Aspects of Spinorial G-Structures

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Abstract

There is a remarkable isomorphism between a pair (metric, spinor), and a certain collection of differential forms on a manifold M. This isomorphism holds for numerous examples from four dimensions to eight dimensions.

In this thesis, we aim to understand reductions of the spin-frame bundle to various orbits of spinors on a spin manifold, called spinorial G-structures, and then develop a schematic for exploring the most "natural" second-order partial differential equation on the collection of differential forms arising from a pair (metric, spinor).

The first part of the thesis deals with real and complexified stabilisers of Weyl spinors. Understanding real stabilisers is done through the study of pure spinors, the real index, and spinor bilinears. These three tools generate a large class of spinorial G-structures, for which we construct a new class called mixed structures. We also apply this machinery to investigate previously unexplored stabilisers in ten and twelve dimensions. Regarding complexified stabilisers of Weyl spinors, we develop an elementary approach using k-simplices and combinatorics. The novelty of this simpler scheme is two-fold. First, the study of the stabiliser of the spinor is shifted to studying the group that leaves the collection of differential forms, arising from the metric and spinor, invariant. Second, this method allows one to examine complexified stabilisers in arbitrarily high dimensions, which classical methods do not allow.

The second part of the thesis explores the most natural second-order partial differential equations on a pair (metric, spinor). We are inspired by Plebanski's theory of gravity in 4 dimensions. He constructs an action functional, which when extremised, results in Einstein conditions on the curvature. In this thesis, we construct all families of diffeomorphism invariant action functionals in the examples of SU(2)- and SU(3)-structures. From our methods, we recover Plebanski's most natural second-order partial differential equations (PDEs) on the collection of differential forms coming from the metric and spinor in the case of SU(2)-structures. Furthermore, we conjecture a schematic for natural second-order PDEs on the collection of differential forms coming from the metric and spinor in the case of SU(3)-structures. In both examples, we show that the linearised action functionals are completely determined by representation theory, and that there is a family of Einstein-Hilbert actions of gravity in a vacuum.

Contents

Chapter 1

Introduction

Gravity is the metric geometry of spacetime influenced by matter, and all matter content is described by spinors. The geometric foundation of this concept lies in a somewhat remarkable isomorphism between a pair (metric, spinor modulo sign) on a manifold M and a specific collection of differential forms on M. We express this isomorphism through the following relation:

$$\text{(metric, spinor modulo sign)} \iff \text{collection of differential forms.} \tag{1.0.1}$$

As detailed in the main body of the thesis, and as is well known within the differential geometry community, this isomorphism holds for numerous cases, see [BK24] in four dimensions, [CS02] in six dimensions, and [Bry87; Kar09] in seven dimensions.

A well-known example of (1.0.1) is presented in [Kar08; Kra24a] for eight dimensions. Let M be an 8-dimensional spin manifold. In this setting, there exists an orbit in the space of 4-forms characterised by the Cayley form, a 4-form stabilised by $\mathrm{Spin}(7) \subset \mathrm{GL}(8,\mathbb{R})$. One can then decompose the space of 4-forms into irreducible representations of $\mathrm{Spin}(7)$, such that certain irreducible components are isomorphic to the spinor orbit stabilised by $\mathrm{Spin}(7)$. Furthermore, it is known that the metric can be algebraically constructed from the Cayley form at each point on the manifold. In this sense, the right-hand side of (1.0.1) holds as a necessary condition. Conversely, if one begins with the spinor orbit stabilised by $\mathrm{Spin}(7)$, it is possible to construct the space of Cayley forms within the space of 4-forms using spinor bilinears and the metric. In this sense, the left-hand side of (1.0.1) holds as a sufficient condition.

A G-structure on an n-dimensional manifold M is the reduction of the principal $\mathrm{GL}(n,\mathbb{R})$-frame bundle. For a given G-structure, the breaking of $\mathrm{GL}(n,\mathbb{R})$ to G can be represented by a collection of tensors that remain invariant under G's action. In this thesis, the "tensors" we refer to are a collection of differential forms and the metric. We show, in our examples, that the metric is algebraically reconstructable from such a suitable collection of differential forms, and thus, the right-hand side of (1.0.1) corresponds to the reduction of the principal $\mathrm{GL}(n,\mathbb{R})$-frame bundle. Conversely, suppose we have an n-dimensional spin manifold M. We introduce the concept of a spinorial G-structure as the reduction of the spin-frame bundle to the stabiliser of a spinor. We investigate cases where G, as the stabiliser of a spinor on the left-hand side of (1.0.1), is isomorphic to a subgroup of $\mathrm{GL}(n,\mathbb{R})$ whose action is invariant on the collection of differential forms on the right-hand side of (1.0.1).

The aim of this thesis is twofold. First, to explore generalisations of (1.0.1) to dimensions higher than eight. Second, to examine how (1.0.1) can be applied to express interesting differential equations on a pair (metric, spinor) as partial differential equations (PDEs) on the collection of differential forms.

In Part I, we gather all known facts related to possible spinor stabilisers in dimensions up to eight and beyond. New techniques are developed to reproduce these known results, and some new findings are discovered. We distinguish between complexified stabilisers of $\mathrm{Spin}(n, \mathbb{C})$ and real stabilisers of $\mathrm{Spin}(p, q)$.

In this part of the thesis, we focus on constructing a suitable collection of differential forms, referred to as the *canonical differential forms*, derived from a spinor and a metric through spinor bilinears, which we call *geometric maps*. In other words, we aim to explore the following direction:

$$(\text{metric, spinor modulo sign}) \implies \text{collection of canonical differential forms.} \tag{1.0.2}$$

As the collection of canonical differential forms emerges from the left-hand side of (1.0.2), the stabiliser associated with the spinor is determined by the group whose action remains invariant on the right-hand side of (1.0.2). This approach forms the basis for determining properties of spinor stabilisers.

Let us begin by presenting the main results concerning $\mathrm{Spin}(n, \mathbb{C})$. Stabilisers of $\mathrm{Spin}(2n)$ over $\mathbb{C}$ have been extensively studied, and classifications have been made up to and including $\mathrm{Spin}(16)$. We provide a classification that relies on straightforward combinatorial techniques. It is known that any Weyl spinor, for $n > 3$, is a linear combination of $k \in \{1, \ldots, 2^n\}$ pure spinors. A significant result, derived through combinatorics based on elementary relations between two or more pure spinors, as constructed in Chapter 6, is that it is unnecessary to consider all k from 1 to 2^n; instead, it suffices to examine up to a much smaller number. This allows for a simpler method, compared to that found in the literature, to categorise spinor stabilisers in dimensions up to and including fourteen. This categorisation is achieved by analysing the invariant action of a group on the canonical differential forms. Furthermore, a novel result we obtain is that this combinatorial approach can be extended beyond $\mathrm{Spin}(14, \mathbb{C})$ to categorise orbits in higher dimensions.

For the real stabilisers of $\mathrm{Spin}(p, q)$, we consider three cases: $\mathrm{Spin}(2n, 0)$, $\mathrm{Spin}(n, n)$, and $\mathrm{Spin}(r, s)$, where $n, r, s \in \mathbb{Z}$, with $r \neq s$ and $r + s \in 2\mathbb{Z}$. It is known that $\mathrm{SU}(n)$ is the stabilising subgroup of a pure spinor for $\mathrm{Spin}(2n)$, $n \in \mathbb{N}$. In fact, for $n \leq 3$, as all Weyl spinors are pure, $\mathrm{SU}(n)$ is the *only* stabilising subgroup. Using geometric maps, one can compute a real symplectic 2-form ω and a complex n-form Ω; these are the canonical differential forms that remain invariant under the action of $\mathrm{SU}(n)$. Alternatively, starting with $\mathrm{Spin}(n, n)$, one has many more choices of pure spinors to consider. The selection of pure spinors, resulting in distinct spinorial G-structures, is governed by an integer known as the *real index*. For $\mathrm{Spin}(2n)$, the real index is *always* zero, which explains why it consistently reduces to $\mathrm{SU}(n)$. Allowing a different real index introduces a broader class of spinorial G-structures. For $n < 4$, all spinors are pure; thus, for example, two non-parallel pure spinors correspond to a paracomplex structure, associated with the reduction of $\mathrm{Spin}(n, n)$ to $\mathrm{SL}(n, \mathbb{R})$. In Chapter 3, we classify all spinorial G-structures constructed for each type of pure spinor. Additionally, we identify a new class of canonical differential forms associated with spinorial G-structures generated by pure spinors in $\mathrm{Spin}(r, s)$, termed mixed structures — a blend of paracomplex and complex structures.

Returning to spinors in $\mathrm{Spin}(2n)$, for $n > 3$, there exist spinors that are no longer pure, known as *impure spinors*. In Chapter 5, we classify the stabiliser of an impure spinor that preserves the canonical differential forms in $\mathrm{Spin}(8)$ using octonions. We observe that in higher even dimensions, there is a way to embed $\mathrm{Spin}(8) \hookrightarrow \mathrm{Spin}(2n)$ for $n = 5$ and $n = 6$. By analysing various spinor orbits for $n = 5$ and $n = 6$, we demonstrate that our methods are comprehensive in determining all real stabilisers of $\mathrm{Spin}(10)$, and we construct a previously undiscovered class of real stabilisers for $\mathrm{Spin}(12)$.

In Part II, we concentrate on the problem of determining "natural" PDEs that can imposed on a system built from a pair (metric, spinor), using (1.0.1). An essential precursor to our discussion is the classification of all possible holonomy groups on a Riemannian manifold. Some of these holonomy groups arise as spinor stabilisers; hence, we define the reduction of the spin-frame bundle to any holonomy group derived from a spinor stabiliser as a *classical* spinorial G-structure. It is well known that the most natural first-order PDEs one can consider involve the vanishing of the exterior derivatives of the collection of differential forms invariant under the action of G[1]. In this thesis, we ask: Are there natural second-order PDEs that can be constructed from the canonical differential forms invariant under the action of G? We approach this question through the lens of action functionals, whereby critical points impose natural second-order PDEs.

Our approach is inspired by Plebanski's formulation of general relativity [Ple77]. In his paper, he utilised an SU(2) gauge connection to construct a 4-dimensional action for gravity. By extremising this functional, one recovers natural second-order PDEs: the Einstein conditions on curvature. Our goal is to generalise his construction to higher dimensions. We achieve this by constructing all independent action functionals that are diffeomorphism invariant and built from the second-order derivatives of canonical differential forms and auxiliary fields. Ordinarily, writing all independent action functionals is not straightforward, as it is not always evident how to parameterise the auxiliary fields. To assist in this regard, we employ *linearised analysis*, a procedure for constructing action functionals from infinitesimal actions of the general linear group on the canonical differential forms. An important aspect of linearised analysis is that it is constrained by representation theory, offering insight, for free, into the general structure of the action functional and the forms the auxiliary fields should take.

In Chapters 8 and 9, we examine SU(2)- and SU(3)-structures. Specifically, we show that the collection of differential forms on the right-hand side of (1.0.1) arises from spinors. Furthermore, we demonstrate that in each case, the metric can be reconstructed from this collection of differential forms. In Chapter 8, we present a non-linear action functional of second-order in derivatives of SU(2)-structures and show its equivalence to Plebanski's formulation of gravity in 4 dimensions. In this way, we recover the most natural second-order PDEs one can formulate with the canonical differential forms: the Einstein conditions on curvature. Finally, in both Chapters 8 and 9, we conduct a linearised analysis for SU(2)- and SU(3)-structures, revealing the novel result that a class of action functionals corresponds to the linearised Einstein-Hilbert action of gravity in a vacuum.

[1] This characterisation acts as a first-order obstruction to an integrable classical spinorial G-structure.

Part I

Geometry & Classification of Spinor Orbits

Chapter 2

Introduction

Spinors, that is, the spin groups and their representations, have been known to mathematicians since the time of Euler, Hamilton, and Clifford [TT94]. However, it was Élie Cartan who first provided a description of what we now understand as the spinor representations of $\mathfrak{spin}(n)$ [Car13]. As is often the case, the popularity of the field and its study surged when physicists began investigating the intrinsic angular momentum (spin) of electrons. This exploration led to the development of the non-relativistic limit electron spin model [Pau27], for which Wolfgang Pauli was awarded the Nobel Prize, given below as:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & -\mathbf{i} \\ \mathbf{i} & 0 \end{pmatrix}, \text{ and } \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \text{ Here } \mathbf{i} \in \mathbb{C} \text{ is the unit imaginary.} \tag{2.0.1}$$

The Pauli matrices (2.0.1) generate the Clifford algebra Cliff_3 and subsequently the spin Lie algebra $\mathfrak{spin}(3)$ that acts on the space of spinors $\mathbb{C}^2$.

Dirac was the first to utilise Pauli matrices in the relativistic limit while attempting to solve the Schrödinger equation for a free relativistic electron, described by the Klein-Gordon equation:

$$(\Box + m^2)\psi = 0, \tag{2.0.2}$$

as elaborated in his foundational paper [A M28]. Solutions to this equation lacked a quantum interpretation; knowing the position of ψ precluded knowledge of its momentum. Dirac proposed a novel solution, the "square root" of this equation, now known as the Dirac equation:

$$(\mathbf{i}\sigma^\mu \partial_\mu - m)\psi = 0, \tag{2.0.3}$$

where $\mu = \{0, 1, 2, 3\}$, with $\sigma^0 = \mathbb{I}$, the identity operator, and $\sigma^{1,2,3}$ as defined in (2.0.1). This theory had twofold consequences: first, the entity ψ was no longer a single function, but a vector of functions. This inadvertently spurred a large field of mathematical study. The Dirac equation (2.0.3) can be analysed on various backgrounds, i.e., different manifolds, leading to the study of harmonic spinors and spinor analysis [Mic13].

Secondly, and more physically, Dirac's theory led to the conjecture of a particle identical in all respects to the electron except for its charge. The positron was discovered just four years later by Carl D. Anderson, earning him a Nobel Prize [HA13]. The connection between the square root of geometry and physics, as elucidated by [Tra93], dates back to [Gra80] regarding the rational rigid representations of rotations. Clifford was the first to generalise these ideas, leading to the development of Clifford algebra over a finite dimensional metric space (V, g) over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

Definition 2.0.0.1. The *Clifford algebra* of (V, g), denoted $\mathrm{Cliff}(V)$, is defined as

$$\mathrm{Cliff}(V) \coloneqq T(V)/\langle\{v \in V \mid v \otimes v - g(v,v)\}\rangle. \tag{2.0.4}$$

Where $T(V) \coloneqq \oplus_{k=1}^{\infty} \otimes^k V$ is the tensor ideal.

It is known that there is a universal property that extends from $V \to A$, where A is an algebra to $\mathrm{Cliff}(V) \to A$ [Har90]. For example, suppose one takes $A = \mathrm{End}(\Lambda V)$, then the universal property is given by Γ-matrices, reminiscent of the same objects in physics, such that its action on elements of ΛV is given by the Chevalley-Kähler formula [Che54], [Käh62], [Tra93]:

$$\Gamma(v) \cdot \psi = v \wedge \psi + v \lrcorner \psi. \tag{2.0.5}$$

Where $\wedge : \Lambda^k V \to \Lambda^{k+1} V$ raises the degree of an element in $\Lambda(V)$, while $\lrcorner : \Lambda^k \to \Lambda^{k-1}$ lowers of the degree of an element in $\Lambda(V)$. Neatly, this information is encoded in the following diagram.

$$\begin{array}{ccc} \mathrm{Cliff}(V) & \longrightarrow & \Lambda V \\ \Gamma \uparrow & \nearrow & \\ V & & \end{array} \tag{2.0.6}$$

Remark 2.0.0.1. As ΛV is a finite dimensional vector space, the naming scheme Γ-matrix makes sense. Indeed, choosing a basis allows $\Lambda V \cong \mathbb{K}^{2^n}$, where $n = \dim(V)$. So $\Gamma \in \mathrm{End}(\Lambda V) \cong \mathrm{M}_{2^n}(\mathbb{K})$. Also, the Γ-matrices don't have to send V to $\mathrm{Cliff}(V)$ exclusively, one can instead take $\mathrm{Cliff}(W)$, where W is the complexification of V.

In this thesis, we explore the implications of Γ-matrices in depth, as these are the backbone of the calculations we perform. For us, ΛV will be generated from an even-dimensional real vector space, with the possibility of mixed signature. We can then choose a basis and explicate (2.0.5). The choice of basis is crucial, as different choices can lead to different representations. This is controlled by a mechanism called the *pure* spinor. Spinors are viewed as the square root of geometry, with pure spinors being more fundamental, carrying the geometric properties of a spinor through their behaviour as *null* vectors.

Consider the simple Pythagorean equation,

$$x^2 + y^2 = z^2, \tag{2.0.7}$$

whose solutions can be elegantly parametrised as:

$$x = p^2 - q^2, \quad y = p^2 + q^2, \quad z = 2pq, \quad \text{for} \quad p, q \in \mathbb{N}. \tag{2.0.8}$$

This can be written as a solution to a matrix equation:

$$\begin{pmatrix} y + x & z \\ z & y - x \end{pmatrix} = 2 \begin{pmatrix} p \\ q \end{pmatrix} \begin{pmatrix} p & q \end{pmatrix}, \tag{2.0.9}$$

with the interpretation that a *null vector* (x, y, z) (the determinant of the matrix above is zero) is the *square* of a spinor (p, q) (the right-hand side of the above is the outer product of two vectors). Cartan's generalisation of this concept came of the form of maximally totally null subspaces. As seen from the formula (2.0.5), null vectors are characterised by the spinors they annihilate.

Following this, we proceed to review the construction of Clifford algebras, and their representations. First, through tensor products of matrix algebras, owed to [BW35]. Then to maximally totally null subspaces, via (2.0.5) owed to [CB68].

2.0.1 Construction through tensor products

The most popular construction of Clifford algebras and their representations on ΛV are produced by exploiting the following theorem and corollary, introduced by Brauer and Weyl [BW35] with a modern treatment given by [Har90].

Theorem 2.0.1.1. Let (V, g) be a metric space with signature (r, s), and define $\mathrm{Cliff}(V) \coloneqq \mathrm{Cliff}_{r,s}$. One has the $\mathrm{Cliff}_{r,s}$ is isomorphic, as an associative and unital algebra, to one of the following matrix algebras

$(r-s)$ mod 8	Matrix algebra
0, 6	$M_N(\mathbb{R})$
2, 4	$M_N(\mathbb{H})$
1, 5	$M_N(\mathbb{C})$
3	$M_N(\mathbb{H}) \oplus M_N(\mathbb{H})$
7	$M_N(\mathbb{R}) \oplus M_N(\mathbb{R})$

where $N = 2^{r+s}$.

Corollary 2.0.1.1.1. Let $\mathrm{Cliff}_{r,s}$ as above, then

$$\begin{aligned} \mathrm{Cliff}_{r+1,s+1} &\cong \mathrm{Cliff}_{r,s} \otimes_{\mathbb{R}} M_2(\mathbb{R}), \\ \mathrm{Cliff}_{s+2,r} &\cong \mathrm{Cliff}_{r,s} \otimes_{\mathbb{R}} M_2(\mathbb{R}), \\ \mathrm{Cliff}_{r,s+2} &\cong \mathrm{Cliff}_{r,s} \otimes_{\mathbb{R}} \mathbb{H}. \end{aligned} \tag{2.0.10}$$

We prove this corollary to highlight the procedure one performs to generate higher dimensional Clifford algebras from lower dimensional ones.

Proof. Begin with generators of γ^I of $\mathrm{Cliff}_{(r,s)}$. Then the matrices

$$\Gamma^I \coloneqq \begin{pmatrix} 0 & \gamma \\ \gamma & 0 \end{pmatrix}, \quad \Gamma^{I+1} \coloneqq \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}, \quad \Gamma^{I+2} \coloneqq \begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix} \tag{2.0.11}$$

anticommute, and $(\Gamma^{I+1})^2 = \mathbb{I}$, $(\Gamma^{I+2})^2 = -\mathbb{I}$. Thus, these matrices generate $\mathrm{Cliff}_{(r+1,s+1)}$.

A suitable modification of this construction gives the second line in (2.0.10). Indeed, we can instead define

$$\Gamma^I \coloneqq \begin{pmatrix} 0 & \gamma \\ -\gamma & 0 \end{pmatrix}, \quad \Gamma^{I+1} \coloneqq \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}, \quad \Gamma^{I+2} \coloneqq \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}. \tag{2.0.12}$$

Then Γ^I generate the Clifford algebra $\mathrm{Cliff}_{(s,r)}$ of opposite signature, and both $\Gamma^{I+1}, \Gamma^{I+2}$ square to plus the identity.

The last line in (2.0.10) is proved by the following construction,

$$\Gamma^I \coloneqq \begin{pmatrix} 0 & \gamma \\ \gamma & 0 \end{pmatrix}, \quad \Gamma^{I+1} \coloneqq \begin{pmatrix} \mathbf{i} & 0 \\ 0 & -\mathbf{i} \end{pmatrix}, \quad \Gamma^{I+2} \coloneqq \begin{pmatrix} \mathbf{j} & 0 \\ 0 & -\mathbf{j} \end{pmatrix}. \tag{2.0.13}$$

Here $\mathbf{i}, \mathbf{j}$ are two imaginary quaternions. These matrices anticommute, and Γ^{I+1}, and Γ^{I+2} square to minus the identity. ◻

This is a recursive relation, implying the knowledge of a base case of Clifford algebras listed below with verifiable isomorphisms [Har90]

$$\begin{aligned} &\mathrm{Cliff}_{0,1} \cong \mathbb{C}, \ \mathrm{Cliff}_{1,0} \cong \mathbb{R} \oplus \mathbb{R}, \ \mathrm{Cliff}_{0,2} \cong \mathbb{H}, \\ &\mathrm{Cliff}_{1,1} \cong M_2(\mathbb{R}), \ \text{and } \mathrm{Cliff}_{2,0} \cong M_2(\mathbb{R}), \end{aligned} \tag{2.0.14}$$

generates a sequence of Clifford algebras up the chain until the desired dimension is reached. Indeed, using (2.0.14) and corollary 2.0.10 one can cumbersomely iterate this until the required dimension is achieved. Writing $2^{r+s} \times 2^{r+s}$ matrices to encode information about Clifford algebras is usually not economical[1]. As this thesis deals with spinors in higher dimensions, we look to other approaches to efficiently encode information about Clifford algebras and their representations.

2.0.2 Construction through maximally totally null subspaces

The treatment of Clifford algebras proposed by Cartan in [CB68] introduced the notion of maximally totally isotropic (null) subspaces, with a modern treatment found in [Har90]. This method is quite abstract at first glance, as there is no concrete "model". For example, Clifford algebras are given by matrices, and their representations are understood as vectors in a 2^n-dimensional vector space. To compensate for this abstraction, we introduce creation and annihilation operators, inspired by their use in physics, to construct models on the space of polyforms (the space of linear combinations of differential forms). By creation and annihilation operators, we refer to $\wedge$ and $\lrcorner$ as defined in (2.0.5).

The advantage of this approach is that it provides efficient descriptions of spinors, especially in higher dimensions. In fact, we will exploit the characterisation of spinors as quaternions and octonions to further streamline this identification.

Abstractly speaking, consider a vector space V and (2.0.6), as before. Now, Γ maps V to $\mathrm{Cliff}(V)$, where W is the complexification of V.

Definition 2.0.2.1. Suppose that $M(\psi) \subset W$ is the subspace that annihilates the spinor $\psi \in \Lambda(W)$ via the Clifford product (2.0.5). The spinor ψ is called *pure* or *simple* if $\dim(M(\psi)) = \dim(V)$.

An important consequence of this definition is that pure spinors are Weyl.

In the construction of Clifford algebras, the complexified vector space W is considered for representations of $\mathrm{Cliff}(W)$. A viewpoint we adopt, first introduced by [TT94], is that pure spinors are real slices of $\mathrm{Cliff}(W)$. Suppose one has the Clifford algebra $\mathrm{Cliff}_{r,s}$ embedded into $\mathrm{Cliff}_{p,p} \otimes \mathbb{C} = \mathrm{Cliff}_{2p} \otimes \mathbb{C}$, here $2p = r + s$. The appropriate real slice of this space corresponds to $\mathrm{Cliff}_{r,s}$. However, this implies that there is more than one choice of $\mathrm{Cliff}_{r,s}$.

Using a tool called the real index, developed by [TT94], and another tool called geometric maps, developed by Cartan, we elucidate what form a real slice takes in terms of creation and annihilation operators. Additionally, we show that this model corresponds to an appropriate geometric structure (complex, paracomplex or mixed structure) determined by a pure spinor and geometric maps. Conversely, if one chooses a pure spinor, this corresponds to one of the many real slices of $\mathrm{Cliff}_{2p} \otimes \mathbb{C}$. This, in turn, corresponds to a choice of geometric structure, resulting in a specific model of creation and annihilation operators [BK22].

Remark 2.0.2.1. The two perspectives: construction by tensor products and construction through maximally totally null spaces, were reconciled and extended by Chevalley [Che54], who removed Cartan's restriction of the base field being $\mathbb{R}$ or $\mathbb{C}$ by viewing spinors as minimal left ideals of Clifford algebras. He rigorously proved several important theorems on the subject and coined the term *pure* spinors[2,3].

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[1] There is an economical way to encode information about Clifford algebras if one uses division algebras such as $\mathbb{H}$, and $\mathbb{O}$ as seen in (2.0.13). In a dimension where this is possible, we will exploit this parametrisation.

[2] The adjective *pure* in algebra is reserved for objects that cannot be represented as a product [Tra93].

[3] Pure spinors, introduced by Chevalley, were initially disliked by physicists because they found it unsettling that Dirac spinors were labelled as "impure" by Chevalley's definition [BT89]—since pure spinors are necessarily Weyl. We do not extend that sentiment and adopt Chevalley's naming conventions.

2.0.3 Chapter Overview

In chapter 3 we apply the above ideas at the linear level, specifically taking $V = \mathbb{R}^{2n}$, $\mathbb{R}^{n,n}$, and $\mathbb{R}^{r,s}$ for $r + s$ even. We begin by defining the Clifford algebra for each V, its representations over polyforms, and the spin Lie algebra. We formally define pure spinors and quantify what a real slice of $\text{Cliff}(W)$ means. Pure spinors give the real slice and correspond to the canonical differential forms of spinorial G-structures via the geometric map, which we also formally define. In part I, it is assumed that we implicitly have access to the metric (1.0.2). We do this because our aims are mainly concerned with classification. Having access to the metric allows one to raise and lower the indices of the canonical differential k-forms. We shall refer to the corresponding endomorphisms as geometric structures. Furthermore, we note that there can only be three types of structures from a pure spinor: complex structures, paracomplex structures, and a new type of structure called a mixed structure.

This chapter serves as a precursor to all other chapters in terms of notation and conventions.

Armed with the general theory from the previous chapter, in chapter 4, we construct examples of spinorial G-structures in low dimensions, specifically in dimensions 2, 4, and 6. Where appropriate, we make comments on the geometry of the orbit in the local coordinate sense. It is well known that spinors in these dimensions are always pure, so there is only one type of orbit in these lower dimensions.

For brevity, we discuss only some examples of complex, paracomplex, or mixed structures arising from general spinors. We encourage readers to refer to the article that this chapter is based on for a complete classification of each type of spinorial G-structure in dimensions 2, 4, and 6: Niren Bhoja and Kirill Krasnov. "Notes on Spinors and Polyforms I: General Case". In: *arXiv* (May 2022). URL: http://arxiv.org/abs/2205.04866.

Eight dimensions mark the first instance where there is more than one type of orbit other than the pure one. This implies a larger class of spinorial G-structures beyond just complex, paracomplex, or mixed. In chapter 5, we aim to classify the orbits of impure spinors (spinors that are not pure). In 8 dimensions, the orbits of spinors are understood through octonions. As we will show, pure spinors are null complexified octonions. We can express any impure spinor as a linear combination of two pure spinors. We then proceed to analyse various cases of the coefficients of each pure spinor. We find that there is only one additional orbit that is not pure, namely $\text{Spin}(7)$.

A similar procedure is applied to $\text{Spin}(4, 4)$, but a much larger class of spinorial G-structures is found. In ten dimensions, one can extend our constructions, and a detailed analysis of real orbits is provided in [Kra22]. For impure spinors, there are $\text{Spin}(7)$-type and $\text{SU}(4)$ orbits. In 12 dimensions, a general classification is not yet known. Nevertheless, using techniques from 8 dimensions, a previously unknown class of orbits has been discovered.

A classification for spinors over $\mathbb{C}$ in 12 dimensions was first done by [Igu70], and similarly, those in 13 dimensions were classified by [GV78]. In 14 dimensions, spinors were classified by [Pop80]; see also [Zhu92] and [Pop77]. The full classification for 16 dimensions was done by [L V82], with special cases involving the use of the exceptional Lie group E_8 by [DR93]. A useful resource is the thesis [Cha97], which provides clear classifications for spinors in 12 and 14 dimensions using pure ones. These classifications are not easy to digest, and many complex ideas from representation theory are used. We propose a simpler approach via pure spinors and combinatorics.

Although the technology we develop can be used to analyse 16 dimensions and higher, it becomes quite unwieldy, and hence we omit these constructions. This is evidenced by the comprehensive classification provided by [L V82], where there are many possible orbits.

The final chapter, chapter 6, splits into two sections. In section 6.1, we aim to simplify the understanding of orbits. Given a reasonable start, we can trace the problem to counting different permutations of simplices. Consider the space $\mathbb{C}^{2n}$, and let $M(\psi_0)$ be a maximally totally null space for a pure spinor ψ_0. Upon choosing a basis in $M(\psi_0)$, one can lift to $\mathrm{Cliff}(\mathbb{C}^n) \subset \mathrm{Cliff}(\mathbb{C}^{2n})$, allowing one to specify a completion of the lifted basis in $\mathrm{Cliff}(\mathbb{C}^{2n})$ — the full lifted basis shall be called a *canonical* basis. This canonical basis can then be used to construct every possible impure spinor in the space S^+, which consists of Weyl spinors of even chirality. Therefore, the problem of classifying spinors in any dimension is reduced to combinatorics by these two facts:

- Any $\psi \in S^+$ can be written as a linear combination of this canonical basis of 2^{n-1} pure spinors.
- **Theorem 3.2.2.4**: Let ψ and ϕ be two (non-parallel) pure spinors. Then $\psi + \phi$ is a pure spinor if, and only if, $M(\psi) \cap M(\phi) = n - 2$.

In this chapter, we provide the general theory based on the two facts above. Diagrams are parameterised by pure spinor representatives as vertices of some k-simplex, weighted by $|M(\psi) \cap M(\phi)|$. The tools we use for counting are occupation box numbers $n_1, \dots, n_i, \dots, n_k$, which count the number of directions shared by i pure spinors. This allows us to define geometric objects such as vertices, edges, and higher-dimensional simplices. Each geometric object carries information; for example, a vertex will be associated with a unique pure spinor, an edge with the number of directions shared by 2 pure spinors, and so on.

This framework allows us to define concepts such as edge intersection numbers e and tetrahedral intersection numbers t. These numbers help us count the ways two spinors can share an edge and the ways four spinors can share a tetrahedron. We develop formulas concerning edge intersection numbers and box occupation numbers to find bounds on the maximal purity of an impure spinor. This generates known results up to $d = 7$ [Cha97], at which point the formula breaks down. We show that considering tetrahedral intersection numbers and box occupation numbers and developing similar formulas to develop stricter bounds on the maximal purity of an impure spinor for $d > 8$. An assumption made in the bounding process is verified to be true in the following chapter.

In section 6.2, we sequentially consider different degrees k of an impure spinor. Normally, one would need to consider k from 1 to 2^{d-1}, but, as per the previous chapter, we only need to consider k up to the bound we develop. In the process of incrementally increasing the dimensions and degree of impurity, we write all unique diagrams, up to a representative for those diagrams, and compare the results to known ones. We also show that they match known results (the simple part of the stabilisers), up to and including dimension 14.

Furthermore, the diagrams are also *uniquely* associated with a canonical differential form that we construct case by case and analyse to understand the orbit type. We demonstrate that by the action of the Cartan subgroup of $\mathrm{Spin}(2n, \mathbb{C})$, one can rotate a spinor to a canonical one, making the differential form an identifier of the orbit. This relates back to part I, where we explicitly constructed geometric structures that characterised orbits of spinors. Consider the following simple example.

Example 2.0.3.1. In 8 dimensions one can construct the diagram

A —— 0 —— B
$$\psi_A = \mathbf{1}, \quad \psi_B = e^1 \wedge e^2 \wedge e^3 \wedge e^4 \tag{2.0.15}$$

Where A, B on the edge refer to the spinors $\psi_{A,B}$. See that the weight 0 is in reference to the fact that $|M(\psi_A) \cap M(\psi_B)| = 0$. This is the unique diagram, up to choice of representative, that characterises the stabiliser subgroup $\mathrm{Spin}(7)$. Upon choosing a representative ψ - general linear combination of ψ_A and ψ_B, one can also write the canonical differential 4-form

$$B_4(\psi) \propto \Omega + \bar{\Omega} + \tfrac{1}{2}\omega^2, \tag{2.0.16}$$

where Ω is the top holomorphic form in 4 complex dimensions, and ω is the symplectic form in 8 real dimensions.

We encounter a phenomenon where diagrams are necessarily unique, even if the spinor is not. In twelve dimensions, there exists an impure spinor of degree 2 that is in the same orbit as a higher-dimensional spinor of degree 4, allowing us to show that the bound constructed from tetrahedral numbers is accurate. This *feature* extrapolates to higher dimensions and serves as a basis for reduction in 14 dimensions for a type of impure spinor of degree 5.

Chapter 3

Preliminaries of Polyforms & Pure Spinors

3.1 Polyform Representations of Spinors

Polyforms are arbitrary linear combinations of differential forms on $\mathbb{R}^{2n}$. This section serves as a preliminary to establishing notation and conventions, facilitating a connection between Clifford algebras over $\mathbb{R}^{2n}$, $\mathbb{R}^{n,n}$, and $\mathbb{R}^{r,s}$, where $r+s$ is even. We explore these connections in terms of creation and annihilation operators on the space of differential forms. Additionally, we discuss their representations over differential forms and introduce Dirac, Weyl, Majorana, and Majorana-Weyl spinors as subspaces of differential forms. Finally, we describe the spin Lie algebra using antisymmetric products of creation and annihilation operators.

3.1.1 Polyform Representations of Spin$(2n)$

Clifford Algebras

Let us start by considering the Clifford Algebra Cliff_{2n}. Let $\mathbb{R}^{2n} \cong V$ be a real $2n$ dimensional vector space with an inner product. We introduce the linear map $J : V \to V$ such that $J^2 = -\mathbb{I}$, that is J is a complex structure. Allowing V to be identified with $W \oplus \bar{W}$, whose real dimensions are n. For our purposes, we consider $W \cong \mathbb{C}^n$ as the $-\mathbf{i}$ eigenspace of J. Let $\{e^1, \dots, e^n\}$ be an orthonormal basis in W, that is $(e^i, e^j) = \delta^{ij}$, and $\Lambda^k(W)$ as the space of $k-$forms, that is the anti-symmetrisation of the $(k,0)$ tensor space $\bigotimes^k W$. Hence, for any $v \in \Lambda^k(W)$ one has

$$v = v_{i_1,\dots,i_k} e^{i_1 \dots i_k}, \text{ for } 1 \le i_1, \dots i_k \le n \text{ and } v_{i_1,\dots i_k} \in \mathbb{C}. \tag{3.1.1}$$

We introduce the notation $e^{i_1 i_2 \cdots}$ above to mean $e^{i_1} \wedge e^{i_2} \wedge \dots$.

Definition 3.1.1.1. A *polyform* ψ is a k-form of varying degrees. Explicitly, $\psi \in \Lambda(W) = \bigoplus_{k=1}^{n} \Lambda^k(W)$.

We remark that $\Lambda(W)$ is a (obviously) graded, associative, anticommuting algebra with unit (and inner product $\langle \ , \ \rangle$ defined later on). We will now introduce the following linear operators on $a_i, a_i^\dagger : \Lambda(W) \to \Lambda(W)$ defined as

$$a_i(\psi) = e^i \lrcorner \psi, \text{ and } a_i^\dagger \psi = e^i \wedge \psi. \tag{3.1.2}$$

Here for a $k-$form $v \in \Lambda^k(W)$, as defined above, the action of $a_j : \Lambda^k \to \Lambda^{k-1}$ on $v \in \Lambda^k(W)$ is given as

$$a_j(v) = e^j \lrcorner \psi := \sum_{m=1}^{k} (-1)^{m-1} \delta_{j i_m} v_{i_1,\ldots,i_m,\ldots,i_k} e^{i_1} \wedge \ldots \widehat{e^{i_m}} \ldots \wedge e^{i_k}. \tag{3.1.3}$$

$\widehat{e^{i_m}}$ means to omit that term from the sum. Again taking $v \in \Lambda^k(W)$ the action of $a_j^\dagger : \Lambda^k \to \Lambda^{k+1}$ is given as

$$a_j^\dagger(v) = e^j \wedge \psi := v_{i_1,\ldots,i_k} e^j \wedge e^{i_1 \ldots i_k}. \tag{3.1.4}$$

The above creation and annihilation operators satisfy the commutation relations

$$a_i a_j^\dagger + a_j^\dagger a_i = \delta_{ij}, \;\; a_i a_j + a_j a_i = 0, \;\text{ and }\; a_i^\dagger a_j^\dagger + a_j^\dagger a_i^\dagger = 0, \;\; \forall\; 1 \le i, j \le n. \tag{3.1.5}$$

To construct the Clifford algebra one takes linear combinations of the above operators defined as

$$\Gamma_i = a_i + a_i^\dagger, \;\text{ and }\; \Gamma_{i+n} = -\mathbf{i}(a_i - a_i^\dagger). \tag{3.1.6}$$

An interesting note that will be useful later is that $\Gamma : V \to \text{Cliff}(W)$ is an algebra homomorphism that maps a vector $v \in V$ to a generator in $\Gamma(v) \in \text{Cliff}(W)$, so $\Gamma_i = \Gamma(e^i)$, see Lemma 9.7 and Remark 9.10 in [Har90]. The action by Γ on $\Lambda(W)$ is referred to as the Clifford action, and the choice of homomorphism Γ is called a model for $\text{Cliff}(W)$. These operators satisfy the Clifford algebra relation,

$$\Gamma_A \Gamma_B + \Gamma_B \Gamma_A = 2\delta_{AB}, \;\text{ for } 1 \le A, B \le 2n. \tag{3.1.7}$$

Hence $\text{Cliff}_{2n} \cong \text{Cliff}(W)$ is generated by the above so-called "gamma matrices". We note that there was a choice of a complex unit $\mathbf{i}$ in the above gamma matrices. If its position were reversed, that is we took

$$\Gamma_i = \mathbf{i}(a_i + a_i^\dagger), \;\text{ and }\; \Gamma_{i+n} = a_i - a_i^\dagger \tag{3.1.8}$$

then the result would have been

$$\Gamma_A \Gamma_B + \Gamma_B \Gamma_A = -2\delta_{AB}, \;\text{ for } 1 \le A, B \le 2n. \tag{3.1.9}$$

This sign is important if our focus was solely on Clifford algebras, however, we will be concentrating on the spin algebra. This makes the sign immaterial.

Spin Lie Algebras, & Weyl Spinors

The spin Lie algebra is generated from the commutators of the gamma matrices generating the Clifford algebra, that is, for any $\Gamma_A, \Gamma_B \in \text{Cliff}_{2n}$, $\mathfrak{spin}(2n)$ is generated by $\Gamma_{[A}\Gamma_{B]}$. An alternative description that can be used is

$$\mathfrak{spin}(2n) \ni \mathcal{A}(X) = \frac{1}{4} \sum_{A<B} X^{AB} \Gamma_A \Gamma_B. \tag{3.1.10}$$

Here X is a real antisymmetric and trace-free $(2,0)$ tensor. Furthermore, A is a Lie algebra homomorphism from $\mathfrak{spin}(2n)$ to $\mathfrak{gl}_{2n}(\mathbb{C})$ given by

$$[\mathcal{A}(X), \mathcal{A}(Y)] = \mathcal{A}([X, Y]) \tag{3.1.11}$$

Here $[X,Y]^{AB} = X^A{}_C Y^{CB} - Y^A{}_C X^{CB}$ (the index B is raised using the metric δ_{AB} on $V \cong \mathbb{R}^{2n}$) is the commutator of two antisymmetric trace-free $2n \times 2n$ matrices. Thus, A is in fact a Lie algebra isomorphism between $\mathfrak{spin}(2n)$ and $\mathfrak{so}(2n)$.

Definition 3.1.1.2. The even/odd subspaces of $\Lambda(W)$ are defined as

$$S_+ \coloneqq \bigoplus_{k\in\Omega_n^{even}} \Lambda^k(W), \text{ and } S_- \coloneqq \bigoplus_{k\in\Omega_n^{odd}} \Lambda^k(W),^{1} \tag{3.1.12}$$

where

$$\Omega_n^{even} \coloneqq \{p \in 2\mathbb{Z} \mid 0 \le p \le n\}, \text{ and } \Omega_n^{odd} \coloneqq \{p \in 2\mathbb{Z}+1 \mid 0 < p \le n\}. \tag{3.1.13}$$

Any vector in $S_\pm$ is called a *Weyl spinor*, and any vector in $\Lambda(W) \coloneqq S = S_+ \oplus S_-$ is called a *(Dirac) spinor.*

The Γ-matrices in Cliff_{2n} map $S_\pm$ to $S_\mp$. As any element of $X \in \mathfrak{spin}(2n)$ is the skew-symmetrised product of any two Γ-matrices, the action of X on $S_\pm$ is an endomorphism $X : S_\pm \to S_\pm$. Thus, the Dirac representation of $\mathfrak{spin}(2n)$ is reducible to the Weyl representations $S_\pm$.

Inner Product

Definition 3.1.1.3. Let $\psi_1, \psi_2 \in S$ then the *inner product on S* is defined as

$$\langle \psi_1, \psi_2 \rangle = \sigma(\psi_1) \wedge \psi_2\big|_{\text{top}}. \tag{3.1.14}$$

Here $\sigma : S \to S$ is the automorphism that reverses the order of each decomposable form in ψ_1, and $\big|_{\text{top}}$ is the projection of the polyform to the coefficient of volume form.

In the above definition, suppose $\lambda_{i_1\ldots i_k} e^{i_1\ldots i_k}$ is a decomposable k-from, such that $\lambda_{i_1\ldots i_k} \in \mathbb{C}$, then

$$\sigma(e^{i_1\ldots i_k}) = e^{i_k\ldots i_1}. \tag{3.1.15}$$

Furthermore, suppose $\psi = \lambda\mathbf{1} + \lambda_{\text{vol}}\text{vol}$, such that $\lambda, \lambda_{\text{vol}} \in \mathbb{C}$, then

$$\psi\big|_{\text{top}} = \lambda_{\text{vol}}. \tag{3.1.16}$$

Proposition 3.1.1.1. Take the inner product given by definition 3.1.1.3, and any $\mathcal{A} \in \mathfrak{spin}(2n)$. Then one has

$$\langle \mathcal{A}\psi_1, \psi_2 \rangle + \langle \psi_1, \mathcal{A}\psi_2 \rangle = 0. \tag{3.1.17}$$

Proof. We shall first prove that a_i and $a_i^\dagger$ are self-adjoint

$$\begin{aligned} \langle a_j^\dagger \psi_1, \psi_2 \rangle &= \sigma(e^j \wedge \psi_1) \wedge \psi_2\big|_{\text{top}} \\ &= \sigma(\psi_1) \wedge e_j \wedge \psi_2\big|_{\text{top}} \\ &= \langle \psi_1, a_j^\dagger \psi_2 \rangle. \end{aligned} \tag{3.1.18}$$

To prove that a_j is self-adjoint it is sufficient to assume that $\psi_1 \in \Lambda^k(W)$ and $\psi_1 \in \Lambda^p(W)$ such that $p + k = n$. Suppose that $\psi_1 = e^{i_1} \wedge \ldots \wedge e^{i_k}$. Then for $a_j\psi_1$ to not vanish, else the inner product vanishes, e^j must appear in ψ_1. This results in $\sigma(a_j\psi_1) = (-1)^j e^{i_k} \wedge \ldots \wedge \widehat{e^j} \wedge \ldots e^{i_1}$. Furthermore, suppose ψ_2 is also decomposable then $\psi_2 = (-1)^j e^j \wedge e^{l_1} \wedge \ldots \wedge e^{l_p}$, if ψ_2 didn't

[1] We also commonly refer to $S_\pm$ as $\Lambda^{\text{even/odd}}(\mathbb{C}^n)$ (as $W \cong \mathbb{C}^n$).

contain e^j then the inner product vanishes.

$$\begin{aligned}\langle a_j\psi_1,\psi_2\rangle &= \sigma(a_j\psi_1)\wedge\psi_2\big|_{\text{top}}\\ &= (-1)^j e^{i_k}\wedge\ldots\wedge\widehat{e^j}\wedge\ldots e^{i_1}\wedge(-1)^j e^j\wedge e^{l_1}\wedge\ldots\wedge e^{l_p}\\ &= e^{i_k}\wedge\ldots\wedge e^j\wedge\ldots e^{i_1}\wedge(-1)^j e^{l_1}\wedge\ldots\wedge e^j\wedge\ldots\wedge e^{l_p}\\ &= \sigma(\psi_1)\wedge a_j\psi_2\big|_{\text{top}}\\ &= \langle\psi_1,a_j\psi_2\rangle.\end{aligned} \tag{3.1.19}$$

This immediately implies that Γ_A for $1\le A\le n$ is self-adjoint. Let $\mathfrak{spin}(2n)\ni\mathcal{A}=\frac{1}{4}X^{AB}\sum_{A<B}\Gamma_A\Gamma_B$ then

$$\begin{aligned}\langle\mathcal{A}\psi_1,\psi_2\rangle &= \sum_{A<B}\frac{1}{4}X^{AB}\langle\Gamma_A\Gamma_B\psi_1,\psi_2\rangle\\ &= \sum_{A<B}\frac{1}{4}X^{AB}\langle\psi_1,\Gamma_B\Gamma_A\psi_2\rangle\\ &= -\sum_{A<B}\frac{1}{4}X^{AB}\langle\psi_1,\Gamma_A\Gamma_B\psi_2\rangle\\ &= -\langle\psi_1,\mathcal{A}\psi_2\rangle.\end{aligned} \tag{3.1.20}$$

On the line 3 in the above equation we use the anti-symmetry of gamma matrices for $A\neq B$. □

Majorana Spinors

We construct two antilinear operators, $\mathfrak{R},\mathfrak{R}':S\to S$ that square to $\pm\mathbb{I}$, and commute or anticommute with Cliff_{2n}. Majorana spinors live in the eigenspace of $\mathfrak{R}$ or $\mathfrak{R}'$ depending on which one squares to the identity and (anti-)commutes with Cliff_{2n}. The reality conditions are defined as

$$\mathfrak{R}=\Gamma_1\ldots\Gamma_n\mathfrak{C},\ \text{and}\ \mathfrak{R}'=\Gamma_{n+1}\ldots\Gamma_{2n}\mathfrak{C}. \tag{3.1.21}$$

Where $\mathfrak{C}$ is the conjugation map on $\mathbb{C}$, taking any complex number to its complex conjugate. We see that $\mathfrak{R}$ is the product of all the "real" gamma matrices and $\mathfrak{R}'$ is the product of all the "imaginary" gamma matrices. We now have the following propositions.

Proposition 3.1.1.2. The operators $\mathfrak{R}$ and $\mathfrak{R}'$ either commute or anticommute with any $\Gamma_C\in\mathrm{Cliff}_{2n}$ for $C\in\{1,\ldots,2n\}$ in the following way

$$\mathfrak{R}\Gamma_C=(-1)^{n-1}\Gamma_C\mathfrak{R}, \tag{3.1.22}$$

and

$$\mathfrak{R}'\Gamma_C=(-1)^{n}\Gamma_C\mathfrak{R}'. \tag{3.1.23}$$

The proof is not conceptually difficult, one simply computes the various cases for Γ_C, $\mathfrak{R}$ and $\mathfrak{R}'$.

Proposition 3.1.1.3. The antilinear maps have square to $\pm\mathbb{I}$, depending on the dimension n, in the following way

$$\mathfrak{R}^2=(-1)^{\frac{n(n-1)}{2}}\mathbb{I},\ \text{and}\ (\mathfrak{R}')^2=(-1)^{\frac{n(n+1)}{2}}\mathbb{I}. \tag{3.1.24}$$

Proof.

$$\begin{aligned}\mathfrak{R}^2 &= \Gamma_1 \dots \Gamma_n \mathfrak{C} \Gamma_1 \dots \Gamma_n \mathfrak{C} \\ &= \Gamma_1 \dots \Gamma_n \Gamma_1 \dots \Gamma_n \mathfrak{C}^2 \\ &= (-1)^{\frac{n(n-1)}{2}} \Gamma_n \dots \Gamma_1 \Gamma_1 \dots \Gamma_n \\ &= (-1)^{\frac{n(n-1)}{2}} \mathbb{I}\end{aligned} \qquad \begin{aligned}(\mathfrak{R}')^2 &= \Gamma_{n+1} \dots \Gamma_{2n} * \Gamma_{n+1} \dots \Gamma_{2n} \mathfrak{C} \\ &= (-1)^n \Gamma_{n+1} \dots \Gamma_{2n} \Gamma_{n+1} \dots \Gamma_{2n} \mathfrak{C}^2 \\ &= (-1)^n (-1)^{\frac{n(n-1)}{2}} \Gamma_{2n} \dots \Gamma_{n+1} \Gamma_{n+1} \dots \Gamma_{2n} \\ &= (-1)^{\frac{n(n+1)}{2}} \mathbb{I}\end{aligned} \tag{3.1.25}$$

□

Definition 3.1.1.4. Suppose $\mathbf{R} \in \{\mathfrak{R}, \mathfrak{R}'\}$ as (3.1.21). The antilinear map $\mathbf{R}$ is called a *reality condition* if it squares to $+\mathbb{I}$ and commutes with $\mathfrak{spin}(2n)$.

What (3.1.1.2) and (3.1.1.3) then show is that for $n = 2 \bmod 4$, no reality conditions can exist because neither operator squares to the identity. For $n = 0 \bmod 4$, both $\mathfrak{R}$ and $\mathfrak{R}'$ are reality conditions. If $n = 1 \bmod 4$ or $n = 3 \bmod 4$, then one of the two reality conditions exists, as one will square to $+\mathbb{I}$, and one to $-\mathbb{I}$.

Theorem 3.1.1.1. $\mathfrak{R}$ and $\mathfrak{R}'$ are the only reality conditions up to phase and ordering.

Proof. Lemma 12.75, and lemma 12.90 in [Har90]. □

In fact the proof found in [Har90] is given for the more general case (r, s), such that $r+s \in 2\mathbb{Z}$. We take $r = 2n, s = 0$ as a special case to prove our results. Thus, when we discuss $\mathrm{Cliff}_{n,n}$, a compatible statement of theorem 3.1.1.1 will be true, i.e. specialise to $r = s = n$.

We now have sufficient data to define a Majorana spinor.

Definition 3.1.1.5. Suppose there exists a reality condition $\mathbf{R}$. Then a spinor $\psi \in S$ is called *Majorana* whenever $\mathbf{R}(\psi) \coloneqq \hat{\psi} = \psi$.

There is also another important definition that we need.

Definition 3.1.1.6. Suppose there exists a reality condition $\mathbf{R}$. Suppose further that $\mathbf{R} : S_\pm \to S_\pm$. Then a spinor $\psi_\pm \in S_\pm$ is called *Majorana-Weyl* whenever $\mathbf{R}(\psi_\pm) \coloneqq \hat{\psi}_\pm = \psi_\pm$.

The above statement is true only when $\mathbf{R}$ is an even number of gamma matrices. Hence since there are no Majorana-Weyl spinors in $n = 2 \bmod 4$. If $n = 1 \bmod 4$ or $n = 3 \bmod 4$ then $\mathbf{R} : S_\pm \to S_\mp$, so chirality is not preserved under this map but Majorana spinors exist. This means that there are only Majorana-Weyl spinors in $n = 0 \bmod 4$.

A Preferred $\mathfrak{u}(n)$ Subalgebra

Recall that in our construction of Cliff_{2n} we introduced a complex structure J that splits the $2n$ real dimension vector space $V = W \oplus \bar{W}$. Where W and W are complex vector spaces with dimension n, and we have chosen $JW = -\mathbf{i}W$ and $J\bar{W} = -\mathbf{i}\bar{W}$. The symmetry of the space then must be preserved by the choice J that was made. That is $\mathrm{SO}(2n)$ contains a $\mathrm{U}(n)$ subgroup that preserves the complex structure J. This is seen more clearly by examining the $\mathfrak{spin}(2n) \cong \mathfrak{so}(2n)$–module S.

Definition 3.1.1.7. Let φ be a k-form. Then the integer k is defined as the *degree* of φ.

There is a preferred $\mathfrak{u}(n)$ subalgebra in $\mathfrak{spin}(2n)$ whose action on $\psi \in S$ fixes the degrees of any decomposable forms that comprise it. In the context of $S = \Lambda(\mathbb{C}^n)$, these are operators in

$\mathfrak{spin}(2n)$ that only contain the product $a_i a_j^\dagger$ for some $i, j \in \{1, \ldots, n\}$. Explicitly, let us construct the $\mathfrak{spin}(2n)$ algebra as

$$\begin{aligned}\mathcal{A}_{\mathfrak{spin}(2n)} &= \sum_{i,j=1}^{n} \frac{1}{2} X^{ij} \Gamma_{[i} \Gamma_{j]} - \frac{1}{2} \Xi^{ij} \Gamma_{[i+n} \Gamma_{j+n]} + Y^{ij} \Gamma_{[i} \Gamma_{j+n]} \\ &= \frac{1}{2} X^{ij} (a_i + a_i^\dagger)(a_j + a_j^\dagger) - \frac{1}{2} \Xi^{ij} (a_i - a_i^\dagger)(a_j - a_j^\dagger) + Y^{ij} (a_i + a_i^\dagger)(a_j - a_j^\dagger).\end{aligned} \tag{3.1.26}$$

It is clear that X^{ij} and Ξ^{ij} are antisymmetric, while Y^{ij} has no symmetry. $\mathcal{A}_{\mathfrak{spin}(2n)}$ can be reduced to $A_{\mathfrak{u}(n)}$ by removing all products of the form $a_i a_j$ and $a_i^\dagger a_j^\dagger$, resulting in the conditions

$$X^{ij} = \Xi^{ij}, \text{ and } Y^{ij} = Y^{ji}. \tag{3.1.27}$$

This gives

$$\mathcal{A}_{\mathfrak{u}(n)} = X^{ij} (a_i a_j^\dagger + a_i^\dagger a_j) - \mathbf{i} Y^{ij} (a_i a_j^\dagger - a_i^\dagger a_j). \tag{3.1.28}$$

Since a and $a^\dagger$ are self-adjoint $A_{\mathfrak{u}(n)}$ is skew-hermitian.

Proposition 3.1.1.4. The vector representation of $\mathfrak{u}(n) \subset \mathfrak{spin}(2n)$ on $\mathbb{R}^{2n}$ is compatible with the complex structure J.

Proof. To prove this we just need to check that the action of (3.1.28) on $\mathbb{R}^{2n}$ preserves the eigenspaces of J. The vector representation of $\mathfrak{spin}(2n)$ is given by considering Cliff_{2n} as a module with the action on it through the commutator. The complex structure J splits $\mathrm{Cliff}_{2n} = \mathrm{span}\{a_i \mid 1 \le i \le n\} \oplus \mathrm{span}\{a_i^\dagger \mid 1 \le i \le n\}$, and hence we consider the vector $y = y^i a_i$ as what we want to be preserved under the bracket. Furthermore, we have the relations between creation and annihilation operators given as

$$[a_i a_j^\dagger, a_k] = 2\delta_{jk} a_i, \ [a_i^\dagger a_j, a_k] = -2\delta_{ik} a_j, \text{ and } [a_i^\dagger a_j^\dagger, a_k] = 2\delta_{jk} a_i^\dagger - 2\delta_{ik} a_j^\dagger. \tag{3.1.29}$$

It is clear that for any $A \in \mathfrak{spin}(2n)$, $[A, y^i a_i] \in \mathrm{span}\{a_i\} \iff A \in \mathfrak{u}(n)$. □

3.1.2 Polyform Representations of Spin(n, n)

Clifford Algebra, Spin Lie Algebra, & Inner product

In a similar vein to the previous section, we study $\mathrm{Cliff}_{n,n}$. We begin by taking a vector space with an inner product of real dimension $2n$, $V \cong \mathbb{R}^{2n}$. Previously, a complex structure was imposed that broke the V into eigenspaces of complex dimension n. However, here we impose a paracomplex structure I, a linear map that squares to $+\mathbb{I}$, and splits V into maximally isotropic subspaces $E^\pm \cong \mathbb{R}^n$. This geometry will be defined more concretely in the next section. For now, we just assume that $V = E^+ \oplus E^-$ exists. The inner product that is compatible with I means that for any vectors $v^+ \in E^+$ and $v^- \in E^-$, $(v^+, v^-) = 0$. Let $\{e^1, \ldots, e^n\}$ be an orthonormal basis in E^+, and let $\Lambda^k(E^+)$ be the space of k–forms, the anti-symmetrisation of $\bigotimes^k E^+$, and $\Lambda(E^+) = \oplus_{k=1}^n \Lambda^k(E^+)$ the space of polyforms. The linear operators $b_i, b_i^\dagger : \Lambda(E^+) \to \Lambda(E^+)$ are defined as

$$b_i \psi = e^i \lrcorner \psi, \text{and } b_i^\dagger \psi = e^i \wedge \psi \tag{3.1.30}$$

The above creation and annihilation operators satisfy the commutation relations

$$b_i a_j^\dagger + b_j^\dagger b_i = \delta_{ij}, \ b_i b_j + b_j b_i = 0, \text{ and } b_i^\dagger b_j^\dagger + b_j^\dagger b_i^\dagger = 0, \ \forall \ 1 \le i, j \le n. \tag{3.1.31}$$

The generators, gamma matrices, for the Clifford algebra $\mathrm{Cliff}_{n,n} \cong \mathrm{Cliff}(E^+)$ is given as

$$\Gamma_i = b_i + b_i^\dagger, \text{ and } \Gamma_{i+n} = b_i - b_i^\dagger \tag{3.1.32}$$

It is easy to check that gamma matrices satisfy

$$\Gamma_A\Gamma_B + \Gamma_B\Gamma_A = \eta_{AB} \text{ for } 1 \le A, B \le 2n. \tag{3.1.33}$$

Here η is a diagonal metric of signature (n, n). Again there is a choice being made here, these models of the Clifford generators aren't unique. One can develop the same Clifford algebra relation by taking appropriate complex linear combinations of b and $b^\dagger$. This is explored more concretely, like the Cliff_{2n} case, in the discussion of pure spinors.

The spin Lie algebra $\mathfrak{spin}(n, n)$ is generated by $\Gamma_{[A}\Gamma_{B]}$ for any $\Gamma_A, \Gamma_B \in \mathrm{Cliff}_{n,n}$. Similar to before, an alternative description used is

$$\mathfrak{spin}(n, n) \ni \mathcal{A}(X) = \frac{1}{4}X^{AB}\sum_{A<B}\Gamma_A\Gamma_B. \tag{3.1.34}$$

Where X is a real antisymmetric and trace-free (2,0) tensor. $\mathcal{A}$ is a Lie algebra homomorphism between $\mathfrak{spin}(n, n)$ and $\mathfrak{gl}_{2n}$, and even more so is a Lie algebra isomorphism between $\mathfrak{spin}(n, n)$ and $\mathfrak{so}(n, n)$. The space $S = \Lambda(E^+)$ is reducible to $S^+ = \Lambda^+(E^+)$ and $S^- = \Lambda^-(E^+)$, the spaces of Weyl spinors.

The $\mathfrak{spin}(n, n)$ inner product is still given by definition (3.1.1.3). This is because the creation and annihilation operators are still self-adjoint.

Finally, as before, there are two antilinear operators $\mathfrak{R}, \mathfrak{R}' : S \to S$ that either square to $\pm\mathbb{I}$, and commute or anticommute with $\mathrm{Cliff}_{n,n}$. Since S is real, the notion of imaginary gamma matrices doesn't exist any more, hence one has

$$\mathfrak{R} = \Gamma_1 \dots \Gamma_{2n}\mathfrak{C}, \text{ and } \mathfrak{R}' = \mathfrak{C}. \tag{3.1.35}$$

Again, $\mathfrak{C}$ is the complex conjugation map on $\mathbb{C}$.

Proposition 3.1.2.1.

$$\mathfrak{R}^2 = (-1)^{n^2}\mathbb{I}, \qquad (\mathfrak{R}')^2 = \mathbb{I} \tag{3.1.36}$$

Proof.

$$\begin{aligned}\mathfrak{R}^2 &= \Gamma_1 \dots \Gamma_{2n}\mathfrak{C}\Gamma_1 \dots \Gamma_{2n}\mathfrak{C}\\ &= \Gamma_1 \dots \Gamma_{2n}\Gamma_1 \dots \Gamma_{2n}\\ &= (-1)^{\frac{n(n-1)}{2}}(-1)^{\frac{n(n-1)}{2}}(-1)^n\mathbb{I}\\ &= (-1)^{n^2}\mathbb{I},\end{aligned} \qquad (\mathfrak{R}')^2 = \mathfrak{C}^2 = \mathbb{I}. \tag{3.1.37}$$

□

Proposition 3.1.2.2.

$$\mathfrak{R}\Gamma_C = -\Gamma_C\mathfrak{R}, \qquad \mathfrak{R}'\Gamma_C = \Gamma_C\mathfrak{R}' \quad \forall\ C \in \{1, \dots 2n\} \tag{3.1.38}$$

Proof. The commutation relations for $\mathfrak{R}'$ are obvious. Suppose that $C \in \{1, \dots n\}$ then $\mathfrak{R}\Gamma_C = (-1)^{2n-1}\Gamma_C\mathfrak{R}$ and suppose $C \in \{n+1, \dots 2n\}$ then $\mathfrak{R}\Gamma_C = (-1)^{2n+1}\Gamma_C\mathfrak{R}$. Since $2n \pm 1$ is always odd, the result follows. □

Theorem 3.1.2.1. $\mathfrak{R}$ and $\mathfrak{R}'$, up to some phase and some ordering, are the only reality condition.

Proof. Lemma 12.75, and lemma 12.90 in [Har90]. □

A Preferred $\mathfrak{gl}(n)$ Subalgebra

In the construction of $\text{Cliff}_{n,n}$ there was a paracomplex structure introduced I that split the $2n$ real dimension vector space as $V = E^+ \oplus E^-$. The symmetry for the choice of I must be preserved. That is $\text{SO}(n,n)$ contains GL_n subgroup that preserves I. Like in the Cliff_{2n} case, we make this clear by examining the $\mathfrak{spin}(n,n)$–module S. There is a preferred $\mathfrak{gl}(n)$ subalgebra in $\mathfrak{spin}(2n)$ whose action on $\psi \in S$ fixes the degrees of any decomposable forms that comprise it. Explicitly the general $\mathfrak{spin}(n,n)$ Lie algebra element can be written as

$$\begin{aligned} A_{\mathfrak{spin}(n,n)} &= \sum_{i,j=1}^{n} \frac{1}{2} X^{ij}\Gamma_{[i}\Gamma_{j]} - \frac{1}{2}\Xi^{ij}\Gamma_{[i+n}\Gamma_{j+n]} + Y^{ij}\Gamma_{[i}\Gamma_{j+n]} \\ &= \frac{1}{2}X^{ij}(b_i + b_i^\dagger)(b_j + b_j^\dagger) - \frac{1}{2}\Xi^{ij}(b_i - b_i^\dagger)(b_j - b_j^\dagger) + Y^{ij}(b_i + b_i^\dagger)(b_j - b_j^\dagger). \end{aligned} \tag{3.1.39}$$

It is clear that X^{ij} and Ξ^{ij} are antisymmetric, while Y^{ij} has no symmetry. $A_{\mathfrak{gl}(n)}$ is then created by removing $b_i b_j$ and $b_i^\dagger b_j^\dagger$ terms in $A_{\mathfrak{spin}(n,n)}$. This results in the conditions

$$X^{ij} = -\Xi^{ij}, \text{ and } Y^{ij} = Y^{ji}. \tag{3.1.40}$$

Giving

$$A_{\mathfrak{gl}(n)} = 2(X^{ij} - Y^{ij})b_i b_j^\dagger + 2Y^{ij}\delta_{ij}\mathbb{I}. \tag{3.1.41}$$

Where X^{ij} is still antisymmetric but now Y^{ij} is symmetric.

Proposition 3.1.2.3. The vector representation of $\mathfrak{gl}(n) \subset \mathfrak{spin}(n,n)$ on $\mathbb{R}^{2n}$ is compatible with the paracomplex structure I.

Proof. The proof is the same structure as (3.1.1.4). Again, we consider vector representation equivalent to the action of $\mathfrak{spin}(n,n)$ on $\text{Cliff}_{n,n}$ via the commutator. Since I splits $\text{Cliff}_{n,n} = \text{span}\{b_i \mid 1 \le i \le n\} \oplus \text{span}\{b_i^\dagger \mid 1 \le i \le n\}$, and the same commutation relations hold for $b \leftrightarrow a$ and $b^\dagger \leftrightarrow a^\dagger$. We see for any $y \in \text{span}\{b_i\}$ and $A \in \mathfrak{spin}(n,n)$, $[A, y^i b_i] \in \text{span}(b_i) \iff A \in \mathfrak{gl}(n)$. □

3.1.3 Polyform Representation of $\text{Spin}(r,s)$

The Maximal Index Model of $\text{Cliff}_{r,s}$

For applications we care about, there is then no distinction between $\text{Spin}(r,s)$ and $\text{Spin}(s,r)$. Therefore, without loss of generality, we can assume $r \ge s$. Furthermore, we are interested in the cases $r + s \in 2\mathbb{Z}$, this is to guarantee that one has access to the Weyl representations, $S^\pm$. We can then write

$$\mathbb{R}^{r,s} = \mathbb{R}^{2n,0} \oplus \mathbb{R}^{s,s}, \text{ such that } n := (r-s)/2. \tag{3.1.42}$$

There is of course some choice in splitting $\mathbb{R}^{r,s}$ in this way, and we discuss the geometry involved in this choice later.

We can now consider a mix of the creation and annihilation constructions on $\Lambda(\mathbb{C}^n)$ and $\Lambda(\mathbb{R}^s)$. We introduce creation and annihilation operators $a_i, a_i^\dagger, i = 1,\dots,n$ as those acting on $\Lambda(\mathbb{C}^n)$. We introduce creation and annihilation operators $b_{\mathsf{I}}, b_{\mathsf{I}}^\dagger, \mathsf{I} = 1,\dots,s$ as those acting on $\Lambda(\mathbb{R}^s)$. The Clifford generators then arise as operators on $\Lambda(\mathbb{C}^n \oplus \mathbb{R}^s)$

$$\begin{aligned} \Gamma_i := a_i + a_i^\dagger,\ \Gamma_{i+n} := \mathbf{i}(a_i - a_i^\dagger),\ \text{for } i = 1,\dots,n, \text{ and,} \\ \Gamma_{\mathsf{I}+2n} := b_{\mathsf{I}} + b_{\mathsf{I}}^\dagger,\ \Gamma_{\mathsf{I}+2n+s} := b_{\mathsf{I}} - b_{\mathsf{I}}^\dagger,\ \text{for } \mathsf{I} = 1,\dots,s. \end{aligned} \tag{3.1.43}$$

These Gamma matrices satisfy the Clifford algebra relations

$$\Gamma_A\Gamma_B + \Gamma_B\Gamma_A = 2g_{AB}\mathbf{1}, \text{ for } 1 \le A, B \le 2n, \text{ where } g = \operatorname{diag}(\underbrace{+1,\dots,+1}_{2n+s \text{ times}}, \underbrace{-1,\dots,-1}_{s \text{ times}}). \tag{3.1.44}$$

The Lie algebra is again generated by all products of pairs of distinct Γ-matrices. Lie algebra acts on spinors, which are elements of the space of all polyforms $S = \Lambda(\mathbb{C}^n \oplus \mathbb{R}^s)$. This splits into the subspaces of even and odd polyforms $S = S_+ \oplus S_-$. The inner product (3.1.1.3) is still an invariant inner product on S.

Reality conditions, Majorana spinors

As in the case of Cliff_{2n}, we now show there are only two antilinear operators (up to a complex multiple) that either commute or anticommute with all Γ-matrices. These operators are obtained by taking the product of all real operators followed by the complex conjugation, or of all imaginary operators again followed by the complex conjugation. Thus, we define

$$\mathfrak{R} = \underbrace{\Gamma_1\dots\Gamma_n}_{n \text{ factors}} \overbrace{\dots\dots\dots\dots}^{n \text{ factors omitted}} \underbrace{\Gamma_{2n+1}\dots\Gamma_{2n+2s}}_{2s \text{ factors}}\mathfrak{C}, \text{ and } \mathfrak{R}' = \underbrace{\Gamma_{n+1}\dots\Gamma_{2n}}_{n \text{ factors}}\mathfrak{C}. \tag{3.1.45}$$

Here, as before, $\mathfrak{C}$ is the complex conjugation map on $\mathbb{C}$. The commutativity properties of these maps are summarised in the proposition

Proposition 3.1.3.1.

$$\mathfrak{R}\Gamma_A = (-1)^{n-1}\Gamma_A\mathfrak{R}, \text{ and } \mathfrak{R}'\Gamma_A = (-1)^n\Gamma_A\mathfrak{R}', \text{ where } A \in \{1,\dots,2(n+s)\}. \tag{3.1.46}$$

Here the formulae are s-independent, and the squares of these maps are captured by the following proposition,

Proposition 3.1.3.2.

$$\mathfrak{R}^2 = (-1)^{\frac{n(n-1)}{2}}\mathbf{1}, \text{ and } (\mathfrak{R}')^2 = (-1)^{\frac{n(n+1)}{2}}\mathbf{1}. \tag{3.1.47}$$

Again the formulae are s-independent. Thus, the existence of the reality conditions and Majorana spinors depends only on the number of complex directions in $\mathbb{C}^n \oplus \mathbb{R}^s$. There are no Majorana spinors when n is even but not a multiple of four. There are Majorana spinors when n is odd, and Majorana-Weyl spinors when $n \in 4\mathbb{Z}$.

