear as the intersection of a 3-dimensional width 2 polytope with a hyperplane, which appears in [AKW17] and which forthcoming work of Bohnert will apply to toric hypersurfaces [Boh22].

Consider polygons with denominator $r \in \mathbb{Z}_{>0}$ and size $k \in \mathbb{Z}_{\geq 0}$. When $r \geq 2$ there are infinitely many such polygons. For example, when a and k are positive integers the polygon $\operatorname{conv}((0,0),(k-1,0),(0,\frac{1}{r}),(\frac{a}{r},\frac{1}{r}))$ has denominator r and size k (see Figure 5.1). However, all but finitely many denominator r size k polygons are part of an infinite family in the following sense.

Definition 5.0.1. A polygon P with denominator r and size k is *infinitely growable* if there exists an infinite sequence of polygons $P_0, P_1, P_2, \ldots$ such that

1. $P_0 = P$,

2. P_i is a proper subset of P_{i+1} for all $i = 0, 1, 2, \ldots$ and

3. P_i is a polygon with denominator r and size k for all $i = 0, 1, 2, \ldots$.

A polygon Q with denominator r and size k is said to be *maximal* if for any point $v \in \frac{1}{r}\mathbb{Z}^2$ not in Q the size of $\mathrm{conv}(Q \cup v)$ is greater than k. We say that a polygon P with denominator r and size k is *finitely growable* if it is not infinitely growable. Notice that a finitely growable polygon is always a subset of some maximal polygon.

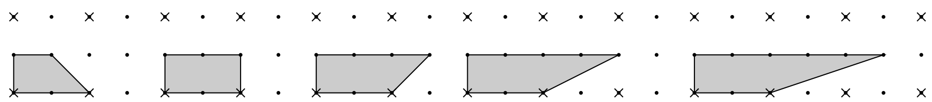


Figure 5.1: *A collection of polygons with denominator 2 and size 2 which can be grown infinitely. Crosses denote points of $\mathbb{Z}^2$, dots denote points of $\frac{1}{2}\mathbb{Z}^2$.*

We prove the following result in Corollary 5.1.5 and Proposition 5.2.6.

Proposition 5.0.2. *Let P be a denominator r, size k polygon, then either P is infinitely growable and is equivalent to a polygon contained in the strip $[0, 1] \times \mathbb{R}$ or P is finitely growable and is one of finitely many exceptions.*

Since the finitely growable denominator r size k polygons are finite in number they are a potential subject for classification. We classify them for small r and k using a growing algorithm. They are enumerated in Table 5.1.

	k					
r	0	1	2	3	4	5
1	0	0	0	1	3	6
2	1	106	1333	8774	40139	?
3	214	?	?	?	?	?

Table 5.1: The number of finitely growable polygons with denominator r and size k.

Using the data obtained from the classification we investigate the Ehrhart theory of denominator 2 polygons. This topic was previously studied in [Her10]. There they approach the classification of the odd components of the Ehrhart quasi-polynomial, whereas we approach the classification of the entire Ehrhart polynomial. There is also interest in *quasi-period collapse*, where the quasi-period of a polytope is strictly smaller than it's denominator. Denominator 2 polygons which exhibit quasi-period collapse have been studied in [MM17] and [Boh24].

The Ehrhart polynomial of a denominator 2 polygon is determined by the number of boundary and interior points in P and $2P$. Therefore, the following result, proven in Section 5.4, places strong restrictions on the possible Ehrhart polynomials of denominator 2 polygons.

Theorem 5.0.3. *Let P be a denominator 2 polygon with b_1 boundary points, i_1 interior points such that the lattice polygon $2P$ has b_2 boundary points and i_2 interior points. For all but finitely many such polygons, the integers b_1, i_1, b_2 and i_2 satisfy one of the following conditions:*

- $i_1 = 0$, $i_2 = 0$ *and* $b_2 \geq \max(3, 2b_1)$,
- $b_1 = 0$, $i_1 = 0$, $b_2 = 4$ *and* $i_2 > 0$,
- $i_1 = 0$, $i_2, b_1 > 0$, $\max(3, 2b_1) \leq b_2 \leq 2b_1 + 4$ *and* $b_2 \leq 2i_2 + 6$ *or*
- $i_1 > 0$, $b_2 \geq \max\{3, 2b_1\}$, $i_2 \geq b_1 + 2i_1 - 1$ *and* $b_2 + i_2 \leq 2b_1 + 6i_1 + 7$.

We continue to use the notation of Section 3.4, so $i(P)$ and $b(P)$ are the number of interior and boundary points of P. The set of tuples (b_1, i_1, b_2, i_2) satisfying the bounds in Theorem 5.0.3 appears to be a close approximation to the set of tuples $(b(P), i(P), b(2P), i(2P))$ for denominator 2 polygons P. When $i(P) = 0$ there are exactly 2 polygons which do not satisfy these bounds, and all tuples which satisfy the bounds are realised by some polygon. We have found no polygon with $i(P) > 0$ which does not satisfy the bounds. Additionally, when the size $k = b_1 + i_1$ is at most 4 and $i(P) > 0$, of the 1201 points satisfying the above inequalities, 968 are realised by some polygon.

In Section 5.1 we discuss infinitely growable polygons in greater generality and show that they are all equivalent to a subset of the strip $[0, 1] \times \mathbb{R}$. In

Section 5.2 we present an algorithm for finding all the finitely growable polygons. This depends on finding a list of 'minimal' polygons and growing them by successively adding points. In Section 5.3 we classify the minimal, denominator r, size k polygons in general. This includes recalling a classification method for lattice polygons of fixed size. Finally, in Section 5.4 we look at the Ehrhart theory of the polygons in our classification and prove Theorem 5.0.3. This includes a complete classification of the tuples $(b(P), i(P), b(2P), i(2P))$ when P has no interior points.

5.1 Infinitely Growable Polygons

In this section we show that a rational polygon is infinitely growable if and only if it is equivalent to a subset of the strip $[0,1] \times \mathbb{R}$. We work in greater generality; instead of $\frac{1}{r}\mathbb{Z}^2$ and $\mathbb{Z}^2$ we consider a lattice L and sublattice $K \subseteq L$ both of rank d. For a rank d lattice K we denote $K \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^d$ by $K_{\mathbb{R}}$. We think of both K and L as sublattices of the Euclidean space $K_{\mathbb{R}} = \mathbb{R}^d$ where K is the integral lattice $\mathbb{Z}^d$ and L is a bigger lattice extended by rational points.

For fixed lattices L and K recall that there exists a basis $\ell_1, \dots, \ell_d$ of L and unique positive integers $a_1, \dots, a_d$ with $a_i \mid a_{i+1}$ for $i = 1, \dots, d-1$ such that $\kappa_1 := a_1\ell_1, \dots, \kappa_d := a_d\ell_d$ is a basis of K. The a_i are usually referred to as *elementary divisors* (or *invariant factors*) of the sublattice $K \subseteq L$. Notice that $a_i \geq 1$ for all $i = 1, \dots, d$ as K has full rank. Furthermore, observe that we have a chain of sublattices $K \subseteq L \subseteq \frac{1}{a_d}K$ where $\frac{1}{a_d}K$ is the sublattice of $L_{\mathbb{R}}$ generated by the basis $\frac{1}{a_d}\ell_1, \dots, \frac{1}{a_d}\ell_d$.

Let us recall some important concepts from [NZ11]. In [NZ11] only one lattice is considered. Since we consider multiple lattices here, we need notation to keep track of the respective lattice.

Definition 5.1.1. Let N be a lattice and P a polytope in $N_{\mathbb{R}}$. We call P an *N-polytope* if the vertices of P are contained in N. We say that P is *N-hollow* if it doesn't contain any points of N in its interior. For $s \in \mathbb{Z}_{>0}$, we say a convex body $P \subseteq N_{\mathbb{R}}$ is *s-N-hollow* if the interior of P doesn't contain any points from the dilated lattice sN. We say that P is *of N-size k* if $|P \cap N| = k$.

The definition of infinitely growable and finitely growable extend naturally to this setting by considering L-polytopes instead of polytopes with denominator r and K-size instead of size.

A *lattice projection* is a surjective affine-linear map $\varphi : K_{\mathbb{R}} \to V$ onto a vector space V of dimension m whose kernel is generated, as a vector space, by elements of K. We get a chain of sublattices

$$\varphi(K) \subseteq \varphi(L) \subseteq \varphi\left(\tfrac{1}{a_d}K\right) = \tfrac{1}{a_d}\varphi(K).$$

We adapt the following result from [NZ11, Theorem 2.1].

Proposition 5.1.2. *There are only finitely many K-hollow d-dimensional L-polytopes (up to affine equivalence) that do not admit a lattice projection φ to a $(d-1)$-dimensional $\varphi(K)$-hollow polytope.*

Proof. Any L-polytope is also a $\frac{1}{a_d}K$-polytope, so it suffices to show the result in the case where $L = \frac{1}{a_d}K$. By replacing K-hollow with a_d-$\frac{1}{a_d}K$-hollow and $\varphi(K)$-hollow with a_d-$\frac{1}{a_d}\varphi(K)$-hollow this is exactly [NZ11, Theorem 2.1]. □

Proposition 5.1.3. *An infinitely growable L-polytope $P \subseteq L_{\mathbb{R}}$ is K-hollow and it admits a lattice projection φ to a $(d-1)$-dimensional $\varphi(K)$-hollow polytope.*

Towards a proof of Proposition 5.1.3, recall that the Euclidean space $\mathbb{R}^d$ can be equipped with the maximum norm which associates to $x \in \mathbb{R}^d$ the maximal absolute value of its coordinates, i.e., $\|x\|_\infty = \max_i |x_i|$. We will use the following well-known result (see, for instance, [Sch91, Theorem 1B]).

Theorem 5.1.4 (Dirichlet's approximation theorem)**.** *For every $w \in \mathbb{R}^d$ and every $N > 1$ there exist $k \in \mathbb{Z}$ with $1 \leq k \leq N^d$ and $x \in \mathbb{Z}^d$ such that $\|kw - x\|_\infty < 1/N$. In other words, for every ray in $\mathbb{R}^d$ there exist lattice points that lie arbitrarily close to it.*

The *recession cone* (or *tail cone*) of a non-empty closed convex set $C \subseteq \mathbb{R}^d$ is defined as $\operatorname{rec}(C) = \{u \in \mathbb{R}^d \mid u + C \subseteq C\}$.

Proof of Proposition 5.1.3. Assume towards a contradiction that there exists $v \in \operatorname{int}(P) \cap K$. Since P can be grown infinitely, there exists a strictly increasing sequence $(P_i)_{i\in\mathbb{N}}$ of L-polytopes that have K-size $|P_i \cap K| = |P \cap K|$ such that $P \subseteq P_i$ and $P_i \subsetneq P_{i+1}$. The closure of the union of the P_i, that is

$$C := \overline{\bigcup_i P_i},$$

is an unbounded, closed, convex set. Hence, its recession cone is non-trivial, i.e. there exists a non-zero point w in $\operatorname{rec}(C)$. By Theorem 5.1.4, there exist lattice points in K that lie arbitrarily close to the ray $\mathbb{R}_{\geq 0}w$. Since $v \in K$, there also exist lattice points in K that lie arbitrarily close to the affine ray $\rho := v + \mathbb{R}_{\geq 0}w$, which is in the interior of C. Notice that either ρ contains a second lattice point of K or v is the only lattice point of K that is contained in ρ. In the first case, ρ contains infinitely many lattice points. In the second case, we can choose a lattice point close enough to ρ that it is in the interior of C, then we can choose an even closer lattice point, and so on infinitely many times to obtain infinite family of lattice points close enough to ρ to be in the interior of C. However, $\operatorname{int}(C) \subseteq \bigcup_i P_i$, and thus, the sizes of the P_i become arbitrarily large. This is a contradiction which proves that P is K-hollow.

Each of the P_i is also infinitely growable using the sequence of polytopes $(P_j)_{j\geq i}$ so the P_i are also K-hollow. By Proposition 5.1.2 there is an $i \in \mathbb{N}$ such that P_i admits a lattice projection φ to a $(d-1)$-dimensional $\varphi(K)$-hollow polytope. Since $P \subseteq P_i$ this projection also takes P to a $(d-1)$-dimensional $\varphi(K)$-hollow polytope. □

From now on we return to the setting where $K = \mathbb{Z}^2$ and $L = \frac{1}{r}\mathbb{Z}^2$.

Corollary 5.1.5. *Let P be a rational polygon with denominator r and size k. Then P is infinitely growable if and only if it is equivalent to a subset of the strip $[0,1] \times \mathbb{R}$ of the plane.*

Proof. Suppose φ is an affine map such that $\varphi(P)$ is a subset of $[0,1] \times \mathbb{R}$. Then for a sufficiently large integer a, the polygons $P_i := \operatorname{conv}(P, \varphi^{-1}(\frac{1}{r}, \frac{a+i}{r}))$ for $i = 0, 1, \ldots$ form an infinite sequence of denominator r size k polygons realising P as infinitely growable.

Now suppose P is infinitely growable. By Proposition 5.1.3 there is a lattice projection $\varphi : \mathbb{R}^2 \to \mathbb{R}$ such that $\varphi(P)$ is $\varphi(\mathbb{Z}^2)$-hollow. By a scaling we may assume that $\varphi(\mathbb{Z}^2) = \mathbb{Z}$ so in other words $\varphi(P)$ is a subset of an interval $[a, a+1]$ for some integer a. The map φ is equal to the map induced by the dual lattice vector $(\varphi(1,0), \varphi(0,1))$. By a change of basis we may assume that $\varphi(0,1) = 0$ (and $\varphi(1,0) \neq 0$). Then, the fact that $\varphi(P) \subseteq [a, a+1]$ can be reinterpreted as: the x-coordinates of P are in the range $\left[\frac{a}{\varphi(1,0)}, \frac{a+1}{\varphi(1,0)}\right]$. By a translation, we may assume that the x-coordinates of P are in $[0, 1]$. $\square$

5.2 Growing Finitely Growable Polygons

We classify the finitely growable polygons of small size and denominator using a growing algorithm. This is done by finding a collection of minimal polygons from which all others can be obtained by successive adding of points.

Definition 5.2.1. A polygon P with denominator r and size k is called *minimal* if for each vertex v of P the polygon

$$\operatorname{conv}\left(\left(P \cap \tfrac{1}{r}\mathbb{Z}^2\right) \setminus \{v\}\right)$$

either has size less than k or is less than two-dimensional.

The growing algorithm is based on the following result.

Proposition 5.2.2. *Let P be a polygon with denominator r and size k. Then there exists a sequence of polygons $P_0, P_1, \ldots, P_s$ with denominator r and size k such that P_0 is minimal, $P_s = P$ and for all $i = 0, \ldots, s-1$:*

1. *$P_{i+1} = \operatorname{conv}(P_i, v_i)$, for some point v_i in $\frac{1}{r}\mathbb{Z}^2$,*
2. *there is exactly one more lattice point in rP_{i+1} than in rP_i (i.e. rv_i) and*
3. *v_i is contained in a hyperplane defined by $u_i \cdot x = h_i + \frac{1}{r}$, where u_i is the outwards pointing normal vector of a facet of P_i and that facet is a subset of the hyperplane $u_i \cdot x = h_i$.*

We call a hyperplane like the one containing v_i in point (3) a hyperplane *adjacent to a facet of P*. To prove the proposition we need the following:

Lemma 5.2.3. *For lattice $N \cong \mathbb{Z}^2$ let P be a rational polygon in $N_{\mathbb{Q}}$. Suppose P contains a point of N in the interior of the half-space*

$$H := \{v \in N_{\mathbb{R}} : u \cdot v \geq h\}$$

where u is a primitive element of the dual space to N. If P also contains a lattice line segment of lattice length 1 in the boundary of H then P contains a lattice point in the hyperplane $\{v : u \cdot v = h + 1\}$.

Proof. By an affine transformation we may assume that $u = (0, 1)$ and $h = 0$ so that H is the set of points with non-negative y-coordinates. By a translation we may assume that P contains the line segment $\operatorname{conv}((0, 0), (1, 0))$. Let $v_0 = (x_0, y_0)$ be the lattice point in $P \cap H^\circ$. If $y_0 = 1$ we are done, otherwise consider the lattice triangle $T_0 = \operatorname{conv}((0, 0), (1, 0), v_0)$ contained in P. This triangle has volume $y_0 > 1$ so by Pick's theorem contains more than three lattice points. Let $v_1 = (x_1, y_1)$ be a lattice point in T_0 which is not a vertex. If $y_1 = 1$ we are done, otherwise replace v_0 with v_1 and repeat this process until we obtain a lattice point in P with y-coordinate 1. This process terminates since each successive vertex v_i has strictly smaller y-coordinate. □

Proof of Proposition 5.2.2. If P is minimal we are done. If P is not minimal let v_{s-1} be a vertex of P such that $P_{s-1} := \operatorname{conv}(P \cap \frac{1}{r}\mathbb{Z}^2 \setminus \{v_{s-1}\})$ is also a denominator r size k polygon. It is immediate that $P = \operatorname{conv}(P_{s-1}, v_{s-1})$ and that there is exactly one more lattice point in rP than in rP_{s-1}. It remains to show that v_{s-1} is contained in a hyperplane adjacent to a facet of P_{s-1}.

We can represent P_{s-1} as the intersection of a finite number of half-spaces, each corresponding to a facet of P_{s-1}. Since $v_{s-1} \notin P_{s-1}$ one of these half-spaces does not contain v_{s-1}. Since a facet of P_{s-1} is contained in the boundary of this half-space P contains both a point in the complementary half-space and a line segment in its boundary. Therefore, by Lemma 5.2.3 there is a point P which falls on a hyperplane adjacent to a facet of P_{s-1}. However, there is only one point of $\frac{1}{r}\mathbb{Z}^2$ in P and not P_{s-1}, that is v_{s-1}. This shows that v_{s-1} satisfies the conditions of the proposition.

If P_{s-1} is minimal we are done, otherwise continue by induction. The process terminates since P contains a finite number of points of $\frac{1}{r}\mathbb{Z}^2$. □

The algorithm will start with the list of minimal polygons and successively add points which lie on hyperplanes adjacent to each of their facets. Based on Proposition 5.2.2 all finitely growable polygons will certainly occur in this manner. However, we must also ensure that the algorithm terminates.

There are infinitely many points on each hyperplane so we need to bound the collection of points which we add to polygons at each growing step. The following definition is useful for this.

Definition 5.2.4. Let $P \subseteq \mathbb{R}^n$ be a polytope with vertices $v_1, \ldots, v_r$ and $v \in \mathbb{Q}^n$ a point. Define the *penumbra* of v with respect to P to be

$$\operatorname{pen}(P, v) := v - \operatorname{cone}(P - v) = \left\{ v - w : w = \sum_{i=1}^{r} \lambda_i (v_i - v), \lambda_i \in \mathbb{R}_{\geq 0} \right\}.$$

Let $\operatorname{pen}(P, w_1, \ldots, w_r)$ denote the union of the regions $\operatorname{pen}(P, w_1), \ldots, \operatorname{pen}(P, w_r)$.

See Figure 5.2 for an example of a penumbra.

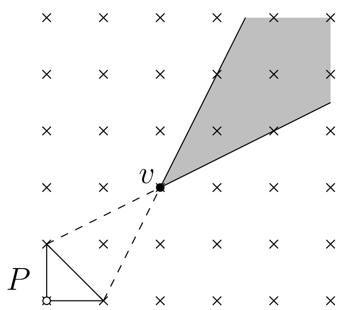


Figure 5.2: *For* $P = \operatorname{conv}((0,0),(1,0),(0,1))$ *and* $v = (2,2)$ *the shaded region is the penumbra of* v *with respect to* P*, that is* $\operatorname{pen}(P, v) = (2,2) + \operatorname{cone}((2,1),(1,2))$.

Proposition 5.2.5. *A point* x *is in* $\operatorname{pen}(P, v)$ *if and only if* v *is in* $\operatorname{conv}(P, x)$.

Proof. Let $x \in \mathrm{pen}(P, v)$. Then by definition there exist non-negative rational numbers $\lambda_1, \ldots, \lambda_r$ such that

$$x = v - \sum_{i=1}^{r} \lambda_i (v_i - v) \quad \Rightarrow \quad v = \frac{1}{1 + \sum_{i=1}^{r} \lambda_i} \left(x + \sum_{i=1}^{r} \lambda_i v_i \right)$$

which means that v meets the conditions to be an element of $\mathrm{conv}(P, x)$.

Let $v \in \mathrm{conv}(P, x)$. Then there exist non-negative rational numbers $\mu_0, \ldots, \mu_r$ such that $\mu_0 + \cdots + \mu_r = 1$ and

$$v = \mu_0 x + \sum_{i=1}^{r} \mu_i v_i \quad \Rightarrow \quad x = \frac{1}{\mu_0} \left(v - \sum_{i=1}^{r} \mu_i v_i \right) = v - \sum_{i=1}^{r} \frac{\mu_i}{\mu_0} (v_i - v)$$

if $\mu_0 \neq 0$ so $x \in \mathrm{pen}(P, v)$. If $\mu_0 = 0$ instead then $v \in P$ so $\mathrm{pen}(P, v)$ is the whole space and therefore contains x. □

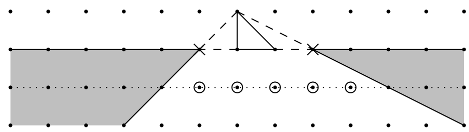


Figure 5.3: *The points which Algorithm 3 adds to* $P = \mathrm{conv}((0,0), (1/2, 0), (0, 1/2))$ *on the hyperplane adjacent to* $\mathrm{conv}((0,0), (1/2, 0))$*. Dots denote points of* $\frac{1}{2}\mathbb{Z}^2$*, crosses denote two points of the hyperplane* $y = 0$ *which we may not include in the new polygon and circles denote points which can be added.*

Suppose we wish to add a point v to P on the hyperplane adjacent to a facet F of P in a way which satisfies the conditions of Proposition 5.2.2. Let H be the half-space containing F in its boundary and not containing P as a subset. Since $\mathrm{conv}(P, v)$ has exactly one additional point, it contains no new points in the boundary of H. In particular, the first point of $\frac{1}{r}\mathbb{Z}^2$ outside of F in either direction, say x_1 and x_2, must not be in $\mathrm{conv}(P, v)$. Thus by Proposition 5.2.5, v cannot be in $\mathrm{pen}(P, x_1, x_2)$. This penumbra is the union of two 2-dimensional cones, both of which are contained in H and have a facet contained in the boundary of H. For this reason, the second facet of each cone is not parallel to the boundary of H, so must intersect the hyperplane adjacent to F in the way depicted in Figure 5.3. This restricts us to a finite collection of points v in the hyperplane adjacent to F. For an example of the points we can add see Figure 5.3. We also exclude lattice points to avoid increasing the size.

For the algorithm to terminate it must also have a finite number of polygons to classify. In particular, there must be a finite number of finitely growable polygons of a given size and denominator.

Proposition 5.2.6. *Let $r \in \mathbb{Z}_{>0}$ and $k \in \mathbb{Z}_{\geq 0}$, then there are finitely many finitely growable polygons with denominator r and size k.*

Proof. By Proposition 5.1.2 it suffices to show that there are finitely many such polygons with interior lattice points. By [LZ91, Theorem 1] the volume of a polygon containing exactly $l \geq 1$ points of $r\mathbb{Z}^2$ in its interior is at most $lr^2(7(lr+1))^{16}$. Therefore, the volume of a polygon containing at most $k-3$ interior points is also bounded by some number V. By [LZ91, Theorem 2] this bound means that the finitely growable polygons with denominator r and size k are equivalent to polygons contained in a lattice square of side length at most $4V$. Therefore, there are finitely many of them. $\square$

Note that a sufficiently small infinitely growable polygon may also be a subset of a maximal polygon. Our growing algorithm cannot exclude infinitely growable polygons, since some minimal polygons are infinitely growable. We still need to classify a finite set so we bound the collection of infinitely growable polygons which we include using the following result.

Proposition 5.2.7. *Let P be a rational polygon with denominator r and size k. Let H be a hyperplane defined by $u \cdot x = \widetilde{h}$ for some non-integral $\widetilde{h} \in \frac{1}{r}\mathbb{Z}$. Let h be the minimum of $(r\widetilde{h} \mod r)$ and $(-r\widetilde{h} \mod r)$. If*

$$\left|P \cap H \cap \tfrac{1}{r}\mathbb{Z}^2\right| \geq r(r-h+1)(k+1)$$

then P is not a subset of a maximal polygon.

Proof. We may assume by an affine map that $u = (1,0)$ and that $0 < \widetilde{h} < 1$. Notice then that $h = r\widetilde{h}$ or $r - r\widetilde{h}$. The line segment $P \cap H$ contains a points from $\frac{1}{r}\mathbb{Z}^2$ for some integer $a \geq r(r+1-h)(k+1)$ so the length of this line segment is at least $\frac{a-1}{r}$.

Assume towards a contradiction that there exists a maximal rational polygon Q with denominator r and size k containing P. Since Q is not infinitely growable it contains a point $v_1 = (x_1, y_1)$ in $\frac{1}{r}\mathbb{Z}^2$ outside of the strip $[0,1] \times \mathbb{R}$. By a reflection in the y-axis followed by a horizontal translation we may assume that $x_1 \geq 1 + \frac{1}{r}$. We claim that there exists a point of $\frac{1}{r}\mathbb{Z}^2$ which is contained in the line segment $Q \cap \{x = 1 + \frac{1}{r}\}$.

Suppose that $x_1 \geq 1 + \frac{2}{r}$ and let T_1 be the triangle $\operatorname{conv}(P \cap H, v_1)$ which is contained in Q. Applying the intercept theorem on the two triangles T_1 and $T_1 \cap \{x \geq 1 + \frac{1}{r}\}$ yields that the length of the line segment $T_1 \cap \{x = 1 + \frac{1}{r}\}$ is at least

$$\frac{a-1}{r} \cdot \frac{rx_1 - r - 1}{rx_1 - \widetilde{rh}}. \tag{5.1}$$

We claim that this lower bound is at least $\frac{1}{r}$ which then guarantees that the line segment $Q \cap \{x = 1 + \frac{1}{r}\}$ contains a rational point in $\frac{1}{r}\mathbb{Z}^2$. Notice that the partial derivative of (5.1) with respect to x_1 is positive:

$$\frac{\partial}{\partial x_1} \frac{a-1}{r} \cdot \frac{rx_1 - r - 1}{rx_1 - h'} = \frac{(a-1)(r+1-h')}{r^2(rx_1 - h')^2} > 0$$

so (5.1) is increasing. Since $x_1 \geq 1 + \frac{2}{r}$, a lower bound for the fraction (5.1) is given by substituting in $x_1 = 1 + \frac{2}{r}$, that is,

$$\frac{a-1}{r} \cdot \frac{rx_1 - r - 1}{rx_1 - h'} \geq \frac{a-1}{r(r+2-\widetilde{rh})}.$$

Thus it suffices to show that $(a-1)/(r(r+2-\widetilde{rh})) \geq \frac{1}{r}$. By definition $a \geq r(r+1-h)(k+1)$ and $k \geq 0$ so it further suffices to show that

$$\frac{r(r+1-h)-1}{r+2-\widetilde{rh}} \geq 1.$$

This can be verified using the facts that $\widetilde{rh} \in \{h, r-h\}$ and $h \leq \frac{r}{2}$. This proves that Q contains a point of $\frac{1}{r}\mathbb{Z}^2$ with x-coordinate $1+\frac{1}{r}$.

From now on, assume that v_1 has x-coordinate $x_1 = 1 + \frac{1}{r}$. Let us denote the a consecutive points of $P \cap H \cap \frac{1}{r}\mathbb{Z}^2$ by $p_1, \dots, p_a$. Let q_i be the rational intersection points of the line segments $\operatorname{conv}(v_1, p_i)$ with $S = Q \cap \{x = 1\}$. By computing the intersection of these line segments in general we see that the y-coordinate of q_i is in $\frac{1}{r(r+1-r\widetilde{h})}\mathbb{Z}$. Thus, S contains at least $r(r+1-h)(k+1)$ points of $\frac{1}{r(r+1-r\widetilde{h})}\mathbb{Z}^2$. At least $k+1$ of these must be integral points which contradicts the size of Q. $\square$

We make the algorithm rigorous in Algorithm 3. We have shown that it starts with finitely many polygons and at each growing step adds finitely many additional polygons with strictly larger volume. Additionally all polygons it produces are either members of the finite set of finitely growable polygons or the finite set of infinitely growable polygons which have a bound, described in Proposition 5.2.7, on the number of colinear points which they can contain. Therefore, the algorithm terminates and classifies all finitely growable polygons of a given size and denominator.

Algorithm 3 does not store any polygon which is infinitely growable. To identify if a denominator r polygon P is infinitely growable we consider the lattice polygon rP. If this has width greater than r then rP has width greater than 1 and cannot be infinitely growable. Otherwise, we consider the finite collection of dual lattice points in $r\mathcal{W}_{rP}$ (where $\mathcal{W}_{rP}$ is as defined in Section 3.1). If P is contained between two adjacent parallel integral hyperplanes then those hyperplanes must be normal to one of these points. We check if this occurs for any point. If it does then P is infinitely growable, otherwise it is not.

Three final adjustments can be made to make the growing algorithm more efficient. First notice that affine equivalent polygons contain the same number of points in $\frac{1}{r}\mathbb{Z}^2$. Also, our algorithm grows polygons by exactly one point of $\frac{1}{r}\mathbb{Z}^2$ at a time. Therefore, we can stratify the growing algorithm by the $\frac{1}{r}\mathbb{Z}^2$-size of the polygons. That is, grow all polygons with $\frac{1}{r}\mathbb{Z}^2$-size n then $n+1$

and so on. The benefit of this is that once all polygons of a given $\frac{1}{r}\mathbb{Z}^2$-size have been classified, and all duplicates and infinitely growable polygons have been removed, they can be stored and deleted from working memory without any danger that we will classify them again later in the iterations.

The next adjustment relates to removing infinitely growable polygons from the final classification. Notice that a finitely growable polygon can never be grown into an infinitely growable polygon. Thus, once we have checked and found that a polygon is finitely growable we should not check whether any polygon grown from it is finitely growable. To avoid this we distinguish between polygons which are infinitely growable and finitely growable throughout the main loop.

Finally, we will show in Section 5.3 that a polygon with denominator $r \geq 1$ and size $k \geq 1$ contains a unique minimal polygon and two such polygons can only be equivalent if their minimal polygons are equivalent (see Remark 5.3.4). As a result, when the size is at least 1 the algorithm grows each minimal polygon independently one after the other rather than simultaneously. This reduces the number of polygons which need to be stored in working memory.

5.3 Minimal polygons

In this section we classify the minimal polygons with denominator $r > 1$ and size $k \geq 0$. Those with size zero are a separate case which we classify first.

Proposition 5.3.1. *The minimal polygons of size zero with denominator r are all affine equivalent to size zero triangles of the form $\frac{1}{r}(\Delta + v)$ where $\Delta \coloneqq \operatorname{conv}((0,0),(1,0),(0,1))$ and v is a lattice point in the square $[0, r-1]^2$.*

First note that not all triangles of the form $\frac{1}{r}(\Delta + v)$ have size zero so the condition 'of size zero' in the proposition places restrictions on v. Secondly, this list may contain duplicates when considered up to affine equivalence. For example, see Figure 5.4.

Algorithm 3: Growing algorithm for polygons with denominator r and size k. All sets are considered modulo affine equivalence.

Data: The set `MinPoly` containing minimal polygons with denominator r and size k.

Result: The set `Final` containing all finitely growable polygons with denominator r and size k.

function `Grow`(P,*r-size*,`IsFin`)

- `NewInf` $\longleftarrow \emptyset$
- `NewFin` $\longleftarrow \emptyset$
- **for** *Each point p on a hyperplane adjacent to a facet of P which we may add to P* **do**
  - $Q \longleftarrow \mathrm{conv}(P, p)$
  - **if** *Q has the correct size, r-size and not too many collinear points according to Proposition 5.2.7* **then**
    - **if** `IsFin` *or Q is finitely growable* **then**
      - `NewFin` $\longleftarrow$ `NewFin` $\cup \{Q\}$
    - **else**
      - `NewInf` $\longleftarrow$ `NewInf` $\cup \{Q\}$
- **if** `IsFin` **then**
  - **return** `NewFin`
- **return** `NewInf`, `NewFin`

`Final` $\longleftarrow \emptyset$

for $P_{min} \in$ `MinPoly` **do**

- **if** $k = 0$ **then**
  - `ToGrowInf` $\longleftarrow$ `MinPoly`
  - `ToGrowFin` $\longleftarrow \emptyset$
- **else if** P_{min} *is infinitely growable* **then**
  - `ToGrowInf` $\longleftarrow \{P_{min}\}$
  - `ToGrowFin` $\longleftarrow \emptyset$
- **else**
  - `ToGrowInf` $\longleftarrow \emptyset$
  - `ToGrowFin` $\longleftarrow \{P_{min}\}$
- r-size $\longleftarrow |P \cap \frac{1}{r}\mathbb{Z}^2|$
- **repeat**
  - r-size $\longleftarrow$ r-size$+1$
  - `ToGrowNextInf` $\longleftarrow \emptyset$
  - `ToGrowNextFin` $\longleftarrow \emptyset$
  - **for** $P \in$ `ToGrowInf` **do**
    - `NewInf`, `NewFin` $\longleftarrow$ `Grow`(P, r-size, false)
    - `ToGrowNextInf` $\longleftarrow$ `ToGrowNextInf` $\cup$ `NewInf`
    - `ToGrowNextFin` $\longleftarrow$ `ToGrowNextFin` $\cup$ `NewFin`
  - **for** $P \in$ `ToGrowFin` **do**
    - `NewFin` $\longleftarrow$ `Grow`(P, r-size, true)
    - `ToGrowNextFin` $\longleftarrow$ `ToGrowNextFin` $\cup$ `NewFin`
  - `Final` $\longleftarrow$ `Final` $\cup$ `ToGrowFin`
  - `ToGrowInf` $\longleftarrow$ `ToGrowNextInf`
  - `ToGrowFin` $\longleftarrow$ `ToGrowNextFin`
- **until** `ToGrowFin` $= \emptyset$ *and* `ToGrowInf` $= \emptyset$

Proof. First we can see that these triangles are minimal since removing any of their vertices makes them one-dimensional.

