# Explicit K-stability of Fano Varieties

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# Abstract

This thesis completes the classification of local stability thresholds ($\delta$-invariant) for smooth del Pezzo surfaces of degree 2 and explores the compactification of K-moduli for Fano 3-folds. In the first part, we show that this invariant is irrational if and only if there is a unique (-1)-curve passing through the point where we are computing the local invariant. This work can be useful for future verification of K-stability in higher dimensions, this is because the computations of $\delta$-invariants of higher dimensional varieties are often reduced to the computations of $\delta$-invariants of del Pezzo surfaces. The irrationality of the local stability threshold also implies the existence of infinitely many local degenerations of the variety, which can lead to interesting further studies. In the second part, we work on the compactification of one-dimensional components of the moduli spaces of Fano 3-folds by studying degenerate objects. The result on K-moduli gives some of the few existing examples of compactifications of components of the K-moduli space for Fano varieties. There are a total of 6 families with one-dimensional moduli. In this thesis, we focus on 3 of those families, we explain the parameterization of each family, the proof of K-polystability of singular elements and the compactification of the K-moduli component by explicitly describing each K-polystable member of the family.

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# Chapter 1

# Introduction

## 1.1 Overview

K-stability stands at the forefront of current research in algebraic geometry, connecting algebraic and differential geometry. A fundamental problem in differential geometry is to detect extremal canonical metrics. In particular, in complex geometry, the natural objects of study are Kähler manifolds, whose optimal choice of canonical metrics are the so-called constant scalar curvature Kähler (cscK) metrics, and a very important subcategory of these are the so-called Kähler-Einstein (KE) metrics. Note that in this thesis we work over $\mathbb{C}$. Algebraic manifolds with constant curvature can be general type, Calabi-Yau, or Fano manifolds. The former two are known to admit KE metrics by the celebrated results of Aubin and Yau [Aub78; Yau78]. However, the existence of such metrics on Fano manifolds is obstructed. Examples of Fano manifolds without a KE metric have been known for a long time, Matsushima gave the first example in [Mat57]. In the latter half of the 20th century, an algebro-geometric approach was introduced to determine whether a Fano manifold admit a KE metric. This is known as the Yau-Tian-Donaldson conjecture, which indicates that the existence of KE metrics is equivalent to an algebro-geometric condition called K-stability (see characterisation in Theorem 2.2.10), and it was recently proven for Fano varieties with at most klt singularities by [CDS15; Tia15; LXZ22].

**Theorem 1.1.1** (Yau-Tian-Donalson Conjecture [Don02; Tia97; Yau82])**.** *Let $X$ be a Fano variety. Then, $X$ admits a Kähler-Einstein metric if and only if $X$ is K-polystable (see characterisation in Theorem 2.2.12).*

K-stability specifically refers to the stability of a variety with respect to test configurations (see Definition 2.2.1) and their associated numerical invariants (like the Donaldson-Futaki invariant). A test configuration is essentially a one-parameter family whose fibres are degenerations of a variety, often leading to simpler or more "degenerate" varieties. The Donaldson-Futaki invariant is a numerical measure that assigns a real number to each test configuration, but it is generally complicated to compute. Nowadays, an explicit verification of K-stability has become more tangible thanks to valuative criteria, which have been achieved via a groundbreaking introduction of birational geometric techniques to study K-stability, see [FO18; BJ20]. Thus K-stability has been reformulated via numerical invariants, the so-called *$\beta$-invariant* and *stability threshold*, $\delta$. In this thesis, these invariants have a substantial impact. In particular the latter will be the focus of study of Chapter 3. The $\delta$-invariant is defined via two numerical invariants for divisors, $E$, over a variety $X$. These are the log discrepancy, $A_X(E)$, a numerical invariant arising from the Minimal Model Program that "measures" singularities on $X$, and the expected vanishing order, $S(-K_X;E)$, a volume average, see Definition 2.2.6. This stability threshold can be computed globally by considering all the divisors over $X$ or locally if we only consider those divisors containing a specific closed subset of $X$ in their centre.

Our interest in the $\delta$-invariant is mainly in the study of its rationality. In their work, Liu, Xu, and Zhuang demonstrated that for K-polystable Fano varieties, the global stability threshold $\delta(X)$ is a rational number, under the condition that $\delta(X) < \frac{1+\dim X}{\dim X}$ [LXZ22]. However, by choosing a closed point $p \in X$, we can define the local stability threshold $\delta_p(X)$. This invariant is defined as $\delta(X)$, but only considering the prime divisors over $X$, $E$, such that $p$ is contained in the centre of $E$ (see Definition 2.2.8). The irrationality of $\delta_p(X)$ for a point $p \in X$ implies the existence of a log canonical place $v$ of $X$ such that the associated graded ring $\mathrm{gr}_v R$ is not finitely generated, even though it was initially conjectured to be. Moreover, as in [LXZ22, §6] one could prove the existence of infinitely many local deformations of a del Pezzo surface of degree 2, $X$, in an open neighbourhood of the points $p \in X$ where $\delta_p(X)$ is irrational, and give the explicit description of these deformations. Unfortunately, checking this rationality presents a more complex challenge. Whether $\delta_p(X)$ is rational remains generally unpredictable. Before the results in this thesis, the only documented example of an irrational local delta invariant was associated with cubic surfaces, given in [AZ22, Lemma A.6]. In this thesis, we prove the irrationality of the local stability threshold in the following case:

**Theorem 1.1.2** (= Theorem 3.0.1)**.** *[Etx24] Let $X \subset \mathbb{P}(1,1,1,2)$ be a smooth del Pezzo surface of degree 2 and let $p_0 \in X$. Assume a unique $(-1)$-curve $L$ is passing through the point $p_0$. Then $\delta_{p_0}(X) = \frac{6}{71}(11 + 8\sqrt{3})$.*

Notice that in a del Pezzo surface of degree 2, there are 56 $(-1)$-curves. So there are infinitely many closed points in $X$ satisfying this condition. Moreover, since delta invariants of higher dimensional Fano varieties can be reduced to computing delta invariants of del Pezzo surfaces (see Theorem 2.2.17), we expect this result will be useful in the future.

One of the other major achievements of K-stability, beyond the Yau-Tian-Donaldson Conjecture is the construction of moduli spaces for Fano varieties. Moduli spaces are a central concept in algebraic geometry since they play an important role in classification problems. They are spaces whose points represent equivalence classes of algebraic objects. Moduli spaces allow geometers to study families of objects as continuous entities rather than as isolated examples and it is convenient if the moduli space itself is an algebraic variety.

Until the development of the theory of K-stability and the introduction of techniques from birational geometry into it, there was no general theory to construct moduli of Fano varieties, which has been a major open problem. Perhaps unsurprisingly from an analytic perspective, given that KE metrics are canonical, the notion of K-polystability leads to a good moduli construction.

Recent advances in K-stability have shown that the compactification of the moduli spaces of KE Fano manifolds obtained by degenerating KE metrics coincides with a compact moduli space of K-polystable $\mathbb{Q}$-Fano varieties, which is a proper projective good moduli space [LWX21; XZ20]. However, their construction is not explicitly defined, and there are only a few examples of detailed descriptions of these K-moduli spaces. For instance, the compactification of the K-moduli space of 2-dimensional Fanos called del Pezzo surfaces is already done by Odaka-Spotti-Sun in [OSS16]. The next natural step is to consider the compactification of the K-moduli of Fano 3-folds denoted by $M_3^{\mathrm{Kps}}$, this compactification is unique by the uniqueness of K-polystable degenerations of Fano varieties. This is the final step of the Calabi problem for Fano 3-folds, which consists of finding all the K-polystable Fano 3-folds in each of 105 deformation families classified by Iskovskikh, Mori, and Mukai in [Isk89; MM82].

Before tackling the compactification, it is necessary to know which deformation families contain K-polystable members; this is solved in [Ara+23]. Then for those families with such elements, we need to check the K-(poly)stability of the smooth members. Although there has been huge progress in this respect, this classification is still incomplete. Then the next step will be to compactify the moduli components of each family. Out of the 78 families with K-polystable members, 24 have 0-dimensional moduli. In this thesis, we focus on the one-dimensional moduli components (6 families) and we prove the following theorem:

**Theorem 1.1.3** (=Main Theorem 4.1.3)**.** *All one-dimensional components of*

$M_3^{\mathrm{Kps}}$ *are isomorphic to* $\mathbb{P}^1$.

We recall that there are already other results on the compactification of different components of $M_3^{\mathrm{Kps}}$ [ADL23; CT23; DeV+24; LZ24; Pap22], and most of them seem to use the geometric invariant theory (GIT) description of the families to assist the construction of the compact K-moduli space. Moreover, in all known cases, the K-moduli space either coincides with some GIT moduli space or is closely related to it by blowing up certain subspaces in the GIT moduli. In our project with Hamid Abban, Ivan Cheltsov, Elena Denisova, Dongchen Jiao, Anne-Sophie Kaloghiros, Jesus Martinez-Garcia, and Theodoros Papazachariou, we also used GIT as a guide but with a new hands-on approach. We first write down a parametrisation of the objects and then examine their limits for K-polystability. Hidden in that approach is the hope that the K-polystable limit has the same description as the smooth objects; in other words, it lives in the same ambient space with similar defining equations — that is to say it follows some GIT principle. This is unfortunately not true in all the cases [Abb+23, §6]. However, in this thesis, we focus on the families where it is true.

## 1.2 Organization and results

This thesis consists of two main parts, in addition to a preliminary chapter (§2) where we review several tools employed throughout the thesis. In the first part (§3), we complete the computations of local stability thresholds for smooth del Pezzo surfaces of degree 2 started in [Ara+23, §2], by proving Theorem 1.1.2. The second part (§4) focuses on the compactification of the one-dimensional components of the K-moduli of Fano 3-folds.

More precisely, in Chapter 3 to prove Theorem 1.1.2, we first take $E_{a,b}$ to be the exceptional divisor of a certain weighted blowup at $p_0$ with weights $(a, b)$, denoted by $\pi_{a,b} : X_{a,b} \to X$. Note that $E_{a,b}$ has two quotient singularities coming from the weights. We resolve these singularities in full generality and we obtain an explicit chain of birational morphisms between $X_{a,b}$ and a weak del Pezzo surface of degree 1. This step is essential to finding an equivalence class for $\pi_{a,b}^*(-K_X)$ as a sum of negative curves (curves with negative self-intersection) and computing the Zariski Decomposition of the divisor $\pi_{a,b}^*(-K_X) - tE_{a,b}$ (§3.2.2). This decomposition splits the divisor into a negative ($N$) and a positive ($P$) part. Once that is done we compute $A_X(E_{a,b})/S(-K_X; E_{a,b})$ (see Definition 2.2.6) that gives us an upper bound for $\delta_{p_0}(X)$ (Theorem 3.3.1). Then, in Theorem 3.3.2 we employ techniques from [AZ22] to establish lower bounds for $\delta_{p_0}(X)$, requiring a precise selection of a minimizer sequence of prime divisors $E_{a_m,b_m}$ over $X$. The choice of $a_m$ and $b_m$ ensures that this bound aligns exactly with the previously found upper bound. In essence, we determine $\delta_{p_0}(X)$.

On the other hand, in Chapter 4, we first introduce the 6 deformation families of Fano 3-folds with 1-dimensional moduli. Two of them were already completely studied in [Ara+23, Section 4.7] and [Pap22], so we just give a short review. In the paper [Abb+23], we study the remaining 4 deformation families. However, this thesis focuses on providing more details, explanations and proofs for 3 of those 4 families.

Chapter 4 is divided into three sections with a similar structure, each dedicated to one of the families. Our goal is to compactify the K-moduli, hence we want to find all the singular K-polystable Fano 3-folds that admit a smoothing to the family and prove Theorem 1.1.3. First, we write an explicit parametrization of the (smooth) members of each family by the set of parameters $\mathbb{A}^1 \backslash \{p_1, ..., p_r\}$, then we find the candidates for the K-polystable limits for each of those points including the infinity point. We give a detailed geometric description of the singular limits, $X$, by describing their automorphism groups, $G = \mathrm{Aut}(X)$, and the $G$-invariant loci. This will simplify the computations to check the K-polystability of $X$. In this process, we use the Abban-Zhuang method [AZ22], and other technical tools such as Zariski Decomposition for 3-folds and surfaces. Once we have done this, we construct a morphism from $\mathbb{P}^1$ to the one-dimensional component of $M_3^{\mathrm{Kps}}$ corresponding to the family. With this we can prove Theorem 1.1.3.

# Chapter 2

# Preliminaries

## 2.1 Basic Notions

This section introduces definitions and some results we mention or use throughout the thesis. We will mainly focus on quotient singularities, valuations, divisors and Zariski decomposition.

### 2.1.1 Singularities and their blowups

Quotient singularities will be present in Chapter 3 and their resolution is a key part of proving its main theorem. Let us start by giving the definitions.

**Definition 2.1.1.** Let $r > 0$ and $a_1, ..., a_n$ be integers and let $x_1, ..., x_n$ be coordinates on $\mathbb{A}^n$. Let $\mu_r$ be the set of all $r$-th roots of the unity and suppose that it acts on $\mathbb{A}^n$ via:

$$x_i \mapsto \varepsilon^{a_i} x_i \text{ for all } i,$$

where $\varepsilon$ is a fixed primitive $r$-th root of unity. A singurality $q \in X$ is a *quotient singularity* of *type* $\frac{1}{r}(a_1, ..., a_n)$ if a neighbourhood of $q$ is isomorphic to an analytic neighbourhood of the origin in $\mathbb{A}^n/\mu_r$.

Quotient singularities are quite common, in particular, we can find them after weighted blowups. Here follows a local definition of weighted blowups. We refer the reader to [ATW23, Definition 3.5] for a more general definition.

**Definition 2.1.2.** Let $X$ be a quasiprojective variety and $\xi \in X$ a nonsingular point, and suppose that $u_1, ..., u_n$ are functions that are regular on a neighbourhood $U$ of $\xi$ on $X$ and

(a) the equations $u_1 = ... = u_n = 0$ have the single solution $\xi$ in $X$;

(b) $u_1, ..., u_n$ form a local system of parameters on $U$.

Consider the product $X \times \mathbb{P}(a_1, ..., a_n)$, where $a_i \in \mathbb{Z}_{>0}$ for all $1 \leq i \leq n$, and the subvariety $Y \subset X \times \mathbb{P}(a_1, ..., a_n)$ consisting of points $(x, [t_1 : ... : t_n])$ with $x \in U$ and $[t_1 : ... : t_n] \in \mathbb{P}(a_1, ..., a_n)$, such that

$$(u_i(x))^{a_j}(t_j)^{a_i} = (t_i)^{a_j}(u_j(x))^{a_i} \text{ for } i, j = 1, ..., n.$$

The regular map $\sigma : Y \to U$ is called the *weighted blowup* of $X$ with centre in $\xi$. It is obtained as the restriction to $Y$ of the projection onto the first component $X \times \mathbb{P}(a_1, ..., a_n) \to X$.

**Remarks 1.**

1. Notice that when $a_1 = ... = a_n = 1$ we will recover the definition of an ordinary blowup of a smooth point.

2. The $Y$ defined above has a quotient singularity lying in the exceptional divisor of the weighted blowup, i.e. in $\sigma^*(\xi)$. For each of $a_i \neq 1$ of type $\frac{1}{a_i}(a_1, ..., \hat{a}_i, ..., a_n)$, the notation $\hat{a}_i$ means that we are removing this coordinate from the tuple, so it is of length $n - 1$ now.

Let us see a simple example of what happens with a weighted blowup at a smooth point in a dimension 2 variety.

**Example 2.1.3.** Let $S$ be an algebraic variety of dimension 2, and let $p \in S$ be a smooth point. Let $(u, v)$ be the local coordinates of $S$ in a neighbourhood of $p$. Let $\pi_{a,b} : Y \to S$ be a weighted blowup at $p$ with weights $wt(u) = a$ and $wt(v) = b$, such that $\gcd(a, b) = 1$. Let $E$ be the $\pi_{a,b}$-exceptional divisor, notice that $E = \mathbb{P}(a, b)$. Let $D_u$ and $D_v$ be two divisors in $S$ passing through $p$, such that their equations in a neighbourhood of $p$ can be written in local coordinates as $D_u := \{u = 0\}$ and $D_v := \{v = 0\}$. After the weighted blowup the intersections will change. Let $\widetilde{D}_u$ and $\widetilde{D}_v$ be the strict transforms of $D_u$ and $D_v$, respectively. Then we have, $\pi^*_{a,b}(D_u) \sim \widetilde{D}_u + aE$ and $\pi^*_{a,b}(D_v) \sim \widetilde{D}_v + bE$. Taking into account that $\widetilde{D}_u \cdot \widetilde{D}_v = \pi^*_{a,b}(D_u) \cdot \pi^*_{a,b}(D_v) - 1$, it is straight forward that $E^2 = -\frac{1}{ab}$, $\widetilde{D}_u \cdot E = \frac{1}{b}$ and $\widetilde{D}_v \cdot E = \frac{1}{a}$. Moreover, as mentioned before we have two quotient singularities in $E$ at the intersection points with $\overline{D}_v$ and $\overline{D}_u$, denoted by $q_1 = \frac{1}{a}(1, b)$ and $q_2 = \frac{1}{b}(1, a)$, respectively.

In Chapter 3 we will see how to do the resolution of singularities of the example 2.1.3. Let me remind you of the definition of resolution of singularities:

**Definition 2.1.4.** Given an algebraic variety $X$, a resolution of singularities is a proper birational morphism $\pi : Y \to X$ such that $Y$ is smooth. In addition, if $Y$ does not have $\pi$-exceptional divisors, then $\pi : Y \to X$ is a small resolution.

There is also another classification of singularities using the discrepancy. However, before introducing this concept let us define log pairs.

**Definition 2.1.5.** A *logarithmic pair* (also called a *log pair* for short) is a pair consisting $(X, D)$ of a normal variety $X$ and a boundary $\mathbb{Q}$-divisor $D$. A boundary divisor $D = \sum d_k D_k$ is a divisor that has all the coefficients between 0 and 1, i.e., $0 \leq d_k \leq 1$, where the $D_k$ are the distinct irreducible components. We call $K_X + D$ the *log canonical divisor* of the log pair $(X, D)$.

**Definition 2.1.6.** Let $(X, D)$ be a log pair consisting of a normal variety $X$ and a boundary $\mathbb{Q}$-divisor $D = \sum d_k D_k$. Assume that $K_X + D$ is $\mathbb{Q}$-Cartier. Let

$f : V \to X$ be a birational morphism from a nonsingular variety $V$. Let $E$ be a prime divisor in $V$, i.e. a prime divisor over $X$, then we define the *discrepancy* of the pair $(X, D)$ at $E$ as

$$a(E; X, D) = \mathrm{ord}_E(K_V - f^*(K_X + D) + f_*^{-1}(D)),$$

where $\mathrm{ord}_E$ gives the the vanishing order along $E$ and $f_*^{-1}(D)$ is the strict transform of $D$. The *log discrepancy* of the pair $(X, D)$ at $E$ is defined to be

$$A_{X,D}(E) = a(E; X, D) + 1,$$

so that

$$K_V + D_V = f^*(K_X + D) + \sum_E A_{X,D}(E)E,$$

where $D_V = \sum_E E + f_*^{-1}(D)$.

There is a more general definition of log discrepancy in [LXZ22, Def. 2.5].

Once we have the definitions above, we can define the most important classes of singularities.

**Definition 2.1.7.** Let $(X, D)$ be a log pair where $X$ is a normal variety of dimension greater or equal to 2 and $D = \sum a_i D_i$ is a sum of distinct prime divisors and $a_i$ are rational numbers such that $0 < a_i \leq 1$. Assume that $m(K_X + D)$ is Cartier for some $m > 0$. We say that $(X, D)$ is

$$\left.\begin{array}{c} \textit{terminal} \\ \textit{canonical} \\ \textit{klt} \\ \textit{plt} \\ \textit{lc} \end{array}\right\} \text{ if } a(E, X, D) \begin{cases} > 0 \text{ for every exceptional } E, \\ \geq 0 \text{ for every exceptional } E, \\ > -1 \text{ for every } E, \\ > -1 \text{ for every exceptional } E, \\ \geq -1 \text{ for every } E. \end{cases}$$

Here klt is short for *'Kawamata log terminal'*, plt for *'purely log terminal'* and lc for *'log canonical'*.

A *lc centre* is an irreducible subvariety $V \subset X$ such that $V$ is the image of $\pi(E)$ for some $E$ (prime divisor) such that $1 - a(E) \leq 0$, where $\pi : Y \to X$ is the log-resolution of $(X, D)$ (see [Laz04, Theorem 4.1.3]).

A *log canonical place* is a valuation corresponding to $E$ as above (see Definition 2.1.12). The *non-lc centre* $\mathrm{Nlc}(X, D)$ of a pair $(X, D)$ is the set of closed points $x \in X$ such that $(X, D)$ is not lc at $x$.

**Definition 2.1.8.** We say that $X$ is *log Fano* if there exists a divisor $D$ such that $-(K_X + D)$ is ample and $(X, D)$ is Kawamata log terminal.

### 2.1.2 Valuation and Divisors

Valuations are essential in the study of K-stability. Moreover, we will see in §2.2.6 that its characterisations involve valuative criteria. Sometimes these valuations are also connected to divisors in our variety. In this section, we will introduce some definitions and properties.

**Definition 2.1.9.** A *discrete valuation* on a field $\mathbb{K}$ is a group homomorphism

$$\nu : \mathbb{K}^* \to \mathbb{Z}$$

such that $\nu(xy) = \nu(x) + \nu(y)$ for $x, y, xy \in \mathbb{K}^*$ and which is onto and satisfies $\nu(x + y) \geq \min(\nu(x), \nu(y))$ when $x, y, x + y \in \mathbb{K}^*$. The corresponding *discrecte valuation ring* is $R = \{x \in \mathbb{K}^* | \nu(x) \geq 0\} \cup \{0\}$.

**Definition 2.1.10.** The *centre* of $v$ on $X$, denoted by $c_X(v)$, is a scheme-theoretic point $\xi \in X$ such that $v \geq 0$ on $\mathcal{O}_{X,\xi}$, and $v > 0$ on the maximal ideal $\mathfrak{m}_{X,\xi}$. Denote by $C_X(v) := \overline{c_X(v)}$.

**Definition 2.1.11.** Let $Y$ and $X$ be two normal varieties. Let $\phi : Y \to X$ be a proper birational morphism and let $p \in Y$ be a point where $Y$ is regular. Let

$\{y_1, ..., y_r\} \in \mathcal{O}_{Y,p}$ be a local coordinate system at $p$, and take $a = (a_1, ..., a_r) \in \mathbb{R}^r_{\geq 0} \setminus \{0\}$. For $f \in \mathcal{O}_{Y,p}$, we write $f = \sum_{b \in \mathbb{Z}^r_{\geq 0}} c_b y^b$ with $c_b \in \hat{\mathcal{O}}_{Y,p}$ either zero or a unit. We define $v_a$ as follows,

$$v_a(f) := \min\{\langle a, b\rangle | c_b \neq 0\}.$$

If a valuation can be written in this form, it is called a *quasi-monomial valuation.*

As mentioned before, we can use divisors over $X$ to define valuations.

**Definition 2.1.12.** Let $\pi : Y \to X$ be a proper morphism where $Y$ is normal. A prime divisor $E$ on $Y$ is called a *prime divisor over* $X$. It induces a valuation $\mathrm{ord}_E : \mathbb{K}(X)^\times \to \mathbb{Z}$ by taking the vanishing order along $E$. A valuation $v \in \mathrm{Val}_X$ is called *divisorial* if $v = c \cdot \mathrm{ord}_E$ for some prime divisor $E$ over $X$ and some $c \in \mathbb{R}_{>0}$.

We also need to introduce some notation for divisors.

**Definition 2.1.13.** Let $L$ be a big and nef divisor on $X$. Then, we define the *stable base locus* of $L$ as

$$\mathrm{BS}(L) = \{x \in X | s(x) = 0 \text{ for all sections } s \in H^0(X, mL) \text{ for all } m \in \mathbb{N}_{>0}\}.$$

Now, let me introduce you to some special divisors, that will appear later on.

**Definition 2.1.14.** Let $(X, D)$ be a log pair and let $F$ be a prime divisor over $X$. When $F$ is a divisor on $X$ we write $D = D_1 + aF$ where $F \nsubseteq \mathrm{Supp}(D_1)$; otherwise let $D_1 = D$.

(i) $F$ is said to be *primitive* over $X$ if there exists a projective birational morphism $\pi : Y \to X$ such that $Y$ is normal and $-F$ is a $\pi$-ample $\mathbb{Q}$-Cartier divisor (see [Laz04, Definition 1.7.1]). We call $\pi : Y \to X$ the *associated prime blowup* (it is uniquely determined by $F$).

(ii) $F$ is said to be of *plt type* if it is primitive over $X$ and the pair $(Y, D_Y + F)$ is plt in a neighbourhood of $F$, where $\pi : Y \to X$ is the associated prime blowup and $D_Y$ is the strict transform of $D_1$ on $Y$. When $(X, D)$ is klt and $F$ is exceptional over $X$, $\pi$ is called a *plt blowup* over $X$.

Next, we introduce the different for divisors on a surface. This appears, for instance, in the K-stability computations when we have some singular points. Let $S$ be a regular surface and $C \subset S$ a regular curve. The classical adjunction formula says that $K_C \sim (K_S+C)|_C$. However, if $S$ is only normal, each singularity of $(S, C)$ leads to a correction term, called the *different*. Since the singularities of $C$ can be complicated, we compute everything on the normalization $\overline{C} \to C$.

**Definition 2.1.15.** Let $S$ be a normal surface and $C \subset S$ a reduced curve with normalization $\overline{C} \to C$. Let $F$ be a $\mathbb{Q}$-divisor on $S$ with no components in common with $C$.

Let $f : Y \to S$ be a log resolution of $(S, C)$ with exceptional curves $E_i$ and $\widetilde{C}, \widetilde{F} \subset Y$ the strict tranforms of $C$, $F$. Note that $\widetilde{C} \simeq \overline{C}$. There is a unique $\Delta(S, Y, C + F) := \sum d_i E_i$ such that

$$(\Delta(S, T, C + F) \cdot E_i) = -((K_Y + \widetilde{C} + \widetilde{F}) \cdot E_i) \;\; \forall i.$$

Define the *different* as

$$\mathrm{Diff}_{\overline{C}}(F) := (\widetilde{F} + \Delta(S, T, C + F))|_{\widetilde{C}}$$

You can see in [Kol13, p. 2.35.1] that this is independent of the choice of $f$.

Next, we will introduce one of the tools we use the most: the Zariski Decomposition for surfaces and 3-folds. This is key to computing the volume of divisors and essential in both Chapters 3 and 4.

### Zariski decomposition

Before defining the Zariski decomposition, we need to introduce some concepts. Let $X$ be a projective variety and $N^1(X)$ the group of Cartier divisors modulo numerical equivalence. We denote by $\mathrm{Nef}(X)$ the cone generated by nef divisors in $N^1(X)_{\mathbb{R}}$. Likewise, we denote by $\overline{\mathrm{Eff}}(X)$ (resp., $\mathrm{Mov}(X)$) the closure of the cone of effective divisors (resp., movable divisors). Similarly, Let $N_1(X)$ be the group of 1-cycles modulo numerical equivalence. We define the Mori cone $\overline{\mathrm{NE}}(X)$ to be the closure of the cone of effective 1-cycles in $N_1(X)_{\mathbb{R}}$.

**Definition 2.1.16.** A *small $\mathbb{Q}$-factorial modification* (SQM) of a normal projective variety $X$ is a small (i.e. isomorphic in codimension one) birational map $X \dashrightarrow Y$ to another normal $\mathbb{Q}$-factorial projective variety $Y$.

**Definition 2.1.17.** A normal projective variety $X$ is called a *Mori Dream Space* (MDS) if the following conditions are satisfied:

(1) $X$ is $\mathbb{Q}$-factorial with $\mathrm{Pic}(X)_{\mathbb{Q}} \cong N^1(X)_{\mathbb{Q}}$;

(2) $\mathrm{Nef}(X)$ is generated by finitely many semiample divisors (see [Laz04, Definition 2.1.15]);

(3) There are finitely many SQMs $f_i : X \dashrightarrow X_i$ such that each $X_i$ satisfies (1) and (2), and $\mathrm{Mov}(X)$ is the union of $f_i^*(\mathrm{Nef}(X_i))$.

Let $f : X \dashrightarrow Y$ be a birational map between normal projective varieties. If $E_1, ..., E_k$ are the prime divisors contracted by $f$, then $E_1, ..., E_k$ are linearly independent in $N^1(X)_{\mathbb{R}}$ and each $E_i$ spans an extremal ray of $\overline{\mathrm{Eff}}(X)$. The effective cone of a MDS also has a decomposition into rational polyhedral cones:

**Proposition 2.1.18.** *[HK00, Prop. 1.11 (2)]. Let $X$ be a MDS. There are finitely many birational contractions $g_i : X \dashrightarrow Y_i$, with $Y_i$ a MDS, such that*

$$\overline{\mathrm{Eff}}(X) = \bigcup_i \mathcal{C}_i, \quad \textit{with} \quad \mathcal{C}_i = g_i^* \mathrm{Nef}(Y_i) + \mathbb{R}_{\geq 0}\{E_1, ..., E_k\},$$

*where $E_1, ..., E_k$ are the prime divisors contracted by $g_i$.*

The cones $\mathcal{C}_i$ are called *Mori chambers* of $X$. This proposition gives us a Zariski decomposition for $X$: for each effective $\mathbb{Q}$-Cartier divisor $D$ on $X$, there exists a birational contraction $g : X \dashrightarrow Y$ (factoring through an SQM and a birational morphism $X \dashrightarrow X' \to Y$) and $\mathbb{Q}$-divisors $P$ and $N$, such that $P$ is nef on $X'$, $N$ is an effective divisor contracted by $g$ and by clearing up the denominator of $N$ and taking $P = g^*g_*(D)$ and $N = D - P$, one gets a canonical inclusion

$$H^0(X, \mathcal{O}(mP)) \hookrightarrow H^0(X, \mathcal{O}(mD))$$

which is surjective for every positive integer $m$ and for a sufficiently large and divisible $m$ is an isomorphism.

**Remarks 2.** (a) If $X$ is a MDS, all birational contractions $X \dashrightarrow Y$ with $\mathbb{Q}$-factorial $Y$, are the ones that appear in Proposition 2.1.18. In particular, any such $Y$ is a MDS.

(b) The SQMs in Definition 2.1.17 are the only SQMs on $X$. In particular, any SQM of a MDS is itself a MDS.

This decomposition of divisors is essential to computing stability thresholds, particularly to compute the volume of divisors. For surfaces, the Zariski Decomposition always exists and we have an explicit description of it:

**Theorem 2.1.19.** ***(Zariski decomposition)***. *[Laz04, Theorem 2.3.19] Let X be a smooth projective surface and let D be a pseudoeffective (i.e. $(D \cdot H) \geq 0$ for every nef divisor H on X) integral divisor on X. Then D can be written uniquely as a sum:*

$$D = P + N$$

*of $\mathbb{Q}$-divisors with the following properties:*

*(i) P is nef;*

*(ii)* $N = \sum_{i=1}^{r} a_i E_i$ *is effective, and if* $N \neq 0$ *then the intersection matrix determined by the components of* $N$*, i.e. the matrix where in the entry* $(i, j)$ *we have the intersection* $E_i \cdot E_j$*, denoted by*

$$(E_i \cdot E_j)_{i,j \in \{1,..,r\}}$$

*is negative definite;*

*(iii)* $P$ *is orthogonal to each of the components of* $N$*, i.e.* $(P \cdot E_i) = 0$ *for every* $1 \leq i \leq r$*.*

$P$ and $N$ are respectively called the *positive* and the *negative* parts of $D$. This definition could be also extended to dimension 3 in some cases where the Zariski decomposition exists (recall that this is not always the case). But as we mentioned before, it exists for MDSs which are the objects we work with in this thesis. We give examples of how to apply this theory in the proof of Theorem 3.3.1 for surfaces and in Chapter 4 for 3-folds.

**Corollary 2.1.20.** *Let* $X$ *be a smooth projective variety of dimension* $n = 2, 3$ *that admits a Zariski Decomposition and let* $D$ *be a pseudoeffective integral divisor on* $X$*. Then,*

$$\mathrm{vol}(D) = P^n$$

*where* $P$ *is the positive part of* $D$*.*

*In particular, the volume of an integral divisor on a surface is always a rational number.*

### 2.1.3 Fano varieties

In this section, we will introduce the objects of study of this thesis.

**Definition 2.1.21.** Let $X$ be an algebraic variety. We say X is *Fano* if it is a normal, projective variety whose anticanonical divisor, $-K_X$, is ample.

These objects are strongly related to one of the three possible terminations of the Minimal Model Program: Fano fibrations. Hence, they are of high interest and have been deeply studied in algebraic geometry. If we focus on smooth Fano varieties, $\mathbb{P}^1$ is the only example in dimension 1. In dimension 2, we have 10 deformation families called del Pezzo surfaces named after the mathematician who encountered them while exploring the surfaces of degree $n$ in $\mathbb{P}^n$ and classified them in [Del87]. These del Pezzo surfaces are classified by the degree $d$. For a del Pezzo surface $X$, the degree is defined by the self-intersection of the anticanonical divisor, $(-K_X)^2 = d$.

**Remark 2.1.22.** All of them but $\mathbb{P}^1 \times \mathbb{P}^1$ can be represented as the blowup of $\mathbb{P}^2$ at $r$ points in general position, for $0 < r < 9$. In these cases, $d = 9 - r$.

In Chapter 3, we focus on studying the K-stability of degree 2 del Pezzo surfaces, and we give more details about this specific surface in section 3.1.1.

On the other hand in dimension 3 we have 105 deformation families classified by Inskovskikh, Mori and Mukai [Isk89; MM82]. And we work with some of these families in Chapter 4.

## 2.2 K-stability

K-stability was first introduced from a differential geometric perspective by Yau and Tian [Yau96; Tia97] and later reformulated by Donaldson [Don02]. The original algebraic definition was given in terms of test configurations and their Donaldson-Futaki invariant [Oda12]. Later, thanks to the use of birational geometry techniques in the study of K-stability, new characterisations were introduced in [FO18; Fuj19a], these are more practical from a computational perspective and easier to verify. In this section, we will introduce K-stability and K-moduli by giving some definitions and outstanding results in the topic, however for a detailed exploration of this topic, please consult [Xu20] or any specific references given throughout the section.

### 2.2.1 Definitions of K-stability

First, recall that all definitions and results in this thesis are over the field $\mathbb{C}$. Before introducing Donaldson's first algebraic definition, let us define test configurations. Stability is typically determined by a numerical criterion based on the degenerations of the object we are studying and test configurations provide the data that encode these degenerations [Don02].

**Definition 2.2.1.** Let $X$ be an $n$-dimensional $\mathbb{Q}$-Fano variety. Assume that $-rK_X$ is Cartier for some fixed $r \in \mathbb{N}$. A *test configuration* of $(X, -rK_X)$ is composed of

- a variety $\mathcal{X}^{\mathrm{tc}}$ with a $\mathbb{G}_m$-action,
- a $\mathbb{G}_m$-equivariant ample line bundle $\mathcal{L}^{\mathrm{tc}} \to \mathcal{X}^{\mathrm{tc}}$,
- a flat $\mathbb{G}_m$-equivariant map $\pi : (\mathcal{X}^{\mathrm{tc}}, \mathcal{L}^{\mathrm{tc}}) \to \mathbb{A}^1$, where $\mathbb{G}_m$ acts on $\mathbb{A}^1$ by multiplication in the standard way $(t, a) \to ta$,

such that for any $t \neq 0$, $(\mathcal{X}^{\mathrm{tc}}_t, \mathcal{L}^{\mathrm{tc}}_t)$ is isomorphic to $(X, -rK_X)$, where $\mathcal{X}^{\mathrm{tc}}_t = \pi^{-1}(t)$ and $\mathcal{L}^{\mathrm{tc}}_t = \mathcal{L}^{\mathrm{tc}}|_{\mathcal{X}^{\mathrm{tc}}_t}$.

We say that $(\mathcal{X}^{\mathrm{tc}}, \mathcal{L}^{\mathrm{tc}})$ is a $\mathbb{Q}$-*test configuration* of $(X, -rK_X)$ for a fixed $r \in \mathbb{Q}_{>0}$ if $\mathcal{L}^{\mathrm{tc}}$ is a $\mathbb{Q}$-Cartier divisor class on $\mathcal{X}^{\mathrm{tc}}$ such that for some integer $s \geq 1$, $(\mathcal{X}^{\mathrm{tc}}, s\mathcal{L}^{\mathrm{tc}})$ yields a test configuration of $(X, -srK_X)$.

**Remark 2.2.2.** In the above case, these test configurations $(\mathcal{X}^{\mathrm{tc}}, \mathcal{L}^{\mathrm{tc}})$ arise naturally from the one-parameter subgroups of $\mathrm{GL}(H^0(X, -srK_X))$ for some integer $s \geq 1$ (see the proof in [RT07, Proposition 3.7]). Moreover, since an arbitrary $\mathbb{G}_m$-equivariant vector bundle over $\mathbb{A}^1$ should be equivariantly trivial ([Don05, Lemma 2]), any (very ample) test configuration can be obtained from a one-parameter subgroup $\mathbb{G}_m \to \mathrm{GL}(H^0(X, -srK_X))$ as indicated in [Oda13c, Proposition 2.2]. Therefore, we can think of test configurations as a geometrization of one-parameter subgroups.

**Example 2.2.3** (Product test configuration)**.** Take the pair $(\mathcal{X} = X \times \mathbb{C}, \mathcal{L} = p_1^*(-K_X))$ such that $X$ has a $\mathbb{G}_m$-action, where $p_1$ is the projection to $X$, this is a test configuration with the $\mathbb{G}_m$-action given by $t \cdot (x, a) \to (t(x), t \cdot a)$.

For the pair $(X, -K_X)$, and a sufficiently divisible $k \in \mathbb{N}$, by Riemann-Roch theorem, the dimension of each vector space $H^0(X, \mathcal{O}_X(-kK_X))$ is given by a Hilbert polynomial of degree $n = \dim X$,

$$d_k = \dim H^0(X, \mathcal{O}_X(-kK_X)) = a_0 k^n + a_1 k^{n-1} + O(k^{n-2})$$

for some $a_0, a_1 \in \mathbb{Q}$. Now, since $\mathbb{G}_m$ acts on the central fibre $(\mathcal{X}_0^{\mathrm{tc}}, \mathcal{L}_0^{\mathrm{tc}})$ which is the restriction of $(\mathcal{X}^{\mathrm{tc}}, \mathcal{L}^{\mathrm{tc}})$ over $\{0\}$, $\mathbb{G}_m$ also acts on the space of holomorphic sections $H^0(\mathcal{X}_0^{\mathrm{tc}}, k\mathcal{L}_0^{\mathrm{tc}})$. The total of the weights of this action is denoted by $w_k$ and by the equivariant Riemann-Roch Theorem, we can rewrite it as

$$w_k = b_0 k^{n+1} + b_1 k^n + O(k^{n-1}).$$

Combining the equations of $d_k$ and $w_k$, we get

$$\frac{w_k}{k d_k} = F_0 + F_1 k^{-1} + O(k^{-2}).$$

Now we can define the Donaldson-Futaki invariant.

**Definition 2.2.4.** [Don02] The (normalized) *Donaldson-Futaki invariant* of the $\mathbb{Q}$-test configuration $(\mathcal{X}^{\mathrm{tc}}, \mathcal{L}^{\mathrm{tc}})$ is defined to be

$$\mathrm{DF}(\mathcal{X}^{\mathrm{tc}}, \mathcal{L}^{\mathrm{tc}}) = -\frac{F_1}{a_0} = \frac{a_1 b_0 - a_0 b_1}{a_0^2}$$

Using this invariant we give the original definition for K-stability of $X$.

**Definition 2.2.5.** A Fano variety $X$ is *K-stable* (resp. *K-semistable*) if and only if $\mathrm{DF}(\mathcal{X}^{\mathrm{tc}}, \mathcal{L}^{\mathrm{tc}}) > 0$ (resp. $\mathrm{DF}(\mathcal{X}^{\mathrm{tc}}, \mathcal{L}^{\mathrm{tc}}) \geq 0$) for any non-trivial test configuration $(\mathcal{X}^{\mathrm{tc}}, \mathcal{L}^{\mathrm{tc}})$ of $(X, -K_X)$. Moreover, we say that $X$ is *K-polystable* if $\mathrm{DF}(\mathcal{X}^{\mathrm{tc}}, \mathcal{L}^{\mathrm{tc}}) \geq 0$ for any non-trivial test configuration of $(\mathcal{X}^{\mathrm{tc}}, \mathcal{L}^{\mathrm{tc}})$, and $DF(\mathcal{X}^{\mathrm{tc}}, \mathcal{L}^{\mathrm{tc}}) = 0$ for product test configurations of $(X, -K_X)$.

There are other expressions for the Donalson-Futaki invariant given in [Wan12; Oda13a] for more explicit cases, however computing this invariant is still complicated and detecting K-stability by using it is even harder. Nevertheless, we obtain other characterisations for K-stability by using birational geometry techniques and invariants. Let us start by defining the latter:

**Definition 2.2.6.** Let $X$ be a $n$-dimensional Fano variety with klt singularities. Let $f : Y \to X$ be a birational morphism and take $E$ a prime divisor on $Y$. We say that $E$ is a prime divisor *over* $X$. If $E$ is $f$-exceptional, we say that $E$ is an exceptional prime divisor over $X$. The subvariety $f(E)$ is called the *centre* of $E$ over $X$ and is denoted by $C_X(E)$. We define the *expected vanishing order* as follows:

$$S(-K_X; E) = \frac{1}{(-K_X)^n} \int_0^\tau \mathrm{vol}(f^*(-K_X) - tE)\, dt$$

where $\mathrm{vol}(f^*(-K_X) - tE)$ denotes the volume of the divisor $f^*(-K_X) - tE$ (see e.g. [Laz04, §2.2.C]) and

$$\tau = \tau(E) = \sup\{t \in \mathbb{Q} | \, \mathrm{vol}(f^*(-K_X) - tE) > 0\}$$

is called the *pseudoeffective threshold* of $E$ with respect to $-K_X$. By [Fuj19c, Proposition 2.1], we have that

$$S(-K_X;E) \leqslant \frac{n}{n+1}\tau. \tag{2.1}$$

Let $\beta(E) = A_X(E) - S(-K_X;E)$, where $A_X(E) = 1 + \mathrm{ord}_E(K_Y - f^*(K_X))$ is the *log discrepancy* of the divisor $E$ (Note that this coincides with Definition 2.1.6).

Let $\pi : Y \to X$ be the log-resolution of $X$ and $E$ a $\pi$-exceptional divisor over $X$, a *log canonical center* is an irreducible subvariety $\pi(E) \subset X$. A *log canonical place* is a valuation corresponding to this $E$, i.e. the valuation $\mathrm{ord}_E$.

The stability threshold is another invariant that characterises K-stability.

**Definition 2.2.7.** Let $X$ be a Fano variety with at most klt singularities and $-K_X$ be its anticanonical divisor. The (adjoint) *stability threshold* (or *$\delta$-invariant*) of $X$ is defined as

$$\delta(X) = \inf_{E/X} \frac{A_X(E)}{S(-K_X;E)} \tag{2.2}$$

where the infimum runs over all divisors $E$ over $X$.

We say that a divisor $E$ over $X$ *computes* $\delta(X)$ if it achieves the infimum in (2.2).

There is also a local version of the stability threshold, which we will study in Chapter 3.

**Definition 2.2.8.** [AZ22] Let $X$ be a Fano variety with at most klt singularities and $-K_X$ be its anticanonical divisor. Let $p$ be a closed point of $X$. We set

$$\delta_p(X) = \inf_{E,\, p\in C_X(E)} \frac{A_X(E)}{S(-K_X;E)}$$

where the infimum runs over all divisors $E$ over $X$ whose center contains $p$.

**Remark 2.2.9.** Notice that $\delta(X) = \inf_{p\in X} \delta_p(X)$. If $\beta(F) \leq 0$ for a divisor $F$ whose centre contains $p$ , then $\delta_p(X) \leq 1$.

Using these two invariants we just defined, we give the following characterisation for K-stability.

**Theorem 2.2.10.** *[LXZ22; Li17; FO18; Fuj19a; CP21; BJ20]. Let $X$ be a Fano variety, then the following statements are equivalent:*

1. *$X$ is K-stable (resp. K-semistable).*
2. *$\beta(E) > 0$ (resp. $\beta(E) \geq 0$) for every prime divisor $E$ over $X$.*
3. *$\delta(X) > 1$ (resp. $\delta(X) \geq 1$).*

This theorem is useful to check the K-unstability of a variety $X$ since you only need to find a divisor $E$ over $X$ that gives you $\beta(E) < 0$. However, to prove that $X$ is K-stable using only this theorem can be a long process. However, there is another result by Zhuang where using a reductive subgroup, $G$, of the automorphism group of $X$ we can characterise K-polystability without having to check all the prime divisors over $X$.

**Remark 2.2.11.** Notice that K-stability implies K-polystability and the latter implies K-semistability.

**Theorem 2.2.12** ([Zhu21, Corollary 4.14])**.** *Let $X$ be a Fano variety with at most klt singularities. Let $G \subset \mathrm{Aut}(X)$ be a reductive subgroup. Suppose that $\beta(E) > 0$ for every $G$-invariant prime divisor $E$ over $X$. Then $X$ is K-polystable.*

Notice this theorem is especially useful when you have a large $G$. For example, take $\mathbb{P}^n$ which is known to not be K-stable, take $G$ to be its automorphism group $G = \mathrm{Aut}(\mathbb{P}^n) = PGL(n+1)$. Then, there are no $G$-invariant subsets of $\mathbb{P}^n$ and in particular, there is no $G$-invariant divisor over $\mathbb{P}^n$. Therefore, by Theorem 2.2.12 we know that $\mathbb{P}^n$ is K-polystable.

Similar invariant notions can be defined in more generality for $\mathbb{N} \times \mathbb{N}^r$-graded linear series with bounded support on $X$. Details can be found in [AZ22, Section 3], but here, we recall the results that we use. These results are part of the so-called Abban-Zhuang method, and its goal is to find a lower bound for the $\delta$-invariant to determine the K-stability.

**Definition 2.2.13.** Let $V$ be a finite dimensional vector space. A *filtration* $\mathcal{F}$ on $V$ is given by a collection of subspaces $\mathcal{F}^\lambda V$ indexed by a totally ordered abelian monoid $\Lambda$ (in which case we also call the filtration a $\Lambda$-*filtration*) such that $\mathcal{F}^{\lambda_0} V = V$, $\mathcal{F}^{\lambda_1} V = 0$ for some $\lambda_0$, $\lambda_1 \in \Lambda$ and $\mathcal{F}^\lambda V \subseteq \mathcal{F}^{\lambda'} V$ whenever $\lambda \geq \lambda'$. For each $\lambda \in \Lambda$, we set $\mathrm{Gr}^\lambda_{\mathcal{F}} V = \mathcal{F}^\lambda V / \cup_{\mu > \lambda} \mathcal{F}^\mu V$. A basis $s_1, ..., s_N$ (where $N = \dim V$) of $V$ is said to be compatible with $\mathcal{F}$ if every $\mathcal{F}^\lambda V$ is the span of some $s_i$.

**Example 2.2.14.** Let $L$ be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on $X$, and let $V \subseteq H^0(X, L)$ be a subspace. Let $\nu$ be a valuation on $X$. Then it induces an $\mathbb{R}$-filtration $\mathcal{F}_\nu$ on $V$ by setting

$$\mathcal{F}^\lambda_\nu V := \{s \in V | \nu(s) \geq \lambda\}.$$

**Definition 2.2.15.** ([LM09, §4.3]). Let $L_1, ..., L_r$ be an ordered sequence of $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors on $X$. An $\mathbb{N}^r$-*graded linear series* $W_{\overrightarrow{\bullet}}$ on $X$ associated to the $L_i$'s consists of finite dimensional subspaces

$$W_{\overrightarrow{a}} \subset H^0(X, \mathcal{O}_X(a_1 L_1 + ... + a_r L_r))$$

for each $\overrightarrow{a} \in \mathbb{N}^r$ such that $W_{\overrightarrow{0}} = \mathbb{C}$ and $W_{\overrightarrow{a_1}} \cdot W_{\overrightarrow{a_2}} \subseteq W_{\overrightarrow{a_1} + \overrightarrow{a_2}}$ for all $\overrightarrow{a_1}, \overrightarrow{a_2} \in \mathbb{N}^r$. The support $\mathrm{Supp}(W_{\overrightarrow{\bullet}}) \subseteq \mathbb{R}^r$ of $W_{\overrightarrow{\bullet}}$ is defined as the closed complex cone spanned by all $\overrightarrow{a} \in \mathbb{N}^r$ such that $W_{\overrightarrow{a}} \neq 0$. We say that $W_{\overrightarrow{\bullet}}$ has *bounded support* if $\mathrm{Supp}(W_{\overrightarrow{\bullet}}) \cap (\{1\} \times \mathbb{R}^{r-1})$ is bounded.

We say that $W_{\overrightarrow{\bullet}}$ *contain an ample series* if the following conditions are satisfied:

(i) $\mathrm{Supp}(W_{\overrightarrow{\bullet}}) \subseteq \mathbb{R}^r$ contains a non-empty interior,

(ii) for any $\overrightarrow{a} \in \mathrm{int}(\mathrm{Supp}(W_{\overrightarrow{\bullet}})) \cap \mathbb{N}^r$, $W_{k\overrightarrow{a}} \neq 0$ for $k \gg 0$,

(iii) there exists some $\overrightarrow{a_0} \in \mathrm{int}(Supp(W_{\overrightarrow{\bullet}})) \cap \mathbb{N}^r$ and a decomposition $\overrightarrow{a_0} \cdot \overrightarrow{L} = \sum_{i=1}^r a_{0i}L_i = A+E$ (where $\overrightarrow{L} = (L_1, ..., L_r)$ and $\overrightarrow{a_0} = (a_{01}, ..., a_{0r})$) with $A$ an ample $\mathbb{Q}$-line bundle and $E$ an effective $\mathbb{Q}$-divisor such that $H^0(X, mA) \subseteq W_{m\overrightarrow{a_0}}$ for all sufficiently divisible $m$.

**Definition 2.2.16.** Let $L_1, ..., L_r$ be Cartier divisors on $X$ and let $V_{\overrightarrow{\bullet}}$ be an $\mathbb{N}^r$-graded linear series associated to the $L_i$'s. Denote $\overrightarrow{L} = (L_1, ..., L_r)$. Let $F$ be a primitive divisor over $X$ with associated prime blowup $\pi : Y \to X$ (see Definition 2.1.14) and let $\mathcal{F}$ be the induced filtration on $V_{\overrightarrow{\bullet}}$ (see Example 2.2.14). Assume that $F$ is either Cartier on $Y$ or of plt type. In the latter case, we define $F|_F$ as the $\mathbb{Q}$-divisor class given by [AZ22, Lemma 2.7]. Then in both cases,

$$W_{\overrightarrow{a},j} = \mathcal{F}^j V_{\overrightarrow{a}} / \mathcal{F}^{j+1} V_{\overrightarrow{a}}$$

can be naturally identified with the image of $\mathcal{F}^j V_{\overrightarrow{a}}$ under the composition

$$\mathcal{F}^j V_{\overrightarrow{a}} \to H^0(Y, \pi^*(\overrightarrow{a} \cdot \overrightarrow{L}) - jF) \to H^0(F, \pi^*(\overrightarrow{a} \cdot \overrightarrow{L})|_F - jF|_F)$$

(this identification is clear when $F$ is Cartier on $Y$; if $F$ is of plt type, we use [AZ22, Lemma 2.7]). It follows that $W_{\overrightarrow{\bullet}}$ is an $\mathbb{N}^{r+1}$-graded linear series on $F$ (associated to the divisors $\pi^*L_1|_F, ..., \pi^*L_r|_F$ and $-F|_F$), called the *refinement* of $V_{\overrightarrow{\bullet}}$ by $F$.

Let us fix a klt pair $(X, \Delta)$, some Cartier divisors $L_1, ..., L_r$ on $X$ and an $\mathbb{N}^r$-graded linear series $V_{\overrightarrow{\bullet}}$ associated to the $L_i$'s such that $V_{\overrightarrow{\bullet}}$ contains an ample series and has bounded support. Let $F$ be a primitive divisor over $X$ with associated prime blowup $\pi : Y \to X$. Assume that $F$ is either Cartier on $Y$ or plt type and let $W_{\overrightarrow{\bullet}}$ be the refinement of $V_{\overrightarrow{\bullet}}$ by $F$.

**Theorem 2.2.17.** *[AZ22, Theorem 3.2] Let $F$ be a primitive divisor over $X$ with* *associated prime blowup $\pi : Y \to X$. Let $Z \subset X$ be a subvariety, and $Z_0$ be an irreducible component of $Z \cap C_X(F)$. Let $\Delta_Y$ be the strict transform of $\Delta$ on $Y$ (but remove the component $F$ as in Definition 2.1.14) and let $\Delta_F = \mathrm{Diff}_F(\Delta_Y)$ be the different so that $(K_Y + \Delta_Y + F)|_F = K_F + \Delta_F$. Then, when $Z \subseteq C_X(F)$, we have*

$$\delta_Z(X, \Delta; V_{\vec{\bullet}}) \geq \min \left\{ \frac{A_{X,\Delta}(F)}{S(V_{\vec{\bullet}}; F)}, \inf_{Z'} \delta_{Z'}(F, \Delta_F; W_{\vec{\bullet}}) \right\}.$$

*See the definition of $\delta_Z(X, \Delta; V_{\vec{\bullet}})$ in [AZ22, Lemma 2.9]. Otherwise*

$$\delta_Z(X, \Delta; V_{\vec{\bullet}}) \geq \inf_{Z'} \delta_{Z'}(F, \Delta_F; W_{\vec{\bullet}}),$$

*where the infimums run over all subvarieties $Z' \subseteq Y$ such that $\pi(Z') = Z_0$.*

This Theorem allows us to simplify the problem of finding a lower bound for the local stability threshold, using local stability thresholds of lower dimensional varieties. In this thesis, I use theorems derived from this one for the specific cases of 2 and 3-dimensional Fano varieties. In our cases we only consider the case where $Z = p$ where $p$ is a closed point in our algebraic variety $X$, and therefore the definition of $\delta_Z(X, \Delta; V_{\vec{\bullet}})$ coincides with the one given in Definition 2.2.8.

The following result is a direct consequence of Theorem 2.2.17, and it is a simplification of the 3-dimensional case in [Ara+23, Remark 1.113]) for surfaces. This result gives a concrete recipe for what the flag of subvarieties of $X$ is, which is a Mori Dream space (Definition 2.1.17 and [HK00]), and we apply it in Theorem 3.3.2 to find a lower bound for the local stability threshold. We will also use the version for 3-folds given in [Ara+23, Remark 1.113] to check the K-stability of some varieties in Chapter 4.

**Theorem 2.2.18.** *Let $p$ be a point in a del Pezzo surface $X$, $\pi : Y \to X$ be a plt blowup of the point $p$, and $E$ be the $\pi$-exceptional divisor. Then $(Y, E)$ has purely log terminal singularities so that there exists an effective divisor $\Delta_E$ defined by $K_E + \Delta_E \sim_{\mathbb{Q}} (K_Y + E)|_E$. For every $t \in [0, \tau(-K_X; E)]$, where $\tau(-K_X; E)$*

*denotes the pseudo-effective threshold as in Definition 2.2.6, let us denote by* $P(t)$ *the positive part of the Zariski decomposition of the divisor* $\pi^*(-K_X) - tE$*, and let us denote by* $N(t)$ *its negative part. Let* $W^E_{\cdot,\cdot}$ *be as in Theorem 2.2.17. Then*

$$\delta_p(X) \geq \min\left\{\frac{A_X(E)}{S(-K_X;E)},\ \min_{x\in E}\frac{1-\mathrm{ord}_x(\Delta_E)}{S(W^E_{\cdot,\cdot};x)}\right\}$$

*where for every* $x \in E$ *we have*

$$S(W^E_{\cdot,\cdot};x) = \frac{2}{(-K_X)^2}\int_0^{\tau(-K_X;E)} h(t)\ dt$$

*where*

$$\begin{aligned} h(t) &= (P(t)\cdot E)\cdot \mathrm{ord}_x\left(N(t)|_E\right) + \int_0^\infty \mathrm{vol}_E\left(P(t)|_E - vx\right)\ dv \\ &= (P(t)\cdot E)\cdot (N(t)\cdot E)_x + \int_0^{P(t)\cdot E}\left(P(t)\cdot E - v\right)\ dv \\ &= (P(t)\cdot E)\cdot (N(t)\cdot E)_x + \frac{(P(t)\cdot E)^2}{2} \end{aligned}$$

### 2.2.2 K-moduli

As previously discussed, a key objective of contemporary algebraic geometry is the construction of moduli spaces for varieties. In addition, we mentioned how K-stability plays an important role in the case of Fano varieties. In this section, we provide an introduction to this concept. First, let us introduce the definition of degeneration of varieties.

**Definition 2.2.19.** A *degeneration* is taking a limit of a family of varieties. Given a flat morphism $\pi : \mathcal{X} \to C$ of a variety to a curve $C$ with origin 0 (e.g., $\mathbb{A}^1$), the fibres $\pi^{-1}(t)$ with $t \in C$ form a family of varieties over $C$. Then the fibre $\pi^{-1}(0)$ may be thought as the limit of $\pi^{-1}(t)$ as $t \to 0$. One then says the *family degenerates* to the special fibre $\pi^{-1}(0)$.

We will say that we have an *isotrivial degeneration*, when $\mathcal{X}$ is an isotrivial family, i.e. if there is an open dense subset $U \subset C$ such that all fibres $\pi^{-1}(t)$ with $t \in U$ are smooth and isomorphic.

Now, let us focus on K-moduli. Let $\mathcal{M}^{\mathrm{Kss}}_{V,n}$ be the moduli functor of K-semistable $\mathbb{Q}$-Fano varieties of dimension $n$ and volume $V$, which sends $S \in \mathrm{Sch}$ to the collection $\mathcal{M}^{\mathrm{Kss}}_{V,n}(S)$ of flat proper morphism $X \to S$, whose geometric fibers are $n$-dimensional K-semistable $\mathbb{Q}$-Fano varieties with volume $V$, satisfying Kollár's condition. Kollár's condition is that the reflexive power $w^{[m]}_{X/S}$ is flat over $S$ and commutes with arbitrary base change for each $m \in \mathbb{Z}$ (see [Kol08, Corollary 24].

It is known now that $\mathcal{M}^{\mathrm{Kss}}_{V,n}$ is represented by an Artin stack of finite type and admits a projective good moduli space $\mathcal{M}^{\mathrm{Kss}}_{V,n} \to M^{\mathrm{Kps}}_{V,n}$ (as in [Alp13, Theorem 13.6]), whose closed points are in bijection with $n$-dimensional K-polystable $\mathbb{Q}$-Fano varieties of volume $V$.

To construct $M^{\mathrm{Kps}}_{V,n}$ it is necessary to prove some concrete statements about families of $\mathbb{Q}$-Fano varieties. For instance, to see that $\mathcal{M}^{\mathrm{Kss}}_{V,n}$ is an Artin stack of finite type and that it is a global quotient, these two properties are necessary:

1. **Boundedness:** $\mathcal{M}^{\mathrm{Kss}}_{V,n}$ is bounded by [Jia20]. This means that there is a positive integer $N = N(n, V)$ such that if $X \in \mathcal{M}^{\mathrm{Kss}}_{V,n}$, then $-NK_X$ is a very ample Cartier divisor.
2. **Zariski openness:** If $\mathcal{X} \to S$ is a family of $\mathbb{Q}$-Fano varieties, then the locus where the fibre is K-semistable is a Zariski open set [BL22; Don15; LWX21; Oda13b].

On the other hand, to prove that $\mathcal{M}^{\mathrm{Kss}}_{V,n}$ admits a projective good moduli space it is essential to prove the following:

3. **Good quotient:** The stack $\mathcal{M}^{\mathrm{Kss}}_{V,n}$ admits a good moduli space. This was proven in [Alp+20] by proving that if $X$ is a K-polystable $\mathbb{Q}$-Fano variety

$X$, then $\mathrm{Aut}(X)$ is reductive.

4. **Separatedness:** Any two K-semistable degenerations of a family of K-semistable $\mathbb{Q}$-Fano varieties over a punctured curve $C^\circ = C \setminus \{0\}$ lie in the same $S$-equivalence class, i.e. they degenerate to a common K-semistable $\mathbb{Q}$-Fano variety via special test configurations, this was proved in [BX19; LWX21; SSY16].

5. **Properness:** It was proven in [LWX21] that any family of K-semistable Fano varieties over a punctured curve $C^\circ = C \setminus \{0\}$ can be filled in over $\{0\}$ to a family of K-semistable $\mathbb{Q}$-Fano varieties over $C$.

6. **Projecitivity:** It was proven in [CP21; LXZ22; XZ20] that a sufficiently divisible multiple of the Chow–Mumford (CM) line bundle yields an ample line bundle on $M^{\mathrm{Kps}}_{V,n}$.

Since we know that good projective compact K-moduli space for Fano varieties exists, the goal is to compactify the different components $M^{\mathrm{Kps}}_{V,n}$. As mentioned in the introduction this is already done in dimension 2 by [OSS16]. In Chapter 4, we will compactify the one dimensional $M^{\mathrm{Kps}}_{V,3}$.

# Chapter 3

# Local stability threshold of degree 2 del Pezzo surfaces

This chapter is based on [Etx24]. Although it contains no results beyond [Etx24], the results and explanations are expanded harmoniously with this thesis.

Here, we focus on the local computation of $\delta$ (see Definition 2.2.8). In particular, we study the irrationality of $\delta_p(X)$, where $X$ is a degree 2 del Pezzo surface and a closed point $p \in X$. In the following, whenever we consider "points" on a variety $X$ we will mean **closed** points on $X$. If $\delta_p(X)$ is irrational for some $p \in X$, then there exists a log canonical place $v$ of $X$ such that the associated graded ring (see Definition 2.2.13 and Example 2.2.14)

$$\operatorname{gr}_v R := \oplus_{m,\lambda} \operatorname{Gr}^{\lambda}_{\mathcal{F}_v} H^0(X, -mK_X)$$

is not finitely generated, which was conjectured initially to be finitely generated for Fano varieties.

On this subject, Liu, Xu, and Zhuang established in [LXZ22] that the global $\delta(X)$ is a rational number for K-polystable Fano varieties, assuming that $\delta(X) < (1 +$

$\dim X)/\dim X$. However, the local stability threshold, $\delta_p(X)$, is more enigmatic. Its rationality is a priory unpredictable. Until the results in this chapter, the only known example where the local delta invariant is irrational was in cubic surfaces (see [AZ22, Lemma A.6]).

When $X$ is a del Pezzo surface of degree strictly greater than 3, it is known that $\delta_p(X)$ is rational for every point $p \in X$ (see [Ara+23, §2]). For degrees 2 and 3, we know the specific points with irrational $\delta_p(X)$ by [Etx24; AZ22], and with these results, the values for all $\delta_p(X)$ are known for every closed point $p$ in a del Pezzo surface of degree greater or equal to two. The fact that we know these specific values is useful when applying the Abban-Zhuang method for higher dimensional Fano varieties (Thoerem 2.2.17) and hence we can use them to bound other delta invariants of Fano varieties that contain del Pezzo surfaces. Currently, the degree 1 case is the only one that remains open amongst the smooth del Pezzo surfaces.

In this chapter, we complete the classification of local stability thresholds for smooth del Pezzo surfaces of degree 2 initiated in [Ara+23, §2]. In particular, we show that this number $\delta_p(X)$ is irrational if and only if there is a unique $(-1)$-curve passing through the point $p \in X$, getting the following theorem:

**Theorem 3.0.1.** *[Etx24, Theorem 1.1] Let $X \subset \mathbb{P}(1,1,1,2)$ be a smooth del Pezzo surface of degree 2 and let $p_0 \in X$. Assume that there is a unique $(-1)$-curve $L$ passing through the point $p_0$. Then $\delta_{p_0}(X) = \frac{6}{71}(11 + 8\sqrt{3})$ and it is computed by taking the limit of a sequence of weighted blow ups at $p_0$ with $\mathrm{wt}(u) = a_m$ and $\mathrm{wt}(v) = b_m$ such that $a_m/b_m \to 2/\sqrt{3}$ when $m \to \infty$, where $u$ and $v$ are local coordinates such that $L = \{u = 0\}$.*

**Sketch of the proof.** To prove that the local stability threshold at $p_0$ as above is irrational, we follow these steps: First, we take $E_{a,b}$ to be the exceptional divisor of a certain weighted blowup at $p_0$ with weights $(a, b)$, denoted by $\pi_{a,b} : X_{a,b} \to X$. The choice of $(a, b)$ becomes clear in §3.3. Using that exceptional divisor we compute an upper bound for $\delta_{p_0}(X) = \inf_{E/X,\, p_0 \in C_X(E)} \frac{A_X(E)}{S(-K_X;E)}$. The difficulty here is to compute the expected vanishing order of the anticanonical divisor of $X$ with respect to $E_{a,b}$, $S(-K_X; E_{a,b})$ (see Definition 2.2.6), which requires a Zariski Decomposition and a careful choice of negative curves in the weighted blowup of $X$. This is the main bulk of the work in section §3.2.2.