Theorem 3.1.3.1. $\mathfrak{R}$ and $\mathfrak{R}'$ are the only reality conditions up to phase and ordering.

Proof. Lemma 12.75, and lemma 12.90 in [Har90]. □

A Preferred Subalgebra of $\mathfrak{spin}(r,s)$

We now look for a subalgebra that does not mix polyforms of different degrees. To this end, it is useful to write a general Lie algebra element in terms of the creation and annihilation operators a, b. We have

$$\begin{aligned} A_{\mathfrak{spin}(r,s)} = {}& \frac{1}{2}X^{ij}(a_i + a_i^\dagger)(a_j + a_j^\dagger) - \frac{1}{2}\tilde{X}^{ij}(a_i - a_i^\dagger)(a_j - a_j^\dagger) + \mathbf{i}Y^{ij}(a_i + a_i^\dagger)(a_j - a_j^\dagger) \\ & + \frac{1}{2}X^{\mathsf{IJ}}(b_\mathsf{I} + b_\mathsf{I}^\dagger)(b_\mathsf{J} + b_\mathsf{J}^\dagger) + \frac{1}{2}\tilde{X}^{\mathsf{IJ}}(b_\mathsf{I} - b_\mathsf{I}^\dagger)(b_\mathsf{J} - b_\mathsf{J}^\dagger) + Y^{\mathsf{IJ}}(b_\mathsf{I} + b_\mathsf{I}^\dagger)(b_\mathsf{J} - b_\mathsf{J}^\dagger) \\ & + Z_{++}^{i\mathsf{I}}(a_i + a_i^\dagger)(b_\mathsf{I} + b_\mathsf{I}^\dagger) + Z_{+-}^{i\mathsf{I}}(a_i + a_i^\dagger)(b_\mathsf{I} - b_\mathsf{I}^\dagger) \\ & + \mathbf{i}Z_{-+}^{i\mathsf{I}}(a_i - a_i^\dagger)(b_\mathsf{I} + b_\mathsf{I}^\dagger) + \mathbf{i}Z_{--}^{i\mathsf{I}}(a_i - a_i^\dagger)(b_\mathsf{I} - b_\mathsf{I}^\dagger). \end{aligned} \tag{3.1.48}$$

Here all parameters $X, \tilde{X}, Y$, and Z are real. The conditions so that the operator above contains both a creation and annihilation operator are

$$X^{ij} = \tilde{X}^{ij},\ Y^{[ij]} = 0,\ X^{\mathsf{IJ}} + \tilde{X}^{\mathsf{IJ}} = 0,\ Y^{[\mathsf{IJ}]} = 0,\ Z^{i\mathsf{J}}_{++} = Z^{i\mathsf{J}}_{+-} = Z^{i\mathsf{J}}_{-+} = Z^{i\mathsf{J}}_{--} = 0. \tag{3.1.49}$$

Clearly this selects $\mathfrak{u}(n) \oplus \mathfrak{gl}(s)$.

3.2 Pure Spinors

It does not take long after introducing the models for Cliff_{2n}, $\mathrm{Cliff}_{n,n}$, and $\mathrm{Cliff}_{r,s}$ in the previous sections to start questioning: why are there certain factors of $\mathbf{i}$ in front of some generators, while others lack these factors? Indeed, in the case of $\mathrm{Spin}(n,n)$, we could have developed a model that preserved the correct split signature but resulted in non-real matrices. The aim of this section is to uncover the hidden choices behind the construction of the Γ-matrices for various signatures of Clifford algebras. These choices stem from pure spinors, which lay the foundation for this entire thesis.

The canonical differential forms, constructed from geometric maps using these spinors, give rise to the spinorial G-structures studied in later parts of the thesis. Furthermore, pure spinors form a basis in the space of Weyl spinors, thus understanding the stabilisers of pure spinors provides insights into the stabilisers of impure spinors—any Weyl spinor that is not pure. This understanding enables classifications of spinor orbits over real and complex numbers.

In this section, we review pure spinors, the geometric maps that construct geometric structures — canonical differential forms with an index raised. We also discuss the real index, which outlines the different types of spinorial G-structures emerging from pure spinors, and present some crucial facts exploited in the remainder of the thesis to comprehend the orbits of Weyl spinors. Additionally, we introduce a new geometric structure called a mixed structure—a structure that is paracomplex on a split signature portion of $\mathrm{Cliff}_{r,s}$ and a complex structure on the compact part of $\mathrm{Cliff}_{r,s}$.

3.2.1 Maximally Totally Null Spaces & Pure spinors

Definition 3.2.1.1. Let V be a vector space with metric η of signature (r,s) and W its complexification. Then

$$W \supset M(\varphi) = \{v \in V \mid \Gamma(v)(\varphi) = 0\} \tag{3.2.1}$$

is called the *null space* of φ. Here Γ is the homomorphism that takes a vector in V to its Clifford generator in $\mathrm{Cliff}(W)$.

So fixing a spinor allows one to pick vector subspaces in W that correspond to it through annihilation via the Clifford action. Furthermore, if $r + s = 2n$, which is the case we will always discuss, then one can show $\dim(M(\varphi)) \leq n$. This motivates the following definition:

Definition 3.2.1.2. $M(\varphi)$ is *maximally totally null (MTN)* or maximally isotropic whenever $\dim(M(\varphi)) = n$. The corresponding spinor φ is then said to be *pure or simple.*

It has been known since Cartan [CB68] that pure spinors are Weyl spinors. Cartan also gives a useful algebraic characterisation of pure spinors. It is convenient to define

Definition 3.2.1.3 (geometric maps). Let φ, ϕ be Weyl spinors of $\mathrm{Spin}(r,s)$ such that $r+s \in 2\mathbb{Z}$, and the same chirality. Then a *geometric map* is defined as

$$\Lambda^k(V) \ni B_k(\varphi,\phi) \coloneqq \langle \varphi, \Gamma_{i_1} \dots \Gamma_{i_k} \phi \rangle \text{ for } 1 \leq i_1 < \dots < i_k \leq n. \tag{3.2.2}$$

As Γ-matrices anticommute with one another, the image of a geometric map is a differential form. In fact, let $\varphi = \phi$ be Weyl spinors, then the image of this geometric map is called a canonical differential k-form, which is invariant under the stabiliser of φ, $G \subset \text{Spin}(r,s)$. In general, the group that stabilises the canonical differential form is *smaller* than G. In the cases we care about, chapters 8 and 9, the stabiliser of the spinor and canonical differential form coincide. As we are concerned with writing theories of gravity, we emphasise the stabiliser of the canonical differential form.

Finally, we give conditions for when a spinor is pure using geometric maps. This was shown by Cartan [CB68] in the following theorem.

Theorem 3.2.1.1. Suppose ϕ is a Weyl spinor of $\text{Spin}(r,s)$ such that $r+s \in 2\mathbb{Z}$, and consider the geometric maps $B_k(\phi,\phi)$. Then ϕ is a pure spinor if, and only if, for any $k < \dim(M(\phi))$, $B_k(\phi,\phi) = 0$. Moreover, if $k = \dim(M(\phi))$

$$B_k(\phi,\phi) \propto f_1 \wedge \ldots \wedge f_k. \tag{3.2.3}$$

Here $\{f_1,\ldots,f_k\}$ is a basis that spans $M(\phi)$.

In sections to come, we freely raise and lower the indices of differential forms coming from geometric maps and define these endomorphisms to be geometric structures. When $r = 2n, s = 0$, these will correspond to complex structures. When $r = s$, these will correspond to paracomplex structures. Finally, when $r \neq s \neq 0$ these correspond to a new type of structure called a *mixed structure*.

3.2.2 The Real Index

Definition 3.2.2.1. If V is a real $2n$ dimensional vector space then W, its complexification, is a complex $2n$ vector space. Hence, the maximum dimension of null isotropic space is n. Given an MTN subspace $N \subset W$, the *real index* r is the dimension of $N \cap V$, i.e. the dimension of the space of real vectors in N.

Using [KT92] one has the following formula to characterise $r = |N \cap V|$,

$$r \in \begin{cases} \{p \in 2\mathbb{Z} \mid 0 < p < n\} & \text{for } n \in 2\mathbb{Z}, \text{ or} \\ \{p \in 2\mathbb{Z}+1 \mid 0 < p < n\} & \text{for } n \in 2\mathbb{Z}+1. \end{cases} \tag{3.2.4}$$

Hence it is clear that for $\mathbb{R}^{r,s}$ with $r \geq s$ the real index can be as large as s. At the same time, in the case $(2\rho, 2\sigma)$ the real index of MTN subspace can be as small as zero, while in the case $(2\rho+1, 2\sigma+1)$ the minimal value of the real index is one. One has the following theorem [KT92].

Theorem 3.2.2.1. The group $\text{SO}(r,s)$ acts transitively on each set of all MTN subspaces of W with a given real index and chirality.

Helicity arises because pure spinors are Weyl, recall there is a one-to-one correspondence between the basis that constitutes the subspace of the Clifford algebra that annihilates pure spinors and MTN subspaces of W. We can now connect the real index to a pure spinor via the following theorems from [KT92], and [BT89].

Theorem 3.2.2.2. Let ϕ, ψ be even pure spinors. Then

- $|M(\phi) \cap M(\psi)| = p := n - 2m$, for some $m \in \mathbb{N}$.
- $|M(\phi) \cap M(\psi)| = p \iff B_k(\phi,\psi) = 0 \;\; \forall\; k \in \mathbb{N}_{<p}$, and $B_p(\phi,\psi) \propto e_1 \wedge e_2 \wedge \ldots \wedge e_p$. Here $\{e_1,\ldots,e_p\}$ is a basis for $M(\phi) \cap M(\psi)$.
- $|M(\phi) \cap M(\psi)| = 0 \iff B_0(\phi,\psi) \neq 0$.

An analogous statement can be made about odd spinors too

Theorem 3.2.2.3. Let ϕ, ψ be odd pure spinors. Then

- $|M(\phi) \cap M(\psi)| = p := n - (2m+1)$, for some $m \in \mathbb{N}$.
- $|M(\phi) \cap M(\psi)| = p \iff B_k(\phi, \psi) = 0 \;\; \forall \; k \in \mathbb{N}_{<p}$, and $B_p(\phi, \psi) \propto e_1 \wedge e_2 \wedge \ldots \wedge e_p$. Here $\{e_1, \ldots, e_p\}$ is a basis for $M(\phi) \cap M(\psi)$.
- $|M(\phi) \cap M(\psi)| = 0 \iff B_0(\phi, \psi) \neq 0$.

Finally, an important theorem due to Cartan [CB68].

Theorem 3.2.2.4. Let ϕ, ψ be two non-parallel pure spinors. If $|M(\phi) \cap M(\psi)| = n - 2$ then $\phi + \psi$ is a pure spinor[2].

The next section now explores how pure spinors generate the models, indexed by r, through the geometric map.

3.2.3 Pure Spinors and Complex Structures in Spin$(2n)$

Obviously, Cliff$_{2n}$ must always have $r = 0$. This is because if there did exist an MTN subspace $N \subset V$, then the metric would no longer be definite, as the real coordinates in $N \cap V$ introduce negative definite signatures. Therefore, the chosen model is the only one for a positive definite signature.

Consider span$\{\Gamma(e^i) + \mathbf{i}\Gamma(e^{i+n}) \mid 1 \leq i \leq n\}$ and span$\{\Gamma(e^i) - \mathbf{i}\Gamma(e^{i+n}) \mid 1 \leq i \leq n\}$, i.e., span$\{a_i \mid 1 \leq i \leq n\}$ and span$\{a_i^\dagger \mid 1 \leq i \leq n\}$, respectively. Thus, one can deduce the pure spinors for this model that characterise it. They are $e_1 \wedge \ldots \wedge e_n$, the volume form, and $\mathbf{1}$, the zero form (or identity form).

The subalgebra of $\mathfrak{spin}(2n)$ that stabilises these spinors is clearly $\mathfrak{su}(n)$. Take the $\mathfrak{u}(n)$ subalgebra (3.1.28) and remove the trace components.

The complex structure can be derived from the following geometric map for a Weyl spinor ψ.

$$M_{AB} := \langle \hat{\psi}, \Gamma_A \Gamma_B \psi \rangle \text{ for } 1 \leq A < B \leq 2n. \tag{3.2.5}$$

Where we recall that $\mathbf{R}(\psi) = \hat{\psi}$ for $\psi \in S$. We see that the insertion in the inner product above to generate M is given by inserting a general Lie algebra element $X \in \mathfrak{spin}(2n)$. If ψ is the volume form or the identity polyform i.e. a pure spinor, then M is the symplectic form that stabilises $\mathfrak{su}(n)$. Raising an index, using δ_{ij} on $\mathbb{R}^{2n}$, gives

$$M_A{}^C M_C{}^B \propto \delta_A{}^B. \tag{3.2.6}$$

Thus rescaling the geometric map appropriately gives a complex structure J that is metric compatible with our hermitian metric g on W

$$g(J\cdot, J\cdot) = g(\cdot, \cdot) \tag{3.2.7}$$

and gives the splitting of Cliff$_{2n}$, i.e. the model that was chosen.

3.2.4 Pure Spinors and Paracomplex Structures in Spin(n, n)

We can now describe the geometry involved in the choice of a pair of maximal totally isotropic subspaces $\mathbb{R}^n$ of $\mathbb{R}^{n,n}$ and thus the described model of Cliff$_{n,n}$. The novelty as compared to the case of Cliff$_{2n}$ is that choosing such a maximal isotropic subspace does not uniquely define its complement in $\mathbb{R}^{n,n}$.

[2]This is an important theorem for chapter 6 in the classification complexified stabilisers of impure spinors in higher dimensions.

As in the case of Cliff_{2n}, the choice of a model can be encoded into a geometric map. In the case of Cliff_{2n}, the geometric map produced a complex structure on $\mathbb{R}^{2n}$ that provided the split of the complexification $\mathbb{R}^{2n} \otimes \mathbb{C}$ into two maximally isotropic subspaces, $\mathbb{C}^n$ and $\bar{\mathbb{C}}^n$. In the split signature case $\mathbb{R}^{n,n}$, the analogue of this is a choice of a paracomplex structure $I \in \mathrm{End}(\mathbb{R}^{n,n})$. This is an operator that squares to plus the identity, $I^2 = +\mathbb{I}$, so that its eigenspaces of eigenvalue ± 1 are real. This operator is also compatible with the split signature metric, but the compatibility condition now involves a sign:

$$\eta(I\cdot, I\cdot) = -\eta(\cdot,\cdot). \tag{3.2.8}$$

As a consequence of this extra minus sign, the eigenspaces of I are totally null. Indeed, if $u, v \in E^+$, where $E^\pm := \{v \in \mathbb{R}^{n,n} : Iv = \pm v\}$, then $\eta(u,v) = -\eta(Iu, Iv) = -\eta(u,v)$, and so E^+ is totally isotropic, of dimension n, and thus maximally totally isotropic. The same holds for E^-. Thus, choosing a metric-compatible paracomplex structure I provides a decomposition $\mathbb{R}^{n,n} = E^+ \oplus E^-$ into two maximal isotropic subspaces.

At this level, the story is analogous to that for $\mathbb{R}^{2n}$ and Cliff_{2n}. The novelty arises because in $\mathbb{R}^{n,n}$, a choice of a maximal totally isotropic subspace E^+ does not define E^-. Thus, a choice of only E^+ is not equivalent to a choice of a paracomplex structure I. The latter carries more information than the former. And it is only E^+ that is in correspondence with pure spinors, as we now discuss.

Similarly to the case of Cliff_{2n}, the described creation and annihilation operator model of $\mathrm{Cliff}_{n,n}$ comes with preferred pure spinors given by $e_1 \wedge \ldots \wedge e_n$ and $\mathbf{1}$, annihilated by $\mathrm{span}\{\Gamma(e^i) + \Gamma(e^{i+n}) \mid 1 \le i \le n\}$ and $\mathrm{span}\{\Gamma(e^i) - \Gamma(e^{i+n}) \mid 1 \le i \le n\}$, respectively.

In the case of Cliff_{2n}, the stabiliser of $e_1 \wedge \ldots \wedge e_n$ is $\mathfrak{su}(n)$. It is clear that the analogous subgroup in the case of $\mathrm{Cliff}_{n,n}$ is $\mathfrak{sl}(n)$, and indeed it is easy to see that $e_1 \wedge \ldots \wedge e_n$ is stabilised by $\mathfrak{sl}(n)$ as in (3.1.41) with $Y^{ij}\delta_{ij} = 0$. The difference with the Cliff_{2n} case is that the stabiliser of $e_1 \wedge \ldots \wedge e_n$ is larger than $\mathfrak{sl}(n)$.

Indeed, it is clear that $e_1 \wedge \ldots \wedge e_n$ is also killed by all transformations (3.1.39) with

$$Y^{ij} = \frac{1}{2}(X^{ij} + \Xi^{ij}), \tag{3.2.9}$$

as these transformations involve the product of two copies of creation operators, and thus kill the pure spinor $e_1 \wedge \ldots \wedge e_n$. Thus, the stabiliser algebra of the pure spinor is the sum $\mathfrak{sl}(n) \oplus N$, where N is a nilpotent subalgebra of dimension $n(n-1)/2$. A similar demonstration can be given for $\mathbf{1}$. It is also stabilised by an $\mathfrak{sl}(n)$ subalgebra as in (3.1.41) with $Y^{ij}\delta_{ij} = 0$, however the nilpotent part is now generated by

$$Y^{ij} = -\frac{1}{2}(X^{ij} + \Xi^{ij}). \tag{3.2.10}$$

Hence, the stabiliser of $\mathbf{1}$ is still isomorphic to $\mathfrak{sl}(n) \oplus N$ because Y is only changed up to a sign.

The novelty in the previous section was subtle, but the map $\mathbf{R}$ takes a pure spinor and gives a pure spinor of the other type, up to a factor of $\mathbf{i}$. For instance, if $\psi = \mathbf{1}$, then taking $\mathbf{R} = \mathfrak{R}$, $\hat{\psi} = e_1 \wedge \ldots \wedge e_n$. What this shows is that to characterise the model for Cliff_{2n}, only one choice of pure spinor is necessary.

The same cannot be said in the case of $\mathrm{Cliff}_{n,n}$. Since all spinors are real here, the notion of $\hat{\psi}$ is redundant. The naive assumption that

$$M_{AB} := \langle \psi, \Gamma_A \Gamma_B \psi \rangle \quad \text{for } 1 \le A < B \le 2n \tag{3.2.11}$$

characterises the model is no longer true. Since $\mathbf{R}$ doesn't map between the zero form and the volume form, and vice versa, we are forced to construct a geometric map involving both of them. Furthermore, since $\mathbf{1} \in E^+$ and $e_1 \wedge \ldots \wedge e_n \in E^-$, we expect our geometric map to be a pairing

$\langle E^+, E^- \rangle$. So, without loss of generality, take any non-zero $\psi_1 \in E^+$ and $\psi_2 \in E^-$. Then one has

$$M_{AB} \coloneqq \langle \psi_1, \Gamma_A \Gamma_B \psi_2 \rangle \quad \text{for } 1 \leq A < B \leq 2n. \tag{3.2.12}$$

Raising one of the indices using η_{ij} on $\mathbb{R}^{n,n}$ gives an operator $M_A{}^B$ whose square is a multiple of the identity. Hence, by rescaling, this gives an appropriate paracomplex structure I.

3.2.5 Mixed structures

A complex structure in $\mathbb{R}^{2n}$ provides a decomposition $\mathbb{R}^{2n} \otimes \mathbb{C} = E^+ \oplus E^-$, where both $E^\pm$ are totally null and arise as the eigenspaces of the complex structure endomorphism. Similarly, a paracomplex structure on $\mathbb{R}^{n,n}$ gives a decomposition $\mathbb{R}^{n,n} = E^+ \oplus E^-$, with $E^\pm$ again totally null, but this time real.

The models of $\mathrm{Cliff}_{r,s}$ we described rely on a structure that is an appropriate mix of complex and paracomplex structures. The purpose of this subsection is to describe such more general structures in geometric terms, with examples provided in the following sections.

We begin with a heuristic of what we would like in our approach to combining complex and paracomplex structures. This leads to an axiomatisation once the principles have been fleshed out.

We begin by imposing that a structure we do create gives an $g-$orthogonal decomposition

$$\mathbb{R}^{r,s} = \mathbb{R}^{2k,2l} \oplus \mathbb{R}^{m,m}. \tag{3.2.13}$$

Second, the structure must select a pair of complementary MTN $E^\pm$ in both $\mathbb{R}^{2k,2l}$ and $\mathbb{R}^{m,m}$. In other words, after a decomposition (3.2.13) is chosen, the structure must select a complex structure J such that $J^2 = -\mathbb{I}$ in $\mathbb{R}^{2k,2l}$, and a paracomplex structure I such that $I^2 = \mathbb{I}$ in $\mathbb{R}^{m,m}$.

Now, working backwards we would like to extend I, J to act on the whole of $\mathbb{R}^{r,s}$, with

$$\mathbb{R}^{2k,2l} = \mathrm{Ker}(I), \text{ and } \mathbb{R}^{m,m} = \mathrm{Ker}(J). \tag{3.2.14}$$

Since both I, J act by projecting on their respective factors, we have $IJ = JI = 0$. Taking the complex linear combination $K \coloneqq I + \mathbf{i}J$ means we have a linear map on $\mathbb{R}^{r,s} \otimes \mathbb{C}$ with the property that

$$K^2 = I^2 + \mathbf{i}(IJ + JI) - J^2 = \mathbb{I}_{\mathbb{R}^{m,m}} + \mathbb{I}_{\mathbb{R}^{2k,2l}} = \mathbb{I}. \tag{3.2.15}$$

That is K is a paracomplex structure on $\mathbb{R}^{r,s}$, apart from the fact that this map is complex-valued. Another property the constructed map has is

$$K\bar{K} = I^2 + J^2 - \mathbf{i}(IJ - JI) = \bar{K}K. \tag{3.2.16}$$

One can rephrase this as $K\bar{K} = \bar{K}K = P$ is real. This provides a reality condition on the map. Note that because $K^2 = \mathbb{I}$ we have also $P^2 = \mathbb{I}$. But now P is a real map and a paracomplex structure that defines the splitting, $\mathbb{R}^{r,s} = \mathbb{R}^{2k,2l} \oplus \mathbb{R}^{m,m}$ being the $P-$eigenspaces of eigenvalue $+1$ and -1 respectively.

Finally, we would like to describe metric compatibility between K and g on $\mathbb{R}^{r,s}$. We have

$$g(KX, KY) = g((I + \mathbf{i}J)X, (I + \mathbf{i}J)Y) = g(IX, IY) - g(JX, JY). \tag{3.2.17}$$

No mixed terms of the type $g(IX, JY)$ arise because both I, J project on to the metric orthogonal components of $\mathbb{R}^{r,s}$, $\mathbb{R}^{2k,2l}$ and $\mathbb{R}^{m,m}$ respectively. Hence, we have

$$g(IX, IY) = -g(X|_2, Y|_2), \text{ and } g(JX, JY) = g(X|_1, Y|_1). \tag{3.2.18}$$

Here $X|_2$ denotes the projection onto $\mathbb{R}^{m,m}$, and $X|_1$ denotes the projection onto $\mathbb{R}^{2k,2l}$. This means that we have

$$g(KX, KY) = -g(X|_2, Y|_2) - g(X|_1, Y|_1) = -g(X, Y). \tag{3.2.19}$$

Thus, the operator K is metric-compatible in the same sense that a paracomplex structure is.

The difference with the usual paracomplex structure is that K is complex-valued, but satisfies the reality condition (3.2.16). It is given by a complex linear combination of a paracomplex and complex structures. We will refer to it as a structure of a mixed type, or a mixed structure for short. It is clear that as constructed, what this operator does is define an orthogonal decomposition of $\mathbb{R}^{r,s}$ as kernels of its real and imaginary parts, as well as define a pair of complex/paracomplex structures on the two factors $\mathbb{R}^{2k,2l} \oplus \mathbb{R}^{m,m}$.

Having constructed an object with the desired properties, let us axiomatise it.

Definition 3.2.5.1. Let (V, g) be a real metric space, and let W be its complexification. Let $K : W \mapsto W$ be a linear map. If K satisfies

$$K^2 = \mathbb{I}, \ K\bar{K} = \bar{K}K, \text{ and } g(KX, KY) = g(\bar{K}X, \bar{K}Y) = -g(X, Y), \tag{3.2.20}$$

then K is called a *mixed structure* on V.

Proposition 3.2.5.1. The eigenvalue ± 1 eigenspaces $E^{\pm} \subset W$ of K are of the same complex dimension and are totally null.

Proof. Since $K^2 = \mathbb{I}$ and $g(KX, KY) = -g(X, Y)$, this shows that eigenspaces $E^{\pm}$ of K are null. The metric g is non-zero only when pairing $E^{\pm}$ with $E^{\mp}$, this guarantees that $E^{\pm}$ are of the same complex dimension. □

These are the same facts as for paracomplex structures, but in the complexified setting. To discuss a "real" version of K we introduce the product structure.

Definition 3.2.5.2. Let (V, g) be a real metric space. Let $P : V \mapsto V$ such that $P^2 = \mathbb{I}$, then P is called a *product structure* on V. Furthermore, if P also satisfies $g(PX, PY) = g(X, Y)$, then P is called an orthogonal product structure on V.

Proposition 3.2.5.2. Let P be an orthogonal product structure, of a real metric space (V, g). Let $V^{\pm}$ be the eigenspaces of P with eigenvalues ± 1, respectively, then $V = V^+ \oplus V^-$.

Proof. It is sufficient to show that V^+ is g–orthogonal to V^-. Indeed, let $X_{\pm} \in V^{\pm}$ then

$$g(X_+, X_-) = -g(PX_+, PX_-) = -g(X_+, X_-). \tag{3.2.21}$$

□

The namesake "product" makes sense now, meaning to factor the space in orthogonal components.

Proposition 3.2.5.3. The operator $P := K\bar{K} = \bar{K}K$, where K is a mixed structure, is an orthogonal product structure on V. That is $P^2 = \mathbb{I}$, and P satisfies $g(PX, PY) = g(X, Y)$.

Proof. Indeed, P is a real operator, and so $P \in \mathrm{End}(V)$. Hence, $P^2 = K\bar{K}K\bar{K} = K\bar{K}\bar{K}K = \mathbb{I}$. Now, using $g(KX, KY) = g(\bar{K}X, \bar{K}Y)$, and $K^2 = \mathbb{I}$,

$$g(KX, KY) = g(\bar{K}X, \bar{K}Y) = g(KK\bar{K}X, KK\bar{K}Y) = g(P(KX), P(KY)). \tag{3.2.22}$$

Then by defining $KX := \tilde{X}$ and $KY := \tilde{Y}$, shows P to be an orthogonal product structure. □

We now use $P = K\bar{K}$ to provide the orthogonal decomposition $V = V_K^- \oplus V_K^+$ into the eigenspaces of P.

Theorem 3.2.5.1. Let K be a mixed structure on V. Let $V = V_K^- \oplus V_K^+$ be the eigenspace decomposition with respect to the orthogonal product structure $P := K\bar{K}$. Then $K|_{V^+}$ is a paracomplex structure and $-\mathbf{i}K|_{V^-}$ is a complex structure.

Proof. To prove this, we introduce the following notations

$$K|_{V^+} := I, \qquad K|_{V^-} := \mathbf{i}J. \tag{3.2.23}$$

These notations are justified by the fact that both I, J are real operators. Indeed, applying K to a vector $X_+ \in V^+$ we have $KX_+ = KK\bar{K}X_+ = \bar{K}X_+$, and so K acts on V^+ as a real operator $I : V^+ \to V^+$. Similarly, acting on a vector $X_- \in V^-$ we have $KX_- = -KK\bar{K}X_- = -\bar{K}X_-$, which means that on V^- the operator K acts as an imaginary operator, or as $\mathbf{i}J$ with $J : V^- \to V^-$.

Some other useful properties can be proven. We have

$$K(V^+) \subset V^+, \qquad K(V^-) \subset V^-. \tag{3.2.24}$$

Indeed, taking $\tilde{X} = KX_+$ and applying P we have $P\tilde{X} = \bar{K}KKX_+ = \bar{K}X_+ = KX_+ = \tilde{X}$, and so $\tilde{X} \in V^+$. Similarly, for $\tilde{X} = KX_-$ we have $P\tilde{X} = \bar{K}KKX_- = \bar{K}X_- = -KX_- = -\tilde{X}$. These properties mean that the maps I, J are linear maps on $V^\pm$ respectively

$$I : V^+ \to V^+, \qquad J : V^- \to V^-. \tag{3.2.25}$$

It remains to show that

$$I^2 = \mathbb{I}_{V^+}, \qquad J^2 = -\mathbb{I}_{V^-}, \tag{3.2.26}$$

where $\mathbb{I}_{V^\pm}$ are the projectors on $V^\pm$ respectively. Indeed, we have $I^2 = (K|_{V^+})^2 = K^2_{V^+}$ because of (3.2.24). Therefore I^2 is the identity operator on V^+. Similarly, $(\mathbf{i}J)^2 = (K|_{V^-})^2 = K^2_{V^-}$, which again equals to the identity. Thus J^2 acts as minus the identity on V^-. □

3.2.6 Pure Spinors in the Maximal Index Case

Given the described model, we have two preferred spinors: the volume form, and the identity polyform in $\Lambda(\mathbb{C}^n \oplus \mathbb{R}^s)$. Their annihilators are $\mathbb{R}^{r,s}$ (complexified) acting by Clifford multiplication have dimension n, and so is maximal. Therefore, they are both pure spinors.

It is interesting to compute the stabiliser of the volume form since the calculation regarding the identity polyform is the same and yields an isomorphic stabiliser, as has been shown for $\mathrm{Cliff}_{n,n}$ and Cliff_{2n}. Its stabiliser subalgebra does not contain terms from (3.1.48) that are built from a pair of annihilation operators. Instead, it contains terms with a pair of creation operators or with a creation and annihilation operator. The last group of terms must be constrained to annihilate the pure spinor. The terms from the first line in (3.1.48) are those acting solely on $\Lambda(\mathbb{C}^n)$. The subset of these terms that annihilates the pure spinor $e_1 \wedge \ldots \wedge e_n$ is $\mathfrak{su}(n)$. The surviving terms in the second line of (3.1.48) generate $\mathfrak{sl}(s)$ plus $s(s-1)/2$ terms satisfying

$$Y^{\mathsf{IJ}} = \frac{1}{2}(X^{\mathsf{IJ}} + \tilde{X}^{\mathsf{IJ}}). \tag{3.2.27}$$

For the last line in (3.1.48), the conditions that there are no terms containing a pair of annihilation operators are

$$Z^{i\mathsf{I}}_{++} = Z^{i\mathsf{I}}_{+-}, \text{ and } Z^{i\mathsf{I}}_{-+} = Z^{i\mathsf{I}}_{--}. \tag{3.2.28}$$

There are thus $2ns$ real such terms. The stabiliser subalgebra is then $\mathfrak{su}(n) \oplus \mathfrak{sl}(s) \oplus A$, where A is the part of the stabiliser generated by $s(s-1)/2 + 2ns$ generators. The general element can be written as

$$A_{\text{stab}} = 2(X^{ij} - \mathbf{i}Y_s^{ij})a_i a_j^\dagger + (X^{\mathsf{IJ}} - \tilde{X}^{\mathsf{IJ}} - 2Y_s^{\mathsf{IJ}})b_{\mathsf{I}}b_{\mathsf{J}}^\dagger + (X^{\mathsf{IJ}} + \tilde{X}^{\mathsf{IJ}})b_{\mathsf{I}}b_{\mathsf{J}} \\ + (Z_+^{i\mathsf{J}}b_{\mathsf{J}} + \mathbf{i}Z_-^{i\mathsf{I}}b_{\mathsf{I}})a_i + (Z_+^{i\mathsf{J}}b_{\mathsf{J}} - \mathbf{i}Z_-^{i\mathsf{I}}b_{\mathsf{I}})a_i^\dagger, \tag{3.2.29}$$

where Y_s^{ij}, Y_s^{IJ} are the symmetric parts of Y's and must be trace-free $Y_s^{ij}\delta_{ij} = 0, Y_s^{IJ}\delta_{IJ} = 0$. The $Z_\pm^{iI}$ are $2ns$ real quantities.

3.2.7 Pure Spinors in the $r = 0$ Case

We now consider the models that become possible when MTN of the real index that is not maximal is chosen. To avoid overcomplicating the notation, we will only treat in the real index r, such that $r = 0$ and $r = 1$.

We start by considering the case $\mathrm{Cliff}_{2\rho,2\sigma}$. In this case, the minimal possible real index is zero. This means that we represent

$$\mathbb{R}^{2\rho,2\sigma} = \mathbb{R}^{2\rho} \oplus \mathbb{R}^{2\sigma}, \tag{3.2.30}$$

and then choose a complex structure in both summands. The corresponding $-\mathbf{i}$ eigenvalue eigenspace is $\mathbb{C}^\rho \oplus \mathbb{C}^\sigma$, and spinors become realised as polyforms in $\Lambda(\mathbb{C}^\rho \oplus \mathbb{C}^\sigma)$.

We again introduce two pairs of creation and annihilation operators $a_i, a_i^\dagger, i = 1, \ldots, \rho$ and $\tilde{a}_I, \tilde{a}_I^\dagger, I = 1, \ldots, \sigma$. We referred to the second set as $\tilde{a}$ rather than b to reserve the name b to operators that act on a number of copies of $\mathbb{R}$ rather than $\mathbb{C}$. The Γ-matrices now become

$$\Gamma_i = a_i + a_i^\dagger,\ \Gamma_{i+\rho} = \mathbf{i}(a_i - a_i^\dagger),\ \Gamma_{I+2\rho} = \mathbf{i}(\tilde{a}_I + \tilde{a}_I^\dagger),\ \text{and } \Gamma_{I+2\rho+\sigma} = \tilde{a}_I - \tilde{a}_I^\dagger. \tag{3.2.31}$$

Note that the placement of the imaginary unit is now opposite in the $\mathbb{C}^\sigma$ factor as compared to the $\mathbb{C}^\rho$ factor. This generates the correct Clifford algebra $\mathrm{Cliff}_{2\rho,2\sigma}$.

The discussion of the Lie algebra, inner product, and Weyl spinors is unchanged from the previous cases. The only novelty is in the available antilinear operators. Again, these arise as the product of either all real or all imaginary Γ-matrices followed by the complex conjugation. Their squares can be deduced using (3.1.1.3). Thus, we define

$$\mathfrak{R} = \underbrace{\Gamma_1 \ldots \Gamma_\rho}_{\rho \text{ times}} \underbrace{\Gamma_{1+2\rho+\sigma} \ldots \Gamma_{2\rho+2\sigma}}_{\sigma \text{ times}} \mathfrak{C},\ \text{and } \mathfrak{R}' = \underbrace{\Gamma_{1+\rho} \ldots \Gamma_{2\rho}}_{\rho \text{ times}} \underbrace{\Gamma_{1+2\rho} \ldots \Gamma_{2\rho+\sigma}}_{\sigma \text{ times}} \mathfrak{C}. \tag{3.2.32}$$

Again, $\mathfrak{C}$ is the complex conjugation map on $\mathbb{C}$. We then have

$$\mathfrak{R}^2 = (-1)^{\rho\sigma}(-1)^{\rho(\rho-1)/2}(-1)^{\sigma(\sigma+1)/2},\ \text{and } (\mathfrak{R}')^2 = (-1)^{\rho\sigma}(-1)^{\rho(\rho+1)/2}(-1)^{\sigma(\sigma-1)/2}. \tag{3.2.33}$$

This can be rewritten as

$$\mathfrak{R}^2 = (-1)^{(\rho-\sigma)(\rho-\sigma-1)/2},\ \text{and } (\mathfrak{R}')^2 = (-1)^{(\rho-\sigma)(\rho-\sigma+1)/2}. \tag{3.2.34}$$

This shows that their properties are controlled only by $\rho - \sigma$. So, the availability of Majorana and Majorana-Weyl spinors depends only on the signature, and not on the model used.

The pure spinor arising in this model is $e_1 \wedge \ldots \wedge e_\rho \wedge e_{1+\rho} \wedge \ldots e_{\sigma+\rho}$. The general Lie algebra element can be written as

$$\begin{aligned}A_{(2\rho,2\sigma)} = &\frac{1}{2}X^{ij}(a_i + a_i^\dagger)(a_j + a_j^\dagger) - \frac{1}{2}\tilde{X}^{ij}(a_i - a_i^\dagger)(a_j - a_j^\dagger) + \mathbf{i}Y^{ij}(a_i + a_i^\dagger)(a_j - a_j^\dagger)\\ &- \frac{1}{2}X^{IJ}(\tilde{a}_I + \tilde{a}_I^\dagger)(\tilde{a}_J + \tilde{a}_J^\dagger) + \frac{1}{2}\tilde{X}^{IJ}(\tilde{a}_I - \tilde{a}_I^\dagger)(\tilde{a}_J - \tilde{a}_J^\dagger) + \mathbf{i}Y^{IJ}(\tilde{a}_I + \tilde{a}_I^\dagger)(\tilde{a}_J - \tilde{a}_J^\dagger)\\ &+ \mathbf{i}Z^{iJ}_{++}(a_i + a_i^\dagger)(\tilde{a}_I + \tilde{a}_I^\dagger) + Z^{iI}_{+-}(a_i + a_i^\dagger)(\tilde{a}_I - \tilde{a}_I^\dagger)\\ &- Z^{iI}_{-+}(a_i - a_i^\dagger)(\tilde{a}_I + \tilde{a}_I^\dagger) + \mathbf{i}Z^{iI}_{--}(a_i - a_i^\dagger)(\tilde{a}_I - \tilde{a}_I^\dagger).\end{aligned} \tag{3.2.35}$$

The terms in the first two lines that kill the pure spinor form the subalgebra $\mathfrak{su}(\rho) \oplus \mathfrak{su}(\sigma)$. The terms in the last line that kill the pure spinor are those that do not have pairs of annihilation operators and thus must satisfy

$$Z^{iI}_{++} + Z^{iI}_{--} = 0, \text{ and } Z^{iI}_{+-} - Z^{iI}_{-+} = 0. \tag{3.2.36}$$

This gives $2\rho\sigma$ real generators.

3.2.8 Pure Spinors in the $r = 1$ Case

In the case $\text{Cliff}_{(2\rho+1,2\sigma+1)}$ the minimal value of the real index is one. This corresponds to the splitting

$$\mathbb{R}^{2\rho+1,2\sigma+1} = \mathbb{R}^{2\rho} \oplus \mathbb{R}^{2\sigma} \oplus \mathbb{R}^{1,1}. \tag{3.2.37}$$

The corresponding MTN subspace is obtained by choosing a complex structure in the first two summands, and a paracomplex structure in the last one. The MTN subspace is then $\mathbb{C}^\rho \oplus \mathbb{C}^\sigma \oplus \mathbb{R}$.

To generate the corresponding model for the Clifford algebra we proceed as in the previous subsection, but append one pair of real creation and annihilation operators $b, b^\dagger$. The Γ-matrices are then

$$\begin{aligned}&\Gamma_i = a_i + a_i^\dagger, && \Gamma_{I+2\rho} = \mathbf{i}(\tilde{a}_I + \tilde{a}_I^\dagger), && \Gamma_{1+2\rho+2\sigma} = b + b^\dagger,\\ &\Gamma_{i+\rho} = \mathbf{i}(a_i - a_i^\dagger), && \Gamma_{I+2\rho+\sigma} = \tilde{a}_I - \tilde{a}_I^\dagger, \text{ and} && \Gamma_{2+2\rho+2\sigma} = b - b^\dagger.\end{aligned} \tag{3.2.38}$$

By taking the products of distinct Γ-matrices, we obtain the Lie algebra. The inner product construction remains unchanged. When we take the products of all real and all imaginary Γ-matrices and subsequently apply complex conjugation, we generate the antilinear operators, from which reality conditions can be established. The subtleties involved become most apparent when considering specific examples; these will be explored in the next chapter. Additionally, we hope that the procedure for writing the stabiliser of a pure spinor, indexed by the real index for arbitrary signatures based on the three special cases we provide, is clear.

Chapter 4

Real Stabilisers of Spinors in Low Dimensions

This chapter explores examples of pure spinors and their corresponding geometric structures. We present a small subset of the comprehensive analysis found in [BK22], upon which this chapter is based. An interesting observation made here is that the geometry arising from Majorana-Weyl spinors is not particularly noteworthy. It is more fruitful to consider the complex Weyl spinor, which is the approach we shall take in studying orbits in higher dimensions. The first interesting examples occur in four dimensions and will be revisited in the latter half of the thesis when studying SU(2)-structures.

4.1 Spinors in 2 Dimensions

4.1.1 Spin(2)

This has a polyform representation over $\Lambda(\mathbb{C})$. Let us denote the complex coordinate on $\mathbb{C}$ by z, and the basis vector in $\Lambda^1(\mathbb{C})$ by $\mathrm{d}z$. We have a generic polyform of the form

$$\Psi = \alpha + \beta \mathrm{d}z \tag{4.1.1}$$

Here $\alpha, \beta \in \mathbb{C}$. The even α and odd $\beta \mathrm{d}z$ parts here are the Weyl spinors. Cliff_2 is generated by the Gamma matrices

$$\Gamma_1 = a + a^\dagger, \text{ and } \Gamma_2 = \mathbf{i}(a - a^\dagger). \tag{4.1.2}$$

Their action on Ψ is

$$\Gamma_1(\alpha + \beta \mathrm{d}z) = \alpha \mathrm{d}z + \beta, \text{ and } \Gamma_2(\alpha + \beta \mathrm{d}z) = \mathbf{i}\alpha \mathrm{d}z - \mathbf{i}\beta. \tag{4.1.3}$$

If we associate with Ψ a 2-component column

$$\Psi = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \tag{4.1.4}$$

the Γ-matrices take the following form

$$\Gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \Gamma_2 = \begin{pmatrix} 0 & -\mathbf{i} \\ \mathbf{i} & 0 \end{pmatrix}. \tag{4.1.5}$$

The Lie algebra is generated by the product $\Gamma_1\Gamma_2$

$$\mathfrak{spin}(2) = \left\{ \begin{pmatrix} \mathbf{i}s & 0 \\ 0 & -\mathbf{i}s \end{pmatrix} \middle| \; s \in \mathbb{R} \right\} \cong \mathfrak{u}(1) \tag{4.1.6}$$

The inner product (3.1.1.3) takes the following form

$$\begin{aligned} \langle \Psi_1, \Psi_2 \rangle &= (\alpha_1 + \beta_1 \mathrm{d}z) \wedge (\alpha_2 + \beta_2 \mathrm{d}z)\Big|_{\text{top}} \\ &= \alpha_1\beta_2 + \beta_1\alpha_2. \end{aligned} \tag{4.1.7}$$

This can be written in matrix form

$$\langle \Psi_1, \Psi_2 \rangle = \Psi_1^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Psi_2. \tag{4.1.8}$$

There are two antilinear operators

$$\mathfrak{R} = \Gamma_1 \mathfrak{C}, \qquad \mathfrak{R}' = \Gamma_2 \mathfrak{C}, \tag{4.1.9}$$

with $\mathfrak{R}^2 = \mathbb{I}, (\mathfrak{R}')^2 = -\mathbb{I}$. Thus, we can use $\mathfrak{R}$ to impose the Majorana reality condition. We have

$$\mathfrak{R} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \beta^* \\ \alpha^* \end{pmatrix}, \tag{4.1.10}$$

which means that Majorana spinors are of the form

$$\Psi_M = \begin{pmatrix} \alpha \\ \alpha^* \end{pmatrix}. \tag{4.1.11}$$

α^* and β^* are complex conjugates of α and β, respectively. The spinor $\mathrm{d}z \in S_-$ is our canonical pure spinor associated with this model. It has a trivial stabiliser. The other canonical pure spinor is $\mathbf{1} \in S_+$. The generic Weyl spinors are multiples of these.

Remark 4.1.1.1. It is also interesting to discuss the Dirac equation. We now promote α, β to functions $\alpha(z,\bar{z}), \beta(z,\bar{z})$, of the complex null coordinates $z, \bar{z}$ on $\mathbb{R}^2$. We take the usual relation

$$z = x_1 + \mathbf{i}x_2, \tag{4.1.12}$$

so that the complex structure acts $J(x_1) = x_2, J(x_2) = -x_1$, and $J(z) = -\mathbf{i}z$. We have

$$\frac{\partial}{\partial x_1} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}, \qquad \frac{\partial}{\partial x_2} = \mathbf{i}\frac{\partial}{\partial z} - \mathbf{i}\frac{\partial}{\partial \bar{z}}. \tag{4.1.13}$$

The Dirac operator is

$$D := \Gamma_1 \frac{\partial}{\partial x_1} + \Gamma_2 \frac{\partial}{\partial x_2} = 2(a\frac{\partial}{\partial z} + a^\dagger \frac{\partial}{\partial \bar{z}}). \tag{4.1.14}$$

This makes it clear that the solution to the Euclidean, massless Dirac equation is of the form

$$\Psi(z,\bar{z}) = \alpha(\bar{z}) + \beta(z)\mathrm{d}z. \tag{4.1.15}$$

4.2 Spinors in 4 Dimensions

Things become much more interesting in dimension four. There are three signatures to consider. The Euclidean, the Lorentzian and split: The Euclidean and Lorentzian cases have just one possible model each. In the case of the split signature there are two possible models, one corresponding to the real index equal to two, the other with real index zero. Thus, there are two types of pure spinors in the split case. For brevity, not all cases are considered, only the $r = 0$ cases.

4.2.1 Spin(4)

We choose a complex structure, thus identifying $\mathbb{R}^4$ with $\mathbb{C}^2$. We will call the arising null complex coordinates $z_{1,2}$, and the corresponding one-forms $\mathrm{d}z_{1,2}$. We introduce two pairs of creation and annihilation operators $a_{1,2}, a^\dagger_{1,2}$. The Γ operators take the following form,

$$\Gamma_1 = -\mathbf{i}(a_2 - a_2^\dagger),\ \Gamma_2 = a_2 + a_2^\dagger,\ \Gamma_3 = -\mathbf{i}(a_1 - a_1^\dagger),\ \text{and } \Gamma_4 = a_1 + a_1^\dagger. \tag{4.2.1}$$

We have adopted the numbering and the signs in the imaginary Γ-matrices that become convenient below. A generic Dirac spinor (general polyform) is given by

$$\Psi = (\alpha + \beta \mathrm{d}z_{12}) + (\gamma \mathrm{d}z_1 + \delta \mathrm{d}z_2), \tag{4.2.2}$$

where $\mathrm{d}z_{12} \coloneqq \mathrm{d}z_1 \wedge \mathrm{d}z_2$ and $\alpha, \beta, \gamma, \delta \in \mathbb{C}$. In matrix notations, the Dirac spinor is 4-component. It is convenient to adopt the 2×2 block notations, in which Weyl spinors are 2-component. Thus, we write

$$\Psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix},\ \text{where } \psi_+ = \begin{pmatrix} \alpha \\ \beta \end{pmatrix},\ \text{and } \psi_- = \begin{pmatrix} \gamma \\ \delta \end{pmatrix}. \tag{4.2.3}$$

The action of the Γ operators is as follows

$$\begin{aligned} &\Gamma_1 \Psi = -\mathbf{i}\beta \mathrm{d}z_1 - \mathbf{i}\alpha \mathrm{d}z_2 + \mathbf{i}\delta + \mathbf{i}\gamma \mathrm{d}z_{12}, && \Gamma_2 \Psi = -\beta \mathrm{d}z_1 + \alpha \mathrm{d}z_2 + \delta - \gamma \mathrm{d}z_{12}, \\ &\Gamma_3 \Psi = -\mathbf{i}\alpha \mathrm{d}z_1 + \mathbf{i}\beta \mathrm{d}z_2 + \mathbf{i}\gamma - \mathbf{i}\delta \mathrm{d}z_{12}, && \text{and } \Gamma_4 \Psi = \alpha \mathrm{d}z_1 + \beta \mathrm{d}z_2 + \gamma + \delta \mathrm{d}z_{12}. \end{aligned} \tag{4.2.4}$$

In matrix notations this becomes

$$\Gamma_4 = \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix},\ \text{and } \Gamma_i = \begin{pmatrix} 0 & \mathbf{i}\sigma^i \\ -\mathbf{i}\sigma^i & 0 \end{pmatrix},\ \text{for } i = 1, 2, 3. \tag{4.2.5}$$

Here σ^i are the usual Pauli matrices. It is this simple form of the resulting Γ-matrices that motivated the choices made in (4.2.1), (4.2.2).

The Lie algebra is generated by products of distinct Γ-matrices. This gives a 4×4 Lie algebra matrix that is block-diagonal. Let us refer to its 2×2 blocks as A, A', where A acts on S_+ and A' on S_- respectively. We have

$$A = \mathbf{i}\left(-\omega^{4i} + \frac{1}{2}\epsilon^{ijk}\omega^{jk}\right)\sigma^i,\ \text{and } A' = \mathbf{i}\left(\omega^{4i} + \frac{1}{2}\epsilon^{ijk}\omega^{jk}\right)\sigma^i. \tag{4.2.6}$$

Here ω^{4i}, ω^{jk} are antisymmetric 2-tensors taking values in $\mathbb{R}$. Both A, A' are skew-hermitian 2×2 matrices. This demonstrates $\mathfrak{spin}(4) = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$.

The invariant inner product is determined by the following computation

$$\begin{aligned} \langle \Psi_1, \Psi_2 \rangle &= (\alpha_1 - \beta_1 \mathrm{d}z_{12} + \gamma_1 \mathrm{d}z_1 + \delta_1 \mathrm{d}z_2) \wedge (\alpha_2 + \beta_2 \mathrm{d}z_{12} + \gamma_2 \mathrm{d}z_1 + \delta_2 \mathrm{d}z_2)\Big|_{\text{top}} \\ &= (\alpha_1\beta_2 - \alpha_2\beta_1) + (\gamma_1\delta_2 - \gamma_2\beta_1). \end{aligned} \tag{4.2.7}$$

Thus (4.2.7) can be restricted to an antisymmetric pairing between $\langle S_+, S_+ \rangle$, or $\langle S_-, S_- \rangle$. In terms of matrices on can write

$$\langle\Psi_1,\Psi_2\rangle = \Psi_1^T\begin{pmatrix}\epsilon & 0\\ 0 & \epsilon\end{pmatrix}\Psi_2,\ \text{where}\ \epsilon := \mathbf{i}\sigma^2 = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}. \tag{4.2.8}$$

For the possible reality conditions, both $\mathfrak{R} = \Gamma_2\Gamma_4\mathfrak{C}$ and $\mathfrak{R}' = \Gamma_1\Gamma_3\mathfrak{C}$ square to minus the identity, and so there are no Majorana spinors in this case. Of them $\mathbf{R} = \mathfrak{R}'$ commutes with all Γ-matrices and defines the hat operator

$$\hat{\psi}_+ = \mathbf{R}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \begin{pmatrix}-\beta^*\\ \alpha^*\end{pmatrix},\ \text{and}\ \hat{\psi}_- = \mathbf{R}\begin{pmatrix}\gamma\\ \delta\end{pmatrix} = \begin{pmatrix}\delta^*\\ -\gamma^*\end{pmatrix}, \tag{4.2.9}$$

which squares to minus the identity. As before, $\alpha^*,\dots\gamma^*$, are complex conjugates of $\alpha,\dots,\gamma$. For later purposes, we define $\psi_\pm^\dagger := \hat{\psi}_\pm^T$.

Two canonical pure spinors come with the model, the identity spinor, $\mathbf{1}$, and the top polyform, $\mathrm{d}z_{12}$. They are both in S_+. The stabiliser of both is the copy of $\mathfrak{su}(2) \subset \mathfrak{spin}(4)$ whose action on S_+ is trivial.

It is clear that a generic Weyl spinor of Spin(4) is also pure. Indeed, one can write an even Weyl spinor of the form $\psi_+ = (\alpha - \beta^*\mathfrak{R}')\mathbf{1}$. Thus, a generic Weyl spinor shares the same stabiliser as $\mathbf{1}$, i.e. SU(2). The group Spin(4) acts transitively on the space of Weyl spinors of fixed norm $\langle\hat{\psi}_+,\psi_+\rangle$. This space is the 3-sphere S^3.

In the case of Cliff_{2n}, pure spinors are in one-to-one correspondence with complex structures. The complex structure on $\mathbb{R}^4$ corresponding to a generic Weyl spinor can be recovered as in (3.2.1.3) via the following proposition.

Proposition 4.2.1.1. Let $\psi_+ \in S_+$ be a pure spinor of Spin(4). Then there exists a complex structure J_{ψ_+} on $\mathbb{R}^4$, corresponding to ψ_+.

Proof. Using definition 3.2.1.3 gives

$$B_2(\psi_+,\hat{\psi}_+) = \mathbf{i}\Sigma^i V^i, \tag{4.2.10}$$

where

$$\Sigma^i = \mathrm{d}x^4 \wedge \mathrm{d}x^i - \frac{1}{2}\epsilon^{ijk}\mathrm{d}x^j \wedge \mathrm{d}x^k \tag{4.2.11}$$

is the basis of self-dual 2-forms and

$$V^i := \mathrm{Tr}\left(\psi_+^\dagger\sigma^i\psi_+\right) = (2\mathrm{Re}(\alpha^*\beta), 2\mathrm{Im}(\alpha^*\beta), |\alpha|^2 - |\beta|^2) \tag{4.2.12}$$

is a 3-vector with squared norm

$$|V|^2 = V^iV^i = (|\alpha|^2 + |\beta|^2)^2 = \langle\hat{\psi}_+,\psi_+\rangle^2. \tag{4.2.13}$$

If one raises an index of $\Sigma^i_{\mu\nu}$, one obtains a triple of endomorphisms of $\mathbb{R}^4$ that satisfy the algebra of the quaternions

$$\Sigma^{i\ \rho}_{\mu}\Sigma^{j\ \nu}_{\rho} = -\delta^{ij}\delta_\mu{}^\nu + \epsilon^{ijk}\Sigma^{k\ \nu}_{\mu}. \tag{4.2.14}$$

Hence object

$$J_{\psi_+} := \frac{1}{|V|}\Sigma^i V^i, \tag{4.2.15}$$

viewed as an endomorphism of $\mathbb{R}^4$, is then a complex structure that corresponds to the pure spinor ψ_+. □

For ψ unit, $B_2(\psi_+,\hat{\psi}_+)$ is an injection $S^1 \hookrightarrow S^3$. Then, one can project to V^i the coefficients of Σ^i in $B_2(\psi_+,\hat{\psi}_+)$. The vector of coefficients, $\vec{V}$, becomes a point on S^2. Therefore, for any unit Weyl spinor, one sees the Hopf fibration given as

$$S^1 \hookrightarrow S^3 \to S^2. \tag{4.2.16}$$

The construction (4.2.14) is very important, as we will see this come again in chapter 8. There we will highlight the quaternionic nature of Spin(4) much more.

4.2.2 Spin(2,2) — $r = 0$ model

To construct this model, we choose an MTN subspace spanned by two complex vectors, obtained as $-\mathbf{i}$ eigenvalue eigenvectors of a complex structure on $\mathbb{R}^{2,2}$. We refer to the corresponding null complex coordinates as $z_{1,2}$, and the corresponding one-forms as $\mathrm{d}z_{1,2}$. The general Dirac spinor is the polyform

$$\Psi = \alpha + \beta \mathrm{d}z_{12} + \gamma \mathrm{d}z_1 + \delta \mathrm{d}z_2, \tag{4.2.17}$$

with $\alpha, \beta, \gamma, \delta \in \mathbb{C}$. We introduce two pairs of creation and annihilation operators $a_{1,2}, a^\dagger_{1,2}$. The Γ-matrices are given by

$$\Gamma_1 = a_1 + a_1^\dagger,\ \Gamma_3 = a_2 - a_2^\dagger,\ \Gamma_2 = -\mathbf{i}(a_1 - a_1^\dagger),\ \text{and}\ \Gamma_4 = \mathbf{i}(a_2 + a_2^\dagger). \tag{4.2.18}$$

Here our choice of $\Gamma_{1,2}$ is motivated to match $\Gamma_{4,3}$ in (4.2.1). The other choices are motivated by the desire to have nicer looking Γ-matrices. The Γ-matrices are then easily recoverable from (4.2.5) and are given by

$$\begin{aligned} \Gamma_1 &= \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix},\ \Gamma_2 = \begin{pmatrix} 0 & \mathbf{i}\sigma^3 \\ -\mathbf{i}\sigma^3 & 0 \end{pmatrix}, \\ \Gamma_3 &= \begin{pmatrix} 0 & -\sigma^1 \\ \sigma^1 & 0 \end{pmatrix},\ \text{and}\ \Gamma_4 = \begin{pmatrix} 0 & -\sigma^2 \\ \sigma^2 & 0 \end{pmatrix}. \end{aligned} \tag{4.2.19}$$

The 2×2 blocks of the Lie algebra element are then

$$\begin{aligned} A &= -\mathbf{i}\sigma^3(\omega^{12} + \omega^{34}) + \sigma^1(\omega^{13} - \omega^{42}) + \sigma^2(\omega^{14} - \omega^{23}),\ \text{and} \\ A' &= \mathbf{i}\sigma^3(\omega^{12} - \omega^{34}) - \sigma^1(\omega^{13} + \omega^{42}) - \sigma^2(\omega^{14} + \omega^{23}). \end{aligned} \tag{4.2.20}$$

Here ω^{ij}, for $1 \le i, j \le 4$ are antisymmetric 2-tensors valued in $\mathbb{R}$. Both A, A' are trace-free matrices with imaginary diagonal and the off-diagonal elements being complex conjugates of each other. These matrices form $\mathfrak{su}(1,1)$, and so the Lie algebra is $\mathfrak{spin}(2,2) = \mathfrak{su}(1,1) \oplus \mathfrak{su}(1,1)$. Thus, this version of the creation and annihilation operator model exhibits the isomorphism $\mathrm{Spin}(2,2) = \mathrm{SU}(1,1) \times \mathrm{SU}(1,1)$. The invariant inner product is still given by (4.2.8).

The novelty as compared to the previous real index two model is that the spinors are now complex. However, there are now two non-trivial antilinear operators that can be constructed, $\mathfrak{R} = \Gamma_1\Gamma_3\mathfrak{C}$ and $\mathfrak{R}' = \Gamma_2\Gamma_4\mathfrak{C}$. Unlike the case of Cliff_4 where their analogues both square to minus the identity, now they both square to plus the identity, and either one of them can be used to define the notion of Majorana spinors. For concreteness, let us use $\mathfrak{R}'$ as the reality condition operator. In matrix form we have

$$\mathfrak{R}' = \begin{pmatrix} \sigma^1\mathfrak{C} & 0 \\ 0 & \sigma^1\mathfrak{C} \end{pmatrix}. \tag{4.2.21}$$

The operator $\mathfrak{R}$ has the same action on S_+, and is minus this on S_-. The action of $\mathfrak{R}'$ preserves S_+ (and S_-), and allows us to define Majorana-Weyl spinors. It is clear that a Majorana-Weyl spinor in both $S_\pm$ is of the form

$$\psi_{MW} = \begin{pmatrix} \alpha \\ \alpha^* \end{pmatrix}. \tag{4.2.22}$$

Thus, a Majorana-Weyl spinor is parametrised by a single complex number, α and its complex conjugate α^*. However, a general spinor in the case of this model is complex 2-dimensional, and we need such complex spinors to recover the complex structure from a pure spinor. Thus, we take a generic Weyl spinor $\psi_+ \in S_+$. This leads to a proposition similar to proposition 4.2.1.1.

Proposition 4.2.2.1. Let $\psi_+ \in S_+$ be a pure spinor of $\mathrm{Spin}(2,2)$. Then there exists a complex structure J on $\mathbb{R}^{2,2}$, corresponding to ψ_+.

Proof. We compute, using definition 3.2.1.3,

$$\begin{aligned} B_2(\mathfrak{R}(\psi_+), \psi_+) = \mathbf{i}(|\alpha|^2 + |\beta|^2)(\mathrm{d}x^1\mathrm{d}x^2 + \mathrm{d}y^1\mathrm{d}y^2) - 2\mathbf{i}\mathrm{Im}(\alpha^*\beta)(\mathrm{d}x^1\mathrm{d}y^1 + \mathrm{d}x^2\mathrm{d}y^2) \\ + 2\mathbf{i}\mathrm{Re}(\alpha^*\beta)(\mathrm{d}x^1\mathrm{d}y^2 - \mathrm{d}x^2\mathrm{d}y^1). \end{aligned} \tag{4.2.23}$$

This is a pure imaginary 2-form, which can be interpreted as a complex structure when one of its indices is raised and it is rescaled appropriately. The best way to do this is to introduce a triple of self-dual 2-forms

$$\Sigma^3 = \mathrm{d}x^1\mathrm{d}x^2 + \mathrm{d}y^1\mathrm{d}y^2,\ \Sigma^1 = \mathrm{d}x^1\mathrm{d}y^2 - \mathrm{d}x^2\mathrm{d}y^1,\ \text{and}\ \Sigma^2 = \mathrm{d}x^1\mathrm{d}y^1 + \mathrm{d}x^2\mathrm{d}y^2. \tag{4.2.24}$$

Then

$$B_2(\mathfrak{R}(\psi_+), \psi_+) = \mathbf{i}\Sigma^i V_i, \tag{4.2.25}$$

where

$$\vec{V} = (2\mathrm{Re}(\alpha^*\beta), -2\mathrm{Im}(\alpha^*\beta), |\alpha|^2 + |\beta|^2). \tag{4.2.26}$$

The objects Σ^i, viewed as endomorphisms of $\mathbb{R}^{2,2}$ satisfy

$$\Sigma^{3\ \rho}_{\mu}\Sigma^{3\ \nu}_{\rho} = -\delta_\mu{}^\nu,\ \Sigma^{1\ \rho}_{\mu}\Sigma^{1\ \nu}_{\rho} = \delta_\mu{}^\nu,\ \text{and}\ \Sigma^{2\ \rho}_{\mu}\Sigma^{2\ \nu}_{\rho} = \delta_\mu{}^\nu. \tag{4.2.27}$$

This shows that

$$J_\mu{}^\nu \coloneqq \frac{1}{|\alpha|^2 - |\beta|^2}\Sigma^{i\ \nu}_{\mu} V_i \tag{4.2.28}$$

squares to minus the identity and is a complex structure. □

In particular, when $\psi_+ = (1,0)$ or $\psi_+ = (0,1)$ this complex structure is plus or minus Σ^3. Thus, complex structures on $\mathbb{R}^{2,2}$ are parametrised by points on the hyperbolic plane H_2, as in the case of $\mathbb{R}^4$ they are parametrised by points of S^2. Once again, it needs to be emphasised that we have access to this complex picture only when we consider general complex-valued Weyl spinors.

Having the antilinear operators $\mathfrak{R}$, and $\mathfrak{R}'$ at our disposal (which agree on S_+) we can compute

$$\langle \mathfrak{R}'(\psi_+), \psi_+ \rangle = |\beta|^2 - |\alpha|^2. \tag{4.2.29}$$

The action of $\mathrm{Spin}(2,2)$ on S_+ viewed as complex 2-component columns preserves this invariant. We note that when ψ_+ is a spinor of a fixed norm (4.2.29), $B_2(\mathfrak{R}(\psi_+), \psi_+)$ is a point on AdS_3 of a fixed radius of curvature $a \in \mathbb{R}_{>0}$ determined as follows

$$(V_1)^2 + (V_2)^2 - (V_3)^2 = -(|\alpha|^2 - |\beta|^2)^2 \coloneqq -a^2. \tag{4.2.30}$$

Projecting to V_i, the coefficients of Σ^i in $B_2(\mathfrak{R}(\psi_+), \psi_+)$, determines a point on the hyperbolic sheet H_2. Thus, we encounter an instance of the non-compact version of the Hopf fibration

$$S^1 \hookrightarrow \mathrm{AdS}_3 \to H_2, \tag{4.2.31}$$

which is a precise analogue of the usual $S^1 \hookrightarrow S^3 \to S^2$ that was encountered in the case of $\mathbb{R}^4$.

One of the two copies of $\mathrm{SU}(1,1)$ does not act on S_+, while the other acts transitively on AdS_3. Thus, the stabiliser of any point on the with fixed radius of curvature $|\beta|^2 - |\alpha|^2 = a^2$ is $\mathrm{SU}(1,1)$ and

$$\mathrm{AdS}_3 = \mathrm{Spin}(2,2)/\mathrm{SU}(1,1) \cong \mathrm{SU}(1,1). \tag{4.2.32}$$

This completely analogous to what we had in the $\mathbb{R}^4$ case where the analogous statement was $S^3 = \mathrm{SU}(2)$. What is worth stressing is that all this becomes possible only in the setting of generic Weyl spinors without any Majorana reality condition imposed.

Remark 4.2.2.1. Other geometric date stored in a generic Weyl spinor are as follows. We compute, using definition 3.2.1.3,

$$\begin{aligned} B_2(\psi_+,\psi_+) = {} & 2\mathbf{i}\alpha\beta(\mathrm{d}x^1\mathrm{d}x^2 + \mathrm{d}y^1\mathrm{d}y^2) + (\alpha^2-\beta^2)(\mathrm{d}x^1\mathrm{d}y^1 + \mathrm{d}x^2\mathrm{d}y^2) \\ & + \mathbf{i}(\alpha^2+\beta^2)(\mathrm{d}x^1\mathrm{d}y^2 - \mathrm{d}x^2\mathrm{d}y^1). \end{aligned} \tag{4.2.33}$$

Here the concatenation of two differential forms above is the suppression of the wedge product[1]. Furthermore, this 2-form is decomposable

$$\begin{aligned} & B_2(\psi_+,\psi_+) = U \wedge \tilde{U}, \text{ where} \\ & \quad U = 2\mathbf{i}\alpha\beta\mathrm{d}y^1 + (\alpha^2-\beta^2)\mathrm{d}x^2 - \mathbf{i}(\alpha^2+\beta^2)\mathrm{d}x^1, \text{ and} \\ & \quad \tilde{U} = \frac{1}{\alpha^2+\beta^2}(-2\alpha\beta\mathrm{d}x^2 + \mathbf{i}(\alpha^2-\beta^2)\mathrm{d}y^1 + (\alpha^2+\beta^2)\mathrm{d}y^2). \end{aligned} \tag{4.2.34}$$

For example, for the canonical spinor $\psi_+ = (0,1)$ we get $B_2(\psi_+,\psi_+) = -(\mathrm{d}x^1 + \mathbf{i}\mathrm{d}x^2)(\mathrm{d}y^1 - \mathbf{i}\mathrm{d}y^2)$, and for $\psi_+ = (1,0)$ we have $B_2(\psi_+,\psi_+) = (\mathrm{d}x^1 - \mathbf{i}\mathrm{d}x^2)(\mathrm{d}y^1 + \mathbf{i}\mathrm{d}y^2)$.

It is interesting to see how much of the above picture survives if we impose the Majorana condition. Indeed, consider the following proposition

Proposition 4.2.2.2. Consider $\psi_{MW} = (\alpha,\alpha^*)$, as in (4.2.22). Then the stabiliser is $\mathrm{SU}(1,1)\times\mathbb{R} \subset \mathrm{Spin}(2,2)$.

Proof. Let us begin by parametrising $\alpha = \mathsf{a} + \mathbf{i}\mathsf{b}$. The Lie algebra (4.2.20), by reparameterising appropriately, is given as

$$A = \begin{pmatrix} \mathbf{i}s & x - \mathbf{i}y \\ x + \mathbf{i}y & s \end{pmatrix}, \text{ for } s,x,y \in \mathbb{R}. \tag{4.2.35}$$

Solving $A\psi_{MW} = 0$ gives the following set of equations

$$\begin{aligned} \mathsf{b}s + \mathsf{a}x - \mathsf{b}y &= 0, \\ \mathsf{a}s + \mathsf{b}x + \mathsf{a}y &= 0. \end{aligned} \tag{4.2.36}$$

Consider the case where $\mathsf{a} \neq 0, \mathsf{b} \neq 0$. Solving for s,x and y in terms of a and b gives

$$A = \frac{y}{(\mathsf{b}^2-\mathsf{a}^2)}\begin{pmatrix} \mathbf{i}(\mathsf{b}^2+\mathsf{a}^2) & 2\mathsf{ab} - \mathbf{i}(\mathsf{b}^2-\mathsf{a}^2) \\ 2\mathsf{ab} + \mathbf{i}(\mathsf{b}^2-\mathsf{a}^2) & \mathbf{i}(\mathsf{b}^2+\mathsf{a}^2) \end{pmatrix}. \tag{4.2.37}$$

A is nilpotent of degree 2. Thus the subgroup stabilising ψ_{MW}, for $\mathsf{a} \neq 0, \mathsf{b} \neq 0$, is

$$\frac{y}{(\mathsf{b}^2-\mathsf{a}^2)}\begin{pmatrix} \mathsf{c}_0 + \mathbf{i}(\mathsf{b}^2+\mathsf{a}^2) & 2\mathsf{ab} - \mathbf{i}(\mathsf{b}^2-\mathsf{a}^2) \\ 2\mathsf{ab} + \mathbf{i}(\mathsf{b}^2-\mathsf{a}^2) & \mathsf{c}_0 + \mathbf{i}(\mathsf{b}^2+\mathsf{a}^2) \end{pmatrix}, \text{ for some } \mathsf{c}_0 \in \mathbb{R}. \tag{4.2.38}$$

[1] We shall continue this notation for the rest of this chapter.