Let P be a minimal polygon of size zero with denominator r. We can find a triangulation of P into triangles containing exactly three points of $\frac{1}{r}\mathbb{Z}^2$. Let T be one of these triangles. Using Pick's theorem we may assume that T is some affine transformation of $\frac{1}{r}\Delta$. By an affine map, we may assume that T is $\frac{1}{r}(\Delta + v)$ for some lattice point v in $[0, r-1]^2$. Any vertex of P not contained in T can be removed without changing the size or dimension so $P = T$. $\square$

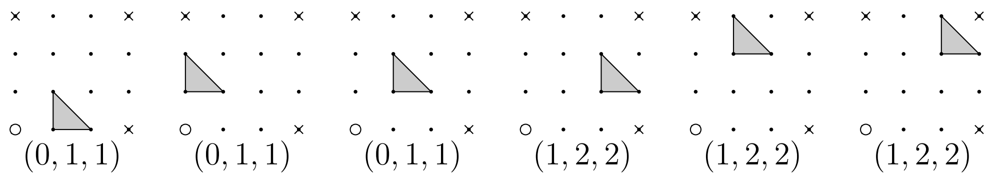


Figure 5.4: *The minimal polygons of size 0 with denominator 3 as described in Proposition 5.3.1, along with the tuples described in Proposition 5.3.2. The first three triangles are affine equivalent, as are the last three.*

Proposition 5.3.2. *There is a bijection between the minimal polygons of size zero with denominator r and the set A of triples $(a_1, a_2, a_3) \in \mathbb{Z}^3_{\geq 0}$ which satisfy $a_1 \leq a_2 \leq a_3 \leq r-1$ and one of the following conditions:*

1. $a_1 + a_2 + a_3 = r - 1$ *and* $0 < a_1 + a_2$
2. $a_1 + a_2 + a_3 = 2r - 1$ *and* $r \leq a_1 + a_2$

For example, see Figure 5.4.

Proof. By Proposition 5.3.1 it suffices to show that there is a bijection between A and the set S of affine equivalence classes $[\frac{1}{r}(\Delta+v)]$ when v is a lattice point.

Let $T := \frac{1}{r}(\Delta + v)$. We can write T as the intersection of three half-spaces $H_i := \{v \in \mathbb{R}^2 : u_i \cdot v \geq \widetilde{h_i}\}$ for $i = 1, 2, 3$ where each $\widetilde{h_i} \in \frac{1}{r}\mathbb{Z}$. We define $h_i = (r\widetilde{h_i} \mod r)$ and, up to relabelling, we may assume that $h_1 \leq h_2 \leq h_3$. By a unimodular map we may assume that $u_1 = (1, 0)$ and $u_2 = (0, 1)$ and by an integral translation we may assume that $r\widetilde{h_1} = h_1$ and $r\widetilde{h_2} = h_2$. This means that $T = \frac{1}{r}((h_1, h_2) + \Delta)$ and so we can explicitly calculate h_3 in terms of h_1

and h_2. That is, $h_3 = r-1-h_1-h_2$ when $h_1+h_2 < r$ and $h_3 = 2r-1-h_1-h_2$ otherwise. Since T contains no lattice points h_1+h_2 is always greater than 0. This shows that $(h_1,h_2,h_3) \in A$.

We claim that the map from S to A taking $[T]$ to (h_1,h_2,h_3) is the desired bijection. This is well-defined as the triple (h_1,h_2,h_3) is invariant under affine maps. It remains to prove that it is injective and surjective. As shown above, if $[T] \mapsto (h_1,h_2,h_3)$ then T is equivalent to the triangle $\frac{1}{r}((h_1,h_2)+\Delta)$ from which we can deduce injectivity. Surjectivity comes from the fact that for any $(a_1,a_2,a_3) \in A$ we have $[\frac{1}{r}((a_1,a_2)+\Delta)] \mapsto (a_1,a_2,a_3)$. $\square$

Proposition 5.3.3. *The minimal polygons with denominator r and size $k > 1$ are exactly the integral polygons of size k and the triangle*

$$T_{r,k} \coloneqq \operatorname{conv}\left((0,0),(k-1,0),\left(0,\tfrac{1}{r}\right)\right)$$

and the only minimal polygon with denominator r and size $k = 1$ is the triangle

$$T_{r,1} \coloneqq \operatorname{conv}\left((0,0),\left(\tfrac{1}{r},0\right),\left(0,\tfrac{1}{r}\right)\right).$$

Proof. The integral polygons of size k are minimal since if we removed any of their vertices the result would have size $k-1$. The triangle $T_{r,k}$ is minimal for all $k \geq 1$, since if we removed an integral vertex from it the result would have size $k-1$ and if we removed a rational vertex the result would be a line.

Suppose P is a minimal polygon with denominator r and size $k > 1$. Let Q be the convex hull of the lattice points in P. This is either a lattice polygon of size k or a line segment of lattice length $k-1$.

If Q is an lattice polygon of size k then any vertex of P not contained in Q can be removed without making the result a smaller size or dimension, therefore $P = Q$ is a lattice polygon of size k.

If Q is a line segment of lattice length $k-1$ then by an affine map we may assume that P contains the line segment between $(0,0)$ and $(k-1,0)$.

By a reflection we may assume that P contains a $\frac{1}{r}$-integral point with positive y-coordinate. By Lemma 5.2.3 P contains a point with y-coordinate $\frac{1}{r}$ which by a shear we may assume is $(0, \frac{1}{r})$. Any point of P not contained in $\mathrm{conv}((0,0),(k-1,0),(0,\frac{1}{r}))$ can be removed without making the result a smaller size or dimension, therefore $P = \mathrm{conv}((0,0),(k-1,0),(0,\frac{1}{r}))$.

If P is a minimal polygon with denominator r and size 1 an affine transformation allows us to assume it contains the points $(0,0)$ and $(\frac{1}{r}, 0)$. A similar argument to the above shows that P is equivalent to $T_{r,1}$. □

Remark 5.3.4. Notice that the minimal polygon contained in a rational polygon of size at least 1 is entirely determined by the convex hull of the lattice points it contains. As a result, two such polygons can only be equivalent if they can be grown from the same minimal polygon.

To compute the minimal polygons, it remains to classify the lattice polygons of a given size. Algorithms to do so have been presented elsewhere, for example in [LZ11] and [BS16]. We use a very similar growing algorithm to the one described in Section 5.2, which adds points on hyperplanes adjacent to the facets of a polygon which is the approach mentioned in [BS16].

5.4 Ehrhart Theory in Denominator 2

As mentioned previously, the Ehrhart polynomial of a lattice polygon determines and is determined by the number of boundary and interior points of that polygon. Thus, the following result gives a complete classification of the Ehrhart polynomials of lattice polygons.

Theorem 5.4.1 ([Sco76, HS09])**.** *Integers b and i are the number of boundary and interior points of a lattice polygon if and only if $b \geq 3$, $i \geq 0$ and one of the following holds*

- *$i = 0$,*
- *$i = 1$ and $b = 9$ or*

- *$i \geq 1$ and $b \leq 2i + 6$.*

The only polygon with one interior point and 9 boundary points is the triangle $\operatorname{conv}((0,0),(3,0),(0,3))$.

Towards a generalisation of this result for rational polygons we first show that the Ehrhart polynomial of a rational polygon can be written in terms of numbers of boundary and interior points of polygons.

Proposition 5.4.2. *Let P be a rational polygon with Ehrhart quasi-polynomial*

$$\operatorname{ehr}_P(n) = a_{2,i}n^2 + a_{1,i}n + a_{0,i} \quad \text{when } n \equiv i \mod r$$

for some positive integer r and $i \in \{0, 1, \dots, r-1\}$. Then

$$\begin{aligned} a_{2,i} &= \tfrac{1}{2}\operatorname{Vol}(P) \\ a_{1,i} &= \tfrac{1}{r}\left(|iP \cap \mathbb{Z}^2| - |(r-i)P^\circ \cap \mathbb{Z}^2| - \tfrac{r(2i-r)}{2}\operatorname{Vol}(P)\right) \\ a_{2,i}i^2 + a_{1,i}i + a_{0,i} &= |iP \cap \mathbb{Z}^2| \end{aligned}$$

for all i where $\operatorname{Vol}(P)$ denotes the normalised volume of P.

Remark 5.4.3. The case $r = 2$ was presented before in [Her10, Lemma 3.3]. We use the same method to extend this to the general quasi-period case.

Proof. The fact that $a_{2,i} = \frac{1}{2}\operatorname{Vol}(P)$ is a known result coming from considering the limit as n tends to infinity. By the definition of $\operatorname{ehr}_P(n)$, for $i = 0, 1, \dots, r-1$

$$a_{2,i}i^2 + a_{1,i}i + a_{0,i} = |iP \cap \mathbb{Z}^2|$$

and by Ehrhart-Macdonald reciprocity (Theorem 2.2.3)

$$a_{2,i}(i-r)^2 + a_{1,i}(i-r) + a_{0,i} = |(r-i)P^\circ \cap \mathbb{Z}^2|.$$

Expanding this second equation and substituting in the first we can simplify to show that

$$a_{1,i} = \frac{1}{r}\left(|iP \cap \mathbb{Z}^2| - |(r-i)P^\circ \cap \mathbb{Z}^2| - \frac{r(2i-r)}{2}\operatorname{Vol}(P)\right).$$

□

Since rP is a lattice polygon we can use Pick's theorem to write the volume of P in terms of the number of boundary and interior points of rP. Similarly to lattice polygons, this shows that the Ehrhart quasi-polynomial of a denominator r polygon is completely encoded by the number of boundary and interior points of $P, \ldots, (r-1)P$ and rP.

For the remainder of this section we study the number of interior and boundary points in P and $2P$ where P is a denominator 2 polygon. Recall, Theorem 5.0.3 states that all but finitely many such polygons P must satisfy one of four given conditions on the number of interior and boundary points of P and $2P$. We prove this in Proposition 5.4.4, 5.4.5, 5.4.10, 5.4.11 and Table 5.2.

5.4.1 Polygons With Zero Interior Points

Infinitely Growable Polygons

We showed in Section 5.1 that infinitely growable denominator 2 polygons are those which are equivalent to a subset of the strip $[0,1] \times \mathbb{R}$. This description allows us to completely classify the Ehrhart polynomials of such polygons.

Proposition 5.4.4. *Let P be an infinitely growable, denominator 2 polygon. Let i_1, b_1, i_2 and b_2 be the number of interior and boundary points of P and $2P$ respectively. Then $i_1 = 0$ and the remaining variables satisfy one of the following conditions:*

1. *$i_2 = 0$ and $b_2 \geq \max(3, 2b_1)$,*

2. *$i_2 > 0$, $b_1 = 0$ and $b_2 = 4$ or*

3. *$i_2, b_1 > 0$, $\max(3, 2b_1) \leq b_2 \leq 2b_1 + 4$ and $b_2 \leq 2i_2 + 6$.*

See Figure 5.5 for plots of pairs (b_2, i_2) when b_1, i_2 and b_2 are small.

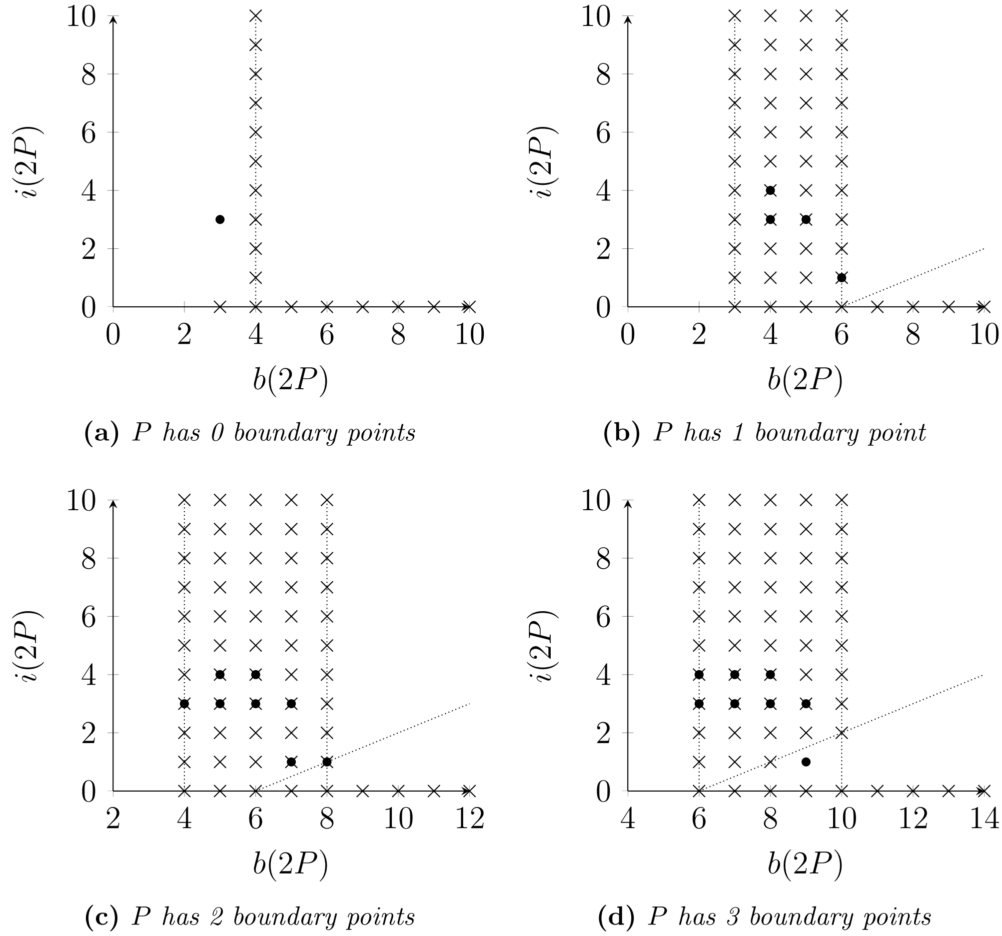
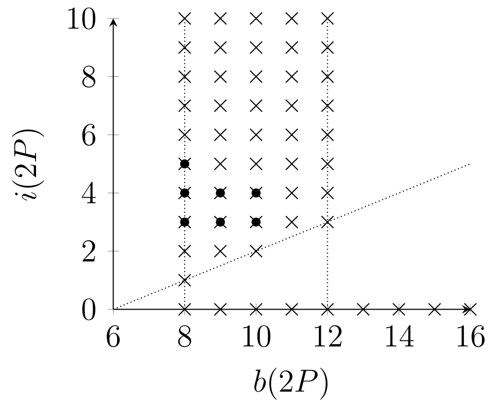


Figure 5.5: *Plots of $(b(2P), i(2P))$ for denominator 2 polygons of size up to 6 with zero interior points. Crosses are realised by infinitely growable polygons and dots by finitely growable polygons. Dotted lines denote the bounds described in Proposition 5.4.4. Continued on next page.*

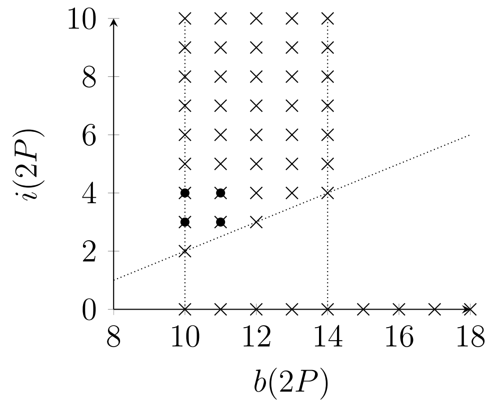
Proof. The fact that P is infinitely growable shows that $i_1 = 0$. It is immediate that exactly one of the following three conditions holds

- $i_2 = 0$,
- $i_2 > 0$ and $b_1 = 0$ or
- $i_2 > 0$ and $b_1 > 0$.

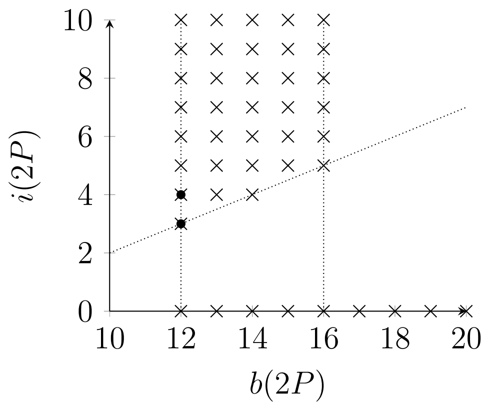
It suffices to show that, in each of these cases, the corresponding condition of the proposition holds.



(e) *P has 4 boundary points*



(f) *P has 5 boundary points*



(g) *P has 6 boundary points*

Figure 5.5: *Plots of* $(b(2P), i(2P))$ *for denominator 2 polygons of size up to 6 with zero interior points continued.*

The integers b_2 and i_2 are the number of boundary and interior points of a lattice polygon so they must satisfy Theorem 5.4.1. The exceptional case where $b_2 = 9$ and $i_2 = 1$ never occurs since the only possible $2P$ in this case has width 3, contradicting the fact that P is infinitely growable. Therefore, $b_2 \geq 3$ and either $i_2 = 0$ or $b_2 \leq 2i_2 + 6$.

For each boundary point v of P we know that $2v$ is a boundary point of $2P$. For each pair of adjacent boundary points of P either they are connected by part of a single edge of P or there is a half-integral vertex between them on the boundary of P. In either case, there is a half-integral point on the boundary of P between each such pair so $b_2 \geq 2b_1$. Therefore, when $i_2 = 0$ condition (1) holds

From now on we assume that $i_2 > 0$. We may assume that P is a subset of

$[0, 1] \times \mathbb{R}$ and consider the number of boundary points $2P$ has on each of the hyperplanes $x = 0, 1$ and 2. Since $i_2 > 0$, $2P$ has interior points on the line $x = 1$, so $2P$ can have at most two boundary points on this line. On the lines $x = 0$ and $x = 2$ together $2P$ can have at most $2b_1 + 2$ boundary points so $b_2 \leq 2b_1 + 4$. Therefore, when i_2 and b_1 are positive, condition (3) holds.

If $b_1 = 0$ and $i_2 > 0$ then $2P$ contains interior lattice points in the line $x = 1$ but contains at most one point in each of the lines $x = 0$ and 2. Therefore, $2P$ contains exactly one boundary point on each of the lines $x = 0$ and 2. Call these v_0 and v_2. The only possible remaining vertices of $2P$ are in the line $x = 1$. If $2P$ has two vertices in the line $x = 1$ then we have shown that $b_2 \geq 4$. Otherwise, the line segment from v_0 to v_2 is an edge of $2P$. It contains a lattice point at its midpoint since it is affine equivalent to $\operatorname{conv}((1, 0), (1, 2))$. Therefore, $2P$ has two boundary points on the line $x = 1$ and $b_2 \geq 4$. Therefore, when i_2 is positive and $b_1 = 0$, condition (2) holds. □

Proposition 5.4.5. *Let b_1, b_2 and i_2 be non-negative integers satisfying one of the conditions in Proposition 5.4.4. Then there is an infinitely growable, denominator 2 polygon P such that P has b_1 boundary points and $2P$ has b_2 boundary points and i_2 interior points.*

Proof. We consider tuples (b_1, b_2, i_2) of non-negative integers which satisfy the conditions of Proposition 5.4.4. We say that a polygon P realises such a tuple if it is infinitely growable, has b_1 boundary points and $2P$ has b_2 boundary points and i_2 interior points. We give examples of infinite families of denominator 2 polygons realising all of these tuples.

The first condition is satisfied by tuples $(0, b_2, 0)$ where $b_2 \geq 3$, $(1, b_2, 0)$ where $b_2 \geq 3$ and $(b_1, b_2, 0)$ where $b_2 \geq 2b_1 \geq 4$. These are realised by the polygons

- $\operatorname{conv}((0, \frac{1}{2}), (\frac{1}{2}, 0), (\frac{1}{2}, \frac{b_2 - 2}{2})$ if $b_1 = 0$, and
- $\operatorname{conv}((0, 0), (0, b_1 - 1), (\frac{1}{2}, 0), (\frac{1}{2}, \frac{b_2 - 2b_1}{2})$ if $b_1 > 0$.

The second condition is satisfied by tuples $(0, 4, i_2)$ where $i_2 > 0$. These are realised by polygons

- $\mathrm{conv}((0, \frac{1}{2}), (1, \frac{1}{2}), (\frac{1}{2}, \frac{i_2+2}{2}))$.

If $b_1 = 1$ the third condition is satisfied by the tuples $(1, b_2, i_2)$ where $3 \leq b_2 \leq 6$ and $b_2 \leq 2i_2 + 6$. These are realised by the polygons

- $\mathrm{conv}((0, 0), (1, \frac{1}{2}), (\frac{1}{2}, \frac{i_2+1}{2}))$ if $b_2 = 3$,
- $\mathrm{conv}((0, 0), (0, \frac{1}{2}), (1, \frac{1}{2}), (\frac{1}{2}, \frac{i_2+1}{2}))$ if $b_2 = 4$,
- $\mathrm{conv}((0, 0), (0, \frac{1}{2}), (\frac{1}{2}, 0), (1, \frac{1}{2}), (\frac{1}{2}, \frac{i_2+1}{2}))$ if $b_2 = 5$, and
- $\mathrm{conv}((0, -\frac{1}{2}), (0, \frac{1}{2}), (1, \frac{1}{2}), (\frac{1}{2}, \frac{i_2+1}{2}))$ if $b_2 = 6$.

If $b_1 \geq 2$ the third condition is satisfied by the tuples (b_1, b_2, i_2) where $2b_1 \leq b_2 \leq 2b_1 + 4$ and $b_2 \leq 2i_2 + 6$. These are realised by the polygons

- $\mathrm{conv}((0, 0), (0, b_1 - 2), (1, 0), (\frac{1}{2}, \frac{i_2+1}{2}))$ if $b_2 = 2b_1$,
- $\mathrm{conv}((0, -\frac{1}{2}), (0, b_1 - 2), (1, 0), (\frac{1}{2}, -\frac{1}{2}), (\frac{1}{2}, \frac{i_2+1}{2}))$ if $b_2 = 2b_1 + 1$,
- $\mathrm{conv}((0, -\frac{1}{2}), (0, b_1 - 2), (1, 0), (1, -\frac{1}{2}), (\frac{1}{2}, \frac{i_2+1}{2}))$ if $b_2 = 2b_1 + 2$,
- $\mathrm{conv}((0, -\frac{1}{2}), (0, b_1 - 2), (1, \frac{1}{2}), (1, -\frac{1}{2}), (\frac{1}{2}, \frac{i_2+1}{2}))$ if $b_2 = 2b_1 + 3$, and
- $\mathrm{conv}((0, -\frac{1}{2}), (0, b_1 - \frac{3}{2}), (1, \frac{1}{2}), (1, -\frac{1}{2}), (\frac{1}{2}, \frac{i_2+1}{2}))$ if $b_2 = 2b_1 + 4$. $\square$

Finitely Growable Polygons

We grow the finitely growable polygons with zero interior points using an adaptation of Algorithm 3. First we identify the minimal polygons they contain.

Proposition 5.4.6. *Let P be a denominator 2, finitely growable polygon with zero interior points. Then P contains a unique minimal polygon P_0 with the* *same size and denominator as P. The polygon P_0 must be one of the following*

$$\begin{array}{rl}
\mathrm{conv}((0,0),(\tfrac{1}{2},0),(0,\tfrac{1}{2})), & \mathrm{conv}((\tfrac{1}{2},\tfrac{1}{2}),(\tfrac{1}{2},1),(1,\tfrac{1}{2})), \\
\mathrm{conv}((0,0),(1,0),(0,\tfrac{1}{2})), & \mathrm{conv}((0,0),(1,0),(0,1)), \\
\mathrm{conv}((0,0),(1,0),(0,1),(1,1)), & \mathrm{conv}((0,0),(2,0),(0,1)), \\
\mathrm{conv}((0,0),(2,0),(0,1),(1,1)), & \mathrm{conv}((0,0),(3,0),(0,1)), \\
\mathrm{conv}((0,0),(3,0),(0,1),(1,1)), & \mathrm{conv}((0,0),(2,0),(0,2)).
\end{array}$$

Proof. Let Q denote the convex hull of the lattice points in P. If Q is empty then P_0 is equivalent to $\mathrm{conv}((\frac{1}{2},\frac{1}{2}),(\frac{1}{2},1),(1,\frac{1}{2}))$ as this is the only size 0 minimal polygon which Proposition 5.3.2 gives when the denominator is 2. The remaining cases depend on the classification of minimal polygons in Proposition 5.3.3. If Q is a point then there is only one minimal polygon of size 1 in any given denominator, so P_0 is equivalent to $\mathrm{conv}((0,0),(\frac{1}{2},0),(0,\frac{1}{2}))$. If Q is a line segment then there is only one minimal polygon whose lattice points are colinear, so P_0 is equivalent to $\mathrm{conv}((0,0),(k,0),(0,\frac{1}{2}))$ for some positive integer k. Finally, if Q is a lattice polygon then it is minimal so $P_0 = Q$.

If Q is a line segment of lattice length at least 2 then we may assume it contains the points $(0,0)$ and $(2,0)$. Since P is finitely growable we may assume that it contains a point with y-coordinate at least $\frac{3}{2}$. However, all such points are contained in the penumbra $\mathrm{pen}(\mathrm{conv}((0,0),(2,0)),(n,1))$ for some integer n, so none can be included in P without contradicting the condition that Q is a line segment. This is enough to complete the cases where Q has dimension less than 2.

Similarly, if Q *contains* a line segment of lattice length at least 4 then, after some affine map, we may assume it contains the points $(0,0)$ and $(4,0)$. Since P is finitely growable we may assume it contains a point with y-coordinate at least $\frac{3}{2}$. However, all such points are contained in the interior of the penumbra $\mathrm{pen}(\mathrm{conv}((0,0),(4,0)),(n,1))$ for some integer n so none can be included in P without contradicting the condition that P has no interior points. Therefore,

Q contains no line segment of lattice length 4 or more.

If Q is a lattice polygon then it is either $\mathrm{conv}((0,0),(2,0),(0,2))$ or it has width 1 as these are the only lattice polygons with no interior points (see for example [Rab89, Theorem 1]). If $Q = \mathrm{conv}((0,0),(2,0),(0,2))$ then it cannot be grown by any point of $\frac{1}{2}\mathbb{Z}^2$ without including a lattice point in the interior since all facets of Q contain an interior point. Therefore, in this case $P = Q$ and we do not attempt to grow Q.

If Q has width 1 then we may assume it is equivalent to a subset of the strip $[0,1] \times \mathbb{R}$ and that P contains some rational vertex with x-coordinate greater than 1. This means that the intersection of Q with the line $x = 1$ must have at most two lattice points. This combined with the fact that Q contains at most 4 collinear lattice points is sufficient to classify the minimal polygons P_0. $\square$

For each minimal polygon we apply Algorithm 3 with the new condition that we discard polygons with interior lattice points. Additionally, suppose a finitely growable polygon P contains 6 colinear points of $\frac{1}{2}\mathbb{Z}^2 \setminus \mathbb{Z}^2$. By an affine transformation we may assume that these points are $(0, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}), \ldots$ and $(\frac{5}{2}, \frac{1}{2})$. Since it is finitely growable P we may assume it contains vertex with y-coordinate greater than 1, however all such points of $\frac{1}{2}\mathbb{Z}^2$ are contained in $\mathrm{pen}(\mathrm{conv}((0, \frac{1}{2}), (\frac{5}{2}, \frac{1}{2})), (n, 1))$ for some integer n. Therefore, our polygons contain at most 5 colinear points in a half-integral hyperplane. We use this to improve compute time. We implement this adjusted algorithm using Magma V2.27 and produce a list of 79 rational polygons P with denominator 2 and $i(P) = 0$. The tuples $(b(P), b(2P), i(2P))$ for these polygons are listed in Table 5.2 and plotted in Figure 5.5.

The two points which do not satisfy the bounds listed in Proposition 5.4.4 are $(0, 3, 3)$ and $(3, 9, 1)$ which are realised by the polygons $\mathrm{conv}((1, 0), (0, 1), (3, 5))$ and $\mathrm{conv}((0, 0), (\frac{3}{2}, 0), (0, \frac{3}{2}))$ respectively. These are notable polygons as the first is the unique finitely growable polygon with denominator 2 and size 0 and the second is $\frac{1}{2}$ times the unique lattice polygon which does not satisfy

Theorem 5.4.1. All other finitely growable, denominator 2 polygons satisfy the bounds of Proposition 5.4.4.

(0, 3, 3)	(1, 4, 3)	(1, 4, 4)	(1, 5, 3)	(1, 6, 1)
(2, 4, 3)	(2, 5, 3)	(2, 5, 4)	(2, 6, 3)	(2, 6, 4)
(2, 7, 1)	(2, 7, 3)	(2, 8, 1)	(3, 6, 3)	(3, 6, 4)
(3, 7, 3)	(3, 7, 4)	(3, 8, 3)	(3, 8, 4)	(3, 9, 1)
(3, 9, 3)	(4, 8, 3)	(4, 8, 4)	(4, 8, 5)	(4, 9, 3)
(4, 9, 4)	(4, 10, 3)	(4, 10, 4)	(5, 10, 3)	(5, 10, 4)
(5, 11, 3)	(5, 11, 4)	(6, 12, 3)	(6, 12, 4)	

Table 5.2: Tuples $(b(P), b(2P), i(2P))$ for all finitely growable denominator 2 polygons with zero interior points.

5.4.2 Polygons With Interior Points

We now consider the polygons with at least one interior point. The number of boundary and interior points of P and $2P$ for all denominator 2 polygons P containing an interior point and up to 4 lattice points are plotted in Figure 5.9. The dashed lines in each plot denote the bounds $b(2P) \geq \max\{3, 2b(P)\}$, $i(2P) \geq b(P) + 2i(P) - 1$ and $b(2P) + i(2P) \leq 2b(P) + 6i(P) + 7$ all of which are satisfied by every polygon in the classification. To prove that these bounds hold in all but finitely many cases we first need some preparatory results.

Lemma 5.4.7. *Let P be a lattice polygon of multi-width (w_1, w_2) written as a subset of the rectangle $Q_{w_1,w_2} \coloneqq [0, w_1] \times [0, w_2]$. Let $h \in [2, w_1 - 2]$ be an integer and let h' be the minimum of h and $w_1 - h$, then P contains at least $h' - 1$ interior points in the line $x = h$.*

Proof. Suppose for contradiction that P contains less than $h'-1$ interior points in the line $x = h$. Then the line segment $P \cap \{x = h\}$ must be a subset of $I_h \coloneqq \{h\} \times [y_1, y_2]$ where y_1 is an integer in $[0, w_2]$ and $y_2 = y_1 + h' - 1$.

Let v be a point of P and u a point of the line $x = h$ not contained in this interval. Then no point of $\mathrm{pen}(v, u)$ can be contained in P. In particular, if v has x-coordinate less than h then all points of P with x-coordinate greater than h must be contained in the affine cone $C_v \coloneqq v + \mathrm{cone}(I_h - v)$ and vice versa when v has x-coordinate greater than h. For example, see Figure 5.6.

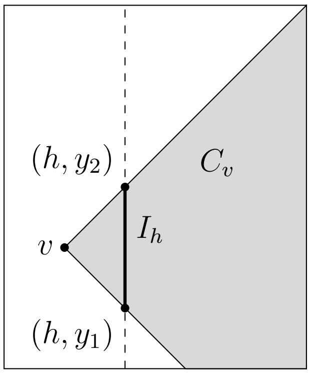


Figure 5.6: *In the proof of Lemma 5.4.7, if $v \in P$ then all points of P with x-coordinate greater than h are contained in the affine cone $C_v := v + \operatorname{cone}(I_h - v)$.*

By the widths of P we know that P has a vertex contained in each edge of the rectangle Q_{w_1,w_2}. Either $y_1 > 0$ or $y_2 < w_2$ so the points in the upper and lower edge of Q_{w_1,w_2} cannot both be in the line $x = h$. By reflections and possibly exchanging the values of h and $w_1 - h$, we may assume that P contains a vertex in the upper edge of Q_{w_1,w_2} with x-coordinate greater than h. Let v be a point of P with x-coordinate less than h, then the vertices of P on the upper and right edges of Q_{w_1,w_2} are contained in C_v. As a result, the point (w_1, w_2) is also contained in C_v, and so $v \in C_{(w_1,w_2)}$.

Suppose towards a contradiction, that P contains a point on the lower edge of Q_{w_1,w_2}, with x-coordinate greater than h (see Figure 5.7a). Then $(w_1, 0)$ must also be contained in C_v as above, so all points of P with x-coordinate less than h are contained in the intersection of $C_{(w_1,0)}$ and $C_{(w_1,w_2)}$. However, the line through $(w_1, 0)$ and (h, y_1) and the line through (w_1, w_2) and (h, y_2) meet at a point with x-coordinate

$$\frac{w_2 h + w_1(y_1 - y_2)}{w_2 - y_2 + y_1} = \begin{cases} \frac{w_2 h - w_1(h-1)}{w_2 - h + 1} & \text{if } h \leq \frac{w_1}{2} \\ \frac{w_2 h - w_1(w_1 - h - 1)}{w_2 - w_1 + h + 1} & \text{if } h > \frac{w_1}{2} \end{cases}$$

which is always greater than 0. This shows that P cannot contain a point in the left edge of Q_{w_1,w_2} which is the desired contradiction.

