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# Radio-Frequency Optically Pumped Magnetometers for Eddy Current Measurements 

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## Abstract

Radio-frequency optically pumped magnetometers (RF OPMs) are capable of measuring oscillating magnetic fields with high sensitivity in the $\mathrm{fT} / \sqrt{\mathrm{Hz}}$ range. Two types of RF OPMs are presented in this thesis. The first is a portable, orientationbased RF OPM with $600 \mathrm{fT} / \sqrt{\mathrm{Hz}}$ sensitivity at 10 kHz in unshielded conditions, a new benchmark for a portable unshielded RF OPM, and $200 \mathrm{fT} / \sqrt{\mathrm{Hz}}$ sensitivity at 10 kHz in shielded conditions, close to the spin projection noise limit. Eddy current measurements were performed with this OPM to remotely detect aluminium disks with diameters as small as 1.5 cm at distances of $\sim 25 \mathrm{~cm}$ from both the excitation coil and the OPM, demonstrating the possibility of using OPMs for remote sensing. Off-axis measurements were performed with this OPM to illustrate how the OPM readout can be interpreted for remote sensing. All aspects of the theory, experimental setup and results relevant for this orientation-based RF OPM and eddy current measurements are presented in this thesis.

The second OPM presented is a table-top alignment-based RF OPM in shielded conditions using a buffer gas cell. The benefit of alignment-based magnetometers over orientation-based RF OPMs is that they require only one laser beam, making them compact and robust. Until now, the alignment-based magnetometer had only been used with hand-blown paraffin-coated cells, but not with buffer gas cells that can be produced on a mass scale using microfabrication techniques. We present here an alignment-based magnetometer using a buffer gas cell (Cs and $\mathrm{N}_{2}$ ). This one-beam RF OPM with a buffer gas cell obtained a sensitivity of $325 \mathrm{fT} / \sqrt{\mathrm{Hz}}$ with an 800 Hz bandwidth to 10 kHz oscillating magnetic fields and is calculated to be close to the spin projection noise limit. The non-linear Zeeman splitting is observed with both the buffer gas cell and a paraffin-coated cell. These results open up the possibility for commercialisation and further miniaturisation of RF OPMs.

We derive a set of equations for the off-axis detection of electrically conductive spheres for the arbitrary positioning of the sphere and magnetometer. The equations are interpreted with a focus on predicting the expected signals for the imaging of the electrical conductivity of the human heart using RF OPMs to potentially help diagnose atrial fibrillation more effectively in the future. The optimal setups are discussed.

Details on how to design compact, low-noise balanced photodetectors are discussed. Several printed circuit board designs are presented and tested. The performance of the balanced photodetector exceeds that of a commercial balanced photodetector at low frequencies, being shot noise limited for powers as low as $3 \mu \mathrm{~W}$ at frequencies of 3 kHz .

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Where else to start than with my supervisor, Kasper. With a brain as awesome as optically pumped magnetometers, I can't help but thank him enough for the countless hours of discussions we have had. His kindness and openness to answer any question at any time (no matter how stupid) make him a great teacher and supervisor to learn from. I am in no doubt that the research group in Nottingham will continue to flourish for as long as Kasper is there.

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## Publications

- L. M. Rushton, T. Pyragius, A. Meraki, L. Elson, and K. Jensen, "Unshielded portable optically pumped magnetometer for the remote detection of conductive objects using eddy current measurements", Review of Scientific Instruments 93, 125103 (2022).
https://doi.org/10.1063/5.0102402
- L. M. Rushton, L. Elson, A. Meraki, and K. Jensen, "Alignment-based optically pumped magnetometer using a buffer gas cell" (2023).
https://doi.org/10.48550/ARXIV.2301.07667
- L. Elson, A. Meraki, L. Rushton, T. Pyragius, and K. Jensen, "Detection and Characterisation of Conductive Objects Using Electromagnetic Induction and a Fluxgate Magnetometer", Sensors 22, 5934 (2022).
https://doi.org/10.3390/s22165934
- E. Fradgley, C. French, L. Rushton, Y. Dieudonné, L. Harrison, J. L. Beckey, H. Miao, C. Gill, P. G. Petrov, and V. Boyer, "Quantum limits of positionsensitive photodiodes", Optics Express 30, 39374 (2022).
https://doi.org/10.1364/OE. 471673
The first publication (Rushton et al., 2022) is presented in Chapter 6, including supplementary data not featured in the publication. I wrote the paper and T. Pyragius, A. Meraki, L. Elson and K. Jensen edited it. I characterised the optically pumped magnetometer (OPM) and setup, and took all the data presented in the paper. T. Pyragius significantly contributed towards the paper, building the robust prototype and laser systems and doing all FPGA and LabView programming. A. Meraki 3D-printed the setup for the eddy current measurements. L. Elson helped obtain the eddy current measurements. K. Jensen was ever-present throughout the process, providing helpful feedback at all times.

A second paper written by myself and edited by L. Elson, A. Meraki, K. Jensen, M. Koźbiał and J. Kolodynski (Rushton et al., 2023) is currently under review and can be found in Chapters 4 and 7, including supplementary theory and experimental results. I took the measurements with L. Elson and we discussed and analysed the results together. A. Meraki built the D1 line laser system. M. Koźbiał and J. Kolodynski contributed towards our understanding of the alignment-based magnetometer theory and provided valuable comments. K. Jensen was involved in all discussions concerning the experimental procedures and results.

I edited the third paper (Elson et al., 2022), not presented in this thesis, and discussed results and experimental procedures throughout the course of the experiments. L. Elson and A. Meraki led all the experimental work, data analysis and COMSOL simulations.

The fourth paper (Fradgley et al., 2022), not presented in this thesis, includes work from my Master's thesis, in which C. French and I designed and tested positionsensitive detectors. These detectors, with significant further work from E. Fradgley and others, were used in the publication.

## List of symbols

| Symbol | Meaning |
| :--- | :--- |
| $\nu, f$ | Frequency |
| $B$ | Magnetic field |
| $B_{0}$ | Static magnetic field |
| $B_{\mathrm{RF}}(t)$ | Oscillating magnetic field |
| $B_{1}(t)$ | Primary magnetic field in eddy current measurements |
| $B_{2}(t)$ | Compensation magnetic field in eddy current measurements |
| $B_{\mathrm{ec}}(t)$ | Induced magnetic field in eddy current measurements |
| RF OPM | Radio-frequency optically pumped magnetometer |
| $\gamma_{\mathrm{Cs}}$ | Cs gryomagnetic ratio: $2 \pi(3.5 \mathrm{kHz} / \mu \mathrm{T})$ |
| $\omega_{L}=\gamma_{\mathrm{Cs}} B$ | Larmor frequency |
| $h$ | Planck's constant |
| $E_{\mathrm{hf}}=h \nu_{\mathrm{hf}}$ | Cs ground state hyperfine splitting: $h(9.193 \mathrm{GHz})$ |
| $\tau_{\mathrm{nat}}$ | Natural lifetime of excited state: $30.5 \mathrm{~ns}(\mathrm{D} 2), 34.9 \mathrm{~ns}(\mathrm{D} 1)$ |
| $\Gamma=1 / \tau_{\mathrm{nat}}$ | Decay rate of excited states via spontaneous emission |
| $Q$ | Quenching factor: $0<Q<1$ |
| $\omega_{\mathrm{RF}}$ | RF frequency |
| $\omega$ | Optical frequency (except in Chapter 5) |
| $T_{1}$ | Longitudinal relaxation time |
| $\Gamma_{1}=1 / T_{1}$ | Longitudinal relaxation rate |
| $T_{2}$ | Transverse relaxation time (spin coherence time) |
| $\gamma=1 / T_{2}$ | Transverse relaxation rate |
| $R_{p}$ | Optical pumping rate |
| BPD | Balanced photodetector |
| PCB | Printed circuit board |
| TIA | Transimpedance amplifier |
| BW | Bandwidth |
| FWHM | Full-width-half-maximum |
| $X$ | In-phase lock-in output |
| $Y$ | Out-of-phase lock-in output |
| $R$ | Magnitude of lock-in output: $\sqrt{X^{2}+Y^{2}}$ |
|  |  |

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## Chapter 1

## Introduction

Magnetic fields are everywhere. The Earth has a big iron core and produces a static magnetic field (25-65 $\mu \mathrm{T}$ or 0.25-0.65 Gauss), which deflects electrically charged particles ejected from the Sun away from the Earth's surface and protects the planet in the process. The first reported magnetic field measurement was by C. Gauss (1833), where the Earth's magnetic field was measured for the first time. Since then, many types of magnetometers have been developed to measure both static, i.e., 0 Hz - bandwidth where the bandwidth is, for example, 100 Hz , and oscillating (AC, changing with time) magnetic fields. These include Hall-effect sensors (Goel et al., 2020), magnetoresistive sensors (Freitas et al., 2007), induction coil sensors (Tumanski, 2007), fluxgate magnetometers (Wei et al., 2021), super-conducting quantum interference devices (SQUIDs) (Fagaly, 2006) and optically pumped magnetometers (OPMs) (Labyt et al., 2022; Auzinsh et al., 2014), among others.

One of the most important properties of a magnetometer is its magnetic field sensitivity. The two most sensitive magnetometers to date have been SQUIDs and OPMs, measuring magnetic fields with $\mathrm{fT} / \sqrt{\mathrm{Hz}}\left(10^{-15} \mathrm{~T} / \sqrt{\mathrm{Hz}}\right)$ and sub-fT $/ \sqrt{\mathrm{Hz}}$ (Kominis et al., 2003; Savukov et al., 2005) sensitivities, respectively. Despite the high sensitivity of SQUIDs, a drawback is the requirement for the sensor to be cyrogenically cooled. This has many added complications, for example when measuring magnetic fields produced by the brain (magnetoencephalography) (Boto et al., 2018; Hill et al., 2020; Boto et al., 2017; Xia et al., 2006), where the sensor cannot fit perfectly around each patient's head due to its immovability, affecting the brain imaging depending on the patient's head size. In addition to this, the requirement to have the patient stationary during measurements can be particularly challenging for children, further hindering diagnoses. Optically pumped magnetometers, on the other hand, have no inherent limitation to be fixed in one place as they do not need to be cyrogenically cooled. This means that OPMs offer far more flexibility than SQUID sensors. As well as measuring magnetic fields produced by the brain, magnetic fields produced by the heart (magnetocardiography) (Jensen et al., 2018; Morales et al., 2017; Bison et al., 2009; Alem et al., 2015) and the nerves (Jensen et al., 2016) have also been measured with OPMs. These signals typically have frequencies $<1 \mathrm{kHz}$.

Optically pumped magnetometers (OPMs), the first demonstration of which came from Bell and Bloom (1957), measure the precession of optically polarised alkali metal atomic spins in magnetic fields (Budker and Romalis, 2007; Seltzer, 2008). Caesium (Cs) is an example of such an alkali metal, which has a finite total angular momentum in its ground state, but no orbital angular momentum, making it highly
sensitive to magnetic fields. Optical pumping, which was demonstrated as far back as the early 1950s (Brossel and Bitter, 1952; Hawkins and Dicke, 1953), allows for angular momentum of the light to be transferred to the atoms, polarising the atomic sample and manipulating the populations of the atomic spin quantum states. Once the state has been prepared, the polarised atomic spins precess around the magnetic field at the Larmor frequency $\omega_{L}=\gamma_{\mathrm{Cs}} B$, where $\gamma_{\mathrm{Cs}}=2 \pi(350 \mathrm{kHz} /$ Gauss $)$ is the gyromagnetic ratio for Cs. These atomic spins are then probed, often with another laser (a probe beam), but sometimes by the same pump beam. This allows for tiny magnetic fields to be measured.

Relaxation processes cause the atomic spins to decohere (relax), due to electron randomisation collisions with the glass walls of the vapour cell containing the Cs, or due to spin-exchange and spin-destructive Cs-Cs collisions, among other relaxation mechanisms (Graf et al., 2005; Ledbetter et al., 2007). It is important to reduce the effects of the relaxation processes to increase the sensitivity of the OPM. The effects of the relaxation processes are typically reduced in two ways: (1) the addition of a buffer gas, such as $\mathrm{N}_{2}$, to a vapour cell slows the diffusion of Cs atoms to the glass walls, extending the relaxation time of the atomic sample; (2) the inside of the vapour cell is coated in an anti-relaxation coating, for example paraffin, which leads to the Cs atoms being able to bounce off the walls thousands of times without undergoing randomisations of the electron spins with the glass walls (Labyt et al., 2022). Both of these methods extends the spin relaxation (spin coherence) time.

The most sensitive OPM that can work close to zero magnetic field is the spin-exchange-relaxation-free (SERF) OPM (Shah and Wakai, 2013; Ledbetter et al., 2008). With SERF OPMs, the vapour cell is heated to high temperatures ( $\sim 100-$ $200^{\circ} \mathrm{C}$ ) to increase the alkali vapour number density. Higher number densities lead to greater collisions between the atoms, where the spins will be randomised upon collisions, i.e., spin relaxation will occur. If this is performed close to zero magnetic field, however, then the spin exchange collisions will be at a far higher frequency than the Larmor precession of the atoms, causing the magnetic field to effectively interact with the average spin of the atoms and inhibiting spin exchange relaxation. This allows for stellar sensitivities to be achieved for the detection of low frequency ( $<1 \mathrm{kHz}$ ) magnetic fields. Buffer gas cells are typically used for SERF OPMs, as the coatings (normally paraffin) in anti-relaxation coated cells have relatively low melting temperatures. SERF OPMs capable of measuring one-, two- or threecomponents of the magnetic field are commercially available (QuSpin 2022; FieldLine Inc 2023; Twinleaf 2022). Alongside commercial SERF OPMs, commercial scalar OPMs with a large enough dynamic range to measure the Earth's magnetic field are also available (QuSpin 2022). Other scalar magnetometers, such as the freeinduction decay OPM (Hunter et al., 2018), also exist which are able to measure large DC magnetic fields.

As well as measuring DC magnetic fields, oscillating (RF) magnetic fields $\mathbf{B}_{\mathrm{RF}}(t)$ in the $\mathrm{kHz}-\mathrm{MHz}$ frequency range are important to detect. One application where oscillating magnetic fields are measured is during eddy current measurements (Griffiths et al., 1999; Griffiths, 2001; Bidinosti et al., 2007; Honke and Bidinosti, 2018). During these measurements, a primary magnetic field $\mathbf{B}_{1}(t)$ induces eddy currents in an electrically conductive object, which in turn generate a secondary magnetic field $\mathbf{B}_{\text {ec }}(t)$ (Wickenbrock et al., 2014; Wickenbrock et al., 2016). This induced field $\mathbf{B}_{\text {ec }}(t)$ can then be measured by a magnetometer and can provide information about
the electrically conductive object, such as its position, shape, electrical conductivity and magnetic permeability (Rushton et al., 2022; Elson et al., 2022). Eddy current measurements are used in many applications, such as in remote sensing (Rushton et al., 2022; Deans et al., 2018b), where magnetic and non-magnetic objects can be detected at a distance, allowing for the position, shape, size, material composition and velocity to be extracted. Eddy current measurements are also used in non-destructive testing for defect detection (Bevington et al., 2019), such as in the detection of cracks in train tracks that are not visible to the eye. The primary magnetic field has a frequency-dependent skin depth, and so varying the frequency of the primary magnetic field can allow for the layers of an object to be extracted (Maddox et al., 2022). Eddy current measurements are also potentially important for medical applications such as non-invasive atrial fibrillation diagnosis (Jensen et al., 2019; Marmugi and Renzoni, 2016; Deans et al., 2020). Atrial fibrillation is a common heart condition where the patient has an irregular heartbeat. The causes of atrial fibrillation are not known in detail (Marmugi and Renzoni, 2016), which is one of the reasons why the treatment of atrial fibrillation via radio-frequency catheter ablation can produce sub-optimal results for the patient (Narayan et al., 2012). It is thought that triggers of atrial fibrillation could be due to permanent anomalies in the electrical conductivity $\sigma$ of the heart, and that creating a 3D map of the electrical conductivity of the human heart via eddy current measurements would improve understanding of the triggers of atrial fibrillation (Marmugi and Renzoni, 2016).

In this thesis, we have developed theory to predict the magnetic fields induced during eddy current measurements in electrically conductive cylinders and spheres, using a similar approach to Griffiths et al. (1999). We generalise the problem to the arbitrary 3D positioning of a magnetometer and a conductive sphere, and derive equations to predict the induced magnetic fields. We analyse the derived equations, placing an emphasis on which setup is most suitable for magnetic induction tomography of the heart using OPMs. We verify the theory using numerical simulations from COMSOL and compare our derived solutions to other eddy current measurement theories (Bidinosti et al., 2007; Honke and Bidinosti, 2018).

The detection of oscillating magnetic fields in the $\mathrm{kHz}-\mathrm{MHz}$ frequency range, such as those induced during eddy current measurements, can be performed with several types of magnetometer, such as with fluxgate magnetometers, pick-up coils and radio-frequency (RF) OPMs. These RF OPMs are the main magnetometer of interest for this thesis. RF OPMs have been able to measure oscillating magnetic fields with sensitivities as high as several $\mathrm{fT} / \sqrt{\mathrm{Hz}}$ in shielded conditions for table-top (Savukov et al., 2005; Ledbetter et al., 2007) and portable (Dhombridge et al., 2022) setups, as well as sub-pT $\sqrt{\mathrm{Hz}}$ sensitivities in unshielded table-top (Yao et al., 2022, Keder et al., 2014. Deans et al., 2018c) and portable (Rushton et al., 2022) setups. These sensitivities exceed those of other RF magnetometers, such as the fluxgate magnetometer which generally obtains sensitivities in the order of $\mathrm{pT} / \sqrt{\mathrm{Hz}}$.

During my PhD we have demonstrated several significant advances in the development of portable orientation-based RF OPMs. We set a benchmark for a portable, unshielded RF OPM, achieving sub-pT/ $\sqrt{\mathrm{Hz}}$ sensitivity to oscillating magnetic fields. In shielded conditions the OPM obtained a sensitivity of $200 \mathrm{fT} / \sqrt{\mathrm{Hz}}$, close to the spin projection noise quantum limit. Using this OPM for eddy current measurements, we demonstrated the remote detection of electrically conductive ob-
jects, detecting aluminium disks with diameters as small as 1.5 cm at distances of $\sim 25 \mathrm{~cm}$ from both the excitation coil and OPM. We demonstrated how the OPM readout could be interpreted for the localisation of electrically conductive objects.

Further on from this, during my PhD we have built, for the first time, a table-top alignment-based RF OPM using a buffer gas cell rather than a paraffin-coated cell (Ledbetter et al., 2007; Zigdon et al., 2010). The advantage of using buffer gas cells is that they can be produced on a mass scale using microfabrication techniques (Shah et al., 2007). Such microfabriaction techniques have so far not been compatible with anti-relaxation coating, which need to be hand-blown. The combination of a simple one-beam RF OPM with mass-producible buffer gas cells open up the possibility for the further miniaturisation (Rushton et al., 2022; Deans et al., 2021; Dhombridge et al., 2022) and commercialisation of RF OPMs in medical physics (Deans et al., 2020; Jensen et al., 2019; Marmugi and Renzoni, 2016), remote sensing (Rushton et al., 2022; Deans et al., 2018b) and non-destructive testing (Bevington et al., 2021; Bevington et al., 2019).

### 1.1 Outline of thesis

- Chapter 2. The atomic structure of Cs is discussed, followed by a review of the vapour cells used in this thesis and the intrinsic sensitivities of OPMs, alongside the mechanisms leading to spin relaxation. The experimental methods used to extract properties of the vapour cell are described, including how to determine the temperature of the cell and how to calculate the buffer gas pressure in buffer gas cells. The atom-magnetic field interactions in small and large static magnetic fields are derived, followed by a derivation of the lightatom interaction. Optical pumping into an oriented state and into an aligned state is then derived using rate equations. We describe how the optical pumping mechanisms change when a quenching gas is present in the vapour cell. The evolution of a density matrix in a magnetic field is then discussed.
- Chapter 33 The theory of an orientation-based optically pumped magnetometer using a pump beam and a probe beam is covered. The evolution of atomic spins in magnetic fields is derived. This is followed by a derivation of how the atomic spins are "read out" by measuring the rotation of a linearly polarised probe beam via Faraday rotation.
- Chapter 4 . The theory of how one-beam alignment-based magnetometers can be used to measure oscillating magnetic fields for both paraffin-coated and buffer gas cells is shown.
- Chapter 5. Our theory on the detection of electrically conductive cylinders and spheres using eddy current measurements is presented, placing a particular emphasis on the feasibility of performing eddy current measurements on the heart to understand the triggers of atrial fibrillation. We verify the derived equations by comparing our theory to another eddy current theory and also to numerical simulations performed in COMSOL.
- Chapter 6. We present results on the development of a portable orientationbased OPM based on the theory from Chapter 3. Details are provided on how
the OPM is characterised in both unshielded and shielded conditions. The sensitivity of the OPM in unshielded and shielded conditions is measured. Eddy current measurements are then performed with the OPM for the remote detection of small aluminium disks in both on-axis and off-axis scenarios.
- Chapter 7. The experimental setup for an alignment-based magnetometer based on the theory from Chapter 4 is described. A paraffin-coated alignmentbased magnetometer is characterised and its sensitivity to oscillating magnetic fields is calculated. Eddy current measurements are performed with the OPM, followed by experimental results of the operation of the alignment-based magnetometer at large magnetic fields where the non-linear Zeeman splitting is observed. Following on from this, a buffer gas containing Cs and 65 Torr $\mathrm{N}_{2}$ is used in an alignment-based magnetometer. The OPM is operated at large magnetic fields to observe the non-linear Zeeman splitting. The OPM is characterised and the sensitivity is extracted.
- Chapter 8: Details are provided on how to design a low-noise balanced photodetector necessary for the operation of sensitive optically pumped magnetometers. Several balanced photodetectors are compared.
- Chapter 9: The findings in this thesis are summarised.
- Appendix A. The code for a machine learning programme to distinguish between electrically conductive spheres with and without a defect is presented.
- Appendix B The hyperfine structure of the Cs D2 line is resolved by performing saturated absorption spectroscopy.
- Appendix C: The rate equations for optical pumping using circularly polarised light are stated.
- Appendix D The rate equations for optical pumping using linearly polarised light are stated.
- Appendix E Calculations for Chapter 3 when measuring Faraday rotation using an OPM are presented.
- Appendix F: Theoretical and experimental data of the paraffin-coated alignment-based magnetometer at high RF amplitudes is presented. The transmission of the laser beam through the paraffin-coated cell is plotted.
- Appendix G: Integrals for Chapter 5 are calculated.
- Appendix H: Calculations for Chapter 6 are presented regarding eddy current calculations in a conductive sphere.


## Chapter 2

## Optically pumped magnetometers

### 2.1 Atomic structure of Caesium

Caesium (Cs) has an atomic number 55 ( 55 protons, 55 electrons) and only has one stable isotope Cs-133 (78 neutrons) (Steck, 2022). Caesium's electron configuration is often referred to as $[\mathrm{Xe}] 6^{2} \mathrm{~S}_{1 / 2}$. Xenon (Xe) is a noble gas and its electrons completely fill the electronic "shells", meaning that Xe is very stable and non-reactive. Caesium, on the other hand, has an extra outer electron, which is described by $6^{2} \mathrm{~S}_{1 / 2}$. The " 6 " describes the principal quantum number $n$. As $n$ gets larger, the electron is less tightly bound to the nucleus. The " S " corresponds to the orbital angular momentum $L$ of the outer electron: " S " corresponds to $L=0$, " P " corresponds to $L=1$. The superscript " 2 " corresponds to $2 S+1$, where $S=1 / 2$ is the intrinsic spin of the outer electron. Finally, the subscript " $1 / 2$ " corresponds to $J$, which is the total electronic angular momentum of the outer electron. The total electronic angular momentum $\mathbf{J}$ is calculated by (Steck, 2022)

$$
\begin{equation*}
\mathbf{J}=\mathbf{L}+\mathbf{S} \tag{2.1}
\end{equation*}
$$

and the possible quantum numbers $J$ lie in the range $|L-S| \leq J \leq L+S$ in increments of 1 . The outer electron of the Cs atom always prefers to be in the lowest possible energy configuration. This corresponds to the level $L=0$, which means that the only possible value of $J$ is $1 / 2$. However, if the outer electron is excited into a state with a larger orbital angular momentum $L=1$, then $J$ can be either $1 / 2$ or $3 / 2$.

In addition to the electron angular momentum, the total nuclear angular momentum $I=7 / 2$ needs to be considered. The total angular momentum $F$ can be calculated by the coupling between the nucleus and the electron from

$$
\begin{equation*}
\mathbf{F}=\mathbf{J}+\mathbf{I} . \tag{2.2}
\end{equation*}
$$

The Cs ground state has $J=1 / 2$ and hence the values of $F$ range from $|J-I| \leq F \leq$ $J+I(3 \leq F \leq 4)$. This coupling between the electron and nuclear angular momenta leads to the hyperfine structure, as shown in Fig. 2.1. The same coupling exists for the lowest excited states, which have an electron orbital angular momentum $L=1$. The possible values of $J$ are therefore $J=1 / 2$ and $J=3 / 2$. For the $J=1 / 2$ state, $F^{\prime}=3$ and $F^{\prime}=4$, whereas for the $J=3 / 2$ state $F^{\prime}=2,3,4,5$. The ground and excited states are notated with $F$ and $F^{\prime}$, respectively.


Figure 2.1: Energy level diagram of Cs, including its ground states $F=3$ and $F=4$, the excited states on the D 2 line $F^{\prime}=2,3,4,5$ and the excited states on the D1 line $F^{\prime}=3,4$.

To excite a Cs atom from its ground state $6^{2} \mathrm{~S}_{1 / 2}(F=3$ or $F=4)$, where the atoms spend almost all of their time, to the excited state $6^{2} \mathrm{P}_{1 / 2}\left(F^{\prime}=3\right.$ or $F^{\prime}=4$ ), an atom must absorb a photon of light with a wavelength of $\sim 895 \mathrm{~nm}$ $(\sim 335 \mathrm{THz})$. This transition from $6^{2} \mathrm{~S}_{1 / 2}$ to $6^{2} \mathrm{P}_{1 / 2}$ is called the "D1" transition. If an atom were to be excited from its ground state $6^{2} \mathrm{~S}_{1 / 2}(F=3$ or $F=4)$ to the excited state $6^{2} \mathrm{P}_{3 / 2}\left(F^{\prime}=2, F^{\prime}=3, F^{\prime}=4\right.$ or $\left.F^{\prime}=5\right)$, an atom must absorb a photon with a wavelength of $\sim 852 \mathrm{~nm}$. This transition from $6^{2} \mathrm{~S}_{1 / 2}$ to $6^{2} \mathrm{P}_{3 / 2}$ is called the "D2" transition. Once an atom has absorbed a photon, it will only remain in the excited state for a very short amount of time, governed by the natural lifetimes $\tau_{\text {nat }} \approx 30.5 \mathrm{~ns}$ and $\tau_{\text {nat }} \approx 34.9 \mathrm{~ns}$ for the $6^{2} \mathrm{P}_{3 / 2}(\mathrm{D} 2)$ and $6^{2} \mathrm{P}_{1 / 2}$ (D1) excited states (Steck, 2022), respectively. The atoms will then decay to either ground state $F=3$ or $F=4$ via the spontaneous emission of a photon, with probabilities governed by the Clebsch-Gordon coefficients (Steck, 2022) in a cell filled only with Cs.

In quantum mechanics, only one of the components $F_{x}, F_{y}$ and $F_{z}$ of the total angular momentum $\mathbf{F}$ can be measured with $100 \%$ certainty. For this example, the component that will be measured with $100 \%$ certainty will be the $F_{z}$ component, which will be called $m$ (also called $m_{F}$ ). The possible values of this component $F_{z}$ are

$$
\begin{equation*}
-F,-F+1, \ldots<m<\ldots, F-1, F \tag{2.3}
\end{equation*}
$$

For the $F=4$ ground state, the possible values of the $F_{z}$ component are therefore $m=-4,-3,-2,-1,0,1,2,3,4$. The components $F_{x}$ and $F_{y}$ can not be known with $100 \%$ certainty, i.e., they do not commute $\left[F_{x}, F_{y}\right]=i \hbar F_{z}$. The spin components $F_{x}, F_{y}, F_{z}$ are quantum mechanical spin operators and should be written as $\hat{F}_{x}$, $\hat{F}_{y}, \hat{F}_{z}$. However, to avoid being cumbersome the "hats" on the operators are left
out. The uncertainties $\Delta F_{x}$ and $\Delta F_{y}$ of measuring $F_{x}$ and $F_{y}$ are governed by the Heisenbery uncertainty principle

$$
\begin{equation*}
\Delta F_{x} \Delta F_{y} \geq \frac{\hbar<F_{z}>}{2} \tag{2.4}
\end{equation*}
$$

### 2.2 Vapour cells and spin relaxation

Several different Cs vapour cells are used in the experiments presented in this thesis. Pictures of some of these are shown in Fig. 2.2. A 7 cm long vapour cell filled with


Figure 2.2: Photos of the vapour cells that are used throughout this thesis. (a) A 7 cm long cell filled with only Cs, which is used as a reference. Another reference cell ( 2 cm long, 1 cm diameter) surrounded by heating wires is also used in this thesis (not pictured). (b) A ( 5 mm$)^{3}$ cubic paraffin-coated vapour cell. The paraffin-coated cell is always operated at room temperature $T \sim 20^{\circ} \mathrm{C}$. (c) A 5 mm long, 5 mm diameter buffer gas vapour cell without any heating elements. (d) A buffer gas vapour cell, similar to the one in (c), but placed in a Shapal ceramic cylinder and surrounded by resistive heating wires, heat insulating aerogel and Kapton tape.

Cs only is included in Fig. 2.2a. This cell, as well as another 2 cm long reference surrounded by heating wires which is not pictured, is used as a reference for many of the experiments presented in this thesis. There are two other types of vapour cells which are the main vapour cells of interest for this thesis. One of the vapour cells is a $(5 \mathrm{~mm})^{3}$ hand-blown cubic paraffin-coated cell and an example of a similar paraffin-coated vapour cell is shown in Fig. 2.2b. The paraffin-coated Cs cell is always operated at room temperature ( $\sim 20^{\circ} \mathrm{C}$ ) throughout this thesis, so no heating elements surround the vapour cell. The other main vapour cell of interest is a 65 Torr $\mathrm{N}_{2}$-Cs cylindrical buffer gas with a 5 mm diameter and 5 mm length. An example of a similar buffer gas cell is shown in Fig. 2.2 c . The buffer gas cell, unlike the paraffincoated cell, is surrounded by heating elements to operate it at temperatures greater than room temperature, as seen in Fig. 2.2d. The 65 Torr $\mathrm{N}_{2}$ - Cs cell is surrounded by a Shapal ceramic cylinder, which is chosen for its high thermal conductivity. The cylinder is wrapped in a non-magnetic resistive twisted wire and wrapped with heat insulator aerogel and Kapton tape. The stem of the cell is not heated so that the solid Cs does not land on the buffer gas cell windows but instead remains in the stem. Besides Cs (Jensen et al., 2019; Rushton et al., 2022; Hunter et al., 2018), other research groups and organisations use rubidium (Ledbetter et al., 2007,

Deans et al., 2018c; Dhombridge et al., 2022), potassium (Savukov et al., 2005) and helium-4 (Beato et al., 2018; Morales et al., 2017). Different types of buffer gases other than $\mathrm{N}_{2}$ can be used, including helium, argon and neon (Kawabata et al., 2010), as well as mixtures of buffer gases. Throughout this thesis, the focus will be on Cs in paraffin-coated cells or Cs with $\mathrm{N}_{2}$ buffer gas.

As described in Sec. 2.1 the Cs atoms have an atomic spin. If the vapour cell is only filled with Cs, as in Fig. 2.2 a , then the atoms would be equally distributed among the 7 magnetic sublevels of $F=3$ and the 9 magnetic sublevels of $F=4$, i.e., $1 / 16$ of the atoms are in each $m$ sublevel (see Fig. 2.3a). In such a situation the atomic sample is "unpolarised". As will be described in more detail later in


Figure 2.3: Example of an unpolarised and a polarised Cs atomic ensemble. (a) The atomic sample is unpolarised, so the populations of the magnetic sublevels in the ground state are equal. This can be pictured as the spins of the vectors in the vapour cell pointing in random directions, such that the total angular momentum vector $\mathbf{J}=0$. (b) The atomic sample is polarised by creating unequal populations of the magnetic sublevels. In this case all of the atoms in the sample $N_{A}$ are pumped into the $F=4, m=4$ sublevel. This can be visualised as each atomic spin $\mathbf{j}$ pointing in the same direction, such that the total angular momentum of the atomic sample is depicted as a vector $\mathbf{J}=\sum_{N_{A}} \mathbf{j}$ pointing in a specific direction.

Sec. 2.6, however, the spins of the Cs atoms can all be prepared via optical pumping to create unequal populations of the magnetic sublevels. An atomic sample where the magnetic sublevels have unequal populations is called a "polarised" sample. The orientation $p_{\text {orient }}$ of the $F=4$ ground state will be defined as

$$
\begin{equation*}
p_{\text {orient }}=\frac{1}{F} \sum_{m=-F}^{F} \rho_{m, m} m, \tag{2.5}
\end{equation*}
$$

where $\rho_{\mathrm{m}, \mathrm{m}}$ is the population of the magnetic sublevel $m$ in the $F$ ground state. If, for example, $100 \%$ of the atoms are pumped into $F=4, m=4$, i.e., $\rho_{4,4}=1$ and $\rho_{3,3}=\rho_{2,2}=\ldots=\rho_{-4,-4}=0$, the orientation of the $F=4$ ground state will be
$p_{\text {orient }}=1$. An unpolarised or aligned distribution has $p_{\text {orient }}=0$. This oriented state can be pictured as a vector $\mathbf{J}$, as it has a favoured direction. This is depicted in Fig. [2.3b. The vector $\mathbf{J}$ is used rather than $\mathbf{F}$ to show that this depiction is semi-classical.

To gain some understanding about how the spin describing the atomic ensemble $\mathbf{J}$ behaves in a magnetic field, an example of free-induction decay is considered. The atomic spin $\mathbf{J}$ is initially polarised along the $z$-axis, parallel to a static magnetic field $\mathbf{B}_{0}=B_{0} \hat{\mathbf{z}}$, as depicted in Fig. 2.4. At $t=0$, a "small" DC magnetic field


Figure 2.4: Free-induction decay demonstration. (a) The spin vector $\mathbf{J}$ representing the atomic ensemble is initially aligned along the $z$-axis parallel to $\mathbf{B}_{0}$. The small transverse magnetic field is applied along the $x$-axis. (b) At $t=0$ a DC magnetic field $\mathbf{B}_{x}$ is applied perpendicular to $\mathbf{B}_{0}$ for a short time $t=t^{\prime}$, represented by the green rectangle in the figure. After the DC pulse is turned off, the atomic spins are monitored and allowed to precess around the static magnetic field with the frequency $\omega_{L} /(2 \pi)=\nu_{L}$. The angle $\theta$ between $\mathbf{B}_{0}=B_{0} \hat{\mathbf{z}}$ and the projection of $\mathbf{J}$ onto the $y-z$ plane is plotted as a function of time. Inset figure: The grey arrow represents $\mathbf{J}$ at $t=t^{\prime}$ and the black arrow represents $\mathbf{J}$ at $t \rightarrow \infty$.
$\mathbf{B}_{x}=B_{x} \hat{\mathbf{x}}$ is applied for a short amount of time $\Delta t=t^{\prime}$ then turned off, as depicted with the green rectangle in Fig. [2.4p. The atomic spin $\mathbf{J}$ is analysed from $t=t^{\prime} \rightarrow$ $t=\infty$. The atomic spins at $t=t^{\prime}$ and $t=\infty$ are represented by a grey vector and a black vector in Fig. 2.4b, respectively. The atomic spin $\mathbf{J}$ is free to precess around the static magnetic field $\mathbf{B}_{0}$ at the Larmor frequency $\omega_{L}=\gamma_{\mathrm{Cs}} B_{0}$, where $\gamma_{\mathrm{Cs}}=2 \pi(350 \mathrm{kHz} /$ Gauss $)$ is the gyromagnetic ratio of Cs. The spiral in the inset figure in Fig. 2.4 b represents the projection of the vector $\mathbf{J}$ onto the $x-y$ plane as a function of time. The variable of interest that is detected here is the angle $\theta$ between $\mathbf{B}_{0}=B_{0} \hat{\mathbf{z}}$ and the projection of $\mathbf{J}$ onto the $y-z$ plane. The angle $\theta$ decreases as a function of time with the form

$$
\begin{equation*}
\theta(t)=A \cos \left(\omega_{L} t+\phi\right) e^{-\frac{\left(t-t^{\prime}\right)}{T_{2}}} \tag{2.6}
\end{equation*}
$$

where $T_{2}$ is the transverse spin relaxation time of the spins $J_{x}$ and $J_{y}, A$ is the amplitude of the decaying sine wave and $\phi$ is the phase. When $t \gg T_{2}, \theta \rightarrow 0$ and the spin will be once again aligned with $\mathbf{B}_{0}$. The relaxation of the atomic polarisation (Graf et al., 2005) is due to the relaxation of the transverse spin components ( $J_{x}$, $J_{y}$ ), governed by the relaxation rate $\gamma=1 / T_{2}$, and the relaxation of the longitudinal
spin component $\left(J_{z}\right)$, governed by the relaxation rate $\Gamma_{1}=1 / T_{1}$. In a vapour cell filled only with Cs, the electron randomisation collisions with the cell walls (Graf et al., 2005) will cause the relaxation of the atomic polarisation. Spin-exchange collisions between Cs atoms also cause relaxation of the atomic polarisation (Graf et al., 2005). Relaxation of the polarisation also occurs due to the exchange of atoms between Cs vapour and solid Cs in the stem.

One of the two most common ways to increase the spin relaxation times $T_{2}$ and $T_{1}$ is to coat the inside of the vapour cell with paraffin (Labyt et al., 2022). The atoms can bounce off the walls thousands of times without the atomic polarisation relaxing through electron randomisation collisions, leading to large relaxation times $T_{1}$ and $T_{2}$, typically on the ms time scale (Labyt et al., 2022, Graf et al., 2005). The second most common way to increase $T_{2}$ and $T_{1}$ is to fill the vapour cell with a buffer gas, such as $\mathrm{N}_{2}$. The Cs atoms diffuse through the buffer gas, increasing the time it takes for the spins to relax. Collisions between the Cs atoms and the buffer gas lead to the relaxation of the atomic spins (Labyt et al., 2022), along with previously mentioned relaxation mechanisms.

### 2.2.1 Quantum noise

There are typically two sources of quantum noise which limit the sensitivity of an OPM: photon shot noise and atomic shot noise (spin projection noise). Photon shot noise originates from the random arrival times of photons hitting a photodiode. A photodiode converts incident light power $P$ into a photocurrent $I$. The variance of photon shot noise $i_{\text {shot }}$ scales linearly with the optical power $P$ hitting the photodiode, such that $i_{\text {shot }} \propto N_{\mathrm{ph}}$, where $N_{\mathrm{ph}}$ is the number of photons hitting the photodiode. Atomic shot noise, also called spin projection noise, arises from the flipping of spins during relaxation processes and is due to the uncertainty relation $\Delta F_{x} \Delta F_{y}=\left|F_{z}\right| / 2$ (Savukov et al., 2005). Ideal OPMs with optimal sensitivities have the atomic noise equal to the photon shot noise (Auzinsh et al., 2004).

If atomic noise dominates over other sources of noise such as photon shot noise, then the sensitivity $\delta B_{\mathrm{spn}}$ of an OPM, such as the orientation-based OPM presented in this thesis, is given by (Savukov et al., 2005)

$$
\begin{equation*}
\delta B_{\mathrm{spn}}=\frac{1}{\gamma_{\mathrm{Cs}}} \sqrt{\frac{8}{F_{z} n V T_{2}}}, \tag{2.7}
\end{equation*}
$$

where $\gamma_{\mathrm{Cs}}=2 \pi(350 \mathrm{kHz} /$ Gauss $), F_{z}=4$ for a fully polarised Cs sample pumped into the $F=4, m=4$ ground state, $n$ is the number density, $T_{2}$ is the transverse relaxation time and $V$ is the volume of the cell that is probed. For a paraffin-coated cell all the atoms in the cell are probed, i.e., $V=V_{\text {cell }}$, whereas for a buffer gas cell only the atoms in the volume of the beam $V_{\text {beam }}$ are probed, i.e., $V=V_{\text {beam }}$. Substituting in some realistic values for OPMs, such as $n=2.2 \times 10^{16} \mathrm{~m}^{-3}$ at $T=18.5^{\circ} \mathrm{C}, V_{\text {cell }}=(5 \mathrm{~mm})^{3}$ and $T_{2}=1 /(\gamma)=8 \mathrm{~ms}$, the sensitivity of such an OPM when limited by atomic noise would be $86 \mathrm{fT} / \sqrt{\mathrm{Hz}}$. This demonstrates the importance of having a long $T_{2}$ time for excellent sensitivities when limited by atomic noise, and also how a larger number density $n$ at higher temperatures can improve the sensitivity of the OPM.

If the photon shot noise dominates over other sources of noise in the orientationbased OPM, then the sensitivity $\delta B_{\mathrm{ph}}$ is given by (Savukov et al., 2005)

$$
\begin{equation*}
\delta B_{\mathrm{ph}}=\frac{4}{\gamma_{\mathrm{Cs}} l r_{e} c f_{\mathrm{osc}} n T_{2} D(\nu) \sqrt{2 \Phi_{\mathrm{pr}} \eta}} \tag{2.8}
\end{equation*}
$$

where $l$ is the length of the vapour cell, $r_{e}$ is the classical electron radius, $c$ is the speed of light, $f_{\text {osc }}$ is the oscillator strength (Steck, 2022), $\Phi_{\mathrm{pr}}$ is the flux of probe beam photons hitting the balanced photodetector and $\eta$ is the quantum efficiency of the photodiode. The term $D(\nu)=\left(\nu-\nu_{0}\right) /\left[\left(\nu-\nu_{0}\right)^{2}+(\Delta \nu / 2)^{2}\right]$, where $\nu$ is the frequency of the probe beam, $\nu_{0}$ is the transition frequency (D1 or D2) and $\Delta \nu$ is the optical full-width-half-maximum (FWHM). The following values are chosen: $l=5 \mathrm{~mm}, r_{e}=2.818 \times 10^{-15} \mathrm{~m}, c=2.998 \times 10^{8} \mathrm{~m} / \mathrm{s}, f_{\text {osc }}=0.715$ on the D2 line, $n=2.2 \times 10^{16} \mathrm{~m}^{-3}, T_{2}=1 / \gamma=8 \mathrm{~ms}, \nu-\nu_{0}=2 \mathrm{GHz}, \Delta \nu=5 \mathrm{MHz}, \eta=0.8$ and $\Phi_{\mathrm{pr}}=5 \mathrm{~mW} /(h \nu A)=6.83 \times 10^{21} \mathrm{~m}^{-2}$, where $A \sim \pi(1 \mathrm{~mm})^{2}$. Using these parameters $\delta B_{\mathrm{ph}}=0.041 \mathrm{fT} / \sqrt{\mathrm{Hz}}$. In this example orientation-based OPM, the atomic noise $\delta B_{\mathrm{spn}}$ would dominate as $\delta B_{\mathrm{spn}} \gg \delta B_{\mathrm{ph}}$.

Most notably, in both cases of $\delta B_{\mathrm{spn}}$ and $\delta B_{\mathrm{ph}}$ an increased transverse relaxation time $T_{2}$ leads to an improved sensitivity. This shows how important having a large $T_{2}$ time is to achieve stellar sensitivities with OPMs. In addition to these sources of quantum noise, it is never desirable to be limited by other forms of technical noise, for example noise from a laser, or electronic noise from a balanced photodetector.

### 2.3 Absorption spectroscopy

To experimentally understand the hyperfine structure in Fig. 2.1, absorption spectroscopy must be performed. A laser is shone through a vapour cell containing Cs atoms and is incident on a photodetector. The photodetector converts the light power incident on it into a voltage that is measured using a data acquisiton card (Spectrum Instrumentation M2p.5932-x4). The setup is depicted in Fig. 2.5. The


Figure 2.5: Experimental setup for absorption spectroscopy. Collimated light is output from an optical fiber. Components include half-wave plates $(\lambda / 2)$, polarising beam splitters (PBS), the vapour cell of interest (Cell), a photodiode (PD) and a data acquisition system.
laser wavelength, for example $\sim 852 \mathrm{~nm}$ for the D2 transition, is ramped up and down as a function of time, such that the energy of the laser light can match the energy difference of each different atomic transition.

The current fed into the laser (Thorlabs DL Pro 850 nm for the D2 transition, custom-made D1 laser for the D1 transition, see Sec. 7.2.1 must be increased to increase its output wavelength. In Fig. 2.6a the laser current is ramped up between 0.069 s and 0.112 s , i.e., half the period of the triangular wave. An increase in current leads to an increase in the laser power incident on the photodetector, explaining the


Figure 2.6: (a) Absorption spectrum of the "ramp-up" unnormalised D2 line with the 7 cm long ( 2.5 cm diameter) reference Cs cell (see Fig. 2.2a). The data was taken at room temperature $\left(\sim 20^{\circ} \mathrm{C}\right)$ and the laser used was a 850 nm Thorlabs DL Pro. (b) Absorption spectrum of the "ramp-down" unnormalised D1 line with a heated 2 cm long reference Cs cell situated inside the custom-made D1 laser box.
rise in the voltage measured by the data acquisition card in Fig. 2.6a. There are two dips present in Fig. 2.6a, which are separated by a frequency $\nu_{\text {hf }}$ of 9.193 GHz (Steck, 2022), which is the energy splitting between $F=3$ and $F=4$ in the ground state (see Fig. 2.1). The energy levels in the excited states are not resolved in the D2 spectrum in Fig. 2.6a, because they are only separated by small energy differences ( $151 \mathrm{MHz}, 201 \mathrm{MHz}$ and 251 MHz , see Fig. 2.1). The reason for this is due to a phenomenon called Doppler broadening, which is discussed in Sec. 2.3.1. The $F=4 \rightarrow F^{\prime}=4$ and $F=4 \rightarrow F^{\prime}=3$ transitions can be resolved for the reference Cs vapour cell data in the D1 absorption spectrum in Fig. 2.6b, because the excited states $F^{\prime}=3$ and $F^{\prime}=4$ are split by a large energy difference of 1.163 GHz (see Fig. 2.1). This data was obtained with a "custom-made" D1 laser, which is described in Sec. 7.2.1.

### 2.3.1 Doppler broadening

Atoms in temperatures above 0 K move with some velocity $\mathbf{v}$. The speed distribution $f(v)$ is governed by the Maxwell-Boltzmann distribution

$$
\begin{equation*}
f(v)=\sqrt{\frac{2}{\pi}}\left(\frac{m_{\mathrm{Cs}}}{k_{B} T}\right)^{3 / 2} v^{2} e^{-\frac{m_{\mathrm{Cs}} v^{2}}{2 k_{B} T}}, \tag{2.9}
\end{equation*}
$$

where $m_{\mathrm{Cs}}=2.21 \times 10^{-25} \mathrm{~kg}$ is the mass of a Cs atom, $k_{B}=1.38 \times 10^{-23} \mathrm{JK}^{-1}$ is the Boltzmann constant, $T$ is the temperature of the environment (room temperature $\sim 293 \mathrm{~K}$ ) and $v$ is the speed of the Cs atom.

This range of velocities which the Cs atoms have give rise to "Doppler broadening". The reason for this is the same as when an ambulance passes you by, producing a higher pitched tone as it comes towards you and a lower pitched tone as it moves away. The is because the frequency of the sound emitted by the ambulance appears higher at your position when the ambulance is travelling towards you, and lower when the ambulance drives away. This is analogous to atoms moving towards or
away from a laser beam, which shifts the frequency of light that the atoms observe and therefore can absorb, given by

$$
\begin{equation*}
\nu\left(v_{y}\right)=\nu_{0}\left(1 \pm \frac{v_{y}}{c}\right), \tag{2.10}
\end{equation*}
$$

where the $y$-axis is picked as the direction of propagation of the laser beam in this example, $c$ is the speed of light and $\nu_{0}$ is the frequency difference between the ground and excited states ( $\nu_{0}=335.1,351.7 \mathrm{THz}$ for the D1 and D2 lines, respectively (Steck, 2022)). The positive and negative terms are when the atoms are moving towards and away from the source, respectively. Due to this Doppler effect, atoms with different $v_{y}$ velocity components therefore absorb different frequency light, meaning that there will be a range of frequencies over which light is absorbed, with a full-width-half-maximum (FWHM) $\Gamma_{G}$ given by

$$
\begin{equation*}
\Gamma_{G}=\frac{2 \nu_{0}}{c} \sqrt{\frac{2 k_{B} T \ln (2)}{m_{\mathrm{Cs}}}} . \tag{2.11}
\end{equation*}
$$

For the D 2 transition, the Doppler broadening at $T=293 \mathrm{~K}$ is $\Gamma_{G}=374 \mathrm{MHz}$. As $\Gamma_{G}$ is larger than $F^{\prime}=2 \rightarrow F^{\prime}=3=151 \mathrm{MHz}, F^{\prime}=3 \rightarrow F^{\prime}=4=201 \mathrm{MHz}$ and $F^{\prime}=4 \rightarrow F^{\prime}=5=251 \mathrm{MHz}$ (see Fig. 2.1), the absorption spectra for the transitions from the ground (for example $F=3$ ) to the excited states ( $F^{\prime}=2,3,4$ ) are not resolved, leading to one single dip instead of three individual dips, as can be observed in Fig. 2.6a. For the D1 line, the excited state frequency separation $(1.168 \mathrm{GHz})$ is greater than the Doppler broadening ( $\Gamma_{G}=356 \mathrm{MHz}$ at $T=293 \mathrm{~K}$ ), which means that the transitions from the ground states to the two excited states can be individually resolved, as can be observed in the absorption spectrum of the D1 line in Fig. [2.6b, Saturated absorption spectroscopy can be used to resolve the hyperfine transitions on the D2 line. An example of this is shown in Appendix B.