Then, we use techniques from [AZ22] to find lower bounds for $\delta_{p_0}(X)$, which requires a delicate choice of a minimizer sequence of prime divisors $E_{a_m,b_m}$ over $X$. The choice of $a_m$, $b_m$ is made so that this bound is exactly the upper bound found earlier by considering $E_{a,b}$. In other words, we compute $\delta_{p_0}(X)$.

## 3.1 Objects of study

In this section, we introduce del Pezzo surfaces of degree 2 and some useful properties they have. These results are used later in subsection §3.2.2, to find an equivalence for $\pi^*_{a,b}(-K_X)$ in terms of negative curves in $X_{a,b}$.

### 3.1.1 Del Pezzo surfaces of degree 2

Let $X$ a degree 2 del Pezzo surface, i.e. $(-K_X)^2 = 2$, it can be realised as the surface in weighted projective space $\mathbb{P}(1,1,1,2)$ with homogeneous coordinates $x$, $y$, $z$, $w$, given by the equation

$$w^2 + wG_2(x,y,z) + G_4(x,y,z) = 0, \tag{3.1}$$

where $G_2(x,y,z)$ and $G_4(x,y,z)$ are weighted homogeneous polynomials of degrees 2 and 4, respectively (see [Kol96, 3.5 Theorem]). When working over $\mathbb{C}$, after a change of coordinates the equation can be simplified to

$$w^2 + G_4(x,y,z) = 0. \tag{3.2}$$

Notice that there exists $\rho : X \to \mathbb{P}^2$ a double cover of $\mathbb{P}^2$ ramified over a quartic curve $R := \{G_4(x, y, z) = 0\}$, which is a canonical model of a curve of genus 3 [Dol12].

**Points in $X$.** Let us uncover some properties of the points in $X$. First, recall that since $X$ is a del Pezzo surface of degree 2, it can be represented as the blowup of $\mathbb{P}^2$ at 7 points in general position $\sigma : X \to \mathbb{P}^2$ (see [Li10, Prop. 1.1.2.]). Looking at this description we can prove the following statement:

**Proposition 3.1.1.** *Let $X$ be a smooth del Pezzo surface of degree 2. Let $p$ be a point in $X$. Then, there are at most four $(-1)$-curves passing through $p$.*

*Proof.* Let $\sigma : X \to \mathbb{P}^2$ be the blowup of $\mathbb{P}^2$ at 7 points in general position denoted by $\{p_1, ..., p_7\} \subset \mathbb{P}^2$. $X$ is a del Pezzo surface of degree 2. First, note that the $(-1)$-curves in $X$ are the following:

- Let $E_i$ be the exceptional divisor corresponding to the blowup of $p_i \in \mathbb{P}^2$, for $i \in \{1, 2, 3, 4, 5, 6, 7\}$.
- Let $L_{i,j}$ be the strict transform of the line in $\mathbb{P}^2$ containing $p_i$ and $p_j$ for $i, j \in \{1, 2, 3, 4, 5, 6, 7\}$ and $i \neq j$ (21 in total).
- Let $C_{i,j}$ be the strict transform of the conic in $\mathbb{P}^2$ that contains 5 of the 7 blowed up points (excluding $p_i$ and $p_j$ for $i, j \in \{1, 2, 3, 4, 5, 6, 7\}$ and $i \neq j$, 21 in total).
- Let $Q_i$ be the strict transform of the cubic in $\mathbb{P}^2$ that contains all the points $p_j$ for $j \in \{1, 2, 3, 4, 5, 6, 7\}$ and has multiplicity 2 at $p_i$ for some $i \in \{1, 2, 3, 4, 5, 6, 7\}$.

Take a close point $p \in X$. We distinguish 2 cases:

(1) $\sigma(p) = p_i$ for some $i \in \{1, 2, 3, 4, 5, 6, 7\}$,

(2) $\sigma(p) \neq p_i$ for all $i \in \{1,2,3,4,5,6,7\}$.

In both cases, it is clear that there is at most one $Q_i$ passing through $p$. First notice that two cubic curves intersect each other at 9 points (counting multiplicity). Take $\{k,j\} \subset \{1,2,3,4,5,6,7\}$ distinct, since $\sigma(Q_j)$ has multiplicity 2 at $p_j$, $\sigma(Q_k)$ has multiplicity 2 at $p_k$, they only intersect at the $p_i$ points for $i \in \{1,2,3,4,5,6,7\}$. Hence, in the case (2) there is at most one $(-1)$-cubic passing through $p$. In case (1), let us assume $p \in Q_j \cap Q_k$ for $k \neq j$ then $\sigma(Q_j)$ and $\sigma(Q_k)$ are tangent at $p_i$. Let us assume without loss of generality that $i = 1$, $j = 2$ and $k = 3$. With a change of coordinates, we can assume $p_1 = (1:0:0)$, $p_2 = (0:1:0)$ and $p_3 = (0:0:1)$. Then we will have the following equations for $Q_2$ and $Q_3$:

$$Q_2 := \{f(x,y,z) = ax^2y + bx^2z + cxyz + dz^2x + ez^2y = 0\},$$
$$Q_3 := \{g(x,y,z) = ax^2y + bx^2z + \alpha xyz + \beta y^2x + \gamma y^2z = 0\}.$$

We want to find $\{a,b,c,d,e,\alpha,\beta,\gamma\} \subset \mathbb{C}$ such that $f(p_i) = g(p_i) = 0$ for every $i \in \{4,...,7\}$. However, the only solution of this equation system is $a = b = c = d = e = \alpha = \beta = \gamma = 0$, hence we get a contadiction.

On the other hand, since the 7 points that we blow up to get $X$ are in general position (i.e. no 3 points in the same line, no 6 points in the same conic) we know there are at most two $(-1)$-lines passing through the same point. There are two possibilities for this to happen at $p \in X$:

(i) There exist distinct $i, j \in \{1,2,3,4,5,6,7\}$ such that $p \in E_i \cap L_{i,j}$. This falls in the case (1) mentioned above.

(ii) There exist distinct $\{i,j\}, \{k,l\} \subset \{1,2,3,4,5,6,7\}$ such that $p \in L_{k,l} \cap L_{i,j}$. This falls in the case (2) mentioned above.

Now, let us see that for both cases there are at most 2 conics with self-intersection

$-1$ passing through $p$. Recall that the equation of a conic in $\mathbb{P}^2$ with coordinates $(x : y : z)$ is as follows:

$$ax^2 + bxy + cy^2 + dxz + eyz + fz^2 = 0, \tag{3.3}$$

such that $(a : b : c : d : e : f) \in \mathbb{P}^5$. Also, since we want this conic to be a $(-1)$-curve, it has to contain 5 of the points we are blowing up. Let us see what happens in each of the cases:

(i) Choose $p_1, p_2, p_3, p_4, p_5 \in \mathbb{P}^2$, these are five of the points that we blow up to get $X$. Assume without loss of generality that $p \in E_1 \cap L_{1,2}$. Define $C$ containing $p_1, p_3, p_4, p_5$. Since we want $\sigma^*(C)$ (the pullback of $C$) to contain $p$, $C$ has to be tangent to $\sigma(L_{1,2})$ at $p_1$ in $\mathbb{P}^2$. Also note, that $C$ can not contain $p_2$ otherwise it will not be irreducible. These conditions completely determine the equation (3.3) and define $C$. Notice that currently, $C$ contains four points that we blow up, but to get a $(-1)$-curve we need one more. Then, choose a point $p_6 \in C$ in general position. With this, we have our first $(-1)$-conic curve $\sigma^*(C)$. Note, that any other irreducible conic in $\mathbb{P}^2$, $C'$, that is tangent to $\sigma(L_{1,2})$ at $p_1$ and such that $\sigma^*(C')$ is a (-1)-curve in $X$, have to contain three points out of $\{p_3, p_4, p_5, p_6\}$. And these conditions will describe the same equation as $C$, hence $C' = C$.

Observe that if $E_1$ is the unique $(-1)$-line containing $p$, we can define up to 2 conic $(-1)$-curves passing through $p$. In this case, we define $C'$, to be the conic, containing $\{p_1, p_2, p_3, p_4\}$ that is tangent to $C$ at $p_1$. These conditions determine the equation (3.3) and define $C'$. Finally for $\sigma^*(C')$ to be a $(-1)$-curve, we choose $p_7 \in C'$ that is in general position to the previous ones. Notice that we can not find more $(-1)$-curves passing through the same point $p \in X = Bl_7\mathbb{P}^2$, the blowup of $\mathbb{P}^2$ at $\{p_1, p_2, p_3, p_4, p_5, p_6, p_7\}$. In this case, we reach a maximum of three $(-1)$-curves passing through $p$.

(ii) Choose $p_1, p_2, p_3, p_4 \in \mathbb{P}^2$ in general position, these are four of the points we blow up to get $X$. We want to define conics that contain $p \in L_{1,2} \cap L_{3,4}$. Define $C$ to be the conic containing $\sigma(p)$, $p_1$, $p_2$, $p_3$ and $p_4$ in $\mathbb{P}^2$, these conditions completely determine the equation (3.3). Notice that $C$ contains four points that we blow up, to get a $(-1)$-curve, choose a point in general position $p_5 \in C$ to be the 5th, then $(\sigma^*(C))^2 = -1$. To define another conic, choose a new point in general position $p_6 \in \mathbb{P}^2$ and let $C'$ be the conic containing $\sigma(p)$, $p_1$, $p_2$, $p_3$ and $p_6$ in $\mathbb{P}^2$, these five points completely determine the equation (3.3). To make it a $(-1)$-curve, choose $p_7 \in C'$ to be the 7th point in general position. Notice that we can not define more conic curves with the desired properties. Then we define $X = Bl_7\mathbb{P}^2$, the blowup of $\mathbb{P}^2$ at $\{p_1, p_2, p_3, p_4, p_5, p_6, p_7\}$ and $p \in L_{1,2} \cap L_{3,4}$ is a point with four $(-1)$-curves passing through it.

Observe that for any point $p \in X \setminus \bigcup_{i=1}^{7} E_i$, we are able to define at most two conic $(-1)$-curves with the same process.

Unitl now, we saw that we have up to 4 (-1)-curves passing through each point $p \in X$ counting lines and conics. We also proved that each point $p \in X$ has at most one (-1)-cubic passing through it. To finish our proof, we need to see that it is not possible to define a (-1)-cubic passing for the point $p$ as defined in (ii).

First, note that a cubic, $Q$, and a conic in $\mathbb{P}^2$ intersect in 6 points. Note that for $\sigma^*(Q)$ to be a $(-1)$-curve containing $p$, $Q$ has to contain all $\{p_1, ..., p_7\}$, with multiplicity 2 at $p_j$ for some $j \in \{1, ..., 7\}$ and $\sigma(p)$. On the other hand, both $C$ and $C'$ contain 5 of the points we are blowing up, in addition to $\sigma(p)$. Hence, $\#(Q \cap C) = 7$ or $\#(Q \cap C') = 7$, which is a contradiction. Therefore, for any closed point $p \in X$ we have at most four $(-1)$-curves passing through $p$.

□

Notice that if a point $p = (x, y, z, 0) \in X$, i.e. if $\rho(p) \in R$, with a change of coordinate we can rewrite $p = (1 : 0 : 0 : 0) \in X$. Recall that $\rho : X \to \mathbb{P}^2$ is a double cover of $\mathbb{P}^2$ ramified over a quartic curve $R := \{G_4(x, y, z) = 0\}$. Let $T_pX$ be a hyperplane in $P(1, 1, 1, 2)$ tangent to $X$ at $p$. Furthermore, we can define a tangent hyperplane section of $X$ by considering $T_pX \cap X$. We can see that in this case the anticanonical divisor is numerically equivalent to the tangent hyperplane section, $-K_X \sim T_pX \cap X$.

**Proposition 3.1.2.** *Let $X$ be a del Pezzo surface of degree 2. Take a point $p \in X$ such that $\rho(p) \in R$ and let $C_p = T_pX \cap X$ be the tangent hyperplane section of $X$ at $p$. Then, $-K_X \sim C_p$.*

*Proof.* Since $\rho(p) \in R$, we can assume $p = (1 : 0 : 0 : 0) \in X$, with $G_4(1, 0, 0) = 0$; in other words, we assume that the coefficient of the monomial $x^4$ on $G_4$ is zero, and since $X$ is smooth $G_4(x, y, z) = x^3(ay + bz) + F_4(x, y, z)$ where $(ay + bz) \neq 0$ and $F_4(x, y, z)$ does not have any monomial containing $x^3$. Then, we change coordinate taking $z = ay + bz$, and we get $G_4'(x, y, z) = x^3z + F_4'(x, y, z)$ where $F_4'(x, y, z)$ does not have any monomial containing $x^3$. It follows from here that the tangent space at the point is $T_pX = \{z = 0\}$, that is $\mathbb{P}(1, 1, 2)$. Moreover, the intersection with $X$ gives $T_pX \cap X = \{w^2 + G_4'(x, y, 0) = 0\}$, which is a quartic in $\mathbb{P}(1, 1, 2)$, and it could be represented generally as follows:

$$w^2 + ax^2y^2 + bxy^3 + cy^4 = 0.$$

However, with another change of coordinates, rewriting $ax^2 + bxy + cy^2 = a(x + \alpha y)^2 + \beta y^2$, and taking $x = x + \alpha y$, we get

$$C_p = T_pX \cap X = \{w^2 + y^2(ax^2 + by^2) = 0)\} \subseteq \mathbb{P}(1, 1, 2). \tag{3.4}$$

Therefore, taking into account that $T_p(X)$ is a plane, we have the following adjunction formula,

$$-K_X \sim (-K_{\mathbb{P}(1,1,1,2)} - X)|_X \sim (5T_p(X) - 4T_p(X))\,|_X = T_p(X) \cap X = C_p,$$

we know that $(C_p)^2 = (-K_X)^2 = 2$. □

**Remark 3.1.3.** From (3.4) we have the following possibilities for $C_p$:

(1) $C_p$ is an irreducible curve that is smooth at $p$, i.e. the ordinary blowup of $X$ at $p$ is a smooth del Pezzo surface of degree 1.

(2) $C_p$ is an irreducible curve with a node at $p$;

(3) $C_p$ is an irreducible curve with a cusp at $p$;

(4) $C_p$ is a union of two $(-1)$-curves that meet transversely at $p$;

(5) $C_p$ is a union of two $(-1)$-curves that are tangent at $p$.

Looking at the previous lemmas and remark, we can say that if $p$ is a closed point in $X$, it satisfies one of the following statements:

(1) If we do an ordinary blowup of $p$ we get a smooth del Pezzo surface of degree 1;

(2) $\rho(p) \in R$, and $C_p$ is an irreducible nodal curve;

(3) $\rho(p) \in R$, and $C_p$ is an irreducible cuspidal curve;

(4) $\rho(p) \in R$, and $C_p$ is a union of two $(-1)$-curves that meet transversally;

(5) $\rho(p) \in R$, and $C_p$ is a union of two $(-1)$-curves that are tangent at $p$;

(6) $\rho(p) \notin R$, and $p$ is contained in exactly one $(-1)$-curve;

(7) $\rho(p) \notin R$, and $p$ is contained in two $(-1)$-curves;

(8) $\rho(p) \notin R$, and $p$ is contained in three $(-1)$-curves;

(9) the point $p$ is a generalized Eckardt point, i.e. $p$ is contained in four $(-1)$-curves..

For each of the cases listed above, $\delta_p$ is computed in [Ara+23, Lemma 2.15] and these are the values it takes:

$$\delta_p(X) = \begin{cases} 36/17 & \text{if } (1) \\ 2 & \text{if } (2) \\ 15/8 & \text{if } (3) \\ 2 & \text{if } (4) \\ 9/5 & \text{if } (5) \\ 48/23 & \text{if } (7) \\ 72/35 & \text{if } (8) \\ 2 & \text{if } (9). \end{cases}$$

Note that the case where $p$ is contained in a unique $(-1)$-curve is not completed. In [Ara+23], the authors bound $\delta_p(X)$, so we know that $\frac{40}{19} \geq \delta_p(X) \geq \frac{60}{31}$, but the exact value is missing. Therefore, we focus on computing $\delta_p(X)$ for this case. To do so, it is important to know a bit more about this point and what happens when we blow it up. That is helpful to find a suitable linear equivalence for the anticanonical divisor $-K_X$.

**Remark 3.1.4.** Let $p \in X$ be a point contained in a unique $(-1)$-curve, $L$. Let $\sigma : Y \to X$ be the ordinary blowup of $X$ at $p$, where $Y$ is a weak del Pezzo surface of degree 1, i.e. $-K_Y$ is big and nef but not ample, and $(-K_Y)^2 = 1$. In such surfaces, we have a birational involution called the Bertini involution, $\iota$. As in the proof of Lemma 2.15 in [Ara+23], the linear system $|-2K_Y|$ gives a morphism $Y \to \mathbb{P}(1,1,2)$ with the following Stein factorization:

$$\begin{array}{ccc} Y & \xrightarrow{\nu} & \widetilde{Y} \\ \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\omega} \\ X & & \mathbb{P}(1,1,2) \end{array}$$

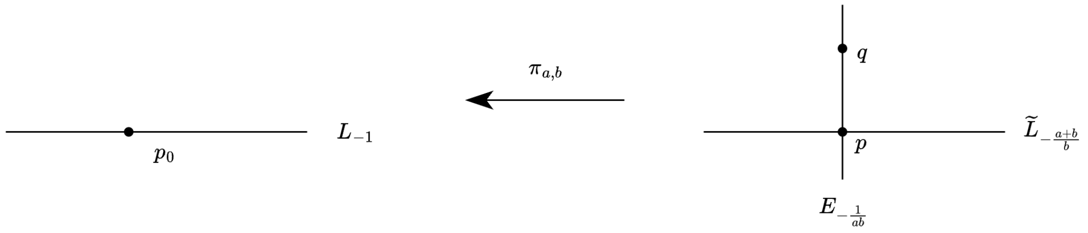
where $\nu$ is a contraction of all $(-2)$-curves in the surface $Y$ (in our case we have a unique $(-2)$-curve, $\widetilde{L}$, which is the strict transform of $L$), $\omega$ is a double cover branched over the union of a sextic curve in $\mathbb{P}(1,1,2)$ and the singular points of $\mathbb{P}(1,1,2)$. The double cover $\widetilde{Y} \to \mathbb{P}(1,1,2)$ induces an involution $\tau \in \mathrm{Aut}(\widetilde{Y})$, and the latter induces the involution $\nu^{-1} \circ \tau \circ \nu = \iota \in \mathrm{Aut}(Y)$ known as the Bertini involution. For detailed equations of the Bertini involution, check [Moo43].

In the next section, we see what happens when we add weights to the blowup of $X$ at the point $p$.

## 3.2 Weighted blowup of $X$

In this section, we introduce some essential technical results for proving Theorem 3.0.1. Let $\pi_{a,b} : X_{a,b} \to X$ be the $(a,b)$-weighted blowup of a del Pezzo surface of degree 2 at a point $p_0 \in X$. As in Theorem 3.0.1, we assume that there is a unique $(-1)$-curve $L$ passing through the point $p_0$, and $\gcd(a,b) = 1$. We choose local coordinates $(u,v)$ in a neighbourhood of $p_0$ such that $L = \{u = 0\}$, which enables us to write down the (local) weighted blowup as $\mathrm{wt}(u) = a$ and $\mathrm{wt}(v) = b$.

Figure 3.1: $(a,b)$-weighted blowup.



In the figure above, the subscripts represent the self-intersections, $E$ is the exceptional divisor of the weighted blowup, and $\widetilde{L}$ is the strict transform of $L$. Notice that since $a$ and $b$ are greater than 0 and coprime, $E$ has at most two singular points, $q$ and $p$, coming for the weights of the blowup.

Once we have this weighted blowup, we use the exceptional divisor of this birational map to give a better bound for $\delta_{p_0}(X)$ by computing $A_X(E)/S(-K_X;E)$, as defined in §2.2. Therefore, we need to compute the Zariski Decomposition of the divisor $\pi^*_{a,b}(-K_X) - tE$.

We start by writing $\pi^*_{a,b}(-K_X) - tE$ as a non-negative combination of negative divisors (see [Laz04, Theorem 2.3.19]). However, currently, we do not have enough negative curves in the picture.

**Next step:** We search for another known algebraic variety, $Y$, that is birational to the weighted blowup $X_{a,b}$ and $X$. Let $\sigma : Y \to X$ be the birational morphism between $Y$ and $X$. The idea is to find a linear equivalence to $\sigma^*(-K_X)$ in terms of $\sigma^*(L)$ in $Y$ with other negative curves, and bring it back to $X_{a,b}$.

### 3.2.1 Notation

In order to simplify the explanations and computations of the resolution of singularities in the weighted blowup $X_{a,b}$ in §3.2.2, here we introduce some numerical definitions.

As in Theorem 3.0.1, let $p_0 = (x_0, y_0, z_0, w_0) \in X$ be a closed point with a unique (-1)-curve passing through it, $L$. Let $X_{a,b}$ be the $(a,b)$-weighted blowup of $X$ at $p_0$. Let $E$ be the exceptional divisor of the weighted blowup. Due to technical reasons affecting the proof of Theorem 1.1.2, we assume $\frac{\sqrt{3}}{2}a \leq b \leq a$ and $\gcd(a,b) = 1$. Therefore, we can rewrite $a = b + \gamma_0$ where $\gamma_0 \in \{1, ..., b-1\}$. Similarly, we can rewrite $b = \gamma_0 \cdot j_0 + \gamma_1$ and $\gamma_0 = \gamma_1 \cdot j_1 + \gamma_2$ where $j_0, j_1 \in \mathbb{N}$, $\gamma_1 \in \{0, 1, ..., \gamma_0 - 1\}$ and $\gamma_2 \in \{0, 1, ..., \gamma_1 - 1\}$. Generalizing this, let $\gamma_k = \gamma_{k+1} \cdot j_{k+1} + \gamma_{k+2}$ where $j_{k+1} \in \mathbb{N}$ and $\gamma_{k+2} \in \{0, 1, ..., \gamma_{k+1} - 1\}$. Let us denote $so_n = \sum_{m=0}^{n} j_{2m+1}$ and $se_n = \sum_{m=0}^{n} j_{2m}$ where $so_{-1} = se_{-1} = 0$.

Since $\{\gamma_k\}$ is a decreasing sequence of natural numbers, there exists a $k_0 \in \mathbb{N}$ where $\gamma_{k_0} = 1$. Furthermore, we can choose $k_0$ such that for any other $k' \in \mathbb{N}$ where $\gamma_{k'} = 1$, we know $k' \geq k_0$.

### 3.2.2 The resolution of $X_{a,b}$ followed by contractions to a weak degree 1 del Pezzo

As mentioned in Example 2.1.3, an $(a, b)$-weighted blowup of a smooth surface has at most two quotient singularities. If $a \neq 1$, denote by $q$ the $\frac{1}{a}(1, c_1)$= $\frac{1}{a}(1, b)$ quotient singularity where $c_1 = -b + n_1 a$ and $n_1 = \left\lceil \frac{b}{a} \right\rceil = 1$, i.e. $c_1 \equiv b \mod a$. On the other hand, if $b \neq 1$, let $p$ be the $\frac{1}{b}(1, d_1)$= $\frac{1}{b}(1, a)$ singularity, where $d_1 = -a + m_1 b$ with $m_1 = \left\lceil \frac{a}{b} \right\rceil = 2$, , i.e. $d_1 \equiv a \mod b$.
Similarly, we define the following sequence of numbers:

- $c_2 = -a + n_2 c_1$, where $n_2 = \left\lceil \frac{a}{c_1} \right\rceil$ and $c_k = -c_{k-2} + n_k c_{k-1}$ where $n_k = \left\lceil \frac{c_{k-2}}{c_{k-1}} \right\rceil$.
- $d_2 = -b + m_2 d_1$, where $m_2 = \left\lceil \frac{b}{d_1} \right\rceil$ and $d_k = -d_{k-2} + m_k d_{k-1}$ where $m_k = \left\lceil \frac{d_{k-2}}{d_{k-1}} \right\rceil$.

The resolution of these singularities is analogous to the Euclidean algorithm. The resolution of the singularity $p$ is achieved for some $i_0 \in \mathbb{N}$ such that $d_{i_0} = 1$ and similarly for $q$, the resolution is complete when we get $c_{l_0} = 1$ for some $l_0 \in \mathbb{N}$.

**Remark 3.2.1.** Each $d_k$ can be represented as $d_k = \mu_k b + \lambda_k a$, where $\mu_k$, $\lambda_k \in \mathbb{Z}$ and their values come from taking backwards $d_k = -d_{k-2} + m_k d_{k-1}$ by substituting $d_{k-2}$ and $d_{k-1}$ with their definitions, and so on until we get to the expression with only $a$ and $b$. Similarly, each $c_k$ can be represented as $c_k = \beta_k b + \alpha_k a$, where $\beta_k$, $\alpha_k \in \mathbb{Z}$. Moreover, notice that the property $\bullet_k = -\bullet_{k-2} + m_k \cdot \bullet_{k-1}$ also holds for the coefficients $\mu_k$ and $\lambda_k$. Likewise, $\bullet_k = -\bullet_{k-2} + n_k \cdot \bullet_{k-1}$ holds for $\alpha_k$ and $\beta_k$.

From now on, we assume neither $a$ nor $b$ are equal to 1. If this is not the case, it is enough to omit the resolution of the singularity that corresponds to the weight which is equal to one. We present an algorithm to determine the resolution of $X_{a,b}$, which consists of repeatedly blowing up the singularities.

**Resolution of $p$.** Let $\sigma_1 : \hat{X}_{a,b} \to X_{a,b}$ be the blowup with suitable (natural) weights of $X_{a,b}$ at the quotient singular point $p$. Let $E^{(1)}$ be the exceptional divisor and let $\hat{L}$ and $\hat{E}$ be the strict transforms of $\widetilde{L}$ and $E$ respectively. Since $p$ is a $\frac{1}{b}(1, d_1)$ quotient singularity, we have that $\sigma_1^*(\widetilde{L}) \sim \hat{L} + \frac{d_1}{b} E^{(1)}$, and $\sigma_1^*(E) \sim \hat{E} + \frac{1}{b} E^{(1)}$. Taking into account that $\hat{L} \cdot \hat{E} = 0$, it is straightforward to check that $(E^{(1)})^2 = -\frac{b}{d_1}$. Therefore, we get the following self-intersections:

$$(\hat{L})^2 = \left(\sigma_1^*(\widetilde{L}) - \frac{d_1}{b} E^{(1)}\right)^2 = (\sigma_1^*(\widetilde{L}))^2 + \left(\frac{d_1}{b} E^{(1)}\right)^2 = -\frac{a+b}{b} - \frac{d_1}{b} = -(1 + m_1),$$
$$(\hat{E})^2 = \left(\sigma_1^*(E) - \frac{1}{b} E^{(1)}\right)^2 = (\sigma_1^*(E))^2 + \left(\frac{1}{b} E^{(1)}\right)^2 = -\frac{1}{ab} - \frac{1}{bd_1} = -\frac{m_1}{ad_1} = -\frac{\mu_1}{ad_1}.$$

Note that in the second equality, we are using the fact that $\sigma_1^*(\widetilde{L}) \cdot E^{(1)} = \widetilde{L} \cdot \ \sigma(E^{(1)}) = 0$ since $\sigma(E^{(1)}) = p$. A similar thing happens when we compute $(\hat{E})^2$. The next step depends on the value of $d_1$. If $d_1 = 1$, the resolution of $p$ is finished and we continue with the resolution of the singularity, $q$, on $\hat{X}_{a,b}$. On the other hand, if $d_1 \neq 1$, $\hat{X}_{a,b}$ has a singularity of type $\frac{1}{d_1}(1, d_2)$ (as well as $q$) in the intersection of $E^{(1)}$ with $\hat{E}$. In the latter case, we iterate the process by blowing the new quotient singularity up. Denote this new blowup with suitable (natural) weights by $\sigma_2 : \dot{X}_{a,b} \to \hat{X}_{a,b}$. Let $E^{(2)}$ be the exceptional divisor of $\sigma_2$, where $(E^{(2)})^2 = -\frac{d_1}{d_2}$. Let $\dot{E}^{(1)}$ and $\dot{E}$ be the strict transforms of $E^{(1)}$ and $\hat{E}$, respectively. We get the following self-intersections:

$$(\dot{E}^{(1)})^2 = \left(\sigma_2^*(E^{(1)}) - \frac{d_2}{d_1}E^{(2)}\right)^2 = (\sigma_2^*(E^{(1)}))^2 + \left(\frac{d_2}{d_1}E^{(2)}\right)^2 = -\frac{b}{d_1} - \frac{d_2}{d_1} = -m_2,$$
$$(\dot{E})^2 = \left(\sigma_2^*(\hat{E}) - \frac{1}{d_1}E^{(2)}\right)^2 = (\sigma_2^*(\hat{E}))^2 + \left(\frac{1}{d_1}E^{(2)}\right)^2 = -\frac{m_1}{ad_1} - \frac{1}{d_1 d_2}$$
$$= -\frac{(m_1 m_2 - 1)d_1}{ad_1 d_2} = -\frac{m_1 m_2 - 1}{ad_2} = -\frac{\mu_2}{ad_2}.$$

Notice that since we chose $a$ and $b$ such that $\gcd(a, b) = 1$, by the Euclidean algorithm there exists an $i_0 \in \mathbb{N}$ such that $d_{i_0} = 1$. Therefore, the resolution of $p$ is achieved after $i_0$ steps. Let us denote $f_p : \check{X}_{a,b} \to X_{a,b}$ the resolution of $p$ defined as $f_p = \sigma_1 \circ \sigma_2 \circ \cdots \circ \sigma_{i_0-1} \circ \sigma_{i_0}$. For each of these blowups of a quotient singular point, let us denote by $E^{(k)}$ the exceptional divisor of $\sigma_k$. Let $\check{E}^{(k)}$, $\check{E}$, and $\check{L}$ be the strict transforms of our divisors after the $i_0$-th blowup. Then, we have the following self-intersections in $\check{X}_{a,b}$:

$$(\check{E}^{(k)})^2 = -m_{k+1} \text{ for } k = 1, ..., i_0 - 1, \quad (E^{(i_0)})^2 = -d_{i_0-1},$$
$$(\check{L})^2 = -(1 + m_1) = -3.$$

**Lemma 3.2.2.** *In the above setting,* $(\check{E})^2 = -\frac{\mu_{i_0}}{a}$.

*Proof.* We previously saw that after the first blowup, $(E)^2 = -\frac{\mu_1}{ad_1}$. By induction, assume it is true for the $k$-th blowup, $(E)^2 = -\frac{\mu_k}{ad_k}$. Let $\overline{E}$ be the strict transform of $E$ after the $k$-th blowup. Let us check that it holds for the $(k+1)$-th blowup.

$$(E)^2 = \left(\sigma_{k+1}^*(\overline{E}) - \frac{1}{d_k}E^{(k+1)}\right)^2 = -\frac{\mu_k}{ad_k} - \frac{1}{d_k d_{k+1}} = -\frac{\mu_k(-d_{k-1} + m_{k+1}d_k) + a}{ad_k d_{k+1}}$$
$$= -\frac{a - \mu_k d_{k-1} + \mu_k m_{k+1} d_k}{ad_k d_{k+1}}.$$

It is enough to show that $a - \mu_k d_{k-1} = -\mu_{k-1} d_k$. Since $a = \mu_{k-1} d_{k-2} - \mu_{k-2} d_{k-1}$, we get the following:

$$\begin{aligned}\mu_k d_{k-1} - \mu_{k-1} d_k &= (-\mu_{k-2} + m_k \mu_{k-1}) d_{k-1} - \mu_{k-1}(-d_{k-2} + m_k d_{k-1}) \\ &= \mu_{k-1} d_{k-2} - \mu_{k-2} d_{k-1} = a.\end{aligned}$$

Therefore, we get $(E)^2 = -\frac{\mu_{k+1}}{ad_{k+1}}$. In particular, $(\check{E})^2 = -\frac{\mu_{i_0}}{a}$, since $d_{i_0} = 1$. □

The values of $i_0$, $d_{i_0-1}$, and $m_k$ can be specified using the notation of subsection 3.2.1. We have two possibilities:

- If $k_0 = 2n_0 + 1$,

$$i_0 = se_{n_0},$$

$$\left(\check{E}^{(se_n+k)}\right)^2 = -m_{(se_n+k+1)} = \begin{cases} -(j_{2n+1}+2) & \text{if } k = 0, \\ -2 & \text{Otherwise.}\end{cases}$$

$$(\check{E}^{(i_0)})^2 = -d_{i_0-1} = -(j_{2n_0+1}+2).$$

- If $k_0 = 2n_0$,

$$i_0 = se_{n_0} - 1,$$

$$\left(\check{E}^{(se_n+k)}\right)^2 = -m_{(se_n+k+1)} = \begin{cases} -(j_{2n+1}+2) & \text{if } k = 0, \\ -2 & \text{Otherwise.}\end{cases}$$

$$(\check{E}^{(i_0)})^2 = -d_{i_0-1} = -2.$$

**Resolution of $q$.** Let $\tau_1 : \ddot{X}_{a,b} \to \check{X}_{a,b}$ be the blowup with suitable (natural) weight of $\check{X}_{a,b}$ at the quotient singular point $q$. Let $F^{(1)}$ be the exceptional divisor and $\ddot{L}$, $\ddot{E}^{(k)}$ and $\ddot{E}$ be the strict transforms of $\check{L}$, $\check{E}^{(k)}$ and $\check{E}$ respectively. Since $q$ is a $\frac{1}{a}(1, c_1)$ quotient singularity, $(F^{(1)})^2 = -\frac{a}{c_1}$. Also, since $q$ is in $\check{E}$, its strict transforms is not isomorphic to the pullback: $\tau_1^*(\check{E}) \sim \ddot{E} + \frac{1}{a}F^{(1)}$. Therefore:

$$(\ddot{E})^2 = \left(\tau_1^*(\check{E}) - \frac{1}{a}F^{(1)}\right)^2 = (\tau_1^*(\check{E}))^2 + \left(\frac{1}{a}F^{(1)}\right)^2 = -\frac{\mu_{i_0}}{a} - \frac{1}{ac_1} = -\frac{n_1\mu_{i_0} + \lambda_{i_0}}{c_1}$$
$$= -\frac{\alpha_1\mu_{i_0} - \beta_1\lambda_{i_0}}{c_1}.$$

After this blowup, $\ddot{X}_{a,b}$ is smooth if $c_1 = 1$ and it is the resolution of $X_{a,b}$. Otherwise, if $c_1 \neq 1$, $\ddot{X}_{a,b}$ has a singularity of type $\frac{1}{c_1}(1, c_2)$ in the intersection of $F^{(1)}$ and $\ddot{E}$. Therefore, we proceed to iterate the process by blowing up the new quotient singularity with a blowup with suitable (natural) weights denoted by $\tau_2 : \dddot{X}_{a,b} \to \ddot{X}_{a,b}$. Let $F^{(2)}$ be the exceptional divisor of the $\tau_2$ blowup, where $(F^{(2)})^2 = -\frac{c_1}{c_2}$. Let $\dddot{F}^{(1)}$ and $\dddot{E}$ be the strict transforms of $F^{(1)}$ and $\ddot{E}$, respectively. We get the following self-intersections:

$$\left(\dddot{F}^{(1)}\right)^2 = \left(\tau_2^*(F^{(1)}) - \frac{c_2}{c_1}F^{(2)}\right)^2 = (\tau_2^*(F^{(1)}))^2 + \left(\frac{c_2}{c_1}F^{(2)}\right)^2 = -\frac{a}{c_1} - \frac{c_1}{c_2} = -n_2,$$
$$\left(\dddot{E}\right)^2 = \left(\tau_1^*(\check{E}) - \frac{1}{a}F^{(1)}\right)^2 = (\tau_1^*(\check{E}))^2 + \left(\frac{1}{c_1}F^{(1)}\right)^2 = -\frac{\mu_{i_0}}{c_1} - \frac{1}{ac_1}$$
$$= -\frac{\alpha_1\mu_{i_0} - \beta_1\lambda_{i_0}}{c_1} - \frac{1}{c_1c_2} = -\frac{\alpha_2\mu_{i_0} - \beta_2\lambda_{i_0}}{c_2}.$$

Recall that there exist $l_0 \in \mathbb{N}$ such that $c_{l_0} = 1$. Therefore, we achieve the resolution of $q$ after $l_0$ steps. Let us denote by $f_q : \overline{X}_{a,b} \to \check{X}_{a,b}$ the resolution of $q$ defined as $f_q = \tau_1 \circ \tau_2 \circ \cdots \circ \tau_{l_0-1} \circ \tau_{l_0}$. For each of these blowups, $\tau_k$, of a quotient singular point, let us denote by $F^{(k)}$ its exceptional divisor. Let $\overline{F}^{(k)}$, $\overline{E}^{(k)}$, $\overline{E}$, and $\overline{L}$ be the strict transforms of our divisors after the $(i_0 + l_0)$-th blowup. Then, we have the following self-intersections:

$$(\overline{F}^{(k)})^2 = -n_{k+1} \text{ for } k = 1, ..., l_0 - 1, \quad (\overline{F}^{(l_0)})^2 = -c_{l_0-1}.$$

**Remark 3.2.3.** Notice that the self-intersections of $\overline{E}^{(k)}$ and $\overline{L}$ are the same ones as their images through $f_q$, this happens because we are not blowing up any point in these divisors in the resolution of $q$.

**Lemma 3.2.4.** *In $\overline{X}_{a,b}$, we have that $(\overline{E})^2 = -1$.*

*Proof.* First we see that in the $(i_0+k)$-th blowup, $(E)^2 = -\frac{\alpha_k\mu_{i_0}-\beta_k\lambda_{i_0}}{c_k}$. And then we see that $\alpha_{l_0}\mu_{i_0} - \beta_{l_0}\lambda_{i_0} = 1$.

We previously saw that after $\tau_1$, we got $(E)^2 = -\frac{\alpha_1\mu_{i_0}-\beta_1\lambda_{i_0}}{c_k}$. By induction on k, assume it is also true for the $i_0+k$-th blowup. Then let us see that it also holds for the $k+1$-th.

$$\begin{aligned}
(E)^2 &= -\frac{\alpha_k\mu_{i_0}-\beta_k\lambda_{i_0}}{c_k} - \frac{1}{c_kc_{k+1}} = -\frac{(\alpha_k\mu_{i_0}-\beta_k\lambda_{i_0})c_{k+1}+1}{c_kc_{k+1}} \\
&= -\frac{(\alpha_k\mu_{i_0}-\beta_k\lambda_{i_0})(-c_{k-1}+n_{k+1}c_k)+1}{c_kc_{k+1}} \\
&= -\frac{1-(\alpha_k\mu_{i_0}-\beta_k\lambda_{i_0})c_{k-1}+(\alpha_k\mu_{i_0}-\beta_k\lambda_{i_0})n_{k+1}c_k}{c_kc_{k+1}}.
\end{aligned}$$

So it is enough to prove that $1-(\alpha_k\mu_{i_0}-\beta_k\lambda_{i_0})c_{k-1} = (\alpha_{k-1}\mu_{i_0}-\beta_{k-1}\lambda_{i_0})c_k$.

$$\begin{aligned}
&(\alpha_k\mu_{i_0}-\beta_k\lambda_{i_0})c_{k-1}+(\alpha_{k-1}\mu_{i_0}-\beta_{k-1}\lambda_{i_0})c_k = ((-\alpha_{k-2}+n_k\alpha_{k-1})\mu_{i_0} \\
&\quad -(-\beta_{k-2}+m_k\beta_{k-1})\lambda_{i_0})c_{k-1}+(\alpha_{k-1}\mu_{i_0}-\beta_{k-1}\lambda_{i_0})(-c_{k-2}+n_kc_{k-1}) \\
&= (\alpha_{k-1}\mu_{i_0}-\beta_{k-1}\lambda_{i_0})c_k = 1.
\end{aligned}$$

Where for the first equation we use the equality $1 = (\alpha_{k-1}\mu_{i_0}-\beta_{k-1}\lambda_{i_0})c_{k-2} + (\alpha_{k-2}\mu_{i_0}-\beta_{k-2}\lambda_{i_0})c_{k-1}$. Therefore, we get, $(E)^2 = -\frac{\alpha_k\mu_{i_0}-\beta_k\lambda_{i_0}}{c_k}$. And in particular, after the last blowup we get $(\overline{E})^2 = -(\alpha_{l_0}\mu_{i_0}-\beta_{l_0}\lambda_{i_0})$, since $c_{l_0} = 1$.

Finally, we need to show that $\alpha_{l_0}\mu_{i_0}-\beta_{l_0}\lambda_{i_0} = 1$. By definition we know that $\alpha_{l_0}a+\beta_{l_0}b = b\mu_{i_0}+a\lambda_{i_0} = 1$. So, we can rewrite $a = n(\mu_{i_0}-\beta_{l_0})$, and $b = n(\alpha_{l_0}-\lambda_{i_0})$, where $n = \frac{b}{(\alpha_{l_0}-\lambda_{i_0})} = \frac{a}{(\mu_{i_0}-\beta_{l_0})}$. However, by the choice of $\mu_{i_0}$, $\lambda_{i_0}$, $\beta_{l_0}$ and $\alpha_{l_0}$, we conclude $n=1$. Then, since $\lambda_{i_0} = \alpha_{l_0}-b$, and $\mu_{i_0} = \beta_{l_0}+a$, we finally get what we wanted,

$$(\overline{E})^2 = -(\alpha_{l_0}\mu_{i_0}-\beta_{l_0}\lambda_{i_0}) = -1.$$

□

Notice that the values of $l_0$, $c_{l_0-1}$ and $n_k$ can be specified using the notation in §3.2.1. We have two possibilities:

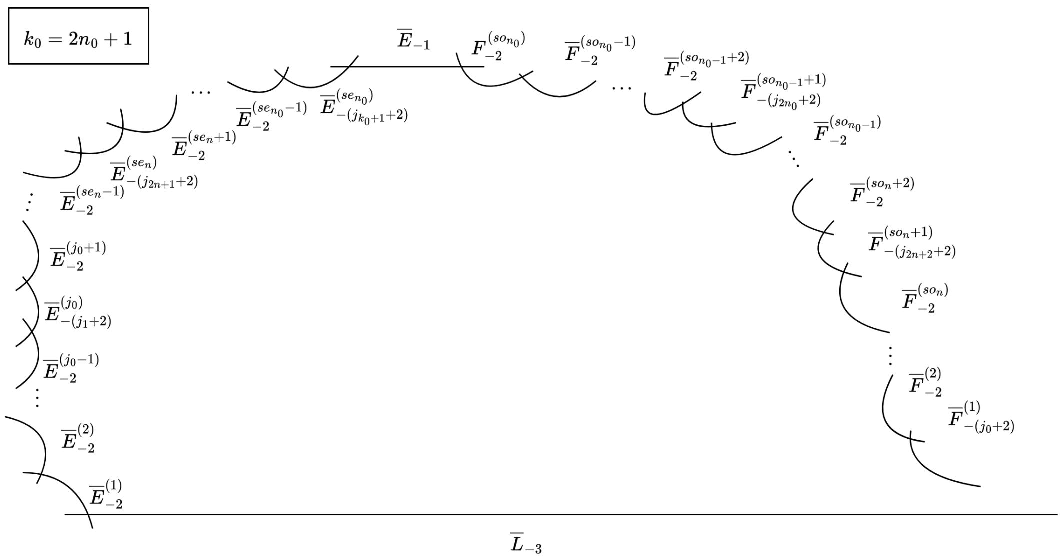
- If $k_0 = 2n_0 + 1$,

$$l_0 = so_{n_0},$$

$$\left(\overline{F}^{(so_n+k)}\right)^2 = -n_{(so_n+k+1)} = \begin{cases} -\left(j_{2n+2}+2\right) & \text{if } k = 1, \\ -2 & \text{Otherwise.} \end{cases}$$

$$(\overline{F}^{(1)})^2 = -n_2 = -(j_0+2), \quad (F^{(l_0)})^2 = -c_{l_0-1} = -2.$$

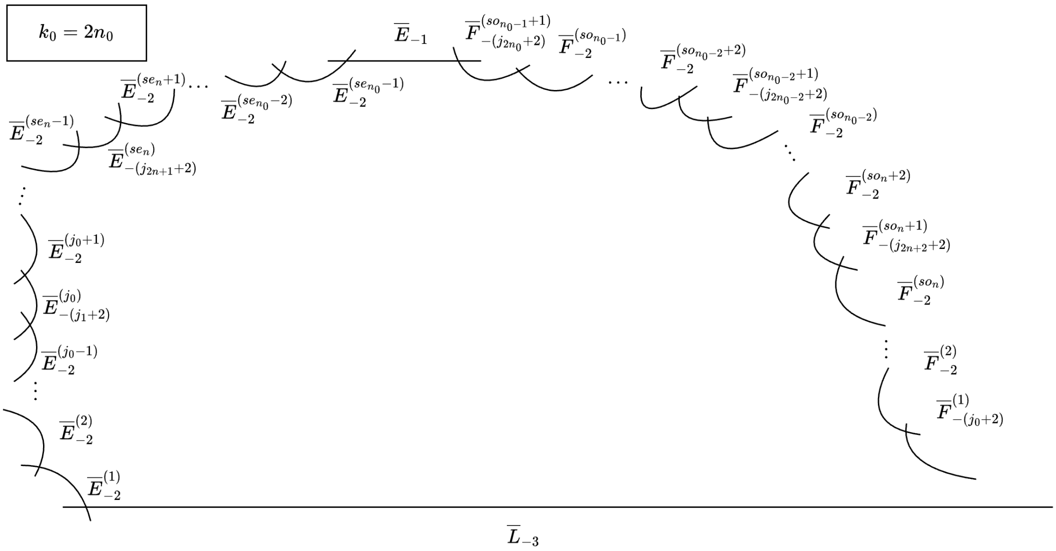
In the following picture, we can see all the divisors involved in the resolution of the singularities $p$ and $q$ when $k_0 = 2n_0 + 1$. Notice that the subscripts represent the self-intersections.



- If $k_0 = 2n_0$,

$$
\begin{aligned}
&l_0 = so_{n_0-1} + 1, \\
&\left(\overline{F}^{(so_n+k)}\right)^2 = -n_{(so_n+k+1)} = \begin{cases} -(j_{2n+2}+2) & \text{if } k = 1, \\ -2 & \text{Otherwise.} \end{cases} \\
&(\overline{F}^{(1)})^2 = -n_2 = -(i+2), \quad (F^{(l_0)})^2 = -c_{l_0-1} = -(j_{2n_0}+2).
\end{aligned}
$$

In the picture below, we see all the divisors involved in the resolution of the singularities $p$ and $q$ when $k_0 = 2n_0$.




Notice that in both cases, we can contract $(-1)$-curves until we get a weak degree one del Pezzo surface. Here you see the order of contractions:

- For $k = 2n_0 + 1$:

  (1) Contract $\overline{E}$,

  (2) Contract $F^{(so_{n_0})}$,..., $\overline{F}^{(so_{n_0-1}+2)}$ ($j_{2n_0+1} - 1$ contractions).

  (3) Contract $\overline{E}^{(se_{n_0})}$,..., $\overline{E}^{(se_{n_0-1}+1)}$ ($j_{2n_0}$ contractions).

  (4) Contract $\overline{F}^{(so_{n_0-1}+1)}$,..., $\overline{F}^{(so_{n_0-1}+2)}$ ($j_{2n_0-1}$ contractions).

  ⋮

$(k_0+1)$ Contract $\overline{E}^{(j_0+j_2)}$,..., $\overline{E}^{(j_0+1)}$ ($j_2$ contractions).

$(k_0+2)$ Contract $\overline{F}^{(j_1+1)}$,..., $\overline{F}^{(2)}$ ($j_1$ contractions).

$(k_0+3)$ Contract $\overline{E}^{(j_0)}$,..., $\overline{E}^{(1)}$ ($j_0$ contractions).

- For $k = 2n_0$:

  (1) Contract $\overline{E}$,

  (2) Contract $\overline{E}^{(se_{n_0}-1)}$,..., $\overline{E}^{(se_{n_0-1}+1)}$ ($j_{2n_0+1}-1$ contractions).

  (3) Contract $F^{(so_{n_0-1}+1)}$,..., $\overline{F}^{(so_{n_0-2}+2)}$ ($j_{2n_0-1}$ contractions).

  (4) Contract $\overline{E}^{(se_{n_0-1})}$,..., $\overline{E}^{(se_{n_0-2}+1)}$ ($j_{2(n_0-1)}$ contractions).

  $\vdots$

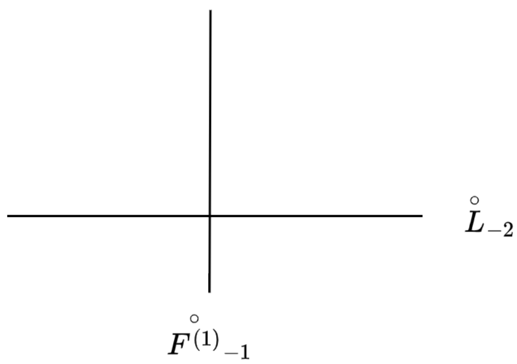
$(k_0+1)$ Contract $\overline{E}^{(j_0+j_2)}$,..., $\overline{E}^{(j_0+1)}$ ($j_2$ contractions).

$(k_0+2)$ Contract $\overline{F}^{(j_1+1)}$,..., $\overline{F}^{(2)}$ ($j_1$ contractions).

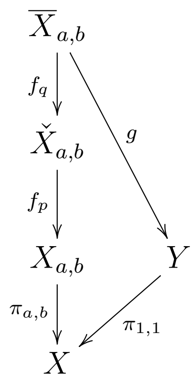
$(k_0+3)$ Contract $\overline{E}^{(j_0)}$,..., $\overline{E}^{(1)}$ ($j_0$ contractions).

Overall, we contract $i_0+l_0 = \sum_{k=0}^{k_0} j_k$ $(-1)$-curves in both cases. Let $g : \overline{X}_{a,b} \to Y$ be the composition of all the contractions. The picture at the end is:

Figure 3.2: On the weak del Pezzo surface of degree 1




Where $\mathring{F}^{(1)} = g(\overline{F^{(1)}})$, and $\mathring{L} = g(\overline{L})$. So we have the following diagram:




Let us define $\pi_{1,1} = \pi_{a,b} \circ f_p \circ f_q \circ g^{-1}$. We prove below that $Y$ is a weak del Pezzo surface of degree 1, and $\pi_{1,1}$ is the ordinary blowup at the point $p_0 \in X$, where $\mathring{F}^{(1)}$ is the $\pi_{1,1}$-exceptional divisor.

**Proposition 3.2.5.** *The surface $Y$ defined above is a weak del Pezzo surface of degree 1.*

*Proof.* As we saw before, with each blowup we do in the resolution of singularities of $X_{a,b}$, we get an exceptional divisor. Moreover, these divisors are added with a nonpositive coefficient to the equivalence class of the anticanonical divisor. For instance, with the $(a, b)$-weighted blowup, we get

$$-K_{X_{a,b}} \sim \pi^*(-K_X) - (a + b - 1)E.$$

Observe that the coefficients of the strict transforms of the current divisors remain unchanged, and this pattern persists through subsequent ordinary blowups as well. The same thing happens when we contract exceptional curves, the coefficients of the remaining exceptional curves do not change in the process. Therefore, to prove that $(-K_Y)^2 = 1$. We need to find the coefficient $e$ of $\mathring{F}^{(1)}$ in $-K_Y \sim \pi_{1,1}^*(-K_X) + e\mathring{F}^{(1)}$.

We got $F^{(1)}$ with the first blowup of the singular point $q \sim \frac{1}{a}(1, c_1)$, where $c_1 = -b + a$.

For simplicity, through this proof, we denote by $\sigma : Y \to X_{a,b}$ the blowup with suitable (natural) weights of $X_{a,b}$ at $q$. This is possible since the coefficient of $F^{(1)}$ is independent of the order of the blowup. Thus, we get the following where $\overline{E}$ is the strict transform of $E$ through $\sigma$:

$$\begin{aligned} -K_Y \sim\ & \sigma^*\left(-K_{X_{a,b}}\right) + \tfrac{a-(c_1+1)}{a}F^{(1)} \\ \sim\ & \sigma^*\left(\pi^*_{a,b}(-K_X) - (a+b-1)E\right) + \tfrac{a-(c_1+1)}{a}F^{(1)} \\ \sim\ & \sigma^*\left(\pi^*_{a,b}(-K_X)\right) - (a+b-1)\bar{E} + \left(-\tfrac{a+b-1}{a} + \tfrac{a-(c_1+1)}{a}\right)F^{(1)}. \end{aligned}$$

Moreover, notice that $e = -1$.

So we have that, $(-K_Y)^2 = \left(\pi^*_{1,1}(-K_X)\right)^2 + \left(\overset{\circ}{F}^{(1)}\right)^2 = 2 - 1 = 1$. It is clear that after contracting $\overset{\circ}{F}^{(1)}$ we are back in the original degree 2 del Pezzo surface we started with. Therefore, we can say that $\pi_{1,1} : Y \to X$ is the ordinary blowup of $p_0 \in X$. □

As mentioned in subsection §3.1.1, after an ordinary blowup of a point in a degree 2 del Pezzo surface, we get a weak degree 1 del Pezzo surface, in our case $Y$. Moreover, $Y$ has a Bertini involution, $\iota$. Notice that in our setting, we only have a (-2)-curve in $Y$, that is $\overset{\circ}{L}$, (see Figure 3.2).

**Lemma 3.2.6.** *Let $\overset{\circ}{D} = \iota(\overset{\circ}{F}^{(1)})$ in $Y$, with the set up described above. Then,*

$$\overset{\circ}{F}^{(1)} + \overset{\circ}{D} + \overset{\circ}{L} \sim -2K_Y.$$

*Proof.* Instead of directly working in the weak degree 1 del Pezzo surface, let us contract the (-2)-curve and get a singular degree 1 del Pezzo surface. We have the following Stein factorisation as mentioned in Remark 3.1.4, where $\nu$ is the contraction of $\overset{\circ}{L}$ to a singular double point.

$$\begin{array}{ccc} Y & \xrightarrow{\ \nu\ } & \widetilde{Y} \\ {\scriptstyle \pi_{1,1}}\downarrow & & \downarrow{\scriptstyle \omega} \\ X & & \mathbb{P}(1,1,2) \end{array}$$

Notice that since $\iota$ is an involution, $\left(\overset{\circ}{D}\right)^2 = \left(\overset{\circ}{F}{}^{(1)}\right)^2 = -1$, and these self-intersections do not change with $\nu$. Let us denote $\overline{D} = \nu\left(\overset{\circ}{D}\right)$ and $\overline{F^{(1)}} = \nu\left(\overset{\circ}{F}{}^{(1)}\right)$. Since $\overline{D} + \overline{F^{(1)}} \in \mathrm{Pic}^{\tau}(\widetilde{Y})$, and the picard number $\rho^{\tau}_{\widetilde{Y}} = \rho_{\mathbb{P}(1,1,2)} = 1$, we know there exist $m \in \mathbb{N}$ such that $-mK_{\widetilde{Y}} \sim \overline{D} + \overline{F^{(1)}}$. Now if we do the self-intersection of this equivalence we get an equation. On the other hand, if we multiply it by $\overline{F^{(1)}}$, we get the following equation system:

$$\begin{cases} m^2 = \left(\overline{F^{(1)}}\right)^2 + 2\overline{F^{(1)}} \cdot \overline{D} + \left(\overline{D}\right)^2 = -1 + 2\overline{F^{(1)}} \cdot \overline{D} - 1, \\ m = \left(\overline{F^{(1)}}\right)^2 + \overline{F^{(1)}} \cdot \overline{D} = -1 + \overline{F^{(1)}} \cdot \overline{D}. \end{cases}$$

Note, that the linear system $|-2K_{\overline{Y}}|$ is represented by the web of sextic curves with eight base points $x_1, ..., x_8$ in $\mathbb{P}^2$, which we blowup to obtain $\overline{Y}$ [Dol12, §8.8.2]. Since $\overline{F^{(1)}}$ is an exceptional curve in $\overline{Y}$, $\omega(\overline{F^{(1)}})$ intersects the branching sextic curve in $\mathbb{P}(1,1,2)$ at least once and it also contains the singular point, hence $\overline{F^{(1)}} \cdot \overline{D} \geq 2$. Therefore, from the equation system we get that $m = 2$ and $\overline{F^{(1)}} \cdot \overline{D} = 3$. Notice that $\overline{F^{(1)}}$ and $\overline{D}$ go through the singular point of $\widetilde{Y}$ with multiplicity 1. Therefore, when we blow up this point to recover $\overset{\circ}{L}$, the intersections change as follows:

$$\overset{\circ}{F}{}^{(1)} \cdot \overset{\circ}{D} = 2, \quad \overset{\circ}{F}{}^{(1)} \cdot \overset{\circ}{L} = \overset{\circ}{L} \cdot \overset{\circ}{D} = 1.$$

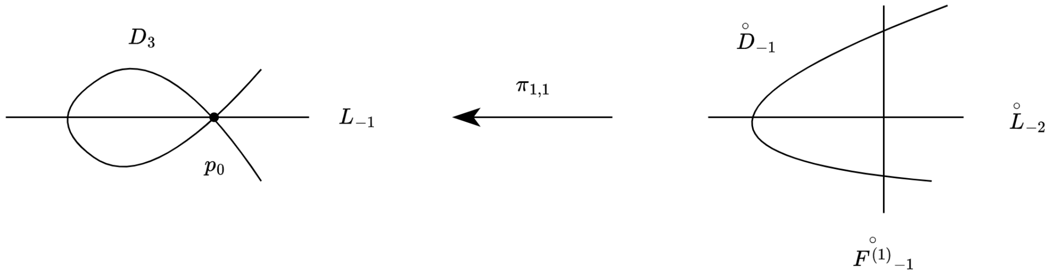
Moreover, since $-2K_{\widetilde{Y}} \sim \overline{D} + \overline{F^{(1)}}$, we know that $-2K_Y \sim \overset{\circ}{D} + \overset{\circ}{F}{}^{(1)} + u\overset{\circ}{L}$, for some $u \in \mathbb{N}$. Furthermore, taking into account the intersections we just mentioned, we can check that $u = 1$. Thus, we have the desired property. □

**Remark 3.2.7.** Note that $\overset{\circ}{D}$, $\overset{\circ}{F}{}^{(1)}$ and $\overset{\circ}{L}$ do not intersect at the same point, i.e. $\overset{\circ}{F}{}^{(1)} \cap \overset{\circ}{D} \cap \overset{\circ}{L} = \emptyset$.

On the other hand, since $\pi_{1,1}$ contracts $\overset{\circ}{F}{}^{(1)}$ to $p_0$, we get that $-2K_X \sim L + D$, where $D = \pi_{1,1}(\overset{\circ}{D})$.

This is the image representing the curves involved after applying $\pi_{1,1}$:

Figure 3.3: Ordinary blowup of $X$ at $p_0$.




In this figure, we can also see how the intersections change. As a consequence, we have the following proposition:

**Proposition 3.2.8.** *Assume that* $3a < 4b$ *and set* $\widetilde{D} = (f_p \circ f_q)(g^*(\mathring{D}))$ *on* $X_{a,b}$. *Then,* $\pi^*_{a,b}(D) = \widetilde{D} + 2bE$ *and* $(\widetilde{D})^2 = 3 - \frac{4b}{a} < 0$.

*Proof.* The idea of this proof is to take the equivalence $\pi^*_{1,1}(D) \sim \mathring{D} + 2\mathring{F}^{(1)}$ in the weak degree 1 del Pezzo surface $Y$, and follow the inverse process of blow ups and contractions to get an equivalence for $\pi^*_{a,b}(D)$ in terms of $\widetilde{D}$ and $E$. Let $e_k$ and $f_k$ be the coefficient of $E^{(k)}$ and $F^{(k)}$ in the equivalence class of $\pi^*_{a,b}(D)$, respectively. Define $e$ to be the final coefficient of $E$. Notice that if $k = 2n_0 + 1$, $e = e_{se_{n_0}} + f_{so_{n_0}}$, otherwise $e = e_{se_{n_0}-1} + f_{so_{n_0-1}+1}$. These become clear when we see that $E$ is obtained as the exceptional divisor of the blowup of the intersection point between $E^{(se_{n_0})}$ and $F^{(so_{n_0})}$, or $E^{(se_{n_0}-1)}$ and $F^{(so_{n_0-1}+1)}$, respectively. In addition, we know for the sequence of blowups that for $n \geq 0$

$$\begin{aligned} e_{se_n+k} = &\quad e_{se_n} + k \cdot f_{so_n+1} \quad \text{for } k = 1, ..., j_{2n+2}, \\ f_{so_n+k+1} = &\quad k \cdot e_{se_{n+1}} + f_{so_n+1} \quad \text{for } k = 1, ..., j_{2n+3}. \end{aligned}$$

Now let us rewrite $b$ in terms of $j_k$ using notation as in section §3.2.1.

$$\begin{aligned} 2b = &\quad 2\gamma_0 \cdot j_0 + 2\gamma_1 = \gamma_0 \cdot e_{j_0} + f_1 \cdot \gamma_1 = \gamma_1(e_{j_0} \cdot j_1 + f_1) + \gamma_2 \cdot e_{j_0} \\ = &\quad \gamma_1 \cdot f_{j_1+1} + \gamma_2 \cdot e_{j_0} = \gamma_2(j_2 \cdot f_{j_1+1} + e_{j_0}) + \gamma_3 \cdot f_{j_1+1} = \gamma_2 \cdot e_{se_1} + \gamma_3 \cdot f_{j_1+1} \\ = &\quad ... = \gamma_{2n_0-1} f_{so_{n_0-1}+1} + \gamma_{2n_0} e_{se_{n_0-1}} = \gamma_{2n_0} e_{se_{n_0}} + \gamma_{2n_0+1} f_{so_{n_0-1}+1} \end{aligned}$$

Here we have two different cases.

- If $k_0 = 2n_0 + 1$, notice that, $\gamma_{2n_0} = j_{2n_0+1}$ and $\gamma_{2n_0+1} = 1$. Therefore,

$$2b = j_{2n_0+1} e_{se_{n_0}} + f_{so_{n_0-1}+1} = f_{so_{n_0}} + e_{se_{n_0}} = e.$$

- If $k_0 = 2n_0$, notice that $\gamma_{2n_0-1} = j_{2n_0}$ and $\gamma_{2n_0} = 1$. Then,

$$2b = j_{2n_0} f_{so_{n_0-1}+1} + e_{se_{n_0-1}} = e_{se_{n_0}-1} + f_{so_{n_0-1}+1} = e.$$

In both cases, as desired, we obtain $e = 2b$. After all the contractions, we get $\pi^*_{a,b}(D) \sim \widetilde{D} + 2b \cdot E$, and

$$\begin{aligned}(\widetilde{D})^2 = (\pi^*_{a,b}(D) - 2bE)^2 &= \left(\pi^*_{a,b}(D)\right)^2 - 4b\left(\pi^*_{a,b}(D)\right) \cdot E + 4b^2E^2 \\ &= 3 - \frac{4b}{a} < 0 \quad \Leftrightarrow \quad 3a < 4b.\end{aligned}$$

□

**Remark 3.2.9.** Following a similar proof and taking into account that $a = b + \gamma_0$, it is easy to check that $\pi^*_{a,b}(L) = \widetilde{L} + aE$, where $\widetilde{L}$ is the strict transform of $L$ after the weighted blowup.

**Corollary 3.2.10.** *Let $\pi_{a,b} : X_{a,b} \to X$ be a $(a, b)$-weighted blowup of the point $p_0 \in X$, as described in Theorem 3.0.1. Let $L$ be the unique $(-1)$-curve passing through the point $p_0$ and $D$ the curve described above. Then,*

$$\pi^*_{a,b}(-K_X) \sim \frac{1}{2}\left(\widetilde{L} + \widetilde{D} + (a + 2b)E\right).$$

## 3.3 Proof of the Main Theorem

In the previous section, we prepared all the ingredients to prove the Main Theorem. Now, we divide the proof into two arguments. First, we take $X_{a,b}$ and we use it to compute an upper bound for $\delta_{p_0}(X)$. Then, we use Theorem 2.2.18 to find lower bounds for $\delta_{p_0}(X)$.