Suppose, towards a contradiction that P contains the point $(h, 0)$ (see Figure 5.7b). Then $y_1 = 0$ and we consider $C_{(w_1,w_2)}$ which must contain all points of P with x-coordinate less than h. However, the line through (w_1, w_2) and (h, y_2) intersects the x-axis at the point

$$\left(\frac{w_2h - w_1y_2}{w_2 - y_2}, 0\right) = \begin{cases} \left(\frac{w_2h-w_1(h-1)}{w_2-(h-1)}, 0\right) & \text{if } h \leq \frac{w_1}{2} \\ \left(\frac{w_2h-w_1(w_1-h-1)}{w_2-(w_1-h-1)}, 0\right) & \text{if } h > \frac{w_1}{2} \end{cases}$$

which is always greater than 0. As above, this prevents P from containing a point in the left edge of Q_{w_1,w_2}, which is the desired contradiction.

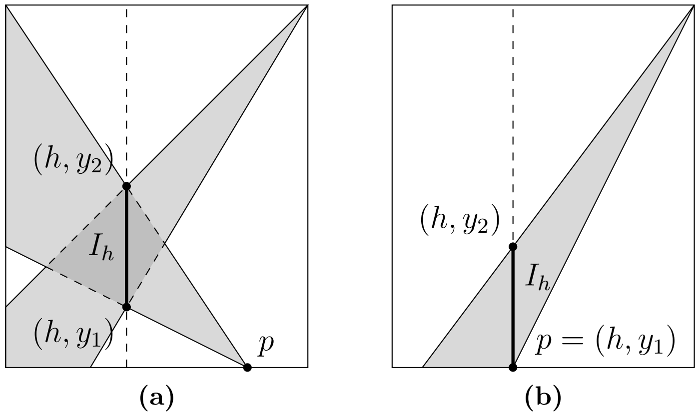


Figure 5.7: *In the proof of Lemma 5.4.7, if P contains a point p on the lower boundary of Q_{w_1,w_2} with x-coordinate greater than h, or the point $p = (h, 0)$, then the points of P with x-coordinate less than h must be contained in the intersection of the depicted cones. However, this prevents P from containing a point in the left boundary of Q_{w_1,w_2} as required.*

Now let v' be a point of P with x-coordinate greater than h. We may assume that a vertex of P in the lower edge of Q_{w_1,w_2} has x-coordinate less than h. The cone $C_{v'}$ must contain both this vertex and some point in the left edge of Q_{w_1,w_2} so $(0, 0)$ is contained in $C_{v'}$. Therefore, v' is contained in $C_{(0,0)}$.

We have shown that P is a subset of the union of $C_{(0,0)}$ and $C_{(w_1,w_2)}$ as depicted in Figure 5.8. We will show that the width of this region with respect to $u = (1, -1)$ is less than w_2 which is the desired contradiction.

By reflections, we can now assume that $h' = h$. Notice that the line $y = \frac{w_2}{w_1}x$ intersects the interval I_h non-trivially. In particular, $w_1y_1 \leq w_2h \leq w_1y_2$. Consider the slope of each edge of $C_{(0,0)}$ and C_{w_1,w_2} compared to the line

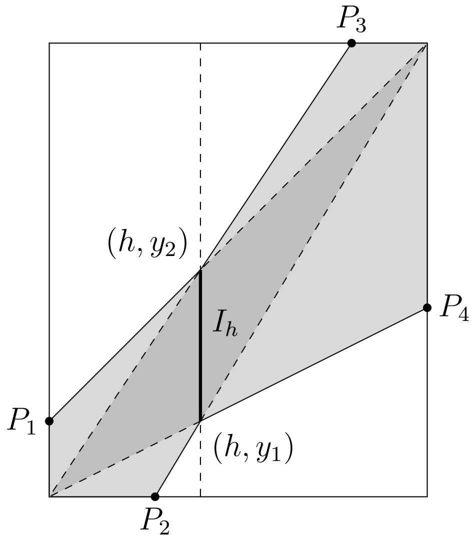


Figure 5.8: *In the proof of Lemma 5.4.7, the polytope P is contained in the union of the two shaded affine cones $C_{(0,0)}$ and $C_{(w_1,w_2)}$ whose points are at $(0,0)$ and (w_1, w_2) respectively.*

$y = \frac{w_2}{w_1}x$. The edges meeting at (h, y_1) and (h, y_2) have gradients such that these two points can never be a vertex of the convex hull

$$P' := \operatorname{conv}\left(\left(C_{(0,0)} \cup C_{(w_1,w_2)}\right) \cap Q_{w_1,w_2}\right)$$

so P' has vertices P_1, P_2, P_3 and P_4, depicted in Figure 5.8, with coordinates

$$\begin{aligned} P_1 &= (0, w_2 - \tfrac{w_2-y_2}{w_1-h}w_1), \quad & P_2 &= (w_1 - \tfrac{w_1-h}{w_2-y_1}w_2, 0), \\ P_3 &= (\tfrac{h}{y_2}w_2, w_2), & P_4 &= (w_1, \tfrac{y_1}{h}w_1). \end{aligned}$$

It suffices to show that the width of P' with respect to u is less than w_2 since P is a subset of P' so $\mathrm{width}_u(P)$ is at most $\mathrm{width}_u(P')$. To bound the width of P' we need only bound the absolute value of $u \cdot (P_i - P_j)$ for each pair of vertices P_i and P_j.

Since the value u takes increases from left to right along horizontal lines and from top to bottom along vertical ones, $u \cdot P_1 \leq u \cdot (0, 0) \leq u \cdot P_2$ and similarly $u \cdot P_3 \leq u \cdot (w_1, w_2) \leq u \cdot P_4$. Therefore, we need only show that the difference between each pair of values of $u \cdot P_i$ is less than w_2.

By considering the value of $u \cdot (0, 0)$ and $u \cdot (w_1, w_2)$ we see that $u \cdot P_1$ and

$u \cdot P_3$ are at most zero and that $u \cdot P_2$ is at least 0. Therefore, $u \cdot (P_1 - P_2)$ and $u \cdot (P_3 - P_2)$ are both negative and thus less than w_2. If $u \cdot P_i$ and $u \cdot P_j$ are both positive or both negative, we need not check $u \cdot (P_i - P_j)$ or $u \cdot (P_j - P_i)$ since they will always have a greater difference with some other vertex (a negative or positive one respectively). This means we need not check $u \cdot (P_1 - P_3)$ or $u \cdot (P_3 - P_1)$. Each of $u \cdot (P_1 - P_4)$, $u \cdot (P_3 - P_4)$ and $u \cdot (P_4 - P_3)$ are either negative or made up of $u \cdot P_i$ and $u \cdot P_j$ with the same sign, so we need not check these either. We now prove that the following are less than w_2 which completes the proof.

$$u \cdot (P_2 - P_1), \quad u \cdot (P_2 - P_3), \quad u \cdot (P_2 - P_4), \quad u \cdot (P_4 - P_1), \quad u \cdot (P_4 - P_3).$$

If $u \cdot (P_2 - P_1) \geq w_2$ then

$$w_1 - \tfrac{w_1 - h}{w_2 - y_1} w_2 + w_2 - \tfrac{w_2 - y_1 - h + 1}{w_1 - h} w_1 \geq w_2.$$

This inequality can be rearranged into the following:

$$w_1 y_1 (2w_2 - w_1 - y_1 + 1) \geq w_2 (w_1 w_2 + w_1 - 2w_1 h + h^2).$$

Since the left-hand side is positive we can use the fact that $w_2 h \geq w_1 y_1$ to say that this is all less than or equal to $w_2 h (2w_2 - w_1 - y_1 + 1)$. Dividing by w_2 and rearranging again shows that

$$h(2w_2 - y_1 + 1 + w_1 - h) \geq w_1 (w_2 + 1).$$

But $w_1 \geq 2h$ so, after cancelling and rearranging again, this implies that $-1 \geq y_1 + h$ which is the desired contradiction.

If $u \cdot (P_2 - P_3) \geq w_2$ then

$$w_1 - \tfrac{w_1 - h}{w_2 - y_1} w_2 - \tfrac{h}{y_1 + h - 1} w_2 + w_2 \geq w_2.$$

This inequality can be rearranged into the following:

$$w_2h(2y_1 - w_2 + h - 1) \geq w_1y_1(y_1 + h - 1).$$

The right-hand side is non-negative so we may assume the left-hand side is too. Using the fact that $w_2h \leq w_1(y_1 + h - 1)$ we can rearrange to get

$$y_1 + h \geq w_2 + 1$$

which is the desired contradiction.

If $u \cdot (P_2 - P_4) \geq w_2$ then

$$w_1 - \tfrac{w_1-h}{w_2-y_1}w_2 - w_1 + \tfrac{y_1}{h}w_1 \geq w_2$$

which we can rearrange into

$$w_1y_1(w_2 - y_1) \geq w_2h(w_2 - y_1 + w_1 - h).$$

Since we know that $w_2h \geq w_1y_1$ and the right-hand side is non-negative then

$$w_2 - y_1 \geq w_2 - y_1 + w_1 - h$$

which is a contradiction since $w_1 > h$.

If $u \cdot (P_4 - P_1) \geq w_2$ then

$$w_1 - \tfrac{y_1}{h}w_1 - \tfrac{w_2-y_1-h+1}{w_1-h}w_1 + w_2 \geq w_2$$

which can be rearranged into

$$h(w_1 - w_2 - 1) \geq y_1(w_1 - 2h).$$

The right-hand side is non-negative and the left-hand side is negative which is the desired contradiction.

If $u \cdot (P_4 - P_3) \geq w_2$ then

$$w_1 - \tfrac{y_1}{h}w_1 - \tfrac{h}{y_1+h-1}w_2 + w_2 \geq w_2$$

which can be rearranged into

$$w_1h(y_1 + h - 1) \geq w_2h^2 + w_1y_1(y_1 + h - 1)$$

but $w_1(y_1 + h - 1) \geq w_2h$ so we get

$$w_1h(y_1 + h - 1) \geq w_2h(h + y_1)$$

which is the desired contradiction. □

We introduce new notation for the following proofs. Let $p_h(P)$, $b_h(P)$ and $i_h(P)$ denote the number of points, boundary points and interior points of P in the line $x = h$.

Lemma 5.4.8. *Let P be a lattice polygon with vertices $(0, y_1)$, $(0, y_2)$, $(2, y_3)$, $(2, y_4)$ where $y_1 \leq y_2$ and $y_3 \leq y_4$, then*

$$p_0(P) + p_2(P) = p_1(P) + i_1(P) + 2.$$

Proof. The normalised volume of P is $2(y_2 - y_1 + y_4 - y_3)$ and the number of boundary points of P is $y_2 - y_1 + y_4 - y_3 + 2 + b_1(P)$. Using Pick's theorem we can combine these to show that the number if interior points of P is $\frac{1}{2}(y_2 - y_1 + y_4 - y_3) - \frac{1}{2}b_1(P)$. The number of points of P in $x = 0$ and $x = 2$ combined is equal to the number of boundary points of P minus $b_1(P)$ so

$$p_0(P) + p_2(P) = y_2 - y_1 + y_4 - y_3 + 2 = 2i(P) + b_1(P) + 2$$

which equals $p_1(P) + i_1(P) + 2$ since $i(P) = i_1(P)$. □

Lemma 5.4.9. *Fix integers $w > 1$, $k > 0$ and $1 \leq h \leq w - 1$. Up to affine equivalence, there are finitely many polygons P contained in the strip $[0, w] \times \mathbb{R}$ with width w and less than k lattice points in the line $x = h$.*

Proof. By a shear about the y-axis and a translation we may assume that the intersection of P and the line $x = h$ is a subset of the interval $I_h := \{h\} \times [0, k]$ and that P has a vertex $v_0 = (0, y_0)$ with $y_0 \in [0, h - 1]$. As in the proof of Lemma 5.4.7 all points of P with x-coordinate greater than h must be contained in the cone $C_{v_0} := v_0 + \operatorname{cone}(I_h - v_0)$. Since P has width w, it must have some vertex $v_w = (w, y_w)$ contained in the interval where C_{v_0} intersects the line $x = w$. Once again, all points of P with x-coordinate less than h must be contained in the cone C_{v_w}.

There are finitely many choices for v_0 and finitely many choices for v_w in each case. The conditions on P described above define a finite region which must contain P. Thus, there are only finitely many such polygons P. $\square$

Proposition 5.4.10. *Let P be a denominator 2, finitely growable polygon. Apart from finitely many exceptions, $p(2P) \leq 2b(P) + 6i(P) + 7$.*

Proof. By a change of basis we may assume that $\operatorname{width}(P) = \operatorname{width}_{(1,0)}(P)$ and that $(1, 0) \cdot P \subseteq [0, w]$ for some integer w such that

(A) $(1, 0) \cdot P = [0, w]$,

(B) $(1, 0) \cdot P = [0, w - \frac{1}{2}]$ or

(C) $(1, 0) \cdot P = [\frac{1}{2}, w - \frac{1}{2}]$.

We prove the bound by bounding the number of points of $2P$ in each line $x = 0, 1, \ldots, 2w$.

First notice that by Lemma 5.4.8, for any $h = 1, \ldots, w - 1$

$$p_{2h-1}(2P) + p_{2h+1}(2P) \leq p_{2h}(2P) + i_{2h}(2P) + 2.$$

This allows us to eliminate positive odd terms from $p(2P) = \sum_{h=0}^{2w} p_h(2P)$ as follows:

$$\begin{aligned} p(2P) &= \sum_{h=0}^{w} p_{2h}(2P) + \sum_{h=1}^{w-1} (p_{2h-1}(2P) + p_{2h+1}(2P)) - \sum_{h=1}^{w-2} p_{2h+1}(2P) \\ &\leq \sum_{h=0}^{w} p_{2h}(2P) + \sum_{h=1}^{w-1} (p_{2h}(2P) + i_{2h}(2P) + 2) - \sum_{h=1}^{w-2} p_{2h+1}(2P). \end{aligned}$$

Now observe that for any integer $0 < h < w$ the number of points $p_{2h}(2P)$ is at most $2i_h(P) + b_h(P) + 1$ and the number of interior points $i_{2h}(2P)$ is at most $2i_h(P) + 1$. We can simplify the previous inequality and substitute these bounds into it to obtain

$$p(2P) \leq p_0(2P) + p_{2w}(2P) + 6i(P) + 5(w-1) + \sum_{h=1}^{w-1} 2b_h(P) - \sum_{h=1}^{w-2} p_{2h+1}(2P).$$

The number of points $p_0(2P)$ and $p_{2w}(2P)$ are bounded by $2b_0(P) + 1$ and $2b_w(P) + 1$ respectively. Applying this to the previous inequality, collecting $b_h(P)$ terms and replacing $\sum_{h=0}^{w} b_h(P)$ with $b(P)$ we get the following:

$$p(2P) \leq 2b(P) + 6i(P) + 5(w-1) + 2 - \sum_{h=1}^{w-2} p_{2h+1}(2P).$$

If $w = 2$ then $5(w-1) + 2 - \sum_{h=1}^{w-2} p_{2h+1}(2P)$ is at most 7, which proves the result in this case. For larger w we find a lower bound for $\sum_{h=1}^{w-2} p_{2h+1}(2P)$ using Lemma 5.4.7. This bound depends on which case out of (A), (B) and (C) we are in. First notice that for any odd integer $3 \leq h \leq w - 3$

$$\text{(A)} \quad p_h(2P) \geq \begin{cases} h - 1 & \text{if } h \leq w \\ (2w - h) - 1 & \text{if } h > w \end{cases}$$

$$\text{(B)} \quad p_h(2P) \geq \begin{cases} h - 1 & \text{if } h < w - \frac{1}{2} \\ (2w - h) - 2 & \text{if } h > w - \frac{1}{2} \end{cases}$$

$$\text{(C)} \quad p_h(2P) \geq \begin{cases} h - 2 & \text{if } h \leq w \\ (2w - h) - 2 & \text{if } h > w. \end{cases}$$

We sum each of these, separating the cases where w is odd and even, to get

(A) $\sum_{h=1}^{w-2} p_{2h+1}(2P) \geq \begin{cases} \frac{1}{2}w^2 - w & \text{if } w \text{ even} \\ \frac{1}{2}w^2 - w + \frac{1}{2} & \text{if } w \text{ odd} \end{cases}$

(B) $\sum_{h=1}^{w-2} p_{2h+1}(2P) \geq \frac{1}{2}w^2 - \frac{3}{2}w + 1$

(C) $\sum_{h=1}^{w-2} p_{2h+1}(2P) \geq \begin{cases} \frac{1}{2}w^2 - 2w + 2 & \text{if } w \text{ even} \\ \frac{1}{2}w^2 - 2w + \frac{5}{2} & \text{if } w \text{ odd.} \end{cases}$

Applying these bounds to $p(2P)$ shows that

(A) $p(2P) \leq 2b(P) + 6i(P) + \begin{cases} -\frac{1}{2}w^2 + 6w - 3 & \text{if } w \text{ even} \\ -\frac{1}{2}w^2 + 6w - \frac{7}{2} & \text{if } w \text{ odd} \end{cases}$

(B) $p(2P) \leq 2b(P) + 6i(P) - \frac{1}{2}w^2 + \frac{13}{2}w - 4$

(C) $p(2P) \leq 2b(P) + 6i(P) + \begin{cases} -\frac{1}{2}w^2 + 7w - 5 & \text{if } w \text{ even} \\ -\frac{1}{2}w^2 + 7w - \frac{11}{2} & \text{if } w \text{ odd} \end{cases}$

The polynomials in w are less than or equal to 7 when

$$\text{(A)} \quad w \geq 10, \quad \text{(B)} \quad w \geq 11, \quad \text{(C)} \quad w \geq 12.$$

Therefore, if $p(2P) > 2b(P) + 6i(P) + 7$ then the width of $2P$ is at most 20 and it contains less than $5(w-1) - 5$ lattice points in the hyperplanes $x = 3, 5, \ldots, 2w - 3$ in total. In particular, $2P$ contains less than $5(w-1) - 5$ points in the hyperplane $x = 3$. By Lemma 5.4.9 this shows finiteness of the exceptions. □

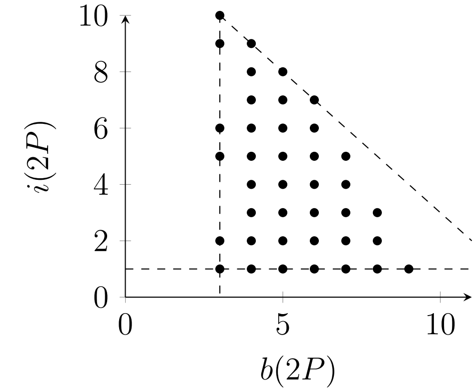
Finally, we prove the last two bounds in Theorem 5.0.3.

Proposition 5.4.11. *Let P be a polygon with denominator 2 and at least one interior point. Then $b(2P) \geq \max\{3, 2b(P)\}$ and $i(2P) \geq b(P) + 2i(P) - 1$.*

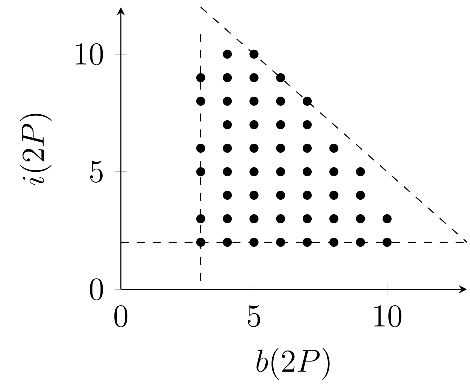
Proof. By definition $2P$ is a lattice polygon so $b(2P) \geq 3$. If $b(P) \geq 2$, consider two adjacent boundary lattice points of P (i.e. one can walk from one to the other along the boundary of P without touching another lattice point). These are either on the same edge of P, or there is at least one half-integral vertex on the boundary between them. In either case, there is a point of $\frac{1}{2}\mathbb{Z}$ on the boundary of P between them. Therefore, $b(2P) \geq 2b(P)$ and the first inequality holds.

Let Q be the convex hull of the interior lattice points of P. If Q is a point or a line then $2Q$ contains $2i(P)-1$ points. Otherwise, $2Q$ contains $3i(P)+i(Q)-3$ points by the considering the Ehrhart polynomial of Q and Pick's theorem. Since Q is a polygon, P has at least three interior points so $3i(P)+i(Q)-3 \geq 2i(P)-1$. Therefore, $2Q$ contains at least $2i(P)-1$ lattice points and all of these are interior points of $2P$.

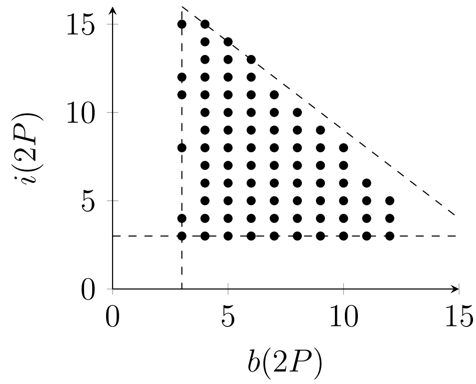
Now, consider subdividing the region $P \setminus Q$ in such a way that each boundary lattice point of P has a corresponding line to a boundary lattice point of Q. Each such line has a half-integral point at its midpoint which will be interior points of $2P$ not contained in $2Q$, when doubled. This gives at least an additional $b(P)$ interior points in $2P$. Thus, the second inequality holds. $\square$



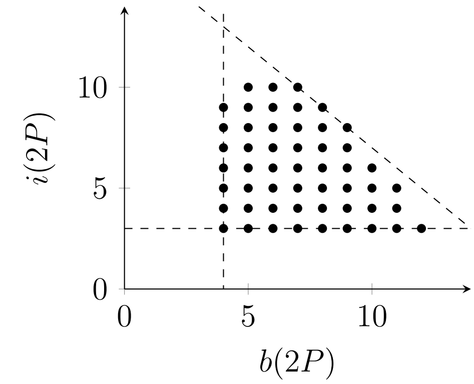
(a) *P has 1 interior point and 0 boundary points*



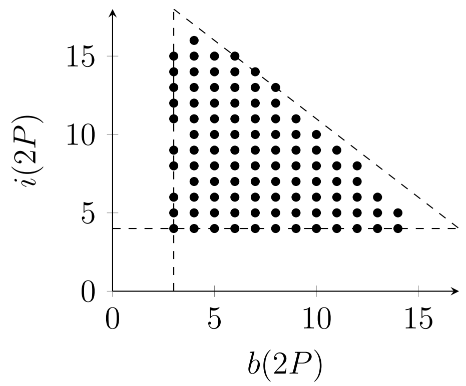
(b) *P has 1 interior point and 1 boundary point*



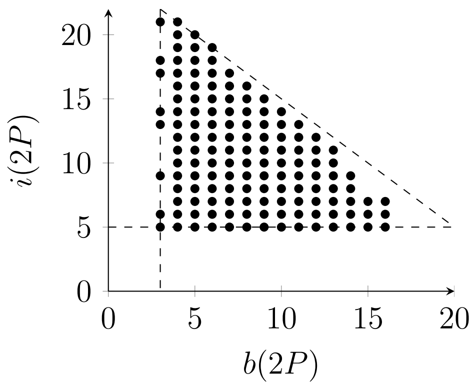
(c) *P has 2 interior points and 0 boundary points*



(d) *P has 1 interior point and 2 boundary points*

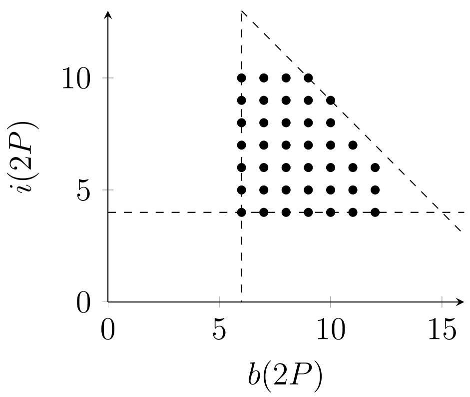


(e) *P has 2 interior points and 1 boundary point*

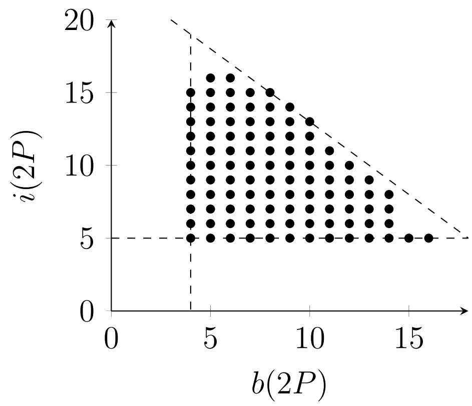


(f) *P has 3 interior points and 0 boundary points*

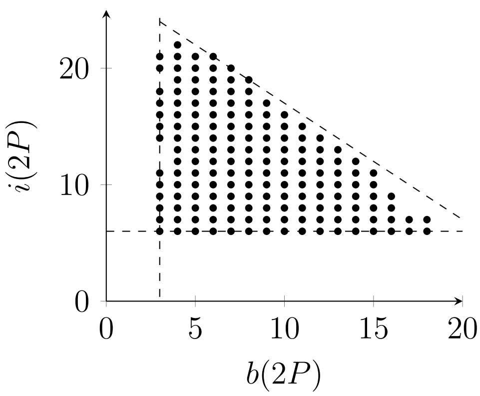
Figure 5.9: *Number of boundary and interior points for P and $2P$ for denominator 2 polygons P of size up to 4. Continued on next page.*



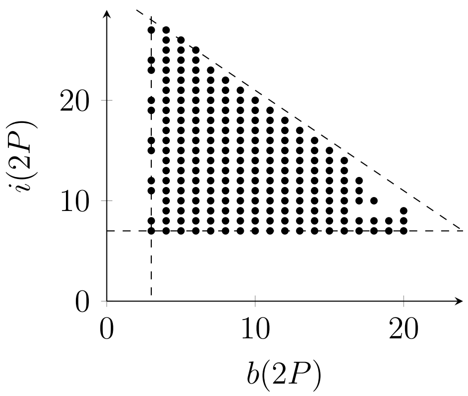
(g) *P has 1 interior point and 3 boundary points*



(h) *P has 2 interior points and 2 boundary points*



(i) *P has 3 interior points and 1 boundary point*



(j) *P has 4 interior points and 0 boundary points*

Figure 5.9: *Ehrhart data for denominator 2 polygons of size up to 4 continued.*

Chapter 6

Isomorphisms of Spherical Varieties

Let G be a connected reductive complex algebraic group. A G-variety is a variety X with an action of G on X. A normal irreducible G-variety X is called a *spherical variety* if there is some point $x_0 \in X$ and some Borel subgroup $B \subseteq G$ such that the B-orbit $B \cdot x_0$ is open and dense in X. A *spherical homogeneous space* is a spherical variety of the form G/H for some subgroup $H \subseteq G$, and such an H is called a *spherical subgroup* of G. In the same way that toric varieties can be characterised by an open dense embedding of a torus $T \hookrightarrow X$, spherical varieties have a G-equivariant open dense embedding of a spherical homogeneous space $G/H \hookrightarrow X$.

A spherical homogeneous space gives us more complicated combinatorics than an algebraic torus. Luna [Lun01] defined a *Luna datum* (also called a *spherical homogeneous datum*), which is a tuple $\mathcal{S} = (\mathcal{M}, \Sigma, S^P, \mathcal{D}^a)$, along with a map ρ, which can be assigned to any spherical homogeneous space G/H. For a full description of this tuple see Section 6.1.2, but for now it suffices to say that $\mathcal{M}$ is a lattice, Σ is a finite set of points in $\mathcal{M}$, S^P is a subset of simple roots of G and $\mathcal{D}^a$ is an abstract set equipped with a map $\rho : \mathcal{D}^a \to \mathcal{N}$ where $\mathcal{N}$ is the lattice dual to $\mathcal{M}$.

Luna formulated a list of axioms which $\mathcal{S}$ satisfies (see Definition 6.1.5), and it has been shown by Bravi, Cupit-Foutou, Loseva and Pezzini that spherical subgroups of G, up to G-equivariant automorphism, are in bijection with Luna data [BP14, CF14, Los09]. Much like toric varieties, spherical varieties have a combinatorial description in terms of lattice fans, but spherical varieties require the additional structure of *colored lattice fans* which are defined with respect to their Luna datum.

Recall, in toric geometry, given two algebraic tori S, T and two toric embeddings $S \hookrightarrow X$ and $T \hookrightarrow Y$, a *toric morphism* $\varphi\colon X \to Y$ is a morphism of algebraic varieties such that $\varphi(S) \subseteq T$ and $\varphi|_S\colon S \to T$ is a morphism of algebraic tori. Toric morphisms are combinatorially described by morphisms of lattice fans, which are a class of lattice morphism between the character lattices of T and S. These play a crucial role in the study of toric varieties. For instance, classification results in toric geometry use the fact that it is enough to consider the corresponding combinatorial objects up to isomorphisms, for example, the classifications of Canonical toric Fano three-folds [Kas10] and Gorenstein toric Fano four-folds [KS00].

Our goal is to extend the understanding of toric isomorphisms to the spherical setting. We use a combinatorial description of epimorphisms of reductive groups described in Section 6.1.1, so from now on, all morphisms of algebraic groups are assumed to be surjective. We generalise toric morphisms as follows.

Definition 6.0.1. Given a spherical G-variety X and a spherical G'-variety X', a pair (F, φ) consisting of a homomorphism of algebraic groups $\varphi\colon G \to G'$ and a morphism of algebraic varieties $F\colon X \to X'$ is called a *twisted equivariant morphism* if $F(g \cdot x) = \varphi(g) \cdot F(x)$ for all $g \in G$ and $x \in X$. We write $(F, \varphi)\colon X \to X'$ when we need to specify the source and image.

In [Kno91, Theorem 4.1], Knop describes exactly when a G-equivariant morphism $(F, \varphi) : G/H \to G/H'$ extends to spherical embeddings $G/H \hookrightarrow X$ and $G/H' \hookrightarrow X'$. The same argument generalises to the twisted equivariant morphisms $(F, \varphi) : G/H \to G'/H'$ which we consider. Naturally, any twisted equivariant morphism $(F, \varphi) : X \to X'$ between spherical embeddings restricts to the corresponding homogeneous spaces. Thus, we restrict our attention to twisted equivariant morphisms between spherical homogeneous spaces.

In general, describing the combinatorial maps induced by twisted equivariant morphisms is difficult, so instead we describe certain lattice automorphisms which are easier to work with, and prove that these are induced by geometric isomorphisms. The first class of lattice automorphisms is defined as follows.

Definition 6.0.2. Let $\mathcal{S}$ be a Luna datum, then an *isomorphism from* $\mathcal{S}$ is a lattice automorphism $\phi\colon \mathcal{M} \to \mathcal{M}$ such that $\phi(\gamma) = \gamma$ for every $\gamma \in \Sigma$. The set of isomorphisms from $\mathcal{S}$ is denoted by $\mathrm{Iso}(\mathcal{S})$.

This is a generalization of [Pas08, Proposition 3.5 and the paragraph before it.] and $\mathrm{Iso}(\mathcal{S})$ is a subgroup of $\mathrm{Aut}(\mathcal{M})$ which has previously appeared in [AB04, Lemma 4.8], though not in the context of Luna data. Notice that these isomorphisms change the Luna datum $\mathcal{S}$, so it is not immediate that there exists a Luna datum that these maps are *to*. The existence of such a datum is proven as part of Theorem 6.0.3 bellow.

In general, isomorphisms from a Luna datum are not induced by a twisted equivariant automorphism, but related in the following sense: in [AB04], Alexeev and Brion show that there exists a bigger connected reductive group $\mathcal{G}$ such that we may regard $G/H \cong \mathcal{G}/\mathcal{H}$ as a spherical homogeneous space with respect to this new group and the new action is *smart* (see Definition 6.1.12). There exists a natural choice of a Borel subgroup $\mathcal{B} \subseteq \mathcal{G}$ and of a maximal torus $\mathcal{T} \subseteq \mathcal{B}$ corresponding to the choice of (B, T) for G. With respect to this natural choice, the Luna datum of $\mathcal{G}/\mathcal{H}$ gets naturally identified with the Luna datum $\mathcal{S}$ of G/H.

Theorem 6.0.3. *For every $\phi \in \mathrm{Iso}(\mathcal{S})$ there is a spherical homogeneous space G'/H' and an adapted twisted equivariant isomorphism $(F, \varphi) : G'/H' \to \mathcal{G}/\mathcal{H}$ such that $\varphi^*|_{\mathcal{M}} = \phi$. The Luna datum of G'/H' is naturally identified with the Luna datum of $\mathcal{G}/\mathcal{H}$.*

(For the definition of *adapted* see Section 6.2).

In Section 6.1 we recall some past results which we will use later. Section 6.1.1 recalls facts about reductive groups and root data. Section 6.1.2 describes the components of a Luna datum in detail and gives the Luna datum axioms. Section 6.1.3 is concerned with the combinatorial description of spherical subgroups of G which contain H. Section 6.1.4 describes smart actions and some of the properties the Luna datum has when the action of G on G/H is smart. In Section 6.2 we refine the correspondence between Luna subdata and spherical subgroups described in Theorem 6.1.10 and use this to combinatorially describe twisted equivariant morphisms. In Section 6.3 we prove Theorem 6.0.3. Finally, in Section 6.4 we describe an algorithm which returns a normal form for rational polytopes in $\mathcal{N}_{\mathbb{Q}}$, under the action of $\mathrm{Iso}(\mathcal{S})$.