### 2.3.2 Calculating the number density of Cs atoms

The temperature in the vapour cell can be determined by measuring the atomic number density $n$ of Cs vapour. To do this, the laser frequency must be swept to incorporate both ground states, i.e., the $F=4 \rightarrow F^{\prime}$ and $F=3 \rightarrow F^{\prime}$ transitions during the absorption spectroscopy measurements. This can be done either on the D1 or D2 line. Once this is done, the spectrum should be normalised such that the intensity of the light after the vapour cell $I(l)$, where $l$ is the length of the vapour cell, is equal to the intensity before the cell $I(0)$, i.e., $I(l) / I(0)=1$ when the light is far off-resonance from the atomic transitions. An example of this being performed on the D1 line with a 65 Torr $\mathrm{N}_{2}$ buffer gas cell can be observed in Figs. 2.7a and 2.7b, where a cell similar to that in Fig. 2.2d is heated. The Beer-Lambert law is given by

$$
\begin{equation*}
I(l)=I(0) e^{-n \sigma(\nu) l} \tag{2.12}
\end{equation*}
$$

where $n$ is the number density of Cs atoms and $\sigma(\nu)$ is the absorption cross-section. The absorption cross-section is given by (Seltzer, 2008)

$$
\begin{equation*}
\sigma(\nu)=\pi r_{e} c f_{\mathrm{osc}} V(\nu) \tag{2.13}
\end{equation*}
$$

where $r_{e}=2.82 \mathrm{fm}$ is the classical electron radius, $f_{\text {osc }}$ is the oscillator strength $\left(f_{\text {osc }}=0.715,0.344\right.$ for the D2 and D1 lines, respectively (Steck, 2022) ) and $V(\nu)$ is


Figure 2.7: (a) Absorption spectrum of a 65 Torr $\mathrm{N}_{2}$ buffer gas cell with a $20 \mu \mathrm{~W}$ power before the cell. A Cs frequency reference is included. (b) The data in (a) normalised to 1 , after which the number density is calculated. (c) The calculated number densities for various optical powers are plotted.
a Voigt profile if both Doppler broadening and pressure broadening (see Sec. 2.3.3) are present. The number density is then calculated to be

$$
\begin{equation*}
n=-\int_{-\infty}^{\infty} \frac{\ln \left(\frac{I(l)}{I(0)}\right)}{\pi r_{e} c f_{\text {osc }} l} d \nu \tag{2.14}
\end{equation*}
$$

The calculated number density of Cs atoms in the buffer gas cell in Fig. 2.7c decreases with increasing optical power. This is because the absorbed photons will excite the atoms, and some of these atoms will decay into dark states (see Sec. 2.6). Due to the long ground state polarisation lifetime in buffer gas cells, the atoms in the dark state can remain there for a relatively long time, and so the atoms pumped into the dark state cannot absorb a photon. The calculated number density of Cs atoms in the vapour cell will therefore be underestimated for higher optical powers. The true number density will be the extrapolated number density at zero optical power, which for the example in Fig. 2.7 c is $\sim 310 \times 10^{16} \mathrm{~m}^{-3}$. This is calculated to correspond to a temperature of $\sim 76^{\circ} \mathrm{C}$ by noting that the number density can be calculated for a given temperature $T$

$$
\begin{equation*}
n=\frac{P_{v}[\mathrm{~Pa}]}{k_{B} T} \tag{2.15}
\end{equation*}
$$

where $k_{B}$ is the Boltzmann constant, $P_{v}$ is the vapour pressure (in Pa ) and $T$ is the temperature (in K). The melting point of Cs is $28.44^{\circ} \mathrm{C}(301.6 \mathrm{~K})$. The pressure $P_{v}$ (in Torr) of the atomic vapour must therefore be written as a piecewise function (Steck, 2022)

$$
\log _{10}\left(P_{v}[\text { Torr }]\right)=\left\{\begin{array}{l}
-219.5+\frac{1089}{T}-0.08336 T+94.89 \log _{10} T<301.6 \mathrm{~K}  \tag{2.16}\\
8.221-\frac{4006}{T}-6.019 \times 10^{-4} T-0.1962 \log _{10} T>301.6 \mathrm{~K}
\end{array}\right.
$$

The pressure can be converted to Pa using the conversion $133.322 \mathrm{~Pa} /$ Torr. The number density $n$ plotted as a function of temperature is shown in Fig. 2.8. The number densities at $(18.5,50.8,55,76.4)^{\circ} \mathrm{C}$ are $\left(2.2 \times 10^{16}, 43.7 \times 10^{16}, 61.8 \times 10^{16}\right.$, $\left.310 \times 10^{16}\right) \mathrm{m}^{-3}$, respectively, to quote some key numbers for this thesis.


Figure 2.8: The atomic number density of Cs vapour is plotted as a function of temperature, using Eq. 2.15 and the piecewise function for the vapour pressure from Eq. 2.16.

### 2.3.3 Pressure broadening in buffer gas cells

The main purpose of buffer gas cells is to slow the diffusion of Cs atoms, leading to a longer $T_{2}$ time (Labyt et al., 2022), as described in Sec. 2.2. Buffer gas cells are vapour cells filled with both Cs and a buffer gas, such as $\mathrm{N}_{2}$. In vapour cells filled only with Cs or in paraffin-coated cells, the natural lifetimes $\tau_{\text {nat }}$ of the excited states $6^{2} \mathrm{P}_{3 / 2}$ (D2) and $6^{2} \mathrm{P}_{1 / 2}(\mathrm{D} 1)$ are $\tau_{\text {nat }} \approx 30.5,34.9 \mathrm{~ns}$, respectively (Steck, 2022). In buffer gas cells, however, the Cs atom in the excited state will rapidly collide with $\mathrm{N}_{2}$ molecules, as the scattering cross section between the outer electron and $\mathrm{N}_{2}$ molecules is much higher in the excited state than in the ground state (Seltzer, 2008). This causes the atoms to be distributed among the magnetic sublevels in the short time they are in the excited state, known as rapid collisional mixing. Now, rather
than having an intrinsic relaxation time on the order of $\sim 30 \mathrm{~ns}$, corresponding to a linewidth of $\sim 5 \mathrm{MHz}$ as in a pure Cs cell or in a paraffin-coated cell, the absorption spectrum has a larger linewidth ( $\sim 1-100 \mathrm{GHz}$ ), often far exceeding that of the Doppler broadening (see Fig. 2.9).


Figure 2.9: Absorption spectrum of the D1 line with a 65(3) Torr $\mathrm{N}_{2} 5 \mathrm{~mm}$ length cell alongside a frequency reference which is a pure Cs cell. The buffer gas cell is heated to $51^{\circ} \mathrm{C}\left(43.7 \times 10^{16} \mathrm{~m}^{-3}\right)$ by applying a 300 mA DC current to the resistive wires (twisted pair). The $F=3 \rightarrow F^{\prime}=3,4$ and $F=4 \rightarrow F^{\prime}=3,4$ transitions are fitted to Voigt profiles and the FWHM $\Gamma_{L}$ and pressure shift are extracted. The Doppler FWHM $\Gamma_{G}$ is fixed to 374 MHz .

The absorption spectrum of a buffer gas cell must be fitted to Voigt profiles using the methods described by Andalkar and Warrington (2002). This is important to do as it allows for the extraction of the pressure broadening and pressure shifts, which are both results of the collisions between $\mathrm{N}_{2}$ buffer gas molecules and Cs atoms. Once the broadening and shift are known, the buffer gas pressure can be calculated. The Voigt profile written in the complex form is given by (Seltzer, 2008)

$$
\begin{equation*}
V\left(\nu-\nu_{0}\right)=\frac{2 \sqrt{\ln 2 / \pi}}{\Gamma_{G}} w\left(\frac{\left.2 \sqrt{\ln 2}\left[\left(\nu-\nu_{0}\right)+i \Gamma_{L} / 2\right)\right]}{\Gamma_{G}}\right), \tag{2.17}
\end{equation*}
$$

where $\Gamma_{G}$ is the Gaussian FWHM due to Doppler broadening, $\Gamma_{L}$ is the Lorentzian FWHM due to pressure broadening and $w(y)$ is

$$
\begin{equation*}
w(y)=e^{-y^{2}}(1-\operatorname{erf}(-i y)), \tag{2.18}
\end{equation*}
$$

where $\operatorname{erf}(-i y)$ is the complex error function. Once $V\left(\nu-\nu_{0}\right)$ has been calculated, one can determine the absorption cross-section

$$
\begin{equation*}
\sigma_{V}(\nu)=\pi r_{e} c f_{\text {osc }} \operatorname{Re}\left[V\left(\nu-\nu_{0}\right)\right] . \tag{2.19}
\end{equation*}
$$

An absorption spectrum of a $\mathrm{N}_{2}$ buffer gas cell is obtained (quoted as 100 Torr by the manufacturer), plotted on top of a pure Cs cell as a frequency reference in Fig. 2.9. The pressure broadening, i.e., the FWHM of the Lorentzian $\Gamma_{L}$, is extracted by fitting the $F=3 \rightarrow F^{\prime}=3$ and $F=3 \rightarrow F^{\prime}=4$ transitions
to two Voigt profiles and using their relative hyperfine strengths ( $1 / 4$ and $3 / 4$, respectively) and then doing the same for the $F=4 \rightarrow F^{\prime}=3$ and $F=4 \rightarrow F^{\prime}=4$ transitions, with hyperfine strengths of $7 / 12$ and $5 / 12$, respectively (Steck, 2022). The FWHM $\Gamma_{G}$ is fixed for Doppler broadening ( 374 MHz FWHM at $51^{\circ} \mathrm{C}$ ). The pressure broadening $\Gamma_{L}$ is fitted to be $1.26(0.05) \mathrm{GHz}$, corresponding to a pressure of $65(3)$ Torr, using the conversion of $19.51 \mathrm{MHz} /$ Torr (Andalkar and Warrington, 2002) for D1 pressure broadening with $\mathrm{N}_{2}$. The pressure can also be extracted from the shift $-0.54(0.01) \mathrm{GHz}$ (Andalkar and Warrington, 2002), which corresponds to a pressure of 65(1) Torr. Further verification of the pressure can be done by comparing the absorption spectrum with that of Andalkar and Warrington (2002), where an 82 Torr $\mathrm{N}_{2}$ cell is analysed and has greater overlap than the absorption spectrum presented here. This indicates that the pressure of this cell is less than 82 Torr, further verifying the pressure of 65(3) Torr.

### 2.4 Atom-magnetic field interaction in a static magnetic field

To understand how an atom behaves in a static magnetic field, a $F=1 \rightarrow F^{\prime}=0$ transition can be considered (see Fig. 2.10a). The ground state sublevels $|F, m\rangle=$ $\{|1,1\rangle,|1,0\rangle,|1,-1\rangle\}$ and the excited state $\left|F^{\prime}, m^{\prime}\right\rangle=\left|0^{\prime}, 0^{\prime}\right\rangle$ can be represented by a set of vectors in the following way (Auzinsh et al., 2014):

$$
\begin{gather*}
|1,1\rangle=\left(\begin{array}{l}
1 \\
0 \\
0 \\
0
\end{array}\right),|1,0\rangle=\left(\begin{array}{l}
0 \\
1 \\
0 \\
0
\end{array}\right), \\
|1,-1\rangle=\left(\begin{array}{l}
0 \\
0 \\
1 \\
0
\end{array}\right),\left|0^{\prime}, 0^{\prime}\right\rangle=\left(\begin{array}{l}
0 \\
0 \\
0 \\
1
\end{array}\right) . \tag{2.20}
\end{gather*}
$$

The Hamiltonian $H$ (which should be written as an operator $\hat{H}$ but is not to avoid being cumbersome) for the magnetic field-atom interaction is

$$
\begin{equation*}
H=H_{0}+H_{B} \tag{2.21}
\end{equation*}
$$

where $H_{0}$ and $H_{B}$ are the unperturbed and light-atom interaction Hamiltonians, respectively. Taking the energy of the lower state to be zero, the unperturbed Hamiltonian $H_{0}$ is given by

$$
\begin{align*}
H_{0} & =\left(\begin{array}{cccc}
\langle 1,1| H_{0}|1,1\rangle & \langle 1,0| H_{0}|1,1\rangle & \langle 1,-1| H_{0}|1,1\rangle & \left\langle 0^{\prime}, 0^{\prime}\right| H_{0}|1,1\rangle \\
\langle 1,1| H_{0}|1,0\rangle & \langle 1,0| H_{0}|1,0\rangle & \langle 1,-1| H_{0}|1,0\rangle & \left\langle 0^{\prime}, 0^{\prime}\right| H_{0}|1,0\rangle \\
\langle 1,1| H_{0}|1,-1\rangle & \langle 1,0| H_{0}|1,-1\rangle & \langle 1,-1| H_{0}|1,-1\rangle & \left\langle 0^{\prime}, 0^{\prime}\right| H_{0}|1,-1\rangle \\
\langle 1,1| H_{0}\left|0^{\prime}, 0^{\prime}\right\rangle & \langle 1,0| H_{0}\left|0^{\prime}, 0^{\prime}\right\rangle & \langle 1,-1| H_{0}\left|0^{\prime}, 0^{\prime}\right\rangle & \left\langle 0^{\prime}, 0^{\prime}\right| H_{0}\left|0^{\prime}, 0^{\prime}\right\rangle
\end{array}\right) \\
& =\left(\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & \hbar \omega_{0}
\end{array}\right), \tag{2.22}
\end{align*}
$$

where $\omega_{0}=2 \pi c / \lambda$ is the transition frequency $(\lambda \sim 895 \mathrm{~nm}, 852 \mathrm{~nm}$ for the Cs D1 and D 2 lines, respectively). If the Hamiltonian $H_{0}$ acts on the excited state $\left|0^{\prime}, 0^{\prime}\right\rangle$, for example, then

$$
H_{0}\left|0^{\prime}, 0^{\prime}\right\rangle=\left(\begin{array}{cccc}
0 & 0 & 0 & 0  \tag{2.23}\\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & \hbar \omega_{0}
\end{array}\right)\left(\begin{array}{l}
0 \\
0 \\
0 \\
1
\end{array}\right)=\hbar \omega_{0}\left(\begin{array}{l}
0 \\
0 \\
0 \\
1
\end{array}\right)
$$

i.e., the energy of the quantum state $\left|0^{\prime}, 0^{\prime}\right\rangle$ is equal to $\hbar \omega_{0}$. Each hyperfine level $F$ contains $m=2 F+1$ sublevels, and these are degenerate in the absence of magnetic fields. However, in the presence of a magnetic field, this degeneracy will be broken. If the magnetic field leads to a splitting of the energy levels that is smaller than the hyperfine splitting, then the magnetic field-atom interaction is given by

$$
\begin{equation*}
H_{B}=-\boldsymbol{\mu} \cdot \mathbf{B}=\frac{g_{F} \mu_{B}}{\hbar} \mathbf{F} \cdot \mathbf{B}=\frac{g_{F} \mu_{B}}{\hbar}\left(F_{x} B_{x}+F_{y} B_{y}+F_{z} B_{z}\right) \tag{2.24}
\end{equation*}
$$

where $\boldsymbol{\mu}=g_{F} \mu_{B}\left(F_{x} \hat{\mathbf{x}}+F_{y} \hat{\mathbf{y}}+F_{z} \hat{\mathbf{z}}\right) / \hbar$ is the Cs atom's magnetic dipole operator, $g_{F}$ is the hyperfine Landé g -factor (Steck, 2022), $\mu_{B}$ is the Bohr magneton, and $B_{x}, B_{y}, B_{z}$ are the magnetic fields (in Tesla) applied along the $x$-, $y$ - and $z$-axes, respectively. With only a non-zero magnetic field component along the $z$-direction with $\mathbf{B}_{0}=B_{0} \hat{\mathbf{z}}, H_{B}$ can therefore be written as

$$
\begin{equation*}
H_{B}=\frac{g_{F} \mu_{B} B_{0}}{\hbar} F_{z} . \tag{2.25}
\end{equation*}
$$

As the quantisation axis is picked as the direction of the static magnetic field, then the projection of $\mathbf{F}$ onto the $z$-axis $F_{z}$ can be calculated using

$$
\begin{equation*}
F_{z}|F, m\rangle=\hbar m|F, m\rangle \tag{2.26}
\end{equation*}
$$

The component $F_{z}$ in matrix form is given by

$$
\begin{aligned}
F_{z} & =\left(\begin{array}{cccc}
\langle 1,1| F_{z}|1,1\rangle & \langle 1,0| F_{z}|1,1\rangle & \langle 1,-1| F_{z}|1,1\rangle & \left\langle 0^{\prime}, 0^{\prime}\right| F_{z}|1,1\rangle \\
\langle 1,1| F_{z}|1,0\rangle & \langle 1,0| F_{z}|1,0\rangle & \langle 1,-1| F_{z}|1,0\rangle & \left\langle 0^{\prime}, 0^{\prime}\right| F_{z}|1,0\rangle \\
\langle 1,1| F_{z}|1,-1\rangle & \langle 1,0| F_{z}|1,-1\rangle & \langle 1,-1| F_{z}|1,-1\rangle & \left\langle 0^{\prime}, 0^{\prime}\right| F_{z}|1,-1\rangle \\
\langle 1,1| F_{z}\left|0^{\prime}, 0^{\prime}\right\rangle & \langle 1,0| F_{z}\left|0^{\prime}, 0^{\prime}\right\rangle & \langle 1,-1| F_{z}\left|0^{\prime}, 0^{\prime}\right\rangle & \left\langle 0^{\prime}, 0^{\prime}\right| F_{z}\left|0^{\prime}, 0^{\prime}\right\rangle
\end{array}\right) \\
& =\left(\begin{array}{cccc}
\hbar & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & -\hbar & 0 \\
0 & 0 & 0 & 0
\end{array}\right) .
\end{aligned}
$$



Figure 2.10: (a) Energy level diagram with a $F=1$ ground state and $F^{\prime}=0$ excited state. The neighbouring groundstate sublevels are each split by the Larmor frequency $\omega_{L}$ when the system is placed in a "small" magnetic field. (b) D1 line of Cs with $F=3,4$ ground states and $F^{\prime}=3,4$ excited states. Small corrections ( $\delta$ ) for the non-linear Zeeman splitting in a "large" magnetic field are shown.

The Hamiltonian $H_{B}$ of the atom-magnetic field interaction can therefore be written as

$$
H_{B}=g_{F} \mu_{B} B_{0}\left(\begin{array}{cccc}
1 & 0 & 0 & 0  \tag{2.28}\\
0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0
\end{array}\right)
$$

A Cs atom in the $F=4$ ground state has $2 F+1=9$ sublevels $|F, m\rangle$ which, when placed in a small magnetic field $B_{0}$, have the energy $E(m)=m h \nu_{L}$ due to the linear Zeeman effect. Here $\nu_{L}$ is the Larmor frequency in Hz. That is to say, the splittings between neighbouring sublevels are all equal to the Larmor frequency

$$
\begin{equation*}
\Delta \nu_{m, m-1} \equiv(E(m)-E(m-1)) / h=\nu_{L} \tag{2.29}
\end{equation*}
$$

In the presence of a "small" magnetic field $B_{0}=1 \mu \mathrm{~T}$, the splitting between the neighbouring magnetic sublevels in the $F=4$ Cs ground state will be $\Delta \nu_{m, m-1}=$ 3.5 kHz .

### 2.4.1 Non-linear Zeeman splitting

For small magnetic fields, the splitting of the energy levels will be much smaller than the hyperfine splitting, which means that Eq. 2.29 is valid. However, when the magnetic field becomes larger such that this is no longer the case, then for a $J=1 / 2$ atom like the alkali atoms $(F=I \pm 1 / 2)$, the Breit-Rabi formula must be used, which is given by (Steck, 2022)

$$
\begin{equation*}
E=-\frac{E_{\mathrm{hf}}}{2(2 I+1)}-g_{I} \mu_{B} m B_{0} \pm \frac{E_{\mathrm{hf}}}{2}\left(1+\frac{4 m x}{(2 I+1)}+x^{2}\right)^{1 / 2} \tag{2.30}
\end{equation*}
$$

where $x=\left(g_{J}-g_{I}\right) \mu_{B} B_{0} / E_{\mathrm{hf}} \sim 2 \mu_{B} B_{0} / E_{\mathrm{hf}}, E_{\mathrm{hf}}=h \nu_{\mathrm{hf}}$ is the hyperfine splitting of the ground state, $g_{I}=\mu_{I} /\left(\mu_{N} I\right)=-3.98 \times 10^{-4}$ is the nuclear g-factor, $\mu_{I}$ is the nuclear magnetic moment and $g_{J} \sim 2$. In most of the experiments presented in this thesis the non-linear Zeeman splitting is negligible, however in the experiments in Sec. 7.3.5 and Sec. 7.4.1 for the alignment-based magnetometers the non-linear Zeeman splitting must be taken into account. In the case of the experiments presented in this thesis, the non-linear Zeeman splitting is calculated for the Cs $F=4$ ground state, meaning that the upper sign term in Eq. 2.30 is used. The lower sign term is for the $F=3$ ground state.

The final term can be expanded to second order in powers of $B_{0}$ as $(1+a x+$ $\left.b x^{2}\right)^{1 / 2}=1+a x / 2+x^{2}\left(4 b-a^{2}\right) / 8$. Substituting in $I=7 / 2$ for Cs we obtain

$$
\begin{equation*}
E=\frac{7 E_{\mathrm{hf}}}{16}+\mu_{B} B_{0}\left(\frac{m}{4}-g_{I}\right)+\frac{\left(\mu_{B} B_{0}\right)^{2}}{16 E_{\mathrm{hf}}}\left(16-m^{2}\right) . \tag{2.31}
\end{equation*}
$$

Generally with the non-linear Zeeman splitting it is interesting to understand the behaviour between adjacent magnetic sublevels, i.e., between $m$ and $m-1$. The energy difference $\Delta E_{\mathrm{m}, \mathrm{m}-1}$ between adjacent sublevels with the same $F$ is thus given by

$$
\begin{equation*}
\Delta E_{\mathrm{m}, \mathrm{~m}-1}=\frac{\mu_{B} B_{0}}{4}+\frac{\left(\mu_{B} B_{0}\right)^{2}}{16 E_{\mathrm{hf}}}(1-2 m) . \tag{2.32}
\end{equation*}
$$

This can be re-written in terms of frequency (in Hz ) as

$$
\begin{equation*}
\Delta \nu_{m, m-1}=\nu_{L}-\delta\left(m-\frac{1}{2}\right), \tag{2.33}
\end{equation*}
$$

where $\delta=2 \nu_{L}^{2} / \nu_{\text {hf }}$. For the $\mathrm{F}=4$ ground state of Cs , the difference in transition frequencies between $m=4 \rightarrow m=3$ and between $m=-3 \rightarrow m=-4$ in a magnetic field is given by

$$
\begin{equation*}
\Delta \nu_{\mathrm{m}=4, \mathrm{~m}=3}-\Delta \nu_{\mathrm{m}=-3, \mathrm{~m}=-4}=-7 \delta . \tag{2.34}
\end{equation*}
$$

These non-linear Zeeman splittings are shown in Fig. 2.10b. If the atoms are placed in a large magnetic field corresponding to a Larmor frequency $\nu_{L}=2 \mathrm{MHz}$, i.e., $B_{0}=$ $2 \mathrm{MHz} /(0.35 \mathrm{MHz} / \mathrm{G})=0.57 \mathrm{mT}$, then $\Delta \nu_{\mathrm{m}=4, \mathrm{~m}=3}-\Delta \nu_{\mathrm{m}=-3, \mathrm{~m}=-4}=-6089 \mathrm{kHz}$. Exploiting the non-linear Zeeman effect and using Eq. 2.34 is very useful when determining which magnetic sublevel the atoms are being optically pumped into, as we will see in Sec. 7.3.5 and in Sec. 7.4.1.

### 2.5 Light-atom interaction and selection rules

To demonstrate the light-atom interaction, an $F=1 \rightarrow F^{\prime}=0$ transition will once again be considered, as depicted in Fig. 2.11. A z-polarised optical electric field $\mathbf{E}$

$$
\prod_{m=-1}^{m^{\prime}=0} \sum_{m=0}^{m_{m=1}^{\prime}=0} \overline{-}_{m=1} F=1
$$

Figure 2.11: $\pi$-polarised light drives transitions between the $F=1$ ground state and the $F^{\prime}=0$ excited state to demonstrate the light-atom interaction. The $z$-polarised light is along the $z$-quantisation axis.
can be written as

$$
\begin{equation*}
\mathbf{E}=E_{0} \cos (\omega t) \hat{\mathbf{z}}, \tag{2.35}
\end{equation*}
$$

where $E_{0}$ is its amplitude and $\omega$ is the optical frequency. The light-atom interaction Hamiltonian $H_{l}$ is (Auzinsh et al., 2014)

$$
\begin{equation*}
H_{l}=-\mathbf{E} \cdot \mathbf{d}, \tag{2.36}
\end{equation*}
$$

where $\mathbf{d}$ is the dipole operator. If the electric field is polarised along the $z$-axis, $H_{l}$ will be given by

$$
\begin{equation*}
H_{l}=-E_{0} \cos (\omega t) d_{z} . \tag{2.37}
\end{equation*}
$$

The dipole operator $\mathbf{d}=d_{x} \hat{\mathbf{x}}+d_{y} \hat{\mathbf{y}}+d_{z} \hat{\mathbf{z}}$ is a vector and therefore has three components. It can be useful to describe such a vector as a first-rank $(\kappa=1)$ tensor $T^{(\kappa=1)}$. A scalar quantity is a zero-rank $(\kappa=0)$ tensor $T^{(\kappa=0)}$. The spherical tensor components $d_{q=1}^{(\kappa=1)}=d_{1}, d_{0}^{(1)}=d_{0}$ and $d_{-1}^{(1)}=d_{-1}$ in terms of the three components of the dipole operator $d_{x}, d_{y}$ and $d_{z}$ (Auzinsh et al., 2014) are

$$
\begin{align*}
& d_{1}^{(1)}=\frac{-\left(d_{x}-i d_{y}\right)}{\sqrt{2}},  \tag{2.38}\\
& d_{0}^{(1)}=d_{z},  \tag{2.39}\\
& d_{-1}^{(1)}=\frac{\left(d_{x}+i d_{y}\right)}{\sqrt{2}}, \tag{2.40}
\end{align*}
$$

which can be re-arranged to give

$$
\begin{align*}
& d_{x}=\frac{\left(d_{-1}^{(1)}-d_{1}^{(1)}\right)}{\sqrt{2}},  \tag{2.41}\\
& d_{y}=-i \frac{\left(d_{-1}^{(1)}-d_{1}^{(1)}\right)}{\sqrt{2}},  \tag{2.42}\\
& d_{z}=d_{0}^{(1)} . \tag{2.43}
\end{align*}
$$

As the polarisation axis $(z)$ of the light and the quantisation axis $(z)$ are the same in this example, then Eq. 2.43 needs to be used. The matrix elements of $d_{0}^{(1)}$ need to be calculated using the Wigner-Eckart Theorem (Auzinsh et al., 2014)

$$
\left\langle F_{1}, m_{1}\right| d_{q}^{(\kappa)}\left|F_{2}, m_{2}\right\rangle=(-1)^{F_{1}-m_{1}}\left(\begin{array}{ccc}
F_{1} & \kappa & F_{2}  \tag{2.44}\\
-m_{1} & q & m_{2}
\end{array}\right)\left\langle F_{1}\right|\left|d^{(\kappa)}\right|\left|F_{2}\right\rangle
$$

where $\left\langle F_{1}\left\|d^{(\kappa)}\right\| F_{2}\right\rangle$ is the reduced matrix element for that transition, and

$$
\left(\begin{array}{ccc}
F_{1} & \kappa & F_{2}  \tag{2.45}\\
-m_{1} & q & m_{2}
\end{array}\right)
$$

is the Wigner-3j symbol, which is related to the Clebsch-Gordon coefficients (Steck, 2022) $\left\langle F_{1} m_{1} \kappa q \mid F_{2}\left(-m_{2}\right)\right\rangle$ by

$$
\left(\begin{array}{ccc}
F_{1} & \kappa & F_{2}  \tag{2.46}\\
m_{1} & q & m_{2}
\end{array}\right)=\frac{(-1)^{F_{1}-\kappa-m_{2}}}{\sqrt{2 F_{2}+1}}\left\langle F_{1} m_{1} \kappa q \mid F_{2}\left(-m_{2}\right)\right\rangle
$$

For the $F=F_{1}=1 \rightarrow F^{\prime}=F_{2}=0$ transition with $z$-linearly polarised light along the quantisation axis, $d_{0}$ is given by

$$
\begin{align*}
d_{0} & =\left(\begin{array}{cccc}
\langle 1,1| d_{0}|1,1\rangle & \langle 1,1| d_{0}|1,0\rangle & \langle 1,1| d_{0}|1,-1\rangle & \langle 1,1| d_{0}\left|0^{\prime}, 0^{\prime}\right\rangle \\
\langle 1,0| d_{0}|1,1\rangle & \langle 1,0| d_{0}|1,0\rangle & \langle 1,0| d_{0}|1,-1\rangle & \langle 1,0| d_{0}\left|0^{\prime}, 0^{\prime}\right\rangle \\
\langle 1,-1| d_{0}|1,1\rangle & \langle 1,-1| d_{0}|1,0\rangle & \langle 1,-1| d_{0}|1,-1\rangle & \langle 1,-1| d_{0}\left|0^{\prime}, 0^{\prime}\right\rangle \\
\left\langle 0^{\prime}, 0^{\prime}\right| d_{0}|1,1\rangle & \left\langle 0^{\prime}, 0^{\prime}\right| d_{0}|1,0\rangle & \left\langle 0^{\prime}, 0^{\prime}\right| d_{0}|1,-1\rangle & \left\langle 0^{\prime}, 0^{\prime}\right| d_{0}\left|0^{\prime}, 0^{\prime}\right\rangle
\end{array}\right)  \tag{2.47}\\
& =\left(\begin{array}{llll}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{array}\right) \frac{\langle 1||d|\left|0^{\prime}\right\rangle}{\sqrt{3}} .
\end{align*}
$$

The light-atom interaction Hamiltonian is therefore

$$
H_{l}=-E_{0} \cos (\omega t) d_{z}=-\frac{E_{0} \cos (\omega t)}{\sqrt{3}}\left(\begin{array}{cccc}
0 & 0 & 0 & 0  \tag{2.48}\\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{array}\right)\langle 1||d|\left|0^{\prime}\right\rangle
$$

Substituting in the Rabi frequency $\Omega_{R}=\langle 1||d||0\rangle E_{0} /(\sqrt{3} \hbar)$ leads to

$$
H_{l}=\hbar \Omega_{R} \cos \omega t\left(\begin{array}{cccc}
0 & 0 & 0 & 0  \tag{2.49}\\
0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0
\end{array}\right)
$$

This example shows how linearly polarised light can cause an atom to change its total angular momentum $F^{\prime}-F=\Delta F=1$ with no change in the projection of angular momentum along the $z$-axis, i.e., $\Delta m=m^{\prime}-m=0$ by the interaction of $z$ polarised light with an atom in this $F=1 \rightarrow F^{\prime}=0$ atomic system. More generally, from the Wigner-3j symbol in Eq. 2.45 it can be calculated that for $F \rightarrow F^{\prime}=F \pm 1$ transitions, $\Delta m=0, \pm 1$ is allowed. For $F \rightarrow F^{\prime}=F$ transitions, $\Delta m= \pm 1$ is permitted, but importantly $\Delta m=0$ is forbidden. These are called selection rules and govern the possible interactions of light with atoms.

### 2.6 Optical pumping

In an "unpolarised" Cs atomic ensemble, the magnetic sublevels are equally populated in the Cs ground states $F=3$ and $F=4$. However, with OPMs it is important to create unequal populations of the magnetic sublevels, creating a "polarised" sample. This is done by optical pumping using laser light. We will consider three optical pumping examples relevant to the experiments in this thesis: (1) optical pumping into an oriented state in a paraffin-coated cell; (2) optical pumping into an aligned state in a paraffin-coated cell; (3) optical pumping into an aligned state in a vapour cell containing a quenching gas and a buffer gas.

### 2.6.1 Optical pumping into an oriented state in a paraffincoated cell

It will now be considered how laser light can be used to prepare an oriented state in a paraffin-coated cell, without any buffer gas or quenching gas present. Circularly polarised $\sigma^{+}$light with a frequency $\omega$ is resonant with the transition between the $F=3$ ground state and the $F^{\prime}=4$ excited state, i.e., $\omega \approx \omega_{0}$. As the excited states on the D1 line are separated by 1.167 GHz , it will be assumed that no atoms will be excited to the $F^{\prime}=3$ state.

Consider an atom in the $F=3, m=-3$ magnetic sublevel. An atom in this state can absorb a photon of light if the energy of the photon is equal to the energy difference between $F=3$ and $F^{\prime}=4$, i.e., $\omega \approx \omega_{0}$, as depicted in Fig. 2.12. The $\sigma^{+}$


Figure 2.12: Schematic of $\sigma^{+}$-circularly polarised light driving transitions between $F=3$ and $F^{\prime}=4$ in a paraffin-coated cell. The potential ways that an atom could decay from $F^{\prime}=4, m^{\prime}=-2$ to the ground state sublevels are included.
photon will transfer its angular momentum to the atom and increase the angular momentum projection from $m=-3$ to $m^{\prime}=-2$. Once in the excited state $F^{\prime}=$ $4, m^{\prime}=-2$, the atom will only remain here for a very short time ( $\tau_{\text {nat }} \approx 30.5 \mathrm{~ns}$ ). After this, the atom will decay to the ground states via six possible ways to any of $F=4, m=-3,-2,-1$ or to any of $F=3, m=-3,-2,-1$, where the probability of each transition is governed by the square of the Clebsch-Gordon coefficients, which we will label as $c_{F, m \leftrightarrow F^{\prime}, m^{\prime}}$ (Steck, 2022). If the atom drops into one of the $F=4, m=-3,-2,-1$ states, then the laser cannot be used to excite the atom
to the excited states again, because the energy of the light is not the same as the energy difference of the required transition. This means that the atom is "stuck" here. These states are called "dark states", because atoms in these states cannot absorb the laser light. If the atom decays into one of $F=3, m=-3,-2,-1$, then the atom can re-absorb another photon and be pumped to the $F^{\prime}=4$ excited state again. Eventually, a point will be reached where the magnetic sublevel with the most atoms in it will be the $F=4, m=4$ state (see Fig. 2.12). It can now be said that, instead of each magnetic sublevel having $1 / 16$ of the population of Cs atoms in the absence of optical pumping, there are now unequal populations in the magnetic sublevels. As described in Sec. 2.2, collisions with the glass walls of the vapour cell and Cs-Cs spin-exchange and destructive collisions cause the atomic spins to relax, i.e., the Cs atoms redistribute themselves among the magnetic sublevels due to the longitudinal relaxation rate $\Gamma_{1}=1 / T_{1}$.

For the equations that are to follow, a paraffin-coated cell without quenching gas will be modelled, where the dominant de-excitation mechanism from the excited state is spontaneous emission. This build-up of a polarised atomic sample can be modelled with rate equations to understand how optical pumping affects the populations of the atoms. The population of each magnetic sublevel will be notated as $p_{\mathrm{F}, \mathrm{m}}$. An example of the rate of change of the population of the magnetic sublevel $F=3, m=-3 d p_{3,-3} / d t$ will be constructed as the following:

$$
\begin{equation*}
\frac{d p_{3,-3}}{d t}=R_{p}\left(-p_{3,-3} c_{3,-3 \leftrightarrow 4^{\prime},-2^{\prime}}+p_{3,-3} c_{3,-3 \leftrightarrow 4^{\prime},-2^{\prime}} c_{3,-3 \leftrightarrow 4^{\prime},-2^{\prime}}\right)-\Gamma_{1} p_{3,-3}+\frac{\Gamma_{1}}{16}, \tag{2.50}
\end{equation*}
$$

where $R_{p}$ is the optical pumping rate, $p_{3,-3}=p_{3,-3}(t)$ is the population of the magnetic sublevel at time $t, c_{3,-3 \leftrightarrow 4^{\prime},-2^{\prime}}$ is the Clebsch-Gordon coefficient squared for the $\sigma^{+}$transition from $F=3, m=-3 \leftrightarrow F^{\prime}=4, m^{\prime}=-2$ and $\Gamma_{1}=1 / T_{1}$ is the longitudinal relaxation rate of the atom. The negative terms depopulate the magnetic sublevel and the positive terms repopulate the sublevel. All 16 terms are provided in Appendix C.

Now that Eqs. C.1 C. 16 have been determined, the set of differential equations can be solved. The 16 populations $p_{F, m}$ are plotted as a function of time $t / R_{p}$, and the populations in the steady state are plotted for three longitudinal relaxation rates $\Gamma_{1} / R_{p}=0$ (Figs. 2.13a and 2.13b), $\Gamma_{1} / R_{p}=1 / 20$ (Figs. 2.13c and 2.13d), $\Gamma_{1} / R_{p}=1 / 2$ (Figs. 2.13e and 2.13f). The magnetic sublevel with the most atoms pumped into it is the $F=4, m=4$ state in each scenario.

The bigger $\Gamma_{1} / R_{p}$, the less polarised the atomic sample in the steady state will be. The orientations of the $F=4$ ground state for $\Gamma_{1}=0, \Gamma_{1}=R_{p} / 20, \Gamma_{1}=R_{0} / 2$ are $p_{\text {orient }}=0.185,0.162,0.065$, respectively (see Eq. 2.5). To pump nearer to $100 \%$ of the atoms into the $F=4, m=4$ state and achieve an orientation $p_{\text {orient }}=1$, a second D2 $\sigma^{+}$beam locked to the $F=4 \rightarrow F^{\prime}=5$ transition (Jensen et al., 2019) could be used.


Figure 2.13: Optical pumping using circularly polarised light. (a), (c), (e) Populations of the $F=3$ and $F=4$ magnetic sublevels of Cs as a function of $t / R_{p}$. (b), (d), (f) The populations in the steady state are plotted. The longitudinal spin relaxation rates are $\Gamma_{1}=1 / T_{1}=0, R_{p} / 20, R_{p} / 2$ for (a, b), (c, d) and (e, f), respectively.

It is also informative to describe the populations in terms of angular momentum probability surfaces. Doing this can help in the visualisation of the experiments, especially when it comes to applying static and oscillating magnetic fields. The density matrix $\rho$, normalised to 1 , when $\Gamma_{1} / R_{p}=0.05$ in the steady state (see

Fig. 2.13 d$)$ of the $F=4$ ground state is given by

$$
\rho=\left(\begin{array}{ccccccccc}
0.159 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0  \tag{2.51}\\
0 & 0.145 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.132 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.120 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.107 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0.096 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.0086 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.079 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.075
\end{array}\right) .
$$

The probability of measuring $m=4,3, \ldots,-3,-4$ are the diagonal elements. The top-left element is $\rho_{4,4}$, i.e., the population of $m=4$, and the bottom-right element is $\rho_{-4,-4}$, the population of $m=-4$. There is therefore a probability of 0.159 of measuring the projection $m=4$, whereas there is a smaller probability of 0.075 of measuring $m=-4$ when the $+z$-axis is taken to be the quantisation axis.

Given that the state with the highest population is $F=4, m=4$ when the projection of angular momentum along the $z$-axis is measured, the probability of measuring $m=4$ along some other quantisation axis can be considered, for example along the $x$-axis. The probability $\rho_{4,4}(\theta, \varphi)$ of measuring $F=4, m=4$ along any quantisation axis will therefore be found. This means that an expression containing the angles $\theta$, the polar angle, and $\varphi$, the azimuthal angle, is required. The probability $\rho_{m=F, m=F}(\theta, \varphi)$ of measuring the projection $m=F$ along any axis is given by (Auzinsh et al., 2014)

$$
\begin{equation*}
\rho_{m=F, m=F}(\theta, \varphi)=\sum_{m, m^{\prime}} D_{m, m^{\prime}=F}^{(F) *}(\varphi, \theta, 0) \rho_{m, m^{\prime}} D_{m^{\prime}, m=F}^{(F)}(\varphi, \theta, 0), \tag{2.52}
\end{equation*}
$$

where

$$
\begin{equation*}
D_{m^{\prime}, m=F}^{(F)}(\varphi, \theta, 0)=e^{-i \varphi m^{\prime}} d_{m^{\prime}, m=F}^{(F)}(\theta) \tag{2.53}
\end{equation*}
$$

and

$$
\begin{align*}
d_{m^{\prime}, m=F}^{(F)}(\theta)= & \sum_{k}(-1)^{\left(k-m+m^{\prime}\right)} \frac{\sqrt{(F+m)!(F-m)!\left(F+m^{\prime}\right)!\left(F-m^{\prime}\right)!}}{(F+m-k)!k!\left(F-k-m^{\prime}\right)!\left(k-m+m^{\prime}\right)!}  \tag{2.54}\\
& \times(\cos (\theta / 2))^{2 F-2 k+m-m^{\prime}}(\sin (\theta / 2))^{2 k-m+m^{\prime}}
\end{align*}
$$

and the possible $k$-values in the summation are those when none of the factorials are negative. The term $\rho_{4,4}(\theta, \varphi)$ can be plotted using the AtomicDensityMatrix software (Rochester, 2022), and is shown in Fig. 2.14a. For comparison, an example is shown when $100 \%$ of the atoms are pumped into $m=4\left(\rho_{4,4}=1, \rho_{3,3}, \rho_{2,2}, \ldots, \rho_{-4,-4}=0\right)$. The corresponding angular momentum probability surface is plotted in Fig. 2.14b, From the "perfect" optical pumping example where $100 \%$ of the atoms are pumped into $F=4, m=4$, it can be seen that there is zero probability of measuring $m=4$ along the $\pm y-, \pm x$ - or $-z$-axes. The angular momentum probability surface has a favoured direction, which in this case is along the $+z$-direction, as $\sigma^{+}$-circularly polarised light propagates along the $+z$-direction, transferring angular momentum to the atoms. These angular momentum probability surfaces can help with the intuition of why an oriented state is often treated as a vector, as depicted in Figs. 2.3b and 2.4.


Figure 2.14: Angular momentum probability surface of (a) the density matrix in Eq. 2.51 and (b) the density matrix when all the atoms are pumped into the $m=4$ sublevel in the $F=4$ ground state.

If $F \rightarrow \infty$, then the angular momentum probability surface looks like the vectors in Figs. 2.3b and 2.4. We will now consider how optical pumping can be used to create a so-called "aligned state" in a paraffin-coated cell.

### 2.6.2 Example of optical pumping into an aligned state in a paraffin-coated cell

Optical pumping in a paraffin-coated cell using linearly polarised light will now be considered, where the dominant de-excitation mechanism from the Cs excited state is spontaneous emission. The light is linearly polarised along the $z$-axis and the quantisation axis is along the $z$-axis. In this example a D1 laser tuned to the $F=4 \rightarrow F^{\prime}=3$ transition is used (see Fig. 2.15). The atoms in the $F=4$ ground state can be driven by $\pi$-transitions, causing no change in the projection of angular momentum, i.e., $\Delta m=0$. The Clebsch-Gordon coefficients once again govern the spontaneous decays from the excited states down to the ground states. Rate equations are generated, as done in Sec. 2.6.1, to describe how the populations


Figure 2.15: Schematic of optical pumping into an aligned state using a linearly polarised laser locked to the $F=4 \rightarrow F^{\prime}=3$ transition. The possible decays of an atom in the $F^{\prime}=3, m=-2$ excited state to the ground states are indicated.


Figure 2.16: Optical pumping into an aligned state with a paraffin-coated cell. (a) Populations of the $F=3$ and $F=4$ magnetic sublevels of Cs as a function of time with the longitudinal relaxation rate $\Gamma_{1}=0$. (b) The populations in the steady state are plotted for $\Gamma_{1}=0$. (c) Populations of the $F=3$ and $F=4$ magnetic sublevels of Cs as a function of time, assuming that there are spin relaxation processes which redistribute the atoms among the magnetic sublevels, where $\Gamma_{1}=R_{p} / 20$. (d) The populations in the steady state for $\Gamma_{1}=R_{p} / 20$ are plotted.
of the magnetic sublevels evolve as a function of time. An example of the rate of change of the population of the magnetic sublevel $F=4, m=3, d p_{\mathrm{F}=4, \mathrm{~m}=3} / d t$, i.e., the diagonal element of the density matrix, is

$$
\begin{align*}
\frac{d p_{4,3}}{d t}= & R_{p}\left(-p_{4,3} c_{4,3 \leftrightarrow 3^{\prime}, 3^{\prime}}+p_{4,2} c_{4,2 \leftrightarrow 3^{\prime}, 2^{\prime}} c_{4,3 \leftrightarrow 3^{\prime}, 2^{\prime}}\right.  \tag{2.55}\\
& \left.+p_{4,3} c_{4,3 \leftrightarrow 3^{\prime}, 3^{\prime}} c_{4,3 \leftrightarrow 3^{\prime}, 3^{\prime}}\right)-\Gamma_{1} p_{4,3}+\Gamma_{1} / 16 .
\end{align*}
$$

All 16 differential equations are included in Appendix D. In a similar fashion to Fig. 2.13, the populations of each ground state magnetic sublevels are plotted in Fig. 2.16 for $\Gamma_{1}=0, R_{p} / 20$. The density matrix of the $F=4$ state (ignoring the


Figure 2.17: Aligned atomic polarisation using (a) the "real" values from Eq. 2.56 for the $F=4$ ground state with a non-zero longitudinal spin relaxation $\left(\Gamma_{1}=R_{p} / 20\right)$ and (b) "unrealistic" values from Fig. 2.16b in the steady state with no longitudinal spin relaxation $\left(\Gamma_{1}=0\right)$.
atoms lost to $F=3$ ) in the steady state from Fig. 2.16d is

$$
\rho=\left(\begin{array}{ccccccccc}
0.299 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0  \tag{2.56}\\
0 & 0.080 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.054 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.045 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.043 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0.045 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.054 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.080 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.299
\end{array}\right) .
$$

We note that $\sum_{m=-4}^{4} \rho_{m, m}=1$. The angular momentum probability surface for the density matrix in Eq. 2.56 is plotted in Fig. 2.17a. An example is also shown when $50 \%$ of the atoms are pumped into $m=-4$ and $50 \%$ in $m=4$ in Fig. 2.17b. The atoms in the $F=4$ sublevels are symmetrically distributed with most atoms in the $m= \pm 4$ sublevels, corresponding to a spin-aligned state.

### 2.6.3 Example of optical pumping into an aligned state in the presence of a quenching and buffer gas

If a quenching gas, such as 65 Torr of $\mathrm{N}_{2}$, is present in a Cs vapour cell without any paraffin coating, as presented in Sec. 7 , then the Cs atoms will mostly decay via quenching rather than via spontaneous emission (Seltzer and Romalis, 2009, Seltzer, 2008), as will now be shown. The many vibrational and rotational states of the quenching gas molecule, in this case $\mathrm{N}_{2}$, mean that when a Cs atom in the excited state collides with a $\mathrm{N}_{2}$ molecule, the Cs atom can de-excite without the emission of a photon, instead transferring its energy to the many vibrational and rotational modes of the $\mathrm{N}_{2}$ molecule. The quenching rate is given by

$$
\begin{equation*}
R_{Q}=n_{Q} \sigma_{Q} v_{\mathrm{Cs}, \mathrm{~N}_{2}}, \tag{2.57}
\end{equation*}
$$

where $n_{Q}=P /\left(k_{B} T\right)=1.91 \times 10^{24} \mathrm{~m}^{-3}$ is the number density of $\mathrm{N}_{2}$ molecules at $T \sim 55^{\circ} \mathrm{C}, P$ is the pressure, $k_{B}$ is the Boltzmann constant, $\sigma_{Q}=5.5 \times 10^{-19} \mathrm{~m}^{2}$
(Seltzer, 2008) is the quenching gas cross-section for Cs and $\mathrm{N}_{2}$ at $100^{\circ} \mathrm{C}$ and $v_{\mathrm{Cs}, \mathrm{N}_{2}}=$ $\sqrt{8 k_{B} T / \pi M}=548 \mathrm{~m} / \mathrm{s}$ is the relative velocity between a Cs atom and $\mathrm{N}_{2}$ molecule. The mass $M=3.84 \times 10^{-26} \mathrm{~kg}$ is the effective mass of a Cs atom and $\mathrm{N}_{2}$ molecule, given by

$$
\begin{equation*}
M=\frac{m_{\mathrm{Cs}} m_{\mathrm{N}_{2}}}{\left(m_{\mathrm{Cs}}+m_{\mathrm{N}_{2}}\right)} . \tag{2.58}
\end{equation*}
$$

The quenching factor $Q$ helps determine the dominant decay mechanism, whether by spontaneous emission $(Q=1)$ or by quenching $(Q=0)$, and is given by (Seltzer, 2008)

$$
\begin{equation*}
Q=\frac{1}{1+R_{Q} \tau_{\text {nat }}} . \tag{2.59}
\end{equation*}
$$

Calculating $R_{Q}=5.9 \times 10^{8} \mathrm{~s}^{-1}$ from the parameters stated above for Cs and 65 Torr $\mathrm{N}_{2}$ and taking the natural lifetime of the D1 excited state to be $\tau_{\text {nat }} \approx 35 \mathrm{~ns}$, then $Q=0.05$. This means that for the 65 Torr $\mathrm{N}_{2}$ buffer gas cell used in this thesis, the dominant de-excitation mechanism from the excited state is quenching. During quenching, the decay probabilities to the ground states are not governed by the Clebsch-Gordon coefficients. Instead the atoms decay with equal probability $1 / 16$ to any of the ground state magnetic sublevels in the $F=3$ and $F=4$ ground states.

The mechanisms occurring in the presence of a buffer and quenching gas will be illustrated, i.e., $\mathrm{N}_{2}$ and a "toy" $F=2 \rightarrow F^{\prime}=1$ transition in Fig. 2.18. The linearly


Figure 2.18: Schematic of optical pumping into an aligned state with a quenching and buffer gas present in a vapour cell with an $F=2$ ground state and $F^{\prime}=1$ excited state. Linearly polarised light drives atoms from the $F=2, m=0, \pm 1$ sublevels into the excited state. In the excited state, rapid collisional mixing between the $\mathrm{N}_{2}$ buffer gas molecules and the Cs atoms cause the redistribution of atoms among the excited state magnetic sublevels. Assuming that the quenching factor $Q=0$, such that no spontaneous emission occurs, the atoms will decay to the ground states equally, i.e., $1 / 5$ to each magnetic sublevel. Spin relaxation mechanisms in the ground state, such as electron randomisation collisions with the walls, lead to spin relaxation in the ground state, tending to redistribute the atoms from $m= \pm 2$ to the other sublevels.
polarised light pumps the atoms to the excited states, where rapid collisional mixing causes the magnetic sublevels of the Cs atoms to be randomised. The atoms then
decay via quenching to the five ground states with equal probability. The majority of atoms accumulate in the $m= \pm 2$ sublevels. Spin relaxation causes some of the atoms in the $m \pm 2$ to be redistributed to the other magnetic sublevels.

An optical pumping example using linearly polarised light resonant on the $F=$ $4 \rightarrow F^{\prime}=3$ transition will be considered, with two longitudinal relaxation rates $\Gamma_{1}=0$ and $\Gamma_{1}=R_{p} / 20$. An example of a rate equation for the population $d p_{4,3} / d t$ of the $F=4, m=3$ magnetic sublevel is given by

$$
\begin{align*}
\frac{d p_{4,3}}{d t}= & R_{p}\left(-p_{4,3} c_{4,3 \leftrightarrow 3^{\prime}, 3^{\prime}}+\frac{1}{16}\left[p_{4,3} c_{4,3 \leftrightarrow 3^{\prime}, 3^{\prime}}+p_{4,2} c_{4,2 \leftrightarrow 3^{\prime}, 2^{\prime}}+p_{4,1} c_{4,1 \leftrightarrow 3^{\prime}, 1^{\prime}}\right.\right. \\
& \left.\left.+p_{4,0} c_{4,0 \leftrightarrow 3^{\prime}, 0^{\prime}}+p_{4,-1} c_{4,-1 \leftrightarrow 3^{\prime},-1^{\prime}}+p_{4,-2} c_{4,-2 \leftrightarrow 3^{\prime},-2^{\prime}}+p_{4,-3} c_{\left.4,-3 \leftrightarrow 3^{\prime},-3^{\prime}\right]}\right]\right) \\
& -\Gamma_{1} p_{4,3}+\frac{\Gamma_{1}}{16} . \tag{2.60}
\end{align*}
$$

The 16 rate equations are solved in the steady state. The figures in Fig. 2.19 when quenching is the dominant de-excitation mechanism $(Q=0)$ look very similar to the figures in Fig. 2.16, where the dominant de-excitation mechanism is spontaneous emission ( $Q=1$ ). We will compare Figs. 2.19b and 2.16b with $\Gamma_{1}=0$. When $Q=1$, $15 \%$ of the atoms are pumped into the $m= \pm 4$ states, whereas when $Q=0,11 \%$ are pumped into $m= \pm 4$. This shows that the optical pumping in a paraffin-coated cell produces slightly better alignment than with a buffer gas cell where quenching is the main decay mechanism. The longitudinal relaxation rate $\Gamma_{1}$ affects the buffer gas cell in the same way as the paraffin-coated cell in the ground state.

### 2.7 Polarised atoms in a static magnetic field

Given that a detailed description of optical pumping has now been described, the behaviour of a polarised sample of atoms in magnetic fields can now be understood (Rochester and Budker, 2001). For an oriented state, as depicted in Fig. 2.14, it has a favoured direction, for example along the $+z$-direction when the circularly polarised light is propagating along the $+z$-axis. The oriented state can thus be modelled as an average angular momentum vector, as done in Sec. 2.2 and in Sec. 3 where the spin of the atomic ensemble is modelled as a vector $\mathbf{J}$. For an aligned state, as depicted in Fig. 2.17, it has a favoured axis, for example along the $z$-axis with $z$-linearly polarised light. The aligned state can thus be modelled as an alignment axis (Rochester and Budker, 2001). When the polarised atomic ensemble (either oriented or aligned) is placed in a static magnetic field, the atomic polarisation (vector for oriented and axis for alignment) will precess around the static magnetic field direction with a precession frequency equal to the Larmor frequency. The only time when the atomic polarisation will not rotate is when it is aligned with the applied static magnetic field $\mathbf{B}_{0}$. The precession of the atomic polarisation results from the different Zeeman sublevels with different populations having different phases, which in turn cause a precession of the atomic polarisation at the Larmor frequency $\omega_{L}=\gamma_{\text {Cs }} B_{0}$, where $\gamma_{\mathrm{Cs}}=2 \pi(3.5 \mathrm{kHz} / \mu \mathrm{T})$. This phenomenon of Larmor precession will now be described using a simple example.