**Theorem 3.3.1.** *Let $X \subset \mathbb{P}(1,1,1,2)$ be a smooth del Pezzo surface of degree 2 and let $p_0 = (x_0, y_0, z_0, w_0)$ be a closed point in $X$. Assume that there is a unique $(-1)$-curve, $L$, passing through the point $p_0$. Then, $\delta_{p_0}(X) \leq \frac{6}{71}(11 + 8\sqrt{3})$.*

*Proof.* For each $a, b > 0$, let $\nu_{a,b}$ be the quasi-monomial valuation over $p_0 \in X$ defined by $\nu_{a,b}(u) = a$ and $\nu_{a,b}(v) = b$, where $(u, v)$ are the local coordinates at $p_0$ such that $L = \{u = 0\}$. Let $\pi : Y = X_{a,b} \to X$ be the weighted blow up at $p_0$ with $\mathrm{wt}(u) = a$ and $\mathrm{wt}(v) = b$. Let $E$ be the exceptional divisor, and let $\widetilde{L}$ and $\widetilde{D}$ be the strict transforms of $L$ and $D$, respectively, where $D$ is the divisor described at the end of the previous section. In order to identify the minimizer of $\frac{A_X(\nu_{a,b})}{S_X(\nu_{a,b})} = \frac{A_X(E)}{S(-K_X;E)}$, we choose coprime integers $a, b > 0$ such that $\frac{\sqrt{3}}{2}a < b < a$.

By definition, $\pi^*(L) = \widetilde{L} + m_L E$ and $\pi^*(D) = \widetilde{D} + m_C E$ where in our case $m_L = \mathrm{mult}_{p_0} L = a$ and $m_D = \mathrm{mult}_{p_0} D = 2b$ as we proved in Theorem 3.2.8. From Example 2.1.3, we known that $E^2 = -\frac{1}{ab}$, so we get: $(\widetilde{L})^2 = -\frac{a+b}{b}$, $(\widetilde{D})^2 = 3 - \frac{4b}{a}$, $\widetilde{L} \cdot E = \frac{1}{b}$, $\widetilde{D} \cdot E = \frac{2}{a}$, and $\widetilde{D} \cdot \widetilde{L} = 1$.

As we saw in the previous section, the stable base locus (see Definition 2.1.13) of

$$\pi^*(-K_X) - tE \sim \frac{1}{2}\left(\widetilde{L} + \widetilde{D}\right) + \left(\frac{a+2b}{2} - t\right) E \tag{3.5}$$

is contained in $\widetilde{D} \cap \widetilde{L}$ for all $0 \leq t \leq \frac{a+2b}{2}$. We want to compute the volume of this divisor to get $S(-K_X; E)$, and therefore we need the Zariski Decomposition (see §2.1.2) of $\pi^*(-K_X) - tE$ for all $0 \leq t \leq \frac{a+2b}{2}$. To get the Zariski Decomposition, we need to separate the positive and negative parts of our divisor. In order to do this, we need to identify the divisors (in this case, curves) that intersect negatively with $\pi^*(-K_X) - tE$, these divisors will be in the negative part. Notice that these divisors can only be those with negative self-intersection that appear in the linear equivalence of $\pi^*(-K_X) - tE$ in (3.5). Hence, we need to intersect

$\pi^*(-K_X) - tE$, with the negative curves $\widetilde{L}$, $\widetilde{D}$ and $E$, and check when this intersections are negative.

$$\begin{aligned}
(\pi^*(-K_X) - tE) \cdot \widetilde{L} &= \tfrac{1}{2}\left(-\tfrac{a+b}{b} + 1\right) + \left(\tfrac{a+2b}{2} - t\right)\tfrac{1}{b} = \tfrac{b-t}{b} \geq 0 &&\Leftrightarrow \quad b \geq t,\\
(\pi^*(-K_X) - tE) \cdot \widetilde{D} &= \tfrac{1}{2}\left(1 + 3 - \tfrac{4b}{a}\right) + \left(\tfrac{a+2b}{2} - t\right)\tfrac{2}{a} = \tfrac{3a-2t}{a} \geq 0 &&\Leftrightarrow \quad \tfrac{3a}{2} \geq t,\\
(\pi^*(-K_X) - tE) \cdot E &= \tfrac{1}{2}\left(\tfrac{1}{b} + \tfrac{2}{a}\right) + \left(\tfrac{a+2b}{2} - t\right)\left(-\tfrac{1}{ab}\right) = \tfrac{t}{ab} \geq 0 &&\quad \forall t.
\end{aligned}$$

For our assumptions on $a$ and $b$, we know that $b \leq \frac{3a}{2}$ and hence, the first negative value appears when we intersect the divisor with $\widetilde{L}$. Therefore we know that $\pi^*(-K_X) - tE$ is nef for $t \in [0, b]$, but for $t > b$ we need to find the smallest $\lambda$ such that $\pi^*(-K_X) - tE - \lambda\widetilde{L}$ is nef. To do so, we repeat the intersections for this new divisor.

$$\begin{aligned}
(\pi^*(-K_X) - tE - \lambda\widetilde{L}) \cdot \widetilde{L} &= \tfrac{\lambda(a+b)+b-t}{b} \geq 0 &&\Leftrightarrow \quad \lambda \geq \tfrac{t-b}{a+b},\\
(\pi^*(-K_X) - tE - \lambda\widetilde{L}) \cdot \widetilde{D} &= \tfrac{3a-2t-a\lambda}{a} \geq 0 &&\Leftrightarrow \quad 3a - 2t \geq \lambda,\\
(\pi^*(-K_X) - tE - \lambda\widetilde{L}) \cdot E &= \tfrac{t-a\lambda}{ab} \geq 0 &&\Leftrightarrow \quad t \geq a\lambda.
\end{aligned}$$

Now we take the smallest $\lambda$ which is $\frac{t-b}{a+b}$, and we repeated the intersections for $\pi^*(-K_X) - tE - \frac{t-b}{a+b}\widetilde{L}$.

$$\begin{aligned}
(\pi^*(-K_X) - tE - \tfrac{t-b}{a+b}\widetilde{L}) \cdot \widetilde{L} &= 0 \geq 0 &&\quad \forall t,\\
(\pi^*(-K_X) - tE - \tfrac{t-b}{a+b}\widetilde{L}) \cdot \widetilde{D} &= \tfrac{3a^2+4ab-(3a+2b)t}{a(a+b)} \geq 0 &&\Leftrightarrow \quad \tfrac{a(3a+4b)}{3a+2b} \geq t,\\
(\pi^*(-K_X) - tE - \tfrac{t-b}{a+b}\widetilde{L}) \cdot E &= \tfrac{a+t}{a(a+b)} \geq 0 &&\quad \forall t.
\end{aligned}$$

Looking at these intersections, we know that $\pi^*(-K_X) - tE - \frac{t-b}{a+b}\widetilde{L}$ is nef for $t \in [b, \frac{a(3a+4b)}{3a+2b}]$, but for $t > \frac{a(3a+4b)}{3a+2b}$ we need to find the smallest $\lambda$ and $\mu$ such that $\pi^*(-K_X) - tE - \lambda\widetilde{L} - \mu\widetilde{D}$ is nef. To do so, we repeat the intersections for this new divisor.

$$(\pi^*(-K_X) - tE - \lambda\widetilde{L} - \mu\widetilde{D}) \cdot \widetilde{L} = \tfrac{\lambda(a+b)+b-t-b\mu}{b} \geq 0 \quad \Leftrightarrow \quad \lambda \geq \tfrac{b\mu+t-b}{a+b},$$
$$(\pi^*(-K_X) - tE - \lambda\widetilde{L} - \mu\widetilde{D}) \cdot \widetilde{D} = \tfrac{3a-2t-a\lambda+(4b-3a)\mu}{a} \geq 0 \quad \Leftrightarrow \quad \mu \geq \tfrac{-3a+2t+a\lambda}{4b-3a},$$
$$(\pi^*(-K_X) - tE - \lambda\widetilde{L} - \mu\widetilde{D}) \cdot E = \tfrac{t-a\lambda-2b\mu}{ab} \geq 0 \quad \Leftrightarrow \quad t \geq a\lambda + 2b\mu.$$

By solving these inequalities, we get that the smallest coefficients for the divisor to be nef are $\lambda = \frac{3t(2b-a)-4b^2}{4b^2-3a^2}$ and $\mu = \frac{t(3a+2b)-a(3a+4b)}{4b^2-3a^2}$. Therefore, we have the following Zariski decomposition:

$$N(\pi^*(-K_X) - tE) = \begin{cases} 0 & 0 \leq t \leq b, \\ \frac{t-b}{a+b}\widetilde{L} & b < t \leq \frac{a(3a+4b)}{3a+2b}, \\ \frac{3t(2b-a)-4b^2}{4b^2-3a^2}\widetilde{L} + \frac{t(3a+2b)-a(3a+4b)}{4b^2-3a^2}\widetilde{D} & \frac{a(3a+4b)}{3a+2b} < t \leq \frac{a+2b}{2}, \end{cases}$$

and

$$P(\pi^*(-K_X) - tE) =$$
$$\begin{cases} \frac{1}{2}\left(\widetilde{L} + \widetilde{D}\right) + \left(\frac{a+2b-2t}{2}\right)E & 0 \leq t \leq b, \\ \frac{a+3b-2t}{2(a+b)}\widetilde{L} + \frac{1}{2}\widetilde{D} + \left(\frac{a+2b-2t}{2}\right)E & b < t \leq \frac{a(3a+4b)}{3a+2b}, \\ \frac{3t(2b-a)(a+2b-2t)}{2(4b^2-3a^2)}\widetilde{L} + \frac{(3a+2b)(a+2b-2t)}{2(4b^2-3a^2)}\widetilde{D} + \left(\frac{a+2b-2t}{2}\right)E & \frac{a(3a+4b)}{3a+2b} < t \leq \frac{a+2b}{2}. \end{cases}$$

Then, by [BFJ09, Theorem B, Example 4.7] and [LM09, Corollary C], we know that

$$\mathrm{vol}_{Y|E}(\pi^*(-K_X) - tE) = P_\sigma(\pi^*(-K_X) - tE) \cdot E = -\frac{1}{2} \cdot \frac{d}{dt}\mathrm{vol}(\pi^*(-K_X) - tE).$$

Thus, we get

$$\mathrm{vol}_{Y|E}(\pi^*(-K_X) - tE) = \begin{cases} \frac{t}{ab} & 0 \leq t \leq b, \\ \frac{a+t}{a(a+b)} & b < t \leq \frac{a(3a+4b)}{3a+2b}, \\ \frac{6(a+2b-2t)}{4b^2-3a^2} & \frac{a(3a+4b)}{3a+2b} < t \leq \frac{a+2b}{2}, \end{cases}$$

and

$$S(-K_X;E)=\frac{2}{(-K_X)^2}\int_0^\infty t\cdot \mathrm{vol}_{Y|E}(\pi^*(-K_X)-tE)dt=\frac{15a^2+34ab+8b^2}{12(3a+2b)}.$$

Since $A_X(E)=a+b$, note that $\frac{A_X(\nu_{a,b})}{S_X(\nu_{a,b})}$ only depends on the ratio $\mu=\frac{a}{b}$, thus by continuity [BLX22, Proposition 2.4] we have:

$$\frac{A_X(\nu_{a,b})}{S_X(\nu_{a,b})}=\frac{12(3\mu+2)(1+\mu)}{15\mu^2+34\mu+8}$$

It achieves its minimum for $\frac{2}{\sqrt{3}}\geq\mu\geq 1$ with the value $\lambda_0=\frac{6}{71}(11+8\sqrt{3})$ at $\mu_0:=\frac{2}{\sqrt{3}}$. In particular, we have $\delta_{p_0}(X)\leq\lambda_0$. □

It remains to show $\delta_{p_0}(X)\geq\lambda_0$. For the next theorem, we use the same method as in [AZ22, Lemma A.6].

**Theorem 3.3.2.** *Let $X\subset\mathbb{P}(1,1,1,2)$ be a smooth del Pezzo surface of degree 2 and let $p_0=(x_0,y_0,z_0,w_0)$ be a closed point in $X$. Assume a unique $(-1)$-curve $L$ is passing through the point $p_0$. Then $\delta_{p_0}(X)\geq\frac{6}{71}(11+8\sqrt{3})$.*

*Proof.* Choose a sequence of coprime integers $a_m,b_m>0$ $(m=1,2,...)$ such that $\mu_m:=\frac{a_m}{b_m}\to\mu_0$ $(m\to\infty)$, where $\frac{2}{3}<\mu_m<\frac{2}{\sqrt{3}}$. Let $\pi_m:Y_m=Y_{a_m,b_m}\to X$ be the corresponding weighted blow up and let $E_m$ be the exceptional divisor. Let $P_1^{(m)}=\widetilde{L}\cap E_m$, $\{P_2^{(m)},P_3^{(m)}\}=\widetilde{D}\cap E_m$, where $P_1^{(m)}$ and $P_2^{(m)}$ are the singular points $p$ and $q$, respectively. Let $W^{E_m}_{\cdot,\cdot}$ be the refinement by $E_m$ of the complete linear series associated to $-K_X$.

Let $\Delta_m=\mathrm{Diff}_{E_m}(0)=(1-\frac{1}{b_m})P_1^{(m)}+(1-\frac{1}{a_m})P_2^{(m)}$, $\lambda_m=\frac{A_X(E_m)}{S(-K_X;E_m)}$. Now using the formula in Theorem 2.2.18 we know that:

$$\delta_p(X)\geq\min\left\{\frac{A_X(E_m)}{S(-K_X;E_m)},\ \min_{x\in E_m}\frac{1-\mathrm{ord}_x(\Delta_{E_m})}{S(W^{E_m}_{\cdot,\cdot};x)}\right\}.$$

We already computed the minimum of $\frac{A_X(E_m)}{S(-K_X;E_m)}$ in Theorem 3.3.1. Therefore, to finish the proof, we need to compute the other minimum. Notice that in $E_m$ we have two singularities, $P_1^{(m)}$ and $P_2^{(m)}$, so we have four types of points $x \in E_m$. Here, we denote by $P(t)$ the positive part of the Zariski decomposition of Theorem 3.3.1 and by $N(t)$ the negative part.

- If $x = P_1^{(m)}$,

$$\frac{1 - \mathrm{ord}_x(\Delta_{E_m})}{S(W^{E_m}_{\cdot,\cdot};x)} = \frac{1 - \left(1 - \frac{1}{b_m}\right)}{S(W^{E_m}_{\cdot,\cdot};P_1^{(m)})} = \frac{12(2b_m + 3a_m)^2}{45a_m^2 + 60a_m b_m + 44b_m^2},$$

where

$$\begin{aligned} S(W^{E_m}_{\cdot,\cdot};P_1^{(m)}) &= \int_0^{\frac{a+2b}{2}} \left( (P(t)\cdot E_m)\cdot (N(t)\cdot E_m)_{P_1^{(m)}} + \frac{(P(t)\cdot E_m)^2}{2} \right) dt \\ &= \frac{45a_m^2 + 60a_m b_m + 44b_m^2}{12b_m(2b_m + 3a_m)^2}. \end{aligned}$$

Now since $\mu_m \to \mu_0$ as $m \to \infty$, let us take the limit

$$\lim_{m\to\infty} \frac{1}{b \cdot S(W^{E_m}_{\cdot,\cdot};P_1^{(m)})} = \frac{6}{47}\left(11 + 3\sqrt{3}\right). \tag{3.6}$$

- If $x = P_2^{(m)}$,

$$\frac{1 - \mathrm{ord}_x(\Delta_{E_m})}{S(W^{E_m}_{\cdot,\cdot};x)} = \frac{1 - \left(1 - \frac{1}{a_m}\right)}{S(W^{E_m}_{\cdot,\cdot};P_2^{(m)})} = \frac{3(2b_m + 3a_m)^2}{18a_m^2 + 12a_m b_m + 4b_m^2},$$

where

$$S(W^{E_m}_{\cdot,\cdot};P_2^{(m)}) = \frac{2(9a_m^2 + 6a_m b_m + 2b_m^2)}{3a_m(2b_m + 3a_m)^2}.$$

Now since $\mu_m \to \mu_0$ as $m \to \infty$, let us take the limit

$$\lim_{m\to\infty} \frac{1}{a \cdot S(W^{E_m}_{\cdot,\cdot};P_2^{(m)})} = \frac{6}{37}\left(8 + 3\sqrt{3}\right). \tag{3.7}$$

- If $x = P_3^{(m)}$,

$$\frac{1-\operatorname{ord}_x(\Delta_{E_m})}{S(W^{E_m}_{\cdot,\cdot};x)} = \frac{1}{S(W^{E_m}_{\cdot,\cdot};P_3^{(m)})} = \frac{3a_m(2b_m+3a_m)^2}{18a_m^2+12a_mb_m+4b_m^2},$$

where

$$S(W^{E_m}_{\cdot,\cdot};P_3^{(m)}) = \frac{2(9a_m^2+6a_mb_m+2b_m^2)}{3a_m(2b_m+3a_m)^2}.$$

Now since $\mu_m \to \mu_0$ as $m \to \infty$, let us take the limit

$$\lim_{m\to\infty} \frac{1}{S(W^{E_m}_{\cdot,\cdot};P_3^{(m)})} = \frac{12}{37}\left(8+3\sqrt{3}\right). \tag{3.8}$$

- If $x \neq P_1^{(m)}$, $P_2^{(m)}$, $P_3^{(m)}$,

$$\frac{1-\operatorname{ord}_x(\Delta_{E_m})}{S(W^{E_m}_{\cdot,\cdot};x)} = \frac{1}{S(W^{E_m}_{\cdot,\cdot};x)} = \frac{18a_m^2+12a_mb_m}{15a_m-2b_m},$$

where

$$S(W^{E_m}_{\cdot,\cdot};x) = \frac{15a_m-2b_m}{18a_m^2+12a_mb_m}.$$

Now since $\mu_m \to \mu$ as $m \to \infty$, let us take the limit

$$\lim_{m\to\infty} \frac{1}{S(W^{E_m}_{\cdot,\cdot};x)} = \frac{12}{37}\left(8+3\sqrt{3}\right). \tag{3.9}$$

If we put it all together, we get that

$$\begin{aligned}\delta_p(X) &\geq \min\left\{\lambda_0, \frac{6}{47}\left(11+3\sqrt{3}\right), \frac{6}{37}\left(8+3\sqrt{3}\right), \frac{12}{37}\left(8+3\sqrt{3}\right)\right\} \\ &= \lambda_0 = \frac{6}{71}(11+8\sqrt{3}).\end{aligned}$$

□

This concludes the proof of the Main Theorem 3.0.1.

# Chapter 4

# One-dimensional components in the K-moduli of smooth Fano 3-folds

This chapter expands upon the paper "One-dimensional components in the K-moduli of smooth Fano 3-folds" which is written in collaboration with Hamid Abban (my advisor), Ivan Cheltsov, Elena Denisova, Dongchen Jiao, Anne-Sophie Kaloghiros, Jesus Martinez-Garcia, and Theodoros Papazachariou [Abb+23]. In the paper, we study one-dimensional components of the K-moduli of Fano 3-folds denoted by $M_3^{\mathrm{Kps}}$. We give a complete description of the six 1-dimensional K-moduli components. As part of this thesis, we show more detailed computations and extended explanations of the description of the families for three of the components, since they are the ones I worked on.

## 4.1 Introduction

Until the development of the theory of K-stability, the subject lacked a unified theory of compact moduli spaces for Fano varieties. Recent advances in K-stability have shown that the compactification of the moduli space of Kähler-Einstein Fano manifolds obtained by degenerating Kähler-Einstein metrics coincides with a compact moduli space of K-polystable $\mathbb{Q}$-Fano varieties; the resulting space, after fixing the dimension $n \in \mathbb{N}$ and the volume $V \in \mathbb{Q}_{>0}$ is a projective variety $M^{\mathrm{Kps}}_{n,V}$ parametrising $n$-dimensional K-polystable smoothable Fano varieties of anticanonical volume $V$ over $\mathbb{C}$ (see [Jia20; LWX21; CP21; BX19; Alp+20; BLX22; Xu20; XZ20; XZ21; Blu+21; LXZ22]).

Much has already been uncovered about the geometry and characteristics of smooth Fano 3-folds. Through the application of techniques from birational geometry, Iskovskikh, Mori, and Mukai achieved the classification of smooth Fano 3-folds into 105 deformation families [Isk89; MM82]. It is natural to test the theory of K-moduli for these varieties.

We also know precisely the deformation families for which a general member of the family is K-stable (see [Ara+23]). In particular, it is known that 78 of those families have K-semistable general members. Of those 78 families with K-(poly)stable elements, 24 have 0-dimensional moduli, meaning they have a unique K-polystable member. This project focuses on the six families with 1-dimensional moduli. In that, the following are the families where the moduli have dimension one (the numbers given are the ones in Mori-Mukai notation [MM82]):

**Family 1:** Divisors of bidegree (1,2) in $\mathbb{P}^2 \times \mathbb{P}^2$ (№2.24). For $\lambda \in \mathbb{P}^1$, let $X_\lambda$ be the 3-fold defined by $\{xu^2 + yv^2 + zw^2 = \lambda(xvw + yuw + zuv)\} \subset \mathbb{P}^2 \times \mathbb{P}^2$, where $([x : y : z], [u : v : w])$ are coordinates on $\mathbb{P}^2 \times \mathbb{P}^2$. Then $X_\lambda$ is smooth if and only if $\lambda^3 \neq 1$ or $\lambda \neq \infty$ and they parametrise the smooth members of this family. By [Ara+23, Lemma 4.70] $X_\lambda$ are K-polystable for all $\lambda \in \mathbb{P}^1$. In particular, if

$\lambda^3 = 1$, then $X_\lambda \cong X_\infty$, and this 3-fold has 3 ordinary double points. Note that the family contains strictly K-semistable smooth members (see [Ara+23, Section 4.7] for details).

**Family 2:** Blowups of $\mathbb{P}^3$ along quartic elliptic curves (№2.25). For $\lambda \in \mathbb{P}^1$, consider the curve $C_\lambda = \left\{x_0^2+x_1^2+\lambda(x_2^2+x_3^2) = 0, \lambda(x_0^2-x_1^2)+x_2^2-x_3^2 = 0\right\} \subset \mathbb{P}^3$, where $[x_0 : x_1 : x_2 : x_3]$ are coordinates on $\mathbb{P}^3$, and let $\pi\colon X_\lambda \to \mathbb{P}^3$ be the blowup along $C_\lambda$. If $\lambda \notin \{0, \pm 1, \pm i, \infty\}$, then $C_\lambda$ is a smooth elliptic curve, and $X_\lambda$ is a smooth K-stable Fano 3-fold [Ara+23, Corollary 4.32]. Moreover, every smooth Fano 3-fold in this family is isomorphic to $X_\lambda$ for some $\lambda \in \mathbb{P}^1$ [Dye77]. If $\lambda \in \{0, \pm 1, \pm i, \infty\}$, then $C_\lambda$ is a union of 4 lines, and $X_\lambda \cong X_0$ is a toric K-polystable smoothable Fano 3-fold; $X_0$ has four singular points, which are ordinary double points (prove in [Pap22]).

**Family 3:** Blowups of $\mathbb{P}^3$ along rational quartic curves (№2.22). We study this family in §4.2.

**Family 4:** Blowups of $\mathbb{P}^3$ along the disjoint union of a twisted cubic and a line (№3.12). We study this family in §4.3.

**Family 5:** Blowups of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ along a curve of degree (1,1,3) (№4.13). We study this family in §4.4.

**Family 6:** Complete intersection of divisors of degree $(1, 1, 0)$, $(1, 0, 1)$ and $(0, 1, 1)$ in $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$ (№3.13). By [Ara+23, Lemma 5.97] we know that every smooth member of Family 6 is isomorphic to $X_\lambda \subset (\mathbb{P}^2)^3$ given by

$$\left\{x_0y_0+x_1y_1+x_2y_2 = 0, y_0z_0+y_1z_1+y_2z_2 = 0, (1+\lambda)x_0z_1+(1-\lambda)x_1z_0-2x_2z_2 = 0\right\}, \tag{4.1}$$

where $\lambda \in \mathbb{P}^1$, and $([x_0 : x_1 : x_2], [y_0 : y_1 : y_2], [z_0 : z_1 : z_2])$ are coordinates on $(\mathbb{P}^2)^3$. If $\lambda \notin \{\pm 1, \infty\}$, then $X_\lambda$ is a smooth K-polystable Fano 3-fold [Ara+23, Lemma 5.99]. For $\lambda \in \{\pm 1, \infty\}$, $X_\lambda$ is K-unstable and singular ($X_{\pm 1}$ has one ordinary double point $p = ([0:1:0],[0:0:1],[1:0:0])$) and $X_\infty$ is singular along the curve defined as $\{([0:0:1],[y_0:y_1:0],[0:0:1])|[y_0:y_1]\in\mathbb{P}^1\}$). However, let

$$X'_\infty = \begin{cases} x_2y_3 - x_3y_2 = 0, \\ y_2z_3 - y_3z_2 = 0, \\ x_2z_3 - x_3z_2 = 0, \\ x_1y_1z_3 + x_1y_3z_1 + x_3y_1z_1 + x_3y_2z_3 = 0, \\ x_1y_1z_2 + x_1y_2z_1 + x_2y_1z_1 + x_2y_3z_2 = 0. \end{cases} \tag{4.2}$$

Then $X'_\infty$ has a unique singular point, which is an ordinary double point, and is a K-polystable limit of elements of this family [Abb+23, §6].

This deformation family also contains some interesting members, which are worth mentioning. It contains a unique strictly K-semistable smooth member, whose automorphism group is isomorphic to $\mathbb{G}_a \rtimes \mathfrak{S}_3$ (see [Ara+23, Lemma 5.98]), i.e. the unique non-trivial semi-direct product of the additive group $\mathbb{G}_a$ and the symmetric group $\mathfrak{S}_3$. Recall that the automorphism group of a K-semistable Fano variety is reductive. The family contains a singular K-polystable toric Fano 3-fold. Furthermore, we can parametrise the family such that $X_\lambda$ degenerates to the singular toric K-polystable Fano 3-fold $\{x_0y_1 = x_1y_0, y_1z_2 = y_2z_1, x_0z_2 = x_2z_0\}$ when $\lambda \to \pm 1$. Recall that a toric variety is K-polystable if and only if the barycenter of its polytope is 0 [Fuj16; WZ04]. Note that it has 3 ordinary double points. Family 6 also contains a non-toric complete intersection in $\mathbb{P}^2\times\mathbb{P}^2\times\mathbb{P}^2$ with one ordinary double point which is K-unstable (see [Abb+23, Example 6.1] for the description).

To explicitly describe the compact moduli space in each of these cases, we proceed as follows: we give an explicit parametrisation of the (smooth) members of each family, which is always $\mathbb{A}^1\setminus\{p_1,..,p_k\}$ for some $k\in\mathbb{N}$, as you can see in Family 1, 2 and 6. Then, we fill the missing points of the 1-dimensional parametrisation to get $\mathbb{P}^1$ with explicit K-polystable singular Fano 3-folds that deform to a smooth member. And finally, we construct a morphism from $\mathbb{P}^1$ to $M_N^{\mathrm{Kps}}$ which is the one-dimensional component of $M_3^{\mathrm{Kps}}$ that contains all smooth K-polystable Fano 3-folds in Family №$N$ in the Mori-Mukai classification. Note that these families are classified by Picard rank and volume $V$, hence, in particular, they are one-dimensional components of $M_{3,V}^{\mathrm{Kps}}$. See in the next Corollary how we construct this morphism for Family 2 and Corollary 4.1.2 for Family 6.

**Corollary 4.1.1.** *The Fano 3-fold $X_\infty$ in Families 1 and 2 are the only singular K-polystable limits of members of the deformation families №2.24, 2.25.*

*Proof.* We only consider Family 2, since the proof is similar for the other family. Denote by $M_{2.25}^{\mathrm{Kps}}$ the one-dimensional component of the K-moduli space $M_3^{\mathrm{Kps}}$ that contains all smooth K-polystable Fano 3-folds in Family 2 (equivalently, all K-polystable elements of Mori-Mukai family №2.25). Above, we described a parametrisation $\mathcal{X} = \left\{X_\lambda; \lambda \in \mathbb{P}^1\right\}$ that is a $\mathbb{Q}$-Gorenstein family (i.e. let $\pi : \mathcal{X} \to \mathbb{P}^1$, then $K_{\mathcal{X}} - \pi^*(K_{\mathbb{P}^1})$ is $\mathbb{Q}$-Cartier), and such that all smooth members of Family 2 are fibres of the family $X_\lambda$ for $\lambda \in \mathbb{P} \setminus \{0, \pm 1, \pm i, \infty\}$. Note that $X_\lambda \cong X_{-\lambda}$ for $\lambda \in \mathbb{P}^1$.

Moreover, it follows from the description of Family 2 above the fact that all objects $X_\lambda$ in the parametrisation are K-polystable. Thus we have a morphism $\mathbb{P}^1 \to \mathcal{M}_{2.25}^{\mathrm{Kss}}$, the moduli stack parametrising K-semistable objects in this family, which descends to a morphism $\phi\colon \mathbb{P}^1 \to M_{2.25}^{\mathrm{Kps}}$ given by $\lambda \mapsto [X_\lambda]$ such that $\phi(0) = \phi(\pm 1) = \phi(\pm i) = \phi(\infty)$, and $\phi(\lambda) = \phi(-\lambda)$ for $\lambda \in \mathbb{P}^1$. Since $M_{2.25}^{\mathrm{Kps}}$ is proper and one-dimensional, we conclude that $\phi$ is surjective, which implies the required assertion. □

**Corollary 4.1.2.** *Singular K-polystable limits of smooth Fano 3-folds in the Mori-Mukai family №3.13 are the toric Fano 3-fold mentioned in Family 6 above and the non-toric Fano 3-fold $X'_\infty$ defined in* (4.2).

*Proof.* Let $M^{\mathrm{Kps}}_{3.13}$ be the one-dimensional component of $M^{\mathrm{Kps}}_{3}$ that contains K-polystable smooth Fano 3-folds in this deformation family. It follows from the description in Family 6 that there exists a $\mathbb{Q}$-Gorenstein family of Fano 3-folds over $\mathbb{P}^1$ such that the fibre $X_\lambda$ over $\lambda \in \mathbb{P}^1$ is the complete intersection in $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$ given by (4.1). This family contains all smooth K-polystable 3-folds in the family №3.13, which are fibres over the points in $\mathbb{P}^1 \setminus \{\pm 1, \infty\}$. Note that $X_\lambda \cong X_{-\lambda}$ for every $\lambda \in \mathbb{P}^1$. As in Corollary 4.1.1, we argue that there is a surjective morphism $\phi\colon \mathbb{P}^1 \to M^{\mathrm{Kps}}_{3.13}$ such that $\phi(\lambda) = [X_\lambda]$ for $\lambda \in \mathbb{P}^1 \setminus \{\pm 1, \infty\}$, and $\phi(\pm 1)$ is the K-polystable toric Fano 3-fold described in Family 6.

For $\lambda \neq \infty$, the K-polystable Fano 3-fold corresponding to $\phi(\lambda)$ is either smooth or has ordinary double points, in particular, $X_\lambda$ has unobstructed $\mathbb{Q}$-Gorenstein deformations. So, it follows from [KP21, Remark 2.4] that $M^{\mathrm{Kps}}_{3.13}$ is smooth at $\phi(\lambda)$ for $\lambda \neq \infty$. It follows from [Abb+23, Main Theorem] that $[X'_\infty] \in M^{\mathrm{Kps}}_{3.13}$, where $X'_\infty$ is the 3-fold (4.2). But $[X'_\infty] \neq \phi(\lambda)$ for $\lambda \neq \infty$, since $X'_\infty \ncong X_\lambda$ for $\lambda \notin \{0, \infty\}$, and $X'_\infty$ is not isomorphic to the toric Fano 3-fold described in Family 6. Thus, we conclude that $\phi(\infty) = [X'_\infty]$, so that $M^{\mathrm{Kps}}_{3.13}$ is smooth at $\phi(\lambda)$, which gives $M^{\mathrm{Kps}}_{3.13} \cong \mathbb{P}^1$. □

In sections §4.2, §4.3, §4.4, we study families 3, 4 and 5, respectively. For each family, we prove that there is a unique K-polystable singular Fano 3-fold that admits a smoothing to a member of the family and as a consequence, we prove the following:

**Main Theorem 4.1.3.** *All one-dimensional components of* $M^{\mathrm{Kps}}_{3}$ *are isomorphic to* $\mathbb{P}^1$.

## 4.2 Family 3

In this section, we study the Fano 3-folds described as blowups of $\mathbb{P}^3$ along rational quartic curves. (№2.22 in Mori-Mukai notation)

### 4.2.1 Parametrisation of the family

From [Ara+23, §7.4.] we have the following parameterization of the Family 3: Define $Q := \{x_0x_3 = x_1x_2\} \subset \mathbb{P}^3$ to be the smooth quadric surface, where $[x_0 : x_1 : x_2 : x_3]$ are coordinates on $\mathbb{P}^3$. Notice that $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$ via the isomorphism given by

$$\Big([u : v], [x : y]\Big) \to \Big[xu : xv : yu : yv\Big],$$

where $([u : v], [x : y])$ are coordinates on $\mathbb{P}^1 \times \mathbb{P}^1$. Let $C_\lambda := \{ux^2(x + \lambda y) = vy^2(y + \lambda x)\}$ be a curve in $Q$, for $\lambda \in \mathbb{P}^1$. Notice that $C_\lambda$ is a smooth rational quartic curve if and only if $\lambda \notin \{\pm 1, \infty\}$. Let $\pi\colon X_\lambda \to \mathbb{P}^3$ be the blowup of $\mathbb{P}^3$ along $C_\lambda$, then $X_\lambda$ is a smoothable Fano 3-fold. Moreover, every (smooth) member of family №2.22 is isomorphic to $X_\lambda$ for some $\lambda \in \mathbb{P}^1$. The 3-fold $X_\lambda$ is K-polystable for $\lambda \notin \{\pm 1, \pm 3, \infty\}$ by [CP22]. On the other hand, $X_{\pm 3}$ is strictly K-semistable, with K-polystable limit $X_0$ by [Ara+23, Lemma 7.5.].

**Lemma 4.2.1.** *The K-polystable limit of $X_{\pm 3}$ is $X_0$.*

*Proof.* Note that since $X_\lambda$ is the blowup of $\mathbb{P}^3$ along $C_\lambda$, it is enough to show that $C_{\pm 3}$ degenerates to $C_0$. We will do the proof for $C_3$, the other one will be similar.

Recall that $C_3$ is defined by equation $\{ux^2(x + 3y) = vy^2(y + 3x)\}$. With a change of coordinates $(x' = x + y)$, we get $\{u(x')^3 = (u + v)(-2y^3 + 3x'y^2)\}$. We change coordinates again taking $v' = v + u$ and $y' = \sqrt[3]{-2}y$, and we get

$\{u(x')^3 = \ v'((y')^3 + \ \frac{3}{(\sqrt[3]{-2})^2}x'(y')^2)\}$. Now we define the family

$$\mathcal{C} := \{\{ux^3 = v(y^3 + txy^2)\}|t \in \mathbb{C}\}.$$

We define $\pi : \mathcal{C} \to \mathbb{C}$ in an obvious way. Notice that for every non-zero $t \in \mathbb{C}$, $\pi^{-1}(t) \cong C_3$. On the other hand, for $\pi^{-1}(0) \ = \{ux^3 = vy^3\} = C_0$. □

We also have a relation between $X_{\pm 1}$ and $X_\infty$.

**Lemma 4.2.2.** *$X_{\pm 1}$ admits an isotrivial degeneration to $X_\infty$ (Definition 2.2.19).*

*Proof.* As in the previous Lemma, it is enough to show that $C_{\pm 1}$ degenerates isotrivially to $C_\infty$. We will do the proof for $C_1$, the other one will be similar.

Recall that $C_1$ is the union of a twisted cubic and a line defined by equation $\{(x + y)(ux^2 - vy^2) = 0\}$. With a change of coordinates ($x' = x + y$), we get $\{x'(u((x')^2 + 2x'y + y^2) - vy^2) = 0\}$. Now we define the family

$$\mathcal{C} := \{\{x(u(tx^2 + 2xy + y^2) - vy^2) = 0\}|t \in \mathbb{A}^1\}.$$

We define $\pi : \mathcal{C} \to \mathbb{A}^1$ in an obvious way. Notice that for every non-zero $t \in \mathbb{A}^1$, $\pi^{-1}(t) \cong C_1$. On the other hand, for $\pi^{-1}(0) \ = \{x(u(2xy + y^2) - vy^2) = 0\}$, we rewrite it as, $\{xy(u2x - (v - u)y) = 0\}$. Then, we do a change of coordinate ($u' = 2u$ and $v' = v - u$) and we get that $\pi^{-1}(0) = C_\infty$. □

As a consequence of Lemma 4.2.2, if $X_\infty$ is K-polystable, then $X_{\pm 1}$ is strictly K-semistable. Note that $X_\infty$ has two ordinary double points which pair with the intersection points between the lines of $C_\infty$.

### 4.2.2 The strategy to prove the K-polystability of $X_\infty$

The strategy to check the K-polystability of the singular members of the three families 3,4 and 5 is the same and it works as follows:

Let $X$ be a singular Fano 3-folds that admits a smoothing to one of the Families 3, 4 or 5 (see sections 4.3 and 4.4 for specific descriptions of the latter two families). For these families, we will present generators of the group $\mathrm{Aut}(X)$, and we will describe basic geometric facts about $X$. For instance, we will see that $X$ has two isolated ordinary double points and that $\mathrm{Aut}(X)$ swaps them. Set $G = \mathrm{Aut}(X)$. Then, using Theorem 2.2.12, we prove that $X$ is K-polystable by showing that $\beta(E) > 0$ for every $G$-invariant prime divisor $E$ over $X$.

Now, let $\varphi\colon \widehat{X} \to X$ be a $G$-equivariant birational morphism with $\widehat{X}$ normal, and let $F$ be a $G$-invariant prime divisor in the 3-fold $\widehat{X}$, and $Z = \varphi(F)$ its centre on $X$. Since $G$ swaps singular points of $X$, we have the following possibilities: $Z$ is a smooth point of $X$, $Z$ is a $G$-invariant irreducible curve, $Z$ is a $G$-invariant irreducible surface.

We replace $X$ with a suitable $G$-equivariant small resolution to simplify the computations. In principle, such a resolution may not exist, but in all cases considered here, it does, leading to a $G$-equivariant commutative diagram

$$\begin{array}{ccc} \widetilde{X} & \dashrightarrow & \overline{X} \\ & \searrow \qquad \swarrow & \\ & X & \end{array}$$

where $\widetilde{X} \to X$ and $\overline{X} \to X$ are small resolutions of singularities of $X$, and $\widetilde{X} \dashrightarrow \overline{X}$ is a composition of two Atiyah flops. Let $Y$ be one of the 3-folds $\widetilde{X}$ or $\overline{X}$, let $\eta\colon Y \to X$ be the corresponding small $G$-equivariant birational morphism, and let $Z_Y$ be the centre of the divisor $F$ on the 3-fold $Y$. Then $-K_Y \sim \eta^*(-K_X)$, which implies that $A_X(F) = A_Y(F)$ and $S(-K_X; F) = S(-K_Y; F)$. Therefore, the $\beta$-invariant that we use to characterise K-stability will not change with a small resolution.

**Remark 4.2.3.** Let $S_1, \ldots, S_r$ be effective divisors on $Y$ such that $S_Y(S_i) < 1$ for every $i$. If every $G$-invariant prime divisor in $Y$ is linearly equivalent to $\sum_{i=1}^{r} n_i S_i$ for some non-negative integers $n_1, \ldots, n_r$, then $\beta(S) > 0$ for every $G$-invariant prime divisor $S$ in $Y$. Using the bound for $S(-K_X; E)$ given in (2.1), we can weaken the condition "every $G$-invariant prime divisor in $Y$ is linearly equivalent to $\sum_{i=1}^{r} n_i S_i$ for some non-negative integers $n_1, \ldots, n_r$" as follows: for every $G$-invariant prime divisor $D \subset Y$ such that $-K_Y \sim_{\mathbb{Q}} \frac{4}{3}D + \Delta$ for some effective $\mathbb{Q}$-divisor $\Delta$ on the 3-fold $Y$, there are non-negative integers $n_1, \ldots, n_r$ such that $D \sim \sum_{i=1}^{r} n_i S_i$. Recall that $\sim$ means numerically equivalent, and for a field $k$, $\sim_k$ is numerically equivalent with coefficients in $k$. Furthermore, using [Fuj19b, Proposition 3.2], we can weaken the latter condition slightly as follows: for every $G$-invariant prime divisor $D \subset Y$ such that $-K_Y \sim_{\mathbb{Q}} \lambda D + \Delta$ for some rational number $\lambda > \frac{4}{3}$ and some effective $\mathbb{Q}$-divisor $\Delta$ on the 3-fold $Y$, there are non-negative integers $n_1, \ldots, n_r$ such that $D \sim \sum_{i=1}^{r} n_i S_i$.

Now, fix a point $p \in Z_Y$. If $\beta(F) \leqslant 0$ for a divisor $F$ whose centre contains $p$, then $\delta_p(Y) \leqslant 1$ (see definitions 2.2.6 and 2.2.8). Quite often, we can use the inductive argument of Abban and Zhuang [AZ22], and its formulation in certain scenarios in [Ara+23], to show that $\delta_p(Y) > 1$. To do this in the cases we deal with in Families 3, 4 and 5, take an admissible flag $p \in C \subset S \subset Y$, where $C$ is a smooth irreducible curve that contains $p$, and $S$ is a smooth irreducible surface in $Y$ that contains $C$. To apply [AZ22; Ara+23], set $\tau = \tau\big(S\big)$ the pseudoeffective threshold defined in Definition 2.2.6. Next, for every $u \in [0, \tau]$, it is required to find the Zariski decomposition

$$-K_Y - uS \sim_{\mathbb{R}} P(u) + N(u),$$

where $P(u)$ is the positive part of the decomposition, and $N(u)$ is the negative part (see §2.1.2). A priori, the Zariski decomposition may not exist on $Y$ for every $u \in [0, \tau]$, but in the cases dealt with here, it exists either for $Y = \widetilde{X}$ or for $Y = \overline{X}$. Hence, we may assume that the required Zariski decomposition exists on $Y$ for every $u \in [0, \tau]$. For $u \in [0, \tau]$, set $d(u) = \mathrm{ord}_C(N(u)|_S)$ and write

$$N(u)\Big|_S = N'(u) + d(u)C,$$

where $N'(u)$ is an effective divisor on $S$ such that $C \not\subset \mathrm{Supp}(N'(u))$. For $u \in [0, \tau]$, set

$$\tau(u) = \sup\Big\{ v \in \mathbb{R}_{\geqslant 0} \;\Big|\; \text{the divisor } P(u)\Big|_S - vC \text{ is pseudo-effective}\Big\}.$$

Then, for every $v \in [0, \tau(u)]$, let $P(u, v)$ be the positive part of the Zariski decomposition of the $\mathbb{R}$-divisor $P(u)|_S - vC$, and let $N(u, v)$ be its negative part. Set

$$S\Big(W^S_{\bullet,\bullet}; C\Big) = \frac{3}{(-K_X)^3}\int\limits_0^\tau d(u)\Big(P(u,0)\Big)^2 du + \frac{3}{(-K_X)^3}\int\limits_0^\tau\int\limits_0^{\tau(u)} \Big(P(u,v)\Big)^2 dv du,$$

which is well defined since the support of $N(u)$ does not contain $S$ for every $u \in [0, \tau]$. If $C = Z_Y$, it follows from Theorem 2.2.17 and[Ara+23] that

$$\frac{A_X(F)}{S_X(F)} = \frac{A_Y(F)}{S_Y(F)} \geqslant \min\left\{\frac{1}{S_Y(S)}, \frac{1}{S\Big(W^S_{\bullet,\bullet}; C\Big)}\right\}. \tag{4.3}$$

Hence, if $C = Z_Y$, $S_Y(S) < 1$ and $S(W^S_{\bullet,\bullet}; C) < 1$, then $\beta(F) > 0$. Using this approach, we can show that $\beta(F) > 0$ if $Z$ is a $G$-invariant irreducible curve.

**Remark 4.2.4.** ([AZ22; Ara+23]) In fact, if $C = Z_Y$, $S_Y(S) < 1$ and $S(W^S_{\bullet,\bullet}; C) \leqslant 1$, then $\beta(F) > 0$.

Now, we observe that $C \not\subset \operatorname{Supp}(N(u,v))$, and set

$$F_P\Big(W^{S,C}_{\bullet,\bullet,\bullet}\Big) = \frac{6}{(-K_X)^3}\int\limits_0^{\tau}\int\limits_0^{\tau(u)}\Big(P(u,v)\cdot C\Big)\cdot \operatorname{ord}_P\Big(N'(u)\Big|_C + N(u,v)\Big|_C\Big)dvdu$$

and

$$S\Big(W^{S,C}_{\bullet,\bullet,\bullet};P\Big) = \frac{3}{(-K_X)^3}\int\limits_0^{\tau}\int\limits_0^{\tau(u)}\Big(P(u,v)\cdot C\Big)^2 dvdu + F_P\Big(W^{S,C}_{\bullet,\bullet,\bullet}\Big).$$

Then it follows from [AZ22; Ara+23] that

$$\frac{A_Y(F)}{S_Y(F)} \geqslant \delta_P\Big(Y\Big) \geqslant \min\left\{\frac{1}{S_Y(S)}, \frac{1}{S\Big(W^{S}_{\bullet,\bullet};C\Big)}, \frac{1}{S\Big(W^{S,C}_{\bullet,\bullet,\bullet};P\Big)}\right\}. \tag{4.4}$$

Thus, if $S_Y(S) < 1$, $S(W^S_{\bullet,\bullet};C) < 1$ and $S(W^{S,C}_{\bullet,\bullet,\bullet};P) < 1$, then $\delta_P(Y) > 1$ and $\beta(F) > 0$.

**Remark 4.2.5** ([AZ22; Ara+23])**.** In fact, if $P = Z_Y$, $S_X(S) < 1$, $S(W^S_{\bullet,\bullet};C) \leqslant 1$ and $S(W^{S,C}_{\bullet,\bullet,\bullet};P) \leqslant 1$, then we also have $\beta(F) > 0$.

Using this approach, we will show in the remaining of this section and Sections 4.3 and 4.4 that $X$ (in the notation of Families 3, 4 and 5) is K-polystable.

### 4.2.3 Geometry of $X_\infty$

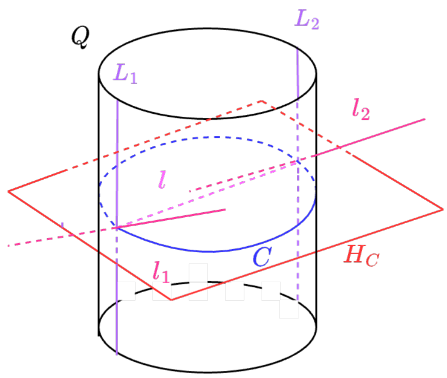
To simplify the notation, let $X = X_\infty$, where $X_\infty$ is described above. Let $C_\infty = C + L_1 + L_2$, where $C = \{x_0 + x_3 = 0, x_0x_3 = x_1x_2\}$, $L_1 = \{x_0 = 0, x_1 = 0\}$ and $L_2 = \{x_2 = 0, x_3 = 0\}$, and let $\gamma\colon V \to \mathbb{P}^3$ be the blowup of the lines $L_1$ and $L_2$. Let $\phi\colon \widetilde{X} \to V$ be the blowup of the proper transform of the conic $C$, and $\varphi\colon W \to \mathbb{P}^3$ be the blowup of the conic $C$, and let $\sigma\colon \overline{X} \to W$ be the blowup of the proper transform of the lines $L_1$ and $L_2$. Then we have the following $G$-equivariant commutative diagram:

$$\begin{array}{ccccc} \widetilde{X} & \longrightarrow & X & \longleftarrow & \overline{X} \\ {\scriptstyle \phi}\downarrow & & {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \sigma} \\ V & \underset{\gamma}{\longrightarrow} & \mathbb{P}^3 & \underset{\varphi}{\longleftarrow} & W \end{array}$$

where $\widetilde{X} \to X$ and $\overline{X} \to X$ are $G$-equivariant small resolutions of $X$. Recall from Section 4.2.2 that $G = \mathrm{Aut}(X)$, and either $Y = \widetilde{X}$ or $Y = \overline{X}$.

Note that $Q = \{x_0x_3 = x_1x_2\} \subset \mathbb{P}^3$, and let $E$, $F_1$, $F_2$ be the $\pi$-exceptional surfaces such that $\pi(E) = C$, $\pi(F_1) = L_1$, $\pi(F_2) = L_2$. Let $H_C = \{x_0 + x_3 = 0\}$, $H_{C'} = \{x_0 - x_3 = 0\}$, and denote by $H$ a general plane in $\mathbb{P}^3$. Note that when we intersect $Q$ with $H_{C'}$ we get another conic, denote it by $C'$. Now we define some of the curves that are relevant for future computations. Let $l_1$ and $l_2$ be the tangent lines to $C'$ at $P_1 = L_1 \cap C$ and $P_2 = L_2 \cap C$, respectively, and $l = H_C \cap H_{C'} = \{x_0 = 0, x_3 = 0\}$ which also contains $P_1$ and $P_2$. Figure 4.1 shows most curves and surfaces mentioned before.

Figure 4.1: Model in $\mathbb{P}^3$ of $X$ Family 3




Now, denote by $\widetilde{E}$, $\widetilde{F}_1$, $\widetilde{F}_2$, $\widetilde{Q}$, $\widetilde{H}_C$, $\widetilde{H}_{C'}$, $\widetilde{H}$ the proper transforms on $\widetilde{X}$ of the surfaces $E$, $F_1$, $F_2$, $Q$, $H_C$, $H_{C'}$, $H$, respectively. Then $\widetilde{Q} \sim 2\widetilde{H} - \widetilde{E} - \widetilde{F}_1 - \widetilde{F}_2$ and $\widetilde{H}_C \sim \widetilde{H} - \widetilde{E}$. This gives

$$-K_{\widetilde{X}} \sim 4\widetilde{H} - \widetilde{E} - \widetilde{F}_1 - \widetilde{F}_2 \sim 2\widetilde{Q} + \widetilde{E} + \widetilde{F}_1 + \widetilde{F}_2 \sim \widetilde{Q} + 2\widetilde{H}_C + 2\widetilde{E}.$$

Note that $(-K_{\widetilde{X}})^3 = (-K_X)^3 = 30$. The divisors $\widetilde{H}$, $\widetilde{E}$, $\widetilde{F}_1$, $\widetilde{F}_2$ generate the group $\mathrm{Pic}(\widetilde{X})$. We have $\widetilde{H}^3 = 1$, $\widetilde{H} \cdot \widetilde{F}_1^2 = \widetilde{H} \cdot \widetilde{F}_2^2 = \widetilde{F}_1 \cdot \widetilde{E}^2 = \widetilde{F}_2 \cdot \widetilde{E}^2 = -1$, $\widetilde{H} \cdot \widetilde{E}^2 = \widetilde{F}_1^3 = \widetilde{F}_2^3 = -2$, $\widetilde{E}^3 = -4$, and all remaining triple intersections are zero.

Similarly, let $\overline{E}$, $\overline{F}_1$, $\overline{F}_2$, $\overline{Q}$, $\overline{H}_C$, $\overline{H}_{C'}$, $\overline{H}$ be the proper transforms on $\overline{X}$ of the surfaces $E$, $F_1$, $F_2$, $Q$, $H_C$, $H_{C'}$, $H$, respectively. Then

$$-K_{\overline{X}} \sim 4\overline{H} - \overline{E} - \overline{F}_1 - \overline{F}_2 \sim 2\overline{Q} + \overline{E} + \overline{F}_1 + \overline{F}_2 \sim \overline{Q} + 2\overline{H}_C + 2\overline{E}.$$

The divisors $\overline{H}$, $\overline{E}$, $\overline{F}_1$, $\overline{F}_2$ generate the group $\mathrm{Pic}(\overline{X})$ and their intersections can be computed as follows: $\overline{H}^3 = 1$, $\overline{F}_1^3 = \overline{F}_2^3 = \overline{F}_1^2 \cdot \overline{H} = \overline{F}_2^2 \cdot \overline{H} = \overline{F}_1^2 \cdot \overline{E} = \overline{F}_2^2 \cdot \overline{E} = -1$, $\overline{E}^2 \cdot \overline{H} = -2$, $\overline{E}^3 = -6$, and all remaining triple intersections are zero.

**Description of the automorphism group** Let us take the following automorphisms in $\mathbb{P}^3$:

$$\begin{cases} \tau : [x_0 : x_1 : x_2 : x_3] \mapsto [x_3 : x_2 : x_1 : x_0], \\ \Gamma := \left\{ [x_0 : x_1 : x_2 : x_3] \mapsto \left[x_0 : \lambda x_1 : \frac{x_2}{\lambda} : x_3\right] \Big| \, \lambda \in \mathbb{C}^* \right\}. \end{cases}$$

Note that $\Gamma$ is the subgroup of $\mathrm{Aut}(\mathbb{P}^3)$. Then $\Gamma$ is a $\mathbb{C}^*$ action, the curve $C_\infty$ is $\langle \tau, \Gamma \rangle$-invariant, and the $\langle \tau, \Gamma \rangle$-action lifts to $X$. Hence, we can identify $\langle \tau, \Gamma \rangle$ with a subgroup in $G = \mathrm{Aut}(X)$. Since $\tau$ is an involution is straightforward to verify that $G = \langle \tau, \Gamma \rangle \cong \mathbb{C}^* \rtimes \mu_2$.

**Description of the $G$-invariant loci.** Here we will study all $G$-invariant subsets of $X$. First, notice that all the invariant subsets in $X$ will come from some $G$-invariant subset in $\mathbb{P}^3$. Let us start by studying the $G$-invariant points on $\mathbb{P}^3$: set $O = [1 : 0 : 0 : 1]$ and $O' = [1 : 0 : 0 : -1]$.

**Lemma 4.2.6.** *The only $G$-fixed points in $\mathbb{P}^3$ are $O$ and $O'$.*

*Proof.* Taking the $\mathbb{C}^*$ action, $\Gamma$, on $\mathbb{P}^3$, we get that any $\Gamma$-invariant point must have $x_1 = x_2 = 0$. And a point of the form $[x_0 : 0 : 0 : x_3]$ to be $\tau$ invariant, we get $x_0 = \pm x_3$. □

Notice that $l$ previously defined, is $G$-invariant. And so are the following curves: $l' = \{x_1 = 0, x_2 = 0\}$, $C_r = \{x_1 x_2 = r x_0 x_3, x_0 + x_3 = 0\}$, $C'_r = \{x_1 x_2 = r x_0 x_3, x_0 = x_3\}$ for $r \in \mathbb{C}^*$. Then $l'$ is the line that passes through $O$ and $O'$. Note that $C_r$ is an irreducible conic in the plane $H_C$, and $C'_r$ is an irreducible conic in the plane $H'_C$, and $C = C_1$, $C' = C'_1$.

**Lemma 4.2.7.** *The curves $l$, $l'$, $C_r$, $C'_r$ are the only $G$-invariant irreducible curves in $\mathbb{P}^3$.*

*Proof.* Let $\mathcal{C}$ be a $G$-invariant irreducible curve in $\mathbb{P}^3$. If $\mathcal{C}$ is pointwise fixed by $\Gamma$, then $\mathcal{C} = l'$. We may assume that $\mathcal{C} \neq l'$. Then $\Gamma$ acts on $\mathcal{C}$ effectively, which implies that $\mathcal{C}$ is rational. Then $\tau$ must fix a point $p \in \mathcal{C}$, which is not fixed by $\Gamma$, which implies that $\mathcal{C} = \overline{\mathrm{Orb}_\Gamma(p)}$. On the other hand, the $\tau$-fixed points are $[b : a : a : b]$ and $[b : a : -a : -b]$ for $[a : b] \in \mathbb{P}^1$, which implies the required assertion. □

Thus, the planes $H_C$ and $H_{C'}$ contain all $G$-invariant irreducible curves in $\mathbb{P}^3$ except $l'$. To complete the description of $G$-invariant curves in $X$, we have to describe $G$-invariant irreducible curves in $E$, which is done in the following lemma:

**Lemma 4.2.8.** *The only $G$-invariant irreducible curves in $\widetilde{E}$ are $\widetilde{E} \cap \widetilde{Q}$ and $\widetilde{E} \cap \widetilde{H}_C$, and the only $G$-invariant irreducible curves in $\overline{E}$ are $\overline{s}_0 = \overline{E} \cap \overline{Q}$ and $\overline{s} = \overline{E} \cap \overline{H}_C$.*

*Proof.* Note that $\sigma(\overline{E}) \cong \mathbb{F}_2$ by [CS19, Lemma 2.6], and $\sigma(\overline{s})$ is the $(-2)$-curve in $\overline{E}$. Let $\varsigma \colon \mathbb{P}^3 \dashrightarrow \mathbb{P}^4$ be the $G$-equivariant map

$$[x_0 : x_1 : x_2 : x_3] \mapsto \big[x_0(x_0+x_3) : x_1(x_0+x_3) : x_2(x_0+x_3) : x_3(x_0+x_3) : x_0x_3 - x_1x_2\big],$$

and let $Y$ be the closure of its image in $\mathbb{P}^4$. Then $Y$ is the quadric $\{tw - tx + wx + yz = 0\}$, where $[x : y : z : t : w]$ are coordinates on $\mathbb{P}^4$. Then $\varsigma$ induces a $G$-equivariant birational map $\theta\colon \mathbb{P}^3 \dashrightarrow Y$ such that there exists the following $G$-equivariant commutative diagram:

$$\begin{array}{ccc} & W & \\ {}^{\varphi}\swarrow & & \searrow^{\upsilon} \\ \mathbb{P}^3 & \overset{\theta}{\dashrightarrow} & Y \end{array}$$

where $\upsilon$ is the contraction of $\sigma(\overline{H}_C)$ to $[0 : 0 : 0 : 0 : 1]$. Set $S_2 = \upsilon \circ \sigma(\overline{E})$. Then

$$S_2 = \Big\{t + x = 0, yz - tx = 0\Big\} \subset Y,$$

and $\upsilon$ induces a $G$-equivariant birational morphism $\sigma(\overline{E}) \to S_2$ that contracts $\sigma(\overline{s})$. Moreover, one can check that the only $G$-invariant irreducible curve in the cone $S_2$ is the conic $\{w = t + x = yz - tx = 0\} = \nu \circ \sigma(\overline{s}_0)$. This implies the required assertion. □

Finally, to use Remark 4.2.3, we need to study the $G$-invariant $G$-irreducible divisors on both $\widetilde{X}$ and $\overline{X}$.

**Lemma 4.2.9.** *Let $S$ be a $G$-invariant $G$-irreducible surface in $\widetilde{X}$ such that $S \neq \widetilde{F}_1 + \widetilde{F}_2$. Then $S \sim a\widetilde{Q} + b\widetilde{H}_C + c\widetilde{E}$ for some non-negative integers $a$, $b$, $c$.*

*Proof.* We may assume that $S \neq \widetilde{E}$, $S \neq \widetilde{H}_C$, $S \neq \widetilde{Q}$. Then $\pi(S)$ is a $G$-invariant surface of degree $d \geqslant 1$, and $S \sim d\widetilde{H} - m\widetilde{E} - n(\widetilde{F}_1 + \widetilde{F}_2)$ for some non-negative integers $m$ and $n$. Let $\ell$ be a general ruling of $\widetilde{Q} \cong \mathbb{P}^1 \times \mathbb{P}^1$ such that $\widetilde{F}_1 \cdot \ell = \widetilde{F}_2 \cdot \ell = 1$. Then $\widetilde{E} \cdot \ell = 1$ and $0 \leqslant S \cdot \ell = d - m - 2n$. Thus, we have $d \geqslant m - 2n$. So, we can let $a = n$, $b = d - 2n$, $c = d - m - n$. □

Using the same technique as above it is straightforward to prove the following:

**Lemma 4.2.10.** *Let $S$ be a $G$-invariant $G$-irreducible surface in $\overline{X}$ such that $S \neq \overline{F}_1 + \overline{F}_2$. Then $S \sim a\overline{Q} + b\overline{H}_C + c\overline{E}$ for some non-negative integers $a$, $b$, $c$.*

### 4.2.4 K-polystability of $X$

We are ready to prove that $X$ is K-polystable using the approach described in Section 4.2.2. Let $F$ be a $G$-invariant prime divisor over $X$, and let $Z$, $\widetilde{Z}$, $\overline{Z}$ be its centres on $X$, $\widetilde{X}$ and $\overline{X}$, respectively. Then it follows from Lemma 4.2.6 that

1. either $Z$ is a $G$-invariant irreducible surface,
2. or $Z$ is a curve described in Lemmas 4.2.7 and 4.2.8,
3. or $Z$ is a point, and $\pi(Z)$ is the point $O$ or $O'$.

We start with the case of a $G$-invariant irreducible surface. Using Remark 4.2.3 and Lemma 4.2.9, we obtain

**Lemma 4.2.11.** *Let $F$ be a $G$-invariant prime divisor on $X$. Then $\beta(F) > 0$.*

*Proof.* By Remark 4.2.3 and Lemma 4.2.9, it is enough to show that $\beta(\widetilde{Q})$, $\beta(\widetilde{H}_C)$, $\beta(\widetilde{E})$ are positive. We will do this using the notations introduced in Section 4.2.2.

We start with $\widetilde{Q}$. Let $Y = \widetilde{X}$ and $S = \widetilde{Q}$. Then, $-K_{\widetilde{X}} - uS \sim_{\mathbb{R}} (2-u)S + \widetilde{E} + \widetilde{F}_1 + \widetilde{F}_2$. This shows that $\tau = 2$.

Recall that a divisor $D$ is nef if the intersection $D \cdot C \geq 0$ is positive for all irreducible curves $C$ in $D$. Also notice that in order to get $(-K_{\widetilde{X}} - uS) \cdot l < 0$, $l$ must to have negative self-intersection in at least in $S$, $\widetilde{E}$, $\widetilde{F}_1$, $\widetilde{F}_2$, or $\widetilde{H}$. Moreover, since $S \cong \widetilde{E} \cong \mathbb{P}^1 \times \mathbb{P}^1$, they do not have negative curves. $\widetilde{F}_i$ has a $(-1)$-curve $\widetilde{g}_i = \widetilde{E}|_{\widetilde{F}_i}$. Finally in $\widetilde{H}$, we have four $(-1)$-curves $(\widetilde{e}_1 + \widetilde{e}_2) = \widetilde{E}|_{\widetilde{H}}$, $\widetilde{f}_i = \widetilde{F}_i|_{\widetilde{H}}$. Note that when we have 3 surfaces $A$, $B$ and $C$, $A|_C \cdot B|_C = A \cdot B \cdot C$ since this number is up to equivalence class. So now we check the following intersections:

$$(-K_{\widetilde{X}} - uS)|_{\widetilde{F}_i} \cdot \widetilde{g}_i = \Big((2-u)S|_{\widetilde{F}_i} + \widetilde{g}_i + \widetilde{F}_i|_{\widetilde{F}_i}\Big) \cdot \widetilde{g}_i = 1 - u \geq 0 \qquad \Leftrightarrow 1 \geq u;$$
$$(-K_{\widetilde{X}} - uS)|_{\widetilde{H}} \cdot \widetilde{f}_i = \Big((2-u)S|_{\widetilde{H}} + \widetilde{e} + \widetilde{f}_1 + \widetilde{f}_2\Big) \cdot \widetilde{f}_i = 1 - u \geq 0 \qquad \Leftrightarrow 1 \geq u;$$
$$(-K_{\widetilde{X}} - uS)|_{\widetilde{H}} \cdot \widetilde{e}_i = \Big((2-u)S|_{\widetilde{H}} + \widetilde{e}_1 + \widetilde{e}_2 + \widetilde{f}_1 + \widetilde{f}_2\Big) \cdot \widetilde{e}_i = 1 - u \geq 0 \quad \Leftrightarrow 1 \geq u.$$

Here we see that $-K_{\widetilde{X}} - uS$ is nef for $u \in [0,1]$. For $u \geq 1$, the intersections with $\widetilde{E}|_{\widetilde{H}}$ and $\widetilde{F}_i|_{\widetilde{H}}$ are negative, therefore we are adding some $\lambda\Big(\widetilde{E} + \widetilde{F}_1 + \widetilde{F}_2\Big)$. In this case it is easy to check that $\lambda = (1-u)$. Hence, we have

$$P(u) \sim_{\mathbb{R}} \begin{cases} (2-u)S + \widetilde{E} + \widetilde{F}_1 + \widetilde{F}_2 & \text{if } 0 \leqslant u \leqslant 1, \\ (2-u)\Big(S + \widetilde{E} + \widetilde{F}_1 + \widetilde{F}_2\Big) & \text{if } 1 \leqslant u \leqslant 2. \end{cases}$$

If $u \in [0,1]$, then $N(u) = 0$. If $u \in [1,2]$, then $N(u) = (u-1)(\widetilde{E} + \widetilde{F}_1 + \widetilde{F}_2)$. Then

$$\Big(P(u)\Big)^3 = \begin{cases} 2u^3 - 6u^2 - 18u + 30 & \text{if } 0 \leqslant u \leqslant 1, \\ 8(2-u)^3 & \text{if } 1 \leqslant u \leqslant 2. \end{cases}$$

Now, integrating $(P(u))^3$, we get $S_Y(S) = \frac{43}{60}$, so that $\beta(\widetilde{Q}) = \frac{17}{60} > 0$.