6.1 Background

6.1.1 Reductive Groups

Understanding reductive groups is key to spherical geometry. They have a combinatorial description in terms of root data which we will recall here. For further details see [Spr09, Chapters 7-10]

Definition 6.1.1. A (reduced) *root datum* is a tuple $\Psi = (\mathfrak{X}, R, \check{\mathfrak{X}}, \check{R})$ where

1. $\mathfrak{X}$ and $\check{\mathfrak{X}}$ are a pair of finite rank lattices with the same rank, and a dual pairing $\langle \cdot, \cdot \rangle \colon \check{\mathfrak{X}} \times \mathfrak{X} \to \mathbb{Z}$, and

2. $R \subseteq \mathfrak{X}$, $\check{R} \subseteq \check{\mathfrak{X}}$ are finite subsets, with a bijection $R \to \check{R}; \alpha \mapsto \check{\alpha}$.

For $\alpha \in R$ we define endomorphisms $s_\alpha \colon \mathfrak{X} \to \mathfrak{X}$ and $s_{\check{\alpha}} \colon \check{\mathfrak{X}} \to \check{\mathfrak{X}}$ by

$$s_\alpha(x) = x - \langle \check{\alpha}, x \rangle \alpha \qquad \text{and} \qquad s_{\check{\alpha}}(y) = y - \langle y, \alpha \rangle \check{\alpha}.$$

The root datum Ψ satisfies the following axioms:

(RD1) $\langle \check{\alpha}, \alpha \rangle = 2$ for all $\alpha \in R$,

(RD2) $s_\alpha(R) \subseteq R$, $s_{\check{\alpha}}(\check{R}) \subseteq \check{R}$ for all $\alpha \in R$, and

(RD3) (Reduced) for all $\alpha \in R$ the only multiples of α in R are $\pm\alpha$.

We can associate a root datum to any reductive group G as follows: Choose a maximal torus $T \subseteq G$ and write $\mathfrak{X}$ for its character lattice and $\check{\mathfrak{X}}$ for its lattice of 1-parameter subgroups. Then the roots R are the weights of the adjoint action of T on the Lie algebra $\mathfrak{g}$ of G and $\check{R}$ can be determined from R by the above axioms.

A set $R^+ \subseteq R$ is called a choice of *positive roots* if, for every $\alpha \in R$, exactly one of α and $-\alpha$ is in R^+, and for every $\alpha, \beta \in R^+$, if $\alpha + \beta$ is a root then $\alpha + \beta \in R^+$. An element of R^+ is called a *simple root* if it cannot be written as a sum of two other elements of R^+ and we denote the set of simple roots in R^+

by S. A choice of Borel subgroup amounts to a choice of positive roots R^+ or a choice of simple roots S. A parabolic subgroup of G is a subgroup containing a Borel subgroup B. There is a natural way to associate a parabolic subgroup P_α, containing B, to a simple root $\alpha \in S$ and any subset of simple roots determines a parabolic subgroup containing B. The main reductive groups which will be relevant here are $(\mathbb{C}^\times)^n$, SL_2, SL_3, SL_4, Sp_4 and products of these. We list choices of simple and positive roots for each of these in Table 6.1 for later convenience.

G	$\mathfrak{X}(T)$	S	$\check{S}$	R^+
$(\mathbb{C}^\times)^n$	$\mathbb{Z}^n$	$\emptyset$	$\emptyset$	$\emptyset$
SL_2	$\mathbb{Z}$	$\alpha_1 = 2$	1	α_1
SL_3	$\mathbb{Z}^2$	$\alpha_1 = (1,-1)$, $\alpha_2 = (1,2)$	$(1,-1)$, $(0,1)$	α_1, α_2, $\alpha_1+\alpha_2$
SL_4	$\mathbb{Z}^3$	$\alpha_1 = (-1,1,0)$, $\alpha_2 = (0,-1,1)$, $\alpha_3 = (-1,-1,-2)$	$(-1,1,0)$ $(0,-1,1)$ $(0,0,-1)$	$\alpha_1, \alpha_2, \alpha_3$, $\alpha_1+\alpha_2$, $\alpha_2+\alpha_3$, $\alpha_1+\alpha_2+\alpha_3$
Sp_4	$\mathbb{Z}^2$	$\alpha_1 = (1,-1)$, $\alpha_2 = (0,2)$	$(1,-1)$, $(0,1)$	α_1, α_2, $\alpha_1+\alpha_2$, $2\alpha_1+\alpha_2$

Table 6.1: Some reductive groups G and a choice of simple and positive roots.

There is a notion of isomorphism of root datum so that two reductive groups are isomorphic if and only if their root data are. This has been extended to a description of epimorphisms of root data by Steinberg [Ste99].

Definition 6.1.2. Let $\Psi = (\mathfrak{X}, R, \check{\mathfrak{X}}, \check{R})$ and $\Psi' = (\mathfrak{X}', R', \check{\mathfrak{X}}', \check{R}')$ be two root data. An *epimorphism* from Ψ onto Ψ' is a lattice homomorphism $\varphi^* \colon \mathfrak{X}' \to \mathfrak{X}$, satisfying the following:

1. φ^* is injective,
2. there exists a partition of sets $R = R_1 \cup R_2$,
3. there exists a bijection $\alpha \mapsto \alpha'$ from R_1 onto R',
4. $\varphi^*(\alpha') = \alpha$, $\varphi_*(\check{\alpha}) = \check{\alpha}'$ for all $\alpha \in R_1$, and

5. $\varphi_*(\check{\alpha}) = 0$ for all $\alpha \in R_2$,

where $\varphi_* \colon \check{\mathfrak{X}} \to \check{\mathfrak{X}}'$ denotes the morphism dual to φ^*.

By [Ste99, 1.5 Isogeny Theorem and Section 5], we get the following.

Theorem 6.1.3 (Epimorphism Theorem)**.** *Let G and G' be reductive groups and let T and T' be maximal tori of them with corresponding character groups $\mathfrak{X}$ and $\mathfrak{X}'$ respectively. To every epimorphism of their root data $\varphi^* \colon \mathfrak{X}' \to \mathfrak{X}$, there exists an epimorphism φ from G onto G' which maps T onto T' and induces φ^*. It is uniquely determined up to composition with the inner automorphism of G effected by an element of T.*

In light of this result, we define an equivalence of epimorphisms so that two epimorphisms are equivalent exactly when they induce the same epimorphism of root data.

Definition 6.1.4. Two epimorphisms $\varphi, \varphi' \colon G \to G'$ are called *equivalent*, if there exists $t \in T$ such that $\varphi' = \varphi \circ \operatorname{Inn}(t)$ where $\operatorname{Inn}(t)$ denotes the inner automorphism of G given by t.

6.1.2 Luna Data

Let G be a reductive group with Borel subgroup B, maximal torus $T \subseteq B$ and spherical subgroup H. We describe the Luna datum $\mathcal{S} = (\mathcal{M}, \Sigma, S^P, \mathcal{D}^a)$.

Note that the character lattices $\mathfrak{X}(B)$ and $\mathfrak{X}(T)$ are isomorphic. $\mathcal{M}$ is the sublattice of $\mathfrak{X}(B)$, of weights $\chi \in \mathfrak{X}(B)$ such that there is a B-semi-invariant rational function $f \in \mathbb{C}(G/H)^{(B)}$ with weight χ, that is $b \cdot f(x) = \chi(b)f(x)$. Such a function f_χ is determined, up to a constant factor, by its weight, so we can write $\mathcal{M} = \mathbb{C}(G/H)^{(B)}/\mathbb{C}^\times$. Let $\mathcal{N} \coloneqq \operatorname{Hom}(\mathcal{M}, \mathbb{Z})$ be the lattice dual to $\mathcal{M}$, with the natural dual pairing $\langle \cdot, \cdot \rangle \colon \mathcal{N} \times \mathcal{M} \to \mathbb{Z}$. Let $\mathcal{N}_\mathbb{Q}$ denote the rational vector-space containing $\mathcal{N}$, that is $\mathcal{N}_\mathbb{Q} \coloneqq \mathcal{N} \otimes_\mathbb{Z} \mathbb{Q}$.

Let $\mathcal{V}$ be the set of G-invariant discrete valuations $\nu : \mathbb{C}(G/H)^* \to \mathbb{Q}$. To such a valuation, we can associate a point $x \in \mathcal{N}_\mathbb{Q}$ by requiring that $\langle x, \chi \rangle =$

$\nu(f_\chi)$ for all $\chi \in \mathcal{M}$. The map $\nu \mapsto x$ is injective so we can think of $\mathcal{V}$ as a subset of $\mathcal{N}_\mathbb{Q}$. It is shown in [Bri90] that $\mathcal{V}$ is a cosimplicial cone, so the negative dual cone $-\mathcal{V}^*$ has up to $\mathrm{rk}(\mathcal{M})$ extremal rays. Σ is defined to be the set of primitive generators of these rays. The elements in Σ are called the *spherical roots* of G/H.

The stabilizer P of the open B-orbit in G/H is a parabolic subgroup of G containing B, hence uniquely determines a subset $S^P \subseteq S$ of simple roots.

A B-invariant prime divisor in G/H is called a *color*, and we write $\mathcal{D}$ for the set of colors of G/H. A color $D \in \mathcal{D}$ induces a valuation denoted by ν_D. We define a map $\rho : \mathcal{D} \to \mathcal{N}$ by requiring that $\langle \rho(D), \chi \rangle = \nu_D(f_\chi)$ for all $\chi \in \mathcal{M}$. For $\alpha \in S$, let $\mathcal{D}(\alpha)$ be the set of colors which are moved by the parabolic subgroup P_α defined in Section 6.1.1. We obtain a map $\varsigma \colon \mathcal{D} \to \mathcal{P}(S)$ by assigning to a color $D \in \mathcal{D}$ the set of simple roots α such that P_α moves D ($\mathcal{P}(S)$ denotes the power set of S). A color $D \in \mathcal{D}(\alpha)$ is said to be of type a if $\alpha \in \Sigma$, of type $2a$ if $2\alpha \in \Sigma$, and of type b otherwise. Every color $D \in \mathcal{D}$ has one of these types and it does not depend on the choice of α, so we can partition $\mathcal{D}$ into subsets $\mathcal{D}^a$, $\mathcal{D}^{2a}$ and $\mathcal{D}^b$ corresponding to each type. This defines the entry $\mathcal{D}^a$ of the Luna datum. It is treated as an abstract set equipped with the restricted map $\rho|_{\mathcal{D}^a}$.

Let Σ_G denote the set of all spherical roots of G, which are simple minimal roots of arbitrary wonderful G-varieties of rank 1 (see for example [BL11] for more detail). We recall the list of Luna datum axioms from [Lun01].

Definition 6.1.5. Given a reductive group G with maximal torus T, a *Luna datum* is a tuple $(\mathcal{M}, \Sigma, S^P, \mathcal{D}^a)$, where $\mathcal{M}$ is a sublattice in $\mathfrak{X}(T)$, $\Sigma \subseteq \Sigma_G$ is a linearly independent set of primitive lattice points in $\mathcal{M}$, S^P is a subset of the simple roots S of G, and $\mathcal{D}^a$ is a finite set equipped with a map $\rho : \mathcal{D}^a \to \mathcal{N}$, satisfying the following axioms:

(A1) $\langle \rho(D), \gamma \rangle \leq 1$ for all $D \in \mathcal{D}^a$ and $\gamma \in \Sigma$. and the equality is reached if and only if $\gamma = \alpha \in \Sigma \cap S$ and $D = D_\alpha^\pm$, where D_α^+ and D_α^- are two distinct elements depending on α.

(A2) $\rho(D_\alpha^+) + \rho(D_\alpha^-) = \alpha^\vee$ on $\mathcal{M}$ for all $\alpha \in \Sigma \cap S$

(A3) $\mathcal{D}^a = \{D_\alpha^\pm : \alpha \in \Sigma \cap S\}$

(Σ1) If $\alpha \in S \cap \frac{1}{2}\Sigma$, then $\langle \alpha^\vee, \mathcal{M} \rangle \subseteq 2\mathbb{Z}$ and $\langle \alpha^\vee, \Sigma \setminus \{2\alpha\} \rangle \leq 0$

(Σ2) If $\alpha, \beta \in S$, $\alpha \perp \beta$, and $\alpha + \beta \in \Sigma \cup 2\Sigma$, then $\alpha^\vee = \beta^\vee$ on $\mathcal{M}$.

(S) $\langle \alpha^\vee, \mathcal{M} \rangle = 0$ for all $\alpha \in S^P$, and the pair (γ, S^P) come from a wonderful variety of rank 1 for any $\gamma \in \Sigma$

The last condition has a reformulation which can be seen in [BL11, Section 1.1.6]. This depends on the fact that all spherical roots are linear combinations of simple roots, and thus have a support supp(γ) which is a collection of roots, denoted by their associated Dynkin diagram.

Definition 6.1.6. We say that a pair (γ, S^P), of a spherical root and a set of simple roots, is *compatible* if

$$S^{pp}(\gamma) \subseteq S^P \subseteq S^P(\gamma)$$

where $S^P(\gamma) = \{\alpha \in S : \langle \check{\alpha}, \gamma \rangle = 0\}$ and

$$S^{pp}(\gamma) = \begin{cases} \operatorname{supp}(\gamma) \cap S^P(\gamma) \setminus \{\alpha_r\} & \text{if } \gamma = \alpha_1 + \cdots + \alpha_r \text{ and has supp. of type } B_r \\ \operatorname{supp}(\gamma) \cap S^P(\gamma) \setminus \{\alpha_1\} & \text{if } \gamma \text{ has supp. of type } C_r \\ \operatorname{supp}(\gamma) \cap S^P(\gamma) & \text{otherwise.} \end{cases}$$

We can replace the condition **(S)** above with

(S) $\langle \alpha^\vee, \mathcal{M} \rangle = 0$ for all $\alpha \in S^P$, and for any spherical root $\gamma \in \Sigma$ the pair (γ, S^P) is compatible.

6.1.3 Containment Relations for Spherical Subgroups

Let us recall from [Hof18] how containment relations among spherical subgroups can be determined combinatorially. Throughout, G is a reductive group with maximal torus and Borel subgroup $T \subseteq B \subseteq G$. Let H be a spherical subgroup of G with Luna datum $\mathcal{S} = (\mathcal{M}, \Sigma, S^P, \mathcal{D}^a)$.

A *colored subspace* is a pair $(\mathcal{N}^1_{\mathbb{Q}}, \mathcal{D}^1)$ where $\mathcal{N}^1_{\mathbb{Q}} \subseteq \mathcal{N}_{\mathbb{Q}}$ is a subspace and $\mathcal{D}^1 \subseteq \mathcal{D}$ is a subset, such that $\mathcal{N}^1_{\mathbb{Q}}$ coincides with the cone spanned by $\mathcal{V} \cap \mathcal{N}^1_{\mathbb{Q}}$ and $\rho(D)$ for all $D \in \mathcal{D}^1$. By Knop [Kno91, Section 4], colored subspaces are in bijective correspondence with spherical subgroups $H' \subseteq G$ containing H such that H'/H is connected. As described by Losev [Los09, Proposition 3.4.3], the Luna datum $\mathcal{S}' = (\mathcal{M}', \Sigma', (S')^P, (\mathcal{D}')^a)$ associated to H' can be determined by the colored subspace as follows.

- $\mathcal{M}' = (\mathcal{N}^1_{\mathbb{Q}})^\perp \cap \mathcal{M}$,
- Σ' is the set of primitive ray generators in $\mathcal{M}'$ of $\operatorname{cone}(\Sigma) \cap (\mathcal{N}^1_{\mathbb{Q}})^\perp$,
- $(S')^P = \{\alpha \in S : \mathcal{D}(\alpha) \subseteq \mathcal{D}^1\}$,
- $(\mathcal{D}')^a = \{D \in \mathcal{D}^a : \varsigma(D) \cap \Sigma' \neq 0\}$ and
- $\rho' = \pi \circ \rho$ where $\pi : \mathcal{N} \to \mathcal{N}'$ is dual to the inclusion of $\mathcal{M}'$ in $\mathcal{M}$.

We write $\mathfrak{X}(R)$ for the subgroup of $\mathfrak{X}$ generated by R, called the *root lattice* of R. We define the set of *distinguished roots* Σ^+ to be the set of all $\gamma \in \Sigma$ such that,

1. γ is in the root lattice $\mathfrak{X}(R)$,
2. there is a spherical subgroup in G with Luna datum $(\mathbb{Z}2\gamma, \{2\gamma\}, S^P, \emptyset)$,
3. if $\gamma \in S$, then $\rho(D^+) = \rho(D^-)$ where $\mathcal{D}(\gamma) = \{D^\pm\}$.

Definition 6.1.7. Let $\widetilde{\mathcal{M}}$ be a subgroup of $\mathcal{M}$ and $\mathcal{D}^1$ a subset of $\mathcal{D}$ such that $\mathcal{C} := (\widetilde{\mathcal{M}}^\perp, \mathcal{D}^1)$ is a colored subspace. Let $\mathcal{S}_0 = (\mathcal{M}_0, \Sigma_0, S^P_0, \mathcal{D}^a_0)$ be the Luna datum of the spherical subgroup corresponding to $\mathcal{C}$ and $\widetilde{\Sigma}$ the set of primitive generators of $\mathrm{cone}(\Sigma) \cap \widetilde{\mathcal{M}}_{\mathbb{Q}}$. Then the pair $(\widetilde{\mathcal{M}}, \mathcal{D}^1)$ is called *distinguished* if $\frac{1}{2}\gamma \in \Sigma_0^+ \cup (\Sigma_0 \setminus \mathfrak{X}(R))$ for every $\gamma \in \widetilde{\Sigma} \setminus \Sigma_0$.

Theorem 6.1.8 ([Hof18, Theorem 1.2])**.** *Let $\widetilde{H} \subseteq G$ be a spherical subgroup containing H and having Luna datum $\widetilde{\mathcal{S}} = (\widetilde{\mathcal{M}}, \widetilde{\Sigma}, \widetilde{S}^p, \widetilde{\mathcal{D}}^a)$. Let $\mathcal{D}^1 \subseteq \mathcal{D}$ be the subset of colors of G/H which get mapped dominantly onto $G/\widetilde{H}$ under the natural map $G/H \to G/\widetilde{H}$. Then $(\widetilde{\mathcal{M}}, \mathcal{D}^1)$ is a distinguished pair and the assignment $\widetilde{H} \mapsto (\widetilde{\mathcal{M}}, \mathcal{D}^1)$ induces a bijection between the spherical subgroups in G containing H and distinguished pairs.*

Theorem 6.1.8 can be rephrased using Luna data.

Definition 6.1.9. A Luna datum $\widetilde{\mathcal{S}} = (\widetilde{\mathcal{M}}, \widetilde{\Sigma}, \widetilde{S}^p, \widetilde{\mathcal{D}}^a)$ is called a *subdatum of* $\mathcal{S}$, if there exists a distinguished pair $(\widetilde{\mathcal{M}}, \mathcal{D}^1)$ such that

1. $\widetilde{\Sigma}$ are the primitive ray generators in $\widetilde{\mathcal{M}}$ of $\mathrm{cone}(\Sigma) \cap \widetilde{\mathcal{M}}_{\mathbb{Q}}$,

2. $\widetilde{S}^p = \{\alpha \in S \colon \mathcal{D}(\alpha) \subseteq \mathcal{D}^1\}$, and

3. $\widetilde{\mathcal{D}}^a = \{D \in \mathcal{D}^a \colon \varsigma(D) \cap \widetilde{\Sigma} \neq \emptyset\}$ equipped with the map $\widetilde{\rho} \colon \widetilde{\mathcal{D}}^a \to \mathrm{Hom}(\widetilde{\mathcal{M}}, \mathbb{Z}) =: \widetilde{\mathcal{N}}; D \mapsto (\pi \circ \rho)(D)$ where $\pi \colon \mathcal{N} \to \widetilde{\mathcal{N}}$ is the map dual to the inclusion $\iota \colon \widetilde{\mathcal{M}} \hookrightarrow \mathcal{M}$.

Theorem 6.1.10 ([Hof18, Theorem 1.4])**.** *The assignment $\widetilde{\mathcal{S}} \mapsto \widetilde{H}$, which associates to a subdatum $\widetilde{\mathcal{S}} \preceq \mathcal{S}$ its corrsponding spherical subgroup $\widetilde{H}$, induces an order-reversing bijection from the set of subdata of $\mathcal{S}$ onto the set of equivalence classes of supergroups of H where equivalence is given by conjugation.*

6.1.4 Smart Actions

We recall the definition of smart actions and some results on them, which we use in the proof of Theorem 6.0.3. In [AB04, Section 4], Alexeev and Brion explain several steps to "improve" the G-action of a spherical embedding $G/H \hookrightarrow X$ into a smart action. The restrictions on smart actions make them much easier to work with. Suppose $G = G^{ss} \times C$ where G^{ss} is semisimple simply connected and C is a central torus, then we define the following:

Definition 6.1.11 ([AB04, Definition 4.3])**.** A G-action on X is called *almost faithful* if its kernel is finite, and C acts faithfully.

Definition 6.1.12 ([AB04, Definition 4.4])**.** A G-action on X is called *smart* when the action is almost faithful, and the natural homomorphism $C \to \mathrm{Aut}^G(X)^\circ$ is an isomorphism, where $\mathrm{Aut}^G(X)^\circ$ is the connected component of the group of G-equivariant automorphisms of X.

By [AB04, Lemma 4.5], for any reductive group G and spherical subgroup $H \subseteq G$, we can find a reductive group $\mathcal{G}$ and spherical subgroup $\mathcal{H} \subseteq \mathcal{G}$ such that $\mathcal{G}/\mathcal{H} \cong G/H$, and the action of $\mathcal{G}$ on $\mathcal{G}/\mathcal{H}$ is smart. Moreover, by [Hof15, Proposition 4.10.0.15], there is a natural identification between the Luna data $\mathcal{S}$ associated to $H \subseteq G$ and $\mathfrak{S}$ associated to $\mathcal{H} \subseteq \mathcal{G}$. The main difference between $\mathcal{S}$ and $\mathfrak{S}$, is how they relate to their ambient character lattices.

From now on assume that $G = G^{ss} \times C$ as above, and that $\mathcal{S}$ is the usual Luna datum of $H \subseteq G$. The Borel subgroup $B \subseteq G = G^{ss} \times C$ satisfies $B = B^{ss} \times C$ for some Borel subgroup $B^{ss} \subseteq G^{ss}$, so that $\mathfrak{X}(B) = \mathfrak{X}(B^{ss}) \oplus \mathfrak{X}(C)$.

Proposition 6.1.13 ([Gag19, Lemma 3.5],[Hof15, Lemma 4.10.0.2])**.** *The central torus C acts faithfully on X if and only if the restriction $\pi_C\colon \mathcal{M} \to \mathfrak{X}(C)$ of the natural projection map $\mathfrak{X}(B) \to \mathfrak{X}(C)$ is surjective.*

We have a natural projection map $\pi^{ss}\colon G \to G^{ss}$. Like the Borel subgroup, the maximal torus $T \subseteq G = G^{ss} \times C$ can be written as $T = T^{ss} \times C$ with $T^{ss} \subseteq B^{ss}$ a maximal torus. The induced set of roots R and the corresponding set of simple roots S with respect to (B, T) and (B^{ss}, T^{ss}) are naturally identified.

Proposition 6.1.14 ([Hof15, Proposition 4.10.0.7])**.** *The Luna datum of $\pi^{ss}(H) \subseteq G^{ss}$, denoted by $\mathcal{S}^{ss} = (\mathcal{M}^{ss}, \Sigma^{ss}, S^{ss,p}, \mathcal{D}^{ss,a})$, satisfies:*

$$\mathcal{M}^{ss} = \mathcal{M} \cap \mathfrak{X}(B^{ss}), \quad \Sigma^{ss} = \Sigma, \quad S^{ss,p} = S^P, \quad \mathcal{D}^{ss,a} = \mathcal{D}^a,$$

where the abstract set $\mathcal{D}^{ss,a}$ is equipped with the map $\rho^{ss}\colon \mathcal{D}^{ss,a} \to \mathcal{N}^{ss} :=$ $\mathrm{Hom}(\mathcal{M}^{ss}, \mathbb{Z})$ given by $\rho^{ss}(D) = \pi \circ \rho(D)$ where $\pi\colon \mathcal{N} \to \mathcal{N}^{ss}$ denotes the map *dual to the inclusion $\mathcal{M}^{ss} \hookrightarrow \mathcal{M}$.*

Proposition 6.1.15 ([Gag19, Lemma 3.6],[Hof15, Lemma 4.10.0.8])**.** *If $G = G^{ss} \times C$ acts almost faithfully on X, then the following are equivalent:*

1. *The G-action is smart.*

2. $|\Sigma| = \mathrm{rk}(\mathcal{M} \cap \mathfrak{X}(B^{ss}))$.

6.2 Adapted Epimorphisms

Throughout this section let G and G' be reductive groups with maximal tori T and T', Borel subgroups B and B', and spherical subgroups H and H' with corresponding Luna data $\mathcal{S}$ and $\mathcal{S}'$. Recall that two epimorphisms $\varphi, \varphi' : G \to G'$ of reductive groups are said to be equivalent if $\varphi' = \varphi \circ \mathrm{Inn}(t)$ for some $t \in T$ where $\mathrm{Inn}(t)$ denotes the inner automorphism given by t (Definition 6.1.4). We define a related notion of equivalence for twisted equivariant morphisms.

Definition 6.2.1. We say that two twisted equivariant morphisms (F_1, φ_1), $(F_2, \varphi_2)\colon G/H \to G'/H'$ are *equivalent*, if there exists $t' \in T'$ such that $\varphi_2 = \mathrm{Inn}(t') \circ \varphi_1$ and a twisted equivariant morphism $(F, \mathrm{Inn}(t')) : G'/H' \to G'/H'$ such that $F_2 = F \circ F_1$.

In this section we obtain a combinatorial description of twisted equivariant morphisms up to equivalence.

By the following result, we can move back and forth between twisted equivariant morphisms and epimorphisms of reductive groups in a meaningful way. As a consequence we are interested in the set of epimorphisms $\varphi\colon G \to G'$ such that $\varphi^{-1}(H')$ contains a conjugate of H, denoted by $\mathrm{Hom}(G/H, G'/H')$.

Proposition 6.2.2. *Let $\varphi : G \to G'$ be an epimorphism of reductive groups. There exists a morphism $F : G/H \to G'/H'$ for which (F, φ) is a twisted equivariant morphism if and only if $\varphi^{-1}(H')$ contains a conjugate of H. Moreover,* *if* $\varphi^{-1}(H')$ *contains* gHg^{-1} *for some* $g \in G$*, then one such* F *is defined by*

$$F : G/H \to G'/H', \quad xH \mapsto \varphi(xg^{-1})H'.$$

Proof. Let $(F, \varphi) : G/H \to G'/H'$ be a twisted equivariant morphism. Then there is some $g' \in G'$ such that $F(eH) = g'H'$. For any $h \in H$ replacing eH with hH does not change this result, so by twisted equivarience:

$$g'H' = F(eH) = F(hH) = \varphi(h)g'H'.$$

Therefore, $g'^{-1}\varphi(H)g'$ is a subset of H', so $\varphi^{-1}(H')$ contains a conjugate of H.

Conversely, let $\varphi : G \to G'$ be an epimorphism of reductive groups such that $\varphi^{-1}(H')$ contains gHg^{-1} for some $g \in G$. Let $F : xH \mapsto \varphi(xg^{-1})H$, then for any $h \in H$ we have $F(xH) = F(xhH)$ so F is well-defined. Therefore, (F, φ) is a twisted equivariant morphism. □

As Luna data depend on the choices of a Borel subgroup and a maximal torus, we say that an epimorphism $\varphi\colon G \to G'$ is *adapted* (to (B, T) and (B', T')) if $\varphi(T) = T'$ and $\varphi(B) = B'$. An epimorphism of root data $\varphi^*\colon \mathfrak{X}' \to \mathfrak{X}$ is called *adapted* (to (B, T) and (B', T')) if $\varphi^*(S') \subseteq S$. A twisted equivariant morphism $(F, \varphi) : G/H \to G'/H'$ is called adapted if φ is adapted. It is straightforward to verify that the bijection between epimorphisms of root data and equivalence classes of epimorphisms of algebraic groups restricts to a bijection between the corresponding adapted morphisms. Note that it is only a mild assumption for an epimorphism of algebraic groups to be adapted. Indeed, as any two Borel subgroups and any two maximal tori in a Borel subgroup are conjugated, it follows that for any epimorphism $\varphi\colon G \to G'$ there exists $g \in G$ such that the epimorphism $\varphi \circ \mathrm{Inn}(g)$ is adapted. We denote the subset of consisting of adapted epimorphisms $\varphi : G \to G'$, such that $\varphi^{-1}(H')$ contains a conjugate of H, by $\mathrm{Hom}_{B,T}^{B',T'}(G/H, G'/H')$.

We now recall two results from Hofscheier's thesis [Hof15].

Proposition 6.2.3 ([Hof15, Proposition 4.2.0.2]). *Let $\varphi^*\colon \mathfrak{X}' \to \mathfrak{X}$ be an adapted epimorphism of root data and let $\varphi\colon G \to G'$ be an adapted epimorphism of algebraic groups inducing φ^*. The Luna datum $\varphi^{-1}(\mathcal{S}') = (\widetilde{\mathcal{M}}, \widetilde{\Sigma}, \widetilde{S}^p, \widetilde{\mathcal{D}}^a)$ of $\varphi^{-1}(H')$ is given as follows:*

$$\widetilde{\mathcal{M}} = \varphi^*(\mathcal{M}'), \qquad \widetilde{\Sigma} = \varphi^*(\Sigma'), \qquad \widetilde{S}^p = \varphi^*((S')^p) \cup S_2, \qquad \widetilde{\mathcal{D}}^a = (\mathcal{D}')^a,$$

where the abstract set $\widetilde{\mathcal{D}}^a$ is equipped with the map $\widetilde{\rho}\colon \widetilde{\mathcal{D}}^a \to \widetilde{\mathcal{N}} := \mathrm{Hom}(\widetilde{\mathcal{M}}, \mathbb{Z})$ given by $(\varphi_* \circ \widetilde{\rho})(D') = \rho'(D')$ for every $D' \in (\mathcal{D}')^a = \widetilde{\mathcal{D}}^a$ where $\varphi_*\colon \widetilde{\mathcal{N}} \to \mathcal{N}'$ is the isomorphism dual to $\varphi^*\colon \mathcal{M}' \to \widetilde{\mathcal{M}}$ and the set S_2 consists of all simple roots $\alpha \in S$ such that the corresponding coroot $\check{\alpha}$ is in the kernel of the dual map $\varphi_*\colon \check{\mathfrak{X}} \to \check{\mathfrak{X}}'$.

Observe that the assumption of being adapted ensures that the preimage $\varphi^{-1}(\mathcal{S}')$ is a Luna datum in terms of the root system R and the set of simple roots S. We get the following description of $\mathrm{Hom}_{B,T}^{B',T'}(G/H, G'/H')$ by the same proof as [Hof15, Proposition 4.8.0.2].

Theorem 6.2.4. *To any adapted epimorphism of root data $\varphi^*\colon \mathfrak{X}' \to \mathfrak{X}$ satisfying $\varphi^{-1}(\mathcal{S}') \preceq \mathcal{S}$, we can associate an adapted epimorphism $\varphi\colon G \to G'$ such that $\varphi^{-1}(H')$ contains a conjugate of H. This assignment induces a bijection:*

$$\mathrm{Hom}_{B,T}^{B',T'}(G/H, G'/H')/\sim\; \leftrightarrow \left\{ \begin{matrix} \textit{adapted epimorphism of root data} \\ \varphi^*\colon \mathfrak{X}' \to \mathfrak{X} \textit{ with } \varphi^{-1}(\mathcal{S}') \preceq \mathcal{S} \end{matrix} \right\}$$

where $\sim$ denotes equivalence of epimorphisms as described in Definition 6.1.4.

We will adapt this theorem to describe twisted equivariant morphisms up to a different notion of equivalence to that described in [Hof15].

By Theorem 6.1.10 there is an inclusion reversing bijection between subdata $\mathcal{S}_1 \preceq \mathcal{S}$ and spherical subgroups H_1 containing H, defined up to conjugation. Using the following lemmas we will show that this is an honest bijection. Throughout, let H_1 be a spherical subgroup of G containing H.

Lemma 6.2.5 (Stein factorization). *With H and H_1 as above, define $H_1' \coloneqq HH_1^\circ$ where H° denotes the connected component of H containing the identity. Then $H \subseteq H_1' \subseteq H_1$, H_1/H_1' is finite and H_1'/H is connected.*

Lemma 6.2.6 ([Hof18, Remark 2.13]). *With H and H_1 as above, if the quotient H_1/H is finite, then normal subgroups $N_G(H^\circ)$, $N_G(H_1^\circ)$, $N_G(H)$ and $N_G(H_1)$ are equal.*

The following is [Hof18, Proposition 4.8] following from [Los09, Lemma 3.1.5].

Lemma 6.2.7 ([Hof18, Los09]). *With H and H_1 as above, let $\mathcal{S}_1$ be the Luna datum corresponding to H_1 and let all components of $\mathcal{S}_1$ be denoted as in $\mathcal{S}$ with an additional subscript 1. If the quotient H_1/H is finite, then*

1. *$\mathcal{M}_1$ is a sublattice of $\mathcal{M}$ of finite index,*
2. *$\operatorname{cone}(\Sigma) = \operatorname{cone}(\Sigma_1)$ and*
3. *$S^P = S_1^P$.*

Proposition 6.2.8. *Fix a reductive group G and spherical subgroup H in G with Luna datum $\mathcal{S} = (\mathcal{M}, \Sigma, S^P, \mathcal{D}^a)$. Then the assignment of a Luna datum to a spherical subgroup induces a bijection*

$$\{H' \subseteq G \text{ spherical} : H' \supseteq H\} \xleftrightarrow{1:1} \{\mathcal{S}' \preceq \mathcal{S}\}.$$

Proof. By Theorem 6.1.10 it suffices to show that any two spherical subgroups H_1 and H_2 containing H, with the same Luna datum $\mathcal{S}_1 = \mathcal{S}_2$, are equal.