It is quite clear that the transformations written above for any $\mathsf{c}_0, \mathsf{a}, \mathsf{b}, y \in \mathbb{R}$ is $\mathrm{SU}(1,1) \times \mathbb{R} \subset \mathrm{Spin}(2,2)$. The reduced cases where $\mathsf{a} = 0, \mathsf{b} \neq 0$, and $\mathsf{a} \neq 0, \mathsf{b} = 0$ follow the same pattern as above, i.e. the transformation preserving ψ_{MW} is $\mathrm{SU}(1,1)$. □

One can quite easily extrapolate this statement to $\psi_{MW} = (\alpha, \alpha^*)$ to see the stabiliser is still $\mathrm{SU}(1,1) \times \mathbb{R}$.

Geometrically, a generic Weyl spinor (of fixed norm) represents a point on AdS_3. But, Majorana-Weyl spinors have zero norm, $|\alpha|^2 - |\alpha^*|^2 = 0$, and thus correspond to points on the light-cone of a point in AdS_3.

Remark 4.2.2.2. To see geometric data encoded by a Majorana-Weyl spinor, we take a general Majorana-Weyl spinor (4.2.22) in S_+, and compute the 2-form $B_2(\psi_{MW}, \psi_{MW})$. We have

$$\begin{aligned} B_2(\psi_{MW}, \psi_{MW}) = 2\mathbf{i}|\alpha|^2(\mathrm{d}x^1\mathrm{d}x^2 + \mathrm{d}y^1\mathrm{d}y^2) + \mathbf{i}(\mathrm{Im}(\alpha^2))(\mathrm{d}x^1\mathrm{d}y^1 + \mathrm{d}x^2\mathrm{d}y^2) \\ + \mathbf{i}(\mathrm{Re}(\alpha^2))(\mathrm{d}x^1\mathrm{d}y^2 - \mathrm{d}x^2\mathrm{d}y^1). \end{aligned} \tag{4.2.39}$$

This 2-form is purely imaginary and decomposable. For example, for $\alpha = 1$, we have

$$B_2(\psi_{MW}, \psi_{MW}) = 2\mathbf{i}(\mathrm{d}x^1 + \mathrm{d}y^1)(\mathrm{d}x^2 + \mathrm{d}y^2).$$

Thus, a Majorana-Weyl spinor only carries information about two real null directions, which are the directions spanning $M(\psi_{MW})$. Here we exhibit the stabiliser of the canonical differential form being smaller than the stabiliser of the spinor. Indeed, the subgroup in $\mathrm{Spin}(2,2)$ that stabilises the 2-form is $\mathrm{SL}(2,\mathbb{R}) \cong \mathrm{SU}(1,1) \subset \mathrm{SU}(1,1) \times \mathbb{R} \subset \mathrm{Spin}(2,2)$.

The two types of Weyl spinors that arise in the case of $\mathrm{Spin}(2,2)$ are only visible when the spinors are complex-valued.

- The spinors of non-zero norm, (4.2.29), are points in AdS_3 of a fixed radius of curvature, and $\mathrm{AdS}_3 \cong \mathrm{SU}(1,1)$. Thus, spinors of the first type are in correspondence with MTN subspaces of $\mathbb{R}^{2,2}$ of real index zero.
- The spinors of zero norm are Majorana-Weyl spinors, and $\mathrm{SU}(1,1)$ acts on this orbit with a non-trivial stabiliser. Thus, spinors of the second type are in correspondence with MTN subspaces of real index two.

Therefore, this example sets the precedent: we lose collapse to a less rich geometry if we impose the Majorana condition. We shall see this again for spinors in 6 dimensions.

4.3 Spinors in 6 Dimensions

4.3.1 Spin(6)

As usual, only the real index zero model is possible in this case. We choose a complex structure on $\mathbb{R}^6$, and introduce 3 complex null coordinates $z_{1,2,3}$, as well as the corresponding one-forms $\mathrm{d}z_{1,2,3}$. We introduce 3 pairs of creation and annihilation operators $a_{1,2,3}, a^\dagger_{1,2,3}$. The Dirac spinor is a polyform

$$\Psi = \alpha_1\mathrm{d}z_{23} + \alpha_2\mathrm{d}z_{31} + \alpha_3\mathrm{d}z_{12} + \alpha_4 + \beta_1\mathrm{d}z_1 + \beta_2\mathrm{d}z_2 + \beta_3\mathrm{d}z_3 - \beta_4\mathrm{d}z_{123}, \tag{4.3.1}$$

with all coefficients complex-valued. The reason the last term is included with the minus sign will become clear when we consider the inner product.

We act upon this Dirac spinor with the following Gamma matrices

$$\begin{aligned} &\Gamma_1 = a_1 + a_1^\dagger, &&\Gamma_2 = a_2 + a_2^\dagger, &&\Gamma_3 = a_3 + a_3^\dagger, \\ &\Gamma_4 = \mathbf{i}(a_1 - a_1^\dagger), &&\Gamma_5 = \mathbf{i}(a_2 - a_2^\dagger), \text{ and } &&\Gamma_6 = \mathbf{i}(a_3 - a_3^\dagger). \end{aligned} \tag{4.3.2}$$

All Γ-matrices work out to be

$$\Gamma_I = \begin{pmatrix} 0 & \gamma_I \\ \gamma_I^\dagger & 0 \end{pmatrix}, \qquad I = 1, \dots, 6, \tag{4.3.3}$$

where γ_I are the following 4×4 matrices

$$\gamma_1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \gamma_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$
$$\gamma_3 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \gamma_4 = \mathbf{i} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \tag{4.3.4}$$
$$\gamma_5 = \mathbf{i} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \text{ and } \gamma_6 = \mathbf{i} \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

They are all antisymmetric. The commutator of these Γ-matrices is block-diagonal, with skew-hermitian trace-free 4×4 blocks on the diagonal. This exhibits the isomorphism $\mathfrak{spin}(6) = \mathfrak{su}(4)$.

The inner product pairs even to odd polyforms, and so is a pairing $\langle S_+, S_- \rangle$. Explicitly, we get

$$\langle \Psi, \tilde{\Psi} \rangle = -\sum_{I=1}^{4} \alpha_I \tilde{\beta}_I + \sum_{I=1}^{4} \beta_I \tilde{\alpha}_I. \tag{4.3.5}$$

It is in order to have the same signs here that we have put the minus sign in the last term in (4.3.1).

There are two antilinear operators that can be constructed, $\mathfrak{R} = \Gamma_1\Gamma_2\Gamma_3\mathfrak{C}$ and $\mathfrak{R}' = \Gamma_4\Gamma_5\Gamma_6\mathfrak{C}$. The first of these squares to minus the identity, while $(\mathfrak{R}')^2 = \mathbb{I}$. So, it is $\mathfrak{R}'$ that gives us a good real structure. It works out to be given by

$$\mathbf{R} = \mathfrak{R}' = \begin{pmatrix} 0 & \mathfrak{C} \\ \mathfrak{C} & 0 \end{pmatrix}. \tag{4.3.6}$$

Given a Weyl spinor $\psi_+ \in S_+$, we can construct

$$\langle \mathbf{R}(\psi_+), \psi_+ \rangle = \sum_{I=1}^{4} |\alpha_I|^2. \tag{4.3.7}$$

Thus, there is a positive-definite Hermitian invariant quadratic form on S_+.

Remark 4.3.1.1. A simple computation shows, for a generic Weyl spinor ψ_+ of fixed norm, the stabiliser is $\mathfrak{su}(3) \subset \mathfrak{spin}(6)$. Hence, one has $\mathrm{Spin}(6) \cong \mathrm{SU}(4)$ acts on the subset in S_+ of spinors of fixed norm squared transitively, with the stabiliser $\mathrm{SU}(3)$. In turn, one has

$$S^7 = \mathrm{SU}(4)/\mathrm{SU}(3). \tag{4.3.8}$$

Given that Weyl spinors are pure in this dimension, and directions of pure spinors define complex structures in $\mathbb{R}^6$, we see that the space of complex structures on $\mathbb{R}^6$ is S^7.

For a Weyl spinor, ψ_+, the object $B_1(\psi_+, \psi_+)$ vanishes because the matrices (4.3.4) are anti-symmetric. The only non-vanishing object that can be constructed without using the operator

$\mathbf{R}$ is $B_3(\psi_+, \psi_+)$. From general grounds, we know that $B_3(\psi_+, \psi_+)$ is given by the wedge product of the three complex null directions in $M(\psi_+)$ (theorem 3.2.1.1). The objects that can be constructed using $\mathbf{R}$ are the norm (4.3.7) as well as $B_2(\mathbf{R}(\psi_+), \psi_+)$. This is a 2-form that gives the complex structure that corresponds to ψ_+ when one of its indices is raised and is rescaled appropriately. For example, for the identity polyform $\mathbf{1} \in S_+$ one gets

$$B_2(\mathbf{R}(\mathbf{1}), \mathbf{1}) = \mathbf{i}(\mathrm{d}x^1 \wedge \mathrm{d}x^4 + \mathrm{d}x^2 \wedge \mathrm{d}x^5 + \mathrm{d}x^3 \wedge \mathrm{d}x^6), \tag{4.3.9}$$

while for $\mathrm{d}z_{123} \in S_-$ the result is the same with the extra minus sign in front. In this case, $B_2(\mathbf{R}(\mathbf{1}), \mathbf{1})$ is invariant under $\mathrm{U}(3)$. Together with a compatible hermitian form, $\mathrm{U}(3)$ reduces to $\mathrm{SU}(3)$. This is one of the few cases where the stabiliser of the spinor coincides with the stabiliser of the differential form (and the compatible hermitian metric).

4.3.2 Spin(3,3)— $r = 1$ model

There are two possible models. A model with real index three, that works with 3 real null coordinates, and has all Γ-matrices built from the creation and annihilation operators with real coefficients. This model is explicitly real, and it is natural to take in it all spinors to be Majorana(-Weyl). The other model has real index one, and takes two complex and one real null direction. As we will show, it is sufficient to only consider the latter model, for which we now spell out the details.

We make a modification of the minimal index case of $\mathrm{Spin}(4,2)$, found in [BK22]. Declaring the complex coordinates to be $z_{1,3}$, and the real null coordinate to be u. There are two pairs of creation and annihilation operators $a_{1,3}, a^\dagger_{1,3}$ and one pair $b, b^\dagger$. The general Dirac spinor is then

$$\Psi = \alpha_1 \mathrm{d}u\mathrm{d}z_3 + \alpha_2 \mathrm{d}z_{31} + \alpha_3 \mathrm{d}z_1 \mathrm{d}u + \alpha_4 + \beta_1 \mathrm{d}z_1 + \beta_2 \mathrm{d}u + \beta_3 \mathrm{d}z_3 - \beta_4 \mathrm{d}z_{31}\mathrm{d}u. \tag{4.3.10}$$

All coefficients are complex-valued. The inner product is then still given by (4.3.5). We take the Γ-matrices to be

$$\begin{aligned} &\Gamma_1 = a_1 + a_1^\dagger, \quad &&\Gamma_2 = b + b^\dagger, \quad &&\Gamma_3 = \mathbf{i}(a_3 + a_3^\dagger), \\ &\Gamma_4 = \mathbf{i}(a_1 - a_1^\dagger), \quad &&\Gamma_5 = b - b^\dagger, \ \text{and} \ &&\Gamma_6 = a_3 - a_3^\dagger. \end{aligned} \tag{4.3.11}$$

The directions $1, 2, 4$ are now positive-definite, while $3, 5, 6$ are negative-definite. The modified Γ-matrices are

$$\Gamma_2 = \begin{pmatrix} 0 & \gamma_2 \\ \gamma_2^\dagger & 0 \end{pmatrix}, \qquad \Gamma_5 = -\mathbf{i}\begin{pmatrix} 0 & \gamma_5 \\ \gamma_5^\dagger & 0 \end{pmatrix}, \tag{4.3.12}$$

where γ_2, γ_5 are still given by (4.3.4).

The complex conjugation operators are given by $\mathfrak{R} = \Gamma_1\Gamma_2\Gamma_5\Gamma_6\mathfrak{C}$ and $\mathfrak{R}' = \Gamma_3\Gamma_4\mathfrak{C}$. They both square to plus the identity, and so either can be used to define Majorana-Weyl spinors. We have

$$\mathbf{R} = \mathfrak{R} = \begin{pmatrix} \rho & 0 \\ 0 & \rho \end{pmatrix}\mathfrak{C}, \qquad \rho = \begin{pmatrix} 0 & -\mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix}. \tag{4.3.13}$$

A simple computation shows any element of the Lie algebra $\mathfrak{spin}(3,3) \cong \mathfrak{sl}(4,\mathbb{R})$ commutes with the real structure defined by $\mathbf{R}$.

As in all cases considered before, the only object that can be constructed from a Weyl spinor $\psi_+ \in S_+$, without involving complex conjugation, is $B_3(\psi_+, \psi_+)$: the product of the three null directions spanning $M(\psi_+)$. There are no invariants of ψ_+ that can be constructed in this signature. Indeed, in 6 dimensions, the invariant pairing is $\langle S_\pm, S_\mp \rangle$. Furthermore, as $\mathbf{R} : S_+ \to S_+$, one can not construct an invariant pairing between ψ_+ and $\mathbf{R}(\psi_+)$. But the action of $\mathrm{Spin}(3,3)$ on S_+, prior to imposing the Majorana-Weyl condition, cannot be transitive. Indeed, we expect two different types of orbits corresponding to two different types of MTN that are possible in this signature.

To see how this arises, let us compute $B_1(\mathbf{R}(\psi_+),\psi_+)$. We get

$$\begin{aligned}B_1(\mathbf{R}(\psi_+),\psi_+) = &(-2\mathrm{Re}(\alpha_1\alpha_2^*) + 2\mathrm{Re}(\alpha_3\alpha_4^*))\mathrm{d}x^1 + (2\mathrm{Re}(\alpha_1\alpha_4^*) - 2\mathrm{Re}(\alpha_2\alpha_3^*))\mathrm{d}x^6\\ &+ (2\mathrm{Im}(\alpha_1\alpha_4^*) - 2\mathrm{Im}(\alpha_2\alpha_3^*))\mathrm{d}x^3 + (2\mathrm{Im}(\alpha_1\alpha_2^*) + 2\mathrm{Im}(\alpha_3\alpha_4^*))\mathrm{d}x^4\\ &+ (|\alpha_1|^2 - |\alpha_2|^2 - |\alpha_3|^2 + |\alpha_4|^2)\mathrm{d}x^2\\ &+ (-|\alpha_1|^2 - |\alpha_2|^2 + |\alpha_3|^2 + |\alpha_4|^2)\mathrm{d}x^5.\end{aligned} \tag{4.3.14}$$

This is a null vector in $\mathbb{R}^{3,3}$, which vanishes when the spinor is Majorana-Weyl, i.e. $\alpha_3 = -\alpha_1^*, \alpha_4 = -\alpha_2^*$. We also note that the canonical spinor $\psi_+ = \mathbf{1}$ is not Majorana, and $B_1(\mathbf{R}(\mathbf{1}),\mathbf{1}) = \mathrm{d}x^2 + \mathrm{d}x^5$. We then explain the two types of spinor in S_+ below.

1. When the spinor ψ_+ is not Majorana-Weyl there is a real direction in $M(\psi_+)$ that can be recovered by computing $B_1(\mathbf{R}(\psi_+),\psi_+)$, as well as two complex directions that are the other two factors in $B_3(\psi_+,\psi_+)$. In this case, $B_3(\psi_+,\psi_+)$ is invariant under $\mathrm{U}(2)\times\mathbb{R}$. Of course, the stabiliser of a generic Weyl spinor is larger than this group. Furthermore, a single generic Weyl spinor, $\psi_+ \in S_+$ defines only its MTN. To recover a complementary subspace, and thus a structure of the mixed type, one needs another spinor ψ_- such that $\langle\psi_+,\psi_-\rangle \neq 0$. For example, we have

$$B_2(-\mathrm{d}z_{31}\mathrm{d}u,\mathbf{1}) = \mathbf{i}(\mathrm{d}x^1\wedge\mathrm{d}x^4 + \mathrm{d}x^3\wedge\mathrm{d}x^6) + \mathrm{d}x^2\wedge\mathrm{d}x^5, \tag{4.3.15}$$

   which is the mixed structure whose null eigenspaces are those on which the model is constructed.

2. When the spinor ψ_+ is Majorana-Weyl the only non-vanishing geometric object that can be constructed is $B_3(\psi_+,\psi_+)$, and it is given by the product of 3 real directions spanning $M(\psi_+)$. In this case, $B_3(\psi_+,\psi_+)$ is invariant under $\mathrm{SL}(3,\mathbb{R})$. Of course, the stabiliser of the Majorana-Weyl spinor is larger than this group. As for a generic Weyl spinor, a Majorana-Weyl spinor, ψ_+, defines only its MTN. To recover a complementary subspace, and thus a paracomplex structure, one needs another spinor ψ_- such that $\langle\psi_+,\psi_-\rangle \neq 0$. For example,

$$B_2(-\mathrm{d}z_{31}\mathrm{d}u - \mathrm{d}u, \mathbf{1} - \mathrm{d}z_{31}) = 2\mathrm{d}x^1\wedge\mathrm{d}x^6 + 2\mathrm{d}x^2\wedge\mathrm{d}x^5 + 2\mathrm{d}x^3\wedge\mathrm{d}x^4, \tag{4.3.16}$$

   which gives a paracomplex structure with 3 real null directions $\mathrm{d}x^1\pm\mathrm{d}x^6, \mathrm{d}x^2\pm\mathrm{d}x^5, \mathrm{d}x^4\pm\mathrm{d}x^3$.

The analysis above shows minimal real index, $r = 1$, model is richer in geometry. This is because in the $r = 3$ model, all spinors are real, and thus one can only generate a paracomplex structure. In the $r = 1$ model, there are both Weyl and Majorana-Weyl spinors, so we can generate mixed and paracomplex structures (complex structures cannot be generated as r can never be zero).

Of course, the metric was assumed in all cases to raise the indices of the canonical differential forms, allowing access to complex, paracomplex, and mixed structures — after all, the goal here is to understand the stabilisers of different (Weyl) spinors. In the latter half of the thesis, we will focus on complex structures only, as a formula exists to generate the metric from the canonical differential forms.

Chapter 5

Real Stabilisers of Spinors in Higher Dimensions

In [BK22], it is demonstrated that Weyl spinors of Cliff_4 and $\mathrm{Cliff}_{2,2}$ correspond to 2-component columns with complex entries, which can be identified with $\mathbb{H}$ and $\mathbb{H}'$, respectively. In contrast, for Cliff_8 and $\mathrm{Cliff}_{4,4}$, the Majorana-Weyl spinors are identified with $\mathbb{O}$ and $\mathbb{O}'$, facilitated by the triality of spinor and vector representations. This identification necessitates imposing a reality condition. To fully understand this, a detailed description of the Majorana-Weyl spinors is required.

Once this identification is established, we can complexify the representation, following the philosophy that the Majorana condition excludes relevant information. It is crucial to note that although there is a classification of spinors in $\mathrm{Spin}(8)$, it pertains to the complexified case. Here, considering $\mathrm{Spin}(8,0)$ — the real case — we find that a pure spinor is stabilised by $\mathrm{SU}(4)$. There are two impure orbits: one represented by a unit real octonion, stabilised by $\mathrm{Spin}(7)$, and another by a general unit complexified octonion, stabilised by $\mathrm{SU}(4)$. The integrability of these pure spinor orbits, corresponding to classical G-structures, is discussed. Later, the discussion extends to the orbits of $\mathrm{Spin}(4,4)$, which present additional complexities. Moreover, the integrability of the $\mathrm{SL}(4)$-structure among $\mathrm{Spin}(4,4)$ stabilisers, which is non-classical, is elaborated in [Kra24b].

A significant outcome is the realisation that impure spinors in 10 and 12 dimensions are derived from the impure spinors of $\mathrm{Spin}(8)$. Pure spinors in these dimensions are associated with classical stabilisers $\mathrm{SU}(5)$ and $\mathrm{SU}(6)$. We demonstrate that the closure of the canonical differential forms derived from pure spinor orbits is equivalent to integrability of a classical $\mathrm{SU}(5)$-structure in 10 dimensions and a classical $\mathrm{SU}(6)$-structure in 12 dimensions. With regard to impure spinors, in 10 dimensions we discuss all stabilisers rooted from impure spinors of $\mathrm{Spin}(8)$. Similarly, in 12 dimensions, using impure spinors of $\mathrm{Spin}(8)$ and imposing certain conditions, we discuss several new stabilisers.

5.1 Spin(8) and Octonions

5.1.1 Creation and annihilation operator construction

To construct Cliff_8 we choose a complex structure, which identifies $\mathbb{R}^8$ with $\mathbb{C}^4$. We denote the complex coordinates by $z_{1,\ldots,4}$, and the coordinate one-forms by $\mathrm{d}z_{1,\ldots,4}$. We introduce four pairs of creation and annihilation operators $a_{\mathsf{I}}, a_{\mathsf{I}}^\dagger, \mathsf{I} = 1,\ldots,4$. We introduce the following Γ-matrices

$$\Gamma_{4+\mathsf{I}} := a_{\mathsf{I}} + a_{\mathsf{I}}^\dagger, \;\; \Gamma_I := -\mathbf{i}(a_{\mathsf{I}} - a_{\mathsf{I}}^\dagger), \;\; \text{where } \mathsf{I} = 1,2,3,4. \tag{5.1.1}$$

Spinors are polyforms, i.e. elements of $\Lambda(\mathbb{C}^4)$, in general with complex coefficients. Weyl spinors are even or odd degree polyforms. Note that we have chosen the same signs as in (4.2.1).

The interesting antilinear operator that arises in this case is given by the product of all four imaginary Γ-matrices with the complex conjugation,

$$\mathbf{R} := \Gamma^1\Gamma^2\Gamma^3\Gamma^4 *, \tag{5.1.2}$$

It is easy to check that $\mathbf{R}$ commutes with all the gamma-matrices. It is also easy to check that $\mathbf{R}^2 = \mathbb{I}$, so $\mathbf{R}$ is a real structure. Since $\mathbf{R}$ is composed of an even number of gamma-matrices, it preserves the spaces of Weyl spinors. Hence, we can define Majorana-Weyl spinors for Spin(8).

5.1.2 Majorana-Weyl spinors explicitly

It is now convenient to choose a basis $e^I = \mathrm{d}z_I$ of basic one-forms. Then a general odd or even polyform, that is also real, can be written as follows

$$\begin{aligned}
\psi^- \equiv \psi^-(u) &= u_1 e^1 + u_1^* e^{234} + u_2 e^2 + u_2^* e^{314} + u_3 e^3 + u_3^* e^{124} + \mathbf{i} u_4^* e^4 + \mathbf{i} u_4 e^{123}, \\
\psi^+ \equiv \psi^+(u) &= u_1 e^{41} + u_1^* e^{23} + u_2 e^{42} + u_2^* e^{31} + u_3 e^{43} + u_3^* e^{12} + \mathbf{i} u_4^* + \mathbf{i} u_4 e^{4123},
\end{aligned} \tag{5.1.3}$$

where the quantities u_{I}, with $\mathsf{I} = 1, \dots, 4$, are complex numbers, and u_{I}^* are the complex conjugates in $\mathbb{C}$. This particular choice of the complex coordinates u_{I}, and in particular the choices made for the coordinate e^4, will become justified below by the desired form of the action of the Γ-matrices.

The inner product is a pairing $\langle S_+, S_+\rangle, \langle S_-, S_-\rangle$. If we take two positive spinors $\psi^+(\tilde{u})$, and $\psi^+(u)$, the product $\langle \psi^+(\tilde{u}), \psi^+(u)\rangle$ is computed by taking the polyform $\psi^+(\tilde{u})$ in the reverse order, wedging with $\psi^+(u)$ and projecting on the top component. A simple computation gives

$$\langle \psi^+(\tilde{u}), \psi^+(u)\rangle = 2\mathrm{Re}\sum_{\mathsf{I}=1}^{4} \tilde{u}_I u_{\mathsf{I}}^*. \tag{5.1.4}$$

Thus, Majorana-Weyl spinors are identified $S_\pm \cong \mathbb{C}^4$, with the invariant pairings on $S_\pm$ being given by the standard definite Hermitian metric on $\mathbb{C}^4$.

The form of the inner product makes it clear that the basis $\{e^{41}, e^{42}, e^{43}, \mathbb{I}\}$ of S^+ in (5.1.3) is totally null. To make contact with octonions that are usually described in a non-null basis, it is necessary to switch to a different parametrisation of polyforms. We parametrise the polyforms by the real and imaginary parts of u_I and write

$$u_1 = \alpha_1 + \mathbf{i}\alpha_5,\ u_2 = \alpha_2 + \mathbf{i}\alpha_6,\ u_3 = \alpha_3 + \mathbf{i}\alpha_7,\ \text{and}\ u_4 = \alpha_0 + \mathbf{i}\alpha_4. \tag{5.1.5}$$

We will denote the components of the positive polyform by α and the negative polyform components by β. We then have

$$\begin{aligned}
\psi^+ &= \alpha_1(e^{41} + e^{23}) + \alpha_2(e^{42} + e^{31}) + \alpha_3(e^{43} + e^{12}) + \alpha_4(1 - e^{4123}) \\
&\quad + \mathbf{i}\alpha_5(e^{41} - e^{23}) + \mathbf{i}\alpha_6(e^{42} - e^{31}) + \mathbf{i}\alpha_7(e^{43} - e^{12}) + \mathbf{i}\alpha_0(1 + e^{4123}),\ \text{and} \\
\psi^- &= \beta_1(e^1 + e^{423}) + \beta_2(e^2 + e^{431}) + \beta_3(e^3 + e^{412}) + \beta_4(e^4 - e^{123}) \\
&\quad + \mathbf{i}\beta_5(e^1 - e^{423}) + \mathbf{i}\beta_6(e^2 - e^{431}) + \mathbf{i}\beta_7(e^3 - e^{412}) + \mathbf{i}\beta_0(e^4 + e^{123}).
\end{aligned} \tag{5.1.6}$$

5.1.3 The action of Γ-matrices

We now introduce a 16-component column

$$\Psi = \begin{pmatrix} \psi^+ \\ \psi^- \end{pmatrix} = \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_7 \\ \beta_0 \\ \beta_1 \\ \vdots \\ \beta_7 \end{pmatrix}. \tag{5.1.7}$$

A computation shows that the Γ-matrices become the following 16×16 matrices

$$\Gamma_0 = \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}, \quad \Gamma_a = \begin{pmatrix} 0 & -E_a \\ E_a & 0 \end{pmatrix} \text{ for } a \in \{1, \dots, 7\}, \tag{5.1.8}$$

where

$$\begin{aligned} &E_1 = -E_{01} + E_{27} - E_{36} + E_{45}, \ E_2 = -E_{02} - E_{17} + E_{35} + E_{46}, \\ &E_3 = -E_{03} + E_{16} - E_{25} + E_{47}, \ E_4 = -E_{04} - E_{15} - E_{26} - E_{37}, \\ &E_5 = -E_{05} + E_{14} + E_{23} - E_{67}, \ E_6 = -E_{06} - E_{13} + E_{24} + E_{57}, \text{ and} \\ &E_7 = -E_{07} + E_{12} + E_{34} - E_{56}. \end{aligned} \tag{5.1.9}$$

5.1.4 Octonions

The space of octonions $\mathbb{O}$ is a normed algebra with the property $|xy| = |x||y|$ (i.e. a composition algebra). The usual octonions (unlike split octonions) also have the property that the norm of every non-zero element is not zero, which makes them into a division algebra. It is non-commutative and non-associative, but alternative, which can be stated as the property that the subalgebra generated by any two elements is associative.

A general octonion is an object

$$q = q_0 \mathbb{I} + \sum_{a=1}^{7} q_a \mathbf{e}^a, \tag{5.1.10}$$

where $\mathbf{e}^a$ are unit imaginary octonions. The unit octonions anticommute and square to minus the identity. The octonion conjugate changes the sign of all the imaginary octonions. The octonionic pairing is

$$(q, q) = |q|^2 = q\overline{q} = (q_0)^2 + \sum_{a=1}^{7} (q_a)^2. \tag{5.1.11}$$

Here, $\overline{q}$ is the octonionic conjuation on $\mathbb{O}$. We encode the octonionic product by the cross-product in the space of imaginary octonions. Thus, we write

$$\mathbb{O} = \mathbb{R} \oplus \operatorname{Im} \mathbb{O}. \tag{5.1.12}$$

Let $\mathbf{e}^{1,\dots,7}$ be a basis in the space of imaginary octonions. The cross-product in $\operatorname{Im}\mathbb{O}$ can be encoded by the following 3-form

$$C = \mathbf{e}^{567} + \mathbf{e}^5(\mathbf{e}^{41} - \mathbf{e}^{23}) + \mathbf{e}^6(\mathbf{e}^{42} - \mathbf{e}^{31}) + \mathbf{e}^7(\mathbf{e}^{43} - \mathbf{e}^{12}). \tag{5.1.13}$$

This encodes the cross-product in the sense that $C(\mathbf{e}^a, \mathbf{e}^b, \mathbf{e}^c) = (\mathbf{e}^a \times \mathbf{e}^b, \mathbf{e}^c)$, where the standard metric on $\mathbb{R}^7$ is used. So, for instance $\mathbf{e}^5 \times \mathbf{e}^6 = \mathbf{e}^7$. C is also referred to as an associative calibration, the Hodge-dual to C in 7 dimensions, $*C$, is referred to as the coassociative calibration [SW10].

5.1.5 Octonionic model for Cliff_8

The octonionic product can also be encoded into 8×8 matrices. To this end, we represent a general octonion as an 8-component column. Then the operators of left multiplication by unit imaginary octonions can be checked to be given by $L_{\mathbf{e}^a} = E_a$, where E_a are precisely the same matrices already encountered in (5.1.9). Coming back to our model for the Clifford algebra, Cliff_8 we see that the general linear combination of the Γ-matrices is,

$$q_0 \Gamma_0 + \sum_{a=1}^{7} q_a \Gamma_a = \begin{pmatrix} 0 & L_{\bar{q}} \\ L_q & 0 \end{pmatrix}. \tag{5.1.14}$$

We thus see that Cliff_8 is generated by matrices (5.1.14) that act on 2-component columns with entries in $\mathbb{O}$. Majorana-Weyl spinors are then identified with copies of $\mathbb{O}$.

5.1.6 Pure Spinor Orbits in Spin(8)

In the creation and annihilation operator model, pure spinors are decomposable polyforms. In particular, the model comes with two preferred pure spinors, the identity polyform, and the top polyform. For Cliff_8, both are in S_+, and both are null spinors. It is also clear from (5.1.3) that only the linear combinations $\mathbf{i}(\mathbf{1} + e^{4123})$ and $\mathbf{1} - e^{4123}$ are Majorana-Weyl spinors. Majorana-Weyl spinors can be identified with $\mathbb{O}$, and are never null. Thus, we cannot see pure spinors if we restrict our attention to Majorana-Weyl spinors.

To describe pure spinors we need complexified real polyforms, which are then identified with complexified octonions. Hence, in the octonionic description of Cliff_8 the generic Weyl spinor is a complexified octonion. In particular, pure spinors are necessarily complexified octonions, because they are null.

Let us see how this works for the two canonical pure spinors. The identity octonion $\mathbb{I} \in \text{Re}(\mathbb{O})$, and purely imaginary unit octonion $\mathbf{e}^4 \in \text{Im}(\mathbb{O})$ are encoded as

$$\mathbb{I} = \mathbf{i}(\mathbf{1} + e^{4123}), \text{ and } \mathbf{u} := \mathbf{e}^4 = \mathbf{1} - e^{4123}\ {}^{1}. \tag{5.1.15}$$

This means that the pure spinors $1, e^{4123} \in S_+$ are given by

$$\mathbf{1} = \frac{1}{2\mathbf{i}}(\mathbb{I} + \mathbf{i}\mathbf{u}), \text{ and } e^{4123} = \frac{1}{2\mathbf{i}}(\mathbb{I} - \mathbf{i}\mathbf{u}). \tag{5.1.16}$$

We have the operation of complex conjugation that reverses the sign in front of $\mathbf{i}$. In fact, this is the complex conjugation operator $\mathbf{R}$, an antilinear operator that acts on polyforms, interpolating between $\mathbf{1}$ and e^{4123}. This operation should not be confused with the octonion conjugation.

For Spin(8), the only possible numbers of insertions of Γ-matrices between two Weyl spinors of the same chirality are zero, two, and four. The product of two Γ-matrices restricted to S_+ is expressible as either E_a or $E_a E_b$ with $a, b = 1, \ldots, 7$. Both are antisymmetric. Thus, $B_2(\psi_+, \psi_+) = 0$. This shows that null spinors with $B_0(\psi_+, \psi_+) = 0$ are pure. Because the inner product of a Majorana-Weyl spinor with itself is the norm squared of the corresponding octonion, the Majorana-Weyl spinors are not pure.

In (5.1.16), we have an example of two canonical pure spinors. In the octonion description, they become complex linear combinations of two unit octonions. From general considerations, we

[1] The convenient notation of denoting $\mathbf{u}$ as the imaginary unit octonion $\mathbf{e}^4$ is used throughout the rest of this chapter.

know that either of these two pure spinors defines a complex structure in $\mathbb{R}^8$ and that $B_4(\psi_+, \psi_+)$ is the product of four null directions spanning one of the eigenspaces of this complex structure. It is interesting to compute both the complex structure and $B_4(\psi_+, \psi_+)$ explicitly for a given representative.

Proposition 5.1.6.1. The symplectic form defined by the pure spinor $\psi_+ = \mathbf{1}$ is proportional to $\mathbb{I} \wedge \mathbf{u} - \omega$. Where

$$\omega \coloneqq e^{15} + e^{26} + e^{37}. \tag{5.1.17}$$

Proof. The complex structure comes from $B_2(\mathbf{R}(\psi_+), \psi_+)$. We have $\mathbf{R}(1) \propto e^{4123}$ and so

$$B_2(\mathbf{R}(\psi_+), \psi_+) \propto \langle (\mathbb{I} + \mathbf{iu}), \Gamma\Gamma(\mathbb{I} - \mathbf{iu}) \rangle. \tag{5.1.18}$$

The components of this 2-form in the $\mathbb{I}, x \in \mathrm{Im}(\mathbb{O})$ directions are

$$\begin{aligned} \langle (\mathbb{I} + \mathbf{iu}), \Gamma_{\mathbb{I}}\Gamma_x(\mathbb{I} - \mathbf{iu}) \rangle &= ((\mathbb{I} + \mathbf{iu}), L_x(\mathbb{I} - \mathbf{iu})), \text{and} \\ \langle (\mathbb{I} + \mathbf{iu}), \Gamma_x\Gamma_y(\mathbb{I} - \mathbf{iu}) \rangle &= ((\mathbb{I} + \mathbf{iu}), L_{\bar{x}}L_y(\mathbb{I} - \mathbf{iu})). \end{aligned} \tag{5.1.19}$$

We also have

$$\begin{aligned} ((\mathbb{I} + \mathbf{iu}), L_x(\mathbb{I} - \mathbf{iu})) &= 2\mathbf{iu}, \\ ((\mathbb{I} + \mathbf{iu}), L_xL_y(\mathbb{I} - \mathbf{iu})) &= 2\mathbf{i}(e^{15} + e^{26} + e^{37}) \coloneqq 2\mathbf{i}\omega. \end{aligned} \tag{5.1.20}$$

Thus, overall, we have the following 2-form in $\mathbb{R}^8$

$$\frac{1}{2\mathbf{i}} \langle (\mathbb{I} + \mathbf{iu}), \Gamma\Gamma(\mathbb{I} - \mathbf{iu}) \rangle = \mathbb{I} \wedge \mathbf{u} - \omega. \tag{5.1.21}$$

□

Hence, raising an index with the metric gives a complex structure reducing Spin(8) to SU(4). The complex structure is present from a choice of complex structure on $\mathbb{O}$. A natural choice is $L_{\mathbf{u}}$ of left multiplication by the unit imaginary octonion $\mathbf{u}$. The eigenvectors of eigenvalue $-\mathbf{i}$ are given by

$$\mathbb{I} + \mathbf{iu},\ \mathbf{e}^1 + \mathbf{ie}^5,\ \mathbf{e}^2 + \mathbf{ie}^6,\ \text{and } \mathbf{e}^3 + \mathbf{ie}^7. \tag{5.1.22}$$

The Kähler form arising is proportional to $\mathbb{I} \wedge \mathbf{u} + \omega$, which is (5.1.21) up to the sign in front of the last term. Consider instead the operator $R_{\mathbf{u}}$ of right multiplication by $\mathbf{u}$. Its $-\mathbf{i}$ eigenvectors are now

$$\mathbb{I} + \mathbf{iu},\ \mathbf{e}^1 - \mathbf{ie}^5,\ \mathbf{e}^2 - \mathbf{ie}^6,\ \text{and } \mathbf{e}^3 - \mathbf{ie}^7, \tag{5.1.23}$$

and the corresponding Kähler form is proportional to (5.1.21). Thus, the sign discrepancy comes from a choice of left *or* right multiplication by unit imaginary octonion $\mathbf{u}$.

We can alternatively recover the complex structure with its $(0,1)$ and $(1,0)$ directions by computing $B_4(\psi_+, \psi_+)$ or $B_4(\mathbf{R}(\psi_+), \mathbf{R}(\psi_+))$. We know that both are decomposable and are given by the product of four $(0,1)$ null directions, or four $(1,0)$ directions.

Proposition 5.1.6.2. The top holomorphic form that encodes the complex structure for $\psi_+ = \mathbf{1}$ is proportional to $(\mathbb{I} + \mathbf{iu}) \wedge \Omega \coloneqq \Omega_8$. Where

$$\Omega \coloneqq (e^1 - \mathbf{i}e^5) \wedge (e^2 - \mathbf{i}e^6) \wedge (e^3 - \mathbf{i}e^7). \tag{5.1.24}$$

Proof. Thus, we want to compute

$$\langle (\mathbb{I} - \mathbf{iu}), \Gamma\Gamma\Gamma\Gamma(\mathbb{I} - \mathbf{iu}) \rangle \in \Lambda^4(\mathbb{R}^8). \tag{5.1.25}$$

The various components of this 4-form that we need are given by

$$\begin{aligned}\langle\psi,\Gamma_{\mathbb{I}}\Gamma_x\Gamma_y\Gamma_z\psi\rangle &= (\psi, L_x L_{\bar{y}} L_z\psi), \text{ and}\\ \langle\psi,\Gamma_x\Gamma_y\Gamma_z\Gamma_w\psi\rangle &= (\psi, L_x L_{\bar{y}} L_z L_{\bar{w}}\psi),\end{aligned} \tag{5.1.26}$$

where on the right-hand side the spinor ψ is interpreted as a (complexified) octonion. A computation gives

$$\begin{aligned}((\mathbb{I}-\mathbf{iu}), L_x L_y L_z(\mathbb{I}-\mathbf{iu})) &= 2\mathbf{i}\Omega,\\ ((\mathbb{I}-\mathbf{iu}), L_x L_y L_z L_w(\mathbb{I}-\mathbf{iu})) &= 2\mathbf{u}\wedge\Omega.\end{aligned} \tag{5.1.27}$$

This means that

$$\frac{\mathbf{i}}{2}\langle(\mathbb{I}-\mathbf{iu}), \Gamma\Gamma\Gamma\Gamma(\mathbb{I}-\mathbf{iu})\rangle = (\mathbb{I}+\mathbf{iu})\wedge\Omega. \tag{5.1.28}$$

□

The 4-form obtained as $\langle(\mathbb{I}+\mathbf{iu}),\Gamma\Gamma\Gamma(\mathbb{I}+\mathbf{iu})\rangle$ is given by the product of the complex conjugate directions. To summarise, we learn that the complex structure on $\mathbb{R}^8 \cong \mathbb{O}$ that corresponds to the complex conjugate pair of pure spinors $\mathbb{I}+\mathbf{iu}$ and $\mathbb{I}-\mathbf{iu}$ is given by $R_{\mathbf{u}}$, the right multiplication by $\mathbf{u}$. Furthermore, the differential forms constructed from the pure spinors are stabilised by SU(4), so there is a reduction from Spin(8) to SU(4) if one chooses a pure spinor orbit. This was an enlightening example for the next section, where we will show that a general complex Weyl spinor corresponds to an SU(4)-structure as well.

5.1.7 Impure Spinor Orbits in Spin(8)

There are two classes of orbits for Spin(8). The first are Majorana-Weyl spinors, or real octonions, with a stabiliser of Spin(7). Then a generic Weyl spinor, or complexified octonions, is considered. This type of spinor will have an SU(4) stabiliser.

We begin by analysing Majorana-Weyl spinors. Such spinors cannot be pure. First, such a spinor has a non-vanishing norm $\langle\psi_M,\psi_M\rangle$, which coincides with the norm squared of the corresponding octonion. Second, the group Spin(8) acts on the space of Majorana-Weyl spinors of fixed norm and of one helicity transitively, with the stabiliser Spin(7). So, we have

$$S^7 = \mathrm{Spin}(8)/\mathrm{Spin}(7). \tag{5.1.29}$$

This can be seen from the fact that the Majorana-Weyl representation is isomorphic to the vector representation $S_+ \cong \mathbb{R}^8$.

A Majorana-Weyl spinor of Spin(8) thus has Spin(7) as the stabiliser, and endows $\mathbb{R}^8$ with a Spin(7)-structure. This is the 4-form $B_4(\psi_M,\psi_M)$ — the Cayley form, whose stabiliser in $\mathrm{GL}(8,\mathbb{R})$ is Spin(7). Any unit Majorana-Weyl spinor is in the Spin(8) orbit of this spinor. It is instructive to compute this 4-form explicitly for a representative.

Proposition 5.1.7.1. The Cayley form for the choice of Majorana-Weyl spinor $\psi_M = \mathbb{I}$ is proportional to $\mathbb{I}\wedge C + *C$, where

$$\begin{aligned}C &:= e^{567} + e^5(e^{41}-e^{23}) + e^6(e^{42}-e^{31}) + e^7(e^{43}-e^{12}), \text{ and}\\ *C &:= e^{1234} + e^{67}(e^{41}-e^{23}) + e^{75}(e^{42}-e^{31}) + e^{56}(e^{43}-e^{12}).\end{aligned} \tag{5.1.30}$$

Note that C is just the associative calibration that encodes the octonion product, and $*C$ is its Hodge dual in $\mathbb{R}^7$, the coassociative calibration.

Proof. The components of this 4-form are given by (5.1.26), hence,

$$(\mathbb{I}, L_x L_y L_z \mathbb{I}) = -C, \text{ and } (\mathbb{I}, L_x L_y L_z L_w \mathbb{I}) = - * C. \tag{5.1.31}$$

Thus, overall, we have

$$\Lambda^4(\mathbb{R}^8) \ni \langle \mathbb{I}, \Gamma\Gamma\Gamma\Gamma\mathbb{I}\rangle = -\mathbb{I}\wedge C - *C. \tag{5.1.32}$$

□

We note, for future use, that

$$C = \mathbf{u}\wedge\omega + \mathrm{Im}(\Omega), \qquad *C = \mathrm{Re}(\Omega)\wedge\mathbf{u} + \frac{1}{2}\omega\wedge\omega. \tag{5.1.33}$$

A generic Weyl spinor of Spin(8) is a complexified octonion. When the spinor is not null, we can always rescale it by a complex number to make it unit. As explained in [Cha97], see page 33, a complex impure spinor defines a certain pure spinor. Indeed, assuming ψ is unit and denoting $\lambda = \langle \mathbf{R}(\psi), \psi\rangle$, consider

$$\tilde{\psi} = \frac{\lambda\psi - \mathbf{R}(\psi)}{\sqrt{\lambda^2 - 1}}. \tag{5.1.34}$$

Then $\langle\tilde{\psi}, \tilde{\psi}\rangle = -1$ and $\psi + \tilde{\psi}$ is null and therefore pure. Because the stabiliser of $\mathbf{R}(\psi)$ is the same as the stabiliser of ψ (since the stabiliser is real), this stabiliser also coincides with that of $\tilde{\psi}$ and thus $\psi + \tilde{\psi}$. This means that the stabiliser of ψ is that of a pure spinor, which is SU(4).

The above discussion suggests that a general complex spinor continues to define a complex structure on $\mathbb{R}^8$. To see this, we begin by showing that ψ (and $\tilde{\psi}$) can take canonical forms.

Lemma 5.1.7.1. Let ψ and $\tilde{\psi}$ unit Weyl spinors as above. Then they can take the following forms.

$$\psi = \cosh\tau\mathbb{I} + \mathbf{i}\sinh\tau\mathbf{u}, \text{ and } \mathbf{R}(\psi) = \cosh\tau\mathbb{I} - \mathbf{i}\sinh\tau\mathbf{u}, \tag{5.1.35}$$

and

$$\tilde{\psi} = \sinh\tau\mathbb{I} + \mathbf{i}\cosh\tau\mathbf{u}, \text{ and } \mathbf{R}(\tilde{\psi}) = \sinh\tau\mathbb{I} - \mathbf{i}\cosh\tau\mathbf{u}. \tag{5.1.36}$$

Here τ is a real parameter.

Proof. A complex spinor has a well-defined real and imaginary parts $\psi = \alpha + \mathbf{i}\beta$, for $\alpha, \beta \in \mathbb{O}$. Two invariant scalars can be constructed

$$\langle\psi, \psi\rangle = |\alpha|^2 - |\beta|^2 + 2\mathbf{i}(\alpha, \beta), \text{ and } \langle\mathbf{R}(\psi), \psi\rangle = |\alpha|^2 + |\beta|^2. \tag{5.1.37}$$

Rescaling the spinor to make it unit, implies $|\alpha|^2 - |\beta|^2 = 1, (\alpha, \beta) = 0$. Also, using the action of Spin(8) we can make α a multiple of $\mathbb{I}$, and then use Spin(7) that stabilises $\mathbb{I}$ to make β (which is orthogonal to α) to be a multiple of $\mathbf{u}$. Thus ψ takes the form,

$$\psi = \cosh\tau\mathbb{I} + \mathbf{i}\sinh\tau\mathbf{u}, \text{ and } \mathbf{R}(\psi) = \cosh\tau\mathbb{I} - \mathbf{i}\sinh\tau\mathbf{u}. \tag{5.1.38}$$

Using $\lambda = \cosh^2\tau + \sinh^2\tau = \cosh(2\tau)$ and (5.1.34) allows $\tilde{\psi}$ to take the form,

$$\tilde{\psi} = \sinh\tau\mathbb{I} + \mathbf{i}\cosh\tau\mathbf{u}, \text{ and } \mathbf{R}(\tilde{\psi}) = \sinh\tau\mathbb{I} - \mathbf{i}\cosh\tau\mathbf{u}. \tag{5.1.39}$$

□

The arguments above imply that

$$\psi + \tilde{\psi} = 2(\cosh\tau + \sinh\tau)(\mathbb{I} + \mathbf{i}\mathbf{u}), \tag{5.1.40}$$

which is the pure spinor we already considered above. Indeed we have the following,

Lemma 5.1.7.2. The Kähler form $B_2(\psi, \mathbf{R}(\psi))$ that encodes the complex structure for ψ, i.e. the complex structure defined by the pure spinor $\mathbb{I} + \mathbf{i}\mathbf{u}$, is given as $\mathbf{i}\sinh(2\tau)\omega_8$, where ω_8 is a symplectic form in 8 dimensions.

Furthermore, the complex 4-form $B_4(\mathbf{R}(\psi), \mathbf{R}(\psi))$ is given as

$$\cosh(2\tau)\mathrm{Im}(\Omega_8) - \mathbf{i}\sinh(2\tau)\mathrm{Re}(\Omega_8) - \frac{1}{2}\omega_8 \wedge \omega_8 \tag{5.1.41}$$

(the so-called complex Cayley form [Kra24b]). Here Ω_8 is a top holomorphic form in 8 dimensions (5.1.28).

Proof. Computing the following

$$(\psi, L_x\mathbf{R}(\psi)) = \mathbf{i}\sinh(2\tau)\mathbf{u}, \text{ and } (\psi, L_xL_y\mathbf{R}(\psi)) = \mathbf{i}\sinh(2\tau)\omega, \tag{5.1.42}$$

this gives

$$\langle\psi, \Gamma\Gamma\mathbf{R}(\psi)\rangle = \mathbf{i}\sinh(2\tau)(\mathbb{I}\wedge\mathbf{u} - \omega) \coloneqq \mathbf{i}\sinh(2\tau)\omega_8. \tag{5.1.43}$$

To compute the 4-form $B_4(\psi, \psi)$ we need the following results

$$\begin{aligned} (\mathbf{R}(\psi), L_xL_yL_z\mathbf{R}(\psi)) &= \mathbf{i}\sinh(2\tau)\mathrm{Re}(\Omega) - \cosh(2\tau)\mathrm{Im}(\Omega) - \mathbf{u}\wedge\omega, \text{ and} \\ (\mathbf{R}(\psi), L_xL_yL_zL_w\mathbf{R}(\psi)) &= -\cosh(2\tau)\mathrm{Re}(\Omega)\wedge\mathbf{u} - \mathbf{i}\sinh(2\tau)\mathrm{Im}(\Omega)\wedge\mathbf{u} - \frac{1}{2}\omega\wedge\omega. \end{aligned} \tag{5.1.44}$$

This means that

$$\begin{aligned} \langle\mathbf{R}(\psi), \Gamma\Gamma\Gamma\Gamma\mathbf{R}(\psi)\rangle &= \cosh(2\tau)(\mathbb{I}\wedge\mathrm{Im}(\Omega) + \mathbf{u}\wedge\mathrm{Re}(\Omega)) \\ &\quad - \mathbf{i}\sinh(2\tau)(\mathbb{I}\wedge\mathrm{Re}(\Omega) - \mathbf{u}\wedge\mathrm{Im}(\Omega)) \\ &\quad - \frac{1}{2}(\mathbb{I}\wedge\mathbf{u} - \omega)\wedge(\mathbb{I}\wedge\mathbf{u} - \omega) \\ &\coloneqq \cosh(2\tau)\mathrm{Im}(\Omega_8) - \mathbf{i}\sinh(2\tau)\mathrm{Re}(\Omega_8) - \frac{1}{2}\omega_8\wedge\omega_8. \end{aligned} \tag{5.1.45}$$

□

The complex Cayley form, $B_4(\mathbf{R}(\psi), \mathbf{R}(\psi))$, is SU(4) invariant. Indeed, because the blocks $\mathbb{I}\wedge\mathrm{Im}(\Omega)+\mathbf{u}\wedge\mathrm{Re}(\Omega)$ and $\mathbb{I}\wedge\mathrm{Re}(\Omega)-\mathbf{u}\wedge\mathrm{Im}(\Omega)$ are the imaginary and real parts of the holomorphic 4-form Ω_8 (5.1.28), they are SU(4) invariant. The last term is the product of the two copies of the Kähler (1,1)-form, which is again SU(4) invariant. This leads to a statement on integrability of SU(4)-structures derived from Weyl spinors.

Proposition 5.1.7.2. The complex Cayley form, $B_4(\mathbf{R}(\psi), \mathbf{R}(\psi))$, is an integrable SU(4)-structure if, and only if, the exterior derivative of $dB_4(\mathbf{R}(\psi), \mathbf{R}(\psi)) = 0$.

This is proved in Proposition 5 of [Kra24b]. The SU(4)-structure here is not generated by a pure spinor, but a complex Weyl spinor[2]. This is one of the few cases where we have understood the integrability of a non-classical spinorial G-structure.

5.2 Spin(4,4) and Split Octonions

5.2.1 Real model

The link to split octonions arises if we consider Majorana-Weyl spinors. These are easiest to describe in the real model, which starts by selecting a paracomplex structure on $\mathbb{R}^{4,4}$. However,

[2] A pure spinor, also generates a SU(4)-structure whose integrability corresponds to being parallel with respect to the Levi-Civita connection.

Majorana-Weyl real spinors do not capture all possible spinor types that arise, as we have witnessed in the previous chapter. For this reason, it is better to develop everything in a complex model from the beginning, and then impose the Majorana condition if needed. Nevertheless, we start by describing the simpler real model and then switch to the complex description. The real model arises by selecting a paracomplex structure on $\mathbb{R}^{4,4}$. Let $\mathbb{R}^4$ be one of the resulting totally null subspaces, and let u_{I}, for $\mathsf{I} = 1,\dots,4$, be the null coordinates and $\mathrm{d}u_{\mathsf{I}}$ the basic one-forms. It is more convenient to use the notation $\mathrm{d}u_{\mathsf{I}} = e_{\mathsf{I}}$. We introduce four pairs of creation and annihilation operators a_{I} and $a_{\mathsf{I}}^\dagger$. We define the Γ-matrices as follows.

$$\begin{aligned} &\Gamma_0 = a_4 + a_4^\dagger,\ \Gamma_1 = a_1 + a_1^\dagger,\ \Gamma_2 = a_2 + a_2^\dagger,\\ &\Gamma_3 = a_3 + a_3^\dagger,\ \Gamma_4 = a_4 - a_4^\dagger,\ \Gamma_5 = a_1 - a_1^\dagger,\\ &\Gamma_6 = a_2 - a_2^\dagger,\ \text{and } \Gamma_7 = a_3 - a_3^\dagger. \end{aligned} \tag{5.2.1}$$

The Majorana-Weyl spinors are even and odd polyforms in $\Lambda(\mathbb{R}^4)$ with real coefficients. The basic polyforms $e_{\mathsf{IJ}\dots} := e_{\mathsf{I}} \wedge e_{\mathsf{J}} \wedge \dots$ are all null with respect to the invariant inner product, which will be described below. For this reason, to establish a link with the split octonions in the usual non-null basis, we introduce a non-null basis in $\Lambda(\mathbb{R}^4)$. This leads us to write,

$$\begin{aligned} S^+ \ni \psi^+ &= \alpha_1(e_{41} - e_{23}) + \alpha_2(e_{42} - e_{31}) + \alpha_3(e_{43} - e_{12}) + \alpha_4(1 - e_{4123})\\ &\quad + \alpha_5(e_{41} + e_{23}) + \alpha_6(e_{42} + e_{31}) + \alpha_7(e_{43} + e_{12}) + \alpha_0(1 + e_{4123}),\ \text{and}\\ S^- \ni \psi^- &= \beta_1(e_1 - e_{423}) + \beta_2(e_2 - e_{431}) + \beta_3(e_3 - e_{412}) + \beta_4(e_4 - e_{123})\\ &\quad + \beta_5(e_1 + e_{423}) + \beta_6(e_2 + e_{431}) + \beta_7(e_3 + e_{412}) + \beta_0(e_4 + e_{123}). \end{aligned} \tag{5.2.2}$$

The invariant inner product is a pairing $\langle S^\pm, S^\pm\rangle$. A simple computation gives

$$\begin{aligned} \langle\psi^+,\psi^+\rangle &= 2((\alpha_0)^2 + (\alpha_1)^2 + (\alpha_2)^2 + (\alpha_3)^2 - (\alpha_4)^2 - (\alpha_5)^2 - (\alpha_6)^2 - (\alpha_7)^2),\ \text{and}\\ \langle\psi^-,\psi^-\rangle &= 2((\beta_0)^2 + (\beta_1)^2 + (\beta_2)^2 + (\beta_3)^2 - (\beta_4)^2 - (\beta_5)^2 - (\beta_6)^2 - (\beta_7)^2). \end{aligned} \tag{5.2.3}$$

We now form a Dirac spinor, which is a 16-component column

$$\Psi = \begin{pmatrix} \psi^+ \\ \psi^- \end{pmatrix} = \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_7 \\ \beta_0 \\ \beta_1 \\ \vdots \\ \beta_7 \end{pmatrix}, \tag{5.2.4}$$

where we put the 8 α_i components of ψ^+ on top and 8 β_j components of ψ^- at the bottom of the column. The order in which the components appear is $0,1,\dots,7$. The Γ-matrices become the following Γ-matrices in this basis

$$\Gamma_0 = \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}, \quad \Gamma_a = \begin{pmatrix} 0 & -\tilde{E}_a \\ \tilde{E}_a & 0 \end{pmatrix} \text{ for } a \in \{1,\dots,7\}, \tag{5.2.5}$$

where

$$\begin{aligned}&\tilde{E}_1 = -E_{01} - E_{23} - E_{45} + E_{67},\ \tilde{E}_2 = -E_{02} - E_{31} - E_{46} + E_{75},\\&\tilde{E}_3 = -E_{03} - E_{12} - E_{47} + E_{56},\ \tilde{E}_4 = S_{04} - S_{15} - S_{26} - S_{37}\\&\tilde{E}_5 = S_{05} + S_{14} - S_{27} + S_{36}\ \tilde{E}_6 = S_{06} + S_{17} + S_{24} - S_{35}, \text{and}\\&\tilde{E}_7 = S_{07} - S_{16} + S_{25} + S_{34}.\end{aligned} \tag{5.2.6}$$

5.2.2 Split octonions

Split octonions $\mathbb{O}'$ form a non-associative normed composition algebra. It is not a division algebra because there are null elements. A split octonion is an object

$$\tilde{q} = \tilde{q}_0\mathbb{I} + \sum_{a=1}^{7} \tilde{q}_a \tilde{\mathbf{e}}^a. \tag{5.2.7}$$

The unit imaginary octonions $\tilde{\mathbf{e}}^a$ anticommute and satisfy

$$(\tilde{\mathbf{e}}^1)^2 = (\tilde{\mathbf{e}}^2)^2 = (\tilde{\mathbf{e}}^3)^2 = -\mathbb{I}, \quad (\tilde{\mathbf{e}}^4)^2 = (\tilde{\mathbf{e}}^5)^2 = (\tilde{\mathbf{e}}^6)^2 = (\tilde{\mathbf{e}}^7)^2 = \mathbb{I}. \tag{5.2.8}$$

Thus, the split octonions $\mathbb{I}, \tilde{\mathbf{e}}^1, \tilde{\mathbf{e}}^2, \tilde{\mathbf{e}}^3$ generate a copy of $\mathbb{H} \subset \mathbb{O}'$. The octonion pairing is given by

$$(\tilde{q}, \tilde{q}) = \tilde{q}\bar{\tilde{q}} = (\tilde{q}_0)^2 + (\tilde{q}_1)^2 + (\tilde{q}_2)^2 + (\tilde{q}_3)^2 - (\tilde{q}_4)^2 - (\tilde{q}_5)^2 - (\tilde{q}_6)^2 - (\tilde{q}_7)^2, \tag{5.2.9}$$

where the conjugation, denoted by a bar, changes the signs of all the imaginary generators.

The product rules are most efficiently encoded into the following 3-form on $\mathrm{Im}(\mathbb{O}') = \mathbb{R}^7$

$$\tilde{C} = \tilde{\mathbf{e}}^{123} - \tilde{\mathbf{e}}^1(\tilde{\mathbf{e}}^{45} - \tilde{\mathbf{e}}^{67}) - \tilde{\mathbf{e}}^2(\tilde{\mathbf{e}}^{46} - \tilde{\mathbf{e}}^{75}) - \tilde{\mathbf{e}}^3(\tilde{\mathbf{e}}^{47} - \tilde{\mathbf{e}}^{56}). \tag{5.2.10}$$

This encodes the vector product via

$$(u \times v, w) = w \lrcorner v \lrcorner u \lrcorner \tilde{C}. \tag{5.2.11}$$

Here $u\lrcorner$ is the operator of insertion of a vector field u into a differential form. For example $\tilde{\mathbf{e}}^1 \times \tilde{\mathbf{e}}^2 = \tilde{\mathbf{e}}^3$, but $\tilde{\mathbf{e}}^1 \times \tilde{\mathbf{e}}^6 = -\tilde{\mathbf{e}}^7$ because the octonion pairing is negative-definite on directions $4, 5, 6, 7$.

5.2.3 Octonionic model for $\mathrm{Cliff}_{4,4}$

One can encode the operators of left multiplication by a unit octonion into 8×8 matrices. Indeed, we encode an octonion into an 8-component column

$$\tilde{q} \to \begin{pmatrix} \tilde{q}_0 \\ \tilde{q}_1 \\ \vdots \\ \tilde{q}_7 \end{pmatrix}. \tag{5.2.12}$$

It is then a straightforward computation to see that the operators of left multiplication by a unit octonion precisely match the matrices in (5.2.6) $L_{\tilde{\mathbf{e}}^a} = \tilde{E}_a$.

$$\tilde{q}_0\Gamma_0 + \sum_{a=1}^{7} \tilde{q}_a\Gamma_a = \begin{pmatrix} 0 & L_{\bar{\tilde{q}}} \\ L_{\tilde{q}} & 0 \end{pmatrix}, \tag{5.2.13}$$

where $L_{\tilde{q}}$ is the operator of left multiplication by a split octonion $\tilde{q} \in \mathbb{O}'$.

5.2.4 The complex model

We now develop the complex model of $\mathrm{Cliff}_{4,4}$. The starting point is a complex structure on $\mathbb{R}^{4,4}$, so that $\mathbb{R}^{4,4}$ gets identified with $\mathbb{C}^4$. Let $z_I, I = 1, \ldots, 4$ be the corresponding complex (null) coordinates, and $e^I = \mathrm{d}z_I$ the basic one-forms. The metric on $\mathbb{R}^{4,4}$ becomes the following indefinite Hermitian metric

$$e^3\bar{e}^3 + e^4\bar{e}^4 - e^1\bar{e}^1 - e^2\bar{e}^2. \tag{5.2.14}$$

A general Dirac spinor is a polyform in $\Lambda(\mathbb{C}^4)$, with complex coefficients. The Γ-matrices that square to plus the identity are given by

$$\Gamma_0 = a_4 + a_4^\dagger,\ \Gamma_3 = \mathbf{i}(a_4 - a_4^\dagger),\ \Gamma_2 = a_3 + a_3^\dagger,\ \text{and}\ \Gamma_1 = \mathbf{i}(a_3 - a_3^\dagger). \tag{5.2.15}$$

Note that these generate a copy of Cliff_4, and act only on the e_3, e_4 polyform directions. The Γ-matrices that square to minus the identity are

$$\Gamma_4 = \mathbf{i}(a_2 + a_2^\dagger),\ \Gamma_7 = a_2 - a_2^\dagger,\ \Gamma_6 = \mathbf{i}(a_1 + a_1^\dagger),\ \text{and}\ \Gamma_5 = a_1 - a_1^\dagger. \tag{5.2.16}$$

The link to split octonions arises if we consider Majorana-Weyl spinors, so we must understand the reality conditions first.

5.2.5 Reality conditions

There are two antilinear operators, the product of all real Γ-matrices followed by the complex conjugation, and the product of the imaginary ones followed by the complex conjugation. Both square to plus the identity, and so give a possible reality condition. They only differ in their action (by a sign) on odd polyforms, and agree on even polyforms. It turns out to be better to use

$$\mathbf{R} = \Gamma_3\Gamma_1\Gamma_4\Gamma_6\mathfrak{C} \tag{5.2.17}$$

as the reality condition. A simple calculation shows that the following polyforms parametrised by $\mathbb{C}^4$ are real

$$\begin{aligned}\psi_- &= u_1e^1 - u_1^*e^{423} + u_2e^2 - u_2^*e^{431} + u_3e^3 + u_3^*e^{412} + \mathbf{i}u_4^*e^4 + \mathbf{i}u_4e^{123},\ \text{and}\\ \psi_+ &= u_1e^{41} - u_1^*e^{23} + u_2e^{42} - u_2^*e^{31} + u_3e^{43} + u_3^*e^{12} + \mathbf{i}u_4^* + \mathbf{i}u_4e^{4123}.\end{aligned} \tag{5.2.18}$$

This should be compared to (5.1.3). Only some signs are different as compared to the Cliff_8 situation. We parametrise the even polyforms by the real and imaginary parts of u_I

$$u_1 = \alpha_5 + \mathbf{i}\alpha_6,\ u_2 = \alpha_7 + \mathbf{i}\alpha_4,\ u_3 = \alpha_2 + \mathbf{i}\alpha_1,\ \text{and}\ u_4 = \alpha_3 + \mathbf{i}\alpha_0. \tag{5.2.19}$$

We get the following real parametrisation of even and odd polyforms

$$\begin{aligned}\psi^+ &= \alpha_5(e^{41} - e^{23}) + \alpha_7(e^{42} - e^{31}) + \alpha_2(e^{43} + e^{12}) + \alpha_0(1 - e^{4123})\\ &\quad + \mathbf{i}\alpha_6(e^{41} + e^{23}) + \mathbf{i}\alpha_4(e^{42} + e^{31}) + \mathbf{i}\alpha_1(e^{43} - e^{12}) + \mathbf{i}\alpha_3(1 + e^{4123}),\ \text{and}\\ \psi^- &= \beta_5(e^1 - e^{423}) + \beta_7(e^2 - e^{431}) + \beta_2(e^3 + e^{412}) + \beta_0(e^4 - e^{123})\\ &\quad + \mathbf{i}\beta_6(e^1 + e^{423}) + \mathbf{i}\beta_4(e^2 + e^{431}) + \mathbf{i}\beta_1(e^3 - e^{412}) + \mathbf{i}\beta_3(e^4 + e^{123}).\end{aligned} \tag{5.2.20}$$

The spinor norms are then

$$\begin{aligned}\frac{1}{2}\langle\psi_+, \psi_+\rangle &= (\alpha_0)^2 + (\alpha_1)^2 + (\alpha_2)^2 + (\alpha_3)^2 - (\alpha_4)^2 - (\alpha_5)^2 - (\alpha_6)^2 - (\alpha_7)^2,\ \text{and}\\ \frac{1}{2}\langle\psi_-, \psi_-\rangle &= (\beta_0)^2 + (\beta_1)^2 + (\beta_2)^2 + (\beta_3)^2 - (\beta_4)^2 - (\beta_5)^2 - (\beta_6)^2 - (\beta_7)^2.\end{aligned} \tag{5.2.21}$$

We now place the components α, β into a 16-component column (5.2.4), and work out the matrix representation of the Γ-matrices. We get precisely the matrices of the form (5.2.5) with (5.2.6), which also justifies the choices for the signs of the Γ-matrices.

5.2.6 Pure Spinors Orbits in Spin(4,4)

In contrast to the Spin(8) case, we now have several types of pure spinors. This is a precursor to the next section, which aims to classify all orbits of Spin(4,4). In this section, we describe certain representatives and compute their stabilisers to show the various geometric structures, based on the complex model.

The complex model was obtained by choosing a complex structure on $\mathbb{R}^{4,4}$, and so in reverse the canonical pure spinors that this model comes with, namely, $\mathbf{1}, e^{4123} \in S_+$ give back this complex structure. To see this, we translate the polyforms into split octonions. As in the case of Spin(8), we need to complexify the octonions to see (at least certain types of) the pure spinors. We have

Proposition 5.2.6.1. The stabiliser of $\tilde{\mathbf{e}}^3 = \mathbf{i}(\mathbf{1} + e^{4123})$, and $\mathbb{I} = (\mathbf{1} - e^{4123})$, hence $\mathbf{1} = \frac{1}{2}(\mathbb{I} - \mathbf{i}\tilde{\mathbf{e}}^3)$, and $e^{4123} = -\frac{1}{2}(\mathbb{I} + \mathbf{i}\tilde{\mathbf{e}}^3)$, is SU(2,2).