Now we deal with $\widetilde{H}_C$. Set $Y = \widetilde{X}$ and $S = \widetilde{H}_C$. Then $-K_{\widetilde{X}} - uS \sim_{\mathbb{R}} (2-u)S + 2\widetilde{E} + \widetilde{Q}$. This gives $\tau = 2$, because $2\widetilde{E} + \widetilde{Q}$ is not big. Moreover, following the same procedure as for $S = \widetilde{Q}$, we have

$$P(u) \sim_{\mathbb{R}} \begin{cases} (2-u)S + 2\widetilde{E} + \widetilde{Q} & \text{if } 0 \leqslant u \leqslant 1, \\ (2-u)S + (3-u)\widetilde{E} + \widetilde{Q} & \text{if } 1 \leqslant u \leqslant 2. \end{cases}$$

If $u \in [0,1]$, then $N(u) = 0$. If $u \in [1,2]$, then $N(u) = (u-1)\widetilde{E}$. We compute

$$\Big(P(u)\Big)^3 = \begin{cases} u^3 - 6u^2 - 12u + 30 & \text{if } 0 \leqslant u \leqslant 1, \\ (2-u)(u^2 - 10u + 22) & \text{if } 1 \leqslant u \leqslant 2, \end{cases}$$

which gives $S_Y(S) = \frac{11}{12}$, so that $\beta(\widetilde{Q}) = \frac{1}{12} > 0$.

Finally, we set $Y = \overline{X}$ and $S = \overline{E}$. Then $-K_{\overline{X}} - uS \sim_{\mathbb{R}} (2-u)S + 2\overline{H}_C + \overline{Q}$. This shows that $\tau = 2$, because $2\overline{H}_C + \overline{Q}$ is not big. Moreover, we have

$$P(u) \sim_{\mathbb{R}} \begin{cases} (2-u)S + 2\overline{H}_C + \overline{Q} & \text{if } 0 \leqslant u \leqslant 1, \\ (2-u)\big(S + \overline{Q} + 2\overline{H}_C\big) & \text{if } 1 \leqslant u \leqslant 2. \end{cases}$$

If $u \in [0,1]$, then $N(u) = 0$. If $u \in [1,2]$, then $N(u) = (u-1)\overline{Q} + 2(u-1)\overline{H}_C$. Then

$$\big(P(u)\big)^3 = \begin{cases} 6u^3 - 6u^2 - 24u + 30 & \text{if } 0 \leqslant u \leqslant 1, \\ 6(2-u)^3 & \text{if } 1 \leqslant u \leqslant 2, \end{cases}$$

which gives $S_Y(S) = \frac{19}{30}$ and $\beta(\widetilde{E}) = \beta(\overline{E}) = \frac{11}{30} > 0$. □

We now show that $\beta(F) > 0$ for $F$ a $G$-invariant prime divisor with small centre on $X$.

**Lemma 4.2.12.** *Suppose that $\widetilde{Z}$ is a $G$-invariant irreducible curve in $\widetilde{H}_C$. Then $\beta(F) > 0$.*

*Proof.* The morphism $\gamma \circ \phi$ induces a birational morphism $\widetilde{H}_C \to H_C$, which is a blowup of the points $H_C \cap L_1$ and $H_C \cap L_2$. Set $\widetilde{f}_1 = \widetilde{F}_1|_{\widetilde{H}_C}$ and $\widetilde{f}_2 = \widetilde{F}_2|_{\widetilde{H}_C}$. Then $\widetilde{f}_1$ and $\widetilde{f}_2$ are exceptional curves of the morphism $\widetilde{H}_C \to H_C$. Let $\widetilde{l}$ be the third $(-1)$-curve in $\widetilde{H}_C$, set $h = \widetilde{H}|_{\widetilde{H}_C}$, and set $\widetilde{C} = \widetilde{E}|_{\widetilde{H}_C}$. Then $\gamma \circ \phi(\widetilde{l}) = l$, so that $\widetilde{l} \sim h - \widetilde{f}_1 - \widetilde{f}_2$. By Lemmas 4.2.7 and 4.2.8, we have the following possible cases:

- $\pi(Z) = l$ and $\widetilde{Z} = \widetilde{l}$,
- $\pi(Z) = C_1 = C$ and $\widetilde{Z} = \widetilde{C} \sim 2h - \widetilde{f}_1 - \widetilde{f}_2$,
- $\pi(Z) = C_r$ with $r \neq 1$, $\widetilde{Z} \not\subset \widetilde{E}$ and $\widetilde{Z} \sim 2h - \widetilde{f}_1 - \widetilde{f}_2$.

Set $Y = \widetilde{X}$, $S = \widetilde{H}_C$, $\mathscr{C} = \widetilde{Z}$. Then it follows from the proof of Lemma 4.2.11 that

$$P(u)\Big|_S \sim_{\mathbb{R}} \begin{cases} (2+u)h - u(\widetilde{f}_1 + \widetilde{f}_2) & \text{if } 0 \leqslant u \leqslant 1, \\ (4-u)h - \widetilde{f}_1 - \widetilde{f}_2 & \text{if } 1 \leqslant u \leqslant 2, \end{cases}$$

and

$$N(u)\Big|_S \sim_{\mathbb{R}} \begin{cases} 0 & \text{if } 0 \leqslant u \leqslant 1, \\ (u-1)\widetilde{C} & \text{if } 1 \leqslant u \leqslant 2. \end{cases}$$

We know from the proof of Lemma 4.2.11 that $S_Y(S) = \frac{11}{12} < 1$. Let us compute $S(W^S_{\bullet,\bullet}; \mathscr{C})$.

Suppose that $\widetilde{Z} = \widetilde{l}$, then

$$P(u)\Big|_S - v\widetilde{Z} \sim_{\mathbb{R}} \begin{cases} (2+u-v)\widetilde{l} + 2(\widetilde{f}_1 + \widetilde{f}_2) & \text{if } 0 \leqslant u \leqslant 1, \\ (4-u-v)\widetilde{l} + (3+u+v)\big(\widetilde{f}_1 + \widetilde{f}_2\big) & \text{if } 1 \leqslant u \leqslant 2, \end{cases}$$

Hence, if $0 \leqslant u \leqslant 1$, then $\tau(u) = 2+u$. If $1 \leqslant u \leqslant 2$, then $\tau(u) = 4-u$. We denote by $P(u,v)$ and $N(u,v)$ the positive and negative parts of the divisor $P(u)\Big|_S - v\widetilde{Z}$ on $S$, respectively. Moreover, if $0 \leqslant u \leqslant 1$, then

$$P(u,v) \sim_{\mathbb{R}} \begin{cases} (2+u-v)h - (u-v)(\widetilde{f}_1 + \widetilde{f}_2) & \text{if } 0 \leqslant v \leqslant u, \\ (2+u-v)h & \text{if } u \leqslant v \leqslant 2+u, \end{cases}$$

and

$$N(u,v) \sim_{\mathbb{R}} \begin{cases} 0 & \text{if } 0 \leqslant v \leqslant u, \\ (v-u)(\widetilde{f}_1 + \widetilde{f}_2) & \text{if } u \leqslant v \leqslant 2+u. \end{cases}$$

Similarly, if $1 \leqslant u \leqslant 2$, then

$$P(u,v) \sim_{\mathbb{R}} \begin{cases} (4-u-v)h - (1-v)(\widetilde{f}_1 + \widetilde{f}_2) & \text{if } 0 \leqslant v \leqslant 1, \\ (4-u-v)h & \text{if } 1 \leqslant v \leqslant 4-u, \end{cases}$$

and

$$N(u,v) \sim_{\mathbb{R}} \begin{cases} 0 & \text{if } 0 \leqslant v \leqslant 1, \\ (v-1)(\widetilde{f}_1+\widetilde{f}_2) & \text{if } 1 \leqslant v \leqslant 4-u. \end{cases}$$

This gives

$$S(W^S_{\bullet,\bullet};\mathscr{C}) = \frac{1}{10}\int_0^1\int_0^u 4-u^2+2uv-v^2+4u-4v\,dvdu+\frac{1}{10}\int_0^1\int_u^{2+u}(2+u-v)^2dvdu+$$
$$+\frac{1}{10}\int_1^2\int_0^1 u^2+2uv-v^2-8u-4v+14\,dvdu+\frac{1}{10}\int_1^2\int_1^{4-u}(u+v-4)^2dvdu = 1.$$

Hence, it follows from Remark 4.2.4 that when $\widetilde{Z} = \widetilde{l}$, $\beta(F) > 0$.

We may assume that $\pi(Z) = C_r$. Then, $\widetilde{Z} \sim 2h - \widetilde{f}_1 - \widetilde{f}_2$. If $0 \leqslant u \leqslant 1$, then $\tau(u) = \frac{2+u}{2}$. Similarly, if $1 \leqslant u \leqslant 2$, then $\tau(u) = \frac{4-u}{2}$. Moreover, if $0 \leqslant u \leqslant 1$, then

$$P(u,v) \sim_{\mathbb{R}} \begin{cases} (2+u-2v)h-(u-v)(\widetilde{f}_1+\widetilde{f}_2) & \text{if } 0 \leqslant v \leqslant u, \\ (2+u-2v)h & \text{if } u \leqslant v \leqslant \frac{2+u}{2}, \end{cases}$$

and

$$N(u,v) \sim_{\mathbb{R}} \begin{cases} 0 & \text{if } 0 \leqslant v \leqslant u, \\ (v-u)(\widetilde{f}_1+\widetilde{f}_2) & \text{if } u \leqslant v \leqslant \frac{2+u}{2}. \end{cases}$$

Similarly, if $1 \leqslant u \leqslant 2$, then

$$P(u,v) \sim_{\mathbb{R}} \begin{cases} (4-u-2v)h-(1-v)(\widetilde{f}_1+\widetilde{f}_2) & \text{if } 0 \leqslant v \leqslant 1, \\ (4-u-2v)h & \text{if } 1 \leqslant v \leqslant \frac{4+u}{2}, \end{cases}$$

and

$$N(u,v) \sim_{\mathbb{R}} \begin{cases} 0 & \text{if } 0 \leqslant v \leqslant 1, \\ (v-1)(\widetilde{f}_1+\widetilde{f}_2) & \text{if } 1 \leqslant v \leqslant \frac{2+u}{2}. \end{cases}$$

Therefore, if $\widetilde{Z}=\widetilde{C}$, then $S(W^S_{\bullet,\bullet};\mathscr{C})$ can be computed as follows:

$$\frac{1}{10}\int\limits_1^2(u-1)(u^2-8u+14)du+\frac{1}{10}\int\limits_0^1\int\limits_0^u(4-u^2+2v^2+4u-8v)dvdu+$$
$$+\frac{1}{10}\int\limits_0^1\int\limits_u^{\frac{2+u}{2}}(2+u-2v)^2dvdu+\frac{1}{10}\int\limits_1^2\int\limits_0^1(u^2+4uv+2v^2-8u-12v+14)dvdu+$$
$$+\frac{1}{10}\int\limits_1^2\int\limits_1^{\frac{4-u}{2}}(u+2v-4)^2dvdu=\frac{53}{80}<1.$$

Similarly, if $\widetilde{Z}\neq\widetilde{C}$, then $S(W^S_{\bullet,\bullet};\mathscr{C})=\frac{39}{80}<1$. Then $\beta(F)>0$ by (4.3). $\square$

Using computations made in the proof of Lemma 4.2.12, we obtain the following result:

**Lemma 4.2.13.** *Suppose that $\pi(Z)$ contains $O$. Then $\beta(F)>0$.*

*Proof.* Let us use the notation introduced in the proof of Lemma 4.2.12. First, we let $Y=\widetilde{X}$. Let $P$ be the preimage on $Y$ of the point $O$. Then $P$ is the unique $G$-fixed point in $\widetilde{H}_C$.

As in the proof of Lemma 4.2.12, we choose $S=\widetilde{H}_C$, and we choose $\mathscr{C}$ to be the curve in the pencil $|h-\widetilde{f}_1|$ that contains $P$. Since $S_Y(S)=\frac{11}{12}$ (see the proof of Lemma 4.2.11), it follows from (4.4) that $\beta(F)>0$ if $S(W^S_{\bullet,\bullet};\mathscr{C})<1$ and $S(W^{S,\mathscr{C}}_{\bullet,\bullet,\bullet};P)<1$.

Let us compute $S(W^S_{\bullet,\bullet};\mathscr{C})$ and $S(W^{S,\mathscr{C}}_{\bullet,\bullet,\bullet};P)$. If $0\leqslant u\leqslant 1$, then $\tau(u)=2$. If $1\leqslant u\leqslant 2$, then $\tau(u)=3-u$. Moreover, if $0\leqslant u\leqslant 1$, then

$$P(u,v)\sim_{\mathbb{R}}\begin{cases}(2+u-v)h-(u-v)\widetilde{f}_1-u\widetilde{f}_2 & \text{if } 0\leqslant v\leqslant u,\\ (2+u-v)h-u\widetilde{f}_2 & \text{if } u\leqslant v\leqslant 2,\end{cases}$$

and

$$N(u,v)\sim_{\mathbb{R}} \begin{cases} 0 & \text{if } 0 \leqslant v \leqslant u, \\ (v-u)\widetilde{f}_1 & \text{if } u \leqslant v \leqslant 2, \end{cases}$$

which gives

$$\Big(P(u,v)\Big)^2 = \begin{cases} 4-u^2+4u-4v & \text{if } 0 \leqslant v \leqslant u, \\ (2-v)(2+2u-v) & \text{if } u \leqslant v \leqslant 2, \end{cases}$$

and

$$P(u,v)\cdot\mathscr{C} = \begin{cases} 2 & \text{if } 0 \leqslant v \leqslant u, \\ 2+u-v & \text{if } u \leqslant v \leqslant 2. \end{cases}$$

Similarly, if $1 \leqslant u \leqslant 2$, then

$$P(u,v) \sim_{\mathbb{R}} \begin{cases} (4-u-v)h-(1-v)\widetilde{f}_1-\widetilde{f}_2 & \text{if } 0 \leqslant v \leqslant 1, \\ (4-u-v)h-\widetilde{f}_2 & \text{if } 1 \leqslant v \leqslant 3-u, \end{cases}$$

and

$$N(u,v)\sim_{\mathbb{R}} \begin{cases} 0 & \text{if } 0 \leqslant v \leqslant 1, \\ (v-1)\widetilde{f}_1 & \text{if } 1 \leqslant 3-u \leqslant 3-u, \end{cases}$$

which implies that

$$\Big(P(u,v)\Big)^2 = \begin{cases} u^2+2uv-8u-6v+14 & \text{if } 0 \leqslant v \leqslant 1, \\ (3-u-v)(5-u-v) & \text{if } 1 \leqslant v \leqslant 3-u, \end{cases}$$

and

$$P(u,v)\cdot\mathscr{C} = \begin{cases} 3-u & \text{if } 0 \leqslant v \leqslant 1, \\ 4-u-v & \text{if } 1 \leqslant v \leqslant 3-u. \end{cases}$$

Observe that $\mathscr{C}$ is not contained in the support of the divisor $N(u)$ for every $u \in [0,2]$, and $P$ is not contained in the support of the divisor $N(u,v)$ for $u \in [0,2]$

and $v \in [0, \tau(u)]$. Now, by integrating we get $S(W^S_{\bullet,\bullet}; \mathscr{C}) = S(W^{S,\mathscr{C}}_{\bullet,\bullet,\bullet}; P) = \frac{47}{60} < 1$, so $\beta(F) > 0$ by (4.4). $\square$

Recall that $H_{C'}$ is the plane in $\mathbb{P}^3$ that contains $O'$ and $C'_r$ for every $r \in \mathbb{C}^*$.

**Lemma 4.2.14.** *Suppose that $\pi(Z) = C'_r$ for some $r \in \mathbb{C}^*$. Then $\beta(F) > 0$.*

*Proof.* As above, we use the notation introduced in section 4.2.2. Let $Y = \widetilde{X}$, and let $S$ be the proper transform on $Y$ of the plane $H_{C'}$. Then $-K_{\widetilde{X}} - uS \sim_{\mathbb{R}} \widetilde{Q} + (2-u)S$. This gives $\tau = 2$. If $u \in [0,1]$, then $N(u) = 0$. If $u \in [1,2]$, then $N(u) = (u-1)\widetilde{Q}$. Thus, we have

$$P(u) \sim_{\mathbb{R}} \begin{cases} \widetilde{Q} + (2-u)S & \text{if } u \in [0,1], \\ (2-u)(\widetilde{Q} + S) & \text{if } u \in [1,2]. \end{cases}$$

Integrating, we get $S_Y(S) = \frac{17}{30}$.

Set $\widetilde{C}' = \widetilde{Q}|_S$. Then $\widetilde{C}'$ is a smooth irreducible $G$-invariant curve, and

$$N(u) \sim_{\mathbb{R}} \begin{cases} 0 & \text{if } u \in [0,1], \\ (u-1)\widetilde{C}' & \text{if } u \in [1,2]. \end{cases}$$

To describe $P(u)|_S$ explicitly, we have to say a few words about the surface $S$.

Set $P_1 = H_{C'} \cap L_1$ and $P_2 = H_{C'} \cap L_2$. Recall that $l$ is the line containing $P_1$ and $P_2$. Let $\mathcal{P}$ be the pencil on $H_{C'}$ generated by $2l$ and $C'$, let $l_1$ and $l_2$ be the lines in $H_{C'}$ that are tangent to $C'$ at the points $P_1$ and $P_2$, respectively. Then

- the base locus of the pencil $\mathcal{P}$ consists of the points $P_1$ and $P_2$,
- the pencil $\mathcal{P}$ contains $l_1 + l_2$ and the conic $C'_r$ for every $r \in \mathbb{C}^*$,
- the conics $2l$ and $l_1 + l_2$ are the only singular curves in $\mathcal{P}$.

The morphism $\gamma \circ \phi$ induces a birational morphism $\xi \colon S \to H_{C'}$ that resolves the base locus of the pencil $\mathcal{P}$. The morphism $\xi$ is a composition of 4 blowups such that we have the following $G$-equivariant commutative diagram:

$$\begin{array}{ccc} & S & \\ {\scriptstyle \xi}\swarrow & & \searrow \\ H_{C'} & \dashrightarrow & \mathbb{P}^1 \end{array}$$

where $H_{C'} \dashrightarrow \mathbb{P}^1$ is the rational map given by $\mathcal{P}$, and $S \to \mathbb{P}^1$ is a surjective morphism. The birational morphism $\xi$ is a composition of the blowup of the points $P_1$ and $P_2$ with the subsequent blowup of two points in the exceptional curves contained in the proper transforms of $l_1$ and $l_2$. Note that $S$ is a weak del Pezzo surface of degree five.

We have $\widetilde{E}|_S = \widetilde{g}_1 + \widetilde{g}_2$, where $\widetilde{g}_1$ and $\widetilde{g}_2$ are two irreducible $\xi$-exceptional $(-1)$-curves such that $\xi(\widetilde{g}_1) = P_1$ and $\xi(\widetilde{g}_2) = P_2$. Let $\widetilde{f}_1$ and $\widetilde{f}_2$ be the remaining $\xi$-exceptional curves that are mapped to the points $P_1$ and $P_2$, respectively, and let $\widetilde{l}$, $\widetilde{l}_1$, $\widetilde{l}_2$, $\widetilde{C}'_r$ be the proper transforms on $S$ of the curves $l$, $l_1$, $l_2$, $C'_r$, respectively. Then $\widetilde{C}' = \widetilde{C}'_1$ and $2\widetilde{l}+\widetilde{f}_1+\widetilde{f}_2 \sim \widetilde{l}_1+\widetilde{l}_2 \sim \widetilde{C}' \sim \widetilde{C}'_r$ for every $r \in \mathbb{C}^*$, and the curves $\widetilde{f}_1$, $\widetilde{f}_2$, $\widetilde{g}_1$, $\widetilde{g}_2$, $\widetilde{l}$, $\widetilde{l}_1$, $\widetilde{l}_2$ generate the Mori cone $\overline{\mathrm{NE}(S)}$ by [CT88, Proposition 8.5].

Note that $\widetilde{F}_1|_S = \widetilde{f}_1 + \widetilde{g}_1$ and $\widetilde{F}_2|_S = \widetilde{f}_2 + \widetilde{g}_2$. Using this, we get

$$P(u)\Big|_S \sim_{\mathbb{R}} \begin{cases} \frac{4-u}{2}\widetilde{C}' + \frac{2-u}{2}\Big(\widetilde{f}_1 + \widetilde{f}_2\Big) + (2-u)\Big(\widetilde{g}_1 + \widetilde{g}_2\Big) & \text{if } 0 \leqslant u \leqslant 1, \\ \frac{6-3u}{2}\widetilde{C}' + \frac{2-u}{2}\Big(\widetilde{f}_1 + \widetilde{f}_2\Big) + (2-u)\Big(\widetilde{g}_1 + \widetilde{g}_2\Big) & \text{if } 1 \leqslant u \leqslant 2. \end{cases} \tag{4.5}$$

Set $\mathscr{C} = \widetilde{Z}$. Let us compute $S(W^S_{\bullet,\bullet};\mathscr{C})$. Recall that $\mathscr{C} \sim \widetilde{C}'$. Then (4.5) gives

$$\tau(u) = \begin{cases} \frac{4-u}{2} & \text{if } 0 \leqslant u \leqslant 1, \\ \frac{6-3u}{2} & \text{if } 1 \leqslant u \leqslant 2. \end{cases}$$

Moreover, if $0 \leqslant u \leqslant 1$, then (4.5) gives

$$P(u,v) \sim_{\mathbb{R}} \begin{cases} \frac{4-u-2v}{2}\mathscr{C} + (2-u)\Big(\widetilde{f}_1+\widetilde{f}_2\Big) + \frac{2-u}{2}\Big(\widetilde{g}_1+\widetilde{g}_2\Big) & \text{if } 0 \leqslant v \leqslant 1, \\ \frac{4-u-2v}{2}\Big(\mathscr{C}+\widetilde{f}_1+\widetilde{f}_2+\widetilde{g}_1+\widetilde{g}_2\Big) & \text{if } 1 \leqslant v \leqslant \frac{4-u}{2}, \end{cases}$$

and

$$N(u,v) \sim_{\mathbb{R}} \begin{cases} 0 & \text{if } 0 \leqslant v \leqslant 1, \\ (v-1)(\widetilde{f}_1+\widetilde{f}_2+2\widetilde{g}_1+2\widetilde{g}_2) & \text{if } 1 \leqslant v \leqslant \frac{4-u}{2}. \end{cases}$$

Similarly, if $1 \leqslant u \leqslant 2$, then (4.5) gives

$$P(u,v) \sim_{\mathbb{R}} \begin{cases} \frac{6-3u-2v}{2}\mathscr{C} + \frac{2-u}{2}\Big(\widetilde{f}_1+\widetilde{f}_2\Big) + (2-u)\Big(\widetilde{g}_1+\widetilde{g}_2\Big) & \text{if } 0 \leqslant v \leqslant 2-u, \\ \frac{6-3u-2v}{2}\Big(\mathscr{C}+\widetilde{f}_1+\widetilde{f}_2+2\widetilde{g}_1+2\widetilde{g}_2\Big) & \text{if } 2-u \leqslant v \leqslant \frac{6-3u}{2}, \end{cases}$$

and

$$N(u,v) \sim_{\mathbb{R}} \begin{cases} 0 & \text{if } 0 \leqslant v \leqslant 2-u, \\ (v+u-2)\Big(\widetilde{f}_1+\widetilde{f}_2+2\widetilde{g}_1+2\widetilde{g}_2\Big) & \text{if } 2-u \leqslant v \leqslant \frac{6-3u}{2}. \end{cases}$$

Therefore, if $\mathscr{C} = \widetilde{C}'$, then we compute $S(W^S_{\bullet,\bullet};\mathscr{C})$ as follows:

$$\frac{1}{10}\int_1^2 (5u^2-20u+20)(u-1)du + \frac{1}{10}\int_0^1\int_0^1 (2-u)(6-u-4v)dvdu + \frac{1}{10}\int_0^1\int_1^{\frac{4-u}{2}} (4-u-2v)^2 dvdu +$$
$$+\frac{1}{10}\int_1^2\int_0^{2-u} (2-u)(10-5u-4v)dvdu + \frac{1}{10}\int_1^2\int_{2-u}^{\frac{6-3u}{2}} (6-3u-2v)^2 dvdu = \frac{43}{60},$$

If $\widetilde{Z} \neq \widetilde{C}'$, similar computations give $S(W^S_{\bullet,\bullet};\mathscr{C}) = \frac{27}{40}$, since $\widetilde{Z} \not\subset \mathrm{Supp}(N(u))$ for $u \in [0,2]$. Hence, it follows from (4.3) that $\beta(F) > 0$, because $S_Y(S) = \frac{17}{30} < 1$. $\square$

The proof of Lemma 4.2.14 implies the following result:

**Lemma 4.2.15.** *Suppose that $\pi(Z) = O'$. Then $\beta(F) > 0$.*

*Proof.* Let us use all assumptions and notations introduced in the proof of Lemma 4.2.14 with one exception: now we let $\mathscr{C} = \widetilde{l}_1$. Set $P = \widetilde{l}_1 \cap \widetilde{l}_2$. Then $P = \widetilde{Z}$ and $\gamma \circ \phi(P) = O'$.

Since $\widetilde{C}' \sim \widetilde{l}_1 + \widetilde{l}_2$ and $\widetilde{l}_1 + \widetilde{f}_1 + 2\widetilde{g}_1 \sim \widetilde{l}_2 + \widetilde{f}_2 + 2\widetilde{g}_2$, it follows from (4.5) that

$$P(u)\big|_S - v\mathscr{C} \sim_{\mathbb{R}} \begin{cases} (3-u-v)\mathscr{C} + \widetilde{l}_2 + (2-u)\widetilde{f}_1 + (4-2u)\widetilde{g}_1 & \text{if } 0 \leqslant u \leqslant 1, \\ (4-2u-v)\mathscr{C} + (2-u)\widetilde{l}_2 + (2-u)\widetilde{f}_1 + (4-2u)\widetilde{g}_1 & \text{if } 1 \leqslant u \leqslant 2. \end{cases}$$

If $u \in [0,1]$, then $\tau(u) = 3-u$. If $u \in [1,2]$, then $\tau(u) = 4-2u$. If $u \in [0,1]$, then

$$P(u,v) \sim_{\mathbb{R}} \begin{cases} (3-u-v)\mathscr{C} + \widetilde{l}_2 + (2-u)\widetilde{f}_1 + (4-2u)\widetilde{g}_1 & \text{if } 0 \leqslant v \leqslant 1, \\ (3-u-v)\Big(\mathscr{C} + \widetilde{f}_1 + 2\widetilde{g}_1\Big) + \widetilde{l}_2 & \text{if } 1 \leqslant v \leqslant 2-u, \\ (3-u-v)\Big(\mathscr{C} + \widetilde{l}_2 + \widetilde{f}_1 + 2\widetilde{g}_1\Big) & \text{if } 2-u \leqslant v \leqslant 3-u, \end{cases}$$

and

$$N(u,v) \sim_{\mathbb{R}} \begin{cases} 0 & \text{if } 0 \leqslant v \leqslant 1, \\ (v-1)\Big(\widetilde{f}_1 + 2\widetilde{g}_1\Big) & \text{if } 1 \leqslant v \leqslant 2-u, \\ (v-1)\Big(\widetilde{f}_1 + 2\widetilde{g}_1\Big) + (v+u-2)\widetilde{l}_2 & \text{if } 2-u \leqslant v \leqslant 3-u, \end{cases}$$

which gives

$$\Big(P(u,v)\Big)^2 = \begin{cases} u^2 + 2uv - v^2 - 8u - 4v + 12 & \text{if } 0 \leqslant v \leqslant 1, \\ u^2 + 2uv + v^2 - 8u - 8v + 14 & \text{if } 1 \leqslant v \leqslant 2-u, \\ 2(3-u-v)^2 & \text{if } 2-u \leqslant v \leqslant 3-u, \end{cases}$$

and

$$P(u,v)\cdot\mathscr{C}=\begin{cases}2-u+v & \text{if } 0\leqslant v\leqslant 1,\\ 4-u-v & \text{if } 1\leqslant v\leqslant 2-u,\\ 6-2u-2v & \text{if } 2-u\leqslant v\leqslant 3-u.\end{cases}$$

Similarly, if $u\in[1,2]$, then

$$P(u,v)\sim_{\mathbb{R}}\begin{cases}(4-2u-v)\mathscr{C}+(2-u)\widetilde{l}_2 & \\ \quad +(2-u)\widetilde{f}_1+(4-2u)\widetilde{g}_1 & \text{if } 0\leqslant v\leqslant 2-u,\\ (4-2u-v)\Big(\mathscr{C}+\widetilde{l}_2+\widetilde{f}_1+2\widetilde{g}_1\Big) & \text{if } 2-u\leqslant v\leqslant 4-2u,\end{cases}$$

and

$$N(u,v)\sim_{\mathbb{R}}\begin{cases}0 & \text{if } 0\leqslant v\leqslant 2-u,\\ (v+u-2)\Big(\widetilde{l}_2+\widetilde{f}_1+2\widetilde{g}_1\Big) & \text{if } 2-u\leqslant v\leqslant 4-2u,\end{cases}$$

which implies that

$$\Big(P(u,v)\Big)^2=\begin{cases}5u^2+2uv-v^2-20u-4v+20 & \text{if } 0\leqslant v\leqslant 2-u,\\ 2(4-2u-v)^2 & \text{if } 2-u\leqslant v\leqslant 4-2u,\end{cases}$$

and

$$P(u,v)\cdot\mathscr{C}=\begin{cases}2-u+v & \text{if } 0\leqslant v\leqslant 2-u,\\ 8-4u-2v & \text{if } 2-u\leqslant v\leqslant 4-2u.\end{cases}$$

Observe that $d(u)=0$ for $u\in[0,2]$, since $\mathscr{C}\not\subset \mathrm{Supp}(N(u))$. Therefore, integrating, we get $S(W^S_{\bullet,\bullet};\mathscr{C})=1$. Similarly, we compute

$$F_P\Big(W^{S,\mathscr{C}}_{\bullet,\bullet,\bullet}\Big)=\frac{1}{5}\int\limits_0^1\int\limits_{2-u}^{3-u}(v+u-2)\Big(P(u,v)\cdot\mathscr{C}\Big)dvdu+$$
$$+\frac{1}{5}\int\limits_1^2\int\limits_{2-u}^{4-2u}(v+u-2)\Big(P(u,v)\cdot\mathscr{C}\Big)dvdu=\frac{1}{12}$$

and $S(W^{S,\mathscr{C}}_{\bullet,\bullet,\bullet};P) = 1$. Thus, it follows from Remark 4.2.5 that $\beta(F) > 0$, since $S_Y(S) < 1$. □

Finally, we prove the following result:

**Lemma 4.2.16.** *Suppose that $\overline{Z}$ be a $G$-invariant irreducible curve in $\overline{E}$. Then $\beta(F) > 0$.*

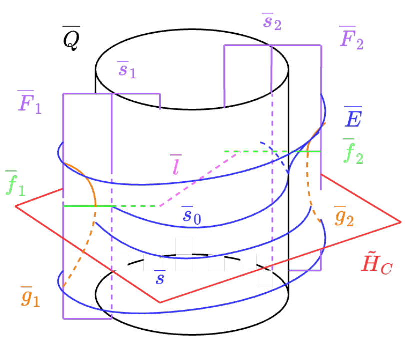
*Proof.* We have $\sigma(\overline{E}) \cong \mathbb{F}_2$, see the proof of Lemma 4.2.8. Set $\overline{s}_0 = \overline{Q} \cap \overline{E}$ and $\overline{s} = \overline{H}_C \cap \overline{E}$. Then $\sigma(\overline{s})$ is the unique $(-2)$-curve in $\sigma(\overline{E})$, and $\sigma(\overline{s}_0)$ is a section of the projection $\sigma(\overline{E}) \to C$ disjoint from $\sigma(\overline{s})$. The morphism $\sigma$ induces a birational map $\xi\colon \overline{E} \to \sigma(\overline{E})$ that blows up two points in $\sigma(\overline{s}_0)$.

Set $\overline{f}_1 = \overline{F}_1 \cap \overline{E}$ and $\overline{f}_2 = \overline{F}_2 \cap \overline{E}$. Observe that $\overline{f}_1$ and $\overline{f}_2$ are the $\xi$-exceptional curves. Let $\overline{g}_1$ and $\overline{g}_2$ be the proper transforms on $\overline{E}$ of the fibres of the projection $\sigma(\overline{E}) \to C$ that pass through $\xi(\overline{f}_1)$ and $\xi(\overline{f}_2)$, respectively. The curves $\overline{f}_1$, $\overline{f}_2$, $\overline{g}_1$, $\overline{g}_2$ are $(-1)$-curves, and the curves $\overline{s}$, $\overline{f}_1$, $\overline{f}_2$, $\overline{g}_1$, $\overline{g}_2$ generates the Mori cone $\overline{\mathrm{NE}}(\overline{E})$. Note that $\overline{s}_0$ is a $(0)$-curve.

The curves $\overline{s}$ and $\overline{s}_0$ are the only $G$-invariant irreducible curves in $\overline{E}$ by Lemma 4.2.8. Moreover, if $\overline{Z} = \overline{s}$, then $\beta(F) > 0$ by Lemma 4.2.12. Thus, we may assume that $\overline{Z} = \overline{s}_0$.

Look at Figure 4.2 to have a rough idea of what $\overline{X}$ looks like.

Figure 4.2: $\overline{X}$ model Family 3



Let $Y = \overline{X}$, $S = \overline{E}$ and $\mathscr{C} = \overline{s}_0$. Then it follows from the proof of Lemma 4.2.11 that

$$P(u)\Big|_S - v\mathscr{C} \sim_{\mathbb{R}} \begin{cases} (1+u-v)\mathscr{C} + (2-u)(\overline{f}_1 + \overline{f}_2) & \\ \quad +(2-2u)(\overline{g}_1 + \overline{g}_2) & \text{if } 0 \leqslant u \leqslant 1, \\ (4-2u-v)\mathscr{C} + (2-u)(\overline{f}_1 + \overline{f}_2) & \text{if } 1 \leqslant u \leqslant 2. \end{cases}$$

Moreover, if $u \in [0,1]$, then $\tau(u) = 1+u$ and $N(u)|_S = 0$. Furthermore, if $u \in [1,2]$, then $\tau(u) = 4-2u$ and $N(u)|_S = (u-1)\mathscr{C} + 2(u-1)\overline{s}$. If $u \in [0,1]$, then

$$P(u,v) \sim_{\mathbb{R}} \begin{cases} (1+u-v)\mathscr{C} + (2-u)(\overline{f}_1 + \overline{f}_2) & \\ \quad +(2-2u)(\overline{g}_1 + \overline{g}_2) & \text{if } 0 \leqslant v \leqslant 1, \\ (1+u-v)\mathscr{C} + (3-u-v)(\overline{f}_1 + \overline{f}_2) & \\ \quad +(2-2u)(\overline{g}_1 + \overline{g}_2) & \text{if } 1 \leqslant v \leqslant 1+u, \end{cases}$$

and

$$N(u,v) \sim_{\mathbb{R}} \begin{cases} 0 & \text{if } 0 \leqslant v \leqslant u, \\ (v-1)(\overline{f}_1 + \overline{f}_2) & \text{if } u \leqslant v \leqslant 1+u. \end{cases}$$

Similarly, if $u \in [1,2]$, then

$$P(u,v) \sim_{\mathbb{R}} \begin{cases} (4-2u-v)\mathscr{C} + (2-u)(\overline{f}_1 + \overline{f}_2) & \text{if } 0 \leqslant v \leqslant 2-u, \\ (4-2u-v)\Big(\mathscr{C} + \overline{f}_1 + \overline{f}_2\Big) & \text{if } 2-u \leqslant v \leqslant 4-2u, \end{cases}$$

and

$$N(u,v) \sim_{\mathbb{R}} \begin{cases} 0 & \text{if } 0 \leqslant v \leqslant 2-u, \\ (v+u-2)(\overline{f}_1 + \overline{f}_2) & \text{if } 2-u \leqslant v \leqslant 4-2u. \end{cases}$$

Now, we compute $S(W^S_{\bullet,\bullet};\mathscr{C})$ as follows:

$$\frac{1}{10}\int\limits_1^2 (u-1)(6u^2-24u+24)du + \frac{1}{10}\int\limits_0^1\int\limits_0^1 (8-6u^2+4uv+4u-8v)dvdu+$$
$$+\frac{1}{10}\int\limits_0^1\int\limits_1^{u+1} 2(5-3u-v)(u+1-v)dvdu+\frac{1}{10}\int\limits_1^2\int\limits_0^{2-u}(6u^2+4uv-24u-8v+24)dvdu+$$
$$+\frac{1}{10}\int\limits_1^2\int\limits_{2-u}^{4-2u} 2(4-2u-v)^2 dvdu = \frac{43}{60}.$$

But $S_Y(S) = \frac{19}{30}$, see the proof of Lemma 4.2.11. Then (4.3) gives $\beta(F) > 0$. $\square$

By Lemmas 4.2.6, 4.2.7, 4.2.11, 4.2.12, 4.2.13, 4.2.14, 4.2.15, 4.2.16, we have $\beta(F) > 0$ in all possible cases except maybe when $\pi(Z) = l'$. But in this case, we have $O \in \pi(Z)$, so $\beta(F) > 0$ by Lemma 4.2.13. Thus, we conclude that $X$ is K-polystable by Theorem 2.2.12.

### 4.2.5 K-moduli component

It is a direct consequence of the following Corollary that the one-dimensional component of $M_3^{\mathrm{Kps}}$ formed by the K-polystable elements of Family 3 is isomorphic to $\mathbb{P}^1$.

**Corollary 4.2.17.** *The Fano 3-fold $X_\infty$ in Family 3 is the only singular K-polystable limit of members of the deformation family 2.22.*

*Proof.* Denote by $M_{2.22}^{\mathrm{Kps}}$ the one-dimensional component of the K-moduli space $M_3^{\mathrm{Kps}}$ that contains all smooth K-polystable Fano 3-folds in Family 3 (equivalently, all K-polystable elements of Mori-Mukai family №2.22). Above, we described a parametrisation $\left\{X_\lambda; \lambda \in \mathbb{P}^1\right\}$ that is a $\mathbb{Q}$-Gorenstein family, and such that all smooth members of Family 3 are fibres of the family $X_\lambda$ for $\lambda \in \mathbb{P}^1 \setminus \{\pm 3, \pm 1, \infty\}$. Note that $X_\lambda \cong X_{-\lambda}$ for $\lambda \in \mathbb{P}^1$.

Moreover, it follows from the description of the Family 3 and the section 4.2.4 that all objects $X_\lambda$ in the parametrisation except for the 3-folds $X_{\pm 3}$ and $X_{\pm 1}$ are K-polystable. As mentioned already, the 3-folds $X_{\pm 3}$ and $X_{\pm 1}$ are K-semistable, and their K-polystable limits are $X_0$ and $X_\infty$, respectively. Thus we have a morphism $\mathbb{P}^1 \to \mathcal{M}^{\mathrm{Kss}}_{2.22}$, the moduli stack parametrising K-semistable objects in this family, which descends to a morphism $\phi\colon \mathbb{P}^1 \to M^{\mathrm{Kps}}_{2.22}$ given by $\lambda \mapsto [X_\lambda]$ such than $\phi(0) = \phi(\pm 3)$, $\phi(\infty) = \phi(\pm 1)$, and $\phi(\lambda) = \phi(-\lambda)$ for $\lambda \in \mathbb{P}^1$. Since $M^{\mathrm{Kps}}_{2.22}$ is proper and one-dimensional, we conclude that $\phi$ is surjective, which implies the required assertion. □

## 4.3 Family 4

Blowups of $\mathbb{P}^3$ along the disjoint union of a twisted cubic and a line. (№3.12 in Mori-Mukai notation)

### 4.3.1 Parametrisation of the family

In the notation of Family 3, we identify $\mathbb{P}^1 \times \mathbb{P}^1$ and $C_\lambda$ with subvarieties of $\mathbb{P}^1 \times \mathbb{P}^2$ via the embedding

$$\big([u:v],[x:y]\big) \mapsto \big([u:v],[x^2:xy:y^2]\big).$$

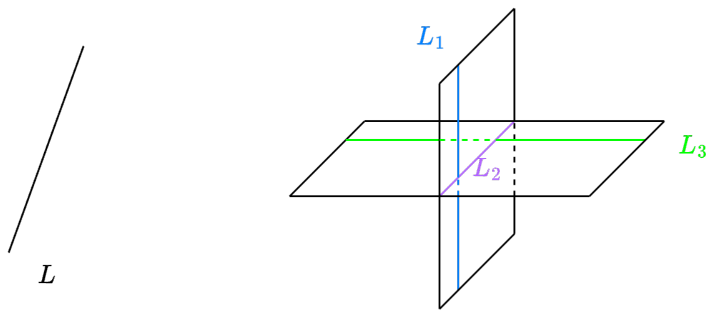
Let $\pi\colon X_\lambda \to \mathbb{P}^1 \times \mathbb{P}^2$ be the blowup along the curve $C_\lambda$, then $X_\lambda$ is a smoothable Fano 3-fold. Further, every (smooth) member of family №3.12 is isomorphic to $X_\lambda$ for some $\lambda \in \mathbb{P}^1 \setminus \{\pm 1, \infty\}$. Moreover, if $\lambda \notin \{\pm 3, \pm 1, \infty\}$, then $X_\lambda$ is K-polystable [Den22]. The smooth Fano 3-fold $X_{\pm 3}$ is strictly K-semistable, with K-polystable limit $X_0$, this proof is identical to the one in Lemma 4.2.1. Since the (singular) 3-fold $X_{\pm 1}$ admits an isotrivial degeneration to $X_\infty$ (proved as in Lemma 4.2.2), if $X_\infty$ is K-polystable, $X_{\pm 1}$ is strictly K-semistable.

### 4.3.2 Geometry of $X_\infty$

In this subsection, we construct a different model of $X_\infty$ from $\mathbb{P}^3$ which is useful when studying its K-stability, let us denote it $X$.

Consider the following lines in $\mathbb{P}^3$: $L = \{x_0 = 0, x_3 = 0\}$, $L_1 = \{x_1 = x_0, x_2 = 0\}$, $L_2 = \{x_1 = 0, x_2 = 0\}$, $L_3 = \{x_2 = x_3, x_1 = 0\}$, where $x_0$, $x_1$, $x_2$, $x_3$ are coordinates on $\mathbb{P}^3$. Then $L$ is disjoint from $L_1$, $L_2$, $L_3$, the lines $L_1$ and $L_3$ are

Figure 4.3: Model in $\mathbb{P}^3$ of $X$ Family 4




disjoint, $L_2 \cap L_1 = [0:0:0:1]$ and $L_2 \cap L_3 = [1:0:0:0]$. Let $\pi\colon X \to \mathbb{P}^3$ be the blowup of the curve $L + L_1 + L_2 + L_3$, and it has two singular points, which are ordinary double points that appear with the blowup of the points $L_2 \cap L_1$ and $L_2 \cap L_3$.

**Lemma 4.3.1.** *The 3-fold $X$ is isomorphic to the 3-fold $X_\infty$ described in the parametrisation §4.3.1.*

*Proof.* Let $\chi\colon \mathbb{P}^3 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^2$ be the dominant rational map given by

$$[x_0 : x_1 : x_2 : x_3] \mapsto \Big([x_0 : x_3], [x_1(x_0 - x_1) : x_1x_2 : x_2(x_3 - x_2)]\Big).$$

Then $\chi$ is undefined along $L \cup L_1 \cup L_2 \cup L_3$, $\pi$ resolves the indeterminacy of $\chi$, and there exists a birational morphism $\eta\colon X \to \mathbb{P}^1 \times \mathbb{P}^2$ that fits in the following commutative diagram:

$$\begin{array}{ccc} & X & \\ {\scriptstyle\pi}\swarrow & & \searrow{\scriptstyle\eta} \\ \mathbb{P}^3 & \overset{\chi}{\dashrightarrow} & \mathbb{P}^1 \times \mathbb{P}^2. \end{array} \tag{4.6}$$

To describe $\eta$, set $H_{12} = \{x_2 = 0\}$ and $H_{23} = \{x_1 = 0\}$, and denote

$$Q = \Big\{x_0x_2 + x_1x_3 - x_0x_3 = 0\Big\}.$$

Then $H_{12}$ is the plane containing the lines $L_1$ and $L_2$, $H_{23}$ is the plane containing $L_2$ and $L_3$, and $Q$ is the unique smooth quadric in $\mathbb{P}^3$ that contains $L$, $L_1$, and $L_3$. Further, $\chi(H_{12})$ is the curve $\mathbb{P}^1 \times \{[1:0:0]\}$, $\chi(H_{23}) = \mathbb{P}^1 \times \{[0:0:1]\}$, and $\chi(Q)$ is the curve parametrised as $([u:v],[u^2:uv:v^2])$, where $[u:v] \in \mathbb{P}^1$. Therefore, we see that $\eta$ contracts the proper transforms of the surfaces $H_{12}$, $H_{23}$, $Q$ to the curves $\chi(H_{12})$, $\chi(H_{23})$, $\chi(Q)$, respectively. Moreover, we have that $C_\infty = \chi(H_{12}) \cup \chi(H_{23}) \cup \chi(Q)$. Note that these curves are contained in the preimage of the smooth conic $\{xz = y^2\}$ via the projection $\mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^2$, where $[x:y:z]$ are coordinates on $\mathbb{P}^2$. Observe also that

$$\chi(Q) = \Big\{uy = vx, vy = uz\Big\} \subset \mathbb{P}^1 \times \mathbb{P}^2.$$

Finally, note that $\chi(H_{12})$ and $\chi(H_{23})$ are the fibres of the projection $\mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^2$ over the points $[1:0:0]$ and $[0:0:1]$, respectively. Hence, $X \cong X_\infty$. □

**Description of the automorphism group** Let us take the following automorphisms in $\mathbb{P}^3$:

$$\begin{cases} \tau : [x_0 : x_1 : x_2 : x_3] \mapsto [x_3 : x_2 : x_1 : x_0], \\ \Gamma := \{\, [x_0 : x_1 : x_2 : x_3] \mapsto [\lambda x_0 : \lambda x_1 : x_2 : x_3] \,|\, \lambda \in \mathbb{C}^* \} \,. \end{cases}$$

Set $G = \langle \tau, \Gamma \rangle$. Then $\Gamma \cong \mathbb{C}^*$ and $G \cong \mathbb{C}^* \rtimes \mu_2$. Since $L + L_1 + L_2 + L_3$ is $G$-invariant, the action of the group $G$ lifts to $X$. Hence, we can identify $G$ with a subgroup in $\mathrm{Aut}(X)$. One can check that $\mathrm{Aut}(X) = G$. Moreover, (4.6) is $G$-equivariant.

**Description of the $G$-invariant loci** Consider the smooth quadric surface $R = \{x_0x_2 = x_1x_3\}$.

**Lemma 4.3.2.** *There are no $G$-fixed point or $G$-invariant plane in $\mathbb{P}^3$. If $S$ is a $G$-invariant irreducible quadric surface in $\mathbb{P}^3$, then $S = R$ or*

$$S = \{ax_0x_3 + bx_1x_2 + c(x_0x_2 + x_1x_3) = 0\}$$

*for some $[a : b : c] \in \mathbb{P}^2$ such that $ab \neq c^2$.*

*Proof.* Taking the $\mathbb{C}^*$ action, $\Gamma$, on $\mathbb{P}^3$, we get that any $\Gamma$-invariant point must have $x_0 = x_1 = 0$. And a point of the form $[0 : 0 : x_2 : x_3]$ to be $\tau$ invariant, we get $x_2 = x_3 = 0$, and this point is not in $\mathbb{P}^3$.

Now, let $S = \left\{\sum_{0\leq i\leq j\leq 3} a_{ij}x_ix_j = 0\right\}$ be a irreducible quadric surface in $\mathbb{P}^3$. If $S$ is $\Gamma$-invariant we have 3 posiblities:

(1) $S = \{a_{22}x_2^2 + a_{23}x_2x_3 + a_{33}x_3^2 = 0\}$, but this is not $\tau$-invariant.

(2) $S = \{a_{00}x_0^2 + a_{01}x_0x_1 + a_{11}x_1^2 = 0\}$, it is not $\tau$-invariant.

(3) $S = \{a_{02}x_0x_2 + a_{03}x_0x_3 + a_{12}x_1x_2 + a_{13}x_1x_3 = 0\}$.

In (3), if $a_{03} = a_{12} = 0$, $S$ is $\tau$-invariant if $a_{02} = \pm a_{13}$. Otherwise, it is $\tau$-invariant if $a_{02} = a_{13}$. For $S$ to be irreducible, notice that $a_{03}a_{12} \neq a_{02}^2$ and we get to the conclusion. □

For $a \in \mathbb{C} \cup \{\infty\}$, set $l_a = \{x_0 = ax_1, x_3 = ax_2\} \subset \mathbb{P}^3$; then $l_a$ is a $G$-invariant line lying on $R$. Note that $l_0 = L$ and $l_\infty = L_2$.

**Lemma 4.3.3.** *If $C \subset \mathbb{P}^3$ is a $G$-invariant irreducible curve, then $C = l_a$ for some $a \in \mathbb{C} \cup \{\infty\}$.*

*Proof.* See the proof of Lemma 4.2.7. □

Denote by $\gamma\colon V \to \mathbb{P}^3$ the blowup of the lines $L$, $L_1$, $L_3$, and by $\phi\colon \widetilde{X} \to V$ the blowup of the proper transform of the line $L_2$.

Let $\varphi\colon W \to \mathbb{P}^3$ be the blowup of the lines $L$ and $L_2$, and $\sigma\colon \overline{X} \to W$ the blowup of the proper transform of the disjoint lines $L_1$ and $L_3$. Then we have a $G$-equivariant commutative diagram:

$$\begin{array}{ccccc} \widetilde{X} & \longrightarrow & X & \longleftarrow & \overline{X} \\ \phi \downarrow & & \pi \downarrow & & \downarrow \sigma \\ V & \xrightarrow{\gamma} & \mathbb{P}^3 & \xleftarrow{\varphi} & W \end{array} \tag{4.7}$$

where $\widetilde{X} \to X$ and $\overline{X} \to X$ are $G$-equivariant small resolutions of the 3-fold $X$.

Let $E_L$, $E_1$, $E_2$, $E_3$ be the $\pi$-exceptional divisors that are mapped to $L$, $L_1$, $L_2$, $L_3$, respectively, let $H_L$ be a general plane in $\mathbb{P}^3$ that contains $L$, let $H_2$ be a general plane in $\mathbb{P}^3$ that contains $L_2$, let $H$ be a general plane in $\mathbb{P}^3$.

Let $\overline{E}_L$, $\overline{E}_1$, $\overline{E}_2$, $\overline{E}_3$, $\overline{Q}$, $\overline{R}$, $\overline{H}_{12}$, $\overline{H}_{23}$, $\overline{H}_L$, $\overline{H}_2$, $\overline{H}$ be the proper transforms on $\overline{X}$ of the surfaces $E_L$, $E_1$, $E_2$, $E_3$, $Q$, $R$, $H_{12}$, $H_{23}$, $H_L$, $H_2$, $H$, respectively.

**Remark 4.3.4.** Note that $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$ via the isomorphism given by

$$\big([u:v],[x:y]\big) \to \Big[ux : \frac{1}{2}u(x+y) : \frac{1}{2}v(x-y) : vx\Big]$$

where $([u:v],[x:y])$ are the coordinates on $\mathbb{P}^1 \times \mathbb{P}^1$. To get $\sigma(\overline{Q})$ we blow up of the points $\{([0:1],[1:1]),([1:0],[1:-1])\} = Q \cap L_2$. Hence, $\overline{Q}$ is a del Pezzo surface of degree 6.

Then $\overline{H}$, $\overline{E}_L$, $\overline{E}_1$, $\overline{E}_2$, $\overline{E}_3$ generate $\mathrm{Pic}(\overline{X})$, and their intersections can be described as follows: $\overline{H}^3 = 1$, $\overline{E}_L^3 = \overline{E}_2^3 = -2$, $\overline{E}_1^3 = \overline{E}_3^3 = -1$, $\overline{E}_L^2 \cdot \overline{H} = \overline{E}_1^2 \cdot \overline{H} = \overline{E}_2^2 \cdot \overline{H} = \overline{E}_3^2 \cdot \overline{H} = \overline{E}_2 \cdot \overline{E}_3^2 = \overline{E}_2 \cdot \overline{E}_1^2 = -1$, and other triple intersections are zero. Note that $-K_{\overline{X}} \sim 4\overline{H} - \overline{E}_L - \overline{E}_1 - \overline{E}_2 - \overline{E}_3$ and

$$\overline{Q} \sim 2\overline{H} - \overline{E}_L - \overline{E}_1 - \overline{E}_3, \qquad \overline{H}_{12} \sim \overline{H} - \overline{E}_1 - \overline{E}_2, \qquad \overline{H}_L \sim \overline{H} - \overline{E}_L,$$
$$\overline{R} \sim 2\overline{H} - \overline{E}_L - \overline{E}_2, \qquad \overline{H}_{23} \sim \overline{H} - \overline{E}_2 - \overline{E}_3, \qquad \overline{H}_2 \sim \overline{H} - \overline{E}_L.$$

Note also that $\overline{E}_L$, $\overline{E}_2$, $\overline{E}_1+\overline{E}_3$, $\overline{Q}$, $\overline{R}$, $\overline{H}_{12}+\overline{H}_{23}$ are $G$-invariant and $G$-irreducible.

**Lemma 4.3.5.** *Let $S$ be a $G$-invariant prime divisor in $\overline{X}$. If $S \neq \overline{E}_L$, then*

$$S \sim a_1\overline{E}_2 + a_2\overline{Q} + a_3\overline{H}_L + a_4\overline{H}_2 + a_5(\overline{H}_{12} + \overline{H}_{23})$$

*for some non-negative integers $a_1$, $a_2$, $a_3$, $a_4$, $a_5$.*

*Proof.* We may assume that $S \neq \overline{E}_L$, $S \neq \overline{E}_2$ and $S \neq \overline{Q}$. Then $\pi(S)$ is a $G$-invariant surface in $\mathbb{P}^3$ of degree $d \geqslant 1$, so that $S \sim d\overline{H} - m\overline{E}_L - r(\overline{E}_1+\overline{E}_3) - s\overline{E}_2$ for some non-negative integers $m$, $r$, $s$.

Let $\ell$ be a general ruling of the quadric $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$ that intersects $L$, $L_1$, $L_3$, let $\overline{\ell}$ be its proper transform on $\overline{X}$. Then $\overline{\ell} \not\subset S$, which gives $0 \leqslant S \cdot \overline{\ell} = d - m - 2r$. Similarly, let $\ell_{12}$ be a general line in the plane $H_{12}$ that passes through the point $H_{12} \cap L$, and let $\overline{\ell}_{12}$ be its proper transform on $\overline{X}$. Then $\overline{\ell}_{12} \not\subset S$, which gives $0 \leqslant S \cdot \overline{\ell}_{12} = d - m - r - s$. Thus, if $m \geqslant r$, we can let $a_1 = d - m - r - s$, $a_2 = r$, $a_3 = m - r$, $a_4 = d - m - r$, $a_5 = 0$. If $m < r$, we can let $a_1 = d - 2m - s$, $a_2 = m$, $a_3 = 0$, $a_4 = d - 2r$, $a_5 = r - m$. ◻

**Lemma 4.3.6.** *Let $S$ be a $G$-invariant prime divisor in $\overline{X}$ such that $-K_{\overline{X}} \sim_{\mathbb{Q}} \lambda S + \Delta$ for some positive rational number $\lambda > \frac{4}{3}$ and some effective $\mathbb{Q}$-divisor $\Delta$ on the 3-fold $\overline{X}$. Then $S = \overline{E}_2$, $S = \overline{E}_L$ or $S = \overline{Q}$.*

*Proof.* Suppose that $S \neq \overline{E}_2$ and $S \neq \overline{E}_L$. Let us show that $S = \overline{Q}$. Since

$S \neq \overline{E}_1 + \overline{E}_3$, we see that $\pi(S)$ is a $G$-invariant irreducible surface of degree $d \geqslant 2$, because $\mathbb{P}^3$ does not contain $G$-invariant planes by Lemma 4.3.2. Then $S \sim d\overline{H} - m\overline{E}_L - r(\overline{E}_1 + \overline{E}_3) - s\overline{E}_2$ for some non-negative integers $m$, $r$, $s$. Then $4 \geqslant \lambda d > \frac{4}{3}d$, so that $d = 2$ and

$$\Delta \sim_{\mathbb{Q}} (4 - 2\lambda)\overline{H} + (m\lambda - 1)\overline{E}_L + (s\lambda - 1)\overline{E}_2 + (r\lambda - 1)(\overline{E}_1 + \overline{E}_3).$$

Let $\ell$ be a general line in $\mathbb{P}^3$ that intersects the lines $L_1$ and $L_2$, and let $\overline{\ell}$ be its proper transform on the 3-fold $\overline{X}$. Then $\overline{\ell} \not\subset \mathrm{Supp}(\Delta)$, so that $0 \leqslant \Delta \cdot \overline{\ell} = 2 - 2\lambda + r\lambda$, which implies that $r \neq 0$. Similarly, intersecting $\Delta$ with the proper transform of a general line in $\mathbb{P}^3$ that intersects $L$ and $L_2$, we see that $(m, s) \neq (0, 0)$.

Since $r \neq 0$, the quadric $\pi(S)$ contains $L_1$ and $L_3$. Hence, using Lemma 4.3.2, we get

$$\pi(S) = \Big\{ax_0x_3 + bx_1x_2 - a(x_0x_2 + x_1x_3) = 0\Big\}$$

for some $[a : b] \in \mathbb{P}^1$ such that $[a : b] \neq [0 : 1]$ and $[a : b] \neq [1 : 1]$. This gives $L_2 \not\subset \pi(S)$, so that $s = 0$. Then $m \neq 0$, so that $L \subset \pi(S)$. Then $[a : b] = [1 : 0]$ and $\overline{S} = \overline{Q}$. □

### 4.3.3 K-polystability of $X$

Here we prove that $X$ is K-polystable. Let $F$ be a $G$-invariant prime divisor over $X$, let $Z$ and $\overline{Z}$ be its centres on $X$ and $Y = \overline{X}$, respectively. Then it follows from Lemmas 4.3.2 and 4.3.3 that one of the following four cases holds:

1. $Z$ is a $G$-invariant irreducible surface,
2. $Z$ is a $G$-invariant irreducible curve in the surface $E_L$,
3. $Z$ is a $G$-invariant irreducible curve in the surface $E_2$,

4. $\pi(Z) = l_a$ for some $a \in \mathbb{C}^*$.

By Theorem 2.2.12, to prove that $X$ is K-polystable, it is enough to show that $\beta(F) > 0$. We use the assumptions and notations introduced in Section 4.2.2, we first consider the case when $Z$ is a surface.

**Lemma 4.3.7.** *Let $S$ be a $G$-invariant prime divisor in $X$. Then $\beta(S) > 0$.*

*Proof.* By Remark 4.2.3 and Lemma 4.3.6, it is enough to show that $\beta(\overline{E}_2)$, $\beta(\overline{E}_L)$, $\beta(\overline{Q})$ are positive. Observe that $\beta(\overline{E}_L) > 0$ follows from the proof of [Den22, Lemma 4.2]. Nevertheless, let us compute $\beta(\overline{E}_L)$. We let $S = \overline{E}_L$. Then

$$-K_Y - uS \sim_{\mathbb{R}} 4\overline{H} - (1+u)S - \overline{E}_1 - \overline{E}_2 - \overline{E}_3 \sim_{\mathbb{R}} \left(\frac{3}{2} - u\right) S + \frac{1}{2}\left(\overline{Q} + \overline{H}_{12} + \overline{H}_{23}\right) + 2\overline{H}_L,$$

Thus, it follows from (4.6) that $\tau = \frac{3}{2}$. Recall that a divisor $D$ is nef if the intersection $D \cdot C \geq 0$ is positive for all irreducible curves $C$ in $D$. Also notice that in order to get $(-K_{\widetilde{X}} - uS) \cdot l < 0$, $l$ must to have negative self-intersection in at least in $S$, $\overline{Q}$, $\overline{H}_{12}$, $\overline{H}_{23}$, or $\overline{H}_L$. Moreover, since $S \cong \mathbb{P}^1 \times \mathbb{P}^1$, it does not have negative curves. As we mentioned in Remark 4.3.4, $\overline{Q}$ is a del Pezzo surface of degree 6, and it has 6 $(-1)$-curves, $\overline{f}_i = \overline{Q} \cap \overline{E_i}$ for $i \in \{1, 3\}$ which are strict transform of $([u : v], [1 : \pm 1])$ fibres, let $\overline{g}_1 = \overline{H}_{23} \cap \overline{Q}$ and $\overline{g}_3 = \overline{H}_{12} \cap \overline{Q}$ be the strict transform of the rulings $([0 : 1], [x : y])$ and $([1 : 0], [x : y])$, respectively; and $\overline{e}_1$ and $\overline{e}_3$ are the strict transforms of the $\varphi$-exceptional divisors in $\sigma(\overline{Q})$, i.e. $\{e_1, e_3\} = \overline{E}_2 \cap \overline{Q}$. Looking at their intersections we have: $\overline{g}_i \cdot \overline{e}_i = \overline{f}_i \cdot \overline{e}_i = 1$ for $i \in \{1, 3\}$ and the rest are 0. In $\overline{H}_{12}$, there is 3 $(-1)$-curves, $\overline{e}_3 = \overline{E}_3 \cap \overline{H}_{12}$, $e_{12} = \overline{E}_L \cap \overline{H}_{12}$ and the strict transform of the line passing trough $\varphi \circ \sigma(e_3)$ and $\varphi \circ \sigma(e_{12})$, notice this is $\overline{g}_3$. In $\overline{H}_{23}$, there is 3 $(-1)$-curves, $\overline{e}_1 = \overline{E}_1 \cap \overline{H}_{23}$, $e_{23} = \overline{E}_L \cap \overline{H}_{23}$ and the strict transform of the line passing trough $\varphi \circ \sigma(e_1)$ and $\varphi \circ \sigma(e_{23})$, notice this is $\overline{g}_1$. $H_L$ is a del Pezzo surface of degree 6, and we have 6 $(-1)$-curves, define $h_i = \overline{H}_L \cap \overline{E}_i$, and the strict transform of the lines $l_{i,j}$ which connect the points $\varphi \circ \sigma(h_i)$ and $\varphi \circ \sigma(h_j)$ $i, j \in \{1, 2, 3\}$ such that $i < j$, note that $\overline{l}_{1,3} = \overline{Q} \cap \overline{H}_L$, $\overline{l}_{1,2} = \overline{H}_{12} \cap \overline{H}_L$ and $\overline{l}_{2,3} = \overline{H}_{23} \cap \overline{H}_L$. All the intersections but the following ones are non-negative for every $u \in [0, \tau]$,

$$(-K_{\overline{X}} - uS)|_{\overline{Q}} \cdot g_i = 1 - u \geq 0 \quad \Leftrightarrow \; 1 \geq u;$$
$$(-K_{\overline{X}} - uS)|_{\overline{H}_L} \cdot \overline{l}_{i,j} = 1 - u \geq 0 \;\; \Leftrightarrow \; 1 \geq u \;\; \forall i,j \in \{1,2,3\} : i < j$$

Here we see that $-K_{\overline{X}} - uS$ is nef for $u \in [0,1]$. For $u \geq 1$, the intersections above are negative, therefore we are adding some $\lambda\overline{Q} + \lambda_1\overline{H}_{12} + \lambda_3\overline{H}_{23}$. To get the values of this coefficients we take $-K_{\overline{X}} - uS - (\lambda\overline{Q} + \lambda_1\overline{H}_{12} + \lambda_3\overline{H}_{23})$, and we do the intersections again.

$$(-K_{\overline{X}} - uS - (\lambda\overline{Q} + \lambda_1\overline{H}_{12} + \lambda_3\overline{H}_{23}))|_{\overline{Q}} \cdot g_i = 1 - u + \lambda_i \geq 0 \quad \Leftrightarrow \; \lambda_i \geq u - 1;$$
$$(-K_{\overline{X}} - uS - (\lambda\overline{Q} + \lambda_1\overline{H}_{12} + \lambda_3\overline{H}_{23}))|_{\overline{H}_L} \cdot \overline{l}_{1,3} = 1 - u + \lambda \geq 0 \;\; \Leftrightarrow \; \lambda \geq u - 1.$$

Therefore, notice we are looking for the smallest coefficients such that our divisor is nef, so we get that $\lambda = \lambda_i = u - 1$, for $i \in \{1,3\}$. Moreover,

$$P(u) \sim_{\mathbb{R}} \begin{cases} \left(\frac{3}{2} - u\right) S + \frac{1}{2}\left(\overline{Q} + \overline{H}_{12} + \overline{H}_{23}\right) + 2\overline{H}_L & \text{if } 0 \leqslant u \leqslant 1, \\ \left(\frac{3}{2} - u\right) \left(S + \overline{Q} + \overline{H}_{12} + \overline{H}_{23}\right) + 2\overline{H}_L & \text{if } 1 \leqslant u \leqslant \frac{3}{2}, \end{cases}$$

and

$$N(u) = \begin{cases} 0 & \text{if } 0 \leqslant u \leqslant 1, \\ (u-1)\left(\overline{Q} + \overline{H}_{12} + \overline{H}_{23}\right) & \text{if } 1 \leqslant u \leqslant \frac{3}{2}. \end{cases}$$

Now, by integrating $(P(u))^3$ we get $\beta(\overline{E}_L) = 1 - S_Y(\overline{E}_L) = 1 - \frac{37}{56} = \frac{19}{56}$.