Define two subgroups $H_i' \coloneqq HH_i^\circ$ as in Lemma 6.2.5 and say their Luna data are denoted by $\mathcal{S}_i'$. We denote each component of the Luna data of $\mathcal{S}_i$ and $\mathcal{S}_i'$ by the same symbol as in $\mathcal{S}$ with an additional subscript or subscript and dash respectively.

Since both groups H_i'/H are connected, both Luna data $\mathcal{S}_i'$ can be defined from $\mathcal{S}$ using some colored subspace $(\mathcal{N}_\mathbb{Q}^i, \mathcal{D}^i)$ as described in Section 6.1.3. In particular, $\mathcal{M}_i' = \mathcal{M} \cap (\mathcal{N}_\mathbb{Q}^i)^\perp$. Since both groups H_i/H_i' are finite, by

Lemma 6.2.7 $\mathcal{M}_i$ is a finite sublattice of $\mathcal{M}'_i$ for each $i = 1, 2$. In particular, the $\mathbb{Q}$-span of $\mathcal{M}_i$ and $\mathcal{M}'_i$ is the same vector space. However, the $\mathbb{Q}$-span of $\mathcal{M}'_i$ is $(\mathcal{N}^i_{\mathbb{Q}})^\perp$ so we have $\mathcal{N}^1_{\mathbb{Q}} = \mathcal{N}^2_{\mathbb{Q}}$. By the description in Section 6.1.3, the lattice $\mathcal{M}'_i$, spherical roots Σ'_i, colors $(\mathcal{D}'_i)^a$ and map $\rho'_i : (\mathcal{D}'_i)^a \to \mathcal{N}'_i$ all depend only on the subspace $\mathcal{N}^i_{\mathbb{Q}}$ so

$$\mathcal{M}'_1 = \mathcal{M}'_2, \quad \Sigma'_1 = \Sigma'_2, \quad (\mathcal{D}'_1)^a = (\mathcal{D}'_1)^a, \quad \rho'_1 = \rho'_2.$$

By Lemma 6.2.7 $(S'_1)^P = (S'_2)^P$ so we have shown that $\mathcal{S}'_1 = \mathcal{S}'_2$.

We are reduced to proving that if H_1 and H_2 are spherical subgroups of G containing H such that H_i/H is finite and $\mathcal{S}_1 = \mathcal{S}_2$ then $H_1 = H_2$. By Theorem 6.1.10 such subgroups would be conjugate to one-another so for some $g \in G$ we have $H_2 = gH_1g^{-1}$. Clearly the connected component H° is contained in both connected components H_i°. Additionally, H° is a finite index, closed subgroup of each H_i, so must contain their connected components (see for example [Hum75, Proposition on page 53]). Thus, the connected components H°, H_1° and $H_2^\circ = gH_1^\circ g^{-1}$ are all equal and g is in the normaliser $N_G(H_1^\circ)$. However, by Lemma 6.2.6 this means g is in the normaliser of H_1, so $H_2 = H_1$. □

Theorem 6.2.9. *To any adapted epimorphism of root data $\varphi^* : \mathfrak{X}' \to \mathfrak{X}$ satisfying $\varphi^{-1}(\mathcal{S}') \preceq \mathcal{S}$, we can associate an adapted epimorphism $\varphi : G \to G'$ such that φ^{-1} contains a conjugate of H and thus there is a twisted equivariant morphism $(F, \varphi) : G/H \to G'/H'$. This assignment induces a bijection:*

$$\left\{\begin{matrix}\textit{adapted t.e.m.} \\ (F,\varphi)\colon G/H \to G'/H'\end{matrix}\right\} / \sim \;\leftrightarrow \left\{\begin{matrix}\textit{adapted epimorphism of root data} \\ \varphi^*\colon \mathfrak{X}' \to \mathfrak{X} \textit{ with } \varphi^{-1}(\mathcal{S}') \preceq \mathcal{S}\end{matrix}\right\}$$

where $\sim$ denotes the equivalence defined in in Definition 6.2.1.

Proof. It is enough to show that any two twisted equivariant morphisms which can be assigned to the same epimorphism of root data are equivalent. Suppose $(F_1, \varphi_1), (F_2, \varphi_2) : G/H \to G'/H'$ are both twisted equivariant morphisms adapted to (B, T) and (B', T'). By Theorem 6.1.3, if $\varphi_1^* = \varphi_2^*$ then $\varphi_2 = \varphi_1 \circ \mathrm{Inn}(t)$ for some $t \in T$.

There are elements g_1 and $g_2 \in G'$ such that $F_i(eH) = g_i H'$. For any $h \in H$ we must have $F_i(eH) = F_i(hH)$ for F_i to be well-defined, and so

$$g_i^{-1}\varphi_i(H)g_i \subseteq H'.$$

We define two subgroups H_1 and H_2 of G by $H_i := \varphi_i^{-1}(g_i H' g_i^{-1})$ both of which contain H. These subgroups are conjugate to one-another, so by Theorem 6.1.10 they have the same Luna datum. Thus, by Proposition 6.2.8 they are equal and so $g_1^{-1}\varphi_1(t^{-1})g_2$ is in the normaliser $N_{G'}(H')$.

Now define $t' := \varphi_1(t)$ and $g' := t' g_1^{-1} t'^{-1} g_2$. By the previous paragraph, $t'^{-1}g' \in N_{G'}(H')$ so, by Proposition 6.2.2, we can define a twisted equivariant morphism $(F', \mathrm{Inn}(t'))$ by $F' : xH' \mapsto \mathrm{Inn}(t')(x)g'H'$. It is a straightforward computation to show that $F'(F_1(xH)) = F_2(xH)$ for all $x \in G$. □

6.3 Proof of Theorem 6.0.3

Throughout this section let G be a reductive group, T a maximal torus, B a Borel subgroup and H a spherical subgroup such that the action of G on G/H is smart. As in Section 6.1.4 we have $G = G^{ss} \times C$ and $B = B^{ss} \times C$ and define the Luna datum $\mathcal{S}^{ss} = (\mathcal{M}^{ss}, \Sigma^{ss}, S^{ss,P}, \mathcal{D}^{ss,a})$ as in Proposition 6.1.14.

Definition 6.3.1. Let $\pi : \mathcal{N} \to \mathcal{N}^{ss}$ be the lattice homomorphism dual to the inclusion $\mathcal{M}^{ss} \hookrightarrow \mathcal{M}$. Then we define

$$\mathrm{Aut}(\mathcal{N}, \pi) := \{\phi_* \in \mathrm{Aut}(\mathcal{N}) : \pi(x) = \pi(\phi_*(x)), \forall x \in \mathcal{N}\}$$

which is a subgroup of the automorphisms of $\mathcal{N}$.

Restricting π to its image $\mathcal{N}_1$ so that it is surjective does not change the group $\mathrm{Aut}(\mathcal{N}, \pi)$. Therefore, the group $\mathrm{Aut}(\mathcal{N}, \pi)$ defined above is equivalent to the $\mathrm{Aut}(\mathcal{N},\pi)$ defined in [AB04, Lemma 4.8] since an automorphism ϕ_* leaves the kernel of π invariant and induces the identity on its image if and only if $\pi(x)=\pi(\phi_*(x))$ for all $x\in\mathcal{N}$.

Lemma 6.3.2. *The set* $\mathrm{Iso}(\mathcal{S})$ *(Definition 6.0.2) is the set of automorphisms of* $\mathcal{M}$ *dual to the automorphisms in* $\mathrm{Aut}(\mathcal{N},\pi)$.

Proof. Recall that all roots in the root datum of $G=G^{ss}\times C$ are in the sublattice $\mathfrak{X}(B^{ss})$ and that spherical roots are linear combinations of these roots (see, for example, [BL11, Section 1.1.6]), contained in $\mathcal{M}$, thus $\Sigma\subseteq\mathcal{M}^{ss}$.

Let $\phi\in\mathrm{Iso}(\mathcal{S})$, so $\phi(\gamma)=\gamma$ for all spherical roots $\gamma\in\Sigma$. By Proposition 6.1.15 the number of spherical roots is equal to the rank of $\mathcal{M}^{ss}$. Since Σ is the set of primitive ray generators of a simplicial cone, it is a linearly independent subset of $\mathcal{M}^{ss}$. Therefore, $\mathrm{span}_{\mathbb{Z}}(\Sigma)$ is a finite index sublattice in $\mathcal{M}^{ss}$, so ϕ fixes all elements of $\mathcal{M}^{ss}$ and the following diagrams commute:

$$\begin{tikzcd} \mathcal{M} & \mathcal{M} \arrow[l, "\phi"'] \\ \mathcal{M}^{ss} \arrow[u, "\pi^*"] & \mathcal{M}^{ss} \arrow[l, "id"] \arrow[u, "\pi^*"'] \end{tikzcd} \quad \text{and its dual} \quad \begin{tikzcd} \mathcal{N} \arrow[r, "\phi_*"] \arrow[d, "\pi"] & \mathcal{N} \arrow[d, "\pi"] \\ \mathcal{N}^{ss} \arrow[r, "id"] & \mathcal{N}^{ss} \end{tikzcd}$$

where π^* is the inclusion of $\mathcal{M}^{ss}$ in $\mathcal{M}$ and $\phi_*:\mathcal{N}\to\mathcal{N}$ is the dual automorphism to ϕ. The second commutative diagram shows that $\pi(x)=\pi(\phi_*(x))$ for all $x\in\mathcal{N}$ so $\phi_*\in\mathrm{Aut}(\mathcal{N},\pi)$.

Now let $\phi_*\in\mathrm{Aut}(\mathcal{N},\pi)$ and let $\phi:\mathcal{M}\to\mathcal{M}$ be its dual. Then by the definition of $\mathrm{Aut}(\mathcal{N},\pi)$ the above two diagrams are once again commutative. Therefore, ϕ fixes elements of $\mathcal{M}^{ss}$ including all spherical roots $\gamma\in\Sigma$. □

Lemma 6.3.3. *Let* π^{ss} *and* π_C *be the natural projection maps from* $\mathfrak{X}(B)$ *onto* $\mathfrak{X}(B^{ss})$ *and* $\mathfrak{X}(C)$*. If* $\phi\in\mathrm{Iso}(\mathcal{S})$*, then there exists a unique injective lattice homomorphism* $\varphi_C:\mathfrak{X}(C)\to\mathfrak{X}(B)$ *such that*

$$\phi(\chi)=\pi^{ss}(\chi)+\varphi_C(\pi_C(\chi)),\quad \textit{for all } \chi\in\mathcal{M}.$$

Proof. By Proposition 6.1.13, the restriction of π_C to $\mathcal{M}$ is surjective. Thus, for any $\chi^C\in\mathfrak{X}(C)$ there is an element $\chi\in\mathcal{M}$ which projects onto χ^C under

π_C. We define $\varphi_C(\chi^C)$ by $\phi(\chi) - \pi^{ss}(\chi)$. It suffices to show that this is an injective lattice homomorphism.

Suppose χ_1 and χ_2 both project onto the same χ^C under π_C. Their difference is contained in $\ker(\pi_C) = \mathcal{M}^{ss}$ and so is fixed by ϕ. Also, since χ_1 and χ_2 have the same $\mathfrak{X}(C)$-component, their difference can be written as the difference between their images under π^{ss}. These two equalities combine to show that $\varphi_C(\chi^C)$ does not depend on our choice of χ_i, so φ_C is a lattice homomorphism.

For injectivity, let χ^C be an element of the kernel of φ_C. By definition of φ_C, for some $\chi \in \mathcal{M}$ projecting onto χ^C, we have $\phi(\chi) = \pi^{ss}(\chi)$. However, both sides of this are in $\mathcal{M}^{ss}$ which is fixed by ϕ so $\chi \in \mathcal{M}^{ss}$ and $\chi^C = \mathbf{0}$. $\square$

Proof of Theorem 6.0.3. Define φ_C as in Lemma 6.3.3, then we can define

$$\varphi^* : \mathfrak{X}(B) \to \mathfrak{X}(B), \quad \chi \mapsto \pi^{ss}(\chi) + \varphi_C(\pi_C(\chi))$$

which restricts to ϕ on $\mathcal{M}$. We claim that this is a lattice automorphism.

For injectivity, let χ be in the kernel of φ^*, then $\varphi_C(\pi_C(\chi))$ must be in $\mathfrak{X}(B^{ss})$. There is some $\chi' \in \mathcal{M}$ with the same $\mathfrak{X}(C)$-component as χ and $\varphi_C(\pi_C(\chi)) = \phi(\chi') - \pi^{ss}(\chi')$. This means that $\phi(\chi')$ and χ' are in $\mathcal{M}^{ss}$. Therefore, $\pi_C(\chi') = \mathbf{0}$, and since χ had this same $\mathfrak{X}(C)$-component, $\chi = \pi^{ss}(\chi)$. This shows that $\pi^{ss}(\chi)$ and thus χ itself are also zero.

For surjectivity, let $\chi \in \mathfrak{X}(B)$. Since $\pi_C|_{\mathcal{M}}$ and ϕ are surjective, there is an element $\chi' \in \mathcal{M}$ such that $\pi_C(\phi(\chi')) = \pi_C(\chi)$. For an appropriate choice of $\chi^{ss} \in \mathfrak{X}(B^{ss})$, the character $\chi^{ss} + \pi_C(\chi')$ maps to χ under φ^*.

Now we define $\Psi' = (\mathfrak{X}', R', \check{\mathfrak{X}}', \check{R}')$ by letting $\mathfrak{X}' = \varphi^*(\mathfrak{X}(B))$, $R' = \varphi^*(R)$, $\check{\mathfrak{X}}' = \varphi_*^{-1}(\check{\mathfrak{X}})$ and $\check{R}' = \varphi_*^{-1}(\check{R})$ where φ_* is the dual lattice automorphism to φ^*. This is a root datum since the root datum axioms are preserved under unimodular maps. Fix the set of simple roots $S' = \varphi^*(S)$, then φ^* is an adapted isomorphism of root data $\varphi^* : \mathfrak{X} \to \mathfrak{X}'$. Note that since φ^* fixes $\mathfrak{X}(B^{ss})$ it also fixes R, though it need not fix $\check{R}$. Let G' be a reductive group with maximal torus T' associated to Ψ', and let and B' be the Borel subgroup determined by the set of simple roots S'. By Theorem 6.1.3, we may fix an adapted isomorphism of reductive groups $\varphi : G' \to G$ which induces $\varphi^* : \mathfrak{X} \to \mathfrak{X}'$.

Define a new Luna datum $\mathcal{S}'$ equal to $\varphi^{-1}(\mathcal{S})$ as defined in Proposition 6.2.3. Let $H' \subseteq G'$ be a spherical subgroup of G' associated to $\mathcal{S}'$. By definition, $\varphi^{-1}(\mathcal{S})$ is a subdatum of $\mathcal{S}'$. Since φ is an isomorphism we can consider its inverse which induces the epimorphism of root data $(\varphi^*)^{-1}$. The preimage $(\varphi^{-1})^{-1}(\mathcal{S}')$ is equal to $\mathcal{S}$ so it is a subdata of $\mathcal{S}$.

By Theorem 6.2.4, $\varphi(H')$ contains a conjugate of H and $\varphi^{-1}(H)$ contains a conjugate of H'. In fact, we obtain the following chain of containments

$$H' \supseteq g_1\varphi^{-1}(H)g_1^{-1} \supseteq g_2H'g_2^{-1} \tag{6.1}$$

for some $g_1, g_2 \in G'$. By Theorem 6.1.10, since $g_2H'g_2^{-1}$ is a conjugate of H' they are spherical subgroups with the same Luna datum. Thus by Proposition 6.2.8, g_2 is in the normaliser $N_{G'}(H')$ and the containments in (6.1) are all equalities. By Proposition 6.2.2, since $\varphi^{-1}(H)$ contains $g_1^{-1}H'g_1$ and $\varphi(H')$ contains $\varphi(g_1)H\varphi(g_1^{-1})$, we can define two twisted equivariant morphisms:

$$\begin{aligned}(F, \varphi) : G'/H' \to G/H, &\quad F(xH') = \varphi(xg_1)H\\ (F', \varphi^{-1}) : G/H \to G'/H', &\quad F'(xH) = \varphi^{-1}(x)g_1^{-1}H'\end{aligned}$$

which are mutually inverse, and so are isomorphisms of G/H and G'/H'.

□

6.4 Generalised Hermite Normal Form

As a practical application of the above, we present an algorithm to find a normal form for rational polytopes containing the origin in their interior up to $\mathrm{Aut}(\mathcal{N}, \pi)$. This is used in the classification found in Chapter 7.

Definition 6.4.1. Let S be a set and A a group acting on S, then a *normal* *form* is a map $\mathrm{NF} : S \to S$ satisfying the following

(NF1) For all $s \in S$ there exists an $a \in A$ such that $\mathrm{NF}(s) = a \cdot s$, and

(NF2) For all $s, t \in S$, $\mathrm{NF}(s) = \mathrm{NF}(t)$ if and only if there exists some $a \in A$ such that $t = a \cdot s$.

In other words, we choose a unique representative of each equivalence class in $S/\sim$ where $s \sim t$ if and only if $t = a \cdot s$ for some $a \in A$. We say that NF is a normal form for S under the action of G.

For example, row Hermite normal form is a normal form for $\mathrm{Mat}_{\mathbb{Z}}(n \times m)$ under multiplication on the left by elements of $\mathrm{GL}_n(\mathbb{Z})$ and column Hermite normal form is a normal form for $\mathrm{Mat}(n \times m)$ under multiplication on the right by elements of $\mathrm{GL}_m(\mathbb{Z})$. We seek an normal form for rational polytopes containing the origin in $\mathcal{N}_{\mathbb{Q}}$, under the action of $\mathrm{Aut}(\mathcal{N}, \pi)$. In fact, it suffices to find a normal form NF for lattice polytopes, then for a denominator d polytope P, define $\mathrm{NF}(P) := \frac{1}{d}\,\mathrm{NF}(dP)$. Therefore, we will discuss only lattice polytopes for the remainder of the section.

We adapt the method used by Kreuzer and Skarke in their Palp software [KS04]. We interpret the vertices of a polytope as rows of a matrix and will define a normal form for such matrices. However, this requires a choice of order of the vertices. A naïve approach would be to define a matrix for every permutation of the vertices, find their normal forms, and choose the 'minimal' such matrix. The normal form of the polytope would then be the convex hull of the rows of this minimal matrix. However, this becomes slow for polytopes with many vertices. Palp improves this approach by using the height of each vertex above each facet to produce a reduced collection of vertex permutations which is invariant under change of basis. We use this portion of Kreuzer and Skarke's algorithm directly. For details on implementation see [GK13].

We are reduced to seeking a normal form for the set $\mathrm{Mat}^n_{\mathbb{Z}}(l \times n)$ of rank n, $l \times n$ matrices under the action of $\mathrm{Aut}(\mathcal{N}, \pi)$, where n is also the rank of $\mathcal{N}$. Our approach is to define three of normal forms NF, NF_m and NF_u, where NF is defined in terms of NF_m and NF_m is defined in terms of NF_u. In these three steps we adapt Hermite normal form into our desired normal form.

The map $\pi : \mathcal{N} \to \mathcal{N}^{ss}$ may not be surjective, so consider instead the restriction to its image $\mathcal{N}_1 \coloneqq \pi(\mathcal{N})$, which does not alter $\mathrm{Aut}(\mathcal{N}, \pi)$. Let m be the rank of $\mathcal{N}_1$ and choose a basis of both $\mathcal{N}$ and $\mathcal{N}_1$ so that we think of their points as row vectors. The map π is defined by multiplication on the right by an $n \times m$ matrix Π. Maps in $\mathrm{Aut}(\mathcal{N}, \pi)$ are defined by multiplication on the right by unimodular matrices so we reinterpret it as a subgroup of $\mathrm{GL}_n(\mathbb{Z})$:

$$\mathrm{Aut}(\mathcal{N}, \pi) = \{U \in \mathrm{GL}_n(\mathbb{Z}) : xU\Pi = x\Pi, \forall x \in \mathcal{N}\}.$$

We additionally define the group of automorphisms

$$\mathrm{Aut}(\mathcal{N}, m) \coloneqq \left\{ \begin{pmatrix} I_m & V_1 \\ 0_{n-m,m} & V_2 \end{pmatrix} \in \mathrm{GL}_n(\mathbb{Z}) \right\}$$

where I_i is the $i \times i$ identity matrix and $0_{i,j}$ is the $i \times j$ zero matrix.

Lemma 6.4.2. *The group* $\mathrm{Aut}(\mathcal{N}, \pi)$ *is conjugate to* $\mathrm{Aut}(\mathcal{N}, m)$*.*

Proof. By the surjectivity of π we know that the rows of Π contain a basis of $\mathcal{N}_1$. Therefore, there exists a change of basis $A \in \mathrm{GL}_n(\mathbb{Z})$ such that

$$A\Pi = \begin{pmatrix} I_m \\ 0_{n-m,m} \end{pmatrix}.$$

This map defines the projection π_m from $\mathcal{N}$ onto the first m coordinates. Notice that the maps in $\mathrm{Aut}(\mathcal{N}, m)$ are exactly the maps which fix the first m coordinates. Therefore $\mathrm{Aut}(\mathcal{N}, m)$ is equal to $\mathrm{Aut}(\mathcal{N}, \pi_m)$ and it suffices to show that $\mathrm{Aut}(\mathcal{N}, \pi) = A^{-1}\,\mathrm{Aut}(\mathcal{N}, \pi_m)A$.

Let $V \in \mathrm{Aut}(\mathcal{N}, \pi_m)$, then we know that $xVA\Pi = xA\Pi$ for all $x \in \mathcal{N}$. Since A is a change of basis we can replace x with xA^{-1} and so $xA^{-1}VA\Pi = x\Pi$ implying that $A^{-1}VA \in \mathrm{Aut}(\mathcal{N}, \pi)$. Now let $V \in \mathrm{Aut}(\mathcal{N}, \pi)$, so $xV\Pi = x\Pi$

for all $x \in \mathcal{N}$. Expand this equation as follows

$$(xA^{-1})(AVA^{-1})(A\Pi) = (xA^{-1})(A\Pi) \quad \text{for all } x \in \mathcal{N}.$$

Since A is a change of basis we can replace xA^{-1} with x which shows that $AVA^{-1} \in \mathrm{Aut}(\mathcal{N}, \pi_m)$. $\square$

Proposition 6.4.3. *If* NF_m *is a normal form for* $\mathrm{Mat}^n_{\mathbb{Z}}(l \times n)$ *under multiplication on the right by* $\mathrm{Aut}(\mathcal{N}, m)$*, then* $\mathrm{NF} : X \mapsto \mathrm{NF}_m(XA^{-1})A$ *is a normal form for* $\mathrm{Mat}^n_{\mathbb{Z}}(l \times n)$ *under multiplication on the right by* $\mathrm{Aut}(\mathcal{N}, \pi)$.

Proof. Let X be a rank n, $l \times n$ matrix. By definition of a normal form, there exists a matrix $U \in \mathrm{Aut}(\mathcal{N}, m)$ such that $\mathrm{NF}_m(XA^{-1}) = XA^{-1}U$. Therefore, $\mathrm{NF}(XA^{-1})$ can be rewritten as $XA^{-1}UA$ and $A^{-1}UA \in \mathrm{Aut}(\mathcal{N}, \pi)$. From this we also see that if $\mathrm{NF}(X) = \mathrm{NF}(X')$ then X' can be obtained from X by multiplication on the right by some element of $\mathrm{Aut}(\mathcal{N}, \pi)$. Conversely, if $X' = XV$ for some $V \in \mathrm{Aut}(\mathcal{N}, \pi)$, then there is a matrix $U \in \mathrm{Aut}(\mathcal{N}, m)$ such that $V = A^{-1}UA$ so in fact $X'A^{-1} = XA^{-1}U$. Therefore, $\mathrm{NF}_m(X'A^{-1}) = \mathrm{NF}_m(XA^{-1})$ and $\mathrm{NF}(X) = \mathrm{NF}(X')$. $\square$

Proposition 6.4.4. *Let* NF_u *be a normal form for* $\mathrm{GL}_n(\mathbb{Z})$ *under multiplication on the left by elements of* $\mathrm{Aut}(\mathcal{N}, m)$ *and let* HNF *denote column Hermite normal form. For any matrix* X *in* $\mathrm{Mat}^n_{\mathbb{Z}}(l \times n)$ *there is a* unique $U \in GL_n(\mathbb{Z})$ *such that* $\mathrm{HNF}(X) = XU$*. The map*

$$\mathrm{NF}_m : X \mapsto XU\,\mathrm{NF}_u(U)^{-1}$$

is a normal form for $\mathrm{Mat}^n_{\mathbb{Z}}(l \times n)$ *under multiplication on the right by* $\mathrm{Aut}(\mathcal{N}, m)$.

Proof. By definition of column Hermite normal form, $\mathrm{HNF}(X)$ is unique and there exists a unimodular matrix U such that $\mathrm{HNF}(X) = XU$. Let $\widetilde{X}$ be a rank n square submatrix of X. Let $\widetilde{XU}$ be the square submatrix of the Hermite normal form made up of the same rows as $\widetilde{X}$, then $\widetilde{X}U = \widetilde{XU}$. Since $\widetilde{X}$ has rank n, it is invertible over $\mathbb{Q}$, so $U = \widetilde{X}^{-1}\widetilde{XU}$ is defined and unique.

By definition of normal form, $\mathrm{NF}_u(U) = VU$ for some $V \in \mathrm{Aut}(\mathcal{N}, m)$. Therefore, we can rewrite $\mathrm{NF}_m(X)$ as XV^{-1}. This also shows that if $\mathrm{NF}_m(X) = \mathrm{NF}_m(Y)$ then Y can be obtained from X by multiplication on the right by some element of $\mathrm{Aut}(\mathcal{N}, m)$. Now suppose $Y = XV$ for some $V \in \mathrm{Aut}(\mathcal{N}, m)$. Then since V is unimodular, X and Y must have the same Hermite normal form. Let U_X and U_Y be the unimodular matrices taking X and Y to their Hermite normal forms, then we have $XU_X = XVU_Y$. By the same argument as above, since X is of rank n, this means that $U_X = VU_Y$ and so $\mathrm{NF}_u(U_X) = \mathrm{NF}_u(U_Y)$. Finally, we can see that $\mathrm{NF}_m(X) = \mathrm{NF}_m(Y)$ by writing $\mathrm{NF}_m(X)$ in terms of Y, U_Y and V and simplifying. □

By the above results it suffices for us to find a normal form for $\mathrm{GL}_n(\mathbb{Z})$ under multiplication on the left by elements of $\mathrm{Aut}(\mathcal{N}, m)$. Let U be a matrix in $\mathrm{GL}_n(\mathbb{Z})$ and let U_1 and U_2 be the submatrices of its first m and last $n-m$ rows respectively. We need to define a matrix V in $\mathrm{Aut}(\mathcal{N}, m)$ such that VU is in normal form. Let V_1 and V_2 be its submatrices as in the definition of $\mathrm{Aut}(\mathcal{N}, m)$. We factorise V as follows

$$V = \begin{pmatrix} I_m & V_1 \\ 0_{n-m,m} & V_2 \end{pmatrix} = \begin{pmatrix} I_m & V_1' \\ 0_{n-m,m} & I_{n-m} \end{pmatrix} \begin{pmatrix} I_m & 0_{m,n-m} \\ 0_{n-m,m} & V_2 \end{pmatrix}$$

where $V_1' = V_1 V_2^{-1}$. Since U is unimodular, U_2 has rank $n-m$ so there is a unique choice of matrix in $\mathrm{GL}_{n-m}(\mathbb{Z})$ sending U_2 to its row Hermite normal form. Define V_2 to be this matrix. Now, in terms of block matrices VU is

$$\begin{pmatrix} I_m & V_1' \\ 0_{n-m,m} & I_{n-m} \end{pmatrix} \begin{pmatrix} I_m & 0_{m,n-m} \\ 0_{n-m,m} & V_2 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} = \begin{pmatrix} U_1 + V_1' U_2' \\ U_2' \end{pmatrix}$$

where $U_2' = V_2 U_2$ is in row Hermite normal form. It remains to fix V_1'.

Let $i_1, \ldots, i_{n-m}$ be the indices of the columns in which each row of U_2' has its first non-zero entry. Note that since U is invertible these are all defined and since U_2' is in Hermite normal form we have $1 \leq i_1 < i_2 < \cdots < i_{n-m} \leq n$.

Let $u_1^{(1)}, \ldots, u_n^{(1)} \in \mathbb{Z}^m$, $u_1^{(2)}, \ldots, u_n^{(2)} \in \mathbb{Z}^{n-m}$ and $c_1, \ldots, c_n \in \mathbb{Z}^m$ denote the columns of U_1, U_2' and $U_1 + V_1'U_2'$ respectively. Then $c_i = u_i^{(1)} + V_1'u_i^{(2)}$.

We fix the columns of V_1' sequentially. Let $i = i_1$, then since only the first entry of $u_i^{(2)}$ is non-zero, only the first column of V_1' impacts the value of c_i. Choose the entries of first column of V_1' such that the entries of c_i are the smallest non-negative integers possible. This choice is unique since it comes from reducing the entries of $u_i^{(1)}$ modulo the non-zero entry of $u_i^{(2)}$. Now suppose $i = i_j$ and we have defined columns $1, \ldots, j-1$ of V_1' then the only entries of V_1' which have not been fixed and have an impact on the value of c_i are those in the j-th column. As before, choose the entries of the j-th column of V_1' such that the entries of c_i are the smallest non-negative integers possible. Again this is unique. From this we can define V_1 and V. Let $\mathrm{NF}_u(U)$ be defined by VU.

Proposition 6.4.5. *The map* NF_u *defined above is a normal form for* $\mathrm{GL}_n(\mathbb{Z})$ *under multiplication on the left by elements of* $\mathrm{Aut}(\mathcal{N}, m)$*.*

Proof. By definition of NF_u, for any $U \in \mathrm{GL}_n(\mathbb{Z})$ there exists a $V \in \mathrm{Aut}(\mathcal{N}, m)$ such that $\mathrm{NF}_u(U) = VU$, and if two matrices U and U' have the same normal form $\mathrm{NF}_u(U)$ then $U' = VU$ for some $V \in \mathrm{Aut}(\mathcal{N}, m)$.

Now suppose $U' = VU$ for some $V \in \mathrm{Aut}(\mathcal{N}, m)$, then expanding these into submatrices we get:

$$\begin{pmatrix} U_1' \\ U_2' \end{pmatrix} = \begin{pmatrix} U_1 + V_1U_2 \\ V_2U_2 \end{pmatrix}.$$

As a consequence, we know that U_2 and U_2' have the same row Hermite normal form and so the last $n-m$ rows of $\mathrm{NF}(U)$ and $\mathrm{NF}(U')$ are the same. We can now assume that U_2 and U_2' are equal and in Hermite normal form.

If the first m rows of $\mathrm{NF}_u(U')$ are given by $U_1' + V_1'U_2$ for some $m \times (n-m)$ matrix V_1', then we rewrite this as $U_1 + (V_1 + V_1')U_2$. This automatically satisfies the minimality conditions required by the construction of NF_u, so is also the first m rows of $\mathrm{NF}_u(U)$. This shows that $\mathrm{NF}_u(U) = \mathrm{NF}_u(U')$. $\square$

Each of the last three results gives rise to a function which together produce a normal form for full dimensional lattice polytopes containing the origin in their interior under the action of $\mathrm{Aut}(\mathcal{N}, \pi)$. In Algorithm 4 we define a function `LeftUnimodNF` which is the normal form NF_u for unimodular matrices under multiplication on the left by elements of $\mathrm{Aut}(\mathcal{N}, m)$. In Algorithm 5 we use a series of functions to transform this into a normal form for lattice polytopes containing the origin in their interior in $\mathcal{N}_{\mathbb{Q}}$ under the action of $\mathrm{Aut}(\mathcal{N}, \pi)$.

Algorithm 4: A normal form for $\mathrm{GL}_n(\mathbb{Z})$ under multiplication on the left by $\mathrm{Aut}(\mathcal{N}, m)$.

Data: A function `RowHermiteNF()` which takes a matrix and returns its row Hermite normal form and a unimodular matrix which realises it.

Result: A function `LeftUnimodNF()` which takes an $n \times n$ unimodular matrix U and an integer $0 \le m \le n$ and returns a normal form for U under multiplication on the left by elements of $\mathrm{Aut}(\mathcal{N}, m)$ and a matrix V such that `LeftUnimodNF(`U`)`$= VU$.

function `LeftUnimodNF(`U,m`)`
- $n \longleftarrow$ number of rows of U
- `/* Split` U `into two submatrices and replace one of them with its Hermite normal form */`
- $U_1 \longleftarrow$ first m rows of U
- $U_2 \longleftarrow$ last $n - m$ rows of U
- $U_2', V_2 \longleftarrow$ `RowHermiteNF(`U_2`)`
- $U' \longleftarrow$ matrix obtained by vertically joining U_1 and U_2'
- `/* If` $m = 0$ `we just want Hermite normal form */`
- **if** $m = 0$ **then**
  - **return** U_2', $U_2'U^{-1}$
- `/* Else, find the entries of` V_1' `one column at a time */`
- $\widetilde{V} \longleftarrow I_m$
- **for** $i \in [1, \ldots, n - m]$ **do**
  - $c \longleftarrow$ smallest index such that $U_2'[i][c]$ is non-zero
  - $\widetilde{VU} \longleftarrow \widetilde{V} \times$ (rows $1, \ldots, m + i - 1$ of column c of U')
  - $newcol \longleftarrow [\ ]$
  - **for** $r \in [1, \ldots, m]$ **do**
    - $d \longleftarrow U_2'[i][c]$
    - Append $\frac{(\widetilde{VU}[r][1] \mod d) - \widetilde{VU}[r][1]}{d}$ to $newcol$
  - $\widetilde{V} \longleftarrow$ horizontally join $\widetilde{V}$ and $newcol^T$
- `/* We now define` V_1' `and` V' `an find the normal form */`
- $V_1' \longleftarrow$ columns $m + 1, \ldots n$ of $\widetilde{V}$
- $V' \longleftarrow \begin{pmatrix} I_m & V_1' \\ 0 & I_{n-m} \end{pmatrix}$
- $normU \longleftarrow V' \times U'$
- **return** $normU$, $normU \times U^{-1}$

Algorithm 5: Normal form for full dimensional lattice polytopes in $\mathcal{N}_{\mathbb{Q}}$ containing the origin in their interior, under the action of $\mathrm{Aut}(\mathcal{N}, \pi)$.