Proof. We now need the following results

$$\begin{aligned} ((\mathbb{I} + \mathbf{i}\tilde{\mathbf{e}}^3), L_x(\mathbb{I} - \mathbf{i}\tilde{\mathbf{e}}^3)) &= 2\mathbf{i}\tilde{\mathbf{e}}^3, \\ ((\mathbb{I} + \mathbf{i}\tilde{\mathbf{e}}^3), L_x L_y(\mathbb{I} - \mathbf{i}\tilde{\mathbf{e}}^3)) &= 2\mathbf{i}(\tilde{\mathbf{e}}^{12} + \tilde{\mathbf{e}}^{74} + \tilde{\mathbf{e}}^{56}). \end{aligned} \tag{5.2.22}$$

This means that

$$\frac{1}{2\mathbf{i}}\langle(\mathbb{I} + \mathbf{i}\tilde{\mathbf{e}}^3), \Gamma\Gamma(\mathbb{I} - \mathbf{i}\tilde{\mathbf{e}}^3)\rangle = \mathbb{I} \wedge \tilde{\mathbf{e}}^3 - (\tilde{\mathbf{e}}^{12} + \tilde{\mathbf{e}}^{74} + \tilde{\mathbf{e}}^{56}). \tag{5.2.23}$$

This is a $(1,1)$-form of the complex structure that this (complex conjugate) pair of pure spinors defines. To see what this complex structure is, let us consider the right multiplication by $\tilde{\mathbf{e}}^3$. The $-\mathbf{i}$ eigenvectors of $R_{\tilde{\mathbf{e}}^3}$ are

$$z_3 := \mathbb{I} + \mathbf{i}\tilde{\mathbf{e}}^3,\ z_4 := \tilde{\mathbf{e}}^1 - \mathbf{i}\tilde{\mathbf{e}}^2,\ z_1 := \tilde{\mathbf{e}}^7 + \mathbf{i}\tilde{\mathbf{e}}^4, \text{ and } z_2 := \tilde{\mathbf{e}}^5 + \mathbf{i}\tilde{\mathbf{e}}^6. \tag{5.2.24}$$

We then have

$$\langle(\mathbb{I} + \mathbf{i}\tilde{\mathbf{e}}^3), \Gamma\Gamma(\mathbb{I} - \mathbf{i}\tilde{\mathbf{e}}^3)\rangle = z_3^* \wedge z_3 + z_4^* \wedge z_4 - z_1^* \wedge z_1 - z_2^* \wedge z_2, \tag{5.2.25}$$

where z_i^* is complex conjugation of the imaginary unit $\mathbf{i}$. Thus, $R_{\tilde{\mathbf{e}}^3}$ is the complex structure that corresponds to the pure spinors $\mathbf{1}$, and e^{4123}.

Let us also state the result for the stabiliser of the spinor $\mathbb{I} - \mathbf{i}\tilde{\mathbf{e}}^3$. The general Lie algebra element on S_+ is

$$A_{\mathfrak{spin}(4,4)} = w^a \tilde{E}_a - w^{ab}\tilde{E}_a\tilde{E}_b. \tag{5.2.26}$$

The stabiliser of $\mathbb{I} - \mathbf{i}\tilde{\mathbf{e}}^3$ is the subalgebra determined by the following 13 constraint equations

$$\begin{aligned} &w^1 = w^{23},\ w^{45} = w^{67},\ w^2 = w^{31},\ w^{46} = w^{75}, \\ &w^4 = w^{73},\ w^{51} = w^{26},\ w^7 = w^{34},\ w^{16} = w^{25}, \\ &w^5 = w^{36},\ w^{14} = w^{27},\ w^6 = w^{53},\ w^{71} = w^{24}, \text{ and} \\ &w^3 = w^{12} + w^{56} + w^{74}. \end{aligned} \tag{5.2.27}$$

The stabiliser is thus $\dim(\mathrm{Spin}(4,4)) - \#\text{constraints} = 28 - 13 = 15$ dimensional. Given that it preserves a complex structure in $\mathbb{R}^{4,4}$, the stabiliser coincides with SU(2,2). □

Similarly for paracomplex structures, one has

Proposition 5.2.6.2. Consider the null split octonion

$$\frac{1}{2}(\mathbb{I}+\tilde{\mathbf{e}}^4)=\mathbf{1}-e^{4123}+\mathbf{i}(e^{42}+e^{31}), \tag{5.2.28}$$

This has stabiliser containing $\mathrm{SL}(4,\mathbb{R})$. The subgroup that stabilises (5.2.28) and

$$\frac{1}{2}(\mathbb{I}-\tilde{\mathbf{e}}^4)=\mathbf{1}-e^{4123}-\mathbf{i}(e^{42}+e^{31}). \tag{5.2.29}$$

is $\mathrm{SL}(4,\mathbb{R})$.

Proof. We have

$$\begin{aligned}((\mathbb{I}-\tilde{\mathbf{e}}^4),L_x(\mathbb{I}+\tilde{\mathbf{e}}^4))&=2\tilde{\mathbf{e}}^4,\ \text{and}\\ ((\mathbb{I}-\tilde{\mathbf{e}}^4),L_xL_y(\mathbb{I}+\tilde{\mathbf{e}}^4))&=-2(\tilde{\mathbf{e}}^{15}+\tilde{\mathbf{e}}^{26}+\tilde{\mathbf{e}}^{37}).\end{aligned} \tag{5.2.30}$$

Therefore

$$\langle(\mathbb{I}-\tilde{\mathbf{e}}^4),\Gamma\Gamma(\mathbb{I}+\tilde{\mathbf{e}}^4)\rangle=2(\mathbb{I}\wedge\tilde{\mathbf{e}}^4+\tilde{\mathbf{e}}^{15}+\tilde{\mathbf{e}}^{26}+\tilde{\mathbf{e}}^{37}). \tag{5.2.31}$$

We thus see that the pair of pure spinors $\mathbb{I}-\tilde{\mathbf{e}}^4,\mathbb{I}+\tilde{\mathbf{e}}^4$ defines the paracomplex structure whose real null eigenvectors are

$$\mathbb{I}+\tilde{\mathbf{e}}^4,\ \tilde{\mathbf{e}}^1+\tilde{\mathbf{e}}^5,\ \tilde{\mathbf{e}}^2+\tilde{\mathbf{e}}^6,\ \text{and}\ \tilde{\mathbf{e}}^3+\tilde{\mathbf{e}}^7. \tag{5.2.32}$$

As an operator on $\mathbb{O}'$ this paracomplex structure is described by $R_{\tilde{\mathbf{e}}^4}$.

Let us state the stabiliser in this case. The stabiliser of $\mathbb{I}+\tilde{\mathbf{e}}^4$ is given by the 7 following constraint equations

$$\begin{aligned}w^1-w^{14}+w^5+w^{45}&=0,\ w^{23}+w^{27}-w^{36}+w^{67}=0,\\ w^2-w^{24}+w^6+w^{46}&=0,\ w^{13}+w^{17}-w^{35}+w^{57}=0,\\ w^3-w^{34}+w^7+w^{47}&=0,\ w^{12}+w^{16}-w^{25}+w^{56}=0,\ \text{and}\\ w^4+w^{15}+w^{26}+w^{37}&=0,\end{aligned} \tag{5.2.33}$$

and so is $\dim(\mathrm{Spin}(4,4))-\#\text{constraints}=28-7=21$ dimensional. This is of course larger than $\dim(\mathrm{SL}(4,\mathbb{R}))=15$. This example displays the case that one has a larger group of transformations stabilising a single pure spinor in the example of split signatures subsection 3.2.4. Now, in addition, if we impose that $\mathbb{I}-\tilde{\mathbf{e}}^4$ is stabilised, we get six new constraint equations

$$\begin{aligned}w^1+w^{14}-w^5+w^{45}&=0,\ w^{23}-w^{27}+w^{36}+w^{67}=0,\\ w^2+w^{24}-w^6+w^{46}&=0,\ w^{13}-w^{17}+w^{35}+w^{57}=0,\\ w^3+w^{34}-w^7+w^{47}&=0,\ w^{12}-w^{16}+w^{25}+w^{56}=0,\end{aligned} \tag{5.2.34}$$

which together with the previous set gives a subalgebra of dimension $\dim(\mathrm{Spin}(4,4))-\#\text{constraints}=28-7-6=15$. Given that one preserves a paracomplex structure in $\mathbb{R}^{4,4}$, the stabiliser coincides with $\mathrm{SL}(4,\mathbb{R})$. □

The above is a special case of $\mathrm{SL}(4,\mathbb{R})$-structures in 8 dimensions. [Kra24b] approaches $\mathrm{SL}(4,\mathbb{R})$ stabilisers in a similar fashion, and then seeks to understand the integrability conditions of these structures in a comparable manner to $\mathrm{SU}(4)$-structures in 8 dimensions. We shall give some details in the next section, when analysing impure orbits.

Now consider the spinors

$$\frac{1}{2}(\mathbb{I}-\mathbf{i}\tilde{\mathbf{e}}^3)-\frac{1}{2}(\tilde{\mathbf{e}}^4+\mathbf{i}\tilde{\mathbf{e}}^7)=\mathbf{1}+\mathbf{i}e^{42},\ \text{and}\ \frac{1}{2}(\mathbb{I}+\mathbf{i}\tilde{\mathbf{e}}^3)+\frac{1}{2}(\tilde{\mathbf{e}}^4-\mathbf{i}\tilde{\mathbf{e}}^7)=-e^{4123}+\mathbf{i}e^{31}. \tag{5.2.35}$$

Both are null spinors, with a non-vanishing inner product between them. The first of them is annihilated by

$$\Gamma_1 - \mathbf{i}\Gamma_2,\ \Gamma_5 + \mathbf{i}\Gamma_6,\ \Gamma_0 + \Gamma_4,\ \text{and}\ \Gamma_3 + \Gamma_7, \tag{5.2.36}$$

and the second by the complement of these four vectors

$$\Gamma_1 + \mathbf{i}\Gamma_2,\ \Gamma_5 - \mathbf{i}\Gamma_6,\ \Gamma_0 - \Gamma_4,\ \text{and}\ \Gamma_3 - \Gamma_7. \tag{5.2.37}$$

So, they are a pair of pure spinors with null subspace spanned by two real and two complex vectors. Note that we can rewrite these pure spinors as

$$\frac{1}{2}(\mathbb{I} - \tilde{\mathbf{e}}^4) - \frac{\mathbf{i}}{2}(\tilde{\mathbf{e}}^3 + \tilde{\mathbf{e}}^7),\ \text{and}\ \frac{1}{2}(\mathbb{I} + \tilde{\mathbf{e}}^4) + \frac{\mathbf{i}}{2}(\tilde{\mathbf{e}}^3 - \tilde{\mathbf{e}}^7). \tag{5.2.38}$$

Both are of the form $\alpha + \mathbf{i}\beta$ where α, β are real pure spinors with $(\alpha, \beta) = 0$. One then has

Lemma 5.2.6.1. The mixed structure defined by the spinors (5.2.35) is given as

$$\mathbb{I} \wedge \tilde{\mathbf{e}}^4 + \tilde{\mathbf{e}}^{37} + \mathbf{i}\tilde{\mathbf{e}}^{12} + \mathbf{i}\tilde{\mathbf{e}}^{56}. \tag{5.2.39}$$

Proof. One has the following facts

$$\begin{aligned} ((\mathbb{I} - \mathbf{i}\tilde{\mathbf{e}}^3 - \tilde{\mathbf{e}}^4 - \mathbf{i}\tilde{\mathbf{e}}^7), L_x(\mathbb{I} + \mathbf{i}\tilde{\mathbf{e}}^3 + \tilde{\mathbf{e}}^4 - \mathbf{i}\tilde{\mathbf{e}}^7)) &= 4\tilde{\mathbf{e}}^4, \\ ((\mathbb{I} - \mathbf{i}\tilde{\mathbf{e}}^3 - \tilde{\mathbf{e}}^4 - \mathbf{i}\tilde{\mathbf{e}}^7), L_x L_y(\mathbb{I} + \mathbf{i}\tilde{\mathbf{e}}^3 + \tilde{\mathbf{e}}^4 - \mathbf{i}\tilde{\mathbf{e}}^7)) &= -4\mathbf{i}(\tilde{\mathbf{e}}^{12} + \tilde{\mathbf{e}}^{56} - \mathbf{i}\tilde{\mathbf{e}}^{37}). \end{aligned} \tag{5.2.40}$$

This means that we have

$$\langle(\mathbb{I} - \mathbf{i}\tilde{\mathbf{e}}^3 - \tilde{\mathbf{e}}^4 - \mathbf{i}\tilde{\mathbf{e}}^7), \Gamma\Gamma(\mathbb{I} + \mathbf{i}\tilde{\mathbf{e}}^3 + \tilde{\mathbf{e}}^4 - \mathbf{i}\tilde{\mathbf{e}}^7)\rangle = 4(\mathbb{I} \wedge \tilde{\mathbf{e}}^4 + \tilde{\mathbf{e}}^{37} + \mathbf{i}\tilde{\mathbf{e}}^{12} + \mathbf{i}\tilde{\mathbf{e}}^{56}). \tag{5.2.41}$$

□

Raising an index of (5.2.39) gives a complex endomorphism on $\mathbb{R}^{4,4}$ whose square is proportional to the identity, and is a sum of a paracomplex structure in the directions $\mathbb{I}, 4, 3, 7$ and the imaginary unit times the complex structure in the directions $1, 2, 5, 6$. The stabilisers associated to the complex and paracomplex structure are characterised as follows.

Proposition 5.2.6.3. The stabiliser of $\frac{1}{2}(\mathbb{I} - \mathbf{i}\tilde{\mathbf{e}}^3) - \frac{1}{2}(\tilde{\mathbf{e}}^4 + \mathbf{i}\tilde{\mathbf{e}}^7)$ and $\frac{1}{2}(\mathbb{I} + \mathbf{i}\tilde{\mathbf{e}}^3) + \frac{1}{2}(\tilde{\mathbf{e}}^4 - \mathbf{i}\tilde{\mathbf{e}}^7)$ is $\mathrm{SU}(1,1) \times \mathrm{SL}(2,\mathbb{R})$.

Proof. The stabilising algebra in $\mathfrak{spin}(4,4)$ of $(\mathbb{I} - \mathbf{i}\tilde{\mathbf{e}}^3 - \tilde{\mathbf{e}}^4 - \mathbf{i}\tilde{\mathbf{e}}^7)$ is determined by the following set of 13 constraint equations

$$\begin{aligned} &w^{16} - w^{25} = 0,\ w^{12} + w^{56} = 0,\ w^{15} + w^{26} = 0,\ w^{37} + w^4 = 0, \\ &w^1 + w^{14} = 0,\ w^{27} - w^{23} = 0,\ w^2 + w^{24} = 0,\ w^{35} + w^{57} = 0, \\ &w^5 - w^{45} = 0,\ w^{36} + w^{67} = 0,\ w^6 - w^{46} = 0,\ w^{13} - w^{17} = 0,\ \text{and} \\ &w^3 + w^{34} + w^{47} - w^7 = 0. \end{aligned} \tag{5.2.42}$$

The dimension of the stabiliser is thus $\dim(\mathrm{Spin}(4,4)) - \#\text{constraints} = 28 - 13 = 15$. Demanding that the complementary spinor $(\mathbb{I} + \mathbf{i}\tilde{\mathbf{e}}^3 + \tilde{\mathbf{e}}^4 - \mathbf{i}\tilde{\mathbf{e}}^7)$ is also stabilised produces 9 more constraint equations

$$\begin{aligned} &w^1 - w^{14} = 0,\ w^{27} + w^{23} = 0,\ w^2 - w^{24} = 0,\ w^{35} - w^{57} = 0, \\ &w^5 + w^{45} = 0,\ w^{36} - w^{67} = 0,\ w^6 + w^{46} = 0,\ w^{13} + w^{17} = 0,\ \text{and} \\ &w^3 - w^{34} + w^{47} + w^7 = 0. \end{aligned} \tag{5.2.43}$$

Simplifying the above 22 constraint equations gives

$$\begin{aligned} w^{16} - w^{25} = 0,\ w^{12} + w^{56} = 0, \\ w^{15} + w^{26} = 0,\ w^{4} = w^{73},\ w^{3} = w^{74}, \\ w^{7} = w^{34},\ w^{1,2,5,6} = 0,\ w^{41} = w^{42} = w^{45} = w^{46} = 0,\ \text{and} \\ w^{27} = w^{23} = w^{35} = w^{57} = w^{36} = w^{67} = w^{13} = w^{17} = 0. \end{aligned} \tag{5.2.44}$$

At the level of the Lie algebra one sees a copy of $\mathfrak{su}(1,1) \subset \mathfrak{spin}(4,4)$ acting on the two complex null directions $\{\tilde{\mathbf{e}}^1 - \mathbf{i}\tilde{\mathbf{e}}^2, \tilde{\mathbf{e}}^5 + \mathbf{i}\tilde{\mathbf{e}}^6\}$, as well as a copy of $\mathfrak{sl}(4,\mathbb{R}) \subset \mathfrak{spin}(4,4)$ acting on the two real null directions $\{\mathbb{I} + \tilde{\mathbf{e}}^4, \tilde{\mathbf{e}}^3 + \tilde{\mathbf{e}}^7\}$. As these Lie subalgebras commute, the stabilising subgroup is $\mathrm{SU}(1,1) \times \mathrm{SL}(2,\mathbb{R})$ □

We have thus described the three representatives of the different types of pure spinors in $\mathrm{Spin}(4,4)$. One gives a complex structure (5.2.6.1), one paracomplex (5.2.6.2), and the third type gives a structure of the mixed type (5.2.6.3). Of these, only the pure spinor giving the paracomplex structure is real. In the next section, we shall construct these geometric structures from general complex Weyl spinors.

5.2.7 Classifying Impure Orbits of Spin(4,4)

We now enter into a less familiar territory, as there seems to be no known classification of the orbits of the real $\mathrm{Spin}(4,4)$ on the complex Weyl spinors, apart from the already considered case of pure and Majorana spinors. This is in contrast to the case of $\mathrm{Spin}(8)$, where there is only one possible type of general spinors, with the stabiliser $\mathrm{SU}(4)$.

We begin with Majorana spinors, which are simply the split octonions. There are three possible types of such spinors: Non-null, spacelike (or timelike), and null. We have determined the stabiliser of a null octonion in (5.2.33). As we have seen in the previous subsection, a null octonion is a pure spinor stabilised by $\mathrm{Spin}(7)$. Similarly, the stabilisers of spacelike or timelike null octonions (pure spinors) are also 21 dimensional, and in both cases are given by $\mathrm{Spin}(4,3)$.

For future use, we compute explicitly the stabiliser subalgebra of $\psi = \mathbb{I}$. It is given by the following 7 constraint equations

$$\begin{aligned} & w^1 = w^{23} - w^{45} + w^{67},\ w^2 = -w^{13} - w^{46} - w^{57},\ w^3 = w^{12} - w^{47} + w^{56}, \\ & w^4 = -w^{15} - w^{26} - w^{37},\ w^5 = w^{14} - w^{27} + w^{36},\ w^6 = w^{17} + w^{25} - w^{35},\ \text{and} \\ & w^7 = -w^{16} + w^{25} + w^{34}. \end{aligned} \tag{5.2.45}$$

To classify general $\mathrm{Spin}(4,4)$ spinors, we use the same idea that worked in the $\mathrm{Spin}(8)$ case. We consider a general complex spinor, which is a complexified split octonion $\psi = \alpha + \mathbf{i}\beta$, $\alpha, \beta \in \mathbb{O}'$, (the relations (5.1.37) are still valid). However, since the signature is mixed, the norm squared does not need to be positive. We can still assume that the spinor is not null (because if it is null, it is pure) and rescale it to unit length. Thus, $|\alpha|^2 - |\beta|^2 = 1$ and $(\alpha, \beta) = 0$.

There are four cases to consider.

1. When $|\beta|^2 > 0$ both α, β have positive norm and can be chosen to be multiples of $\mathbb{I}$, and $\tilde{\mathbf{e}}^3$. The unit spinor can be chosen to be

$$\psi = \cosh\tau\,\mathbb{I} + \mathbf{i}\sinh\tau\,\tilde{\mathbf{e}}^3. \tag{5.2.46}$$

Here, τ is a real parameter. It is clear that the analysis in the case of $\mathrm{Spin}(8)$ is unchanged, and this spinor defines a pure spinor that is a multiple of $\mathbb{I} + \mathbf{i}\tilde{\mathbf{e}}^3$, whose stabiliser is $\mathrm{SU}(2,2)$. Thus, the general spinor of this type still defines a complex structure, and its stabiliser is $\mathrm{SU}(2,2)$.

2. Take $-1 < |\beta|^2 < 0$. This means that $|\alpha|^2 > 0$, and we can still choose α to be a multiple of $\mathbb{I}$. This leads us to consider the unit spinor

$$\psi = \cos\theta \mathbb{I} + \mathbf{i}\sin\theta\, \tilde{\mathbf{e}}^4. \tag{5.2.47}$$

Here θ is a real parameter, and we have $\langle \mathbf{R}(\psi), \psi\rangle = \cos^2\theta - \sin^2\theta = \cos(2\theta)$. Denoting $\lambda = \cos(2\theta)$ we can form the complement

$$\tilde{\psi} = \frac{\lambda\psi - \mathbf{R}(\psi)}{\mathbf{i}\sqrt{1-\lambda^2}} = \mathbf{i}\sin\theta\,\mathbb{I} + \cos\theta\,\tilde{\mathbf{e}}^4. \tag{5.2.48}$$

This is again a spinor of norm minus one, and so

$$\psi \pm \tilde{\psi} = (\cos\theta + \mathbf{i}\sin\theta)(\mathbb{I} \pm \tilde{\mathbf{e}}^4) \tag{5.2.49}$$

are both null and thus pure spinors. They are (complex) multiples of the spinors $(\mathbb{I} \pm \tilde{\mathbf{e}}^4)$ that we have already encountered before. Recall that a paracomplex structure can be constructed only when we have access to a pair of pure spinors with a non-vanishing inner product. As ψ defines these *two* pure spinors $(\mathbb{I} \pm \tilde{\mathbf{e}}^4)$, we have the stabiliser $\mathrm{SL}(4,\mathbb{R})$. When $\theta = \frac{\pi}{4}$, one recovers Lorentzian Cayley forms, whose integrability is discussed in [Kra24b].

3. Suppose $|\beta|^2 < -1$. This means that both α, β have negative norms. For example, we can choose $\alpha \propto \tilde{\mathbf{e}}^4, \beta \propto \tilde{\mathbf{e}}^7$. This leads us to consider the unit spinor

$$\psi = \sinh\tau\,\tilde{\mathbf{e}}^4 + \mathbf{i}\cosh\tau\,\tilde{\mathbf{e}}^7, \tag{5.2.50}$$

with real parameter τ. Then $\lambda = \langle \mathbf{R}(\psi), \psi\rangle = -\cosh(2\tau)$ and the complement is given as

$$\tilde{\psi} = \frac{\lambda\psi - \mathbf{R}(\psi)}{\sqrt{\lambda^2 - 1}} = -\cosh\tau\,\tilde{\mathbf{e}}^4 - \mathbf{i}\sinh\tau\,\tilde{\mathbf{e}}^7. \tag{5.2.51}$$

This is again a spinor of norm minus one, and the spinor

$$\psi + \tilde{\psi} = (\sinh\tau - \cosh\tau)(\tilde{\mathbf{e}}^4 - \mathbf{i}\tilde{\mathbf{e}}^7) \tag{5.2.52}$$

is pure. This spinor defines a complex structure on $\mathbb{R}^{4,4}$ with the Kähler form

$$\frac{1}{2\mathbf{i}}\langle(\tilde{\mathbf{e}}^4 - \mathbf{i}\tilde{\mathbf{e}}^7), \Gamma\Gamma(\tilde{\mathbf{e}}^4 + \mathbf{i}\tilde{\mathbf{e}}^7)\rangle = \mathbb{I}\wedge\tilde{\mathbf{e}}^3 + \tilde{\mathbf{e}}^{12} + \tilde{\mathbf{e}}^{47} + \tilde{\mathbf{e}}^{56}. \tag{5.2.53}$$

Thus, the general spinor of this type still defines a complex structure, and its stabiliser is $\mathrm{SU}(2,2)$.

4. Finally, if β is null, this forces the octonion α to be unit, and so we are led to consider the spinor

$$\psi = \mathbb{I} + \mathbf{i}(\tilde{\mathbf{e}}^3 + \tilde{\mathbf{e}}^4). \tag{5.2.54}$$

(5.2.54) is a complex spinor whose real part is an identity octonion, and the imaginary part is a null octonion. Then, $\lambda = \langle \mathbf{R}(\psi), \psi\rangle = 1$, and the construction of $\tilde{\psi}$ is no longer applicable. It is clear that the geometric map arising in this case knows both about the geometry related to $\mathbb{I}$, and that of the real pure spinor $\tilde{\mathbf{e}}^3 + \tilde{\mathbf{e}}^4$. The identity octonion defines the $\mathrm{Spin}(4,3)$ invariant 4-form $-\mathbb{I}\wedge\tilde{C} + *\tilde{C}$ on $\mathbb{R}^{4,4}$. Here $\tilde{C}$ is the 3-form, invariant under $\mathrm{Spin}(4,3)$ transformations, built from a null purely imaginary basis in $\mathrm{Im}(\mathbb{O}')$. The pure spinor $\tilde{\mathbf{e}}^3 + \tilde{\mathbf{e}}^4$ defines its null subspace that can be seen to be spanned by

$$\mathbb{I} - \tilde{\mathbf{e}}^7,\ \tilde{\mathbf{e}}^3 + \tilde{\mathbf{e}}^4,\ \tilde{\mathbf{e}}^2 - \tilde{\mathbf{e}}^5,\ \text{and}\ \tilde{\mathbf{e}}^1 + \tilde{\mathbf{e}}^6. \tag{5.2.55}$$

To understand the geometry arising better, we compute

$$\langle(\mathbb{I}+\mathbf{i}\tilde{\mathbf{e}}^3+\mathbf{i}\tilde{\mathbf{e}}^4),\Gamma\Gamma(\mathbb{I}-\mathbf{i}\tilde{\mathbf{e}}^3-\mathbf{i}\tilde{\mathbf{e}}^4)\rangle = 2\mathbf{i}((\mathbb{I}+\tilde{\mathbf{e}}^7)\wedge(\tilde{\mathbf{e}}^3-\tilde{\mathbf{e}}^4)+(\tilde{\mathbf{e}}^2+\tilde{\mathbf{e}}^5)\wedge(\tilde{\mathbf{e}}^1-\tilde{\mathbf{e}}^6)). \quad (5.2.56)$$

The 2-form that arises is thus a sum of two decomposable pieces, each built entirely from null vectors that are complementary to those in (5.2.55). This has stabiliser $\mathfrak{sl}(4,\mathbb{R})\oplus N$. Where N is a 6 dimensional nilpotent algebra. Which means, at the level of the Lie algebra, the stabiliser is contained in $\mathfrak{spin}(4,3)\cap(\mathfrak{sl}(4,\mathbb{R})\oplus N)$.

It is also interesting to compute the stabiliser of $\mathbb{I}+\mathbf{i}\tilde{\mathbf{e}}^3+\mathbf{i}\tilde{\mathbf{e}}^4$. It is clear that the stabiliser of ψ is in the intersection of the stabilisers of the pure spinors $\mathbb{I}$ and $\tilde{\mathbf{e}}^3+\tilde{\mathbf{e}}^4$. Both are stabilised by two different copies of $\mathfrak{spin}(4,3)\subset\mathfrak{spin}(4,4)$ of dimension 21. An explicit calculation shows that this intersection is given by (5.2.45) supplemented by the following 4 constraint equations,

$$\begin{gathered} w^{16}=w^{25},\; w^{12}+w^{15}+w^{26}+w^{56}=0, \\ w^{23}-w^{24}-w^{35}+w^{45}=0,\text{ and } -w^{13}+w^{14}-w^{36}+w^{46}=0. \end{gathered} \quad (5.2.57)$$

Hence the stabilising algebra is $\dim(\mathrm{Spin}(4,4))-\dim(\mathrm{Spin}(4,3))-\#\text{constraints}=28-7-4=17$ dimensional.

Again, to summarise the more complicated results of this section. There are several types of impure spinor controlled by the parameter $|\beta|^2$.

Theorem 5.2.7.1. Let ψ be a unit Weyl (impure) spinor of Spin(4,4). Then one can write $\psi=\alpha+\mathbf{i}\beta\in\mathbb{O}'\otimes\mathbb{C}$. The stabilisers of ψ for various values of $|\beta|^2$ are given as:

- $|\beta|^2>0$ leads to an SU(2,2) stabiliser.
- $-1<|\beta|^2<0$ leads to an SL(4, $\mathbb{R}$) stabiliser.
- $|\beta|^2<-1$ leads to an SU(2,2) stabiliser.
- $|\beta|^2=0$ leads to a 17 dimensional stabiliser inside $\mathfrak{spin}(4,3)\cap(\mathfrak{sl}(4,\mathbb{R})\oplus N)$ subject to the constraint equation (5.2.45) and (5.2.57).

5.3 Spin(10) and Octonions

In the discussion of 10 dimensions, we take the approach of parametrising the space spinors using complexified octonions. This results in a reduction to 8 dimensions. We see that there are two types of stabilisers of impure spinors, given by: SU(4) × U(1) and Spin(7) × U(1). In addition, we discuss the integrability of the pure spinor orbit stabilised by SU(5).

5.3.1 creation and annihilation operator construction

To construct Cliff_{10} we choose a complex structure, which identifies $\mathbb{R}^{10}\cong\mathbb{C}^5$. We denote the complex coordinates by $z_{1,\ldots,5}$, and the coordinate one-forms by $\mathrm{d}z_{1,\ldots,5}$. We introduce five pairs of creation and annihilation operators $a_{\mathsf{I}},a^\dagger_{\mathsf{I}},\mathsf{I}=1,\ldots,5$. We also introduce the following Γ-matrices

$$\Gamma_{4+\mathsf{I}}:=a_{\mathsf{I}}+a^\dagger_{\mathsf{I}},\; \Gamma_{\mathsf{I}}:=-\mathbf{i}(a_{\mathsf{I}}-a^\dagger_{\mathsf{I}}),\text{ where } \mathsf{I}=1,2,3,4,5. \quad (5.3.1)$$

Notice that $\mathrm{Cliff}_8\subset\mathrm{Cliff}_{10}$, and hence $\Lambda(\mathbb{C}^4)\subset\Lambda(\mathbb{C}^5)$. So there is a choice of $\Lambda(\mathbb{C}^4)\cong(\mathbb{O}\otimes\mathbb{C})^2$ (given explicitly in section 5.1). Therefore, Weyl spinors in 10 dimensions have the following parametrisation,

$$\psi_+=\alpha+\beta\wedge\mathrm{d}z_5,\text{ and } \psi_-=\gamma\wedge\mathrm{d}z_5+\delta. \quad (5.3.2)$$

Where $\alpha, \gamma \in S_4^+ \cong \mathbb{O} \otimes \mathbb{C}$ and $\beta, \delta \in S_4^- \cong \mathbb{O} \otimes \mathbb{C}$. $S_4^\pm$ are the even and odd Weyl spaces of $\Lambda(\mathbb{C}^4)$, respectively. We therefore identify $\Gamma_{1,\dots,8}$, via (5.1.14), through L_x on $\Lambda^{\text{odd}}(\mathbb{C}^4)$ and $L_{\bar{x}}$ on $\Lambda^{\text{even}}(\mathbb{C}^4)$, where $x \in \mathbb{O}$, and $\bar{x}$ is the octonionic conjugate. Writing the Dirac spinor (5.3.2) as a vector, one has $\psi = \begin{pmatrix} \psi_+ & \psi_- \end{pmatrix}^{\mathrm{T}} = \begin{pmatrix} \alpha & \beta & \gamma & \delta \end{pmatrix}^{\mathrm{T}}$.

The $\Gamma-$matrices can be seen as real endomorphisms on $(\mathbb{O} \otimes \mathbb{C})^4 \cong \mathbb{C}^{32}$

$$\Gamma_x = \begin{pmatrix} 0 & 0 & 0 & L_{\bar{x}} \\ 0 & 0 & L_x & 0 \\ 0 & L_{\bar{x}} & 0 & 0 \\ L_x & 0 & 0 & 0 \end{pmatrix}, \ \Gamma_9 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \ \text{and } \Gamma_{10} = \begin{pmatrix} 0 & 0 & -\mathbf{i} & 0 \\ 0 & 0 & 0 & -\mathbf{i} \\ \mathbf{i} & 0 & 0 & 0 \\ 0 & \mathbf{i} & 0 & 0 \end{pmatrix}. \tag{5.3.3}$$

In 10 dimensions, spinors of opposite chirality are paired, that is the spinor bilinear is of the form $\langle S_\pm, S_\mp \rangle$. The conjugation operator is defined as the following

$$\mathbf{R} = \Gamma_1 \dots \Gamma_4 \Gamma_9 \mathfrak{C}. \tag{5.3.4}$$

$\mathbf{R}$ commutes with all the other $\Gamma-$matrices and squares to the identity. Since it is a product of an odd number of generators, one can take a spinor $\psi_+ \in S_+$, and can construct $\langle \mathbf{R}(\psi_+), \psi_+ \rangle$. Concretely,

$$\psi_+ = \begin{pmatrix} \alpha_1 + \mathbf{i}\alpha_1 \\ \beta_2 + \mathbf{i}\beta_2 \end{pmatrix} \in S_+, \ \text{then } \mathbf{R}(\psi_+) = \begin{pmatrix} \alpha_1 - \mathbf{i}\alpha_1 \\ \beta_2 - \mathbf{i}\beta_2 \end{pmatrix} \in S_-, \tag{5.3.5}$$

for $\alpha_{1,2}, \beta_{1,2} \in \mathbb{O}$. Hence

$$\langle \mathbf{R}(\psi_+), \psi_+ \rangle = |\alpha_1|^2 + |\alpha_2|^2 + |\beta_1|^2 + |\beta_2|^2. \tag{5.3.6}$$

Before we begin the analysis, we would like to introduce a different characterisation of Weyl spinors in Spin(8) to that of section 5.1, only to make notation more compact. As before, let $\psi = \alpha_1 + \mathbf{i}\alpha_2$ where $\alpha_{1,2} \in \mathbb{O}$. Also as previously, one imposes $|\psi|^2 = 1$, so that $|\alpha_1|^2 - |\alpha_2|^2 = 1$ and $(\alpha_1, \alpha_2) = 0$. Hence,

$$|\alpha_1| = \cosh(\tau), \ \text{and } |\alpha_2| = \sinh(\tau). \tag{5.3.7}$$

Now the notational change: ψ can be rewritten as

$$\psi = e^\tau \Psi + e^{-\tau} \widehat{\Psi}, \ \text{for } \tau \in \mathbb{R}. \tag{5.3.8}$$

Here $\Psi \coloneqq \frac{1}{2}\left(\frac{\alpha_1}{|\alpha_1|} + \mathbf{i}\frac{\alpha_2}{|\alpha_2|} \right)$ is a pure spinor, i.e. a null complexified octonion (see [Kra22] for more details on the construction) and $\widehat{\Psi} \coloneqq \frac{1}{2}\left(\frac{\alpha_1}{|\alpha_1|} - \mathbf{i}\frac{\alpha_2}{|\alpha_2|} \right)$ is the complex conjugate.

5.3.2 Pure Spinors in 10 Dimensions

It is covered in [Kra22] what pure spinors are in 10 dimensions. But briefly speaking, one can parametrise ψ^+ as

$$\psi^+ = \begin{pmatrix} \alpha_1 + \mathbf{i}\alpha_2 \\ \beta_1 + \mathbf{i}\beta_2 \end{pmatrix}, \tag{5.3.9}$$

where $\alpha_{1,2}, \beta_{1,2} \in \mathbb{O}$. Then by theorem 3.2.2.3 one derives algebraic conditions from $B_1(\psi^+, \Gamma\psi^+) = X + \mathbf{i}Y = 0$. Resulting in

$$\begin{gathered} \beta_1\bar{\alpha}_1 - \beta_2\bar{\alpha}_2 = 0, \ \beta_1\bar{\alpha}_2 - \beta_2\bar{\alpha}_1 = 0, \ (\alpha_1, \alpha_2) = (\beta_1, \beta_2) = 0, \\ |\alpha_1|^2 = |\alpha_2|^2 \text{ and } |\beta_1|^2 = |\beta_2|^2. \end{gathered} \tag{5.3.10}$$

The last 4 conditions are familiar, they imply that α and β are pure spinors of Spin(8).

5.3.3 Impure Spinor Orbits in 10 Dimensions

By (5.3.10) we can see that a canonical impure spinor in 10 dimensions is of the form

$$\tilde{\psi}^{+} = \begin{pmatrix} e^{\tau_\alpha}\Psi_\alpha + e^{-\tau_\alpha}\widehat{\Psi}_\alpha \\ t_\beta e^{\tau_\beta}\Psi_\beta + t_\beta^{*} e^{-\tau_\beta}\widehat{\Psi}_\beta \end{pmatrix} \text{ for } \tau_{\alpha,\beta} \in \mathbb{R}, \text{ and } t_\beta \in \mathbb{C}\backslash\{0\}. \tag{5.3.11}$$

We have no complex parameter for Ψ_α by rotating ψ^+ to a unit vector. Furthermore, assuming that t_β is non-zero allows us to impose that $|t_\beta|^2 = 1$. We now invoke a powerful result by [Bry20] that shows one can rotate Ψ to kill the β component, reducing it to

Lemma 5.3.3.1. Let $\tilde{\psi}^+$ as (5.3.11). Then one can transform $\tilde{\psi}^+$, by the action of Spin(10) to

$$\psi^{+} = \begin{pmatrix} e^{\tau_\alpha}\Psi_\alpha + e^{-\tau_\alpha}\widehat{\Psi}_\alpha \\ 0 \end{pmatrix} \text{ for } \tau_\alpha \in \mathbb{R}. \tag{5.3.12}$$

This means that understanding the orbit structure of Spin(10) reduces to understanding the orbits of (5.3.12), which is already understood because this is an impure spinor in 8 dimensions.

To find the stabilisers, let us compute the 5-form in various cases for the value of τ_α. First, consider a generic impure spinor, i.e. $\tau_\alpha \neq 0$.

$$B_5(\psi^+,\psi^+) = \left(\cosh(2\tau)\mathrm{Re}(\Omega_8) - \frac{1}{2}\omega_8 \wedge \omega_8 + \mathbf{i}\sinh(2\tau)\mathrm{Im}(\Omega_8)\right) \wedge \mathrm{d}\bar{z}_5. \tag{5.3.13}$$

The subgroup that stabilises this must then contain SU(4) × U(1). Here again Ω_8 is the (4,0)-form ψ^+ along with ω_8 is the Kähler form in 8 dimensions, that produces a compatible hermitian metric reducing Spin(10) to SU(4). The $\mathrm{d}\bar{z}_5$ is stabilised by U(1) and is the complex coordinate spanning the direction orthogonal to $\mathbb{O}$.

Now suppose that $\tau_\alpha = 0$, the resulting spinor is real and the 5-form is

$$B_5(\psi^+,\psi^+) = \left(\mathrm{Re}(\Omega_8) - \frac{1}{2}\omega_8 \wedge \omega_8\right) \wedge \mathrm{d}\bar{z}_5. \tag{5.3.14}$$

If we choose $\Psi_\alpha = \frac{1}{2}(\mathbb{I}+\mathbf{iu})$, i.e. ψ^+ is a unit octonion, we can see the manifest Spin(7) symmetry. Thus, the stabiliser contains Spin(7) × U(1). We can summarise the above as follows

Theorem 5.3.3.1. Let ψ^+ be of the form (5.3.12). Then, for various values of τ_α, one can compute the following stabilisers.

- $\tau_\alpha \neq 0$ leads to a stabiliser containing SU(4) × U(1).
- $\tau_\alpha = 0$ leads to a stabiliser containing Spin(7) × U(1).

Finally take $\psi^+ = \Psi_\alpha$, since this is pure we expect the 5-form to be

$$B_5(\psi^+,\psi^+) = \Omega_8 \wedge \mathrm{d}\bar{z}_5 \coloneqq \Omega_{10}. \tag{5.3.15}$$

This is the top holomorphic form in 10 dimensions and hence stabilised by SU(5). We also note that

$$B_2(\mathbf{R}(\psi^+),\psi^+) = 2\mathbf{i}\omega_{10}. \tag{5.3.16}$$

Where ω_{10} is the Kähler form in 10 dimensions.

Proposition 5.3.3.1. Let M be a spin manifold of 10 dimensions, and suppose that ψ^+ is a spinor stabilised by SU(5) ⊂ Spin(10). The set of canonical differential forms associated to the SU(5)-structure is given as $B_5(\psi^+,\psi^+) = \Omega_{10}$, and $\frac{1}{2\mathbf{i}}B_2(\mathbf{R}(\psi^+),\psi^+) = \omega_{10}$. The SU(5)-structure is integrable if, and only if, the exterior derivatives of its canonical differential forms vanish.

Proof. The structure of the proof resembles those found in [CS02] and [Kra24b]. Indeed, If the above structures are integrable, then $\nabla\Omega_{10} = \nabla\omega_{10} = 0$, hence the projection to $\mathrm{d}\Omega_{10}$, and $\mathrm{d}\omega_{10}$ vanish.

In the other direction, from general principles, it is known that integrability of SU(5) structures is controlled by the vanishing of intrinsic torsion, which takes values in

$$\begin{aligned}\Lambda^1 \otimes \mathfrak{su}(5)^\perp &= [[\Lambda^{1,0} \otimes \Lambda^{2,0}]] \oplus \Lambda^1 \\ &\cong \underbrace{\mathcal{W}_1}_{\cong[[\Lambda^{3,0}]]} \oplus \mathcal{W}_2 \oplus \underbrace{\mathcal{W}_3}_{\cong[[\Lambda_0^{2,1}]]} \oplus \underbrace{\mathcal{W}_4}_{\cong\Lambda^1} \oplus \underbrace{\mathcal{W}_5}_{\cong\Lambda^1}.\end{aligned} \tag{5.3.17}$$

The exterior derivatives of Ω_{10}, and ω_{10} take values in

$$\begin{aligned}&\mathrm{d}\Omega_{10} \in \Lambda^{6,0} \oplus \Lambda^{5,1} \text{ and} \\ &\mathrm{d}\omega_{10} \in \Lambda^3 \cong [[\Lambda^{3,0}]] \oplus [[\Lambda_0^{2,1}]] \oplus \Lambda^1.\end{aligned} \tag{5.3.18}$$

The Nijenhuis tensor, valued in $\mathcal{W}_1 \oplus \mathcal{W}_2$ that controls the integrability of the complex structure and so vanishes when $\mathrm{d}B_5(\psi^+, \psi^+) = 0 \implies \mathrm{d}\Omega_{10} = 0$. Furthermore, the $\Lambda^{5,1} \cong \Lambda^1$ part of $\mathrm{d}\Omega_{10}$ vanishing implies that $\mathcal{W}_5 \cong \Lambda^1$ vanishes too. Again, it is known, see [Sal89], and discussion in [CS02], that the components of $\mathrm{d}\omega_{10}$ control $\mathcal{W}_{1,3,4}$. As $\mathrm{d}B_2(\mathbf{R}(\psi), \psi) = 0 \implies \mathrm{d}\omega_{10} = 0$, $\mathcal{W}_{1,3,4}$ vanish. □

In the proof above we have used standard notation for real complexified differential forms and standard results decomposition of the intrinsic torsion space (5.3.17), found in, for example, chapter 3 of [Sal89], and [CS02].

5.4 Spin(12) and Octonions

In 12 dimensions, much less is known about the real orbits. In 10 dimensions, the classification was made by [Bry20], and in the language of pure spinors, it was rediscovered by [Kra24b]. The analysis of real stabilisers of Weyl spinors in 12 dimensions is difficult to approach unless certain assumptions about the spinor are made. We shall show it is possible to parametrise a Weyl spinor in 12 dimensions via even and odd Weyl spinors in 10 dimensions. After imposing Majorana type constraints, we show there is a canonical class of spinors orbits one can consider.

5.4.1 creation and annihilation operator construction

Extending the case above, one identifies $\{w^{\mathsf{I}} \mid 1 \le \mathsf{I} \le 12\}$ be real coordinates of $\mathbb{R}^{12}$. We then endow a complex structure that identifies $\mathbb{R}^{12} \cong \mathbb{C}^6$: $z^{\mathsf{I}} = w^{\mathsf{I}} + \mathbf{i}w^{\mathsf{I}+4}$, $z = w^9 + \mathbf{i}w^{11}$, and $\tilde{z} = w^{10} + \mathbf{i}w^{12}$. The one-forms are then given as $\mathrm{d}z_{1,2,3,4}, \mathrm{d}z, \mathrm{d}\tilde{z}$. We introduce 6 pairs of creation and annihilation operators $a_{\mathsf{I}}, a_{\mathsf{I}}^\dagger, a, a^\dagger, \tilde{a}, \tilde{a}^\dagger$, $\mathsf{I} = 1, \ldots, 4$. We also introduce the Γ-matrices that generate Cliff_{12} as,

$$\begin{aligned}&\Gamma_{4+\mathsf{I}} \coloneqq a_{\mathsf{I}} + a_{\mathsf{I}}^\dagger,\ \Gamma_{\mathsf{I}} \coloneqq -\mathbf{i}(a_{\mathsf{I}} - a_{\mathsf{I}}^\dagger),\ \text{for } i = 1,2,3,4,\ \Gamma_9 = a + a^\dagger, \\ &\Gamma_{10} = -\mathbf{i}(a - a^\dagger),\ \Gamma_{11} = \tilde{a} + \tilde{a}^\dagger,\ \text{and } \Gamma_{12} = -\mathbf{i}(\tilde{a} - \tilde{a}^\dagger).\end{aligned} \tag{5.4.1}$$

Similar to 10 dimensions, the coordinates are chosen suggestively to induce octonions. Indeed, $\mathrm{Cliff}_8 \subset \mathrm{Cliff}_{12}$, hence $(\mathbb{O}\otimes\mathbb{C})^2 \cong \Lambda(\mathbb{C}^4) \subset \Lambda(\mathbb{C}^6)$. In fact, the Clifford algebra has a representation over $(\mathbb{O} \otimes \mathbb{C})^8$. We choose to characterise the Weyl representations as follows.

$$S^+ = \left\{ \psi^+ = \alpha \mathrm{d}z + \beta \mathrm{d}\tilde{z} + \gamma + \delta \mathrm{d}z \wedge \mathrm{d}\tilde{z} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} \middle| \alpha, \beta \in S_4^-,\ \gamma, \delta \in S_4^+ \right\}, \text{ and}$$
$$S^- = \left\{ \psi^- = \tilde{\alpha} \mathrm{d}z + \tilde{\beta} \mathrm{d}\tilde{z} + \tilde{\gamma} + \tilde{\delta} \mathrm{d}z \wedge \mathrm{d}\tilde{z} = \begin{pmatrix} \tilde{\alpha} \\ \tilde{\beta} \\ \tilde{\gamma} \\ \tilde{\delta} \end{pmatrix} \middle| \tilde{\alpha}, \tilde{\beta} \in S_4^+,\ \tilde{\gamma}, \tilde{\delta} \in S_4^- \right\}. \tag{5.4.2}$$

Identifying the orientation of a Dirac spinor as $\psi = \begin{pmatrix} \psi^+ & \psi^- \end{pmatrix}^{\mathrm{T}}$. In addition, the Lie algebra action on Weyl spinors $S^+ \cong (\mathbb{O} \otimes \mathbb{C})^4$ is given as

$$\begin{pmatrix} A + \mathbf{i}s & p^* + \mathbf{i}q^* & -L_{x_1} - \mathbf{i}L_{y_1} & L_{x_2} - \mathbf{i}L_{y_2} \\ -p - iq & A - \mathbf{i}s & -L_{x_2} - \mathbf{i}L_{y_2} & -L_{x_1} + \mathbf{i}L_{y_1} \\ L_{\bar{x}_1} - \mathbf{i}L_{\bar{y}_1} & -L_{\bar{x}_2} - \mathbf{i}L_{\bar{y}_2} & A' + \mathbf{i}t & -p + \mathbf{i}q \\ -L_{\bar{x}_2} - \mathbf{i}L_{\bar{y}_2} & L_{\bar{x}_1} + \mathbf{i}L_{\bar{y}_1} & p^* - \mathbf{i}q^* & A' - \mathbf{i}t \end{pmatrix}, \text{ where} \tag{5.4.3}$$

$A, A' \in \mathfrak{spin}(8),\ p, q \in \mathbb{C},\ x_{1,2}, y_{1,2} \in \mathbb{O},\ s, t \in \mathbb{R}$.

5.4.2 Pure Spinors in 12 Dimensions

To characterise pure spinors in 12 dimensions, we need a similar approach to that in 10 dimensions. In 12 dimensions, $\langle S^\pm, S^\pm \rangle$ is the appropriate pairing. Hence, one can only insert 0, 2, 4, or 6 Γ-matrices in the pairing. Furthermore, by antisymmetry, $B_0(\psi^+, \psi^+) = B_4(\psi^+, \psi^+) = 0$. Thus, the only non-trivial differential forms one can construct are $B_2(\psi^+, \psi^+)$ and $B_6(\psi^+, \psi^+)$. If ψ^+ is pure, then the 2-form $B_2(\psi^+, \psi^+)$ vanishes. First, let us compute the components of $B_2(\psi^+, \psi^+)$ in a basis,

$$\begin{aligned}
\langle \psi^+, \Gamma_9 \Gamma_{10} \psi^+ \rangle &= -(\alpha, \alpha) - (\beta, \beta) - (\gamma, \gamma) - (\delta, \delta), \\
\langle \psi^+, \Gamma_{11} \Gamma_{12} \psi^+ \rangle &= -(\alpha, \alpha) - (\beta, \beta) + (\gamma, \gamma) + (\delta, \delta), \\
\langle \psi^+, \Gamma_9 \Gamma_{12} \psi^+ \rangle &= \mathbf{i} \left[-(\alpha, \alpha) + (\beta, \beta) - (\gamma, \gamma) + (\delta, \delta) \right], \\
\langle \psi^+, \Gamma_{10} \Gamma_{11} \psi^+ \rangle &= \mathbf{i} \left[-(\alpha, \alpha) + (\beta, \beta) + (\gamma, \gamma) - (\delta, \delta) \right], \\
\langle \psi^+, \Gamma_9 \Gamma_{11} \psi^+ \rangle &= 2\mathbf{i} \left[(\alpha, \beta) + (\gamma, \delta) \right], \\
\langle \psi^+, \Gamma_{10} \Gamma_{12} \psi^+ \rangle &= -2\mathbf{i} \left[(\alpha, \beta) - (\gamma, \delta) \right], \text{ and} \\
\langle \psi^+, \Gamma_x \Gamma_y \psi^+ \rangle &= (\alpha, (L_x L_{\bar{y}} - L_y L_{\bar{x}}) \beta) - (\gamma, (L_{\bar{x}} L_y - L_{\bar{y}} L_x) \delta).
\end{aligned} \tag{5.4.4}$$

$(-,-)$ in the above formulas is the natural bilinear on $\mathbb{O} \otimes \mathbb{C}$ between two complexified octonions, see 5.1.37. In addition, the components that are not written, like $\langle \psi^+, \Gamma_x \Gamma_9 \psi^+ \rangle$, $\langle \psi^+, \Gamma_x \Gamma_{10} \psi^+ \rangle$ etc., are exactly zero. Writing

$$\psi^+ = \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} = \begin{pmatrix} \alpha_1 + \mathbf{i}\alpha_2 \\ \beta_1 + \mathbf{i}\beta_2 \\ \gamma_1 + \mathbf{i}\gamma_2 \\ \delta_1 + \mathbf{i}\delta_2 \end{pmatrix}, \text{ for } \alpha_{1,2}, \beta_{1,2}, \gamma_{1,2}, \delta_{1,2} \in \mathbb{O}, \tag{5.4.5}$$

a spinor in 12 dimensions is pure if $B_2(\psi^+, \psi^+) = 0$ (theorem 3.2.1.1), thus one has,

$$\begin{aligned}
|\alpha_1|^2 = |\alpha_2|^2,\ |\beta_1|^2 = |\beta_2|^2,\ |\gamma_1|^2 = |\gamma_2|^2,\ |\delta_1|^2 = |\delta_2|^2, \text{ and} \\
(\alpha_1, \alpha_2) = (\beta_1, \beta_2) = (\gamma_1, \gamma_2) = (\delta_1, \delta_2) = 0.
\end{aligned} \tag{5.4.6}$$

These are exactly the conditions for the components $\alpha, \beta, \gamma, \delta \in \mathbb{O} \otimes \mathbb{C}$ to be pure spinors in 8 dimensions.

5.4.3 Orbits of Spin(8)-type Spinors in Spin(12)

As mentioned previously, the strategy of using spinors in 8 dimensions as roots of analysis allows us to generate a new class of orbits in 12 dimensions. This is done by considering the following even Weyl spinor in 12 dimensions. Let Ψ be an even spinor in 12 dimensions, characterised by (5.4.2). Then Ψ takes the form

$$\Psi = \alpha \mathrm{d}z + \beta \mathrm{d}\tilde{z} + \gamma + \delta \mathrm{d}z \wedge \mathrm{d}\tilde{z} = (\gamma + \beta \mathrm{d}\tilde{z}) + (\alpha - \delta \mathrm{d}\tilde{z}) \wedge \mathrm{d}z, \tag{5.4.7}$$

where $\alpha, \beta \in S_4^-$, $\gamma, \delta \in S_4^+$. Imposing a modified Majorana constraint, $\mathbf{R}(\gamma + \alpha \mathrm{d}z) = -(\beta - \delta \mathrm{d}z) \wedge \mathrm{d}\tilde{z}$[3] (Where $\mathbf{R}$ is the product of all real Γ-matrices in 12 dimensions), implies one can then consider

$$\Psi_\theta = \begin{pmatrix} 0 \\ 0 \\ \cos(\theta)\mathbb{I} + \mathbf{i}\sin(\theta)\mathbf{u} \\ \cos(\theta)\mathbb{I} - \mathbf{i}\sin(\theta)\mathbf{u} \end{pmatrix}. \tag{5.4.8}$$

Indeed, we see that the first spinor $\gamma + \alpha \mathrm{d}z$ is an even spinor of 10 dimensions, and $\beta + \delta \mathrm{d}z$ is an odd spinor in 10 dimensions. In 12 dimensions, there is no concept of Majorana-Weyl spinors. However, since the action of $\mathbf{R}$ interpolates between Weyl spinors of 10 dimensions (this exactly why we wrote Ψ in the form (5.4.7)), it is fruitful to consider instead: $\mathbf{R}(\gamma + \alpha \mathrm{d}z) = -(\beta - \delta \mathrm{d}z) \wedge \mathrm{d}\tilde{z}$. This condition implies that $\delta = \gamma^*$ and $-\beta = \alpha^*$. Following that, at least at the Lie algebra level, it is not difficult to show there exists a subalgebra of $\mathfrak{spin}(12)$ (5.4.3) that can kill α, and $-\alpha^* = \beta$ simultaneously. This results in

$$\Psi = \gamma + \gamma^* \mathrm{d}z \wedge \mathrm{d}\tilde{z}. \tag{5.4.9}$$

The final simplification that we make is to use a canonical parameterisation of a generic impure spinor in Spin(8), again given in [Bry20], i.e. $\gamma = \cos(\theta)\mathbb{I} + \mathbf{i}\sin(\theta)\mathbf{u}$[4]. Thus, Ψ takes the desired form (5.4.8).

Although Ψ_θ is complex, the inner product $\langle \Psi_\theta, \Psi_\theta \rangle = |\gamma|^2$ is real. This implies that any differential forms we compute will be real (or purely imaginary). Indeed,

Proposition 5.4.3.1. Consider Ψ_θ such that $\theta \neq 2\pi k$, $\theta \neq \pi/4 + \pi k$ for $k \in \mathbb{Z}$. Then the stabiliser of ψ_θ contains $\mathrm{SU}(4) \times \mathrm{SU}(2)$. Furthermore, the real symplectic form is given as

$$B_2(\Psi_\theta, \Psi_\theta) = \mathbf{i}\sin(2\theta)\omega_8 + \mathbf{i}\cos(2\theta)(X \wedge Y + \bar{X} \wedge \bar{Y}). \tag{5.4.10}$$

Proof. The non-zero, and unique components of the 2-form are given as,

$$\begin{aligned} B_2(\Psi_\theta, \Psi_\theta)_{x,y} &= -\mathbf{i}\sin(2\theta)(\mathbf{u}x, y), \\ B_2(\Psi_\theta, \Psi_\theta)_{9,10} &= -2\mathbf{i}, \\ B_2(\Psi_\theta, \Psi_\theta)_{11,12} &= 2\mathbf{i}, \\ B_2(\Psi_\theta, \Psi_\theta)_{9,11} &= 2\mathbf{i}\cos(2\theta), \text{ and} \\ B_2(\Psi_\theta, \Psi_\theta)_{10,12} &= -2\mathbf{i}\cos(2\theta) \end{aligned} \tag{5.4.11}$$

One then computes the following 2-form as

$$B_2(\Psi_\theta, \Psi_\theta) = \mathbf{i}\sin(2\theta)\omega_8 + 2\mathbf{i}(-\mathrm{d}w^9\mathrm{d}w^{10} + \mathrm{d}w^{11}\mathrm{d}w^{12} + \cos(2\theta)\mathrm{d}w^9\mathrm{d}w^{11} - \cos(2\theta)\mathrm{d}w^{10}\mathrm{d}w^{12}). \tag{5.4.12}$$

[3]The minus sign is to ensure that overall sign of the complexified octonions in Ψ_θ are positive.
[4]One is also free to use the parameterisation for an impure spinor given by (5.3.8).

Here ω_8 is the Kähler form in 8 dimensions, and the concatenation of decomposable differential forms is the suppression of the wedge product. At first glance, it is unclear the stabiliser of the directions $\{\mathrm{d}w^9, \mathrm{d}w^{10}, \mathrm{d}w^{11}, \mathrm{d}w^{12}\}$. However, one can consider the following coordinate transformation

$$\begin{aligned} X &= \sec(2\theta)\mathrm{d}w^9 + \mathbf{i}\tan(2\theta)\mathrm{d}w^{11} - \mathrm{d}w^{12}, \\ Y &= \sec(2\theta)\mathrm{d}w^{11} - \mathbf{i}\tan(2\theta)\mathrm{d}w^9 - \mathrm{d}w^{10}, \\ \bar{X} &= \sec(2\theta)\mathrm{d}w^9 - \mathbf{i}\tan(2\theta)\mathrm{d}w^{11} - \mathrm{d}w^{12}, \\ \bar{Y} &= \sec(2\theta)\mathrm{d}w^{11} + \mathbf{i}\tan(2\theta)\mathrm{d}w^9 - \mathrm{d}w^{10}. \end{aligned} \tag{5.4.13}$$

Therefore, $B_2(\Psi_\theta, \Psi_\theta)$ is the desired result. Furthermore, notice, by raising an index with the metric, there are two complex structures present. One reduces 8 of the 12 real dimensions to 4 complex dimensions, and the remaining 4 directions reduce 2 complex dimensions independently. Hence, the stabiliser must contain $\mathrm{SU}(4) \times \mathrm{SU}(2)$. □

Proposition 5.4.3.2. Consider Ψ_θ such that $\sin(\theta) = \cos(\theta)$, that is $\theta = \pi/4 + \pi k$ for $k \in \mathbb{Z}$. Then the stabiliser of ψ_θ contains $\mathrm{SU}(6)$, one can also compute the 2-form $B_2(\Psi_\theta, \Psi_\theta)$ as

$$B_2(\Psi_\theta, \Psi_\theta) = 2\mathbf{i}\omega_8 + 2\mathbf{i}(-\mathrm{d}w^9\mathrm{d}w^{10} + \mathrm{d}w^{11}\mathrm{d}w^{12}). \tag{5.4.14}$$

Proof. It is not difficult to compute $B_2(\Psi_\theta, \Psi_\theta)$ above. Notice (5.4.14) is symplectic form in 12 dimensions. Raising the index of the symplectic form gives us a complex structure, hence the stabiliser must contain $\mathrm{SU}(6)$. □

Even further,

Proposition 5.4.3.3. Consider Ψ_0, that is Ψ_θ such that $\theta = 0$. Then the stabiliser of ψ_θ contains $\mathrm{Spin}(7) \times \mathrm{SU}(2)$.

Proof. If $\theta = 0$, then the spinor is a real octonion. This means that it is stabilised by $\mathrm{Spin}(7) \subset \mathrm{Spin}(8) \subset \mathrm{Spin}(12)$. Furthermore, computing the 2-form gives

$$B_2(\Psi_0, \Psi_0) = \mathbf{i}\cos(2\theta)(X \wedge Y + \bar{X} \wedge \bar{Y}). \tag{5.4.15}$$

The above is a symplectic form in the 4 directions $\{\mathrm{d}w^9, \mathrm{d}w^{10}, \mathrm{d}w^{11}, \mathrm{d}w^{12}\}$, raising an index with the metric converts it to a complex structure in 4 dimensions. Hence, the stabiliser of a real octonion is $\mathrm{Spin}(7) \times \mathrm{SU}(2)$. □

Finally, take $\psi^+ = \Psi_\alpha$ and the rest of the components to be zero, i.e. a pure spinor. The 6-form $B_6(\psi^+, \psi^+)$ is calculated to be

$$B_6(\psi^+, \psi^+) = \Omega_{12}. \tag{5.4.16}$$

This is the top holomorphic form in 10 dimensions and hence stabilised by $\mathrm{SU}(6)$. We also note,

$$B_2(\psi^+, \psi^+) = 2\mathbf{i}\omega_{12}. \tag{5.4.17}$$

Here ω_{12} is the symplectic form in 12 dimensions. As in the 10 dimensional case, we can make a statement concerning integrability of a SU(6)-structures derived from pure spinors.

Proposition 5.4.3.4. Let M be a spin manifold of 12 dimensions, and suppose that ψ^+ is a spinor stabilised by $\mathrm{SU}(6) \subset \mathrm{Spin}(12)$. The set of canonical differential forms associated to the SU(6)-structure is given as $B_6(\psi^+, \psi^+) = \Omega_{12}$, and $\frac{1}{2\mathbf{i}} B_2(\mathbf{R}(\psi^+), \psi^+) = \omega_{12}$. The SU(6)-structure is integrable if, and only if, the exterior derivatives of its canonical differential forms vanish.

Let us summarise the stabilisers we have found in the section, in the following table.

Stabiliser	Spinor Representative
$SU(4)\times SU(2)$	Ψ_θ such that $\theta \neq 2\pi k$, $\theta \neq \pi/4+\pi k$ for $k\in\mathbb{Z}$.
$SU(6)$	Ψ_θ such that $\theta = \pi/4+\pi k$ for $k\in\mathbb{Z}$.
$\mathrm{Spin}(7)\times SU(2)$	Ψ_θ such that $\theta = 0$.
$SU(6)$	Ψ is a pure spinor.

(5.4.18)

The classification of $\mathrm{Spin}(12,\mathbb{C})$ (see section 6.2) tells us there are three types of complexified stabilisers for impure orbits: $\mathrm{Spin}(7,\mathbb{C})\times SL(2,\mathbb{C})$, $SL(6,\mathbb{C})$, and $Sp(6,\mathbb{C})$, and the complexified stabiliser for the pure spinor orbit is $SL(6,\mathbb{C})$.

The real stabilisers we computed in table (5.4.18) correspond to the complexified stabilisers as follows:

- In the impure cases.
  1. $\mathrm{Spin}(7,\mathbb{C})\times SL(2,\mathbb{C})$ relates to $\mathrm{Spin}(7)\times SU(2)$.
  2. $SL(6,\mathbb{C})$ relates to $SU(4)\times SU(2)$, and $SU(6)$.
- In the pure case, the complex spinor stabiliser corresponds to $SU(6)$.

Notice that the complexified stabiliser $Sp(6,\mathbb{C})$ is omitted above. This omission is due to uncertainty regarding whether this stabiliser persists in the real setting, and if it does, what its realisations are. This situation is likely a result of relaxing the modified Majorana constraint we imposed on a spinor in 12 dimensions and attempting to analyse stabilisers from that point onwards. Furthermore, the real stabilisers mentioned above, corresponding to their respective complexifications, represent only a single "slice" of all possibilities. We are unsure if there are additional real stabilisers not captured by the class of spinors we have considered, or if these real stabilisers are the only realisations of the complexified stabilisers.

Chapter 6

Complexified Stabilisers of Impure Spinors in Higher Dimensions

6.1 Setting up the combinatorial problem and applications

Consider the vector space $\mathbb{C}^{2d}$ and the associated Clifford algebra $\mathrm{Cliff}(\mathbb{C}^{2d})$. Suppose we choose a pure spinor ψ_0. Then the maximally totally null space of ψ_0, $M(\psi_0)$, is the space spanned by a space isomorphic to $\mathbb{C}^d$. Let $\{e_i \mid 1 \le i \le d\}$ be a null basis in this space where $g(e_i, e_j) = 0$. This null basis allows for a lift to $\mathrm{Cliff}(\mathbb{C}^d) \subset \mathrm{Cliff}(\mathbb{C}^{2d})$, corresponding to the generators that annihilate ψ_0. We define these operators using the standard annihilation operator notation a_i. Additionally, there exists another maximally totally null space such that $\mathrm{Cliff}(\mathbb{C}^d) \oplus \mathrm{Cliff}(\overline{\mathbb{C}}^d) = \mathrm{Cliff}(\mathbb{C}^{2d})$. This null space can be generated from ψ_0 and corresponds to a choice of null basis $\{\bar{e}_i \mid 1 \le i \le d\} \subset \overline{\mathbb{C}}^d$, where $g(\bar{e}_i, \bar{e}_j) = 0$ and $g(e_i, \bar{e}_j) = \delta_{ij}$. These operators are defined using the standard creation operator notation $a_i^\dagger$.

Now consider the space

$$S = \mathrm{Span}\left\{ \prod_{i_1 < i_2 < \ldots i_k} a_{i_1}^\dagger a_{i_2}^\dagger \ldots a_{i_k}^\dagger \psi_0 | k \in \mathbb{N}_{<n} \right\}. \tag{6.1.1}$$

Then consider the subspace

$$S^+ = \mathrm{Span}\left\{ \prod_{i_1 < i_2 < \ldots i_k} a_{i_1}^\dagger a_{i_2}^\dagger \ldots a_{i_k}^\dagger \psi_0 | k \in 2\mathbb{N}_{<n} \right\}. \tag{6.1.2}$$

It is suggestive, but also quite clear, that $S \cong \Lambda(\mathbb{C}^d)$ and $S^+ \cong \Lambda^{\mathrm{even}}(\mathbb{C}^d)$ above is the irreducible Weyl representation. Taking motivation from physics, we define the vacuum state to be ψ_0 in (6.1.2), as every other state is generated from this one.

It is also clear that these 2^{d-1} vectors of S^+ are pure spinors, and so any spinor in S^+ can be written as a linear combination of at most 2^{d-1} pure spinors. This motivates the following definition,

Definition 6.1.0.1. Let ψ be a Weyl spinor in S^+, equivalently a polyform in $\Lambda^{\mathrm{even}}(\mathbb{C}^d)$. Then ψ is called an *impure spinor of degree k* if it can be written as a linear combination of pure spinors

$$\psi = \sum_{\alpha=1}^{k} c_\alpha \psi_\alpha, \tag{6.1.3}$$

such that ψ_α are pure spinors, $c_\alpha \in \mathbb{C}$, and for any ψ_α and ψ_β, $|M(\psi_\alpha) \cap M(\psi_\beta)| < d - 4$.

The condition on the dimension of the intersection of maximally totally null subspaces, as discussed by Cartan and Chevalley [CB68; Che54], states that if ψ and ϕ are two (non-parallel) pure spinors, then $\psi + \phi$ is a pure spinor if, and only if, $M(\psi) \cap M(\phi) = n - 2$. This was why we imposed the condition $|M(\psi_\alpha) \cap M(\psi_\beta)| < d - 4$ in definition 6.1.0.1.

The degree of an impure spinor will be very important in calculations to come. We will explain later how stabilisers of impure spinors are understood through the intersection of maximally totally null (MTN) spaces of the pure spinors that constitute an impure one. We want to minimise the number of MTN spaces analysed, thus we shall try to bound the degree k.

For convenience, we also define the following

Definition 6.1.0.2. Let ψ_α be a pure spinor as above, then we define the concatenation of directions that constitute $M(\psi_\alpha)$ as the spinor ψ.

Example 6.1.0.1. Consider $M(\psi_3) = \mathrm{Span}\{a_1, a_2, a_3^\dagger, a_4^\dagger\}$ in $d = 4$. Then we will exploitatively write,

$$\psi_3 = a_1 a_2 a_3^\dagger a_4^\dagger. \tag{6.1.4}$$

With these foundations, we are equipped to start counting possible orbits. ψ can be represented, according to definition 6.1.0.1, by a k-simplex with edges weighted by the dimension of the intersection of maximally totally null subspaces. A powerful corollary of these techniques is the computability of orbits. However, in dimensions larger than 14 classifying diagrams is no longer tractable. For example, the classification of spinors in 16 dimensions [L V82] is notably lengthy, reflecting the fact that the number of diagrams one can generate is substantially larger than in fourteen dimensions. These computations are omitted in this thesis.

As was mentioned before, we associated to each diagram a canonical differential form. We review the geometric map here again, because we are using a slightly different convention,

$$B_k(\Psi, \Psi) := \frac{1}{k!} \langle \Psi, \Gamma_{\mathsf{I}_1} \dots \Gamma_{\mathsf{I}_k} \Psi \rangle M^{\mathsf{I}_1} \wedge \dots \wedge M^{\mathsf{I}_k}. \tag{6.1.5}$$

Here $M^{\mathsf{I}_1}, \dots, M^{\mathsf{I}_{2d}} \in \Lambda^1(\mathbb{C}^{2d})$ is a basis in the space of 1−forms in $\mathbb{C}^{2d}$.

It is useful to recast this calculation as one in terms of creation and annihilation operators. We write,

$$M^i = \frac{m^i + \bar{m}^i}{2}, \text{ and } M^{i+d} = \frac{m^i - \bar{m}^i}{2\mathbf{i}}. \tag{6.1.6}$$

Then the geometric map takes the form

$$B_k(\Psi, \Psi) = \frac{1}{k!} \langle \Psi, (a_{i_1} m^{i_1} + a_{i_1}^\dagger \bar{m}^{i_1}) \dots (a_{i_k} m^{i_k} + a_{i_k}^\dagger \bar{m}^{i_k}) \Psi \rangle. \tag{6.1.7}$$

Following [BK23], the goal of this chapter is to use the machinery of geometric maps to understand parts of stabilising algebras of (Weyl) spinors found through combinatorial means. As the combinatorics is exhaustive, it is sufficient to pick a representative in a spinor orbit and compute the canonical differential forms. Analysing the group actions that leave the forms invariant allows us to access part of the stabiliser. Then, as we already have a list of all possible stabilisers, [Cha97; Pop80], we can match our stabilisers to the literature.