Next, we deal with $\overline{Q}$. Set $S = \overline{Q}$. Since $\overline{Q} \sim 2\overline{H} - \overline{E}_1 - \overline{E}_L - \overline{E}_3$, we have

$$-K_Y - uS \sim_{\mathbb{R}} \left(\frac{3}{2} - u\right) S + \frac{1}{2}\left(\overline{E}_L + \overline{E}_1 + \overline{E}_2\right) + \frac{1}{2}\overline{H}_2,$$

so that $\tau = \frac{3}{2}$. By using a similar procedure as before we get that if $0 \leqslant u \leqslant 1$, then $N(u) = 0$. Similarly, if $1 \leqslant u \leqslant \frac{3}{2}$, then $N(u) = (u-1)(\overline{E}_L + \overline{E}_1 + \overline{E}_2)$.

Then

$$P(u) \sim_{\mathbb{R}} \begin{cases} \left(\frac{3}{2}-u\right)S+\frac{1}{2}\left(\overline{E}_L+\overline{E}_1+\overline{E}_2\right)+\frac{1}{2}\overline{H}_2 & \text{if } 0 \leqslant u \leqslant 1, \\ \left(\frac{3}{2}-u\right)\left(S+\overline{E}_L+\overline{E}_1+\overline{E}_2\right)+\frac{1}{2}\overline{H}_2 & \text{if } 1 \leqslant u \leqslant \frac{3}{2}. \end{cases}$$

Now, by integrating we get $\beta(\overline{Q}) = 1 - S_Y(\overline{Q}) = 1 - \frac{129}{224} > 0$.

Finally, we proceed to study $\overline{E}_2$. Let $S = \overline{E}_2$. Then $\tau = 2$, since

$$-K_Y - uS \sim_{\mathbb{R}} (2-u)S + \frac{3}{2}\left(\overline{H}_{12}+\overline{H}_{23}\right)+\frac{1}{2}\left(\overline{E}_1+\overline{E}_3\right).$$

Moreover, if $u \in [0,1]$, then $N(u) = 0$. If $u \in [1,2]$, then $N(u) = (u-1)(\overline{H}_{12} + \overline{H}_{23})$, so

$$P(u) \sim_{\mathbb{R}} \begin{cases} (2-u)S+\frac{3}{2}\left(\overline{H}_{12}+\overline{H}_{23}\right)+\frac{1}{2}\left(\overline{E}_1+\overline{E}_3\right) & \text{if } 0 \leqslant u \leqslant 1, \\ (2-u)S+\frac{5-u}{2}\left(\overline{H}_{12}+\overline{H}_{23}\right)+\frac{1}{2}\left(\overline{E}_1+\overline{E}_3\right) & \text{if } 1 \leqslant u \leqslant 2. \end{cases}$$

Integrating, leads to $S_Y(\overline{E}_2) = \frac{51}{56}$, so that $\beta_{\overline{X}}(\overline{E}_2) > 0$. □

Now let us assume the centre of $F$ is small on $X$ and show that $\beta(F) > 0$.

**Lemma 4.3.8** ([Den22, Lemma 4.2])**.** *Suppose that $Z$ is a curve in $E_L$. Then $\beta(F) > 0$.*

*Proof.* Note that $\overline{E}_L \cong \mathbb{P}^1 \times \mathbb{P}^1$. Let $s$ be a section of the natural projection $\overline{E}_L \to L$ such that $s^2 = 0$, and let $l$ be a fibre of this projection. Then $\overline{E}_L|_{\overline{E}_L} \sim -s + l$ and $\overline{H}|_{\overline{E}_L} \sim l$.

Set $C_Q = \overline{Q} \cap \overline{E}_L$ and $C_R = \overline{R} \cap \overline{E}_L$. Then $C_Q$ and $C_R$ are smooth irreducible $G$-invariant curves. Furthermore, these are the only $G$-invariant irreducible curves in the surface $\overline{E}_L$. Hence, we have two options either $\overline{Z} = C_Q$ or $\overline{Z} = C_R$. Also, note that $C_Q \sim C_R \sim s + l$.

Take $S = \overline{E}_L$, $\mathscr{C} = \overline{Z}$. Then $S_Y(S) = \frac{37}{56} < 1$, see the proof of Lemma 4.3.7.

Let us compute $S(W^S_{\bullet,\bullet};\mathscr{C})$. We recall from the proof of Lemma 4.3.7 that $\tau = \frac{3}{2}$ and it also follows that

$$P(u)\Big|_S - v\mathscr{C} \sim_{\mathbb{R}} \begin{cases} (1+u-v)s + (3-u-v)l & \text{if } 0 \leqslant u \leqslant 1, \\ (2-v)s + (6-4u-v)l & \text{if } 1 \leqslant u \leqslant \frac{3}{2}, \end{cases}$$

and

$$N(u)\Big|_S = \begin{cases} 0 & \text{if } 0 \leqslant u \leqslant 1, \\ (u-1)\big(C_Q + l_{12} + l_{23}\big) & \text{if } 1 \leqslant u \leqslant \frac{3}{2}, \end{cases}$$

where $l_{12} = \overline{H}_{12} \cap \overline{E}_L$ and $l_{23} = \overline{H}_{23} \cap \overline{E}_L$ are fibres of the natural projection $\overline{E}_L \to L$ over the points $L \cap H_{12}$ and $L \cap H_{13}$, respectively.

We have $P(u,v) \sim_{\mathbb{R}} P(u)|_S - v\mathscr{C}$ and $N(u,v) = 0$ for $v \in [0, \tau(u)]$, where

$$\tau(u) = \begin{cases} 1+u & \text{if } 0 \leqslant u \leqslant 1, \\ 6-4u & \text{if } 1 \leqslant u \leqslant \frac{3}{2}. \end{cases}$$

Thus, if $\mathscr{C} = C_Q$, then

$$S(W^S_{\bullet,\bullet};\mathscr{C}) = \frac{3}{28}\int\limits_1^{\frac{3}{2}} (u-1)(24-16u)dvdu +$$
$$+\frac{3}{28}\int\limits_0^1\int\limits_0^{1+u} 2(3-u-v)(1+u-v)dvdu + \frac{3}{28}\int\limits_1^{\frac{3}{2}}\int\limits_0^{6-4u} 2(2-v)(6-4u-v)dvdu = \frac{159}{224}.$$

Similarly, if $\mathscr{C} = C_R$, then $S(W^S_{\bullet,\bullet};\mathscr{C}) = \frac{151}{224}$. So, it follows from (4.3) that $\beta(F) > 0$. □

Now, we study the $G$-invariant irreducible curves on $\overline{E}_2$.

**Lemma 4.3.9.** *Suppose that $\pi(Z) = L_\infty = L_2$; then $\beta(F) > 0$.*

*Proof.* Observe that $\overline{Z} \subset \overline{E}_2$. It is clear that $\overline{Z}$ is a smooth $G$-irreducible curve, and by Lemma 4.3.7 $S_{\overline{X}}(\overline{E}_2) = \frac{51}{56}$.

As in Lemma 4.3.2, let $S = \{x_0x_2 + x_1x_3 = 0\}$, and let $\overline{S}$ be its proper transform on $\overline{X}$. Set $C_S = \overline{S}|_{\overline{E}_2}$ and $C_R = \overline{R}|_{\overline{E}_2}$. Using the map $\theta$ defined by $[x_0 : x_1 : x_2 : x_3] \mapsto ([x_0 : x_1], [x_1 : x_2])$, we can $G$-equivariantly identify $\sigma(\overline{E}_2) = \mathbb{P}^1 \times \mathbb{P}^1$ with coordinates $([x_0 : x_3], [x_1 : x_2])$ such that the involution $\tau$ acts as $([x_0 : x_3], [x_1 : x_2]) \mapsto ([x_3 : x_0], [x_2 : x_1])$, and $\Gamma \cong \mathbb{C}^*$ acts as

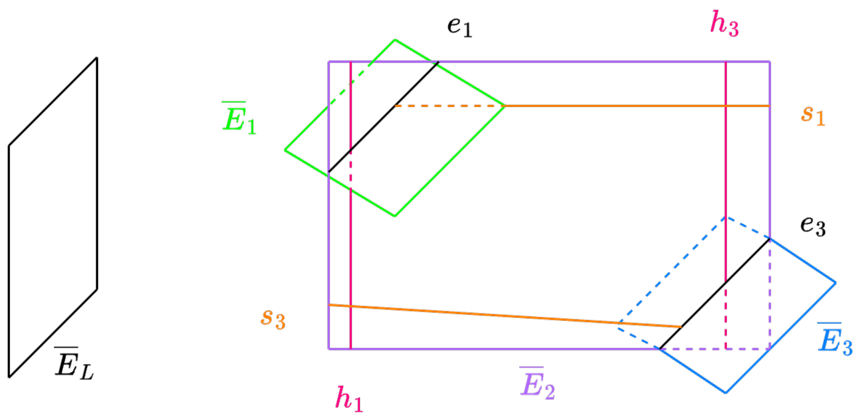
$$\big([x_0 : x_3], [x_1 : x_2]\big) \mapsto \big([\lambda x_0 : x_3], [\lambda x_1 : x_2]\big),$$

where $\lambda \in \mathbb{C}^*$. Therefore, $\sigma(C_S) = \{x_0x_2+x_1x_3 = 0\}$ and $\sigma(C_R) = \{x_0x_2-x_1x_3 = 0\}$ are the only $G$-invariant irreducible curves in the surface $\sigma(\overline{E}_2)$. Hence, $C_S$ and $C_R$ are the only $G$-invariant irreducible curves in $\overline{E}_2$, so that $\mathscr{C} = C_S$ or $\mathscr{C} = C_R$.

The morphism $\sigma$ in (4.7) induces a $G$-equivariant birational morphism $\theta\colon S \to \sigma(\overline{E}_2)$ that blows up the points $([0 : 1], [1 : 0])$ and $([1 : 0], [0 : 1])$, which are not contained in the curves $\sigma(C_S)$ and $\sigma(C_R)$. In particular, we see that $\overline{E}_2$ is a del Pezzo surface of degree 6.

Set $e_1 = \overline{E}_1|_{\overline{E}_2}$ and $e_3 = \overline{E}_3|_{\overline{E}_2}$. Then $e_1$ and $e_3$ are the $\theta$-exceptional curves such that $\theta(e_1) = ([0 : 1], [1 : 0])$ and $\theta(e_3) = ([1 : 0], [0 : 1])$. Let $s_1 = \overline{H}_{12}|_{\overline{E}_2}$ and $s_3 = \overline{H}_{23}|_{\overline{E}_2}$ be the proper transforms on $\overline{E}_2$ of the curves $\{x_2 = 0\}$ and $\{x_1 = 0\}$, and $h_1 = \overline{H}_1|_{\overline{E}_2}$ and $h_3 = \overline{H}_3|_{\overline{E}_2}$ be the proper transforms of the curves $\{x_0 = 0\}$ and $\{x_3 = 0\}$, respectively, where $\overline{H_i}$ is the strict transform of a general hyperplane containing $L_i$. Then $\theta(s_1)$ and $\theta(s_3)$ are the sections of the natural projection $\theta(\overline{E}_2) \to L_2$ that pass through the points $\theta(e_1)$ and $\theta(e_3)$, respectively, and $\theta(h_1)$ and $\theta(h_3)$ are the fibres of this projection that pass through the points $\theta(e_1)$ and $\theta(e_3)$, respectively (check Figure 4.4). Then $C_S \sim C_R \sim s_1 + h_1 + 2e_1 \sim s_3 + h_3 + 2e_3$.

Figure 4.4: Model for $\overline{X}$ Family 4




Recall that $e_1$, $e_3$, $s_1$, $s_3$, $h_1$, $h_3$ are all the $(-1)$-curves in $\overline{E}_2$, and they generate the Mori cone $\overline{\mathrm{NE}(\overline{E}_2)}$. Note that $\overline{H}|_{\overline{E}_2} \sim h_1 + e_1$ and $\overline{E}_2|_{\overline{E}_2} \sim -s_1 + h_1$. So, using the description of $P(u)$ and $N(u)$ obtained in the proof of Lemma 4.3.7, we get

$$P(u)\big|_{\overline{E}_2} \sim_{\mathbb{R}} \begin{cases} 3e_1 - e_3 + (3-u)h_1 + (u+1)s_1 & \text{if } 0 \leqslant u \leqslant 1, \\ (4-u)e_1 + (u-2)e_3 + (3-u)h_1 + (3-u)s_1 & \text{if } 1 \leqslant u \leqslant 2, \end{cases}$$

and

$$N(u)\big|_{\overline{E}_2} \sim_{\mathbb{R}} \begin{cases} 0 & \text{if } 0 \leqslant u \leqslant 1, \\ (u-1)(s_1+s_3) & \text{if } 1 \leqslant u \leqslant 2. \end{cases}$$

In particular, we see that $\mathscr{C} \not\subset \mathrm{Supp}(N(u)|_{\overline{E}_2})$ for every $u \in [0,2]$.

Now, intersecting $P(u)|_{\overline{E}_2} - v\mathscr{C}$ with $e_1$, $e_3$, $s_1$, $s_3$, $h_1$, $h_3$, we find $P(u,v)$ and $N(u,v)$ for $u \in [0,2]$ and $v \in [0,\tau(u)]$. If $u \in [0,1]$, then $\tau(u) = 1$,

$$P(u,v) \sim_{\mathbb{R}} \begin{cases} (3-2v)e_1 - e_3 + (3-u-v)h_1 + (u-v+1)s_1 & \text{if } 0 \leqslant v \leqslant u, \\ (3-2v)e_1 - e_3 + (3-2v)h_1 + (u-v+1)s_1 & \text{if } u \leqslant v \leqslant 1, \end{cases}$$

and

$$N(u,v)\sim_{\mathbb{R}}\begin{cases}0 & \text{if } 0\leqslant v\leqslant u,\\ (v-u)\big(h_1+h_3\big) & \text{if } u\leqslant v\leqslant 1.\end{cases}$$

If $u\in[1,2]$, then $\tau(u)=\frac{1}{2}$, $N(u,v)=0$ and $P(u,v)\sim_{\mathbb{R}} P(u)|_{\overline{E}_2}-v\mathscr{C}$. This gives

$$\big(P(u,v)\big)^2\sim_{\mathbb{R}}\begin{cases}4-2u^2+2v^2+4u-8v & \text{if } u\in[0,1], v\in[0,u],\\ 4(1-v)(1+u-v) & \text{if } u\in[0,1], v\in[u,1],\\ 2(1-2v)(5-2u-2v) & \text{if } u\in[1,2], v\in[0,0.5].\end{cases}$$

Now, by integrating we get $S(W^{\overline{E}_2}_{\bullet,\bullet};\mathscr{C})=\frac{9}{28}$, so that $\beta(F)>0$ by (4.3). □

Now we want to study the other $G$-invariant curves. Hence, for $a\in\mathbb{C}^*$, define $\Pi_a=\{x_0-ax_1=x_3-ax_2\}\subset\mathbb{P}^3$. Note that $l_a\subset\Pi_a$, the plane $\Pi_a$ does not contain $L$, $L_1$, $L_2$, $L_3$. Let $P_1=\Pi_a\cap L_1$, $P_2=\Pi_a\cap L_2$, $P_3=\Pi_a\cap L_3$, $P_4=\Pi_a\cap L$. Let $\overline{\Pi}_a$ be the preimage on $\overline{X}$ of the plane $\Pi_a$. Then $\varphi\circ\sigma$ in (4.7) induces a birational morphism $\overline{\Pi}_a\to\Pi_a$ that is a blowup of $P_1$, $P_2$, $P_3$, $P_4$.

**Lemma 4.3.10.** *If $a\notin\{1,2\}$, no three of $P_1,P_2,P_3$, and $P_4$ are collinear, and none of them lies on $l_a$.*
*When $a=1$, no three of $P_1,P_2,P_3$, and $P_4$ are collinear, $P_1$ and $P_3$ lie on $l_1$, but $P_2$ and $P_4$ do not. When $a=2$, then $P_1,P_3$ and $P_4$ lie on the line $\Pi_2\cap\{x_0-x_1+x_2=0\}$, but $P_2$ does not, and none of $P_1,P_2,P_3$ or $P_4$ lies on $l_2$.*

*Proof.* Note that $P_1=[1:1:0:(1-a)]$, $P_2=[1:0:0:1]$, $P_3=[(1-a):0:1:1]$ and $P_4=[0:1:1:0]$. Then, we define $\ell^a_{ij}$ to be the lines in $\Pi_a$ containing the points $P_i$ and $P_j$ as follows, $\ell^a_{12}:=\Pi_a\cap\{x_2=0\}$, $\ell^a_{13}:=\Pi_a\cap\{x_0+x_3=(2-a)(x_1+x_2)\}$, $\ell^a_{14}:=\Pi_a\cap\{x_0+x_2=x_1\}$, $\ell^a_{23}:=\Pi_a\cap\{x_1=0\}$, $\ell^a_{24}:=\Pi_a\cap\{x_0=x_3\}$ and $\ell^a_{12}:=\Pi_a\cap\{x_1+x_3=x_2\}$. If $a\neq 1,2$, the points $P_i$ and $P_j$ are the only ones contained in $\ell^a_{ij}$ for $i,j\in\{1,2,3,4\}$ and $i<j$, and none of the points lies in $l_a=\{x_0=ax_1,x_3=ax_2\}$.

When $a=1$, we still have that $P_k, P_t \notin \ell^1_{ij}$ for $\{k,t\} \neq \{i,j\}$. Moreover, $\ell^1_{13} = l_1$. For $a=2$, $P_1$, $P_3$ and $P_4$ are collinear it is straightforward to see that $P_4 \in \ell^2_{13}$, but $P_2 \notin \ell^2_{13}$, and none of the points lies in $l_2 = \{x_0 = 2x_1, x_3 = 2x_2\}$. □

Thus, if $a \neq 2$, $\overline{\Pi}_a$ is del Pezzo surface of degree 5, while if $a=2$, $\overline{\Pi}_a$ is a weak del Pezzo surface of degree 5. In both cases, we let $Y = \overline{X}$ and $S = \overline{\Pi}_a$. Then, since $\overline{\Pi}_a \sim \overline{H}$,

$$-K_Y - uS \sim_{\mathbb{R}} \left(\frac{3}{2} - u\right) S + \frac{1}{2}\Big(\overline{Q} + \overline{H}_{12} + \overline{H}_{23}\Big) + \frac{1}{2}\overline{H}_L.$$

Therefore, $\tau = \frac{3}{2}$. Moreover, by checking the intersections with all the negative curves in these surfaces we see that if $0 \leqslant u \leqslant 1$, then $N(u) = 0$. Furthermore, if $1 \leqslant u \leqslant \frac{3}{2}$, then $N(u) = (u-1)(\overline{Q} + \overline{H}_{12} + \overline{H}_{23})$. Thus, we have

$$P(u) \sim_{\mathbb{R}} \begin{cases} \left(\frac{3}{2} - u\right) S + \frac{1}{2}\Big(\overline{Q} + \overline{H}_{12} + \overline{H}_{23}\Big) + \frac{1}{2}\overline{H}_L & \text{if } 0 \leqslant u \leqslant 1, \\ \left(\frac{3}{2} - u\right)\Big(S + \overline{Q} + \overline{H}_{12} + \overline{H}_{23}\Big) + \frac{1}{2}\overline{H}_L & \text{if } 1 \leqslant u \leqslant \frac{3}{2}, \end{cases}$$

By integrating we obtain $S_Y(S) = \frac{227}{448}$.

Now, let $e_1$, $e_2$, $e_3$, $e_4$ be exceptional curves of the blowup $S \to \Pi_a$ that are mapped to the points $P_1$, $P_2$, $P_3$, $P_4$, respectively. Then, $\overline{E}_1|_S = e_1$, $\overline{E}_2|_S = e_2$, $\overline{E}_3|_S = e_3$, $\overline{E}_L|_S = e_4$. Set $h = \overline{H}|_S$. Then

$$P(u)\Big|_S \sim_{\mathbb{R}} \begin{cases} (4-u)h - e_1 - e_2 - e_3 - e_4 & \text{if } 0 \leqslant u \leqslant 1, \\ (8-5u)h - (3-2u)\Big(e_1 + e_2 + e_3\Big) - (2-u)e_4 & \text{if } 1 \leqslant u \leqslant \frac{3}{2}. \end{cases}$$

Note that $l_a \not\subset H_{12} \cup H_{23}$ for every $a \in \mathbb{C}^*$. Moreover, the only one contained in $Q$ is $l_2$.

**Lemma 4.3.11.** *Suppose that $\pi(Z) = l_a$ for $a \in \mathbb{C} \setminus \{0,1,2\}$. Then $\beta(F) > 0$.*

*Proof.* Let $\mathscr{C} = \overline{Z}$. Note that $\mathscr{C} \sim h$. As in the proof of [Den22, Lemma 4.1], we get

$$\tau(u) = \begin{cases} 2-u & \text{if } 0 \leqslant u \leqslant 1, \\ \frac{5-3u}{2} & \text{if } 1 \leqslant u \leqslant \frac{7}{5}, \\ 6-4u & \text{if } \frac{7}{5} \leqslant u \leqslant \frac{3}{2}. \end{cases}$$

Furthermore, if $0 \leqslant u \leqslant 1$, then $P(u,v) \sim_{\mathbb{R}} (4-u-v)h - e_1 - e_2 - e_3 - e_4$ for $v \in [0, 2-u]$. Similarly, if $1 \leqslant u \leqslant \frac{3}{2}$ and $0 \leqslant v \leqslant 3-2u$, then

$$P(u,v) \sim_{\mathbb{R}} (8-5u-v)h - (3-2u)\big(e_1+e_2+e_3\big) - (u-2)e_4.$$

Finally, if $1 \leqslant u \leqslant \frac{3}{2}$ and $3-2u \leqslant v \leqslant \tau(u)$, then

$$P(u,v) \sim_{\mathbb{R}} (17-11u-4v)h - (6-4u-v)\big(e_1+e_2+e_3\big) - (11-7u-3v)e_4.$$

This gives $S(W^S_{\bullet,\bullet};\mathscr{C}) = \frac{753}{1120}$, so that $\beta(F) > 0$ by (4.3), since $S_Y(S) = \frac{227}{448}$. □

**Lemma 4.3.12.** *Suppose that $\pi(Z) = l_1$. Then $\beta(F) > 0$.*

*Proof.* Let $\mathscr{C} = \overline{Z}$. Then $\mathscr{C} \sim h - e_1 - e_3$. Moreover, if $0 \leqslant u \leqslant 1$, then $\tau(u) = 3-u$. Similarly, if $1 \leqslant u \leqslant \frac{3}{2}$, then $\tau(u) = 6-4u$. Furthermore, if $0 \leqslant u \leqslant 1$, then

$$P(u,v) \sim_{\mathbb{R}} \begin{cases} (4-u-v)h + (v-1)\big(e_1+e_3\big) - e_2 - e_4 & \text{if } 0 \leqslant v \leqslant 1, \\ (4-u-v)h - e_2 - e_4 & \text{if } 1 \leqslant v \leqslant 2-u, \\ (3-2u-2v)\big(2h - e_2 - e_4\big) & \text{if } 2-u \leqslant v \leqslant 3-u. \end{cases}$$

Similarly, if $1 \leqslant u \leqslant \frac{3}{2}$ and $0 \leqslant v \leqslant 3-2u$, then

$$P(u,v) \sim_{\mathbb{R}} (8-5u-v)h - (3-2u-v)\big(e_1+e_3\big) - (3-2u)e_2 - (u-2)e_4.$$

Finally, if $1 \leqslant u \leqslant \frac{3}{2}$ and $3-2u \leqslant v \leqslant 6-4u$, then

$$P(u,v) \sim_{\mathbb{R}} (11-7u-2v)h-(6-4u-v)e_2-(5-3u-v)e_4.$$

Therefore, we have

$$\begin{aligned}S\Big(W^S_{\bullet,\bullet};\mathscr{C}\Big) &= \frac{3}{28}\int\limits_0^1\int\limits_0^1 u^2+2uv-v^2-8u-4v+12dvdu+\\ &+\frac{3}{28}\int\limits_1^{2-u}\int\limits_1^{2-u} u^2+2uv+v^2-8u-8v+14dvdu+\frac{3}{28}\int\limits_{2-u}^{3-u}\int\limits_{2-u}^{3-u}2(3-u-v)^2dvdu+\\ &+\frac{3}{28}\int\limits_1^{\frac{3}{2}}\int\limits_0^{3-2u}12u^2+2uv-v^2-40u-4v+33dvdu+\\ &+\frac{3}{28}\int\limits_1^{\frac{3}{2}}\int\limits_{3-2u}^{6-4u}2(6-4u-v)(5-3u-v)dvdu=\frac{31}{32}.\end{aligned}$$

Thus, it follows from (4.3) that $\beta(F)>0$, because $S_Y(S)=\frac{227}{448}$. □

**Lemma 4.3.13.** *Suppose that $\pi(Z)=l_2$. Then $\beta(F)>0$.*

*Proof.* Let $\overline{\ell}_{ij}$ be the proper transforms of the lines $\ell^2_{ij}$ in $\Pi_2$ for $i,j\in\{1,2,3,4\}$ such that $i<j$. Notice that $\overline{\ell}_{23}=\overline{\ell}_{14}=\overline{\ell}_{34}$, since $P_1$, $P_3$ and $P_4$ are collinear.

On $S$, we have $\overline{\ell}_{12}\sim h-e_1-e_2$, $\overline{\ell}_{23}\sim h-e_2-e_3$. $\overline{\ell}_{24}\sim h-e_2-e_4$, $\overline{\ell}_{13}\sim h-e_1-e_3-e_4$. Note that $e_1$, $e_2$, $e_3$, $e_4$, $\overline{\ell}_{12}$, $\overline{\ell}_{23}$, $\overline{\ell}_{24}$ are all $(-1)$-curves in $S$, and $\overline{\ell}_{13}$ is the unique $(-2)$-curve in the surface $S$. By [CT88, Proposition 8.5], these curves generate the Mori cone $\overline{\mathrm{NE}(S)}$.

Now, let $\mathscr{C} = \overline{Z}$. Then $\mathscr{C} \sim h$ and we can rewrite $P(u)\big|_S - vh$ as follows:

$$P(u)\Big|_S - vh \sim_{\mathbb{R}} \begin{cases} (\frac{7}{3} - u - v)h + \frac{1}{3}(\overline{\ell}_{12} + \overline{\ell}_{23} + \overline{\ell}_{24}) + \frac{2}{3}\overline{\ell}_{13} & \text{if } 0 \leqslant u \leqslant 1, \\ (\frac{10-6u}{3} - v)h + \frac{4-3u}{3}\Big(\overline{\ell}_{12} + \overline{\ell}_{13} + \overline{\ell}_{23}\Big) + \\ \qquad +(u-1)(e_1 + e_3) + \frac{4}{3}\overline{\ell}_{24} + \frac{1}{3}e_2 & \text{if } 1 \leqslant u \leqslant \frac{3}{2}, \\ (6 - 4u - v)h + \frac{(4-3u)}{2}\Big(e_1 + e_2 + e_3\Big) + \\ \qquad + \frac{(2-u)}{2}\Big(\overline{\ell}_{13} + \overline{\ell}_{24}\Big) & \text{if } \frac{4}{3} \leqslant u \leqslant \frac{3}{2}. \end{cases}$$

From here, we get that,

$$\tau(u) = \begin{cases} \frac{7-3u}{3} & \text{if } 0 \leqslant u \leqslant 1, \\ \frac{10-6u}{3} & \text{if } 1 \leqslant u \leqslant \frac{4}{3}, \\ 6 - 4u & \text{if } \frac{4}{3} \leqslant u \leqslant \frac{3}{2}. \end{cases}$$

Moreover, intersecting the divisors under consideration with the curves $e_1$, $e_2$, $e_3$, $e_4$, $\overline{\ell}_{12}$, $\overline{\ell}_{23}$, $\overline{\ell}_{24}$, $\overline{\ell}_{13}$, we see that if $0 \leqslant u \leqslant 1$, then

$$P(u,v) \sim_{\mathbb{R}} \begin{cases} (4 - u - v)h - e_1 - e_2 - e_3 - e_4 & \text{if } 0 \leqslant v \leqslant 1 - u, \\ \frac{3-u-v}{2}\Big(3h - e_1 - e_3 - e_4\Big) - e_2 & \text{if } 1 - u \leqslant v \leqslant 2 - u, \\ \frac{7-3u-3v}{2}\Big(3h - e_1 - 2e_2 - e_3 - e_4\Big) & \text{if } 2 - u \leqslant v \leqslant \frac{7-3u}{3}, \end{cases}$$

and

$$N(u,v) \sim_{\mathbb{R}} \begin{cases} 0 & \text{if } 0 \leqslant v \leqslant 1 - u, \\ \frac{v+u-1}{2}\overline{\ell}_{13} & \text{if } 1 - u \leqslant v \leqslant 2 - u, \\ \frac{v+u-1}{2}\overline{\ell}_{13} + (u + v - 2)\Big(\overline{\ell}_{12} + \overline{\ell}_{23} + \overline{\ell}_{24}\Big) & \text{if } 2 - u \leqslant v \leqslant \frac{7-3u}{3}. \end{cases}$$

Similarly, if $1 \leqslant u \leqslant \frac{4}{3}$, then $P(u,v)$ is $\mathbb{R}$-rationally equivalent to

$$\begin{cases} \frac{16-10u-3v}{2}h - \frac{6-4u-v}{2}\Big(e_1+e_3\Big) - (3-2u)e_2 - \frac{4-2u-v}{2}e_4 & \text{if } 0 \leqslant v \leqslant 3-2u, \\ \frac{22-14u-5v}{2}h - \frac{6-4u-v}{2}\Big(e_1+2e_2+e_3\Big) - \frac{10-6u-3v}{2}e_4 & \text{if } 3-2u \leqslant v \leqslant 2-u, \\ \frac{10-6u-3v}{2}\Big(h-e_1+2e_2-e_3+e_4\Big) & \text{if } 2-u \leqslant v \leqslant \frac{10-6u}{3}, \end{cases}$$

and

$$N(u,v) \sim_{\mathbb{R}} \begin{cases} \frac{v}{2}\overline{\ell}_{13} & \text{if } 0 \leqslant v \leqslant 3-2u, \\ \frac{v}{2}\overline{\ell}_{13} + (v+2u-3)\overline{\ell}_{24} & \text{if } 3-2u \leqslant v \leqslant 2-u, \\ \frac{v}{2}\overline{\ell}_{13} + (v+2u-3)\overline{\ell}_{24} + & \\ \quad +(u+v-2)\Big(\overline{\ell}_{12}+\overline{\ell}_{23}\Big) & \text{if } 2-u \leqslant v \leqslant \frac{10-6u}{3}. \end{cases}$$

Likewise, if $\frac{4}{3} \leqslant u \leqslant \frac{3}{2}$ and $0 \leqslant v \leqslant 3-2u$, then

$$P(u,v) \sim_{\mathbb{R}} \frac{16-10u-3v}{2}h - \frac{6-4u-v}{2}\Big(e_1+e_3\Big) - (3-2u)e_2 - \frac{4-2u-v}{2}e_4$$

and $N(u,v) \sim_{\mathbb{R}} \frac{v}{2}\overline{\ell}_{13}$. Finally, if $\frac{4}{3} \leqslant u \leqslant \frac{3}{2}$ and $3-2u \leqslant v \leqslant 6-4u$, then

$$P(u,v) \sim_{\mathbb{R}} \frac{22-14u-5v}{2}h - \frac{6-4u-v}{2}\Big(e_1+2e_2+e_3\Big) - \frac{10-6u-3v}{2}e_4$$

and $N(u,v) \sim_{\mathbb{R}} \frac{v}{2}\overline{\ell}_{13} + (v+2u-3)\overline{\ell}_{24}$.

If $1 \leqslant u \leqslant \frac{3}{2}$, then $\mathscr{C} \subset \mathrm{Supp}(N(u))$ and $\mathrm{ord}_{\mathscr{C}}(N(u)|_S) = (u-1)$. Thus, we have

$$
\begin{aligned}
S\Big(W^S_{\bullet,\bullet};\mathscr{C}\Big) = \frac{3}{28}\int_1^{\frac{3}{2}}(12u^2-40u+33)(u-1)du+ \\
+\frac{3}{28}\int_0^1\int_0^{1-u} u^2+2uv+v^2-8u-8v+12dvdu+ \\
+\frac{3}{28}\int_0^1\int_{1-u}^{2-u}\frac{3u^2+6uv+3v^2-18u-18v+25}{2}dvdu+ \\
+\frac{3}{28}\int_0^1\int_{2-u}^{\frac{7-3u}{3}}\frac{(7-3u-3v)^2}{2}dvdu+ \\
+\frac{3}{28}\int_1^{\frac{4}{3}}\int_0^{3-2u}\frac{24u^2+20uv+3v^2-80u-32v+66}{2}dvdu+ \\
+\frac{3}{28}\int_1^{\frac{4}{3}}\int_{3-2u}^{2-u}\frac{(14-8u-5v)(6-4u-v)}{2}dvdu+ \\
+\frac{3}{28}\int_1^{\frac{4}{3}}\int_{2-u}^{\frac{10-6u}{3}}\frac{(10-6u-3v)^2}{2}dvdu+ \\
+\frac{3}{28}\int_{\frac{4}{3}}^{\frac{3}{2}}\int_0^{3-2u}\frac{24u^2+20uv+3v^2-80u-32v+66}{2}dvdu+ \\
+\frac{3}{28}\int_{\frac{4}{3}}^{\frac{3}{2}}\int_{3-2u}^{6-4u}\frac{(14-8u-5v)(6-4u-v)}{2}dvdu.
\end{aligned}
$$

This gives $S(W^S_{\bullet,\bullet};\mathscr{C}) = \frac{2885}{4032} < 1$. Then $\beta(F) > 0$ by (4.3), since $S_Y(S) = \frac{227}{448} < 1$. $\square$

This finishes the proof that $X$ is K-polystable.

### 4.3.4 K-moduli component

It is a direct consequence of the following Corollary that the one-dimensional component of $M_3^{\mathrm{Kps}}$ formed by the K-polystable elements of Family 4 is isomorphic to $\mathbb{P}^1$.

**Corollary 4.3.14.** *The Fano 3-fold $X_\infty$ in Family 4 is the only singular K-polystable limit of members of the deformation family 3.12.*

*Proof.* Denote by $M^{\mathrm{Kps}}_{3.12}$ the one-dimensional component of the K-moduli space $M^{\mathrm{Kps}}_{3}$ that contains all smooth K-polystable Fano 3-folds in Family 4 (equivalently, all K-polystable elements of Mori-Mukai family №3.12). In §4.3.1 we descrived a parametrisation $\left\{X_\lambda | \lambda \in \mathbb{P}^1\right\}$ that is a $\mathbb{Q}$-Gorenstein family, and such that all smooth members of Family 4 are fibres of the family $X_\lambda$ for $\lambda \in \mathbb{P}^1 \setminus \{\pm 3, \pm 1, \infty\}$. Note that $X_\lambda \cong X_{-\lambda}$ for $\lambda \in \mathbb{P}^1$.

Moreover, it follows from the description of the Family and §4.3.3, where we prove the K-polystability of $X_\infty$, that all objects $X_\lambda$ in the parametrisation except for the 3-folds $X_{\pm 3}$ and $X_{\pm 1}$ are K-polystable. As mentioned already, the 3-folds $X_{\pm 3}$ and $X_{\pm 1}$ are K-semistable, and their K-polystable limits are $X_0$ and $X_\infty$, respectively. Thus we have a morphism $\mathbb{P}^1 \to \mathcal{M}^{\mathrm{Kss}}_{3.12}$, the moduli stack parametrising K-semistable objects in this family, which descends to a morphism $\phi\colon \mathbb{P}^1 \to M^{\mathrm{Kps}}_{3.12}$ given by $\lambda \mapsto [X_\lambda]$ such than $\phi(0) = \phi(\pm 3)$, $\phi(\infty) = \phi(\pm 1)$, and $\phi(\lambda) = \phi(-\lambda)$ for $\lambda \in \mathbb{P}^1$. Since $M^{\mathrm{Kps}}_{3.12}$ is proper and one-dimensional, we conclude that $\phi$ is surjective, which implies the required assertion. □

## 4.4 Family 5

The members of this family are the blowups of $V = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ along a curve of degree $(1, 1, 3)$. (№4.13 in Mori-Mukai notation).

### 4.4.1 Parametrisation of the family

Let $[x_0 : x_1] \times [y_0 : y_1] \times [z_0 : z_1]$ be the coordinates in $V$. We have three natural projections $p_i\colon V \to \mathbb{P}^1$, $i = 1, 2, 3$ which define three natural divisor classes of

$V$, namely $H_i = p_i^*(p)$ where $p \in \mathbb{P}^1$ and $i = 1, 2, 3$. A member $X$ of Family 5 is defined as the blow-up $\breve{\pi}\colon X \to V$ along a curve $C$ of tri-degree $(1, 1, 3)$ in $V$, i.e. $C \cdot H_1 = C \cdot H_2 = 1$ and $C \cdot H_3 = 3$. Note that $X$ is smooth if and only if $C$ is smooth.

**Theorem 4.4.1.** *Let $C$ be a smooth curve of tri-degree $(1, 1, 3)$ curve on $V$. Then, $C$ can be expressed in one of the following forms:*

$$C = C_\lambda := \begin{cases} x_0y_1 - x_1y_0 = 0 \\ x_0^3z_0 - x_1^3z_1 + \lambda(x_0x_1^2z_0 - x_1x_0^2z_1) = 0 \end{cases} \qquad \lambda \in \mathbb{C}\backslash\{\pm 1\} \tag{4.8}$$

*or*

$$C := \begin{cases} x_0y_1 - x_1y_0 = 0 \\ x_0^3z_0 - x_1^3z_1 + x_0x_1^2z_0 = 0. \end{cases} \tag{4.9}$$

*Proof.* Take $C = \{G(x, y, z) = F(x, y, z) = 0\}$ where $x = [x_0 : x_1]$ and similarly for $y$ and $z$. Since $C \cdot H_1 = 1$, we can assume that fixing $a \in \mathbb{P}^1$ we get that $G(a, y, z) = s_0(a)y_0 + s_1(a)y_1 + t_0(a)z_0 + t_1(a)z_1$. Likewise, since $C \cdot H_2 = 1$, we have a similar picture when we fix $a \in \mathbb{P}^1$ for $G(x, a, z)$. In particular, when we combine both conditions and make a change of coordinates we get that $G(x, y, z) = x_0y_1 - x_1y_0$. From this first equation, $G(x, y) = 0$, we get that $[x_0 : x_1] = [y_0 : y_1]$, so we may assume that $F(x, y, z) = F(x, z) = h_0(x)z_0 - h_1(x)z_1$. Note that $\{F = 0\}$ is a $(1, 3)$ curve in $\mathbb{P}^1 \times \mathbb{P}^1$. In particular, it is left to prove that any $(1, 3)$-curve in $\mathbb{P}^1 \times \mathbb{P}^1$ can be expressed as $\{x_0^3z_0 - x_1^3z_1 + \lambda(x_0x_1^2z_0 - x_1x_0^2z_1) = 0\}$ or $\{x_0^3z_0 - x_1^3z_1 + x_0x_1^2z_0 = 0\}$.

Let $W := H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(1, 3))$ be the vector space of polynomials that define degree $(1, 3)$-curves in $\mathbb{P}^1 \times \mathbb{P}^1$ with coordinates $([x_1 : x_2], [z_1 : z_2])$. Hence, we have,

$$W := \left\{ \begin{matrix} a_0 z_0 x_0^3 + a_1 z_0 x_0^2 x_1 + a_2 z_0 x_0 x_1^2 + a_3 z_0 x_1^3 + \\ b_0 z_1 x_0^3 + b_1 z_1 x_0^2 x_1 + b_2 z_1 x_0 x_1^2 + b_3 z_1 x_1^3 \end{matrix} \middle| a_i, b_i \in \mathbb{C} \right\}.$$

Let us define $W^V := \langle a_0, ..., a_3, b_0, ..., b_3 \rangle$ the vector space generated by the coefficients of $W$. Now define

$$J := \{\nu z_0 x_0^3 + \lambda z_0 x_0 x_1^2 + \lambda z_1 x_0^2 x_1 + \nu z_1 x_1^3 | \lambda, \nu \in \mathbb{C}\}$$

a subspace of $W$. We want to show that any smooth element $F \in W$ can be written as an element in $J$ unless $\{F = 0\}$ is isomorphic to $\{x_0^3 z_0 - x_1^3 z_1 + x_0 x_1^2 z_0 = 0\}$. We will distinguish the elements of $W$ depending on their discriminant; if $\mathrm{Disc}_z(\mathrm{Disc}_x(F)) = 0$ we get specific equations for $F$; otherwise, we will use geometric invariant theory (GIT) to transform $F$ into an element of $J$. Consider SL generated as follows,

$$\left\langle \left\{ \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \middle| c \in \mathbb{C} \right\}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle.$$

Note that $G = \mathrm{SL} \times \mathrm{SL}$ acts in $\mathbb{P}^1 \times \mathbb{P}^1$ and also acts naturally on $W$ and $W^V$.

Take $F \in W$, then the discriminant of $F$ in the variables $[x_0 : x_1]$, denoted by $\mathrm{Disc}_x(F)$, is a degree 4 polynomial in variables $[z_0 : z_1]$ (see equation in Appendix A.2). Hence, counting with multiplicities, we have 4 points in $\mathbb{P}^1$ that are the solutions of $\mathrm{Disc}_x(F) = 0$. Moreover, if we take the projection from $\mathbb{P}^1 \times \mathbb{P}^1$ to the $\mathbb{P}^1$ with coordinates $[z_0 : z_1]$, $F$ gives us a $3 : 1$ cover of $\mathbb{P}^1$ that branches on the points where $\mathrm{Disc}_x(F) = 0$. In addition, $\mathrm{Disc}_x(F)$ is a quartic elliptic curve giving a $2 : 1$ cover of $\mathbb{P}^1$ with 4 branched points.

Let $S(W^V)$ be the *symmetric algebra* of $W^V$, the quotient of the tensor algebra $T(W^V) = \bigoplus_{d \geq 0} (W^V)^{\otimes d}$ by the two-sided ideal generated by tensors of the form $v \otimes \ w - w \otimes v$ for $v, w \in W^V$. The symmetric algebra is a graded commutative algebra, and its graded components $S^dW^V$ are the images of $(W^V)^{\otimes d}$ in the quotient. By classical results in invariant theory [Dol12, Chapter 1], original theory in [Ell95; Hil93; Sch68], there exist $s \in S^2W^V$ and $t \in S^6W^V$ such that the subspace of G-invariant elements of the symmetric algebra of $W^V$ is generated by $s$ and $t$, $(S(W^V))^G = \mathbb{C}[s,t]$. Using computational tools we can get their specific equations (see Appendix A.1):

$$\begin{aligned} s :=\ & 3a_0b_3 - a_1b_2 + a_2b_1 - 3, a_3b_0 \\ t :=\ & a_0^3b_3^3 - a_0^2a_1b_2b_3^2 - 2a_0^2a_2b_1b_3^2 + a_0^2a_2b_2^2b_3 - 3a_0^2a_3b_0b_3^2 + 3a_0^2a_3b_1b_2b_3 - a_0^2a_3b_2^3 \\ & +a_0a_1^2b_1b_3^2 + 3a_0a_1a_2b_0b_3^2 - a_0a_1a_2b_1b_2b_3 - a_0a_1a_3b_0b_2b_3 - 2a_0a_1a_3b_1^2b_3 \\ & +a_0a_1a_3b_1b_2^2 - 2a_0a_2^2b_0b_2b_3 + a_0a_2^2b_1^2b_3 + a_0a_2a_3b_0b_1b_3 + 2a_0a_2a_3b_0b_2^2 \\ & -a_0a_2a_3b_1^2b_2 + 3a_0a_3^2b_0^2b_3 - 3a_0a_3^2b_0b_1b_2 + a_0a_3^2b_1^3 - a_1^3b_0b_3^2 + a_1^2a_2b_0b_2b_3 \\ & +2a_1^2a_3b_0b_1b_3 - a_1^2a_3b_0b_2^2 - a_1a_2^2b_0b_1b_3 - 3a_1a_2a_3b_0^2b_3 + a_1a_2a_3b_0b_1b_2 \\ & +2a_1a_3^2b_0^2b_2 - a_1a_3^2b_0b_1^2 + a_2^3b_0^2b_3 - a_2^2a_3b_0^2b_2 + a_2a_3^2b_0^2b_1 - a_3^3b_0^3. \end{aligned}$$

Using Hilbert-Munford criteria and classical results of invariant theory [Dol12, Chapter 1] we can define a map from the GIT semistable elements of the projectivisation of $W^V$ to the GIT-quotient that is isomorphic to $\mathbb{P}^1$ with coordinates $[s^3, t]$:

$$\begin{aligned} (\mathbb{P}_*(W^V))^{ss} &\xrightarrow{\phi} \mathbb{P}_*(W^V)^{ss}//G \cong \mathbb{P}^1 \\ [a_0 : \dots : a_3 : b_0 : \dots : b_3] &\mapsto [s^3 : t] \end{aligned}$$

Note we have the same map for $W$. We can check that $\mathrm{Disc}_z(\mathrm{Disc}_x(F)) = 256 \cdot t \cdot (s^3 - 27t)^3$ (explicit code in Appendix A.2).

**Lemma 4.4.2.** *Assume $F \in W$ is smooth, but $\mathrm{Disc}_z(\mathrm{Disc}_x(F)) = 0$ (i.e. at least 2 of the branch points of the cover to $\mathbb{P}^1$ merge). Then, up to a twist by $\mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C})$, $F$ is either*

*(i) $z_0x_0^3 + z_1x_0x_1^2 + z_1x_1^3$ (if only 1 point has multiplicity 2).*

*(ii) $z_0x_0^3 + z_1x_1^3$ (if there are 2 points with multiplicity 2).*

*Proof.* As we mentioned previously, the projection $([x_0 : x_1], [z_0 : z_1]) \mapsto [z_0 : z_1]$ gives a triple cover $C \to \mathbb{P}^1$, with 4 ramification points (counting multiplicity). Here we assumed that at least one of these points has multiplicity 2. Up to a change of coordinates, we can assume that two of the ramification points are $[1 : 0]$ and $[0 : 1]$. In particular, we can assume that the point $[z_0 : z_1] = [0 : 1]$ has multiplicity 2, then after a change of coordinate on $[x_0, x_1]$ we can assume $F|_{z_0=0} = -x_1^3$ and $F|_{z_1=0} = x_0^2(ax_0 + bx_1)$. From here we have 2 possibilities if $[z_0 : z_1] = [1 : 0]$ has multiplicity 1, we recover (i) after a change of coordinates, and if it has multiplicity 2 we get (ii). □

**Remark 4.4.3.** Note that for (i) $s(F) = 3$ and $t(F) = 1$, therefore $[F] \in \mathbb{P}_*(W^{ss})$ and $\phi(F) = [27 : 1]$. For (ii), note that $F \in J$ by taking $[\nu : \lambda] = [1 : 0]$.

**Remark 4.4.4.** Take $F_{[\nu:\lambda]} \in J$, then $s(F_{[\nu:\lambda]}) = 3\nu^2 + \lambda^2$ and $t(F_{[\nu:\lambda]}) = \nu^2(\nu^2 - \lambda^2)^2$. Notice that either $s(F_{[\nu:\lambda]}) \neq 0$ or $t(F_{[\nu:\lambda]}) \neq 0$ if $(\nu, \lambda) \neq (0, 0)$. Hence, $[F_{[\nu,\lambda]}] \in \mathbb{P}_*(W^{ss})$ and $\phi(F_{[\nu,\lambda]}) = [(3\nu^2+\lambda^2)^3 : \nu^2(\nu^2-\lambda^2)^2]$. Furthermore, $F_{[\nu:\lambda]}$ is singular if and only if $[\nu : \lambda] = [0 : 1]$ or $[1, \pm 1]$ and in this cases $\phi(F_{[\nu:\lambda]}) = [1 : 0]$. Also notice that, $\phi^{-1}([27 : 1])|_J = \{F_{[1:0]}, F_{[1:\pm 3]}\}$.

**Lemma 4.4.5.** *Assume $F \in W$ is smooth and* $\mathrm{Disc}_z(\mathrm{Disc}_x(F)) \neq 0$*. Then, $F$ is GIT-stable with respect to $G$ acting on $W$.*

*Proof.* Let us assume $F$ is not GIT-stable. Then there exists a 1-parameter subgroup on $G$

$$\sigma = \left\langle \begin{pmatrix} t^{r_1} & 0 \\ 0 & t^{-r_1} \end{pmatrix}, \begin{pmatrix} t^{r_2} & 0 \\ 0 & t^{-r_2} \end{pmatrix} \right\rangle,$$

acting on $W$, and therefore in $W^V$, such that $r_1, r_2 \geq 0$, $(r_1, r_2) \neq (0, 0)$ and $\mu(F, \sigma) = \max\{\mathrm{wt}(a_i), \mathrm{wt}(b_i) | i \in \{0, 1, 2, 3\}\} \leq 0$ (see [New78, Theorem 29] for characterisation of GIT-stable). The weights on $W^V$ are the following:

$$\begin{aligned} \text{wt}(a_0) &= r_1 + 3r_2, & \text{wt}(b_0) &= -r_1 + 3r_2, \\ \text{wt}(a_1) &= r_1 + r_2, & \text{wt}(b_1) &= -r_1 + r_2, \\ \text{wt}(a_2) &= r_1 - r_2, & \text{wt}(b_2) &= -r_1 - r_2, \\ \text{wt}(a_3) &= r_1 - 3r_2, & \text{wt}(b_3) &= -r_1 - 3r_2. \end{aligned}$$

Since we assumed $\mu(F,\sigma) \leq 0$ we know that $a_0 = a_1 = 0$. If $3r_2 > r1$, then $b_0 = 0$, but then $F$ is reducible and we get a contradiction. If $3r_2 \leq r_1$, in particular $r_1 > r_2$, hence $a_2 = 0$ and $F|_{z_1=0} = a_3x_1^3$. In this case, $\text{Disc}_z(\text{Disc}_x(F)) = 0$ wich is against our assumption. Therefore, $F$ is GIT-stable. □

Now take any smooth $F \in W$. If $\text{Disc}_z(\text{Disc}_x(F)) = 0$, then by Lemma 4.4.2, we know that either $F = z_0x_0^3 + z_1x_0x_1^2 + z_1y_1^3$ or $F = F_{[1:0]}$. In the former case, take $C = \{G(x,y) = F(x,z) = 0\}$ we recover (4.9), similarly, the latter case gives us $C_0$ from (4.8). Hence, let us assume that $\text{Disc}_z(\text{Disc}_x(F)) \neq 0$. This is true if and only if $256t(s^3 - 27t)^3 \neq 0$, which is equivalent of saying that $p = \phi(F) = [s^3 : t] \neq [1 : 0], [27 : 1]$. We also know by Lemma 4.4.5 that $F$ is GIT-stable. On the other hand, by Remark 4.4.4 there exist $[\nu : \lambda] \notin \{[1 : 0], [1 : \pm 3], [0 : 1], [1, \pm 1]\}$ such that $\phi(F_{[\nu:\lambda]}) = p$. Again by Remark 4.4.4, this means that $F_{[\nu:\lambda]}$ is smooth with $\text{Disc}_z(\text{Disc}_x(F_{[\nu:\lambda]})) \neq 0$ and by Lemma 4.4.5 it is GIT-stable. Hence, we proved that $F_{[\nu:\lambda]} \in G \cdot [F] \subset (\mathbb{P}_*W)^{ss}$ and we recover $C_{\lambda/\nu} = \{G(x,y) = F_{[1:\lambda/\nu]}(x,z) = 0\}$ from (4.8).

□

Let $X_\lambda$ be the blow-up of $V$ along $C_\lambda$ in (4.8). By [Ara+23, Theorem 5.115, Corollary 5.111] we know that $X_\lambda$ is K-polystable, and the blow-up of $V$ along (4.9) is strictly K-semistable. Moreover, the latter admits a degeneation to $X_0$.

**Lemma 4.4.6.** *Let $X$ the blow-up of $V$ along (4.9). Then it admits an isotrivial degeneration to $X_\infty$.*

*Proof.* It is enough to show that $C$ described in (4.9) degenerates isotrivially to $C_0$. We define the family

$$\mathcal{C} := \{\{x_0y_1 - x_1y_0 = x_0^3z_0 - x_1^3z_1 + tx_0x_1^2z_0 = 0\} | t \in \mathbb{A}^1\}.$$

We define $\pi : \mathcal{C} \to \mathbb{A}^1$ in an obvious way. Notice that for every non-zero $t \in \mathbb{A}^1$, $\pi^{-1}(t) \cong C$. On the other hand, for $\pi^{-1}(0) = \{x_0y_1 - x_1y_0 = x_0^3z_0 - x_1^3z_1 = 0\}$, so we clearly have $\pi^{-1}(0) = C_0$. □

Furthermore, every smooth family member is isomorphic to one of these 3-folds. If we allow, $\lambda = \pm 1, \infty$ in (4.8), we get singular members of the family. Moreover, we see that $X_{\pm 1} \cong X_\infty$.

**Lemma 4.4.7.** *Let $X_{\pm 1}$, $X_\infty$ be described as above. Then, $X_{\pm 1} \cong X_\infty$.*

*Proof.* Consider the following linear action on sections of $H_i$ on $V$:

$$[x_0 : x_1] \mapsto [x_0 + x_1 : -i(x_0 - x_1)]$$

$$[y_0 : y_1] \mapsto [y_0 + y_1 : i(y_1 - y_0)]$$

$$[z_0 : z_1] \mapsto [-i(z_0 - z_1) : z_0 + z_1]$$

One can check that

$$(x_0y_1 - x_1y_0) \mapsto 2i(x_0y_1 - x_1y_0)$$

$$(x_0^3z_0 - x_1^3z_1) + (x_0x_1^2z_0 - x_1x_0^2z_1) \mapsto 8i(x_0x_1^2z_0 - x_1x_0^2z_1)$$

So we get $X_1 \cong X_\infty$. Similarly we have $X_{-1} \cong X_\infty$. □

Notice that $C_\infty$ is the union of 3 lines, $l_{3,1} = \{x_0 = y_0 = 0\}$, $l_{3,2} = \{x_1 = y_1 = 0\}$ and $\Delta = \{x_0y_1 - x_1y_0 = x_0z_1 - x_1z_0 = 0\}$. Moreover, $\Delta$ and $l_{3,i}$ intersect in a point $p_i$ for $i \in \{1, 2\}$. Therefore, when we blow up $V$ along $C_\infty$ to obtain the threefold $X_\infty$, we get two ordinary double points coming from the blowup of $p_1$ and $p_2$. In the next section, we study the geometry of this member of the family.

### 4.4.2 Geometry of $X_\infty$

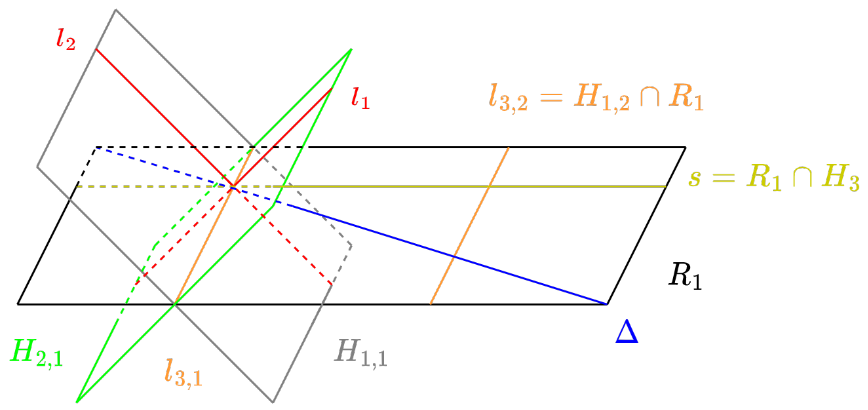
We aim to prove that $X_\infty$ is K-polystable. We will first define some surfaces and curves on $V$ we will use in the following proofs:

$$\begin{cases} H_{1,i} := \{x_{i-1} = 0\} \\ H_{2,i} := \{y_{i-1} = 0\} \text{ for } i \in \{1,2\} \\ H_{3,0} := \{z_0 = 0\} \end{cases} \qquad \begin{cases} R_1 := \{x_0y_1 - x_1y_0 = 0\} \\ R_2 := \{x_0z_1 - x_1z_0 = 0\} \\ R_3 := \{y_0z_1 - y_1z_0 = 0\} \end{cases}$$

$$\begin{cases} l_1 := H_{3,0} \cap H_{2,1} = \{z_0 = y_0 = 0\} \\ l_2 := H_{3,0} \cap H_{1,1} = \{z_0 = x_0 = 0\} \\ \Delta := R_1 \cap R_2 \cap R_3 = \{x_0y_1 - x_1y_0 = x_0z_1 - x_1z_0 = 0\} \\ \Delta' := \{x_0y_1 - x_1y_0 = x_0z_1 + x_1z_0 = 0\} \\ l_{3,1} = R_1 \cap H_{1,1} = R_1 \cap H_{2,1} = \{x_0 = y_0 = 0\}, \\ l_{3,2} = R_1 \cap H_{1,2} = R_1 \cap H_{2,2} = \{x_1 = y_1 = 0\}, \\ s := H_{3,0} \cap R_1 = \{z_0 = x_0y_1 - x_1y_0 = 0\}. \end{cases}$$

Let $H_1$, $H_2$, $H_3$ be general fibres of the fibrations $\mathrm{pr}_1 \circ \pi$, $\mathrm{pr}_2 \circ \pi$, $\mathrm{pr}_3 \circ \pi$, respectively, where $\mathrm{pr}_1$, $\mathrm{pr}_2$, $\mathrm{pr}_3$, are projections of $(\mathbb{P}^1)^3$ to the first, second, third factor, respectively. Notice that $H_{j,i} \in |H_j|$ for $i \in \{0,1,2\}$ and $j \in \{1,2,3\}$. Also notice that $R_1 \sim H_1 + H_2$, similarly we can find linear equivalences for $R_2$ and $R_3$ in terms of $H_j$. In the image below you can visualize some of these curves and surfaces.

Figure 4.5: Model in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ Family 5




Notice that $X_\infty$ is the blow up of $V$ along $C_\infty = l_{3,1} \cup l_{3,2} \cup \Delta$. Let us take the following automorphisms in $V$:

$$\iota_1 : [x_0 : x_1] \times [y_0 : y_1] \times [z_0 : z_1] \mapsto [x_1 : x_0] \times [y_1 : y_0] \times [z_1 : z_0],$$

$$\iota_2 : [x_0 : x_1] \times [y_0 : y_1] \times [z_0 : z_1] \mapsto [y_0 : y_1] \times [x_0 : x_1] \times [z_0 : z_1],$$

$$\tau_s : [x_0 : x_1] \times [y_0 : y_1] \times [z_0 : z_1] \mapsto [sx_0 : x_1] \times [sy_0 : y_1] \times [sz_0 : z_1],$$

where $s \in \mathbb{C}^*$. Now we consider the group $G$ generated by $\iota_1, \iota_2, \tau_s$. Notice that $C_\infty$ is invariant under $G$. Therefore, the action of $G$ on $V$ can be lifted to $X$ and we have $G \cong (\mathbb{C}^* \rtimes \mu_2) \rtimes \mu_2 \subset \mathrm{Aut}(X)$.

**Description of the $G$-invariant loci.** For the $G$-invariant structure on $V$, we have the following lemma:

**Lemma 4.4.8.** *There is no point of $V$ fixed by $G$. The only $G$-invariant irreducible curves in $V$ are $\Delta$ and $\Delta'$.*

*Proof.* Notice that there are only 8 points on $V$ fixed by $\tau_s$: $\{p_i \times p_j \times p_k,\ i, j, k \in \{1, 2\}\}$, where $p_1 = [1 : 0]$, $p_2 = [0 : 1]$. Moreover, none of these points is fixed by $\iota_1$, hence $V$ does not contain any $G$-invariant points.

On the other hand, let $C$ be a $G$-invariant irreducible curve on $V$. Notice that $C$ cannot be pointwise fixed by $\tau_s$, hence $C$ is rational. Then $\iota_1$ must fix a point

$p \in C$, which is not fixed by $\tau_s$. Since there are only 8 $\iota_1$-fixed points $[1 : \pm 1] \times [1 : \pm 1] \times [1 : \pm 1]$, $C$ must be the closure of the $\mathbb{C}^*$-orbit of one of those points. There are only 4 possible curves if we take the orbits: $\{x_0 y_1 \pm x_1 y_0 = x_0 z_1 \pm x_1 z_0 = 0\}$. Since $C$ is also $\iota_2$-fixed, we only have two possible invariant curves: $\Delta$ and $\Delta'$. $\square$

Notice that, $R_1$, $R_2$ and $R_3$ are also $G$-invariant and $G$-irreducible and $\Delta$ is their intersection. Let $\varphi : W \to V$ be the blowup of the curve $\Delta$, let $\sigma : \widetilde{X} \to W$ be the blowup of the strict transforms of the curves $l_{3,1}$ and $l_{3,2}$, let $\gamma : U \to V$ be the blowup of $l_{3,1}$ and $l_{3,2}$, and let $\phi \colon \widetilde{X} \to U$ be the blowup of the proper transform of $\Delta$. Then we have a $G$-equivariant commutative diagram

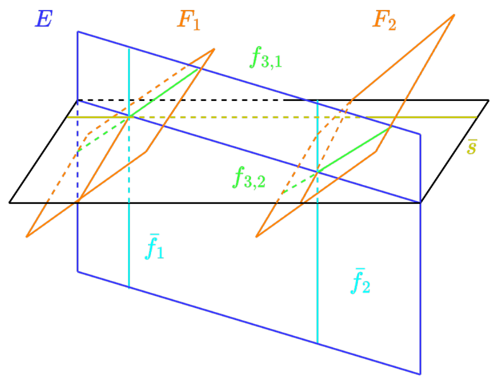
$$\begin{array}{ccccc} \widetilde{X} & \longrightarrow & X_\infty & \longleftarrow & \overline{X} \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \sigma} \\ U & \xrightarrow[\gamma]{} & V & \xleftarrow[\varphi]{} & W \end{array} \tag{4.10}$$

where $\widetilde{X} \to X_\infty$ and $\overline{X} \to X_\infty$ are $G$-equivariant small resolutions of the 3-fold $X$. Recall from Section 4.2.2 that either $Y = \widetilde{X}$ or $Y = \overline{X}$.

Let $\overline{H}_{i,j}, \overline{H}_3, \overline{R}_k$ be the strict transforms of $H_{i,j}, H_3, R_k$ on $\overline{X}$, for $i, j = 1, 2$ and $k = 1, 2, 3$. Let $F_i$ be the exceptional divisor on $X$ over $l_{3,i}$ for $i = 1, 2$, and $E$ be the exceptional divisor on $X$ over $\Delta$. Then, denote by $\overline{F}_1$, $\overline{F}_2$, $\overline{E}$ their strict transforms on $\overline{X}$.

Let $\bar{l}_{3,1}, \bar{l}_{3,2}, \bar{s}, \overline{\Delta}, \overline{\Delta'}$ be the corresponding sections of $l_{3,1}, l_{3,2}, s, \Delta, \Delta'$ on $\overline{R}_1$. Denote by $\bar{l}_3$ the strict transformon $\overline{R}_1$ of a general curve $l_3 \in |l_{3,1}| = |l_{3,2}|$ in $R_1 \subset V$. Let $f_{3,1}$ and $f_{3,2}$ be in the class of a $\sigma$-fibre in $\overline{F}_1$ and $\overline{F}_2$ such that $f_{3,i} = \overline{E} \cap \overline{F}_i$, respectively. Denote by $\overline{f}_1$ and $\overline{f}_2$, the proper transform on $\overline{E}$ of the fibres of this projection that pass through the points $\sigma(\bar{l}_{3,1})$ and $\sigma(\bar{l}_{3,2})$, respectively.