To calculate how the atomic polarisation will change as a function of time when just a static magnetic field is applied perpendicular to the atomic polarisation, the


Figure 2.19: Optical pumping into an aligned state in the presence of a quenching gas. (a) Populations of the $F=3$ and $F=4$ magnetic sublevels of Cs as a function of time with the longitudinal relaxation rate $\Gamma_{1}=0$. (b) The populations in the steady state are plotted for $\Gamma_{1}=0$. (c) Populations of the $F=3$ and $F=4$ magnetic sublevels of Cs as a function of time, assuming that there are spin relaxation processes which redistribute the atoms among the magnetic sublevels, where $\Gamma_{1}=R_{p} / 20$. (d) The populations in the steady state for $\Gamma_{1}=R_{p} / 20$ are plotted.

Heisenberg equation of motion

$$
\begin{equation*}
\frac{d \rho}{d t}=\frac{1}{i \hbar}\left[\rho, H_{B}\right] \tag{2.61}
\end{equation*}
$$

must be solved. The Hamiltonian $H_{B}$ must also be known to calculate the evolution of the density matrix elements as a function of time. For an $F=1$ ground state as in Sec. 2.4 this is given by

$$
H_{B}=\hbar g_{F} \mu_{0} B_{0}\left(\begin{array}{ccc}
1 & 0 & 0  \tag{2.62}\\
0 & 0 & 0 \\
0 & 0 & -1
\end{array}\right)=\hbar \omega_{L}\left(\begin{array}{ccc}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & -1
\end{array}\right)
$$

where the static magnetic field is aligned along the $z$-axis, and $z$ is treated as the quantisation axis. Substituting the values of $\rho$ and $H_{B}$ into Eq. 2.61,

$$
\frac{d \rho}{d t}=\frac{\omega_{L}}{i}\left(\begin{array}{ccc}
0 & -\rho_{1,0} & -2 \rho_{1,-1}  \tag{2.63}\\
\rho_{0,1} & 0 & -\rho_{0,-1} \\
2 \rho_{-1,1} & \rho_{-1,0} & 0
\end{array}\right)
$$

is obtained and thus the 9 differential equations can be solved. The evolution of the density matrix elements as a function of time are given by

$$
\begin{align*}
\rho_{1,1}(t) & =\rho_{1,1}(0)  \tag{2.64}\\
\rho_{1,0}(t) & =\rho_{1,0}(0) e^{-\frac{\omega_{L}}{i} t}  \tag{2.65}\\
\rho_{1,-1}(t) & =\rho_{1,-1}(0) e^{-2 \frac{\omega_{L}}{i} t},  \tag{2.66}\\
\rho_{0,1}(t) & =\rho_{0,1}(0) e^{\frac{\omega_{L}}{i} t}  \tag{2.67}\\
\rho_{0,0}(t) & =\rho_{0,0}(0)  \tag{2.68}\\
\rho_{0,-1}(t) & =\rho_{0,-1}(0) e^{-\frac{\omega_{L}}{i} t}  \tag{2.69}\\
\rho_{-1,1}(t) & =\rho_{-1,1}(0) e^{2 \frac{\omega_{L}}{i} t}  \tag{2.70}\\
\rho_{-1,0}(t) & =\rho_{-1,0}(0) e^{\frac{\omega_{L}}{i} t}  \tag{2.71}\\
\rho_{-1,-1}(t) & =\rho_{-1,-1}(0) \tag{2.72}
\end{align*}
$$

Now the initial conditions must be dealt with. If the circularly polarised pump beam is propagating along the quantisation axis then the atoms could be pumped into the $F=1, m=1$ state. If at $t=0$ the atomic sample is fully polarised, then the ground state density matrix $\rho$ of an $F=1$ state is given by

$$
\rho(t=0)=\left(\begin{array}{ccc}
\rho_{1,1}(0) & \rho_{1,0}(0) & \rho_{1,-1}(0)  \tag{2.73}\\
\rho_{0,1}(0) & \rho_{0,0}(0) & \rho_{0,-1}(0) \\
\rho_{-1,1}(0) & \rho_{-1,0}(0) & \rho_{-1,-1}(0)
\end{array}\right)=\left(\begin{array}{lll}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}\right) .
$$

Substituting these initial values will lead to no change in any density matrix element. If another example is considered where the circularly polarised pump beam propagates along the $y$-direction, then the density matrix with a $z$-quantisation axis will be

$$
\rho(t=0)=\left(\begin{array}{ccc}
\frac{1}{4} & -\frac{i}{2 \sqrt{2}} & -\frac{1}{4}  \tag{2.74}\\
\frac{i}{2 \sqrt{2}} & \frac{1}{2} & -\frac{i}{2 \sqrt{2}} \\
-\frac{1}{4} & \frac{i}{2 \sqrt{2}} & \frac{1}{4}
\end{array}\right) .
$$

From the solved differential equations in Eqs. 2.64] 2.72 it is clear that the populations of the $F=1, m=1, F=1, m=0$ and $F=1, m=-1$ states will not change as a function of time. However, the phases of the off-diagonal elements, the coherences, will change as a function of time due to the exponents. The density matrix with these initial conditions is then given by

$$
\rho(t)=\left(\begin{array}{ccc}
\frac{1}{4} & -\frac{i}{2 \sqrt{2}} e^{-\frac{\omega_{L}}{i} t} & -\frac{1}{4} e^{-2 \frac{\omega_{L}}{i} t}  \tag{2.75}\\
\frac{i}{2 \sqrt{2}} e^{\frac{\omega_{L}}{i} t} & \frac{1}{2} & -\frac{i}{2 \sqrt{2}} e^{-\frac{\omega_{L}}{i} t} \\
-\frac{1}{4} e^{2 \frac{\omega_{L}}{i} t} & \frac{i}{2 \sqrt{2}} e^{\frac{\omega_{L}}{i} t} & \frac{1}{4}
\end{array}\right) .
$$

From these equations, the angular momentum probability surfaces can be plotted as a function of time. Beginning with the spin aligned along the $y$-axis at $t=0$,


Figure 2.20: Oriented atomic polarisation undergoing Larmor precession in a static magnetic field $\mathbf{B}_{0}=B_{0} \hat{\mathbf{z}}$. Snapshots at different times are taken: (a) $t=0=1 / \nu_{L}$, (b) $t=1 /\left(4 \nu_{L}\right)$, (c) $t=2 /\left(4 \nu_{L}\right)$, (d) $t=3 /\left(4 \nu_{L}\right)$.
the atomic polarisation will precess around the $z$-axis with a frequency equal to the Larmor frequency. The evolution of the atomic polarisation is depicted in Fig. 2.20. Similarly, if there is an aligned state with its axis along the $x$-direction, then a static field applied along the $z$-axis will cause the aligned atomic polarisation to precess around the $z$-axis. This is shown in Fig. 2.21 with snapshots as a function of time. The aligned state is symmetric every $t=1 /\left(2 \nu_{L}\right)$, unlike with the oriented state which is symmetric every $t=1 / \nu_{L}$.

(a)

(b)

(c)

(d)

Figure 2.21: Aligned atomic polarisation undergoing Larmor precession in a static magnetic field $\mathbf{B}_{0}=B_{0} \hat{\mathbf{z}}$. Snapshots at different times are taken: (a) $t=0=1 / \nu_{L}$, (b) $t=1 /\left(4 \nu_{L}\right)$, (c) $t=2 /\left(4 \nu_{L}\right)$, (d) $t=3 /\left(4 \nu_{L}\right)$.

The evolution of a density matrix in a static magnetic field has therefore been shown. This in turn illustrates how the atomic polarisation vector for an oriented state and the atomic polarisation axis for an aligned state evolve in magnetic fields. Based on the work presented in this chapter, we will now be able to understand the theory of two types of radio-frequency optically pumped magnetometers, namely an orientation-based RF OPM in Sec. 3 and an alignment-based RF OPM in Sec. 4 .

## Chapter 3

## Theory of an orientation-based optically pumped magnetometer

To detect oscillating magnetic fields using an orientation-based radio-frequency optically pumped magnetometer (RF OPM), the theory of how a spin-polarised sample behaves in static and oscillating magnetic fields must be understood. This theory is essential to understand how the portable orientation-based RF OPM presented in Sec. 6 operates.

### 3.1 Preparing the state using a pump beam with optical pumping

For this theory, it is assumed that all of the atoms are pumped into the $F=4, m=4$ sublevel (see Fig. 3.1b). Close to $100 \%$ spin polarisation in $F=4, m=4$ is only possible using two circularly polarised beams (Jensen et al., 2019), as described in Sec. 2.6.1. In the case of Cs, a configuration with two circularly polarised laser beams propagating along the $z$-direction, with a pump beam resonant with the D1 $F=4 \rightarrow F^{\prime}=4$ transition, and a repump beam resonant with the D2 $F=3 \rightarrow$ $F^{\prime}=4$ transition (Jensen et al., 2019), will pump almost all of the atoms into the $6^{2} S_{1 / 2}|F=4, m=4\rangle$ sublevel. When every atom is pumped into the same state, the total spin (or total angular momentum) $\mathbf{F}$ of all the $N_{A}$ atoms in the vapour cell is given by $\mathbf{F}=4 N_{A} \hbar \hat{\mathbf{z}}=J_{\max } \hbar \hat{\mathbf{z}}$. Due to the state being oriented and its angular momentum probability surface favouring an axis and direction, as shown in Fig. 2.14b, the atomic spins can be represented by a classical vector $\mathbf{J}$ (see Fig. 2.3b), which represents the mean angular momentum vector of the spins. The notation $\mathbf{J}$ instead of $\mathbf{F}$ is used to show that this is a semi-classical approach to understanding the spins of the Cs atoms. Both the unpolarised (no pump beam) and polarised (with pump beam) states are shown in Figs. 3.1a and 3.1b, respectively.

### 3.2 Placing the polarised sample in magnetic fields

In addition to having a pump beam oriented along the $z$-axis, a constant magnetic field $\mathbf{B}_{0}$ is required and is also oriented along the $z$-axis. When it comes to imagining
the atomic spins as a vector, it is important to realise that the static magnetic field does not affect the vector if it is aligned with the direction of propagation of the circularly polarised light. This is visualised in Fig. 3.1k.
(a)

(b)

(c)


Figure 3.1: "Part one" of a step-by-step guide to visualise how an orientation-based RF OPM operates. The experimental approach (top), the spins of the Cs atoms (middle) and the populations in the ground states (bottom) are included to show the different ways that one can visualise an orientation-based RF OPM. (a) An unpolarised sample. (b) A circularly polarised pump beam propagating along the $z$ direction polarises the atomic sample, pumping all of the atoms into the $F=4, m=$ sublevel. The spin $\mathbf{j}$ of each Cs atom therefore points along the same direction. (c) The polarised atomic sample is placed in a static magnetic field $\mathbf{B}_{0}$, leading to Zeeman splitting (see Sec. 2.4).

Now consider additionally applying an oscillating magnetic field $\mathbf{B}_{\mathrm{RF}}(t)=$ $\mathbf{B}_{1}(t)=\left(B_{1, \mathrm{c}} \cos \left(\omega_{\mathrm{RF}} t\right)+B_{1, \mathrm{~s}} \sin \left(\omega_{\mathrm{RF}} t\right)\right) \hat{\mathbf{x}}$, where $B_{1, \mathrm{c}}$ and $B_{1, \mathrm{~s}}$ are the amplitudes of the in-phase and out-of-phase components of the RF field. This oscillating magnetic field is along the $x$-axis, perpendicular to the static magnetic field $\mathbf{B}_{0}$. This will tilt the spins away from the $z$-axis, and the spins will precess around the $z$-axis at a frequency $\omega_{\text {RF }}$ (see Fig. 3.2 a). To understand this mathematically, a differential equation describing the time evolution of the sample's atomic spins $\mathbf{J}$ is constructed.

The semi-classical differential equation (Jensen et al., 2019) can be written as

$$
\begin{align*}
\frac{d \mathbf{J}}{d t}= & \gamma_{\mathrm{Cs}} \mathbf{J} \times\left(B_{0} \hat{\mathbf{z}}+\left(B_{1, \mathrm{c}} \cos \left(\omega_{\mathrm{RF}} t\right)+B_{1, \mathrm{~s}} \sin \left(\omega_{\mathrm{RF}} t\right)\right) \hat{\mathbf{x}}\right)+R_{p} J_{\max } \hat{\mathbf{Z}}  \tag{3.1}\\
& -\left(R_{p}+\Gamma_{\mathrm{pr}}+\Gamma_{\text {dark }}\right) \mathbf{J},
\end{align*}
$$

where $\mathbf{J}$ is the angular momentum vector describing the atomic spins, $\gamma_{\mathrm{Cs}}$ is the gyromagnetic ratio of Cs, $B_{0}$ is the static magnetic field, $B_{1, c} \cos \left(\omega_{\mathrm{RF}} t\right)$ and $B_{1, s} \sin \left(\omega_{\mathrm{RF}} t\right)$ are the in-phase and out-of-phase components of the RF field $\mathbf{B}_{1}(t), R_{p}$ is the optical pumping rate, $\Gamma_{\mathrm{pr}}$ is the relaxation rate due to the probe beam and $\Gamma_{\text {dark }}$ is the relaxation rate due to effects not due to the light, such as Cs-Cs spin-exchange collisions and electron randomisation collisions with the glass walls (Graf et al., 2005; Labyt et al., 2022).

The different terms in the equation will now be explained. If the spin is aligned along the $z$-axis, i.e., $\mathbf{J}=J \hat{\mathbf{z}}$, and a static magnetic field $B_{0}$ is applied along the


Figure 3.2: "Part two" of a step-by-step guide to visualise how an orientation-based RF OPM operates. (a) An oscillating magnetic field $\mathbf{B}_{\mathrm{RF}}(t)$ applied perpendicular to $\mathbf{B}_{0}$ causes the atomic spins $N_{A} \mathbf{j}$ to precess around the static magnetic field $\mathbf{B}_{0}$. The RF field causes coherences between the neighbouring magnetic sublevels. (b) The spin properties of the atoms are transferred to an experimental observable by probing the atoms with a probe beam. The linearly polarised probe beam is rotated due to the Faraday effect and its polarisation oscillates at the Larmor frequency. The polarisation rotation is extracted with the help of a half-wave plate $(\lambda / 2)$, a polarising beam splitter (PBS) and a balanced photodetector (BPD). The output of the BPD is a voltage oscillating at the Larmor frequency. (c) The BPD output is fed into a lock-in amplifier and the signal is demodulated, producing in-phase ( $X$ ) and out-of-phase $(Y)$ signals, alongside the magnitude $R=\sqrt{X^{2}+Y^{2}}$. When the RF frequency is swept over the Larmor frequency then absorption-Lorentzian and dispersive-Lorentzian plots are obtained.
$z$-axis, then $\mathbf{J}$ will remain unchanged as $\gamma_{\text {Cs }} J \hat{\mathbf{z}} \times B_{0} \hat{\mathbf{z}}=0$. If an oscillating magnetic field $B_{1, \mathrm{c}} \cos \left(\omega_{\mathrm{RF}} t\right) \hat{\mathbf{x}}$ is applied along the $x$-axis, then $\gamma_{\mathrm{Cs}} \mathbf{J} \times B_{1, \mathrm{c}} \cos \left(\omega_{\mathrm{RF}} t\right) \hat{\mathbf{x}}$ will cause $\mathbf{J}$ to move towards the $y$-direction at $t=0$. Optical pumping is taken into account by the term $J_{\max } \hat{\mathbf{z}}$, which keeps pumping the atoms towards the $+z$-direction. Without this term, the atomic sample would become unpolarised with a distribution of atoms between the magnetic sublevels and hence would lose the polarisation along the $+z$-axis. Finally, there are terms which cause the atomic spins to relax (Balabas et al., 2010a) as mentioned in Sec. 2.2, causing $\mathbf{J}$ to contract. We note that typically the relaxation time $T_{2}$ of the transverse components $J_{x}$ and $J_{y}$ is shorter than the longitudinal relaxation time of the atomic spin component $J_{z}$, which has a relaxation time $T_{1}$ (Julsgaard, 2003). However, we do not distinguish between these different relaxation times in Eq. 3.1. Finally, we note that in paraffin-coated cells all atoms in the cell are probed (as they can bounce off the walls thousands of times), whereas inside buffer gas cells only the atoms inside the beam are probed, as once they hit the glass wall the spin coherence is destroyed.

### 3.3 Solving the differential equations

To solve Eq. 3.1, the three components of the differential equation $d \mathbf{J} / d t$ can be written as

$$
\begin{align*}
\frac{d J_{x}}{d t} & =\gamma_{\mathrm{Cs}}\left(J_{y} B_{z}-J_{z} B_{y}\right)-\left(R_{p}+\Gamma_{\mathrm{pr}}+\Gamma_{\mathrm{dark}}\right) J_{x}  \tag{3.2}\\
\frac{d J_{y}}{d t} & =\gamma_{\mathrm{Cs}}\left(J_{z} B_{x}-J_{x} B_{z}\right)-\left(R_{p}+\Gamma_{\mathrm{pr}}+\Gamma_{\mathrm{dark}}\right) J_{y}  \tag{3.3}\\
\frac{d J_{z}}{d t} & =\gamma_{\mathrm{Cs}}\left(J_{x} B_{y}-J_{y} B_{x}\right)-\left(R_{p}+\Gamma_{\mathrm{pr}}+\Gamma_{\mathrm{dark}}\right) J_{z}+R J_{\mathrm{max}} \tag{3.4}
\end{align*}
$$

Implementing $B_{x}=B_{\mathrm{RF}}(t)=B_{\mathrm{RF}}, B_{y}=0, B_{z}=B_{0}$ and $\delta \omega=R_{p}+\Gamma_{\mathrm{pr}}+\Gamma_{\mathrm{dark}}$, the equations become

$$
\begin{align*}
\frac{d J_{x}}{d t} & =\gamma_{\mathrm{Cs}} J_{y} B_{0}-\delta \omega J_{x}  \tag{3.5}\\
\frac{d J_{y}}{d t} & =\gamma_{\mathrm{Cs}}\left(J_{z} B_{\mathrm{RF}}-J_{x} B_{0}\right)-\delta \omega J_{y}  \tag{3.6}\\
\frac{d J_{z}}{d t} & =-\gamma_{\mathrm{Cs}} J_{y} B_{\mathrm{RF}}-\delta \omega J_{z}+R_{p} J_{\mathrm{max}} \tag{3.7}
\end{align*}
$$

In the lab frame $J_{x}(t), J_{y}(t)$ and $J_{z}(t)$ will be time-dependent, as the atomic spins will be rotating around $\mathbf{B}_{0}$. It is much easier to go to a rotating frame where $d J_{x^{\prime}} / d t=d J_{y^{\prime}} / d t=d J_{z^{\prime}} / d t=0$. The chosen frame is one that rotates at the frequency of the primary magnetic field $\omega_{\mathrm{RF}}$ around the $z$-axis.

The conversion from the time-dependent quantities $J_{x}(t), J_{y}(t)$ and $J_{z}(t)$ to a frame where $J_{x^{\prime}}, J_{y^{\prime}}$ and $J_{z^{\prime}}$ are time-independent requires the rotation matrix

$$
\left[\begin{array}{c}
J_{x^{\prime}}  \tag{3.8}\\
J_{y^{\prime}} \\
J_{z^{\prime}}
\end{array}\right]=\left[\begin{array}{ccc}
\cos \omega_{\mathrm{RF}} t & -\sin \omega_{\mathrm{RF}} t & 0 \\
\sin \omega_{\mathrm{RF}} t & \cos \omega_{\mathrm{RF}} t & 0 \\
0 & 0 & 1
\end{array}\right]\left[\begin{array}{l}
J_{x} \\
J_{y} \\
J_{z}
\end{array}\right],
$$

such that

$$
\begin{align*}
& J_{x^{\prime}}=J_{x} \cos \omega_{\mathrm{RF}} t-J_{y} \sin \omega_{\mathrm{RF}} t,  \tag{3.9}\\
& J_{y^{\prime}}=J_{x} \sin \omega_{\mathrm{RF}} t+J_{y} \cos \omega_{\mathrm{RF}} t . \tag{3.10}
\end{align*}
$$

It is useful to write $J_{x}$ and $J_{y}$ in terms of $J_{x^{\prime}}, J_{y^{\prime}}$ and $J_{z^{\prime}}$ :

$$
\begin{align*}
J_{x^{\prime}} \cos \omega_{\mathrm{RF}} t & =J_{x} \cos ^{2} \omega_{\mathrm{RF}} t-J_{y} \sin \omega_{\mathrm{RF}} t \cos \omega_{\mathrm{RF}} t,  \tag{3.11}\\
J_{x^{\prime}} \sin \omega_{\mathrm{RF}} t & =J_{x} \cos \omega_{\mathrm{RF}} t \sin \omega_{\mathrm{RF}} t-J_{y} \sin ^{2} \omega_{\mathrm{RF}} t,  \tag{3.12}\\
J_{y^{\prime}} \cos \omega_{\mathrm{RF}} t & =J_{y} \cos ^{2} \omega_{\mathrm{RF}} t+J_{x} \sin \omega_{\mathrm{RF}} t \cos \omega_{\mathrm{RF}} t,  \tag{3.13}\\
J_{y^{\prime}} \sin \omega_{\mathrm{RF}} t & =J_{y} \cos \omega_{\mathrm{RF}} t \sin \omega_{\mathrm{RF}} t+J_{x} \sin ^{2} \omega_{\mathrm{RF}} t . \tag{3.14}
\end{align*}
$$

The components $J_{x}$ and $J_{y}$ written as a function of $J_{x^{\prime}}$ and $J_{y^{\prime}}$ are therefore

$$
\begin{gather*}
J_{x}=J_{x^{\prime}} \cos \omega_{\mathrm{RF}} t+J_{y^{\prime}} \sin \omega_{\mathrm{RF}} t,  \tag{3.15}\\
J_{y}=J_{y^{\prime}} \cos \omega_{\mathrm{RF}} t-J_{x^{\prime}} \sin \omega_{\mathrm{RF}} t . \tag{3.16}
\end{gather*}
$$

The derivatives $d \mathbf{J}^{\prime} / d t$ in the rotating frame must then be found, given by

$$
\begin{align*}
\frac{d J_{x^{\prime}}}{d t} & =\frac{d J_{x}}{d t} \cos \omega_{\mathrm{RF}} t-\frac{d J_{y}}{d t} \sin \omega_{\mathrm{RF}} t-\omega_{\mathrm{RF}} J_{y^{\prime}},  \tag{3.17}\\
\frac{d J_{y^{\prime}}}{d t} & =\frac{d J_{y}}{d t} \cos \omega_{\mathrm{RF}} t+\frac{d J_{x}}{d t} \sin \omega_{\mathrm{RF}} t+\omega_{\mathrm{RF}} J_{x^{\prime}},  \tag{3.18}\\
\frac{d J_{z^{\prime}}}{d t} & =\frac{d J_{z}}{d t} . \tag{3.19}
\end{align*}
$$

Expressions for $d J_{x} / d t, d J_{y} / d t, d J_{z} / d t$ (Eqs. 3.5.|3.7), $J_{x}$ (Eq. 3.15) and $J_{y}$ (Eq. 3.16) can now be substituted into Eqs. 3.17-3.19. The substitutions $\omega_{L}=\gamma_{\mathrm{Cs}} B_{0}$ and $\Delta_{\mathrm{RF}}=\omega_{\mathrm{RF}}-\omega_{L}$ are used to give

$$
\begin{align*}
\frac{d J_{x^{\prime}}}{d t} & =-\delta \omega J_{x^{\prime}}-\Delta_{\mathrm{RF}} J_{y^{\prime}}-\gamma_{\mathrm{Cs}} B_{\mathrm{RF}} J_{z^{\prime}} \sin \omega_{\mathrm{RF}} t  \tag{3.20}\\
\frac{d J_{y^{\prime}}}{d t} & =\Delta_{\mathrm{RF}} J_{x^{\prime}}-\delta_{\omega} J_{y^{\prime}}+\gamma_{\mathrm{Cs}} B_{\mathrm{RF}} J_{z^{\prime}} \cos \omega_{\mathrm{RF}} t  \tag{3.21}\\
\frac{d J_{z^{\prime}}}{d t} & =R_{p} J_{\mathrm{max}}+\gamma_{\mathrm{Cs}} B_{\mathrm{RF}} J_{x^{\prime}} \sin \omega_{\mathrm{RF}} t-\gamma_{\mathrm{Cs}} B_{\mathrm{RF}} J_{y^{\prime}} \cos \omega_{\mathrm{RF}} t-\delta \omega J_{z^{\prime}} \tag{3.22}
\end{align*}
$$

In the steady state in the rotating frame $d J_{x^{\prime}} / d t=d J_{y^{\prime}} / d t=d J_{z^{\prime}} / d t=0$, leading to

$$
\begin{align*}
& 0=-\delta \omega J_{x^{\prime}}-\Delta_{\mathrm{RF}} J_{y^{\prime}}-\gamma_{\mathrm{CS}} B_{\mathrm{RF}} J_{z^{\prime}} \sin \omega_{\mathrm{RF}} t,  \tag{3.23}\\
& 0=\Delta_{\mathrm{RF}} J_{x^{\prime}}-\delta \omega J_{y^{\prime}}+\gamma_{\mathrm{Cs}} B_{\mathrm{RF}} J_{z^{\prime}} \cos \omega_{\mathrm{RF}} t,  \tag{3.24}\\
& 0=R_{p} J_{\mathrm{max}}+\gamma_{\mathrm{Cs}} B_{\mathrm{RF}} J_{x^{\prime}} \sin \omega_{\mathrm{RF}} t-\gamma_{\mathrm{Cs}} B_{\mathrm{RF}} J_{y^{\prime}} \cos \omega_{\mathrm{RF}} t-\delta \omega J_{z^{\prime}} . \tag{3.25}
\end{align*}
$$

There are now three equations (Eqs. 3.233 .25 ) with three unknowns $J_{x^{\prime}}, J_{y^{\prime}}$ and $J_{z^{\prime}}$. A variable $J_{\mathrm{ss}}=R_{p} J_{\max } / \delta \omega$ is defined. As $J_{x^{\prime}}, J_{y^{\prime}}$ and $J_{z^{\prime}}$ have no time dependence, the average values of the following expressions must be obtained, including

$$
\begin{align*}
\left\langle\cos \omega_{\mathrm{RF}} t \sin \omega_{\mathrm{RF}} t\right\rangle & =\left\langle\frac{\sin 2 \omega_{\mathrm{RF}} t}{2}\right\rangle=\frac{1}{T} \int_{0}^{T} \frac{\sin 2 \omega_{\mathrm{RF}} t}{2} d t \sim 0,  \tag{3.26}\\
\left\langle\sin ^{2} \omega_{\mathrm{RF}} t\right\rangle & =\left\langle\frac{1-\cos 2 \omega_{\mathrm{RF}} t}{2}\right\rangle=\frac{1}{T} \int_{0}^{T} \frac{1-\cos 2 \omega_{\mathrm{RF}} t}{2} d t \sim \frac{1}{2},  \tag{3.27}\\
\left\langle\cos ^{2} \omega_{\mathrm{RF}} t\right\rangle & =\left\langle\frac{1+\cos \omega_{\mathrm{RF}} t}{2}\right\rangle=\frac{1}{T} \int_{0}^{T} \frac{1+\cos \omega_{\mathrm{RF}} t}{2} d t \sim \frac{1}{2} . \tag{3.28}
\end{align*}
$$

This allows us to calculate

$$
\begin{align*}
\left\langle B_{\mathrm{RF}}^{2}\right\rangle & =\left\langle B_{1, \mathrm{c}}^{2} \cos ^{2} \omega_{\mathrm{RF}} t+2 B_{1, \mathrm{c}} B_{1, \mathrm{~s}} \cos \omega_{\mathrm{RF}} t \sin \omega_{\mathrm{RF}} t+B_{1, \mathrm{~s}}^{2} \sin ^{2} \omega_{\mathrm{RF}} t\right\rangle \\
& =\frac{B_{1, \mathrm{c}}^{2}+B_{1, \mathrm{~s}}^{2}}{2} . \tag{3.29}
\end{align*}
$$

This is true when integrating for an integer number of periods or for a long time $T \gg 2 \pi / \omega_{\text {RF }}$. Solving the three sets of simultaneous equations gives rise to $J_{x^{\prime}}, J_{y^{\prime}}$
and $J_{z^{\prime}}$ :

$$
\begin{align*}
& J_{x^{\prime}}=-J_{\mathrm{ss}} \frac{\gamma_{\mathrm{Cs}}\left(B_{1, \mathrm{c}} \Delta_{\mathrm{RF}}+B_{1, \mathrm{~s}} \delta \omega\right) / 2}{\delta \omega^{2}+\gamma_{\mathrm{Cs}}^{2}\left(B_{1, \mathrm{c}}^{2}+B_{1, \mathrm{~s}}^{2}\right) / 2+\Delta_{\mathrm{RF}}^{2}},  \tag{3.30}\\
& J_{y^{\prime}}=J_{\mathrm{ss}} \frac{\gamma_{\mathrm{Cs}}\left(\delta \omega B_{1, \mathrm{c}}-B_{1, \mathrm{~s}} \Delta_{\mathrm{RF}}\right) / 2}{\delta \omega^{2}+\gamma_{\mathrm{Cs}}^{2}\left(B_{1, \mathrm{c}}^{2}+B_{1, \mathrm{~s}}^{2}\right) / 2+\Delta_{\mathrm{RF}}^{2}},  \tag{3.31}\\
& J_{z^{\prime}}=J_{\mathrm{ss}} \frac{\delta \omega^{2}+\Delta_{\mathrm{RF}}^{2}}{\delta \omega^{2}+\gamma_{\mathrm{Cs}}^{2}\left(B_{1, \mathrm{c}}^{2}+B_{1, \mathrm{~s}}^{2}\right) / 2+\Delta_{\mathrm{RF}}^{2}} . \tag{3.32}
\end{align*}
$$

To understand these expressions, consider letting $B_{1, \mathrm{~s}}=0$. This would be the case if just an in-phase RF field is present. The components would thus simplify to

$$
\begin{align*}
& J_{x^{\prime}}=-\frac{J_{\mathrm{ss}} \gamma_{\mathrm{Cs}} B_{1, \mathrm{c}}}{2} \frac{\Delta_{\mathrm{RF}}}{\delta \omega^{2}+\gamma_{\mathrm{Cs}}^{2} B_{1, \mathrm{c}}^{2} / 2+\Delta_{\mathrm{RF}}^{2}},  \tag{3.33}\\
& J_{y^{\prime}}=\frac{J_{\mathrm{ss}} \gamma_{\mathrm{Cs}} B_{1, \mathrm{c}} \delta \omega}{2} \frac{1}{\delta \omega^{2}+\gamma_{\mathrm{Cs}}^{2} B_{1, \mathrm{c}}^{2} / 2+\Delta_{\mathrm{RF}}^{2}},  \tag{3.34}\\
& J_{z^{\prime}}=J_{\mathrm{ss}} \frac{\delta \omega^{2}+\Delta_{\mathrm{RF}}^{2}}{\delta \omega^{2}+\gamma_{\mathrm{Cs}}^{2} B_{1, \mathrm{c}}^{2} / 2+\Delta_{\mathrm{RF}}^{2}} . \tag{3.35}
\end{align*}
$$

These equations can be further understood by varying $\omega_{\mathrm{RF}}$, which is the frequency of the primary field $\mathbf{B}_{1}(t)$, and hence $\Delta_{\mathrm{RF}}=\omega_{R F}-\omega_{L}$. The lineshape of $J_{x^{\prime}}$ is dispersive-Lorentzian, as it has the form $-\Delta_{\mathrm{RF}} /\left(1+\Delta_{\mathrm{RF}}^{2}\right)$ and the lineshape of $J_{y^{\prime}}$ is absorption-Lorentzian, as it has the form $1 /\left(1+\Delta_{\mathrm{RF}}^{2}\right)$. These are plotted in Fig. 3.3. We note that in the experimental data obtained in Sec. 6, the in-phase output of the lock-in amplifier $X \propto J_{y^{\prime}}$ and the out-of-phase output $Y \propto J_{x^{\prime}}$. When


Figure 3.3: A dispersive-Lorentzian lineshape representing $J_{x^{\prime}}=\Delta_{\mathrm{RF}} /\left(1+\Delta_{\mathrm{RF}}^{2}\right)$ and an absorption-Lorentzian lineshape $J_{y^{\prime}}=1 /\left(1+\Delta_{\mathrm{RF}}^{2}\right)$ from Eqs. 3.33 and 3.34, respectively, are plotted as a function of $\Delta_{\mathrm{RF}}=\omega_{\mathrm{RF}}-\omega_{L}$.
an oscillating magnetic field $B_{\mathrm{RF}}(t) \hat{\mathbf{x}}$ is applied along the $x$-axis, then the spin $J_{y^{\prime}}$ in the rotating frame will be non-zero, with $J_{x^{\prime}}=0$ when $\Delta_{\mathrm{RF}}=0$, i.e., $\omega_{L}=\omega_{\mathrm{RF}}$. The width of the Lorentzian lineshapes $J_{x^{\prime}}$ and $J_{y^{\prime}}$ is defined as the detuning $\Delta_{\mathrm{RF}}$ when $J_{y^{\prime}}=J_{y^{\prime}}\left(\Delta_{\mathrm{RF}}=0\right) / 2$. This occurs when $\Delta_{\mathrm{RF}}=\omega_{\mathrm{RF}}-\omega_{L}=\sqrt{\delta \omega^{2}+\gamma_{\mathrm{Cs}}^{2} B_{1, \mathrm{c}}^{2} / 2}$ and
shows that the half-width-half-maximum (HWHM) depends on the applied RF field $B_{1, \mathrm{c}}$ and on the properties of the pump beam $R_{p}$, probe beam $\Gamma_{\mathrm{pr}}$ and the vapour cell itself $\Gamma_{\text {dark }}$, which is the intrinsic linewidth of the OPM.

### 3.4 Faraday rotation

To experimentally extract the atomic spins as an observable, a probe beam will propagate through the vapour cell along the $y$-axis and will be strongly linearly polarised along the $z$-axis. The experimental setup is depicted in Fig. 3.4. For this


Figure 3.4: Simplified experimental setup of the orientation-based RF OPM described in this chapter. The probe beam is strongly $z$-polarised before the vapour cell and propagates along the $y$-axis. The static magnetic field $\mathbf{B}_{0}$ is applied along the $z$-axis. The circularly polarised pump beam propagates along the $z$-axis. An oscillating magnetic field $\mathbf{B}_{\mathrm{RF}}(t)$ is applied along the $x$-direction at the position of the vapour cell. Faraday rotation of the detuned linearly polarised probe beam is extracted with help from a half-wave plate ( $\lambda / 2$ ), a polarising beam splitter (PBS) and a balanced photodetector (BPD). The half-wave plate after the vapour cell rotates the $z$-polarised light by $45^{\circ}$.
quantum mechanical treatment, just a single atom with $j=1 / 2, m= \pm 1 / 2$ ground states and $j^{\prime}=1 / 2, m^{\prime}= \pm 1 / 2$ excited states will be considered. No magnetic fields will be present and no pump beam will be present. It will be shown how the spin $j_{y}$ of a single atom along the propagation direction of the probe beam can be extracted via polarisation rotation of a linearly polarised probe beam, using the formalism in (Julsgaard, 2003; Sherson, 2006). The quantisation axis in this quantum mechanical treatment will be along the direction of the probe direction, which in this case is along the $y$-axis. As the $z$-polarised light is not along the quantisation axis, $\Delta m= \pm 1$ transitions between the ground and excited states will be driven. If the quantisation axis was along the $z$-axis then $\Delta m=0$ transitions would be driven.

The polarisation states of light in terms of the Stokes vectors are given by

$$
\begin{align*}
S_{0} & =\frac{1}{2}\left[n_{\mathrm{ph}}(z)+n_{\mathrm{ph}}(x)\right],  \tag{3.36}\\
S_{1} & =\frac{1}{2}\left[n_{\mathrm{ph}}(z)-n_{\mathrm{ph}}(x)\right],  \tag{3.37}\\
S_{2} & =\frac{1}{2}\left[n_{\mathrm{ph}}\left(45^{\circ}\right)-n_{\mathrm{ph}}\left(-45^{\circ}\right)\right],  \tag{3.38}\\
S_{3} & =\frac{1}{2}\left[n_{\mathrm{ph}}\left(\sigma^{+}\right)-n_{\mathrm{ph}}\left(\sigma^{-}\right)\right], \tag{3.39}
\end{align*}
$$

where $n_{\mathrm{ph}}(z)$ and $n_{\mathrm{ph}}(x)$ are the number of photons in a pulse of $z$ - and $x$-polarised light, respectively, $n_{\mathrm{ph}}\left(+45^{\circ}\right)$ and $n_{\mathrm{ph}}\left(-45^{\circ}\right)$ are the number of linearly polarised photons at a $\pm 45^{\circ}$ angle from the $z$-axis in the $x-z$ plane, and $n_{\mathrm{ph}}\left(\sigma^{+}\right)$and $n_{\mathrm{ph}}\left(\sigma^{-}\right)$are the number of right- and left- circularly polarised photons around the propagation direction of the probe beam ( $y$-axis).

Faraday rotation will be understood by using a two level scheme, with two magnetic sublevels $m= \pm 1 / 2$ in the ground and excited states, as depicted in Fig. 3.5. The probe beam will have a detuning $\Delta=\omega_{0}-\omega$ from the atomic energy level difference $\hbar \omega_{0}$. The $\sigma^{+}$light couples the $|1\rangle \rightarrow|4\rangle$ transition and the $\sigma^{-}$light couples the $|2\rangle \rightarrow|3\rangle$ transition. The density matrix elements will be written as, for example, $\rho_{1,1}=|1\rangle\langle 1|$, such that $\rho_{4,1} \rho_{1,4}=|4\rangle\langle 1 \mid 1\rangle\langle 4|=|4\rangle\langle 4|=\rho_{4,4}$. The population of the sublevel $|i\rangle$ is $\rho_{i, i}$ and the coherences between the sublevels $|i\rangle$ and $|j\rangle$ are notated by $\rho_{i, j}$ when $i \neq j$. The number of $\sigma^{+}$and $\sigma^{-}$photons are counted by $a_{+}^{\dagger} a_{+}$and $a_{-}^{\dagger} a_{-}$, respectively.


Figure 3.5: Level scheme with two magnetic sublevels in the ground and excited states. The decomposition of the linearly polarised light into $\sigma^{+}$- and $\sigma^{-}$-polarised light leads to detuned $\sigma^{+}$light driving transitions from $j=1 / 2, m=-1 / 2$ to $j^{\prime}=1 / 2, m^{\prime}=1 / 2$. The $\sigma^{-}$light drives transitions to the $j^{\prime}=1 / 2, m^{\prime}=-1 / 2$ sublevel.

The Hamiltonian $H$ describing the energy of the system in Fig. 3.5 is (Sherson, 2006)

$$
\begin{align*}
H= & \hbar \omega\left(a_{+}^{\dagger} a_{+}+a_{-}^{\dagger} a_{-}\right)+\hbar \omega_{0}\left(\rho_{3,3}+\rho_{4,4}\right)+\hbar g\left(a_{+}^{\dagger} e^{i \omega t} \rho_{1,4}+\rho_{4,1} a_{+} e^{-i \omega t}\right)  \tag{3.40}\\
& +\hbar g\left(a_{-}^{\dagger} e^{i \omega t} \rho_{2,3}+\rho_{3,2} a_{-} e^{-i \omega t}\right),
\end{align*}
$$

where $g=-d \sqrt{\omega /\left(2 \hbar \varepsilon_{0} V\right)}, d=\langle i| e r|j\rangle$ is the dipole matrix element for the transition $|i\rangle \rightarrow|j\rangle, \omega$ is the optical frequency, $\varepsilon_{0}$ is the vacuum permittivity and $V$ is the
quantisation volume. The coherences $\rho_{1,4}, \rho_{4,1}, \rho_{2,3}$ and $\rho_{3,2}$ in Eq. 3.40 will first be determined by solving the Heisenberg equation of motion for each coherence. For $d \rho_{1,4} / d t$ the Heisenberg equation of motion is given by

$$
\begin{equation*}
\frac{d \rho_{1,4}}{d t}=\frac{1}{i \hbar}\left[\rho_{1,4}, H\right]=\frac{1}{i \hbar}\left(\rho_{1,4} H-H \rho_{1,4}\right)=-i\left[\omega_{0} \rho_{1,4}+g\left(\rho_{1,1}-\rho_{4,4}\right) a_{+} e^{-i \omega t}\right] . \tag{3.41}
\end{equation*}
$$

A slowly varying operator $\tilde{\rho}_{1,4}=\rho_{1,4} e^{i \omega t}$ is defined and substituted into Eq. 3.41, giving rise to

$$
\begin{equation*}
\frac{d \tilde{\rho}_{1,4}}{d t}=-i\left[\Delta \tilde{\rho}_{1,4}+g\left(\rho_{1,1}-\rho_{4,4}\right) a_{+}\right] . \tag{3.42}
\end{equation*}
$$

Some assumptions are now made about the system (Sherson, 2006; Julsgaard, 2003). As the probe is detuned from the atomic transition, the populations in the excited states will be assumed to be zero, i.e., $\rho_{4,4}=\rho_{3,3}=0$. It is also assumed that as the detuning of the probe is large, whenever there is a change in the ground state population, $\rho_{1,1}$, the coherences, $\tilde{\rho}_{1,4}$, will rapidly reach the steady state, i.e., $d \tilde{\rho}_{1,4} / d t=0$. This allows for $\tilde{\rho}_{1,4}$ to be written as

$$
\begin{equation*}
\tilde{\rho}_{1,4}=-\frac{g a_{+} \rho_{1,1}}{\Delta} . \tag{3.43}
\end{equation*}
$$

The same method is used for solving $\tilde{\rho}_{4,1}, \tilde{\rho}_{2,3}$ and $\tilde{\rho}_{3,2}$ in Appendix. E. These are calculated to be

$$
\begin{align*}
& \tilde{\rho}_{4,1}=-\frac{g a_{+}^{\dagger} \rho_{1,1}}{\Delta},  \tag{3.44}\\
& \tilde{\rho}_{2,3}=-\frac{g a_{-} \rho_{2,2}}{\Delta},  \tag{3.45}\\
& \tilde{\rho}_{3,2}=-\frac{g a_{-}^{\dagger} \rho_{2,2}}{\Delta} . \tag{3.46}
\end{align*}
$$

Letting $H=H_{0}+H_{\text {eff }}$, where $H_{0}=\hbar \omega\left(a_{+}^{\dagger} a_{+}+a_{-}^{\dagger} a_{-}\right)+\hbar \omega_{0}\left(\rho_{3,3}+\rho_{4,4}\right)$, the effective Hamiltonian $H_{\text {eff }}$ is given by

$$
\begin{equation*}
H_{\mathrm{eff}}=\hbar g\left(a_{+}^{\dagger} e^{i \omega t} \rho_{1,4}+\rho_{4,1} a_{+} e^{-i \omega t}+a_{-}^{\dagger} e^{i \omega t} \rho_{2,3}+\rho_{3,2} a_{-} e^{-i \omega t}\right) \tag{3.47}
\end{equation*}
$$

Substituting in $\rho_{1,4}=\tilde{\rho}_{1,4} e^{-i \omega t}, \rho_{4,1}=\tilde{\rho}_{4,1} e^{i \omega t}, \rho_{2,3}=\tilde{\rho}_{2,3} e^{-i \omega t}, \rho_{3,2}=\tilde{\rho}_{3,2} e^{i \omega t}$, followed by the substitutions of Eqs. E.10.E. 13

$$
\begin{align*}
H_{\mathrm{eff}} & =\hbar g\left(a_{+}^{\dagger} \tilde{\rho}_{1,4}+\tilde{\rho}_{4,1} a_{+}+a_{-}^{\dagger} \tilde{\rho}_{2,3}+\tilde{\rho}_{3,2} a_{-}\right) \\
& =-\frac{2 \hbar g^{2}}{\Delta}\left(a_{+}^{\dagger} a_{+} \rho_{1,1}+a_{-}^{\dagger} a_{-} \rho_{2,2}\right) . \tag{3.48}
\end{align*}
$$

The effective Hamiltonian can be re-written as

$$
\begin{equation*}
H_{\mathrm{eff}}=-\frac{2 \hbar g^{2}}{\Delta}\left[\left(\rho_{1,1}-\rho_{2,2}\right)\left(a_{+}^{\dagger} a_{+}-a_{-}^{\dagger} a_{-}\right)+\left(\rho_{1,1}+\rho_{2,2}\right)\left(a_{+}^{\dagger} a_{+}+a_{-}^{\dagger} a_{-}\right)\right] . \tag{3.49}
\end{equation*}
$$

The Hamiltonian $H_{\text {eff }}$ will now be written in terms of the Stokes operator $S_{3}=$ $\left(a_{+}^{\dagger} a_{+}-a_{-}^{\dagger} a_{-}\right) / 2$ from Eq. 3.39 and in terms of the spin component $j_{z}$ along the probe direction. It is noted that $j_{y}|1\rangle=-1 / 2|1\rangle$ and $j_{y}|2\rangle=1 / 2|2\rangle$ (see Fig. 3.5), such that $j_{y}=\left(\rho_{22}-\rho_{11}\right) / 2$. Substituting in $2 S_{3}=\left(a_{+}^{\dagger} a_{+}-a_{-}^{\dagger} a_{-}\right), \rho_{11}+\rho_{22}=1$, $2 j_{y}=\left(\rho_{22}-\rho_{11}\right)$ and $a_{+}^{\dagger} a_{+}+a_{-}^{\dagger} a_{-}=N_{\mathrm{ph}}$, where $N_{\mathrm{ph}}$ is the total number of photons, the effective Hamiltonian $H_{\text {eff }}$ for one atom is given by

$$
\begin{equation*}
H_{\mathrm{eff}}=\frac{2 \hbar g^{2}}{\Delta}\left[4 j_{y} S_{3}-1 \times N_{\mathrm{ph}}\right] \tag{3.50}
\end{equation*}
$$

The final term " $1 \times N_{\text {ph }}$ " provides a Stark shift in overall energy and can therefore be ignored. The term of interest is $j_{y} S_{3}$. The commutation relations are $\left[j_{x}, j_{y}\right]=i \hbar j_{z}$, $\left[j_{y}, j_{z}\right]=i \hbar j_{x}$ and $\left[j_{z}, j_{x}\right]=i \hbar j_{y}$. The term will be understood by calculating the Heisenberg equations of motion for the atomic spins $j_{x}, j_{y}$ and $j_{z}$ :

$$
\begin{align*}
& \frac{d j_{x}}{d t}=\frac{1}{i \hbar}\left[j_{x}, H_{\mathrm{eff}}\right]=-i \frac{8 g^{2} S_{3}}{\Delta}\left(j_{x} j_{y}-j_{y} j_{x}\right)=\frac{8 \hbar g^{2} S_{3}}{\Delta} j_{z}  \tag{3.51}\\
& \frac{d j_{y}}{d t}=\frac{1}{i \hbar}\left[j_{y}, H_{\mathrm{eff}}\right]=0,  \tag{3.52}\\
& \frac{d j_{z}}{d t}=\frac{1}{i \hbar}\left[j_{z}, H_{\mathrm{eff}}\right]=-i \frac{8 g^{2} S_{3}}{\Delta}\left(j_{z} j_{y}-j_{y} j_{z}\right)=-\frac{8 \hbar g^{2} S_{3}}{\Delta} j_{x} \tag{3.53}
\end{align*}
$$

The equation of motion for a Stokes operator $S_{i}$ is assumed to be (Sherson, 2006)

$$
\begin{equation*}
c \frac{\partial S_{i}(y, t)}{\partial y}=\frac{1}{i \hbar}\left[S_{i}(y, t), H_{\mathrm{eff}}\right], \tag{3.54}
\end{equation*}
$$

where $\partial S_{i} / \partial y$ is the rate of change of the Stokes operator through the vapour cell. The uncertainty relations of the Stokes vector can be shown to be $\left[S_{1}, S_{2}\right]=i S_{3}$, $\left[S_{2}, S_{3}\right]=i S_{1}$ and $\left[S_{3}, S_{1}\right]=i S_{2}$, leading to

$$
\begin{align*}
& c \frac{\partial S_{1}(y, t)}{\partial y}=\frac{1}{i \hbar}\left[S_{1}(y, t), H_{\mathrm{eff}}\right]=-i \frac{8 g^{2} j_{y}}{\Delta}\left(S_{1} S_{3}-S_{3} S_{1}\right)=-\frac{8 g^{2} j_{y}}{\Delta} S_{2}  \tag{3.55}\\
& c \frac{\partial S_{2}(y, t)}{\partial y}=\frac{1}{i \hbar}\left[S_{2}(y, t), H_{\mathrm{eff}}\right]=-i \frac{8 g^{2} j_{y}}{\Delta}\left(S_{2} S_{3}-S_{3} S_{2}\right)=\frac{8 g^{2} j_{y}}{\Delta} S_{1}  \tag{3.56}\\
& c \frac{\partial S_{3}(y, t)}{\partial y}=-i\left[S_{3}(y, t), H_{\mathrm{eff}}\right]=0 . \tag{3.57}
\end{align*}
$$

The experimental setup in Sec. 3.3 and Fig. 3.4 will now be reconsidered, where an ensemble of atoms were studied. The atoms are pumped along the $z$-direction, creating a "classical" vector $J_{z}$ along the $z$-direction. As one component is now being treated as a classical component, we will explicitly write $\hat{J}_{x}$ and $\hat{J}_{y}$ for the ensemble of atoms to emphasise that the transverse components of the atomic spins have quantum, not classical, behaviour. It is assumed that the collective spin along the $z$-direction of the atomic ensemble $J_{z}$ will not change, such that $d J_{z} / d t=0$. Similarly, the light is strongly polarised along the $z$-direction, and so it is also assumed that $S_{1}$ is a classical parameter, not a quantum operator. As polarisation changes along $x$ will make negligible differences to $S_{1}$, it is assumed that $\partial S_{1} / \partial y=0$.

The only non-zero derivatives are therefore $d \hat{J}_{x} / d t$ and $\partial \hat{S}_{2} / \partial y$. The observable of interest in this oriented OPM setup will therefore be the change of polarisation
$\hat{S}_{2}$, which needs to be integrated over the length of the cell $y_{\text {cell }}$, i.e.,

$$
\begin{equation*}
\int_{\hat{S}_{2, \text { in }}}^{\hat{S}_{2, \text { out }}} \partial \hat{S}_{2}(y, t)=\int_{0}^{y_{\mathrm{cell}}} \frac{8 g^{2} \hat{J}_{y}}{c \Delta} S_{1} \partial y . \tag{3.58}
\end{equation*}
$$

The observable which is read out from the experiment is therefore

$$
\begin{equation*}
\hat{S}_{2, \text { out }}=\hat{S}_{2, \text { in }}+\frac{8 g^{2} \hat{J}_{y} S_{1} y_{\text {cell }}}{c \Delta} \tag{3.59}
\end{equation*}
$$

The linearly polarised light, which we make perfectly $z$-polarised before the vapour cell such that $\hat{S}_{2 \text {, in }}=0$ as $n_{+45^{\circ}}=n_{-45^{\circ}}$, will therefore be rotated by a small amount after leaving the vapour cell such that $n_{+45^{\circ}} \neq n_{-45^{\circ}}$. This therefore means that after passing through the cell of length $y_{\text {cell }}$, the linearly polarised probe beam has been rotated by a small angle. This is called Faraday rotation. The Faraday rotation will be larger if the cell is longer, if the detuning is smaller and if the spin $\hat{J}_{y}$ of the atomic ensemble along the probe beam direction is larger. This demonstrates how we detect Faraday rotation in our orientation-based optically pumped magnetometer experiments, allowing us to extract $\hat{J}_{y}$, which can be non-zero in our experiments by the application of an oscillating magnetic field $\mathbf{B}_{\mathrm{RF}}(t)$ transverse to the static magnetic field $\mathbf{B}_{0}$.