6.1.1 Occupation numbers

Let us consider each of the $2d$ basis vectors $\{a_i, a_i^\dagger\}$ in $\mathbb{C}^{2d}$ as a box. Each of the spinors ψ_α is a pure spinor whose null subspace $M(\psi_\alpha)$ is d-dimensional, and thus occupies precisely d of the $2d$ available boxes. This observation motivates the definition.

Definition 6.1.1.1. The *box occupation numbers* are defined as the number of boxes occupied by $0, 1, \ldots, k$ pure spinors in d dimensions, and denoted by

$$n_0, n_1, \ldots, n_k. \tag{6.1.8}$$

Each box is occupied by some number (possibly zero) of pure spinors, and thus each is counted. Another way to restate this definition is that n_k counts the number of shared directions in (6.1.2). For example, if $n_2 = 5$, there are 5 null directions shared by 2 pure spinors. Or $n_3 = 4$, implies there are 4 null directions shared by 3 pure spinors. A consequence of definition 6.1.1.1 is,

Proposition 6.1.1.1. Let n_i be the box occupation number, k the degree of an impure spinor in d dimensions, then

$$n_k + \ldots + n_0 = 2d, \tag{6.1.9}$$

Proof. Fix, without loss of generality, $n_k \in \{0, \ldots, d\}$, i.e. there are k pure spinors that share n_k directions, pure spinors are Weyl and so can't share more than d directions. This means there are $2d - n_k$ directions to allocate for $n_{k-1} \ldots, n_0$. Without loss of generality, fix n_{k-1}. Then as previously there are $2d - n_k - n_{k-1}$ directions left to allocate between for $n_{k-2} \ldots, n_0$. Eventually one will no longer have any directions left to allocate and so $2d - n_k - \ldots - n_0 = 0$, giving the desired result. □

We now begin to write formulas that follow a consistent pattern. Consider n_1, which simply represents the dimension of the Maximally Totally Null (MTN) space of a pure spinor. Geometrically, we can visualise this as a vertex. n_2 extends this concept by querying the number of shared directions between two pure spinors, effectively connecting the vertices with an edge whose weight corresponds to the number of directions shared. Progressing further, n_3 pertains to information about three pure spinors. This can be visualised as a triangle, where the weight represents the number of shared directions. It is evident that embedded within n_3 are three edges (incorporating information about n_2) and three vertices (incorporating information about n_1). Within a k-simplex, it is relevant to ask about the counts of lower-dimensional simplices embedded within it. This consideration leads us to explore the counting formula.

Proposition 6.1.1.2. Let n_i be the box occupation number, k the degree of an impure spinor in d dimensions, then

$$kn_k + (k-1)n_{k-1} + \ldots n_1 = kd. \tag{6.1.10}$$

Proof. The formula relates the number of total number of boxes that can be occupied to how they are distributed in terms of pure spinors. Fix n_k, meaning there will be k spinors sharing n_k directions. For each direction, one has $\binom{k}{1}$ boxes filled, but there are n_k lots of these, so $n_k \times \binom{k}{1} = kn_k$ occupied boxes contributing from these k pure spinors. Continuing along, we find the total number of boxes occupied given by the sum

$$\binom{k}{1} n_k + \ldots + \binom{k-r}{1} n_{k-r} + \ldots + \binom{1}{1} n_1. \tag{6.1.11}$$

Now the total number of boxes is occupied is not difficult to see being $k \times d$, there are k pure spinors and each pure spinor spans d directions. The result follows from equating kd and (6.1.11). □

Finally, there is a duality relation between occupation number. Indeed, If a pure spinor occupies some box, it does not occupy the dual to its box. For example, if $a_i^\dagger$ is occupied then a_i cannot be occupied by the same pure spinor, because the space spanned by the boxes occupied by some pure spinor must be totally null. Hence,

Proposition 6.1.1.3. Let n_i be the box occupation number, k the degree of an impure spinor in d dimensions, then

$$n_k = n_0, \; n_{k-1} = n_1, \text{ etc.} \tag{6.1.12}$$

In view of this duality, the two relations (6.1.9) and (6.1.10) are equivalent, thus we only need one of them, e.g. (6.1.9).

Example 6.1.1.1. Let us consider a simple example in dimension eight of a spinor that is not pure. This spinor is a sum of two pure spinors whose null subspaces do not intersect. Let one of these pure spinors, say ψ_A, be the vacuum state in (6.1.2) from which all other possible basis pure spinors are built. It is convenient to denote pure spinors by their corresponding null subspaces. And so we take

$$\psi_A = a_1 a_2 a_3 a_4. \tag{6.1.13}$$

Let us consider another spinor defined as

$$\psi_B = a_1^\dagger a_2^\dagger a_3^\dagger a_4^\dagger, \tag{6.1.14}$$

where, similarly, we represent a pure spinor by its null subspace. In this case, $d = 4$, and the occupation numbers are $n_2 = n_0 = 0$ and $n_1 = 8$. This implies that there are no directions (boxes) that are occupied by either zero or two pure spinors, and every box is occupied by precisely one pure spinor. Another useful interpretation, highlighting the concept of duality and the idea of *boxes*, is demonstrated through the following table:

Pure Spinor	a_1	a_2	a_3	a_4	$a_1^\dagger$	$a_2^\dagger$	$a_3^\dagger$	$a_4^\dagger$
ψ_A	■	■	■	■				
ψ_B					■	■	■	■

A *filled* box corresponds to a direction that the spinor spans, and an empty box, quite intuitively, indicates a direction the spinor does not span. When stating $n_2 = n_0 = 0$, it means no column has two filled boxes or is completely empty, while $n_1 = 8$ indicates that there are eight columns each with only one box filled. This visualisation not only clarifies duality but also shows that if one fills horizontally four boxes, as is the case with ψ_A and ψ_B, then any empty column in one spinor corresponds directly to a filled column in the other, thus illustrating the combinatorial construction clearly.

Example 6.1.1.2. Let us consider a more complex example of a general spinor in eight dimensions, represented by a linear combination of all pure spinors listed in (6.1.2). Here, $k = 8$, indicating that there are eight different possible states in (6.1.2). It is straightforward to observe that each of the available 8 boxes is occupied by exactly 4 different pure spinors, so $n_4 = 8$. Another way to express this is to note that in a table of the type presented previously, 4 boxes are highlighted in every column, satisfying all relationships. However, at this stage, such a table would be too large to illustrate, and the corresponding diagram too complex to draw. We will later see that there is no need to consider $k = 8$, based on some straightforward analyses.

6.1.2 Edge intersection numbers

Definition 6.1.2.1. Let ψ be an impure spinor constituted by k pure spinors. Introduce an index $I \in \{1, \dots, k(k-1)/2\}$ that labels edges with a label I that connects vertices α, β. The *edge intersection dimension* is defined as the dimension of the intersection subspace of the null subspaces $M(\psi_\alpha)$, and $M(\psi_\beta)$. Explicitly,

$$e_I \equiv e_{\widehat{\alpha\beta}} = \dim(M(\psi_\alpha) \cap M(\psi_\beta)). \tag{6.1.15}$$

Define $e \coloneqq \sum_I e_I$ as the *edge intersection number.*

As we know the possible edge intersection dimensions are $d-2, d-4, \ldots$, where we assume that the pure spinors in the sum (6.1.3) are distinct, and so their null subspaces cannot intersect in d, and $d-2$ dimensions. Below, it will be useful how to take this constraint into account.

There is a relation between the occupation numbers and the sum of the intersection numbers.

Theorem 6.1.2.1. Let n_i be the box occupation number, and k the degree of an impure spinor in d dimensions. Then

$$\frac{k(k-1)}{2} n_k + \frac{(k-1)(k-2)}{2} n_{k-1} + \ldots + n_2 = e. \tag{6.1.16}$$

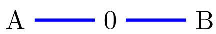
Proof. Similarly to the proof of theorem 6.1.1.2, we are counting the total contribution from each edge of a k-simplex from k pure spinors. The right-hand side is obvious as there are $\binom{k}{2}$ edges each carrying some weight e_I so the total contribution is just the sum. On the other hand, if one fixes n_k, meaning there will be k spinors sharing n_k directions, then for each direction there are $\binom{k}{2}$ shared edges. Of course there are n_k lots of these and so for k pure spinors one contributes $n_k \times \binom{n}{k}$. Continuing along, one finds the total contribution of edges given by the sum

$$\binom{k}{2} n_k + \ldots + \binom{k-r}{2} n_{k-r} + \ldots + \binom{2}{2} n_2. \tag{6.1.17}$$

The result follows from equating (6.1.17) to the total contribution from each edge, i.e. the right-hand side of (6.1.16). ☐

Let us illustrate the relation (6.1.16) on the two examples already considered.

Example 6.1.2.1. For the spinor of purity $k=2$ in eight dimensions the two pure constituents $\psi_A = a_1 a_2 a_3 a_4$, and $\psi_B = a_1^\dagger a_2^\dagger a_3^\dagger a_4^\dagger$ have intersection number zero. So, we represent their linear combination of a simplex consisting of two vertices, and the edge connecting them. The edge has intersection number zero. This corresponds to a diagram,



$$\psi_A = a_1 a_2 a_3 a_4, \quad \psi_B = a_1^\dagger a_2^\dagger a_3^\dagger a_4^\dagger. \tag{6.1.18}$$

The only non-vanishing occupation number is $n_1 = 8$, which does not appear in (6.1.16). The right-hand side is also zero, as the only intersection number is zero in this case.

Example 6.1.2.2. Let us now consider the spinor obtained as the general linear combination of all possible pure spinors in 8 dimensions. In this case, we get a simplex with 8 vertices, and the occupation numbers are $n_4 = 8$. There are 28 edges. On one hand, the left-hand side of the formula dictates that one must have $\left(4 \times \frac{3}{2}\right) \times 8 = 48$. On the other hand, if we consider from each vertex, there emanates an edge with intersection number zero, as well as six edges with intersection number two. Then there are 24 edges with intersection number two. Hence, both sides of equation (6.1.16) match.

6.1.3 Tetrahedral intersection numbers

Definition 6.1.3.1. Let ψ be an impure spinor constituted by k pure spinors. Introduce an index $J \in \{1, \ldots, k(k-1)(k-2)(k-3)/24\}$ that labels the tetrahedra, with a label J that connects four vertices $\alpha, \beta, \gamma, \delta$. The *tetrahedral intersection dimension* is defined as the dimension of the intersection subspace of the null subspaces $M(\psi_\alpha)$, $M(\psi_\beta)$, $M(\psi_\gamma)$, and $M(\psi_\delta)$. Explicitly,

$$t_J \coloneqq t_{\overline{\alpha\beta\gamma\delta}} = \dim(M(\psi_\alpha) \cap M(\psi_\beta) \cap M(\psi_\gamma) \cap M(\psi_\delta)). \tag{6.1.19}$$

Define $t \coloneqq \sum_J t_J$ as the *tetrahedral intersection number.*

We now have a relation similar to (6.1.16)

Theorem 6.1.3.1. Let n_i be the box occupation number, and k the degree of an impure spinor in d dimensions. Then

$$\frac{k(k-1)(k-2)(k-3)}{24} n_k + \frac{(k-1)(k-2)(k-3)(k-4)}{24} n_{k-1} + \ldots + n_4 = t \tag{6.1.20}$$

We omit the proof of theorem 6.1.3.1, as this is a simple extension of proofs given in theorems 6.1.1.2 and 6.1.2.1.

Example 6.1.3.1. In $d = 4$ there are $2^{2-1} = 2$ Weyl spinors in (6.1.2), so no tetrahedron can be constructed, thus we consider $d = 6$. This is the minimum dimension that a tetrahedron can be constructed as there are $2^{3-1} = 4$ Weyl spinors. It is constructive to consider an example via a table

pure spinor	a_1	a_2	a_3	$a_1^\dagger$	$a_2^\dagger$	$a_3^\dagger$
ψ_1	■	■	■			
ψ_2			■	■	■	
ψ_3		■		■		■
ψ_4	■				■	■

Without loss of generality, ψ_1 can be declared the vacuum state, so we are left to play with $\psi_{2,3,4}$. These spinors must span the whole Weyl space and so, up to permutation, the contents of the table are all that one can write. Now, reading the table, one sees that the there are 2 boxes always filled per column, in other words $n_2 = 6$. Furthermore, we see that the edges intersect in 1 dimensions each, so the sum of all edge intersection dimensions is 6. We also notice that $n_4 = 0$, hence, there is no shared direction between 4 pure spinors. This in turn implies the tetrahedral intersection number is zero.

Example 6.1.3.2. Consider the complete set of pure spinors in eight dimensions, the next minimal case after. There are eight pure spinors and $n_4 = 8$. There are in total 70 tetrahedra, but most have intersection number zero. There are eight tetrahedra with a non-zero intersection number, and they are given by

$$\begin{array}{l}
a_1a_2a_3a_4,\ a_2^\dagger a_3^\dagger a_1a_4,\ a_2^\dagger a_4^\dagger a_1a_3,\ a_3^\dagger a_4^\dagger a_1a_2,\ \text{all share direction } a_1. \\
a_1a_2a_3a_4,\ a_1^\dagger a_3^\dagger a_2a_4,\ a_1^\dagger a_4^\dagger a_2a_3,\ a_3^\dagger a_4^\dagger a_1a_2,\ \text{all share direction } a_2. \\
a_1a_2a_3a_4,\ a_1^\dagger a_2^\dagger a_3a_4,\ a_1^\dagger a_4^\dagger a_2a_3,\ a_2^\dagger a_4^\dagger a_1a_3,\ \text{all share direction } a_3. \\
a_1a_2a_3a_4,\ a_1^\dagger a_2^\dagger a_3a_4,\ a_1^\dagger a_3^\dagger a_2a_4,\ a_2^\dagger a_3^\dagger a_1a_4,\ \text{all share direction } a_4. \\
a_1^\dagger a_2^\dagger a_3^\dagger a_4^\dagger,\ a_1^\dagger a_2^\dagger a_3a_4,\ a_1^\dagger a_3^\dagger a_2a_4,\ a_1^\dagger a_4^\dagger a_2a_3,\ \text{all share direction } a_1^\dagger. \\
a_1^\dagger a_2^\dagger a_3^\dagger a_4^\dagger,\ a_1^\dagger a_2^\dagger a_3a_4,\ a_2^\dagger a_3^\dagger a_1a_4,\ a_2^\dagger a_4^\dagger a_1a_3,\ \text{all share direction } a_2^\dagger. \\
a_1^\dagger a_2^\dagger a_3^\dagger a_4^\dagger,\ a_1^\dagger a_3^\dagger a_3a_4,\ a_2^\dagger a_3^\dagger a_1a_4,\ a_3^\dagger a_4^\dagger a_1a_2,\ \text{all share direction } a_3^\dagger. \\
a_1^\dagger a_2^\dagger a_3^\dagger a_4^\dagger,\ a_1^\dagger a_4^\dagger a_2a_3,\ a_2^\dagger a_4^\dagger a_1a_3,\ a_3^\dagger a_4^\dagger a_1a_2,\ \text{all share direction } a_4^\dagger.
\end{array} \tag{6.1.21}$$

Thus, there are eight tetrahedra with $t = 1$, and (6.1.20) holds.

What we have shown is that theorem 6.1.3.1 provides bounds for our counts. Indeed, we must have $t \leq d-7$ in order to obtain a linear combination of pure spinors that cannot be further reduced, indeed, for $d=8$, $t=8-7=1$. As simplicies embed themselves into higher dimensional ones, for any dimensions higher than $d=8$, one always expects this relation to hold.

We will now analyse what all the equations imply together with the constraints on the edge and tetrahedral intersection numbers. We consider the case of k odd and even separately.

6.1.4 Bounds on The Degree of Impure Spinors

We now analyse the bounds on the degree of impurity of a spinor k. It was noted that considering $k \in \{0,\ldots,2^{d-1}\}$ is superfluous due to additional conditions on edge intersection numbers, for example. We begin by analysing the formula in theorem 6.1.2.1. Other formulas we have generated help us understand the relationships between k and the underlying dimension d. We will find that for $d<8$, there is a neat bound that reproduces standard results in the literature. For $d \geq 8$, the formulas for bounds on purity either break down or become trivially true.

We then appeal to the formula for tetrahedral intersection numbers in theorem 6.1.3.1, similar to our approach with edge intersection numbers. These formulas are necessary to analyse spinor orbits in the next chapter 6.2 and need to be divided into two types. The reason is that intersection numbers can only be of the form $d-2p$ where $p \in \{0,\ldots,\lfloor \frac{d}{2} \rfloor\}$. Therefore, either one reduces to a minimal edge number of 0, or 1.

Bounds via Edge Intersection Numbers

Beginning with odd purity, we derive a bound on k dependent on the underlying dimension d as follows,

Proposition 6.1.4.1. Let n_i be the box occupation number, and k the degree of an impure spinor in d dimensions, such that $k>1$ is odd. Then one has, for $d<8$,

$$k \leq \frac{d}{8-d}. \tag{6.1.22}$$

Proof. As k is odd, $k+1$ is even. Furthermore, as all occupation numbers are in pairs related by the duality, proposition (6.1.1.3), equation (6.1.9) becomes

$$n_k + n_{k-1} + \ldots + n_{(k+1)/2} = d. \tag{6.1.23}$$

Then using (6.1.23) with the relation (6.1.16), one can write the formula in (6.1.2.1) in terms of the higher occupation numbers

$$\begin{aligned} e = {} & \frac{k(k-1)}{2} n_k + \frac{(k-1)(k-2)}{2} n_{k-1} + \left(\frac{(k-2)(k-3)}{2} + 1 \right) n_{k-2} \\ & + \ldots + \left(\frac{(k-a)(k-a-1)}{2} + \frac{a(a-1)}{2} \right) n_{k-a} + \ldots + \frac{(k-1)^2}{4} n_{(k+1)/2}. \end{aligned} \tag{6.1.24}$$

Here a goes up to the maximal value $a_{max} = (k-1)/2$. Substituting

$$n_{(k+1)/2} = d - n_k - n_{k-1} - \ldots - n_{(k+3)/2}. \tag{6.1.25}$$

in the above equation gives,

$$e - \frac{(k-1)^2}{4}d = \left(\frac{k(k-1)}{2} - \frac{(k-1)^2}{4}\right)n_k + \left(\frac{(k-1)(k-2)}{2} - \frac{(k-1)^2}{4}\right)n_{k-1} \\ + \ldots + \left(\frac{(k-a)(k-a-1)}{2} + \frac{a(a-1)}{2} - \frac{(k-1)^2}{4}\right)n_{k-a} + \ldots \tag{6.1.26}$$

The $n_{(k+1)/2}$ term has been subtracted, so the last term on the right-hand side is one containing $n_{(k+3)/2}$. One can check that the coefficients in front of all the terms on the right-hand side are positive, and so the left-hand side is a non-negative number. On the other hand, as we have already discussed, the maximal value of the edge intersection dimensions in $2d$ dimensions is $d-4$. This is so that no sum of the impure spinor is pure. And so the right-hand side of the above is less than or equal to

$$\frac{k(k-1)}{2}(d-4) - \frac{(k-1)^2}{4}d = \frac{(k-1)}{4}((k+1)d - 8k). \tag{6.1.27}$$

This must be non-negative, because the sum on the left-hand side of (6.1.26) is non-negative, which gives, for $k > 1$

$$(k+1)d - 8k \geq 0 \implies k(d-8) + d \geq 0. \tag{6.1.28}$$

Rearranging the equation above to make d the subject achieves the desired result. □

The condition that $d < 8$ is imposed is to ensure k has a value, for example $d = 8$, and for k to be positive, for example $d > 8$. We will incorporate bounds for $d \geq 8$ in the next section.

Example 6.1.4.1. For $d = 6$, one has $k \leq \frac{6}{8-6} \leq 3$. This means that one must consider $k = 3$ and $k = 1$. For $d = 7$, $k \leq \frac{7}{8-7} \leq 7$ so one must consider $k = 7, 5, 3$ and 1. Finally, for $d \geq 8$ the formula breaks, k must be a positive integer, and there is no condition for the bounds on k.

We now perform a similar analysis for the even number of pure spinors. We have

Proposition 6.1.4.2. Let n_i be the box occupation number, and k the degree of an impure spinor in d dimensions, such that k is even and $d < 7$. Then one has, for $d < 8$,

$$k \leq \frac{8}{8-d}. \tag{6.1.29}$$

Proof. Using duality, like was done in the odd case, and (6.1.1.1) one obtains

$$2n_k + 2n_{k-1} + \ldots + 2n_{k/2+1} + n_{k/2} = 2d. \tag{6.1.30}$$

The expression for the intersection number in terms of the independent occupation numbers is now

$$e = \frac{k(k-1)}{2}n_k + \frac{(k-1)(k-2)}{2}n_{k-1} + \left(\frac{(k-2)(k-3)}{2} + 1\right)n_{k-2} \\ + \ldots + \left(\frac{(k-a)(k-a-1)}{2} + \frac{a(a-1)}{2}\right)n_{k-a} + \ldots + \frac{k(k-2)}{8}n_{k/2}. \tag{6.1.31}$$

Substituting

$$n_{k/2} = 2d - n_k - \ldots - 2n_{k/2+1} \tag{6.1.32}$$

and taking the factor of $dk(k-2)/4$ to the left-hand side gives

$$e - \frac{k(k-2)}{4}d = \left(\frac{(k-1)(k-2)}{2} - \frac{k(k-2)}{4}\right)n_{k-1} + \ldots \tag{6.1.33}$$

and using the fact that the maximal possible value of the edge intersection dimension is $d-4$, hence we get the maximal possible value of the left-hand side to be

$$\frac{k(k-1)}{2}(d-4)-\frac{k(k-2)}{4}d=\frac{k}{4}(k(d-8)+8). \tag{6.1.34}$$

Thus, as the right-hand side of (6.1.26) is positive,

$$k(d-8)+8\geq 0. \tag{6.1.35}$$

Rearranging for k gives the desired result. □

$d<8$ was imposed for the same reason as the k odd case. This condition will be relaxed in the next section.

Example 6.1.4.2. For $d=4,5$ this gives $k\leq 2$, which tells us that there the only impure spinors are those consisting of two pure spinors in these numbers of dimensions. This is a known result, which we have reproduced by our method.

Example 6.1.4.3. For $d=6$ we get $k\leq 4$, and for $d=7$ we get $k\leq 8$.

Bounds via Tetrahedral Intersection Numbers

As was explained, to understand $d\geq 8$ one appeals to tetrahedral intersection numbers. Let us now analyse the consequences

Theorem 6.1.4.1. Let n_i be the box occupation number, and k the degree of an impure spinor in d dimensions, such that $k>3$ is odd. Then one has

$$k^2(28-3d)-56k+15d\leq 0. \tag{6.1.36}$$

Proof. As was done before use duality and rewrite (6.1.20) as

$$\begin{aligned} t=&\frac{k(k-1)(k-2)(k-3)}{24}n_k+\frac{(k-1)(k-2)(k-3)(k-4)}{24}n_{k-1}\\ &+\frac{(k-2)(k-3)(k-4)(k-5)}{24}n_{k-2}+\frac{(k-3)(k-4)(k-5)(k-6)}{24}n_{k-3}\\ &+\left(\frac{(k-4)(k-5)(k-6)(k-7)}{24}+1\right)n_{k-4}+\dots\\ &+\left(\frac{(k-a)(k-a-1)(k-a-2)(k-a-3)}{24}+\frac{a(a-1)(a-2)(a-3)}{24}\right)n_{k-a}\\ &+\dots+\frac{(k-1)(k-3)^2(k-5)}{96}n_{(k+1)/2}. \end{aligned} \tag{6.1.37}$$

We then again use (6.1.25). All terms on the right-hand side will contain an occupation number, except for $n_{(k+1)/2}$. We then move this term over to the left-hand side, and leave all other terms, containing the occupation numbers, on the right-hand side. This results in the following left-hand side of the above equation:

$$t-\frac{(k-1)(k-3)^2(k-5)}{96}d. \tag{6.1.38}$$

This must be non-negative. Recall, the largest value of each tetrahedral intersection dimension that does not lead to a reduction in the number of pure spinors is $d-7$. As the number of tetrahedra is $k(k-1)(k-2)(k-3)/24$, the following inequality must be satisfied

$$\frac{k(k-1)(k-2)(k-3)}{24}(d-7) - \frac{(k-1)(k-3)^2(k-5)}{96}d \geq 0. \tag{6.1.39}$$

For $k > 3$ this is equivalent to

$$4k(k-2)(d-7) - (k-3)(k-5)d \geq 0, \tag{6.1.40}$$

or

$$k^2(28-3d) - 56k + 15d \leq 0. \tag{6.1.41}$$

□

We do not need to consider the dimensions up to and including $d = 6$, because we already know that the maximal value of k in $d = 6$ is $k = 3$.

Example 6.1.4.4. The new constraints first arise in $d = 7$, where we get that $k \leq 5$. This is a constraint stronger than the one we had previously by considering only edge intersection numbers.

Example 6.1.4.5. A completely new case is $d = 8$, where we had no constraint coming from edge intersection numbers. In this case, we have an equivalent inequality $4k^2 - 56k + 120 \leq 0 \iff (k-7)^2 - 19 < 0$. The largest such integer that satisfies this inequality is $k = 11$, which gives us a new bound on the number of pure spinors in the case of sixteen dimensions.

A repetition for k even, as was done with k odd, regarding tetrahedral intersection numbers, can also be performed. However, this results in a weaker bound. We state and prove it for completeness.

Theorem 6.1.4.2. Let n_i be the box occupation number, and k the degree of an impure spinor in d dimensions, such that $k > 2$ is even. Then one has

$$k^2(28-3d) - k(112-6d) + 12d + 84 \leq 0. \tag{6.1.42}$$

Proof. We first rewrite (6.1.20) using duality

$$\begin{aligned} t = {} & \frac{k(k-1)(k-2)(k-3)}{24}n_k + \frac{(k-1)(k-2)(k-3)(k-4)}{24}n_{k-1} \\ & + \frac{(k-2)(k-3)(k-4)(k-5)}{24}n_{k-2} + \frac{(k-3)(k-4)(k-5)(k-6)}{24}n_{k-3} \\ & + \left(\frac{(k-4)(k-5)(k-6)(k-7)}{24} + 1\right)n_{k-4} + \ldots \\ & + \left(\frac{(k-a)(k-a-1)(k-a-2)(k-a-3)}{24} + \frac{a(a-1)(a-2)(a-3)}{24}\right)n_{k-a} \\ & + \ldots + \frac{k(k-2)(k-4)(k-6)}{192}n_{k/2}. \end{aligned} \tag{6.1.43}$$

We then substitute the value of $n_{k/2}$ from (6.1.30), and take the term containing the dimension $2d$ to the right-hand side. We use the fact that the maximal value of t for each tetrahedron is $t = d - 7$. This gives the following inequality that must be satisfied

$$\frac{k(k-1)(k-2)(k-3)}{24}(d-7) - \frac{k(k-2)(k-4)(k-6)}{96}d \geq 0. \tag{6.1.44}$$

For $k > 2$ this is equivalent to

$$4(k-1)(k-3)(d-7) - d(k-4)(k-6) \geq 0. \tag{6.1.45}$$

expanding above and rearranging gives the desired result. □

For $d = 7$ this gives $k \leq 6$, and for $d = 8$ this gives $k \leq 12$. Of course, these are weaker conditions to what we have already found.

6.1.5 Rotating a spinor

Before stating representatives of orbits, we shall show that for a generic linear combination of pure spinors, there is a Cartan subalgebra in $\mathfrak{spin}(2d, \mathbb{C})$ that allows us to rescale the spinor to an overall factor. We introduced the spin Lie algebra in chapter 3, via creation and annihilation operators. The Cartan subalgebra in $\mathfrak{spin}(2d) \cong \mathfrak{so}(2d)$ is the maximally commuting subalgebra generated by d one-dimensional generators for $d \geq 4$. The Cartan algebra will be taken as $\mathfrak{h} = \{\lambda_i(a_i a_i^\dagger - a_i^\dagger a_i) \mid i \in \{1, \ldots, d\}\}$. Since the algebra is commuting, exponentials of the generators also commute. The generic element has the form

$$X_{\mathfrak{h}} = \sum_i^d \exp(\lambda_i(a_i a_i^\dagger - a_i^\dagger a_i)) = \prod_{i=1}^d \cosh(\lambda_i)\mathbb{I} + \sinh(\lambda_i)(a_i a_i^\dagger - a_i^\dagger a_i) \tag{6.1.46}$$

Consider a generic spinor $\psi = \sum_{\alpha=1}^k c_\alpha \psi_\alpha$, where ψ_α is a pure spinor. The action of $X_{\mathfrak{h}}$ on ψ will only rescale the factors in front of each pure spinor, that is,

$$X_{\mathfrak{h}}\psi = \sum_{\alpha=1}^k \rho_\alpha c_\alpha \psi_\alpha. \tag{6.1.47}$$

However, ρ_α can only be of the form $\cosh(\lambda_i) \pm \sinh(\lambda_i) = e^{\pm\lambda_i}$, with the sign dependent on the spinor. We shall show, case-by-case, that one can pick a subset $\mathfrak{h}' \subset \mathfrak{h}$, usually $k-1$ generators of the d generators such that

$$X_{\mathfrak{h}'}\psi = \Lambda \sum_{i=1}^k \psi_\alpha \coloneqq \tilde{\psi}. \tag{6.1.48}$$

The reason that want to be able to rescale the coefficients c_α is to make computations of the geometric maps, $B_k(\psi)$ easier, and thus the analyses of the simple parts of their stabilisers.

6.2 Classification and Geometry up to and including 14 dimensions

We now explicitly solve the combinatorial problem in each given dimension, listing the possible graphs allowed by the combinatorial analysis, demonstrating a possible spinor representing each graph, and finding the (simple parts of the) stabiliser. In this section, we reproduce the known classification of orbits up to and including fourteen dimensions.

6.2.1 Eight dimensions

It is well-known that every Weyl spinor in dimensions two, four, and six is pure, with stabiliser $\mathfrak{sl}(d, \mathbb{C})$. The first non-trivial problem arises in dimension eight. From the general analysis above, we expect to be able to construct an impure spinor of purity two in this case. We are thus looking at $k = 2$. Proposition 6.1.1.2 gives $2n_2 + n_1 = 8$. Note that as we only have two pure spinors, so $n_2 = e$, where e is the intersection number. But given that the largest value of e is

$d-4=4-4=0$, we must have $n_2=0$. So, the only solution is given by two pure spinors whose null subspaces are complementary

$$\psi_1 \;\text{———}\; 0 \;\text{———}\; \psi_2 \qquad \begin{aligned}\psi_1 &= a_1a_2a_3a_4,\\ \psi_2 &= a_1^\dagger a_2^\dagger a_3^\dagger a_4^\dagger.\end{aligned} \tag{6.2.1}$$

Define $\psi = c_1\psi_1 + c_2\psi_2$, for $\psi_{1,2}$ as above. Then one has

Proposition 6.2.1.1. Take ψ in (6.2.1). Then the stabiliser of $\tilde{\psi}$ contains $\mathrm{Spin}(7,\mathbb{C})$.

Proof. First, we show there exists a Cartan subalgebra $X_{\mathfrak{h}} \subset \mathfrak{spin}(8,\mathbb{C})$ such that $\psi \mapsto \tilde{\psi} = \Lambda(\psi_1+\psi_2)$,for some $\Lambda \in \mathbb{C}$ to be determined. Choose $\mathfrak{h}' = \lambda_1(a_1a_1^\dagger - a_1^\dagger a_1)$. Acting $X_{\mathfrak{h}'}$ on ψ and imposing that it rescaled by an overall factor Λ, one has the following system of equations

$$\lambda_1 = \ln(\Lambda/c_1), \text{ and } -\lambda_1 = \ln(\Lambda/c_2). \tag{6.2.2}$$

Solving the above gives $\Lambda^2 = c_1c_2$. More specifically, $\lambda_1 = \ln(\sqrt{c_2}/\sqrt{c_1})$ or $\lambda_1 = \ln(\sqrt{c_1}/\sqrt{c_2})$ depending on the sign of the square root of Λ. Hence, for a choice of λ_1, we can determine the factor Λ.

To understand the simple part of stabiliser of this spinor, the most convenient strategy is to compute all non-vanishing spinor bilinears $B_k(\psi)$. In eight dimensions only $B_4(\psi_{1,2}) \neq 0$. Each of these 4-forms is decomposable and is given by the product of the null directions spanning the null subspace of $\psi_{1,2}$

$$B_4(\psi_1) = m_1 \wedge m_2 \wedge m_3 \wedge m_4, \quad B_4(\psi_2) = \bar{m}_1 \wedge \bar{m}_2 \wedge \bar{m}_3 \wedge \bar{m}_4. \tag{6.2.3}$$

Recall, (6.1.7), m_i are the coordinates representing the annihilation operator in the basis expansion, and similarly, $\bar{m}_i$ are the coordinates representing the creation operator. Given two pure spinors as above, we also have the following non-vanishing images of the geometrics maps: $B_0(\psi_1,\psi_2)$, $B_2(\psi_1,\psi_2)$, $B_4(\psi_1,\psi_2)$ as,

$$B_0(\psi_1,\psi_2) = \Lambda^2, \quad B_2(\psi_1,\psi_2) = \Lambda^2\omega, \quad B_4(\psi_1,\psi_2) = \frac{\Lambda^2}{2}\omega\wedge\omega, \tag{6.2.4}$$

where

$$\omega = m_1\wedge\bar{m}_1 + m_2\wedge\bar{m}_2 + m_3\wedge\bar{m}_3 + m_4\wedge\bar{m}_4. \tag{6.2.5}$$

This shows that

$$B_4(\tilde{\psi}) = \Lambda^2\left(m_1\wedge m_2\wedge m_3\wedge m_4 + \bar{m}_1\wedge\bar{m}_2\wedge\bar{m}_3\wedge\bar{m}_4 + \frac{1}{2}\omega\wedge\omega\right). \tag{6.2.6}$$

It is not difficult to see that ψ is a multiple of a unit spinor whose B_4 is the Cayley form in 8 dimension. Therefore, the stabiliser of ψ contains $\mathrm{Spin}(7,\mathbb{C})$. □

6.2.2 Ten dimensions

Again, we know that the only possible impure spinor is one with $k=2$. We have $2n_2+n_1=10$ and $n_2=e$. The only possible value for e is $d-4=1$ (the intersection number must take odd values for odd d). Thus, the only possible solution is $n_2=1$ and $n_1=8$. A possible solution representing this is two pure spinors whose null subspaces are

$$\psi_1 \;\text{———}\; 1 \;\text{———}\; \psi_2 \qquad \begin{aligned}\psi_1 &= a_1a_2a_3a_4a_5,\\ \psi_2 &= a_1^\dagger a_2^\dagger a_3^\dagger a_4^\dagger a_5.\end{aligned} \tag{6.2.7}$$

We have chosen the null direction common to both pure spinors to be a_5. It is clear that the spinor given by a linear combination of these two pure spinors is rooted in the unique impure spinor in 8 dimensions, as described in proposition 6.2.1.1. The additional dimension does not play any role. The rotation of this spinor to one with an overall scale factor is identical to the one described earlier.

Below, we will see that there is a similar impure spinor of purity two in any dimension. Its stabiliser contains Spin(7), which leaves the Cayley form in 8 dimensions invariant.

6.2.3 Twelve Dimensions

We expect to consider the degree of impure spinors, in twelve dimensions, $k \le 3$. However, we also need to consider the case $k = 4$, as the elimination of this case, as not independent, is what leads to the bound provided by the tetrahedral intersection number, $t \le d - 7$, in the first place.

Impurity of degree 2

The equation relating the occupation numbers is $2n_2 + n_1 = 12$. We also have $n_2 = e$, and the dimensions of intersecting edges are $d - 4 = 6 - 4 = 2$, and $d - 6 = 6 - 6 = 0$. When the edge intersection number is $e = 2$, we have $n_2 = n_0 = 2, n_1 = 8$. A possible representative of such a pair of pure spinors is

$$\psi_1 \;\text{———}\; 2 \;\text{———}\; \psi_2 \qquad \begin{aligned} \psi_1 &= a_1 a_2 a_3 a_4 a_5 a_6, \\ \psi_2 &= a_1^\dagger a_2^\dagger a_3^\dagger a_4^\dagger a_5 a_6. \end{aligned} \tag{6.2.8}$$

Define $\psi_A = \sum_{\alpha=1}^{2} c_\alpha \psi_\alpha$, for $\psi_{1,2}$ as above. Then one has

Proposition 6.2.3.1. Take ψ_A in (6.2.8). Then, the stabiliser of ψ_A contains $\mathrm{Spin}(7, \mathbb{C}) \times \mathrm{SL}(2, \mathbb{C})$.

Proof. First, there exists a Cartan subalgebra $X_{\mathfrak{h}} \subset \mathfrak{spin}(12, \mathbb{C})$ such that $\psi_A \mapsto \tilde{\psi}_A = \Lambda \sum_{\alpha=1}^{2} \psi_\alpha$, for some $\Lambda \in \mathbb{C}$. Now, computing $B_6(\tilde{\psi}_A)$ shows it proportional to the product of the Cayley form, Φ, invariant under the action of Spin(7), in the 8-dimensional space spanned by $m_{1,2,4,5}$ and $\bar{m}_{1,2,3,4}$, and the wedge product of the null directions $\bar{m}_{5,6}$ invariant under the action $\mathrm{SL}(2, \mathbb{C})$. Finally, at the level of the Lie algebra, the algebras commute. Thus, the stabiliser contains $\mathrm{Spin}(7, \mathbb{C}) \times \mathrm{SL}(2, \mathbb{C})$. □

When the dimension of the intersection is zero, we have $n_1 = 12$ and so the pure spinors are complementary,

$$\psi_1 \;\text{———}\; 0 \;\text{———}\; \psi_2 \qquad \begin{aligned} \psi_1 &= a_1 a_2 a_3 a_4 a_5 a_6, \\ \psi_2 &= a_1^\dagger a_2^\dagger a_3^\dagger a_4^\dagger a_5^\dagger a_6^\dagger. \end{aligned} \tag{6.2.9}$$

Define $\psi_B = \sum_{\alpha=1}^{2} c_\alpha \psi_\alpha$, for $\psi_{1,2}$ as above. Then one has

Proposition 6.2.3.2. Take ψ_B in (6.2.9). Then, its stabiliser contains $\mathfrak{sl}(6, \mathbb{C})$.

Proof. First, there exists a Cartan subalgebra $X_{\mathfrak{h}} \subset \mathfrak{spin}(12, \mathbb{C})$ such that $\psi_B \mapsto \tilde{\psi}_B = \Lambda \sum_{\alpha=1}^{2} \psi_\alpha$, for some $\Lambda \in \mathbb{C}$. Construct the geometric map $B_2(\tilde{\psi}_B) = \Lambda^2 (m_1 \wedge \bar{m}_1 + \ldots + m_6 \wedge \bar{m}_6)$. Now, any $g \in \mathrm{SL}(6, \mathbb{C})$ acts as $g \cdot m_i$ for $i \in \{1, \ldots, 6\}$, and simultaneously, $g^{-1} \in \mathrm{SL}(6, \mathbb{C})$ acts as $g^{-1} \cdot \bar{m}_i$ for $i \in \{1, \ldots, 6\}$. Thus, $B_2(\psi_B)$ remains invariant under the overall action of $\mathrm{SL}(6, \mathbb{C})$. □

One sees that the spinors representatives above are the only two possible solutions with purity $k = 2$.

Impurity of Degree 3

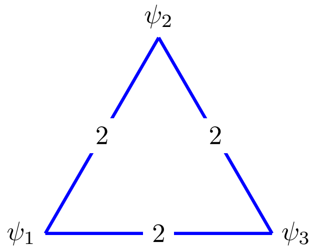
Now consider possible solutions arising with a triple of pure spinors. The occupation numbers are related via $n_3 + n_2 = 6$. If we denote the edge intersection dimensions by e_1, e_2, e_3, we have

$$3n_3 + n_2 = e_1 + e_2 + e_3. \tag{6.2.10}$$

Using the relation between n_2, n_3 one can write

$$n_3 = \frac{1}{2}\left(\sum_{I=1}^{3} e_I - 6\right). \tag{6.2.11}$$

At the same time, the maximal possible edge intersection number in this number of dimensions is $e = 2$. This means that there is a unique solution in the case of $k = 3$ — all edge intersection dimensions having value $e_{1,2,3} = 2$. This gives $n_3 = n_0 = 0$, so that $n_2 = n_1 = 6$. A possible representative is the following list of 3 pure spinors



$$\begin{aligned} \psi_1 &= a_1 a_2 a_3 a_4 a_5 a_6, \\ \psi_2 &= a_1^\dagger a_2^\dagger a_3^\dagger a_4^\dagger a_5 a_6, \\ \psi_3 &= a_1 a_2 a_3^\dagger a_4^\dagger a_5^\dagger a_6^\dagger. \end{aligned} \tag{6.2.12}$$

Define $\psi = \sum_{\alpha=1}^{3} c_\alpha \psi_\alpha$ where ψ_α are given as the pure spinors above. One then has

Proposition 6.2.3.3. Take ψ in (6.2.12). Then, its stabiliser contains $\mathrm{Sp}(6, \mathbb{C})$.

Proof. First, there exists a Cartan subalgebra $X_{\mathfrak{h}} \subset \mathfrak{spin}(12, \mathbb{C})$ such that $\psi \mapsto \tilde{\psi} = \Lambda \sum_{\alpha=1}^{2} \psi_\alpha$,for some $\Lambda \in \mathbb{C}$ to be determined. We shall choose the subalgebra of the Cartan subalgebra $\mathfrak{h}' \subset \mathfrak{h}$ generated by $\{\lambda_1(a_1 a_1^\dagger - a_1^\dagger a_1), \lambda_5(a_5 a_5^\dagger - a_5^\dagger a_5)\}$. Solving $X_{\mathfrak{h}'}\psi = \tilde{\psi}$, one derives the conditions

$$\Lambda^2 = c_2 c_3, \;\; \lambda_1 = \frac{1}{2}\ln\left(\frac{c_2}{c_1}\right), \;\text{ and }\; \lambda_5 = \frac{1}{2}\ln\left(\frac{c_3}{c_1}\right). \tag{6.2.13}$$

Thus, $B_2(\tilde{\psi})$ is given as

$$B_2(\tilde{\psi}) = -2\Lambda^2(\bar{m}_5 \wedge \bar{m}_6 + \bar{m}_1 \wedge \bar{m}_2 + m_1 \wedge m_2). \tag{6.2.14}$$

$B_2(\tilde{\psi})$ is a symplectic form constructed from a basis spanned by $\{m_1, \bar{m}_1, m_2, \bar{m}_2, \bar{m}_5, \bar{m}_6\}$, hence the subgroup that leaves this differential form invariant is $\mathrm{Sp}(6, \mathbb{C})$. □

Impurity of Degree 4

The relation between the occupation numbers is $2n_4 + 2n_3 + n_2 = 12$. Furthermore, the relation between the occupation numbers and the edge intersection number of the resulting tetrahedron is

$$6n_4 + 3n_3 + n_2 = \sum_{I=1}^{6} e_I. \tag{6.2.15}$$

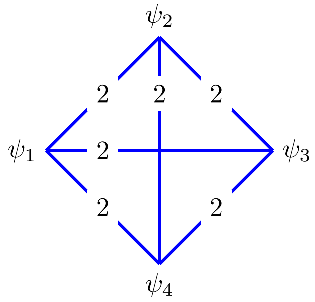
We now express n_2 via n_4, n_3 and substitute into the above equation to get

$$4n_4 + n_3 = \sum_{i=1}^{6} e_i - 12, \tag{6.2.16}$$

The maximal possible value of the edge intersection number is $e = 2$, and so the maximal value of the sum of the intersection numbers is 12. We thus see that there is a single solution of the combinatorial problem in this case

$$n_4 = n_0 = 0, \;\; n_3 = n_1 = 0, \;\text{ and }\; n_2 = 12. \tag{6.2.17}$$

A possible representative of this solution is the following list of four pure spinors



$$\begin{aligned}\psi_1 &= a_1a_2a_3a_4a_5a_6,\\ \psi_2 &= a_1^\dagger a_2^\dagger a_3^\dagger a_4^\dagger a_5a_6,\\ \psi_3 &= a_1a_2a_3^\dagger a_4^\dagger a_5^\dagger a_6^\dagger,\\ \psi_4 &= a_1^\dagger a_2^\dagger a_3a_4a_5^\dagger a_6^\dagger.\end{aligned} \tag{6.2.18}$$

To understand the geometry of this orbit, we compute $B_2(\psi)$, where $\psi = \sum_{\alpha=1}^4 c_\alpha\psi_\alpha$.

Proposition 6.2.3.4. Take ψ in 6.2.18. Then its stabiliser contains $\mathrm{SL}(6,\mathbb{C})$.

Proof. Given that ψ is a linear combination of 4 pure spinors, and B_2 vanishes for each pure spinor (only B_6 is different from zero), the non-zero contributions to $B_2(\psi)$ come from pairs of different pure spinors. However, each such pair intersects precisely in two dimensions, and $B_2(\psi_\alpha,\psi_\beta)$, for two pure spinors that intersection in two dimensions, is a multiple of the wedge product of the corresponding null directions. It is then clear that $B_2(\psi)$ is given by the following six terms

$$B_2(\psi) = c_1c_3a_1a_2 + c_1c_4a_3a_4 + c_1c_2a_5a_6 + c_2c_4a_1^\dagger a_2^\dagger + c_2c_3a_3^\dagger a_4^\dagger + c_3c_4a_5^\dagger a_6^\dagger. \tag{6.2.19}$$

Let us consider the terms involving only the directions $1,2$, i.e., $c_1c_3a_1a_2 + c_2c_4a_1^\dagger a_2^\dagger$. We want to show that there is a different canonically normalised null basis $b_1, b_2, b_1^\dagger, b_2^\dagger$ in the space spanned by $a_1, a_2, a_1^\dagger, a_2^\dagger$, such that

$$c_1c_3a_1a_2 + c_2c_4a_1^\dagger a_2^\dagger = \lambda(b_1b_1^\dagger + b_2b_2^\dagger), \tag{6.2.20}$$

where λ is some constant. We define

$$b_1 = \alpha a_1 + \beta a_2^\dagger, \quad b_1^\dagger = \gamma a_2 + \delta a_1^\dagger, b_2 = \alpha a_2 - \beta a_1^\dagger, \quad b_2^\dagger = \delta a_2^\dagger - \gamma a_1,$$

which satisfies

$$g(b_1,b_2) = g(b_1^\dagger,b_2^\dagger) = 0, \qquad g(b_1,b_1^\dagger) = g(b_2,b_2^\dagger) = \alpha\delta + \beta\gamma. \tag{6.2.21}$$

It is then easy to check that the 2-form $b_1b_1^\dagger + b_2b_2^\dagger$, where the wedge product of directions is assumed, does not have any $a_1a_1^\dagger, a_2a_2^\dagger$ terms when $\alpha\delta = \beta\gamma$. Hence,

$$b_1b_1^\dagger + b_2b_2^\dagger = 2\alpha\gamma a_1a_2 - 2\beta\delta a_1^\dagger a_2^\dagger. \tag{6.2.22}$$

We then choose $\delta = (2\alpha)^{-1}, \gamma = (2\beta)^{-1}$ to have the canonical normalisations. Then (6.2.20) holds for

$$\frac{\alpha^2}{\beta^2} = -\frac{c_1c_3}{c_2c_4}, \qquad \lambda_2 = -c_1c_2c_3c_4. \tag{6.2.23}$$

Thus, (6.2.20) is indeed possible for an appropriate choice of a canonically normalised null basis in the space spanned by $a_1, a_2, a_1^\dagger, a_2^\dagger$. This means that there exists another canonical basis $b_1, b_1^\dagger, \ldots, b_6, b_6^\dagger$ in $\mathbb{C}^{12}$ such that

$$B_2(\psi) \propto b_1 \wedge b_1^\dagger + \ldots + b_6 \wedge b_6^\dagger. \tag{6.2.24}$$

Thus $B(\psi)$ as in (6.2.24) is invariant under the action of $\mathfrak{sl}(6,\mathbb{C})$, see proposition 6.2.3.2. □

This shows that the purity four spinor in twelve dimensions is in the same Spin(12) orbit as the purity two spinor (6.2.9) with the stabiliser $\mathfrak{sl}(6,\mathbb{C})$. Which implies that the tetrahedral intersection number in 6 dimensions must be zero. This is important in the reduction of orbits that we will find in higher dimensions. Keep this in mind as we look for something similar in fourteen dimensions.

6.2.4 Fourteen Dimensions

We now consider the more difficult case of fourteen dimensions. As we know, just taking the edge intersection number constraints, we have the upper bound $k \le 8$. However, this is not optimal because there is also the tetrahedral intersection number constraint. This arises because the maximal tetrahedral intersection number for $d = 7$ is zero. Taking into account these constraints, we have seen that $k \le 6$ in this number of dimensions. As before, we consider different values of k case by case.

Impurity of Degree 2

The occupation numbers are n_2, n_1, n_0, which satisfy $n_2 = n_0$. The only equation one can write is $2n_2 + n_1 = 14$. We also have $n_2 = e$, the dimension of the intersecting null subspace. This dimension can take values $7 - 4 = 3$ and $7 - 6 = 1$. In the first case, we have $n_2 = n_0 = 3, n_1 = 8$. A possible representative of the solution is

$$\psi_1 \;\text{———}\; 3 \;\text{———}\; \psi_2 \qquad \begin{array}{l}\psi_1 = a_1a_2a_3a_4a_5a_6a_7,\\ \psi_2 = a_1^\dagger a_2^\dagger a_3^\dagger a_4^\dagger a_5a_6a_7.\end{array} \tag{6.2.25}$$

Define $\psi_A = \sum_{\alpha=1}^2 c_\alpha\psi_\alpha$, for $\psi_{1,2}$ as above. Then one has

Proposition 6.2.4.1. Take ψ_A in (6.2.25). Then, its stabiliser contains $\mathrm{Spin}(7,\mathbb{C}) \times \mathrm{SL}(3,\mathbb{C})$.

Proof. First, there exists a Cartan subalgebra $X_\mathfrak{h} \subset \mathfrak{spin}(14,\mathbb{C})$ such that $\psi_A \mapsto \tilde{\psi}_A = \Lambda\sum_{\alpha=1}^2 \psi_\alpha$, for some $\Lambda \in \mathbb{C}$. Notice that ψ_A is the same type of spinor as in 8 dimension, (6.2.1.1), except there are an extra 3 null directions. Hence, $B_7(\tilde{\psi}_A)$ is the wedge product of the Cayley form, invariant under Spin(7), in the directions $m_{1,2,3,4}$, and $\bar{m}_{1,2,3,4}$, and the wedge product of the null directions $m_{5,6,7}$, invariant under the action of $\mathrm{SL}(3,\mathbb{C})$. Thus, the stabiliser contains $\mathrm{Spin}(7,\mathbb{C}) \times \mathrm{SL}(3,\mathbb{C})$. □

When the dimension of the intersecting null subspace is $e = 1$ we have $n_2 = n_0 = 1$ and $n_1 = 12$. A possible pair of spinors realising this solution is

$$\psi_1 \;\text{———}\; 1 \;\text{———}\; \psi_2 \qquad \begin{array}{l}\psi_1 = a_1a_2a_3a_4a_5a_6a_7,\\ \psi_2 = a_1^\dagger a_2^\dagger a_3^\dagger a_4^\dagger a_5^\dagger a_6^\dagger a_7.\end{array} \tag{6.2.26}$$

Define $\psi_B = \sum_{\alpha=1}^2 c_\alpha\psi_\alpha$, for $\psi_{1,2}$ as above. Then one has

Proposition 6.2.4.2. Take ψ_B in (6.2.26). Then, its stabiliser contains $\mathrm{SL}(6,\mathbb{C})$.

Proof. First, there exists a Cartan subalgebra $X_\mathfrak{h} \subset \mathfrak{spin}(14,\mathbb{C})$ such that $\psi_B \mapsto \tilde{\psi}_B = \Lambda\sum_{\alpha=1}^2 \psi_\alpha$, for some $\Lambda \in \mathbb{C}$. Notice that ψ_B is the same type of spinor as the 12 dimensional spinor (6.2.9), except there is an extra null oscillator, a_7, present, that plays no role. Therefore, the stabiliser contains $\mathrm{SL}(6,\mathbb{C})$. □

Impurity of Degree 3

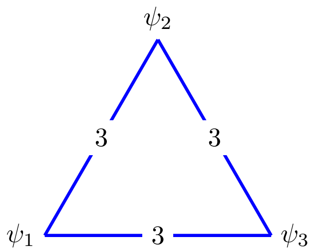
There is a single independent equation $n_3 + n_2 = 7$ in this case, which in particular implies $n_2 \leq 7$. As before, we envisage drawing a triangle with the 3 spinors sitting at its vertices, and putting the pairwise intersection numbers on the edges. It is clear that

$$3n_3 + n_2 = \sum_{i=1}^{3} e_i \coloneqq e, \tag{6.2.27}$$

which, rewritten in terms of n_3, becomes

$$n_3 = \frac{1}{2}\left(\sum_{i=1}^{3} e_i - 7\right). \tag{6.2.28}$$

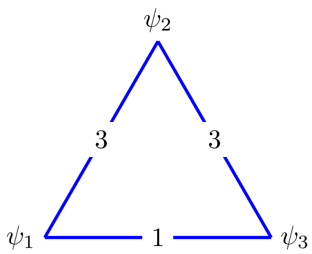
Suppose that $e = 9$. Then the maximal possible value of each edge intersection dimension is 3, and so the maximal value of n_3 is 1, implying $n_2 = 6$. A possible triple of pure spinors realising this solution is



$$\begin{aligned} \psi_1 &= a_1 a_2 a_3 a_4 a_5 a_6 a_7, \\ \psi_2 &= a_1^\dagger a_2^\dagger a_3^\dagger a_4^\dagger a_5 a_6 a_7, \\ \psi_3 &= a_1 a_2 a_3^\dagger a_4^\dagger a_5^\dagger a_6^\dagger a_7. \end{aligned} \tag{6.2.29}$$

It is clear that this is just the 12 dimensional solution (6.2.12), with one extra oscillator added. Hence, the stabiliser contains $\mathrm{Sp}(6, \mathbb{C})$.

The only other possible solution in the purity three case is $e_1 = e_2 = 3, e_3 = 1$, implying the edge intersection number to be 7. This gives $n_3 = n_0 = 0$, and thus $n_2 = n_1 = 7$. A possible representative of this solution is



$$\begin{aligned} \psi_1 &= a_1 a_2 a_3 a_4 a_5 a_6 a_7, \\ \psi_2 &= a_1^\dagger a_2^\dagger a_3^\dagger a_4^\dagger a_5 a_6 a_7, \\ \psi_3 &= a_1^\dagger a_2^\dagger a_3^\dagger a_4 a_5^\dagger a_6^\dagger a_7^\dagger. \end{aligned} \tag{6.2.30}$$

Define $\psi = \sum_{\alpha=1}^{3} c_\alpha \psi_\alpha$, where ψ_α are the pure spinors above. One then has

Proposition 6.2.4.3. Take ψ in (6.2.30). Then, its stabiliser contains $\mathrm{SL}(3, \mathbb{C}) \times \mathrm{SL}(3, \mathbb{C})$.

Proof. First, there exists a Cartan subalgebra $X_{\mathfrak{h}} \subset \mathfrak{spin}(14, \mathbb{C})$ such that $\psi \mapsto \tilde{\psi} = \Lambda \sum_{\alpha=1}^{2} \psi_\alpha$,for some $\Lambda \in \mathbb{C}$ to be determined. We shall choose the subalgebra of the Cartan subalgebra $\mathfrak{h}' \subset \mathfrak{h}$ generated by $\{\lambda_1(a_1 a_1^\dagger - a_1^\dagger a_1), \lambda_5(a_5 a_5^\dagger - a_5^\dagger a_5)\}$. Solving $X_{\mathfrak{h}'}\psi = \tilde{\psi}$, one derives the conditions,

$$\Lambda^2 = c_1 c_3, \ \ \lambda_1 = \frac{1}{2}\ln\left(\frac{c_1}{c_2}\right), \ \text{ and } \lambda_5 = \frac{1}{2}\ln\left(\frac{c_3}{c_2}\right). \tag{6.2.31}$$

Computing $B_3(\tilde{\psi})$ gives

$$B_3(\tilde{\psi}) = -2\Lambda^2\left(\bar{m}_5 \wedge \bar{m}_6 \wedge \bar{m}_7 + m_1 \wedge m_2 \wedge m_3 - \frac{1}{2}\bar{m}_4 \wedge \lambda\right). \tag{6.2.32}$$

Where $\lambda = \bar{m}_1 \wedge m_1 + \ldots + \bar{m}_7 \wedge m_7$. It is clear that λ is stabilised by $\mathrm{SL}(6,\mathbb{C})$. But notice that $\bar{m}_5 \wedge \bar{m}_6 \wedge \bar{m}_7$ and $m_1 \wedge m_2 \wedge m_3$, individually, are top forms in 3 dimensions. Thus, they are both stabilised by two separate $\mathrm{SL}(3,\mathbb{C})$s inside $\mathrm{SL}(6,\mathbb{C})$. Hence, the stabiliser contains $\mathrm{SL}(3,\mathbb{C}) \times \mathrm{SL}(3,\mathbb{C})$. □

Impurity of Degree 4

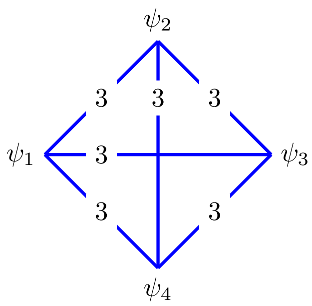
The equation relating the occupation numbers in this case reads $2n_4 + 2n_3 + n_2 = 14$. The relation to the edge intersection number is given by

$$6n_4 + 3n_3 + n_2 = \sum_{i=1}^{6} e_i \coloneqq e. \tag{6.2.33}$$

Expressing $n_2 = 14 - 2n_3 - 2n_4$, we get

$$4n_4 + n_3 = \sum_{i=1}^{6} e_i - 14. \tag{6.2.34}$$

When the edge intersection dimensions are maximal, all have value 3, we have two possible solutions. In the first case $n_4 = n_0 = 0, n_3 = n_1 = 4$, and so $n_2 = 6$. A possible representative of this solution is



$$\begin{aligned} \psi_1 &= a_1 a_2 a_3 a_4 a_5 a_6 a_7, \\ \psi_2 &= a_1^\dagger a_2^\dagger a_3^\dagger a_4^\dagger a_5 a_6 a_7, \\ \psi_3 &= a_1^\dagger a_2 a_3^\dagger a_4 a_5 a_6^\dagger a_7^\dagger, \\ \psi_4 &= a_1 a_2^\dagger a_3^\dagger a_4 a_5^\dagger a_6^\dagger a_7. \end{aligned} \tag{6.2.35}$$

Define $\psi_A = \sum_{\alpha=1}^{4} c_\alpha \psi_\alpha$, where ψ_α are the pure spinors above. One then has

Proposition 6.2.4.4. Take ψ_A in (6.2.35). Then, its stabiliser contains $\mathrm{SL}(4,\mathbb{C})$.

Proof. First, there exists a Cartan subalgebra $X_{\mathfrak{h}} \subset \mathfrak{spin}(14,\mathbb{C})$ such that $\psi_A \mapsto \tilde{\psi}_A = \Lambda \sum_{\alpha=1}^{2} \psi_\alpha$,for some $\Lambda \in \mathbb{C}$ to be determined. We shall choose the subalgebra of the Cartan subalgebra $\mathfrak{h}' \subset \mathfrak{h}$ generated by $\{\lambda_1(a_1 a_1^\dagger - a_1^\dagger a_1), \lambda_2(a_2 a_2^\dagger - a_2^\dagger a_2), \lambda_3(a_3 a_3^\dagger - a_3^\dagger a_3)\}$. Solving $X_{\mathfrak{h}'}\psi_A = \tilde{\psi}_A$, one derives the conditions

$$\Lambda^2 = c_1 c_2, \ \lambda_1 = \frac{1}{2}\ln\left(\frac{c_2}{c_4}\right), \ \lambda_2 = \frac{1}{2}\ln\left(\frac{c_2}{c_3}\right), \text{ and } \lambda_3 = \frac{1}{2}\ln\left(\frac{c_3 c_4}{c_1 c_2}\right). \tag{6.2.36}$$

Computing $B_3(\tilde{\psi}_A)$ gives

$$\begin{aligned} B_3(\tilde{\psi}_A) = -2\Lambda^2(&\bar{m}_5 \wedge \bar{m}_6 \wedge \bar{m}_7 - \bar{m}_2 \wedge \bar{m}_4 \wedge \bar{m}_5 + \bar{m}_1 \wedge \bar{m}_4 \wedge \bar{m}_7 \\ &- m_1 \wedge m_3 \wedge \bar{m}_5 - m_2 \wedge m_3 \wedge \bar{m}_7 + n_3 \wedge m_6 \wedge \bar{m}_4). \end{aligned} \tag{6.2.37}$$

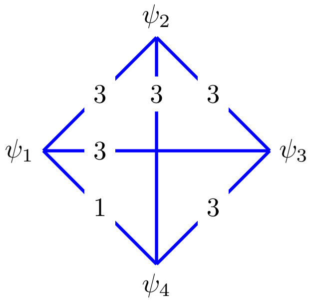
Consider the following vectors,

$$
\begin{aligned}
v^T &= \left(m_3 \wedge \bar{m}_5 \quad m_3 \wedge \bar{m}_7 \quad -m_3 \wedge \bar{m}_4 \quad -\bar{m}_4 \wedge \bar{m}_7 \quad \bar{m}_4 \wedge \bar{m}_5 \quad \bar{m}_5 \wedge \bar{m}_7\right), \text{and} \\
w^T &= \left(m_1 \quad m_2 \quad m_6 \quad \bar{m}_1 \quad \bar{m}_2 \quad \bar{m}_6.\right)
\end{aligned} \tag{6.2.38}
$$

Then $v \in \Lambda^2(\mathbb{C}^4) \cong \mathbb{C}^6$ and $w \in \mathbb{C}^6$. One can act on both vectors by the same transformation in, $\mathfrak{sl}(4,\mathbb{C}) \cong \mathfrak{so}(6,\mathbb{C})$ and thus $v^T w = B_3(\tilde{\psi}_A)$ remains invariant under the action. So the stabiliser contains $\mathrm{SL}(4,\mathbb{C})$. □

In the other case, $n_4 = 1$, $n_3 = 0$, implying $n_2 = 12$. It is clear that in this solution, all four pure spinors share a common null direction. In other words, the tetrahedral intersection number, t, has value 1. However, this is not allowed by our tetrahedral intersection constraint $t \leq d - 7$. Alternatively, we are effectively dealing with a purity four spinor in 12 dimensions. As we know, this purity four spinor is actually a purity two spinor. Therefore, the solution in this case is in the same Spin(14) orbit as the purity two solution with an intersection number of one. Consequently, we will not consider it any further.

Let us also find solutions with the edge intersection numbers less than maximal. When there is a single edge intersection dimension with value one, we have $\sum_{i=1}^{6} e_i = 16$. In this case, we necessarily have $n_4 = 0$, implying that $n_3 = 2, n_2 = 10$. A possible representative of this solution is



$$
\begin{aligned}
\psi_1 &= a_1 a_2 a_3 a_4 a_5 a_6 a_7, \\
\psi_2 &= a_1^\dagger a_2^\dagger a_3^\dagger a_4 a_5 a_6 a_7^\dagger, \\
\psi_3 &= a_1 a_2 a_3^\dagger a_4^\dagger a_5^\dagger a_6 a_7^\dagger, \\
\psi_4 &= a_1^\dagger a_2^\dagger a_3^\dagger a_4^\dagger a_5^\dagger a_6^\dagger a_7.
\end{aligned} \tag{6.2.39}
$$

Define $\psi_B = \sum_{\alpha=1}^{4} c_\alpha \psi_\alpha$, where ψ_α are the pure spinors above. One then has

Proposition 6.2.4.5. Take ψ_B in (6.2.26). Then, its stabiliser contains $\mathrm{SL}(2,\mathbb{C}) \times \mathrm{Sp}(4,\mathbb{C})$.

Proof. First, there exists a Cartan subalgebra $X_{\mathfrak{h}} \subset \mathfrak{spin}(14,\mathbb{C})$ such that $\psi_B \mapsto \tilde{\psi}_B = \Lambda \sum_{\alpha=1}^{2} \psi_\alpha$, for some $\Lambda \in \mathbb{C}$ to be determined. We shall choose the subalgebra of the Cartan subalgebra $\mathfrak{h}' \subset \mathfrak{h}$ generated by $\{\lambda_2(a_2 a_2^\dagger - a_2^\dagger a_2), \lambda_3(a_3 a_3^\dagger - a_3^\dagger a_3), \lambda_4(a_4 a_4^\dagger - a_4^\dagger a_4)\}$. Solving $X_{\mathfrak{h}'}\psi = \tilde{\psi}_B$, one derives the conditions

$$
\Lambda^2 = c_1 c_4, \;\; \lambda_2 = \frac{1}{2}\ln\left(\frac{c_4}{c_3}\right), \;\; \lambda_3 = \frac{1}{2}\ln\left(\frac{c_2 c_3}{c_1 c_4}\right), \;\; \text{and } \lambda_4 = \frac{1}{2}\ln\left(\frac{c_4}{c_2}\right). \tag{6.2.40}
$$

Now, consider the subalgebra $\chi \subset \mathfrak{spin}(14,\mathbb{C})$ that stabilises $\tilde{\psi}_B$. By computing, $\chi\tilde{\psi}_B = 0$ one finds the stabiliser to split as $\chi = \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sp}(4,\mathbb{C}) \oplus \mathfrak{h}$. $\mathfrak{h}$ is a remaining part of the stabiliser that isn't $\mathfrak{sl}(2,\mathbb{C})$ or $\mathfrak{sp}(4,\mathbb{C})$. Even further, $\mathfrak{sl}(2,\mathbb{C})$ and $\mathfrak{sp}(4,\mathbb{C})$ are given as creation and annihilation operators as:

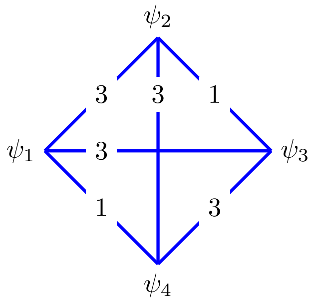
$$
\begin{aligned}
\mathfrak{sp}(4,\mathbb{C}) =&\mathrm{Span}\{a_1 a_4^\dagger - a_5 a_2^\dagger, a_4 a_1^\dagger - a_2 a_5^\dagger, a_5 a_1^\dagger + a_2 a_4^\dagger, a_1 a_5^\dagger + a_4 a_2^\dagger, a_1 a_1^\dagger - a_2 a_2^\dagger, a_4 a_4^\dagger - a_5 a_5^\dagger, \\
& a_1 a_2^\dagger, a_2 a_1^\dagger, a_4 a_5^\dagger, a_5 a_4^\dagger\}, \text{ and} \\
\mathfrak{sl}(2,\mathbb{C}) =&\mathrm{Span}\{a_1 a_1^\dagger + a_2 a_2^\dagger + a_4 a_4^\dagger + a_5 a_5^\dagger - 2a_3 a_3^\dagger - 2a_6 a_6^\dagger, 2a_6 a_7^\dagger - a_7 a_3 + a_1^\dagger a_2^\dagger + a_4^\dagger a_5^\dagger, \\
& - a_7 a_6^\dagger + 2a_3^\dagger a_7^\dagger + a_1 a_2 + a_4 a_5\}.
\end{aligned} \tag{6.2.41}
$$

It can easily be shown that these two subalgebras commute. On top of that, computing $B_3(\tilde{\psi}_B)$ gives

$$B_3(\tilde{\psi}_B) = -2\Lambda^2((n^1 \wedge n^2 + n^4 \wedge n^5) \wedge n^3 - (\bar{n}^1 \wedge \bar{n}^2 + \bar{n}^4 \wedge \bar{n}^5) \wedge \bar{n}^6 + \bar{n}^6 \wedge n^3 \wedge n^7 + \bar{n}^7 \wedge \lambda). \quad (6.2.42)$$

Where $\lambda = \bar{n}_1 n_1 + \ldots + \bar{n}_7 n_7$. It is not difficult to check that the actions by the generators in $\mathfrak{sl}(2,\mathbb{C})$, or $\mathfrak{sp}(4,\mathbb{C})$ kill $B_3(\tilde{\psi}_B)$. Thus, the simple part of the stabilising subgroup is $\mathrm{SL}(2,\mathbb{C}) \times \mathrm{Sp}(4,\mathbb{C})$. □

Finally, when $\sum_{i=1}^{6} e_i = 14$, then there are two edge intersection dimension with value one, i.e. $n_4 = n_3 = 0, n_2 = 14$ with possible representative of this solution as



$$\begin{aligned} \psi_1 &= a_1 a_2 a_3 a_4 a_5 a_6 a_7, \\ \psi_2 &= a_1^\dagger a_2^\dagger a_3^\dagger a_4^\dagger a_5 a_6 a_7, \\ \psi_3 &= a_1 a_2 a_3 a_4^\dagger a_5^\dagger a_6^\dagger a_7^\dagger, \\ \psi_4 &= a_1^\dagger a_2^\dagger a_3^\dagger a_4 a_5^\dagger a_6^\dagger a_7^\dagger. \end{aligned} \quad (6.2.43)$$

Define $\psi_C = \sum_{\alpha=1}^{4} c_\alpha \psi_\alpha$, where ψ_α are the pure spinors above. One then has

Proposition 6.2.4.6. Take ψ_C in (6.2.43). Then, its stabiliser contains $G_2 \times G_2$.