Figure 4.6: Model for $\overline{X}$ Family 5




**Lemma 4.4.9.** *The effective cone of* $\overline{X}$, $\mathrm{Eff}(\overline{X})$, *is generated by* $\overline{H}_{i,j}$ *(for* $i, j \in \{1,2\}$*),* $\overline{H}_3$, $\overline{R}_i$ *(for* $i \in \{1,2,3\}$*),* $\overline{F}_1$, $\overline{F}_2$ *and* $\overline{E}$*.*

*Proof.* Suppose $\pi(S) \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is the surface of degree $(d_1, d_2, d_3)$ in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Then, we have

$$S \sim d_1\pi^*(\overline{H}_1) + d_2\pi^*(\overline{H}_2) + d_3\pi^*(\overline{H}_3) - m\overline{E} - m_1\overline{F}_1 - m_2\overline{F}_2,$$

where $m$ is the multiplicity of $\pi(S)$ in $\Delta$, $m_i$ is the multiplicity of $\pi(S)$ in $l_{3,i}$ for $i \in \{1,2\}$. Suppose that $S \neq \overline{E}$, $S \neq \overline{F}_i$ for $i \in \{1,2\}$, $S \neq \overline{H}_{i,j}$, $S \neq \overline{H}_3$, $S \neq \overline{R}_i$ for $i \in \{1,2,3\}$. Now, we intersect $S$ with $f_{3,1}$, $f_{3,2}$, $\overline{f}_2 + f_{3,2}$, $\bar{l}_3$, $\overline{s}$, $\overline{s}_i = \overline{H}_3 \cdot \overline{R}_i$ for $i \in \{2,3\}$ and $t_{i,j} = \overline{H}_3 \cdot (\overline{H}_i - \overline{F}_j)$ for $i, j \in \{1,2\}$. It is straighforward to check that $\overline{E}^3 = -4$, $\overline{F}_k^3 = 1$, $\overline{H}_1\overline{H}_2\overline{H}_3 = 1$, $\overline{H}_i^2\overline{H}_j = \overline{H}_i\overline{H}_j\overline{F}_k = \overline{H}_i\overline{H}_j\overline{E} = \overline{H}_i\overline{F}_k^2 = 0$ and $\overline{F}_k^2\overline{E} = \overline{F}_k^2\overline{H}_3 = \overline{E}^2\overline{H}_i = -1$.

$$\begin{cases} S \cdot f_{3,1} = m_1 \geq 0, & S \cdot \overline{s}_i = d_i - m \geq 0 \quad \text{for } i \in \{2,3\} \\ S \cdot f_{3,2} = m_2 \geq 0, & S \cdot \bar{l}_3 = d_3 - m \geq 0, \\ S \cdot (f_{3,2} + \overline{f}_2) = m \geq 0, & S \cdot t_{i,j} = d_i - m_j \geq 0 \quad \text{for } i, j \in \{1,2\}. \\ S \cdot \overline{s} = d_1 + d_2 - m - m_1 - m_2 \geq 0, & \end{cases}$$

Now we want to find the integer positive solutions for:

$$d_1\pi^*(\overline{H}_1) + d_2\pi^*(\overline{H}_2) + d_3\pi^*(\overline{H}_3) - m\overline{E} - m_1\overline{F}_1 - m_2\overline{F}_2 = \sum_{i,j=1,2} h_{i,j}\overline{H}_{ij} + h_3\overline{H}_3 + \sum_{i=1}^{3} r_i\overline{R}_i + \sum_{i=1}^{2} f_i\overline{F}_i + e\overline{E}$$

Comparing the coefficients we get the system:

$$\begin{cases} e = r_1 + r_2 + r_3 - m, \quad h_3 = d_3 - r_2 - r_3 \\ f_1 = -r_1 - r_2 - r_3 - h_{1,2} + d_1 - h_{2,2} + d_2 - m_1, \\ f_2 = r_1 + h_{1,2} + h_{2,2} - m_2, \quad h_{1,1} = d_1 - r_1 - r_2 - h_{1,2}. \end{cases}$$

The non-negative solution to this system can be given by

$$\begin{cases} e = 0, \quad f_1 = 0, \quad r_2 = m, & r_1 = 0, \\ h_3 = d_3 - m, \quad r_3 = 0, & h_{2,2} = d_2 - m_1, \\ h_{1,1} = 0, \quad h_{1,2} = d_1 - m, & h_{2,1} = m_1, \\ f_2 = d_1 + d_2 - m_1 - m_2 - m. \end{cases}$$

Thus, the cone of effective divisors over $\mathbb{Z}$ is generated by $\overline{H}_{ij}$ for $i, j \in \{1, 2\}$, $\overline{H}_3$, $\overline{R}_i$ for $i \in \{1, 2, 3\}$, $\overline{F}_1$, $\overline{F}_2$ and $\overline{E}$. □

Then, we can rewrite $-K_{\overline{X}}$ as the sum of effective divisors,

$$-K_{\overline{X}} \sim 2\overline{R}_1 + \overline{E} + \overline{F}_1 + \overline{F}_2 + 2\overline{H}_3 \sim \overline{R}_1 + \overline{R}_2 + \overline{R}_3 + 2\overline{E}.$$

Note that the divisors $\overline{E}$, $\overline{F}_1 + \overline{F}_2$, $\overline{R}_1$, $\overline{R}_2 + \overline{R}_3$ are $G$-invariant and $G$-irreducible, but the pencils $|\overline{H}_1|$, $|\overline{H}_2|$, $|\overline{H}_3|$ do not contain $G$-invariant members.

### 4.4.3 K-polystability of $X_\infty$

We now prove that $X = X_\infty$ is K-polystable. Suppose that $X$ is not K-polystable. By Theorem 2.2.12, there exists a $G$-invariant prime divisor $F$ over $X$ such that

$$\beta(F) = A_X\big(F\big) - S_X\big(F\big) \leqslant 0. \tag{4.11}$$

Let $Z$ and $\overline{Z}$ be its centres on $X$ and $\overline{X}$, respectively. We first consider the case when $Z$ is a divisor.

**Lemma 4.4.10.** *Let $S$ be a $G$-invariant prime divisor in $\overline{X}$. Then $\beta(S) > 0$.*

*Proof.* If $\beta(S) \leqslant 0$, then the divisor $-K_{\overline{S}} - S$ is big by (2.1). On the other hand, arguing as in the proof of [Ara+23, Lemma 5.113], we see that $-K_{\overline{X}} - S$ is big only if $S = \overline{R}_1$ or $S = \overline{E}$. Thus, to complete the proof, it is enough to show that $\beta(\overline{R}_1) > 0$ and $\beta(\overline{E}) > 0$.

Set $S = \overline{R}_1$. Then $\tau = 2$. Arguing as in the proof of [Ara+23, Lemma 5.114], we see that $N(u) = 0$ for $u \in [0,1]$, and $N(u) = (u-1)(\overline{E} + \overline{F}_1 + \overline{F}_1)$ for $u \in [1,2]$. Then

$$P(u) \sim_{\mathbb{R}} \begin{cases} (2-u)\overline{R}_1 + \overline{E} + \overline{F}_1 + \overline{F}_2 + 2\overline{H}_3 & \text{if } 0 \leqslant u \leqslant 1, \\ (2-u)\Big(\overline{H}_1 + \overline{H}_2\Big) + 2\overline{H}_3 & \text{if } 1 \leqslant u \leqslant 2. \end{cases}$$

A direct computation then shows that

$$\begin{aligned} S_{\overline{X}}(\overline{R}_1) &= \frac{1}{(-K_{\overline{X}})^3} \int_0^\tau \mathrm{vol}(-K_{\overline{X}} - u\overline{R}_1) du \\ &= \frac{1}{26}\bigg( \int_0^1 (-2u^3 - 6u^2 - 6u + 26) du + \int_1^2 (12u^2 - 48u + 48) du \bigg) \\ &= \frac{49}{52} < 1 \end{aligned}$$

Now, we compute $\beta(E)$. First notice that we have $-K_{\overline{X}} - u\overline{E} \sim (2-u)\overline{E} + \overline{R}_1 + \overline{R}_2 + \overline{R}_3$, where $\overline{R}_1 + \overline{R}_2 + \overline{R}_3$ is not big. Therefore, the divisor $-K_{\overline{X}} - u\overline{E}$ is nef for $0 \leq u \leq 1$ and pseudoeffective for $0 \leq u \leq 2$. We have the following positive and negative parts:

$$P(-K_{\overline{X}} - u\overline{E}) = \begin{cases} -K_{\overline{X}} - u\overline{E} & \text{if } 0 \leq u \leq 1, \\ -K_{\overline{X}} - u\overline{E} - (u-1)(\overline{R}_1 + \overline{R}_2 + \overline{R}_3) & \text{if } 1 \leq u \leq 2, \end{cases}$$

and

$$N(-K_{\overline{X}} - u\overline{E}) = \begin{cases} 0 & \text{if } 0 \leq u \leq 1, \\ (u-1)(\overline{R}_1 + \overline{R}_2 + \overline{R}_3) & \text{if } 1 \leq u \leq 2. \end{cases}$$

Now we compute

$$\begin{aligned} S_{\overline{X}}(\overline{E}) &= \frac{1}{(-K_{\overline{X}})^3} \int_0^{\tau} \mathrm{vol}(-K_{\overline{X}} - u\overline{E}) du \\ &= \frac{1}{26} \left( \int_0^1 (4u^3 - 6u^2 - 18u + 26) du + \int_1^2 (-6u^3 + 36u^2 - 72u + 48) du \right) \\ &= \frac{35}{52} < 1. \end{aligned}$$

Hence, $\beta(\overline{E}) > 0$ and we get the desired contradiction. □

We now consider the case when $Z$ and $\overline{Z}$ are small. By Lemma 4.4.8, we only need to consider curves, and either $\pi(Z) = \Delta$ and $\overline{Z} \subset \overline{E}$, or $\pi(Z) = \Delta'$ and $\overline{Z} \subset \overline{R}_1$.

**Lemma 4.4.11.** *The curve $\overline{Z}$ is not contained in the surface $\overline{R}_1$.*

*Proof.* The proof is very similar to the proof of [Ara+23, Lemma 5.114]. Suppose that $\overline{Z} \subset \overline{R}_1$. Let $\overline{\Delta} = \overline{E} \cap \overline{R}_1$, and let $\overline{\Delta}'$ be the proper transform on the 3-fold $\overline{X}$ of the curve $\Delta'$. Then it follows from Lemma 4.4.8 that either $\overline{Z} = \overline{\Delta}$ or $\overline{Z} = \overline{\Delta}'$.

Recall from the proof of Lemma 4.4.10 that $S_{\overline{X}}(\overline{R}_1) = \frac{49}{52}$. Thus, it follows from

(4.3) that $S(W^{\overline{R}_1}_{\bullet,\bullet};\overline{Z}) \geqslant 1$. Let us compute it.

Notice that $\bar{l}_3$ and $\bar{s}$ the rulings of $\overline{R}_1 \cong \mathbb{P}^1 \times \mathbb{P}^1$ such that $\mathrm{pr}_1 \circ \pi$ and $\mathrm{pr}_2 \circ \pi$ contract $\bar{l}_3$, and $\mathrm{pr}_3 \circ \pi$ contracts $\bar{s}$. Moreover, $\overline{\Delta} \sim_{\overline{R}_1} \bar{l}_3 + \bar{s}$. Then it follows from the proof of Lemma 4.4.10 that

$$P(u)\Big|_{\overline{R}_1} - v\overline{Z} \sim_{\mathbb{R}} \begin{cases} (1+u-v)\bar{l}_3 + (1+u-v)\bar{s} & 0 \leqslant u \leqslant 1, \\ (4-2u-v)\bar{l}_3 + (2-v)\bar{s} & 1 \leqslant u \leqslant 2, \end{cases}$$

and

$$N(u)\Big|_{\overline{R}_1} = \begin{cases} 0 & 0 \leqslant u \leqslant 1, \\ (u-1)\Big(\overline{\Delta} + \bar{l}_{3,1} + \bar{l}_{3,2}\Big) & 1 \leqslant u \leqslant 2, \end{cases}$$

where $\bar{l}_{3,1} = \overline{F}_1|_{\overline{R}_1}$ and $\bar{l}_{3,2} = \overline{F}_2|_{\overline{R}_1}$ as defined in the previous subsection. Thus, if $0 \leqslant u \leqslant 1$, then $\tau(u) = 1+u$. Similarly, if $1 \leqslant u \leqslant 2$, then $\tau(u) = 4-2u$. We have $P(u,v) \sim_{\mathbb{R}} P(u)\Big|_{\overline{R}_1} - v\overline{Z}$ for every $v \in [0, \tau(u)]$. Hence, if $\overline{Z} = \overline{\Delta}$, then $S(W^{\overline{R}_1}_{\bullet,\bullet};\overline{Z})$ can be computed as follows:

$$\frac{3}{26}\int_1^2 (16-8u)(u-1)du + \frac{3}{26}\int_0^1\int_0^{1+u} 2(1+u-v)^2 dvdu$$
$$+\frac{3}{26}\int_1^2\int_0^{4-2u} 2(4-2u-v)(2-v)dvdu = \frac{35}{52} < 1.$$

Similarly, if $\overline{Z} = \overline{\Delta}'$, then $S(W^{\overline{E}}_{\bullet,\bullet};\overline{Z}) = \frac{27}{52} < 1$. This is a contradiction. $\square$

Thus, we see that $\pi(Z) = \Delta$, $\overline{Z} \subset \overline{E}$ and $\overline{Z} \neq \overline{E} \cap \overline{R}_1$. Observe that $\sigma\Big(\overline{E}\Big) \cong \mathbb{P}^1 \times \mathbb{P}^1$. Recall $f_{3,1} = \overline{F}_1 \cap \overline{E}$ and $f_{3,2} = \overline{F}_2 \cap \overline{E}$. Then $\sigma$ in (4.10) induces a birational morphism $\overline{E} \to \sigma(\overline{E})$ that contracts $f_{3,1}$ and $f_{3,2}$ to two distinct points on the curve $\sigma(\overline{\Delta})$, which is a section the natural projection $\sigma(\overline{E}) \to \Delta$. Denote by $\overline{f}_1$ and $\overline{f}_2$ the proper transforms on $\overline{E}$ of the fibres of this projection that pass through the points $\sigma(f_{3,1})$ and $\sigma(f_{3,2})$, respectively. Note that $\overline{E}$ is a weak del Pezzo surface, the curves $f_{3,1}$, $f_{3,2}$, $\overline{f}_1$, $\overline{f}_2$ are all $(-1)$-curves in $\overline{E}$, and $\overline{\Delta}$ is the only $(-2)$-curve in $\overline{E}$. By [CT88, Proposition 8.3], the curves $\overline{\Delta}$, $f_{3,1}$, $f_{3,2}$, $\overline{f}_1$, $\overline{f}_2$ generate the Mori cone $\overline{\mathrm{NE}(\overline{E})}$.

Now, let us focus on $\overline{E}$. Then it follows from the proof of Lemma 4.4.10 that

$$P(u)\Big|_{\overline{E}} - v\overline{Z} \sim_{\mathbb{R}} \begin{cases} (1+u-v)\overline{\Delta} + (2-v)\Big(\overline{f}_{3,1} + \overline{f}_{3,2}\Big) + \\ \quad +(2-u)\Big(\overline{f}_1 + \overline{f}_2\Big) & \text{if } 0 \leqslant u \leqslant 1, \\ (4-2u-v)\Big(\overline{\Delta} + \overline{f}_{3,1} + \overline{f}_{3,2}\Big) + \\ \quad +(2-u)\Big(\overline{f}_1 + \overline{f}_2\Big) & \text{if } 1 \leqslant u \leqslant 2, \end{cases}$$

and

$$N(u)\Big|_{\overline{E}} = \begin{cases} 0 & \text{if } 0 \leqslant u \leqslant 1, \\ (u-1)\Big(\overline{\Delta} + \overline{\Delta}_2 + \overline{\Delta}_3\Big) & \text{if } 1 \leqslant u \leqslant 2, \end{cases}$$

where $\overline{\Delta}_2 = \overline{E} \cap \overline{R}_2$ and $\overline{\Delta}_3 = \overline{E} \cap \overline{R}_3$. Thus, if $0 \leqslant u \leqslant 1$, then $\tau(u) = 1+u$. Similarly, if $1 \leqslant u \leqslant 2$, then $\tau(u) = 4-2u$.

Now in order to do the Zariski decomposition of $P(u)\Big|_{\overline{E}} - v\overline{Z}$ on $\overline{E}$ we need to intersect it with the generators of $\overline{\mathrm{NE}(\overline{E})}$. Recall that we have the following intersections between the generators in $\overline{E}$: $\overline{\Delta} \cdot f_{3,i} = 1$, $\overline{f}_i \cdot f_{3,i} = 1$ for $i \in \{1,2\}$ and the rest are 0. Moreover, if $0 \leqslant u \leqslant 1$, then

$$\begin{aligned} &(P(u)\Big|_{\overline{E}} - v\overline{Z}) \cdot \overline{\Delta} = 2-2u \geq 0 \quad \forall v; \\ &(P(u)\Big|_{\overline{E}} - v\overline{Z}) \cdot f_{3,i} = 1 \geq 0 \qquad \forall v; \qquad i \in \{1,2\} \\ &(P(u)\Big|_{\overline{E}} - v\overline{Z}) \cdot \overline{f}_i = u-v \geq 0 \quad \Leftrightarrow u \geq v. \end{aligned}$$

Hence, $P(u)\Big|_{\overline{E}} - v\overline{Z}$ is nef for $0 \leq v \leq u$. For $u \leq v \leq 1+u$ the intersection with $\overline{f}_i$ is negative, therefore we are adding some $\lambda_1\overline{f}_1 + \lambda_2\overline{f}_2$. Now, we want the smallest $\lambda_i$ such that $(1+u-v)\overline{\Delta} + (2-v)\Big(\overline{f}_{3,1} + \overline{f}_{3,2}\Big) + (2-u-\lambda_1)\overline{f}_1 + (2-u-\lambda_2)\overline{f}_2$ is nef.

$$\begin{aligned}
&(P(u)\big|_{\overline{E}} - v\overline{Z} - \lambda_1\overline{f}_1 + \lambda_2\overline{f}_2)\cdot\overline{\Delta} = 2-2u \geq 0 && \forall v;\\
&(P(u)\big|_{\overline{E}} - v\overline{Z} - \lambda_1\overline{f}_1 + \lambda_2\overline{f}_2)\cdot f_{3,i} = 1-\lambda_i \geq 0 && \Leftrightarrow 1 \geq \lambda_i \;\; i\in\{1,2\}\\
&(P(u)\big|_{\overline{E}} - v\overline{Z} - \lambda_1\overline{f}_1 + \lambda_2\overline{f}_2)\cdot \overline{f}_i = u-v-\lambda_i \geq 0 && \Leftrightarrow u-v \leq \lambda_i.
\end{aligned}$$

So the smallest $\lambda_i = u - v \leq 1$. Therefore, we get the following Zariski decomposition:

$$P(u,v) \sim_{\mathbb{R}} \begin{cases} (1+u-v)\overline{\Delta} + (2-v)\big(f_{3,1}+f_{3,2}\big)+ & \\ \qquad +(2-u)\big(\overline{f}_1+\overline{f}_2\big) & \text{if } 0 \leqslant v \leqslant u,\\ (1+u-v)\overline{\Delta} + (2-v)\big(f_{3,1}+f_{3,2}+\overline{f}_1+\overline{f}_2\big) & \text{if } u \leqslant v \leqslant 1+u, \end{cases}$$

and

$$N(u,v) = \begin{cases} 0 & \text{if } 0 \leqslant v \leqslant u,\\ (v-u)\big(\overline{f}_1+\overline{f}_2\big) & \text{if } u \leqslant v \leqslant 1+u, \end{cases}$$

so that

$$\big(P(u,v)\big)^2 = \begin{cases} 7-3u^2+2uv+v^2+6u-10v & \text{if } 0 \leqslant v \leqslant u,\\ (1+u-v)(7-u-3v) & \text{if } u \leqslant v \leqslant 1+u. \end{cases}$$

Similarly, if $1 \leqslant u \leqslant 2$, we get this other Zariski decomposition,

$$P(u,v) \sim_{\mathbb{R}} \begin{cases} (4-2u-v)\big(\overline{\Delta}+f_{3,1}+f_{3,2}\big)+ & \\ \qquad +(2-u)\big(\overline{f}_1+\overline{f}_2\big) & \text{if } 0 \leqslant v \leqslant 2-u,\\ (4-2u-v)\big(\overline{\Delta}+f_{3,1}+f_{3,2}+\overline{f}_1+\overline{f}_2\big) & \text{if } 2-u \leqslant v \leqslant 4-2u, \end{cases}$$

and

$$N(u,v) = \begin{cases} 0 & \text{if } 0 \leqslant v \leqslant 2-u,\\ (v+u-2)\big(\overline{f}_1+\overline{f}_2\big) & \text{if } 2-u \leqslant v \leqslant 4-2u, \end{cases}$$

so that

$$\left(P(u,v)\right)^2 = \begin{cases} 10u^2 + 8uv + v^2 - 40u - 16v + 40 & \text{if } 0 \leqslant v \leqslant 2-u, \\ 3(4-2u-v)^2 & \text{if } 2-u \leqslant v \leqslant 4-2u. \end{cases}$$

Note that $\overline{Z} \not\subset \operatorname{Supp}(N(u))$ for $u \in [0,2]$. Thus, integrating, we get $S(W^{\overline{E}}_{\bullet,\bullet};\overline{Z}) = \frac{87}{104}$. On the other hand, we already know from the proof of Lemma 4.4.10 that $S_Y(\overline{E}) = \frac{35}{52} < 1$. Then $\beta(F) > 0$ by (4.3), which contradicts (4.11). This shows that $X$ is K-polystable.

### 4.4.4 K-moduli component

It is a direct consequence of the following Corollary that the one-dimensional component of $M_3^{\mathrm{Kps}}$ formed by the K-polystable elements of Family 5 is isomorphic to $\mathbb{P}^1$.

**Corollary 4.4.12.** *The Fano 3-fold $X_\infty$ in Family 5 is the only singular K-polystable limit of members of the deformation family 4.13.*

*Proof.* Denote by $M_{4.13}^{\mathrm{Kps}}$ the one-dimensional component of the K-moduli space $M_3^{\mathrm{Kps}}$ that contains all smooth K-polystable Fano 3-folds in Family 5 (equivalently, all K-polystable elements of Mori-Mukai family №4.13). In §4.4.1 we descrived a parametrisation $\left\{X_\lambda; \lambda \in \mathbb{P}^1\right\}$ that is a $\mathbb{Q}$-Gorenstein family, and such that all smooth members of Family 4 are fibres of the family $X_\lambda$ for $\lambda \in \mathbb{P}^1 \setminus \{\pm 1, \infty\}$. Note that $X_\lambda \cong X_{-\lambda}$ for $\lambda \in \mathbb{P}^1$.

Moreover, it follows from the description of the Family and §4.4.3, where we prove the K-polystability of $X_\infty$, that all objects $X_\lambda$ in the parametrisation except for the 3-folds $X_{\pm 1}$ are K-polystable. As mentioned already, the 3-folds $X_{\pm 1}$ are K-semistable, and their K-polystable limit is $X_\infty$. Thus we have a morphism $\mathbb{P}^1 \to \mathcal{M}_{4.13}^{\mathrm{Kss}}$, the moduli stack parametrising K-semistable objects in this family, which descends to a morphism $\phi\colon \mathbb{P}^1 \to M^{\mathrm{Kps}}_{4.13}$ given by $\lambda \mapsto [X_\lambda]$ such than $\phi(\infty) = \phi(\pm 1)$, and $\phi(\lambda) = \phi(-\lambda)$ for $\lambda \in \mathbb{P}^1$. Since $M^{\mathrm{Kps}}_{4.13}$ is proper and one-dimensional, we conclude that $\phi$ is surjective, which implies the required assertion. $\square$

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# Appendix A

# Computations of Parametrisation of Family 5

## A.1 Invariant polynomials

### A.1.1 Invariant polynomials using Magma

This command in MAGMA reduces the possible monomials:

```
K<w>:= FiniteField(227);
G := MatrixGroup<8,K |
    [1,0,0,0,0,0,0,0,
 3*w,1,0,0,0,0,0,0,
   3*w^2,2*w,1,0,0,0,0,0,
   w^3,w^2,w,1,0,0,0,0,
   0,0,0,0,1,0,0,0,
  0,0,0,0,3*w,1,0,0,
   0,0,0,0,3*w^2,2*w,1,0,
   0,0,0,0,w^3,w^2,w,1],
  [0,0,0,1,0,0,0,0,
  0,0,-1,0,0,0,0,0,
   0,1,0,0,0,0,0,0,
   -1,0,0,0,0,0,0,0,
   0,0,0,0,0,0,0,1,
  0,0,0,0,0,0,-1,0,
   0,0,0,0,0,1,0,0,
   0,0,0,0,-1,0,0,0],
[1,0,0,0,0,0,0,0,
  0,1,0,0,0,0,0,0,
   0,0,1,0,0,0,0,0,
   0,0,0,1,0,0,0,0,
   w,0,0,0,1,0,0,0,
   0,w,0,0,0,1,0,0,
   0,0,w,0,0,0,1,0,
   0,0,0,w,0,0,0,1]>;
InvariantsOfDegree(G, 2);
```

```
Output:
[
   x1*x8 + 151*x2*x7 + 76*x3*x6 + 226*x4*x5
]
```

Since this is only a degree 2 polinomial, we can optain it directly using magma and check we get the same result.

---

```
K<w>:= FiniteField(227);
G := MatrixGroup<8,K |
    [1,0,0,0,0,0,0,0,
 3*w,1,0,0,0,0,0,0,
   3*w^2,2*w,1,0,0,0,0,0,
   w^3,w^2,w,1,0,0,0,0,
   0,0,0,0,1,0,0,0,
  0,0,0,0,3*w,1,0,0,
   0,0,0,0,3*w^2,2*w,1,0,
   0,0,0,0,w^3,w^2,w,1],
  [0,0,0,1,0,0,0,0,
  0,0,-1,0,0,0,0,0,
   0,1,0,0,0,0,0,0,
   -1,0,0,0,0,0,0,0,
   0,0,0,0,0,0,0,1,
  0,0,0,0,0,0,-1,0,
```

```
   0,0,0,0,0,1,0,0,
   0,0,0,0,-1,0,0,0],
[1,0,0,0,0,0,0,0,
  0,1,0,0,0,0,0,0,
   0,0,1,0,0,0,0,0,
   0,0,0,1,0,0,0,0,
   w,0,0,0,1,0,0,0,
   0,w,0,0,0,1,0,0,
   0,0,w,0,0,0,1,0,
   0,0,0,w,0,0,0,1],
[0,0,0,0,1,0,0,0,
   0,0,0,0,0,1,0,0,
   0,0,0,0,0,0,1,0,
   0,0,0,0,0,0,0,1,
-1,0,0,0,0,0,0,0,
  0,-1,0,0,0,0,0,0,
   0,0,-1,0,0,0,0,0,
   0,0,0,-1,0,0,0,0]>;
InvariantsOfDegree(G, 6);
```

Output:

```
[
   x1^3*x8^3 + 226*x1^2*x2*x7*x8^2 + 76*x1^2*x3*x7^2*x8 + 224*x1^2*x4*x5*x8^2 +
      x1^2*x4*x6*x7*x8 + 151*x1^2*x4*x7^3 + 76*x1*x2^2*x6*x8^2 +
      202*x1*x2^2*x7^2*x8 + x1*x2*x3*x5*x8^2 + 201*x1*x2*x3*x6*x7*x8 +
      x1*x2*x4*x5*x7*x8 + 75*x1*x2*x4*x6^2*x8 + 76*x1*x2*x4*x6*x7^2 +
      75*x1*x3^2*x5*x7*x8 + 51*x1*x3^2*x6^2*x8 + 226*x1*x3*x4*x5*x6*x8 +
      152*x1*x3*x4*x5*x7^2 + 151*x1*x3*x4*x6^2*x7 + 3*x1*x4^2*x5^2*x8 +
      226*x1*x4^2*x5*x6*x7 + 76*x1*x4^2*x6^3 + 151*x2^3*x5*x8^2 + 28*x2^3*x7^3
      + 76*x2^2*x3*x5*x7*x8 + 143*x2^2*x3*x6*x7^2 + 152*x2^2*x4*x5*x6*x8 +
      176*x2^2*x4*x5*x7^2 + 151*x2*x3^2*x5*x6*x8 + 84*x2*x3^2*x6^2*x7 +
      226*x2*x3*x4*x5^2*x8 + 26*x2*x3*x4*x5*x6*x7 + 151*x2*x4^2*x5*x6^2 +
      76*x3^3*x5^2*x8 + 199*x3^3*x6^3 + 151*x3^2*x4*x5^2*x7 +
      25*x3^2*x4*x5*x6^2 + x3*x4^2*x5^2*x6 + 226*x4^3*x5^3,
   x1^2*x3*x6*x8^2 + 151*x1^2*x3*x7^2*x8 + 226*x1^2*x4*x6*x7*x8 +
      76*x1^2*x4*x7^3 + 151*x1*x2^2*x6*x8^2 + 101*x1*x2^2*x7^2*x8 +
      226*x1*x2*x3*x5*x8^2 + 101*x1*x2*x3*x6*x7*x8 + x1*x2*x4*x5*x7*x8 +
      152*x1*x2*x4*x6^2*x8 + 151*x1*x2*x4*x6*x7^2 + 152*x1*x3^2*x5*x7*x8 +
      25*x1*x3^2*x6^2*x8 + 226*x1*x3*x4*x5*x6*x8 + 75*x1*x3*x4*x5*x7^2 +
      76*x1*x3*x4*x6^2*x7 + x1*x4^2*x5*x6*x7 + 151*x1*x4^2*x6^3 +
      76*x2^3*x5*x8^2 + 14*x2^3*x7^3 + 151*x2^2*x3*x5*x7*x8 +
      185*x2^2*x3*x6*x7^2 + 75*x2^2*x4*x5*x6*x8 + 202*x2^2*x4*x5*x7^2 +
      76*x2*x3^2*x5*x6*x8 + 42*x2*x3^2*x6^2*x7 + x2*x3*x4*x5^2*x8 +
      126*x2*x3*x4*x5*x6*x7 + 226*x2*x4^2*x5^2*x7 + 76*x2*x4^2*x5*x6^2 +
      151*x3^3*x5^2*x8 + 213*x3^3*x6^3 + 76*x3^2*x4*x5^2*x7 +
      126*x3^2*x4*x5*x6^2
]
```

Changing to Finite field 257 we get:

```
x1^3*x8^3 + 256*x1^2*x2*x7*x8^2 + 86*x1^2*x3*x7^2*x8 + 254*x1^2*x4*x5*x8^2 +
      x1^2*x4*x6*x7*x8 + 171*x1^2*x4*x7^3 + 86*x1*x2^2*x6*x8^2 +
      143*x1*x2^2*x7^2*x8 + x1*x2*x3*x5*x8^2 + 142*x1*x2*x3*x6*x7*x8 +
```

```
    x1*x2*x4*x5*x7*x8 + 85*x1*x2*x4*x6^2*x8 + 86*x1*x2*x4*x6*x7^2 +
    85*x1*x3^2*x5*x7*x8 + 229*x1*x3^2*x6^2*x8 + 256*x1*x3*x4*x5*x6*x8 +
    172*x1*x3*x4*x5*x7^2 + 171*x1*x3*x4*x6^2*x7 + 3*x1*x4^2*x5^2*x8 +
    256*x1*x4^2*x5*x6*x7 + 86*x1*x4^2*x6^3 + 171*x2^3*x5*x8^2 +
    184*x2^3*x7^3 + 86*x2^2*x3*x5*x7*x8 + 219*x2^2*x3*x6*x7^2 +
    172*x2^2*x4*x5*x6*x8 + 28*x2^2*x4*x5*x7^2 + 171*x2*x3^2*x5*x6*x8 +
    38*x2*x3^2*x6^2*x7 + 256*x2*x3*x4*x5^2*x8 + 115*x2*x3*x4*x5*x6*x7 +
    171*x2*x4^2*x5*x6^2 + 86*x3^3*x5^2*x8 + 73*x3^3*x6^3 +
    171*x3^2*x4*x5^2*x7 + 114*x3^2*x4*x5*x6^2 + x3*x4^2*x5^2*x6 +
    256*x4^3*x5^3,
  x1^2*x3*x6*x8^2 + 171*x1^2*x3*x7^2*x8 + 256*x1^2*x4*x6*x7*x8 +
    86*x1^2*x4*x7^3 + 171*x1*x2^2*x6*x8^2 + 200*x1*x2^2*x7^2*x8 +
    256*x1*x2*x3*x5*x8^2 + 200*x1*x2*x3*x6*x7*x8 + x1*x2*x4*x5*x7*x8 +
    172*x1*x2*x4*x6^2*x8 + 171*x1*x2*x4*x6*x7^2 + 172*x1*x3^2*x5*x7*x8 +
    114*x1*x3^2*x6^2*x8 + 256*x1*x3*x4*x5*x6*x8 + 85*x1*x3*x4*x5*x7^2 +
    86*x1*x3*x4*x6^2*x7 + x1*x4^2*x5*x6*x7 + 171*x1*x4^2*x6^3 +
    86*x2^3*x5*x8^2 + 92*x2^3*x7^3 + 171*x2^2*x3*x5*x7*x8 +
    238*x2^2*x3*x6*x7^2 + 85*x2^2*x4*x5*x6*x8 + 143*x2^2*x4*x5*x7^2 +
    86*x2*x3^2*x5*x6*x8 + 19*x2*x3^2*x6^2*x7 + x2*x3*x4*x5^2*x8 +
    57*x2*x3*x4*x5*x6*x7 + 256*x2*x4^2*x5^2*x7 + 86*x2*x4^2*x5*x6^2 +
    171*x3^3*x5^2*x8 + 165*x3^3*x6^3 + 86*x3^2*x4*x5^2*x7 +
    57*x3^2*x4*x5*x6^2
```

We check that for both cases we have the same monomials, so we replace the numbers for coefficients and we continue our computations in magma by also changing the notation to the one we are using in this thesis.

```
l*a0^3*b3^3 + (-l)*a0^2*a1*b2*b3^2 + a*a0^2*a2*b2^2*b3 +k*a0^2*a3*b0*b3^2 +
    l*a0^2*a3*b1*b2*b3 + (-a)*a0^2*a3*b2^3 + a*a0*a1^2*b1*b3^2 +
    b*a0*a1^2*b2^2*b3 + l*a0*a1*a2*b0*b3^2 + c*a0*a1*a2*b1*b2*b3 +
    l*a0*a1*a3*b0*b2*b3 + d*a0*a1*a3*b1^2*b3 + a*a0*a1*a3*b1*b2^2 +
    d*a0*a2^2*b0*b2*b3 + e*a0*a2^2*b1^2*b3 + (-l)*a0*a2*a3*b0*b1*b3 +
    (-d)*a0*a2*a3*b0*b2^2 + (-a)*a0*a2*a3*b1^2*b2 + (-k)*a0*a3^2*b0^2*b3 +
    (-l)*a0*a3^2*b0*b1*b2 + a*a0*a3^2*b1^3 + (-a)*a1^3*b0*b3^2 +
    f*a1^3*b2^3 + a*a1^2*a2*b0*b2*b3 + j*a1^2*a2*b1*b2^2 +
    (-d)*a1^2*a3*b0*b1*b3 + (-e)*a1^2*a3*b0*b2^2 + (-a)*a1*a2^2*b0*b1*b3 +
    (-j)*a1*a2^2*b1^2*b2 + (-l)*a1*a2*a3*b0^2*b3 + (-c)*a1*a2*a3*b0*b1*b2 +
    (-a)*a1*a3^2*b0*b1^2 + a*a2^3*b0^2*b3 + (-f)*a2^3*b1^3 +
    (-a)*a2^2*a3*b0^2*b2 + (-b)*a2^2*a3*b0*b1^2 + l* a2*a3^2*b0^2*b1 +
    (-l)*a3^3*b0^3,
  l* a0^2*a2*b1*b3^2 + (-a)*a0^2*a2*b2^2*b3 + (-l)*a0^2*a3*b1*b2*b3 +
    a*a0^2*a3*b2^3 + (-a)*a0*a1^2*b1*b3^2 + g*a0*a1^2*b2^2*b3 +
    (-l)*a0*a1*a2*b0*b3^2 + g*a0*a1*a2*b1*b2*b3 + l*a0*a1*a3*b0*b2*b3 +
    (-d)*a0*a1*a3*b1^2*b3 + (-a)*a0*a1*a3*b1*b2^2 + (-d)*a0*a2^2*b0*b2*b3 +
    (-b)*a0*a2^2*b1^2*b3 + (-l)*a0*a2*a3*b0*b1*b3 + d*a0*a2*a3*b0*b2^2 +
    a*a0*a2*a3*b1^2*b2 + l*a0*a3^2*b0*b1*b2 + (-a)*a0*a3^2*b1^3 +
    a*a1^3*b0*b3^2 + h*a1^3*b2^3 + (-a)*a1^2*a2*b0*b2*b3 +
    i*a1^2*a2*b1*b2^2 + d*a1^2*a3*b0*b1*b3 + b*a1^2*a3*b0*b2^2 +
    a*a1*a2^2*b0*b1*b3 + (-i)*a1*a2^2*b1^2*b2 + l*a1*a2*a3*b0^2*b3 +
    (-g)*a1*a2*a3*b0*b1*b2 + (-l)*a1*a3^2*b0^2*b2 + a*a1*a3^2*b0*b1^2 +
    (-a)*a2^3*b0^2*b3 + (-h)*a2^3*b1^3 + a*a2^2*a3*b0^2*b2 +
    (-g)*a2^2*a3*b0*b1^2
```

### A.1.2 Invariant polynomials using Maple

# First we start by checking how does G act in $W^V$

```
collect(expand(subs({x0 = y0 + c·y1, x1 = y1}, a0·z0·x0^3 + a1·z0·x0^2·x1 + a2·z0·x0·x1^2 + a3·z0·x1^3 + b0·z1·x0^3 + b1·z1·x0^2·x1 + b2·z1·x0·x1^2 + b3·z1·x1^3)), {z0, z1, y0, y1})
```

$$(a0\,z0 + b0\,z1)\,y0^3 + ((3\,a0\,c + a1)\,z0 + (3\,b0\,c + b1)\,z1)\,y1\,y0^2 + ((3\,a0\,c^2 + 2\,a1\,c + a2)\,z0 + (3\,b0\,c^2 + 2\,b1\,c + b2)\,z1)\,y1^2\,y0 + ((a0\,c^3 + a1\,c^2 + a2\,c + a3)\,z0 + (b0\,c^3 + b1\,c^2 + b2\,c + b3)\,z1)\,y1^3 \quad (1)$$

```
collect(expand(subs({x0 = -y1, x1 = y0}, a0·z0·x0^3 + a1·z0·x0^2·x1 + a2·z0·x0·x1^2 + a3·z0·x1^3 + b0·z1·x0^3 + b1·z1·x0^2·x1 + b2·z1·x0·x1^2 + b3·z1·x1^3)), {z0, z1, y0, y1})
```

$$(a3\,z0 + b3\,z1)\,y0^3 + (-a2\,z0 - b2\,z1)\,y1\,y0^2 + (a1\,z0 + b1\,z1)\,y1^2\,y0 + (-a0\,z0 - b0\,z1)\,y1^3 \quad (2)$$

```
collect(expand(subs({z0 = y0 + c·y1, z1 = y1}, a0·z0·x0^3 + a1·z0·x0^2·x1 + a2·z0·x0·x1^2 + a3·z0·x1^3 + b0·z1·x0^3 + b1·z1·x0^2·x1 + b2·z1·x0·x1^2 + b3·z1·x1^3)), {x0, x1, y0, y1})
```

$$(a0\,y0 + (a0\,c + b0)\,y1)\,x0^3 + (a1\,y0 + (a1\,c + b1)\,y1)\,x1\,x0^2 + (a2\,y0 + (a2\,c + b2)\,y1)\,x1^2\,x0 + (a3\,y0 + (a3\,c + b3)\,y1)\,x1^3 \quad (3)$$

```
collect(expand(subs({z0 = -y1, z1 = y0}, a0·z0·x0^3 + a1·z0·x0^2·x1 + a2·z0·x0·x1^2 + a3·z0·x1^3 + b0·z1·x0^3 + b1·z1·x0^2·x1 + b2·z1·x0·x1^2 + b3·z1·x1^3)), {z0, z1, y0, y1})
```

$$(b0\,x0^3 + b1\,x0^2\,x1 + b2\,x0\,x1^2 + b3\,x1^3)\,y0 + (-a0\,x0^3 - a1\,x0^2\,x1 - a2\,x0\,x1^2 - a3\,x1^3)\,y1 \quad (4)$$

#Now I want to find the G invariant s

```
collect(expand(subs({a0 = a0, a1 = (3 a0 c + a1), a2 = (3 a0 c^2 + 2 a1 c + a2), a3 = (a0 c^3 + a1 c^2 + a2 c + a3), b0 = b0, b1 = (3 b0 c + b1), b2 = (3 b0 c^2 + 2 b1 c + b2), b3 = (b0 c^3 + b1 c^2 + b2 c + b3)}, a0·(d00·b0 + d01·b1 + d02·b2 + d03·b3) + a1·(d10·b0 + d11·b1 + d12·b2 + d13·b3) + a2·(d20·b0 + d21·b1 + d22·b2 + d23·b3) + a3·(d30·b0 + d31·b1 + d32·b2 + d33·b3))), {a0, a1, a2, a3, b0, b1, b2, b3})
```

$$((c^6\,d33 + 3\,c^5\,d23 + 3\,c^5\,d32 + 3\,c^4\,d13 + 9\,c^4\,d22 + 3\,c^4\,d31 + c^3\,d03 + 9\,c^3\,d12 + 9\,c^3\,d21 + c^3\,d30 + 3\,c^2\,d02 + 9\,c^2\,d11 + 3\,c^2\,d20 + 3\,c\,d01 + 3\,c\,d10 + d00)\,b0 + (c^5\,d33 + 3\,c^4\,d23 + 2\,c^4\,d32 + 3\,c^3\,d13 + 6\,c^3\,d22 + c^3\,d31 + c^2\,d03 + 6\,c^2\,d12 + 3\,c^2\,d21 + 2\,c\,d02 + 3\,c\,d11 + d01)\,b1 + (c^4\,d33 + 3\,c^3\,d23 + c^3\,d32 + 3\,c^2\,d13 + 3\,c^2\,d22 + c\,d03 + 3\,c\,d12 + d02)\,b2 + (c^3\,d33 + 3\,c^2\,d23 + 3\,c\,d13 + d03)\,b3)\,a0 + ((c^5\,d33 + 2\,c^4\,d23 + 3\,c^4\,d32 + c^3\,d13 + 6\,c^3\,d22 + 3\,c^3\,d31 + 3\,c^2\,d12 + 6\,c^2\,d21 + c^2\,d30 + 3\,c\,d11 + 2\,c\,d20 + d10)\,b0 + (c^4\,d33 + 2\,c^3\,d23 + 2\,c^3\,d32 + c^2\,d13 + 4\,c^2\,d22 + c^2\,d31 + 2\,c\,d12 + 2\,c\,d21 + d11)\,b1 + (c^3\,d33 + 2\,c^2\,d23 + c^2\,d32 + c\,d13 + 2\,c\,d22 + d12)\,b2 + (c^2\,d33 + 2\,c\,d23 + d13)\,b3)\,a1 + ((c^4\,d33 + c^3\,d23 + 3\,c^3\,d32 + 3\,c^2\,d22 + 3\,c^2\,d31 + 3\,c\,d21 + c\,d30 + d20)\,b0 + (c^3\,d33 + c^2\,d23 + 2\,c^2\,d32 + 2\,c\,d22 + c\,d31 + d21)\,b1 + (c^2\,d33 + c\,d23 + c\,d32 + d22)\,b2 + (c\,d33 + d23)\,b3)\,a2 + ((c^3\,d33 + 3\,c^2\,d32 + 3\,c\,d31 + d30)\,b0 + (c^2\,d33 + 2\,c\,d32 + d31)\,b1 + (c\,d33 + d32)\,b2 \quad (5)$$

$+ d33\, b3)\, a3$

$$solve(\{(c^6\, d33 + 3\, c^5\, d23 + 3\, c^5\, d32 + 3\, c^4\, d13 + 9\, c^4\, d22 + 3\, c^4\, d31 + c^3\, d03 + 9\, c^3\, d12 + 9\, c^3\, d21 + c^3\, d30 + 3\, c^2\, d02 + 9\, c^2\, d11 + 3\, c^2\, d20 + 3\, c\, d01 + 3\, c\, d10 + d00) = d00, (c^5\, d33 + 3\, c^4\, d23 + 2\, c^4\, d32 + 3\, c^3\, d13 + 6\, c^3\, d22 + c^3\, d31 + c^2\, d03 + 6\, c^2\, d12 + 3\, c^2\, d21 + 2\, c\, d02 + 3\, c\, d11 + d01) = d01, (c^4\, d33 + 3\, c^3\, d23 + c^3\, d32 + 3\, c^2\, d13 + 3\, c^2\, d22 + c\, d03 + 3\, c\, d12 + d02) = d02, (c^3\, d33 + 3\, c^2\, d23 + 3\, c\, d13 + d03) = d03, (c^5\, d33 + 2\, c^4\, d23 + 3\, c^4\, d32 + c^3\, d13 + 6\, c^3\, d22 + 3\, c^3\, d31 + 3\, c^2\, d12 + 6\, c^2\, d21 + c^2\, d30 + 3\, c\, d11 + 2\, c\, d20 + d10) = d10, (c^4\, d33 + 2\, c^3\, d23 + 2\, c^3\, d32 + c^2\, d13 + 4\, c^2\, d22 + c^2\, d31 + 2\, c\, d12 + 2\, c\, d21 + d11) = d11, (c^3\, d33 + 2\, c^2\, d23 + c^2\, d32 + c\, d13 + 2\, c\, d22 + d12) = d12, (c^2\, d33 + 2\, c\, d23 + d13) = d13, (c^4\, d33 + c^3\, d23 + 3\, c^3\, d32 + 3\, c^2\, d22 + 3\, c^2\, d31 + 3\, c\, d21 + c\, d30 + d20) = d20, (c^3\, d33 + c^2\, d23 + 2\, c^2\, d32 + 2\, c\, d22 + c\, d31 + d21) = d21, (c^2\, d33 + c\, d23 + c\, d32 + d22) = d22, (c\, d23 + d23) = d23, (c^3\, d33 + 3\, c^2\, d32 + 3\, c\, d31 + d30) = d30, (c^2\, d33 + 2\, c\, d32 + d31) = d31, (c\, d33 + d32) = d32, d33 = d33\}, \{d00, d01, d02, d03, d10, d11, d12, d13, d20, d21, d22, d23, d30, d31, d32, d33\})$$

$$\left\{d00 = d00, d01 = -d10, d02 = -\frac{3\, d11}{2}, d03 = 3\, d21, d10 = d10, d11 = d11, d12 = -d21, d13 = 0, d20 = -\frac{3\, d11}{2}, d21 = d21, d22 = 0, d23 = 0, d30 = -3\, d21, d31 = 0, d32 = 0, d33 = 0\right\} \quad \textbf{(6)}$$

$$subs\left(\left\{d00 = d00, d01 = -d10, d02 = -\frac{3\, d11}{2}, d03 = 3\, d21, d10 = d10, d11 = d11, d12 = -d21, d13 = 0, d20 = -\frac{3\, d11}{2}, d21 = d21, d22 = 0, d23 = 0, d30 = -3\, d21, d31 = 0, d32 = 0, d33 = 0\right\}, a0 \cdot (d00 \cdot b0 + d01 \cdot b1 + d02 \cdot b2 + d03 \cdot b3) + a1 \cdot (d10 \cdot b0 + d11 \cdot b1 + d12 \cdot b2 + d13 \cdot b3) + a2 \cdot (d20 \cdot b0 + d21 \cdot b1 + d22 \cdot b2 + d23 \cdot b3) + a3 \cdot (d30 \cdot b0 + d31 \cdot b1 + d32 \cdot b2 + d33 \cdot b3)\right)$$

$$a0\left(d00\, b0 - b1\, d10 - \frac{3\, b2\, d11}{2} + 3\, b3\, d21\right) + a1\,(d10\, b0 + d11\, b1 - b2\, d21) + a2\left(-\frac{3\, b0\, d11}{2} + d21\, b1\right) - 3\, a3\, b0\, d21 \quad \textbf{(7)}$$

*#After one of the condition we reduced the equation to this.*

$$collect\left(expand\left(subs\left(\{a0 = a3, a1 = -a2, a2 = a1, a3 = -a0, b0 = b3, b1 = -b2, b2 = b1, b3 = -b0\}, a0\left(d00\, b0 - b1\, d10 - \frac{3\, b2\, d11}{2} + 3\, b3\, d21\right) + a1\,(d10\, b0 + d11\, b1 - b2\, d21) + a2\left(-\frac{3\, b0\, d11}{2} + d21\, b1\right) - 3\, a3\, b0\, d21\right)\right), \{a0, a1, a2, a3, b0, b1, b2, b3\}\right)$$

$$3\, a0\, b3\, d21 + a1\left(-\frac{3\, b3\, d11}{2} - b2\, d21\right) + (d21\, b1 + b2\, d11 - b3\, d10)\, a2 + a3\,\Big(d00\, b3 \quad \textbf{(8)}$$

$+ b2\, d10 - \frac{3\, d11\, b1}{2} - 3\, b0\, d21\Big)$

$solve\Big(\Big\{3 \cdot d21 = 3 \cdot d21, -\frac{3\ d11}{2} = 0, -d21 = -d21, d21 = d21, d11 = 0, -d10 = 0, d00 = 0, d10 = 0, -\frac{3}{2} \cdot d11 = 0, -3 \cdot d21 = -3 \cdot d21\Big\}, \{d00, d10, d11, d21\}\Big)$

$$\{d00 = 0, d10 = 0, d11 = 0, d21 = d21\} \tag{9}$$

$s := subs\Big(\{d00 = 0, d10 = 0, d11 = 0, d21 = 1\}, a0\,\Big(d00\, b0 - b1\, d10 - \frac{3\, b2\, d11}{2} + 3\, b3\, d21\Big) + a1\,(d10\, b0 + d11\, b1 - b2\, d21) + a2\,\Big(-\frac{3\, b0\, d11}{2} + d21\, b1\Big) - 3\, a3\, b0\, d21\Big)$

$$s := 3\, a0\, b3 - a1\, b2 + a2\, b1 - 3\, a3\, b0 \tag{10}$$

#With this we already now the equation of S, but just in case we will double check that it is invariant with the last action icontent

$collect(expand(subs(\{a0 = a0, a1 = a1, a2 = a2, a3 = a3, b0 = (a0\, c + b0), b1 = (a1\, c + b1), b2 = (a2\, c + b2), b3 = (a3\, c + b3)\}, s)), \{a0, a1, a2, a3, b0, b1, b2, b3\})$

$$3\, a0\, b3 - a1\, b2 + a2\, b1 - 3\, a3\, b0 \tag{11}$$

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#Notice that magma returns two polynomials of degree 6, let us start by studying the first one. Let us check if it is invariant under the actions

$collect\big(expand\big(subs\big(\{a0 = a3, a1 = -a2, a2 = a1, a3 = -a0, b0 = b3, b1 = -b2, b2 = b1, b3 = -b0\}, a\, a0^2\, a2\, b2^2\, b3 - a\, a0^2\, a3\, b2^3 + a\, a0\, a1^2\, b1\, b3^2 + a\, a0\, a1\, a3\, b1\, b2^2 - a\, a0\, a2\, a3\, b1^2\, b2 + a\, a0\, a3^2\, b1^3 - a\, a1^3\, b0\, b3^2 + a\, a1^2\, a2\, b0\, b2\, b3 - a\, a1\, a2^2\, b0\, b1\, b3 - a\, a1\, a3^2\, b0\, b1^2 + a\, a2^3\, b0^2\, b3 - a\, a2^2\, a3\, b0^2\, b2 + l\, a0^3\, b3^3 - l\, a0^2\, a1\, b2\, b3^2 + k\, a0^2\, a3\, b0\, b3^2 + l\, a0^2\, a3\, b1\, b2\, b3 + b\, a0\, a1^2\, b2^2\, b3 + l\, a0\, a1\, a2\, b0\, b3^2 + c\, a0\, a1\, a2\, b1\, b2\, b3 + l\, a0\, a1\, a3\, b0\, b2\, b3 + d\, a0\, a1\, a3\, b1^2\, b3 + d\, a0\, a2^2\, b0\, b2\, b3 + e\, a0\, a2^2\, b1^2\, b3 - l\, a0\, a2\, a3\, b0\, b1\, b3 - d\, a0\, a2\, a3\, b0\, b2^2 - k\, a0\, a3^2\, b0^2\, b3 - l\, a0\, a3^2\, b0\, b1\, b2 + f\, a1^3\, b2^3 + j\, a1^2\, a2\, b1\, b2^2 - d\, a1^2\, a3\, b0\, b1\, b3 - e\, a1^2\, a3\, b0\, b2^2 - j\, a1\, a2^2\, b1^2\, b2 - l\, a1\, a2\, a3\, b0^2\, b3 - c\, a1\, a2\, a3\, b0\, b1\, b2 - f\, a2^3\, b1^3 - b\, a2^2\, a3\, b0\, b1^2 + l\, a2\, a3^2\, b0^2\, b1 - l\, a3^3\, b0^3\big)\big), \{a0, a1, a2, a3, b0, b1, b2, b3\}\big)$

$$l\, a0^3\, b3^3 + \big(-b2\, b3^2\, l\, a1 + a\, b2^2\, b3\, a2 + \big(-a\, b2^3 + b3^2\, k\, b0 + b2\, b3\, l\, b1\big)\, a3\big)\, a0^2 + \big(\big(a\, b3^2\, b1 + b\, b3\, b2^2\big)\, a1^2 + \big(\big(b3^2\, l\, b0 + b1\, b2\, b3\, c\big)\, a2 + \big(a\, b2^2\, b1 + b2\, b3\, l\, b0 + b3\, d\, b1^2\big)\, a3\big)\, a1 + \big(b2\, b3\, d\, b0 + b1^2\, b3\, e\big)\, a2^2 + \big(\big(-b3\, l\, b1 - b2^2\, d\big)\, b0 - a\, b1^2\, b2\big)\, a3\, a2 + \big(a\, b1^3 - b3\, k\, b0^2 - b1\, b2\, l\, b0\big)\, a3^2\big)\, a0 + \big(-a\, b3^2\, b0 + b2^3\, f\big)\, a1^3 + \big(\big(a\, b3\, b2\, b0 + b2^2\, j\, b1\big)\, a2 + \big(-b3\, d\, b1 - b2^2\, e\big)\, b0\, a3\big)\, a1^2 + \big(\big(-a\, b1\, b3\, b0 - b2\, j\, b1^2\big)\, a2^2 + \big(-b3\, l\, b0^2 - b1\, b2\, c\, b0\big)\, a3\, a2 - a\, a3^2\, b0\, b1^2\big)\, a1 + \big(a\, b3\, b0^2 - b1^3\, f\big)\, a2^3 + \big(-a\, b2\, b0^2 - b\, b1^2\, b0\big)\, a3\, a2^2 + l\, a2\, a3^2\, b0^2\, b1 - l\, a3^3\, b0^3 \tag{12}$$

$collect\big(expand\big(a\, a0^2\, a2\, b2^2\, b3 - a\, a0^2\, a3\, b2^3 + a\, a0\, a1^2\, b1\, b3^2 + a\, a0\, a1\, a3\, b1\, b2^2$

$- a\, a0\, a2\, a3\, b1^2\, b2 + a\, a0\, a3^2\, b1^3 - a\, a1^3\, b0\, b3^2 + a\, a1^2\, a2\, b0\, b2\, b3 - a\, a1\, a2^2\, b0\, b1\, b3 - a\, a1\, a3^2\, b0\, b1^2 + a\, a2^3\, b0^2\, b3 - a\, a2^2\, a3\, b0^2\, b2 + l\, a0^3\, b3^3 - l\, a0^2\, a1\, b2\, b3^2 + k\, a0^2\, a3\, b0\, b3^2 + l\, a0^2\, a3\, b1\, b2\, b3 + b\, a0\, a1^2\, b2^2\, b3 + l\, a0\, a1\, a2\, b0\, b3^2 + c\, a0\, a1\, a2\, b1\, b2\, b3 + l\, a0\, a1\, a3\, b0\, b2\, b3 + d\, a0\, a1\, a3\, b1^2\, b3 + d\, a0\, a2^2\, b0\, b2\, b3 + e\, a0\, a2^2\, b1^2\, b3 - l\, a0\, a2\, a3\, b0\, b1\, b3 - d\, a0\, a2\, a3\, b0\, b2^2 - k\, a0\, a3^2\, b0^2\, b3 - l\, a0\, a3^2\, b0\, b1\, b2 + f\, a1^3\, b2^3 + j\, a1^2\, a2\, b1\, b2^2 - d\, a1^2\, a3\, b0\, b1\, b3 - e\, a1^2\, a3\, b0\, b2^2 - j\, a1\, a2^2\, b1^2\, b2 - l\, a1\, a2\, a3\, b0^2\, b3 - c\, a1\, a2\, a3\, b0\, b1\, b2 - f\, a2^3\, b1^3 - b\, a2^2\, a3\, b0\, b1^2 + l\, a2\, a3^2\, b0^2\, b1 - l\, a3^3\, b0^3 ), \{a0, a1, a2, a3, b0, b1, b2, b3\})$

$$l\, a0^3\, b3^3 + \left(-b2\, b3^2\, l\, a1 + a\, b2^2\, b3\, a2 + \left(-a\, b2^3 + b3^2\, k\, b0 + b2\, b3\, l\, b1\right) a3\right) a0^2 + \left(\left(a\, b3^2\, b1 + b\, b3\, b2^2\right) a1^2 + \left(\left(b3^2\, l\, b0 + b1\, b2\, b3\, c\right) a2 + \left(a\, b2^2\, b1 + b2\, b3\, l\, b0 + b3\, d\, b1^2\right) a3\right) a1 + \left(b2\, b3\, d\, b0 + b1^2\, b3\, e\right) a2^2 + \left(\left(-b3\, l\, b1 - b2^2\, d\right) b0 - a\, b1^2\, b2\right) a3\, a2 + \left(a\, b1^3 - b3\, k\, b0^2 - b1\, b2\, l\, b0\right) a3^2\right) a0 + \left(-a\, b3^2\, b0 + b2^3\, f\right) a1^3 + \left(\left(a\, b3\, b2\, b0 + b2^2\, j\, b1\right) a2 + \left(-b3\, d\, b1 - b2^2\, e\right) b0\, a3\right) a1^2 + \left(\left(-a\, b1\, b3\, b0 - b2\, j\, b1^2\right) a2^2 + \left(-b3\, l\, b0^2 - b1\, b2\, c\, b0\right) a3\, a2 - a\, a3^2\, b0\, b1^2\right) a1 + \left(a\, b3\, b0^2 - b1^3\, f\right) a2^3 + \left(-a\, b2\, b0^2 - b\, b1^2\, b0\right) a3\, a2^2 + l\, a2\, a3^2\, b0^2\, b1 - l\, a3^3\, b0^3 \quad \textbf{(13)}$$

#Notifce that this action does not change anything

$collect(expand(subs(\{a0 = b0, a1 = b1, a2 = b2, a3 = b3, b0 = -a0, b1 = -a1, b2 = -a2, b3 = -a3\}, l\, a0^3\, b3^3 + \left(-b2\, b3^2\, l\, a1 + a\, b2^2\, b3\, a2 + \left(-a\, b2^3 + b3^2\, k\, b0 + b2\, b3\, l\, b1\right) a3\right) a0^2 + \left(\left(a\, b3^2\, b1 + b\, b3\, b2^2\right) a1^2 + \left(\left(b3^2\, l\, b0 + b1\, b2\, b3\, c\right) a2 + \left(a\, b2^2\, b1 + b2\, b3\, l\, b0 + b3\, d\, b1^2\right) a3\right) a1 + \left(b2\, b3\, d\, b0 + b1^2\, b3\, e\right) a2^2 + \left(\left(-b3\, l\, b1 - b2^2\, d\right) b0 - a\, b1^2\, b2\right) a3\, a2 + \left(a\, b1^3 - b3\, k\, b0^2 - b1\, b2\, l\, b0\right) a3^2\right) a0 + \left(-a\, b3^2\, b0 + b2^3\, f\right) a1^3 + \left(\left(a\, b3\, b2\, b0 + b2^2\, j\, b1\right) a2 + \left(-b3\, d\, b1 - b2^2\, e\right) b0\, a3\right) a1^2 + \left(\left(-a\, b1\, b3\, b0 - b2\, j\, b1^2\right) a2^2 + \left(-b3\, l\, b0^2 - b1\, b2\, c\, b0\right) a3\, a2 - a\, a3^2\, b0\, b1^2\right) a1 + \left(a\, b3\, b0^2 - b1^3\, f\right) a2^3 + \left(-a\, b2\, b0^2 - b\, b1^2\, b0\right) a3\, a2^2 + l\, a2\, a3^2\, b0^2\, b1 - l\, a3^3\, b0^3)), \{a0, a1, a2, a3, b0, b1, b2, b3\})$

$$l\, a0^3\, b3^3 + \left(-b2\, b3^2\, l\, a1 + a\, b2^2\, b3\, a2 + \left(-a\, b2^3 + b3^2\, k\, b0 + b2\, b3\, l\, b1\right) a3\right) a0^2 + \left(\left(a\, b3^2\, b1 + b\, b3\, b2^2\right) a1^2 + \left(\left(b3^2\, l\, b0 + b1\, b2\, b3\, c\right) a2 + \left(a\, b2^2\, b1 + b2\, b3\, l\, b0 + b3\, d\, b1^2\right) a3\right) a1 + \left(b2\, b3\, d\, b0 + b1^2\, b3\, e\right) a2^2 + \left(\left(-b3\, l\, b1 - b2^2\, d\right) b0 - a\, b1^2\, b2\right) a3\, a2 + \left(a\, b1^3 - b3\, k\, b0^2 - b1\, b2\, l\, b0\right) a3^2\right) a0 + \left(-a\, b3^2\, b0 + b2^3\, f\right) a1^3 + \left(\left(a\, b3\, b2\, b0 + b2^2\, j\, b1\right) a2 + \left(-b3\, d\, b1 - b2^2\, e\right) b0\, a3\right) a1^2 + \left(\left(-a\, b1\, b3\, b0 - b2\, j\, b1^2\right) a2^2 + \left(-b3\, l\, b0^2 - b1\, b2\, c\, b0\right) a3\, a2 - a\, a3^2\, b0\, b1^2\right) a1 + \left(a\, b3\, b0^2 - b1^3\, f\right) a2^3 + \left(-a\, b2\, b0^2 - b\, b1^2\, b0\right) a3\, a2^2 + l\, a2\, a3^2\, b0^2\, b1 - l\, a3^3\, b0^3 \quad \textbf{(14)}$$

#No changes again, lets see if it changes with the $|C^*$ actions

$collect(expand(subs(\{a0 = a0, a1 = a1, a2 = a2, a3 = a3, b0 = (a0\,\text{lambda} + b0), b1 = (a1\,\text{lambda} + b1), b2 = (a2\,\text{lambda} + b2), b3 = (a3\,\text{lambda} + b3)\}, l\, a0^3\, b3^3 + \left(-b2\, b3^2\, l\, a1 + a\, b2^2\, b3\, a2 + \left(-a\, b2^3 + b3^2\, k\, b0 + b2\, b3\, l\, b1\right) a3\right) a0^2 + \left(\left(a\, b3^2\, b1 + b\, b3\, b2^2\right) a1^2$