## Chapter 4

## Theory of an alignment-based optically pumped magnetometer

### 4.1 Motivation

The theory described in Sec. 3 is for an orientation-based OPM with a pump and a probe beam. The alignment-based magnetometer (Ledbetter et al., 2007; Zigdon et al., 2010) only has a single laser beam which both "pumps" and "probes" the atoms. Throughout this chapter there is emphasis on how this theory works for both paraffin-coated cells and for buffer gas cells, as both of these types of cells are used experimentally in Sec. 7.

### 4.2 Hamiltonian to describe the system

The theory underpinning the alignment-based magnetometer (Auzinsh et al., 2014 Zigdon et al., 2010; Rochester, 2010; Rochester, 2023) will now be revised and discussed. Consider atoms with a $F=1 \rightarrow F^{\prime}=0$ optical transition with groundstate sublevels $|F, m\rangle=\{|1,1\rangle,|1,0\rangle,|1,-1\rangle\}$ and an excited state $\left|F^{\prime}, m^{\prime}\right\rangle=|0,0\rangle$ as shown in Fig. 4.1a. A single laser beam is used for optical pumping and probing of the atoms. Assume the light is linearly $(\pi)$ polarised along the direction of a static magnetic field $B_{0} \hat{\mathbf{z}}$. In this case, the atoms will be optically pumped into the $m= \pm 1$ sublevels with equal probability, creating a so-called "spin-aligned state". This is a dark state, such that with perfect optical pumping, the light will be fully transmitted through the atomic vapour. Now assume further that there is a transverse oscillating (RF) magnetic field which we would like to detect. That RF field will affect the optical pumping and thereby the transmitted light which can be detected by measuring its intensity or polarisation.

The total Hamiltonian $H$ which describes our system is given by

$$
\begin{equation*}
H=H_{0}+H_{l}+H_{B}, \tag{4.1}
\end{equation*}
$$

where $H_{0}, H_{l}$ and $H_{B}$ are the unperturbed, light-atom interaction and magnetic field-atom interaction Hamiltonians, respectively. The unperturbed Hamiltonian $H_{0}$ is given by


Figure 4.1: (a) Energy level diagram with an $F=1$ ground state and $F^{\prime}=0$ excited state. For "small" $\mathbf{B}_{0}$ the neighbouring sublevels are each split by $\Delta E=\hbar \omega_{L}$. The light is $z$-linearly polarised and drives transitions between $m=0$ and $m^{\prime}=0$. A perfectly aligned state would have $50 \%$ of the atoms in $m=1$ and $50 \%$ of the atoms in $m=-1$, as depicted by the bar graphs. (b) D1 line of $\mathrm{Cs}(F=3,4$ ground states and $F^{\prime}=3,4$ excited states). Linearly polarised $(\pi)$ light drives transitions between $F=4$ and $F^{\prime}=3$.

$$
H_{0}=\left(\begin{array}{cccc}
0 & 0 & 0 & 0  \tag{4.2}\\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & \hbar \omega_{0}
\end{array}\right),
$$

where $\omega_{0}=2 \pi c / \lambda$ is the optical transition frequency, $\lambda$ its wavelength and $c$ the speed of light. The light-atom interaction is governed by

$$
\begin{equation*}
H_{l}=-\mathbf{E} \cdot \mathbf{d} \tag{4.3}
\end{equation*}
$$

where $\mathbf{d}$ is the dipole operator and $\mathbf{E}=E_{0} \cos (\omega t) \hat{\mathbf{z}}$ is the electric field of the light. The light-atom interaction Hamiltonian $H_{l}$ is derived in Sec. 2.5 for this configuration to be

$$
H_{l}=\hbar \Omega_{R} \cos \omega t\left(\begin{array}{cccc}
0 & 0 & 0 & 0  \tag{4.4}\\
0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0
\end{array}\right)
$$

where $\Omega_{R}=\langle 1||d|\left|0^{\prime}\right\rangle E_{0} /(\sqrt{3} \hbar)$ is the Rabi frequency and $\langle 1||d|\left|0^{\prime}\right\rangle$ is the transition dipole matrix element. Substituting $E_{0}=\sqrt{2 I_{\text {ave }} /\left(c \varepsilon_{0}\right)}$ into the Rabi frequency gives rise to

$$
\begin{equation*}
\Omega_{R}=\frac{\langle 1||d|\left|0^{\prime}\right\rangle}{\sqrt{3} \hbar} \sqrt{\frac{2 I_{\mathrm{ave}}}{c \varepsilon_{0}}} \tag{4.5}
\end{equation*}
$$

where $I_{\text {ave }}$ is the average intensity across the cell for the paraffin-coated cell and average intensity across the beam for the buffer gas cell and $\varepsilon_{0}$ is the vacuum permittivity.

Now only $H_{B}$ needs to be determined. An example was derived in Sec. 2.4 for an $F=1 \rightarrow F^{\prime}=0$ system including only a static magnetic field $\mathbf{B}_{0}=B_{0} \hat{\mathbf{z}}$ and with a $z$-quantisation axis. As we are interested in the detection of oscillating magnetic fields with the alignment-based RF OPM, an oscillating magnetic field $\mathbf{B}_{\mathrm{RF}}(t)$ must also be included in the calculations. The magnetic field-atom interaction is given by

$$
\begin{equation*}
H_{B}=-\boldsymbol{\mu} \cdot \mathbf{B} \tag{4.6}
\end{equation*}
$$

and assuming $B_{y}=0$,

$$
\begin{equation*}
H_{B}=\frac{g_{F} \mu_{B}}{\hbar}\left(F_{x} B_{x}+F_{z} B_{z}\right)=\frac{g_{F} \mu_{B}}{\hbar}\left(F_{x} B_{\mathrm{RF}} \cos \omega_{\mathrm{RF}} t+F_{z} B_{0}\right), \tag{4.7}
\end{equation*}
$$

where $\boldsymbol{\mu}=g_{F} \mu_{B}\left(F_{x} \hat{\mathbf{x}}+F_{y} \hat{\mathbf{y}}+F_{z} \hat{\mathbf{z}}\right) / \hbar$ is the Cs atom's magnetic dipole operator, $g_{F}$ is the hyperfine Landé g -factor (Steck, 2022), and $\mu_{B}$ is the Bohr magneton.

In this example, there are two magnetic fields (static and RF), so one will not be along the quantisation axis. Radio-frequency coupling (coherences) is expected between the ground state sublevels $|1,1\rangle$ and $|1,0\rangle$, as well as between $|1,0\rangle$ and $|1,-1\rangle$. The component $F_{x}$ can be calculated using

$$
\begin{equation*}
F_{x}=\frac{F_{+}+F_{-}}{2}, \tag{4.8}
\end{equation*}
$$

where $F_{+}$is the raising operator and $F_{-}$is the lowering operator. The following two equations can be used to calculate $F_{+}$and $F_{-}$

$$
\begin{align*}
F_{+}|F, m\rangle & =\hbar \sqrt{(F-m)(F+m+1)}|F, m+1\rangle  \tag{4.9}\\
& =\hbar \sqrt{F(F+1)-m(m+1)}|F, m+1\rangle \\
F_{-}|F, m\rangle & =\hbar \sqrt{(F+m)(F-m+1)}|F, m-1\rangle  \tag{4.10}\\
& =\hbar \sqrt{F(F+1)-m(m-1)}|F, m-1\rangle .
\end{align*}
$$

Therefore,

$$
\begin{align*}
F_{x}|F, m\rangle= & \frac{F_{+}|F, m\rangle+F_{-}|F, m\rangle}{2} \\
= & \frac{\hbar}{2}(\sqrt{F(F+1)-m(m+1)}|F, m+1\rangle  \tag{4.11}\\
& +\sqrt{F(F+1)-m(m-1)}|F, m-1\rangle) .
\end{align*}
$$

The component $F_{z}$ can be calculated using

$$
\begin{equation*}
F_{z}|F, m\rangle=\hbar m|F, m\rangle \tag{4.12}
\end{equation*}
$$

which gives rise to the matrix

$$
F_{z}=\left(\begin{array}{cccc}
\hbar & 0 & 0 & 0  \tag{4.13}\\
0 & 0 & 0 & 0 \\
0 & 0 & -\hbar & 0 \\
0 & 0 & 0 & 0
\end{array}\right) .
$$

The matrix $F_{x}$ is calculated to be

$$
F_{x}=\left(\begin{array}{cccc}
0 & \hbar / \sqrt{2} & 0 & 0  \tag{4.14}\\
\hbar / \sqrt{2} & 0 & \hbar / \sqrt{2} & 0 \\
0 & \hbar / \sqrt{2} & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}\right)
$$

The magnetic field-atom interaction can thus be constructed

$$
\begin{align*}
H_{B} & =\frac{g_{F} \mu_{B}}{\hbar}\left[\left(\begin{array}{cccc}
0 & \hbar / \sqrt{2} & 0 & 0 \\
\hbar / \sqrt{2} & 0 & \hbar / \sqrt{2} & 0 \\
0 & \hbar / \sqrt{2} & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}\right) B_{\mathrm{RF}} \cos \omega_{\mathrm{RF}} t+\left(\begin{array}{cccc}
\hbar & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & -\hbar & 0 \\
0 & 0 & 0 & 0
\end{array}\right) B_{0}\right] \\
& =g_{F} \mu_{B}\left(\begin{array}{cccc}
B_{0} & \frac{B_{\mathrm{RF}} \cos \omega_{\mathrm{RF}} t}{\sqrt{2}} & 0 & 0 \\
\frac{B_{\mathrm{RF}} \cos \omega_{\mathrm{RF}} t}{\sqrt{2}} & 0 & \frac{B_{\mathrm{RF}} \cos \omega_{\mathrm{RF}} t}{\sqrt{2}} & 0 \\
0 & \frac{B_{\mathrm{RF}} \cos \omega_{\mathrm{RF}} t}{\sqrt{2}} & -B_{0} & 0 \\
0 & 0 & 0 & 0
\end{array}\right) . \tag{4.15}
\end{align*}
$$

Letting the strength of the RF field to be $\Omega_{\mathrm{RF}}=g_{F} \mu_{B} B_{\mathrm{RF}} / \hbar$, the total Hamiltonian $H=H_{0}+H_{B}+H_{l}$ is then given by

$$
H=\hbar\left(\begin{array}{cccc}
\omega_{L} & \frac{\Omega_{\mathrm{RF}} \cos \omega_{\mathrm{RF}} t}{\sqrt{2}} & 0 & 0  \tag{4.16}\\
\frac{\Omega_{\mathrm{RF}} \cos \omega_{\mathrm{RF}} t}{\sqrt{2}} & 0 & \frac{\Omega_{\mathrm{RF}} \cos \omega_{\mathrm{RF}} t}{\sqrt{2}} & -\frac{\Omega_{R} \cos \omega t}{\sqrt{3}} \\
0 & \frac{\Omega_{\mathrm{RF}} \cos \omega_{\mathrm{RF}} t}{\sqrt{2}} & -\omega_{L} & 0 \\
0 & -\frac{\Omega_{R} \cos \omega t}{\sqrt{3}} & 0 & \omega_{0}
\end{array}\right) .
$$

### 4.3 Rotating-wave approximation

The time-dependence in $H$ can be removed by using the rotating-wave approximation (Auzinsh et al., 2014). The rotating-wave approximation is valid when the optical frequency $\omega$ is close to the atomic transition frequency $\omega_{0}$. This is true for an alignment-based magnetometer as the optical frequency $\omega$ is tuned to $\omega_{0}$. The Hamiltonian in the "lab frame" (Eq. 4.16) can be converted to one rotating at the optical frequency $\omega$ by using a rotation matrix $U$. Such a rotation matrix $U$ must be unitary, i.e., $U U^{\dagger}=U^{\dagger} U=1$. We therefore use the rotation matrix

$$
U=\left(\begin{array}{cccc}
1 & 0 & 0 & 0  \tag{4.17}\\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & e^{i \omega t}
\end{array}\right)
$$

Typically, a Hamiltonian is transformed by the operator $H_{\text {eff }}=U^{\dagger} H U$. However, $H_{\text {eff }}$ is not inertial and so a second term must be added. The effective Hamiltonian is given by

$$
\begin{equation*}
H_{\mathrm{eff}}=U^{\dagger} H U+i \hbar U^{\dagger} \frac{\partial U}{\partial t} \tag{4.18}
\end{equation*}
$$

The first and second terms are calculated to be

$$
\begin{align*}
U^{\dagger} H U & =\hbar\left(\begin{array}{cccc}
\omega_{L} & \frac{\Omega_{\mathrm{RF}} \cos \omega_{\mathrm{RF}} t}{\sqrt{2}} & 0 & 0 \\
\frac{\Omega_{\mathrm{RF}} \cos \omega_{\mathrm{RF}} t}{\sqrt{2}} & 0 & \frac{\Omega_{\mathrm{RF}} \cos \omega_{\mathrm{RF}} t}{\sqrt{2}} & -\frac{\Omega_{R} \cos (\omega t) e^{i \omega t}}{\sqrt{3}} \\
0 & \frac{\Omega_{\mathrm{RF}} \cos \omega_{\mathrm{RF}} t}{\sqrt{2}} & -\omega_{L} & 0 \\
& 0 & -\frac{\Omega_{R} \cos (\omega t) e^{-i \omega t}}{\sqrt{3}} & 0 \\
\omega_{0}
\end{array}\right),  \tag{4.19}\\
i \hbar U^{\dagger} \frac{\partial U}{\partial t} & =\hbar\left(\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\omega
\end{array}\right), \tag{4.20}
\end{align*}
$$

respectively, leading to the effective Hamiltonian

$$
H_{\mathrm{efF}}=\hbar\left(\begin{array}{cccc}
\omega_{L} & \frac{\Omega_{\mathrm{RF}} \cos \omega_{\mathrm{RF}} t}{\sqrt{2}} & 0 & 0  \tag{4.21}\\
\frac{\Omega_{\mathrm{RF}} \cos \omega_{\mathrm{RF}} t}{\sqrt{2}} & 0 & \frac{\Omega_{\mathrm{RF}} \cos \omega_{\mathrm{RF}} t}{\sqrt{2}} & -\frac{\Omega_{R}\left(1+e^{2 i \omega t}\right)}{2 \sqrt{3}} \\
0 & \frac{\Omega_{\mathrm{RF}} \cos \omega_{\mathrm{RF}} t}{\sqrt{2}} & -\omega_{L} & 0 \\
0 & -\frac{\Omega_{R}\left(1+e^{-2 i \omega t}\right)}{2 \sqrt{3}} & 0 & \omega_{0}-\omega
\end{array}\right) .
$$

As the light frequency $\omega$ is tuned to the atomic transition $\omega_{0}$ in an alignment-based magnetometer, the first component is kept and the second component, detuned by $2 \omega$ from the first, is ignored. This is the rotating-wave approximation. The effective Hamiltonian is then given by

$$
H_{\mathrm{eff}}=\hbar\left(\begin{array}{cccc}
\omega_{L} & \frac{\Omega_{\mathrm{RF}} \cos \omega_{\mathrm{RF}} t}{\sqrt{2}} & 0 & 0  \tag{4.22}\\
\frac{\Omega_{\mathrm{RF}} \cos \omega_{\mathrm{RF}} t}{\sqrt{2}} & 0 & \frac{\Omega_{\mathrm{RF}} \cos \omega_{\mathrm{RF}} t}{\sqrt{2}} & -\frac{\Omega_{R}}{2 \sqrt{3}} \\
0 & \frac{\Omega_{\mathrm{RF}} \cos \omega_{\mathrm{RF}} t}{\sqrt{2}} & -\omega_{L} & 0 \\
0 & -\frac{\Omega_{R}}{2 \sqrt{3}} & 0 & \omega_{0}-\omega
\end{array}\right) .
$$

There is still time-dependence of the RF field, which will be removed in a similar way. A second unitary transformation can be performed to go to another rotating frame. This transformation matrix $U_{\mathrm{RF}}$ will be written as

$$
U_{\mathrm{RF}}=\left(\begin{array}{cccc}
e^{-i \omega_{\mathrm{RF}} t} & 0 & 0 & 0  \tag{4.23}\\
0 & 1 & 0 & 0 \\
0 & 0 & e^{i \omega_{\mathrm{RF}} t} & 0 \\
0 & 0 & 0 & 1
\end{array}\right)
$$

The unitary transformation to a second rotating frame is performed

$$
\begin{equation*}
H_{\mathrm{RF}}=U_{\mathrm{RF}}^{\dagger} H_{\mathrm{eff}} U_{\mathrm{RF}}-i \hbar U_{\mathrm{RF}}^{\dagger} \frac{\partial U_{\mathrm{RF}}}{\partial t} \tag{4.24}
\end{equation*}
$$

and the Hamiltonian $\tilde{H}=H_{\mathrm{RF}}$ is calculated to be

$$
\tilde{H}=\hbar\left(\begin{array}{cccc}
\omega_{L}-\omega_{\mathrm{RF}} & \frac{\Omega_{\mathrm{RF}}\left(1+e^{2 i \omega_{\mathrm{RF}} t}\right)}{2 \sqrt{2}} & 0 & 0  \tag{4.25}\\
\frac{\Omega_{\mathrm{RF}}\left(1+e^{-2 i \omega_{\mathrm{RF}} t}\right)}{2 \sqrt{2}} & 0 & \frac{\Omega_{\mathrm{RF}}\left(1+e^{2 i \omega_{\mathrm{RF}} t}\right)}{2 \sqrt{2}} & -\frac{\Omega_{R}}{2 \sqrt{3}} \\
0 & \frac{\Omega_{\mathrm{RF}}\left(1+e^{\left.-2 i \omega_{\mathrm{RF}^{t}}\right)}\right.}{2 \sqrt{2}} & -\omega_{L}+\omega_{\mathrm{RF}} & 0 \\
0 & -\frac{\Omega_{R}}{2 \sqrt{3}} & 0 & \omega_{0}-\omega
\end{array}\right)
$$

Neglecting the fast oscillating terms at $2 \omega_{\mathrm{RF}}$ and letting $\Delta=\omega-\omega_{0}$ and $\Delta_{\mathrm{RF}}=$ $\omega_{\mathrm{RF}}-\omega_{L}$, the effective Hamiltonian $\tilde{H}$ is

$$
\tilde{H}=\hbar\left(\begin{array}{cccc}
-\Delta_{\mathrm{RF}} & \frac{\Omega_{\mathrm{RF}}}{2 \sqrt{2}} & 0 & 0  \tag{4.26}\\
\frac{\Omega_{\mathrm{RF}}}{2 \sqrt{2}} & 0 & \frac{\Omega_{\mathrm{RF}}}{2 \sqrt{2}} & -\frac{\Omega_{R}}{2 \sqrt{3}} \\
0 & \frac{\Omega_{\mathrm{RF}}}{2 \sqrt{2}} & \Delta_{\mathrm{RF}} & 0 \\
0 & -\frac{\Omega_{R}}{2 \sqrt{3}} & 0 & -\Delta
\end{array}\right) .
$$

Note that $\tilde{H}$ can be obtained using the "Nonlinear Magneto-Optical Rotation in a Radio-Frequency Field" AtomicDensityMatrix Mathematica package (Rochester, 2023), a very useful tool when determining analytical and numerical solutions for atomic density matrix calculations.

### 4.4 Relaxation and repopulation

The atoms in the excited state decay to the ground state sublevels at a rate $\Gamma$ and are assumed to repopulate the three ground state sublevels in equal proportion $\Gamma \rho_{0^{\prime}, 0^{\prime}} / 3$, where $\rho_{0^{\prime}, 0^{\prime}}$ is the population of the excited state. For a paraffin-coated cell, the excited atoms decay via spontaneous emission (at the rate $2 \pi(4.56 \mathrm{MHz}$ ) (Steck, 2022) for Cs atoms in the first excited state). For a buffer gas cell with a relatively high buffer gas pressure such as the 65 Torr $\mathrm{N}_{2}$ cell used in this thesis, the excited atoms mainly decay via quenching (see Sec. 2.6.3). Note that the model above only includes one excited state and therefore does not describe rapid collisional mixing between multiple excited states (see Sec. 2.6.3). The atoms also have a transverse spin relaxation rate $\gamma=1 / T_{2}$, where $T_{2}$ is the spin coherence time. In a buffer gas cell, the alkali atoms diffuse slowly due to collisions with the buffer gas, increasing $T_{2}$. The spins relax when the alkali atoms hit the glass walls due to electron randomisation collisions or via spin-exchange or spin-destruction collisions between two alkali atoms (Graf et al., 2005; Ledbetter et al., 2007; Labyt et al., 2022). In a paraffin-coated cell the alkali atoms can bounce off the walls thousands of times before spin relaxation occurs (Balabas et al., 2010b). The relaxation $\Gamma^{\prime}$ and the repopulation $\Lambda$ matrices are therefore given by

$$
\begin{align*}
\Gamma^{\prime} & =\left(\begin{array}{llll}
\gamma & 0 & 0 & 0 \\
0 & \gamma & 0 & 0 \\
0 & 0 & \gamma & 0 \\
0 & 0 & 0 & \gamma+\Gamma
\end{array}\right),  \tag{4.27}\\
\Lambda & =\left(\begin{array}{cccc}
\frac{\gamma+\Gamma \rho_{0^{\prime}, 0^{\prime}}}{3} & 0 & 0 & 0 \\
0 & \frac{\gamma+\Gamma \rho_{0^{\prime}, 0^{\prime}}}{3} & 0 & 0 \\
0 & 0 & \frac{\gamma+\Gamma \rho_{0^{\prime}, 0^{\prime}}}{3} & 0 \\
0 & 0 & 0 & 0
\end{array}\right) . \tag{4.28}
\end{align*}
$$

### 4.5 Liouville equation

To understand what is happening to each atomic state in the rotating frame, a density matrix $\tilde{\rho}$ is constructed, given by

$$
\tilde{\rho}=\left(\begin{array}{cccc}
\tilde{\rho}_{1,1} & \tilde{\rho}_{1,0} & \tilde{\rho}_{1,-1} & \tilde{\rho}_{1,0^{\prime}}  \tag{4.29}\\
\tilde{\rho}_{0,1} & \tilde{\rho}_{0,0} & \tilde{\rho}_{0,-1} & \tilde{\rho}_{0,0^{\prime}} \\
\tilde{\rho}_{-1,1} & \tilde{\rho}_{-1,0} & \tilde{\rho}_{-1,-1} & \tilde{\rho}_{-1,0^{\prime}} \\
\tilde{\rho}_{0^{\prime}, 1} & \tilde{\rho}_{0^{\prime}, 0} & \tilde{\rho}_{0^{\prime},-1} & \tilde{\rho}_{0^{\prime}, 0^{\prime}}
\end{array}\right) .
$$

The diagonal elements ( $\tilde{\rho}_{1,1}, \tilde{\rho}_{0,0}, \tilde{\rho}_{-1,-1}$ and $\tilde{\rho}_{0^{\prime}, 0^{\prime}}$ ) represent the populations of each state in the rotating frame. The off-diagonal elements represent the coherences between the different sublevels. These matrix elements can be calculated by solving the Liouville equation for the rotating-frame density matrix:

$$
\begin{align*}
& i \hbar \frac{\partial \tilde{\rho}}{\partial t}=[\tilde{H}, \tilde{\rho}]-i \hbar \frac{1}{2}(\Gamma \tilde{\rho}+\tilde{\rho} \Gamma)+i \hbar \Lambda \\
& =\hbar\left(\begin{array}{cccc}
-\Delta_{\mathrm{RF}} & \frac{\Omega_{\mathrm{RF}}}{2 \sqrt{2}} & 0 & 0 \\
\frac{\Omega_{\mathrm{RF}}}{2 \sqrt{2}} & 0 & \frac{\Omega_{\mathrm{RF}}}{2 \sqrt{2}} & -\frac{\Omega_{R}}{2 \sqrt{3}} \\
0 & \frac{\Omega_{\mathrm{RF}}}{2 \sqrt{2}} & \Delta_{\mathrm{RF}} & 0 \\
0 & -\frac{\Omega_{R}}{2 \sqrt{3}} & 0 & -\Delta
\end{array}\right)\left(\begin{array}{cccc}
\tilde{\rho}_{1,1} & \tilde{\rho}_{1,0} & \tilde{\rho}_{1,-1} & \tilde{\rho}_{1,0^{\prime}} \\
\tilde{\rho}_{0,1} & \tilde{\rho}_{0,0} & \tilde{\rho}_{0,-1} & \tilde{\rho}_{0,0^{\prime}} \\
\tilde{\rho}_{-1,1} & \tilde{\rho}_{-1,0} & \tilde{\rho}_{-1,-1} & \tilde{\rho}_{-1,0^{\prime}} \\
\tilde{\rho}_{0^{\prime}, 1} & \tilde{\rho}_{0^{\prime}, 0} & \tilde{\rho}_{0^{\prime},-1} & \tilde{\rho}_{0^{\prime}, 0^{\prime}}
\end{array}\right) \\
& -\hbar\left(\begin{array}{cccc}
\tilde{\rho}_{1,1} & \tilde{\rho}_{1,0} & \tilde{\rho}_{1,-1} & \tilde{\rho}_{1,0^{\prime}} \\
\tilde{\rho}_{0,1} & \tilde{\rho}_{0,0} & \tilde{\rho}_{0,-1} & \tilde{\rho}_{0,0^{\prime}} \\
\tilde{\rho}_{-1,1} & \tilde{\rho}_{-1,0} & \tilde{\rho}_{2-1,-1} & \tilde{\rho}_{-1,0^{\prime}} \\
\tilde{\rho}_{0^{\prime}, 1} & \tilde{\rho}_{0^{\prime}, 0} & \tilde{\rho}_{0^{\prime},-1} & \tilde{\rho}_{0^{\prime}, 0^{\prime}}
\end{array}\right)\left(\begin{array}{cccc}
-\Delta_{\mathrm{RF}} & \frac{\Omega_{\mathrm{RF}}}{2 \sqrt{2}} & 0 & 0 \\
\frac{\Omega_{\mathrm{RF}}}{2 \sqrt{2}} & 0 & \frac{\Omega_{\mathrm{RF}}}{2 \sqrt{2}} & -\frac{\Omega_{R}}{2 \sqrt{3}} \\
0 & \frac{\Omega_{\mathrm{RF}}}{2 \sqrt{2}} & \Delta_{\mathrm{RF}} & 0 \\
0 & -\frac{\Omega_{R}}{2 \sqrt{3}} & 0 & -\Delta
\end{array}\right) \\
& -i \hbar \frac{1}{2}\left[\left(\begin{array}{cccc}
\gamma & 0 & 0 & 0 \\
0 & \gamma & 0 & 0 \\
0 & 0 & \gamma & 0 \\
0 & 0 & 0 & \gamma+\Gamma
\end{array}\right)\left(\begin{array}{cccc}
\tilde{\rho}_{1,1} & \tilde{\rho}_{1,0} & \tilde{\rho}_{1,-1} & \tilde{\rho}_{1,0^{\prime}} \\
\tilde{\rho}_{0,1} & \tilde{\rho}_{0,0} & \tilde{\rho}_{0,-1} & \tilde{\rho}_{0,0^{\prime}} \\
\tilde{\rho}_{-1,1} & \tilde{\rho}_{-1,0} & \tilde{\rho}_{-1,-1} & \tilde{\rho}_{-1,0^{\prime}} \\
\tilde{\rho}_{0^{\prime}, 1} & \tilde{\rho}_{0^{\prime}, 0} & \tilde{\rho}_{0^{\prime},-1} & \tilde{\rho}_{0^{\prime}, 0^{\prime}}
\end{array}\right)\right.  \tag{4.30}\\
& \left.+\left(\begin{array}{cccc}
\tilde{\rho}_{1,1} & \tilde{\rho}_{1,0} & \tilde{\rho}_{1,-1} & \tilde{\rho}_{1,0^{\prime}} \\
\tilde{\rho}_{0,1} & \tilde{\rho}_{0,0} & \tilde{\rho}_{0,-1} & \tilde{\rho}_{0,0^{\prime}} \\
\tilde{\rho}_{-1,1} & \tilde{\rho}_{-1,0} & \tilde{\rho}_{-1,-1} & \tilde{\rho}_{-1,0^{\prime}} \\
\tilde{\rho}_{0^{\prime}, 1} & \tilde{\rho}_{0^{\prime}, 0} & \tilde{\rho}_{0^{\prime},-1} & \tilde{\rho}_{0^{\prime}, 0^{\prime}}
\end{array}\right)\left(\begin{array}{cccc}
\gamma & 0 & 0 & 0 \\
0 & \gamma & 0 & 0 \\
0 & 0 & \gamma & 0 \\
0 & 0 & 0 & \gamma+\Gamma
\end{array}\right)\right] \\
& +i \hbar\left(\begin{array}{cccc}
\frac{\gamma+\Gamma \rho_{0^{\prime}, 0^{\prime}}}{3} & 0 & 0 & 0 \\
0 & \frac{\gamma+\Gamma \rho_{\rho^{\prime}, 0^{\prime}}}{3} & 0 & 0 \\
0 & 0 & \frac{\gamma+\Gamma \rho_{0^{\prime}, 0^{\prime}}}{3} & 0 \\
0 & 0 & 0 & 0
\end{array}\right) .
\end{align*}
$$

Some examples of density matrix elements $d \tilde{\rho}_{1,1} / d t, d \tilde{\rho}_{0^{\prime}, 0^{\prime}} / d t, d \tilde{\rho}_{1,0} / d t$ and $d \tilde{\rho}_{0,0^{\prime}} / d t$ are

$$
\begin{align*}
\frac{d \tilde{\rho}_{1,1}}{d t} & =\frac{i \Omega_{\mathrm{RF}}}{2 \sqrt{2}}\left(\tilde{\rho}_{1,0}-\tilde{\rho}_{0,1}\right)-\gamma \tilde{\rho}_{1,1}+\frac{\gamma+\Gamma \rho_{0^{\prime}, 0^{\prime}}}{3}  \tag{4.31}\\
\frac{d \tilde{\rho}_{0^{\prime}, 0^{\prime}}}{d t} & =\frac{i \Omega_{\mathrm{R}}}{2 \sqrt{3}}\left(\tilde{\rho}_{0,0^{\prime}}-\tilde{\rho}_{0^{\prime}, 0}\right)-(\gamma+\Gamma) \tilde{\rho}_{0^{\prime}, 0^{\prime}},  \tag{4.32}\\
\frac{d \tilde{\rho}_{1,0}}{d t} & =i\left(\Delta_{\mathrm{RF}} \tilde{\rho}_{1,0}-\frac{\Omega_{\mathrm{RF}}}{2 \sqrt{2}}\left(\tilde{\rho}_{0,0}-\tilde{\rho}_{1,1}-\tilde{\rho}_{1,-1}\right)-\frac{\Omega_{R}}{2 \sqrt{3}} \tilde{\rho}_{1,0^{\prime}}\right)-\gamma \tilde{\rho}_{1,0}  \tag{4.33}\\
\frac{d \tilde{\rho}_{0,0^{\prime}}}{d t} & =-i\left(\frac{\Omega_{\mathrm{RF}}}{2 \sqrt{2}}\left(\tilde{\rho}_{1,0^{\prime}}+\tilde{\rho}_{-1,0^{\prime}}\right)+\frac{\Omega_{R}}{2 \sqrt{3}}\left(\tilde{\rho}_{0,0}-\tilde{\rho}_{0^{\prime}, 0^{\prime}}\right)+\Delta \tilde{\rho}_{0,0^{\prime}}\right)-(2 \gamma+\Gamma) \frac{\tilde{\rho}_{0,0^{\prime}}}{2} . \tag{4.34}
\end{align*}
$$

The rate of change of the $F=1, m=1$ sublevel population $d \tilde{\rho}_{1,1}(t) / d t$, for example, depends on the strength of the RF field $\Omega_{\mathrm{RF}}$, as the RF field will cause coherences between the $m=1$ and $m=0$ sublevels. There will be a depopulation term $-\tilde{\rho}_{1,1} \gamma$ which reduces the population due to, for example, Cs-Cs spin-exchange collisions and electron randomisation collisions with the glass wall (Ledbetter et al., 2007; Graf et al., 2005). In the case of buffer gas cells there are also spin-exchange collisions between the buffer gas, for example $\mathrm{N}_{2}$, and the Cs atoms. The repopulation term $\left(\gamma+\Gamma \tilde{\rho}_{0^{\prime}, 0^{\prime}}\right) / 3$ is due to the atomic spin being randomly distributed among the magnetic sublevels due to the relaxation rate $\gamma$, and due to the decays from the excited state.

The 16 differential equations can be solved via the "Nonlinear Magneto-Optical Rotation in a Radio-Frequency Field" AtomicDensityMatrix Mathematica package (Rochester, 2023), allowing for the 16 density matrix elements in the rotating frame to be extracted.

### 4.6 Observables

The "observable" in the alignment-based magnetometer experiments presented in Sec. 7 is the rotation of polarisation of the linearly polarised beam by the angle $\phi$, where $\phi$ is the angle subtended between the $z$-axis and the polarisation axis of the beam (see Fig. 4.2). The light is strongly linearly polarised along the $z$-axis and the quantisation axis is also the $z$-axis. To theoretically determine how this angle $\phi$ depends on the atomic polarisation $\mathbf{P}$ of the medium induced by the light, we will initially write $\mathbf{P}$ using the parameterisation (Auzinsh et al., 2014)

$$
\begin{equation*}
\left.\mathbf{P}=\operatorname{Re}\left[e^{i(\mathbf{k} \cdot \mathbf{r}+\alpha)} e^{-i \omega t}\left[\left(P_{1}-i P_{2}\right) \hat{\mathbf{z}}+\left(P_{3}-i P_{4}\right) \hat{\mathbf{x}}\right)\right]\right] \tag{4.35}
\end{equation*}
$$

where $P_{1}$ and $P_{2}$ are the in-phase and quadrature components of the atomic polarisation along the $z$-direction, respectively, $P_{3}$ and $P_{4}$ are the in-phase and quadrature components of the atomic polarisation along the $x$-direction, respectively, $\mathbf{k}=k \hat{\mathbf{y}}$ is the wave vector, $\omega$ is the optical frequency and $\alpha$ is the phase. The polarisation $\mathbf{P}$ can be substituted into the wave equation (Auzinsh et al., 2014)

$$
\begin{equation*}
\frac{\partial^{2} \mathbf{E}}{\partial l^{2}}+k^{2} \mathbf{E}=-4 \pi k^{2} \mathbf{P} \tag{4.36}
\end{equation*}
$$



Figure 4.2: Rotation of a linearly polarised beam in an alignment-based magnetometer in the laboratory frame with a precessing atomic polarisation. The light is linearly polarised along the $z$-axis before the vapour cell and propagates along the $y$-axis, represented by the propagation vector $\mathbf{k}$. The static magnetic field $\mathbf{B}_{0}$ is directed along the $z$-axis and an oscillating magnetic field $\mathbf{B}_{\mathrm{RF}}(t)$ is applied along the $x$-axis. The electric field vector $\mathbf{E}_{0}$ is rotated after the cell. The axis of the angular momentum probability surface can be considered to be the polarisation of light transmitted through the cell. Due to the Larmor precession the outgoing polarisation will oscillate.
along with the electric field vector $\mathbf{E}$ (Auzinsh et al., 2014)

$$
\begin{align*}
\mathbf{E}(\mathbf{r}, t)= & \operatorname{Re}\left[E_{0} e^{i(\mathbf{k} \cdot \mathbf{r}+\alpha)} e^{-\omega t}[(\cos \phi \cos \epsilon-i \sin \phi \sin \epsilon) \hat{\mathbf{z}}\right.  \tag{4.37}\\
& +(\sin \phi \cos \epsilon+i \cos \phi \sin \epsilon) \hat{\mathbf{x}}]],
\end{align*}
$$

where $\epsilon$ is the ellipticity of the light. If Eqs. 4.37 and 4.35 are subbed into Eq. 4.36 and some approximations are made (Auzinsh et al., 2014), the rate of change of the angle per unit length

$$
\begin{equation*}
\frac{\partial \phi}{\partial l}=\frac{2 \pi \omega}{\varepsilon_{0} c} P_{4} \tag{4.38}
\end{equation*}
$$

can be found. To extract $P_{4}$, we must now calculate the polarisation of the medium induced by the light, given by

$$
\begin{equation*}
\mathbf{P}=n \operatorname{Tr}[\rho \mathbf{d}], \tag{4.39}
\end{equation*}
$$

where $n$ is the number density, $\rho$ is the density matrix in the lab frame and $\mathbf{d}=$ $d_{x} \hat{\mathbf{x}}+d_{y} \hat{\mathbf{y}}+d_{z} \hat{\mathbf{z}}$ is the dipole moment operator. From Sec. 2.5 the dipole moment
operators $d_{x}, d_{y}$ and $d_{z}$ with a $z$-quantisation axis are

$$
\begin{align*}
& d_{x}=\frac{d_{-1}-d_{1}}{\sqrt{2}}=\left(\begin{array}{cccc}
0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
-1 & 0 & 1 & 0
\end{array}\right) \frac{\langle 1||d|\left|0^{\prime}\right\rangle}{\sqrt{6}},  \tag{4.40}\\
& d_{y}=-i \frac{\left(d_{-1}+d_{1}\right)}{\sqrt{2}}=-i\left(\begin{array}{cccc}
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
-1 & 0 & -1 & 0
\end{array}\right) \frac{\langle 1||d|\left|0^{\prime}\right\rangle}{\sqrt{6}},  \tag{4.41}\\
& d_{z}=d_{0}=\left(\begin{array}{llll}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{array}\right) \frac{\langle 1||d|\left|0^{\prime}\right\rangle}{\sqrt{3}} . \tag{4.42}
\end{align*}
$$

The density matrix $\rho$ in the lab frame will be written in terms of the rotating frame density matrix $\tilde{\rho}$ in Eq. 4.29. The density matrix in the rotating frame is rotated back to the lab frame by the relation $\rho=U U_{\mathrm{RF}} \tilde{\rho} U_{\mathrm{RF}}^{\dagger} U^{\dagger}$, where $U$ and $U_{\mathrm{RF}}$ are defined in Eqs. 4.17 and 4.23, respectively. The density matrix in the lab frame is

$$
\rho=\left(\begin{array}{cccc}
\tilde{\rho}_{1,1} & \tilde{\rho}_{1,0} e^{-i \omega_{\mathrm{RF}} t} & \tilde{\rho}_{1,-1} e^{-i 2 \omega_{\mathrm{RF}} t} & \tilde{\rho}_{1,0^{\prime}} e^{-i \omega t} e^{-i \omega_{\mathrm{RF}} t}  \tag{4.43}\\
\tilde{\rho}_{0,1} e^{i \omega_{\mathrm{RF}} t} & \tilde{\rho}_{0,0} & \tilde{\rho}_{0,-1} e^{-i \omega_{\mathrm{RF}} t} & \tilde{\rho}_{0,0^{\prime}} e^{-i \omega t} \\
\tilde{\rho}_{-1,1} e^{i \omega_{\mathrm{RF}} t} & \tilde{\rho}_{-1,0} e^{i \omega_{\mathrm{RF}} t} & \tilde{\rho}_{-1,-1} & \tilde{\rho}_{-1,0^{\prime}} e^{-i \omega t} e^{i \omega_{\mathrm{RF}} t} \\
\tilde{\rho}_{0^{\prime}, 1} e^{i \omega_{\mathrm{RF}} t} e^{i \omega t} & \tilde{\rho}_{0^{\prime}, 0} e^{i \omega t} & \tilde{\rho}_{0^{\prime},-1} e^{-i \omega_{\mathrm{RF}} t} e^{i \omega t} & \tilde{\rho}_{0^{\prime}, 0^{\prime}}
\end{array}\right) .
$$

Using $\tilde{\rho}_{0^{\prime}, 1} e^{i \omega t} e^{i \omega_{\mathrm{RF}} t}=\tilde{\rho}_{1,0^{\prime}} e^{-i \omega t} e^{-i \omega_{\mathrm{RF}} t}$ (Auzinsh et al., 2014), the trace elements $\operatorname{Tr}\left(\rho d_{x}\right), \operatorname{Tr}\left(\rho d_{y}\right)$ and $\operatorname{Tr}\left(\rho d_{z}\right)$ are calculated to be

$$
\begin{align*}
\operatorname{Tr}\left(\rho d_{x}\right)= & \frac{\langle 1||d|\left|0^{\prime}\right\rangle}{\sqrt{6}}\left(-\tilde{\rho}_{1,0^{\prime}} e^{-i \omega t} e^{-i \omega_{\mathrm{RF}} t}+\tilde{\rho}_{-1,0^{\prime}} e^{-i \omega t} e^{i \omega_{\mathrm{RF}} t}-\tilde{\rho}_{0^{\prime}, 1} e^{i \omega_{\mathrm{RF}} t} e^{i \omega t}\right. \\
& \left.+\tilde{\rho}_{0^{\prime},-1} e^{-i \omega_{\mathrm{RF}} t} e^{i \omega t}\right)  \tag{4.44}\\
= & \sqrt{\frac{2}{3}}\langle 1||d|\left|0^{\prime}\right\rangle e^{-i \omega t}\left(\tilde{\rho}_{-1,0^{\prime}} e^{i \omega_{\mathrm{RF}} t}-\tilde{\rho}_{1,0^{\prime}} e^{-i \omega_{\mathrm{RF}} t}\right), \\
\operatorname{Tr}\left(\rho d_{y}\right)= & -i \frac{\langle 1|\left|d \| 0^{\prime}\right\rangle}{\sqrt{6}}\left(-\tilde{\rho}_{1,0^{\prime}} e^{-i \omega t} e^{-i \omega_{\mathrm{RF}} t}-\tilde{\rho}_{-1,0^{\prime}} e^{-i \omega t} e^{i \omega_{\mathrm{RF}} t}\right. \\
& \left.+\tilde{\rho}_{0^{\prime}, 1} e^{i \omega_{\mathrm{RF}} t} e^{i \omega t}+\tilde{\rho}_{0^{\prime},-1} e^{-i \omega_{\mathrm{RF}} t} e^{i \omega t}\right)  \tag{4.45}\\
= & 0, \\
\operatorname{Tr}\left(\rho d_{z}\right)= & \frac{\langle 1||d|\left|0^{\prime}\right\rangle}{\sqrt{3}}\left(\tilde{\rho}_{0,0^{\prime}} e^{-i \omega t}+\tilde{\rho}_{0^{\prime}, 0} e^{i \omega t}\right)=\frac{2}{\sqrt{3}}\left\langle 1\|d\| 0^{\prime}\right\rangle e^{-i \omega t} \tilde{\rho}_{0,0^{\prime}} . \tag{4.46}
\end{align*}
$$

The polarisation $\mathbf{P}$ of the medium induced by the light is therefore

$$
\begin{equation*}
\mathbf{P}=n \sqrt{\frac{2}{3}}\langle 1||d|\left|0^{\prime}\right\rangle \operatorname{Re}\left[e^{-i \omega t}\left(\sqrt{2} \tilde{\rho}_{0,0^{\prime}} \hat{\mathbf{z}}+\left(\tilde{\rho}_{-1,0^{\prime}} e^{i \omega_{\mathrm{RF}} t}-\tilde{\rho}_{1,0^{\prime}} e^{-i \omega_{\mathrm{RF}} t}\right) \hat{\mathbf{x}}\right)\right] \tag{4.47}
\end{equation*}
$$

Comparing this with

$$
\begin{equation*}
\left.\mathbf{P}=\operatorname{Re}\left[e^{i(\mathbf{k} \cdot \mathbf{r}+\alpha)} e^{-i \omega t}\left[\left(P_{1}-i P_{2}\right) \hat{\mathbf{z}}+\left(P_{3}-i P_{4}\right) \hat{\mathbf{x}}\right)\right]\right] \tag{4.48}
\end{equation*}
$$

we then see that, in the lab frame,

$$
\begin{equation*}
P_{4}=-n \sqrt{\frac{2}{3}}\langle 1||d|\left|0^{\prime}\right\rangle \operatorname{Im}\left[\tilde{\rho}_{-1,0^{\prime}} e^{i \omega_{\mathrm{RF}} t}-\tilde{\rho}_{1,0^{\prime}} e^{-i \omega_{\mathrm{RF}} t}\right] . \tag{4.49}
\end{equation*}
$$

To get the form in (Zigdon et al., 2010) we assume $\tilde{\rho}_{-1,0^{\prime}} e^{i \omega_{\mathrm{RF}} t}=\tilde{\rho}_{0^{\prime},-1} e^{-i \omega_{\mathrm{RF}} t}$. It is important to get rid of the time-dependence of the RF field, which is done by going to a rotating frame. We can write $e^{ \pm i \omega_{\mathrm{RF}} t}=\cos \omega_{\mathrm{RF}} t \pm i \sin \omega_{\mathrm{RF}} t$ such that

$$
\begin{equation*}
P_{4}=-n \sqrt{\frac{2}{3}}\langle 1||d|\left|0^{\prime}\right\rangle \operatorname{Im}\left[\cos \omega_{\mathrm{RF}} t\left(\tilde{\rho}_{0^{\prime},-1}-\tilde{\rho}_{0^{\prime}, 1}\right)-i \sin \omega_{\operatorname{RF}} t\left(\tilde{\rho}_{0^{\prime},-1}+\tilde{\rho}_{0^{\prime}, 1}\right)\right] . \tag{4.50}
\end{equation*}
$$

With a lock-in detector with a reference signal oscillating at $\omega_{\mathrm{ref}}=\omega_{\mathrm{RF}}$, the in-phase and out-of-phase components are therefore

$$
\begin{align*}
\frac{\partial \phi^{\text {in }}}{\partial l} & \propto-\operatorname{Im}\left(\tilde{\rho}_{0^{\prime},-1}-\tilde{\rho}_{0^{\prime}, 1}\right),  \tag{4.51}\\
\frac{\partial \phi^{\text {out }}}{\partial l} & \propto \operatorname{Re}\left(\tilde{\rho}_{0^{\prime},-1}+\tilde{\rho}_{0^{\prime}, 1}\right) . \tag{4.52}
\end{align*}
$$

The terms $\tilde{\rho}_{-1,0^{\prime}}$ and $\tilde{\rho}_{1,0^{\prime}}$ are optical coherences between the ground state magnetic sublevels $|1,-1\rangle$ and $|1,1\rangle$ and the excited state $\left|0^{\prime}, 0^{\prime}\right\rangle$, respectively. The terms for $\tilde{\rho}_{-1,0^{\prime}}$ and $\tilde{\rho}_{1,0^{\prime}}$, calculated from the AtomicDensityMatrix package (Zigdon et al., 2010, Rochester, 2023), are substituted into Eqs. 4.51 and 4.52 . This allows for expressions for the rate of change of optical rotation $\partial \phi / \partial l$ with respect to the length $l$ of the vapour cell in the rotating frame to be obtained. The expressions for the in-phase $\partial \phi^{\text {in }} / \partial l$ and quadrature $\partial \phi^{\text {out }} / \partial l$ values are

$$
\begin{align*}
\frac{\partial \phi^{\text {in }}}{\partial l} & =\frac{n \Delta_{\mathrm{RF}} \lambda^{2} \Omega_{\mathrm{RF}}\left(2 \gamma^{2}+8 \Delta_{\mathrm{RF}}^{2}-\Omega_{\mathrm{RF}}^{2}\right) \Omega_{R}^{2}}{36 \pi \Gamma \gamma\left(\gamma^{2}+4 \Delta_{\mathrm{RF}}^{2}+\Omega_{\mathrm{RF}}^{2}\right)\left[4\left(\gamma^{2}+\Delta_{\mathrm{RF}}^{2}\right)+\Omega_{\mathrm{RF}}^{2}\right]}  \tag{4.53}\\
& \approx \frac{n \lambda^{2}}{72 \pi} \cdot \frac{\Omega_{R}^{2}}{\Gamma} \cdot \Omega_{\mathrm{RF}} \cdot \frac{\Delta_{\mathrm{RF}} / \gamma}{\Delta_{\mathrm{RF}}^{2}+\gamma^{2}} \text { for } \Omega_{\mathrm{RF}}^{2} \ll \gamma^{2}, \\
\frac{\partial \phi^{\text {out }}}{\partial l} & =\frac{n \lambda^{2} \Omega_{\mathrm{RF}}\left(4 \gamma^{2}+16 \Delta_{\mathrm{RF}}^{2}+\Omega_{\mathrm{RF}}^{2}\right) \Omega_{R}^{2}}{72 \pi \Gamma\left(\gamma^{2}+4 \Delta_{\mathrm{RF}}^{2}+\Omega_{\mathrm{RF}}^{2}\right)\left[4\left(\gamma^{2}+\Delta_{\mathrm{RF}}^{2}\right)+\Omega_{\mathrm{RF}}^{2}\right]}  \tag{4.54}\\
& \approx \frac{n \lambda^{2}}{72 \pi} \cdot \frac{\Omega_{R}^{2}}{\Gamma} \cdot \Omega_{\mathrm{RF}} \cdot \frac{1}{\Delta_{\mathrm{RF}}^{2}+\gamma^{2}} \text { for } \Omega_{\mathrm{RF}}^{2} \ll \gamma^{2},
\end{align*}
$$

where $n$ is the Cs atomic density, $\Delta_{\mathrm{RF}}=\omega_{\mathrm{RF}}-\omega_{L}$ is the detuning of the RF frequency $\omega_{\mathrm{RF}}$ from the Larmor frequency $\omega_{L}, \lambda$ is the transition wavelength, $\Omega_{R}$ is the Rabi frequency of the light, $\Gamma$ is the decay rate of the excited state, $\Omega_{\mathrm{RF}}=g_{F} \mu_{B} B_{\mathrm{RF}} / \hbar$ is the strength of the RF field ( 20 nT corresponds to $\Omega_{\mathrm{RF}}=2 \pi(70 \mathrm{~Hz})$ ) and $\gamma$ is the transverse relaxation rate.