Proof. First, there exists a Cartan subalgebra $X_{\mathfrak{h}} \subset \mathfrak{spin}(14,\mathbb{C})$ such that $\psi_C \mapsto \tilde{\psi}_C = \Lambda \sum_{\alpha=1}^{2} \psi_\alpha$,for some $\Lambda \in \mathbb{C}$ to be determined. We shall choose the subalgebra of the Cartan subalgebra $\mathfrak{h}' \subset \mathfrak{h}$ generated by $\{\lambda_3(a_3 a_3^\dagger - a_3^\dagger a_3, \lambda_4(a_4 a_4^\dagger - a_4^\dagger a_4, \lambda_5(a_5 a_5^\dagger - a_5^\dagger a_5)\}$. Solving $X_{\mathfrak{h}'} \psi_C = \tilde{\psi}_C$, one derives the conditions

$$\Lambda^4 = \frac{c_1 c_2^2 c_3}{c_4}, \ \lambda_4 = \frac{1}{2} \ln\left(\frac{c_3 c_2}{c_1 c_4}\right), \ \lambda_5 = \frac{1}{2} \ln\left(\frac{c_3 c_4}{c_1 c_2}\right), \text{ and } \lambda_3 = \frac{1}{2} \ln\left(\frac{\Lambda^2}{c_1 c_3}\right). \quad (6.2.44)$$

Computing $B_3(\tilde{\psi}_C)$ gives

$$\begin{aligned} B_3(\tilde{\psi}_C) = 2\Lambda^2 \Big(&\frac{1}{2} m_4 \wedge \lambda + m_1 \wedge m_2 \wedge m_3 + m_5 \wedge m_6 \wedge m_7 \\ &- \frac{1}{2} \bar{m}_4 \wedge \lambda - \bar{m}_1 \wedge \bar{m}_2 \wedge \bar{m}_3 - \bar{m}_5 \wedge \bar{m}_6 \wedge \bar{m}_7 \Big). \end{aligned} \quad (6.2.45)$$

One now makes the following transformation,

$$m_i \coloneqq x_i + \bar{x}_i, \text{ and } \bar{m}_i \coloneqq x_i - \bar{x}_i \text{ for } i \neq 4. \quad (6.2.46)$$

This results in the following:

$$\begin{aligned} B_3(\tilde{\psi}_C) = \Lambda^2 (&\bar{x}_1 \wedge \bar{x}_2 \wedge \bar{x}_3 + \bar{x}_1 \wedge (x_2 \wedge x_3 - \bar{m}_4 \wedge x_1) \\ &+ \bar{x}_2 \wedge (x_3 \wedge x_1 - \bar{m}_4 \wedge x_2) + \bar{x}_3 \wedge (x_1 \wedge x_2 - \bar{m}_4 \wedge x_3) \\ &+ x_5 \wedge x_6 \wedge x_7 + x_5 \wedge (\bar{x}_6 \wedge \bar{x}_7 + m_4 \wedge \bar{x}_5) \\ &+ x_6 \wedge (\bar{x}_7 \wedge \bar{x}_5 + m_4 \wedge \bar{x}_6) + x_7 \wedge (\bar{x}_5 \wedge \bar{x}_6 + m_4 \wedge \bar{x}_7)). \end{aligned} \quad (6.2.47)$$

The first line exhibits a G_2 structure, while the last line exhibits a G_2 structure of a different type. This shows that the stabiliser contains given by $G_2 \times G_2$. □

We have exhausted all combinations for degree 4 spinors. We are left now only to discuss $k = 5$.

Impurity of Degree 5

The relation between the occupation numbers is $n_5 + n_4 + n_3 = 7$. The relation with the edge intersection number is

$$10n_5 + 6n_4 + 3n_3 + n_2 = \sum_{i=1}^{10} e_i, \tag{6.2.48}$$

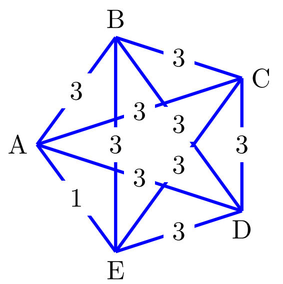
where of course $n_2 = n_3$ by duality. Eliminating $n_3 = 7 - n_5 - n_4$, we get

$$6n_5 + 2n_4 = \sum_{i=1}^{10} e_i - 28. \tag{6.2.49}$$

This immediately shows, for the right-hand-side above to be non-negative, the edge intersection number must have value at least 28.

The largest possible value of the edge intersection number is 30. This means that necessarily $n_5 = 0$. We then have a possible solution with all 10 edge intersection dimensions with value 3 and $n_4 = 1$. However, $n_4 = 1$ means that there are four pure spinors that all share a single common null direction. As we know, this means that these four pure spinors effectively live in 12 dimensions, where they correspond to a reducible configuration of purity two. So, this is not a new orbit.

The other possibility is when there are 9 edges with an edge intersection dimension value of 3 , and the last remaining one with edge intersection dimension with value 1. This gives the solution $n_5 = n_4 = n_0 = n_1 = 0, n_3 = n_2 = 7$. A possible representative of this solution is



$$\begin{aligned}
\psi_A &= a_1 a_2 a_3 a_4 a_5 a_6 a_7,\\
\psi_B &= a_1^\dagger a_2^\dagger a_3^\dagger a_4^\dagger a_5 a_6 a_7,\\
\psi_C &= a_1^\dagger a_2 a_3 a_4^\dagger a_5 a_6^\dagger a_7^\dagger,\\
\psi_D &= a_1 a_2^\dagger a_3 a_4^\dagger a_5^\dagger a_6 a_7^\dagger,\\
\psi_E &= a_1^\dagger a_2^\dagger a_3^\dagger a_4 a_5^\dagger a_6^\dagger a_7^\dagger.
\end{aligned} \tag{6.2.50}$$

Define $\psi = \sum_{\alpha=1}^{5} c_\alpha \psi_\alpha$, where ψ_α are the pure spinors above. One then has

Proposition 6.2.4.7. Take ψ in (6.2.50). Then its stabiliser contains G_2.

Proposition 6.2.4.8.

Proof. First, there exists a Cartan subalgebra $X_{\mathfrak{h}} \subset \mathfrak{spin}(14, \mathbb{C})$ such that $\psi \mapsto \tilde{\psi} = \Lambda \sum_{\alpha=1}^{2} \psi_\alpha$,for some $\Lambda \in \mathbb{C}$ to be determined. We shall choose the subalgebra of the Cartan subalgebra $\mathfrak{h}' \subset \mathfrak{h}$ generated by $\{\lambda_1(a_1 a_1^\dagger - a_1^\dagger a_1), \lambda_2(a_2 a_2^\dagger - a_2^\dagger a_2), \lambda_3(a_3 a_3^\dagger - a_3^\dagger a_3), \lambda_4(a_4 a_4^\dagger - a_4^\dagger a_4)\}$. Solving $X_{\mathfrak{h}'}\psi = \tilde{\psi}$, one derives the conditions

$$\begin{aligned}
\Lambda^2 = c_1 c_2,\ \ \lambda_1 = \frac{1}{2}\ln\left(\frac{c_2 c_4}{c_5^2}\right),\ \ \lambda_2 = \frac{1}{2}\ln\left(\frac{c_4 c_5}{c_1 c_3}\right),\\
\lambda_3 = \frac{1}{2}\ln\left(\frac{c_1 c_2}{c_4 c_5}\right),\ \text{and},\ \ \lambda_4 = \frac{1}{2}\ln\left(\frac{c_2}{c_5}\right).
\end{aligned} \tag{6.2.51}$$

Computing $B_3(\tilde{\psi})$ gives

$$B_3(\tilde{\psi}) = 2\Lambda^2\bigg(\frac{1}{2}\bar{m}_4 \wedge \lambda - \bar{m}_5 \wedge \bar{m}_6 \wedge \bar{m}_7 + \bar{m}_2 \wedge \bar{m}_3 \wedge \bar{m}_5 - \bar{m}_1 \wedge \bar{m}_3 \wedge \bar{m}_6$$
$$- m_1 \wedge m_2 \wedge m_3 + m_6 \wedge m_7 \wedge m_1 - m_5 \wedge m_7 \wedge m_2 \tag{6.2.52}$$
$$+ m_4 \wedge (m_7 \wedge \bar{m}_3 + n_1 \wedge \bar{m}_5 + m_2 \wedge \bar{m}_6)\bigg).$$

Where $\lambda = \bar{m}_1 \wedge m_1 + \ldots + \bar{m}_7 \wedge m_7$. Introduce the following coordinate change

$$x_i = m_i + m_{i+4},\ \ x_{i+4} = x_i = m_i - m_{i+4},\ \ \bar{x}_i = \bar{m}_i + \bar{m}_{i+4},\ \ \bar{x}_{i+4} = \bar{m}_i = m_i - \bar{m}_{i+4}$$
$$\text{for } i \in \{1,2,3\},\ \text{ and } t = m_4 + \bar{m}_4,\ \text{ and } \bar{t} = m_4 - \bar{m}_4. \tag{6.2.53}$$

In this basis, the 3-form becomes

$$B_3(\tilde{\psi}) = \Lambda^2\bigg(\bar{t} \wedge (\bar{x}_1 \wedge x_1 + \bar{x}_2 \wedge x_2 + \bar{x}_3 \wedge x_3) + x_1 \wedge x_2 \wedge x_3 + \bar{x}_1 \wedge \bar{x}_2 \wedge \bar{x}_3$$
$$- t \wedge (\bar{x}_5 \wedge x_5 + \bar{x}_6 \wedge x_6 + \bar{x}_7 \wedge x_7) + x_5 \wedge x_6 \wedge x_7 + \bar{x}_5 \wedge \bar{x}_6 \wedge \bar{x}_7 \tag{6.2.54}$$
$$- \frac{1}{4}(x_1 - x_5) \wedge (x_2 - x_6) \wedge (x_3 - x_7) - \frac{1}{4}(\bar{x}_1 + \bar{x}_5) \wedge (\bar{x}_2 + \bar{x}_6) \wedge (\bar{x}_3 + \bar{x}_7)\bigg)$$

The 3-form above shows $\tilde{\psi}$ is stabilised by a Lie subalgebra in $\mathfrak{g}_2 \times \mathfrak{g}_2 \coloneqq \mathfrak{g}_2 \times \mathfrak{g}_2'$. Furthermore, $\mathfrak{g}_2 = \mathfrak{sl}(3,\mathbb{C}) \oplus \mathfrak{sl}(3,\mathbb{C})^\perp$ ($\mathfrak{g}_2' = \mathfrak{sl}(3,\mathbb{C})' \oplus \mathfrak{sl}(3,\mathbb{C})'^\perp$), where $\mathfrak{sl}(3,\mathbb{C})^\perp$ ($\mathfrak{sl}(3,\mathbb{C})'^\perp$) is the orthogonal 6-dimensional Lie algebra inside $\mathfrak{g}_2$ ($\mathfrak{g}_2'$). The last line in $B_3(\tilde{\psi})$ shows that $\mathfrak{sl}(3,\mathbb{C}) \subset \mathfrak{g}_2$ must be identified with $\mathfrak{sl}(3,\mathbb{C})' \subset \mathfrak{g}_2'$. Therefore, the Lie subalgebra that stabilises $B_3(\tilde{\psi})$ is isomorphic to $\mathfrak{h} = \mathfrak{sl}(3,\mathbb{C}) \oplus \mathfrak{sl}(3,\mathbb{C})^\perp \oplus \mathfrak{sl}(3,\mathbb{C})'^\perp$. However, $\mathfrak{h}$ is not a Lie subalgebra unless one identifies $\mathfrak{sl}(3,\mathbb{C})^\perp \subset \mathfrak{g}_2$ with $\mathfrak{sl}(3,\mathbb{C})'^\perp \subset \mathfrak{g}_2'$. This results in $\mathfrak{h} \cong \mathfrak{sl}(3,\mathbb{C}) \oplus \mathfrak{sl}(3,\mathbb{C})^\perp$, that is to say, a diagonal copy of $\tilde{\mathfrak{g}}_2 \subset \mathfrak{g}_2 \times \mathfrak{g}_2'$. □

The G_2 stabiliser contains all other stabilisers unique to Spin(14), except the $G_2 \times G_2$ stabiliser. Indeed, remove the edge with weight 1 and the two vertices sharing this edge in (6.2.50), this recovers the SL(6) stabiliser. If, however, one removes one of the edges with weight 3 and the two vertices sharing this edge in (6.2.50), then one recovers the $\mathrm{Sp}(4) \times \mathrm{SL}(2)$ stabiliser. There is no way to construct a degeneration to $G_2 \times G_2$ via the method of removing an edge and corresponding vertices.

Impurity of Degree 6

The relation between the occupation numbers gives $n_3 = 14 - 2n_6 - 2n_5 - 2n_4$. Relating this to the edge intersection number gives

$$15n_6 + 10n_5 + 7n_4 + 3n_3 = \sum_{i=1}^{15} e_i, \tag{6.2.55}$$

where $n_4 = n_2$ by duality. Substituting the expression for n_3 in terms of $n_{4,5,6}$ we get

$$9n_6 + 4n_5 + n_4 = \sum_{i=1}^{15} e_i - 42. \tag{6.2.56}$$

The largest possible value of the edge intersection number is 45. This shows that $n_5 = n_6 = 0$. When the sum of the edge intersection dimensions is 45 we get $n_4 = n_2 = 3$ and $n_3 = 8$. However, as we already know from the previous considerations, we cannot have n_4 different from zero, as then the tetrahedral intersection number is different from zero. So, the sum of edge intersection number must be 42. However, this is not possible, because its possible values are $45, 43, 41, \ldots$ So, there is no solution with $k = 6$ that gives rise to an irreducible configuration of pure spinors in this case.

Part II

G-structures & Plebanski-Formalism

Chapter 7

Introduction

The goal of this part of the thesis is to study classical G-structures to construct natural second-order partial differential equations using canonical differential forms, inspired by Plebanksi. In low enough dimensions, the only spinorial G-structures we can construct are classical ones, since all Weyl spinors are pure. Indeed, we shall consider SU(2)- and SU(3)-structures in 4 and 6 dimensions, respectively. The integrability of these structures is equivalent to parallel spinors on the manifold, which is equivalent to covariantly constant canonical differential forms of the spinorial G-structures, which is equivalent to holonomy group contained in a specific $G \subset \mathrm{Spin}(2n)$. The aforementioned is measured through a tool called intrinsic torsion.

7.0.1 G-structures, Holonomy & Intrinsic Torsion

We define spinorial G-structures, and how one can link them to the study of integrable G-structures through the use of intrinsic torsion.

Definition 7.0.1.1. Let M be a spin manifold of dimension n. A *spinorial G-structure*, on a manifold M, is the reduction from the principal $\mathrm{Spin}(n)$-spin-frame bundle to a $\mathrm{Spin}(n) \supset G$-spin-frame subbundle P. Then, let $\rho : G \to GL(S)$ be the spin representation of G on a finite dimensional space S. Since, G acts freely on $P \times S$ on the right, naturally by the principal bundle action on the first coordinate and by ρ on the second coordinate,

$$(u, \psi) \cdot g = (u \circ g, \rho(g^{-1}) \cdot \psi), \text{ for } u \in P, \text{ and } \psi \in S. \tag{7.0.1}$$

The quotient space, $\rho(P) \coloneqq (P \times S)/G$, is called the *associated spinor bundle* to the spinorial G-structure.

Indeed, $P/G \cong M$, via the free principal bundle action right-action. Hence, the map $\pi : \rho(P) \coloneqq (P \times V)/G \to P/G$ is just a projection from $\rho(P)$ to M, with fibres the finite dimensional representation S.

In this thesis, we take $S = \Lambda\left(\mathbb{C}^{\lfloor \frac{n}{2} \rfloor}\right)$, called the Dirac representation. Whenever possible, we instead take the Weyl, Majorana or Majorana-Weyl representations inside the Dirac representation in the definition above.

Holonomy, Parallel Spinors, and Integrability

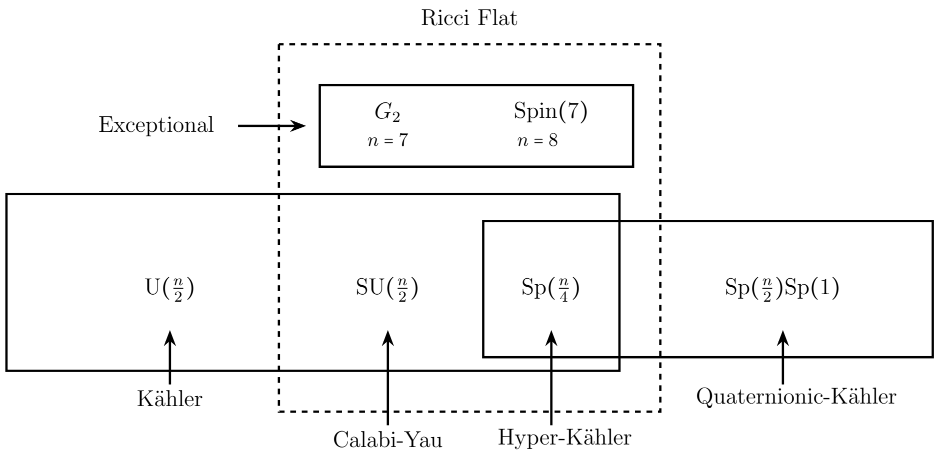
We begin by recalling Berger's celebrated classification of Holonomy groups on an oriented Riemannian manifold [Ber60].

Theorem 7.0.1.1. Let M, be an oriented simply-connected, irreducible (not locally a product space), and non-symmetric (not locally a symmetric space) n-dimensional Riemannian manifold.

Then G must equal

$$\mathrm{SO}(n),\ \mathrm{U}\left(\frac{n}{2}\right),\ \mathrm{SU}\left(\frac{n}{2}\right),\ \mathrm{Sp}\left(\frac{n}{4}\right)\mathrm{Sp}(1),\ \mathrm{Sp}\left(\frac{n}{4}\right),\ G_2\ (n=7),\ \text{or}\ \mathrm{Spin}(7)\ (n=8). \tag{7.0.2}$$

This is a list of not just groups but representations too — the action of G on its tangent spaces. The groups we are most in this thesis: $\mathrm{SU}(\frac{n}{2})$, for $n = 4$, and $n = 6$[1]. This is because we shall construct canonical differential forms using the Weyl representations. In doing so, we can discuss Einstein conditions on the curvature of a spin manifold, using these integrability conditions of these differential forms.



(7.0.3)

Definition 7.0.1.2. S is called a *tensor* if it is a section of $\bigotimes^s TM \otimes \bigotimes^t T^*M \coloneqq \mathcal{T}^{s,t}(M)$.

We can then relate holonomy group G to the canonical tensors invariant under the G-action.

Proposition 7.0.1.1. Let ∇ be a connection on TM of a manifold M. Then, ∇ induces a connection $\mathcal{T}^{s,t}(M)$ and any subbundle, for example $\Lambda^k(T^*M)$.

Definition 7.0.1.3. Let S be tensor. Then S is covariantly constant, or parallel, if $\nabla S = 0$.

We can now link holonomy to parallel tensors.

Proposition 7.0.1.2. Let M be a manifold from (7.0.3), ∇ a Riemannian connection on TM, and $H = \mathrm{Hol}_x(\nabla) \subset \mathrm{GL}(T_xM)$ the holonomy group. Then, S is covariantly constant if, and only if, S on each fibre of E is fixed by the natural action of H.

[Wan95] and [MS00] gave a classification for the holonomy group to be contained in $\mathrm{Spin}(n)$ rather than $\mathrm{SO}(n)$, using the condition of parallel spinors (theorem 7.0.1.2). That is, given a covariant derivative ∇ induced from the Riemannian one, a spinor ψ is parallel if, and only if, $\nabla\psi \equiv 0$.

Theorem 7.0.1.2. Let (M, g) be a simply connected spin manifold whose holonomy representation is irreducible. Then M carries a parallel spinor if, and only if,

Holonomy Group $H \subset \mathrm{Spin}(n)$	Dimensions of spin manifold M
$\mathrm{SU}(m)$	$m \in 2\mathbb{N}$ (Calabi-Yau)
$\mathrm{Sp}(k)$	$k \in 4\mathbb{N}$ (Hyper-Kähler)
G_2	$n = 7$ (Exceptional)
$\mathrm{Spin}(7)$	$n = 8$ (Exceptional)

[1]We shall make some comments concerning G_2 in 7 dimensions, and Spin(7) in 8 dimensions throughout chapters 8 and 9.

This leads us to define,

Definition 7.0.1.4. Let M be a spin manifold. A spinorial G-structure is called classical if it a $\mathrm{SU}(m)$-structure on M with dimension $m \in 2\mathbb{N}$, a $\mathrm{Sp}(k)$-structure on M with dimension $k \in 4\mathbb{N}$, a G_2-structure on M with dimension 7, or a $\mathrm{Spin}(7)$-structure on M dimension with dimension 8.

Furthermore, recall if a classical spinorial G-structure is integrable then it is Ricci-flat. Indeed, it is proven in [Hit74] that if M admits a parallel spinor, then M is Ricci-flat. Subsequently [MS00],

Theorem 7.0.1.3. Let M be an oriented connected manifold of dimensions n, and suppose H is a holonomy subgroup. Suppose there exists an embedding ϕ from H into $\mathrm{Spin}(n)$. If $\pi \circ \phi = \iota$, (where π is the projection from $\mathrm{Spin}(n) \to \mathrm{SO}(n)$, and ι is the inclusion of the holonomy group into $\mathrm{SO}(n)$), then M carries a spin structure whose holonomy group is isomorphic to H.

The above theorem shows there is a subclass of spinorial G-structures amenable to integrability via parallel tensors from holonomy considerations. Indeed, in this thesis we show that for a $\mathrm{SU}(2)$-structure on a spin manifold of dimension 4, and a $\mathrm{SU}(3)$-structure on a spin manifold of dimension 6, using the geometric map one can compute a set of canonical differential forms from a generic Weyl spinor. Then computing the covariant derivative of these differential forms, we show that imposing they are parallel is equivalent to integrable $\mathrm{SU}\left(\frac{n}{2}\right)$-structure (for $n = 4$ and $n = 6$).

This leads us to intrinsic torsion, the tool that allows one to parameterise the covariant derivative of canonical differential forms. Consequently, this give a concrete representation of the covariantly constant differential forms.

To reiterate, for an oriented manifold, the allowed G-structures must come from Berger's classification by way of holonomy considerations. G-structures are integrable when they are parallel with respect to an induced connection on TM. Integrability of spinorial G-structures can be studied through parallel tensors if one restricts to classical G-structures. In that case, the set of canonical differential forms are constructed from the geometric map and the metric from algebraic relations between the canonical differential forms. Classical G-structures are then integrable whenever the set of canonical differential forms is parallel with respect to the metric connection induced from TM. From this discussion, it is reasonable to drop the adjective "classical" and refer to classical spinorial G-structures as G-structures when there is no ambiguity.

Intrinsic Torsion

Intrinsic torsion is a tool one can use to understand integrability. By this we mean, the covariant derivatives of the canonical differential is the intrinsic torsion space. Thus, the intrinsic torsion vanishes if, and only if, the G-structure is integrable. Intrinsic torsion is intimately linked to G-structures, and in fact it is connection independent, i.e. one does not need to fret over a choice of connection used to covariantly differentiate the canonical differential forms.

Definition 7.0.1.5. Let M be a manifold and ∇ a connection on TM. The *torsion* is a linear operator defined as[2]

$$T(\nabla) : \mathrm{End}(TM) \otimes T^*M \to T^*M \otimes \Lambda^2(TM),\ (X,Y) \mapsto \nabla_X Y - \nabla_Y X - [X,Y]. \tag{7.0.4}$$

Torsion can be shown to be tensorial, and is then a section of the bundle $TM \otimes \Lambda^2 T^*M$. A section $T(\nabla)$ of $TM \otimes \Lambda^2 T^*M$ is said to be torsion free if $T(\nabla) = 0$. Suppose ∇ is fixed,

[2]This definition only makes sense on TM as we have the action defined on vector fields in TM, this isn't the case for a general vector bundle E over M. However, in our case as there is a one-to-one correspondence between connections on TM and P, so we don't need to fret over these details.

heuristically, one can ask how far away is ∇ from being torsion-free. Consider, then, another connection ∇' on TM, or equivalently on the P the principal fibre bundle over M. Now, define the tensor $\alpha \coloneqq \nabla - \nabla'$. Furthermore, α is a section of $\mathrm{ad}(P) \otimes T^*M$ from the view point of connections on P. But $\mathrm{ad}(P) \subset TM \otimes T^*M \cong \mathrm{End}(\mathfrak{g})$. Hence, α is a section of $TM \otimes T^*M \otimes T^*M$, and can be written as $\alpha^a_{bc} = \nabla^a_{bc} - \nabla'^a_{bc}$. Then using the definition of torsion, one has

$$T(\nabla)^a_{bc} - T(\nabla')^a_{bc} = \nabla^a_{bc} - \nabla^a_{cb} - \nabla'^a_{bc} + \nabla'^a_{cb} = -\alpha^a_{bc} + \alpha^a_{cb}. \tag{7.0.5}$$

Thus the torsion free-condition for a varying ∇' is that $T(\nabla)^a_{bc} = \alpha^a_{bc} - \alpha^a_{cb}$. However, what we wanted is an obstruction, intrinsic torsion, to the torsion-free condition without needing dependence on a connection, ∇. This is possible if one considers the following linear map σ and various linear spaces under the image σ, see [Joy00].

Definition 7.0.1.6. Let G be a Lie group, $\mathfrak{g}$ its Lie algebra, and define $V \coloneqq \mathbb{R}^n$. Hence, G acts faithfully on V, and define,

$$\sigma : \mathfrak{g} \otimes V^* \hookrightarrow \underbrace{(V \otimes V^*)}_{\cong \mathrm{End}(V)} \otimes V^* \to V \otimes \Lambda^2(V^*), \; \alpha^a_{bc} \mapsto \alpha^a_{bc} - \alpha^a_{cb}. \tag{7.0.6}$$

Finally, define the following vector spaces.

$$W_1 \coloneqq V \otimes \Lambda^2(V^*), \; W_2 \coloneqq \mathrm{Im}(\sigma), \; W_3 \coloneqq V \otimes \Lambda^2(V^*)/\mathrm{Im}(\sigma), \text{ and } W_4 \coloneqq \mathrm{Ker}(\sigma). \tag{7.0.7}$$

$W_{2,3,4}$ fit into the short exact sequence

$$0 \longrightarrow W_4 \longrightarrow g \otimes V^* \xrightarrow{\sigma} W_2 \longrightarrow W_3 \longrightarrow 0. \tag{7.0.8}$$

Let $\rho_i : G \to \mathrm{GL}(W_i)$ be the natural representations of G on W_i. If M is a manifold and P a principal G-frame bundle, then $\rho_i(P)$ are vector bundles over M. One can now make the key observations

- $T(\nabla)$ is a section of $\rho_1(P)$.
- If ∇ and ∇' are two different connections on TM, then $T(\nabla) - T(\nabla')$ is a section of $\rho_2(P)$, a vector subbundle of $\rho_1(P)$.
- Most importantly, if ∇ and ∇' are two different connections on TM, then $T(\nabla)$ and $T(\nabla')$ project to the *same* section T^0 of the quotient bundle $\rho_3(P) = \rho_1(P)/\rho_2(P)$.
- The difference of any two torsion-free connections is a section of $\rho_4(P)$. If $T^0 = 0$, then the space of torsion-free connections are in one-to-one correspondence with sections of $\rho_4(P)$. Furthermore, if $\mathrm{Ker}(\sigma) = \{0\}$, there is only *one* unique torsion-free connection ∇.

The second to last bullet point states there is a quantity, T^0, that is *independent* of the connection chosen and solely depends on the G-structure, i.e. intrinsic torsion[3].

Consider a spin manifold M, and a classical spinorial H-structure. Define $\mathfrak{h}$ to be the Lie algebra of H, and $\mathfrak{h}^\perp$ to be its orthogonal complement in $\mathfrak{spin}(n) \cong \mathfrak{so}(n)$. Then, because $\Lambda^2(V^*) \cong \mathfrak{so}(n)$, and $W_2 \coloneqq \mathrm{Im}(\sigma) \cong V \otimes \mathfrak{h}$,

$$W_3 \cong (V \otimes \mathfrak{so}(n))/(V \otimes \mathfrak{h}) \cong V \otimes \mathfrak{h}^\perp. \tag{7.0.9}$$

Thus, for any point $p \in P$, $T^0(p)$ takes values in $V \otimes \mathfrak{h}^\perp \cong \Lambda^1(V^*) \otimes \mathfrak{h}^\perp$.

[3] As we are discussing G-structures in this thesis we shall drop the adjective intrinsic when it is not ambiguous to do so.

From holonomy considerations, we know that the integrability is characterised by the covariant derivative of all H-invariant tensors. [Sal89], together with other propositions we have considered above, states

Theorem 7.0.1.4. Let M be an spin manifold, equipped with a classical G-structure that stabilises a spinor ψ. Then, pointwise on the G-spin-frame subbundle, the covariant derivatives of set the canonical differential forms take values in $\Lambda^1(T^*M) \otimes \mathfrak{h}^{\perp}$.

Thus, intrinsic torsion and the covariant derivatives take values in the same space, $\Lambda^1(T^*M)\otimes \mathfrak{h}^{\perp}$. So if all canonical differential forms are parallel, then intrinsic torsion vanishes too, and vice versa.

7.0.2 Geometric Flow as a Gradient Flow

Let M be a 7-dimensional compact manifold. The question of when a G_2-structure can be deformed to a torsion-free one (integrable and hence Ricci flat) is not easy to answer. [Lot20] reviews the types of flows that have been constructed for the problem, but we will only briefly outline one. Karigiannis [Kar05; Kar09; Kar08] initiated the idea of performing a gradient flow of the action of $\mathrm{GL}(7,\mathbb{R})$ and $\mathrm{GL}(8,\mathbb{R})$ on the tangent space to G_2- and Spin(7)-structures in the direction of intrinsic torsion. This philosophy was concretised in [DGK23] by writing all possible independent diffeomorphism-invariant and only second-order in derivatives G_2- (Spin(7)- [Dwi24]) invariant tensors, and performing a gradient flow on these instead.

Inspiration from Flowing G_2-structures

Suppose ϕ is a G_2-structure that is not parallel. It decomposes the spaces of differential forms on a compact 7-dimensional manifold M. Then one can take the $\mathrm{GL}(7,\mathbb{R})$ action on ϕ to determine the infinitesimal action on the space of 3-forms [Kar08; Kar05; Bry05; Bry87]

$$D : \mathrm{GL}(7,\mathbb{R}) \to \Omega^3(M), \quad A \mapsto \left.\frac{d}{dt}\right|_{t=0} e^{At} \cdot \phi. \tag{7.0.10}$$

As $\mathrm{GL}(7,\mathbb{R})$ is the space of matrices, it can be split into symmetric and antisymmetric parts. h is the symmetric part of A that determines the torsion, and X is the vector field corresponding to the antisymmetric part of A that determines the torsion. These are profoundly simple techniques to parameterise the torsion space. A more general flow equation for ϕ can then be written down using h and X

$$\frac{\partial}{\partial t}\phi_{ijk} = 3h_{[i}{}^{l}\phi_{|l|jk]} + X^l \psi_{lijk}. \tag{7.0.11}$$

A similar set of equations can be written for the Hodge-dual 4-form ψ, the metric, and the volume form. We wish to mimic these ideas for SU(2)- and SU(3)-structures to generate all pieces to parameterise the intrinsic torsion in terms of tensors within $\mathrm{GL}(n,\mathbb{R})$. Recall:

- The torsion of a G_2 structure ϕ is characterised by $\mathrm{d}\phi$ and $\mathrm{d}(*\phi)$ [FG82].
- The torsion of a Spin(7) structure Φ is characterised by $\mathrm{d}\Phi$ [Fer86].

The fact that integrability can be measured by exterior derivatives of the canonical differential forms is what we wish to exploit in the thesis to write higher dimensional linear theories of gravity. However, we shall still discuss the metric connection, ∇, as it allows us to explore higher dimensional non-linear theories of gravity.

7.0.3 Plebanksi-formalism

Normally, the most general theory is not simple to write down, as it is not always obvious what the parametrisation of the auxiliary field(s) should be. Let us go back to the work of Plebanksi on Einstein structures in 4 dimensions [Ple77] to elucidate this. The unique guage and diffeomorphism invariant action that can be written is

$$S[\Sigma, A, \Psi] = \int_M \vec{\Sigma}^T \left(\mathrm{d}A + \frac{1}{2} A \wedge A \right) - \frac{1}{2} \vec{\Sigma}^T \Psi \vec{\Sigma} - \frac{\Lambda}{6} \vec{\Sigma}^T \vec{\Sigma}. \tag{7.0.12}$$

Here A is and SU(2) gauge connection, Ψ is a symmetric 3×3 matrix, $\vec{\Sigma} = (\Sigma^1, \Sigma^2, \Sigma^3)$ is a vector of the triple of self-dual 2-forms invariant under the action SU(2), and Λ is the cosmological constant. Varying with respect to the vector $\vec{\Sigma}$, gives

$$F(A) \coloneqq \mathrm{d}A + \frac{1}{2} A \wedge A = \left(\Psi + \frac{\Lambda}{3} \mathbb{I} \right) \vec{\Sigma}. \tag{7.0.13}$$

F is the curvature of A. The quantity in the brackets above is the Ricci curvature tensor, by algebraically moving $\vec{\Sigma}$ to the left-hand side of the equation implies that $R_{\mu\nu} = \Lambda g_{\mu\nu}$, i.e. Einstein conditions.

We now try to generalise the approach Plebanski took. We define the following schematic

Definition 7.0.3.1. *Plebanski-formalism* is the procedure of constructing the linear combination of all independent diffeomorphism invariant action functionals built from canonical differential forms, their exterior derivatives, and auxiliary fields one wishes to extremise.

Remark 7.0.3.1. We wish to impose more invariance on the action functional, however diffeomorphism invariance is a good starting place. As we see in the example below. It is unclear how to impose additional invariance such that one constrains to Einstein conditions in Plebanski's original work.

We briefly discuss Plebanksi-formalism in 8 dimensions, via Spin(7) structures. This gives the blueprint to the approach we take with SU(2)- and SU(3)-structures. Inspired by Karigiannis et al. we hope the torsion can be written in terms of some symmetric and antisymmetric tensors inside $GL(8, \mathbb{R})$. Indeed, this was already done in [Kar08]. However, one can also see that torsion is completely determined by 3-forms and so there is a compact parameterisation via exterior derivatives of Φ, the canonical tensor of the Spin(7)-structure

$$\nabla_i \Phi_{abcd} = 4 T_{i[a}{}^p \Phi_{|p|bcd]}, \quad \text{where} \ \ast \mathrm{d}\Phi = \frac{2}{5} J_3(T). \tag{7.0.14}$$

Here $J_3 : \Omega^3(M) \to \Omega^3(M)$ is an invertible operator defined in equation (2.13) of [Kra24a]. This parameterisation of the torsion leads to application of the Plebanski-formalism as follows,

$$S[\Phi, C] = \int_M \Phi \wedge (\mathrm{d}C - 6C \wedge_\Phi C) + \frac{\kappa}{6} (C)^2 v_\Phi + \text{constraint terms on curvature.} \tag{7.0.15}$$

Here C is an auxiliary 3-form, $v_\Phi = \frac{1}{14} \Phi \wedge \Phi$, and $(C \wedge_\Phi C)_{abcd} = C_{abp} C_{cdq} g^{pq}$. By constraint terms, we mean the terms we deem necessary to extremise to Ricci flat, or Einstein conditions, these are fixed by hopefully choosing an appropriate value of κ and auxiliary function C. Thus the above is the a one-parameter family of diffeomorphism-invariant Lagrangians that are second order in derivatives of Φ but first order in C. For $\kappa = 0$, this corresponds to C being the torsion, and the gradient flow of the functional is elliptic (modulo gauge), ensuring that short-time and unique solutions exist. If $\kappa = -2$, this corresponds to the non-linear completion of the linearised theory. Unlike the Plebanski's formulation of gravity in 4 dimensions, there is no obvious way to fix C such that one extremises to Einstein conditions on the curvature, and this remains an open problem.

Finally, the tangent space to canonical differential forms and the metric can be modelled by an infinitesimal action of 2-tensors in $\mathrm{GL}(n,\mathbb{R}) \cong TM \otimes T^*M$. This leads to following

Definition 7.0.3.2. Define *linearised analysis* as the procedure of constructing *all* independent action functionals built from two derivatives of 2-tensors in $\mathrm{GL}(n,\mathbb{R}) \cong TM \otimes T^*M$, and their products, from the representation theory of the subgroup G, the invariant differential forms are invariant under.

Furthermore, define the *non-linear completion* to be the subspace of action functionals built from Plebanski-formalism that linearise to the action functionals built from a linearised analysis.

Hence, linearised analysis gives insight in to how one *should* write a theory of gravity in higher dimensions.

7.0.4 Chapter Overview

Chapter 8 is dedicated to SU(2)-structures, which provide the most complete picture we have in terms of formulating a philosophy aligned with Plebanski-formalism. The study of SU(2)-structures in 4 dimensions is challenging yet simultaneously easier than in other dimensions with other G-structures. The remarkable accidental isomorphism

$$\mathfrak{su}(2) \oplus \mathfrak{su}(2) \cong \mathfrak{so}(4) \cong \mathfrak{spin}(4), \tag{7.0.16}$$

dictates the role that intrinsic torsion plays, the orthogonal complement to $\mathfrak{su}(2)$ is also a Lie algebra. Generally, the intrinsic torsion space is merely a vector space. However, as $\mathfrak{su}(2)^{\perp} \cong \mathfrak{su}(2)$, one must keep tract of this additional algebraic structure when discussing intrinsic torsion.

In the work by [FS24], the second copy of $\mathfrak{su}(2)$ that acts covariantly on the SU(2)-structures is not tracked. However, this thesis, along with [BK24], aims to use it to characterise the torsion. By the holonomy principal, the covariant derivative of the tensors characterising SU(2)-structures provides access to the full torsion space. Thus, we consider tensor products of irreducible SU(2) representations as pieces of the covariant derivatives of SU(2)-structures in these representations.

For future purposes, we exploit that the torsion from G-structures of this type can be written as exterior derivatives of SU(2)-structures, a weaker condition than the full covariant derivative. Upon completing this exercise, we can proceed to characterise parts of the Riemann curvature tensor that torsion gives access to (by taking another covariant derivative). We find that it provides access to the Ricci tensor and scalar curvature, but not the anti-self-dual part of the curvature.

Motivated by the ideas of [Kar08] and [DGK23], we then consider a diffeomorphism-invariant and second-order in derivatives theory of SU(2)-structures.

To this end, we perform a linearised analysis to construct a second-order in derivatives diffeomorphism-invariant Lagrangian from perturbed SU(2)-structures and the perturbed metric they algebraically construct. This calculation shows that the space of second-order derivatives diffeomorphism-invariant Lagrangian splits into two pieces: a linearised theory of gravity in a vacuum (developed solely from the metric perturbation) and a second piece that couples the metric with an extra field allowed by the representation theory of SU(2).

If one imposes gauge invariance, the second Lagrangian cannot exist, as it is no longer gauge invariant. This means that only the gravitational theory is allowed. Finally, we propose that the action functional in (8.5.6.0.1) is the best second-order theory, as its critical points are the Einstein conditions, equivalent to the first-order theory constructed by Plebanski.

The techniques explored in SU(3)-structures are extended to SU(3)-structures in 6 dimensions in chapter 9. The approach deviates in two places. Firstly, for SU(3)-structures, the intrinsic torsion is characterised by a vector space, unlike in the 4-dimensional case. Indeed,

$$\mathfrak{spin}(6) \cong \mathfrak{su}(3) \oplus \mathfrak{su}(3)^{\perp} \cong \mathfrak{su}(3) \oplus (\mathbb{R} \oplus \mathbb{R}^6). \tag{7.0.17}$$

On one hand, torsion does not carry additional algebraic structure. On the other hand, this makes torsion less easily characterisable. To simplify the identification, we transition to representations over complex forms,

$$\Lambda^{p,q}(M) = \Lambda^{p,0}(M) \otimes \Lambda^{0,q}(M), \tag{7.0.18}$$

of holomorphic and anti-holomorphic types. We decompose the torsion space accordingly and then characterise it in the complex setting. Upon completing this exercise, we interpret the torsion in terms of exterior derivatives of the SU(3)-structures. Much work has already been done on the real front, see [BV07] and [CS02]. Finally, we understand which parts of the Riemann curvature tensor are accessible and express the full Ricci tensor and scalar curvature in terms of torsion.

As in the previous chapter, we perform a linearised analysis of the space of second-order diffeomorphism-invariant Lagrangians. Although we construct a real theory, the notation we use invokes complex coordinates, as this is more compact. In previous linearised theories, such as SU(2) or Spin(7), the most general second-order diffeomorphism-invariant Lagrangian is built from two Lagrangians: one resembling general relativity and one where the extra field of the theory couples to the metric. In 6 dimensions, although there is an exclusively linearised theory of gravity in a vacuum, there are many extra fields that couple to the metric and themselves, making the basis of independent Lagrangians seven-dimensional, from the neat one-dimensional case of SU(2). Unlike the SU(2) theory, and much like the Spin(7) theory, there is no clear understanding of gauge fixing that can be exploited. Furthermore, for similar reasons, the non-linear completion is not understood and is therefore omitted in this thesis.

Chapter 8

SU(2)-structures in 4 dimensions

8.1 Geometry from Spinors

We begin by reminding the reader of the Clifford algebra Cliff_4, and the geometric maps that one can compute for a general Weyl spinor. The generators of the Clifford algebra are given as

$$\begin{aligned}\Gamma_1 &= -\mathbf{i}(a_2 - a_2^\dagger),\ \Gamma_2 = a_2 + a_2^\dagger,\\ \Gamma_3 &= -\mathbf{i}(a_1 - a_1^\dagger),\ \Gamma_4 = a_1 + a_1^\dagger.\end{aligned} \tag{8.1.1}$$

Taking the generic Weyl spinor $\psi = \alpha + \beta \mathrm{d}x^{12}$, the geometric maps $B_2(\hat{\psi}, \psi)$ and $B_2(\psi,\psi)$ are computed as

$$B_2(\hat{\psi}, \psi) = \mathbf{i}\Sigma^i V^i, \text{ and } B_2(\psi,\psi) = \mathbf{i}\Sigma^i W^i. \tag{8.1.2}$$

Here

$$\Sigma^i = \mathrm{d}x^4 \wedge \mathrm{d}x^i - \frac{1}{2}\varepsilon^{ijk}\mathrm{d}x^i \wedge \mathrm{d}x^j \tag{8.1.3}$$

are 2-forms in 4 dimensions, and $V \in \mathbb{R}^3$ and $W \in \mathbb{C}^3$ are given by

$$V = (\mathrm{Re}(\alpha\beta^*), \mathrm{Im}(\alpha\beta^*), |\alpha|^2 - |\beta|^2),\ W = (\alpha^2 - \beta^2, \alpha^2 + \beta^2, 2\alpha\beta). \tag{8.1.4}$$

It is clear the that the stabiliser of ψ is the $\mathfrak{su}(2) \subset \mathfrak{spin}(4)$ that annihilates even spinors. Setting $\alpha = 1$ and $\beta = 0$ one generates

$$-\mathbf{i}B_2(\hat{\psi}, \psi) = \Sigma^3, \text{ and } -\mathbf{i}B_2(\psi,\psi) = \Sigma^1 - \mathbf{i}\Sigma^2. \tag{8.1.5}$$

Since every spinor in 4 dimensions is pure, it is possible to rotate to this canonical spinor. Then, there is a set of intriguing conditions given as follows

$$\begin{aligned}\Sigma^i \wedge \Sigma^j &= \frac{1}{2}\delta^{ij}\mathrm{d}x^{1234}, \text{ equivalently } \Omega\wedge\Omega = 0,\\ \Omega \wedge \omega &= 0, \text{ and } \Omega \wedge \bar{\Omega} = \frac{1}{2}\omega\wedge\omega.\end{aligned} \tag{8.1.6}$$

Here ω is Kähler 2-form in 4 dimensions, and Ω is the top holomorphic form in 4 dimensions. The description above is very specific, there are conditions on the 2-forms that need to be specified, these are 5 conditions on the 2-forms, thus the space of admissible 2-forms has dimensions $18 - 5 = 13$. This is of course the dimensions of $\mathrm{GL}(4,\mathbb{R})/\mathrm{SU}(2)$ the space of SU(2)-structures. The geometry of non-parallel SU(2)-structures in 4 dimensions is given by Ω and ω and on each fibre the metric can be recovered by the formula

$$g_{\omega,\Omega}(X,Y) = \omega(JX,Y). \tag{8.1.7}$$

Here J is extracted from the $\mathrm{Re}(\Omega)$ by the formula

$$J^i{}_j = 4\epsilon^{il_1l_2l_3}\mathrm{Re}(\Omega)_{ja_1}\mathrm{Re}(\Omega)_{l_2l_3}, \tag{8.1.8}$$

ϵ is the antisymmetric tensor that is non-zero whenever the indices are not the same. Analogously, the image of the geometric maps $B_2(\psi,\hat{\psi})$, and $B_2(\psi,\psi)$ in 4 dimensions declare a subspace of self-dual 2-forms Σ^i that satisfy certain algebraic relations (the imaginary quaternion algebra) with respect to a metric given by the formula

$$g_\Sigma(X,Y)\mathrm{vol}_g = -\frac{1}{6}\varepsilon^{ijk}\iota_X\Sigma^i \wedge \iota_Y\Sigma^j \wedge \Sigma^k. \tag{8.1.9}$$

We briefly explain now SU(2)-structures as equivariant maps. Define E be the bundle over M such that the pullback of each point on M is $(\mathbb{R}^3, \langle,\rangle, \wedge)$, that is, the vector space $\mathbb{R}^3$ with a standard inner product and a cross product. Notice that $(\mathbb{R}^3, \langle,\rangle, \wedge) \cong \mathfrak{su}(2)$.

Next, consider the following bundle over M, such that the pullback of each point over the manifold is $(\Lambda^2(M), g_\Sigma)$. Here g_Σ is the metric defined in (8.1.9). Notice that the self-dual 2-forms, via g_Σ, $\Lambda^+ \subset \Lambda^2(M)$ can be identified with $\mathfrak{su}(2)$.

Then an SU(2)-structure the bundle map $\Sigma : E \to \Lambda^2(M)$, for $p \in M$, that is an isometry on to its image. Similarly, for any $p \in M$

$$\Lambda^+ \cong \mathfrak{su}(2) \subset \mathfrak{so}(4) = \Lambda^2(M). \tag{8.1.10}$$

In turn one has

$$\Sigma : E \to \Lambda^+, \tag{8.1.11}$$

is a bundle isomorphism preserving the Lie algebra structure of $\mathfrak{su}(2)$.

Remark 8.1.0.1. There is an identification between E and Λ^- (the space of anti-self-dual 2-forms), this comes from a different metric $g_{\Sigma'}$ as explained in [BK24].

This leads to our discussion of SU(2)-structures being different from those that have been explained even as recently as [FS24]. This is because we keep *both* representations $\mathrm{SU}(2) \times \mathrm{SU}(2) \subset \mathrm{Spin}(4)$, where they do not. The upshot of keeping the algebraic structure of $\mathfrak{g}^\perp$ means one need not break $\mathrm{GL}(4,\mathbb{R})$ to $\mathrm{O}(4,\mathbb{R})$, a unique case not studied among other G-structures. The only drawback is decomposing tensors with respect to both SU(2)s.

We now have the necessary tools to begin analysing the intrinsic torsion, as we identify $\mathfrak{su}(2) \cong E \cong \mathfrak{g}^\perp \subset \mathfrak{so}(4)$[1]. It is clear that $\mathfrak{g}^\perp$ is a Lie algebra, which in this setting is unique. G-structures in general do *not* follow this pattern and are only linear structures (see chapter 9). This makes the analysis below much richer, as one has to take into account the extra structure carried by E.

8.2 Decomposition of E-valued Differential Forms

Again, in our case, $\mathfrak{g}^\perp \cong \mathfrak{su}(2) \cong E$, and so the intrinsic torsion, by the holonomy principal, must be an object with values in $\Lambda^1(M) \otimes E$. The differential forms that defines the SU(2)-structure take values in $\Lambda^2(M) \otimes E$, and so do their covariant derivatives $\nabla_X \Sigma$ in any direction $X \in TM$. For this reason, we need to understand the decomposition of the spaces $\Lambda^1(M) \otimes E$, and $\Lambda^2(M) \otimes E$ into irreducible representations of $\mathrm{SU}(2) \times \mathrm{SU}(2)$.

[1] We abuse notation but by $\mathfrak{su}(2)$ we mean the bundle over M such that the pullback at each point is the Lie algebra $\mathfrak{su}(2)$.

Irreducible representations of SU(2) are the spin $k/2$ representations that we denote by S^k. They are of dimension $\dim(S^k) = k+1$. As we have previously discussed, there are two different SU(2)'s in the game. One SU(2) is the group with respect to which the 2-forms Σ^i are invariant. This motivates the following definition,

Definition 8.2.0.1. Let Σ^i be the self-dual 2-forms defined by equivariant map defined in (8.1.11).

The copy of $\mathrm{SU}(2) \subset \mathrm{Spin}(4)$ that stabilises Σ^i shall be denoted $\mathrm{SU}_-(2)$. We define the irreducible representations of $\mathrm{SU}_-(2)$ as S_-^k.

On the other hand, the copy of $\mathrm{SU}(2) \subset \mathrm{Spin}(4)$ that acts covariantly on Σ^i shall be denoted as $\mathrm{SU}_+(2)$. We define the irreducible representations of $\mathrm{SU}_+(2)$ as S_+^k.

By these definitions, one has

$$\Lambda^1(M) = S_+ \otimes S_-, \;\; \Lambda^2(M) = S_+^2 \oplus S_-^2, \text{ such that } E \cong S_+^2\big|_p, \tag{8.2.1}$$

leading to the following lemma.

Lemma 8.2.0.1. The decomposition of $\Lambda^1(M) \otimes E$, and $\Lambda^2(M) \otimes E$ into irreducible representations is given as

$$\begin{aligned} &\Lambda^1(M) \otimes E \cong (S_+^3 \otimes S_-) \oplus (S_+ \otimes S_-), \text{ and} \\ &\Lambda^2(M) \otimes E \cong S_+^4 \oplus S_+^2 \oplus C^\infty(M) \oplus (S_+^2 \otimes S_-^2). \end{aligned} \tag{8.2.2}$$

Proof. A standard exercise in looking up the decomposition of tensor products of irreducible SU(2) representations, see for example tables in [FKS20]. □

8.2.1 Algebra of Σs

To obtain explicit formulas for the irreducible parts of E-valued differential forms, we need some identities satisfied by the 2-forms Σ^i. Bryant introduced algebraic formulae for G_2-structures in [Bry05] called ϵ-identities. A similar procedure has been completed by [Kar08] for Spin(7)-structures ([Kar09] and [DGK23] for G_2 structures), where a larger program of algebraic identities are considered and recently for SU(3)-structures [BV07], as we will see in chapter 9. Although these formulae can be written index-free, we choose to consider their indicial variants, which are most suited for our type of calculations, where Σ^i are complex structures that satisfy the quaternionic algebra.

First, one of the two indices of these differential forms can be raised with the metric they define, to convert these into objects in End(TM). We then have a triple of such endomorphisms of the tangent bundle, satisfying the algebra of the imaginary quaternions

$$\Sigma^{i\;\alpha}_{\mu}\Sigma^{j\;\nu}_{\alpha} = -\delta^{ij}\delta_{\mu}{}^{\nu} + \epsilon^{ijk}\Sigma^{k\;\nu}_{\mu}. \tag{8.2.3}$$

There are also useful relations

$$\Sigma^i_{\mu\nu}\Sigma^i_{\rho\sigma} = g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho} + \epsilon_{\mu\nu\rho\sigma}, \tag{8.2.4}$$

and

$$\epsilon^{ijk}\Sigma^j_{\mu\nu}\Sigma^k_{\rho\sigma} = -2\Sigma^i_{[\mu|\rho|}g_{\nu]\sigma} + 2\Sigma^i_{[\mu|\sigma|}g_{\nu]\rho}. \tag{8.2.5}$$

Here, throughout this chapter (and next chapter 9), the square bracket on indices means one antisymmetrises the indices. For example, let A be a (2,0) tensor, then

$$A_{[\mu\nu]} = \frac{1}{2}(A_{\mu\nu} - A_{\nu\mu}). \tag{8.2.6}$$

Similarly, if one uses circular brackets on indices that means one symmetrises the indices. For example, let A be as (8.2.6), then

$$A_{(\mu\nu)} = \frac{1}{2}(A_{\mu\nu} + A_{\nu\mu}). \tag{8.2.7}$$

8.2.2 Decomposition of $\Lambda^1(M) \otimes E$

One can decompose $\Lambda^1(M) \otimes E$ using a self-adjoint operator on it and then considering its spectrum. Therefore, one defines

Definition 8.2.2.1. Let Σ^i be SU(2)-structures, and $A^i_\mu \in \Lambda^1(M) \otimes E$. Define the pointwise linear operator $J_\Sigma : \Lambda^1(M) \otimes E \to \Lambda^1(M) \otimes E$ as, $J_\Sigma(A)^i_\mu := \epsilon^{ijk}\Sigma^{j\ \alpha}_{\mu} A^k_\alpha$.

It is not difficult to show that J_Σ is pointwise self-adjoint with respect to the metric on $\Lambda^1(M) \otimes E$, $\tilde{g} = g \otimes g_\Sigma$. Here g is the standard inner product on Λ^k, and g_Σ is the metric given on E, see equation (8.1.9). Furthermore, a simple calculation using (8.2.3) shows that

$$J^2_\Sigma = 2\mathbb{I} + J_\Sigma. \tag{8.2.8}$$

This means that the eigenvalues of J_Σ are $2, -1$. The eigenspaces of J_Σ are precisely the irreducible representations appearing in the first line in (8.2.2). It is also easy to check that objects of the form

$$\xi^\alpha \Sigma^i_{\alpha\mu} \in \Lambda^1(M) \otimes E \tag{8.2.9}$$

are eigenvectors of eigenvalue 2. The dimension of the subspace spanned by $\{\xi^\alpha \Sigma^i_{\alpha\mu} \mid \xi \in \Lambda^1(M)\}$ is 4-dimensional. Acting J_Σ on $\xi^\alpha \Sigma^i_{\alpha\mu}$ scales tensor by a value of 2. Thus the following characterisation can be made:

$$(\Lambda^1(M) \otimes E)_4 := (S_+ \otimes S_-) \cong \{\xi^\alpha \Sigma^i_{\alpha\mu}, \xi \in TM\}. \tag{8.2.10}$$

Using (8.2.2), the remaining space 8-dimensional irreducible representation must be identified as follow

$$(\Lambda^1(M) \otimes E)_8 \cong (S^3_+ \otimes S_-). \tag{8.2.11}$$

Objects in this space are in the orthogonal complement of (8.2.10) in $\Lambda^1(M) \otimes E$, that scale by a factor of -1 when acted upon by J_Σ.

8.2.3 Decomposition of $\Lambda^2(M) \otimes E$

We can also describe the irreducible subspaces of $\Lambda^2(M) \otimes E$ as eigenspaces of a certain operator in E-valued 2-forms, similar to how we used J_Σ to decompose $\Lambda^1 \otimes E$.

Definition 8.2.3.1. Let Σ^i be SU(2)-structures, and $B^i_{\mu\nu} \in \Lambda^2 \otimes E$. Define the pointwise linear operator $J_2 : \Lambda^2 \otimes E \to \Lambda^2 \otimes E$ as, $J_2(B)^i_{\mu\nu} = \epsilon^{ijk}\Sigma^{j\ \alpha}_{[\mu} B^k_{|\alpha|\nu]}$.

As previously seen, a similar pattern holds for J_2. It is a self-adjoint operator on $\Lambda^2 \otimes E \to \Lambda^2 \otimes E$ with respect to the metric $\tilde{g} = g \otimes g_\Sigma$. In addition, a computation gives

$$\begin{aligned}
J^2_2(B)^i_{\mu\nu} &= \frac{1}{2}B^i_{\mu\nu} + \frac{1}{2}\epsilon_{\mu\nu}{}^{\alpha\beta}B^i_{\alpha\beta} + \frac{1}{2}J_2(B)^i_{\mu\nu} + \frac{1}{2}\Sigma^{i\ \alpha}_{[\mu}\Sigma^{j\ \beta}_{\nu]}B^j_{\alpha\beta}, \\
J^3_2(B)^i_{\mu\nu} &= \frac{1}{2}\epsilon_{\mu\nu}{}^{\alpha\beta}B^i_{\alpha\beta} + 2J_2(B)^i_{\mu\nu} + \Sigma^{i\ \alpha}_{[\mu}\Sigma^{j\ \beta}_{\nu]}B^j_{\alpha\beta}, \text{ and} \\
J^4_2(B)^i_{\mu\nu} &= \frac{1}{2}B^i_{\mu\nu} + \frac{3}{2}\epsilon_{\mu\nu}{}^{\alpha\beta}B^i_{\alpha\beta} + \frac{5}{2}J_2(B)^i_{\mu\nu} + \frac{5}{2}\Sigma^{i\ \alpha}_{[\mu}\Sigma^{j\ \beta}_{\nu]}B^j_{\alpha\beta}.
\end{aligned} \tag{8.2.12}$$

This implies

$$J_2^4 - 2J_2^3 - J_2^2 + 2J_2 = 0 \text{ or } J_2(J_2-2)(J_2-1)(J_2+1) = 0, \tag{8.2.13}$$

which shows that the eigenvalues of J_2 are $2, 1, -1, 0$.

Lemma 8.2.3.1. Consider the decomposition of $\Lambda^2(M) \otimes E$ given in lemma 8.2.0.1. Let $(\Lambda^2(M) \otimes E)_k$ be an eigenspace of J_2, where k is the pointwise dimension of the eigenspace. Then

$$(\Lambda^2(M) \otimes E)_5 \cong S_+^4,\ (\Lambda^2(M) \otimes E)_3 \cong S_+^2,\ (\Lambda^2(M) \otimes E)_1 \cong C^\infty(M), \\ \text{and } (\Lambda^2(M) \otimes E)_9 \cong S_+^2 \otimes S_-^2. \tag{8.2.14}$$

In the proof below we shall explicitly identify the eigenvalues associated to each eigenspace of J_2 listed above.

Proof. To characterise the eigenspaces we consider an arbitrary 3×3 matrix $M^{ij} = M_s^{ij} + M_a^{ij}, M_s^{ij} = M_s^{(ij)}, M_a^{ij} = M_a^{[ij]}$ and compute

$$J_2(M^{ij}\Sigma^j_{\mu\nu}) = \mathrm{Tr}(M)\Sigma^i_{\mu\nu} - M^{ji}\Sigma^j_{\mu\nu} = \mathrm{Tr}(M)\Sigma^i_{\mu\nu} - M_s^{ij}\Sigma^j_{\mu\nu} + M_a^{ij}\Sigma^j_{\mu\nu}. \tag{8.2.15}$$

This means that the eigenspace of J_2 of eigenvalue 2 is $C^\infty(M)$ spanned by multiples of $\Sigma^i_{\mu\nu}$, take $M \propto \mathbb{I}_{3\times3}$. The eigenspace of eigenvalue 1 is S_+^2 spanned by $M_a^{ij}\Sigma^j_{\mu\nu}$. The eigenspace of eigenvalue -1 is S_+^4 spanned by $M_s^{ij}\Sigma^j_{\mu\nu}$ with $\mathrm{Tr}(M_s) = 0$.

We can also apply the operator J_2 to objects of the type $h_{[\mu}{}^\alpha\Sigma^i_{|\alpha|\nu]}$. We get

$$J_2(h_{[\mu}{}^\alpha\Sigma^i_{|\alpha|\nu]}) = \frac{1}{2}h_\alpha{}^\alpha\Sigma^i_{\mu\nu}. \tag{8.2.16}$$

If one takes $h \propto \mathbb{I}_{4\times4}$, then the eigenspace of eigenvalue 2 is $C^\infty(M)$ spanned by multiples of $\Sigma^i_{\mu\nu}$. This means that the space $S_+^2 \otimes S_-^2$ spanned by $h_{[\mu}{}^\alpha\Sigma^i_{|\alpha|\nu]}$, with $\mathrm{Tr}(h) = 0$, is the eigenspace of J_2 of eigenvalue 0. Thus, the desired result is achieved. □

8.3 Intrinsic Torsion and Curvature

8.3.1 Intrinsic Torsion

From general principles, it follows that the torsion of a G-structure should be described by an object valued in $\Lambda^1(M) \otimes \mathfrak{g}^\perp$, which in our case is $\Lambda^1(M) \otimes E$. At the same time, the intrinsic torsion quantifies non-integrability of the G-structure, and thus the failure of the canonical differential forms defining this G-structure to be parallel with respect to the Levi-Civita connection. Thus, we expect that $\nabla_\mu\Sigma^i_{\alpha\beta}$ should be expressible via the intrinsic torsion $A^i_\mu \in \Lambda^1(M) \otimes E$. The following proposition is a statement to this effect

Theorem 8.3.1.1. There exists a set of objects $A^i_\mu \in \Lambda^1(M) \otimes E$ such that

$$\nabla_\mu\Sigma^i_{\alpha\beta} = -\epsilon^{ijk}A^j_\mu\Sigma^k_{\alpha\beta}. \tag{8.3.1}$$

Proof. Comparing the right-hand side of the formula (8.3.1) with the set of objects that appear in lemma 8.2.3.1, we see that equation (8.3.1) is equivalent to saying there are no contributions from the $S_+^4, C^\infty(M)$, and $S_+^2 \otimes S_-^2$ irreducible components in $X^\mu\nabla_\mu\Sigma^i_{\alpha\beta} \in \Lambda^2(M) \otimes E,\ \forall X^\mu \in TM$. The $S_+^4, C^\infty(M)$ components are extracted as

$$2\Sigma^{(i|\alpha\beta|}\nabla_\mu\Sigma^{j)}_{\alpha\beta} = \nabla_\mu(\Sigma^{i|\alpha\beta|}\Sigma^j_{\alpha\beta}) = 4\nabla_\mu\delta^{ij} = 0. \tag{8.3.2}$$

Here we have used the fact that the operation of raising-lowering of the indices commutes with ∇_μ. Similarly, the $S^2_+ \otimes S^2_-$ component is extracted as

$$2\Sigma^i_{(\mu}{}^\alpha \nabla_\rho \Sigma^i_{\nu)\alpha} = \nabla_\rho \Sigma^i_\mu{}^\alpha \Sigma^i_{\nu\alpha} = 3\nabla_\rho g_{\mu\nu} = 0. \tag{8.3.3}$$

This shows that no undesired components are present in $\nabla_\mu \Sigma^i_{\alpha\beta}$ and that (8.3.1) holds. □

8.3.2 Bianchi Identity

Establishing a version of the formula (8.3.1) is one of the more laborious parts of the analysis of a non-integrable G-structure. The rest of the analysis is much more algorithmic. Let us consider the projection of the Riemannian curvature tensor by our G-structures Σ^i, this is given by the Bianchi identity as

Corollary 8.3.2.0.1. Let $R_{\mu\nu\rho\sigma}$ be a Riemann curvature tensor on a compact 4-dimensional spin manifold M, and let Σ^i be SU(2)-structures. One then has

$$\pi_\Sigma(R)^i_{\mu\nu\rho\sigma} \coloneqq R_{\mu\nu[\rho}{}^\alpha \Sigma^i_{|\alpha|\sigma]} = -\frac{1}{2}\epsilon^{ijk} F^j_{\mu\nu} \Sigma^k_{\rho\sigma}, \tag{8.3.4}$$

where we have defined the *curvature* of the connection A^i_μ as

$$F^i_{\mu\nu} \coloneqq 2\nabla_{[\mu} A^i_{\nu]} + \epsilon^{ijk} A^j_\mu A^k_\nu. \tag{8.3.5}$$

M is spin because the construction of Σ comes from a generic Weyl spinor in 4 dimensions.

Proof. One computes using (8.3.1),

$$\begin{aligned} 2\nabla_{[\mu}\nabla_{\nu]}\Sigma^i_{\rho\sigma} &= -2\epsilon^{ijk}(\nabla_{[\mu} A^j_{\nu]}\Sigma^k_{\rho\sigma} + A^j_{[\nu}\nabla_{\mu]}\Sigma^k_{\rho\sigma}) \\ &= -2\epsilon^{ijk}(\nabla_{[\mu} A^j_{\nu]}\Sigma^k_{\rho\sigma} + A^j_{[\mu}\epsilon^{klm} A^l_{\nu]}\Sigma^l_{\rho\sigma}) \\ &= -\epsilon^{ijk} F^j_{\mu\nu}\Sigma^k_{\rho\sigma}. \end{aligned} \tag{8.3.6}$$

The result then follows from the identity

$$2R_{\mu\nu[\rho}{}^\alpha \Sigma^i_{|\alpha|\sigma]} = 2\nabla_{[\mu}\nabla_{\nu]}\Sigma^i_{\rho\sigma}. \tag{8.3.7}$$

□

We observe that, in the case of SU(2)-structures in four dimensions, the intrinsic torsion assembles itself into an $\mathfrak{su}(2)$-valued one-form, or an SU(2) connection. This is exactly why one can define a curvature 2-form (8.3.5).

Since the left-hand side in (8.3.4) is the projection of the Riemann tensor inside $\Lambda^2(M) \odot \Lambda^2(M)$ onto E. Here, and throughout, $\odot$ is the symmetric product. There is no loss of information if we multiply both sides of (8.3.4) with $\epsilon^{ijk}\Sigma^{j\rho\sigma}$ to get

$$R_{\mu\nu}{}^{\rho\sigma}\Sigma^k_{\rho\sigma} = 2F^k_{\mu\nu}. \tag{8.3.8}$$

This is the most useful form of the "Bianchi identity" (8.3.4), using the terminology of [DGK23]. In words, the self-dual part of the Riemann curvature $R_{\mu\nu\rho\sigma}$ with respect to the pair of indices $\{\rho,\sigma\}$ equals a multiple of the curvature tensor $F^i_{\mu\nu}$, which is also the curvature of the intrinsic torsion A^i_μ. The fact that the intrinsic torsion assembles itself into an SU(2) connection does not have analogues in the case of other G-structures, see chapter 9, and [Kra24a]. This is because the intrinsic torsion is valued in $\mathfrak{g}^\perp \otimes \Lambda^1(M)$, and only in this case is $\mathfrak{g}^\perp$ a lie algebra.

8.3.3 Ricci tensor and Curvature

We can extract the Ricci tensor from (8.3.8) via

$$\Sigma^i_\mu{}^\alpha R_{\alpha\nu\rho\sigma}\Sigma^i_{\rho\sigma} = (g_{\mu\rho}g^\alpha{}_\sigma - g_{\mu\sigma}g^\alpha{}_\rho + \epsilon_\mu{}^\alpha{}_{\rho\sigma})R_{\alpha\nu\rho\sigma} = -2R_{\mu\nu}, \tag{8.3.9}$$

where we used (8.2.4). On the other hand, applying this to the right-hand side of (8.3.8) we get

$$R_{\mu\nu} = -\Sigma^i_\mu{}^\alpha F^i_{\alpha\nu}. \tag{8.3.10}$$

Thus, in particular,

$$R = \Sigma^{i\mu\nu}F^i_{\mu\nu}. \tag{8.3.11}$$

If one contracts both sides by Σ one can write F as a function of Σ and the Ricci scalar R,

$$F^i_{\mu\nu} = \Psi^{ij}\Sigma^j_{\mu\nu} - \frac{R}{6}\Sigma^i_{\mu\nu} + R_{[\mu}{}^\alpha\Sigma^i_{|\alpha|\nu]}. \tag{8.3.12}$$

Here Ψ^{ij} is the matrix of components of the chiral half of the Weyl curvature. Using $R_{\mu\nu} = \tilde{R}_{\mu\nu} + \frac{1}{4}Rg_{\mu\nu}$, where $\tilde{R}_{\mu\nu}$ is the trace-free part of the Ricci curvature, we can also rewrite this as

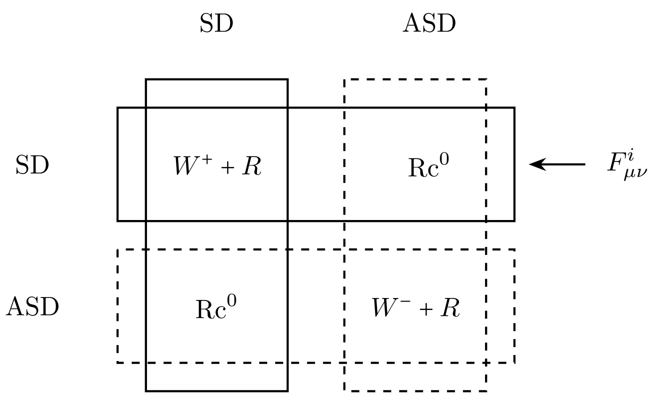
$$F^i_{\mu\nu} = \Psi^{ij}\Sigma^j_{\mu\nu} + \frac{R}{12}\Sigma^i_{\mu\nu} + \tilde{R}_{[\mu}{}^\alpha\Sigma^i_{|\alpha|\nu]}. \tag{8.3.13}$$

The first two terms here are self-dual as 2-forms, the last is anti-self-dual.