Data: A lattice polytope $P \subseteq \mathcal{N}_{\mathbb{Q}}$ of full dimension containing the origin in its interior, with vertices $v_1, \ldots, v_l$ and a surjective lattice projection $\pi : \mathcal{N} \to \mathcal{N}_1$. A column Hermite normal form function `ColHermiteNF`, `LeftUnimodNF` as defined in Algorithm 4 and `PALPPerm` which takes a polytope and returns a set of permutations of its vertices.

Result: A polytope Q, equivalent to P and in normal form.

`/* A normal form for` $\mathrm{Mat}^n_{\mathbb{Z}}(l \times n)$ `under multiplication on the right by` $\mathrm{Aut}(\mathcal{N}, m)$ `as in Proposition 6.4.4 */`

function `RightMatNFm(`M, m`)`

- $\_, U \longleftarrow$ `ColHermiteNF(`M`)`
- $\_, V \longleftarrow$ `LeftUnimodNF(`U,m`)`
- **return** MV^{-1}, V^{-1}

`/* A normal form for` $\mathrm{Mat}^n_{\mathbb{Z}}(l \times n)$ `under multiplication on the right by` $\mathrm{Aut}(\mathcal{N}, \pi)$ `as in Proposition 6.4.3 */`

function `RightMatNFpi(`M,π`)`

- $\Pi \longleftarrow$ `DefiningMatrix(`π`)`
- $A, B \longleftarrow$ matrices such that $A\Pi B$ is in Smith normal form
- $m \longleftarrow$ number of columns of Π
- $A \longleftarrow \begin{pmatrix} B^{-1} & 0 \\ 0 & I_{n-m} \end{pmatrix}^{-1} A$
- $norm, V \longleftarrow$ `RightMatNFm(`MA^{-1}, m`)`
- **return** $normA$, $A^{-1}VA$

`/* A normal form for polytopes from the normal form for matrices obtained in the same way as Kreuzer and Skarke's PALP algorithm */`

$NormalForms \longleftarrow \{\}$

for $\sigma \in$ `PALPPerm(`P`)` **do**

- $M \longleftarrow \begin{pmatrix} v_{\sigma(1)} \\ \vdots \\ v_{\sigma(l)} \end{pmatrix}$
- $NormalForms \longleftarrow NormalForms \cup \{$`RightMatNFm(`$M$, π`)`$\}$

$M_Q \longleftarrow$ minimum matrix in $NormalForms$

$Q \longleftarrow$ convex hull of the rows of M_Q

Chapter 7

Classification of Spherical Canonical Fano Four-Folds

In this chapter we describe how to classify the (non-toric) spherical canonical Fano four-folds by classifying polytopes which correspond to them. This sits within the rich history of classifying families of Fano varieties using convex geometry. In particular, our classification includes the spherical Gorenstein Fano four-folds and so extends Kreuzer and Skarke's classification of the toric Gorenstein Fano four-folds [KS00].

In his thesis, Hofscheier completed the classification of spherical Gorenstein Fano three-folds [Hof15]. Similar methods were recently used by Delcroix and Montagard, to classify the spherical locally factorial Fano varieties with dimension up to 4 and rank of $\mathcal{M}$ up to 2 [DM23]. We follow a similar approach to these classifications, but we allow the rank of $\mathcal{M}$ to be up to 3, which is the maximum rank it can have for non-toric spherical four-folds. As a result, our classification is far too large to complete by hand in the way these previous classifications were. Instead we use algorithmic methods to classify the polytopes and interpret them as spherical varieties.

Toric canonical Fano varieties have been classified in dimension 3 [Kas10] and we use the corresponding list of polytopes as a key component of our classification. However, it is worth noting that toric canonical Fano four-folds have not been classified, due to the expected massive size of this list. Therefore, we do not present a classification of *all* spherical canonical Fano four-folds, only those which are spherical under the action of some non-toric reductive group G. This may incidentally include some toric varieties, but without identifying an action of $(\mathbb{C}^{\times})^4$ on them or a fan associated to them.

Our approach uses the fact that spherical canonical Fano varieties are in correspondence with a certain class of polytope which we call G/H-canonical polytopes. We classify the possible Luna data of spherical homogeneous spaces G/H for which the action of G on G/H is smart and such that G/H has an embedding in a spherical canonical Fano four-fold. Each Luna datum $\mathcal{S}$ indicates the type of polytope which can be G/H-canonical with respect to $\mathcal{S}$ so we get a list of families of polytopes which we then classify. For example, we classify the rational polygons with denominator 3, exactly one interior point and exactly one non-lattice vertex. We do this using a combination of the methods of Chapter 5 and [Kas10]. Finally, we describe a way to combine the polytopes and Luna data in every possible way to obtain a list of G/H-canonical polytopes for each Luna datum.

In Section 7.1, we define colored fans, describe how we can obtain a polytope associated to a $\mathbb{Q}$-Gorenstein spherical Fano variety and define G/H-canonical polytopes. In Section 7.2, we classify the Luna data of all spherical canonical Fano four-folds. In Section 7.3, we classify all but one family of polytopes which may be G/H-canonical with respect to one of the Luna data we have found. For the final family, consisting of three-dimensional denominator 2 polytopes with one interior lattice point and one non-lattice vertex, we sketch the approach we will use to classify them. We expect the algorithm to take months to run, so do not present the classification itself at this time. Finally, in Section 7.4 we describe an algorithm which, given a polytope P and a Luna datum $\mathcal{S}$, returns all polytopes (up to equivalence) which are G/H-canonical with respect to $\mathcal{S}$ and are unimodularly equivalent to P.

7.1 Spherical Canonical Fano Varieties

In this section we discuss the fans and polytopes associated to spherical varieties and define the G/H-canonical polytopes which we will classify. Throughout fix a reductive group G with a Borel subgroup B, maximal torus $T \subseteq B$ and spherical subgroup H. Let $\mathcal{S} = (\mathcal{M}, \Sigma, S^P, \mathcal{D}^a)$ be the Luna datum associated to H with components as defined in Section 6.1.2. Let $\mathcal{V}$ be the valuation cone, that is, the set of G-invariant discrete valuations $\nu : \mathbb{C}(G/H)^* \to \mathbb{Q}$.

Definition 7.1.1. A *colored cone* is a pair $(\mathcal{C}, \mathcal{F})$ with $\mathcal{C} \subseteq \mathcal{N}_\mathbb{Q}$ and $\mathcal{F} \subseteq \mathcal{D}$ such that $\mathcal{C}$ is a cone generated by $\rho(\mathcal{F})$ and finitely many elements of $\mathcal{V}$, and such that $\mathcal{C}^\circ \cap \mathcal{V} \neq \emptyset$. A colored cone is called *strictly convex* if $\mathcal{C}$ is strictly convex and $\mathbf{0} \notin \rho(\mathcal{F})$.

A *face* of a colored cone $(\mathcal{C}, \mathcal{F})$ is a colored cone $(\mathcal{C}', \mathcal{F}')$ such that $\mathcal{C}'$ is a face of $\mathcal{C}$ and $\mathcal{F}' = \mathcal{F} \cap \rho^{-1}(\mathcal{C}')$.

A *colored fan* is a nonempty finite collection $\mathfrak{F}$ of strictly convex colored cones such that for every $(\mathcal{C}, \mathcal{F}) \in \mathfrak{F}$ every face of $(\mathcal{C}, \mathcal{F})$ is also in $\mathfrak{F}$ and for every $v \in \mathcal{N}_\mathbb{Q}$ there is at most one $(\mathcal{C}, \mathcal{F}) \in \mathfrak{F}$ with $v \in \mathcal{C}^\circ$. A colored fan $\mathfrak{F}$ is called *complete* if $\operatorname{supp}(\mathfrak{F}) \coloneqq \bigcup_{(\mathcal{C},\mathcal{F})\in\mathfrak{F}} \mathcal{C} \supseteq \mathcal{V}$.

Theorem 7.1.2 ([Kno91, Theorems 3.3 and 4.2])**.** *Colored fans are in bijective correspondence with isomorphism classes of spherical embeddings $G/H \hookrightarrow X$. Moreover, X is complete if and only if the corresponding colored fan is complete.*

Example 7.1.3. Let $G = \mathrm{SL}_2$, B the upper triangular matrices and T the diagonal matrices all acting on $X = \mathbb{C}^2$ by matrix multiplication. The open dense B-orbit is $B \cdot (0,1) = \{(x,y) : y \neq 0\}$ and the G-orbit containing this is $G\dot(0,1) = \mathbb{C}^2 \setminus \{(0,0)\} \cong G/N(T)$, so this is our spherical homogeneous space. Let the character lattice $\mathfrak{X}(T)$ be generated by ω_1, then the Luna datum corresponding to $G/N(T)$ has components $\mathcal{M} = \langle 2\omega_1 \rangle$, $\Sigma = \{2\omega_1\}$, $S^P = \emptyset$ and $\mathcal{D} = \{D_1\}$ with $\rho(D_1) = 1$. The colored fan associated to X is

$$\mathfrak{F}_X = \{(0, \emptyset), (\operatorname{cone}(1), \{D_1\})\}$$

where the colored cone $(0, \emptyset)$ corresponds to the open dense orbit $G \cdot (1,0)$ and the colored cone $(\text{cone}(1), \{D_1\})$ corresponds to the orbit $G \cdot (0,0)$. The color $D_1 = B \cdot (1,0)$ is associated to the orbit $G \cdot (0,0)$ because this orbit is contained in the closure of the color.

Example 7.1.4. With the same G, B and T as in the previous example let G act on $X = \text{Bl}_{(0,0)}(\mathbb{C}^2) = \{((x,y),[z:w]) : xw = yz\}$ by matrix multiplication on $\left(\begin{smallmatrix} x & z \\ y & w \end{smallmatrix}\right)$. The open dense B-orbit is $\{((x,y),[z:w]) \in X : y, w \neq 0\}$. The G-orbit containing this is $\{((x,y),[z:w]) \in X : x, y \neq 0\} \cong G/N(T)$, so we have the same spherical homogeneous space and Luna datum as above. The colored fan associated to X is

$$\mathfrak{F}_X = \{(0, \emptyset), (\text{cone}(1), \emptyset)\}$$

where the colored cone $(0, \emptyset)$ corresponds to the orbit $G \cdot ((0,1),[0:1])$ and the colored cone $(\text{cone}(1), \emptyset)$ corresponds to the orbit $G \cdot ((1,0),[1:0])$. Notice that the color $D_1 = B \cdot ((1,0),[1:0])$ is not associated to any of these orbits since its closure contains none of them as a subset.

For our classification it is easier to work with polytopes than colored fans. To associate a polytope to a spherical variety, we first need to define some constants associated to the colors. For a spherical embedding $G/H \hookrightarrow X$ let $X_1, \ldots, X_r$ be the G-invariant prime divisors in X, then Brion [Bri97] showed that the anti-canonical divisor of X can be written as

$$-K_X = \sum_{i=1}^{r} X_i + \sum_{D \in \mathcal{D}} m_D D$$

where the m_D are positive integers depending on G/H. For each color D, m_D can be computed as follows:

$$m_D = \begin{cases} 1 & \text{if } D \text{ is a color of type } a \text{ or } 2a \\ \langle \check{\alpha}, \kappa_P \rangle & \text{if } D \text{ is a color of type } b \end{cases}$$

where α is a simple root such that P_α moves D and κ_P is the sum of the positive roots generated by S minus the sum of the positive roots generated by S^P [Lun97, Section 3.6].

Definition 7.1.5. Let $G/H \hookrightarrow X$ be a complete spherical embedding with G-invariant divisors $X_1, \ldots, X_r$ and associated valuations $\nu_{X_i} \in \mathcal{N}_{\mathbb{Q}}$. Then we define the polytope associated to X to be the following rational polytope

$$Q_X \coloneqq \operatorname{conv}\left(\tfrac{1}{m_{\mathcal{D}}}\rho(\mathcal{D}), \nu_{X_1}, \ldots, \nu_{X_r}\right) \subseteq \mathcal{N}_{\mathbb{Q}}$$

where $\frac{1}{m_{\mathcal{D}}}\rho(\mathcal{D})$ denotes the set of points $\frac{1}{m_D}\rho(D)$ for each color $D \in \mathcal{D}$.

Definition 7.1.6. Given a rational polytope $Q \subseteq \mathcal{N}_{\mathbb{Q}}$ containing the origin in its interior, we can define the following colored fan associated to Q:

$$\mathfrak{F}_Q \coloneqq \{(\operatorname{cone}(F), \rho^{-1}(F)) : F \text{ is a proper face of } Q \text{ s.t. } F^\circ \cap \mathcal{V} \neq \emptyset\}.$$

This is a colored fan called the *face fan* of Q and we denote the associated spherical variety by X_Q.

Like toric Fano polytopes, the polytopes of the form Q_X, satisfy a collection of conditions which are identified by Gagliardi and Hofscheier in [GH15].

Definition 7.1.7 ([GH15, Definition 7.1])**.** A polytope $Q \subseteq \mathcal{N}_{\mathbb{Q}}$ is called $\mathbb{Q}$-G/H-*reflexive* if the following conditions are satisfied:

1. $\frac{1}{m_D}\rho(D) \in Q$ for every $D \in \mathcal{D}$,
2. $\mathbf{0} \in Q^\circ$ and
3. Every vertex of Q is in $\frac{1}{m_{\mathcal{D}}}\rho(\mathcal{D})$ or is a primitive element of $\mathcal{N} \cap \mathcal{V}$.

Proposition 7.1.8 ([GH15, Proposition 7.4])**.** *The assignments $X \mapsto Q_X$ and $Q \mapsto X_Q$ define a bijection between isomorphism classes of $\mathbb{Q}$-Gorenstein spherical Fano embeddings of G/H and $\mathbb{Q}$-G/H-reflexive polytopes.*

They use this to define a spherical analouge for reflexive polytopes:

Definition 7.1.9 ([GH15, Definition 1.8])**.** A $\mathbb{Q}$-G/H-reflexive polytope $Q \subseteq \mathcal{N}_{\mathbb{Q}}$ is called *G/H-reflexive* if $v \in \mathcal{M}$ for every vertex v of Q^* associated to a facet of Q whose interior intersects non-trivially with $\mathcal{V}$.

Proposition 7.1.10 ([GH15, Theorem 1.9])**.** *The assignment $X \mapsto Q_X$ induces a bijection between isomorphism classes of Gorenstein spherical Fano embeddings $G/H \hookrightarrow X$ and G/H-reflexive polytopes.*

We adjust this definition to describe canonical spherical varieties:

Definition 7.1.11. A $\mathbb{Q}$-G/H-reflexive polytope $Q \subseteq \mathcal{N}_{\mathbb{Q}}$ is called *G/H-canonical* if it contains no non-zero interior points in $\mathcal{V}$.

We use the following result of Pasquier to prove that this is the correct description.

Proposition 7.1.12 ([Pas17, Proposition 5.2])**.** *Let G/H be a spherical homogeneous space. Let $G/H \hookrightarrow X$ be a $\mathbb{Q}$-Gorenstein G/H-embedding associated to a colored fan $\mathfrak{F}_X$. For any colored cone $(\mathcal{C}, \mathcal{F})$ of $\mathfrak{F}_X$, write $h_{\mathcal{C}}$ for the linear function such that for any color $D \in \mathcal{F}$, $h_{\mathcal{C}}(\rho(D)) = m_D$ and, for any primitive element u of an edge of $\mathcal{C}$ (not generated by some $\rho(D)$ with $D \in \mathcal{F}$), $h_{\mathcal{C}}(u) = 1$. Then X is canonical if and only if for any colored cone $(\mathcal{C}, \mathcal{F})$ of $\mathfrak{F}_X$, for any $x \in \mathcal{C} \cap \mathcal{N} \cap \mathcal{V}$, $h_{\mathcal{C}}(x) \geq 1$.*

Proposition 7.1.13. *The assignment $X \mapsto Q_X$ induces a bijection between isomorphism classes of canonical spherical Fano embeddings $G/H \hookrightarrow X$ and G/H-canonical polytopes.*

Proof. Given a $\mathbb{Q}$-Gorenstein G/H-embedding X and its associated $\mathbb{Q}$-G/H-reflexive polytope Q_X the function $h_{\mathcal{C}}$ is exactly the outwards pointing normal vector to the face of Q_X which generates the cone $\mathcal{C}$. In other words, an interior point $x \in Q_X$ satisfies $h_{\mathcal{C}}(x) < 1$. Therefore, X is canonical if and only if no non-zero point of $\mathcal{N} \cap \mathcal{V}$ is contained in the interior of Q_X. □

7.2 Classification of Luna Data

In this section we classify the Luna data $\mathcal{S}$ corresponding to spherical homogeneous spaces G/H for which the action of G on G/H is smart and G/H can be embedded in a spherical canonical Fano four-fold.

A *spherical system* is a triple $(\Sigma, S^P, \mathcal{D}^a)$ along with a map $\rho : \mathcal{D}^a \to \mathrm{span}_{\mathbb{Z}}(\Sigma)^*$ such that $(\mathrm{span}_{\mathbb{Z}}(\Sigma), \Sigma, S^P, \mathcal{D}^a)$ is a Luna datum. Notice that any Luna datum determines a spherical system, by replacing the lattice $\mathcal{M}$ with the sublattice generated by Σ. Therefore, we can obtain any Luna datum by extending the lattice of some spherical system.

Spherical systems have a combinatorial description in terms of *Luna diagrams.* A Luna diagram is a decoration of the Dynkin diagram of G. Since our action is smart we have $G = G^{ss} \times C$ so this is also the Dynkin diagram of G^{ss}. Recall that the vertices of a Dynkin diagram correspond to simple roots. To obtain a Luna diagram from a spherical system we first apply the markings associated to each spherical root. The full list of such markings can be found in [BL11] but all the ones we will need are in Table 7.1. Then we circle any root which is not in S^P and does not yet have a circle above, below or around it. If $S \cap \Sigma$ is empty then we are done. Otherwise, by the Luna datum axioms, each root α in $S \cap \Sigma$ has a corresponding pair of colors D_α^+ and D_α^- of type a in $\mathcal{D}^a$. We assign D_α^+ to the circle above α and D_α^- to the circle bellow α, then join circles which correspond to the same element of $\mathcal{D}^a$ with a line. Finally, for every spherical root γ not orthogonal to α such that $\langle \rho(D_\alpha^+), \gamma \rangle = -1$, we add an arrow of the form $<$ or $>$, starting from the circle corresponding to D_α^+, and pointing toward γ.

Using the Luna datum axioms in Definition 6.1.5 we can retrieve the spherical system from a Luna diagram. In fact, the diagram associated to a Luna datum tells us the full set of colors and the map $\rho : \mathcal{D} \to \mathcal{N}$. Each circle corresponds to a color, it has type a if there are circles above and bellow a root, type $2a$ if there is just a circle bellow that root and type b if the circle is around the

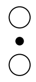
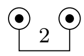
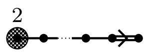
diagram	spherical root
	α_1
	$2\alpha_1$
	$\alpha_1 + \alpha_1'$
	$\alpha_1 + \cdots + \alpha_r$
	$2\alpha_1 + \cdots + 2\alpha_r$

Table 7.1: Spherical roots which appear in spherical canonical Fano four-folds and their corresponding marking of a Luna diagram.

root. When D is of type a, $\rho(D)$ is determined by Definition 6.1.5. Otherwise

$$\rho(D) = \begin{cases} \frac{1}{2}\check{\alpha}|_{\mathcal{M}} & \text{if } D \text{ is of type } 2a \\ \check{\alpha}|_{\mathcal{M}} & \text{if } D \text{ is of type } b \end{cases}$$

where α is a simple root such that P_α moves D.

The Luna diagrams which can be associated to spherical canonical Fano four-folds were classified by Hofscheier using the Luna datum axioms and dimension bounds on G/P where P is the parabolic subgroup which stabilizes the open dense Borel orbit in G/H. These diagrams appear in column 1 of Tables 7.2-7.5. Proof of this classification will appear in a forthcoming paper. It remains to find all Luna data which give each of these Luna diagrams.

First we determine the reductive group $G = G^{ss} \times C$. The semi-simple part G^{ss} is determined by the Dynkin diagram the Luna diagram is based on. These are products of the reductive groups in Table 6.1. We label the vertices of the diagram and their corresponding simple roots $\alpha_1, \ldots, \alpha_k$ from left to right and assume that the simple roots are the corresponding products of those listed in Table 6.1. Since we assume a smart action, $\mathfrak{X}(B) = \mathfrak{X}(B^{ss}) \oplus \mathfrak{X}(C)$, $\mathrm{rk}(\mathcal{M} \cap \mathfrak{X}(B^{ss})) = |\Sigma|$ and $\mathcal{M}$ surjects onto $\mathfrak{X}(C)$ under the natural projection map $\pi_C : \mathfrak{X}(B) \to \mathfrak{X}(C)$. We can choose a basis $b_1, \ldots, b_n$ of $\mathcal{M}$ such that $\pi_C(b_{k+1}), \ldots, \pi_C(b_n)$ is a basis of $X(C)$ and $\pi_C(b_i) = 0$ for $i \leq k$. Therefore, the rank of $\mathcal{M}$ is the sum of the number of spherical roots and rank of $\mathfrak{X}(C)$. By [BLV86], the rank of $\mathcal{M}$ is $\dim(G/H) - \dim(G/P)$, which determines the dimension of the torus C, so we have found G.

The set of simple roots S^P can be written down directly from the diagram by recording the roots which do not have a circle, above, bellow or around them. The set of spherical roots can be found using Table 7.1 and the set $\mathcal{D}^a$ comes from the spherical roots which are also simple roots. The map $\rho : \mathcal{D}^a \to \mathcal{N}$ depends on the choice of sublattice $\mathcal{M}$ but once $\mathcal{M}$ is fixed it follows from the above. Therefore, the main step is finding all sublattices $\mathcal{M}$ for each diagram.

Proposition 7.2.1. *Let G/H be a spherical homogeneous space such that the action of G on G/H is smart and there is some spherical embedding $G/H \hookrightarrow X$ such that X is a canonical Fano four-fold. Then the sublattice $\mathcal{M} \subseteq \mathfrak{X}(B)$ is listed in the second column of one of Tables 7.2-7.5.*

Table 7.2: Luna data for spherical canonical Fano four-folds with $\dim(G/P) = 4$.

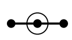
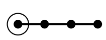
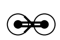
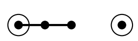
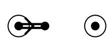
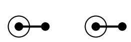
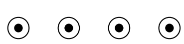
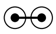
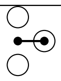
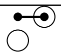
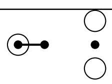
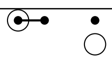
Luna diagram	$\mathcal{M}$	$\Sigma \subseteq \mathcal{M}$	S^P	$\rho(\mathcal{D}) \subseteq \mathcal{N}_{\mathbb{Q}}$	Data ID
	$\langle 0 \rangle$	$\emptyset$	$\{\alpha_1, \alpha_3\}$	$\{0\}$	1
	$\langle 0 \rangle$	$\emptyset$	$\{\alpha_2, \alpha_3, \alpha_4\}$	$\{0\}$	2
	$\langle 0 \rangle$	$\emptyset$	$\emptyset$	$\{0, 0\}$	3
	$\langle 0 \rangle$	$\emptyset$	$\{\alpha_2, \alpha_3\}$	$\{0, 0\}$	4
	$\langle 0 \rangle$	$\emptyset$	$\emptyset$	$\{0, 0, 0\}$	5
	$\langle 0 \rangle$	$\emptyset$	$\{\alpha_2\}$	$\{0, 0\}$	6
	$\langle 0 \rangle$	$\emptyset$	$\{\alpha_1\}$	$\{0, 0\}$	7
	$\langle 0 \rangle$	$\emptyset$	$\{\alpha_2, \alpha_4\}$	$\{0, 0\}$	8
	$\langle 0 \rangle$	$\emptyset$	$\{\alpha_2\}$	$\{0, 0, 0\}$	9
	$\langle 0 \rangle$	$\emptyset$	$\emptyset$	$\{0, 0, 0, 0\}$	10

Table 7.3: Luna data for spherical canonical Fano four-folds with $\dim(G/P) = 3$.

Luna diagram	$\mathcal{M}$	$\Sigma \subseteq \mathcal{M}$	S^P	$\frac{1}{m_{\mathcal{D}}}\rho(\mathcal{D}) \subseteq \mathcal{N}_{\mathbb{Q}}$	Data ID
	$\langle b_1\omega_1 + \chi_1\rangle$ where $b_1 \in \{0, 1, 2, 3, 4\}$	$\emptyset$	$\{\alpha_2, \alpha_3\}$	$\{\frac{1}{4}(-b_1)\}$	1-5
	$\langle b_1\omega_1 + b_2\omega_2 + \chi_1\rangle$ where $b_2 \geq b_1$ and $b_2 - b_1, b_2$ $\in \{0, \pm 1, \pm 2\}$	$\emptyset$	$\emptyset$	$\{\frac{1}{2}(b_2 - b_1), \frac{1}{2}b_2\}$	6-20
	$\langle \omega_1 - \omega_2\rangle$	$\{1\}$	$\emptyset$	$\{\underline{\tfrac{1}{1}1}, \underline{\tfrac{1}{1}1}, -\frac{1}{2}1\}$	21
	$\langle 2\omega_1 - 2\omega_2\rangle$	$\{1\}$	$\emptyset$	$\{\frac{1}{1}2, -\frac{1}{2}2\}$	22
	$\langle 2\omega_1 + \omega_2\rangle$	$\{1\}$	$\emptyset$	$\{\frac{1}{2}1, \frac{1}{2}1\}$	23
	$\langle b_2\omega_2 + \chi_1\rangle$ where $b_2 \in \{0, 1, 2, 3\}$	$\emptyset$	$\{\alpha_2\}$	$\{\frac{1}{3}b_2\}$	24-27
	$\langle 2\omega_1 + 2\omega_2\rangle$	$\{1\}$	$\{\alpha_2\}$	$\{\frac{1}{3}2\}$	28
	$\langle b_1\omega_1 + \chi_1\rangle$ where $b_1 \in \{0, \ldots, 4\}$	$\emptyset$	$\{\alpha_1\}$	$\{\frac{1}{4}b_1\}$	29-33
	$\langle b_1\omega_1 + b_3\omega_3 + \chi_1\rangle$ where $b_1 \in \{0, \ldots, 3\}$ and $b_3 \in \{0, \pm 1, \pm 2\}$	$\emptyset$	$\{\alpha_2\}$	$\{\frac{1}{3}b_1, \frac{1}{2}b_3\}$	34-51
	$\langle 2\omega_3\rangle$	$\{1\}$	$\{\alpha_2\}$	$\{0, \underline{\tfrac{1}{1}1}, \underline{\tfrac{1}{1}1}\}$	52
	$\langle 4\omega_3\rangle$	$\{1\}$	$\{\alpha_2\}$	$\{0, \frac{1}{1}2\}$	53

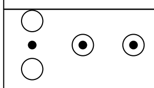
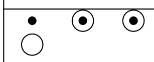
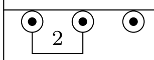
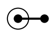
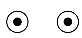
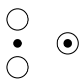
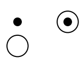
Luna diagram	$\mathcal{M}$	$\Sigma \subseteq \mathcal{M}$	S^P	$\frac{1}{m_{\mathcal{D}}}\rho(\mathcal{D}) \subseteq \mathcal{N}_{\mathbb{Q}}$	Data ID
	$\langle b_1\omega_1 + b_2\omega_2 + b_3\omega_3 + \chi_1\rangle$ where $b_i \in \{0, \pm 1, \pm 2\}$, $b_1 \geq 0$ and $b_1 \geq b_2 \geq b_3$	$\emptyset$	$\emptyset$	$\{\frac{1}{2}b_1, \frac{1}{2}b_2, \frac{1}{2}b_3\}$	54-84
	$\langle 2\omega_1\rangle$	$\{1\}$	$\emptyset$	$\{\underline{\tfrac{1}{1}1}, \underline{\tfrac{1}{1}1}, 0, 0\}$	85
	$\langle 4\omega_1\rangle$	$\{1\}$	$\emptyset$	$\{\frac{1}{1}2, \frac{1}{2}0, \frac{1}{2}0\}$	86
	$\langle 2\omega_1 + 2\omega_2\rangle$	$\{1\}$	$\emptyset$	$\{\frac{1}{2}2, \frac{1}{2}2, \frac{1}{2}0\}$	87

Table 7.4: Luna data for spherical canonical Fano four-folds with $\dim(G/P) = 2$. The basis of $\mathcal{M}$ is given in terms of the coordinates of $\mathfrak{X}(B)$ and written as the columns of a matrix. All other coordinates are given in terms of this basis of $\mathcal{M}$ and its dual basis of $\mathcal{N}$.

Luna diagram	Basis of $\mathcal{M}$	$\Sigma \subseteq \mathcal{M}$	S^P	$\frac{1}{m_{\mathcal{D}}}\rho(\mathcal{D}) \subseteq \mathcal{N}_{\mathbb{Q}}$	Data ID
	$\begin{pmatrix} b_1 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\emptyset$	$\{\alpha_2\}$	$\{\frac{1}{3}(b_1, 0)\}$ where $b_1 \in \{0, 1, 2, 3\}$	1-4
	$\begin{pmatrix} b_{11} & 0 \\ b_{21} & b_{22} \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$ where $\begin{pmatrix} b_{11} & 0 \\ b_{21} & b_{22} \end{pmatrix}$ is in column HNF	$\emptyset$	$\emptyset$	$\{\frac{1}{2}(b_{11}, 0), \frac{1}{2}(b_{21}, b_{22})\}$ where $b_{11}, \gcd(b_{21}, b_{22}) \in \{0, 1, 2\}$	5-17
	$\begin{pmatrix} 2 & b_1 \\ 0 & b_2 \\ 0 & 1 \end{pmatrix}$	$\{\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)\}$	$\emptyset$	$\{\underline{\tfrac{1}{1}(1, d_1^{\pm})}, \frac{1}{2}(0, b_2)\}$ where $d_1^+ + d_1^- = b_1 \in \{0, 1\}$ and $b_2 \in \{0, 1, 2\}$	18-35
	$\begin{pmatrix} 4 & 2b_1 \\ 0 & b_2 \\ 0 & 1 \end{pmatrix}$	$\{\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)\}$	$\emptyset$	$\{\frac{1}{1}(2, b_1), \frac{1}{2}(0, b_2)\}$ where $b_1 \in \{0, 1\}$ and $b_2 \in \{0, 1, 2\}$	36-41

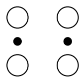
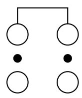
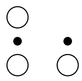
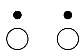
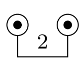
Luna diagram	Basis of $\mathcal{M}$	$\Sigma \subseteq \mathcal{M}$	S^P	$\frac{1}{m_{\mathcal{D}}}\rho(\mathcal{D}) \subseteq \mathcal{N}_{\mathbb{Q}}$	Data ID
	$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$	$\{\binom{1}{0}, \binom{0}{1}\}$	$\emptyset$	$\frac{1}{1}\{\underline{(1, d_1^+)}, \underline{(1, -d_1^+)}, \uwave{(d_2^+, 1)}, \uwave{(-d_2^+, 1)}\}$	42-48
	$\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$	$\{\binom{1}{0}, \binom{-1}{2}\}$	$\emptyset$	$\frac{1}{1}\{\underline{(1, d_1^+)}, \underline{(1, 1 - d_1^+)}, \uwave{(2d_2^+ - 1, d_2^+)}, \uwave{(1 - 2d_2^+, 1 - d_2^+)}\}$	49-54
	$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$	$\{\binom{1}{0}, \binom{0}{1}\}$	$\emptyset$	$\frac{1}{1}\{\uwave{(1, 1)}, \underline{(1, -1)}, \uwave{(-1, 1)}\}$	55
	$\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$	$\{\binom{1}{0}, \binom{-1}{2}\}$	$\emptyset$	$\frac{1}{1}\{\uwave{(1, 1)}, \underline{(1, 0)}, \uwave{(-1, 0)}\}$	56
	$\begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$	$\{\binom{1}{0}, \binom{0}{1}\}$	$\emptyset$	$\frac{1}{1}\{\underline{(1, d_1^+)}, \underline{(1, -d_1^+)}, (0, 2)\}$ where $d_1^+ \in [0, 2]$	57-59
	$\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$	$\{\binom{1}{0}, \binom{-1}{2}\}$	$\emptyset$	$\frac{1}{1}\{\underline{(1, d_1^+)}, \underline{(1, 1 - d_1^+)}, (0, 1)\}$ where $d_1^+ \in [1, 2]$	60-61
	$\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$	$\{\binom{1}{0}, \binom{0}{1}\}$	$\emptyset$	$\frac{1}{1}\{(2, 0), (0, 2)\}$	62
	$\begin{pmatrix} 4 & 2 \\ 0 & 2 \end{pmatrix}$	$\{\binom{1}{0}, \binom{-1}{2}\}$	$\emptyset$	$\frac{1}{1}\{(2, 1), (0, 1)\}$	63
	$\begin{pmatrix} 2 & b_1 \\ 2 & b_2 \\ 0 & 1 \end{pmatrix}$ where $b_1 \in \{0, 1\}$ and $b_2 \geq b_1$	$\{\binom{1}{0}\}$	$\emptyset$	$\{\frac{1}{2}(2, b_1), \frac{1}{2}(2, b_2)\}$	64-80

Table 7.5: Luna data for spherical canonical Fano four-folds with $\dim(G/P) = 1$. The basis of $\mathcal{M}$ is given in terms of the coordinates of $\mathfrak{X}(B)$ and written as the columns of a matrix. All other coordinates are given in terms of this basis of $\mathcal{M}$ and its dual basis of $\mathcal{N}$.