In the limit when $\Omega_{\mathrm{RF}}^{2} \ll \gamma^{2}$, as is the case throughout this thesis apart from Appendix F.1, $\partial \phi^{\text {in }} / \partial l$ and $\partial \phi^{\text {out }} / \partial l$ are proportional to the RF magnetic field amplitude, i.e., $B_{\mathrm{RF}} \propto \Omega_{\mathrm{RF}}$, and have dispersive- and absorptive-Lorentzian lineshapes, respectively, when varying the RF detuning $\Delta_{\mathrm{RF}}$. The light polarisation rotation


Figure 4.3: Alignment-based magnetometer theoretical results when $\Omega_{\mathrm{RF}}<\gamma$. (a) $\partial \phi^{\text {in }} / \partial l(Y)$ and $\partial \phi^{\text {out }} / \partial l(X)$ from Eqs. 4.53 and 4.54 are plotted as a function of $\Delta_{\mathrm{RF}}$ for three different RF amplitudes. (b) The magnitude $R=\sqrt{X^{2}+Y^{2}}$ is plotted as a function of $\Delta_{\mathrm{RF}}$ for several more RF amplitudes. (c) The maximum signal in $R$ from (b) is plotted as a function of $B_{\text {RF }}$. (d) The FWHM of $R$ is plotted as a function of $B_{\mathrm{RF}}$.
is measured using a balanced photodetector and lock-in detection, yielding the inphase $X$ and out-of-phase $Y$ lock-in outputs, which can be written as

$$
\begin{align*}
& X \propto \frac{\partial \phi^{\text {out }}}{\partial l} \propto B_{\mathrm{RF}} \cdot \frac{1}{\left(\omega_{\mathrm{RF}}-\omega_{L}\right)^{2}+\gamma^{2}},  \tag{4.55}\\
& Y \propto \frac{\partial \phi^{\mathrm{in}}}{\partial l} \propto B_{\mathrm{RF}} \cdot \frac{\left(\omega_{\mathrm{RF}}-\omega_{L}\right) / \gamma}{\left(\omega_{\mathrm{RF}}-\omega_{L}\right)^{2}+\gamma^{2}},  \tag{4.56}\\
& R=\sqrt{X^{2}+Y^{2}}=|X+i Y| \propto B_{\mathrm{RF}} \cdot\left|\frac{1+i\left(\omega_{\mathrm{RF}}-\omega_{L}\right) / \gamma}{\left(\omega_{\mathrm{RF}}-\omega_{L}\right)^{2}+\gamma^{2}}\right| . \tag{4.57}
\end{align*}
$$

To understand Eqs. 4.53 and $4.54, \partial \phi^{\text {in }} / \partial l \propto Y$ and $\partial \phi^{\text {out }} / \partial l \propto X$ are plotted as a function of the RF detuning $\Delta_{\mathrm{RF}}$ in Fig. 4.3 a for three different RF amplitudes: $2 \mathrm{nT}\left(\Omega_{\mathrm{RF}}=2 \pi(7 \mathrm{~Hz})\right), 8 \mathrm{nT}\left(\Omega_{\mathrm{RF}}=2 \pi(14 \mathrm{~Hz})\right), 14 \mathrm{nT}\left(\Omega_{\mathrm{RF}}=2 \pi(20 \mathrm{~Hz})\right)$. The magnitude $R=\sqrt{X^{2}+Y^{2}}$ is plotted in Fig. 4.3b for more RF amplitudes. The max signal in $R$ is plotted as a function of $B_{\mathrm{RF}}$ in Fig. 4.3 d and the FWHM of $R$ is plotted as a function of $B_{\mathrm{RF}}$ in Fig. 4.3d. This data is observed experimentally in

Sec. 7. Equations 4.53 and 4.54 are also valid when $\gamma<\Omega_{\mathrm{RF}}<\omega_{L}$. This is explored in Appendix F.1, but is not included here as experimentally we generally want $X, Y \propto B_{\mathrm{RF}}$, which is only true when $\Omega_{\mathrm{RF}} \ll \gamma$. The amplitudes of $\partial \phi^{\mathrm{in}} / \partial l \propto Y$ and $\partial \phi^{\text {out }} / \partial l \propto X$ are normalised to 1 , because the theory here is valid for a nonphysical $F=1 \rightarrow F^{\prime}=0$ transition. It is therefore not possible to predict the expected rotation of the linearly polarised light for a Cs $F=4 \rightarrow F^{\prime}=3$ transition. Obtaining correct values of $\partial \phi / \partial l$ for the $F=4 \rightarrow F^{\prime}=3$ transition would require further work to generalise to larger $F$ and $F^{\prime}$ values. Experimentally, the angle of rotation $\phi$ is $\ll 1 \mathrm{rad}$. For example, in the work done by Ledbetter et al. (2007), a rotation of $\phi \sim 0.3 \mathrm{mrad}$ was measured with a large spherical paraffin-coated vapour cell with a 3.5 cm diameter, i.e., $\partial \phi / \partial l \sim 0.0086 \mathrm{rad} / \mathrm{m}$.

As depicted in Fig. 4.2, the transmission axis of the light is along the alignment axis of the angular momentum probability surface. As the $m=0$ state has a smaller population than the $m= \pm 1$ states, then the $z$-polarised light will be transmitted whereas $x$-polarised light will be absorbed. This phenomenon is described as linear dichroism, where orthogonal polarisation components of the light have different absorption coefficients. The angular momentum probability surfaces are plotted in the lab frame in Fig. 4.4, where a small RF field $\Omega_{\mathrm{RF}} \ll \gamma$ is applied along the $x$-axis and a static magnetic field is applied along the $z$-axis.

(a)

(e)

(b)

(f)

(c)

(g)

(d)

(h)

Figure 4.4: Aligned atomic polariation undergoing Larmor precession in a static magnetic field $\mathbf{B}_{0}=B_{0} \hat{\mathbf{z}}$ and a small oscillating magnetic field $\mathbf{B}_{\mathrm{RF}}(t)=B_{\mathrm{RF}}(t) \hat{\mathbf{x}}$, where $\Omega_{\mathrm{RF}} \ll \gamma$. Snapshots at different times are taken: (a) $t=0=1 / \nu_{L}$, (b) $t=1 /\left(8 \nu_{L}\right)$, (c) $t=2 /\left(8 \nu_{L}\right)$, (d) $t=3 /\left(8 \nu_{L}\right)$, (e) $t=4 /\left(8 \nu_{L}\right)$, (f) $t=5 /\left(8 \nu_{L}\right)$, (g) $t=6 /\left(8 \nu_{L}\right)$, (h) $t=7 /\left(8 \nu_{L}\right)$. The angles are exaggerated to illustrate the precession of the polarisation.

The aligned state precesses around the $z$-axis in the lab frame. Whenever the polarisation tilts into the $x-z$ plane, optical rotation can be induced in the light by the atoms, i.e., at $t=0$ in Fig.4.4a no optical rotation is induced in the light, i.e., the outgoing light from the vapour cell remains $z$-polarised. At $t=1 /\left(8 \nu_{L}\right)$, however, a
small amount of optical rotation will be induced in the light. The maximum optical rotation, i.e., $\theta=\left|\theta_{\max }\right|$ will be generated at $t=1 /\left(4 \nu_{L}\right)$ and $t=3 /\left(4 \nu_{L}\right)$, where the maximum angle between the axis of the aligned state in the $x-z$ plane and the $z$-axis is subtended. The Larmor precession leads to an oscillating rotation of the linearly polarised beam. It is the demodulation of this oscillating optical rotation using a lock-in amplifier which leads to Eqs. 4.53 and 4.54. The theory presented in this chapter provides an excellent framework for understanding how alignmentbased magnetometers work and is the theory that underpins all the experimental work that is presented in Sec. 7 .

## Chapter 5

## Eddy current measurements theory

### 5.1 Motivation

When an electrically conductive object is placed in a primary oscillating magnetic field $\mathbf{B}_{1}(t)$, eddy currents are induced in the object, which in turn produce a secondary magnetic field $\mathbf{B}_{\text {ec }}(t)$ that can be measured (Wait, 1951; Luquire et al., 1970). These measurements, so-called "eddy current measurements", can be used to create a 2D or 3D map of the electrical conductivity of an object. This technique is important for generating a map of the electrical conductivity of the human heart to try and understand the triggers of atrial fibrillation (Jensen et al., 2019; Marmugi and Renzoni, 2016) and in the diagnosis of brain haemorrhages through magnetic induction tomography (MIT) (Klein et al., 2020; Wei and Soleimani, 2012; Luquire et al., 1970; Ma and Soleimani, 2017; Mamatjan, 2014). In the latter area of research, typical experiments involve electrically conductive objects being imaged whilst surrounded by excitation coils and magnetometers (Mamatjan, 2014, Ma and Soleimani, 2017).

One way that the secondary magnetic field $\mathbf{B}_{\text {ec }}$ is calculated is with the use of Maxwell's equations (Bidinosti et al., 2007; Honke and Bidinosti, 2018; Nagel, 2018b). Another approach that can be used to calculate the secondary field was demonstrated by Griffiths et al. (1999). They developed an expression for the secondary magnetic field induced in a solid disk using an approach whereby individual current loops are calculated. This approach forms the basis of our calculations due to its intuitive nature, which we generalise to the arbitrary 3D positioning of an electrically conductive sphere and a magnetometer.

Our theory is used to predict expected signals from MIT of the heart measurements using OPMs, taking technical considerations of the OPM into account. We use feasible parameters based on the experimental setups from (Jensen et al., 2019; Deans et al., 2020), where salt-water containers with low electrical conductivities were imaged. In the work done by Jensen et al. (2019) the excitation coil producing the oscillating magnetic field $\mathbf{B}_{1}(t)$ and the vapour cell are on the same side of the salt-water container, as depicted in Fig. 5.1a. In the work by Deans et al. (2020) the excitation coil is on the other side of the salt-water container from the OPM, as depicted in Fig. 5.1b, Note that in both of these experimental setups there was a compensation coil, which is required for the optimal operation of the OPM.


Figure 5.1: Two potential diagnosis methods. In both scenarios the excitation coil (black ellipse, notated as " E ") produces $\mathbf{B}_{1}(t)$ and will induce eddy currents in the heart. The compensation coil (red ellipse, notated as "C") produces $\mathbf{B}_{2}(t)$, such that the total field measured by the OPM is only the induced magnetic field $\mathbf{B}_{\text {ec }}(t)$, i.e., $\mathbf{B}_{\text {tot }}(t)=\mathbf{B}_{1}(t)+\mathbf{B}_{2}(t)+\mathbf{B}_{\mathrm{ec}}(t)=\mathbf{B}_{\text {ec }}(t)$. The compensation coil is far from the heart so as to induce minimal eddy currents. (a) The excitation coil is on the same side of the heart as the vapour cell. (b) The excitation coil is on the other side of the heart from the vapour cell. The compensation coil in this case could be a Helmholtz coil to minimise magnetic field inhomogeneities.

We begin in Sec. 5.2 by predicting the "on-axis" induced magnetic fields in a low-conductivity ( $\sigma=1 \mathrm{~S} / \mathrm{m}$ ) cylinder. Feasible parameters are used for MIT of the heart measurements using OPMs. The theoretical results are verified using COMSOL, a software package for physics simulations. A detailed description of the COMSOL simulations can be found in the work done by Elson et al. (2022). In Sec. 5.2 .2 and Sec. 5.3 the induced magnetic fields in a sphere for both on-axis and off-axis cases are predicted, respectively. The off-axis case, in particular, will likely prove useful in future MIT of the heart measurements with arrays of OPMs, and also for the diagnosis of brain haemorrhages using MIT (Klein et al., 2020; Wei and Soleimani, 2012; Luquire et al., 1970; Ma and Soleimani, 2017; Mamatjan, 2014). Heatmaps of the induced magnetic fields are presented for the off-axis case, which are also very applicable to the remote sensing measurements performed in Sec. 6 . We use these heatmaps to discuss optimal MIT setups using OPMs.

The theory for eddy current measurements by Bidinosti et al. (2007) and Honke and Bidinosti (2018) is also presented, which is valid for the detection of metallic objects with high electrical conductivities. Our theory is only valid in the low frequency limit. Our off-axis theory is verified by comparing it to the theory of Bidinosti et al. (2007) and to COMSOL simulations. Finally, in Appendix A a simple machine learning programme was developed to distinguish between a sphere with and without a defect. This will become especially important in years' time when advanced techniques will have to be used to determine defects in the electrical conductivity of the human heart of patients suffering from atrial fibrillation.

### 5.2 Determining the induced magnetic field in an on-axis, electrically conductive solid cylinder

The vector potential $\mathbf{A}_{1}(\mathbf{r})$ of a magnetic dipole is given by

$$
\begin{equation*}
\mathbf{A}_{1}(\mathbf{r})=\frac{\mu_{0}}{4 \pi r^{2}} \frac{\mathbf{m} \times \mathbf{r}}{r} \tag{5.1}
\end{equation*}
$$

where $\mathbf{m}=I_{0} e^{i \omega t^{\prime}} \pi r_{w}^{2} \hat{\mathbf{x}}$ is the magnetic moment of the primary coil aligned along the $x$-axis (see Fig. 5.2), r is the vector from the centre of the primary coil to a point $P$ in space (see Fig. 5.2), $I_{0}$ is the amplitude of the current flowing through the primary coil and $\omega$ is the frequency of the alternating current in the coil. Note that in the rest of this thesis $\omega=\omega_{\mathrm{RF}}$, however as the optical frequency, notated normally by $\omega$, is not relevant to this chapter, we use $\omega$ to describe the frequency of the RF field. A coil can be treated as a magnetic dipole when the radius $r_{w}$ of the coil is much less than $r$. The strength of the magnetic field generated by the


Figure 5.2: An illustration of the primary magnetic field (red stream lines) generated by a magnetic dipole. An induced eddy current $d I_{\text {ec }}$ with a radius $\rho^{\prime}$ in an electrically conductive cylinder of radius $\rho$ is included. The circular eddy current will flow in such a direction that the secondary magnetic field it produces will be in the opposite direction to the change in flux of the primary field. The distance $a$ from the centre of the coil to the centre of the cylinder is included, as well as the distance $r$ from the centre of the coil to a point on the cylinder, where $\theta$ is the angle between the vectors $\mathbf{r}$ and $\mathbf{a}$. The vapour cell, notated by Cs, is near the excitation coil in this example. The compensation coil is not included.
primary coil is given by

$$
\begin{equation*}
\mathbf{B}_{1}(\mathbf{r})=\nabla \times \mathbf{A}_{1}(\mathbf{r})=\frac{\mu_{0}}{4 \pi}\left(\frac{3 \hat{\mathbf{r}}(\mathbf{m} \cdot \hat{\mathbf{r}})-\mathbf{m}}{r^{3}}\right) . \tag{5.2}
\end{equation*}
$$

With the magnetic moment of the primary coil aligned with the $x$-axis and using $\hat{\mathbf{x}}=\cos (\theta) \hat{\mathbf{r}}-\sin (\theta) \hat{\boldsymbol{\theta}}$, where $\theta$ is the angle subtended from the $x$-axis to the point of interest in space (see Fig. 5.2), the equation

$$
\begin{equation*}
\mathbf{B}_{1}(\mathbf{r})=\frac{\mu_{0} m}{4 \pi r^{3}}(2 \cos (\theta) \hat{\mathbf{r}}+\sin (\theta) \hat{\boldsymbol{\theta}})=B_{1, \mathrm{r}} \hat{\mathbf{r}}+B_{1, \theta} \hat{\boldsymbol{\theta}} \tag{5.3}
\end{equation*}
$$

is obtained. Expressions for the magnitudes $B_{1, \mathrm{r}}$ and $B_{1, \theta}$ of a dipole field in spherical coordinates are therefore given by

$$
\begin{align*}
& B_{1, \mathrm{r}}=\frac{\mu_{0} m}{4 \pi r^{3}} 2 \cos \theta,  \tag{5.4}\\
& B_{1, \theta}=\frac{\mu_{0} m}{4 \pi r^{3}} \sin \theta . \tag{5.5}
\end{align*}
$$

The magnetic field along the $x$-direction can be calculated by projecting the magnitudes $B_{1, \mathrm{r}}$ and $B_{1, \theta}$ onto the $x$-axis, which leads to $B_{1, \mathrm{x}}=B_{1, \mathrm{r}} \cos \theta-B_{1, \theta} \sin \theta$, i.e.,

$$
\begin{equation*}
B_{1, \mathrm{x}}=\frac{\mu_{0} m}{4 \pi r^{3}}\left(2 \cos ^{2} \theta-\sin ^{2} \theta\right) \tag{5.6}
\end{equation*}
$$

For this on-axis case, only $B_{1, \mathrm{x}}$ is calculated, not $B_{1, \mathrm{y}}$ and $B_{1, \mathrm{z}}$. To turn the expression into one dependent on the radius $\rho^{\prime}$ of the eddy current and the distance $a$ from the centre of the coil to the centre of the cylinder (see Fig. 5.2), the substitutions $\cos \theta=a / r, \sin \theta=\rho^{\prime} / r$ and $r=\sqrt{a^{2}+\rho^{\prime 2}}$ lead to

$$
\begin{equation*}
B_{1, \mathrm{x}}=\frac{\mu_{0} m}{4 \pi\left(a^{2}+\rho^{\prime 2}\right)^{5 / 2}}\left(2 a^{2}-\rho^{\prime 2}\right) \tag{5.7}
\end{equation*}
$$

Using this expression for $B_{1, \mathrm{x}}\left(\rho^{\prime}\right)$, the induced magnetic field $B_{\mathrm{ec}, \mathrm{x}}$ in a cylinder can be calculated. The first step is to calculate the magnitude of the current $d I_{\text {ec }}$ from one eddy current loop with a radius $\rho^{\prime}$, as illustrated in Fig. 5.2 (Griffiths et al., 1999). This is given by the expression

$$
\begin{equation*}
d I_{\mathrm{ec}}=J\left(\rho^{\prime}\right) d \rho^{\prime} d \tau^{\prime}=\sigma E_{\text {eddy }} d \rho^{\prime} d \tau^{\prime} \tag{5.8}
\end{equation*}
$$

where $J\left(\rho^{\prime}\right)$ is the current density at a radius $\rho^{\prime}$ in the cylinder, $d \rho^{\prime}$ is the width of the eddy current in the radial direction, $d \tau^{\prime}$ is the width of the eddy current in the axial $(x)$ direction, $\sigma$ is the electrical conductivity of the cylinder and $E_{\text {eddy }}$ is the magnitude of the induced electric field around the eddy current loop (which causes a current to flow). The induced e.m.f $\varepsilon$ around the current loop is calculated from Faraday's law by

$$
\begin{equation*}
\varepsilon=-\frac{d \Phi}{d t} \tag{5.9}
\end{equation*}
$$

The induced electric field is then calculated by

$$
\begin{equation*}
E_{\text {eddy }}=-\frac{1}{2 \pi \rho^{\prime}} \frac{d \Phi}{d t} \tag{5.10}
\end{equation*}
$$

where $d \Phi / d t$ is the change in magnetic flux through the eddy current loop of radius $\rho^{\prime}$. The magnetic flux $\Phi$ is given by

$$
\begin{equation*}
\Phi=\int_{0}^{\rho^{\prime}} B_{1, \mathrm{x}}\left(\rho^{\prime}\right) 2 \pi \rho^{\prime} d \rho^{\prime}=\frac{\mu_{0} m}{2} \int_{0}^{\rho^{\prime}} \frac{\left(2 a^{2} \rho^{\prime}-\rho^{\prime 3}\right)}{\left(a^{2}+\rho^{\prime 2}\right)^{5 / 2}} d \rho^{\prime} \tag{5.11}
\end{equation*}
$$

The integral is completed step-by-step in Appendix G. The result is

$$
\begin{equation*}
\Phi\left(\rho^{\prime}\right)=\frac{\mu_{0} m}{2} \frac{\rho^{\prime 2}}{\left(a^{2}+\rho^{\prime 2}\right)^{3 / 2}}=\frac{\mu_{0} I_{0} e^{i \omega t^{\prime}} \pi r_{w}^{2}}{2} \frac{\rho^{\prime 2}}{\left(a^{2}+\rho^{\prime 2}\right)^{3 / 2}} . \tag{5.12}
\end{equation*}
$$

The electric field around the eddy current of radius $\rho^{\prime}$ is therefore given by

$$
\begin{equation*}
E_{\text {eddy }}\left(\rho^{\prime}\right)=-\frac{1}{2 \pi \rho^{\prime}} \frac{d \Phi}{d t}=-i \frac{\omega}{2 \pi \rho^{\prime}} \frac{\mu_{0} I_{0} e^{i \omega t^{\prime}} \pi r_{w}^{2}}{2} \frac{\rho^{\prime 2}}{\left(a^{2}+\rho^{\prime 2}\right)^{3 / 2}} \tag{5.13}
\end{equation*}
$$

The current flowing in the loop of width $d \rho^{\prime}$ in the radial direction and $d \tau^{\prime}$ in the axial direction of the cylinder is

$$
\begin{equation*}
d I_{\mathrm{ec}}\left(\rho^{\prime}\right)=J\left(\rho^{\prime}\right) d \rho^{\prime} d \tau^{\prime}=-i \sigma \frac{\omega}{2 \pi \rho^{\prime}} \frac{\mu_{0} I_{0} e^{i \omega t^{\prime}} \pi r_{w}^{2}}{2} \frac{\rho^{\prime 2}}{\left(a^{2}+\rho^{\prime 2}\right)^{3 / 2}} d \rho^{\prime} d \tau^{\prime} \tag{5.14}
\end{equation*}
$$

Now that the induced current in a radius $\rho^{\prime}$ has been determined, the induced magnetic field $d B_{\mathrm{ec}, \mathrm{x}}\left(\rho^{\prime}\right)$ due to this current loop can be found. The on-axis case of a coil is then used to determine $d B_{\mathrm{ec}, \mathrm{x}}\left(\rho^{\prime}\right)$ such that

$$
\begin{equation*}
d B_{\mathrm{ec}, \mathrm{x}}\left(\rho^{\prime}\right)=d I_{\mathrm{ec}} \frac{\mu_{0} \rho^{\prime 2}}{2\left(\rho^{\prime 2}+d^{2}\right)^{3 / 2}}=-\frac{i \sigma \omega \mu_{0}^{2} I_{0} e^{i \omega t^{\prime}} r_{w}^{2}}{8} \frac{\rho^{\prime 3}}{\left(a^{2}+\rho^{\prime 2}\right)^{3 / 2}\left(\rho^{\prime 2}+d^{2}\right)^{3 / 2}} d \rho^{\prime} d \tau^{\prime} \tag{5.15}
\end{equation*}
$$

where $d$ is the distance from the cylinder to the OPM and $a$ is the distance from the primary coil to the cylinder (see Fig. 5.1).

It is important to realise that $d B_{\mathrm{ec}, \mathrm{x}}$ and $B_{1, \mathrm{x}}$ have different phases, indicated by the imaginary number $i$ in the expression for $d B_{\mathrm{ec}, \mathrm{x}}$. This comes from Faraday's law, where the derivative of the magnetic flux $d \Phi / d t$ is taken. This means that the induced magnetic field is $90^{\circ}$ out of phase with the primary field. This is only true when the thickness $t$ of the cylinder is less than the skin depth $\delta$ of the primary field, which is given by $\left.\delta=\sqrt{2 /\left(\omega \mu_{0} \sigma\right.}\right)$. The human heart ( $\sigma \sim 1 \mathrm{~S} / \mathrm{m}$ ) has a size of roughly $\sim 5 \mathrm{~cm}$ and so Faraday's law, and therefore this theory, is valid for frequencies below $\sim 101 \mathrm{MHz}$. To calculate the total induced magnetic field from every eddy current loop, the cylinder must be integrated along the radial direction in the limits $(0, \rho)$ and along the axial direction in the limits $(0, t)$

$$
\begin{equation*}
B_{\mathrm{ec}, \mathrm{x}}=-\frac{i \sigma \omega \mu_{0}^{2} I_{0} e^{i \omega t^{\prime}} r_{w}^{2}}{8} \int_{0}^{t} d \tau^{\prime} \int_{0}^{\rho} \frac{\rho^{\prime 3}}{\left(a^{2}+\rho^{\prime 2}\right)^{3 / 2}\left(\rho^{\prime 2}+d^{2}\right)^{3 / 2}} d \rho^{\prime} \tag{5.16}
\end{equation*}
$$

If $d=a$ then Eq. 5.16 can be simplified and the on-axis induced magnetic field is calculated to be

$$
\begin{equation*}
B_{\mathrm{ec}, \mathrm{x}}=-\frac{i \sigma \omega \mu_{0}^{2} I_{0} e^{i \omega t^{\prime}} r_{w}^{2}}{32} t \frac{\rho^{4}}{a^{2}\left(a^{2}+\rho^{2}\right)^{2}}=-\frac{i \mu_{0} I_{0} e^{i \omega t^{\prime}} r_{w}^{2} t}{16 \delta^{2}} \frac{\rho^{4}}{a^{2}\left(a^{2}+\rho^{2}\right)^{2}} \tag{5.17}
\end{equation*}
$$

In the limit where $a \gg \rho$, the expression for $B_{\mathrm{ec}, \mathrm{x}}$ will reduce down to

$$
\begin{equation*}
B_{\mathrm{ec}, \mathrm{x}, \mathrm{a} \gg \rho}=-\frac{i \mu_{0} I_{0} e^{i \omega t^{\prime}} r_{w}^{2} t}{16 \delta^{2}} \frac{\rho^{4}}{a^{6}} . \tag{5.18}
\end{equation*}
$$

However in the more general case when $d \neq a$ then the induced magnetic field $B_{\mathrm{ec}, \mathrm{x}}$ is

$$
\begin{equation*}
B_{\mathrm{ec}, \mathrm{x}}=-\frac{i \sigma \omega \mu_{0}^{2} I_{0} e^{i \omega t^{\prime}} r_{w}^{2}}{8} t\left(\frac{a^{2}\left(2 d^{2}+\rho^{2}\right)+d^{2} \rho^{2}}{\left(a^{2}-d^{2}\right)^{2} \sqrt{a^{2}+\rho^{2}} \sqrt{d^{2}+\rho^{2}}}-\frac{2 a d}{\left(a^{2}-d^{2}\right)^{2}}\right) \tag{5.19}
\end{equation*}
$$

### 5.2.1 Analysis

We have therefore calculated the induced on-axis magnetic field $B_{\text {ec, }, \mathrm{x}}$ in a solid electrically conductive cylinder, which is valid when $\nu<1 /\left(\pi \delta^{2} \sigma \mu_{0}\right)$. To verify Eq. 5.19 a situation is considered like in Fig. 5.1a, where the excitation coil is on the same side of the heart as the vapour cell, in a similar layout to the work done by Jensen et al. (2019) and also presented in Sec. 7.3.4 for the detection of an aluminium disk using an alignment-based magnetometer. The full analytical solution $B_{\mathrm{ec}, \mathrm{x}}$ for $d \neq a$ (Eq. 5.19) is plotted in Fig. 5.3a as a function of the separation $a$ between the excitation coil and the cylinder $(d=a+1.7 \mathrm{~cm})$. The distance from the excitation coil to the vapour cell $d-a$ will remain constant. The parameters used are those of an envisaged MIT setup: $\rho=0.5 \mathrm{~cm}, t=0.5 \mathrm{~cm}, d-a=1.7 \mathrm{~cm}, r_{w}=1 \mathrm{~cm}$, $N_{\text {turns }}=20, I_{0}=1 \mathrm{~A}, 1 \mathrm{~cm}<a<20 \mathrm{~cm}, \sigma=1 \mathrm{~S} / \mathrm{m}, \nu=2 \mathrm{MHz}$. The theory was verified by comparing the results to simulations performed with COMSOL in Fig. 5.3a. When the distance $a$ is sufficiently large ( $a>3 \mathrm{~cm}$ ), the COMSOL data agrees with the theory. The discrepancy at smaller distances is expected, because the theory predicts the coil to be a magnetic dipole, and so does not take the finite radius of the excitation coil into account. The experimental setup when $a=3 \mathrm{~cm}$ is shown to scale in Fig. 5.3c. If $r_{w}>1 \mathrm{~cm}$ or $a<3 \mathrm{~cm}$ then the theory and COMSOL will begin to disagree more substantially. The induced magnetic field is $B_{\mathrm{ec}}=1 \mathrm{pT}$ when $a=3 \mathrm{~cm}$ and $B_{\text {ec }}=0.1 \mathrm{pT}$ when $a=5 \mathrm{~cm}$.

When working with OPMs experimentally, it is challenging to detect small induced magnetic fields $\mathbf{B}_{\mathrm{ec}}(t)$. From an experimental/technical perspective, two things limit how well a conductive object can be detected: (1) how small $B_{\text {ec, } x}$ is (for example $B_{\text {ec }}=0.1 \mathrm{pT}$ when $a=5 \mathrm{~cm}$ ) and (2) how small the secondary field is relative to the primary field $B_{\mathrm{ec}, \mathrm{x}} / B_{1, \mathrm{x}}$ (at the OPM position). Experimentally, a "compensation coil" producing an oscillating magnetic field $\mathbf{B}_{2}(t)$ is also included in the setup to null the primary field at the position of the OPM, i.e., $\mathbf{B}_{\text {tot }}(t)=\mathbf{B}_{1}(t)+\mathbf{B}_{2}(t) \sim 0$. This is necessary to ensure that the measured magnetic field by the OPM is $\mathbf{B}_{\text {tot }}=\mathbf{B}_{1}(t)+\mathbf{B}_{2}(t)+\mathbf{B}_{\text {ec }}(t) \sim \mathbf{B}_{\text {ec }}(t)$ during eddy current measurements. However, it is not possible to perfectly cancel the primary magnetic field. In (Jensen et al., 2019) low conductivity objects were detected for $B_{\mathrm{ec}, \mathrm{x}} / B_{1, \mathrm{x}}$ as small as $1 \times 10^{-5}$. In Sec. 6.4 objects were detected with $B_{\text {ec, } \mathrm{x}} / B_{1, \mathrm{x}}=2 \times 10^{-4}$. The measurement system was predicted to detect a ratio as small as $B_{\mathrm{ec}, \mathrm{x}} / B_{1, \mathrm{x}}=2 \times 10^{-5}$. The induced magnetic field $B_{\mathrm{ec}, \mathrm{x}}$ is therefore assumed to be undetectable if $B_{\mathrm{ec}, \mathrm{x}} / B_{1, \mathrm{x}} \leq 10^{-5}$. This experimental limit is included in Fig. 5.3b, The fraction $B_{\mathrm{ec}, \mathrm{x}} / B_{1, \mathrm{x}}$ is plotted for the experimental setups depicted in Figs. 5.3 c and 5.3 d .


Figure 5.3: (a) The magnitude of the induced magnetic field $B_{\text {ec }, \mathrm{x}}$ (Eq. 5.19) is plotted as a function of the separation $a$ between the centre of the coil and the centre of the cylinder. A COMSOL simulation of the setup was performed. Details of the COMSOL settings can be found in (Elson et al., 2022). An experimental limit of $100 \mathrm{fT} / \sqrt{\mathrm{Hz}}$ is included. The parameters used are $\sigma=1 \mathrm{~S} / \mathrm{m}, \nu=2 \mathrm{MHz}$, $t=0.5 \mathrm{~cm}, \rho=0.5 \mathrm{~cm}, x_{\text {coil }}=0 \mathrm{~cm}, m=6.3 \mathrm{mAm}^{2}, x_{\mathrm{OPM}}=-1.7 \mathrm{~cm}$ and $r_{w}=1.0 \mathrm{~cm}$. (b) The fraction $B_{\mathrm{ec}, \mathrm{x}} / B_{1, \mathrm{x}}$ is plotted for two scenarios: one when the OPM is only 1.7 cm from the excitation coil as in (c), i.e., $|d-a|=1.7 \mathrm{~cm}$, and the other when the OPM is $2 a+1.7 \mathrm{~cm}$ as in (d), i.e., $|d-a|=2 a+1.7 \mathrm{~cm}$, from the excitation coil. The experimental limit is around $10^{-5}$, as obtained by Jensen et al. (2019). (c) Experimental setup (to scale) with a 1 cm radius excitation coil at $x=0$, the OPM at $x=-1.7 \mathrm{~cm}$ and the cylinder at $x=3 \mathrm{~cm}$. (d) Experimental setup (to scale) with the OPM at $x=7.7 \mathrm{~cm}$ instead of $x=-1.7 \mathrm{~cm}$.


Figure 5.4: (a) Potential diagnosis method with the excitation coil on the same side of the heart (sphere) as the vapour cell. (b) Potential diagnosis method with the excitation coil on the other side of the heart (sphere) from the vapour cell.

The distances $d$ and $a$ are the same in Figs. 5.3 d and 5.3d, but in Fig. 5.3d the excitation coil is on the other side of the cylinder from the vapour cell, such that the distance between the excitation coil and vapour cell is $d+a$. This contrasts with the setup in Fig. 5.3c when the vapour cell is closer to the excitation coil and the separation between them is $d-a$. As the excitation coil is further from the OPM in Fig. 5.3d, it is easier to cancel $\mathbf{B}_{1}(t)$ with a compensation coil at the position of the OPM, hence making it a more desirable configuration if the main limitation is the inability to extract $\mathbf{B}_{\text {ec }}(t)$ from the (nulled) primary magnetic field $\mathbf{B}_{1}(t)+\mathbf{B}_{2}(t) \sim 0$.

The maximum detectable distance when the OPM is close to the excitation coil is $a=1.2 \mathrm{~cm}$, but when the OPM is far from the excitation coil the maximum detectable distance is $a=3.0 \mathrm{~cm}$. Note that both of these distances are smaller than the experimental limitation if the sensitivity of the OPM is $100 \mathrm{fT} / \sqrt{\mathrm{Hz}}$, which would lead to a maximum detection distance of $a=5.0 \mathrm{~cm}$. Our theory and COMSOL simulations show that the limitation in MIT of the heart using this configuration would be the inability to perfectly null the primary field at the position of the OPM. This shows how the setup in Fig. 5.3d would most likely be the most promising setup for MIT of the heart, as it reduces the effect of the primary magnetic field on the Cs atoms. This setup was used experimentally by Deans et al. (2020). This is the same setup that will be shown in Sec. 6 for the remote detection of conductive objects.

### 5.2.2 Determining the induced magnetic field in an on-axis, electrically conductive sphere

Now that $B_{\text {ec, } \mathrm{x}}$ for a solid cylinder has been found in Sec. 5.2. similar methodology can be used to determine $B_{\text {ec, }, \mathrm{x}}$ that is induced in a solid sphere. A solid sphere can be constructed out of solid cylinders stacked on top of one another, as depicted in Fig. 5.4a. Therefore, the induced magnetic fields from one solid cylinder will be found, after which the solid cylinders can be summed, or integrated, to form a
sphere. To determine $B_{\text {ec, } \mathrm{x}}$ at the position of the OPM, we begin at Eq. 5.15

$$
\begin{equation*}
d B_{\mathrm{ec}, \mathrm{x}}\left(\rho^{\prime}\right)=-\frac{i \sigma \omega \mu_{0}^{2} I_{0} e^{i \omega t^{\prime}} r_{w}^{2}}{8} \frac{\rho^{\prime 3}}{\left(a^{2}+\rho^{\prime 2}\right)^{3 / 2}\left(d^{2}+\rho^{\prime 2}\right)^{3 / 2}} d \rho^{\prime} d \tau^{\prime} \tag{5.20}
\end{equation*}
$$

The sphere will be treated as a stack of cylinders with different radii. Each cylinder now has a thickness $d \tau$ in the axial direction, instead of $t$ as used earlier. The induced magnetic field from a cylinder with axial thickness $d \tau$ is given by

$$
\begin{equation*}
B_{\mathrm{ec}, \mathrm{x}}=-\frac{i \sigma \omega \mu_{0}^{2} I_{0} e^{i \omega t^{\prime}} r_{w}^{2}}{8}\left(\frac{a^{2}\left(2 d^{2}+\rho^{2}\right)+d^{2} \rho^{2}}{\left(a^{2}-d^{2}\right)^{2} \sqrt{a^{2}+\rho^{2}} \sqrt{d^{2}+\rho^{2}}}-\frac{2 a d}{\left(a^{2}-d^{2}\right)^{2}}\right) d \tau \tag{5.21}
\end{equation*}
$$

In the limit that $n_{\text {cyl }} \rightarrow \infty$, the total induced magnetic field is now given by

$$
\begin{array}{r}
B_{\mathrm{ec}, \mathrm{x}}(\rho)=-\int_{-r_{s}}^{r_{s}} \frac{i \sigma \omega \mu_{0}^{2} I_{0} e^{i \omega t^{\prime}} r_{w}^{2}}{8}\left[\frac{a^{2}\left(2 d^{2}+\left(r_{s}^{2}-y^{2}\right)\right)+d^{2}\left(r_{s}^{2}-y^{2}\right)}{\left(a^{2}-d^{2}\right)^{2} \sqrt{a^{2}+r_{s}^{2}-y^{2}} \sqrt{d^{2}+r_{s}^{2}-y^{2}}}\right.  \tag{5.22}\\
\left.-\frac{2 a d}{\left(a^{2}-d^{2}\right)^{2}}\right] d y
\end{array}
$$

which can be solved numerically. The numerical solution will be used in the next section.

### 5.3 Determining the induced magnetic field in an off-axis electrically conductive sphere

Now that the on-axis cases of a conductive cylinder and sphere have been considered, the induced magnetic fields in a sphere that is off-axis to the magnetic moment of the coil and OPM position will be determined. This is depicted in Fig. 5.5. This is important because future potential atrial fibrillation diagnoses would likely be done with multiple OPMs at different positions (Marmugi and Renzoni, 2016). A sphere, not a cylinder like in Sec. 5.2, is used because of the simplification of the 3D problem due to the rotational symmetry of a sphere compared with a cylinder, as will become clear shortly.

The magnetic field $\mathbf{B}_{1}(\mathbf{r})$ generated by the excitation coil, which is positioned at $\mathbf{r}_{0}=(0,0,0)$ and has a magnetic moment $\mathbf{m}$, at the position $\mathbf{r}=x_{s} \hat{\mathbf{x}}+y_{s} \hat{\mathbf{y}}+z_{s} \hat{\mathbf{z}}$ of an electrically conductive sphere is given by

$$
\begin{equation*}
\mathbf{B}_{1}(\mathbf{r})=\frac{\mu_{0}}{4 \pi}\left(\frac{3 \mathbf{r}(\mathbf{m} \cdot \mathbf{r})}{r^{5}}-\frac{\mathbf{m}}{r^{3}}\right) . \tag{5.23}
\end{equation*}
$$

A similar expression can be used to determine the induced magnetic field $\mathbf{B}_{\mathrm{ec}}\left(r^{\prime}\right)$ that is detected at the position of the OPM, given by

$$
\begin{equation*}
\mathbf{B}_{\mathrm{ec}}\left(\mathbf{r}^{\prime}\right)=\frac{\mu_{0}}{4 \pi}\left(\frac{3 \mathbf{r}^{\prime}\left(\mathbf{m}_{\mathrm{ec}} \cdot \mathbf{r}^{\prime}\right)}{r^{\prime 5}}-\frac{\mathbf{m}_{\mathrm{ec}}}{r^{\prime 3}}\right), \tag{5.24}
\end{equation*}
$$

where $\mathbf{r}^{\prime}$ is the vector from the sphere to the OPM and $\mathbf{m}_{\text {ec }}$ is the induced magnetic moment in the sphere.


Figure 5.5: Treating the human heart as an electrically conductive sphere (green circle) at the position $\left(x_{s}, y_{s}, z_{s}\right)$. The excitation coil has a magnetic moment $\mathbf{m}$ oriented along the $x$-axis and is at $(0,0,0)$. The OPM is at the position $\left(x_{\mathrm{OPM}}, y_{\mathrm{OPM}}, z_{\mathrm{OPM}}\right)$. The excitation coil and sphere are connected by the vector $\mathbf{r}$. The OPM and sphere are connected by the vector $\mathbf{r}$ '. The compensation coil in the OPM head is a homogeneous Helmholtz coil.

The direction of the induced magnetic moment $\mathbf{m}_{\mathrm{ec}}$ is equivalent to the direction of $-\mathbf{B}_{1}\left(x_{s}, y_{s}, z_{s}\right)$ at the position of the sphere. This is illustrated in Fig. 5.6 for two different positions of the sphere. It is assumed that the primary field is constant throughout the sphere and is equal to $\mathbf{B}_{1}\left(x_{s}, y_{s}, z_{s}\right)$. This approximation only holds if $r_{s} \ll r^{\prime}$ and $r_{s} \ll r$. The magnitude of the induced magnetic moment of an eddy current loop $d m_{\text {ec }}$ of radius $\rho^{\prime}$ (see Fig. 5.2) is given by

$$
\begin{equation*}
d m_{\mathrm{ec}}=d I_{\mathrm{ec}} \pi \rho^{\prime 2} \tag{5.25}
\end{equation*}
$$

where $d I_{\text {ec }}\left(\rho^{\prime}\right)$ is the induced eddy current with a radius $\rho^{\prime}$ and is given by

$$
\begin{equation*}
d I_{\mathrm{ec}}\left(\rho^{\prime}\right)=J\left(\rho^{\prime}\right) d \rho^{\prime} d \tau^{\prime}=\sigma E_{\text {eddy }}\left(\rho^{\prime}\right) d \rho^{\prime} d \tau^{\prime} \tag{5.26}
\end{equation*}
$$

where $\sigma$ is the conductivity of the sphere, $J\left(\rho^{\prime}\right)$ is the current density at a radius $\rho^{\prime}$, $d \rho^{\prime}$ is the width of the eddy current in the radial direction, $d \tau^{\prime}$ is the width of the eddy current in the axial direction and $E_{\text {eddy }}\left(\rho^{\prime}\right)$ is the induced electric field around the current loop. $E_{\text {eddy }}\left(\rho^{\prime}\right)$ can be calculated by

$$
\begin{equation*}
E_{\text {eddy }}\left(\rho^{\prime}\right)=-\frac{1}{2 \pi \rho^{\prime}} \frac{d \Phi}{d t}, \tag{5.27}
\end{equation*}
$$

where $\Phi$ is the flux that passes through an eddy current of radius $\rho^{\prime}$. The flux $\Phi$ through a ring of radius $\rho^{\prime}$ therefore needs to be calculated. As the object under consideration is a sphere, this means that the sphere can be considered to consist of $n_{\text {cyl }}$ cylinders, all with a thickness $s$ in the axial direction. The cylinders have different radii ranging from $\rho^{\prime}=0$, up to $\rho^{\prime}=r_{s}$ and back to $\rho^{\prime}=0$. These constitute a sphere when stacked upon each other (see Fig. 5.4). This means that the normal vector from each cylinder $\hat{\mathbf{n}}$ can be treated as parallel to the primary magnetic field at the sphere, but in the opposite direction, such that $\hat{\mathbf{n}} \cdot \hat{\mathbf{B}}_{1}=-B_{1}$.

The flux through a ring of radius $\rho^{\prime}$ is given by


Figure 5.6: Diagrams of the direction of the primary magnetic fields (blue streamlines) and the secondary magnetic field induced in a sphere (orange streamlines). The positions of the sphere (red circle), the primary coil (black circle) and the array of OPMs (black squares) are included. (a) The sphere is on-axis. (b) The sphere is off-axis.

$$
\begin{align*}
\Phi & =\int_{0}^{\rho^{\prime}} \mathbf{B}_{1} \cdot \hat{\mathbf{n}} 2 \pi \rho^{\prime} d \rho^{\prime}=-\int_{0}^{\rho^{\prime}} B_{1}\left(x_{s}, y_{s}, z_{s}\right) 2 \pi \rho^{\prime} d \rho^{\prime} \\
& =-\frac{\mu_{0}}{4 \pi} \int_{0}^{\rho^{\prime}} \sqrt{\left(\frac{3 \mathbf{r}(\mathbf{m} \cdot \mathbf{r})}{r^{5}}-\frac{\mathbf{m}}{r^{3}}\right) \cdot\left(\frac{3 \mathbf{r}(\mathbf{m} \cdot \mathbf{r})}{r^{5}}-\frac{\mathbf{m}}{r^{3}}\right)} 2 \pi \rho^{\prime} d \rho^{\prime}  \tag{5.28}\\
& =-\frac{\mu_{0}}{4} \sqrt{\frac{3(\mathbf{m} \cdot \mathbf{r})^{2}}{r^{8}}+\frac{m^{2}}{r^{6}}} \rho^{\prime 2} .
\end{align*}
$$

The induced electric field around a loop of radius $\rho^{\prime}$ is found to be

$$
\begin{align*}
E_{\text {eddy }} & =-\frac{1}{2 \pi \rho^{\prime}} \frac{d \Phi}{d t}=\frac{\mu_{0} \rho^{\prime}}{8 \pi} \frac{d}{d t}\left(e^{i \omega t^{\prime}} \sqrt{\frac{3\left(\hat{\mathbf{m}} \cdot \mathbf{r} m_{c}\right)^{2}}{r^{8}}+\frac{m_{c}^{2}}{r^{6}}}\right)  \tag{5.29}\\
& =\frac{i \omega e^{i \omega t^{\prime}} \mu_{0} \rho^{\prime}}{8 \pi} \sqrt{\frac{3\left(\hat{\mathbf{m}} \cdot \mathbf{r} m_{c}\right)^{2}}{r^{8}}+\frac{m_{c}^{2}}{r^{6}}}
\end{align*}
$$

where $\mathbf{m}=\pi r_{w}^{2} I_{0} e^{i \omega t^{\prime}} \hat{\mathbf{m}}=m_{c} e^{i \omega t^{\prime}} \hat{\mathbf{m}}, r_{w}$ is the radius of the primary coil, $I_{0}$ is the amplitude of the current flowing through the primary coil, and $m_{c}$ is the constant, non time-dependent part of the magnetic moment. The current is oscillating at a frequency $\omega$, so a sinusoidal, time-dependent part is included. The induced current with a radius $\rho^{\prime}$ is therefore

$$
\begin{equation*}
d I_{\mathrm{ec}}\left(\rho^{\prime}\right)=\sigma E_{\text {eddy }}\left(\rho^{\prime}\right) d \rho^{\prime} d \tau^{\prime}=\sigma \frac{i \omega e^{i \omega t^{\prime}} \mu_{0} \rho^{\prime}}{8 \pi} \sqrt{\frac{3\left(\hat{\mathbf{m}} \cdot \mathbf{r} m_{c}\right)^{2}}{r^{8}}+\frac{m_{c}^{2}}{r^{6}}} d \rho^{\prime} d \tau^{\prime} \tag{5.30}
\end{equation*}
$$

The magnitude of the induced magnetic moment $d m_{\text {ec }}$ is

$$
\begin{equation*}
d m_{\mathrm{ec}}=d I_{\mathrm{ec}}\left(\rho^{\prime}\right) \pi \rho^{\prime 2}=\sigma \frac{i \omega e^{i \omega t^{\prime}} \mu_{0} \rho^{\prime 3}}{8} \sqrt{\frac{3\left(\hat{\mathbf{m}} \cdot \mathbf{r} m_{c}\right)^{2}}{r^{8}}+\frac{m_{c}^{2}}{r^{6}}} d \rho^{\prime} d \tau^{\prime} . \tag{5.31}
\end{equation*}
$$

The direction $d \hat{\mathbf{m}}_{\text {ec }}$ is equal to the opposite direction of $\mathbf{B}_{1}$ at the position of the sphere (see Fig. 5.6), given by the coordinates $\left(x_{s}, y_{s}, z_{s}\right)$. The induced magnetic moment $d \mathbf{m}_{\text {ec }}$ is calculated to be

$$
\begin{equation*}
d \mathbf{m}_{\mathrm{ec}}=-d m_{\mathrm{ec}} \hat{\mathbf{B}}_{1}\left(x_{s}, y_{s}, z_{s}\right) \tag{5.32}
\end{equation*}
$$

The induced magnetic field from a singular eddy current loop is

$$
\begin{equation*}
d \mathbf{B}_{\mathrm{ec}}\left(\mathbf{r}^{\prime}\right)=\frac{\mu_{0} d m_{\mathrm{ec}}}{4 \pi}\left(-\frac{3 \mathbf{r}^{\prime}\left(\hat{\mathbf{B}}_{1}\left(x_{s}, y_{s}, z_{s}\right) \cdot \mathbf{r}^{\prime}\right)}{r^{\prime 5}}+\frac{\hat{\mathbf{B}}_{1}\left(x_{s}, y_{s}, z_{s}\right)}{r^{\prime 3}}\right) \tag{5.33}
\end{equation*}
$$

where the vector $\mathbf{r}^{\prime}=\mathbf{r}_{\mathrm{OPM}}-\mathbf{r}_{s}$ is written as

$$
\begin{equation*}
\mathbf{r}^{\prime}=\left(x_{\mathrm{OPM}}-x_{s}\right) \hat{\mathbf{x}}+\left(y_{\mathrm{OPM}}-y_{s}\right) \hat{\mathbf{y}}+\left(z_{\mathrm{OPM}}-z_{s}\right) \hat{\mathbf{z}} \tag{5.34}
\end{equation*}
$$

and the magnitude $r^{\prime}=\sqrt{\left(x_{\mathrm{OPM}}-x_{s}\right)^{2}+\left(y_{\mathrm{OPM}}-y_{s}\right)^{2}+\left(z_{\mathrm{OPM}}-z_{s}\right)^{2}}$.
Now, the only unknown in Eq. 5.33 is the direction vector $\hat{\mathbf{B}}_{1}\left(x_{s}, y_{s}, z_{s}\right)$, which is calculated by

$$
\begin{equation*}
\hat{\mathbf{B}}_{1}\left(x_{s}, y_{s}, z_{s}\right)=\frac{\mathbf{B}_{1}\left(x_{s}, y_{s}, z_{s}\right)}{B_{1}\left(x_{s}, y_{s}, z_{s}\right)} \tag{5.35}
\end{equation*}
$$

Assuming that the magnetic moment of the primary coil is oriented along the positive $x$-direction, then $\mathbf{m}=m \hat{\mathbf{x}}$. Substituting $\mathbf{r}=x_{s} \hat{\mathbf{x}}+y_{s} \hat{\mathbf{y}}+z_{s} \hat{\mathbf{z}}$ and $\mathbf{m}=m \hat{\mathbf{x}}$ into Eq. 5.23, $\mathbf{B}_{1}\left(x_{s}, y_{s}, z_{s}\right)$ is given by

$$
\begin{equation*}
\mathbf{B}_{1}\left(x_{s}, y_{s}, z_{s}\right)=\frac{\mu_{0} m}{4 \pi r^{5}}\left(\left(3 x_{s}^{2}-r^{2}\right) \hat{\mathbf{x}}+3 y_{s} x_{s} \hat{\mathbf{y}}+3 x_{s} z_{s} \hat{\mathbf{z}}\right) \tag{5.36}
\end{equation*}
$$

Therefore,

$$
\begin{equation*}
\hat{\mathbf{B}}_{1}\left(x_{s}, y_{s}, z_{s}\right)=\frac{\left(3 x_{s}^{2}-r^{2}\right) \hat{\mathbf{x}}+3 y_{s} x_{s} \hat{\mathbf{y}}+3 z_{s} x_{s} \hat{\mathbf{z}}}{\sqrt{\left(3 x_{s}^{2}-r^{2}\right)^{2}+9 y_{s}^{2} x_{s}^{2}+9 z_{s}^{2} x_{s}^{2}}} \tag{5.37}
\end{equation*}
$$

A full expression for the induced magnetic field $d \mathbf{B}_{\text {ec }}$ from an eddy current of radius $\rho^{\prime}$ in the sphere has been calculated (Eq. 5.33). Now the total induced magnetic field $\mathbf{B}_{\text {ec }}$ in the sphere needs to be determined. To obtain the total induced magnetic moment $m_{\mathrm{ec}}$ in the sphere, the substitution $\rho^{\prime}=\sqrt{r_{s}^{2}-y^{2}}$ is used, where $y$ is the height of the cylinder as done in Sec. 5.2.2. An equatorial cylinder has a height $y=0$, whilst a cylinder with a radius $\rho^{\prime}=0$ in the sphere will have $y=r_{s}$ (see Fig. 5.4.