As is well-known, the Riemann curvature can be viewed as a symmetric endomorphism of $\Lambda^2(M)$. Then one can decompose curvature into its self-dual and anti-self-dual blocks. This can be usefully captured by the following matrix representation

$$\text{Curvature} = \begin{pmatrix} W^+ + R & \text{Rc}^0 \\ \text{Rc}^0 & W^- + R \end{pmatrix}. \tag{8.3.14}$$

Here $W^\pm$ are the two chiral halves of the Weyl curvature, and Rc^0 is the trace-free part of the Ricci tensor. The trace-part is denoted by R and is the scalar curvature. We re-characterise the above matrix to highlight properties of self-duality (SD) and anti-self-duality (ASD)



(8.3.15)

The first and second rows correspond in (8.3.15) to the $\{\rho, \sigma\}$ indices in the curvature tensor $R_{\mu\nu\rho\sigma}$. As these correspond to objects in $\Lambda^2(M)$, they split as self-dual and anti-self dual parts. Similarly, the first and second columns in (8.3.15) correspond to the $\{\mu, \nu\}$ indices in the curvature tensor $R_{\mu\nu\rho\sigma}$. This again splits in the same way as SD and ASD parts.

Thus, we see that the curvature $F^i_{\mu\nu}$ of the intrinsic torsion encodes precisely the first row of the matrix (8.3.14). This means one only has access to the self-dual part of the Weyl curvature,

W^+, but one does have access to *all* the Ricci curvature.

8.3.4 Einstein Condition

It is now clear that the Einstein condition can be encoded as one on the curvature $F^i_{\mu\nu}$. Indeed, the Einstein equation states

$$R_{\mu\nu} = \Lambda g_{\mu\nu}. \tag{8.3.16}$$

We then propose

Proposition 8.3.4.1. The Ricci curvature is Einstein, in the sense of (8.3.16), $\iff$ $F^i_{\mu\nu}$ is self-dual $\iff$ $\mathrm{Rc}^0 = 0$.

Proof. Using equation (8.3.10), one satisfies the Einstein condition (8.3.16) if, and only if,

$$F^i_{\mu\nu} = \left(\Psi^{ij} + \frac{R}{36}\delta^{ij}\right)\Sigma^j_{\mu\nu}. \tag{8.3.17}$$

This is because the Ricci tensor must vanish, see 8.3.15, else the ASD part remains. Here Ψ^{ij} is an arbitrary symmetric trace-free 3×3 matrix, which encodes the W^+ part of the curvature, and R the Ricci scalar. Comparing (8.3.17) to the general form of $F^i_{\mu\nu}$, as in (8.3.13), one sees the Einstein condition is equivalent to the anti-self-dual part, Rc^0 vanishing. Finally, if one takes $R = 12\Lambda$, then one recovers (8.3.16). □

8.4 Linearised Analysis

The purpose of this section is to consider perturbations of SU(2)-structures, and construct the most general diffeomorphism invariant Lagrangian for such perturbations. This linearised story provides a very good intuition for the non-linear story in the next section.

8.4.1 Perturbation of SU(2)-Structures

The tangent space to the $\mathrm{GL}(4,\mathbb{R})$ orbit of Σ^i contains irreducible representations $(\Lambda^2\otimes E)_{3+1+9}$, the subscript here is notation for the decomposition of the 12-dimensional space into an irreducible 1-dimensional trace part, an irreducible 9-dimensional trace-free symmetric part, and an irreducible 3-dimensional antisymmetric part. We can parametrise these spaces as

$$(\Lambda^2\otimes E)_{1+9} \ni 2h_{[\mu}{}^{\alpha}\Sigma^i_{|\alpha|\nu]}, \text{ and } (\Lambda^2\otimes E)_3 \ni 2\epsilon^{ijk}\Sigma^j_{\mu\nu}\xi^k, \tag{8.4.1}$$

with $h_{\mu\nu}$ being a symmetric tensor and $\xi^i \in E$. The role of the numerical factors chosen is to simplify some formulas that follow. This means that perturbations of $\Sigma^i_{\mu\nu}$, which we denote by $\delta\Sigma^i_{\mu\nu} \coloneqq \sigma^i_{\mu\nu}$ can be parametrised as

$$\sigma^i_{\mu\nu} = 2h_{[\mu}{}^{\alpha}\Sigma^i_{|\alpha|\nu]} + 2\epsilon^{ijk}\Sigma^j_{\mu\nu}\xi^k. \tag{8.4.2}$$

The inverse is given by

$$h_{\mu\nu} = -\frac{1}{2}\sigma^i_{(\mu}{}^{\alpha}\Sigma^i_{|\alpha|\nu)} - \frac{1}{12}\eta_{\mu\nu}\Sigma^{i\rho\sigma}\sigma^i_{\rho\sigma}, \text{ and } \xi^i = -\frac{1}{16}\epsilon^{ijk}\Sigma^{j\mu\nu}\sigma^k_{\mu\nu}. \tag{8.4.3}$$

Transformation Properties under Diffeomorphisms

Let us consider a background triple of 2-forms Σ^i. The diffeomorphisms act $\delta_X\Sigma^i = \mathcal{L}_X\Sigma^i = i_X d\Sigma^i + d i_X\Sigma^i$. In the case of a constant triple of 2-forms we get, $\delta_X\sigma^i = d i_X\Sigma^i$. In index notation

$$\delta_X \sigma^i_{\mu\nu} = 2\partial_{[\mu} X^\alpha \Sigma^i_{|\alpha|\nu]}. \tag{8.4.4}$$

This means that

$$\delta_X h_{\mu\nu} = \partial_{(\mu} X_{\nu)}, \text{ and } \delta_X \xi^i = \frac{1}{4}\Sigma^{i\mu\nu}\partial_\mu X_\nu. \tag{8.4.5}$$

Transformation Properties under SU(2)

In addition to diffeomorphisms, we can also consider how quantities transform under the SU(2) transformations that rotate Σ^i. The infinitesimal version of these transformations is

$$\delta_\phi \sigma^i_{\mu\nu} = 2\epsilon^{ijk}\Sigma^j_{\mu\nu}\phi^k. \tag{8.4.6}$$

Under these transformations

$$\delta_\phi h_{\mu\nu} = 0, \text{ and } \delta_\phi \xi^i = \phi^i. \tag{8.4.7}$$

8.4.2 Second-order Action Functional

We now determine the most general diffeomorphism invariant linear action functional that can be written in terms of fields $h_{\mu\nu}$ and ξ^i, subject to the transformation properties (8.4.5). We first write the general linear combination of all possible terms. The types of terms are dictated by simple representation theory. So that one is free to integrate by parts, later in this section, it is assumed that the 4-dimensional spin manifold, M, in question is compact.

Theorem 8.4.2.1. The most general linear second-order in derivatives Lagrangian that one can write is

$$\mathcal{L} = \frac{\rho}{2}(\partial_\mu h_{\nu\rho})^2 + \frac{\alpha}{2}(\partial_\mu h)^2 - \beta h \partial^\mu\partial^\nu h_{\mu\nu} - \gamma(\partial^\mu h_{\mu\nu})^2 + \frac{\lambda}{2}(\partial_\mu \xi^i)^2 + \mu(\partial^\mu h_{\mu\nu})(\partial^\alpha \xi^i)\Sigma^{i\ \nu}_{\alpha}. \tag{8.4.8}$$

Proof. First, we can write the most general linear combination of terms that can be constructed solely from $h_{\mu\nu}$. This is standard and independent of the dimension. We write this as

$$\frac{\rho}{2}(\partial_\mu h_{\nu\rho})^2 + \frac{\alpha}{2}(\partial_\mu h)^2 - \beta h \partial^\mu\partial^\nu h_{\mu\nu} - \gamma(\partial^\mu h_{\mu\nu})^2. \tag{8.4.9}$$

We then need to determine all possible terms involving two copies of ξ^i, as well as $h\xi$ terms. The fields h and ξ are characterised as $h_{\mu\nu} \in (S^2_+ \otimes S^2_-) \oplus C^\infty(M)$, and $\xi^i \in S^2_+$. We have the following tensor products,

$$(S^2_+ \otimes S^2_-) \otimes S^2_+ = (S^4_+ \otimes S^2_-) \oplus (S^2_+ \otimes S^2_-) \oplus S^2_-, \text{ and } S^2_+ \odot S^2_+ = S^4_+ \oplus C^\infty(M). \tag{8.4.10}$$

We need to combine these irreducible pieces with those arising from the symmetrised product of two partial derivatives, which is in $S^2_+ \otimes S^2_- \oplus C^\infty(M)$. This makes it clear that the only term that can be constructed from two copies of ξ^i is $(\partial_\mu \xi^i)^2$. There is also just a single term that can be constructed from $h_{\mu\nu}$ and ξ^i, which is

$$(\partial^\mu h_{\mu\nu})(\partial^\alpha \xi^i)\Sigma^{i\ \nu}_{\alpha}. \tag{8.4.11}$$

The result then follows from summing all independent pieces. □

We now calculate the effect of a diffeomorphism on $\mathcal{L}$, and find the following,

Corollary 8.4.2.1.1. Let $\mathcal{L}$ be the most general linear second order theory as written in theorem 8.4.2.1. The most general diffeomorphism invariant Lagrangian is

$$\mathcal{L} = \rho\mathcal{L}_{GR} + \mu\mathcal{L}', \text{ for } \rho, \mu \in \mathbb{R}. \tag{8.4.12}$$

Such that

$$\begin{aligned}\mathcal{L}_{GR} &= \frac{1}{2}(\partial_\mu h_{\nu\rho})^2 - \frac{1}{2}(\partial_\mu h)^2 - h\partial^\mu\partial^\nu h_{\mu\nu} - (\partial^\mu h_{\mu\nu})^2,\\ \mathcal{L}' &= -\frac{1}{4}(\partial_\mu h)^2 - \frac{1}{2}h\partial^\mu\partial^\nu h_{\mu\nu} - \frac{1}{4}(\partial^\mu h_{\mu\nu})^2 - (\partial_\mu \xi^i)^2 + (\partial^\mu h_{\mu\nu})(\partial^\alpha \xi^i)\Sigma^{i\ \nu}_{\alpha}.\end{aligned} \tag{8.4.13}$$

Proof. We integrate by parts where necessary (using the assumption that M is compact), and use the symbol $\approx$ to denote equality up to modulo integration by parts. Then computing the diffeomorphism of $\mathcal{L}$ one has

$$\begin{aligned}\delta_X\mathcal{L} \approx{}& (\rho - \gamma + \frac{\mu}{4})\partial^2(\partial^\mu h_{\mu\nu})X^\nu - (\alpha+\beta)\partial^2 h(\partial X)\\ &+ (-\beta + \gamma + \frac{\mu}{4})(\partial X)(\partial^\mu\partial^\nu h_{\mu\nu}) - (\frac{\lambda}{4} + \frac{\mu}{2})\partial^2\xi^i\Sigma^{i\mu\nu}\partial_\mu X_\nu.\end{aligned} \tag{8.4.14}$$

Equating the coefficients in front of the independent terms to zero, and parametrising the solution by ρ, μ we have

$$\beta = -\alpha = \rho + \frac{\mu}{2}, \ \gamma = \rho + \frac{\mu}{4}, \ \text{and } \lambda = -2\mu. \tag{8.4.15}$$

□

What we see is that the space of diffeomorphism invariant theories is two-dimensional. There is a classical part, the linearised theory of the background metric seen from general relativity. Furthermore, there is a non-classical part due to the extra field ξ inherent from discussing SU(2)-structures. It can couple to both the metric and itself, resulting in a modified theory of general relativity.

We note that this is a very similar story to what happens in the case of Spin(7)-structures in eight dimensions, see [Kra24a]. In that context, as here, the most general diffeomorphism invariant Lagrangian is also given by a linear combination of two terms.

We further comment that in SU(3), see chapter 9, the most general theory is no longer a sum of two diffeomorphism invariant Lagrangians. In SU(2) and Spin(7), only a vector field can couple to the classical theory, but in 6 dimensions there are many more fields that can couple.

The Lagrangian (8.4.12) is diffeomorphism invariant (modulo integration by parts). It is also invariant under global (i.e. rigid) SO(3) rotations acting on E. However, it is clear that it is also possible to demand local SO(3) invariance. From (8.4.7) we see that these transformations act only on ξ^i. It is clear that the Lagrangian $\mathcal{L}'$ is not invariant under such local transformations. Therefore, only $\mathcal{L}_{GR}$ is both diffeomorphism *and* SU(2) gauge invariant. It is therefore to be expected that there exists a unique non-linear Lagrangian for $\Sigma^i_{\mu\nu}$, which is second order in derivatives, with both diffeomorphism and SU(2) gauge invariance. We can also expect this non-linear action to have critical points that are Einstein metrics. This is exactly what happens, as we shall now verify.

8.5 Action Functionals

In preparation to the construction of the action, we will first show that the intrinsic torsion is completely determined by the exterior derivative $\mathrm{d}\Sigma^i$.

8.5.1 Torsion as Exterior Derivatives

In (8.3.1) we have related the torsion A^i_μ to the covariant derivative $\nabla_\mu\Sigma^i_{\alpha\beta}$ of the 2-forms Σ^i. We now explain that the knowledge of the exterior derivative is sufficient,

Theorem 8.5.1.1. The intrinsic torsion is determined by the exterior derivatives of the 2-forms Σ^i. Specifically, we have

$$A = \frac{1}{4}(J_\Sigma - \mathbb{I})(*\mathrm{d}\Sigma^i), \tag{8.5.1}$$

where J_Σ is the operator from definition (8.2.2.1) and $*\mathrm{d}\Sigma^i$ is the Hodge dual of the 3-form $\mathrm{d}\Sigma^i$.

Proof. We project the equation (8.3.1) to the space of 3-forms, anti-symmetrising over all 3 indices. We have

$$\partial_{[\nu}\Sigma^i_{\alpha\beta]} = -\epsilon^{ijk} A^j_{[\nu}\Sigma^k_{\alpha\beta]}. \tag{8.5.2}$$

We can write this in index-free differential form notations as

$$\mathrm{d}\Sigma^i + \epsilon^{ijk} A^j \Sigma^k = 0. \tag{8.5.3}$$

To solve this, we multiply with the ϵ tensor and use the self-duality of $\Sigma^i_{\mu\nu}$

$$\epsilon^{\mu\nu\alpha\beta}\partial_\nu\Sigma^i_{\alpha\beta} = -2\epsilon^{ijk}A^j_\nu\Sigma^{k\mu\nu} = 2(J_\Sigma(A))^{i\mu}, \tag{8.5.4}$$

where J_Σ is the operator on $\Lambda^1(M)\otimes E$ that was introduced in (8.2.2.1). We can write this in an index-free way as

$$*\mathrm{d}\Sigma^i = 2J_\Sigma(A). \tag{8.5.5}$$

The J_Σ operator is invertible, with the inverse given by

$$J_\Sigma^{-1} = \frac{1}{2}(J_\Sigma - \mathbb{I}). \tag{8.5.6}$$

This establishes (8.5.1). □

We have an immediate, well-known corollary.

Corollary 8.5.1.1.1. An SU(2)-structure is integrable if, and only if, $\mathrm{d}\Sigma^i = 0$.

Finally, let us use (8.5.1) to construct an indicial expression as follows

Corollary 8.5.1.1.2. Let $A \in \Lambda^1(M)\otimes E$ be the intrinsic torsion from a set of SU(2)-structures Σ^i, then

$$A^i_\mu = -\frac{1}{4}\epsilon_\mu{}^{\alpha\beta\gamma}\partial_\alpha\Sigma^i_{\beta\gamma} - \frac{1}{4}\epsilon^{ijk}\Sigma^{j\alpha\beta}\partial_\mu\Sigma^k_{\alpha\beta} - \frac{1}{2}\epsilon^{ijk}\Sigma^{j\alpha\beta}\partial_\beta\Sigma^k_{\mu\alpha}. \tag{8.5.7}$$

Proof. We have

$$(*\mathrm{d}\Sigma)^i_\mu = \epsilon_\mu{}^{\alpha\beta\gamma}\partial_\alpha\Sigma^i_{\beta\gamma}, \tag{8.5.8}$$

and

$$A^i_\mu = \frac{1}{4}(J_\Sigma - \mathbb{I})(*\mathrm{d}\Sigma)^i_\mu = -\frac{1}{4}\epsilon_\mu{}^{\alpha\beta\gamma}\partial_\alpha\Sigma^i_{\beta\gamma} + \frac{1}{4}\epsilon^{ijk}\Sigma^j_\mu{}^\alpha\epsilon_\alpha{}^{\beta\gamma\delta}\partial_\beta\Sigma^k_{\gamma\delta}. \tag{8.5.9}$$

We can simplify the last term using

$$\epsilon^{\mu\nu\rho\sigma}\Sigma^i_{\alpha\sigma} = 3\delta^{[\rho}_\alpha\Sigma^{i\mu\nu]}. \tag{8.5.10}$$

□

We shall find this expression is useful for the action functionals described below.

8.5.2 Bianchi Identity

One can now derive a useful Bianchi identity in terms of the curvature F and torsion A

Corollary 8.5.2.0.1. Let the intrinsic torsion be parameterised by $A \in \Lambda^1(M) \otimes E$. One can write

$$\epsilon^{ijk} F^j \Sigma^k = 0, \tag{8.5.11}$$

where,

$$\Lambda^2(M) \otimes E \ni F^i = \mathrm{d}A^i + \frac{1}{2}\epsilon^{ijk} A^j A^k. \tag{8.5.12}$$

Proof. A useful consequence of (8.5.3), obtained by taking its exterior derivative is

$$\epsilon^{ijk}\mathrm{d}A^j \Sigma^k - \epsilon^{ijk} A^j \mathrm{d}\Sigma^k = 0. \tag{8.5.13}$$

We now substitute $\mathrm{d}\Sigma^k$ from (8.5.3) as $\mathrm{d}\Sigma^k = -\epsilon^{klm} A^l \Sigma^m$, we then use $A^j A^l = (1/2)\epsilon^{jls}\epsilon^{spq} A^p A^q$ to rewrite

$$\epsilon^{ijk} A^j \epsilon^{klm} A^l \Sigma^m = \epsilon^{ijk}\left(\frac{1}{2}\epsilon^{jlm} A^l A^m\right)\Sigma^k. \tag{8.5.14}$$

The result follows from rearranging the above expression, using identities for contractions of the ϵ tensor and that the curvature of A can be written $F^i = \mathrm{d}A^i + \frac{1}{2}\epsilon^{ijk} A^j A^k$. □

We note that (8.5.11) can be interpreted as the statement that there is no S^2_+ component in the decomposition of the $F \in E \otimes \Lambda^2$ into its irreducible components.

8.5.3 Transformation Properties under Diffeomorphisms

Under infinitesimal diffeomorphisms along a vector field X, $\delta_X \Sigma^i = \mathrm{d}i_X \Sigma^i + i_X \mathrm{d}\Sigma^i$. We now assume that the intrinsic torsion solves (8.5.3) and determine how it transforms under diffeomorphisms. We have the following theorem,

Theorem 8.5.3.1. Let $A \in \Lambda^1(M) \otimes E$ be the intrinsic torsion of the SU(2)-structure $\{\Sigma^i\}$. Then for a vector field $X \in TM$, A transforms covariantly under diffeomorphisms i.e.

$$\delta_X A^i = \mathrm{d}i_X A^i + i_X \mathrm{d}A^i. \tag{8.5.15}$$

Proof. Taking the variation of (8.5.3) we have

$$\mathrm{d}(i_X \mathrm{d}\Sigma^i) + \epsilon^{ijk}\delta_X A^j \Sigma^k + \epsilon^{ijk} A^j(\mathrm{d}i_X \Sigma^i + i_X \mathrm{d}\Sigma^i) = 0. \tag{8.5.16}$$

We can also insert the vector field X into (8.5.3) to get

$$i_X \mathrm{d}\Sigma^i + \epsilon^{ijk}(i_X A^j)\Sigma^k - \epsilon^{ijk} A^j i_X \Sigma^k = 0. \tag{8.5.17}$$

Substituting $i_X \mathrm{d}\Sigma^i$ from here into (8.5.16) we have

$$\begin{aligned} \mathrm{d}(\epsilon^{ijk} A^j i_X \Sigma^k - \epsilon^{ijk}(i_X A^j)\Sigma^k) + \epsilon^{ijk}\delta_X A^j \Sigma^k + \epsilon^{ijk} A^j d i_X \Sigma^k \\ + \epsilon^{ijk} A^j(\epsilon^{klm} A^l i_X \Sigma^m - \epsilon^{klm}(i_X A^l)\Sigma^m) = 0. \end{aligned} \tag{8.5.18}$$

The terms in the first line become

$$\epsilon^{ijk}\mathrm{d}A^j i_X \Sigma^k - \epsilon^{ijk}\mathrm{d}(i_X A^j)\Sigma^k + \epsilon^{ijk}(i_X A^j)\epsilon^{klm} A^l \Sigma^m + \epsilon^{ijk}\delta_X A^j \Sigma^k, \tag{8.5.19}$$

where we have used (8.5.3) again. The first term in the second line can also be simplified. We again use $A^j A^l = (1/2)\epsilon^{jls}\epsilon^{spq} A^p A^q$ to get

$$\epsilon^{ijk} A^j \epsilon^{klm} A^l i_X \Sigma^m = \epsilon^{ijk}(\frac{1}{2}\epsilon^{jlm} A^l A^m) i_X \Sigma^k. \tag{8.5.20}$$

This means that (8.5.18) can be rewritten as

$$\epsilon^{ijk} F^j i_X \Sigma^k + \epsilon^{ijk}(\delta_X A^j - \mathrm{d}(i_X A^j))\Sigma^k \\ + \epsilon^{ijk}(i_X A^j)\epsilon^{klm} A^l \Sigma^m - \epsilon^{ilk} A^l \epsilon^{kjm}(i_X A^j)\Sigma^m = 0, \tag{8.5.21}$$

where we changed the names of the dummy indices suggestively. The last two terms can be simplified using the identity

$$\epsilon^{ijk}\epsilon^{klm} + \epsilon^{ilk}\epsilon^{kmj} + \epsilon^{imk}\epsilon^{kjl} = 0. \tag{8.5.22}$$

This gives,

$$\epsilon^{ijk} F^j i_X \Sigma^k + \epsilon^{ijk}(\delta_X A^j - \mathrm{d}(i_X A^j) - \epsilon^{jpq} A^p (i_X A^q))\Sigma^k = 0. \tag{8.5.23}$$

We can finally insert X into (8.5.11) to rewrite the first term here as $-\epsilon^{ijk} i_X F^j \Sigma^k$. Overall, this produces terms that are all of the type of operator J_Σ acting on an E-valued 1-form. The operator J_Σ is invertible, which allows us to write

$$\delta_X A^i = \mathrm{d}(i_X A^i) + \epsilon^{ijk} A^j (i_X A^k) + i_X F^i. \tag{8.5.24}$$

The first two terms here assemble into the covariant derivative of $i_X A^i$, computed using the connection A^i. The last term is the insertion of X into the curvature F^i. Using the formula (8.5.12) for F^i, and noting a cancellation, one achieves the desired result. □

It should be noted that (8.5.24) is a very useful formula that we will need below.

8.5.4 Transformation Properties under SU(2) Gauge Transformations

Let us also determine how the torsion transforms under the local SU(2) gauge transformations.

Theorem 8.5.4.1. Let $A \in \Lambda^1(M) \otimes E$ be the intrinsic torsion of the SU(2)-structures Σ^i. Then under infinitesimal gauge transformations, $\delta_\phi \Sigma^i = \epsilon^{ijk}\phi^k \Sigma^k$, A transforms in the usual way with the parameter $-\phi^i$ as

$$\delta_\phi A^i = -\mathrm{d}\phi^i - \epsilon^{ijk} A^j \phi^k, \tag{8.5.25}$$

Proof. Taking the variation of (8.5.3) we have

$$\mathrm{d}(\epsilon^{ijk}\phi^j \Sigma^k) + \epsilon^{ijk}\delta_\phi A^j \Sigma^k + \epsilon^{ijk} A^j \epsilon^{klm}\phi^l \Sigma^m = 0. \tag{8.5.26}$$

The first term gives a contribution containing $\mathrm{d}\phi^i$, as well as one with $\mathrm{d}\Sigma^i$. The latter can be transformed using (8.5.3). This gives

$$\epsilon^{ijk}(\delta_\phi A^j + \mathrm{d}\phi^j)\Sigma^k + \epsilon^{ijk} A^j \epsilon^{klm}\phi^l \Sigma^m - \epsilon^{ijk}\phi^j \epsilon^{klm} A^l \Sigma^m = 0. \tag{8.5.27}$$

The last two terms can again be transformed using (8.5.22). This puts all terms in the same form of J_Σ acting on an E-valued 1-form. Because J_Σ is invertible, we get the desired result under the usual gauge transformation with parameter $-\phi^i$. □

One then has the following corollary regarding gauge transformations of the curvature F.

Corollary 8.5.4.1.1. Let $F \in \Lambda^2(M) \otimes E$ be the curvature of the torsion, $A \in \Lambda^1(M) \otimes E$. Then F^i transforms covariantly, i.e.

$$\delta_\phi F^i = \epsilon^{ijk}\phi^j F^k. \tag{8.5.28}$$

8.5.5 Diffeomorphism Invariant Action

We have confirmed that the torsion transforms covariantly under diffeomorphisms. This means that any action that is schematically of the type $\int A^2$ is diffeomorphism invariant. Now, the representation theoretic decomposition (8.2.10), (8.2.11) of $A \in \Lambda^1 \otimes E$ shows that there are two irreducible components of the intrinsic torsion. This means that there are only two quadratic invariants that can be constructed from A. One can always take as a basis of such invariants the quantities $(A^i_\mu)^2$ and $A^i_\mu J_\Sigma(A)^{i\mu}$. This leads to the following theorem,

Theorem 8.5.5.1. The most general diffeomorphism invariant action that can be constructed is

$$S[A(\Sigma)] = a \int (A^i_\mu)^2 + b \int A^i_\mu J_\Sigma(A)^{i\mu}, \tag{8.5.29}$$

where $a, b \in \mathbb{R}$.

It can be confirmed that the linearisation of this general diffeomorphism invariant action (8.5.29) coincides with the linearised action (8.4.12).

8.5.6 Diffeomorphism and SU(2) Invariant Action

Let us now impose the requirement that the action is both diffeomorphism and SU(2) gauge invariant. At the linearised level, we have seen that this has the effect that only one of the two diffeomorphism invariant terms survives, and one gets linearised Einstein-Hilbert action. It is clear that from the two terms A^2 and $AJ_\Sigma(A)$ the first one is not gauge invariant. Using physics terminology, this term is a mass term for the connection, which cannot be gauge invariant. Let us discuss the other term.

Corollary 8.5.6.0.1. The action,

$$S[\Sigma] = \int A(\Sigma)^i_\mu J_\Sigma(A(\Sigma))^{i\mu}, \tag{8.5.30}$$

is diffeomorphism *and* SU(2) invariant.

Proof. To see this, it is best to rewrite it using some integration by parts identities. Consider $\int \Sigma^i \mathrm{d}A^i$. Integrating by parts, assuming M is compact, we have

$$\int \Sigma^i \mathrm{d}A^i \approx -\int \mathrm{d}\Sigma^i A^i = -\int \epsilon^{ijk} A^i A^j \Sigma^k. \tag{8.5.31}$$

In the last equality we have used (8.5.3). The quantity on the right-hand side is a multiple of $AJ_\Sigma(A)$. This means that

$$\int \Sigma^i F^i = \int \Sigma^i (\mathrm{d}A^i + \frac{1}{2}\epsilon^{ijk} A^j A^k) \approx -\frac{1}{2}\int \Sigma^i \epsilon^{ijk} A^j A^k. \tag{8.5.32}$$

The integrand on the left is built from objects that transform covariantly under local SU(2) gauge transformations and is invariant under them. The integral is then both diffeomorphism and gauge invariant. This means that this is also the case for the object on the left-hand side. This establishes that there is a unique action for SU(2)-structures in dimension four that is both diffeomorphism and SU(2) gauge invariant. □

Our action has the schematic of torsion squared, and expanding the definition of $J_\Sigma(A)$ gives

$$S[\Sigma] = -\frac{1}{2}\int_M \Sigma^i \epsilon^{ijk} A^j(\Sigma) A^k(\Sigma). \tag{8.5.33}$$

As we shall see below (8.5.33) is the *best* action one to write. *Best*, in the sense that, it is the *unique* diffeomorphism *and* gauge invariant action, and, as we shall see below, its critical points give Einstein conditions on the curvature. One can substitute the expression for $A(\Sigma)$ given by

(8.5.7) to obtain a second order in derivatives action for Σ. If one supplements (8.5.33) with a constraint term

$$-\frac{1}{2}\left(\Psi^{ij}+\frac{\Lambda}{3}\delta^{ij}\right)\Sigma^i\Sigma^j, \tag{8.5.34}$$

then this action is exactly the same as Plebanski's action [Ple77], see below. Hence,(8.5.33) is a reformulation of his original theory, again see below, theorem (8.5.6.1).

Plebanski Action and Einstein Condition

We can now discuss the Plebanski action, which is a first-order in derivatives version of (8.5.33). The idea is to write an action that is a functional of both Σ^i and an independent E-valued one-form field A^i, such that the Euler-Lagrange equations for A^i coincide with (8.5.3). A suitable candidate is $\int \Sigma^i F^i$. One can then to supplement by a constraint terms that guarantee that Σ^i satisfy the quaterion algebra. One is also free to add to this action the volume form with an arbitrary coefficient. Plebanski [Ple77] was the first to consider such an action given as,

$$S[\Sigma,A,\Psi]=\int_M \Sigma^i(\mathrm{d}A^i+\frac{1}{2}\epsilon^{ijk}A^jA^k)-\frac{1}{2}\left(\Psi^{ij}+\frac{\Lambda}{3}\delta^{ij}\right)\Sigma^i\Sigma^j. \tag{8.5.35}$$

Here, Ψ^{ij} is an arbitrary traceless symmetric 3×3 matrix, whose components serve as Lagrange multipliers to impose the constraints $\Sigma^i\Sigma^j\sim\delta^{ij}$. Indeed, the variation with respect to the field Ψ^{ij} gives $\Sigma^i\Sigma^j\sim\delta^{ij}$, which are the algebraic conditions that need to be satisfied by an SU(2)-structure defining 2-forms Σ^i. It is also not difficult to see that its Euler-Lagrange equation arising by varying with respect to A^i is precisely (8.5.3), and the Euler-Lagrange equation arising by varying with respect to Σ^i is precisely (8.3.17).

This establishes the following

Theorem 8.5.6.1. The critical points of (8.5.33), or equivalently of (8.5.35), are SU(2)-structures whose associated metric is Einstein.

Chapter 9

SU(3)-structures in 6 dimensions

In 6 dimensions, every spinor is pure. This corresponds to the same orbit SU(3) (see Section 4.3). We shall not reconstruct the geometric map from scratch, as was done in Chapter 8. This is because, in 4 dimensions, we wanted to highlight the Hyper-Kähler nature of SU(2)-structures.

In 6 dimensions, there are two differential forms that characterise the SU(3)-structure: ω, the symplectic form, and Ω, the complex volume form in $\Lambda^{3,0}(M)$. [Koe11] gives a formula to generate the complex structure regardless of the dimension d as

$$J^i{}_j = c\varepsilon^{il_1\dots l_{d-1}}\mathrm{Re}(\Omega)_{jl_1\dots l_{(d/2)-1}}\mathrm{Re}(\Omega)_{l_{d/2}\dots l_{d-1}}. \tag{9.0.1}$$

c is an appropriate normalisation, and ε is the antisymmetric symbol that is ± 1. As Ω is constructed from a general Weyl spinor, its open orbits are the full space of 3-forms. In 6 dimensions, as seen through [Hit00a], this is given as $\mathrm{GL}(6,\mathbb{R})/\mathrm{SU}(3)$. In $d > 6$, this construction is still possible as seen by (9.0.1), however, the orbit of $\rho = \mathrm{Re}(\Omega)$ cannot be be open, similar to Spin(7)-structures in 8 dimensions [Kar08]. Concretely, in 6 dimensions define a basis $\{\mathrm{d}w^1,\dots,\mathrm{d}w^6\}$ on T^*M. This allows our almost symplectic structure and complex volume form to be specified as follows

$$\begin{aligned}\omega_0 &= \mathrm{d}w^1\wedge\mathrm{d}w^2 + \mathrm{d}w^3\wedge\mathrm{d}w^4 + \mathrm{d}w^5\wedge\mathrm{d}w^6, \text{ and}\\ \Omega_0 &= (\mathrm{d}w^1 - \mathbf{i}\mathrm{d}w^2)\wedge(\mathrm{d}w^3 - \mathbf{i}\mathrm{d}w^4)\wedge(\mathrm{d}w^5 - \mathbf{i}\mathrm{d}w^6).\end{aligned} \tag{9.0.2}$$

Using (9.0.1) the almost complex structure J_0 is found to be,

$$J_0(\mathrm{d}w^1) = \mathrm{d}w^2,\ J_0(\mathrm{d}w^3) = \mathrm{d}w^4, \text{ and } J_0(\mathrm{d}w^5) = \mathrm{d}w^6. \tag{9.0.3}$$

Leading to the identification,

$$\Omega_0 = \mathrm{Re}(\Omega_0) + \mathbf{i}\mathrm{Im}(\Omega_0) = \mathrm{Re}(\Omega_0) + \mathbf{i}J_0\mathrm{Re}(\Omega_0). \tag{9.0.4}$$

Finally, on each fibre we define the metric to be

$$g_0(X,Y) \coloneqq \omega_0(X, JY). \tag{9.0.5}$$

Like in the previous chapter, we also assume that our spin manifold is compact so that we may exploit integration by parts.

9.0.1 ϵ identities

We shall now define $\mathrm{Re}(\Omega_0) \coloneqq \mathcal{C}_0$ and $J_0\mathrm{Re}(\Omega_0) \coloneqq \hat{\mathcal{C}}_0$. The following identities are found in [BV07] and are used tacitly. We give them here for completeness and because we change notation. Define the following

$$\mathcal{C}_0 = \mathcal{C}_{ijk}\mathrm{d}w^{ijk},\ \hat{\mathcal{C}}_0 = \hat{\mathcal{C}}_{ijk}\mathrm{d}w^{ijk},\ \text{and } \omega_0 = \omega_{ij}\mathrm{d}w^{ij}. \tag{9.0.6}$$

Here $w^{i_1 i_2 \cdots} \coloneqq w^{i_1} \wedge w^{i_2} \ldots$ This gives the following identities

$$\begin{aligned}
&\mathcal{C}_{ipq}\omega_{pq} = 0,\ \hat{\mathcal{C}}_{ipq}\mathcal{C}_{jpq} = -4\omega_{ij},\ \omega_{ip}\omega_{pj} = -\delta_{ij},\ \mathcal{C}_{ijp}\omega_{pr} = \hat{\mathcal{C}}_{ijr},\\
&\hat{\mathcal{C}}_{ijp}\mathcal{C}_{klp} = -\omega_{ik}\delta_{jl} + \omega_{jk}\delta_{il} + \omega_{il}\delta_{jk} - \omega_{jl}\delta_{ik},\\
&\hat{\mathcal{C}}_{ijp}\omega_{pr} = -\mathcal{C}_{ijr},\ \text{and } \mathcal{C}_{ijp}\mathcal{C}_{klp} = \hat{\mathcal{C}}_{ijp}\hat{\mathcal{C}}_{klp} = -\omega_{ik}\omega_{jl} + \omega_{jk}\omega_{il} - \delta_{il}\delta_{jk} + \delta_{jl}\delta_{ik}.
\end{aligned} \tag{9.0.7}$$

9.1 Decomposition of Exterior Algebra

Let M with the data (ω, Ω) be a 6-dimensional manifold. Then Ω allows the construction of an almost complex structure J. Hence, the 6-dimensional tangent bundle TM can be split into $\pm\mathbf{i}$ eigenspaces of J, labelled $T^{1,0}M$ and $T^{0,1}M$, respectively[1]. Explicitly if $\left\{\frac{\partial}{\partial w^i} \mid i \in \{1,2,3,4,5,6\}\right\}$ is a real basis for a local section in TM then locally J is an endomorphism such that

$$J\left(\frac{\partial}{\partial w^l}\right) = \frac{\partial}{\partial w^{l+3}},\ \text{and } J\left(\frac{\partial}{\partial w^{i+3}}\right) = -\frac{\partial}{\partial w^i},\ \text{for } l = 1,2,3. \tag{9.1.1}$$

It is not hard to see that it squares to minus the identity map. Define in this neighbourhood $z^\mu \coloneqq x^\mu + \mathbf{i}y^\mu$, and $\bar{z}^\mu$ the complex conjugate. Then define

$$\frac{\partial}{\partial z^\mu} \coloneqq \frac{1}{2}\left(\frac{\partial}{\partial w^\mu} - \mathbf{i}\frac{\partial}{\partial w^{\mu+3}}\right) \implies \frac{\partial}{\partial \bar{z}^\mu} \coloneqq \frac{1}{2}\left(\frac{\partial}{\partial w^\mu} + \mathbf{i}\frac{\partial}{\partial w^{\mu+3}}\right), \tag{9.1.2}$$

so that

$$J\left(\frac{\partial}{\partial z^\mu}\right) = \mathbf{i}\frac{\partial}{\partial z^\mu},\ \text{and } J\left(\frac{\partial}{\partial \bar{z}^\mu}\right) = -\mathbf{i}\frac{\partial}{\partial \bar{z}^\mu}. \tag{9.1.3}$$

The data $\{\omega, \Omega\}$ gives rise to a hermitian metric by (9.0.5). So we are able to write, in complex coordinates,

$$g = J_\alpha{}^\mu \omega_{\mu\bar{\beta}}(\mathrm{d}z^\alpha \otimes \mathrm{d}\bar{z}^\beta + \mathrm{d}\bar{z}^\beta \otimes \mathrm{d}z^\alpha) \coloneqq g_{\alpha\bar{\beta}}\mathrm{d}z^\alpha \odot \mathrm{d}\bar{z}^\beta, \tag{9.1.4}$$

The symmetry $g_{\alpha\bar{\beta}} = g_{\beta\bar{\alpha}}$, and the relation $g = \mathbf{i}\omega$ is apparent. In other words, lowering an index by g is equivalent to interchanging between barred to unbarred coordinates, similarly the inverse form, g^{-1} i.e. $g^{\bar{\alpha}\beta}$, raises the index, performing the same interpolation. We give the following definition.

Definition 9.1.0.1. Consider $A \in \mathcal{T}^{0,2} \cong T^*M \otimes T^*M$. This space is reducible under J. We then define the symmetric, antisymmetric, hermitian and skew-hermitian pieces of A as

$$\begin{aligned}
A = &M_{\mu\nu}\mathrm{d}z^\mu \wedge \mathrm{d}z^\nu + M_{\bar{\mu}\bar{\nu}}\mathrm{d}\bar{z}^\mu \wedge \mathrm{d}\bar{z}^\nu + N_{\mu\nu}\mathrm{d}z^\mu\mathrm{d}z^\nu + N_{\bar{\mu}\bar{\nu}}\mathrm{d}\bar{z}^\mu\mathrm{d}\bar{z}^\nu\\
&+ C_{\mu\bar{\nu}}\mathrm{d}z^\mu \wedge \mathrm{d}\bar{z}^\nu + S_{\mu\bar{\nu}}\mathrm{d}z^\mu\mathrm{d}\bar{z}^\nu.
\end{aligned} \tag{9.1.5}$$

Here, $M \in \Lambda^{2,0}$, $\bar{M} \in \Lambda^{0,2}$, $N \in S^{2,0}$, $\bar{N} \in S^{0,2}$, $C \in [[\Lambda^{1,1}]]$, and $S \in [[S^{1,1}]]$. $\Lambda^{p,q}$ are complexified antisymmetric (p,q)-tensors, while $S^{p,q}$ are complexified symmetric (p,q)-tensors, and $[[W]]$ denotes the real part of a complexified vector space W.

The last line in (9.1.5) is the most interesting to analyse. In the space of hermitian forms, there is the line element given by the "trace", i.e., the hermitian metric g. This separates reduced hermitian forms into "trace-free" hermitian forms plus the metric, i.e.,

$$S = (s_{\mu\bar{\nu}} + g_{\mu\bar{\nu}})\,\mathrm{d}z^\mu\mathrm{d}\bar{z}^\nu \in S_0^{1,1} \oplus \mathbb{R}g. \tag{9.1.6}$$

[1] Similar notation throughout this chapter is used and commonly associated with complex geometry, see [Sal89].

Similarly, noting that $\omega = -\mathbf{i}g$ is skew-hermitian, a similar decomposition can be made. This skew-hermitian tensor is the $(1,1)$-form compatible with the complex structure, J. Hence,

$$C = (c_{\mu\bar{\nu}} + \mathbf{i}g_{\mu\bar{\nu}})\,\mathrm{d}z^\mu \wedge \mathrm{d}\bar{z}^\nu \in \Lambda^{1,1}_0 \oplus \mathbb{R}\omega. \tag{9.1.7}$$

This further decomposition comes from the reduction of $\mathfrak{u}(3)$ to $\mathfrak{su}(3)$ structures. These specify a complex 3-form Ω and its complex conjugate $\bar{\Omega}$. Thus, there is a refinement to the decomposition of $\mathcal{T}^{0,2}$:

$$\mathcal{T}^{0,2} = \Lambda^{2,0} \oplus S^{2,0} \oplus \Lambda^{1,1}_0 \oplus S^{1,1}_0 \oplus \mathbb{R}g \oplus \mathbb{R}\omega. \tag{9.1.8}$$

Lemma 9.1.0.1. Consider the M, N, etc. as defined in (9.1.5). Then one has

$$\begin{aligned}
M &= \frac{1}{2}M_{\mu\nu}\left(g^{\nu\bar{\sigma}}\mathrm{d}z^\mu \otimes \frac{\partial}{\partial\bar{z}^\sigma} - g^{\mu\bar{\sigma}}\mathrm{d}z^\nu \otimes \frac{\partial}{\partial\bar{z}^\sigma}\right),\\
\bar{M} &= \frac{1}{2}M_{\bar{\mu}\bar{\nu}}\left(g^{\bar{\nu}\sigma}\mathrm{d}\bar{z}^\mu \otimes \frac{\partial}{\partial z^\sigma} - g^{\bar{\mu}\sigma}\mathrm{d}\bar{z}^\nu \otimes \frac{\partial}{\partial z^\sigma}\right),\\
N &= \frac{1}{2}N_{\mu\nu}\left(g^{\nu\bar{\sigma}}\mathrm{d}z^\mu \otimes \frac{\partial}{\partial\bar{z}^\sigma} + g^{\mu\bar{\sigma}}\mathrm{d}z^\nu \otimes \frac{\partial}{\partial\bar{z}^\sigma}\right),\\
\bar{N} &= \frac{1}{2}N_{\bar{\mu}\bar{\nu}}\left(g^{\bar{\nu}\sigma}\mathrm{d}\bar{z}^\mu \otimes \frac{\partial}{\partial z^\sigma} + g^{\bar{\mu}\sigma}\mathrm{d}\bar{z}^\nu \otimes \frac{\partial}{\partial z^\sigma}\right),\\
C &= \frac{1}{2}C_{\mu\bar{\nu}}\left(g^{\bar{\nu}\sigma}\mathrm{d}z^\mu \otimes \frac{\partial}{\partial z^\sigma} - g^{\mu\bar{\sigma}}\mathrm{d}z^\nu \otimes \frac{\partial}{\partial\bar{z}^\sigma}\right), \text{ and}\\
S &= \frac{1}{2}S_{\mu\bar{\nu}}\left(g^{\bar{\nu}\sigma}\mathrm{d}z^\mu \otimes \frac{\partial}{\partial z^\sigma} + g^{\mu\bar{\sigma}}\mathrm{d}z^\nu \otimes \frac{\partial}{\partial\bar{z}^\sigma}\right).
\end{aligned} \tag{9.1.9}$$

This lemma is a simple consequence of lowering indices with the inverse of the hermitian metric. The above then become a model for the decomposition of $\mathfrak{gl}(6,\mathbb{R})$ with respect to $\{\omega, \Omega\}$.

9.1.1 Decomposition of 2-forms

To decompose the space of 2-forms we consider the infinitesimal action of $\mathrm{GL}(6,\mathbb{R})$ on ω mimicking [Kar09] as follows[2]. Let $A \in \mathfrak{gl}(6,\mathbb{R})$, and then $e^{At} \in \mathrm{GL}(6,\mathbb{R})$, the action is then given as

$$e^{At}\cdot\omega = -\mathbf{i}g_{\alpha\bar{\beta}}(e^{At}\mathrm{d}z^\alpha) \wedge \left(e^{At}\mathrm{d}\bar{z}^\beta\right). \tag{9.1.10}$$

We know from general theory the orbit of ω is generic, so one should be able to recover $\Lambda^2(M)$ by the action above. Indeed,

Proposition 9.1.1.1. Consider the action of $A \in \mathfrak{gl}(6,\mathbb{R})$ on ω. Then the orbit of ω is $\Lambda^2(M)$, explicitly,

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} e^{At}\cdot\omega = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} e^{(M+\bar{M}+S))t}\cdot\omega = \mathbf{i}(M-\bar{M}) - \mathbf{i}S \tag{9.1.11}$$

Proof. Differentiating with respect to t and then setting it to zero one has

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} e^{At}\cdot\omega = -\mathbf{i}g_{\alpha\bar{\beta}}\left[(A\mathrm{d}z^\alpha)\wedge\mathrm{d}\bar{z}^\beta + \mathrm{d}z^\alpha \wedge \left(A\mathrm{d}\bar{z}^\beta\right)\right]. \tag{9.1.12}$$

For the rest of the proof we use lemma 9.1.0.1 implictly.

[2] We shall use the same techniques when decomposing the space of 3-forms via Ω and $\bar{\Omega}$ in the next section.

Consider N as above (9.1.5), then

$$\begin{aligned}\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} e^{Nt}\cdot\omega &= -\frac{\mathbf{i}}{2}N_{\mu\nu}\mathrm{d}z^{\alpha}\wedge\left(g_{\alpha\bar{\beta}}g^{\nu\bar{\sigma}}\mathrm{d}z^{\mu}\frac{\partial}{\partial\bar{z}^{\sigma}}\left(\mathrm{d}\bar{z}^{\beta}\right)+g_{\alpha\bar{\beta}}g^{\mu\bar{\sigma}}\mathrm{d}z^{\nu}\frac{\partial}{\partial\bar{z}^{\sigma}}\left(\mathrm{d}\bar{z}^{\beta}\right)\right)\\ &= -\frac{\mathbf{i}}{2}N_{\mu\nu}\mathrm{d}z^{\alpha}\wedge\left(\delta_{\alpha}{}^{\nu}\mathrm{d}z^{\mu}+\delta_{\alpha}{}^{\mu}\mathrm{d}z^{\nu}\right)\\ &= \frac{\mathbf{i}}{2}N_{\mu\nu}\left(\mathrm{d}z^{\mu}\wedge\mathrm{d}z^{\nu}+\mathrm{d}z^{\nu}\wedge\mathrm{d}z^{\mu}\right)=0.\end{aligned}\tag{9.1.13}$$

Similar to the above, one can show

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} e^{\bar{N}t}\cdot\omega = 0.\tag{9.1.14}$$

The reason that N and $\bar{N}$ vanish is due to the fact they are symmetric, that is the + between the first and second terms on the first line of (9.1.13). This means that if one considered the orbits of M and $\bar{M}$ they would be non-vanishing, they would map to themselves with a factor of $\pm\mathbf{i}$ in front. That is to say

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} e^{Mt}\cdot\omega = \mathbf{i}M,\ \text{and}\ \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} e^{\bar{M}t}\cdot\omega = -\mathbf{i}\bar{M}.\tag{9.1.15}$$

Next, consider the hermitian and skew-hermitian spaces. Taking C, as above (9.1.5),

$$\begin{aligned}\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} e^{Ct}\cdot\omega &= -\frac{\mathbf{i}}{2}C_{\mu\bar{\nu}}g_{\alpha\bar{\beta}}\left(g^{\bar{\nu}\sigma}\mathrm{d}z^{\mu}\frac{\partial}{\partial z^{\sigma}}\left(\mathrm{d}z^{\alpha}\right)\wedge\mathrm{d}\bar{z}^{\beta}-g^{\mu\bar{\sigma}}\mathrm{d}z^{\alpha}\wedge\mathrm{d}z^{\nu}\frac{\partial}{\partial\bar{z}^{\sigma}}\left(\mathrm{d}\bar{z}^{\beta}\right)\right)\\ &= -\frac{\mathbf{i}}{2}\left(g_{\lambda\bar{\beta}}C_{\alpha\bar{\nu}}g^{\bar{\nu}\lambda}-g_{\alpha\bar{\tau}}C_{\mu\bar{\beta}}g^{\mu\bar{\tau}}\right)\mathrm{d}z^{\alpha}\wedge\mathrm{d}\bar{z}^{\beta}\\ &= -\frac{\mathbf{i}}{2}\left(C_{\alpha\bar{\beta}}-C_{\alpha\bar{\beta}}\right)\mathrm{d}z^{\alpha}\wedge\mathrm{d}\bar{z}^{\beta}=0.\end{aligned}\tag{9.1.16}$$

The reason that this term vanishes is that it is skew-hermitian. This means that if one considers S it is non-vanishing, i.e.

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} e^{St}\cdot\omega = -\mathbf{i}S_{\mu\bar{\nu}}\mathrm{d}z^{\mu}\wedge\mathrm{d}\bar{z}^{\nu}.\tag{9.1.17}$$

□

Equation (9.1.16) makes sense because the extra factor of $\mathbf{i}$ changes C from skew-hermitian to hermitian and S from hermitian to skew-hermitian, and skew-symmetric objects are zero under skew-symmetric tensor products. So if one considers the orbit of the infinitesimal action of $A\in\mathrm{GL}(6,\mathbb{R})$, then the kernel of this map is the space of symmetric complex tensors $S^{2,0}\oplus S^{0,2}$ and skew-hermitian tensors $[[\Lambda^{1,1}]]$. Furthermore, the image of this map is contained in the space of 2-forms, which by the complex structure J splits as $\{\pm1\}$−eigenspaces. Under this map, the image of skew-symmetric complex tensors $\Lambda^{2,0}\oplus\Lambda^{0,2}$ is sent to itself (the −1 eigenspace) in the space of 2-forms. Furthermore, this map sends hermitian tensors to skew-hermitian tensors (the +1 eigenspace). Counting dimensions, these are both real 9-dimensions, so they are isomorphic to one another via (9.1.12).

9.1.2 Decomposition of 3-forms

To decompose the space of 3-forms we need to consider the infinitesimal action of $\mathrm{GL}(6,\mathbb{R})$ on Ω and $\bar{\Omega}$. Let $A \in \mathfrak{gl}(6,\mathbb{R})$, then $e^{At} \in \mathrm{GL}(6,\mathbb{R})$, and the action on Ω is given as

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} e^{At}\cdot\Omega = 3\Omega_{\alpha\beta\gamma}(A\mathrm{d}z^{\alpha})\wedge\mathrm{d}z^{\beta}\wedge\mathrm{d}z^{\gamma}, \tag{9.1.18}$$

and

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} e^{At}\cdot\bar{\Omega} = 3\Omega_{\bar{\alpha}\bar{\beta}\bar{\gamma}}(A\mathrm{d}z^{\bar{\alpha}})\wedge\mathrm{d}z^{\bar{\beta}}\wedge\mathrm{d}z^{\bar{\gamma}}. \tag{9.1.19}$$

Now the following proposition,

Proposition 9.1.2.1. Consider the action of $A \in \mathfrak{gl}(6,\mathbb{R})$ on Ω and $\bar{\Omega}$. Then the orbit of Ω and $\bar{\Omega}$ is $\Lambda^{3,0}\oplus\Lambda^{2,1}$ and $\Lambda^{0,3}\oplus\Lambda^{1,2}$, explicitly,

$$\begin{aligned}
&\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} e^{(g+\omega)t}\cdot\Omega \hookrightarrow \Lambda^{3,0},\ \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} e^{\bar{N}t}\cdot\Omega \hookrightarrow \Lambda_0^{2,1},\ \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} e^{\bar{M}t}\cdot\Omega \hookrightarrow \Lambda^{0,1},\\
&\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} e^{(g+\omega)t}\cdot\bar{\Omega} \hookrightarrow \Lambda^{0,3},\ \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} e^{Nt}\cdot\bar{\Omega} \hookrightarrow \Lambda_0^{1,2},\ \text{and}\ \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} e^{Mt}\cdot\bar{\Omega} \hookrightarrow \Lambda^{1,0}.
\end{aligned} \tag{9.1.20}$$

Proof. As was done in the previous proposition, we use lemma 9.1.0.1 implicitly. Now, it is clear from the beginning that the action (9.1.18) on barred coordinates is zero. So N, and M are in the kernel i.e.

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} e^{Mt}\cdot\Omega = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} e^{Nt}\cdot\Omega = 0. \tag{9.1.21}$$

Then consider $\bar{N}$ and $\bar{M}$, under (9.1.18) they become

$$\begin{aligned}
&\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} e^{\bar{M}t}\cdot\Omega = 3M_{\bar{\alpha}\bar{\nu}}g^{\bar{\nu}\sigma}\Omega_{\sigma\beta\gamma}\mathrm{d}\bar{z}^{\alpha}\wedge\mathrm{d}z^{\beta}\wedge\mathrm{d}z^{\gamma}, \text{and}\\
&\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} e^{\bar{N}t}\cdot\Omega = 3N_{\bar{\alpha}\bar{\nu}}g^{\bar{\nu}\sigma}\Omega_{\sigma\beta\gamma}\mathrm{d}\bar{z}^{\alpha}\wedge\mathrm{d}z^{\beta}\wedge\mathrm{d}z^{\gamma}.
\end{aligned} \tag{9.1.22}$$

Then consider c, an element of $[[\Lambda_0^{1,1}]]$

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} e^{ct}\cdot\Omega = \frac{3}{2}c_{\mu\bar{\nu}}g^{\bar{\nu}\sigma}\Omega_{\sigma\beta\gamma}\mathrm{d}z^{\mu}\wedge\mathrm{d}z^{\beta}\wedge\mathrm{d}z^{\gamma}. \tag{9.1.23}$$

Using

$$c_{\mu\bar{\nu}} = J_{\mu}{}^{\alpha}J_{\bar{\nu}}{}^{\bar{\beta}}C_{\alpha\bar{\beta}},\ \text{and}\ g^{\bar{\nu}\sigma} = J^{\bar{\nu}}{}_{\bar{\lambda}}J^{\sigma}{}_{\rho}g^{\bar{\lambda}\rho} \tag{9.1.24}$$

(9.1.23) becomes

$$\begin{aligned}
\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} e^{ct}\cdot\Omega &= \frac{3}{2}c_{\mu\bar{\nu}}g^{\bar{\nu}\sigma}\Omega_{\sigma\beta\gamma}\mathrm{d}z^{\mu}\wedge\mathrm{d}z^{\beta}\wedge\mathrm{d}z^{\gamma}\\
&= -\frac{3}{2}c_{\mu\bar{\nu}}g^{\bar{\nu}\sigma}\Omega_{\sigma\beta\gamma}\mathrm{d}z^{\mu}\wedge\mathrm{d}z^{\beta}\wedge\mathrm{d}z^{\gamma}.
\end{aligned} \tag{9.1.25}$$

Hence

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} e^{ct}\cdot\Omega = 0. \tag{9.1.26}$$

The information we relied on was (9.1.24), i.e. the action of J on 2-tensors with a barred and an unbarred index is the same as the identity transformation. This means that for $s \in [[S_0^{1,1}]]$ the same result will hold

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} e^{st} \cdot \Omega = 0. \tag{9.1.27}$$

If we then consider the elements in $[[\Lambda^{1,1}]]$ and $[[S^{1,1}]]$ one has

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} e^{gt} \cdot \Omega = 3\Omega, \text{ and } \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} e^{\omega t} \cdot \Omega = 3\mathbf{i}\Omega. \tag{9.1.28}$$

Under the map, (9.1.18), the kernel is $\Lambda^{2,0} \oplus S^{2,0} \oplus [[S_0^{1,1}]] \oplus [[\Lambda_0^{1,1}]]$. The same exercise for $\bar{\Omega}$, that is the orbit for $A \in \mathfrak{gl}(6, \mathbb{R})$ yields,

$$\begin{aligned}
&\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} e^{Mt} \cdot \bar{\Omega} = 3M_{\alpha\nu} g^{\nu\bar{\sigma}} \Omega_{\bar{\sigma}\bar{\beta}\bar{\gamma}} \mathrm{d}z^\alpha \wedge \mathrm{d}\bar{z}^\beta \wedge \mathrm{d}\bar{z}^\gamma, \ \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} e^{Nt} \cdot \bar{\Omega} = 3N_{\alpha\nu} g^{\nu\bar{\sigma}} \Omega_{\bar{\sigma}\bar{\beta}\bar{\gamma}} \mathrm{d}z^\alpha \wedge \mathrm{d}\bar{z}^\beta \wedge \mathrm{d}\bar{z}^\gamma, \\
&\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} e^{gt} \cdot \bar{\Omega} = 3\bar{\Omega}, \ \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} e^{\omega t} \cdot \bar{\Omega} = -3\mathbf{i}\bar{\Omega}, \ \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} e^{\bar{M}t} \cdot \bar{\Omega} = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} e^{\bar{N}t} \cdot \bar{\Omega} = 0, \text{ and} \\
&\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} e^{ct} \cdot \bar{\Omega} = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} e^{st} \cdot \bar{\Omega} = 0.
\end{aligned} \tag{9.1.29}$$

One can see that the kernel is $\Lambda^{0,2} \oplus S^{0,2} \oplus [[S_0^{1,1}]] \oplus [[\Lambda_0^{1,1}]]$. To make the characterisation with the space of 3-forms, we consider $M \in \Lambda^{1,0}$ and $\bar{M} \in \Lambda^{0,1}$. Then

$$\xi_\mu g^{\mu\bar{\sigma}} \Omega_{\bar{\sigma}\bar{\beta}\bar{\gamma}} \mathrm{d}\bar{z}^\beta \wedge \mathrm{d}\bar{z}^\gamma \hookrightarrow \Lambda^{0,2}, \text{ and } \xi_{\bar{\mu}} g^{\bar{\mu}\sigma} \Omega_{\sigma\beta\gamma} \mathrm{d}z^\beta \wedge \mathrm{d}z^\gamma \hookrightarrow \Lambda^{2,0}. \tag{9.1.30}$$

Counting dimensions gives an isomorphism

$$\Lambda^{2,0} \cong \Lambda^{0,1}, \text{ and } \Lambda^{0,2} \cong \Lambda^{1,0}. \tag{9.1.31}$$

From standard representation theory, the space of real 3-forms splits as

$$\Lambda^3 \cong \Lambda^{3,0} \oplus \Lambda^{0,3} \oplus \Lambda_0^{2,1} \oplus \Lambda_0^{1,2} \oplus \Lambda^{1,0} \oplus \Lambda^{0,1}. \tag{9.1.32}$$

From dimension counting one sees that for $A \in \mathfrak{gl}(6, \mathbb{R})$

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} e^{At} \cdot \Omega \hookrightarrow \Lambda^{3,0} \oplus \Lambda_0^{2,1} \oplus \Lambda^{0,1}, \text{ and } \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} e^{At} \cdot \bar{\Omega} \hookrightarrow \Lambda^{0,3} \oplus \Lambda_0^{1,2} \oplus \Lambda^{1,0}. \tag{9.1.33}$$

The images of (9.1.18) and (9.1.19), and (9.1.31) prove the proposition. □

Thus, we have recovered that the orbit of $\mathrm{Re}(\Omega) = \frac{1}{2}(\Omega + \bar{\Omega})$ is generic in Λ^3.

9.2 Characterising the Torsion Space

The torsion space is given by $\mathfrak{g}^\perp \otimes [[\Lambda^{1,0}]] \cong [[\Lambda^{1,1}]] \oplus [[S^{1,1}]] \oplus [[\Lambda^{2,1}]] \oplus [[\Lambda^{1,0}]]$. These are irreducible representations for $\mathfrak{su}(3)$. We re-write in another form

$$\mathfrak{g}^\perp \otimes [[\Lambda^{1,0}]] = (\mathbf{1} \oplus \mathbf{8}) \oplus (\mathbf{1} \oplus \mathbf{8}) \oplus (\mathbf{3} \oplus \bar{\mathbf{6}}) \oplus (\bar{\mathbf{3}} \oplus \mathbf{6}) \oplus (\mathbf{3} \oplus \bar{\mathbf{3}}). \tag{9.2.1}$$

We write it this way to highlight that the total dimension of the torsion space is 42, we also wish to highlight that there are 10 irreducible subspaces. This is an important count for the linearised theory, as this number dictates the number of free independent Lagrangians one expects to write. We parameterise the torsion as follows. First consider, $A \in [[\Lambda_0^{1,1}]]$, and $B \in [[S_0^{1,1}]]$. $JB \in [[\Lambda^{1,1}]]$, thus can instead use a complex (1,1)-form η and parameterise the space $(\mathbf{1}\oplus\mathbf{8})\oplus(\mathbf{1}\oplus\mathbf{8})$, in (9.2.1), using $\mathrm{Re}(\eta) \coloneqq A$ and $\mathrm{Im}(\eta) \coloneqq JB$. Next, the space $(\mathbf{3}\oplus\bar{\mathbf{6}})\oplus(\bar{\mathbf{3}}\oplus\mathbf{6})$ in (9.2.1) can be described through a one-form $\kappa \in [[\Lambda^{1,0}]] \cong \mathbf{3}\oplus\bar{\mathbf{3}}$ and real symmetric 2-tensor $h \in [[S^{2,0}]] \cong \mathbf{6}\oplus\bar{\mathbf{6}}$. Finally, the space $(\mathbf{3}\oplus\bar{\mathbf{3}})$ can be characterised by another real one-form $\lambda \in [[\Lambda^{1,0}]]$.

9.2.1 Intrinsic Torsion of SU(3)-structures

To be able to use the power of the techniques developed by [Kar09] we need to convert the SU(3)-structures to real ones. ω is already real, and so we start there. The covariant derivative $\nabla\omega$ lies in the real version of the decomposition of 2-forms. We have shown that there is no component of $\nabla\omega$ lying in $[[\Lambda^{1,1}]]$, and $S^{2,0}$ (hence $S^{0,2}$). We proceed to show the following theorem,

Theorem 9.2.1.1. Let X be a vector field on the 6-dimensional spin manifold M with an SU(3)-structure $\{\omega, \Omega\}$. Then one has

$$X^\alpha \nabla_\alpha \omega_{\mu\nu} \mathrm{d}w^{\mu\nu} = -\tau_{\mu\lambda} J^\lambda{}_\nu \mathrm{d}w^{\mu\nu}. \tag{9.2.2}$$

Here $\tau \in [[\Lambda^{2,0}]]$.

Proof. One must consider a symmetric tensor in $[[S^{1,1}]]$, then its real action on ω has the form

$$S = J_\mu{}^\rho S_{\rho\nu} \mathrm{d}w^{\mu\nu}. \tag{9.2.3}$$

We show that this component is orthogonal to the direction of $\nabla_X\omega = X^\mu \nabla_\mu \omega_{\alpha\beta} \mathrm{d}w^{\alpha\beta}$ by taking the inner product with (9.2.3)

$$2X^\mu \nabla_\mu \omega_{\alpha\beta} J^\alpha{}_\rho S^{\rho\beta} = 2X^\mu S^{\rho\beta} g_{\rho\kappa} \nabla_\mu \omega_{\alpha\beta} \omega^{\alpha\kappa}. \tag{9.2.4}$$

We have used metric compatibility of the connection to commute g, and now use

$$0 = \nabla_\mu(\delta_\alpha{}^\kappa) = \nabla_\mu(\omega_{\alpha\beta}\omega^{\alpha\kappa}) = \nabla_\mu \omega_{\alpha\beta} \omega^{\alpha\kappa} + \omega_{\alpha\beta} \nabla_\mu \omega^{\alpha\kappa}. \tag{9.2.5}$$

The above shows that $\nabla_\mu \omega_{\alpha\beta} \omega^{\alpha\kappa}$ is antisymmetric. This, in turn, shows that the contraction on the right-hand-side of (9.2.4) is zero as a symmetric tensor is completely contracted with an antisymmetric tensor. As objects in $[[S^{2,0}]]$, and $[[\Lambda^{1,1}]]$ do not appear in the decomposition of 2-forms, the only space that $\nabla_X\omega$ can be in is $[[\Lambda^{2,0}]]$. ☐

Similarly, we now prove the following statement for the real and imaginary parts of Ω, $\mathcal{C}$ and $\hat{\mathcal{C}} = J\mathcal{C}$.

Theorem 9.2.1.2. Let X be a vector field on the 6-dimensional spin manifold M with an SU(3)-structure $\{\omega, \Omega\}$. Then one has

$$X^\rho \nabla_\rho \mathcal{C}_{\alpha\beta\gamma} \mathrm{d}w^{\alpha\beta\gamma} = (\tilde{\tau}_{\alpha\mu} \mathcal{C}^\mu{}_{\beta\gamma} + \lambda \hat{\mathcal{C}}_{\alpha\beta\gamma}) \mathrm{d}w^{\alpha\beta\gamma}. \tag{9.2.6}$$

Here $\tilde{\tau} \in [[\Lambda^{2,0}]]$ and $\lambda \in [[\Lambda^0]]$. Furthermore,

$$X^\rho \nabla_\rho \hat{\mathcal{C}}_{\alpha\beta\gamma} \mathrm{d}w^{\alpha\beta\gamma} = (*(\tilde{\tau}_{\alpha\mu} \hat{\mathcal{C}}^\mu{}_{\beta\gamma}) - \lambda \mathcal{C}_{\alpha\beta\gamma}) \mathrm{d}w^{\alpha\beta\gamma}. \tag{9.2.7}$$

Proof. The second equation (9.2.7) follows from the first and taking the Hodge dual. Recall that the space $\Lambda_0^{2,1} \subset \Lambda^3$ is parameterised by a complex symmetric 2-tensor $\bar{N}$ by

$$\Lambda_0^{2,1} \ni N_{\bar{\alpha}\bar{\lambda}} g^{\bar{\lambda}\nu} \Omega_{\nu\beta\gamma} \mathrm{d}z^{\bar{\alpha}\beta\gamma}. \tag{9.2.8}$$

Taking then the real part of this equation gives us access to the parameterisation in $[[\Lambda_0^{2,1}]]$. Let $N = h - \mathbf{i}Jh$ for some real function $h \in [[S_0^{2,0}]]$, one has

$$[[\Lambda_0^{2,1}]] \ni h_{\alpha\lambda}\mathcal{C}^{\lambda}{}_{\beta\gamma}\mathrm{d}w^{\alpha\beta\gamma}. \tag{9.2.9}$$

This component is orthogonal to the direction $\nabla_X\mathcal{C} = X^\sigma\nabla_\sigma\mathcal{C}_{\mu\nu\rho}\mathrm{d}w^{\mu\nu\rho}$ by taking the inner product with (9.2.9)

$$6X^\sigma h^{\mu\lambda}\nabla_\sigma\mathcal{C}_{\mu\nu\rho}\mathcal{C}_\lambda{}^{\nu\rho}. \tag{9.2.10}$$

Now we use

$$0 = \nabla_\sigma(g_{\mu\lambda}) = \nabla_\sigma(\mathcal{C}_{\mu\nu\rho}\mathcal{C}_\lambda{}^{\nu\rho}) = \nabla_\sigma\mathcal{C}_{\mu\nu\rho}\mathcal{C}_\lambda{}^{\nu\rho} + \mathcal{C}_{\mu\nu\rho}\nabla_\sigma\mathcal{C}_\lambda{}^{\nu\rho}. \tag{9.2.11}$$

The above shows that $\nabla_\sigma\mathcal{C}_{\mu\nu\rho}\mathcal{C}_\lambda{}^{\nu\rho}$ is antisymmetric. Next, consider the space Λ^0, parametrised by real smooth function a and b in $C^\infty(M)$ as

$$\Lambda^0 \ni (ag_{\mu\bar{\lambda}} + \mathbf{i}b\omega_{\mu\bar{\lambda}})g^{\bar{\lambda}\rho}\Omega_{\rho\tau\sigma}\mathrm{d}w^{\mu\tau\sigma}. \tag{9.2.12}$$

Taking the real part one then has

$$[[\Lambda^0]] \ni (a\mathcal{C}_{\mu\nu\rho} + b\hat{\mathcal{C}}_{\mu\nu\rho})\mathrm{d}w^{\mu\nu\rho}. \tag{9.2.13}$$

Finally, using

$$0 = \nabla_\sigma(g_{\mu\lambda}) = \nabla_\sigma(\mathcal{C}_{\mu\nu\rho}\mathcal{C}^{\lambda\nu\rho}) = 2\nabla_\sigma\mathcal{C}_{\mu\nu\rho}\mathcal{C}^{\lambda\nu\rho}. \tag{9.2.14}$$

Thus $\nabla_X\mathcal{C}$ is orthogonal to $\mathcal{C}$ in (9.2.13). The rest of the cases need not be considered as they do not appear in the decomposition of real 3-forms. □

9.2.2 Reparametrising & Extracting Torsion

Let us begin with $\nabla_X\mathcal{C}$. In this first instance, we recall that for 2-form in $\Lambda^{2,0}$, one can write it as a vector field in $\Lambda^{0,1}$ contracted with Ω (9.1.31). This means in the real case one can write

$$\tau \coloneqq X^\gamma\mathcal{C}_{\gamma\alpha\beta}\mathrm{d}w^{\alpha\beta}, \text{ and } \tilde{\tau} \coloneqq Y^\gamma\mathcal{C}_{\gamma\alpha\beta}\mathrm{d}w^{\alpha\beta}, \text{ for some } X, Y \in \Lambda^1(M). \tag{9.2.15}$$

Then using the relation that $J\mathcal{C} = \hat{\mathcal{C}}$ one can write the $\nabla_X\omega$ in component form

$$\nabla_X\omega_{\mu\nu} = X^\lambda\hat{\mathcal{C}}_{\lambda\mu\nu}, \tag{9.2.16}$$

and in a similar vein one can write

$$\nabla_X\mathcal{C}_{\alpha\beta\gamma} = Y^\lambda\mathcal{C}_{\lambda[\alpha|\mu|}C^\mu{}_{\beta\gamma]} + \lambda_\sigma\hat{\mathcal{C}}_{\alpha\beta\gamma}. \tag{9.2.17}$$

We now have the following lemma

Lemma 9.2.2.1. Let X and Y as above, then $X = \frac{3}{2}Y$.