Luna diagram	Basis of $\mathcal{M}$	$\Sigma \subseteq \mathcal{M}$	S^P	$\frac{1}{m_{\mathcal{D}}}\rho(\mathcal{D}) \subseteq \mathcal{N}_{\mathbb{Q}}$	Data ID
$\odot$	$\begin{pmatrix} b & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ where $b \in \{0, 1, 2\}$	$\emptyset$	$\emptyset$	$\{\frac{1}{2}(b, 0, 0)\}$	1-3
$\circ$ $\bullet$ $\circ$	$\begin{pmatrix} 2 & b & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ where $b \in \{0, 1\}$	$\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right\}$	$\emptyset$	$\{\frac{1}{1}(1, d_1^{\pm}, 0)\}$ where $d_1^+ + d_1^- = b$ and $d_1^+ - d_1^- \in [0, 12]$	4-16
$\bullet$ $\circ$	$\begin{pmatrix} 4 & 2b & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ where $b \in \{0, 1\}$	$\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right\}$	$\emptyset$	$\{\frac{1}{1}(2, b, 0)\}$	17-18

In the proof we need an extended definition of lattice length. Let $\operatorname{len}(P)$ denote the lattice length of a lattice line segment P, then we define the lattice length of a denominator r line segment to be $\frac{1}{r}\operatorname{len}(rP)$.

Proof. The parabolic subgroup P is the one determined by the simple roots S^P, so the dimension of G/P can be determined from the Luna diagram. The tables are divided by dimension of G/P and so also by the rank of $\mathcal{M}$. We will mainly use the facts that at least one G/H-canonical polytope exists and that automorphisms of the form $\varphi : \mathcal{M} \to \mathcal{M}, \chi \mapsto \pi^{ss}(\chi) + \varphi_C(\pi_C(\chi))$, where φ_C is an automorphism of $\mathfrak{X}(C)$, are induced by twisted equivariant isomorphisms. The later is shown by the same proof as Theorem 6.0.3 and allows us to remove some Luna data with equivalent spherical subgroups.

Throughout, assume that the basis of $\mathfrak{X}(B^{ss})$ is $\omega_1, \ldots, \omega_k$ and the basis of $\mathfrak{X}(C)$ is $\chi_1, \ldots, \chi_{n-m}$ where k is the number of simple roots, m is the number of spherical roots and n is the rank of $\mathcal{M}$.

Dimension of G/P is 4:

The rank of $\mathcal{M}$ is 0 so $\mathcal{M}$ can only be the trivial lattice $\langle \mathbf{0} \rangle$.

Dimension of G/P is 3:

The rank of $\mathcal{M}$ is 1. If there is one spherical root then it is required to be a primitive point of $\mathcal{M}$ so we may assume it is the one basis vector of $\mathcal{M}$. This determines $\mathcal{M}$ for entries 21-23, 28, 52, 53 and 85-87 of Table 7.3.

The coefficients of the basis vector of $\mathcal{M}$ in entries 1-5, 24-27 and 29-51 are all restricted by considering the spherical subgroups H which can induce them. This is done in [DM23, Proposition 3.20]. In all of these cases there are no spherical roots so the valuation cone is $\mathcal{V} = \mathcal{N}_{\mathbb{Q}}$ and we can have no interior points in a G/H-canonical polytope. Therefore, for each color $D \in \mathcal{D}$ the lattice length of $\operatorname{conv}(\mathbf{0}, \frac{1}{m_D}\rho(D))$ must be at most 1. This bounds the coefficients of the basis vector of $\mathcal{M}$ to finitely many possibilities.

In the remaining cases, 6-20 and 54-84, since we know that $\mathcal{M}$ surjects onto $\mathfrak{X}(C)$ under the natural projection π_C, we may assume that $\mathcal{M}$ is generated by $b_1\omega_1 + \cdots + b_k\omega_k + \chi_1$ for some integers $b_1, \ldots, b_k$. In both of these cases there are no spherical roots so as above the lattice length of $\operatorname{conv}(\mathbf{0}, \frac{1}{m_D}\rho(D))$ must be at most 1. This, as well as symmetry of the diagram in case 54-84, gives the bounds on the b_i described in Table 7.3.

Dimension of G/P is 2:

The rank of $\mathcal{M}$ is 2. The coefficients of the basis vector of $\mathcal{M}$ in entries 1-4 are restricted by considering the spherical subgroups H which can induce them. This is done in [DM23, Proposition 3.18]. There are no spherical roots, so for each color $D \in \mathcal{D}$ the lattice length of $\operatorname{conv}(\mathbf{0}, \frac{1}{m_D}\rho(D))$ is at most 1. Therefore, we may assume the basis of $\mathcal{M}$ is $b_1\omega_1 + \chi_1$, χ_2 for some $b_1 \in \{0, 1, 2, 3\}$.

Entries 5-17 of Table 7.4 have no spherical roots, so we may assume that $\mathcal{M}$ has a basis $a_{11}\omega_1 + a_{21}\omega_2 + \chi_1$, $a_{12}\omega_1 + a_{22}\omega_2 + \chi_2$ for some integers a_{ij}. There is an equivalent basis of $\mathcal{M}$ of the form $b_{11}\omega_1 + b_{21}\omega_2 + \chi_1'$, $b_{22}\omega_2 + \chi_2'$ where $\left(\begin{smallmatrix} b_{11} & 0 \\ b_{21} & b_{22} \end{smallmatrix}\right)$ is in column Hermite normal form and χ_1', χ_2' is some other basis of $\mathfrak{X}(C)$. Therefore, by applying an automorphism of $\mathfrak{X}(C)$ to $\mathfrak{X}(B)$, we may assume that $\mathcal{M}$ has basis $b_{11}\omega_1 + b_{21}\omega_2 + \chi_1$, $b_{22}\omega_2 + \chi_2$. A G/H-canonical polytope P can have no interior points since there are no spherical roots, and has denominator 2 since that is the worst denominator of the set $\frac{1}{m_{\mathcal{D}}}\rho(\mathcal{D})$. In Section 7.3 we classify all denominator 2 polygons with one interior point, and by examining this classification we can list all pairs of points $\frac{1}{2}(b_{11}, 0)$, $\frac{1}{2}(b_{21}, b_{22})$ which can be contained in such a polygon to a finite list:

If $b_{11} = 0$ then we may assume (b_{21}, b_{22}) is one of

$$(0,0), \quad (1,0), \quad (2,0)$$

If $b_{11} = 1$ then we may assume that (b_{21}, b_{22}) is one of

$$\begin{array}{ccccccccc} (-1,0) & (0,1) & (1,0) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) & (2,3) \\ (2,9) & (3,4) & (3,5) & (3,8) & (4,7) & (5,8) & (5,9) & (7,12) & \end{array}$$

If $b_{11} = 2$ then we may assume that (b_{21}, b_{22}) is one of

$$\begin{array}{ccccccc} (0,0) & (0,1) & (1,2) & (1,3) & (1,4) & (1,5) & (2,0) \\ (2,2) & (2,3) & (2,4) & (2,5) & (2,6) & (2,7) & (2,8) \end{array}.$$

Entries 18-35 in Table 7.4 have one spherical root $2\omega_1$, so we may assume that this is the first basis vector of $\mathcal{M}$. Since $\mathcal{M}$ surjects onto $\mathfrak{X}(C)$ we may assume the second basis vector has the form $b_1\omega_1 + b_2\omega_2 + \chi_1$ for some integers b_1 and b_2. We may assume that $b_1 \in \{0, 1\}$ and add or subtract $2\omega_1$ to the second basis vector until this is true otherwise. The point $(0, b_2)$ is the image of a color D under ρ and is in the valuation cone. Since $m_D = 2$ and a G/H-canonical polytope cannot have interior points in the valuation cone $b_2 \in \{0, 1, 2\}$.

The matrices in entries 36-41 are obtained in a very similar way with the additional restriction that, since $\rho(D)$ is a lattice point of $\mathcal{N}$ for all colors D, the leading coefficient of the second basis vector must be even.

The matrices in entries 64-80 are also obtained in a similar way but here we can also use the symmetry of the Luna diagram to assume that $b_2 \geq b_1$. Let x_1

and x_2 be the two primitive lattice point in the boundary of $\mathcal{V}$, then a G/H-canonical polygon cannot contain these in its interior. Therefore, P cannot contain any point in the interior of the penumbra of x_1 or x_2 with respect to $\operatorname{conv}(\frac{1}{m_{\mathcal{D}}}\rho(\mathcal{D}))$. Since P contains the origin in its interior it must have some vertex with negative x-coordinate. Combined, these give the bound $b_2 \leq 8$.

Entries 42-54 of Table 7.4 have two spherical roots, $2\omega_1$ and $2\omega_2$, both of which need to be primitive points of $\mathcal{M}$. We may assume that $2\omega_1$ is the first basis vector of $\mathcal{M}$. The second basis vector has the form $b_1\omega_1 + b_2\omega_2$ where we may assume $b_2 \in \{1, 2\}$ in order for $2\omega_2$ to be a point of $\mathcal{M}$. We may also assume that $b_1 \in \{0, 1\}$ otherwise we can add or subtract $2\omega_1$ from the second basis vector until this is the case. Checking this finite collection of choices, we find that the only possible second basis vectors are $2\omega_2$ and $\omega_1 + \omega_2$. The rank 2 Luna data with IDs 55-63 all follow in a very similar way.

Dimension of G/P is 1:

The rank of $\mathcal{M}$ is 3. Entries 1-3 of Table 7.5 have no spherical roots so we may assume that the basis of $\mathcal{M}$ is $b_1\omega_1 + \chi_1$, $b_2\omega_1 + \chi_2$, $b_3\omega_1 + \chi_3$ for some integers b_1, b_2 and b_3. There is an equivalent basis of $\mathcal{M}$ of the form $\gcd(b_1, b_2, b_3)\omega_1 + \chi_1'$, χ_2', χ_3' where χ_1', χ_2', χ_3' is some other basis of $\mathfrak{X}(C)$. Therefore, possibly after applying an automorphism of $\mathfrak{X}(C)$ to $\mathfrak{X}(B)$, we may assume that $\mathcal{M}$ has basis $b\omega_1 + \chi_1$, χ_2, χ_3 for some non-negative integer b. Since there are no spherical roots, the line segment $\operatorname{conv}(\mathbf{0}, \frac{1}{2}(b, 0, 0))$ must have lattice length at most 1 so $b \in \{0, 1, 2\}$.

Entries 4-16 of Table 7.5 have one spherical root $2\omega_1$, which we may assume is the first basis vector of $\mathcal{M}$. As in the previous case we may assume the remaining basis vectors are $b\omega_1 + \chi_1$ and χ_2 for some non-negative integer b. We may assume that $b \in \{0, 1\}$ otherwise we can add or subtract $2\omega_1$ to the second basis vector until this is so.

The lattice $\mathcal{M}$ in entries 17 and 18 of Table 7.5 can be found in the same way as the previous case with the additional restriction that, since $\rho(D)$ must be a lattice point, the leading coefficient of the second basis vector is even. □

We refer to the rank of $\mathcal{M}$ as the rank of the Luna datum. When a Luna datum has rank at least 2 and some color of type a there may be infinitely many possibilities for the map $\rho : \mathcal{D} \to \mathcal{N}$ which we now bound using the condition that a G/H-canonical polytope exists. In the rank 2 cases 18-35, 42-54 and 57-61, let x_1 and x_2 be the two distinct primitive lattice points in the boundary of $\mathcal{V}$. A G/H-canonical polytope can have no non-zero interior point in the valuation cone. In particular, P can have no point in the interior of the penumbra of x_1 or x_2 with respect to $\mathrm{conv}(\frac{1}{m_\mathcal{D}}\rho(\mathcal{D}))$. This condition bounds the values of d_1^+, d_1^- and d_2^+ to finite possibilities in all of these cases.

In Section 7.3 we will show that G/H-canonical polytopes for the rank 3 Luna data 4-16 are all canonical lattice polytopes and that the line segment connecting $(1, d_1^+, 0)$ and $(1, d_1^-, 0)$ is an edge of such a polytope. These have been classified previously, so we bound $d_1^\pm$ using this classification.

Tables 7.2-7.5 also list the spherical roots and sets $\frac{1}{m_\mathcal{D}}\rho(\mathcal{D})$ for each datum in terms of the basis of $\mathcal{M}$. Since these are what the definition of G/H-canonical depends on they will be useful when we classify the G/H-canonical polytopes in the following sections.

7.3 Classification of Polytopes

In this section, we classify all polytopes which can be G/H-canonical with respect to some Luna datum in Tables 7.2–7.5.

In rank 0 this classification is trivial as there is only one rank 0 polytope. For the rank 1 Luna data 22, 52 and 85 the only G/H-canonical polytope is $\mathrm{conv}(-1, 2)$. For the remaining Luna data all points $\frac{1}{m_D}\rho(D)$ have lattice length at most 1 so the only interior point of a G/H-canonical polytope is the origin. Notice that the denominator of a G/H-canonical polytope is at worst the lowest common multiple of the denominators of $\frac{1}{m_D}\rho(D)$ for each color D. Therefore, it is enough to classify all line segments of denominator $1, 2, 3, 4$ and 6 with one interior point. Not all of these line segments need be G/H-

canonical for any rank 1 Luna datum, but the list will include all line segments which we are looking for. This will be the case in the remaining classifications also, then in Section 7.4 we will describe how to determine which polytopes are G/H-canonical for which Luna data.

7.3.1 Rank 2, Denominator 1

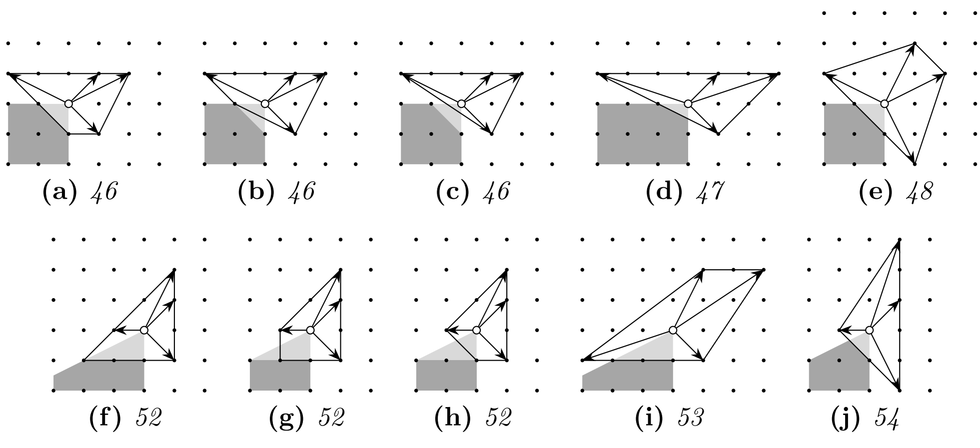
For many of the Luna data $\mathcal{S}$ in Table 7.4, the set $\frac{1}{m_{\mathcal{D}}}\rho(\mathcal{D})$ contains only lattice points, so all G/H-canonical polygons P are lattice polygons. If $\mathcal{S}$ has no spherical roots then, by Definition 7.1.11, the only interior point of P is the origin. Otherwise, let x_1 and x_2 be the two primitive lattice points on the boundary of the valuation cone $\mathcal{V}$. Any non-zero interior lattice point of P must be contained in the convex hull of x_1, x_2 and $\frac{1}{m_{\mathcal{D}}}\rho(\mathcal{D})$ and not in $\mathcal{V}$.

In this way we can show that, for several of the Luna data in Table 7.4, the only polygons which are G/H-canonical are canonical lattice polygons. There is a well-known classification of these 16 polygons. These Luna data include the rank 2 Luna data with ID 55, 56, 60, 61 and 63 as well as

- 1-4 for which $b_1 \in \{0, 3\}$,
- 5-17 for which $b_{11}, \gcd(b_{21}, b_{22}) \in \{0, 2\}$,
- 18-35 for which $b_2 \in \{0, 2\}$,
- 36-41 for which $b_1 = 1$ and $b_2 \in \{0, 2\}$,
- 42-48 for which $(d_1^+, d_2^+) \in \{(0,0), (0,1), (0,2), (1,1)\}$,
- 49-54 for which $(d_1^+, d_3^+) \in \{(1,1), (1,2), (2,1)\}$ and
- 64-80 for which $b_1 = 0$ and b_2 is even.

Let P be a polytope which is G/H-canonical for a Luna data out of 36-41 for which $b_1 = 0$ and $b_2 \in \{0, 2\}$. This is a lattice polygon with vertex $(2, 0)$ and the origin in its interior. Let Q be the convex hull of the vertices of P except replacing $(2,0)$ with $(1,0)$. Then Q is a lattice polygon, and its only interior point is the origin. Therefore, we can classify such polygons P, by doubling each vertex of each canonical lattice polygon and removing equivalent polygons. The result is a list of 38 lattice polygons.

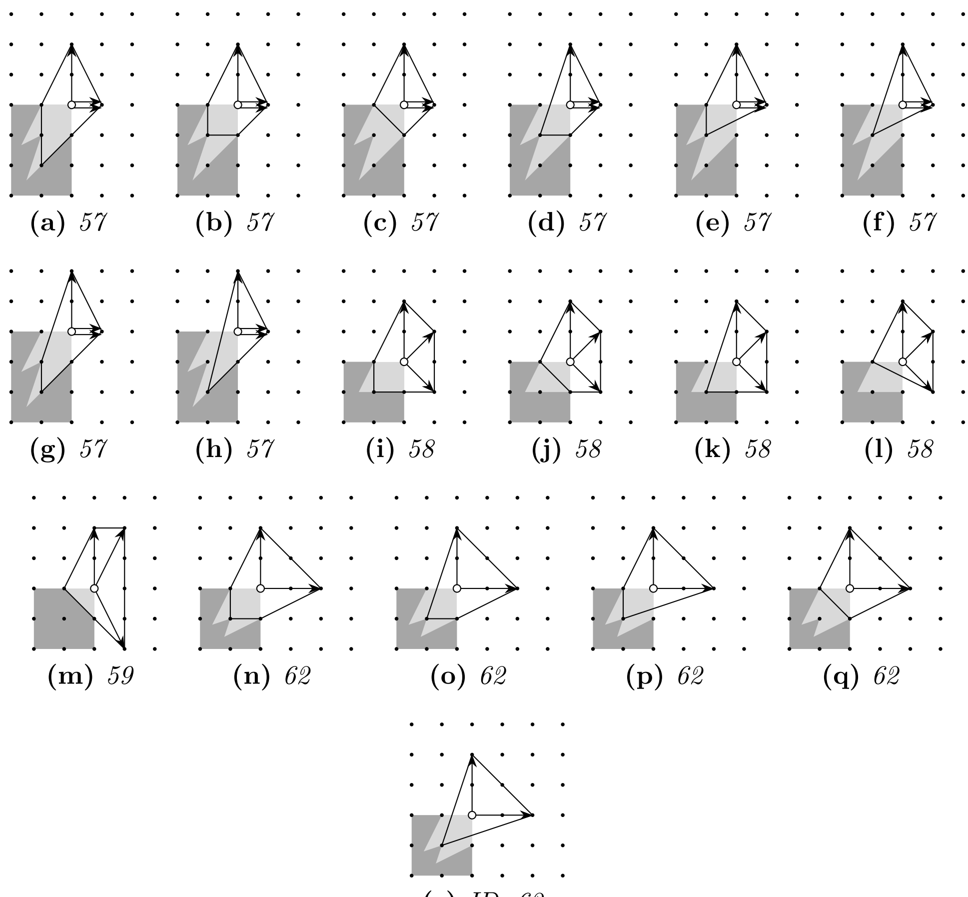
The remaining rank 2 Luna data for which all G/H-canonical polygons are lattice polygons are best approached by hand. These are the remaining rank 2 Luna data out of 42-54, 57-59 and 62. Let $\mathcal{S}$ be one of these, and let Q be the convex hull of the origin and $\frac{1}{m_{\mathcal{D}}}\rho(\mathcal{D})$. Let P be a lattice polygon which is G/H-canonical with respect to $\mathcal{S}$, then P contains Q and contains no non-zero interior points in $\mathcal{V}$. Let x_1 and x_2 be the two primitive lattice points in the boundary of $\mathcal{V}$, then x_1, x_2 and $(-1,-1)$ cannot be interior points of P. In particular, no vertex of P is in the interior of the penumbra of x_1, x_2 or $(-1,-1)$ with respect to Q. For any of these Luna data, only finitely many lattice points in $\mathcal{V}$ remain which can be in P, so we can classify such polygons. See Figures 7.1 and 7.2 for the full list.



(a) *46* **(b)** *46* **(c)** *46* **(d)** *47* **(e)** *48*

(f) *52* **(g)** *52* **(h)** *52* **(i)** *53* **(j)** *54*

Figure 7.1: *All G/H-canonical polytopes for the rank 2 Luna data with ID:*
46, 47, 48: $\left(\left(\begin{smallmatrix}2&0\\0&2\end{smallmatrix}\right), \{\alpha_1,\alpha_2\}, \emptyset, \{D_1^+, D_1^-, D_2^+, D_2^-\}\right)$
52, 53, 54: $\left(\left(\begin{smallmatrix}2&1\\0&1\end{smallmatrix}\right), \{\alpha_1,\alpha_2\}, \emptyset, \{D_1^+, D_1^-, D_2^+, D_2^-\}\right)$
where the points of $\frac{1}{m_{\mathcal{D}}}\rho(\mathcal{D})$ are depicted by arrows. The shaded cone is $\mathcal{V}$ and the penumbra of forbidden points in $\mathcal{V}$ is shaded darker.



(a) *57* **(b)** *57* **(c)** *57* **(d)** *57* **(e)** *57* **(f)** *57*

(g) *57* **(h)** *57* **(i)** *58* **(j)** *58* **(k)** *58* **(l)** *58*

(m) *59* **(n)** *62* **(o)** *62* **(p)** *62* **(q)** *62*

(r) *ID: 62*

Figure 7.2: *All G/H-canonical polytopes for the rank 2 Luna data with ID: 57, 58, 59:* $\left(\left(\begin{smallmatrix} 2 & 0 \\ 0 & 4 \end{smallmatrix}\right), \{\alpha_1, 2\alpha_2\}, \emptyset, \{D_1^+, D_1^-\}\right)$ *and 62:* $\left(\left(\begin{smallmatrix} 4 & 0 \\ 0 & 4 \end{smallmatrix}\right), \{2\alpha_1, 2\alpha_2\}, \emptyset, \emptyset\right)$ *where the points of* $\frac{1}{m_{\mathcal{D}}}\rho(\mathcal{D})$ *are depicted by arrows. The shaded cone is* $\mathcal{V}$ *and the penumbra of forbidden points in* $\mathcal{V}$ *is shaded darker.*

7.3.2 Rank 2, Denominator 2

Several of the Luna data in Table 7.4 can only be realised by denominator 2 polygons with exactly one interior point and up to two rational vertices. These are the rank 2 Luna data with ID 5-35 and 64-80 as well as the Luna data out of 36-41 for which $b_1 = 1$. This list repeats some Luna data considered in the previous section, which we can therefore ignore. The proof that these data are only realised by the type of polygon described is identical to the one for canonical lattice polygons in the previous section. Note that these data may have canonical lattice polygons which are G/H-canonical, however we already have a classification of these so need only classify the rational polygons.

To classify these polygons we classify *all* denominator 2 polygons containing one interior point, by adapting the algorithm introduced in Section 5.2. We first adjust our definition of minimal polytope:

Definition 7.3.1. Let P be a denominator r polytope whose only interior lattice point is the origin. We call P *minimal* if for any vertex v of P the polytope

$$\mathrm{conv}(P \cap \tfrac{1}{r}\mathbb{Z}^2 \setminus \{v\})$$

is either of dimension less than $\dim(P)$ or contains no interior points.

Notice that this is similar to but different from Definition 5.2.1 since we place a condition on the number of interior points rather than the size of a polytope.

We can describe these minimal rational polytopes in terms of minimal canonical lattice polytopes.

Proposition 7.3.2. *A denominator r polytope P, whose only interior lattice point is the origin, is minimal if and only if $P = \frac{1}{r}Q$ for some minimal canonical lattice polytope Q.*

Proof. Let Q be a minimal canonical lattice polytope. Consider $P = \frac{1}{r}Q$. Suppose there is a vertex v of P such that $P' = \mathrm{conv}(Q \cap \frac{1}{r}\mathbb{Z}^2 \setminus \{v\})$ is a polytope with the same dimension and number of interior points as P. Then rP' is a lattice polytope of the correct dimension with the origin in its interior and is strictly contained in P. This is a contradiction so $\frac{1}{r}P$ is minimal.

Let P be a minimal denominator r polytope containing only the origin in its interior. Then much as above, $Q = rP$ must be a minimal canonical lattice polytope. □

Therefore, the minimal denominator r polytopes containing only the origin in their interior are:

$$\frac{1}{r}\,\mathrm{conv}((1,0),(0,1),(-1,-1))$$
$$\frac{1}{r}\,\mathrm{conv}((1,0),(0,1),(-1,0),(0,-1))$$
$$\frac{1}{r}\,\mathrm{conv}((1,0),(0,1),(-1,-2)).$$

It is a minor adjustment of Algorithm 3 to grow these polygons by one point of $\frac{1}{r}\mathbb{Z}^2$ at a time, until all denominator r-polygons containing one interior point have been found. We need only change the conditions for when we continue to grow a polygon. We run this algorithm in denominator 2 and store those polygons which have one or two rational vertices. The result is a list of 114 polygons with exactly one non-integral vertex and 624 with exactly two.

The only remaining Luna datum for which denominator 2 polygons are G/H-canonical, is the datum out of 36-41 for which $b_1 = 0$ and $b_2 = 1$. Let P be a G/H-canonical polygon with respect to this datum, then let Q be the convex hull of the vertices of P except replacing $(2, 0)$ with $(1, 0)$. Then Q has exactly one interior point and at most one rational vertex. Therefore, we can classify such polygons P by taking the list of polygons with one interior point and up to one half-integral vertex and doubling each of their integral vertices, then removing equivalent polygons. The result is a list of 321 denominator 2 polygons and the 38 lattice polygons classified in the previous section.

7.3.3 Rank 2, Denominator 3

If a polytope P is G/H-canonical with respect to the rank 2 Luna datum 2 or 3, then it has at most one $\frac{1}{3}$-integral vertex and one interior point, since there are no spherical roots. We could classify these in the same way as the denominator 2 polygons, but this requires that we classify many more polygons than we need to. Instead we adapt a growing algorithm of Borisov and Borisov described in detail in [Kas10].

We want to classify rational polygons P whose only interior point is the origin and whose vertices are all lattice points except for the vertex v_0 which is equivalent to either $(\frac{1}{3}, 0)$ or $(\frac{2}{3}, 0)$. Unless otherwise stated we assume P satisfies this. Once again, we redefine minimal polygons:

Definition 7.3.3. We say that a rational polytope P with exactly one interior lattice point and up to one rational vertex v_0 is *minimal* if, for all lattice point vertices v of P, the polytope $\mathrm{conv}(P \cap \mathbb{Z}^n \setminus \{v\}, v_0)$ either has no interior points or is of dimension smaller than P.

The majority of these minimal polygons are triangles, which we classify later. The rest are given by the following.

Proposition 7.3.4. *If P is minimal then either P is a triangle or P is equivalent to one of the following polygons:*

- $\mathrm{conv}((\frac{1}{3},0),(-1,0),(0,1),(0,-1))$,
- $\mathrm{conv}((\frac{2}{3},0),(-1,0),(0,1),(0,-1))$ *and*
- $\mathrm{conv}((\frac{2}{3},0),(-1,0),(0,1),(1,-1))$.

Proof. Suppose $P = \mathrm{conv}(v_0,\ldots,v_k)$ is not a triangle. By a unimodular map we may assume that $v_0 = (\frac{1}{3},0)$ or $(\frac{2}{3},0)$. Consider the triangulation of P obtained by drawing a line between v_0 and each other vertex of P. One of these triangles contains the origin and by minimality of P it contains the origin in a facet. After a possible relabelling we may assume that $v_1 = (-1,0)$ is the other vertex of this facet.

P contains at least two more vertices, one with positive and one with negative y-coordinate. In fact, we may assume that P contains exactly two more lattice points v_2 and v_3 by Lemma 5.2.3, and that these points have y-coordinates 1 and -1 respectively. By a shear we may assume that $v_2 = (0,1)$. To keep v_0, v_1 and v_2 as vertices, v_3 must be one of $(-1,-1)$, $(0,-1)$ or $(1,-1)$. Removing non-minimal cases leaves us with the desired list. □

Proposition 7.3.5. *The denominator 3 triangles with exactly one non-lattice vertex in $\frac{1}{3}\mathbb{Z}^2$ and exactly one interior point are those equivalent to one of:*

- $\mathrm{conv}(v_0,(-1,-1),(0,-1))$ *where* $3v_0 \in \{(1,2),(1,3),(1,4),(2,4),(1,5),(2,5),(2,6),(2,7),(3,7),(3,8),(4,9),(4,10),(4,11),(5,12),(5,13)\}$,
- $\mathrm{conv}(v_0,(-1,-1),(1,-1))$ *where* $3v_0 \in \{(0,1),(0,2),(1,2),(1,3),(2,3),(1,4),(2,4),(3,4),(1,5),(2,5),(3,5),(2,6),(2,7),(3,7),(3,8)\}$,

- $\operatorname{conv}(v_0, (-2,-1), (1,-1))$ *where* $3v_0 \in \{(0,1), (1,2), (2,2), (3,2), (1,3), (1,4), (2,4), (2,5)\}$,
- $\operatorname{conv}(v_0, (-2,-1), (2,-1))$ *where* $3v_0 \in \{(0,1), (1,1), (0,2), (1,2)\}$,
- $\operatorname{conv}(v_0, (-3,-1), (2,-1))$ *where* $3v_0 \in \{(0,1), (1,1), (1,2)\}$,
- $\operatorname{conv}(v_0, (-3,-1), (3,-1))$ *where* $3v_0 \in \{(0,1), (1,1)\}$,
- $\operatorname{conv}((0, \frac{1}{3}), (-4,-1), (3,-1))$ *or*
- $\operatorname{conv}((0, \frac{1}{3}), (-4,-1), (4,-1))$

Proof. Let the vertices of T be v_0, v_1 and v_2 and suppose v_0 is non-integral. Then the convex hull of v_1, v_2 and the origin is a lattice triangle with no interior points and no lattice points on two of its boundaries, thus it is affine equivalent to $\operatorname{conv}((0,0), (k,0), (0,1))$ for some positive integer k. By a unimodular map we may assume that $v_1 = (-\lceil \frac{k}{2} \rceil, -1)$ and $v_2 = (\lfloor \frac{k}{2} \rfloor, -1)$.

Now v_0 must be chosen such that the origin is in the interior of T. In other words, v_0 must be in the interior of $\operatorname{pen}(\operatorname{conv}(v_1, v_2), (0,0))$. Additionally, no other lattice point can be in the interior of T so v_0 must not be in the interior of $\operatorname{pen}(\operatorname{conv}(v_1, v_2), x)$ for any lattice point $x \notin \operatorname{conv}((0,0), v_1, v_2)$. In practice it is enough to consider $x \in \{(0,1), (1,1), (1,2), (1,0), (-1,0)\}$. The collection of points of $\frac{1}{3}\mathbb{Z}^2$ which remains is finite and can be identified using Figure 7.3. Equivalent triangles can be removed by considering shears about the x-axis and reflections in the y-axis. $\square$

Not all these triangles are minimal but they certainly contain all the minimal triangles so we can initialise the growing algorithm with them and the non-triangular minimal polygons. We will grow them by adding *lattice points* to them rather than points of $\frac{1}{3}\mathbb{Z}^2$, so the method of growing in Chapter 5 no longer works. Instead we use the following result which is based on an argument in [Kas10, Section 1].

Proposition 7.3.6. *Let P and Q be denominator 3 polygons with the same, single rational vertex: either $(\frac{1}{3}, 0)$ or $(\frac{2}{3}, 0)$. Suppose $Q = \mathrm{conv}(P, v)$ for some lattice point v, then there is a lattice simplex S, with at most one $\frac{1}{3}$-integral vertex and containing exactly one interior point, such that S is the convex hull of v and a vertex or edge of P.*

Proof. Consider the line through v and the origin. This intersects a face of P at a point λv for a negative rational number λ. Let F be the inclusion minimal face of P with this property. The convex hull of F and v is the simplex S. $\square$

The simplex S in the above proposition is either one of the triangles classified in Proposition 7.3.5 or one of the line segments $\mathrm{conv}(-1, 1)$, $\mathrm{conv}(-1, \frac{1}{3})$ and $\mathrm{conv}(-1, \frac{2}{3})$. The *weights* of a simplex $\mathrm{conv}(v_0, \ldots, v_n)$ are a tuple of integers $(\lambda_0, \ldots, \lambda_n)$ with greatest common divisor 1, such that $\lambda_0 v_0 + \ldots \lambda_n v_n$ is the origin. For example, the weights of the above line segments are $(1, 1)$, $(1, 3)$ and $(2, 3)$ respectively. The weights of all denominator 3 triangles with one interior point and up to one rational vertex are listed in Table 7.6.