For a cylinder, it is of interest to integrate from $\rho^{\prime}=0$ to $\rho^{\prime}=\rho$ and from $\tau^{\prime}=0$ to $\tau^{\prime}=s$, giving rise to

$$
\begin{align*}
m_{\mathrm{ec}, \mathrm{cyl}} & =\sigma \frac{i \omega e^{i \omega t^{\prime}} \mu_{0}}{8} \sqrt{\frac{3\left(\hat{\mathbf{m}} \cdot \mathbf{r} m_{c}\right)^{2}}{r^{8}}+\frac{m_{c}^{2}}{r^{6}}} \int_{0}^{\rho} \rho^{\prime 3} d \rho^{\prime} \int_{0}^{s} d \tau^{\prime}  \tag{5.38}\\
& =\sigma \frac{i \omega e^{i \omega t^{\prime}} \mu_{0}}{8} \sqrt{\frac{3\left(\hat{\mathbf{m}} \cdot \mathbf{r} m_{c}\right)^{2}}{r^{8}}+\frac{m_{c}^{2}}{r^{6}}} \frac{\rho^{4}}{4} s
\end{align*}
$$

The sphere will consist of $n$ cylinders, each with a different radius $\rho_{n}$. The radius $\rho_{n}$ will be rewritten as $\rho_{n}=\sqrt{r_{s}^{2}-y_{n}^{2}}$, where $y_{n}$ is the height of the cylinder in the sphere. The total magnetic moment of all the cylinders that constitute the sphere will therefore be given by

$$
\begin{equation*}
m_{\mathrm{ec}}=-\sum_{n=1}^{n_{\mathrm{cy} 1}} \sigma \frac{i \omega e^{i \omega t^{\prime}} \mu_{0}}{8} \sqrt{\frac{3\left(\hat{\mathbf{m}} \cdot \mathbf{r} m_{c}\right)^{2}}{r^{8}}+\frac{m_{c}^{2}}{r^{6}} \frac{\left(r_{s}^{2}-y_{n}^{2}\right)^{2}}{4}} s \tag{5.39}
\end{equation*}
$$

In the limit that $n_{\mathrm{cyl}} \rightarrow \infty$, then this summation can be rewritten as an integral, such that

$$
\begin{align*}
m_{\mathrm{ec}} & =-\int_{-r_{s}}^{r_{s}} \sigma \frac{i \omega e^{i \omega t^{\prime}} \mu_{0}}{8} \sqrt{\frac{3\left(\hat{\mathbf{m}} \cdot \mathbf{r} m_{c}\right)^{2}}{r^{8}}+\frac{m_{c}^{2}}{r^{6}} \frac{\left(r_{s}^{2}-y^{2}\right)^{2}}{4} d y}  \tag{5.40}\\
& =-\sigma \frac{i \omega e^{i \omega t^{\prime}} \mu_{0}}{32} \sqrt{\frac{3\left(\hat{\mathbf{m}} \cdot \mathbf{r} m_{c}\right)^{2}}{r^{8}}+\frac{m_{c}^{2}}{r^{6}}}\left(\frac{16 r_{s}^{5}}{15}\right) .
\end{align*}
$$

Every vector is now known in $\mathbf{B}_{\text {ec }}$, which is given by

$$
\begin{equation*}
\mathbf{B}_{\mathrm{ec}}\left(\mathbf{r}^{\prime}\right)=\frac{\mu_{0} m_{\mathrm{ec}}}{4 \pi}\left(-\frac{3 \mathbf{r}^{\prime}\left(\hat{\mathbf{B}}_{1}\left(x_{s}, y_{s}, z_{s}\right) \cdot \mathbf{r}^{\prime}\right)}{r^{\prime 5}}+\frac{\hat{\mathbf{B}}_{1}\left(x_{s}, y_{s}, z_{s}\right)}{r^{\prime 3}}\right) . \tag{5.41}
\end{equation*}
$$

The three components $B_{\mathrm{ec}, \mathrm{x}}, B_{\mathrm{ec}, \mathrm{y}}$ and $B_{\mathrm{ec}, \mathrm{z}}$ are determined to be

$$
\begin{align*}
B_{\mathrm{ec}, \mathrm{x}}= & \frac{\mu_{0} m_{\mathrm{ec}}}{4 \pi}\left(-\frac{3\left(x_{\mathrm{OPM}}-x_{s}\right)^{2}\left(3 x_{s}^{2}-r^{2}\right)}{r^{\prime 5} \sqrt{\left(3 x_{s}^{2}-r^{2}\right)^{2}+9 y_{s}^{2} x_{s}^{2}+9 z_{s}^{2} x_{s}^{2}}}\right.  \tag{5.42}\\
& \left.+\frac{3 x_{s}^{2}-r^{2}}{r^{\prime 3} \sqrt{\left(3 x_{s}^{2}-r^{2}\right)^{2}+9 y_{s}^{2} x_{s}^{2}+9 z_{s}^{2} x_{s}^{2}}}\right), \\
B_{\mathrm{ec}, \mathrm{y}}= & \frac{\mu_{0} m_{\mathrm{ec}}}{4 \pi}\left(-\frac{9\left(y_{\mathrm{OPM}}-y_{s}\right)^{2} y_{s} x_{s}}{r^{\prime 5} \sqrt{\left(3 x_{s}^{2}-r^{2}\right)^{2}+9 y_{s}^{2} x_{s}^{2}+9 z_{s}^{2} x_{s}^{2}}}\right.  \tag{5.43}\\
& \left.+\frac{3 y_{s} x_{s}}{r^{\prime 3} \sqrt{\left(3 x_{s}^{2}-r^{2}\right)^{2}+9 y_{s}^{2} x_{s}^{2}+9 z_{s}^{2} x_{s}^{2}}}\right), \\
B_{\mathrm{ec}, \mathrm{Z}}= & \frac{\mu_{0} m_{\mathrm{ec}}}{4 \pi}\left(-\frac{9\left(z_{\mathrm{OPM}}-z_{s}\right)^{2} z_{s} x_{s}}{r^{\prime 5} \sqrt{\left(3 x_{s}^{2}-r^{2}\right)^{2}+9 y_{s}^{2} x_{s}^{2}+9 z_{s}^{2} x_{s}^{2}}}\right.  \tag{5.44}\\
& \left.+\frac{3 z_{s} x_{s}}{r^{\prime 3} \sqrt{\left(3 x_{s}^{2}-r^{2}\right)^{2}+9 y_{s}^{2} x_{s}^{2}+9 z_{s}^{2} x_{s}^{2}}}\right),
\end{align*}
$$

where

$$
\begin{equation*}
m_{\mathrm{ec}}=-\sigma \frac{i \omega e^{i \omega t^{\prime}} \mu_{0}}{32} \sqrt{\frac{3\left(\hat{\mathbf{m}} \cdot \mathbf{r} m_{c}\right)^{2}}{r^{8}}+\frac{m_{c}^{2}}{r^{6}}}\left(\frac{16 r_{s}^{5}}{15}\right) \tag{5.45}
\end{equation*}
$$

and $\left(x_{\text {OPM }}, y_{\text {OPM }}, z_{\text {OPM }}\right)$ is the position of the OPM, $\left(x_{\mathrm{s}}, y_{\mathrm{s}}, z_{\mathrm{s}}\right)$ is the position of the conductive sphere, the coil is at the position $(0,0,0), r=\sqrt{x_{s}^{2}+y_{2}^{2}+z_{s}^{2}}$, $r^{\prime}=\sqrt{\left(x_{\mathrm{OPM}}-x_{s}\right)^{2}+\left(y_{\mathrm{OPM}}-y_{s}\right)^{2}+\left(z_{\mathrm{OPM}}-z_{s}\right)^{2}}$ and $m_{c}=I_{0} \pi r_{w}^{2}$ is the magnetic moment of the primary coil. Here, we have derived the three secondary magnetic fields induced in an electrically conductive sphere of radius $r_{s}$ and electrical conductivity $\sigma$ in the presence of an oscillating magnetic field with a frequency $\omega$.

### 5.3.1 Analysis

To visualise how the components in Eqs. 5.425 .44 vary as a function of the position of the sphere, heatmaps of $B_{\mathrm{ec}, \mathrm{x}}$ and $B_{\mathrm{ec}, \mathrm{y}}$ are plotted in Figs. 5.7a and 5.7b, respectively. The induced magnetic field $B_{\mathrm{ec}, \mathrm{z}}=0$, as the sphere is in the $x-y$ plane in this example. A situation is envisaged where the sphere is between the excitation coil and the OPM array, as depicted in Fig. 5.4b. The OPM and excitation coil are separated by a distance of 10 cm . This distance is used because the first eddy current measurements on a real heart may well be performed in a similar manner to the work done by Jensen et al. (2018), where a guinea pig heartbeat was detected. The excitation coil is envisaged to be on the other side of the heart from the vapour cell (see Fig. 5.5). The conductive sphere is small with a radius $r_{s}=1 \mathrm{~cm}$. The other parameters are $\sigma=1 \mathrm{~S} / \mathrm{m}, \nu=2 \mathrm{MHz}, x_{\mathrm{OPM}}=10 \mathrm{~cm}, x_{\text {coil }}=0 \mathrm{~cm}, m=6.3 \mathrm{mAm}^{2}$, $r_{w}=1 \mathrm{~cm}$. This can be considered to be a more realistic setup for the non-invasive imaging of a human heart than the detection of salt-water containers described in Sec. 5.2.

The polarity of $B_{\mathrm{ec}, \mathrm{x}}$, its magnitude $B_{\mathrm{ec}, \mathrm{x}}$ and ratio $B_{\mathrm{ec}, \mathrm{x}} / B_{1, \mathrm{x}}$ at the position of the OPM are plotted in Fig. 5.7a, with the corresponding data for $B_{\text {ec, }, \text { s }}$ shown in Fig. 5.7b The peak in signal of $B_{\text {ec }, \mathrm{x}}$ occurs when the sphere is close to the OPM or to the excitation coil. The heatmap of $B_{\text {ec,y }}$ is distinctly different, with one key attribute being that when the sphere is on-axis, i.e., $y_{s}=0$, then $B_{\text {ec, } \mathbf{y}}=0$ as expected because the induced magnetic field $\mathbf{B}_{\mathrm{ec}}(t)$ will only have an on-axis component $B_{\text {ec }, \mathrm{x}}$.

To understand how the model in Sec. 5.2 .2 in Eq. 5.19 compares with the model in this section, the two models are compared for the on-axis case, i.e., $y_{s}=0$. When the primary coil and OPM are at the same position, it can be seen in Figs. 5.8a and 5.8 b that the models disagree close to the excitation coil or OPM. The reason for this disagreement at small distances is because the off-axis model, as described here, assumes that the primary field is constant across the sphere. This is a false assumption to make at small distances, as the primary field will vary across the sphere. This is why the model diverges at very small distances (see Figs. 5.8a and 5.8 b ). When the non-constant primary field is taken into account, as is done with the model in Sec. 5.2.2, no divergence is observed at very small distances. The model with a non-constant magnetic field across the object is still not perfect, however, as the primary coil is treated as a magnetic dipole. At small distances, when $r_{w} \approx x_{s}$, both models will not be valid. This is the same assumption as in Sec. 5.4 for the theory presented by Bidinosti et al. (2007).

Using the assumption that the OPM has a noise floor of $100 \mathrm{fT} / \sqrt{\mathrm{Hz}}$ during the eddy current measurements and that $B_{\mathrm{ec}, \mathrm{x}} / B_{1, \mathrm{x}} \geq 10^{-5}$ is necessary, it is predicted that in the on-axis scenario in Figs. 5.8 a and 5.8 b , the limitation on the detection of the low-conductivity object is on $B_{\mathrm{ec}, \mathrm{x}} / B_{1, \mathrm{x}}$ being too small. Even with a $1 \mathrm{pT} / \sqrt{\mathrm{Hz}}$


Figure 5.7: (a) (b) Heatmaps of the induced magnetic fields $B_{\mathrm{ec}, \mathrm{x}}$ and $B_{\mathrm{ec}, \mathrm{y}}$ as a function of the position of the sphere, respectively. The top heatmaps represent the polarity of the induced magnetic fields, with yellow being positive and blue being negative. The middle heatmaps plot $B_{\mathrm{ec}, \mathrm{x}}$ and $B_{\mathrm{ec}, \mathrm{y}}$, respectively. The bottom heatmaps plot $B_{\text {ec }, \mathrm{x}} / B_{1, x}$ and $B_{\mathrm{ec}, \mathrm{y}} / B_{1, x}$, respectively. The OPM in this case is placed at $x_{\text {OPM }}=10 \mathrm{~cm}$ and the coil at $x_{\text {coil }}=0 \mathrm{~cm}$. Other parameters include $\sigma=1 \mathrm{~S} / \mathrm{m}, \nu=2 \mathrm{MHz}, r_{s}=1 \mathrm{~cm}, m=6.3 \mathrm{mAm}^{2}$.
sensitivity the inability to perfectly null $\mathbf{B}_{1}(t)$ will be the limiting factor, according to the parameters picked in this analysis. For this configuration the conductive object can be detected almost everywhere between the coil and OPM, however $B_{\mathrm{ec}, \mathrm{x}} / B_{1, \mathrm{x}}$ drops below $10^{-5}$ at $x_{\mathrm{s}} \sim 5 \mathrm{~cm}$. This proves once again how important it is to have the excitation coil on the other side of the conductive object from the vapour cell for the detection of the on-axis induced magnetic field, as found in Sec. 5.2. The case when the vapour cell is next to the excitation coil is not shown, as the detection distance is significantly worse.

We now consider the detection of low-conductivity spheres that are off-axis to the primary coil and OPM. The heatmaps in Figs. 5.7a and 5.7b are used for this investigation. The induced magnetic fields $B_{\mathrm{ec}, \mathrm{x}}\left(x_{s}=2.5 \mathrm{~cm}, y_{s}\right)$ and $B_{\text {ec, } \mathrm{y}}\left(x_{s}=2.5 \mathrm{~cm}, y_{s}\right)$ are plotted in Fig. 5.8 c for all values of $y_{s}$ in the heatmap. The change in polarity of the signals can be understood from the polarity plots in Figs. 5.7 and 5.7 b . It can be seen that fraction of the maximum induced magnetic fields $\operatorname{Max}\left[B_{\mathrm{ec}, \mathrm{x}}\right] / / \operatorname{Max}\left[B_{\mathrm{ec}, \mathrm{y}}\right]=[-18.4 \mathrm{pT}] /[-5.83 \mathrm{pT}] \sim 3.2$. Even though smaller magnetic fields are induced in the off-axis components from the primary magnetic field, the benefit of measuring $B_{\mathrm{ec}, \mathrm{y}}$ instead of $B_{\mathrm{ec}, \mathrm{x}}$ is that, in theory, no primary magnetic field needs to be nulled, as $B_{1, \mathrm{y}}=0$. However, alignment issues would mean that $B_{1, \mathrm{y}}=0$ would not be possible, but the fraction $\operatorname{Max}\left[B_{\mathrm{ec}, \mathrm{y}}\right] / B_{1, \mathrm{y}}$ could potentially be less than $10^{-5}$. For the measurement of $B_{\mathrm{ec}, \mathrm{y}}$ instead of $B_{\mathrm{ec}, \mathrm{x}}$, the experimental setup in Bevington et al. (Bevington et al., 2020) would be suitable. The static magnetic field $\mathbf{B}_{0}$, which is required for the operation of an RF OPM (see


Figure 5.8: (a) (b) Plots of the induced magnetic field $B_{\text {ec,x }}$ and $B_{\text {ec }, \mathrm{x}} / B_{1, \mathrm{x}}$, respectively, as a function of position of the sphere $x_{s}$. The blue data points are from the model when the primary field is considered constant across the sphere, as described in Sec. 5.3. The orange data points are from the model when the primary field varies across the sphere, as described in Sec. 5.2.2. The parameters used are $r_{s}=1 \mathrm{~cm}$, $\sigma=1 \mathrm{~S} / \mathrm{m}, \nu=2 \mathrm{MHz}, x_{\mathrm{OPM}}=10 \mathrm{~cm}, x_{\text {coil }}=0 \mathrm{~cm}, m=6.3 \mathrm{mAm}^{2}$ and $r_{w}=1 \mathrm{~cm}$. (c) Off-axis measurements of $B_{\mathrm{ec}, \mathrm{x}}$ and $B_{\mathrm{ec}, \mathrm{y}}$ where $x_{\mathrm{s}}=2.5 \mathrm{~cm}$. At $x_{\mathrm{s}}=2.5 \mathrm{~cm}$, $B_{\text {ec }, \mathrm{x}} / B_{1, x}=1.4 \times 10^{-5}$, very close to the supposed experimental limit of $10^{-5}$.

Sec. 3 and Sec. (4), would be along the $x$-axis, such that the total magnetic field along the $x$-axis would be $B_{x}=B_{0}+B_{1}(t)$. In such a case, the OPM is only sensitive to the measurement of $B_{\mathrm{ec}, \mathrm{y}}$, assuming $B_{\mathrm{ec}, \mathrm{z}}=0$ when $z_{s}=0$. The conductive object would have to be off-axis to measure a non-zero off-axis signal which may, or may not, be convenient depending on the experimental setup. This configuration has not yet been used for MIT of the heart and so needs to be experimentally investigated before conclusions about its feasibility are made.

### 5.4 Eddy current measurements theory for higher conductivity samples

When performing eddy current measurements with objects where the skin depth is comparable to or less than the thickness of the object such as in the detection of
metallic objects in Sec. 6.4, the induced magnetic field cannot just be considered to be $90^{\circ}$ out of phase with the primary field. Instead, the following theory by Bidinosti et al. (2007) must be used. This theory is important for the imaging of metallic samples, where the skin depth for a 10 kHz oscillating magnetic field in aluminium ( Al ) ( $\mu_{r}=1, \sigma=25 \mathrm{MS} / \mathrm{m}$ ) is equal to $\delta=\sqrt{1 /\left(\pi \sigma \nu \mu_{r} \mu_{0}\right)}=1 \mathrm{~mm}$. For the imaging of samples thicker than 1 mm , the theory developed in Sec. 5.2 and Sec .5 .3 would not be valid, as is the case for the eddy current measurements performed in Sec. 6 using an OPM.

To begin with, the secondary magnetic field $\mathbf{B}_{\text {ec }}$ induced in a solid sphere is calculated (Bidinosti et al., 2007). The total field $\mathbf{B}_{t}=B_{\mathrm{t}, \mathrm{r}} \hat{\mathbf{r}}+B_{\mathrm{t}, \theta} \hat{\boldsymbol{\theta}}=\mathbf{B}_{1}+\mathbf{B}_{\mathrm{ec}}$ is written in spherical coordinates, with its radial and angular components given by

$$
\begin{align*}
& B_{\mathrm{t}, \mathrm{r}}=B_{1, \mathrm{r}}+B_{\mathrm{ec}, \mathrm{r}}=\left(B_{1}+\frac{2 \mu_{0} m_{\mathrm{ec}}}{4 \pi r^{3}}\right) \cos \theta  \tag{5.46}\\
& B_{\mathrm{t}, \theta}=B_{1, \theta}+B_{\mathrm{ec}, \theta}=\left(-B_{1}+\frac{\mu_{0} m_{\mathrm{ec}}}{4 \pi r^{3}}\right) \sin \theta \tag{5.47}
\end{align*}
$$

where $B_{1}$ is the magnitude of the primary field measured by the OPM at the position in space, $m_{\mathrm{ec}}$ is the induced magnetic moment in the sphere, $r$ is the distance from the coil to the sphere and the distance from the sphere to the OPM (equivalent) and $\theta$ is the angle between the $x$-axis and the sphere, assuming the magnetic moment of the coil is pointing along the $x$-direction. This setup is depicted in Fig. 5.9. The induced magnetic moment $m_{\mathrm{ec}}$ is given by

OPM


Figure 5.9: A diagram of the setup for Eqs. 5.46 and 5.47. The magnetic moment is aligned with the $x$-axis and the sphere can be off-axis at an angle $\theta$. The OPM and primary coil are co-located at $(0,0,0)$.

$$
\begin{equation*}
m_{\mathrm{ec}}=\frac{2 \pi r_{s}^{3} B_{1}}{\mu_{0}} \frac{\left.\left(2\left(\mu-\mu_{0}\right) j_{0}\left(k r_{s}\right)\right)+\left(2 \mu+\mu_{0}\right) j_{2}\left(k r_{s}\right)\right)}{\left(\left(\mu+2 \mu_{0}\right) j_{0}\left(k r_{s}\right)+\left(\mu-\mu_{0}\right) j_{2}\left(k r_{s}\right)\right)}, \tag{5.48}
\end{equation*}
$$

where $r_{s}$ is the radius of the sphere and $\mu=\mu_{r} \mu_{0}$ is the relative permeability ( $\mu_{r}=1$ for $\mu_{0}$ ). The propagation constant $k$ is

$$
\begin{equation*}
k=\sqrt{\mu \varepsilon \omega^{2}+i \mu \sigma \omega}, \tag{5.49}
\end{equation*}
$$

and the spherical Bessel functions of order $n=0$ and $n=1$ are

$$
\begin{align*}
& j_{0}\left(k r_{s}\right)=\frac{\sin k r_{s}}{k r_{s}}  \tag{5.50}\\
& j_{2}\left(k r_{s}\right)=\left(\frac{3}{\left(k r_{s}\right)^{3}}-\frac{1}{k r_{s}}\right) \sin k r_{s}-\frac{3}{\left(k r_{s}\right)^{2}} \cos k r_{s} \tag{5.51}
\end{align*}
$$

The theory developed by Bidinosti et al. (2007) is valid for a homogeneous sphere in a uniform excitation field, i.e., when the sphere is far away from the primary coil, such that the field is constant across the sphere and can be approximated as a magnetic dipole. Moreover, the long-wavelength limit must apply, i.e., $\lambda \gg r_{s}$ and charge separation is ignored, i.e., $\nu \ll \sigma / \varepsilon_{0}$. We consider a setup in Sec. 6.4 for the remote detection of conductive objects. In those measurements $r_{s} \sim 2.5 \mathrm{~cm}$, $\sigma=25 \mathrm{MS} / \mathrm{m}$ and $\nu=10 \mathrm{kHz}$ are reasonable parameters. We therefore calculate $\lambda=c / \nu \sim 10^{8} \mathrm{~m}$ and $\sigma / \varepsilon_{0} \sim 10^{18} \mathrm{~Hz}$. This theory is therefore valid for the eddy current measurements presented in Sec. 6.


Figure 5.10: Comparison of (a) our theory from Sec. 5.3 for low frequency eddy current measurements involving a solid sphere, (b) COMSOL simulations using the methods from (Elson et al., 2022) and (c) the theory of Bidinosti et al. (2007) for both the in-phase and the out-of-phase signals. The low- and high-frequency ranges are split by the skin depth, which is included in the figure. The parameters used are $\sigma=25 \mathrm{MS} / \mathrm{m}, r_{s}=2.5 \mathrm{~cm}, x_{\mathrm{OPM}}=50 \mathrm{~cm}, x_{s}=25 \mathrm{~cm}, x_{\text {coil }}=0 \mathrm{~cm}$, $m=0.025 \mathrm{Am}^{2}, \mu_{r}=1$.

The theory of Bidinosti, our theory (Eq. 5.42) presented in Sec. 5.3, and COMSOL simulations are now compared. A situation is envisaged where a 2.5 cm radius Al sphere is detected 25 cm from the excitation coil and 25 cm from the OPM, foreshadowing some results that will be presented in Sec. 6 for the remote detection of conductive objects. The theories and COMSOL completely agree in the low frequency regime, i.e., when $t<\delta$, as demonstrated in Fig. 5.10. Above this frequency the real component $B_{\text {ec,re }}$ dominates, as is modelled in the work done by Bidinosti et al. (2007). This clearly demonstrates how the theory developed in this chapter can be directly compared to other theories in the low frequency regime focused on predicting the eddy currents induced in electrically conductive spheres and cylinders.

### 5.5 Conclusion

We present theory in this chapter which predicts the induced magnetic fields $\mathbf{B}_{\text {ec }}$ in eddy current measurements. We derive an off-axis theory for the arbitrary positioning of an electrically conductive sphere and a magnetometer, deriving equations for $B_{\mathrm{ec}, \mathrm{x}}, B_{\mathrm{ec}, \mathrm{y}}$ and $B_{\mathrm{ec}, \mathrm{z}}$ (Eqs. 5.425 .44 ). Our theories for an on-axis cylinder and an off-axis sphere are verified by comparing the data with COMSOL simulations and also with the work presented by Bidinosti et al. (2007). We first note that the equations (Eqs. 5.425 .44 for the off-axis sphere, Eq. 5.19 for the on-axis cylinder) and theory presented in this chapter can be tailored to the reader's wishes. It is hoped that the intuitive nature of the equations developed in this section, and the analysis of them, can prove useful to those working on diagnosing atrial fibrillation using eddy current measurements and OPMs, and in the pursuit of diagnosing brain haemorrhages using MIT (Klein et al., 2020; Wei and Soleimani, 2012; Luquire et al., 1970 Ma and Soleimani, 2017).

We take the derived equations and analyse them in detail with a particular emphasis on using them to understand the advantages and disadvantages of various experimental setups for magnetic induction tomography of the heart using OPMs. For the measurement of on-axis secondary magnetic fields, there are two possible configurations that could be used. These are depicted in Figs. 5.4a and 5.4b, In both of these configurations the on-axis induced magnetic fields measured by the OPM are the same. In the configuration in Fig. 5.4a, however, the RF power broadening of the primary magnetic field has a bigger impact than for the configuration in Fig. 5.4b, as the primary magnetic field is more difficult to null for the configuration in Fig. 5.4a. Our theoretical analysis shows that the limiting factor in both cases seems to be the inability to null the primary field adequately. The detection distance is therefore found to be improved with the configuration in Fig. 5.4b. We note that this configuration has been used for the detection of sub- $\mathrm{Sm}^{-1}$ electrically conductive objects (Deans et al., 2020).

For the detection of larger conductive objects, or where it is important to have the excitation coil, vapour cell and compensation coil in the same OPM head (see Fig. 5.4a), then it may not be possible to use the configuration in Fig. 5.4b. For the measurement of the on-axis induced magnetic field, it may be important to develop techniques to be able to null the primary field to a ratio better than $B_{\text {ec }} / B_{1}=10^{-5}$ to increase the detection distance. Alternatively, it could be possible to use a slightly different setup and detect the off-axis induced magnetic fields, where the static magnetic field $\mathbf{B}_{0}=B_{0} \hat{\mathbf{x}}$ is along the same direction as the primary magnetic field $\mathbf{B}_{1}(t)=B_{1}(t) \hat{\mathbf{x}}$ (Bevington et al., 2019). In this case the OPM would be sensitive to the off-axis induced magnetic field components $B_{\mathrm{ec}, \mathrm{y}}$ and $B_{\mathrm{ec}, \mathrm{z}}$. It is possible that in this configuration it could be easier to avoid the effects of the primary magnetic field, however we stress that this setup has not been used experimentally for MIT of the heart and so firm conclusions about its feasibility are not possible to make.

## Chapter 6

## Building a portable orientation-based optically pumped magnetometer

### 6.1 Introduction

Radio-frequency optically pumped magnetometers (RF OPMs) can detect oscillating magnetic fields with frequencies ranging from kHz to a few MHz (Savukov et al., 2005; Wasilewski et al., 2010; Chalupczak et al., 2012). Such RF OPMs can be used for detecting electrically conductive objects (Wickenbrock et al., 2014; Wickenbrock et al., 2016). Portable RF magnetometers in unshielded conditions have been developed with sensitivities to small oscillating magnetic fields as high as $19 \mathrm{pT} / \sqrt{\mathrm{Hz}}$ (Deans et al., 2021). Using the principle of electromagnetic induction, an excitation coil producing a primary oscillating magnetic field $\mathbf{B}_{1}(t)$ induces eddy currents in the object, which in turn produce a secondary oscillating magnetic field $\mathbf{B}_{\text {ec }}(t)$ that can be measured (Griffiths et al., 1999). These eddy current measurements can be useful for imaging conductive objects with low conductivity (Feldkamp and Quirk, 2019; Jensen et al., 2019; Deans et al., 2020) including the human heart (Marmugi and Renzoni, 2016) with the potential of helping those suffering from heart diseases such as atrial fibrillation. Other applications include characterising rechargeable batteries (Zhang et al., 2021), non-destructive testing (Bevington et al., 2019; Bevington et al., 2021; Deans et al., 2021), and remotely detecting and localising conductive objects for security applications (Deans et al., 2018b; Das et al., 1990; Verre et al., 2021; Elson et al., 2022).

In this chapter, we present a portable RF OPM working in unshielded conditions with sub- $\mathrm{pT} / \sqrt{\mathrm{Hz}}$ sensitivity to small oscillating magnetic fields. When detecting eddy currents, we use a differential technique (Jensen et al., 2019) and in that case our magnetometer achieves a sensitivity of $2-6 \mathrm{pT} / \sqrt{\mathrm{Hz}}$. We use this highperformance sensor to demonstrate a new benchmark for the long-range detection of conductive objects using a portable OPM. Here, long-range means that an object with a dimension $\sim a$ is detected at a far distance $r \gg a$ from both the excitation coil and the OPM. To be specific, we demonstrate detection with a good signal-tonoise ratio (SNR) of a 1.5 cm diameter aluminium (Al) disk at a remote distance of $\sim 25 \mathrm{~cm}$ from both the excitation coil and the OPM, which exceeds the previous benchmark of a $\sim 10 \mathrm{~cm}$ size Al square plate being detected $\sim 10 \mathrm{~cm}$ away (Deans
et al., 2018b from both the OPM and the excitation coil. The fact that our RF OPM can detect metallic objects at a relatively large distance makes it promising for remote sensing applications. Optically pumped magnetometers are promising alternatives to, for example, fluxgate magnetometers due to their superior magnetic field sensitivity. We note that total-field OPMs, which are based on measuring the Larmor frequency $\omega_{L} \propto|\mathbf{B}|$ (where $|\mathbf{B}|$ is the magnitude of the total magnetic field), can be used for the detection of magnetic objects and have recently been mounted on an underwater glider (Page et al., 2021) and on an airborne drone (Kolster et al., 2022). In contrast, RF OPMs can be used for the detection of both magnetic and non-magnetic conductive objects. Extracting the size, location and motion of an object are important measurements in the field of remote sensing. As a step towards using RF OPMs for remote sensing, we here experimentally detect an Al disk moving along a linear path using our single RF OPM. We analyse the RF OPM response in order to extract two spatial components of the induced magnetic field which are correlated with the position of the disk along its path. Additional work would be needed to fully demonstrate the potential of high sensitivity RF OPMs for remote sensing, for example by simultaneously recording data from multiple RF OPMs, by placing one or more RF OPMs on a moving platform, and developing algorithms for extracting information from the recorded signals.

### 6.2 Experimental setup

### 6.2.1 OPM head

The experimental setup for detecting conductive objects is shown in Fig. 6.1a. The setup includes an excitation coil and our unshielded portable OPM which is placed inside a cylindrical coil system. The OPM sensor head (see Figs. 6.1a and 6.1b) contains a cubic $(5 \mathrm{~mm})^{3}$ caesium (Cs) vapour cell (see Fig. 2.2 b in Sec. 2.2), optics, a balanced photodetector and a small compensation coil inside a 3D-printed housing. Two optical fibers provide laser light, and one cable provides electrical connections to the OPM sensor head. The vapour cell is paraffin-coated on the inside and is kept at room temperature ( $\sim 19^{\circ} \mathrm{C}$ ).

The caesium atoms are optically pumped into the $F=4$ hyperfine ground state manifold and spin-polarised in the $z$-direction by a circularly-polarised 0.1 mW pump beam resonant with the $\mathrm{D} 2 F=3 \rightarrow F^{\prime}$ transition and propagating along the $z$-direction. Here, $F$ and $F^{\prime}$ are hyperfine quantum numbers for the caesium ground and excited states, respectively. We refer the reader to Sec. 2.6.1 to understand how to generate an oriented state via optical pumping. A static field $\mathbf{B}_{0}$ is oriented along the $z$-axis. The atomic spins precess about the direction of the static field when an oscillating magnetic field $\mathbf{B}_{\mathrm{RF}}(t)$ is applied along the $x$-direction, with a maximum signal occurring when the frequency of the oscillating magnetic field $\omega_{\mathrm{RF}}$ equals the Larmor frequency $\omega_{L}=\gamma_{\mathrm{Cs}} B_{0}$, where $\gamma_{\mathrm{Cs}}=2 \pi(3.5 \mathrm{kHz} / \mu \mathrm{T})$ is the gyromagnetic ratio for caesium. Details of the behaviour of an oriented state in static and oscillating magnetic fields are described in detail in Sec. 3 .

To detect the precession of the atomic spins, a 5 mW probe beam propagating along the $y$-axis and linearly polarised along the $z$-axis passes through the vapour cell. The probe beam is $\sim 1.8 \mathrm{GHz}$ blue-detuned from the $\mathrm{D} 2 F=4 \rightarrow F^{\prime}$ transition. The polarisation of the beam rotates due to the Faraday effect and hence oscillates

(a)

(b)

Figure 6.1: (a) Experimental setup. The OPM and fluxgate are placed in a cylindrical coil system consisting of a flexible printed circuit board (PCB) cosine-theta coils which produce homogeneous transverse magnetic fields along the $y$ - and $z$-directions and also by a solenoid which produce a magnetic field along the $x$-direction. Inset: excitation coil is placed at $x=0$, the conductive object (with radius $a$ ) at $x=r$ and the OPM at $x=r+r^{\prime}$. (b) Schematic of the portable OPM head. Components include half-wave plates $(\lambda / 2)$, a quarter-wave plate $(\lambda / 4)$, a polarising beam splitter (PBS), linear polarisers (LP) and a balanced photodetector (BPD).
at a frequency $\omega=\omega_{\mathrm{RF}}$ when an oscillating magnetic field is present. Details of the Faraday effect can be found in Sec. 3.4. The light is split into its horizontal and vertical polarisation components by a polarising beam splitter, and each beam is incident on a balanced photodetector (BPD). The BPD used for this portable OPM is discussed in Sec.8.2.1. The oscillating BPD signal has an amplitude proportional to $B_{\mathrm{RF}}$, assuming that the OPM is being operated in the low-RF amplitude regime. The signal is demodulated using a lock-in amplifier, which produces DC values for the in-phase $X$ and quadrature $Y$ signals, providing information about the amplitude $R=\sqrt{X^{2}+Y^{2}}$ and phase of the oscillating magnetic field $\mathbf{B}_{\mathrm{RF}}(t)$.

### 6.2.2 Magnetic field stabilisation

A stable DC field $\mathbf{B}_{0}$ oriented along the $z$-axis is required for the operation of our OPM. Operating the OPM at 10.5 kHz , which is an appropriate frequency for the detection of our Al samples, requires the DC field to have an amplitude of $3.00 \mu \mathrm{~T}$. The Earth's magnetic field is $30-60 \mu \mathrm{~T}$ and needs to be compensated for in order


Figure 6.2: (a) Photo of the portable OPM with a ruler next to it for scale. (b) Photo of the inside of the portable OPM.
to have a stable field along the $z$-axis. A 3-axis fluxgate magnetometer (Bartington Mag690) is used to measure the ambient field and its detection point is 6.25 cm from the centre of the vapour cell. This 3-axis fluxgate has a bandwidth of 1.5 kHz and can measure magnetic fields up to $\pm 100 \mu \mathrm{~T}$ in the $x$-, $y$ - and $z$-directions, which makes it suitable for measuring the Earth's field as well as 50 Hz magnetic field noise. The two magnetometers are placed inside a 3D-printed cylinder, which is surrounded by flexible printed circuit board cosine-theta coils capable of producing magnetic fields along the $y$ - and $z$-directions and also by a solenoid which can produce a magnetic field along the $x$-direction. The $x$-, $y$ - and $z$-fluxgate outputs (100 $\mathrm{mV} / \mu \mathrm{T}$ ) are fed into the analogue inputs of a field-programmable gate array (FPGA, sbRIO-9627). A proportional-integral-derivative (PID) controller implemented on the FPGA outputs a voltage to a current feedback amplifier (LT1210) and a current is sent through the coils, producing magnetic fields to cancel the Earth's field at the position of the fluxgate in the $x$ - and $y$-directions, whilst keeping the $z$-static field fixed to $B_{0}=3.00 \mu \mathrm{~T}$. Without the PID in place, the 50 Hz noise measured by the fluxgate along the $z$-axis is $\sim 21.2 \mathrm{nT}_{\mathrm{p} \text {-p }}$ (corresponding to $\sim 74 \mathrm{~Hz}$ when converting to Hz using the caesium gyromagnetic ratio). With the PID in place, the 50 Hz noise is reduced by at least an order of magnitude down to $\sim 2.5 \mathrm{nT}_{\mathrm{p}-\mathrm{p}}$ (corresponding to $\sim 9 \mathrm{~Hz}$ precession frequency), reducing the 50 Hz noise to below the linewidth of the magnetic resonance $(40 \mathrm{~Hz})$. Further noise reduction can potentially be achieved with the implementation of an active noise control system for magnetic fields (Pyragius and Jensen, 2021).

### 6.3 Characterisation of OPM

### 6.3.1 Calibrating the compensation coil

Before characterising the OPM in unshielded conditions, the OPM was placed in a magnetic shield (Twinleaf MS-2). The compensation coil, positioned next to the vapour cell, produces a magnetic field which needs to be calibrated based on the voltage produced by the voltage source (for example the output of a SR830
lock-in amplifier or a RIGOL function generator DG1032Z, both with $50 \Omega$ output impedances). The way this is done is by using two RF coils, one which needs to be calibrated (the compensation coil) and another one which simply applies a small oscillating magnetic field, i.e., a low RF amplitude sine wave. Any residual DC magnetic fields in the $x$ - and $y$-components were nulled by adjusting the currents applied to the $x$ - and $y$-coils to decrease $\omega_{L}\left(\propto \sqrt{B_{x}^{2}+B_{y}^{2}+B_{z}^{2}}\right)$ such that $\omega_{L}=\gamma B_{z}$ only. Once this was done, it was possible to calibrate the compensation coil. A range of DC voltages from -3 V to +3 V were applied to the RF coil from the RIGOL function generator, and after each DC voltage a magnetic resonance signal was obtained (see Fig. 6.3a). The Larmor frequency is then plotted as a function of the applied DC voltage. It is important to remember from Ohm's law that any resistors added in series to the coil (1-2 $\Omega$ ) and function generator ( $50 \Omega$ ) will lead to a different calibration of the RF coil. For these measurements there was no extra resistor.

Assuming $B_{y}=0$, the Larmor frequency $\omega_{L}$ is given by

$$
\begin{equation*}
\omega_{L}=\gamma \sqrt{B_{x}^{2}+B_{z}^{2}} . \tag{6.1}
\end{equation*}
$$

Assuming that the range of $B_{x}$ applied to the RF coil is much smaller than $B_{z}$, then Eq. 6.1 can undergo a Binomial expansion $(1+x)^{n}=1+n x$ to first order when $x \ll 1$. Assuming that $\omega_{L}(0) /(2 \pi)=\nu_{0}$ is when $B_{x}=B_{y}=0$, we can write

$$
\begin{equation*}
\omega_{L} /(2 \pi)=\nu_{L}(0)\left(1+\frac{1}{2} \frac{\left(k V_{\mathrm{DC}}\right)^{2}}{\nu_{L}(0)^{2}}\right), \tag{6.2}
\end{equation*}
$$

where $V_{\mathrm{DC}}$ is the DC voltage applied to the RF coil being calibrated and $k$ is the conversion to Larmor frequency. Comparing this to a quadratic $y=b\left(x-x_{0}\right)^{2}+c$, $b$ is therefore equal to

$$
\begin{equation*}
b=\frac{k^{2}}{2 \nu_{L}(0)}, \tag{6.3}
\end{equation*}
$$

where $b$ must have units of $\mathrm{Hz} / \mathrm{V}_{\mathrm{DC}}$. Following on from this, we can rearrange for $k$ to give

$$
\begin{equation*}
k=\sqrt{2 \nu_{L}(0) b_{0}} . \tag{6.4}
\end{equation*}
$$

If RF coils with large inductances are used, then it is important to consider its inductance for higher frequency applications, as this calibration is calculated at 0 Hz . The compensation coil used in this experiment had an inductance of $\sim 0.2 \mu \mathrm{H}$. For frequencies $\gtrsim R / L=50 \Omega / 0.2 \mu \mathrm{H} \approx 30 \mathrm{MHz}$ then the inductance of the coil needs to be taken into account. Given the small diameter of the coil and the small number of windings, the inductance does not play a role in these measurements and the calibration here is valid for all the experiments presented at 10 kHz .

### 6.3.2 Unshielded conditions

The first part of the characterisation of the OPM in unshielded conditions involves measuring the magnetic resonance, which is done by sweeping the frequency of an applied oscillating magnetic field $\mathbf{B}_{2}(t)$ produced by the compensation coil $(5 \mathrm{~mm}$ diameter) inside the OPM head adjacent to the vapour cell. The amplitude of the RF field $B_{2}$ is $3.36 \mathrm{nT}_{\mathrm{rms}}$. The peak value of 4.32 V in Fig. 6.4 can be used to


Figure 6.3: RF coil calibration for the compensation coil in the OPM head in shielded conditions. (a) Some example magnetic resonances are shown for various DC voltages applied to the compensation coil. (b) The Larmor frequency $\omega_{L} /(2 \pi)$ is plotted as a function of the applied DC voltage.
calculate a conversion ( $1.285 \mathrm{~V} / \mathrm{nT}_{\mathrm{rms}}$ ) between the lock-in amplifier output and the amplitude of the oscillating magnetic field. The FWHM of $X$, equivalent to the bandwidth of the OPM, is 40 Hz .

Once this step is done, the RF frequency is fixed to where there is maximum signal in $X$, which in this case is at 10.5 kHz . A 240 s time trace was taken (see Fig. 6.5a for $X$ and $Y$, followed by a time trace with both coils off (see Fig. 6.5b). In each time trace the averaged signals $\bar{X}$ and $\bar{Y}$ with 1 s integration times are also plotted, along with the calculated standard deviation SD of these averaged time traces. The Allan deviation of the time traces is plotted in Fig. 6.6a ("RF on", "RF off"), which calculates the minimum detectable field $B_{\text {min }}$ for different averaging (or integration/gate) times. With the RF field on, the minimum detectable field for a $\tau=1 \mathrm{~s}$ integration time is $\approx 6 \mathrm{pT}$ for $X$ and $\approx 35 \mathrm{pT}$ for $Y$, while without any RF field the minimum detectable field is $\approx 0.6 \mathrm{pT}$, i.e., there is more noise when a large RF magnetic field is applied. This could be because the applied RF field or the laser powers are not perfectly stable or because low-frequency magnetic noise gets converted to high-frequency RF noise by the OPM (Jensen et al., 2019). The increased noise in $Y$ compared to $X$ implies that the static field $\mathbf{B}_{0}$ is noisy and a lower-noise current source or better magnetic field stabilisation should decrease the noise in $Y$. The time constant of the lock-in amplifier is 10 ms , which leads to a drop in the Allan deviation at small gate times. In any case, the sensitivity $\approx B_{\min } \sqrt{\tau}$ of an OPM is typically defined as the sensitivity to small signals, and it can therefore be stated that the sensitivity of our OPM (to small oscillating magnetic fields with a 10.5 kHz frequency) is $\approx 0.6 \mathrm{pT} / \sqrt{\mathrm{Hz}}$ in unshielded conditions. The long-term stability of the OPM is also demonstrated in Fig.6.6a, where the minimum detectable field at an integration time of 100 s is $30-60 \mathrm{fT}$.

The characterisation of the OPM so far has been achieved by applying an oscillating magnetic field $\mathbf{B}_{2}(t)$ using the small compensation coil placed inside the OPM. During eddy current measurements, the excitation coil which produces the primary oscillating magnetic field $\mathbf{B}_{1}(t)$ is also used. We employ a differential method (Jensen et al., 2019) where the amplitudes and phases of the primary and


Figure 6.4: Magnetic resonance signal. The frequency of the RF field (produced by the compensation coil) is swept over the Larmor frequency $\omega_{L}=2 \pi(10.5 \mathrm{kHz})$ in this data set.
compensation fields are adjusted such that $\mathbf{B}_{1}\left(t, \mathbf{r}_{\mathrm{OPM}}\right)+\mathbf{B}_{2}\left(t, \mathbf{r}_{\mathrm{OPM}}\right)=0$ at the vapour cell position, as can be seen in Fig. 6.5 c where a 240 s time trace is taken. Note that the OPM and the excitation coil are placed on opposite sides of the conductive object. This is to minimise any effects of the primary magnetic field on the OPM (see Sec. 5 for further discussions about OPM setups). When a conductive object is placed between the coils, the total oscillating field at the OPM position is then $\mathbf{B}_{\text {tot }}\left(t, \mathbf{r}_{\mathrm{OPM}}\right)=\mathbf{B}_{1}\left(t, \mathbf{r}_{\mathrm{OPM}}\right)+\mathbf{B}_{2}\left(t, \mathbf{r}_{\mathrm{OPM}}\right)+\mathbf{B}_{\mathrm{ec}}\left(t, \mathbf{r}_{\mathrm{OPM}}\right) \approx \mathbf{B}_{\mathrm{ec}}\left(t, \mathbf{r}_{\mathrm{OPM}}\right)$, where $\mathbf{B}_{\text {ec }}\left(t, \mathbf{r}_{\mathrm{OPM}}\right)$ is the secondary magnetic field induced in the object. The differential technique improves the SNR and thereby allows for the detection of small objects at a remote distance, because it allows for the detection of the small signal $\mathbf{B}_{\text {ec }}\left(t, \mathbf{r}_{\mathrm{OPM}}\right)$ on a zero background. Without the differential technique, one would measure the signal from the conductive object on top of the large primary magnetic field, i.e., $\mathbf{B}_{\text {tot }}\left(t, \mathbf{r}_{\mathrm{OPM}}\right)=\mathbf{B}_{1}\left(t, \mathbf{r}_{\mathrm{OPM}}\right)+\mathbf{B}_{\text {ec }}\left(t, \mathbf{r}_{\mathrm{OPM}}\right)$, which for OPMs lead to nonlinearities and additional noise. The measurement shown in Fig. 6.5 c was done with 38 times larger oscillating fields than when just one RF coil was on (see Fig. 6.5a). Despite the larger applied RF fields, the Allan deviation (at a gate time of 1 s ) of the OPM is around a factor of four better with both coils on than with only one coil on. Taking into account the larger amplitude, this demonstrates that the differential method would give a factor of $38 \times 4 \approx 150$ improvement in SNR when detecting conductive objects. Even higher RF amplitudes would further improve the SNR.

### 6.3.3 Shielded conditions

The intrinsic sensitivity of the OPM was tested by placing the OPM in a magnetic shield (Twinleaf MS-2). Time traces with the compensation coil on and off were taken, from which the Allan deviation was calculated and plotted in Fig. 6.6b, The sensitivity is $200 \mathrm{fT} / \sqrt{\mathrm{Hz}}$ in shielded conditions (using $B_{\min } \approx 0.2 \mathrm{pT}$ for $\tau=1 \mathrm{~s}$ ), due to the fact that the coil system for the transverse fields did not have to be connected, reducing the magnetic noise at 10.5 kHz . We note that in Sec. 2.2 the spin projection noise, or atomic noise, was calculated to be $\delta B_{\mathrm{spn}}=$


Figure 6.5: Unshielded characterisation. Three sets of 240 s time traces at a frequency of 10.5 kHz . (a) Compensation coil $\mathbf{B}_{2}(t)$ on with an amplitude $B_{2}=$ $3.36 \mathrm{nT}_{\mathrm{rms}}$. (b) Excitation and compensation coils both disconnected, i.e., $B_{1}=$ $B_{2}=0$. (c) Both on with amplitudes $B_{1}=B_{2}=127.7 \mathrm{nT}_{\mathrm{rms}}$ at the position of the vapour cell such that $\mathbf{B}_{1}\left(t, \mathbf{r}_{\mathrm{OPM}}\right)+\mathbf{B}_{2}\left(t, \mathbf{r}_{\mathrm{OPM}}\right)=0$.
$\sqrt{8 /\left(\gamma_{\mathrm{Cs}}^{2} F_{z} n V_{\text {cell }} T_{2}\right)}=86 \mathrm{fT} / \sqrt{\mathrm{Hz}}$ from Eq. 2.7 for this setup with $F_{z}=4$, a number density $n=2.2 \times 10^{16} \mathrm{~m}^{-3}$ at $T=18.5^{\circ} \mathrm{C}, V_{\text {cell }}=(5 \mathrm{~mm})^{3}, T_{2}=1 /(\pi(40 \mathrm{~Hz}))=8 \mathrm{~ms}$ and where all the atoms are assumed to be pumped into $F=4, m=4$. We attribute the factor of $\sim 2$ difference in sensitivity between $\delta B_{\mathrm{spn}}=86 \mathrm{fT} / \sqrt{\mathrm{Hz}}$ and the experimentally obtained $200 \mathrm{fT} / \sqrt{\mathrm{Hz}}$ mainly down to the fact that only one pump beam was used and not all the atoms are pumped into $F=4, m=4$, effectively decreasing $F_{z}$. The sensitivity of the OPM could be improved by: (1) using a second pump beam resonant on the $F=4 \rightarrow F^{\prime}=4$ transition as in (Jensen et al., 2019) to pump more of the atoms into $F=4, m=4$, (2) increasing the number density by heating up the cell and (3) optimising the detuning of the probe beam, as well as the powers of the pump and probe beams for shielded conditions, which would reduce the linewidth and increase $T_{2}$. These changes would improve the sensitivity of this OPM to $<200 \mathrm{fT} / \sqrt{ } \mathrm{Hz}$.


Figure 6.6: Allan deviation plots. (a) Unshielded calculations when the compensation coil is on and the excitation coil is disconnected (RF on), when both compensation and excitation coils are connected (both on) and when both coils are disconnected (RF off). (b) Shielded calculations with RF on and RF off.

### 6.3.4 Shielded conditions at 0 Hz

As well as higher frequency data, we also wanted to demonstrate the low frequency functionality of our magnetometer, relevant for magnetocardiography (Jensen et al., 2018; Morales et al., 2017; Bison et al., 2009; Alem et al., 2015) and magnetoencephalography (Boto et al., 2018; Hill et al., 2020; Boto et al., 2017; Xia et al., 2006) measurements. To do this, we placed the OPM in the magnetic shield and, unlike the 10 kHz data, ignored the lock-in amplifier. The magnetic fields were nulled using the methods described by Arnbak (Arnbak, 2018). For this data, we simply took a 1 s time trace of the output from the transimpedance amplifier (Thorlabs AMP102, $100 \mathrm{kV} /$ A gain). The transimpedance amplifier is connected to the output of the balanced photodetector (see Fig. 8.2.1 for details of the BPD used for this experiment) and converts the current from the BPD to a voltage which we can then analyse using our data acquisition card (Spectrum Instrumentation M2p.5932-x4). An example of a time trace when the RF frequency was 7 Hz is included in Fig. 6.7a, We emphasise how interesting Fig. 6.7a is. This is a raw time trace of the Faraday rotation of the linearly polarised probe beam, as predicted from Eq. 3.59. This method of taking a time trace was repeated for different RF frequencies ranging from 7 Hz up to 100 Hz . The Fourier transform of each time trace was taken. The conversion of the Fourier transform from $\mathrm{V} / \sqrt{\mathrm{Hz}}$ to $\mathrm{pT} / \sqrt{\mathrm{Hz}}$ was done by dividing the Fourier transform in $\mathrm{V} / \sqrt{\mathrm{Hz}}$ by the height of a peak within the bandwidth of the magnetometer in $\mathrm{V}_{\mathrm{RMS}} / \sqrt{\mathrm{Hz}}$ (i.e., 7 Hz or 17 Hz in Fig. 6.7b, but not 27 Hz as the peak is lower), then multiplying by the calibration of the RF coil ( $112.1 \mathrm{nT} / \mathrm{V}$ ) and the RF amplitude ( $14.1 \mathrm{mV}_{\mathrm{RMS}}$ ), giving rise to a calibrated Fourier transform in $\mathrm{pT} / \sqrt{\mathrm{Hz}}$. The bandwidth of the magnetometer is $\sim 30 \mathrm{~Hz}$ here. The sensitivity at 30 Hz is $260 \mathrm{fT} / \sqrt{\mathrm{Hz}}$ and at 10 Hz is $640 \mathrm{fT} / \sqrt{\mathrm{Hz}}$. This shows that this OPM can be used effectively at zero-field with a good sensitivity. Improved sensitivity could be achieved by heating the vapour cell (Ledbetter et al., 2008).


Figure 6.7: Shielded characterisation at zero Larmor frequency. (a) A 1 second time trace measured by the data acquisition card when a 7 Hz RF field was exposed to the Cs atoms. (b) Fourier transforms of the 1 second time traces for different RF frequencies ranging from 7 Hz up to 100 Hz .

### 6.4 Eddy current measurements

### 6.4.1 Detection of aluminium disks with varying diameters

Results are now presented on the detection of Al (grade 6061 with conductivity $\sigma \approx 25 \mathrm{MS} / \mathrm{m})$ disks of 4 mm thickness with various diameters ( $1 \mathrm{~cm} ; 1.5 \mathrm{~cm}$; $2 \mathrm{~cm} ; 3 \mathrm{~cm}$ and 5 cm ) in unshielded conditions using a frequency of 10.5 kHz . The excitation coil and the OPM (with the compensation coil right next to the vapour cell) are separated by 50.3 cm . One set of measurements was taken with the Al disks only 6.4 cm from the excitation coil ( 43.9 cm from the OPM), and a second set of measurements taken with the disks roughly halfway between the excitation coil and the OPM ( 26.4 cm from disk to excitation coil, 23.9 cm from OPM to disk). The compensation coil was used throughout these measurements.