Proof. Firstly, we use the relation $\mathcal{C}\wedge\omega = 0$, and $\mathcal{C}\wedge\hat{\mathcal{C}} = (2/3)\omega\wedge\omega\wedge\omega$ to show

$$\begin{aligned}\nabla_X\mathcal{C}\wedge\omega + \mathcal{C}\wedge\nabla_X\omega = 0 \implies Y^\lambda\mathcal{C}_{\lambda[\alpha|\mu|}C^\mu{}_{\beta\gamma}\omega_{\sigma\rho]} + X^\lambda\hat{\mathcal{C}}_{\lambda[\sigma\rho}\mathcal{C}_{\alpha\beta\gamma]} &= 0\\ Y^\lambda\omega_{\lambda[\alpha}\omega_{\beta\gamma}\omega_{\sigma\rho]} + X^\lambda\hat{\mathcal{C}}_{\lambda[\sigma\rho}\mathcal{C}_{\alpha\beta\gamma]} &= 0\\ (Y\lrcorner\omega)\wedge\omega\wedge\omega + (W\lrcorner\hat{\mathcal{C}})\wedge\mathcal{C} &= 0.\end{aligned} \tag{9.2.18}$$

On the other hand,

$$\mathcal{C}\wedge\hat{\mathcal{C}} = \frac{2}{3}\omega\wedge\omega\wedge\omega \implies (X\lrcorner\hat{\mathcal{C}})\wedge\mathcal{C} + \frac{2}{3}(X\lrcorner\omega)\wedge\omega\wedge\omega = 0 \tag{9.2.19}$$

This must mean that $X = \frac{3}{2}Y$. □

This completes the picture for what the torsion should look like in index form.

Theorem 9.2.2.1. Let ω and Ω be SU(3)-structures, then the covariant derivatives, i.e. torsion, is given component-wise as

$$\nabla_\sigma \omega_{\mu\nu} = \frac{3}{2} T_{\sigma;\lambda} \hat{\mathcal{C}}^\lambda{}_{\mu\nu}, \ \nabla_\sigma \mathcal{C}_{\alpha\beta\gamma} = T_{\sigma;}{}^\lambda \omega_{\lambda[\alpha}\omega_{\beta\gamma]} + \lambda_\sigma \hat{\mathcal{C}}_{\alpha\beta\gamma}, \text{ and} \\ \nabla_\sigma \hat{\mathcal{C}}_{\alpha\beta\gamma} = \frac{1}{2} T_{\sigma;[\alpha}\omega_{\beta\gamma]} - \lambda_\sigma \mathcal{C}_{\alpha\beta\gamma}. \tag{9.2.20}$$

Here a tensor of the type $T_{\sigma;\lambda}$ refers to a an object in $\Lambda^1(M) \otimes \Lambda^1(M)$. The semicolon is to emphasise there are is no symmetry in the indices. Also, for the last expression we recall that $\nabla_X \hat{\mathcal{C}}$ is the Hodge dual to $\nabla_X \mathcal{C}$, and that $*\omega = (1/2)\omega \wedge \omega$. One has that $T \in \Lambda^1(M) \otimes \Lambda^1(M)$, and $\lambda \in \Lambda^1(M)$ so

$$T = A + B + \kappa \lrcorner \mathcal{C} + h + ag + b\omega \in \Lambda^1(M) \otimes \Lambda^1(M). \tag{9.2.21}$$

Where $A \in [[\Lambda_0^{1,1}]]$, $B \in [[S_0^{1,1}]]$, $a, b \in [[\Lambda^0]]$, $\kappa, \lambda \in [[\Lambda^{1,0}]]$, and $h \in [[S^{2,0}]]$. We now proceed to extract the torsion terms above in terms of exterior derivatives of ω, C and $\hat{\mathcal{C}}$, we exploit the characterisation made by [CS02] to write

$$(\mathrm{d}w)_{\mu\nu\rho} = -\frac{3}{2} b\mathcal{C}_{\mu\nu\rho} + \frac{3}{2} a\hat{\mathcal{C}}_{\mu\nu\rho} + \lambda_{[\rho}\omega_{\mu\nu]} + h_{\lambda[\rho}\mathcal{C}^\lambda{}_{\mu\nu]}, \\ (\mathrm{d}\mathcal{C})_{\sigma\rho\mu\nu} = a\omega_{[\sigma\rho}\omega_{\mu\nu]} + \kappa_{[\sigma}\mathcal{C}_{\rho\mu\nu]} - A_{[\sigma\rho}\omega_{\mu\nu]}, \text{ and} \\ (\mathrm{d}\hat{\mathcal{C}})_{\sigma\rho\mu\nu} = b\omega_{[\sigma\rho}\omega_{\mu\nu]} - \kappa^\lambda \omega_{\lambda[\sigma}\mathcal{C}_{\rho\mu\nu]} - B_{[\sigma}{}^\lambda \omega_{|\lambda|\rho}\omega_{\mu\nu]}. \tag{9.2.22}$$

It is convenient to take the hodge duals of the second two terms written

$$*(\mathrm{d}\mathcal{C})_{\mu\nu} = 2a\omega_{\mu\nu} + \kappa^\lambda \hat{\mathcal{C}}_{\lambda\mu\nu} - \frac{1}{2} A_{\mu\nu}, \text{ and} \\ *(\mathrm{d}\hat{\mathcal{C}})_{\mu\nu} = 2b\omega_{\mu\nu} - \kappa^\lambda \mathcal{C}_{\lambda\mu\nu} + \frac{1}{2} J_{[\mu}{}^\lambda B_{\nu]\lambda}. \tag{9.2.23}$$

It is very easy to extract the tensors from the exterior derivatives by contracting appropriately with the background forms ω, $\mathcal{C}$ and $\hat{\mathcal{C}}$.

Lemma 9.2.2.2. Let $a, b \in [[\Lambda^0]]$, and $\kappa, \lambda \in [[\Lambda^{1,0}]]$ be components of the torsion, (9.2.21), then one has

$$a = \frac{1}{36}(\mathrm{d}w)_{\mu\nu\rho}\hat{\mathcal{C}}^{\mu\nu\rho}, \ b = -\frac{1}{36}(\mathrm{d}w)_{\mu\nu\rho}\mathcal{C}^{\mu\nu\rho}, \ \lambda_\sigma = \frac{3}{4}(\mathrm{d}w)_{\mu\nu\sigma}\omega^{\mu\nu}, \text{ and} \\ \kappa_\sigma = \frac{1}{4} * (\mathrm{d}\mathcal{C})_{\mu\nu}\hat{\mathcal{C}}^{\mu\nu}{}_\sigma. \tag{9.2.24}$$

Proposition 9.2.2.1. Let $A \in [[\Lambda_0^{1,1}]]$, $B \in [[S_0^{1,1}]]$, and $h \in [[S^{2,0}]]$ be components of the torsion, (9.2.21), then one has

$$\begin{aligned} h_{\lambda[\rho}\mathcal{C}^{\lambda}{}_{\mu\nu]} &= \frac{1}{24}(\mathrm{d}w)_{\alpha\beta\gamma}\mathcal{C}^{\alpha\beta\gamma}\mathcal{C}_{\mu\nu\rho} - \frac{1}{24}(\mathrm{d}w)_{\alpha\beta\gamma}\hat{\mathcal{C}}^{\alpha\beta\gamma}\hat{\mathcal{C}}_{\mu\nu\rho} \\ &\quad - \frac{3}{4}(\mathrm{d}w)_{\alpha\beta[\rho}\omega^{\alpha\beta}\omega_{\mu\nu]} + (\mathrm{d}w)_{\mu\nu\rho}, \\ A_{\mu\nu} &= \frac{2}{9}(\mathrm{d}w)_{\alpha\beta\gamma}\hat{\mathcal{C}}^{\alpha\beta\gamma}\omega_{\mu\nu} + \frac{1}{2} * (\mathrm{d}\mathcal{C})_{\alpha\beta}\hat{\mathcal{C}}^{\alpha\beta\lambda}\hat{\mathcal{C}}_{\lambda\mu\nu} - 2 * (\mathrm{d}\mathcal{C})_{\mu\nu}, \text{ and} \\ J_{[\mu}{}^{\lambda}B_{\nu]\lambda} &= \frac{2}{9}(\mathrm{d}w)_{\alpha\beta\gamma}\mathcal{C}^{\alpha\beta\gamma}\omega_{\mu\nu} + \frac{1}{2} * (\mathrm{d}\mathcal{C})_{\alpha\beta}\hat{\mathcal{C}}^{\alpha\beta\lambda}\mathcal{C}_{\lambda\mu\nu} - 2 * (\mathrm{d}\hat{\mathcal{C}})_{\mu\nu}. \end{aligned} \tag{9.2.25}$$

Remark 9.2.2.1. We do not extract the symmetric parts, because they are more useful in these forms, in keeping with Plebanski-formalism. Indeed, the goal is to be able to write an action functional, hence one requires the auxiliary fields to be differential forms, which inherently are antisymmetric. The tensors B and h are not antisymmetric *unless* they are contracted with J and $\mathcal{C}$, respectively.

We can now state the following corollary,

Corollary 9.2.2.1.1. ω and Ω are parallel SU(3)-structures if, and only if, $\mathrm{d}\omega$, $\mathrm{d}\mathcal{C}$ and $\mathrm{d}\hat{\mathcal{C}}$ vanish if, and only if, the tensors characterising the torsion space $\mathfrak{g}^{\perp} \otimes [[\Lambda^{1,0}]]$ vanish.

9.3 Curvature

The space of curvature tensors $\{R_{ijkl}\} \in \mathcal{K} \subset \Lambda^2(M) \odot \Lambda^2(M)$ satisfy the Bianchi identity,

$$R_{ijkl} + R_{iljk} + R_{iklj} = 0. \tag{9.3.1}$$

The decomposition of $\Lambda^2(M) \odot \Lambda^2(M)$ into SU(3) irreducible representations is given as

$$\begin{aligned} \Lambda^2(M) \odot \Lambda^2(M) &= S^2(\mathbf{1} \oplus \mathbf{3} \oplus \bar{\mathbf{3}} \oplus \mathbf{8}) \\ &= (\mathbf{1} \oplus \mathbf{3} \oplus \bar{\mathbf{3}} \oplus \mathbf{8}) \oplus (\mathbf{1} \oplus \mathbf{6} \oplus \bar{\mathbf{6}} \oplus \mathbf{8}) \\ &\quad \oplus (\mathbf{1} \oplus \mathbf{3} \oplus \bar{\mathbf{3}} \oplus \mathbf{8} \oplus \mathbf{15} \oplus \overline{\mathbf{15}} \oplus \mathbf{27} \oplus \mathbf{6} \oplus \bar{\mathbf{6}}). \\ &:= \Lambda^4(V) \oplus \text{Ricci} \oplus \text{Weyl}. \end{aligned} \tag{9.3.2}$$

We now have the decomposition of Riemann tensors given as the direct sum of Ricci and Weyl. Next, recall that

$$\mathfrak{g}^{\perp} = \mathbf{1} \oplus \mathbf{3} \oplus \bar{\mathbf{3}}, \text{ and } \mathfrak{g} = \mathbf{8}. \tag{9.3.3}$$

One can ask how the products of each sit in the space of Riemann tensors, more specifically what components lie in the Ricci and Weyl parts.

$$\begin{aligned} S(\mathfrak{g}^{\perp} \otimes \mathfrak{g}^{\perp}) &= 2(\mathbf{1}) \oplus \mathbf{3} \oplus \bar{\mathbf{3}} \oplus \mathbf{8} \oplus \mathbf{6} \oplus \bar{\mathbf{6}}, \\ S(\mathfrak{g} \otimes \mathfrak{g}) &= \mathbf{27} \oplus \mathbf{8} \oplus \mathbf{1}, \text{ and} \\ \mathfrak{g}^{\perp} \otimes \mathfrak{g} &= \mathbf{15} \oplus \overline{\mathbf{15}} \oplus \mathbf{6} \oplus \bar{\mathbf{6}} \oplus \mathbf{3} \oplus \bar{\mathbf{3}} \oplus \mathbf{8}. \end{aligned} \tag{9.3.4}$$

The above shows that the decomposition of the Ricci tensor lies in the $+\mathbf{i}$ eigen-subspace with respect to J, which is $\mathbf{1} \oplus \mathbf{8}$, and in the $-\mathbf{i}$ eigen-subspace with respect to J, which is $\mathbf{6} \oplus \bar{\mathbf{6}}$.

There are two ways to project the Riemann curvature tensor: either through the background real 2-form ω or the real background 3-forms C and $\hat{\mathcal{C}}$. We wish to characterise the portion of the curvature that can be accessed by the torsion, which is facilitated by the following theorem.

Theorem 9.3.0.1. [**Bianchi Identity**] One can access the parts of the curvature tensor R by projecting onto the background forms ω, $\mathcal{C}$ and $\hat{\mathcal{C}}$ as follows

$$\begin{aligned}\pi_\omega(R)_{\rho\sigma;\mu\nu} &\coloneqq R_{\rho\sigma}{}^{\tau}{}_{[\mu}\omega_{|\tau|\nu]} = \frac{3}{2}\nabla_{[\rho}T_{\sigma]}{}^{\tau}\mathcal{C}_{\tau\mu\nu} + \frac{3}{4}T_{[\rho}{}^{\tau}T_{\sigma][\tau}\omega_{\mu\nu]} \\ &\quad + T_{[\rho}{}^{\tau}\lambda_{\sigma]}\mathcal{C}_{\tau\mu\nu}, \\ \pi_{\mathcal{C}}(R)^{\rho\sigma;}{}_{\alpha\beta\gamma} &\coloneqq R^{\rho\sigma\tau}{}_{[\alpha}\mathcal{C}_{|\tau|\beta\gamma]} = \nabla^{[\rho}T^{\sigma]\tau}\omega_{\tau[\alpha}\omega_{\beta\gamma]} + \nabla^{[\rho}\lambda^{\sigma]}\mathcal{C}_{\alpha\beta\gamma} \\ &\quad - \frac{1}{2}\lambda^{[\rho}T^{\sigma]}{}_{[\alpha}\omega_{\beta\gamma]}, \text{ and} \\ \pi_{\hat{\mathcal{C}}}(R)^{\rho\sigma;}{}_{\alpha\beta\gamma} &\coloneqq R^{\rho\sigma\tau}{}_{[\alpha}\hat{\mathcal{C}}_{|\tau|\beta\gamma]} = \frac{1}{2}\nabla^{[\rho}T^{\sigma]}{}_{[\alpha}\omega_{\beta\gamma]} - \frac{1}{2}T^{[\rho}{}_{[\alpha}\nabla^{\sigma]}\omega_{\beta\gamma]} \\ &\quad - \nabla^{[\rho}\lambda^{\sigma]}\mathcal{C}_{\alpha\beta\gamma} + \lambda^{[\rho}T^{\sigma]\tau}\omega_{\tau[\alpha}\omega_{\beta\gamma]}.\end{aligned} \tag{9.3.5}$$

Where have suppressed the semicolon in the expression above for T. It isn't difficult to compute the above using algebraic identities involving C and ω, for example $C \wedge \omega = 0$, the identity for curvature tensors on a k-form β

$$R_{ab}{}^{c}{}_{[\mu_1}\beta_{|c|\dots\mu_k]} = \nabla_{[a}\nabla_{b]}\beta_{\mu_1\dots\mu_k}, \tag{9.3.6}$$

and theorem (9.2.2.1). What we can proceed to show is that information about curvature is encoded only in orbit of ω. Indeed, if one contracts $\pi_{\mathcal{C},\hat{\mathcal{C}}}(R)$ with $\hat{\mathcal{C}}$, and applies appropriate identities, one has the resulting corollary

Corollary 9.3.0.1.1. Let $\pi_{\mathcal{C},\hat{\mathcal{C}}}(R)$ as above. Then under the following maps one has

$$\begin{aligned}\pi_{\mathcal{C}}(R)^{\rho\sigma;}{}_{[\alpha|\beta\gamma|}\hat{\mathcal{C}}^{\beta\gamma}{}_{\delta]} &= R^{\rho\sigma\tau}{}_{[\alpha}\mathcal{C}_{|\tau\beta\gamma|}\hat{\mathcal{C}}^{\beta\gamma}{}_{\delta]} = 2\pi_\omega(R)^{\rho\sigma;}{}_{\alpha\delta}, \text{ and} \\ \pi_{\hat{\mathcal{C}}}(R)^{\rho\sigma;}{}_{[\alpha|\beta\gamma|}\hat{\mathcal{C}}^{\beta\gamma}{}_{\delta]} &= R^{\rho\sigma\tau}{}_{[\alpha}\hat{\mathcal{C}}_{|\tau\beta\gamma|}\hat{\mathcal{C}}^{\beta\gamma}{}_{\delta]} = 0\end{aligned} \tag{9.3.7}$$

If one chose to contract both $\pi_{\mathcal{C},\hat{\mathcal{C}}}(R)$ with $\mathcal{C}$ the same result holds.

The curvature tensor can be considered as an object in $(\Lambda^2 \odot \Lambda^2) - \Lambda^4$. We want to understand are the parts of the torsion that can be accessed by these projections $\pi_{\omega,\mathcal{C},\hat{\mathcal{C}}}(R)$. Corollary 9.3.0.1.1 shows that, to access the projections of the curvature tensor parameterised by intrinsic torsion, it is only useful to consider $\pi_\omega(R)$. Recall, the orbit of ω in Λ^2 reduces is $[[\Lambda^{2,0}]]$, theorem (9.2.1.1). Therefore,

$$\pi_\omega : (\Lambda^2 \odot \Lambda^2) - \Lambda^4 \to \Lambda^2 \otimes [[\Lambda^{2,0}]], \tag{9.3.8}$$

writing the above in terms of irreducible representations, one gets

$$\begin{aligned}\Lambda^2 \otimes ([[\Lambda^{2,0}]]) &\cong (\mathbf{3} \oplus \bar{\mathbf{3}} \oplus \mathbf{8} \oplus \mathbf{1}) \otimes (\mathbf{3} \oplus \bar{\mathbf{3}}) \\ &\cong \mathbf{1} \oplus \mathbf{8} \oplus 4(\mathbf{3} \oplus \bar{\mathbf{3}}) \\ &\quad \oplus 2(\mathbf{6} \oplus \bar{\mathbf{6}}) \oplus (\mathbf{15} \oplus \overline{\mathbf{15}}).\end{aligned} \tag{9.3.9}$$

Compare this result with (9.3.4), and notice there are an extra three copies of $\mathbf{3} \oplus \bar{\mathbf{3}}$. One expects that in expanding the torsion T in terms of its irreducible representations via the formulas in theorem (9.3.0.1), these extra copies of $\mathbf{3} \oplus \bar{\mathbf{3}}$ don't appear. Furthermore, from this analysis, one sees that $\mathbf{27}$, a copy of $\mathbf{1}$, and a copy of $\mathbf{8}$ do not appear. These are exactly the parts of the Weyl curvature we don't have access to. Let us now write the Ricci tensor and Ricci scalar from $\pi_\omega(R)$.

Theorem 9.3.0.2. The Ricci tensor, in local coordinates, is given as

$$\begin{aligned} R_{\rho\nu} \coloneqq \pi_\omega(R)_{(\rho|\sigma;\mu|\nu)} g^{\sigma\mu} = &\frac{3}{8}(\nabla^\tau T_\rho{}^\mu \mathcal{C}_{\tau\mu\nu} + \nabla^\tau T_\nu{}^\mu \mathcal{C}_{\tau\mu\rho}) \\ &- \frac{3}{8}(\nabla_\rho T^{\tau\mu}\mathcal{C}_{\tau\mu\nu} + \nabla_\nu T^{\tau\mu}\mathcal{C}_{\tau\mu\rho}) \\ &+ \frac{3}{16}(T_\rho{}^\tau T^\mu{}_{[\tau}\omega_{\mu\nu]} + T_\nu{}^\tau T^\mu{}_{[\tau}\omega_{\mu\rho]}) \\ &+ \frac{3}{16}(T^{\tau\mu}T_{\rho[\tau}\omega_{\mu\nu]} + T^{\tau\mu}T_{\nu[\tau}\omega_{\mu\rho]}) \\ &+ \frac{1}{4}(T_\rho{}^\tau\lambda^\mu\mathcal{C}_{\tau\mu\nu} + T_\nu{}^\tau\lambda^\mu\mathcal{C}_{\tau\mu\rho}) \\ &+ \frac{1}{4}(T^{\tau\mu}\lambda_\rho\mathcal{C}_{\tau\mu\nu} + T^{\tau\mu}\lambda_\nu\mathcal{C}_{\tau\mu\rho}). \end{aligned} \tag{9.3.10}$$

The Ricci scalar, in local coordinates, is given as

$$R \coloneqq R_{\rho\nu}g^{\rho\nu} = -\frac{3}{2}\nabla^\tau T^{\mu\nu}\mathcal{C}_{\tau\mu\nu} + \frac{3}{4}T^{\tau\mu}T^\nu{}_{[\tau}\omega_{\mu\nu]} + \frac{1}{4}T^{\tau\mu}\lambda^\nu\mathcal{C}_{\tau\mu\nu}. \tag{9.3.11}$$

The torsion completely determines (9.3.9), the Ricci tensor sits inside this decomposition and is characterised by the equations above. This leads to the following corollary.

Corollary 9.3.0.2.1. Let M be a spin manifold in 6 dimensions equipped with an SU(3)-structure $\{\omega, \Omega\}$. M is Ricci flat if, and only if, the intrinsic torsion space completely vanishes.

Furthermore, we then have a characterisation that an SU(3) structure is parallel, or integrable if, and only if, the Ricci tensor is vanishing. So M is an SU(3) manifold if, and only if, the intrinsic torsion vanishes if, and only if, it is Ricci flat.

9.4 Linearised Analysis

We conclude this chapter by constructing an action that is linearised in the background fields: the metric and the SU(3)-structures ω and Ω – similar to SU(2). To begin to understand what constitutes a linearised diffeomorphism invariant Lagrangian, one needs to consider the tangent space to the orbits of these structures. We write $\partial^a \coloneqq p^a$ ($\bar\partial^a \coloneqq \bar p^a$) as notational convenience for now. This is because we understand the behaviour of the derivative, locally, and how the symmetric product of two derivatives can pair with themselves and other other tensors. We begin by decomposing:

$$\Lambda^1 \odot \Lambda^1 = \mathbf{6} \oplus \bar{\mathbf{6}} \oplus \mathbf{8} \oplus \mathbf{1}. \tag{9.4.1}$$

It is understood that $\mathbf{1}$ corresponds to $p_a\bar p^a$, $\mathbf{8}$ corresponds to $p_a\bar p_b$ (without the trace), $\mathbf{6}$ corresponds to $p_a p_b$, and $\bar{\mathbf{6}}$ corresponds to $\bar p_a \bar p_b$. To understand what field content we need to consider the tangent space to it, it is not hard to see that the tangent space to the SU(3)-structures consist of irreducible representations $\mathbf{3}$, $\bar{\mathbf{3}}$, $\mathbf{6}$, $\bar{\mathbf{6}}$, $\mathbf{8}$, $\mathbf{1}$, and $\tilde{\mathbf{1}}$. Here $\tilde{\mathbf{1}} \cong \mathbf{1}$ and is written as such to emphasise there is another field present different to that lying in $\mathbf{1}$. As we are taking symmetric pairings of the fields we only need to consider

$$\begin{aligned} &\mathbf{6}\odot\mathbf{6} = \bar{\mathbf{6}}\oplus\mathbf{15}, && \mathbf{8}\odot\mathbf{8} = \mathbf{1}\oplus\mathbf{8}\oplus\mathbf{27}, \\ &\mathbf{6}\otimes\bar{\mathbf{6}} = \mathbf{1}\oplus\mathbf{8}\oplus\mathbf{27}, && \mathbf{8}\otimes\mathbf{3} = \mathbf{3}\oplus\bar{\mathbf{6}}\oplus\mathbf{15}, \\ &\mathbf{6}\otimes\mathbf{3} = \mathbf{8}\oplus\mathbf{10} && \mathbf{3}\odot\mathbf{3} = \mathbf{6}, \\ &\mathbf{6}\otimes\bar{\mathbf{3}} = \mathbf{3}\oplus\mathbf{15}, && \mathbf{3}\otimes\bar{\mathbf{3}} = \mathbf{1}\oplus\mathbf{8}. \\ &\mathbf{6}\otimes\mathbf{8} = \bar{\mathbf{3}}\oplus\mathbf{6}\oplus\overline{\mathbf{15}}\oplus\mathbf{24}, && \end{aligned} \tag{9.4.2}$$

Of course, there are also the products with the trivial representation $\mathbf{1}$ and $\tilde{\mathbf{1}}$. Furthermore, any missing decompositions are complex conjugates of the above.

9.4.1 Complex Lagrangian

Armed with the representation theory above, we can write terms that one expects in the complex linearised Lagrangian. Now,

$$\mathbf{3} \cong \Lambda^{2,0} \cong \Lambda^{0,1}, \ \mathbf{6} \cong S^{0,2} \cong \Lambda_0^{2,1}, \ \mathbf{8} \cong S_0^{1,1} \cong \Lambda_0^{1,1}, \text{ and } \tilde{\mathbf{1}} \cong \mathbf{1} \cong C^\infty(M)g \cong C^\infty(M)\omega. \tag{9.4.3}$$

along with any complex conjugates. Our tensor fields will be constructed as follows

- Let $N_{\alpha\beta}\mathrm{d}z^\alpha \odot \mathrm{d}z^\beta$ be a tensor fields in $\bar{\mathbf{6}}$, and let $M_{\alpha\beta}\mathrm{d}z^\alpha \wedge \mathrm{d}z^\beta$ be a tensor fields in $\mathbf{3}$. Then define $H = M + N$ be a field with no symmetries in $\mathbf{3} \oplus \bar{\mathbf{6}}$. This means $\bar{H}$ is a field in $\bar{\mathbf{3}} \oplus \mathbf{6}$.

- Let h be a field in $\mathbf{8} \oplus \mathbf{1}$, h is a real tensor field.

- Finally, let c be a real field in $\tilde{\mathbf{1}} \cong \mathbf{1}$.

One now couples all the fields and using representation theory we can determine all possible kinetic terms in the linearised theory

- $H\bar{H}$ corresponds to decomposition $(\mathbf{3} \oplus \bar{\mathbf{6}}) \otimes (\bar{\mathbf{3}} \oplus \mathbf{6}) = 4(\mathbf{8}) \oplus 2(\mathbf{1}) \oplus \mathbf{10} \oplus \bar{\mathbf{10}} \oplus \mathbf{27}$. Hence, there are 4 terms with derivatives of type $p_a\bar{p}^b$ and 2 terms with derivatives of type p^2. It should be noted that since there are representations and their complex conjugate (consider NN+MM terms) there are derivatives with 4 real degrees of freedom and a single derivative with 1 complex degree of freedom.

- HH corresponds to the decomposition $(\mathbf{3} \oplus \bar{\mathbf{6}}) \odot (\mathbf{3} \oplus \bar{\mathbf{6}}) = \mathbf{3} \oplus \bar{\mathbf{3}} \oplus 2(\mathbf{6}) \oplus 2(\mathbf{15})$. Hence, there are 2 terms with derivatives of type $p_a p_b$. Since there are complex conjugate terms, one will have derivatives with 2 complex degrees of freedom.

- Hh corresponds to the decomposition $(\mathbf{1} \oplus \mathbf{8}) \otimes (\mathbf{3} \oplus \bar{\mathbf{6}}) = 3(\mathbf{3}) \oplus 3(\bar{\mathbf{6}}) \oplus \mathbf{15} \oplus \bar{\mathbf{15}} \oplus \mathbf{24}$. Hence, there are 3 derivatives of the types $\bar{p}_a\bar{p}_b$. Since there are complex conjugate terms, there will be derivatives with 3 complex degrees of freedom.

- Hc corresponds to the decomposition $\mathbf{3} \oplus \bar{\mathbf{6}}$. There is a derivative of type $\bar{p}_a\bar{p}_b$: since there are complex conjugate terms, the derivative will have 1 complex degree of freedom.

- hh corresponds to the decomposition $(\mathbf{1} \oplus \mathbf{8}) \odot (\mathbf{1} \oplus \mathbf{8}) = 2(\mathbf{1}) \oplus 2(\mathbf{8}) \oplus \mathbf{27}$. There are 2 derivatives of type $p_a\bar{p}_b$ and 2 of the type p^2. Hence, the derivatives will have 4 real degrees of freedom.

- hc corresponds to the decomposition $(\mathbf{1} \oplus \mathbf{8}) \otimes \tilde{\mathbf{1}}$. Hence, there is a derivative of type p^2 and $p_a\bar{p}_b$ and hence 2 real degrees of freedom.

- c^2 corresponds to $\tilde{\mathbf{1}} \cong \mathbf{1}$, and so the derivative has 1 real degree of freedom.

Therefore, there are 7 complex derivatives (and their conjugates) and 11 real derivatives constituting the kinetic terms in the linear Lagrangian.

Diffeomorphism transformation properties

We wish to determine the transformation rules for the fields that we have introduced. Let A be the perturbation of a real 2-form parameterised as follows

$$A = A_{\mu\nu}\mathrm{d}z^{\mu\nu} + A_{\mu\bar{\nu}}\mathrm{d}z^{\mu\bar{\nu}} + \text{complex conjugate (c.c.)}. \tag{9.4.4}$$

Since it is tangent to ω it has the form

$$A = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} e^{(M+S)t}\cdot\omega + \text{c.c.} \tag{9.4.5}$$

This characterisation was made in the previous section and these pieces have been described already. Explicitly

$$A = \omega_{\mu\bar{\lambda}}g^{\bar{\lambda}\rho}M_{\rho\nu}\mathrm{d}z^{\mu\nu} + \omega_{\mu\bar{\lambda}}g^{\bar{\lambda}\sigma}S_{\sigma\bar{\nu}}\mathrm{d}z^{\mu\bar{\nu}}. \tag{9.4.6}$$

Under diffeomorphism the various pieces of A transform differently, we have

$$\delta A = \delta A_{\mu\nu}\mathrm{d}z^{\mu\nu} + \delta A_{\mu\bar{\nu}}\mathrm{d}z^{\mu\bar{\nu}} + \text{c.c.} \tag{9.4.7}$$

Knowing that ω is a constant background 2-form, under diffeomorphism, i.e. the Lie derivative & Cartan's magic formula, that variation in $\mathbf{3}$ and $\mathbf{8}\oplus\mathbf{1}$ is given as

$$\delta A_{\alpha\beta} = \delta\mathbf{i}M_{\alpha\beta} = \partial_{[\alpha}\xi_{|\mu|}\omega_{\beta]\bar{\nu}}g^{\mu\bar{\nu}},\ \text{and}\ \delta A_{\alpha\bar{\beta}} = \delta\mathbf{i}S_{\alpha\bar{\beta}} = \partial_{[\alpha}\xi_{|\bar{\mu}|}\omega_{\bar{\beta}]\nu}g^{\bar{\mu}\nu}. \tag{9.4.8}$$

Here ξ generates the diffeomorphisms. Simplifying the above gives,

$$\delta M_{\alpha\beta} = \partial_{[\alpha}\xi_{\beta]}\ \text{and}\ \delta h_{\alpha\bar{\beta}} = \partial_{(\alpha}\xi_{\bar{\beta})}. \tag{9.4.9}$$

The diffeomorphism of the trace is extracted by tracing the second equation giving

$$\delta h = \frac{1}{2}(\partial\xi + \overline{\partial\xi}). \tag{9.4.10}$$

Where we define $\partial_\mu\xi^\mu \coloneqq \partial\xi$, and $\partial_{\bar{\mu}}\xi^{\bar{\mu}} \coloneqq \overline{\partial\xi}$. As we will see further on, δh is the real part of the perturbation of a complex function f. This is why we see two copies of $\mathbf{1}$. There is a perturbation associated with the imaginary part of f. This is extracted by considering B, a non-degenerate perturbation of a complex 3-form of the following kind

$$B = B_{\alpha\beta\gamma}\mathrm{d}z^{\alpha\beta\gamma} + B_{\bar{\alpha}\beta\gamma}\mathrm{d}z^{\bar{\alpha}\beta\gamma}. \tag{9.4.11}$$

It is tangent to Ω and so has the form

$$B = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} e^{(M+N+g+\omega)t}\cdot\Omega. \tag{9.4.12}$$

Taking the variation of the various components B

$$\delta B_{\bar{\alpha}\beta\gamma} = \partial_{\bar{\alpha}}\xi_{\bar{\mu}}\Omega_{\nu\beta\gamma}g^{\bar{\mu}\nu},\ \text{and}\ \delta B_{\alpha\beta\gamma} = 3\partial_\alpha\xi_{\bar{\mu}}\Omega_{\nu\beta\gamma}g^{\bar{\mu}\nu}. \tag{9.4.13}$$

Consider the variation $B_{\bar{\alpha}\beta\gamma} \in \Lambda^{2,1}$. B can be written as the sum of a complex symmetric (0,2) tensor N, and an complex antisymmetric (0,2) tensor M which have a single index contracted with the complex (3,0) form Ω, i.e. $B_{\bar{\alpha}\beta\gamma} = (N_{\bar{\alpha}\bar{\mu}} + M_{\bar{\alpha}\bar{\mu}})\Omega_{\nu\beta\gamma}g^{\bar{\mu}\nu}$. Then taking the variation and recalling (9.4.13), one has

$$\partial_{\bar{\alpha}}\xi_{\bar{\mu}}\Omega_{\nu\beta\gamma}g^{\bar{\mu}\nu} = \delta B_{\bar{\alpha}\beta\gamma} = (\delta N_{\bar{\alpha}\bar{\mu}} + \delta M_{\bar{\alpha}\bar{\mu}})\Omega_{\nu\beta\gamma}g^{\bar{\mu}\nu}. \tag{9.4.14}$$

The antisymmetric part has already been extracted in (9.4.9), this means that

$$\delta N_{\bar{\mu}\bar{\nu}} = \partial_{(\bar{\mu}}\xi_{\bar{\nu})}. \tag{9.4.15}$$

Since $B_{\alpha\beta\gamma} = f\Omega_{\alpha\beta\gamma}$, for the function $f = h + \mathbf{i}c$, the respective traces of the metric and the (1,1)-form.

$$\delta B_{\alpha\beta\gamma} = \delta f \Omega_{\alpha\beta\gamma} = 3\partial_\alpha \xi_{\bar{\mu}}\Omega_{\nu\beta\gamma}g^{\bar{\mu}\nu}. \tag{9.4.16}$$

Now multiplying both sides by $\Omega_{\bar{\lambda}\bar{\tau}\bar{\varphi}}$, and contracting indices totally on the left-hand side gives

$$\delta f \Omega_{\alpha\beta\gamma}\Omega_{\bar{\lambda}\bar{\tau}\bar{\varphi}}g^{\bar{\lambda}\alpha}g^{\bar{\tau}\beta}g^{\bar{\varphi}\gamma} = 3\partial_\alpha\xi_{\bar{\mu}}\Omega_{\nu\beta\gamma}g^{\bar{\mu}\nu}\Omega_{\bar{\lambda}\bar{\tau}\bar{\varphi}}g^{\bar{\lambda}\alpha}g^{\bar{\tau}\beta}g^{\bar{\varphi}\gamma}. \tag{9.4.17}$$

We now use the following identities

$$\Omega_{\alpha\beta\gamma}\Omega_{\bar{\lambda}\bar{\tau}\bar{\varphi}}g^{\bar{\lambda}\alpha}g^{\bar{\tau}\beta}g^{\bar{\varphi}\gamma} = 12, \text{ and } \Omega_{\nu\beta\gamma}\Omega_{\bar{\lambda}\bar{\tau}\bar{\varphi}}g^{\bar{\tau}\beta}g^{\bar{\varphi}\gamma} = 4(g_{\nu\bar{\lambda}} - \omega_{\nu\bar{\lambda}}), \tag{9.4.18}$$

one has

$$\delta f = \partial_\alpha\xi_{\bar{\mu}}(g^{\alpha\bar{\mu}} - \omega^{\alpha\bar{\mu}}). \tag{9.4.19}$$

The above recovers (9.4.10), but one also uncovers the perturbation

$$\delta c = \frac{1}{2\mathbf{i}}(\partial\xi - \overline{\partial\xi}). \tag{9.4.20}$$

This is the other field **1**. In summary,

Lemma 9.4.1.1. Consider the tensor fields parameterising the $\mathfrak{gl}(3,\mathbb{C})$ orbit to the SU(3)-structures and metric. Then the diffeomorphisms along a holomorphic vector field ξ are given as

$$\delta H_{\alpha\beta} = \partial_\alpha\xi_\beta,\ \delta h_{\alpha\bar{\beta}} = \partial_\alpha\xi_{\bar{\beta}},\ \delta h = \frac{1}{2}(\partial\xi + \overline{\partial\xi}),\ \text{and } \delta c = \frac{1}{2\mathbf{i}}(\partial\xi - \overline{\partial\xi}). \tag{9.4.21}$$

9.4.2 Diffeomorphism Invariant Linearised Lagrangian

We now vary the 25 terms that can be written in the Lagrangian. As stated previously, 7 are complex derivatives, so we write down 18 terms with an overall scaling factor Λ_i. We shall vary all 18 terms and write them below, using integration by parts liberally (we assume our 6-dimensional manifold is compact), and assuming total derivatives vanish.

$$\begin{aligned}
\Lambda_1 &: \ \partial_{\bar{\sigma}}\partial_\kappa\delta(H^{\bar{\sigma}\bar{\beta}}H^{\kappa}{}_{\bar{\beta}}) = \partial^{\bar{\alpha}}\xi^{\bar{\beta}}\partial^2 H_{\bar{\alpha}\bar{\beta}} + \partial^\alpha\xi^\beta\partial^2 H_{\alpha\beta},\\
\Lambda_2 &: \ \partial_{\bar{\sigma}}\partial_\kappa\delta(H^{\bar{\beta}\bar{\sigma}}H^{\kappa}{}_{\bar{\beta}}) + \text{c.c} = \partial\xi\partial^{\bar{\alpha}}\partial^{\bar{\beta}}H_{\bar{\alpha}\bar{\beta}} + \partial^\beta\xi^\alpha\partial^2 H_{\alpha\beta} + \text{c.c},\\
\Lambda_3 &: \ \partial_{\bar{\sigma}}\partial_\kappa\delta(H^{\bar{\alpha}\bar{\sigma}}H_{\bar{\alpha}}{}^{\kappa}) = \partial\xi\partial^{\bar{\alpha}}\partial^{\bar{\beta}}H_{\bar{\alpha}\bar{\beta}} + \overline{\partial\xi}\partial^\alpha\partial^\beta H_{\alpha\beta},\\
\Lambda_4 &: \ \partial^2\delta(H^{\bar{\alpha}\bar{\beta}}H_{\bar{\alpha}\bar{\beta}}) = \partial^{\bar{\alpha}}\xi^{\bar{\beta}}\partial^2 H_{\bar{\alpha}\bar{\beta}} + \partial^\alpha\xi^\beta\partial^2 H_{\alpha\beta},\\
\Lambda_5 &: \ \partial^2\delta(H^{\bar{\beta}\bar{\alpha}}H_{\bar{\alpha}\bar{\beta}}) = \partial^{\bar{\beta}}\xi^{\bar{\alpha}}\partial^2 H_{\bar{\alpha}\bar{\beta}} + \partial^\beta\xi^\alpha\partial^2 H_{\alpha\beta},\\
\Lambda_6 &: \ \partial_\sigma\partial_\kappa\delta(H_{\mu\nu}H_{\alpha\beta})\Omega^{\sigma\mu\nu}\Omega^{\kappa\alpha\beta} + \text{c.c} = -2\partial_\sigma\partial_\mu\xi_\nu\partial_\kappa H_{\alpha\beta}\Omega^{\sigma\mu\nu}\Omega^{\kappa\alpha\beta} + \text{c.c} = 0,
\end{aligned}$$

$$
\begin{aligned}
\Lambda_7:&\quad \partial_\sigma\partial_\kappa\delta(H_{\mu\nu}H_{\alpha\beta})\Omega^{\sigma\mu\alpha}\Omega^{\kappa\nu\beta}+\text{c.c}=2\partial_\sigma\partial_\mu\xi_\alpha\partial_\kappa H_{\nu\beta}\Omega^{\sigma\mu\nu}\Omega^{\kappa\alpha\beta}+\text{c.c}=0,\\
\Lambda_8:&\quad \partial_\sigma\partial_{\bar\kappa}\delta(h^{\sigma\bar\beta}h^{\bar\kappa}{}_{\bar\beta})=\frac{1}{2}(\partial\xi+\overline{\partial\xi})\partial^\mu\partial^{\bar\nu}h_{\mu\bar\nu}+\partial^\mu\xi^{\bar\nu}\partial^2h_{\mu\bar\nu},\\
\Lambda_9:&\quad \partial_\sigma\partial_{\bar\kappa}\delta(h^{\bar\kappa\sigma}h)=\frac{1}{2}(\partial\xi+\overline{\partial\xi})(\partial^2h+\partial^\mu\partial^{\bar\nu}h_{\mu\bar\nu}),\\
\Lambda_{10}:&\quad \partial^2\delta(h^{\alpha\bar\beta}h_{\alpha\bar\beta})=2\partial^\mu\xi^{\bar\nu}\partial^2h_{\mu\bar\nu},\\
\Lambda_{11}:&\quad \partial^2\delta(h^2)=(\partial\xi+\overline{\partial\xi})\partial^2h,\\
\Lambda_{12}:&\quad \partial_{\bar\sigma}\partial_{\bar\kappa}\delta(H^{\bar\sigma\bar\nu}h^{\bar\kappa}{}_{\bar\nu})+\text{c.c}=-\partial^\mu\xi^{\bar\nu}\partial^2h_{\mu\bar\nu}+\partial^\alpha\xi^\beta\partial^2H_{\alpha\beta}+\overline{\partial\xi}\partial^\alpha\partial^\beta H_{\alpha\beta}+\text{c.c},\\
\Lambda_{13}:&\quad \partial_{\bar\sigma}\partial_{\bar\kappa}\delta(H^{\bar\nu\bar\sigma}h^{\bar\kappa}{}_{\bar\nu})+\text{c.c}=\overline{\partial\xi}\partial^\mu\partial^{\bar\nu}h_{\mu\bar\nu}+\partial^\beta\xi^\alpha\partial^2H_{\alpha\beta}+\overline{\partial\xi}\partial^\alpha\partial^\beta H_{\alpha\beta}+\text{c.c},\\
\Lambda_{14}:&\quad \partial_{\bar\sigma}\partial_{\bar\kappa}\delta(H^{\bar\sigma\bar\kappa}h)+\text{c.c}=\overline{\partial\xi}\partial^2h+\frac{1}{2}(\partial\xi+\overline{\partial\xi})\partial^\alpha\partial^\beta H_{\alpha\beta}+\text{c.c},\\
\Lambda_{15}:&\quad \partial_{\bar\sigma}\partial_{\bar\kappa}\delta(H^{\bar\sigma\bar\kappa}c)+\text{c.c}=\overline{\partial\xi}\partial^2c+\frac{1}{2\mathbf{i}}(\partial\xi-\overline{\partial\xi})\partial^\alpha\partial^\beta H_{\alpha\beta}+\text{c.c},\\
\Lambda_{16}:&\quad \partial_{\bar\sigma}\partial_\kappa\delta(h^{\bar\sigma\kappa}c)=\frac{1}{2}(\partial\xi+\overline{\partial\xi})\partial^2c+\frac{1}{2\mathbf{i}}(\partial\xi-\overline{\partial\xi})\partial^\mu\partial^{\bar\nu}h_{\mu\bar\nu},\\
\Lambda_{17}:&\quad \partial^2\delta(hc)=\frac{1}{2\mathbf{i}}(\partial\xi-\overline{\partial\xi})\partial^2h+\frac{1}{2}(\partial\xi+\overline{\partial\xi})\partial^2c,\\
\Lambda_{18}:&\quad \partial^2\delta(c^2)=-\mathbf{i}(\partial\xi-\overline{\partial\xi})\partial^2c.
\end{aligned}
$$

Where $\partial^2 \coloneqq \partial_\alpha\partial^{\bar\alpha}$. Collecting like terms gives the following set of equations that must be set to zero

$$
\begin{aligned}
\partial^{\bar\alpha}\xi^{\bar\beta}\partial^2H_{\bar\alpha\bar\beta}&:\Lambda_1+\Lambda_4+\bar\Lambda_{12}, & \partial\xi\partial^\mu\partial^{\bar\nu}h_{\mu\bar\nu}&:\frac{1}{2}\Lambda_8+\frac{1}{2}\Lambda_9+\bar\Lambda_{13}+\frac{1}{2\mathbf{i}}\Lambda_{16},\\
\partial^\alpha\xi^\beta\partial^2H_{\alpha\beta}&:\Lambda_1+\Lambda_4+\Lambda_{12}, & \overline{\partial}\xi\partial^\mu\partial^{\bar\nu}h_{\mu\bar\nu}&:\frac{1}{2}\Lambda_8+\frac{1}{2}\Lambda_9+\Lambda_{13}-\frac{1}{2\mathbf{i}}\Lambda_{16},\\
\partial^{\bar\beta}\xi^{\bar\alpha}\partial^2H_{\bar\alpha\bar\beta}&:\bar\Lambda_2+\Lambda_5+\bar\Lambda_{13}, & \partial^\mu\xi^{\bar\nu}\partial^2h_{\mu\bar\nu}&:\Lambda_8+2\Lambda_{10}-\Lambda_{12}-\bar\Lambda_{12},\\
\partial^\beta\xi^\alpha\partial^2H_{\alpha\beta}&:\Lambda_2+\Lambda_5+\Lambda_{13}, & \partial\xi\partial^2h&:\frac{1}{2}\Lambda_9+\Lambda_{11}+\bar\Lambda_{14}+\frac{1}{2\mathbf{i}}\Lambda_{17},\\
\partial\xi\partial^{\bar\alpha}\partial^{\bar\beta}H_{\bar\alpha\bar\beta}&:\Lambda_2+\Lambda_3+\bar\Lambda_{12}+\bar\Lambda_{13}, & \overline{\partial\xi}\partial^2h&:\frac{1}{2}\Lambda_9+\Lambda_{11}+\Lambda_{14}-\frac{1}{2\mathbf{i}}\Lambda_{17},\\
\overline{\partial\xi}\partial^\alpha\partial^\beta H_{\alpha\beta}&:\bar\Lambda_2+\Lambda_3+\Lambda_{12}+\Lambda_{13}, & \partial\xi\partial^2c&:\bar\Lambda_{15}+\frac{1}{2}\Lambda_{16}+\frac{1}{2\mathbf{i}}\Lambda_{17}-\mathbf{i}\Lambda_{18},\\
\overline{\partial\xi}\partial^{\bar\alpha}\partial^{\bar\beta}H_{\bar\alpha\bar\beta}&:\frac{1}{2}\bar\Lambda_{14}+\frac{1}{2\mathbf{i}}\bar\Lambda_{15}, & \overline{\partial\xi}\partial^2c&:\Lambda_{15}+\frac{1}{2}\Lambda_{16}-\frac{1}{2\mathbf{i}}\Lambda_{17}+\mathbf{i}\Lambda_{18}.\\
\partial\xi\partial^\alpha\partial^\beta H_{\alpha\beta}&:\frac{1}{2}\Lambda_{14}-\frac{1}{2\mathbf{i}}\Lambda_{15}, & &
\end{aligned}
\tag{9.4.22}
$$

Re-writing the above equations gives

$$
\begin{aligned}
\Lambda_1+\Lambda_4+\mathrm{Re}(\Lambda_{12})&=0, & \mathrm{Im}(\Lambda_{12}+\Lambda_{13}+\Lambda_2)&=0, & 2\mathrm{Re}(\Lambda_{12})-\Lambda_8-2\Lambda_{10}&=0,\\
\mathrm{Im}(\Lambda_{12})&=0, & \mathrm{Re}(\Lambda_{14})+\mathrm{Im}(\Lambda_{15})&=0, & 2\mathrm{Re}(\Lambda_{14})+\Lambda_9+2\Lambda_{11}&=0,\\
\mathrm{Re}(\Lambda_2+\Lambda_{13})+\Lambda_5&=0, & \mathrm{Im}(\Lambda_{14})-\mathrm{Re}(\Lambda_{15})&=0, & 2\mathrm{Im}(\Lambda_{14})-\Lambda_{17}&=0,\\
\mathrm{Im}(\Lambda_{13}+\Lambda_2)&=0, & 2\mathrm{Re}(\Lambda_{13})+\Lambda_8+\Lambda_9&=0, & 2\mathrm{Re}(\Lambda_{15})+\Lambda_{16}&=0,\\
\mathrm{Re}(\Lambda_{12}+\Lambda_{13}+\Lambda_2)+\Lambda_3&=0, & 2\mathrm{Re}(\Lambda_{13})-\Lambda_{16}&=0, & \Lambda_{17}+2\Lambda_{18}&=0.
\end{aligned}
\tag{9.4.23}
$$

The solutions are then

$$
\begin{aligned}
\mathrm{Im}(\Lambda_{12}) = \mathrm{Im}(\Lambda_{13}) = \mathrm{Im}(\Lambda_{2}) &= 0,\\
\mathrm{Re}(\Lambda_{12}) &= -\Lambda_1 - \Lambda_4,\\
\mathrm{Re}(\Lambda_{2}) &= -\frac{1}{2}\Lambda_{16} - \Lambda_3 + \Lambda_1 + \Lambda_4,\\
\mathrm{Re}(\Lambda_{14}) &= -\mathrm{Im}(\Lambda_{15}),\\
\Lambda_{11} &= \frac{1}{2}\Lambda_{16} - (\Lambda_1 + \Lambda_4 + \Lambda_{10}) - \mathrm{Re}(\Lambda_{14}),\\
-\mathrm{Re}(\Lambda_{13}) = \mathrm{Re}(\Lambda_{15}) = \mathrm{Im}(\Lambda_{14}) &= \frac{1}{2}\Lambda_{17} = -\frac{1}{2}\Lambda_{16} = -\Lambda_{18},\\
\Lambda_5 &= \Lambda_3 - \Lambda_1 - \Lambda_4,\\
\Lambda_8 &= -2(\Lambda_1 + \Lambda_4 + \Lambda_{10}),\\
\Lambda_9 &= -\Lambda_{16} + 2(\Lambda_1 + \Lambda_4 + \Lambda_{10}).
\end{aligned}
\tag{9.4.24}
$$

There are 15 equations above that reduce 21 of the 25 parameters to 6 free ones

$$\{\Lambda_1, \Lambda_3, \Lambda_4, \Lambda_{10}, \mathrm{Re}(\Lambda_{14}), \Lambda_{16}\}. \tag{9.4.25}$$

Furthermore, as the terms with $\Lambda_{6,7}$ (and their complex conjugates) vanish under diffeomorphisms, there are 2 more complex free parameters, which means there are 10 free parameters in the theory:

$$\{\Lambda_1, \Lambda_3, \Lambda_4, \Lambda_6, \Lambda_7, \Lambda_{10}, \mathrm{Re}(\Lambda_{14}), \Lambda_{16}\}. \tag{9.4.26}$$

We now conclude this chapter with the following theorem.

Theorem 9.4.2.1. The space of linearised diffeomorphism invariant Lagrangians on a compact 6-dimensional spin manifold M with SU(3)-structure, $\{\omega, \Omega\}$, is constructed from a basis of 8 pieces: 6 pieces parameterised by real parameters and 2 pieces parameterised by complex parameters, as follows

$$\mathcal{L} \approx \mathcal{L}_{GR} + \mathcal{L}_{H} + \mathcal{L}_{H'} + \mathcal{L}_{16} + \mathcal{L}_{14} + \mathcal{L}_{4} + \mathcal{L}_{3} + \mathcal{L}_{1}. \tag{9.4.27}$$

Here each piece is given as

$$
\begin{aligned}
\mathcal{L}_{GR} &\approx \Lambda_{10}\left(\partial^2 h^{\alpha\bar{\beta}} h_{\alpha\bar{\beta}} - 2\partial_\sigma\partial_{\bar{\kappa}} h^{\sigma\bar{\beta}} h^{\bar{\kappa}}{}_{\bar{\beta}} + 2h\partial_\sigma\partial_{\bar{\kappa}} h^{\sigma\bar{\kappa}} - \partial^2 h\right),\\
\mathcal{L}_{H} &\approx \Lambda_6\left(\partial_\sigma\partial_\kappa H_{\mu\nu} H_{\alpha\beta}\Omega^{\sigma\mu\nu}\Omega^{\kappa\alpha\beta}\right),\\
\mathcal{L}_{H'} &\approx \Lambda_7\left(\partial_\sigma\partial_\kappa H_{\mu\nu} H_{\alpha\beta}\Omega^{\sigma\mu\alpha}\Omega^{\kappa\nu\beta}\right),\\
\mathcal{L}_{16} &\approx \Lambda_{16}\Bigg(\left(\partial_{\bar{\sigma}}\partial_{\bar{\kappa}} H^{\bar{\nu}\bar{\sigma}} h^{\bar{\kappa}}{}_{\bar{\nu}} + \partial_\sigma\partial_\kappa H^{\nu\sigma} h^{\kappa}{}_{\nu}\right) + \left(\partial_{\bar{\sigma}}\partial_\kappa H^{\bar{\beta}\bar{\sigma}} H^{\kappa}{}_{\bar{\beta}} + \partial_\sigma\partial_{\bar{\kappa}} H^{\beta\sigma} H^{\bar{\kappa}}{}_{\beta}\right)\\
&\qquad - \mathbf{i}\left((h - \mathbf{i}c)\partial_{\bar{\sigma}}\partial_{\bar{\kappa}} H^{\bar{\sigma}\bar{\kappa}} - (h + \mathbf{i}c)\partial_\sigma\partial_\kappa H^{\sigma\kappa}\right) - \frac{1}{2}\partial^2(h-c)^2 - (h-c)\partial_\sigma\partial_{\bar{\kappa}} h^{\sigma\bar{\kappa}}\Bigg),\\
\mathcal{L}_{14} &\approx \frac{\Lambda_{14}}{2}\left((h - \mathbf{i}c)\,\partial_{\bar{\sigma}}\partial_{\bar{\kappa}} H^{\bar{\sigma}\bar{\kappa}} + (h + \mathbf{i}c)\,\partial_\sigma\partial_\kappa H^{\sigma\kappa} - 2\partial^2 h^2\right),\\
\mathcal{L}_{4} &\approx \frac{\Lambda_4}{2}\left(h\partial_\sigma\partial_{\bar{\kappa}} h^{\sigma\bar{\kappa}} - \partial_\sigma\partial_{\bar{\kappa}} h^{\sigma\bar{\beta}} h^{\bar{\kappa}}{}_{\bar{\beta}} + 2\partial^2 H^{\alpha\beta}\left(H_{\alpha\beta} - H_{\beta\alpha}\right)\right),\\
\mathcal{L}_{3} &\approx \frac{\Lambda_3}{2}\left(\partial^2 H^{\alpha\beta} H_{\alpha\beta} + \left(\partial_{\bar{\sigma}}\partial_\kappa H^{\bar{\beta}\bar{\sigma}} H^{\kappa}{}_{\bar{\beta}} + 2\partial_\sigma\partial_{\bar{\kappa}} H^{\beta\sigma} H^{\bar{\kappa}}{}_{\beta}\right) + \partial_{\bar{\sigma}}\partial_\kappa H^{\bar{\alpha}\bar{\sigma}} H_{\bar{\alpha}}{}^{\kappa}\right),
\end{aligned}
$$

$$\mathcal{L}_1 \approx \Lambda_1 \Bigg(2h\partial_\sigma \partial_{\bar{\kappa}} h^{\sigma\bar{\kappa}} - 2\partial_\sigma \partial_{\bar{\kappa}} h^{\sigma\bar{\beta}} h^{\bar{\kappa}}{}_{\bar{\beta}} - \partial^2 h^2 - \partial^2 H^{\alpha\beta} H_{\beta\alpha} + \frac{1}{2}\left(\partial_{\bar{\sigma}} \partial_\kappa H^{\bar{\beta}\bar{\sigma}} H^{\alpha}{}_{\bar{\beta}} + \partial_\sigma \partial_{\bar{\kappa}} H^{\beta\sigma} H^{\bar{\kappa}}{}_{\beta} \right)$$
$$- \frac{1}{2}\left(\partial_{\bar{\sigma}\bar{\kappa}} H^{\bar{\sigma}\bar{\nu}} h^{\bar{\kappa}}{}_{\bar{\nu}} + \partial_\sigma \partial_\kappa H^{\sigma\nu} h^{\kappa}{}_{\nu} \right) \Bigg).$$

Remark 9.4.2.1. $\mathcal{L}_{GR}$ is the linearisation of the Einstein-Hilbert Lagrangian. One can exhibit its canonical form by further integrating by parts $\mathcal{L}_{GR}$, and rescaling $\Lambda_{10} \to -\frac{1}{2}\Lambda_{10}$.

Chapter 10

Discussions

We now make some concluding remarks about the work completed in this thesis, and possible avenues of study one could pursue.

10.1 Non-Classical Spinorial G-structures

10.1.1 Classification of Real Spinors and Integrability

The stabilisers of the impure orbits of Spin(8) were SU(4) and Spin(7). By embedding Spin(8) into Spin(10) and Spin(12), we parameterised impure orbits using octonions. This yielded the stabilisers SU(4) × U(1) and Spin(7) × U(1) within Spin(10), and SU(4) × SU(2), SU(6), and Spin(7) × SU(2) within Spin(12). However, many more stabilisers in 12 dimensions, at least, cannot be accessed using Spin(8) alone.

In the analysis of 12 dimensions, we used a modified Majorana constraint to reduce the difficulty of the problem. This neat trick, effective in 10 dimensions, and a convenient parameterisation of spinors in 8 dimensions, allowed us to access new stabilisers. If one relaxes the modified Majorana constraint, the classification of stabilisers becomes too difficult, as we can no longer eliminate components with the aforementioned trick. Hence, the best way to move forward in classifying all real stabilisers in higher dimensions remains uncertain.

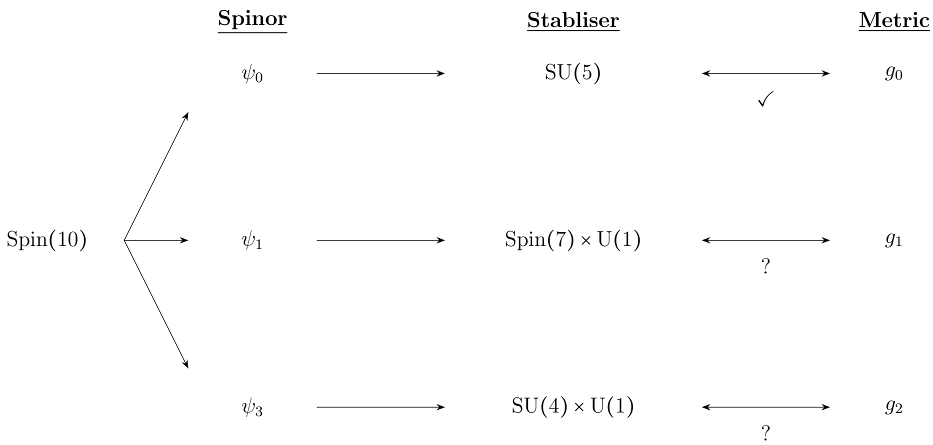


Figure 10.1: $\psi_{0,1,2}$ are representatives of an orbit, with stabiliser SU(5), Spin(7) × U(1), or SU(4) × U(1). The metric at each point is generated only for ψ_0, $g_{1,2}$ are not known for now.

Next, let us consider the integrability of non-classical spinorial G-structures. Recall that there are three types of spinorial G-structures in 10 dimensions: SU(5), SU(4) × U(1), and Spin(7) × U(1). Consider the set of canonical differential forms associated with the classical SU(5)-structure that stabilises a pure spinor in 10 dimensions, ψ_0, denoted as $\{\omega_{10}, \Omega_{10}\}$. One can algebraically reconstruct the metric using the formula $g(X,Y) = \omega_{10}(J_{10}X, Y)$ (where J_{10} is the complex structure derived from Ω_{10} via the formula (9.0.1)), and then evaluate the covariant derivatives of ω_{10} and Ω_{10} with respect to the Levi-Civita connection, ∇. If they are parallel with respect to ∇, then the SU(5)-structure is integrable.

We have shown in the main text (see proposition 5.3.3.1) that the exterior derivatives are both necessary and sufficient for the integrability of the SU(5)-structure. On the other hand, as shown in figure 10.1, we have not yet provided a formula to generate a metric in 10 dimensions for SU(4)×U(1)- or Spin(7)×U(1)-structures. Consequently, the conditions for integrability of these new structures with respect to the Levi-Civita connection remain unclear. However, we can compute the closure of canonical differential forms associated with SU(4)×U(1)- and Spin(7)×U(1)-structures. Work in this direction has been initiated in [Kra24b] regarding SL(4, $\mathbb{R}$)-structures in 8 dimensions. These speculations lead one to make the following conjecture:

Conjecture on Closure. Let M be a compact spin manifold of dimension n, and let ψ be a spinor be stabilised by $G \subset \mathrm{Spin}(2n)$. A non-classical spinorial G-structure is integrable (the associated canonical differential forms are parallel with respect to a metric connection built from algebraic relations between the canonical differential forms) if and only if each differential form in the set of canonical differential forms is closed.

Finally, the examples exhibited from Spin(n, n) are even more mysterious. Hitchin's generalised geometry, as discussed in [Hit00b], appears to be the most natural framework to study spinorial G-structures where $G \subset \mathrm{Spin}(n,n)$. We hope in the future to at least motivate the reconstruction of the metric from a collection of differential forms for non-classical spinorial G-structures in Spin($2n$), and to better understand Hitchin's generalised geometry to apply it to spinorial G-structures in Spin(n, n). It would be fantastic if there was a generalised framework that could then be applied to spinorial G-structures inside Spin(r, s).

10.1.2 Classification of Complex spinors

The beauty of chapter 6 lies in the elementary nature of its design. It begins with the consideration of pure spinors and swiftly connects these to geometric shapes and counting principles. This method recovers all classical results for simple parts of stabilisers, and moreover, these results are extracted via geometric maps, as introduced in previous chapters. However, this method encounters issues in higher dimensions, inherent to both the approach taken and the objects studied. In 14 dimensions, the space of Weyl spinors is $2^7 = 128$, while the dimension of the group is $7 \times 13 = 91$. Here, the space of spinors exceeds that of the group, thus constraining the orbits.

In 16 dimensions, the space of Weyl spinors amounts to $2^8 = 256$, but the dimension of the group is $8 \times 15 = 120$. The larger space of spinors means that orbits are no longer constrained. [L V82] has meticulously documented *all* the stabilisers. This significant achievement has, unfortunately, not been translated into English; nonetheless, the results are outstanding. One does not need to read or understand the Cyrillic script to appreciate the extensive list provided.

We verified many of the orbits using brute force code, constrained by the occupation number formulas, as outlined in [Wel23]. A preliminary analysis through the code demonstrates its capability to reproduce the same results as those found in [DR93], which constructed *some* of the orbits in [L V82] using the symmetries of $E_8 \supset \mathrm{Spin}(16)$. Further work is necessary to refine the brute force approach and/or to enhance our algebraic understanding of spinors for $d > 8$.

10.2 Integrability and Higher Dimensional Gravity

The Plebanski formalism, as explained in chapter 7 and elaborated in part II, appears to be the *most* natural generalisation for gravity-type theories one could propose. This is hopefully demonstrated through the discussion of SU(2)-structures in chapter 8. The story for SU(3), however, is less than complete. While all the ingredients to write a Plebanski-type action functional are present, they have not yet coalesced into a full theory due to the challenging selection of suitable auxiliary field(s). We hope to revisit this in the future and further investigate the Plebanski formalism for SU(3)-structures, aiming to achieve an Einstein condition by extremising the action.

The analysis of the Plebanski formalism required unique approaches for SU(2) and SU(3). However, one hopes there is an underlying theory that extends to all SU(m)-structures for $m \in \mathbb{Z}_{\geq 4}$. For instance, in 8 dimensions, using the tools developed throughout this thesis, it can be shown that the GL$(8,\mathbb{R})$ action is not generic on SU(4)-structures. Indeed, the orbit of the action on the real part of the holomorphic top form is contained within $[[\Lambda^{4,0}]] \oplus [[\Lambda^{3,1}]] \subset \Lambda^4$. This analysis resembles that of Spin(7), where the space of admissible 4-forms is also not generic in Λ^4, [Bry87; Kar09]. Furthermore, we were able to show that the most natural second-order in derivative conditions on the canonical differential forms in 4 dimensions are Einstein conditions. However, we were not able to do the same in 6 dimensions. This is a difficult question because SU(2) gauge invariance gave a unique action function to extremise. While in the case of SU(3) one has an 8-parameter family of action functionals, even at the linearised level. There is no clear gauge one can impose, like was done in 4 dimensions, to construct the most natural set of second order PDEs.

Turning to a more physically conceptualised approach, even the linearised theory performed in 6 dimensions presents challenges, due to a plethora of fields coupled to themselves and the metric. G_2 provides a more illustrative example case to understand the symmetries of the theory. This is because, in the current studies of the linearised theory, only a vector field needs to be coupled to the metric. Additionally, there is an inherent link between SU(3) and G_2 — one can calibrate a G_2 structure to exhibit SU(3)-structures. Specifically, by selecting a *time* direction in 7 dimensions, a generic 3-form in 7 dimensions decomposes into $\mathrm{Re}(\Omega)$ and ω, while its Hodge dual breaks into $\mathrm{Im}(\Omega)$ and $\omega \wedge \omega$ (where Ω and ω represent a complex 3-form and a real symplectic 2-form, respectively, in the remaining 6 dimensions). Work is currently ongoing to study the linearised theory of G_2-structures, such that SU(3)-structures are clearly exhibited.

10.3 Flows of G-structures

A large scheme of interesting ideas that have not been explored is geometric flows as gradient flows from the second-order diffeomorphism invariant Lagrangians that we have written for SU(2), and could write, for SU(3). This seems like a fruitful path from the level of Ricci flow. Recall for(M, g), a Riemannian manifold, the Ricci flow is defined as

$$\frac{\partial}{\partial t} g = -2\mathrm{Ric}(g). \tag{10.3.1}$$

Here Ric is the Ricci tensor. The intuition for this equation comes from being able to deform the metric in the direction in the direction of its curvature in hopes of extremising to the most "natural" metric. As we are inspired by physics, we instead consider the following flow

$$S[g] = \int_M \mathrm{R}\sqrt{-g}, \implies \frac{\partial}{\partial t} g = -\mathrm{Ric} + \frac{1}{2}\mathrm{R}g \tag{10.3.2}$$

where R is the Ricci scalar, and the right-hand-side is the Einstein-Hilbert action for gravity in a vacuum. This does not behave as a good system of differential equations. However, this can be remedied by Perelman's trick, [Per02] exhibiting this flow as a coupled system of differential equations, consisting of a Ricci-flow and heat-flow equation. Indeed, if one considers the action functional,

$$S[g] = \int_M (\mathrm{R} + |\nabla f|^2) e^{-f} \sqrt{-g}, \implies \begin{aligned} &\frac{\partial}{\partial t} g = -2\mathrm{Ric}, \text{ and} \\ &\frac{\partial f}{\partial t} = -\Delta f + |\nabla f|^2 - R, \end{aligned} \tag{10.3.3}$$

then this system does have short-time, and unique solutions. An application that we have yet to apply but is a promising first step is to take the gradient flow of

$$S[\Sigma] = -\frac{1}{2} \int_M \Sigma^i \epsilon^{ijk} A^j(\Sigma) A^k(\Sigma), \tag{10.3.4}$$

and see if one can apply a Perelman-type trick. We hope that other theories that we construct, as a generalisation of the Einstein-Hilbert theory, are also obtainable in a similar fashion.

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