$+ \left( \left( b3^2\, l\, b0 + b1\, b2\, b3\, c \right) a2 + \left( a\, b2^2\, b1 + b2\, b3\, l\, b0 + b3\, d\, b1^2 \right) a3 \right) a1 + \left( b2\, b3\, d\, b0 + b1^2\, b3\, e \right) a2^2 + \left( \left( -b3\, l\, b1 - b2^2\, d \right) b0 - a\, b1^2\, b2 \right) a3\, a2 + \left( a\, b1^3 - b3\, k\, b0^2 - b1\, b2\, l\, b0 \right) a3^2 \right) a0 + \left( -a\, b3^2\, b0 + b2^3\, f \right) a1^3 + \left( \left( a\, b3\, b2\, b0 + b2^2\, j\, b1 \right) a2 + \left( -b3\, d\, b1 - b2^2\, e \right) b0\, a3 \right) a1^2 + \left( \left( -a\, b1\, b3\, b0 - b2\, j\, b1^2 \right) a2^2 + \left( -b3\, l\, b0^2 - b1\, b2\, c\, b0 \right) a3\, a2 - a\, a3^2\, b0\, b1^2 \right) a1 + \left( a\, b3\, b0^2 - b1^3\, f \right) a2^3 + \left( -a\, b2\, b0^2 - b\, b1^2\, b0 \right) a3\, a2^2 + l\, a2\, a3^2\, b0^2\, b1 - l\, a3^3\, b0^3 \right) \right), \{a0, a1, a2, a3, b0, b1, b2, b3\} \right)$

$$\left( \left( k\lambda^2 + 3\, l\lambda^2 \right) b3\, a3^2 + \left( k\lambda + 3\, l\lambda \right) b3^2\, a3 + b3^3\, l \right) a0^3 + \left( -b2\, b3^2\, l\, a1 + \left( 2\, a\lambda^2 + d\lambda^2 \right) b3\, a2^3 + \left( \left( -2\, a\lambda^2 - d\lambda^2 \right) b2\, a3 + \left( 2\, a\lambda + d\lambda \right) b3\, b2 \right) a2^2 + \left( \left( -2\, a\lambda - d\lambda \right) b2^2\, a3 + a\, b2^2\, b3 \right) a2 + \left( -k\lambda^2 - 3\, l\lambda^2 \right) b0\, a3^3 + \left( -a\, b2^3 + b3^2\, k\, b0 + b2\, b3\, l\, b1 \right) a3 \right) a0^2 + \left( \left( \left( \left( b\lambda^2 + c\lambda^2 + e\lambda^2 \right) b3\, a2^2 + \left( \left( 2\, a\lambda^2 + 2\, b\lambda^2 - 2\, e\lambda^2 \right) b2\, a3 + \left( a\lambda + 2\, b\lambda + c\lambda \right) b3\, b2 \right) a2 + \left( 2\, a\lambda^2 + d\lambda^2 \right) b1\, a3^2 + \left( \left( 2\, a\lambda + d\lambda \right) b3\, b1 + \left( a\lambda + b\lambda - e\lambda \right) b2^2 \right) a3 + a\, b3^2\, b1 + b\, b3\, b2^2 \right) a1^2 + \left( \left( \left( \left( -2\, a\lambda^2 - 2\, b\lambda^2 + 2\, e\lambda^2 \right) b1\, a3 + \left( -a\lambda + c\lambda + 2\, e\lambda \right) b3\, b1 \right) a2^2 + \left( b3^2\, l\, b0 + b1\, b2\, b3\, c \right) a2 + \left( 2\, a\lambda + d\lambda \right) b1^2\, a3^2 + \left( a\, b2^2\, b1 + b2\, b3\, l\, b0 + b3\, d\, b1^2 \right) a3 \right) a1 + \left( 2\, a\lambda + d\lambda \right) b3\, b0\, a2^3 + \left( \left( \left( -2\, a\lambda - d\lambda \right) b2\, b0 + \left( -a\lambda - b\lambda + e\lambda \right) b1^2 \right) a3 + b2\, b3\, d\, b0 + b1^2\, b3\, e \right) a2^2 + \left( \left( -b3\, l\, b1 - b2^2\, d \right) b0 - a\, b1^2\, b2 \right) a3\, a2 + \left( -k\lambda - 3\, l\lambda \right) b0^2\, a3^3 + \left( a\, b1^3 - b3\, k\, b0^2 - b1\, b2\, l\, b0 \right) a3^2 \right) a0 + \left( \left( 3\, f\lambda^2 + j\lambda^2 \right) b2\, a2^2 + \left( 3\, f\lambda + j\lambda \right) b2^2\, a2 + \left( -2\, a\lambda^2 - d\lambda^2 \right) b0\, a3^2 + \left( -2\, a\lambda - d\lambda \right) b3\, b0\, a3 - a\, b3^2\, b0 + b2^3\, f \right) a1^3 + \left( \left( -3\, f\lambda^2 - j\lambda^2 \right) b1\, a2^3 + \left( -b\lambda^2 - c\lambda^2 - e\lambda^2 \right) b0\, a3\, a2^2 + \left( \left( a\lambda - c\lambda - 2\, e\lambda \right) b2\, b0\, a3 + a\, b3\, b2\, b0 + b2^2\, j\, b1 \right) a2 + \left( -2\, a\lambda - d\lambda \right) b1\, b0\, a3^2 + \left( -b3\, d\, b1 - b2^2\, e \right) b0\, a3 \right) a1^2 + \left( \left( -3\, f\lambda - j\lambda \right) b1^2\, a2^3 + \left( \left( -a\lambda - 2\, b\lambda - c\lambda \right) b1\, b0\, a3 - a\, b1\, b3\, b0 - b2\, j\, b1^2 \right) a2^2 + \left( -b3\, l\, b0^2 - b1\, b2\, c\, b0 \right) a3\, a2 - a\, a3^2\, b0\, b1^2 \right) a1 + \left( a\, b3\, b0^2 - b1^3\, f \right) a2^3 + \left( -a\, b2\, b0^2 - b\, b1^2\, b0 \right) a3\, a2^2 + l\, a2\, a3^2\, b0^2\, b1 - l\, a3^3\, b0^3 \tag{15}$$

$solve\left( \left\{ \left( 2\, a\lambda^2 + d\lambda^2 \right) = 0, \left( 2\, a\lambda + d\lambda \right) = 0, \left( b\lambda^2 + c\lambda^2 + e\lambda^2 \right) = 0, \left( 2\, a\lambda^2 + 2\, b\lambda^2 - 2\, e\lambda^2 \right) = 0, \left( a\lambda + 2\, b\lambda + c\lambda \right) = 0, \left( 2\, a\lambda^2 + d\lambda^2 \right) = 0, \left( 2\, a\lambda + d\lambda \right) = 0, \left( a\lambda + b\lambda - e\lambda \right) = 0, \left( -a\lambda + c\lambda + 2\, e\lambda \right) = 0, \left( -a\lambda - b\lambda + e\lambda \right) = 0, \left( 3\, f\lambda^2 + j\lambda^2 \right) = 0, \left( 3\, f\lambda + j\lambda \right) = 0, \left( k\lambda + 3\, l\lambda \right) = 0 \right\}, \{a, b, c, d, e, f, g, h, i, j, k, l\} \right)$

$$\{a = c + 2\, e, b = -c - e, c = c, d = -2\, c - 4\, e, e = e, f = f, g = g, h = h, i = i, j = -3\, f, k = -3\, l, l = l\} \tag{16}$$

$collect\left( expand\left( subs\left( \{a = c + 2\, e, b = -c - e, c = c, d = -2\, c - 4\, e, e = e, f = f, g = g, h = h, i = i, j = -3\, f, k = -3\, l, l = l\}, l\, a0^3\, b3^3 + \left( -b2\, b3^2\, l\, a1 + a\, b2^2\, b3\, a2 + \left( -a\, b2^3 + b3^2\, k\, b0 + b2\, b3\, l\, b1 \right) a3 \right) a0^2 + \left( \left( a\, b3^2\, b1 + b\, b3\, b2^2 \right) a1^2 + \left( \left( b3^2\, l\, b0 + b1\, b2\, b3\, c \right) a2 + \left( a\, b2^2\, b1 + b2\, b3\, l\, b0 + b3\, d\, b1^2 \right) a3 \right) a1 + \left( b2\, b3\, d\, b0 + b1^2\, b3\, e \right) a2^2 + \left( \left( -b3\, l\, b1 - b2^2\, d \right) b0 - a\, b1^2\, b2 \right) a3\, a2 + \left( a\, b1^3 - b3\, k\, b0^2 - b1\, b2\, l\, b0 \right) a3^2 \right) a0 + \left( -a\, b3^2\, b0 \right.$

$$+ b2^3 f) a1^3 + \left(\left(a\, b3\, b2\, b0 + b2^2 j\, b1\right) a2 + \left(-b3\, d\, b1 - b2^2 e\right) b0\, a3\right) a1^2 + \left(\left(-a\, b1\, b3\, b0 - b2\, j\, b1^2\right) a2^2 + \left(-b3\, l\, b0^2 - b1\, b2\, c\, b0\right) a3\, a2 - a\, a3^2\, b0\, b1^2\right) a1 + \left(a\, b3\, b0^2 - b1^3 f\right) a2^3 + \left(-a\, b2\, b0^2 - b\, b1^2\, b0\right) a3\, a2^2 + l\, a2\, a3^2\, b0^2\, b1 - l\, a3^3\, b0^3 \big)\big), \{a0, a1, a2, a3, b0, b1, b2, b3\}\big)$$

$$l\, a0^3\, b3^3 + \left(-b2\, b3^2\, l\, a1 + (c + 2e)\, b3\, b2^2\, a2 + \left(-3\, b3^2\, l\, b0 + b2\, b3\, l\, b1 + (-c - 2e)\, b2^3\right) a3\right) a0^2 + \Big(\left((c + 2e)\, b3^2\, b1 + (-c - e)\, b3\, b2^2\right) a1^2 + \Big(\left(b3^2\, l\, b0 + b1\, b2\, b3\, c\right) a2 + \left(b2^2\, (c + 2e)\, b1 + b2\, b3\, l\, b0 + (-2c - 4e)\, b3\, b1^2\right) a3\Big) a1 + \Big((-2c - 4e)\, b3\, b2\, b0 + b1^2\, b3\, e\Big) a2^2 + \Big(\left(-b3\, l\, b1 + (2c + 4e)\, b2^2\right) b0 + (-c - 2e)\, b2\, b1^2\Big) a3\, a2 + \left(b1^3\, (c + 2e) + 3\, b3\, l\, b0^2 - b1\, b2\, l\, b0\right) a3^2\Big) a0 + \left((-c - 2e)\, b3^2\, b0 + b2^3 f\right) a1^3 + \Big(\left((c + 2e)\, b3\, b2\, b0 - 3\, b1\, b2^2 f\right) a2 + \left((2c + 4e)\, b3\, b1 - b2^2 e\right) b0\, a3\Big) a1^2 + \Big(\left((-c - 2e)\, b3\, b1\, b0 + 3\, b1^2\, b2 f\right) a2^2 + \left(-b3\, l\, b0^2 - b1\, b2\, c\, b0\right) a3\, a2 + (-c - 2e)\, b1^2\, b0\, a3^2\Big) a1 + \left((c + 2e)\, b3\, b0^2 - b1^3 f\right) a2^3 + \left((-c - 2e)\, b2\, b0^2 + (c + e)\, b1^2\, b0\right) a3\, a2^2 + l\, a2\, a3^2\, b0^2\, b1 - l\, a3^3\, b0^3 \quad \textbf{(17)}$$

*#After this action we already simplify the equation, let us check the other \C* action now.*

$$collect\Big(expand\Big(subs\Big(\{a0 = a0,\, a1 = (3\, a0\, \text{lambda} + a1),\, a2 = \left(3\, a0\, \text{lambda}^2 + 2\, a1\, \text{lambda} + a2\right),\, a3 = \left(a0\, \text{lambda}^3 + a1\, \text{lambda}^2 + a2\, \text{lambda} + a3\right),\, b0 = b0,\, b1 = (3\, b0\, \text{lambda} + b1),\, b2 = \left(3\, b0\, \text{lambda}^2 + 2\, b1\, \text{lambda} + b2\right),\, b3 = \left(b0\, \text{lambda}^3 + b1\, \text{lambda}^2 + b2\, \text{lambda} + b3\right)\},$$

$$l\, a0^3\, b3^3 + \left(-b2\, b3^2\, l\, a1 + (c + 2e)\, b3\, b2^2\, a2 + \left(-3\, b3^2\, l\, b0 + b2\, b3\, l\, b1 + (-c - 2e)\, b2^3\right) a3\right) a0^2 + \Big(\left((c + 2e)\, b3^2\, b1 + (-c - e)\, b3\, b2^2\right) a1^2 + \Big(\left(b3^2\, l\, b0 + b1\, b2\, b3\, c\right) a2 + \left(b2^2\, (c + 2e)\, b1 + b2\, b3\, l\, b0 + (-2c - 4e)\, b3\, b1^2\right) a3\Big) a1 + \Big((-2c - 4e)\, b3\, b2\, b0 + b1^2\, b3\, e\Big) a2^2 + \Big(\left(-b3\, l\, b1 + (2c + 4e)\, b2^2\right) b0 + (-c - 2e)\, b2\, b1^2\Big) a3\, a2 + \left(b1^3\, (c + 2e) + 3\, b3\, l\, b0^2 - b1\, b2\, l\, b0\right) a3^2\Big) a0 + \left((-c - 2e)\, b3^2\, b0 + b2^3 f\right) a1^3 + \Big(\left((c + 2e)\, b3\, b2\, b0 - 3\, b1\, b2^2 f\right) a2 + \left((2c + 4e)\, b3\, b1 - b2^2 e\right) b0\, a3\Big) a1^2 + \Big(\left((-c - 2e)\, b3\, b1\, b0 + 3\, b1^2\, b2 f\right) a2^2 + \left(-b3\, l\, b0^2 - b1\, b2\, c\, b0\right) a3\, a2 + (-c - 2e)\, b1^2\, b0\, a3^2\Big) a1 + \left((c + 2e)\, b3\, b0^2 - b1^3 f\right) a2^3 + \left((-c - 2e)\, b2\, b0^2 + (c + e)\, b1^2\, b0\right) a3\, a2^2 + l\, a2\, a3^2\, b0^2\, b1 - l\, a3^3\, b0^3 \Big)\Big), \{a0, a1, a2, a3, b0, b1, b2, b3\}\Big)$$

$$\Big(\Big(-4c\lambda^6 + e\lambda^6 + 27 f\lambda^6 - 3 l\lambda^6\Big) b1^3 + \Big(\left(-12 c\lambda^5 + 3 e\lambda^5 + 81 f\lambda^5 - 9 l\lambda^5\right) b2 + \left(6 c\lambda^4 + 21 e\lambda^4 - 7 l\lambda^4\right) b3\Big) b1^2 + \Big(\left(-15 c\lambda^4 - 3 e\lambda^4 + 81 f\lambda^4 - 8 l\lambda^4\right) b2^2 + \left(3 c\lambda^3 + 24 e\lambda^3 - 11 l\lambda^3\right) b3\, b2 + \left(9 c\lambda^2 + 18 e\lambda^2 - 3 l\lambda^2\right) b3^2\Big) b1 + \left(-7 c\lambda^3 - 5 e\lambda^3 + 27 f\lambda^3 - 2 l\lambda^3\right) b2^3 + \left(-6 c\lambda^2 - 3 e\lambda^2 - 3 l\lambda^2\right) b3\, b2^2 + b3^3\, l\Big) a0^3 + \Big(\Big(\Big(\Big(12 c\lambda^6 - 3 e\lambda^6 - 81 f\lambda^6 + 9 l\lambda^6\Big) b1^2 + \Big(\left(24 c\lambda^5 - 6 e\lambda^5 - 162 f\lambda^5 + 18 l\lambda^5\right) b2 + \left(-12 c\lambda^4 - 42 e\lambda^4 + 14 l\lambda^4\right) b3\Big) b1 + \left(15 c\lambda^4 + 3 e\lambda^4 - 81 f\lambda^4 + 8 l\lambda^4\right) b2^2 + \left(-3 c\lambda^3 - 24 e\lambda^3 + 11 l\lambda^3\right) b3\, b2 + \left(-9 c\lambda^2 - 18 e\lambda^2 + 3 l\lambda^2\right) b3^2\Big) b0 + \Big(\left(-2 c\lambda^4 + 2 e\lambda^4 + 27 f\lambda^4 - 2 l\lambda^4\right) b2 + \left(6 c\lambda^3 + 12 e\lambda^3 - 2 l\lambda^3\right) b3\Big) b1^2 \quad \textbf{(18)}$$

$$+ ((-7 c \lambda^3 - 2 e \lambda^3 + 54 f \lambda^3 - 3 l \lambda^3) b2^2 + (5 c \lambda^2 + 16 e \lambda^2 - 5 l \lambda^2) b3\, b2 + (6 c \lambda$$

$$+ 12 e \lambda - 2 l \lambda) b3^2) b1 + (-5 c \lambda^2 - 4 e \lambda^2 + 27 f \lambda^2 - l \lambda^2) b2^3 + (-4 c \lambda - 2 e \lambda$$

$$- 2 l \lambda) b3\, b2^2 - b3^2\, b2\, l) a1 + ((((12 c \lambda^5 - 3 e \lambda^5 - 81 f \lambda^5 + 9 l \lambda^5) b1^2 + ((30 c \lambda^4$$

$$+ 6 e \lambda^4 - 162 f \lambda^4 + 16 l \lambda^4) b2 + (-3 c \lambda^3 - 24 e \lambda^3 + 11 l \lambda^3) b3) b1 + (21 c \lambda^3$$

$$+ 15 e \lambda^3 - 81 f \lambda^3 + 6 l \lambda^3) b2^2 + (12 c \lambda^2 + 6 e \lambda^2 + 6 l \lambda^2) b3\, b2) b0 + (2 c \lambda^4$$

$$- 2 e \lambda^4 - 27 f \lambda^4 + 2 l \lambda^4) b1^3 + ((7 c \lambda^3 + 2 e \lambda^3 - 54 f \lambda^3 + 3 l \lambda^3) b2 + (4 c \lambda^2$$

$$+ 2 e \lambda^2 + 2 l \lambda^2) b3) b1^2 + ((5 c \lambda^2 + 4 e \lambda^2 - 27 f \lambda^2 + l \lambda^2) b2^2 + (7 c \lambda + 8 e \lambda$$

$$+ l \lambda) b3\, b2) b1 + (c + 2 e) b3\, b2^2) a2 + (((( -6 c \lambda^4 - 21 e \lambda^4 + 7 l \lambda^4) b1^2 + (($$

$$-3 c \lambda^3 - 24 e \lambda^3 + 11 l \lambda^3) b2 + (-18 c \lambda^2 - 36 e \lambda^2 + 6 l \lambda^2) b3) b1 + (6 c \lambda^2 + 3 e \lambda^2$$

$$+ 3 l \lambda^2) b2^2 - 3 b3^2 l) b0 + (-6 c \lambda^3 - 12 e \lambda^3 + 2 l \lambda^3) b1^3 + ((-9 c \lambda^2 - 18 e \lambda^2$$

$$+ 3 l \lambda^2) b2 + (-6 c \lambda - 12 e \lambda + 2 l \lambda) b3) b1^2 + ((-3 c \lambda - 6 e \lambda + l \lambda) b2^2$$

$$+ b3\, b2\, l) b1 + (-c - 2 e) b2^3) a3) a0^2 + (((((-12 c \lambda^6 + 3 e \lambda^6 + 81 f \lambda^6 - 9 l \lambda^6) b1$$

$$+ (-12 c \lambda^5 + 3 e \lambda^5 + 81 f \lambda^5 - 9 l \lambda^5) b2 + (6 c \lambda^4 + 21 e \lambda^4 - 7 l \lambda^4) b3) b0^2$$

$$+ (((4 c \lambda^4 - 4 e \lambda^4 - 54 f \lambda^4 + 4 l \lambda^4) b2 + (-12 c \lambda^3 - 24 e \lambda^3 + 4 l \lambda^3) b3) b1$$

$$+ (7 c \lambda^3 + 2 e \lambda^3 - 54 f \lambda^3 + 3 l \lambda^3) b2^2 + (-5 c \lambda^2 - 16 e \lambda^2 + 5 l \lambda^2) b3\, b2 + (-6 c \lambda$$

$$- 12 e \lambda + 2 l \lambda) b3^2) b0 + ((-c \lambda^2 - e \lambda^2 + 9 f \lambda^2) b2^2 + (c + 2 e) b3^2) b1 + (-c \lambda$$

$$- e \lambda + 9 f \lambda) b2^3 + (-c - e) b3\, b2^2) a1^2 + (((((-24 c \lambda^5 + 6 e \lambda^5 + 162 f \lambda^5$$

$$- 18 l \lambda^5) b1 + (-30 c \lambda^4 - 6 e \lambda^4 + 162 f \lambda^4 - 16 l \lambda^4) b2 + (3 c \lambda^3 + 24 e \lambda^3$$

$$- 11 l \lambda^3) b3) b0^2 + ((-4 c \lambda^4 + 4 e \lambda^4 + 54 f \lambda^4 - 4 l \lambda^4) b1^2 + (-13 c \lambda^2 - 20 e \lambda^2$$

$$+ l \lambda^2) b3\, b1 + (10 c \lambda^2 + 8 e \lambda^2 - 54 f \lambda^2 + 2 l \lambda^2) b2^2 + (c \lambda - 4 e \lambda + 3 l \lambda) b3\, b2$$

$$+ b3^2 l) b0 + (2 c \lambda^2 + 2 e \lambda^2 - 18 f \lambda^2) b2\, b1^2 + ((2 c \lambda + 2 e \lambda - 18 f \lambda) b2^2$$

$$+ b3\, c\, b2) b1) a2 + (((12 c \lambda^4 + 42 e \lambda^4 - 14 l \lambda^4) b1 + (3 c \lambda^3 + 24 e \lambda^3$$

$$- 11 l \lambda^3) b2 + (18 c \lambda^2 + 36 e \lambda^2 - 6 l \lambda^2) b3) b0^2 + ((12 c \lambda^3 + 24 e \lambda^3 - 4 l \lambda^3) b1^2$$

$$+ (13 c \lambda^2 + 20 e \lambda^2 - l \lambda^2) b2\, b1 + (7 c \lambda + 8 e \lambda + l \lambda) b2^2 + b3\, b2\, l) b0 + (-2 c$$

$$- 4 e) b3\, b1^2 + b2^2 (c + 2 e) b1) a3) a1 + ((((-15 c \lambda^4 - 3 e \lambda^4 + 81 f \lambda^4 - 8 l \lambda^4) b1$$

$$+ (-21 c \lambda^3 - 15 e \lambda^3 + 81 f \lambda^3 - 6 l \lambda^3) b2 + (-6 c \lambda^2 - 3 e \lambda^2 - 3 l \lambda^2) b3) b0^2 + (($$

$$-7 c \lambda^3 - 2 e \lambda^3 + 54 f \lambda^3 - 3 l \lambda^3) b1^2 + ((-10 c \lambda^2 - 8 e \lambda^2 + 54 f \lambda^2 - 2 l \lambda^2) b2 + ($$

$$-7 c \lambda - 8 e \lambda - l \lambda) b3) b1 + (-2 c - 4 e) b3\, b2) b0 + (-c \lambda^2 - e \lambda^2 + 9 f \lambda^2) b1^3$$

$$+ ((-c \lambda - e \lambda + 9 f \lambda) b2 + b3\, e) b1^2) a2^2 + ((((3 c \lambda^3 + 24 e \lambda^3 - 11 l \lambda^3) b1 + ($$

$$-12\,c\,\lambda^{2}-6\,e\,\lambda^{2}-6\,l\,\lambda^{2}\big)\,b2\big)\,b0^{2}+\Big(\big(5\,c\,\lambda^{2}+16\,e\,\lambda^{2}-5\,l\,\lambda^{2}\big)\,b1^{2}+\Big(\big(-c\,\lambda+4\,e\,\lambda$$
$$-3\,l\,\lambda\big)\,b2-b3\,l\Big)\,b1+(2\,c+4\,e)\,b2^{2}\Big)\,b0+(-c-2\,e)\,b2\,b1^{2}\Big)\,a3\,a2+\Big(\Big(\big(9\,c\,\lambda^{2}$$
$$+18\,e\,\lambda^{2}-3\,l\,\lambda^{2}\big)\,b1+3\,b3\,l\Big)\,b0^{2}+\Big(\big(6\,c\,\lambda+12\,e\,\lambda-2\,l\,\lambda\big)\,b1^{2}-b2\,b1\,l\Big)\,b0$$
$$+b1^{3}\,(c+2\,e)\Big)\,a3^{2}\Big)\,a0+\Big(\big(4\,c\,\lambda^{6}-e\,\lambda^{6}-27\,f\,\lambda^{6}+3\,l\,\lambda^{6}\big)\,b0^{3}+\Big(\big(-2\,c\,\lambda^{4}+2\,e\,\lambda^{4}$$
$$+27\,f\,\lambda^{4}-2\,l\,\lambda^{4}\big)\,b2+\big(6\,c\,\lambda^{3}+12\,e\,\lambda^{3}-2\,l\,\lambda^{3}\big)\,b3\Big)\,b0^{2}+\Big(\big(c\,\lambda^{2}+e\,\lambda^{2}$$
$$-9\,f\,\lambda^{2}\big)\,b2^{2}+(-c-2\,e)\,b3^{2}\Big)\,b0+b2^{3}\,f\Big)\,a1^{3}+\Big(\Big(\Big(\big(12\,c\,\lambda^{5}-3\,e\,\lambda^{5}-81\,f\,\lambda^{5}$$
$$+9\,l\,\lambda^{5}\big)\,b0^{3}+\Big(\big(2\,c\,\lambda^{4}-2\,e\,\lambda^{4}-27\,f\,\lambda^{4}+2\,l\,\lambda^{4}\big)\,b1+\big(-7\,c\,\lambda^{3}-2\,e\,\lambda^{3}+54\,f\,\lambda^{3}$$
$$-3\,l\,\lambda^{3}\big)\,b2+\big(9\,c\,\lambda^{2}+18\,e\,\lambda^{2}-3\,l\,\lambda^{2}\big)\,b3\Big)\,b0^{2}+\Big(\big(-2\,c\,\lambda^{2}-2\,e\,\lambda^{2}+18\,f\,\lambda^{2}\big)\,b2\,b1$$
$$+\big(c\,\lambda+e\,\lambda-9\,f\,\lambda\big)\,b2^{2}+(c+2\,e)\,b3\,b2\Big)\,b0-3\,b1\,b2^{2}\,f\Big)\,a2+\Big(\big(-6\,c\,\lambda^{4}-21\,e\,\lambda^{4}$$
$$+7\,l\,\lambda^{4}\big)\,b0^{3}+\Big(\big(-6\,c\,\lambda^{3}-12\,e\,\lambda^{3}+2\,l\,\lambda^{3}\big)\,b1+\big(-4\,c\,\lambda^{2}-2\,e\,\lambda^{2}-2\,l\,\lambda^{2}\big)\,b2$$
$$+\big(6\,c\,\lambda+12\,e\,\lambda-2\,l\,\lambda\big)\,b3\Big)\,b0^{2}+\Big((2\,c+4\,e)\,b3\,b1-b2^{2}\,e\Big)\,b0\Big)\,a3\Big)\,a1^{2}$$
$$+\Big(\Big(\big(15\,c\,\lambda^{4}+3\,e\,\lambda^{4}-81\,f\,\lambda^{4}+8\,l\,\lambda^{4}\big)\,b0^{3}+\Big(\big(7\,c\,\lambda^{3}+2\,e\,\lambda^{3}-54\,f\,\lambda^{3}+3\,l\,\lambda^{3}\big)\,b1$$
$$+\big(-5\,c\,\lambda^{2}-4\,e\,\lambda^{2}+27\,f\,\lambda^{2}-l\,\lambda^{2}\big)\,b2+\big(3\,c\,\lambda+6\,e\,\lambda-l\,\lambda\big)\,b3\Big)\,b0^{2}+\Big(\big(c\,\lambda^{2}+e\,\lambda^{2}$$
$$-9\,f\,\lambda^{2}\big)\,b1^{2}+\Big(\big(-2\,c\,\lambda-2\,e\,\lambda+18\,f\,\lambda\big)\,b2+(-c-2\,e)\,b3\Big)\,b1\Big)\,b0+3\,b1^{2}\,b2\,f\Big)\,a2^{2}$$
$$+\Big(\big(-3\,c\,\lambda^{3}-24\,e\,\lambda^{3}+11\,l\,\lambda^{3}\big)\,b0^{3}+\Big(\big(-5\,c\,\lambda^{2}-16\,e\,\lambda^{2}+5\,l\,\lambda^{2}\big)\,b1+\big(-7\,c\,\lambda$$
$$-8\,e\,\lambda-l\,\lambda\big)\,b2-b3\,l\Big)\,b0^{2}-b1\,b2\,c\,b0\Big)\,a3\,a2+\Big(\big(-9\,c\,\lambda^{2}-18\,e\,\lambda^{2}+3\,l\,\lambda^{2}\big)\,b0^{3}$$
$$+\big(-6\,c\,\lambda-12\,e\,\lambda+2\,l\,\lambda\big)\,b1\,b0^{2}+(-c-2\,e)\,b1^{2}\,b0\Big)\,a3^{2}\Big)\,a1+\Big(\big(7\,c\,\lambda^{3}+5\,e\,\lambda^{3}$$
$$-27\,f\,\lambda^{3}+2\,l\,\lambda^{3}\big)\,b0^{3}+\Big(\big(5\,c\,\lambda^{2}+4\,e\,\lambda^{2}-27\,f\,\lambda^{2}+l\,\lambda^{2}\big)\,b1+(c+2\,e)\,b3\Big)\,b0^{2}$$
$$+\big(c\,\lambda+e\,\lambda-9\,f\,\lambda\big)\,b1^{2}\,b0-b1^{3}\,f\Big)\,a2^{3}+\Big(\big(6\,c\,\lambda^{2}+3\,e\,\lambda^{2}+3\,l\,\lambda^{2}\big)\,b0^{3}+\Big(\big(4\,c\,\lambda$$
$$+2\,e\,\lambda+2\,l\,\lambda\big)\,b1+(-c-2\,e)\,b2\Big)\,b0^{2}+(c+e)\,b1^{2}\,b0\Big)\,a3\,a2^{2}+l\,a2\,a3^{2}\,b0^{2}\,b1$$
$$-l\,a3^{3}\,b0^{3}$$

$$solve\Big(\Big\{\big(-4\,c\,\lambda^{6}+e\,\lambda^{6}+27\,f\,\lambda^{6}-3\cdot l\cdot\lambda^{6}\big)=0,\ \big(6\,c\,\lambda^{4}+21\,e\,\lambda^{4}-7\,l\,\lambda^{4}\big)=0,\ \big(-15\,c\,\lambda^{4}-3\,e\,\lambda^{4}+81\,f\,\lambda^{4}-8\cdot l\cdot\lambda^{4}\big)=0,\ \big(3\,c\,\lambda^{3}+24\,e\,\lambda^{3}-11\cdot l\cdot\lambda^{3}\big)=0,\ \big(9\,c\,\lambda^{2}+18\,e\,\lambda^{2}-3\cdot l\cdot\lambda^{2}\big)=0,\ \big(-7\,c\,\lambda^{3}-5\,e\,\lambda^{3}+27\,f\,\lambda^{3}-2\cdot l\cdot\lambda^{3}\big)=0,\ \big(6\,c\,\lambda^{2}+3\,e\,\lambda^{2}+3\,l\,\lambda^{2}\big)=0\Big\},\ \{c,e,f,g,h,i,l\}\Big)$$

$$\left\{c=\frac{63\,f}{2},\ e=-\frac{45\,f}{2},\ f=f,\ g=g,\ h=h,\ i=i,\ l=-\frac{81\,f}{2}\right\} \tag{19}$$

$$collect\Bigg(expand\Bigg(subs\Bigg(\left\{c=\frac{63\,f}{2},\ e=-\frac{45\,f}{2},\ f=f,\ g=g,\ h=h,\ i=i,\ l=-\frac{81\,f}{2}\right\},\ 2\cdot\Big(l\,a0^{3}\,b3^{3}+\Big(-b2\,b3^{2}\,l\,a1+(c+2\,e)\,b3\,b2^{2}\,a2+\big(-3\,b3^{2}\,l\,b0+b2\,b3\,l\,b1+(-c-2\,e)\,b2^{3}\big)\,a3\Big)\,a0^{2}+\Big(\Big((c+2\,e)\,b3^{2}\,b1+(-c-e)\,b3\,b2^{2}\Big)\,a1^{2}+\Big(\big(b3^{2}\,l\,b0+b1\,b2\,b3\,c\big)\,a2+\big(b2^{2}\,(c+2\,e)\,b1+b2\,b3\,l\,b0+(-2\,c-4\,e)\,b3\,b1^{2}\big)\,a3\Big)\,a1+\Big((-2\,c-4\,e)\,b3\,b2\,b0$$

$$+ b1^2\, b3\, e\big)\, a2^2 + \Big(\big(-b3\, l\, b1 + (2\, c + 4\, e)\, b2^2\big)\, b0 + (-c - 2\, e)\, b2\, b1^2\Big)\, a3\, a2 + \big(b1^3\, (c + 2\, e) + 3\, b3\, l\, b0^2 - b1\, b2\, l\, b0\big)\, a3^2\Big)\, a0 + \big((-c - 2\, e)\, b3^2\, b0 + b2^3\, f\big)\, a1^3 + \Big(\big((c + 2\, e)\, b3\, b2\, b0 - 3\, b1\, b2^2\, f\big)\, a2 + \big((2\, c + 4\, e)\, b3\, b1 - b2^2\, e\big)\, b0\, a3\Big)\, a1^2 + \Big(\big((-c - 2\, e)\, b3\, b1\, b0 + 3\, b1^2\, b2\, f\big)\, a2^2 + \big(-b3\, l\, b0^2 - b1\, b2\, c\, b0\big)\, a3\, a2 + (-c - 2\, e)\, b1^2\, b0\, a3^2\Big)\, a1 + \big((c + 2\, e)\, b3\, b0^2 - b1^3\, f\big)\, a2^3 + \big((-c - 2\, e)\, b2\, b0^2 + (c + e)\, b1^2\, b0\big)\, a3\, a2^2 + l\, a2\, a3^2\, b0^2\, b1 - l\, a3^3\, b0^3\Big)\Big)\Big),\ \{a0, a1, a2, a3, b0, b1, b2, b3\}\Big)$$

$$-81\, f\, a0^3\, b3^3 + \Big(81\, b2\, b3^2\, f\, a1 - 27\, f\, b3\, b2^2\, a2 + \big(243\, b3^2\, f\, b0 - 81\, b2\, b3\, f\, b1 + 27\, b2^3\, f\big)\, a3\Big)\, a0^2 + \Big(\big(-27\, f\, b3^2\, b1 - 18\, f\, b3\, b2^2\big)\, a1^2 + \Big(\big(-81\, b3^2\, f\, b0 + 63\, b2\, b3\, f\, b1\big)\, a2 + \big(-81\, b2\, b3\, f\, b0 + 54\, f\, b3\, b1^2 - 27\, b1\, b2^2\, f\big)\, a3\Big)\, a1 + \big(54\, b2\, b3\, f\, b0 - 45\, f\, b3\, b1^2\big)\, a2^2 + \Big(\big(81\, b3\, f\, b1 - 54\, f\, b2^2\big)\, b0 + 27\, b1^2\, b2\, f\Big)\, a3\, a2 + \big(-243\, b3\, f\, b0^2 + 81\, b1\, b2\, f\, b0 - 27\, b1^3\, f\big)\, a3^2\Big)\, a0 + \big(27\, b3^2\, f\, b0 + 2\, b2^3\, f\big)\, a1^3 + \Big(\big(-27\, b2\, b3\, f\, b0 - 6\, b1\, b2^2\, f\big)\, a2 + \big(-54\, b3\, f\, b1 + 45\, f\, b2^2\big)\, b0\, a3\Big)\, a1^2 + \Big(\big(27\, f\, b3\, b1\, b0 + 6\, b1^2\, b2\, f\big)\, a2^2 + \big(81\, b3\, f\, b0^2 - 63\, b1\, b2\, f\, b0\big)\, a3\, a2 + 27\, f\, b1^2\, b0\, a3^2\Big)\, a1 + \big(-27\, b3\, f\, b0^2 - 2\, b1^3\, f\big)\, a2^3 + \big(27\, f\, b2\, b0^2 + 18\, f\, b1^2\, b0\big)\, a3\, a2^2 - 81\, f\, a2\, a3^2\, b0^2\, b1 + 81\, f\, a3^3\, b0^3 \qquad \textbf{(20)}$$

*#Let us define then t1*

$$t1 := collect\Big(expand\Big(subs\Big(\Big\{f = -\frac{1}{81}\Big\},\ -81\, f\, a0^3\, b3^3 + \big(81\, b2\, b3^2\, f\, a1 - 27\, f\, b3\, b2^2\, a2 + \big(243\, b3^2\, f\, b0 - 81\, b2\, b3\, f\, b1 + 27\, b2^3\, f\big)\, a3\big)\, a0^2 + \Big(\big(-27\, f\, b3^2\, b1 - 18\, f\, b3\, b2^2\big)\, a1^2 + \Big(\big(-81\, b3^2\, f\, b0 + 63\, b2\, b3\, f\, b1\big)\, a2 + \big(-81\, b2\, b3\, f\, b0 + 54\, f\, b3\, b1^2 - 27\, b1\, b2^2\, f\big)\, a3\Big)\, a1 + \big(54\, b2\, b3\, f\, b0 - 45\, f\, b3\, b1^2\big)\, a2^2 + \Big(\big(81\, b3\, f\, b1 - 54\, f\, b2^2\big)\, b0 + 27\, b1^2\, b2\, f\Big)\, a3\, a2 + \big(-243\, b3\, f\, b0^2 + 81\, b1\, b2\, f\, b0 - 27\, b1^3\, f\big)\, a3^2\Big)\, a0 + \big(27\, b3^2\, f\, b0 + 2\, b2^3\, f\big)\, a1^3 + \Big(\big(-27\, b2\, b3\, f\, b0 - 6\, b1\, b2^2\, f\big)\, a2 + \big(-54\, b3\, f\, b1 + 45\, f\, b2^2\big)\, b0\, a3\Big)\, a1^2 + \Big(\big(27\, f\, b3\, b1\, b0 + 6\, b1^2\, b2\, f\big)\, a2^2 + \big(81\, b3\, f\, b0^2 - 63\, b1\, b2\, f\, b0\big)\, a3\, a2 + 27\, f\, b1^2\, b0\, a3^2\Big)\, a1 + \big(-27\, b3\, f\, b0^2 - 2\, b1^3\, f\big)\, a2^3 + \big(27\, f\, b2\, b0^2 + 18\, f\, b1^2\, b0\big)\, a3\, a2^2 - 81\, f\, a2\, a3^2\, b0^2\, b1 + 81\, f\, a3^3\, b0^3\Big)\Big),\ \{a0, a1, a2, a3, b0, b1, b2, b3\}\Big)$$

$$t1 := a0^3\, b3^3 + \left(\left(-3\, b0\, b3^2 - \frac{1}{3}\, b2^3 + b1\, b2\, b3\right) a3 - a1\, b2\, b3^2 + \frac{a2\, b2^2\, b3}{3}\right) a0^2 + \left(\left(\frac{1}{3}\, b1\, b3^2 + \frac{2}{9}\, b2^2\, b3\right) a1^2 + \left(\left(b0\, b3^2 - \frac{7}{9}\, b1\, b2\, b3\right) a2 + \left(b0\, b2\, b3 - \frac{2}{3}\, b1^2\, b3 + \frac{1}{3}\, b1\, b2^2\right) a3\right) a1 + \left(-\frac{2}{3}\, b0\, b2\, b3 + \frac{5}{9}\, b1^2\, b3\right) a2^2 + \left(\left(-b1\, b3 + \frac{2\, b2^2}{3}\right) b0 - \frac{b1^2\, b2}{3}\right) a3\, a2 + \left(3\, b0^2\, b3 + \frac{1}{3}\, b1^3 - b0\, b1\, b2\right) a3^2\right) a0 + \left(-\frac{b0\, b3^2}{3} - \frac{2\, b2^3}{81}\right) a1^3 + \left(\left(\frac{1}{3}\, b0\, b2\, b3 + \frac{2}{27}\, b1\, b2^2\right) a2 + \left(\frac{2\, b1\, b3}{3}\right.\right. \qquad \textbf{(21)}$$

$- \frac{5\, b2^2}{9}\Big) b0\, a3\Big) a1^2 + \Big(\Big(-\frac{1}{3}\, b0\, b1\, b3 - \frac{2}{27}\, b1^2\, b2\Big) a2^2 + \Big(-b0^2\, b3$
$+ \frac{7}{9}\, b0\, b1\, b2\Big) a3\, a2 - \frac{b1^2\, b0\, a3^2}{3}\Big) a1 + \Big(\frac{b0^2\, b3}{3} + \frac{2\, b1^3}{81}\Big) a2^3 + \Big(-\frac{1}{3}\, b0^2\, b2$
$- \frac{2}{9}\, b0\, b1^2\Big) a3\, a2^2 + a2\, a3^2\, b0^2\, b1 - a3^3\, b0^3$

*#Now we will do exactly the same procedure for the other invariant*

$collect(expand(subs(\{a0 = a3, a1 = -a2, a2 = a1, a3 = -a0, b0 = b3, b1 = -b2, b2 = b1, b3 = -b0\},$
$-a\, a0^2\, a2\, b2^2\, b3 + a\, a0^2\, a3\, b2^3 - a\, a0\, a1^2\, b1\, b3^2 - a\, a0\, a1\, a3\, b1\, b2^2 + a\, a0\, a2\, a3\, b1^2\, b2$
$- a\, a0\, a3^2\, b1^3 + a\, a1^3\, b0\, b3^2 - a\, a1^2\, a2\, b0\, b2\, b3 + a\, a1\, a2^2\, b0\, b1\, b3 + a\, a1\, a3^2\, b0\, b1^2$
$- a\, a2^3\, b0^2\, b3 + a\, a2^2\, a3\, b0^2\, b2 + l\, a0^2\, a2\, b1\, b3^2 - l\, a0^2\, a3\, b1\, b2\, b3 + g\, a0\, a1^2\, b2^2\, b3$
$- l\, a0\, a1\, a2\, b0\, b3^2 + g\, a0\, a1\, a2\, b1\, b2\, b3 + l\, a0\, a1\, a3\, b0\, b2\, b3 - d\, a0\, a1\, a3\, b1^2\, b3$
$- b\, a0\, a2^2\, b1^2\, b3 - d\, a0\, a2^2\, b0\, b2\, b3 - l\, a0\, a2\, a3\, b0\, b1\, b3 + d\, a0\, a2\, a3\, b0\, b2^2$
$+ l\, a0\, a3^2\, b0\, b1\, b2 + h\, a1^3\, b2^3 + i\, a1^2\, a2\, b1\, b2^2 + b\, a1^2\, a3\, b0\, b2^2 + d\, a1^2\, a3\, b0\, b1\, b3$
$- i\, a1\, a2^2\, b1^2\, b2 + l\, a1\, a2\, a3\, b0^2\, b3 - g\, a1\, a2\, a3\, b0\, b1\, b2 - l\, a1\, a3^2\, b0^2\, b2 - h\, a2^3\, b1^3$
$- g\, a2^2\, a3\, b0\, b1^2)), \{a0, a1, a2, a3, b0, b1, b2, b3\})$

$((-a\, b2^2\, b3 + b3^2\, l\, b1)\, a2 + (a\, b2^3 - b2\, b3\, l\, b1)\, a3)\, a0^2 + ((-a\, b3^2\, b1 + b2^2\, b3\, g)\, a1^2$ **(22)**
$+ ((-b3^2\, l\, b0 + b1\, b2\, b3\, g)\, a2 + (-a\, b2^2\, b1 + b2\, b3\, l\, b0 - b3\, d\, b1^2)\, a3)\, a1 + ($
$-b\, b1^2\, b3 - b2\, b3\, d\, b0)\, a2^2 + ((-b3\, l\, b1 + b2^2\, d)\, b0 + a\, b1^2\, b2)\, a3\, a2 + (-a\, b1^3$
$+ b1\, b2\, l\, b0)\, a3^2)\, a0 + (a\, b3^2\, b0 + b2^3\, h)\, a1^3 + ((-a\, b3\, b2\, b0 + b1\, b2^2\, i)\, a2 + (b\, b2^2$
$+ b3\, d\, b1)\, b0\, a3)\, a1^2 + ((a\, b1\, b3\, b0 - b1^2\, b2\, i)\, a2^2 + (b3\, l\, b0^2 - b1\, b2\, g\, b0)\, a3\, a2$
$+ (a\, b1^2\, b0 - b2\, l\, b0^2)\, a3^2)\, a1 + (-a\, b3\, b0^2 - b1^3\, h)\, a2^3 + (a\, b2\, b0^2$
$- b1^2\, g\, b0)\, a3\, a2^2$

$collect(expand(-a\, a0^2\, a2\, b2^2\, b3 + a\, a0^2\, a3\, b2^3 - a\, a0\, a1^2\, b1\, b3^2 - a\, a0\, a1\, a3\, b1\, b2^2$
$+ a\, a0\, a2\, a3\, b1^2\, b2 - a\, a0\, a3^2\, b1^3 + a\, a1^3\, b0\, b3^2 - a\, a1^2\, a2\, b0\, b2\, b3 + a\, a1\, a2^2\, b0\, b1\, b3$
$+ a\, a1\, a3^2\, b0\, b1^2 - a\, a2^3\, b0^2\, b3 + a\, a2^2\, a3\, b0^2\, b2 + l\, a0^2\, a2\, b1\, b3^2 - l\, a0^2\, a3\, b1\, b2\, b3$
$+ g\, a0\, a1^2\, b2^2\, b3 - l\, a0\, a1\, a2\, b0\, b3^2 + g\, a0\, a1\, a2\, b1\, b2\, b3 + l\, a0\, a1\, a3\, b0\, b2\, b3$
$- d\, a0\, a1\, a3\, b1^2\, b3 - b\, a0\, a2^2\, b1^2\, b3 - d\, a0\, a2^2\, b0\, b2\, b3 - l\, a0\, a2\, a3\, b0\, b1\, b3$
$+ d\, a0\, a2\, a3\, b0\, b2^2 + l\, a0\, a3^2\, b0\, b1\, b2 + h\, a1^3\, b2^3 + i\, a1^2\, a2\, b1\, b2^2 + b\, a1^2\, a3\, b0\, b2^2$
$+ d\, a1^2\, a3\, b0\, b1\, b3 - i\, a1\, a2^2\, b1^2\, b2 + l\, a1\, a2\, a3\, b0^2\, b3 - g\, a1\, a2\, a3\, b0\, b1\, b2$
$- l\, a1\, a3^2\, b0^2\, b2 - h\, a2^3\, b1^3 - g\, a2^2\, a3\, b0\, b1^2), \{a0, a1, a2, a3, b0, b1, b2, b3\})$

$((-a\, b2^2\, b3 + b3^2\, l\, b1)\, a2 + (a\, b2^3 - b2\, b3\, l\, b1)\, a3)\, a0^2 + ((-a\, b3^2\, b1 + b2^2\, b3\, g)\, a1^2$ **(23)**
$+ ((-b3^2\, l\, b0 + b1\, b2\, b3\, g)\, a2 + (-a\, b2^2\, b1 + b2\, b3\, l\, b0 - b3\, d\, b1^2)\, a3)\, a1 + ($
$-b\, b1^2\, b3 - b2\, b3\, d\, b0)\, a2^2 + ((-b3\, l\, b1 + b2^2\, d)\, b0 + a\, b1^2\, b2)\, a3\, a2 + (-a\, b1^3$
$+ b1\, b2\, l\, b0)\, a3^2)\, a0 + (a\, b3^2\, b0 + b2^3\, h)\, a1^3 + ((-a\, b3\, b2\, b0 + b1\, b2^2\, i)\, a2 + (b\, b2^2$
$+ b3\, d\, b1)\, b0\, a3)\, a1^2 + ((a\, b1\, b3\, b0 - b1^2\, b2\, i)\, a2^2 + (b3\, l\, b0^2 - b1\, b2\, g\, b0)\, a3\, a2$

$+ \left( \left( a\, b1^{2}\, b0 - b2\, l\, b0^{2} \right) a3^{2} \right) a1 + \left( -a\, b3\, b0^{2} - b1^{3}\, h \right) a2^{3} + \left( a\, b2\, b0^{2} - b1^{2}\, g\, b0 \right) a3\, a2^{2}$

*#This is still ok, it does not change anything,*

$collect(expand(subs(\{a0 = b0, a1 = b1, a2 = b2, a3 = b3, b0 = -a0, b1 = -a1, b2 = -a2, b3 = -a3\}, ((-a\, b2^{2}\, b3 + b3^{2}\, l\, b1)\, a2 + (a\, b2^{3} - b2\, b3\, l\, b1)\, a3)\, a0^{2} + ((-a\, b3^{2}\, b1 + b2^{2}\, b3\, g)\, a1^{2} + ((-b3^{2}\, l\, b0 + b1\, b2\, b3\, g)\, a2 + (-a\, b2^{2}\, b1 + b2\, b3\, l\, b0 - b3\, d\, b1^{2})\, a3)\, a1 + (-b\, b1^{2}\, b3 - b2\, b3\, d\, b0)\, a2^{2} + ((-b3\, l\, b1 + b2^{2}\, d)\, b0 + a\, b1^{2}\, b2)\, a3\, a2 + (-a\, b1^{3} + b1\, b2\, l\, b0)\, a3^{2})\, a0 + (a\, b3^{2}\, b0 + b2^{3}\, h)\, a1^{3} + ((-a\, b3\, b2\, b0 + b1\, b2^{2}\, i)\, a2 + (b\, b2^{2} + b3\, d\, b1)\, b0\, a3)\, a1^{2} + ((a\, b1\, b3\, b0 - b1^{2}\, b2\, i)\, a2^{2} + (b3\, l\, b0^{2} - b1\, b2\, g\, b0)\, a3\, a2 + (a\, b1^{2}\, b0 - b2\, l\, b0^{2})\, a3^{2})\, a1 + (-a\, b3\, b0^{2} - b1^{3}\, h)\, a2^{3} + (a\, b2\, b0^{2} - b1^{2}\, g\, b0)\, a3\, a2^{2})), \{a0, a1, a2, a3, b0, b1, b2, b3\})$

$$((-a\, b2^{2}\, b3 + b3^{2}\, l\, b1)\, a2 + (a\, b2^{3} - b2\, b3\, l\, b1)\, a3)\, a0^{2} + ((-a\, b3^{2}\, b1 + b2^{2}\, b3\, g)\, a1^{2} + ((-b3^{2}\, l\, b0 + b1\, b2\, b3\, g)\, a2 + (-a\, b2^{2}\, b1 + b2\, b3\, l\, b0 - b3\, d\, b1^{2})\, a3)\, a1 + (-b\, b1^{2}\, b3 - b2\, b3\, d\, b0)\, a2^{2} + ((-b3\, l\, b1 + b2^{2}\, d)\, b0 + a\, b1^{2}\, b2)\, a3\, a2 + (-a\, b1^{3} + b1\, b2\, l\, b0)\, a3^{2})\, a0 + (a\, b3^{2}\, b0 + b2^{3}\, h)\, a1^{3} + ((-a\, b3\, b2\, b0 + b1\, b2^{2}\, i)\, a2 + (b\, b2^{2} + b3\, d\, b1)\, b0\, a3)\, a1^{2} + ((a\, b1\, b3\, b0 - b1^{2}\, b2\, i)\, a2^{2} + (b3\, l\, b0^{2} - b1\, b2\, g\, b0)\, a3\, a2 + (a\, b1^{2}\, b0 - b2\, l\, b0^{2})\, a3^{2})\, a1 + (-a\, b3\, b0^{2} - b1^{3}\, h)\, a2^{3} + (a\, b2\, b0^{2} - b1^{2}\, g\, b0)\, a3\, a2^{2} \quad \textbf{(24)}$$

*#This does not change anything either.*

$collect(expand(subs(\{a0 = a0, a1 = a1, a2 = a2, a3 = a3, b0 = (a0\, \mathrm{lambda} + b0), b1 = (a1\, \mathrm{lambda} + b1), b2 = (a2\, \mathrm{lambda} + b2), b3 = (a3\, \mathrm{lambda} + b3)\}, ((-a\, b2^{2}\, b3 + b3^{2}\, l\, b1)\, a2 + (a\, b2^{3} - b2\, b3\, l\, b1)\, a3)\, a0^{2} + ((-a\, b3^{2}\, b1 + b2^{2}\, b3\, g)\, a1^{2} + ((-b3^{2}\, l\, b0 + b1\, b2\, b3\, g)\, a2 + (-a\, b2^{2}\, b1 + b2\, b3\, l\, b0 - b3\, d\, b1^{2})\, a3)\, a1 + (-b\, b1^{2}\, b3 - b2\, b3\, d\, b0)\, a2^{2} + ((-b3\, l\, b1 + b2^{2}\, d)\, b0 + a\, b1^{2}\, b2)\, a3\, a2 + (-a\, b1^{3} + b1\, b2\, l\, b0)\, a3^{2})\, a0 + (a\, b3^{2}\, b0 + b2^{3}\, h)\, a1^{3} + ((-a\, b3\, b2\, b0 + b1\, b2^{2}\, i)\, a2 + (b\, b2^{2} + b3\, d\, b1)\, b0\, a3)\, a1^{2} + ((a\, b1\, b3\, b0 - b1^{2}\, b2\, i)\, a2^{2} + (b3\, l\, b0^{2} - b1\, b2\, g\, b0)\, a3\, a2 + (a\, b1^{2}\, b0 - b2\, l\, b0^{2})\, a3^{2})\, a1 + (-a\, b3\, b0^{2} - b1^{3}\, h)\, a2^{3} + (a\, b2\, b0^{2} - b1^{2}\, g\, b0)\, a3\, a2^{2})), \{a0, a1, a2, a3, b0, b1, b2, b3\})$

$$((-2\, a\, \lambda^{2} - d\, \lambda^{2})\, b3\, a2^{3} + ((2\, a\, \lambda^{2} + d\, \lambda^{2})\, b2\, a3 + (-2\, a\, \lambda - d\, \lambda)\, b3\, b2)\, a2^{2} + ((2\, a\, \lambda + d\, \lambda)\, b2^{2}\, a3 - a\, b2^{2}\, b3 + b3^{2}\, l\, b1)\, a2 + (a\, b2^{3} - b2\, b3\, l\, b1)\, a3)\, a0^{2} + (((-b\, \lambda^{2} + 2\, g\, \lambda^{2})\, b3\, a2^{2} + ((-2\, a\, \lambda^{2} + 2\, b\, \lambda^{2} + 2\, g\, \lambda^{2})\, b2\, a3 + (-a\, \lambda + 3\, g\, \lambda)\, b3\, b2)\, a2 + (-2\, a\, \lambda^{2} - d\, \lambda^{2})\, b1\, a3^{2} + ((-2\, a\, \lambda - d\, \lambda)\, b3\, b1 + (-a\, \lambda + b\, \lambda + g\, \lambda)\, b2^{2})\, a3 - a\, b3^{2}\, b1 + b2^{2}\, b3\, g)\, a1^{2} + (((2\, a\, \lambda^{2} - 2\, b\, \lambda^{2} - 2\, g\, \lambda^{2})\, b1\, a3 + (a\, \lambda - 2\, b\, \lambda + g\, \lambda)\, b3\, b1)\, a2^{2} + (-b3^{2}\, l\, b0 + b1\, b2\, b3\, g)\, a2 + (-2\, a\, \lambda - d\, \lambda)\, b1^{2}\, a3^{2} + (-a\, b2^{2}\, b1 + b2\, b3\, l\, b0 - b3\, d\, b1^{2})\, a3)\, a1 + (-2\, a\, \lambda - d\, \lambda)\, b3\, b0\, a2^{3} + (((2\, a\, \lambda + d\, \lambda)\, b2\, b0 + (a\, \lambda - b\, \lambda - g\, \lambda)\, b1^{2})\, a3 - b\, b1^{2}\, b3 - b2\, b3\, d\, b0)\, a2^{2} + ((-b3\, l\, b1 + b2^{2}\, d)\, b0 \quad \textbf{(25)}$$

$$+ a\,b1^2\,b2)\,a3\,a2 + (-a\,b1^3 + b1\,b2\,l\,b0)\,a3^2)\,a0 + ((3\,h\,\lambda^2 + i\,\lambda^2)\,b2\,a2^2 + (3\,h\,\lambda + i\,\lambda)\,b2^2\,a2 + (2\,a\,\lambda^2 + d\,\lambda^2)\,b0\,a3^2 + (2\,a\,\lambda + d\,\lambda)\,b3\,b0\,a3 + a\,b3^2\,b0 + b2^3\,h)\,a1^3 + ((-3\,h\,\lambda^2 - i\,\lambda^2)\,b1\,a2^3 + (b\,\lambda^2 - 2\,g\,\lambda^2)\,b0\,a3\,a2^2 + ((-a\,\lambda + 2\,b\,\lambda - g\,\lambda)\,b2\,b0\,a3 - a\,b3\,b2\,b0 + b1\,b2^2\,i)\,a2 + (2\,a\,\lambda + d\,\lambda)\,b1\,b0\,a3^2 + (b\,b2^2 + b3\,d\,b1)\,b0\,a3)\,a1^2 + ((-3\,h\,\lambda - i\,\lambda)\,b1^2\,a2^3 + ((a\,\lambda - 3\,g\,\lambda)\,b1\,b0\,a3 + a\,b1\,b3\,b0 - b1^2\,b2\,i)\,a2^2 + (b3\,l\,b0^2 - b1\,b2\,g\,b0)\,a3\,a2 + (a\,b1^2\,b0 - b2\,l\,b0^2)\,a3^2)\,a1 + (-a\,b3\,b0^2 - b1^3\,h)\,a2^3 + (a\,b2\,b0^2 - b1^2\,g\,b0)\,a3\,a2^2$$

$$solve(\{(2\,a\,\lambda^2 + d\,\lambda^2) = 0, (2\,a\,\lambda + d\,\lambda) = 0, (-2\,a\,\lambda^2 + 2\,b\,\lambda^2 + 2\,g\,\lambda^2) = 0, (-a\,\lambda + 3\,g\,\lambda) = 0, (-b\,\lambda^2 + 2\,g\,\lambda^2) = 0, (-a\,\lambda + b\,\lambda + g\,\lambda) = 0, (3\,h\,\lambda^2 + i\,\lambda^2) = 0, (3\,h\,\lambda + i\,\lambda) = 0, (-a\,\lambda + 2\,b\,\lambda - g\,\lambda) = 0\}, \{a, b, c, d, e, f, g, h, i, j, k, l\})$$

$$\{a = 3\,g, b = 2\,g, c = c, d = -6\,g, e = e, f = f, g = g, h = h, i = -3\,h, j = j, k = k, l = l\} \tag{26}$$

$$collect(expand(subs(\{a = 3\,g, b = 2\,g, c = c, d = -6\,g, e = e, f = f, g = g, h = h, i = -3\,h, j = j, k = k, l = l\}, ((-a\,b2^2\,b3 + b3^2\,l\,b1)\,a2 + (a\,b2^3 - b2\,b3\,l\,b1)\,a3)\,a0^2 + ((-a\,b3^2\,b1 + b2^2\,b3\,g)\,a1^2 + ((-b3^2\,l\,b0 + b1\,b2\,b3\,g)\,a2 + (-a\,b2^2\,b1 + b2\,b3\,l\,b0 - b3\,d\,b1^2)\,a3)\,a1 + (-b\,b1^2\,b3 - b2\,b3\,d\,b0)\,a2^2 + ((-b3\,l\,b1 + b2^2\,d)\,b0 + a\,b1^2\,b2)\,a3\,a2 + (-a\,b1^3 + b1\,b2\,l\,b0)\,a3^2)\,a0 + (a\,b3^2\,b0 + b2^3\,h)\,a1^3 + ((-a\,b3\,b2\,b0 + b1\,b2^2\,i)\,a2 + (b\,b2^2 + b3\,d\,b1)\,b0\,a3)\,a1^2 + ((a\,b1\,b3\,b0 - b1^2\,b2\,i)\,a2^2 + (b3\,l\,b0^2 - b1\,b2\,g\,b0)\,a3\,a2 + (a\,b1^2\,b0 - b2\,l\,b0^2)\,a3^2)\,a1 + (-a\,b3\,b0^2 - b1^3\,h)\,a2^3 + (a\,b2\,b0^2 - b1^2\,g\,b0)\,a3\,a2^2)), \{a0, a1, a2, a3, b0, b1, b2, b3\})$$

$$((b3^2\,l\,b1 - 3\,b2^2\,b3\,g)\,a2 + (-b2\,b3\,l\,b1 + 3\,b2^3\,g)\,a3)\,a0^2 + ((-3\,b3^2\,g\,b1 + b2^2\,b3\,g)\,a1^2 + ((-b3^2\,l\,b0 + b1\,b2\,b3\,g)\,a2 + (b2\,b3\,l\,b0 + 6\,b3\,g\,b1^2 - 3\,b2^2\,g\,b1)\,a3)\,a1 + (6\,b2\,b3\,g\,b0 - 2\,b3\,g\,b1^2)\,a2^2 + ((-b3\,l\,b1 - 6\,b2^2\,g)\,b0 + 3\,g\,b1^2\,b2)\,a3\,a2 + (b1\,b2\,l\,b0 - 3\,b1^3\,g)\,a3^2)\,a0 + (3\,b3^2\,g\,b0 + b2^3\,h)\,a1^3 + ((-3\,b2\,b3\,g\,b0 - 3\,b1\,b2^2\,h)\,a2 + (-6\,b3\,g\,b1 + 2\,b2^2\,g)\,b0\,a3)\,a1^2 + ((3\,b3\,g\,b1\,b0 + 3\,b1^2\,b2\,h)\,a2^2 + (b3\,l\,b0^2 - b1\,b2\,g\,b0)\,a3\,a2 + (-b2\,l\,b0^2 + 3\,b1^2\,g\,b0)\,a3^2)\,a1 + (-3\,b3\,g\,b0^2 - b1^3\,h)\,a2^3 + (3\,b2\,g\,b0^2 - b1^2\,g\,b0)\,a3\,a2^2 \tag{27}$$

$$collect(expand(subs(\{a0 = a0, a1 = (3\,a0\,\mathrm{lambda} + a1), a2 = (3\,a0\,\mathrm{lambda}^2 + 2\,a1\,\mathrm{lambda} + a2), a3 = (a0\,\mathrm{lambda}^3 + a1\,\mathrm{lambda}^2 + a2\,\mathrm{lambda} + a3), b0 = b0, b1 = (3\,b0\,\mathrm{lambda} + b1), b2 = (3\,b0\,\mathrm{lambda}^2 + 2\,b1\,\mathrm{lambda} + b2), b3 = (b0\,\mathrm{lambda}^3 + b1\,\mathrm{lambda}^2 + b2\,\mathrm{lambda} + b3)\}, ((b3^2\,l\,b1 - 3\,b2^2\,b3\,g)\,a2 + (-b2\,b3\,l\,b1 + 3\,b2^3\,g)\,a3)\,a0^2 + ((-3\,b3^2\,g\,b1 + b2^2\,b3\,g)\,a1^2 + ((-b3^2\,l\,b0 + b1\,b2\,b3\,g)\,a2 + (b2\,b3\,l\,b0 + 6\,b3\,g\,b1^2 - 3\,b2^2\,g\,b1)\,a3)\,a1 + (6\,b2\,b3\,g\,b0 - 2\,b3\,g\,b1^2)\,a2^2 + ((-b3\,l\,b1 - 6\,b2^2\,g)\,b0 + 3\,g\,b1^2\,b2)\,a3\,a2 + (b1\,b2\,l\,b0 - 3\,b1^3\,g)\,a3^2)\,a0 + (3\,b3^2\,g\,b0 + b2^3\,h)\,a1^3 + ((-3\,b2\,b3\,g\,b0 - 3\,b1\,b2^2\,h)\,a2 + (-6\,b3\,g\,b1 + 2\,b2^2\,g)\,b0\,a3)\,a1^2 + ((3\,b3\,g\,b1\,b0 + 3\,b1^2\,b2\,h)\,a2^2 + (b3\,l\,b0^2 - b1\,b2\,g\,b0)\,a3\,a2 + (-b2\,l\,b0^2 + 3\,b1^2\,g\,b0)\,a3^2)\,a1 + ($$

$$\begin{array}{l}
-3\,b3\,g\,b0^2 - b1^3\,h)\,a2^3 + (3\,b2\,g\,b0^2 - b1^2\,g\,b0)\,a3\,a2^2)),\ \{a0, a1, a2, a3, b0, b1, b2, b3\})
\end{array}$$

$$\begin{array}{l}
((-6\,g\,\lambda^6 + 27\,h\,\lambda^6 + l\,\lambda^6)\,b1^3 + ((-18\,g\,\lambda^5 + 81\,h\,\lambda^5 + 3\,l\,\lambda^5)\,b2 + (-36\,g\,\lambda^4 \qquad \textbf{(28)}\\
+ 4\,l\,\lambda^4)\,b3)\,b1^2 + ((-9\,g\,\lambda^4 + 81\,h\,\lambda^4 + 2\,l\,\lambda^4)\,b2^2 + (-45\,g\,\lambda^3 + 5\,l\,\lambda^3)\,b3\,b2 + (\\
-27\,g\,\lambda^2 + 3\,l\,\lambda^2)\,b3^2)\,b1 + (3\,g\,\lambda^3 + 27\,h\,\lambda^3)\,b2^3)\,a0^3 + ((((18\,g\,\lambda^6 - 81\,h\,\lambda^6\\
- 3\,l\,\lambda^6)\,b1^2 + ((36\,g\,\lambda^5 - 162\,h\,\lambda^5 - 6\,l\,\lambda^5)\,b2 + (72\,g\,\lambda^4 - 8\,l\,\lambda^4)\,b3)\,b1 + (9\,g\,\lambda^4\\
- 81\,h\,\lambda^4 - 2\,l\,\lambda^4)\,b2^2 + (45\,g\,\lambda^3 - 5\,l\,\lambda^3)\,b3\,b2 + (27\,g\,\lambda^2 - 3\,l\,\lambda^2)\,b3^2)\,b0 + ((\\
-6\,g\,\lambda^4 + 27\,h\,\lambda^4 + l\,\lambda^4)\,b2 + (-18\,g\,\lambda^3 + 2\,l\,\lambda^3)\,b3)\,b1^2 + ((-3\,g\,\lambda^3 + 54\,h\,\lambda^3\\
+ l\,\lambda^3)\,b2^2 + (-27\,g\,\lambda^2 + 3\,l\,\lambda^2)\,b3\,b2 + (-18\,g\,\lambda + 2\,l\,\lambda)\,b3^2)\,b1 + (3\,g\,\lambda^2\\
+ 27\,h\,\lambda^2)\,b2^3)\,a1 + (((18\,g\,\lambda^5 - 81\,h\,\lambda^5 - 3\,l\,\lambda^5)\,b1^2 + ((18\,g\,\lambda^4 - 162\,h\,\lambda^4\\
- 4\,l\,\lambda^4)\,b2 + (45\,g\,\lambda^3 - 5\,l\,\lambda^3)\,b3)\,b1 + (-9\,g\,\lambda^3 - 81\,h\,\lambda^3)\,b2^2)\,b0 + (6\,g\,\lambda^4\\
- 27\,h\,\lambda^4 - l\,\lambda^4)\,b1^3 + (3\,g\,\lambda^3 - 54\,h\,\lambda^3 - l\,\lambda^3)\,b2\,b1^2 + ((-3\,g\,\lambda^2 - 27\,h\,\lambda^2)\,b2^2\\
+ (-9\,g\,\lambda + l\,\lambda)\,b3\,b2 + b3^2\,l)\,b1 - 3\,b2^2\,b3\,g)\,a2 + (((36\,g\,\lambda^4 - 4\,l\,\lambda^4)\,b1^2\\
+ ((45\,g\,\lambda^3 - 5\,l\,\lambda^3)\,b2 + (54\,g\,\lambda^2 - 6\,l\,\lambda^2)\,b3)\,b1)\,b0 + (18\,g\,\lambda^3 - 2\,l\,\lambda^3)\,b1^3\\
+ ((27\,g\,\lambda^2 - 3\,l\,\lambda^2)\,b2 + (18\,g\,\lambda - 2\,l\,\lambda)\,b3)\,b1^2 + ((9\,g\,\lambda - l\,\lambda)\,b2^2 - b3\,b2\,l)\,b1\\
+ 3\,b2^3\,g)\,a3)\,a0^2 + ((((-18\,g\,\lambda^6 + 81\,h\,\lambda^6 + 3\,l\,\lambda^6)\,b1 + (-18\,g\,\lambda^5 + 81\,h\,\lambda^5\\
+ 3\,l\,\lambda^5)\,b2 + (-36\,g\,\lambda^4 + 4\,l\,\lambda^4)\,b3)\,b0^2 + (((12\,g\,\lambda^4 - 54\,h\,\lambda^4 - 2\,l\,\lambda^4)\,b2\\
+ (36\,g\,\lambda^3 - 4\,l\,\lambda^3)\,b3)\,b1 + (3\,g\,\lambda^3 - 54\,h\,\lambda^3 - l\,\lambda^3)\,b2^2 + (27\,g\,\lambda^2 - 3\,l\,\lambda^2)\,b3\,b2\\
+ (18\,g\,\lambda - 2\,l\,\lambda)\,b3^2)\,b0 + ((g\,\lambda^2 + 9\,h\,\lambda^2)\,b2^2 - 3\,b3^2\,g)\,b1 + (g\,\lambda + 9\,h\,\lambda)\,b2^3\\
+ b2^2\,b3\,g)\,a1^2 + ((((-36\,g\,\lambda^5 + 162\,h\,\lambda^5 + 6\,l\,\lambda^5)\,b1 + (-18\,g\,\lambda^4 + 162\,h\,\lambda^4\\
+ 4\,l\,\lambda^4)\,b2 + (-45\,g\,\lambda^3 + 5\,l\,\lambda^3)\,b3)\,b0^2 + ((-12\,g\,\lambda^4 + 54\,h\,\lambda^4 + 2\,l\,\lambda^4)\,b1^2\\
+ (27\,g\,\lambda^2 - 3\,l\,\lambda^2)\,b3\,b1 + (-6\,g\,\lambda^2 - 54\,h\,\lambda^2)\,b2^2 + (9\,g\,\lambda - l\,\lambda)\,b3\,b2 - b3^2\,l)\,b0\\
+ (-2\,g\,\lambda^2 - 18\,h\,\lambda^2)\,b2\,b1^2 + ((-2\,g\,\lambda - 18\,h\,\lambda)\,b2^2 + b2\,b3\,g)\,b1)\,a2 + (((\\
-72\,g\,\lambda^4 + 8\,l\,\lambda^4)\,b1 + (-45\,g\,\lambda^3 + 5\,l\,\lambda^3)\,b2 + (-54\,g\,\lambda^2 + 6\,l\,\lambda^2)\,b3)\,b0^2 + ((\\
-36\,g\,\lambda^3 + 4\,l\,\lambda^3)\,b1^2 + (-27\,g\,\lambda^2 + 3\,l\,\lambda^2)\,b2\,b1 + (-9\,g\,\lambda + l\,\lambda)\,b2^2 + b3\,b2\,l)\,b0\\
+ 6\,b3\,g\,b1^2 - 3\,b2^2\,g\,b1)\,a3)\,a1 + (((-9\,g\,\lambda^4 + 81\,h\,\lambda^4 + 2\,l\,\lambda^4)\,b1 + (9\,g\,\lambda^3\\
+ 81\,h\,\lambda^3)\,b2)\,b0^2 + ((-3\,g\,\lambda^3 + 54\,h\,\lambda^3 + l\,\lambda^3)\,b1^2 + ((6\,g\,\lambda^2 + 54\,h\,\lambda^2)\,b2\\
+ (9\,g\,\lambda - l\,\lambda)\,b3)\,b1 + 6\,b2\,b3\,g)\,b0 + (g\,\lambda^2 + 9\,h\,\lambda^2)\,b1^3 + ((g\,\lambda + 9\,h\,\lambda)\,b2\\
- 2\,b3\,g)\,b1^2)\,a2^2 + ((-45\,g\,\lambda^3 + 5\,l\,\lambda^3)\,b1\,b0^2 + ((-27\,g\,\lambda^2 + 3\,l\,\lambda^2)\,b1^2 + ((\\
-9\,g\,\lambda + l\,\lambda)\,b2 - b3\,l)\,b1 - 6\,b2^2\,g)\,b0 + 3\,g\,b1^2\,b2)\,a3\,a2 + ((-27\,g\,\lambda^2\\
+ 3\,l\,\lambda^2)\,b1\,b0^2 + ((-18\,g\,\lambda + 2\,l\,\lambda)\,b1^2 + b2\,b1\,l)\,b0 - 3\,b1^3\,g)\,a3^2)\,a0 + ((6\,g\,\lambda^6
\end{array}$$