Figure 6.8 shows 110 s time traces of $X$ and $Y$ when the 5 cm diameter disk is placed 6.4 cm from the excitation coil for $\sim 10 \mathrm{~s}$ (e.g. 10-22 s), then being removed for $\sim 10 \mathrm{~s}(\mathrm{e} . \mathrm{g} .23-30 \mathrm{~s})$. The disk was placed in the setup five times. The in-phase secondary magnetic field is 2600 pT and the out-of-phase secondary magnetic field is 164 pT . We observe that $|X| \gg|Y|$ meaning that the secondary magnetic field is almost completely out-of-phase ( $180^{\circ}$ ) with the primary magnetic field. This is expected (Honke and Bidinosti, 2018; Bidinosti et al., 2007; Elson et al., 2022) as the skin-depth in Al for a 10.5 kHz RF field is $\delta=1 / \sqrt{\pi \nu_{\mathrm{RF}} \mu_{0} \sigma} \approx 1.0 \mathrm{~mm}$ which is much smaller than the 4 mm thickness of the disk. Here $\nu=10.5 \mathrm{kHz}$ is the excitation frequency and $\mu_{0}$ is the magnetic permeability of free space.

Time traces of the eddy current measurements are shown in Fig. 6.9, when Al disks with $5 \mathrm{~cm}, 3 \mathrm{~cm}, 2 \mathrm{~cm}$ and 1.5 cm diameters are placed 26.4 cm from the excitation coil ( 23.9 cm from the vapour cell). The spikes in the time traces arise when the disk is in the process of being placed in front of the excitation coil. The signal remains stable, before the object is removed. The data in the stable region was used for the calculation of the signal size.

From such time traces we can calculate the induced field in pT as a function


Figure 6.8: Example eddy current measurement. 110 s time traces of $X$ and $Y$. The time segment $0-12 \mathrm{~s}$ is when the 5 cm diameter Al disk is removed and $12-22 \mathrm{~s}$ is when the Al disk is placed 6.4 cm from the excitation coil.
of disk diameter for the two disk positions (see Fig. 6.10). We can also calculate the standard deviation, SD , of the 1 s integrated time traces when the object is not present, permitting for the $\mathrm{SNR}=$ signal/SD to be calculated for each diameter disk. The calculated values of the SDs agree with the Allan deviation in Fig. 6.6a, where the smallest detectable field with a 1 s integration time is $\sim 2 \mathrm{pT}$ for $X$ and $\sim 7 \mathrm{pT}$ for $Y$. When a 1.5 cm diameter disk is placed midway between the excitation coil and the OPM, the SNR is $\sim 20$ in $X$ and $\sim 2$ in $Y$, meaning that the disk is easily detectable with a good SNR.

Our experimental results are compared to analytical formulae calculated from a model based on the work by Bidinosti et al. (2007) and Honke and Bidinosti (2018) and to the outcome of numerical simulations carried out in COMSOL. As detailed below, we find a good agreement on the scaling of the induced magnetic field with the diameter of the disks, and the predicted values for the induced field agree well with the experimentally measured ones.

In the work done by Honke and Bidinosti (2018), $B_{\text {ec }} / B_{1}$ is calculated for all frequencies for a non-magnetic, conductive sphere with radius $a$ in a uniform magnetic field. In Appendix H.1 we calculate the secondary magnetic field for certain positions of the excitation coil, object and OPM. If the sphere is a distance $r$ from the excitation coil and a distance $r^{\prime}$ from the OPM (see inset in Fig. 6.1a), and the high frequency limit is considered, then

$$
\begin{equation*}
\frac{B_{\mathrm{ec}}}{B_{1}}=\frac{a^{3}\left(r+r^{\prime}\right)^{3}}{r^{3} r^{\prime 3}} \tag{6.5}
\end{equation*}
$$

at the position of the OPM. If the object is exactly halfway between the excitation coil and the OPM, i.e., $r=r^{\prime}$, Eq. 6.5 further simplifies to

$$
\begin{equation*}
\frac{B_{\mathrm{ec}}}{B_{1}}=\frac{(2 a)^{3}}{r^{3}} . \tag{6.6}
\end{equation*}
$$

The experimental data sets in Fig. 6.10 are fitted to the function $\log \left(B_{\text {ec }}\right)=\log (c)+$ $3 \log (D)$, corresponding to the power law dependence $B_{\text {ec }}=c D^{3}$ as in Eq. 6.6.


Figure 6.9: Time traces of the eddy current measurements for (a) 5 cm , (b) 3 cm , (c) 2 cm , (d) 1.5 cm diameter Al disks, all with 4 mm thicknesses. The disks were placed 26.4 cm from the excitation coil ( 23.9 cm from the vapour cell).

Here $D=2 a$ is the diameter $D$ of the disks in cm . The constant $c$ is equal to $B_{1}\left(r+r^{\prime}\right)^{3} /\left(8 r^{3} r^{\prime 3}\right)$ when $r \neq r^{\prime}$ and equal to $B_{1} / r^{3}$ when $r=r^{\prime}$ (see Eqs. 6.5 and 6.6).

The fitted constant $c_{\text {exp }}$ in Fig. 6.10 when the disk is 26.4 cm from the excitation coil is $6.0 \mathrm{pT} / \mathrm{cm}^{3}$, whereas the theoretical value $c_{\text {theory }}$ is $8.1 \mathrm{pT} / \mathrm{cm}^{3}$, i.e., $34 \%$ higher than $c_{\exp }$ (using $r=26.4 \mathrm{~cm}, r^{\prime}=23.9 \mathrm{~cm}$ and $B_{1}=127.7 \mathrm{nT}_{\mathrm{rms}}$ ). When the disk is close to the excitation coil $c_{\exp }=29.6 \mathrm{pT} / \mathrm{cm}^{3}$, whereas $c_{\text {theory }}=91.6 \mathrm{pT} / \mathrm{cm}^{3}$, i.e., $210 \%$ higher than $c_{\exp }$ (using $r=6.4 \mathrm{~cm}, r^{\prime}=43.9 \mathrm{~cm}$ ).

There is a larger discrepancy when the disk is closer to the excitation coil where the radius of the excitation coil $R_{c}(5 \mathrm{~cm})$ is similar to the distance $r$ from the centre of the coil to the disk $(6.4 \mathrm{~cm})$. Equation 6.5 and hence the calculation of $c_{\text {theory }}$ assumes that the primary magnetic field is a magnetic dipole. The primary magnetic field $B_{1}(x=r)$ at the position of the disk is a factor of $\left(r^{2}+R_{c}^{2}\right)^{3 / 2} / r^{3}=2.04$ smaller (see Eqs. H.1 and H.12) if the primary magnetic field from a coil with a finite radius $R_{c}$ is used instead of the primary magnetic field from a magnetic dipole. This reduces $c_{\text {theory }}$ by a factor of 0.49 down to $44.9 \mathrm{pT} / \mathrm{cm}^{3}$, around $52 \%$ higher than $c_{\exp }$. For the 26.4 cm disk position the constant $c_{\text {theory }}$ is only affected slightly as $R_{c} \ll r$, with a correction from $8.1 \mathrm{pT} / \mathrm{cm}^{3}$ to $7.7 \mathrm{pT} / \mathrm{cm}^{3}$, indicating a $27 \%$ overestimation of $c_{\text {theory }}$ versus $c_{\text {exp }}$.

To investigate the discrepancy between theory and experiment further, numerical simulations of the experimental setup were performed in COMSOL using the methods used by Elson et al. (2022). The data points from the simulations are included in Fig. 6.10. Uncertainties in the positioning of the disks ( $\pm 1 \mathrm{~cm}$ ) were included in the error bars in the COMSOL data. The finite thickness ( 2 cm ) of the coil and the uncertainty on the OPM position were not taken into account, although these would also contribute to uncertainties in the numerical simulations. With regard to experimental uncertainties, we calculate the standard deviation of the induced field from 5 repeated measurements (see Fig. 6.8). Furthermore, the eddy current measurements were taken over the course of several hours ( $r=6.4 \mathrm{~cm}$ data followed by $r=26.4 \mathrm{~cm}$ data) where the lab temperature gradually increased throughout this period of time. This led to an increased number density of caesium atoms throughout the day. Due to the room-temperature operation of this OPM, the resonance signal amplitude of the OPM increased between the beginning (4.32 V in Fig. 6.4) and end $(5.56 \mathrm{~V})$ of the day by $\sim 30 \%$. A temperature increase of $2.5^{\circ} \mathrm{C}$ will lead to an increase in the atomic density by $30 \%$ (Steck, 2022). The data in Figs. 6.4 and 6.5 for the sensitivity measurements was obtained within minutes of each other at the beginning of the day and so temperature changes will have had little impact on these measurements. The calibration at the beginning of the day was used for the eddy current measurements, meaning that in fact smaller $B_{\text {ec }}$ values were being measured than in the stated calibrated pT values in Figs. 6.8 and 6.9. Including this uncertainty in the error bars on the experimental data in Fig. 6.10 means that the experimental data and the COMSOL data are in agreement with each other. The differences between experiment/COMSOL and theory are most likely due to the fact that the theory is true for a solid sphere in a uniform RF field, while in the experiment/COMSOL simulations we detected a solid disk (Nagel, 2018b; Nagel, 2018a).

The ratio $B_{\text {ec }} / B_{1}$ measured at the OPM position can be used as a figure of merit for the remote detection of conductive objects. For the 1.5 cm diameter disk which was clearly detectable, we have $B_{\text {ec }} / B_{1} \approx 2 \times 10^{-4}$. Using the noise level of 2 pT for $X$ from the Allan deviation calculations, the smallest detectable diameter should be around 0.7 cm (see Fig. 6.10), leading to a ratio as small as $B_{\text {ec }} / B_{1} \approx 2 \times 10^{-5}$. For comparison, a 2 cm diameter coin $(87 \% \mathrm{Cu})$ is detected 7.5 cm from the excitation/sensing coil in (Verre et al., 2021) with a good SNR, giving a ratio of $B_{\text {ec }} / B_{1} \approx 2 \times 10^{-2}$. We are able to detect a small ratio and therefore able to detect small objects at relatively large distances for two reasons: firstly, our OPM (at $x=r+r^{\prime}$ ) and excitation coil (at $x=0$ ) are placed on opposite sides of the disk (at $x \approx r$ ), which means that the ratio $B_{\text {ec }} / B_{1}$ is improved by a factor of $1 /\left[R_{c}^{3} /\left(8 r^{3}\right)\right] \approx 500$, where $R_{c}$ is the radius of the excitation coil (see Appendix H.2), compared to the case where the OPM and excitation coil are co-located; secondly, by implementing the differential technique we achieved an improvement in SNR by a factor of 150 .

### 6.4.2 Detection of a moving aluminium disk

To illustrate the potential of using RF OPMs for remote sensing, we have detected the 5 cm diameter disk as it was moved off-axis along a linear path from $y=-22.5 \mathrm{~cm}$ to $y=22.5 \mathrm{~cm}$ at a fixed $x=6.4 \mathrm{~cm}$ position. The disk was moved by hand with


Figure 6.10: The secondary magnetic field $B_{\text {ec }}$ is plotted as a function of the Al disk diameter when (i) the Al disks are 6.4 cm from the excitation coil $(43.9 \mathrm{~cm}$ from the OPM) and (ii) the Al disks are 26.4 cm from the excitation coil ( 23.9 cm from the OPM). The experimental results ("Exp") are plotted alongside theory ("The") curves given by Eq. 6.6 and results of COMSOL simulations ("COM"), together with fits to the function $B_{\mathrm{ec}}=c D^{3}$.
an approximately constant velocity on an orthogonal rail (not shown) parallel to the table which was added to the setup in Fig. 6.1a to steer the motion. As the disk is being moved in the $x-y$ plane, $B_{\mathrm{ec}, \mathrm{z}}=0$ due to symmetry. The $B_{\mathrm{ec}, \mathrm{x}}$ and $B_{\text {ec, } \mathrm{y}}$ components are in general non-zero when the disk is placed in the $x-y$ plane, however for the specific case of the object being on-axis (i.e., placed on the $x$-axis), the induced magnetic field only has a $B_{\text {ec, } \mathrm{x}}$ component at the magnetometer position.

RF OPMs are sensitive to oscillating magnetic fields perpendicular to the direction of the static field $\mathbf{B}_{0}$, which in our case are the $x$ - and $y$-directions. The measured secondary field can be written as

$$
\begin{equation*}
\mathbf{B}_{\mathrm{ec}}\left(t, \mathbf{r}_{\mathrm{OPM}}\right)=\left[B_{\mathrm{ec}, \mathrm{x}}(t) \widehat{\mathbf{x}}+B_{\mathrm{ec}, \mathrm{y}}(t) \widehat{\mathbf{y}}\right] \cos \left(\omega_{\mathrm{RF}} t+\theta\right), \tag{6.7}
\end{equation*}
$$

where $\widehat{\mathbf{x}}$ and $\widehat{\mathbf{y}}$ are unit vectors along the $x$ - and $y$-directions. For a moving disk, the amplitudes of the induced field at the magnetometer position, $B_{\mathrm{ec}, \mathrm{x}}(t)$ and $B_{\mathrm{ec}, \mathrm{y}}(t)$, will vary slowly as a function of time due to the changing position of the disk. Overall, the induced field is oscillating at the excitation frequency $\omega_{\mathrm{RF}}$ and with a phase $\theta$ which here is defined as the phase relative to the compensation field (which is $180^{\circ}$ out-of-phase with the primary field). The phase $\theta$ should not depend on the position of the disk. When the thickness $t$ of the disk is much larger than the skin depth $\delta$, or equivalently the excitation frequency $\nu=2 \pi \omega_{\mathrm{RF}} \gg 1 /\left(\pi t^{2} \mu_{0} \sigma\right)$, then the secondary field will be $180^{\circ}$ out-of-phase with the primary field (Elson et al., 2022), corresponding to a phase $\theta=0$. From the Bloch equations describing an RF OPM (Jensen et al., 2019), one can show that the recorded lock-in magnetometer
signals for an off-axis disk are

$$
\begin{align*}
& X(t) \propto B_{\mathrm{ec}, \mathrm{x}}(t) \cos (\theta)-B_{\mathrm{ec}, \mathrm{y}}(t) \sin (\theta) \\
& Y(t) \propto-B_{\mathrm{ec}, \mathrm{x}}(t) \sin (\theta)-B_{\mathrm{ec}, \mathrm{y}}(t) \cos (\theta) . \tag{6.8}
\end{align*}
$$

In the above we assumed that the amplitudes $B_{\mathrm{ec}, \mathrm{x}}(t)$ and $B_{\mathrm{ec}, \mathrm{y}}(t)$ vary slowly in time compared to the oscillation period $1 / \nu_{\mathrm{RF}}$, the inverse of the magnetometer bandwidth $(1 /(40 \mathrm{~Hz}))$ and the lock-in time constant of 10 ms . From Eq. 6.8 we see that the lock-in outputs $X(t)$ and $Y(t)$ from the RF OPM depend on the $x$ - and $y$-components of the induced magnetic field, $B_{\text {ec, } \mathrm{x}}(t)$ and $B_{\mathrm{ec}, \mathrm{y}}(t)$, respectively, as well as its phase $\theta$. When the object is placed on-axis, the measured secondary field only has an $x$-component and the lock-in signals are

$$
\begin{align*}
X(t) & \propto B_{\mathrm{ec}, \mathrm{x}}(t) \cos (\theta)  \tag{6.9}\\
Y(t) & \propto-B_{\mathrm{ec}, \mathrm{x}}(t) \sin (\theta) . \tag{6.10}
\end{align*}
$$

Figure 6.11 shows the magnitude $R=\sqrt{X^{2}+Y^{2}}$ and phase $\phi=\arctan (Y / X)$ of the recorded signals when the disk is moved off-axis along the described linear path. The largest signal in $R \approx 1560 \mathrm{pT}$ occurs at $t=8.1(0.2) \mathrm{s}$ when the disk is located on-axis, i.e., at the position $x=6.4 \mathrm{~cm}$ and $y=0$. At that point, the recorded phase $\phi=-\theta \approx-0.04(0.02) \mathrm{rad}=-2(1)^{\circ}$ is close to zero, as expected. We note that in the experiment the RF field could be slightly detuned from the atomic resonance due to small drifts in the bias magnetic field, which would lead to a small phase offset as well.

In our experiment the phase $\theta$ is close to zero. In a more general situation, however, the phase $\theta$ will be non-zero and will depend on the object's size and shape, its electrical conductivity and magnetic permeability, and the excitation frequency (Elson et al., 2022). However, the phase $\theta$ should not depend on the position of the object. For the localisation of an object, it can therefore be useful to remove the dependence on the phase $\theta$ by rotating the lock-in outputs $X$ and $Y$ from the RF OPM (see Eq. 6.8) by the angle $-\theta$, giving rise to the rotated variables

$$
\begin{align*}
X^{\prime}(t) & \propto B_{\mathrm{ec}, \mathrm{x}}(t)  \tag{6.11}\\
Y^{\prime}(t) & \propto-B_{\mathrm{ec}, \mathrm{y}}(t) . \tag{6.12}
\end{align*}
$$

Based on the geometry of our experimental setup and the fact that the Al disk is moving parallel to the $y$-axis, we expect that $B_{\mathrm{ec}, \mathrm{x}} \propto X^{\prime}$ is symmetric around $y=0$ (equivalent to $8.1(0.2) \mathrm{s}$ in Fig.6.11) as a function of $y$-position, and that $B_{\text {ec, } \mathrm{y}} \propto Y^{\prime}$ is asymmetric around $y=0$ as a function of $y$-position. We note that this behaviour was predicted in our derived off-axis eddy current measurements theory in Fig. 5.8c. Within reasonably good agreement, we find experimentally (see Fig. 6.11) that $X^{\prime}$ is symmetric and $Y^{\prime}$ is asymmetric, as expected. Any small discrepancies are expected to be due to small positioning errors/misalignment. We also note that for every position of the disk along its particular linear path there is a corresponding unique $\left(X^{\prime}, Y^{\prime}\right)$ value measured by the RF OPM, meaning that the position of the disk along its particular linear path and the direction of motion can be extracted. We also refer the reader to Fig. 5.8 c in Sec. 5.3 , where our eddy current theory predicts the induced magnetic fields $B_{\mathrm{ec}, \mathrm{x}}$ and $B_{\mathrm{ec}, \mathrm{y}}$ for off-axis eddy current measurements, as is presented here.

Our method of detecting conductive objects using RF OPMs can potentially be extended to localising unknown conductive objects moving along arbitrary paths. As a single RF OPM only provides two measurements $X(t)$ and $Y(t)$ at each instance of time, more RF OPMs would be needed to uniquely determine the position of the object in real time. Furthermore, one would need to develop algorithms for extracting the location of the object based on the recorded data. Also, localisation of stationary conductive objects using one or more RF OPMs could be done by placing the RF OPMs on a moving platform and recording data while the platform is moving over some area.


Figure 6.11: Off-axis example. The 5 cm Al disk is moved from $y=-22.5 \mathrm{~cm}$ to $y=22.5 \mathrm{~cm}$ at a distance of $x=6.4 \mathrm{~cm}$ from the excitation coil. The magnitude $(R)$, rotated in-phase $X^{\prime}$ and quadrature $Y^{\prime}$ components are plotted in (a), with a zoomed-in section in (b). The phase is plotted in (c), and the regions furthest from $y=0$ are excluded as $X$ and $Y$ become very small, making the calculated phase less insightful.

### 6.5 Conclusion

In conclusion, we have developed a portable sub-pT $/ \sqrt{\mathrm{Hz}}$ radio-frequency optically pumped magnetometer (RF OPM), which works in unshielded/ambient conditions, setting a new benchmark for the sensitivity of a portable RF OPM in unshielded conditions. Using electromagnetic induction, we have demonstrated remote detection of electrically conductive objects far from both the excitation coil and the magnetometer. We detected a $2 a=1.5 \mathrm{~cm}$ diameter Al disk at a remote distance
of $r \approx 25 \mathrm{~cm}$ from both the OPM and the excitation coil, i.e., at a distance $r \gg a$ much larger than the object size. This detection distance could be further extended using larger primary magnetic fields or by improving the sensitivity of the OPM. To illustrate the potential of high sensitivity RF OPMs for remote sensing applications, we detected a moving Al disk using our RF OPM. We analysed the magnetometer signals to extract two spatial components of the induced magnetic field which depend on the position of the disk. Using this principle with multiple OPMs and an extraction algorithm should allow for the location and motion of conductive objects to be determined in the future.

## Chapter 7

## Building a table-top alignment-based optically pumped magnetometer

### 7.1 Motivation

Optically pumped magnetometers (OPMs) (Labyt et al., 2022; Auzinsh et al., 2014, Budker and Romalis, 2007) based on spin-polarised atoms (for example alkali atoms such as caesium (Cs) or rubidium) can measure magnetic fields with high sensitivity in the $\mathrm{fT} / \sqrt{\mathrm{Hz}}$ range (Kominis et al., 2003; Wasilewski et al., 2010; Chalupczak et al., 2012; Yao et al., 2022). Current commercial OPMs (QuSpin 2022; FieldLine Inc 2023; Twinleaf 2022) are operated close to zero magnetic field in the spinexchange relaxation-free (SERF) regime measuring one, two or three components of the magnetic field, or in the Earth's field as scalar magnetometers measuring the total magnetic field amplitude. These OPMs use one or two beams of circularly polarised light generated from a single laser diode inside the OPM, making the sensors compact and robust. The circularly polarised light effectively generates spin-orientation along the light propagation, i.e., the atomic spins point in a certain direction, which responds to magnetic fields and can be measured by detecting the transmitted light. When detecting oscillating magnetic fields in the $\mathrm{kHz}-\mathrm{MHz}$ frequency range, radio-frequency (RF) OPMs (Savukov et al., 2005; Auzinsh et al., 2014; Rochester, 2010; Deans et al., 2018c; Deans et al., 2021; Dhombridge et al., 2022) must be used. One type of RF OPM using only a single laser beam is the alignment-based magnetometer (Ledbetter et al., 2007; Zigdon et al., 2010; Rochester, 2023), which uses linearly polarised light capable of effectively aligning the atoms in the direction perpendicular to its propagation. As a result, as the RF field affects such alignment being created, its presence can be sensed directly by measuring properties of the same beam.

High sensitivity optical magnetometry requires a long atomic spin coherence time. This can be achieved using vapour cells coated on the inside with an antirelaxation coating (for example paraffin), such that the moving alkali atoms can bounce off the inner glass walls of the vapour cell many times without losing their spin coherence (Balabas et al., 2010b; Li et al., 2017). Alternatively, a long coherence time can be achieved by filling the vapour cell with buffer gas (for example $\mathrm{N}_{2}$ ). Rapid collisions between the buffer gas atoms and the alkali atoms make the alkali
atoms diffuse slowly, which mitigates the effects of spin-destroying wall collisions. Alkali vapour cells for magnetometry are typically hand-blown, however buffer gas cells for magnetometry can be produced on a mass scale using microfabrication techniques (Shah et al., 2007, Dyer et al., 2022). Such microfabrication techniques have not, as of yet, been compatible with anti-relaxation coatings.

So far, alignment-based optical magnetometry has been demonstrated using hand-blown, anti-relaxation coated cells (Ledbetter et al., 2007; Zigdon et al., 2010). The presence of buffer gas leads to pressure broadening of the alkali vapour absorption spectrum, reducing the light-atom coupling and affecting the optical pumping preparing the aligned state. The buffer gas $\mathrm{N}_{2}$ is also a quenching gas (Seltzer, 2008) which causes the alkali atoms not to de-excite via spontaneous emission. Rapid collisional mixing in the excited state (Seltzer, 2008) also occurs in buffer gas cells, but not in paraffin-coated cells. We show here that, despite these complexities, it is possible to realise an alignment-based magnetometer using a buffer gas cell. We experimentally demonstrate an alignment-based magnetometer using a Cs alkali vapour and 65 Torr $\mathrm{N}_{2}$ buffer gas cell with a sensitivity of $325 \mathrm{fT} / \sqrt{\mathrm{Hz}}$ to oscillating magnetic fields at 10 kHz . We also demonstrate an alignment-based magnetometer with a paraffin-coated cell placed in the same experimental setup to verify the methods and for comparison. Our results open up the possibility for miniaturisation (Dhombridge et al., 2022; Deans et al., 2021; Rushton et al., 2022) and commercialisation of RF OPMs, with potential impact in areas such as medical physics (Deans et al., 2020; Jensen et al., 2019; Marmugi and Renzoni, 2016), remote sensing (Rushton et al., 2022; Deans et al., 2018b) and non-destructive testing (Bevington et al., 2021; Bevington et al., 2019).

### 7.2 Experimental setup

A schematic of the experimental setup is shown in Fig. 7.1a. The custom-made D1 laser system, described in Sec. 7.2.1, outputs light resonant with the $F=4 \rightarrow F^{\prime}=3$ transition on the Cs D1 line ( 895 nm ). This is passed through an optical fiber and is collimated at its output. The beam then propagates along the $y$-axis. The linearly polarised light at the output of the fiber is rotated by a half-wave plate $(\lambda / 2)$ to make it horizontally polarised, such that all of the light passes through the polarising beam splitter (PBS) directly after the half-wave plate. Doing this ensures that small changes to the polarisation of the light output from the fiber, for example from temperature fluctuations in the lab, corresponds to the minimum change in the power of the horizontally polarised light coming out of the PBS. After the first PBS another half-wave plate and PBS are used to permit for the power of the D1 laser light to be varied. For the alignment-based magnetometer experiments presented in this chapter low light powers $(\leq 100 \mu \mathrm{~W})$ are generally used. The $z$-polarised light then passes through the vapour cell, which for these experiments was either paraffin-coated or was filled with $\mathrm{N}_{2}$ buffer gas.

The light passing out of the vapour cell passes through another half-wave plate, which rotates the light by around $45^{\circ}$ such that when the light is incident on another PBS, half the light is reflected (vertically polarised light) and the other half is transmitted (horizontally polarised light). Each beam of light is incident on a separate photodiode on a balanced photodetector (BPD). Each photodiode converts the incident power into a current (based on the responsivity of the photodiode),

(a)

(b)

Figure 7.1: (a) Schematic of an alignment-based magnetometer. The laser light propagates along the $y$-direction and is $z$-polarised. Components include half-wave plates $(\lambda / 2)$, polarising beam splitters (PBS), a paraffin-coated or buffer gas Cs vapour cell (Cell), a balanced photodetector (BPD) and a static $\mathbf{B}_{0}=B_{0} \hat{\mathbf{z}}$ and an oscillating magnetic field $\mathbf{B}_{\mathrm{RF}}(t)=B_{\mathrm{RF}}(t) \hat{\mathbf{x}}$ at the position of the vapour cell. (b) Picture of the experimental setup in the lab, including the magnetic shield (Twinleaf MS-1). A picture of the paraffin-coated vapour cell inside the shield is shown in Fig. 7.8a.
after which the difference of the photocurrents is amplified by a transimpedance amplifier. This converts the resultant photocurrent into a voltage with the multiplication factor $G$. The majority of the table-top experiments that will be presented used a commercial balanced photodetector (Thorlabs PDB210A/M) with a 1 MHz bandwidth, however some of the data was obtained with a custom-made balanced photodetector, which is discussed in more detail in Sec. 8. The output voltage from the BPD oscillates at the RF frequency (see Sec. (4) and is demodulated using a lock-in amplifier, producing in-phase $(X)$ and out-of-phase $(Y)$ components.

A static magnetic field $\mathbf{B}_{0}$ is applied along the $z$-axis using the in-built coils in the
magnetic shield (Twinleaf MS-1). For small $\mathbf{B}_{0}$ magnetic fields a DM Technologies (DM Technologies 2022) current source was used to supply the current. Currents above 320 mA for large $B_{0}$ magnetic fields were supplied by a DC Twinleaf current source (Twinleaf 2022). The oscillating magnetic field $\mathbf{B}_{\mathrm{RF}}(t)$ for the paraffin-coated cell was produced by a small 3 mm diameter coil situated 1.5 cm above the vapour cell (see Fig. 7.8b). The oscillating magnetic field for the buffer gas cell was produced by a square Helmholtz coil (see Fig. 7.15a).

### 7.2.1 Laser system

The laser system is shown in Fig. 7.2. The key component of the D1 laser system is a butterfly DBR laser (Thorlabs DBR895PN). This laser is controlled by a CTL200 Koheron laser controller. The fiber-coupled output of the butterfly laser is separated by a fiber splitter into three individual fibers ( $25 \%, 25 \%, 50 \%$ power splitting), custom-made by Thorlabs (see Fig. 7.2b). One of the fibers is sent to perform absorption spectroscopy to lock the laser to the $F=4 \rightarrow F^{\prime}=3$ transition, and the other two are used for experiments with a $50 \% / 25 \%$ splitting in the powers of the fibers. The outputs of the fibers can be seen in Fig. 7.2a, along with the Cs cell wrapped in yellow Kapton tape which is used for absorption spectroscopy.


Figure 7.2: The custom-made D1 laser system. (a) Inside of the laser box. (b) Custom-made 895 nm Thorlabs optical fibers with three outputs.

To lock the lasers, the "spectroscopy" fiber is passed through a pure Cs cylindrical cell ( 12.7 mm diameter, 20 mm length) and is incident on a photodiode (Thorlabs SM05PD1A). The resulting photocurrent is amplified using a transimpedance amplifier (Koheron PD10TIA). The laser controller, transimpedance amplifier and the cell heater are powered by a voltage supply (Koheron SPS100). The amplified signal is sent to the input of a Red Pitaya (Red Pitaya 2022). The Red Pitaya is an alternative for many lab instruments, such as a replacement for a function generator,
oscilloscope and lock-in amplifier, among others. The Red Pitaya provides DC and AC (normally 618 kHz ) modulations to the laser controller. The laser is locked to the transition $F=4 \rightarrow F^{\prime}=3$ using the Red Pitaya, accessible via internet connection. The laser system is encased in a box with a size of $30 \mathrm{~cm} \times 20 \mathrm{~cm} \times 10 \mathrm{~cm}$, weighs $\leq 5 \mathrm{~kg}$ and is easily transportable.

### 7.3 Alignment-based magnetometer using a paraffin-coated cell <br> 



Figure 7.3: (a) Magnetic resonance signals ( $R$ as a function of the RF frequency $\omega_{\mathrm{RF}}$ ) are obtained for several RF amplitudes for the paraffin-coated cell at room temperature. The laser power before the cell is $10 \mu \mathrm{~W}$. The data is fitted to absorption-Lorentzian lineshapes (black dotted lines). (b) The amplitude $A$, fullwidth half-maximum FWHM and $A /$ FWHM are extracted from the fits in (a) and plotted.
the ideal light power to maximise sensitivity, it is important to get signals with as large amplitudes $A$ as possible whilst keeping the FWHM $\propto 1 / T_{2}$ as small as possible. From Eqs. 2.7 and 2.8 in Sec. 2.2 it can be seen that both atomic noise limited and photon shot noise limited OPMs achieve better sensitivities with larger $T_{2}$ times and larger number densities $(n \propto A)$. The light power which maximises $A / \mathrm{FWHM}$ is therefore chosen, which in this case was $10 \mu \mathrm{~W}$. With a $10 \mu \mathrm{~W}$ optical power the FWHM of $R$ is 400 Hz , corresponding to a $400 \mathrm{~Hz} / \sqrt{3}=230 \mathrm{~Hz}$ OPM bandwidth (Zigdon et al., 2010), where the bandwidth in this case is considered to be the FWHM of $X$. It is also insightful to notice that when the RF amplitude and light power are both small then the FWHM of $R$ is $\sim 250 \mathrm{~Hz}$. This means that the intrinsic linewidth of the OPM is $250 \mathrm{~Hz} / \sqrt{3} \sim 145 \mathrm{~Hz}$.

### 7.3.2 Calibrating the RF coil

It is now important to calibrate the RF field $B_{\mathrm{RF}}(t)$ that is being applied to the atoms. This is described in detail in Sec.6.3.1. The coil that was calibrated was the small 3 mm diameter "excitation" coil, which is pictured in Figs. 7.8a and 7.8b, Any residual DC magnetic fields in the $x-$ and $y$-components were nulled by adjusting the currents supplied to the $x$ - and $y$-coils (in this case to a MS-1 Twinleaf shield) to decrease the Larmor frequency $\left(\propto \sqrt{B_{x}^{2}+B_{y}^{2}+B_{z}^{2}}\right)$ such that $\omega_{L}=\gamma_{\mathrm{Cs}} B_{z}$ only. A range of DC voltages from -1 V to +1 V were applied to the excitation coil with no resistor in series. For each DC voltage a magnetic resonance signal was obtained. The oscillating magnetic field was produced by the "compensation coil" in Fig. 7.8b. The Larmor frequency is plotted as a function of the applied DC voltage in Fig. 7.5. Based on the analysis in Sec. 6.3.1, the calibration of the excitation coil requires the equation $k=\sqrt{2 \nu_{0} b_{0}}$. Substituting in $\nu_{0}=1740.1 \mathrm{~Hz}$ and $b=156.4 \mathrm{~Hz} / \mathrm{V}_{\mathrm{DC}}$ from the quadratic fit in Fig. 7.5, the conversion from applied voltage to Tesla is $k=737.85 \mathrm{~Hz} / \mathrm{V}_{\mathrm{DC}}=210.8 \mathrm{nT} / \mathrm{V}_{\mathrm{DC}}$, where the gyromagnetic ratio of $3.5 \mathrm{~Hz} / \mathrm{nT}$

(a)

(b)

Figure 7.4: Varying the light power in the paraffin-coated alignment-based magnetometer at room temperature with a $20 \mathrm{mV}_{\mathrm{RMS}} \mathrm{RF}$ amplitude. (a) The magnetic resonance signals of some light powers (10 averages each) are plotted. (b) The parameters $A$, FWHM and $A /$ FWHM of the Lorentzians fitted to the data in (a) are plotted, including extra data not included in (a).


Figure 7.5: Calibrating the RF coil. A DC voltage is applied to the coil, shifting the Larmor frequency. The Larmor frequency is plotted as a function of the applied DC voltage, with the quadratic fit included in the plot.
is used (Steck, 2022). The reverse procedure was performed for the compensation coil and the calibration was calculated to be $208.6 \mathrm{nT} / \mathrm{V}_{\mathrm{DC}}$. The calibrations of the two coils are almost identical as they have the same number of windings (20) and are the same distance from the vapour cell.

### 7.3.3 Sensitivity

Given that the paraffin-coated alignment-based magnetometer has been characterised, it is possible to calculate the sensitivity of the RF OPM. To do this, four separate measurements were performed, as demonstrated in Fig. 7.6. Figure 7.6 a shows an RF frequency sweep with $\omega_{L}=2 \pi(10.25 \mathrm{kHz})$. A $4.216 \mathrm{nT}_{\mathrm{RMS}}$ ( $20 \mathrm{mV}_{\text {RMS }}$ ) oscillating magnetic field is applied. The peak of the resonance signal is extracted and divided by the applied oscillating magnetic field to give a conversion


Figure 7.6: Sensitivity measurement of a paraffin-coated alignment-based magnetometer at a Larmor frequency of $\omega_{L}=2 \pi(10.25 \mathrm{kHz})$ using a Thorlabs PDB210A/M as the BPD. (a) Magnetic resonance with the RF frequency swept over the Larmor frequency. (b) A 240 s time trace with the RF frequency fixed to the Larmor frequency, i.e., "RF on". (c) A 240 s time trace with the RF amplitude set to zero $\left(B_{\mathrm{RF}}=0\right)$, i.e., "RF off". (d) A 240 s time trace with the balanced photodetector blocked, i.e., the intrinsic noise of the BPD. (e) Allan deviation calculations of the three sets of time traces in (b), (c) and (d).
between the lock-in amplifier readout and the corresponding RF field amplitude $B_{\mathrm{RF}}$ $(1.24 \mathrm{~V} / \mathrm{nT})$. The RF frequency is then fixed to the Larmor frequency, i.e., $\omega_{\mathrm{RF}}=\omega_{L}$, and a 4 minute time trace obtained in Fig. 7.6b. The RF amplitude is then set to zero amplitude, i.e., "RF off", and another time trace in Fig. 7.6 c is obtained. The light hitting the balanced photodetector (Thorlabs PDB210A/M) is then completely blocked and another time trace is obtained in Fig. 7.6d, Allan deviation calculations of the time traces are shown in Fig. 7.6e,

The sensitivity to small oscillating magnetic fields, i.e., "RF off", is $480 \mathrm{fT} / \sqrt{\mathrm{Hz}}$ for $X$ and $460 \mathrm{fT} / \sqrt{\mathrm{Hz}}$ for $Y$. The noise with the light blocked is only just below at $410 \mathrm{fT} / \sqrt{\mathrm{Hz}}$ for $X$ and $380 \mathrm{fT} / \sqrt{\mathrm{Hz}}$ for $Y$. The non-electronic noise is therefore $\sqrt{480^{2}-410^{2}}=250 \mathrm{fT}$ for $X$ and 260 fT for $Y$. This means that the electronic noise of the balanced photodetector is the dominant noise source (with a small contribution from the data acquisition card), which is not good for a high sensitivity OPM. It will now be shown that the non-electronic noise is dominated by photon shot noise.

In Fig. 8.2 the Thorlabs BPD is characterised. The data obtained in Fig. 7.6 was taken with a $10 \mu \mathrm{~W}$ beam. As the transmission of this paraffin-coated cell was $77 \%$ (see Appendix F.2), $7.7 \mu \mathrm{~W}$ hits the BPD. The electronic noise of the BPD from Fig. 8.2a is $\sim 20 \times 10^{-14} \mathrm{~V}^{2} / \mathrm{Hz}$ at 10 kHz and the photon shot noise of a $7.7 \mu \mathrm{~W}$ beam is $\sim 7.7 \times 10^{-14} \mathrm{~V}^{2} / \mathrm{Hz}$. The fraction $7.7 \times 10^{-14} /\left(20 \times 10^{-14}\right)=0.39$ from Fig. 8.2a is very similar to $(250 / 410)^{2}=0.37$ calculated from the data in Figs. 7.6 c and 7.16 d . This therefore means that the non-electronic noise is dominated by photon shot noise.

To make the photon shot noise the dominant noise source rather than the electronic noise of the Thorlabs BPD, our custom-made detector was used instead. Our custom-made BPD (see Sec. 8) had the least electronic noise at 4 kHz (see Fig. 8.11a), lower than that of the Thorlabs BPD (see Fig. 8.2a). This frequency was therefore picked to illustrate an improved sensitivity. The measurements in Fig. 7.7 were performed in the same manner as in Fig. 7.6 , albeit with $\omega_{L}=2 \pi(4.1 \mathrm{kHz})$ instead of $\omega_{L}=2 \pi(10.2 \mathrm{kHz})$. This required the transverse fields $B_{x}$ and $B_{y}$ to be nulled. This was done by adjusting the currents supplied to the $B_{x}$ and $B_{y}$ coils to minimise the Larmor frequency, as $\omega_{L} \propto \sqrt{B_{x}^{2}+B_{y}^{2}+B_{z}^{2}}$. With the custom-made balanced photodetector the sensitivity to oscillating magnetic fields improved to $360 \mathrm{fT} / \sqrt{\mathrm{Hz}}$ for $X$ and $370 \mathrm{fT} / \sqrt{\mathrm{Hz}}$ for $Y$ and the electronic noise to $200 \mathrm{fT} / \sqrt{\mathrm{Hz}}$ and $210 \mathrm{fT} / \sqrt{\mathrm{Hz}}$ for $X$ and $Y$, respectively. This means that the photon shot noise is now the dominant noise source at $300 \mathrm{fT} / \sqrt{\mathrm{Hz}}$ for both $X$ and $Y$. The signal size and thereby the sensitivity could be improved by heating the vapour cell (Ledbetter et al., 2007; Seltzer, 2008) and using a larger vapour cell.

A fundamental limit to the sensitivity is given by the spin projection noise (Ledbetter et al., 2007; Graf et al., 2005)

$$
\begin{equation*}
\delta B_{\mathrm{spn}}=\frac{2 \hbar}{g_{F} \mu_{B} \sqrt{n V T_{2}}} \tag{7.4}
\end{equation*}
$$

where $g_{F}=1 / 4$ for the $F=4$ Cs ground state, $n \sim 2.2 \times 10^{16} \mathrm{~m}^{-3}\left(T \sim 18.5^{\circ} \mathrm{C}\right)$ is the number density of Cs atoms, $T_{2} \sim 1 /(\pi(230 \mathrm{~Hz})) \sim 1.4 \mathrm{~ms}$ is the transverse relaxation time and $V=(5 \mathrm{~mm})^{3}$ is the volume of the whole cell, as all the atoms in the cell are probed. The sensitivity is estimated to be $\delta B_{\mathrm{spn}} \sim 50 \mathrm{fT} / \sqrt{\mathrm{Hz}}$ using the numbers above. A balanced photodetector with reduced electronic noise would help


Figure 7.7: Sensitivity measurement of a paraffin-coated alignment-based magnetometer at a Larmor frequency of $\omega_{L}=2 \pi(4 \mathrm{kHz})$ using our custom-made balanced photodetector (BPD V2 in Sec. 8). (a) Magnetic resonance with the RF frequency swept over the Larmor frequency. (b) A 240 s time trace when the RF frequency is fixed to $\omega_{\mathrm{RF}}=\omega_{L}$, i.e., "RF on". (c) Time trace with the RF amplitude set to zero ("RF off") i.e., the intrinsic noise floor of the OPM. (d) Noise floor of our custom-made balanced photodetector (BPD V2). (e) Allan deviation calculations of the three sets of data ( $X$ and $Y$ for each).
us get closer to this quantum-limited sensitivity. Another contribution as to why we do not appear to reach this limit is because many of the atoms decay to the $F=3$ ground state during optical pumping (see Fig. 2.16d), meaning that these atoms are not probed, effectively reducing the number density (Ledbetter et al., 2007).

### 7.3.4 Eddy current measurements

Eddy current measurements were then taken with the alignment-based magnetometer using a paraffin-coated vapour cell. The primary magnetic field $\mathbf{B}_{1}(t)$ for the eddy current measurements is produced by the same coil that was used to produce $\mathbf{B}_{\mathrm{RF}}(t)$ in the analysis so far. As discussed in Sec. 6 , it is beneficial to use a compensation coil ( 3 mm diameter, 20 windings) producing an oscillating magnetic field $\mathbf{B}_{2}(t)$ such that the total RF field $\mathbf{B}_{\text {tot }}(t)$ at the position of the vapour cell is equal to $\mathbf{B}_{\text {tot }}=\mathbf{B}_{1}(t)+\mathbf{B}_{2}(t)=0$ when no magnetic fields are induced in an electrically conductive object. This configuration ensures that the only magnetic field detected by the OPM during eddy current measurements is $\mathbf{B}_{\text {tot }}=\mathbf{B}_{1}(t)+\mathbf{B}_{2}(t)+\mathbf{B}_{\mathrm{ec}}(t)=\mathbf{B}_{\mathrm{ec}}(t)$.

A 2 cm diameter, 4 mm thickness Al disk was scanned 0.5 cm above the excitation coil, 2 cm above the vapour cell and 3.5 cm above the compensation coil, as pictured in Figs. 7.8a and 7.8b, The disk was scanned along the $+z$-direction over the excitation coil using a 1D translation stage, then back along the $-z$-direction. In Fig. 7.8 c the primary magnetic field at the position of the vapour cell would be $B_{1}=40 \mathrm{mV}$ RMS $\times 210.8 \mathrm{nT} / \mathrm{V}=8.43 \mathrm{nT}$ without the compensation coil producing an equal and opposite magnetic field. The maximum induced magnetic field of $B_{\text {ec }}=2.33 \mathrm{nT}$ in Fig. 7.8 c leads to $B_{\text {ec }} / B_{1} \sim 0.28$. This fraction is large due to the close proximity of the conductive object to the excitation coil and vapour cell, especially when compared with the remote detection of conductive objects in Sec. 6.4, where $B_{\text {ec }} / B_{1}$ was as small as $2 \times 10^{-4}$ for the detection of a 1.5 cm diameter Al disk $\sim 25 \mathrm{~cm}$ from both the vapour cell and excitation coil. When the primary magnetic field $\mathbf{B}_{1}(t)$ was increased by a factor of two in Fig. 7.8d, $\mathbf{B}_{\text {ec }}(t)$ also increased by a factor of $\sim 2$. The improvement to the signal-to-noise-ratio with a bigger primary magnetic field $\mathbf{B}_{1}(t)$ is evident from the time traces.

This is the first demonstration of using an alignment-based magnetometer for the detection of electrically conductive objects. The setup presented here is very similar to the setup used by Jensen et al. (2019), where salt-water samples with electrical conductivities as low as $4 \mathrm{~S} / \mathrm{m}$ were detected. This presents the possibility of using an alignment-based magnetometer for eddy current measurements of lowconductivity objects such as the human heart. The measurements by Jensen et al. (2019), however, were taken at $\omega_{L}=2 \pi(2 \mathrm{MHz})$, whereas these were taken at $\omega_{L}=2 \pi(10 \mathrm{kHz})$. In the next section we will analyse the behaviour of the alignment-based magnetometer at a similarly large magnetic field.

### 7.3.5 Non-linear Zeeman splitting

Operating at large $\mathbf{B}_{0}$ magnetic fields is important to understand optical pumping. A square Helmholtz coil was designed and placed around the paraffin-coated cell in the MS-1 Twinleaf shield (see Fig. 7.8a). No magnetic field gradients $\left(d B_{z} / d z=\right.$ $d^{2} B_{z} / d z^{2}=0$ ) were applied due to the homogeneity of the Helmholtz coil.


Figure 7.8: Eddy current measurements with an alignment-based magnetometer and a paraffin-coated cell. (a) Picture of the vapour cell in its 3D-printed holder, surrounded by a 3D-printed square Helmholtz coil. The excitation coil is on top of the holder and the compensation coil is on the bottom (both 1.5 cm from the vapour cell). The 2 cm diameter Al disk is attached to a 3D-printed arm which travels along the $z$-axis by being pushed and pulled by a 1D translation stage. (b) Sketch of the setup in (a). (c) The induced magnetic field $B_{\text {ec }}$ is plotted as a function of time as the disk is moved over the vapour cell along the $+z$-axis, and then moved backwards along the $-z$-axis. (d) The same measurement as (c) but with twice as big a primary magnetic field, and therefore twice as big a compensation magnetic field such that $\mathbf{B}_{\text {tot }}(t)=\mathbf{B}_{1}(t)+\mathbf{B}_{2}(t)=0$ when the disk is not directly over the excitation coil.

A Cs atom in the $F=4$ ground state has $2 F+1=9$ sublevels $|F, m\rangle$ which, when placed in a small magnetic field $B_{0}$, have the energy $E(m)=m h \nu_{L}$ due to the linear Zeeman effect. Here $\nu_{L}$ is the Larmor frequency in Hz . That is to say, the splittings between neighbouring sublevels are all equal to the Larmor frequency $\Delta \nu_{m, m-1} \equiv(E(m)-E(m-1)) / h=\nu_{L}$. In this case, a single magnetic resonance will be observed when sweeping the RF frequency $\nu_{\mathrm{RF}}$ (in Hz ) across the Larmor frequency $\nu_{L}$ and measuring the polarisation rotation of the transmitted light (see Eqs. 7.1.7.3). However, at larger magnetic fields, the splittings between sublevels are slightly different due to the non-linear Zeeman effect (see Sec. 2.4.1). We calculate
(Julsgaard, 2003; Bao et al., 2018; Steck, 2022)

$$
\begin{equation*}
\Delta \nu_{m, m-1}=\nu_{L}-\delta\left(m-\frac{1}{2}\right) \tag{7.5}
\end{equation*}
$$

where the non-linear Zeeman splitting (in Hz ) is

$$
\begin{equation*}
\delta=\frac{2 \nu_{L}^{2}}{\nu_{\mathrm{hf}}} \tag{7.6}
\end{equation*}
$$

as illustrated in Fig. 4.1(b). In particular, the difference in transition frequencies between $\Delta \nu_{4,3}$ and $\Delta \nu_{-3,-4}$ is

$$
\begin{equation*}
\left|\Delta \nu_{4,3}-\Delta \nu_{-3,-4}\right|=7 \delta \tag{7.7}
\end{equation*}
$$

In other words, at larger magnetic fields a total of 8 magnetic resonances should be observed when sweeping the RF field across the Larmor frequency with the outermost resonances split by $7 \delta$.

The optical pumping of an aligned state can be experimentally verified by exploiting the non-linear Zeeman effect. These measurements were done at a relatively large static magnetic field $\left(B_{0}=5.84 \mathrm{G}\right)$ corresponding to a Larmor frequency close to 2 MHz . When the RF frequency was swept over the range $2.037-2.051 \mathrm{MHz}$, we observe a magnetic resonance spectrum with several peaks (see Fig. 7.9). The two largest peaks correspond to the transitions $m=4 \rightarrow m=3$ and $m=-3 \rightarrow m=-4$ with transition frequencies $\Delta \nu_{4,3}$ and $\Delta \nu_{-3,-4}$, respectively. The difference in transition frequencies $\left|\Delta \nu_{4,3}-\Delta \nu_{-3,-4}\right|$ is experimentally found to be $6.38(0.02) \mathrm{kHz}$, agreeing with the value $7 \delta=6.37 \mathrm{kHz}$ calculated from Eqs. 7.6 and 7.7 , confirming that we are observing the non-linear Zeeman splitting. This difference in transition frequencies was extracted by fitting the data of $R$ in Fig. 7.9 to the function (Julsgaard, 2003)

$$
\begin{equation*}
R=\left|\sum_{m=-3}^{4} \frac{A_{m, m-1}\left[1+i\left(\nu_{\mathrm{RF}}-\nu_{m, m-1}\right) / \tilde{\gamma}\right]}{\left(\nu_{\mathrm{RF}}-\nu_{m, m-1}\right)^{2}+\tilde{\gamma}^{2}}\right| \tag{7.8}
\end{equation*}
$$

which is a sum of eight magnetic resonances with resonance frequencies $\nu_{m, m-1}=$ $\nu_{L}-\delta\left(m-\frac{1}{2}\right)$ and half width at half maximum (HWHM) $\tilde{\gamma}=1 /\left(2 \pi T_{2}\right)$ (in Hz ) as seen by comparison with Eq. 7.3 and illustrated in Fig. 4.1. The data was fitted with seven free parameters: four amplitudes $A_{4,3}, A_{3,2}, A_{2,1}, A_{1,0}$ (as the magnetic resonance spectrum is symmetric such that $A_{0,-1}=A_{1,0}, A_{-1,-2}=A_{2,1}, A_{-2,-3}=$ $A_{3,2}, A_{-3,-4}=A_{4,3}$, the Larmor frequency $\nu_{L}$, the non-linear Zeeman splitting $\delta$, and the width $\tilde{\gamma}$.

In total, the spectrum has eight peaks, although the middle two are hardly visible in Fig. 7.9 due to their smaller height. The height of the individual peaks corresponding to $A_{m, m-1} / \tilde{\gamma}^{2}$ in Eq. 7.8 are proportional to the difference in populations of neighbouring magnetic sublevels (Julsgaard, 2003). This is why there are eight peaks in the non-linear Zeeman splitting, but nine populations in Fig. 7.11a. As the outermost peaks are largest and have equal height, we conclude that an aligned state is created in the $F=4$ ground state, with the majority of the atoms pumped into the $F=4, m= \pm 4$ states. The optical pumping is not perfect as some of the atoms are pumped into the other magnetic sublevels. This is due to the non-zero longitudinal relaxation rate $\Gamma_{1}$.


Figure 7.9: Non-linear Zeeman splitting of the magnetic resonances using a paraffincoated cell. The magnitude $R$ is fitted to Eq. 7.8. The fit is included as a dotted line. The magnetic resonances for $m=4 \rightarrow m=3$ and $m=-3 \rightarrow m=-4$, with different Larmor frequencies, are indicated.

### 7.4 Alignment-based magnetometer using a buffer gas cell

Until now, alignment-based magnetometers have only used paraffin-coated vapour cells to extend the spin relaxation lifetimes of the alkali vapour. The drawback of these cells is that they are hand-blown and are therefore time-intensive, and somewhat unreliable, to produce. Buffer gas cells, on the other hand, can be manufactured on a mass scale using microfabrication techniques. The alignment-based magnetometer uses a single laser beam for optical pumping and probing and could potentially allow for more rapid commercialisation of radio-frequency OPMs, due to the robustness of the one-beam geometry and the potential for mass-scale microfabrication of buffer gas cells.