Given a polygon $P = \mathrm{conv}(v_0, \ldots, v_k)$, whose one rational vertex is v_0, we can only add one of the following points to grow P:

- $-3v_0$ if v_0 is equivalent to $(\frac{1}{3}, 0)$,
- $-\frac{3}{2}v_0$ if v_0 is equivalent to $(\frac{2}{3}, 0)$,
- $-v_i$ where $i = 1, \ldots, k$, or
- $\frac{\lambda_1}{\lambda_0}v_i + \frac{\lambda_2}{\lambda_0}v_j$ for any two distinct vertices v_i and v_j, where some permutation of $(\lambda_0, \lambda_1, \lambda_2)$ is in Table 7.6.

We grow polygons with a vertex equivalent to $(\frac{1}{3}, 0)$ and $(\frac{2}{3}, 0)$ separately to reduce the collection of possible points we can add at each step. Like in Section 5.2, we stratify the growing steps by the size of P to improve compute time. To grow a polygon P, we compute the list of possible points v we can add and compute $Q = \mathrm{conv}(P, v)$ for each v which is a lattice point. If Q is a

Weights of triangles with a vertex equivalent to $(\frac{1}{3}, 0)$.					
(1, 1, 3)	(1, 1, 6)	(1, 2, 3)	(1, 2, 9)	(1, 3, 3)	(1, 3, 4)
(1, 3, 6)	(1, 3, 12)	(1, 5, 6)	(1, 5, 9)	(1, 6, 7)	(2, 3, 3)
(2, 3, 5)	(2, 3, 15)	(3, 3, 4)	(3, 3, 5)	(3, 4, 5)	(3, 4, 7)
(3, 4, 21)	(3, 5, 6)	(3, 5, 7)	(3, 5, 8)	(3, 5, 12)	(3, 6, 7)
(4, 5, 9)	(5, 6, 9)	(5, 6, 11)	(5, 7, 9)	(7, 8, 9)	
Weights of triangles with a vertex equivalent to $(\frac{2}{3}, 0)$.					
(1, 1, 3)	(1, 3, 3)	(2, 2, 3)	(2, 3, 4)	(2, 4, 9)	(3, 4, 6)
Weights of lattice triangles.					
(1, 1, 1)	(1, 1, 2)	(1, 2, 3)			

Table 7.6: Weights of denominator 3 triangles with one interior lattice point and up to one non-integral vertex.

denominator 3 polygon with exactly one rational vertex, exactly one interior lattice point and size equal to one more than the size of P, then we save Q and grow it in the next iteration. The result is a list of 238 polygons with a vertex equivalent to $(\frac{1}{3}, 0)$ and 88 polygons with a vertex equivalent to $(\frac{2}{3}, 0)$.

7.3.4 Rank 3, Denominator 1

The rank 3 Luna data, apart from that with ID 2, are lattice polytopes. Polytopes which are G/H-canonical for Luna data 1 and 3 must be canonical polytopes since there are no spherical roots. Polytopes P which are G/H-canonical for Luna data 4-16 must also be canonical since all points of P outside of the valuation cone are contained in the hyperplane $x = 1$, thus P can have no interior points inside or outside of $\mathcal{V}$.

Let P be a G/H-canonical polytope for the rank 3 Luna datum with ID 17. Then, let Q be the convex hull of points of P except replacing $(2, 0, 0)$ with $(1, 0, 0)$. Q is a lattice polytope containing the origin and no other lattice point in its interior. In fact, $\mathrm{conv}(S)$ is G/H-canonical with respect to the rank 3 Luna datum with ID 4. Therefore, we can obtain all such polytopes by doubling a vertex of a canonical lattice polytope.

Let P be a G/H-canonical polytope for the rank 3 Luna datum with ID 18. Suppose V is some non-zero interior lattice point of P, so v is contained in the interior of the affine cone $(2, 1, 0) + \mathrm{conv}(P - (2, 1, 0))$ and is not in $\mathcal{V}$. The

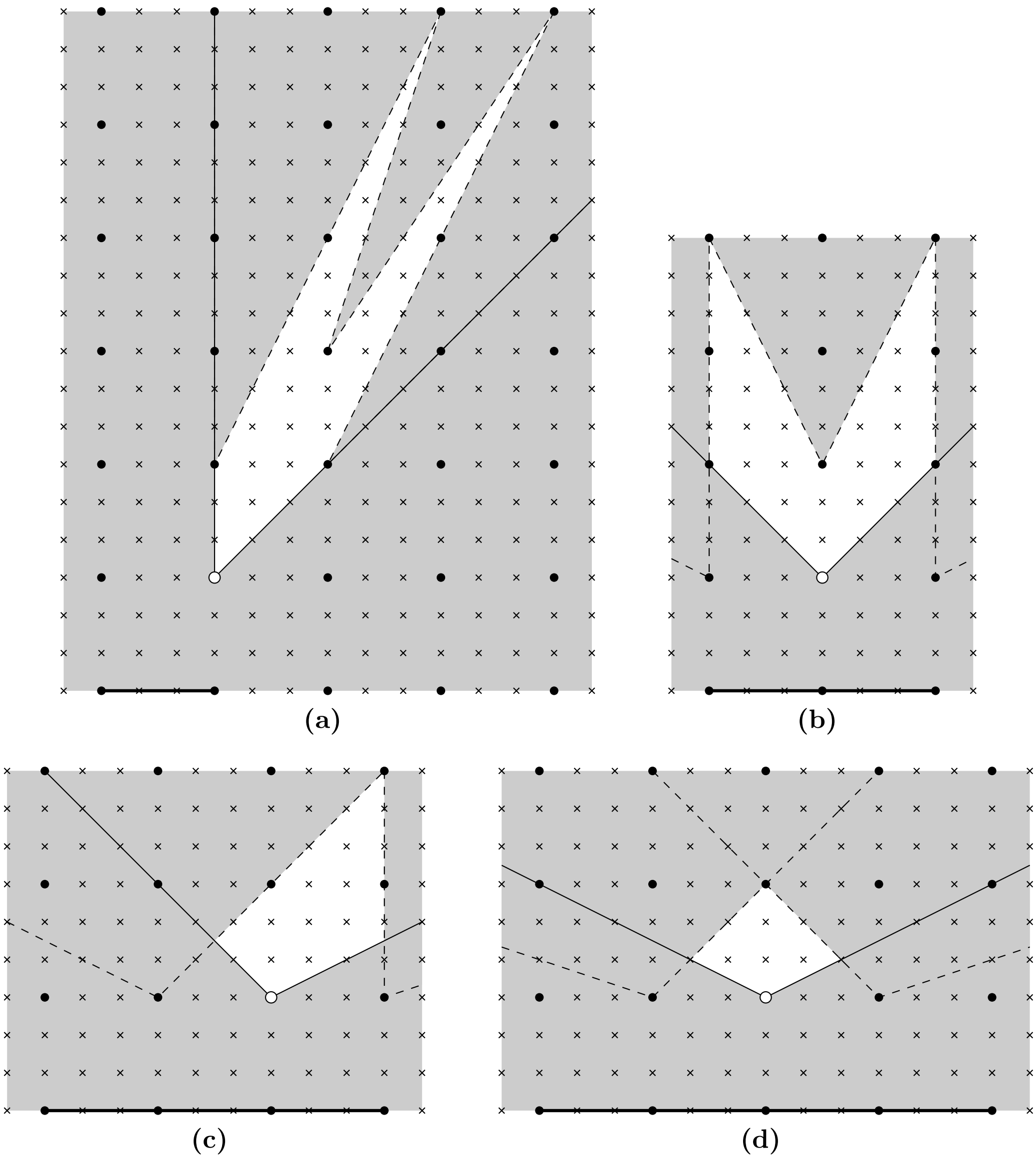


Figure 7.3: *Points in the non-shaded region and its dashed boundaries are the points of $\frac{1}{3}\mathbb{Z}^2$ which may be the third vertex of a triangle with exactly one $\frac{1}{3}$-integral vertex and exactly one interior lattice point.*

x-coordinate of v is 1 so there is a non-zero lattice point $2v - (2, 1, 0)$ which is both in the interior of P and in $\mathcal{V}$. This is a contradiction, so P has only one interior point and is also a canonical polytope.

Recall that the three-dimensional canonical polytopes have been classified in [Kas10], so the G/H-canonical polytopes of rank 3 can almost all be obtained from this classification.

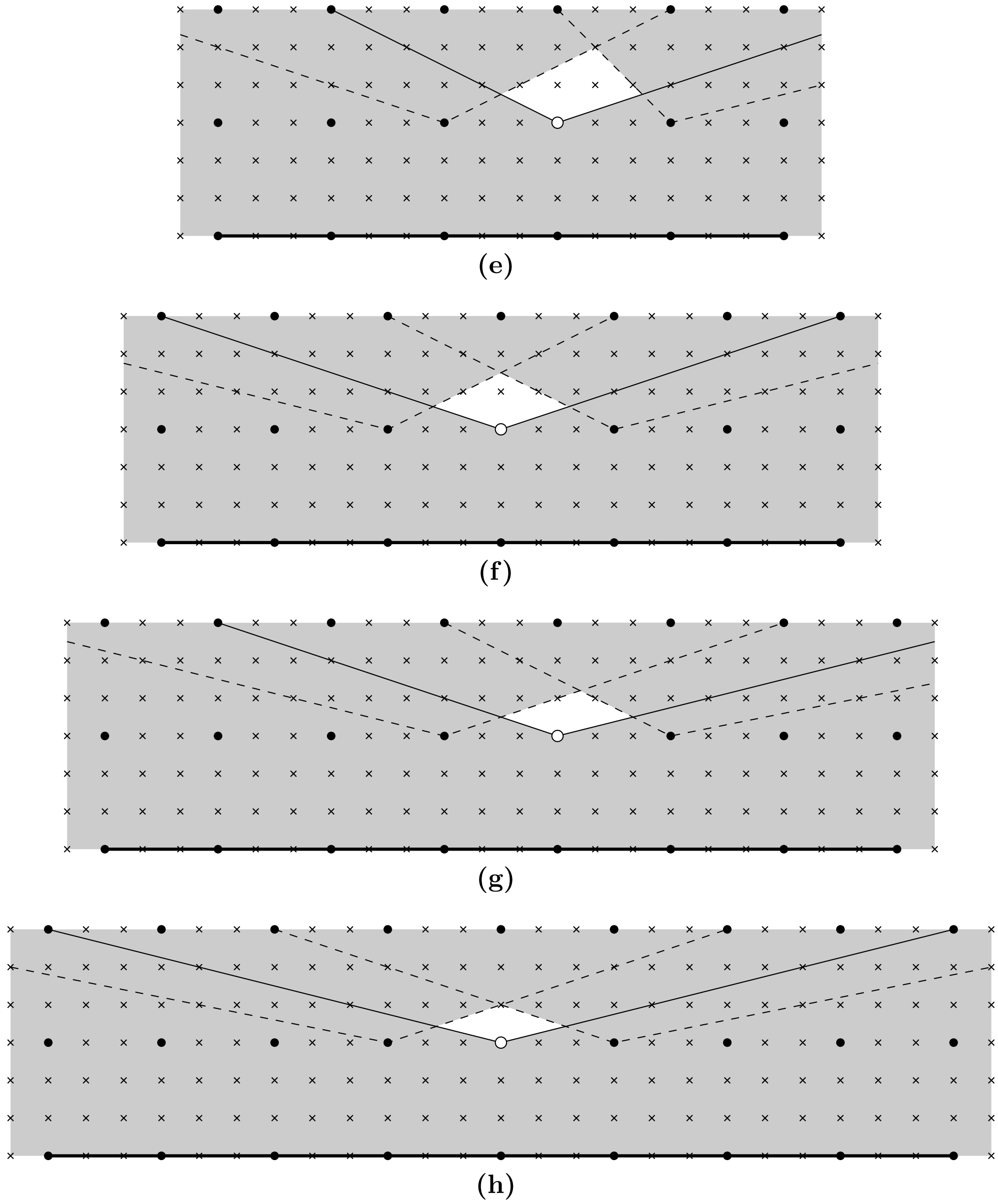


Figure 7.3: *Points which may be the vertex of a denominator 3 triangle containing one interior point continued.*

7.3.5 Rank 3, Denominator 2

Let $\mathcal{S}$ be the rank 3 Luna datum with ID 2 which has one color D for which the denominator of $\frac{1}{m_D}\rho(D)$ is 2. Since $\mathcal{S}$ has no spherical roots, any G/H-canonical polytope P has up to one interior lattice point. When P has no rational vertices it is a 3-dimensional canonical lattice polytope, which have already been classified. Therefore, it remains to classify the rational polygons with exactly one interior point and exactly one rational vertex, which is equivalent to $(\frac{1}{2}, 0, 0)$. As in Section 7.3.3, we adapt the growing method of Borisov and Borisov, detailed in [Kas10]. In this section we present a sketch of the approach we will use to complete the classification at a later date.

We continue with the definition of minimal introduced in Section 7.3.3.

Proposition 7.3.7. *Let P be a 3-dimensional denominator 2 polytope with one interior lattice point and exactly one rational vertex which is equivalent to $(\frac{1}{2}, 0, 0)$. If P is not a simplex then $P = \operatorname{conv}(S, P')$ where S is a k-dimensional minimal simplex and P' is a $n - k + r$-dimensional minimal polytope where $0 \leq r < k < 3$. Moreover, $\dim(S \cap P') \leq r$, and r equals the number of common vertices of S and P'.*

Proof. The proof is the same as the proof of [Kas10, Proposition 2.2]. □

Notice that P' and S may be lattice polytopes or rational polytopes and both have dimension less than P. Therefore, we have classified all polytopes needed to build the 3-dimensional non-simplex minimal polytopes in Section 7.3.2. The method will be the same as that used in [Kas10, Section 3] with the addition that we will use an algorithmic approach to help deal with the increased number of cases to consider.

To classify the three-dimensional simplices we return to the idea of the weights of a simplex. Using the methods of [Con02] we can obtain a list of all lattice simplices with a given collection of weights. Multiplying these simplicies by $\frac{1}{2}$ preserves their weights and gives the denominator 2 simplices we seek. Therefore, it suffices for us to bound the collection of possible weights which denominator 2 tetrahedra can have.

To do this we will use the Barycentric coordinates with respect to a simplex. Given an n-dimensional simplex $P = \operatorname{conv}(v_0, \ldots, v_n)$, we can write any point $x \in \mathbb{Q}^n$ uniquely in the form $x = \alpha_0 v_0 + \cdots + \alpha_n v_n$ where $\alpha_0 + \cdots + \alpha_n = 1$. We call $(\alpha_0, \ldots, \alpha_n)$ the Barycentric coordinates of x and note that they are all positive if and only if x is in the interior of P. We use the following two results to bound the sum of the weights.

Lemma 7.3.8. *Let $P = \mathrm{conv}(v_0, \ldots, v_n)$ be a denominator 2 simplex and let w be an interior point of P with Barycentric coordinates $(\alpha_0, \ldots, \alpha_n)$. Then*

$$\mathrm{vol}(2P) \leq \frac{2^n}{n!\alpha_1 \ldots \alpha_n} |P^\circ \cap \mathbb{Z}^n|$$

Proof. This is a special case of [Pik01, Lemma 5]. □

Lemma 7.3.9. *Let P be a denominator 2 simplex with weights $\lambda_0, \ldots, \lambda_n$. Then*

$$\sum_{i=0}^{n} \lambda_i \leq n! \,\mathrm{vol}(2P).$$

Proof. This follows from [Kas09, Equation (2.3)]. □

Proposition 7.3.10. *Let P be a denominator 2 tetrahedron whose only interior point is the origin. Then the sum of the weights of P is at most* 1152*.*

Proof. Let $v_0, \ldots, v_3$ be the vertices of P and let $\alpha_0, \ldots, \alpha_3$ be the Barycentric coordinates of the origin, that is

$$\alpha_0 + \cdots + \alpha_3 = 1 \quad \text{and} \quad \alpha_0 v_0 + \cdots + \alpha_3 v_3 = \mathbf{0}.$$

After a possible relabelling we may assume that $0 < \alpha_0 \leq \cdots \leq \alpha_3$ where positivity comes from the fact that the origin is in the interior of P. If v_3 is the rational vertex of P then the point $-2v_3$ is a non-zero lattice point and hence is not contained in the interior of P. It has Barycentric coordinates $(3\alpha_0, 3\alpha_1, 3\alpha_2, 3\alpha_3 - 2)$ one of which must be non-positive since it is not in the interior of P. However, $3\alpha_i > 0$ for $i = 0, 1, 2, 3$ so we have $\alpha_3 \leq \frac{2}{3}$. In a similar way the points $-2v_2 - 2v_3$, $-2v_1 - 2v_2 - 2v_3$ and $-v_2 - 2v_3$ give us the bounds $\alpha_2 \leq \frac{2}{5}$, $\alpha_1 \leq \frac{2}{7}$ and either $\alpha_2 \leq \frac{1}{4}$ or $\alpha_3 \leq \frac{1}{2}$. Combining the inequalities for α_2 and α_3 we can show that $\alpha_2 + \alpha_3 \leq \frac{11}{12}$ and since $2\alpha_1 + \alpha_2 + \alpha_3 \geq 1$, we then know that $\alpha_1 \geq \frac{1}{2}(1 - \alpha_2 - \alpha_3) \geq \frac{1}{24}$.

The collection of linear bounds we have defined on the α_i describe a 3-dimensional polytope of possible points $(\alpha_1, \alpha_2, \alpha_3)$ in the interior of the cube

$[0,1]^3 \subseteq \mathbb{Q}^3$. The minimum value which $\alpha_1\alpha_2\alpha_3$ can take is realised on a vertex of this polytope so we can show that $\alpha_1\alpha_2\alpha_3 \geq \frac{1}{144}$.

Using the same procedure we obtain a similar collection of linear bounds on the Barycentric coordinates when each of the other vertices is the rational vertex of P. The lattice points we need to consider and the bounds they give are listed in Table 7.7. However, the bounds we obtain on $\alpha_1\alpha_2\alpha_3$ are all stronger than the one we have already computed, so whichever vertex is rational $\alpha_1\alpha_2\alpha_3 \geq \frac{1}{144}$. Thus, by Lemma 7.3.8, the volume of $2P$ is at most 192 and by Lemma 7.3.9, the sum of the weights of P is at most 1152. $\square$

Lattice Point	Barycentric coordinates	Inequality
	When v_0 is the rational vertex.	
$-v_3$	$(2\alpha_0, 2\alpha_1, 2\alpha_2, 2\alpha_3 - 1)$	$\alpha_3 \leq \frac{1}{2}$
$-v_2 - v_3$	$(3\alpha_0, 3\alpha_1, 3\alpha_2 - 1, 3\alpha_3 - 1)$	$\alpha_2 \leq \frac{1}{3}$
$-v_1 - v_2 - v_3$	$(4\alpha_0, 4\alpha_1 - 1, 4\alpha_2 - 1, 4\alpha_3 - 1)$	$\alpha_1 \leq \frac{1}{4}$
$-v_2 - 2v_3$	$(4\alpha_0, 4\alpha_1, 4\alpha_2 - 1, 4\alpha_3 - 2)$	$\alpha_2 \leq \frac{1}{4}$ or $\alpha_3 \leq \frac{1}{2}$
	When v_1 is the rational vertex.	
$-v_3$	$(2\alpha_0, 2\alpha_1, 2\alpha_2, 2\alpha_3 - 1)$	$\alpha_3 \leq \frac{1}{2}$
$-v_2 - v_3$	$(3\alpha_0, 3\alpha_1, 3\alpha_2 - 1, 3\alpha_3 - 1)$	$\alpha_2 \leq \frac{1}{3}$
$-2v_2 - 2v_2 - 2v_3$	$(7\alpha_0, 7\alpha_1 - 2, 7\alpha_2 - 2, 7\alpha_3 - 2)$	$\alpha_1 \leq \frac{2}{7}$
$-v_2 - 2v_3$	$(4\alpha_0, 4\alpha_1, 4\alpha_2 - 1, 4\alpha_3 - 2)$	$\alpha_2 \leq \frac{1}{4}$ or $\alpha_3 \leq \frac{1}{2}$
	When v_2 is the rational vertex.	
$-v_3$	$(2\alpha_0, 2\alpha_1, 2\alpha_2, 2\alpha_3 - 1)$	$\alpha_3 \leq \frac{1}{2}$
$-2v_2 - 2v_3$	$(5\alpha_0, 5\alpha_1, 5\alpha_2 - 2, 5\alpha_3 - 2)$	$\alpha_2 \leq \frac{2}{5}$
$-2v_1 - 2v_2 - 2v_3$	$(7\alpha_0, 7\alpha_1 - 2, 7\alpha_2 - 2, 7\alpha_3 - 2)$	$\alpha_1 \leq \frac{2}{7}$
$-2v_2 - 3v_3$	$(6\alpha_0, 6\alpha_1, 6\alpha_2 - 2, 6\alpha_3 - 3)$	$\alpha_2 \leq \frac{1}{3}$ or $\alpha_3 \leq \frac{1}{2}$
	When v_3 is the rational vertex.	
$-2v_3$	$(3\alpha_0, 3\alpha_1, 3\alpha_2, 3\alpha_3 - 2)$	$\alpha_3 \leq \frac{2}{3}$
$-2v_2 - 2v_3$	$(5\alpha_0, 5\alpha_1, 5\alpha_2 - 2, 5\alpha_3 - 2)$	$\alpha_2 \leq \frac{2}{5}$
$-2v_1 - 2v_2 - 2v_3$	$(7\alpha_0, 7\alpha_1 - 2, 7\alpha_2 - 2, 7\alpha_3 - 2)$	$\alpha_1 \leq \frac{2}{7}$
$-v_2 - 2v_3$	$(4\alpha_0, 4\alpha_1, 4\alpha_2 - 1, 4\alpha_3 - 2)$	$\alpha_2 \leq \frac{1}{4}$ or $\alpha_3 \leq \frac{1}{2}$

Table 7.7: Lattice points which witness the bounds on the Barycentric coordinates of the origin with respect to a denominator 2 tetrahedron with exactly one interior point and one rational vertex.

We are yet to classify the weights and simplices.

7.4 Embedding Polytopes in $\mathcal{N}$

From the previous two sections we have a classification of Luna data and a classification of the polytopes which may appear as G/H-canonical polytopes with respect to these data. In this section we describe how to combine these to produce a list of spherical canonical Fano four-folds. Throughout we assume that G/H is a spherical homogeneous space corresponding to one of the Luna data $\mathcal{S}$ in Tables 7.2-7.5. Consequently, the action of G on G/H is smart.

First, we must be able to identify equivalent G/H-canonical polytopes. The normal form of Section 6.4 may change the Luna datum, which makes polytopes difficult to compare, so we define a new class of automorphisms.

Definition 7.4.1. An *automorphism* of a Luna datum $\mathcal{S}$ is a lattice automorphism $\phi\colon \mathcal{M} \to \mathcal{M}$ such that

1. $\phi(\gamma) = \gamma$ for every spherical root $\gamma \in \Sigma$, and
2. $\phi_*(\rho(\mathcal{D}(\alpha))) = \rho(\mathcal{D}(\alpha))$ for every $\alpha \in S \setminus S^P$, where $\phi_*\colon \mathcal{N} \to \mathcal{N}$ is the dual to ϕ, and $\mathcal{D}(\alpha)$ is the set of colors moved by P_α.

The set of automorphisms of $\mathcal{S}$, denoted by $\mathrm{Aut}(\mathcal{S})$, forms a group under composition and is contained in $\mathrm{Iso}(\mathcal{S})$. We say two G/H-canonical polytopes P and P' are *equivalent* if there is an automorphism ϕ of $\mathcal{S}$ with $P' = \phi_*(P)$.

In other words, automorphisms of $\mathcal{S}$ are isomorphisms from $\mathcal{S}$ which fix $\rho(D)$ for each color D of type $2a$ and b and which preserve the sets $\{\rho(D^+), \rho(D^-)\}$ where D^+ and D^- are a pair of type a colors associated to the same simple root. Therefore, by Theorem 6.0.3 automorphisms of $\mathcal{S}$ are induced by isomorphisms of spherical varieties so if two G/H-canonical polytopes are equivalent they are associated to the same spherical canonical Fano variety. We write $P \sim_{aut} P'$ if P and P' are equivalent.

We introduce a new normal form for G/H-canonical polytopes which allows us to determine whether they are equivalent under automorphisms of $\mathcal{S}$. This is done by 'marking' P with a tuple t, then finding a normal form for this marked polytope (P, t). If $\mathcal{D} = \{D_1, \ldots, D_k\}$ is the set of colors of G/H then we define $\rho_i \in \mathcal{N}_{\mathbb{Q}}$ to be $\frac{1}{m_{D_i}}\rho(D_i)$ for each $i = 1, \ldots, k$. Let t be the tuple which first lists, in order of i, the ρ_i such that D_i is a color of type $2a$ or b, then lists, in lexicographic order of (i, j), the sets $\{\rho_i, \rho_j\}$ such that $i < j$ and D_i and D_j are colors of type a associated to the same simple root.

Given total orders on sets $S_1, \ldots, S_r$, we can define the lexicographic order on the Cartesian product $S_1 \times \cdots \times S_r$, which is itself a total order. Lexicographic order defines a total order on $\mathcal{N}_{\mathbb{Q}}$. We order sets $\{v_1, v_2\}$ of two distinct points in $\mathcal{N}_{\mathbb{Q}}$, by treating them as elements of the subset of $\mathcal{N}_{\mathbb{Q}}^2$ where $v_1 < v_2$, then using lexicographic order. In this way we obtain a total order on the set of tuples t which first list k_1 points of $\mathcal{N}_{\mathbb{Q}}$ then list k_2 two-point sets of points in $\mathcal{N}_{\mathbb{Q}}$. For brevity we refer to this as lexicographic order.

Now let P be a G/H-canonical polytope and let π be the dual to the inclusion of $\mathcal{M}^{ss}$ in $\mathcal{M}$. Let $\phi \in \mathrm{Aut}(\mathcal{N}, \pi)$ be an automorphism of $\mathcal{N}$ such that $\phi(P) = \mathrm{NF}(P)$ is the normal form of P defined in Section 6.4. We get a modified tuple $\phi(t)$, whose entries are the image of the entries of t under ϕ. The intersection of $\mathrm{Aut}(\mathcal{N}, \pi)$ and the group $\mathrm{Aut}(P)$ of automorphisms of $\mathcal{N}$ which preserve P is non-empty and finite. We choose $\psi \in \mathrm{Aut}(\mathcal{N}, \pi) \cap \mathrm{Aut}(P)$ which lexicographically minimises the tuple $\psi(\phi(t))$. We define $\mathrm{NF}^{aut}(P, t)$ to be the pair $(\phi(P), \psi(\phi(t)))$.

Notice that NF^{aut} is not strictly a normal form in the sense of Section 6.4, since $\phi(P)$ need not be G/H-canonical. However, it serves the purpose of a normal form in the following sense.

Proposition 7.4.2. *Suppose P and P' are two G/H-canonical polytopes, then $P \sim_{aut} P'$ if and only if $\mathrm{NF}^{aut}(P, t) = \mathrm{NF}^{aut}(P', t)$.*

Proof. $(\Rightarrow)$ Suppose $P \sim_{aut} P'$, so there is an automorphism of $\mathcal{S}$ which maps P to P'. Therefore, the normal forms $\mathrm{NF}(P) = \phi(P)$ and $\mathrm{NF}(P') = \phi'(P')$ are equal and, since all automorphisms of $\mathcal{S}$ fix the tuple t, the minimised tuples $\psi(\phi(t))$ and $\psi'(\phi'(t))$ are also equal.

($\Leftarrow$) Conversely, suppose $\mathrm{NF}^{aut}(P,t) = \mathrm{NF}^{aut}(P',t)$. Then there is and isomorphism ϕ of $\mathcal{S}$, such that $\mathrm{NF}^{aut}(P,t) = (\phi(P), \phi(t))$ and a similar isomorphism ϕ' for P'. By our assumption, $\phi(P) = \phi'(P')$ and $\phi(t) = \phi'(t)$, so the isomorphism of $\mathcal{S}$ given by $\phi'^{-1} \circ \phi$ maps P to P' and fixes t. Therefore, $\phi'^{-1} \circ \phi$ is an automorphism of $\mathcal{S}$ and $P \sim_{aut} P'$. □

Now we consider combining the polytopes and Luna data. Let $\mathcal{S}$ be the Luna datum associated G/H, and say $\mathrm{rk}(\mathcal{M}) = n$. Suppose $\Sigma = \{\gamma_1, \ldots, \gamma_m\}$ and $\mathcal{D} = \{D_1, \ldots, D_k\}$ and define ρ_i as above. Let $P_0 \subseteq \mathcal{N}_{\mathbb{Q}}$ be a rational polytope of dimension n containing the origin in its interior. We find all G/H-canonical polytopes, up to $\sim_{aut}$, which are unimodularly equivalent to P_0.

First we make a choice of points $a_1, \ldots, a_k$ in P which we will eventually map to the points $\rho_1, \ldots, \rho_k$. These are chosen so that the lattice length of each line segment $\mathrm{conv}(a_i, \mathbf{0})$ matches the length of $\mathrm{conv}(\rho_i, \mathbf{0})$ and so that the convex hull of $\{\mathbf{0}, a_1, \ldots, a_k\}$ is unimodularly equivalent to the convex hull of $\{\mathbf{0}, \rho_1, \ldots, \rho_k\}$. There are finitely many choices for the a_i. If all vertices of P are either lattice points or points a_i we proceed with this selection of a_i's.

For each color D_i and each spherical root γ_j, the value of $\rho_i \cdot \gamma_j$ is fixed. We choose linearly independent primitive points $\sigma_1, \ldots, \sigma_m$ of $\mathcal{M}$ such that

$$a_i \cdot \sigma_j = \rho_i \cdot \gamma_j, \quad \text{for all } i = 1, \ldots, k \text{ and } j = 1, \ldots, m$$

and such that, for all vertices v of P which do not equal any a_i, we have $v \cdot \sigma_j \leq 0$. Note that for each vertex v of P we require $v \cdot \sigma_j \leq r$ for some non-negative rational number r, therefore σ_i is in a polytope obtained by moving some facets of P^* in or out, without them passing the origin. In particular, there are finitely many choices for each σ_j. We choose linearly independent σ_i such that the convex hull of $\{\mathbf{0}, \sigma_1, \ldots, \sigma_m\}$ is equivalent to the convex hull of $\{\mathbf{0}, \gamma_1, \ldots, \gamma_m\}$.

Recall that the definition of G/H-canonical depends only on the points $\rho_1, \ldots, \rho_k$ and $\gamma_1, \ldots, \gamma_m$. Therefore, we can meaningfully ask if P is G/H-

canonical with respect to $\{a_1, \ldots, a_k\}$ and $\{\sigma_1, \ldots, \sigma_m\}$, even if we do not know that there is a Luna datum with this collection of colors and spherical roots. If P is not G/H-canonical with respect to $\{a_1, \ldots, a_k\}$ and $\{\sigma_1, \ldots, \sigma_m\}$, then there is no unimodular φ map taking a_i to ρ_i and σ_j to γ_j such that $\varphi(P_0)$ is G/H-canonical with respect to $\mathcal{S}$, so our choice of a_i and σ_j was not valid. Otherwise, we proceed.

Now we have a polygon and two finite lists of points to map to the colors and spherical roots respectively. We map the σ_i to the spherical roots first. If $m = 0$ the identity map suffices. If $m = 2$ then we can only be in rank 2, so there is exactly one linear map which takes both σ_i to their respective γ_i. If this map is in $\mathrm{GL}_2(\mathbb{Z})$ we proceed, otherwise our choice of a_i and σ_j was not valid. If $m = 1$, then by our choice of basis of $\mathcal{M}$, $\gamma_1 = (1, 0, \ldots, 0)$ (see Tables 7.3-7.5). Note that σ_1 is primitive so we can use the Euclidean algorithm to find a point on which σ_1 evaluates to 1. Combining this with a basis of the kernel of σ_1 gives a new basis of $\mathcal{N}$. The map taking this basis to the standard basis has dual taking σ_1 to γ_1.

We have found a unimodular map whose dual takes the σ_i to the γ_i. Let P_1 be the image of P_0 under this map and let b_i be the image of a_i for each $i = 1, \ldots, k$. Now P_1 is G/H-canonical with respect to $\{b_1, \ldots, b_k\}$ and $\{\gamma_1, \ldots, \gamma_2\}$. From now on we want to preserve the spherical roots so can only use isomorphisms of $\mathcal{S}$ to adjust P_1. We want an isomorphism ϕ of $\mathcal{S}$, which takes each b_i to its corresponding ρ_i. By our earlier check, we know there is a unimodular map φ taking the convex hull of $b_1, \ldots, b_k$ and the origin to the convex hull of $\rho_1, \ldots, \rho_k$ and the origin. However, there may be more than one such map, and there no guarantee that any of them is an isomorphism of $\mathcal{S}$ or maps b_i to ρ_i for all $i = 1, \ldots, k$. Therefore, we consider each automorphism $\psi \in \mathrm{Aut}(\mathrm{conv}(\mathbf{0}, \rho_1, \ldots, \rho_k))$, and check if $\psi \circ \varphi$ is the isomorphism of $\mathcal{S}$ we seek. If we find none then our choice of a_i and σ_j was not valid. If we find any then let P_2 be the image of P_1 under such a map and we have found a polytope P_2 which is G/H-canonical and equivalent to P_0. We will repeat this process for each Luna datum and each polytope which could be G/H-canonical with respect to that datum and use the normal form described above to remove polytopes which are equivalent under the action of $\mathrm{Aut}(\mathcal{S})$.

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