$- 27\,h\,\lambda^6 - l\,\lambda^6)\,b0^3 + ((-6\,g\,\lambda^4 + 27\,h\,\lambda^4 + l\,\lambda^4)\,b2 + (-18\,g\,\lambda^3 + 2\,l\,\lambda^3)\,b3)\,b0^2$
$+ ((-g\,\lambda^2 - 9\,h\,\lambda^2)\,b2^2 + 3\,b3^2\,g)\,b0 + b2^3\,h)\,a1^3 + (((18\,g\,\lambda^5 - 81\,h\,\lambda^5$
$- 3\,l\,\lambda^5)\,b0^3 + ((6\,g\,\lambda^4 - 27\,h\,\lambda^4 - l\,\lambda^4)\,b1 + (-3\,g\,\lambda^3 + 54\,h\,\lambda^3 + l\,\lambda^3)\,b2 + ($
$-27\,g\,\lambda^2 + 3\,l\,\lambda^2)\,b3)\,b0^2 + ((2\,g\,\lambda^2 + 18\,h\,\lambda^2)\,b2\,b1 + (-g\,\lambda - 9\,h\,\lambda)\,b2^2$
$- 3\,b2\,b3\,g)\,b0 - 3\,b1\,b2^2\,h)\,a2 + ((36\,g\,\lambda^4 - 4\,l\,\lambda^4)\,b0^3 + ((18\,g\,\lambda^3 - 2\,l\,\lambda^3)\,b1 + ($
$-18\,g\,\lambda + 2\,l\,\lambda)\,b3)\,b0^2 + (-6\,b3\,g\,b1 + 2\,b2^2\,g)\,b0)\,a3)\,a1^2 + (((9\,g\,\lambda^4 - 81\,h\,\lambda^4$
$- 2\,l\,\lambda^4)\,b0^3 + ((3\,g\,\lambda^3 - 54\,h\,\lambda^3 - l\,\lambda^3)\,b1 + (3\,g\,\lambda^2 + 27\,h\,\lambda^2)\,b2 + (-9\,g\,\lambda$
$+ l\,\lambda)\,b3)\,b0^2 + ((-g\,\lambda^2 - 9\,h\,\lambda^2)\,b1^2 + ((2\,g\,\lambda + 18\,h\,\lambda)\,b2 + 3\,b3\,g)\,b1)\,b0$
$+ 3\,b1^2\,b2\,h)\,a2^2 + ((45\,g\,\lambda^3 - 5\,l\,\lambda^3)\,b0^3 + ((27\,g\,\lambda^2 - 3\,l\,\lambda^2)\,b1 + (9\,g\,\lambda - l\,\lambda)\,b2$
$+ b3\,l)\,b0^2 - b1\,b2\,g\,b0)\,a3\,a2 + ((27\,g\,\lambda^2 - 3\,l\,\lambda^2)\,b0^3 + ((18\,g\,\lambda - 2\,l\,\lambda)\,b1$
$- b2\,l)\,b0^2 + 3\,b1^2\,g\,b0)\,a3^2)\,a1 + ((-3\,g\,\lambda^3 - 27\,h\,\lambda^3)\,b0^3 + ((-3\,g\,\lambda^2$
$- 27\,h\,\lambda^2)\,b1 - 3\,b3\,g)\,b0^2 + (-g\,\lambda - 9\,h\,\lambda)\,b1^2\,b0 - b1^3\,h)\,a2^3 + (3\,b2\,g\,b0^2$
$- b1^2\,g\,b0)\,a3\,a2^2$

$solve(\{(-6\,g\,\lambda^6 + 27\,h\,\lambda^6 + l\,\lambda^6) = 0, (-36\,g\,\lambda^4 + 4\,l\,\lambda^4) = 0, (-9\,g\,\lambda^4 + 81\,h\,\lambda^4 + 2\,l\,\lambda^4), (-45\,g\,\lambda^3 + 5\,l\,\lambda^3) = 0, (-27\,g\,\lambda^2 + 3\,l\,\lambda^2), (-g\,\lambda - 9\,h\,\lambda) = 0\}, \{c, e, f, g, h, j, k, l\})$

$$\{c = c, e = e, f = f, g = -9\,h, h = h, j = j, k = k, l = -81\,h\} \quad \textbf{(29)}$$

$collect(expand(subs(\{c = c, e = e, f = f, g = -9\,h, h = h, j = j, k = k, l = -81\,h\}, ((b3^2\,l\,b1$
$- 3\,b2^2\,b3\,g)\,a2 + (-b2\,b3\,l\,b1 + 3\,b2^3\,g)\,a3)\,a0^2 + ((-3\,b3^2\,g\,b1 + b2^2\,b3\,g)\,a1^2 + (($
$-b3^2\,l\,b0 + b1\,b2\,b3\,g)\,a2 + (b2\,b3\,l\,b0 + 6\,b3\,g\,b1^2 - 3\,b2^2\,g\,b1)\,a3)\,a1 + (6\,b2\,b3\,g\,b0$
$- 2\,b3\,g\,b1^2)\,a2^2 + ((-b3\,l\,b1 - 6\,b2^2\,g)\,b0 + 3\,g\,b1^2\,b2)\,a3\,a2 + (b1\,b2\,l\,b0$
$- 3\,b1^3\,g)\,a3^2)\,a0 + (3\,b3^2\,g\,b0 + b2^3\,h)\,a1^3 + ((-3\,b2\,b3\,g\,b0 - 3\,b1\,b2^2\,h)\,a2 + ($
$-6\,b3\,g\,b1 + 2\,b2^2\,g)\,b0\,a3)\,a1^2 + ((3\,b3\,g\,b1\,b0 + 3\,b1^2\,b2\,h)\,a2^2 + (b3\,l\,b0^2$
$- b1\,b2\,g\,b0)\,a3\,a2 + (-b2\,l\,b0^2 + 3\,b1^2\,g\,b0)\,a3^2)\,a1 + (-3\,b3\,g\,b0^2 - b1^3\,h)\,a2^3$
$+ (3\,b2\,g\,b0^2 - b1^2\,g\,b0)\,a3\,a2^2))), \{a0, a1, a2, a3, b0, b1, b2, b3\})$

$$((-81\,b3^2\,h\,b1 + 27\,b2^2\,b3\,h)\,a2 + (81\,b2\,b3\,h\,b1 - 27\,b2^3\,h)\,a3)\,a0^2 + ((27\,b3^2\,h\,b1 - 9\,b2^2\,b3\,h)\,a1^2 + ((81\,b3^2\,h\,b0 - 9\,b2\,b3\,h\,b1)\,a2 + (-81\,b2\,b3\,h\,b0 - 54\,b3\,h\,b1^2 + 27\,b1\,b2^2\,h)\,a3)\,a1 + (-54\,b2\,b3\,h\,b0 + 18\,b3\,h\,b1^2)\,a2^2 + ((81\,b3\,h\,b1 + 54\,b2^2\,h)\,b0 - 27\,b1^2\,b2\,h)\,a3\,a2 + (-81\,b1\,b2\,h\,b0 + 27\,b1^3\,h)\,a3^2)\,a0 + (-27\,b3^2\,h\,b0 + b2^3\,h)\,a1^3 + ((27\,b2\,b3\,h\,b0 - 3\,b1\,b2^2\,h)\,a2 + (54\,b3\,h\,b1 - 18\,b2^2\,h)\,b0\,a3)\,a1^2 + ((-27\,b3\,h\,b1\,b0 + 3\,b1^2\,b2\,h)\,a2^2 + (-81\,b3\,h\,b0^2 + 9\,b1\,b2\,h\,b0)\,a3\,a2 + (81\,b2\,h\,b0^2 - 27\,b1^2\,h\,b0)\,a3^2)\,a1 + (27\,b3\,h\,b0^2 - b1^3\,h)\,a2^3 + (-27\,b2\,h\,b0^2 + 9\,b1^2\,h\,b0)\,a3\,a2^2 \quad \textbf{(30)}$$

*#Let us define t2*

$$t2 := collect\left(expand\left(subs\left(\left\{h = -\frac{1}{81}\right\}, \left(\left(-81\, b3^2\, h\, b1 + 27\, b2^2\, b3\, h\right) a2 + \left(81\, b2\, b3\, h\, b1 - 27\, b2^3\, h\right) a3\right) a0^2 + \left(\left(27\, b3^2\, h\, b1 - 9\, b2^2\, b3\, h\right) a1^2 + \left(\left(81\, b3^2\, h\, b0 - 9\, b2\, b3\, h\, b1\right) a2 + \left(-81\, b2\, b3\, h\, b0 - 54\, b3\, h\, b1^2 + 27\, b1\, b2^2\, h\right) a3\right) a1 + \left(-54\, b2\, b3\, h\, b0 + 18\, b3\, h\, b1^2\right) a2^2 + \left(\left(81\, b3\, h\, b1 + 54\, b2^2\, h\right) b0 - 27\, b1^2\, b2\, h\right) a3\, a2 + \left(-81\, b1\, b2\, h\, b0 + 27\, b1^3\, h\right) a3^2\right) a0 + \left(-27\, b3^2\, h\, b0 + b2^3\, h\right) a1^3 + \left(\left(27\, b2\, b3\, h\, b0 - 3\, b1\, b2^2\, h\right) a2 + \left(54\, b3\, h\, b1 - 18\, b2^2\, h\right) b0\, a3\right) a1^2 + \left(\left(-27\, b3\, h\, b1\, b0 + 3\, b1^2\, b2\, h\right) a2^2 + \left(-81\, b3\, h\, b0^2 + 9\, b1\, b2\, h\, b0\right) a3\, a2 + \left(81\, b2\, h\, b0^2 - 27\, b1^2\, h\, b0\right) a3^2\right) a1 + \left(27\, b3\, h\, b0^2 - b1^3\, h\right) a2^3 + \left(-27\, b2\, h\, b0^2 + 9\, b1^2\, h\, b0\right) a3\, a2^2\right)\right), \{a0, a1, a2, a3, b0, b1, b2, b3\}\right)$$

$$t2 := \left(\left(b1\, b3^2 - \frac{1}{3}\, b2^2\, b3\right) a2 + \left(-b1\, b2\, b3 + \frac{1}{3}\, b2^3\right) a3\right) a0^2 + \left(\left(-\frac{1}{3}\, b1\, b3^2 + \frac{1}{9}\, b2^2\, b3\right) a1^2 + \left(\left(-b0\, b3^2 + \frac{1}{9}\, b1\, b2\, b3\right) a2 + \left(b0\, b2\, b3 + \frac{2}{3}\, b1^2\, b3 - \frac{1}{3}\, b1\, b2^2\right) a3\right) a1 + \left(\frac{2}{3}\, b0\, b2\, b3 - \frac{2}{9}\, b1^2\, b3\right) a2^2 + \left(\left(-b1\, b3 - \frac{2\, b2^2}{3}\right) b0 + \frac{b1^2\, b2}{3}\right) a3\, a2 + \left(b0\, b1\, b2 - \frac{1}{3}\, b1^3\right) a3^2\right) a0 + \left(\frac{b0\, b3^2}{3} - \frac{b2^3}{81}\right) a1^3 + \left(\left(\frac{1}{27}\, b1\, b2^2 - \frac{1}{3}\, b0\, b2\, b3\right) a2 + \left(-\frac{2\, b1\, b3}{3} + \frac{2\, b2^2}{9}\right) b0\, a3\right) a1^2 + \left(\left(-\frac{1}{27}\, b1^2\, b2 + \frac{1}{3}\, b0\, b1\, b3\right) a2^2 + \left(b0^2\, b3 - \frac{1}{9}\, b0\, b1\, b2\right) a3\, a2 + \left(-b0^2\, b2 + \frac{1}{3}\, b0\, b1^2\right) a3^2\right) a1 + \left(-\frac{b0^2\, b3}{3} + \frac{b1^3}{81}\right) a2^3 + \left(\frac{1}{3}\, b0^2\, b2 - \frac{1}{9}\, b0\, b1^2\right) a3\, a2^2 \tag{31}$$

*#Now recall that s and t have to be algebraically independent, so let us define t as a convination of t1 and t2*

$$t := collect(expand(1 \cdot t1 + (-2) \cdot t2), \{a0, a1, a2, a3, b0, b1, b2, b3\})$$

$$t := a0^3\, b3^3 + \left(-a1\, b2\, b3^2 + \left(-2\, b1\, b3^2 + b2^2\, b3\right) a2 + \left(-3\, b0\, b3^2 + 3\, b1\, b2\, b3 - b2^3\right) a3\right) a0^2 + \left(a1^2\, b1\, b3^2 + \left(\left(3\, b0\, b3^2 - b1\, b2\, b3\right) a2 + \left(-b0\, b2\, b3 - 2\, b1^2\, b3 + b1\, b2^2\right) a3\right) a1 + \left(-2\, b0\, b2\, b3 + b1^2\, b3\right) a2^2 + \left(\left(b1\, b3 + 2\, b2^2\right) b0 - b1^2\, b2\right) a3\, a2 + \left(3\, b0^2\, b3 - 3\, b0\, b1\, b2 + b1^3\right) a3^2\right) a0 - a1^3\, b0\, b3^2 + \left(b3\, a2\, b0\, b2 + \left(2\, b1\, b3 - b2^2\right) b0\, a3\right) a1^2 + \left(-b3\, b1\, b0\, a2^2 + \left(-3\, b0^2\, b3 + b0\, b1\, b2\right) a3\, a2 + \left(2\, b0^2\, b2 - b0\, b1^2\right) a3^2\right) a1 + a2^3\, b0^2\, b3 - a3\, a2^2\, b0^2\, b2 + a2\, a3^2\, b0^2\, b1 - a3^3\, b0^3 \tag{32}$$

*#Notice that we can write t2 in terms of s and t*

$$collect\left(\left(simplify\left(\frac{(27\, t - s^3)}{-81}\right)\right), \{a0, a1, a2, a3, b0, b1, b2, b3\}\right);$$

$$\left(\left(b1\, b3^2 - \frac{1}{3}\, b2^2\, b3\right) a2 + \left(-b1\, b2\, b3 + \frac{1}{3}\, b2^3\right) a3\right) a0^2 + \left(\left(-\frac{1}{3}\, b1\, b3^2\right.\right. \tag{33}$$

$$+ \frac{1}{9} b2^2 b3\Big) a1^2 + \Big(\Big(-b0\, b3^2 + \frac{1}{9} b1\, b2\, b3\Big) a2 + \Big(b0\, b2\, b3 + \frac{2}{3} b1^2 b3$$

$$- \frac{1}{3} b1\, b2^2\Big) a3\Big) a1 + \Big(\frac{2}{3} b0\, b2\, b3 - \frac{2}{9} b1^2 b3\Big) a2^2 + \Big(\Big(-b1\, b3 - \frac{2\, b2^2}{3}\Big) b0$$

$$+ \frac{b1^2 b2}{3}\Big) a3\, a2 + \Big(b0\, b1\, b2 - \frac{1}{3} b1^3\Big) a3^2\Big) a0 + \Big(\frac{b0\, b3^2}{3} - \frac{b2^3}{81}\Big) a1^3$$

$$+ \Big(\Big(\frac{1}{27} b1\, b2^2 - \frac{1}{3} b0\, b2\, b3\Big) a2 + \Big(-\frac{2\, b1\, b3}{3} + \frac{2\, b2^2}{9}\Big) b0\, a3\Big) a1^2 + \Big(\Big($$

$$-\frac{1}{27} b1^2 b2 + \frac{1}{3} b0\, b1\, b3\Big) a2^2 + \Big(b0^2 b3 - \frac{1}{9} b0\, b1\, b2\Big) a3\, a2 + \Big(-b0^2 b2$$

$$+ \frac{1}{3} b0\, b1^2\Big) a3^2\Big) a1 + \Big(-\frac{b0^2 b3}{3} + \frac{b1^3}{81}\Big) a2^3 + \Big(\frac{1}{3} b0^2 b2 - \frac{1}{9} b0\, b1^2\Big) a3\, a2^2$$

*# Basically we proved that t2* $= \frac{1}{-81}\left(27\, t - s^3\right)$

## A.2 Discriminant of $F$

*#We start with the discriminat Disc x (F), We previously computed the Res(F,F') using a matrix calculator*

```
collect(expand(simplify(-1/(a·x + b·y) (4*a^2*f^3*x^5 - a*c^2*f^2*x^5 + 27*a^3*h^2*x
  ^5 + 4*a*c^3*h*x^5 - 18*a^2*c*f*h*x^5 + 4*b^2*g^3*y^5 - b*d^2*g^2*y^5
  + 27*b^3*j^2*y^5 + 4*b*d^3*j*y^5 - 18*b^2*d*g*j*y^5 + 8*a*b*g^3*x*y^4
  - a*d^2*g^2*x*y^4 - 2*b*c*d*g^2*x*y^4 + 12*b^2*f*g^2*x*y^4 - 2*b*d^2
  *f*g*x*y^4 + 4*b*d^3*h*x*y^4 - 18*b^2*d*g*h*x*y^4 + 81*a*b^2*j^2*x*y
  ^4 + 4*a*d^3*j*x*y^4 + 12*b*c*d^2*j*x*y^4 - 18*b^2*d*f*j*x*y^4 - 18*b
  ^2*c*g*j*x*y^4 - 36*a*b*d*g*j*x*y^4 + 54*b^3*h*j*x*y^4 - b*d^2*f^2*x
  ^2*y^3 + 4*a^2*g^3*x^2*y^3 - b*c^2*g^2*x^2*y^3 - 2*a*c*d*g^2*x^2*y^3
  + 24*a*b*f*g^2*x^2*y^3 + 12*b^2*f^2*g*x^2*y^3 - 2*a*d^2*f*g*x^2*y^3
  - 4*b*c*d*f*g*x^2*y^3 + 27*b^3*h^2*x^2*y^3 + 4*a*d^3*h*x^2*y^3 + 12
  *b*c*d^2*h*x^2*y^3 - 18*b^2*d*f*h*x^2*y^3 - 18*b^2*c*g*h*x^2*y^3
  - 36*a*b*d*g*h*x^2*y^3 + 81*a^2*b*j^2*x^2*y^3 + 12*a*c*d^2*j*x^2*y
  ^3 + 12*b*c^2*d*j*x^2*y^3 - 18*b^2*c*f*j*x^2*y^3 - 36*a*b*d*f*j*x^2*y
  ^3 - 36*a*b*c*g*j*x^2*y^3 - 18*a^2*d*g*j*x^2*y^3 + 162*a*b^2*h*j*x^2
  *y^3 + 4*b^2*f^3*x^3*y^2 - a*d^2*f^2*x^3*y^2 - 2*b*c*d*f^2*x^3*y^2 - a
  *c^2*g^2*x^3*y^2 + 12*a^2*f*g^2*x^3*y^2 + 24*a*b*f^2*g*x^3*y^2 - 2*b
  *c^2*f*g*x^3*y^2 - 4*a*c*d*f*g*x^3*y^2 + 81*a*b^2*h^2*x^3*y^2 + 12*a
  *c*d^2*h*x^3*y^2 + 12*b*c^2*d*h*x^3*y^2 - 18*b^2*c*f*h*x^3*y^2 - 36
  *a*b*d*f*h*x^3*y^2 - 36*a*b*c*g*h*x^3*y^2 - 18*a^2*d*g*h*x^3*y^2
  + 27*a^3*j^2*x^3*y^2 + 4*b*c^3*j*x^3*y^2 + 12*a*c^2*d*j*x^3*y^2 - 36
  *a*b*c*f*j*x^3*y^2 - 18*a^2*d*f*j*x^3*y^2 - 18*a^2*c*g*j*x^3*y^2 + 162
  *a^2*b*h*j*x^3*y^2 + 8*a*b*f^3*x^4*y - b*c^2*f^2*x^4*y - 2*a*c*d*f^2
  *x^4*y + 12*a^2*f^2*g*x^4*y - 2*a*c^2*f*g*x^4*y + 81*a^2*b*h^2*x^4*y
  + 4*b*c^3*h*x^4*y + 12*a*c^2*d*h*x^4*y - 36*a*b*c*f*h*x^4*y - 18*a^2
  *d*f*h*x^4*y - 18*a^2*c*g*h*x^4*y + 4*a*c^3*j*x^4*y - 18*a^2*c*f*j*x
  ^4*y + 54*a^3*h*j*x^4*y))), {x, y})
```

$$\left(-27\,a^2h^2+18\,acfh-4\,af^3-4\,c^3h+c^2f^2\right)x^4+\left(-54\,a^2hj-54\,ah^2b+18\,acfj+18\,acgh+18\,adfh-12\,af^2g+18\,bcfh-4\,bf^3-4\,c^3j-12\,c^2dh+2\,c^2gf+2\,cdf^2\right)yx^3+\left(-27\,a^2j^2-108\,abhj+18\,acgj+18\,dfaj+18\,dagh-12\,afg^2-27\,b^2h^2+18\,bcfj+18\,bcgh+18\,dfbh-12\,bf^2g-12\,c^2dj+c^2g^2-12\,cd^2h+4\,cdfg+d^2f^2\right)y^2x^2+\left(-54\,abj^2+18\,dagj-4\,ag^3-54\,b^2hj+18\,bcgj+18\,bdfj+18\,bdgh-12\,bfg^2-12\,cd^2j+2\,cdg^2-4\,d^3h+2\,d^2fg\right)y^3x+\left(-27\,b^2j^2+18\,bdgj-4\,bg^3-4\,d^3j+d^2g^2\right)y^4 \tag{1}$$

*#now we change the coefficients to the ones we use*

```
collect(expand((-27*a0^2*a3^2 + 18*a0*a1*a2*a3 - 4*a0*a2^3 - 4*a1^3*a3 + a1^2
  *a2^2)*x^4 + (-4*a1^3*b3 - 54*((2*b1*a3)/9 - b2*a2/27)*a1^2 - 54*(-b1*a2
  ^2/27 + (-a0*b3/3 - b0*a3/3)*a2 - a0*b2*a3/3)*a1 + 18*a0*b1*a2*a3 - 4*b0
  *a2^3 - 12*a0*a2^2*b2 - 54*a0*a3*(a0*b3 + b0*a3))*y*x^3 + (-27*((4*b1
  *b3)/9 - b2^2/27)*a1^2 - 27*((4*b1^2*a3)/9 - (4*b1*a2*b2)/27 - (2*b0*a2
  *b3)/3 - (2*b2*(a0*b3 + b0*a3))/3)*a1 + b1^2*a2^2 - 27*((-(2*a0*b3)/3 - (2
  *b0*a3)/3)*a2 - (2*a0*b2*a3)/3)*b1 - 12*a0*a2*b2^2 - 12*b0*a2^2*b2 - 27
  *b0^2*a3^2 - 27*a0^2*b3^2 - 108*a0*b0*a3*b3)*y^2*x^2 + (-54*(-1/3*b0
```

```
* b2 * b3 + 2/9 * b1^2 * b3 - 1/27 * b1 * b2^2) * a1 - 4 * b1^3 * a3 + 2 * b1^2 * a2 * b2 - 54
* (-b0 * a2 * b3/3 - b2 * (a0 * b3 + b0 * a3)/3) * b1 - 54 * a0 * b0 * b3^2 - 4 * a0 * b2^3 - 54
* b0^2 * a3 * b3 - 12 * b0 * a2 * b2^2) * y^3 * x + (-27 * b0^2 * b3^2 + 18 * b0 * b1 * b2 * b3
- 4 * b0 * b2^3 - 4 * b1^3 * b3 + b1^2 * b2^2) * y^4), {x, y})
```

$$\begin{aligned}
&(-27\, a0^2\, a3^2 + 18\, a0\, a1\, a2\, a3 - 4\, a0\, a2^3 - 4\, a1^3\, a3 + a1^2\, a2^2)\, x^4 + (-54\, a0^2\, a3\, b3 \\
&\quad + 18\, a0\, a1\, a2\, b3 + 18\, a0\, a1\, a3\, b2 - 12\, a0\, a2^2\, b2 + 18\, a0\, b1\, a2\, a3 - 54\, a0\, a3^2\, b0 \\
&\quad - 4\, a1^3\, b3 + 2\, a1^2\, b2\, a2 - 12\, a1^2\, a3\, b1 + 2\, a1\, b1\, a2^2 + 18\, a1\, a2\, a3\, b0 - 4\, b0\, a2^3)\, y\, x^3 \\
&\quad + (-27\, a0^2\, b3^2 + 18\, a0\, a1\, b2\, b3 + 18\, b1\, a2\, a0\, b3 - 12\, a0\, a2\, b2^2 - 108\, a0\, b0\, a3\, b3 \\
&\quad + 18\, b1\, a0\, b2\, a3 - 12\, a1^2\, b1\, b3 + a1^2\, b2^2 + 18\, a1\, a2\, b0\, b3 + 4\, a1\, a2\, b1\, b2 \\
&\quad + 18\, a1\, a3\, b0\, b2 - 12\, a1\, a3\, b1^2 - 12\, b0\, a2^2\, b2 + b1^2\, a2^2 + 18\, b1\, a2\, b0\, a3 - 27\, b0^2\, a3^2) \\
&\quad y^2\, x^2 + (-54\, a0\, b0\, b3^2 + 18\, b1\, a0\, b2\, b3 - 4\, a0\, b2^3 + 18\, a1\, b0\, b2\, b3 - 12\, a1\, b1^2\, b3 \\
&\quad + 2\, a1\, b1\, b2^2 + 18\, a2\, b0\, b1\, b3 - 12\, b0\, a2\, b2^2 + 2\, b1^2\, a2\, b2 - 54\, b0^2\, a3\, b3 \\
&\quad + 18\, a3\, b0\, b1\, b2 - 4\, b1^3\, a3)\, y^3\, x + (-27\, b0^2\, b3^2 + 18\, b0\, b1\, b2\, b3 - 4\, b0\, b2^3 - 4\, b1^3\, b3 \\
&\quad + b1^2\, b2^2)\, y^4
\end{aligned} \qquad \textbf{(2)}$$

*#Now we will compute the Disc z Disc x (F)*

*with(LinearAlgebra) :*

*#We first compute the Res(Disc x(F), (Disc x(F))')*

$$\begin{aligned}
&resz := Matrix([[(-27\, a0^2\, a3^2 + 18\, a0\, a1\, a2\, a3 - 4\, a0\, a2^3 - 4\, a1^3\, a3 + a1^2\, a2^2), 0, 0, 4\cdot( \\
&\quad -27\, a0^2\, a3^2 + 18\, a0\, a1\, a2\, a3 - 4\, a0\, a2^3 - 4\, a1^3\, a3 + a1^2\, a2^2), 0, 0, 0], [(-54\, a0^2\, a3\, b3 \\
&\quad + 18\, a0\, a1\, a2\, b3 + 18\, a0\, a1\, a3\, b2 - 12\, a0\, a2^2\, b2 + 18\, a0\, b1\, a2\, a3 - 54\, a0\, a3^2\, b0 - 4\, a1^3\, b3 \\
&\quad + 2\, a1^2\, b2\, a2 - 12\, a1^2\, a3\, b1 + 2\, a1\, b1\, a2^2 + 18\, a1\, a2\, a3\, b0 - 4\, b0\, a2^3), (-27\, a0^2\, a3^2 \\
&\quad + 18\, a0\, a1\, a2\, a3 - 4\, a0\, a2^3 - 4\, a1^3\, a3 + a1^2\, a2^2), 0, 3\cdot(-54\, a0^2\, a3\, b3 + 18\, a0\, a1\, a2\, b3 \\
&\quad + 18\, a0\, a1\, a3\, b2 - 12\, a0\, a2^2\, b2 + 18\, a0\, b1\, a2\, a3 - 54\, a0\, a3^2\, b0 - 4\, a1^3\, b3 + 2\, a1^2\, b2\, a2 \\
&\quad - 12\, a1^2\, a3\, b1 + 2\, a1\, b1\, a2^2 + 18\, a1\, a2\, a3\, b0 - 4\, b0\, a2^3), 4\cdot(-27\, a0^2\, a3^2 + 18\, a0\, a1\, a2\, a3 \\
&\quad - 4\, a0\, a2^3 - 4\, a1^3\, a3 + a1^2\, a2^2), 0, 0], [(-27\, a0^2\, b3^2 + 18\, a0\, a1\, b2\, b3 + 18\, b1\, a2\, a0\, b3 \\
&\quad - 12\, a0\, a2\, b2^2 - 108\, a0\, b0\, a3\, b3 + 18\, b1\, a0\, b2\, a3 - 12\, a1^2\, b1\, b3 + a1^2\, b2^2 + 18\, a1\, a2\, b0\, b3 \\
&\quad + 4\, a1\, a2\, b1\, b2 + 18\, a1\, a3\, b0\, b2 - 12\, a1\, a3\, b1^2 - 12\, b0\, a2^2\, b2 + b1^2\, a2^2 + 18\, b1\, a2\, b0\, a3 \\
&\quad - 27\, b0^2\, a3^2), (-54\, a0^2\, a3\, b3 + 18\, a0\, a1\, a2\, b3 + 18\, a0\, a1\, a3\, b2 - 12\, a0\, a2^2\, b2 \\
&\quad + 18\, a0\, b1\, a2\, a3 - 54\, a0\, a3^2\, b0 - 4\, a1^3\, b3 + 2\, a1^2\, b2\, a2 - 12\, a1^2\, a3\, b1 + 2\, a1\, b1\, a2^2 \\
&\quad + 18\, a1\, a2\, a3\, b0 - 4\, b0\, a2^3), (-27\, a0^2\, a3^2 + 18\, a0\, a1\, a2\, a3 - 4\, a0\, a2^3 - 4\, a1^3\, a3 \\
&\quad + a1^2\, a2^2), 2\cdot(-27\, a0^2\, b3^2 + 18\, a0\, a1\, b2\, b3 + 18\, b1\, a2\, a0\, b3 - 12\, a0\, a2\, b2^2 \\
&\quad - 108\, a0\, b0\, a3\, b3 + 18\, b1\, a0\, b2\, a3 - 12\, a1^2\, b1\, b3 + a1^2\, b2^2 + 18\, a1\, a2\, b0\, b3 + 4\, a1\, a2\, b1\, b2 \\
&\quad + 18\, a1\, a3\, b0\, b2 - 12\, a1\, a3\, b1^2 - 12\, b0\, a2^2\, b2 + b1^2\, a2^2 + 18\, b1\, a2\, b0\, a3 - 27\, b0^2\, a3^2), 3 \\
&\quad \cdot(-54\, a0^2\, a3\, b3 + 18\, a0\, a1\, a2\, b3 + 18\, a0\, a1\, a3\, b2 - 12\, a0\, a2^2\, b2 + 18\, a0\, b1\, a2\, a3 \\
&\quad - 54\, a0\, a3^2\, b0 - 4\, a1^3\, b3 + 2\, a1^2\, b2\, a2 - 12\, a1^2\, a3\, b1 + 2\, a1\, b1\, a2^2 + 18\, a1\, a2\, a3\, b0 \\
&\quad - 4\, b0\, a2^3), 4\cdot(-27\, a0^2\, a3^2 + 18\, a0\, a1\, a2\, a3 - 4\, a0\, a2^3 - 4\, a1^3\, a3 + a1^2\, a2^2), 0], [( \\
&\quad -54\, a0\, b0\, b3^2 + 18\, b1\, a0\, b2\, b3 - 4\, a0\, b2^3 + 18\, a1\, b0\, b2\, b3 - 12\, a1\, b1^2\, b3 + 2\, a1\, b1\, b2^2 \\
&\quad + 18\, a2\, b0\, b1\, b3 - 12\, b0\, a2\, b2^2 + 2\, b1^2\, a2\, b2 - 54\, b0^2\, a3\, b3 + 18\, a3\, b0\, b1\, b2 - 4\, b1^3\, a3), ( \\
&\quad -27\, a0^2\, b3^2 + 18\, a0\, a1\, b2\, b3 + 18\, b1\, a2\, a0\, b3 - 12\, a0\, a2\, b2^2 - 108\, a0\, b0\, a3\, b3
\end{aligned}$$

$+ 18\, b1\, a0\, b2\, a3 - 12\, a1^2\, b1\, b3 + a1^2\, b2^2 + 18\, a1\, a2\, b0\, b3 + 4\, a1\, a2\, b1\, b2 + 18\, a1\, a3\, b0\, b2$
$- 12\, a1\, a3\, b1^2 - 12\, b0\, a2^2\, b2 + b1^2\, a2^2 + 18\, b1\, a2\, b0\, a3 - 27\, b0^2\, a3^2) ,\ ( -54\, a0^2\, a3\, b3$
$+ 18\, a0\, a1\, a2\, b3 + 18\, a0\, a1\, a3\, b2 - 12\, a0\, a2^2\, b2 + 18\, a0\, b1\, a2\, a3 - 54\, a0\, a3^2\, b0 - 4\, a1^3\, b3$
$+ 2\, a1^2\, b2\, a2 - 12\, a1^2\, a3\, b1 + 2\, a1\, b1\, a2^2 + 18\, a1\, a2\, a3\, b0 - 4\, b0\, a2^3) ,\ ( -54\, a0\, b0\, b3^2$
$+ 18\, b1\, a0\, b2\, b3 - 4\, a0\, b2^3 + 18\, a1\, b0\, b2\, b3 - 12\, a1\, b1^2\, b3 + 2\, a1\, b1\, b2^2 + 18\, a2\, b0\, b1\, b3$
$- 12\, b0\, a2\, b2^2 + 2\, b1^2\, a2\, b2 - 54\, b0^2\, a3\, b3 + 18\, a3\, b0\, b1\, b2 - 4\, b1^3\, a3), 2\cdot( -27\, a0^2\, b3^2$
$+ 18\, a0\, a1\, b2\, b3 + 18\, b1\, a2\, a0\, b3 - 12\, a0\, a2\, b2^2 - 108\, a0\, b0\, a3\, b3 + 18\, b1\, a0\, b2\, a3$
$- 12\, a1^2\, b1\, b3 + a1^2\, b2^2 + 18\, a1\, a2\, b0\, b3 + 4\, a1\, a2\, b1\, b2 + 18\, a1\, a3\, b0\, b2 - 12\, a1\, a3\, b1^2$
$- 12\, b0\, a2^2\, b2 + b1^2\, a2^2 + 18\, b1\, a2\, b0\, a3 - 27\, b0^2\, a3^2), 3\cdot( -54\, a0^2\, a3\, b3 + 18\, a0\, a1\, a2\, b3$
$+ 18\, a0\, a1\, a3\, b2 - 12\, a0\, a2^2\, b2 + 18\, a0\, b1\, a2\, a3 - 54\, a0\, a3^2\, b0 - 4\, a1^3\, b3 + 2\, a1^2\, b2\, a2$
$- 12\, a1^2\, a3\, b1 + 2\, a1\, b1\, a2^2 + 18\, a1\, a2\, a3\, b0 - 4\, b0\, a2^3), 4\cdot( -27\, a0^2\, a3^2 + 18\, a0\, a1\, a2\, a3$
$- 4\, a0\, a2^3 - 4\, a1^3\, a3 + a1^2\, a2^2)],\ [( -27\, b0^2\, b3^2 + 18\, b0\, b1\, b2\, b3 - 4\, b0\, b2^3 - 4\, b1^3\, b3$
$+ b1^2\, b2^2),\ ( -54\, a0\, b0\, b3^2 + 18\, b1\, a0\, b2\, b3 - 4\, a0\, b2^3 + 18\, a1\, b0\, b2\, b3 - 12\, a1\, b1^2\, b3$
$+ 2\, a1\, b1\, b2^2 + 18\, a2\, b0\, b1\, b3 - 12\, b0\, a2\, b2^2 + 2\, b1^2\, a2\, b2 - 54\, b0^2\, a3\, b3 + 18\, a3\, b0\, b1\, b2$
$- 4\, b1^3\, a3),\ ( -27\, a0^2\, b3^2 + 18\, a0\, a1\, b2\, b3 + 18\, b1\, a2\, a0\, b3 - 12\, a0\, a2\, b2^2$
$- 108\, a0\, b0\, a3\, b3 + 18\, b1\, a0\, b2\, a3 - 12\, a1^2\, b1\, b3 + a1^2\, b2^2 + 18\, a1\, a2\, b0\, b3 + 4\, a1\, a2\, b1\, b2$
$+ 18\, a1\, a3\, b0\, b2 - 12\, a1\, a3\, b1^2 - 12\, b0\, a2^2\, b2 + b1^2\, a2^2 + 18\, b1\, a2\, b0\, a3 - 27\, b0^2\, a3^2), 0,$
$( -54\, a0\, b0\, b3^2 + 18\, b1\, a0\, b2\, b3 - 4\, a0\, b2^3 + 18\, a1\, b0\, b2\, b3 - 12\, a1\, b1^2\, b3 + 2\, a1\, b1\, b2^2$
$+ 18\, a2\, b0\, b1\, b3 - 12\, b0\, a2\, b2^2 + 2\, b1^2\, a2\, b2 - 54\, b0^2\, a3\, b3 + 18\, a3\, b0\, b1\, b2 - 4\, b1^3\, a3) , 2$
$\cdot( -27\, a0^2\, b3^2 + 18\, a0\, a1\, b2\, b3 + 18\, b1\, a2\, a0\, b3 - 12\, a0\, a2\, b2^2 - 108\, a0\, b0\, a3\, b3$
$+ 18\, b1\, a0\, b2\, a3 - 12\, a1^2\, b1\, b3 + a1^2\, b2^2 + 18\, a1\, a2\, b0\, b3 + 4\, a1\, a2\, b1\, b2 + 18\, a1\, a3\, b0\, b2$
$- 12\, a1\, a3\, b1^2 - 12\, b0\, a2^2\, b2 + b1^2\, a2^2 + 18\, b1\, a2\, b0\, a3 - 27\, b0^2\, a3^2), 3\cdot( -54\, a0^2\, a3\, b3$
$+ 18\, a0\, a1\, a2\, b3 + 18\, a0\, a1\, a3\, b2 - 12\, a0\, a2^2\, b2 + 18\, a0\, b1\, a2\, a3 - 54\, a0\, a3^2\, b0 - 4\, a1^3\, b3$
$+ 2\, a1^2\, b2\, a2 - 12\, a1^2\, a3\, b1 + 2\, a1\, b1\, a2^2 + 18\, a1\, a2\, a3\, b0 - 4\, b0\, a2^3)],\ [0,\ ( -27\, b0^2\, b3^2$
$+ 18\, b0\, b1\, b2\, b3 - 4\, b0\, b2^3 - 4\, b1^3\, b3 + b1^2\, b2^2),\ ( -54\, a0\, b0\, b3^2 + 18\, b1\, a0\, b2\, b3$
$- 4\, a0\, b2^3 + 18\, a1\, b0\, b2\, b3 - 12\, a1\, b1^2\, b3 + 2\, a1\, b1\, b2^2 + 18\, a2\, b0\, b1\, b3 - 12\, b0\, a2\, b2^2$
$+ 2\, b1^2\, a2\, b2 - 54\, b0^2\, a3\, b3 + 18\, a3\, b0\, b1\, b2 - 4\, b1^3\, a3), 0 , 0,\ ( -54\, a0\, b0\, b3^2$
$+ 18\, b1\, a0\, b2\, b3 - 4\, a0\, b2^3 + 18\, a1\, b0\, b2\, b3 - 12\, a1\, b1^2\, b3 + 2\, a1\, b1\, b2^2 + 18\, a2\, b0\, b1\, b3$
$- 12\, b0\, a2\, b2^2 + 2\, b1^2\, a2\, b2 - 54\, b0^2\, a3\, b3 + 18\, a3\, b0\, b1\, b2 - 4\, b1^3\, a3), 2\cdot( -27\, a0^2\, b3^2$
$+ 18\, a0\, a1\, b2\, b3 + 18\, b1\, a2\, a0\, b3 - 12\, a0\, a2\, b2^2 - 108\, a0\, b0\, a3\, b3 + 18\, b1\, a0\, b2\, a3$
$- 12\, a1^2\, b1\, b3 + a1^2\, b2^2 + 18\, a1\, a2\, b0\, b3 + 4\, a1\, a2\, b1\, b2 + 18\, a1\, a3\, b0\, b2 - 12\, a1\, a3\, b1^2$
$- 12\, b0\, a2^2\, b2 + b1^2\, a2^2 + 18\, b1\, a2\, b0\, a3 - 27\, b0^2\, a3^2)],\ [0, 0,\ ( -27\, b0^2\, b3^2$
$+ 18\, b0\, b1\, b2\, b3 - 4\, b0\, b2^3 - 4\, b1^3\, b3 + b1^2\, b2^2), 0 , 0, 0,\ ( -54\, a0\, b0\, b3^2 + 18\, b1\, a0\, b2\, b3$
$- 4\, a0\, b2^3 + 18\, a1\, b0\, b2\, b3 - 12\, a1\, b1^2\, b3 + 2\, a1\, b1\, b2^2 + 18\, a2\, b0\, b1\, b3 - 12\, b0\, a2\, b2^2$
$+ 2\, b1^2\, a2\, b2 - 54\, b0^2\, a3\, b3 + 18\, a3\, b0\, b1\, b2 - 4\, b1^3\, a3)]])$

$resz := [[ -27\, a0^2\, a3^2 + 18\, a0\, a1\, a2\, a3 - 4\, a0\, a2^3 - 4\, a1^3\, a3 + a1^2\, a2^2, 0, 0, -108\, a0^2\, a3^2$ **(3)**
$+ 72\, a0\, a1\, a2\, a3 - 16\, a0\, a2^3 - 16\, a1^3\, a3 + 4\, a1^2\, a2^2, 0, 0, 0],$
$[ -54\, a0^2\, a3\, b3 + 18\, a0\, a1\, a2\, b3 + 18\, a0\, a1\, a3\, b2 - 12\, a0\, a2^2\, b2 + 18\, a0\, a2\, a3\, b1$
$- 54\, a0\, a3^2\, b0 - 4\, a1^3\, b3 + 2\, a1^2\, a2\, b2 - 12\, a1^2\, a3\, b1 + 2\, a1\, a2^2\, b1 + 18\, a1\, a2\, a3\, b0$

$- 4\, a2^{3}\, b0,\ -27\, a0^{2}\, a3^{2} + 18\, a0\, a1\, a2\, a3 - 4\, a0\, a2^{3} - 4\, a1^{3}\, a3 + a1^{2}\, a2^{2},\ 0,$

$-162\, a0^{2}\, a3\, b3 + 54\, a0\, a1\, a2\, b3 + 54\, a0\, a1\, a3\, b2 - 36\, a0\, a2^{2}\, b2 + 54\, a0\, a2\, a3\, b1$

$- 162\, a0\, a3^{2}\, b0 - 12\, a1^{3}\, b3 + 6\, a1^{2}\, a2\, b2 - 36\, a1^{2}\, a3\, b1 + 6\, a1\, a2^{2}\, b1 + 54\, a1\, a2\, a3\, b0$

$- 12\, a2^{3}\, b0,\ -108\, a0^{2}\, a3^{2} + 72\, a0\, a1\, a2\, a3 - 16\, a0\, a2^{3} - 16\, a1^{3}\, a3 + 4\, a1^{2}\, a2^{2},\ 0,\ 0\, ],$

$[\, -27\, a0^{2}\, b3^{2} + 18\, a0\, a1\, b2\, b3 + 18\, a0\, a2\, b1\, b3 - 12\, a0\, a2\, b2^{2} - 108\, a0\, a3\, b0\, b3$

$+ 18\, a0\, a3\, b1\, b2 - 12\, a1^{2}\, b1\, b3 + a1^{2}\, b2^{2} + 18\, a1\, a2\, b0\, b3 + 4\, a1\, a2\, b1\, b2$

$+ 18\, a1\, a3\, b0\, b2 - 12\, a1\, a3\, b1^{2} - 12\, a2^{2}\, b0\, b2 + a2^{2}\, b1^{2} + 18\, a2\, a3\, b0\, b1 - 27\, a3^{2}\, b0^{2},$

$-54\, a0^{2}\, a3\, b3 + 18\, a0\, a1\, a2\, b3 + 18\, a0\, a1\, a3\, b2 - 12\, a0\, a2^{2}\, b2 + 18\, a0\, a2\, a3\, b1$

$- 54\, a0\, a3^{2}\, b0 - 4\, a1^{3}\, b3 + 2\, a1^{2}\, a2\, b2 - 12\, a1^{2}\, a3\, b1 + 2\, a1\, a2^{2}\, b1 + 18\, a1\, a2\, a3\, b0$

$- 4\, a2^{3}\, b0,\ -27\, a0^{2}\, a3^{2} + 18\, a0\, a1\, a2\, a3 - 4\, a0\, a2^{3} - 4\, a1^{3}\, a3 + a1^{2}\, a2^{2},\ -54\, a0^{2}\, b3^{2}$

$+ 36\, a0\, a1\, b2\, b3 + 36\, a0\, a2\, b1\, b3 - 24\, a0\, a2\, b2^{2} - 216\, a0\, a3\, b0\, b3 + 36\, a0\, a3\, b1\, b2$

$- 24\, a1^{2}\, b1\, b3 + 2\, a1^{2}\, b2^{2} + 36\, a1\, a2\, b0\, b3 + 8\, a1\, a2\, b1\, b2 + 36\, a1\, a3\, b0\, b2$

$- 24\, a1\, a3\, b1^{2} - 24\, a2^{2}\, b0\, b2 + 2\, a2^{2}\, b1^{2} + 36\, a2\, a3\, b0\, b1 - 54\, a3^{2}\, b0^{2},\ -162\, a0^{2}\, a3\, b3$

$+ 54\, a0\, a1\, a2\, b3 + 54\, a0\, a1\, a3\, b2 - 36\, a0\, a2^{2}\, b2 + 54\, a0\, a2\, a3\, b1 - 162\, a0\, a3^{2}\, b0$

$- 12\, a1^{3}\, b3 + 6\, a1^{2}\, a2\, b2 - 36\, a1^{2}\, a3\, b1 + 6\, a1\, a2^{2}\, b1 + 54\, a1\, a2\, a3\, b0 - 12\, a2^{3}\, b0,$

$-108\, a0^{2}\, a3^{2} + 72\, a0\, a1\, a2\, a3 - 16\, a0\, a2^{3} - 16\, a1^{3}\, a3 + 4\, a1^{2}\, a2^{2},\ 0\, ],$

$[\, -54\, a0\, b0\, b3^{2} + 18\, a0\, b1\, b2\, b3 - 4\, a0\, b2^{3} + 18\, a1\, b0\, b2\, b3 - 12\, a1\, b1^{2}\, b3 + 2\, a1\, b1\, b2^{2}$

$+ 18\, a2\, b0\, b1\, b3 - 12\, a2\, b0\, b2^{2} + 2\, a2\, b1^{2}\, b2 - 54\, a3\, b0^{2}\, b3 + 18\, a3\, b0\, b1\, b2 - 4\, a3\, b1^{3},$

$-27\, a0^{2}\, b3^{2} + 18\, a0\, a1\, b2\, b3 + 18\, a0\, a2\, b1\, b3 - 12\, a0\, a2\, b2^{2} - 108\, a0\, a3\, b0\, b3$

$+ 18\, a0\, a3\, b1\, b2 - 12\, a1^{2}\, b1\, b3 + a1^{2}\, b2^{2} + 18\, a1\, a2\, b0\, b3 + 4\, a1\, a2\, b1\, b2$

$+ 18\, a1\, a3\, b0\, b2 - 12\, a1\, a3\, b1^{2} - 12\, a2^{2}\, b0\, b2 + a2^{2}\, b1^{2} + 18\, a2\, a3\, b0\, b1 - 27\, a3^{2}\, b0^{2},$

$-54\, a0^{2}\, a3\, b3 + 18\, a0\, a1\, a2\, b3 + 18\, a0\, a1\, a3\, b2 - 12\, a0\, a2^{2}\, b2 + 18\, a0\, a2\, a3\, b1$

$- 54\, a0\, a3^{2}\, b0 - 4\, a1^{3}\, b3 + 2\, a1^{2}\, a2\, b2 - 12\, a1^{2}\, a3\, b1 + 2\, a1\, a2^{2}\, b1 + 18\, a1\, a2\, a3\, b0$

$- 4\, a2^{3}\, b0,\ -54\, a0\, b0\, b3^{2} + 18\, a0\, b1\, b2\, b3 - 4\, a0\, b2^{3} + 18\, a1\, b0\, b2\, b3 - 12\, a1\, b1^{2}\, b3$

$+ 2\, a1\, b1\, b2^{2} + 18\, a2\, b0\, b1\, b3 - 12\, a2\, b0\, b2^{2} + 2\, a2\, b1^{2}\, b2 - 54\, a3\, b0^{2}\, b3$

$+ 18\, a3\, b0\, b1\, b2 - 4\, a3\, b1^{3},\ -54\, a0^{2}\, b3^{2} + 36\, a0\, a1\, b2\, b3 + 36\, a0\, a2\, b1\, b3 - 24\, a0\, a2\, b2^{2}$

$- 216\, a0\, a3\, b0\, b3 + 36\, a0\, a3\, b1\, b2 - 24\, a1^{2}\, b1\, b3 + 2\, a1^{2}\, b2^{2} + 36\, a1\, a2\, b0\, b3$

$+ 8\, a1\, a2\, b1\, b2 + 36\, a1\, a3\, b0\, b2 - 24\, a1\, a3\, b1^{2} - 24\, a2^{2}\, b0\, b2 + 2\, a2^{2}\, b1^{2}$

$+ 36\, a2\, a3\, b0\, b1 - 54\, a3^{2}\, b0^{2},\ -162\, a0^{2}\, a3\, b3 + 54\, a0\, a1\, a2\, b3 + 54\, a0\, a1\, a3\, b2$

$- 36\, a0\, a2^{2}\, b2 + 54\, a0\, a2\, a3\, b1 - 162\, a0\, a3^{2}\, b0 - 12\, a1^{3}\, b3 + 6\, a1^{2}\, a2\, b2 - 36\, a1^{2}\, a3\, b1$

$+ 6\, a1\, a2^{2}\, b1 + 54\, a1\, a2\, a3\, b0 - 12\, a2^{3}\, b0,\ -108\, a0^{2}\, a3^{2} + 72\, a0\, a1\, a2\, a3 - 16\, a0\, a2^{3}$

$- 16\, a1^{3}\, a3 + 4\, a1^{2}\, a2^{2}\, ],$

$$[-27\,b0^2\,b3^2 + 18\,b0\,b1\,b2\,b3 - 4\,b0\,b2^3 - 4\,b1^3\,b3 + b1^2\,b2^2, -54\,a0\,b0\,b3^2 + 18\,a0\,b1\,b2\,b3 - 4\,a0\,b2^3 + 18\,a1\,b0\,b2\,b3 - 12\,a1\,b1^2\,b3 + 2\,a1\,b1\,b2^2 + 18\,a2\,b0\,b1\,b3 - 12\,a2\,b0\,b2^2 + 2\,a2\,b1^2\,b2 - 54\,a3\,b0^2\,b3 + 18\,a3\,b0\,b1\,b2 - 4\,a3\,b1^3,$$

$$-27\,a0^2\,b3^2 + 18\,a0\,a1\,b2\,b3 + 18\,a0\,a2\,b1\,b3 - 12\,a0\,a2\,b2^2 - 108\,a0\,a3\,b0\,b3 + 18\,a0\,a3\,b1\,b2 - 12\,a1^2\,b1\,b3 + a1^2\,b2^2 + 18\,a1\,a2\,b0\,b3 + 4\,a1\,a2\,b1\,b2 + 18\,a1\,a3\,b0\,b2 - 12\,a1\,a3\,b1^2 - 12\,a2^2\,b0\,b2 + a2^2\,b1^2 + 18\,a2\,a3\,b0\,b1 - 27\,a3^2\,b0^2,$$

$$0, -54\,a0\,b0\,b3^2 + 18\,a0\,b1\,b2\,b3 - 4\,a0\,b2^3 + 18\,a1\,b0\,b2\,b3 - 12\,a1\,b1^2\,b3 + 2\,a1\,b1\,b2^2 + 18\,a2\,b0\,b1\,b3 - 12\,a2\,b0\,b2^2 + 2\,a2\,b1^2\,b2 - 54\,a3\,b0^2\,b3 + 18\,a3\,b0\,b1\,b2 - 4\,a3\,b1^3,$$

$$-54\,a0^2\,b3^2 + 36\,a0\,a1\,b2\,b3 + 36\,a0\,a2\,b1\,b3 - 24\,a0\,a2\,b2^2 - 216\,a0\,a3\,b0\,b3 + 36\,a0\,a3\,b1\,b2 - 24\,a1^2\,b1\,b3 + 2\,a1^2\,b2^2 + 36\,a1\,a2\,b0\,b3 + 8\,a1\,a2\,b1\,b2 + 36\,a1\,a3\,b0\,b2 - 24\,a1\,a3\,b1^2 - 24\,a2^2\,b0\,b2 + 2\,a2^2\,b1^2 + 36\,a2\,a3\,b0\,b1 - 54\,a3^2\,b0^2, -162\,a0^2\,a3\,b3 + 54\,a0\,a1\,a2\,b3 + 54\,a0\,a1\,a3\,b2 - 36\,a0\,a2^2\,b2 + 54\,a0\,a2\,a3\,b1 - 162\,a0\,a3^2\,b0 - 12\,a1^3\,b3 + 6\,a1^2\,a2\,b2 - 36\,a1^2\,a3\,b1 + 6\,a1\,a2^2\,b1 + 54\,a1\,a2\,a3\,b0 - 12\,a2^3\,b0],$$

$$[0, -27\,b0^2\,b3^2 + 18\,b0\,b1\,b2\,b3 - 4\,b0\,b2^3 - 4\,b1^3\,b3 + b1^2\,b2^2, -54\,a0\,b0\,b3^2 + 18\,a0\,b1\,b2\,b3 - 4\,a0\,b2^3 + 18\,a1\,b0\,b2\,b3 - 12\,a1\,b1^2\,b3 + 2\,a1\,b1\,b2^2 + 18\,a2\,b0\,b1\,b3 - 12\,a2\,b0\,b2^2 + 2\,a2\,b1^2\,b2 - 54\,a3\,b0^2\,b3 + 18\,a3\,b0\,b1\,b2 - 4\,a3\,b1^3,$$

$$0, 0, -54\,a0\,b0\,b3^2 + 18\,a0\,b1\,b2\,b3 - 4\,a0\,b2^3 + 18\,a1\,b0\,b2\,b3 - 12\,a1\,b1^2\,b3 + 2\,a1\,b1\,b2^2 + 18\,a2\,b0\,b1\,b3 - 12\,a2\,b0\,b2^2 + 2\,a2\,b1^2\,b2 - 54\,a3\,b0^2\,b3 + 18\,a3\,b0\,b1\,b2 - 4\,a3\,b1^3, -54\,a0^2\,b3^2 + 36\,a0\,a1\,b2\,b3 + 36\,a0\,a2\,b1\,b3 - 24\,a0\,a2\,b2^2 - 216\,a0\,a3\,b0\,b3 + 36\,a0\,a3\,b1\,b2 - 24\,a1^2\,b1\,b3 + 2\,a1^2\,b2^2 + 36\,a1\,a2\,b0\,b3 + 8\,a1\,a2\,b1\,b2 + 36\,a1\,a3\,b0\,b2 - 24\,a1\,a3\,b1^2 - 24\,a2^2\,b0\,b2 + 2\,a2^2\,b1^2 + 36\,a2\,a3\,b0\,b1 - 54\,a3^2\,b0^2],$$

$$[0, 0, -27\,b0^2\,b3^2 + 18\,b0\,b1\,b2\,b3 - 4\,b0\,b2^3 - 4\,b1^3\,b3 + b1^2\,b2^2, 0, 0, 0, -54\,a0\,b0\,b3^2 + 18\,a0\,b1\,b2\,b3 - 4\,a0\,b2^3 + 18\,a1\,b0\,b2\,b3 - 12\,a1\,b1^2\,b3 + 2\,a1\,b1\,b2^2 + 18\,a2\,b0\,b1\,b3 - 12\,a2\,b0\,b2^2 + 2\,a2\,b1^2\,b2 - 54\,a3\,b0^2\,b3 + 18\,a3\,b0\,b1\,b2 - 4\,a3\,b1^3]]$$

$$t := a0^3\,b3^3 + (-b3^2\,b2\,a1 + (-2\,b3^2\,b1 + b3\,b2^2)\,a2 + (-3\,b3^2\,b0 + 3\,b3\,b2\,b1 - b2^3)\,a3)\,a0^2 + (a1^2\,b3^2\,b1 + ((3\,b3^2\,b0 - b3\,b2\,b1)\,a2 + (-b3\,b2\,b0 - 2\,b1^2\,b3 + b2^2\,b1)\,a3)\,a1 + (-2\,b3\,b2\,b0 + b1^2\,b3)\,a2^2 + ((b3\,b1 + 2\,b2^2)\,b0 - b2\,b1^2)\,a3\,a2 + (3\,b3\,b0^2 - 3\,b2\,b1\,b0 + b1^3)\,a3^2)\,a0 - a1^3\,b3^2\,b0 + (b2\,b3\,b0\,a2 + (2\,b3\,b1 - b2^2)\,b0\,a3)\,a1^2 + (-a2^2\,b3\,b1\,b0 + (-3\,b3\,b0^2 + b2\,b1\,b0)\,a3\,a2 + (2\,b2\,b0^2 - b1^2\,b0)\,a3^2)\,a1 + a2^3\,b3\,b0^2 - a3\,a2^2\,b2\,b0^2 + a2\,a3^2\,b0^2\,b1 - a3^3\,b0^3;$$

$$t := a0^3\, b3^3 + \left(-b3^2\, b2\, a1 + \left(-2\, b3^2\, b1 + b3\, b2^2\right) a2 + \left(-3\, b3^2\, b0 + 3\, b3\, b2\, b1 - b2^3\right) a3\right) a0^2 + \left(a1^2\, b3^2\, b1 + \left(\left(3\, b3^2\, b0 - b3\, b2\, b1\right) a2 + \left(-b3\, b0\, b2 - 2\, b3\, b1^2 + b1\, b2^2\right) a3\right) a1 + \left(-2\, b3\, b0\, b2 + b3\, b1^2\right) a2^2 + \left(\left(b3\, b1 + 2\, b2^2\right) b0 - b1^2\, b2\right) a3\, a2 + \left(3\, b3\, b0^2 - 3\, b1\, b2\, b0 + b1^3\right) a3^2\right) a0 - a1^3\, b3^2\, b0 + \left(b3\, b0\, b2\, a2 + \left(2\, b3\, b1 - b2^2\right) b0\, a3\right) a1^2 + \left(-a2^2\, b3\, b1\, b0 + \left(-3\, b3\, b0^2 + b1\, b2\, b0\right) a3\, a2 + \left(2\, b2\, b0^2 - b1^2\, b0\right) a3^2\right) a1 + a2^3\, b3\, b0^2 - a3\, a2^2\, b2\, b0^2 + a2\, a3^2\, b0^2\, b1 - a3^3\, b0^3 \qquad \textbf{(4)}$$

$s := 3\, a0\, b3 - a1\, b2 + a2\, b1 - 3\, a3\, b0$

$$s := 3\, b3\, a0 - a1\, b2 + a2\, b1 - 3\, a3\, b0 \qquad \textbf{(5)}$$

*#Now we proof that Disc z(Disc x (F))=256·t·(s*$^3 - 27\,t)^3$

$collect\left(\left(simplify\left(1 \big/ \left(\left(-27\, a0^2\, a3^2 + 18\, a0\, a1\, a2\, a3 - 4\, a0\, a2^3 - 4\, a1^3\, a3 + a1^2\, a2^2\right) \cdot 256 \cdot t \cdot \left(s^3 - 27\, t\right)^3\right) \cdot Determinant(resz)\right)\right), \{a0, a1, a2, a3, b0, b1, b2, b3\}\right);$

$$1 \qquad \textbf{(6)}$$