### 7.4.1 Optical pumping

An absorption spectrum of a $N_{2}$ buffer gas cell is obtained, plotted on top of a pure Cs cell as a frequency reference in Fig. 7.10. The detailed method of how to extract the Doppler broadening $\Gamma_{G}$ and the pressure shift is described in Sec. 2.3.3. The pressure broadening $\Gamma_{L}$ is $1.26(0.05) \mathrm{GHz}$, corresponding to a pressure of $65(3)$ Torr, using the conversion of $19.51 \mathrm{MHz} /$ Torr from (Andalkar and Warrington, 2002) for the D1 pressure broadening with $\mathrm{N}_{2}$. The pressure can also be extracted from the shift $-0.54(0.01) \mathrm{GHz}$, which corresponds to a pressure of $65(1)$ Torr.

Our alignment-based magnetometer uses $\pi$-polarised light resonant with the $F=$ $4 \rightarrow F^{\prime}=3$ transition (see Fig. 4.1b), as in this case, the $F=4, m= \pm 4$ states are dark states and atoms become optically pumped into those states with equal probability, creating the spin alignment, as depicted in Fig. 7.11b, We refer the reader to Sec. 2.6.3 to understand how an aligned state is created in the presence


Figure 7.10: Duplicate of Fig. 2.9. Absorption spectrum of the D1 line with a 65(3) Torr $\mathrm{N}_{2}$ cell alongside a frequency reference. In this figure the buffer gas cell is heated to $\sim 51^{\circ} \mathrm{C}\left(43.7 \times 10^{16} \mathrm{~m}^{-3}\right)$ by applying a 300 mA DC current (Twinleaf) to the resistive wires (twisted pair). The $F=3 \rightarrow F^{\prime}=3,4$ and $F=4 \rightarrow F^{\prime}=3,4$ transitions are fitted to Voigt profiles and the FWHM $\Gamma_{L}$ and pressure shift are extracted. The Doppler FWHM $\Gamma_{G}$ is 374 MHz .
of a quenching gas such as $\mathrm{N}_{2}$. Note that for $\pi$-polarised light resonant with the $F=4 \rightarrow F^{\prime}=4$ transition, the $F=4, m=0$ sublevel will be a dark state instead. With buffer gas pressure broadening, the $F=4 \rightarrow F^{\prime}=3$ and $F=4 \rightarrow F^{\prime}=4$ resonances begin to overlap. From our fit, we deduce that the overlap is only $\sim 10 \%$ for our pressure of 65 Torr $\mathrm{N}_{2}$ (see Fig. 7.10 and the thin dotted vertical line). At higher pressures the two transitions will overlap even more. This is problematic for an alignment-based magnetometer as the light in this case will drive both $F=4 \rightarrow$ $F^{\prime}=3$ and $F=4 \rightarrow F^{\prime}=4$ transitions at the same time. The $F=4, m= \pm=4$ are then not dark states and significantly less spin alignment is created.

To verify whether optical pumping into the $F=4, m= \pm 4$ states is possible with the 65 Torr $\mathrm{N}_{2}$ buffer gas cell where the excited hyperfine states partially overlap ( $\sim 10 \%$ ) and where quenching is the main de-excitation mechanism (see Sec. 2.6.3), once again the static field is adjusted to be large ( $B_{0}=8.38 \mathrm{G}$ ) and a magnetic resonance spectrum is recorded (see Fig. 7.12). Again we see the magnetic resonances split due to the non-linear Zeeman effect, and the two outermost resonances have the largest and equal heights. The frequency difference between the $m=4 \rightarrow m=3$ transition and the $m=-3 \rightarrow m=-4$ transition is found experimentally to be $\left|\Delta \nu_{4,3}-\Delta \nu_{-3,-4}\right|=13.2(0.1) \mathrm{kHz}$ from a fit of the data in Fig. 7.12 to Eq. 7.8 , which agrees well with the value $7 \delta=13.1 \mathrm{kHz}$ calculated from Eqs. 7.6 and 7.7 .

This experimentally demonstrates that it is possible to generate a spin-aligned state in the 65 Torr $\mathrm{N}_{2}$ buffer gas cell by optically pumping more Cs atoms into the $m= \pm 4$ states than the other magnetic sublevels in the $F=4$ ground state. It is expected that better optical pumping into the $m= \pm 4$ states will be achieved if a smaller buffer gas pressure is used, as there will be less unwanted pumping to the $F=4 \rightarrow F^{\prime}=4$ transition. A higher ratio $R_{p} / \Gamma_{1}$ (see Eq. 2.60) will


Figure 7.11: Optical pumping from $F=4 \rightarrow F^{\prime}=3$ with $\pi$-polarised light. The populations of the $F=3$ and $F=4$ ground state magnetic sublevels in the steady state are plotted, with a longitudinal relaxation rate $\Gamma_{1}=R_{p} / 20$ for (a) a paraffincoated cell where the dominant de-excitation mechanism from the excited state is spontaneous emission (see Sec. 2.6 .2 for more details), and (b) a buffer gas cell where the dominant de-excitation mechanism is quenching (see Sec. 2.6 .3 for more details).
also increase pumping into the $m= \pm 4$ states. The drawback of a lower buffer gas pressure, however, is that the atoms will diffuse more quickly to the walls, leading to a smaller $T_{2}$ time and hence a less sensitive OPM. These two processes compete and need to be taken into consideration when selecting the optimal buffer gas pressure for an alignment-based magnetometer.

### 7.4.2 Characterising cell

The 65 Torr $\mathrm{N}_{2}$ buffer gas cell will now be characterised, using the same methods as described for the paraffin-coated cell in Sec. 7.3 . The 65 Torr buffer gas cell needs to be heated, unlike the paraffin-coated cell which is operated at room temperature. The cell is heated to $\sim 55^{\circ} \mathrm{C}\left(n \sim 62 \times 10^{16} \mathrm{~m}^{-3}\right)$ by applying a 320 mA DC current to the twisted-pair resistive wires (see Fig. 2.2 d ). The light power through the vapour cell was varied between $0 \mu \mathrm{~W}$ and $60 \mu \mathrm{~W}$ and several magnetic resonance signals are plotted in Fig. 7.13a, Each data set was fitted to an absorption-Lorentzian lineshape. The FWHM and amplitude of each fit were extracted and plotted as a function of light power in Fig. 7.13b The sensitivity is optimal between $20-60 \mu \mathrm{~W}(30 \mu \mathrm{~W}$ was picked). This was performed with a low RF amplitude ( 4 mV RMS ). When varying the RF amplitude in Fig. 7.14, the magnetic resonance signal increases linearly with the RF amplitude up to around $30 \mathrm{mV}_{\text {RMS }}$. This data was taken with a low light power of $2 \mu \mathrm{~W}$, which is why the linewidths tend to the same value at low light powers and low RF amplitudes in both Figs. 7.13b and 7.14b.

The RF coil with a $3 \mathrm{k} \Omega$ resistor in series was calibrated ( $128.9 \mathrm{nT} / \mathrm{V}$ ) using the methods described in Sec. 7.3.1. The Larmor frequency is plotted as a function of the applied DC voltage to the RF coil and resistor in Fig. 7.15b. A square Helmholtz coil (see Fig. 7.15a) was used to produce the oscillating magnetic fields for all measurements taken with the 65 Torr buffer gas cell.


Figure 7.12: Non-linear Zeeman splitting of the magnetic resonances using a 65 Torr $\mathrm{N}_{2}$ buffer gas cell heated to $\sim 55^{\circ} \mathrm{C}$. The magnitude $R$ is fitted to Eq. 7.8. The magnetic resonances for $m=4 \rightarrow m=3$ and $m=-3 \rightarrow m=-4$ are indicated.


Figure 7.13: (a) Varying the light power through the 65 Torr buffer gas cell and plotting the magnetic resonance signal for each light power. The RF amplitude is $4 \mathrm{mV}_{\mathrm{RMS}}$ and the temperature of the cell is $\sim 55^{\circ} \mathrm{C}$ ( 320 mA DC current). (b) The data in (a) is fitted to absorption-Lorentzian lineshapes and the FWHM and amplitude $A$ are extracted.

### 7.4.3 Sensitivity

Given that the optimal light power was found to be $30 \mu \mathrm{~W}$, the sensitivity of the OPM could be found. A magnetic resonance signal at 10 kHz was obtained with the 65 Torr cell in Fig. 7.16a. A 240 s time trace with the RF frequency fixed to the Larmor frequency, i.e., $\omega_{\mathrm{RF}}=\omega_{L}$ is shown in Fig. 7.16b, A 240 s time trace with $B_{\mathrm{RF}}=0$, i.e., the intrinsic noise of the OPM, is shown in Fig. 7.16 c , along with the intrinsic noise of the BPD with no light hitting the detector (Thorlabs PDB210A/M) in Fig. 7.16d. The Allan deviation calculations of each of these time traces are plotted in Fig. 7.16e,

(a)

(b)

Figure 7.14: Varying the RF amplitude applied to the RF coil in the 65 Torr buffer gas cell. The light power is $2 \mu \mathrm{~W}$ and the temperature is $\sim 55^{\circ} \mathrm{C}(320 \mathrm{~mA} \mathrm{DC}$ current). (a) The magnetic resonance signals are plotted for each RF amplitude. (b) The data in (a) is fitted to absorption-Lorentzian lineshapes and the extracted parameters are plotted in (b).


Figure 7.15: RF coil calibration for the 65 Torr buffer gas cell. (a) Picture of the vapour cell holder, where the square Helmholtz coils were calibrated and used throughout the experiments with a $3 \mathrm{k} \Omega$ in series between the function generator (DG1032Z) and the RF coil. (b) The Larmor frequency $\omega_{L} /(2 \pi)$ is plotted as a function of the DC voltage applied to the Helmholtz coil. The oscillating magnetic field applied to produce the resonance signal was produced by the $y$-coil inside the Twinleaf MS-1 shield.

The sensitivity of the OPM, defined as the Allan deviation of the "RF off" time trace when the gate time $=1 \mathrm{~s}$, is $290 \mathrm{fT} / \sqrt{\mathrm{Hz}}$ for $X$ and $340 \mathrm{fT} / \sqrt{\mathrm{Hz}}$ for $Y$. These values are very similar to the SD calculations of $310 \mathrm{fT} / \sqrt{\mathrm{Hz}}$ and $340 \mathrm{fT} / \sqrt{\mathrm{Hz}}$ in Fig. 7.16 c for $X$ and $Y$, respectively. The electronic noise of the BPD is $183 \mathrm{fT} / \sqrt{\mathrm{Hz}}$ in $X$ and $176 \mathrm{fT} / \sqrt{\mathrm{Hz}}$ in $Y$. The non-electronic noise is dominated by photon shot noise and is $225 \mathrm{fT} / \sqrt{\mathrm{Hz}}$ for $X$ and $290 \mathrm{fT} / \sqrt{\mathrm{Hz}}$ for $Y$. In an ideal world, the atomic noise would equal the photon shot noise (Auzinsh et al., 2004), with the electronic noise being negligible.


Figure 7.16: Sensitivity measurement of the 65 Torr $\mathrm{N}_{2}$ cell. (a) Magnetic resonance with $\omega_{L}=2 \pi(10 \mathrm{kHz})$. (b) RF frequency fixed to $\nu_{L}$. (c) RF turned off (just electronic noise and shot noise). (d) Electronic noise of the balanced photodetector. (e) Allan deviation calculations of the 3 sets of 240 s time traces in (b), (c) and (d).

We use Eq. 7.4 to calculate the predicted quantum-limited spin projection noise. The number density $n=60 \times 10^{16} \mathrm{~m}^{-3}$ at $T=55^{\circ} \mathrm{C}$ and $T_{2}=1 /(\pi(800 \mathrm{~Hz}))$. In a buffer gas cell only the atoms inside the beam are probed, unlike in a paraffin-coated cell where all the atoms in the cell are probed. We therefore use the volume inside the beam $V=V_{\text {beam }}=3.9 \times 10^{-9} \mathrm{~m}^{3}$, where the diameter of the beam is $\sim 1 \mathrm{~mm}$ and length of the cell is 5 mm . Inserting the numbers above, we estimate the atomic noise to be $\delta B_{\mathrm{spn}} \sim 100 \mathrm{fT} / \sqrt{\mathrm{Hz}}$. We therefore calculate that we are only a factor of $\sim 3$ away from the spin projection noise limit. A better sensitivity could be obtained by increasing the diameter and length of the cell, whilst increasing the size of the beam. If a 5 mm diameter beam was used, probing the whole cell, the atomic noise is estimated to be $\delta B_{\mathrm{spn}} \sim 20 \mathrm{fT} / \sqrt{\mathrm{Hz}}$. Note that many atoms are lost to the $F=3$ ground state (see Fig. 7.11b), reducing the number of Cs atoms that are probed. Using a second laser beam (typically called a repumper) bringing the atoms out of $F=3$ and back into $F=4$ would also increase the number of probed atoms, improving the sensitivity of the RF OPM.

### 7.5 Conclusion

The results presented in this chapter demonstrate the first implementation of a onebeam radio-frequency optically pumped magnetometer (RF OPM), the alignmentbased magnetometer, being used with a buffer gas cell. The sensitivity of the alignment-based magnetometer with Cs alkali vapour and 65 Torr $\mathrm{N}_{2}$ buffer gas was $325 \mathrm{fT} / \sqrt{\mathrm{Hz}}$ to 10 kHz oscillating magnetic fields. This sensitivity could be further improved upon by using a balanced photodetector with lower electronic noise. Further studies could investigate the optimal vapour cell size, operating temperature and buffer gas pressure. Although our experiments were carried out using hand-blown vapour cells, we expect similar performance with microfabricated buffer gas cells. Our work opens up the possibility of the commercialisation of compact, robust and portable RF OPMs using only one laser beam with buffer gas cells, a much more scalable and commercially viable option than using paraffin-coated vapour cells.

## Chapter 8

## Designing and testing a low-noise balanced photodetector

### 8.1 Motivation

A portable optically pumped magnetometer (OPM), such as the one described in Sec. 6, requires a small balanced photodetector (BPD) which can fit in the OPM head. The measurements described in Sec. 6 for the portable orientation-based OPM required only a 10 kHz bandwidth. For measurements where eddy current measurements on the human heart are performed, however, the operational frequency of the OPM will be more like $1-2 \mathrm{MHz}$ (Jensen et al., 2019; Deans et al., 2020) (see Sec. 5), requiring a similar bandwidth of the BPD. There are two types of OPMs we have been developing over the past few years, an orientation-based OPM which is described in Sec. 6, and an alignment-based OPM which is described in Sec. 7 . The power of the probe beam for the orientation-based OPM was 5 mW , whereas the power of the beam in the alignment-based OPM was $\sim 10-30 \mu \mathrm{~W}$. Using a very low-powered laser beam in an OPM is challenging when it comes to designing a BPD. The main challenge is that the variance of shot noise increases linearly with light power, and so a very low-power beam means that it is difficult not to be limited by the electronic noise of the BPD rather than being limited by shot noise, which is what is required for the operation of a very sensitive RF OPM (see Sec. 2.2). The variance $i_{\text {shot }}$ for an average photocurrent $I$ is equal to

$$
\begin{equation*}
i_{\text {shot }}=2 e I=2 e R P, \tag{8.1}
\end{equation*}
$$

where $e$ is the electron charge, $R$ is the responsivity of the photodiode and $P$ is the power of the beam hitting the photodiode. Details on how to build a low-noise BPD limited by shot noise will be provided in this chapter, alongside experimental results.

An example of a high-performance table-top BPD is the Thorlabs PDB210A/M ( $R=0.58$ and $0.6 \mathrm{~A} / \mathrm{W}$ at 852 nm and 895 nm , respectively). This was used for most of the measurements described in Sec. 7 for the table-top alignment-based magnetometer. As described from Eq. 8.1, the power spectral density (PSD) of a time trace should increase linearly with optical power if the detector is shot noise limited. The laser power $P$ was split into two beams hitting two photodiodes $(P / 2$ onto each), as depicted in Fig. 8.1a. A time trace was taken for each power with a high sample rate to avoid aliasing and the PSD of the time trace calculated. This


Figure 8.1: Sketch (a) and photo (b) of the experimental setup for most of the measurements included in this chapter using a 850 nm Toptica DL Pro and BPD V2, as will be described in Sec. 8.4. Components in the setup include half-wave plates $(\lambda / 2)$, polarising beam splitters (PBS) and a balanced photodetector (BPD).
was repeated for several powers up to 3 mW . The PSD for each optical power is plotted in Fig. 8.2a. The increase in noise at low frequencies is due to $1 / f$ noise. The averages of the data in certain frequency ranges of interest were then taken and plotted as a function of optical power in Fig. 8.2b. The PSD increases linearly with optical power, indicating that the detector is limited by shot noise, not by other sources of noise. The $3-\mathrm{dB}$ bandwidth is $\sim 1 \mathrm{MHz}$ (see Fig. 8.2a), as specified by the manufacturer. Figure 8.2 c plots the laser power at which point the shot noise equals the electronic noise for different frequencies. Above this laser power the detector is shot noise limited. At 100 kHz , for example, the Thorlabs PDB210A/M is shot noise limited at $\sim 10 \mu \mathrm{~W}$. Due to the bandwidth of the BPD this laser power "threshold" increases with frequency. This performance is roughly what will be expected of the BPD that must be designed for both the orientation-based and alignment-based OPMs for MIT of the heart. Ideally, the $3-\mathrm{dB}$ bandwidth would be $1-2 \mathrm{MHz}$, as previous MIT measurements (Jensen et al., 2019, Deans et al., 2020) have been performed at 2 MHz .

### 8.2 Considerations for designing a low-noise balanced photodetector

### 8.2.1 Design of a BPD for low-frequency applications

The most basic implementation of a BPD is having two photodiodes, such as two S8729 photodiodes from Hamamatsu (Si Pin photodiode S8729 2023) (see Fig. 8.3c for photos of the photodiodes), and subtracting the photocurrents produced by each photodiode when a laser beam is incident on each photodiode. This leads to a resultant current $I_{\text {out }}$, as depicted in Fig. 8.3a. The conversion of light power into a photocurrent for a given photodiode is governed by its responsivity, which is $0.73 \mathrm{~A} / \mathrm{W}$ at 895 nm for the S 8729 photodiodes. This most simple type of BPD design is used in the portable OPM in Sec. 6. The BPD printed circuit board (PCB), designed in Altium, that was used in the portable OPM head is included in


Figure 8.2: Shot noise measurements of the Thorlabs PDB210A/M and an 850 nm Toptica DL Pro. (a) $100 \times 0.1 \mathrm{~s}$ time traces with a 10 MHz sample rate are obtained for several powers in the range $0-3 \mathrm{~mW}$. The power spectral density (PSD) of each time trace is obtained, after which the 100 power spectral densities are averaged. (b) The average PSD in several frequency ranges $(317-347 \mathrm{kHz}, 990-1010 \mathrm{kHz}, 2-$ 2.2 MHz ) are plotted as a function of optical power. (c) The optical power when the shot noise $\geq$ electronic noise is plotted as a function of frequency.

Fig. 8.3b. As data acquisition cards and lock-in amplifiers analyse input voltages, the resultant current $I_{\text {out }}$ must be converted to a voltage. For the portable OPM in Sec. 6. $I_{\text {out }}$ was passed through a commercial (Thorlabs AMP102) transimpedance amplifier (TIA) via the ethernet connector (RJ45, through hole mount) found in Fig. 8.3b, which converts the current $I_{\text {out }}$ into a voltage $V=I_{\text {out }} G$, where $G$ is the transimpedance gain, normally expressed in $\mathrm{kV} / \mathrm{A}: 1 \mathrm{kV} / \mathrm{A}, 10 \mathrm{kV} / \mathrm{A}$ and $100 \mathrm{kV} / \mathrm{A}$ are the three possible gains for the AMP102. The data in Sec. 6 was obtained with a gain of $100 \mathrm{kV} / \mathrm{A}$. The circuit in Fig. 8.3a works nicely at low frequencies, however it cannot operate at high frequencies, as the photodiodes are not reverse-biased and the capacitance of the photodiode $C_{D}$ is high ( $>50 \mathrm{pF}$ at $V_{R}=0 \mathrm{~V}$ for the S8729). If $V_{R}>0$, then $C_{D}$ decreases ( 16 pF when $V_{R}=5 \mathrm{~V}$ for the S8729), which in turn increases the bandwidth of the BPD.

As the measurements with the portable OPM in Sec. 6 were taken at low frequencies ( 10 kHz ), then the requirement for a large bandwidth was not necessary and hence $V_{R}=0$ was used. However, when designing a BPD to operate $\geq 1 \mathrm{MHz}$,


Figure 8.3: Unbiased BPD suitable for low-frequency applications. (a) A schematic of the simplest type of balanced photodetector which is used in Sec. 6. (b) The PCB designed for the BPD in Sec. 6. (c) Picture of the soldered PCB. The two pairs of twisted wires provide voltages to the RF coils in the OPM head.
as is the case for MIT of the heart experiments, then $V_{R}>0$. The fact that low frequency applications can be performed with no biasing of the photodiodes and that the power of the probe was "high" at several mW in Sec. 6 meant that the transimpedance amplification could be done outside of the OPM head by a Thorlabs AMP102 TIA. Operating an orientation-based or alignment-based magnetometer at several MHz , however, will require biasing of the photodiodes. As well as this, the amplification of the photocurrents will have to be done on the PCB inside the OPM head, not externally as was done at low frequencies. Therefore, thought has to go into how to amplify the photocurrents inside the OPM head. This will now be discussed.

### 8.2.2 Design of a BPD for higher frequency applications

An integral component of a transimpedance amplifier (TIA) is the op-amp, such as the OPA657 (OPA657 2022). An example of the simplest circuit for a TIA is provided in Fig. 8.4. Decoupling capacitors, components which clean the voltages supplied to the OPA657 by voltage regulators, are not included in this circuit. In this example the feedback resistor $R_{f}=20 \mathrm{k} \Omega$, corresponding to a transimpedance gain of $G=20 \mathrm{kV} / \mathrm{A}$. This gain is achieved because all of the current flows through


Figure 8.4: Simple example of two reverse-biased photodiodes and a transimpedance amplification stage using an OPA657. The feedback resistor R1 is $20 \mathrm{k} \Omega$. Decoupling capacitors are not included in this schematic.
the feedback resistor $R_{f}$, as no current is allowed to pass through the inverting (-) input of the TIA due to its high input impedance. The non-inverting ( + ) input is fixed to ground $(0 \mathrm{~V})$, which means that the inverting input is also fixed to 0 V . This means that the output voltage $V_{\text {out }}$ will be equal to $V_{\text {out }}=I_{\text {out }} R_{f}$. Both photodiodes are reverse-biased by 5 V to reduce each photodiode capacitance $C_{D}$.

If detecting small resultant photocurrents $I_{\text {out }}$, then the output voltage $V_{\text {out }}=$ $I_{\text {out }} G$ can be swallowed up by voltage noise $V_{\text {out, } \mathrm{n}}$ (in $\mathrm{V} / \sqrt{\mathrm{Hz}}$ ) ( Lu et al., 2019). There are four main sources of noise which can find themselves in $V_{\text {out, } n \text { OPA657 }}$ 2022). It is important to understand the contribution of each noise source when designing a BPD. The inverting (-) input of the op-amp has some input current noise $I_{-}(1.3 \mathrm{fA} / \sqrt{\mathrm{Hz}})$, which will flow through $R_{f}$ to produce voltage noise $I_{-} R_{f}$ (in $\mathrm{V} / \sqrt{\mathrm{Hz}}$ ). Another contribution is the Johnson noise $V_{J}=\sqrt{4 k_{B} T R_{f}}$ of the feedback resistor. Finally, there exists a noise source which is due to the amplification of the input voltage noise $V_{-}(4.8 \mathrm{nV} / \sqrt{\mathrm{Hz}})$. In a voltage amplifier circuit there is typically a gain resistor $R_{G}$ which sets the voltage gain to be $1+R_{f} / R_{G}$. No gain resistor $R_{G}$ is present in a TIA circuit, but a gain still exists due to the resistor being in parallel to the impedance $-j /\left(\omega C_{T}\right)$, where $C_{T}=C_{\mathrm{c}-\mathrm{m}}+C_{\mathrm{d}-\mathrm{m}}+2 C_{D}+C_{R}+C_{\mathrm{PCB}}$. From Table 8.1 the total input capacitance is $C_{T}=37.7 \mathrm{pF}$ for the OPA657 and

| $\boldsymbol{C}$ | Description | Typical value $(\mathbf{p F})$ |
| :---: | :---: | :---: |
| $C_{\mathrm{c}-\mathrm{m}}$ | op-amp common-mode input | $0.7($ OPA657) |
| $C_{\mathrm{d}-\mathrm{m}}$ | op-amp differential-mode input | 4.5 (OPA657) |
| $C_{\mathrm{D}}$ | photodiode capacitance | $16($ S8729 $)$ |
| $C_{R}$ | $R_{f}$ parasitic capacitance | $\sim 0.2$ |
| $C_{\mathrm{PCB}}$ | PCB capacitance | $50 \mathrm{fF}-$ several pF |
| $C_{T}$ | $C_{\mathrm{c}-\mathrm{m}}+C_{\mathrm{d}-\mathrm{m}}+2 C_{D}+C_{R}+C_{\mathrm{PCB}}$ | $\sim 37.7$ |

Table 8.1: Typical contributions to the total capacitance $C_{T}$.
two S 8729 photodiodes. This leads to a gain of $1+j 2 \pi \nu R_{f} C_{T}$ for $V_{-}$. The output
voltage noise $V_{\text {out, } \mathrm{n}}$ (in $\mathrm{V} / \sqrt{\mathrm{Hz}}$ ) is therefore given by

$$
\begin{equation*}
V_{\mathrm{out}, \mathrm{n}}=\sqrt{\left(I_{-} R_{f}\right)^{2}+4 k_{B} T R_{f}+V_{-}^{2}\left(1+\frac{\left(2 \pi \nu_{-3 \mathrm{~dB}} R_{f} C_{T}\right)^{2}}{3}\right)} . \tag{8.2}
\end{equation*}
$$

Assuming that $R_{f}=20 \mathrm{k} \Omega$ and $T=293 \mathrm{~K},\left(I_{-} R_{F}\right)^{2}=\left(6.76 \times 10^{-22}\right) \mathrm{V}^{2} / \mathrm{Hz}$, $4 k_{B} T R_{f}=3.24 \times 10^{-16} \mathrm{~V}^{2} / \mathrm{Hz}, V_{-}^{2}=2.30 \times 10^{-17} \mathrm{~V}^{2} / \mathrm{Hz}$ and $\left(1+\left(2 \pi \nu R_{f} C_{T}\right)^{2} / 3\right)=$ $(1+7.48)$. From this analysis it can be seen that the limiting noise source is the Johnson noise when a $R_{f}=20 \mathrm{k} \Omega$ is used as the feedback resistor with an OPA657 in a TIA circuit. However, it must be noted that when detecting very small signals (as will be shown shortly), higher gains may be necessary, as cables can pick up noise and be the dominant noise source, so larger voltages can be beneficial in this scenario, requiring even larger feedback resistors.

Another consideration in the design of the BPD is that the op-amp has a bias current, also called the leakage current (Huang et al., 2013). Ideally, no current should flow into the input op-amp terminals. However, in reality a small current flows into the terminals, which is called the bias current ( 2 pA for OPA657), which can lead to offsets and additional noise (Lu et al., 2019; Huang et al., 2013). There are ways to limit this, such as interdicting the bias current using a JFET (Lu et al., 2019; Huang et al., 2013), such that the bias current cannot flow through the feedback resistor and be a source of voltage noise. JFETs, however, were not used in these designs so as to reduce the physical size of the PCB such that it can fit into the OPM head.

It is important to consider how to make a BPD circuit stable. This generally involves including some capacitors into the design. The combined input capacitance $C_{T}$ was earlier calculated to be $\sim 37.7 \mathrm{pF}$. The Butterworth response should be satisfied to have a stable circuit. The suitable value of the feedback capacitor $C_{f}$ (C1 in Fig. 8.4) to achieve this is

$$
\begin{equation*}
C_{f}=\frac{1}{2 \pi R_{f}} \sqrt{\frac{4 \pi R_{f} C_{T}}{\mathrm{GBP}}} \tag{8.3}
\end{equation*}
$$

where GBP ( 1600 MHz for OPA657) is the gain-bandwidth product of the op-amp. Therefore, a $\sim 0.6 \mathrm{pF}$ feedback capacitor is calculated to achieve stability when $R_{f}=20 \mathrm{k} \Omega$. This is a small capacitance and is roughly comparable to the parasitic capacitance of a well-designed PCB. Excellently designed PCBs can have parasitic capacitances as low as 0.05 pF (OPA657|2022). The optimal response of the TIA can be understood by varying the capacitance and seeing how this affects the stability and bandwidth of the BPD. In the circuits presented in this thesis the feedback capacitors were not used, however they will be needed in future circuits to control the bandwidth of the BPD, as will be discussed shortly. Assuming that $C_{f}=0.6 \mathrm{pF}$ for these calculations, however, the $3-\mathrm{dB}$ cut-off frequency of this circuit should therefore be given by

$$
\begin{equation*}
f_{-3 \mathrm{~dB}}=\sqrt{\frac{\mathrm{GBP}}{2 \pi R_{f} C_{T}}}, \tag{8.4}
\end{equation*}
$$

which in this case is equal to $\sim 18 \mathrm{MHz}$, far greater than the bandwidth required for MIT of the heart. If $R_{f}=200 \mathrm{k} \Omega$, then $C_{f}=0.2 \mathrm{pF}$ and $f_{-3 \mathrm{~dB}}=5.8 \mathrm{MHz}$, also greater than the bandwidth required for MIT of the heart.

Finally, it must be decided whether or not a ground plane should be used. Ground planes are made of copper, which can therefore have eddy currents induced in them during eddy current measurements, affecting the magnetometer signals. In principle, the ground plane can be removed leaving only copper traces, however for fast op-amps like the OPA657 it is essential to have well-designed ground planes matching the data sheet specifications (OPA657 2022). For this reason whole layer ground planes were used in all designs of the PCBs presented. This may be a problem in future experiments, where low conductivity objects are imaged. It is possible that the eddy currents induced in the copper could produce a bigger magnetic field than from the object being imaged, for example the human heart. This would lead to added complications during the eddy current measurements.

A PCB was designed based on the above considerations and a schematic is shown in Fig. 8.5a. Decoupling capacitors ( C 17 and $\mathrm{C} 20,0.1 \mu \mathrm{~F}$ ) on the cathode and anode


Figure 8.5: (a) Schematic of BPD V1 with only one stage amplification (BPD V1a), the transimpedance amplification stage. The decoupling capacitors are included in the schematic. The voltage regulator circuit, which was on a separate PCB, is not included. (b) Photo of the soldered PCB. The SMA connector is disconnected, but normally carries away the signal to the data acquisition card. The hanging PCB is the voltage regulator PCB , connected via a ribbon cable to the BPD PCB.
of the two diodes are included to get rid of high-frequency noise and keep the $\pm 5 \mathrm{~V}$
supplies as clean as possible. Decoupling capacitors are also included on the noninverting $(+)$ input of the OPA657, along with decoupling capacitors on the $\pm 5 \mathrm{~V}$ supplies to the OPA657. Careful consideration is given to the layout of the OPA657, following the PCB requirements on the spec sheet (OPA657 2022). The circuit is shown in Fig. 8.5a. Notably, the feedback resistor in this example is $20 \mathrm{k} \Omega$ or $200 \mathrm{k} \Omega$, as both feedback resistors were used to compare performance.


Figure 8.6: BPD V1a with a $20 \mathrm{k} \Omega$ feedback resistor, based on the schematic in Fig. 8.5a, using the custom-made D1 laser in Sec. 7.2.1. The sample rate was 10 MHz . (a) The PSD for each light power is plotted. (b) The PSD vs light power for the frequency range $75-120 \mathrm{kHz}$ is plotted.

### 8.3 Balanced photodetector V1 results

A PCB based on the schematic in Fig. 8.5a was manufactured, soldered with components and tested by varying the laser power $P$, which was split $50 / 50$ such that $P / 2$ power hit each photodiode. If the detector is shot noise limited, then according to Eq. 8.1 the variance of the shot noise should increase linearly with laser power. A shot noise limited BPD is one where the shot noise $\geq$ electronic noise. Shot noise is white noise so is constant across all frequencies. To begin with a feedback resistor $R_{f}=20 \mathrm{k} \Omega$ was used and the power spectral densities are analysed in Fig. 8.6. It is clear to see that for the optical powers analysed the BPD is not shot noise limited.

To explore this further, the feedback resistor was increased to $200 \mathrm{k} \Omega$ in Fig. 8.7. In Fig. 8.7a, the power of the light was increased in 0.2 mW increments in the $0-1 \mathrm{~mW}$ range. In this data a 1 MHz low-pass filter (EF508) was placed between the output of the BPD and the data acquisition card, which is why there is such a drastic cut-off after 1 MHz . The shot noise remains flat from around $0.001-1 \mathrm{MHz}$ for all powers, increasing in magnitude for each light power. Different frequency regions are plotted in Fig 8.7b. This detector, BPD V1a (balanced photodetector version 1), is shot noise limited for all powers above $7.5 \mu \mathrm{~W}$ between $15-20 \mathrm{kHz}$ and above $16.5 \mu \mathrm{~W}$ between $83-122 \mathrm{kHz}$. At higher frequencies, however, the many spikes in the data mean that there are several frequency ranges where BPD V1a does not work optimally. When the 1 MHz LPF was removed, it can be seen that there is a roll off of the PSD at around 2 MHz from Figs. 8.7c and 8.7d. This


Figure 8.7: BPD V1a with $R_{f}=200 \mathrm{k} \Omega$, based on the schematic in Fig. 8.5a and using the custom-made D1 laser in Sec. 7.2.1. (a) Power spectral density plotted as a function of frequency and (b) the power spectral densities in certain frequency ranges are plotted as a function of light power with a 1 MHz low-pass filter between the output of BPD V1a and the data acquisition card. (c), (d) Exact same data as (a) and (b) but without the 1 MHz low-pass filter present, such that the bandwidth of BPD V1a can be extracted. The sample rate for all the data was 10 MHz .
means that the bandwidth of this detector is around 2 MHz . Although this had very good performance in some aspects, especially at lower frequencies $\leq 100 \mathrm{kHz}$, several improvements were required: (i) The amount of spikes in the data should be reduced, which was thought to be achievable through an improved PCB layout and with voltage regulators on the same PCB as the BPD; (ii) Increase the 0.1 dB flat frequency range to higher frequencies, i.e., $1-2 \mathrm{MHz}$; (iii) Make the BPD shot noise limited for a $2-50 \mu \mathrm{~W}$ beam at $1-2 \mathrm{MHz}$, which was not achieved in this setup due to the rising electronics noise floor at higher frequencies.

To try and combat some of these issues, a second stage amplification using a THS4021 was incorporated into the PCB design (BPD V1b). It could have been that the electronic noise/spikes in the data were being incorporated after the first stage amplification, due to the small signals from the output of the OPA657 picking up noise in the lab before the data acquisition card. If this was the case, then implementing the THS4021 second stage amplification should lead to a shot noiselimited BPD at lower powers. The circuit incorporating a second stage amplification


Figure 8.8: BPD V1b with $R_{f}=200 \mathrm{k} \Omega$ and a gain of 11 in the second stage amplification ( 10 MHz sample rate, custom-made D1 laser in Sec. 7.2.1). (a) Schematic of the circuit. (b) Several power spectral densities are plotted for different optical powers. (c) Average PSD in some frequency ranges plotted versus optical power.
is shown in Fig. 8.8a, alongside the power spectral densities as a function of optical power in some frequency ranges. The introduction of the second stage did not help lifting the shot noise level off the electronic noise floor for $1-2 \mathrm{MHz}$ frequencies. For this reason, it seemed clear that a re-design of BPD V1 was necessary.

For the design of this first BPD design (BPD V1), the voltage regulators were on a separate PCB from the PCB in the OPM head. This can be seen in Fig. 8.5b, where the voltage regulator PCB is attached via a ribbon cable from the main PCB. Not having the voltage regulators on the PCB in the OPM head was likely introducing significant noise into the circuit, because the ribbon cables that the $\pm 5 \mathrm{~V}$ and $\pm 12 \mathrm{~V}$ voltages were coming through were likely picking up noise from the lab. For this reason, it was decided that a new PCB design BPD V2 had to be made which had the voltage regulators on the PCB in the OPM head to reduce the peaks in the spectra.

### 8.4 Balanced photodetector V2 results

The schematic of the new PCB is shown in Fig. 8.9a, alongside a photo of the soldered PCB in Fig. 8.9b, as well as the traces (from Altium) in Fig. 8.9c. Almost all of the traces were on the front of the PCB to ensure that a large ground plane was present on the back of the PCB. The PCB schematic and design for the breakout


Figure 8.9: Balanced photodetector version 2 (BPD V2). (a) Schematic of the PCB. (b) Front-side of the PCB design. (c) The PCB traces on the front-side are shown in red and the ground-plane on the back is in blue.
box for BPD V2 are shown in Fig. 8.10.
As in the analysis of BPD V1, the light power was varied and the PSD of each dataset plotted in Fig. 8.11. There are no spikes present in the PSD in Fig. 8.11b, The main crucial difference between BPD V1 and BPD V2 was that BPD V2 had the voltage regulators on the same PCB as the BPD , not on a separate PCB as in BPD V1. It is likely that this was one of the main causes of the many spikes observed in the data in BPD V1 in Fig. 8.7. The increase in the electronic noise of BPD V2 is predicted in the data sheet for the OPA657 (OPA657 2022). It could be


Figure 8.10: Breakout box for BPD V2 with connections for the balanced photodetector, as well as for heating elements and current supplies to coils, provided by an ethernet connector to the OPM head.
that a change of op-amp could improve performance further. The response for BPD V2 is flat for frequencies up to around 200 kHz , at which point the signal starts to roll off due to the bandwidth of the BPD. This detector with a $150 \mathrm{k} \Omega$ feedback resistor is shot noise limited, i.e., shot noise $\geq$ electronic noise, at 3 kHz for light powers $\sim 3 \mu \mathrm{~W}$, as is plotted in Fig. 8.11c. This is a stellar performance at these low frequencies, as commercial BPDs like the Thorlabs PDB210A/M are only shot noise limited for powers $\geq 10 \mu \mathrm{~W}$ in Fig. 8.2 c at 3 kHz .

For MIT of the heart, however, improvements to the bandwidth to frequencies $\geq 1 \mathrm{MHz}$ are necessary. The photodiodes were removed and the breakout box (see Fig. 8.10) was bypassed. Oscillating voltages ( $1 \mathrm{~V}_{\mathrm{p}-\mathrm{p}}$ ) with frequencies between 10 kHz and 6 MHz from a RIGOL function generator over a $1 \mathrm{M} \Omega$ resistor, i.e., $1 \mu \mathrm{~A}$ currents, were applied at the position of the photodiodes (see Fig. 8.12a). The response of the balanced photodetector without the photodiodes is flat up to $\sim 1 \mathrm{MHz}$, at which point the gain begins to increase all the way up to 6 MHz . The maxima of the peaks in Fig. 8.12b (and extra data not included in Fig. 8.12b) are plotted in Fig. 8.12c. The increase in the gain of the OPA657 is not a problem - this will be adjusted in the future by including a feedback capacitor to create a $1-2 \mathrm{MHz}$ low-pass filter. We contrast Fig. 8.11a, where the bandwidth is $\sim 200 \mathrm{kHz}$, with Fig. 8.12b where the bandwidth is $>6 \mathrm{MHz}$. The difference in the bandwidth implies that either the breakout box (see Appendix 8.10), or the use of photodiodes, is causing an undesired roll-off at 200 kHz in the BPD V2 response. We note that the same model of S8729 photodiodes is used with BPD V1 in Fig. 8.7a which has a bandwidth of $\sim 2 \mathrm{MHz}$. The cause of the small bandwidth is believed to be due to improper impedance matching in the breakout box. The twisted pairs in the


Figure 8.11: Shot noise measurements using BPD V2 and a 850 nm Toptica DL Pro. The low-pass filter capacitor C8 in Fig. 8.9a is 220 pF LPF for this measurement, the feedback resistor $R_{f}=150 \mathrm{k} \Omega$ and the gain of the second stage amplification is $1+3 \mathrm{k} \Omega / 300 \Omega=11$. The sample rate was 40 MHz . (a) Power spectral density plotted as a function of frequency for several optical powers. (b) The PSD is plotted as a function of optical power for two frequency ranges. (c) The laser power above which the detector is shot noise limited is plotted as a function of frequency.
ethernet connector have a $100 \Omega$ impedance, but the SMA connectors have a $50 \Omega$ impedance. We believe this causes reflections leading to a smaller-than-expected bandwidth. A new breakout box is being designed with a unity gain buffer driver to be able to deal with the change of impedance and will be tested in the near future.

### 8.5 Conclusion

Several different designs of balanced photodetectors (BPDs) are presented in this chapter. One is an unbiased BPD for low-frequency applications which is used in Sec. 6 and the others are biased BPDs for higher-frequency applications, which will be used in future experiments with alignment-based OPMs and for MIT of the heart measurements. A bandwidth of 200 kHz with a stellar performance of being shot noise limited for light powers $\geq 3 \mu \mathrm{~W}$ at low frequencies of around 3 kHz is obtained with the custom-made balanced photodetector. This is especially important for


Figure 8.12: Testing the BPD without photodiodes and without the breakout box. (a) Photo of the PCB with power supplies (red, green, black wires), the applied voltage from the BNC on the right-hand-side over a $1 \mathrm{M} \Omega$ resistor, and the BNC on the bottom which is directly connected to the data acquisiton card. The sample rate was 40 MHz . (b) Example power spectral density calculations for some of the data sets. (c) Plotting the maxima of the peaks in Fig. 8.12b (including extra data sets) up to several MHz.
the alignment-based magnetometers we have worked on, as these have light powers $\leq 30 \mu \mathrm{~W}$ hitting the BPD . Despite a re-design of the breakout box being required, it is expected that the BPD designs presented in this chapter will provide a solid foundation for further development of balanced photodetectors for high-performance optically pumped magnetometers.

## Chapter 9

## Conclusion

In this thesis the two main areas of focus are (i) the development of radio-frequency optically pumped magnetometers and (ii) how to use these magnetometers for eddy current measurements.

We have developed theory on eddy current measurements in Chapter 5. Equations are derived for the arbitrary positioning of an electrically conductive sphere and a magnetometer. These equations are analysed, with a particular emphasis on understanding the optimal configuration for magnetic induction tomography of the heart using optically pumped magnetometers. As well as being useful for magnetic induction tomography of the heart, we also predict these equations being useful for other research groups working on magnetic induction tomography.

There are details in Chapter 8 about how to design compact, low-noise balanced photodetectors for sensitive optically pumped magnetometers. A balanced photodetector is designed which is shot-noise limited at $\sim 3 \mathrm{kHz}$ frequencies for light powers as low as $3 \mu \mathrm{~W}$. The performance of this balanced photodetector surpasses a commercial balanced photodetector at these frequencies.

Two types of radio-frequency optically pumped magnetometers (RF OPMs) are presented, namely an orientation-based OPM and an alignment-based OPM. The theory underpinning the orientation-based OPM is shown in Chapter 3. The theory underpinning the alignment-based magnetometer is shown in Chapter 4.

With the portable orientation-based OPM we obtained a sub-pT $/ \sqrt{\mathrm{Hz}}$ sensitivity in unshielded conditions using a room-temperature Cs paraffin-coated vapour cell, setting a new benchmark for an unshielded portable RF OPM. We achieved a sensitivity of $200 \mathrm{fT} / \sqrt{\mathrm{Hz}}$ in shielded conditions at room temperature, close to the spin projection noise limit. Heating the vapour cell would have led to further improved sensitivity. We used this RF OPM for eddy current measurements, where we managed to remotely detect aluminium disks with diameters as small as 1.5 cm at a distance of $\sim 25 \mathrm{~cm}$ from both the excitation coil and the OPM, setting a new benchmark for the remote detection of conductive objects using OPMs. Offaxis measurements of a 5 cm diameter aluminium disk are presented, demonstrating how to interpret the OPM readout for future location extraction algorithms.

For the first time, a buffer gas cell $\left(\mathrm{Cs}, \mathrm{N}_{2}\right)$ is used in an alignment-based magnetometer. The results are compared with a paraffin-coated alignment-based magnetometer. A sensitivity of $325 \mathrm{fT} / \sqrt{\mathrm{Hz}}$ is obtained with a 65 Torr $\mathrm{N}_{2}$ alignmentbased magnetometer. This one-beam RF OPM has the potential to be miniaturised compared with two- or three-beam orientation-based RF OPMs, making it suit-
able for arrays of RF OPMs in close proximity to one another, such as in magnetic induction tomography of the heart measurements. Using buffer gas vapour cells instead of paraffin-coated cells is favourable, because paraffin-coated cells can only be hand-blown and therefore have variable quality coatings. Buffer gas cells, on the other hand, can be produced on a mass scale using microfabrication techniques. We therefore believe that the developments presented in this thesis will accelerate the miniaturisation of RF OPMs.

Future experiments in our research group will strive towards further miniaturisation and further improvement in the sensitivity of optically pumped magnetometers. In the short term, magnetic induction tomography of the heart will be the main focus of the research group, where we will work on improving the robustness of prototypes and miniaturising the electronics. Our prototype OPMs will hopefully be brought over to the Niels Bohr Institute in Copenhagen, where MIT of the heart measurements will hopefully be performed on animal hearts and salt water containers. We look forward to further collaboration between our research group and the research groups under Eugene Polzik in Copenhagen and under Janek Kolodynski in Warsaw. The future of the research group under Kasper Jensen in Nottingham looks bright and, fingers crossed, stellar results will continue to come out of the lab in the next few years!

## Appendix A

## Using machine learning for defect detection using Tensorflow and Keras

Throughout this thesis there is plenty of focus on eddy current measurements. A primary magnetic field $\mathbf{B}_{1}(t)$ induces eddy currents in an object of interest, which in turn produce a secondary magnetic field $\mathbf{B}_{\text {ec }}(t)$. This method can be used to detect defects in an object. For example, this could be used to detect a hole in an otherwise solid ball, or for detecting regions of abnormal electrical conductivity in the human heart, which is thought to be a possible reason for the condition of atrial fibrillation (Marmugi and Renzoni, 2016). We will use machine learning in this section to differentiate between objects with and without a defect (Deans et al., 2018a).

Using the theory of eddy current measurements in Sec. 5, some data was generated of the secondary magnetic field $\mathbf{B}_{\text {ec }}(t)$ that is induced in a solid, electrically conductive sphere. The sphere and excitation coil are kept at a fixed position and the magnetometer is moved around relative to the object. An example of a simulation is shown in Fig. A.1. The code will be presented, alongside descriptions and comments on the figures. The code is included, because the programming is significantly different to other data analysis done in our lab and in the rest of this thesis.

## A. 1 Importing data and picking the parameters

```
import tensorflow as tf
import numpy as np
import matplotlib.pyplot as plt
from scipy import fftpack
from tensorflow.keras import regularizers
from tensorflow.keras import layers
import glob
# Specify parameters for machine learning
activation = 'relu' # Pick activation: e.g. sigmoid, relu, elu
optimiser = 'Adam' # Adam, Adamax, Adadelta, Adagrad, RMSprop
```

```
number_training = 150 # Number of training datasets
number_testing = 30 # Number of testing datasets
l2_regulariser = np.array([1.778279410038923e-07]) # Regulariser
dropout = np.array([0.5]) # Dropout
learning_rate = np.array([0.0005623413251903491]) # Learning rate
num_epochs = 9 # Number of epochs
# Paths to data
train_images_path = 'D:/TrainingData_Images'
train_labels_path = 'D:/TrainingData_Labels.txt'
test_images_path = 'D:/TestData_Images'
test_labels_path = 'D:/TestData_Labels.txt'
# Import training data images and labels
train_images = []
for i in range(number_training):
    train_images_generated = np.loadtxt(1%s/%s.txt' \
    % (train_images_path, i))
    train_images.append(train_images_generated)
train_images = np.array(train_images)
train_labels = np.loadtxt(train_labels_path)
train_labels = [int(a) for a in list(train_labels)]
# Import test data images and labels
test_images = []
for i in range(number_testing):
    test_images_generated = np.loadtxt( }1%\mathrm{ s/% s.txt' \
    % (test_images_path, i))
    test_images.append(test_images_generated)
test_images = np.array(test_images)
test_labels = np.loadtxt(test_labels_path)
test_labels = [int(a) for a in list(test_labels)]
# No defect (0) or Defect (1)
class_names = ['0', '1']
```


## A. 2 Plotting some of the training images

Some training images are plotted, some with a defect (hole in the centre) and some without (no hole in the centre) in Fig. A.2. It is clear that the defect samples look very similar to one another, however there are subtle differences in the images depending on the strength of the induced magnetic field. This is meant to be only a trivial example to illustrate the potential of using machine learning - this could of course be applied to more complex defects and cracks in the future. The challenge will be to provide a large enough range of data for the machine to train on for more
complex problems.

```
# Preparing the figure
plt.figure()
plt.imshow(np.abs(train_images[40]))
plt.colorbar()
plt.grid(False)
plt.xlabel('x')
plt.ylabel('y')
plt.show()
```



Figure A.1: A simulation of our magnetometer being scanned over a spherical object, measuring the secondary magnetic field induced in the object, and plotting it in a heatmap.

```
# Plotting some of the training images
plt.figure(figsize = (10,10))
for i in range(60, 85, 1):
    plt.subplot(5,5,i-60+1)
    plt.xticks([])
    plt.yticks([])
    plt.grid(False)
    plt.imshow(train_images[i], cmap=plt.cm.binary)
    plt.xlabel(class_names[train_labels[i]])
plt.show()
```

The arrays for the training and test data sets are then flattened and plotted as a function of pixel position. Flattening the array shows more intuitively how machine learning can be applied to basically any data set. In Fig. A.3, for the pixels 0-100 (corresponding to the first 4 rows of 25 pixels in each row), the pixel value is $\sim 0$. This means that the image is white, or the lowest value, which is what we see in the image above. Then, we begin to see some periodicity emerging between 300 and 500 pixels, which is when there is a region of higher induced magnetic field. This is simply because the magnetometer is close to the object that is being probed.


Figure A.2: Simulations of eddy current measurements of objects with a defect (1) and without a defect (0).


Figure A.3: Plot of the pixel value as a function of pixel number, to help illustrate how machine learning works. This could just as well be any other form of data.

## A. 3 Setting up layers for the model

In order to build a neural network we need to configure layers for the model (Monga et al., 2021), then compile the model. The basic building block of a neural network is a layer. Layers extract representations from the data fed to them. It is important
to realise that we cannot force a layer to concentrate on a specific property. The programme analyses the data and fits the data to some functions, using different layers to extract different representations.

Before we can train the model, there are a few settings we need to include when compiling the model:

- We need a "loss function", which measures how accurate the model is during training. We want to minimise this loss to "steer" the model in the correct direction. However, it is important not to train your model too much (hence steer it too much), because the model will begin overfitting to the noise in the data, not to the general properties of the data. This makes it worse at correctly guessing whether a defect is present or not.
- An "optimiser" is needed. This is how the model is updated based on the data it sees and its loss function. Optimisers include: Adam, Adamax, RMSprop, among others.
- We need "metrics", which are used to compare the training and testing steps. In this example we use accuracy, which indicates the fraction of the images that are correctly classified.


## A. 4 Machine learning part of code

The model learns to fit itself to the training data. The number of runs, or "epochs" can be specified. After each epoch, the model gets better at fitting itself to the data. The most important parameter to extract from this cell is the "test accuracy". This is the fraction of the test images which are correctly classified by the machine learning model. This example is so simple that the machine learning model gets $100 \%$ correct classification. In real defect detection, the defects will come in all shapes and sizes and this will not be the case.

```
# Machine learning part of the code
i = 0
i2 = 0
i3 = 0
model = tf.keras.Sequential([
    tf.keras.layers.Flatten(input_shape=(28, 28)),
    tf.keras.layers.Dense(256, activation=activation,
    kernel_regularizer=regularizers.l2(l2_regulariser[i2])),
    layers.Dropout(dropout[i]),
    tf.keras.layers.Dense(2)
])
if optimiser == 'Adam':
    opt = tf.keras.optimizers.Adam(learning_rate=learning_rate[i3])
elif optimiser == 'Adamax':
    opt = tf.keras.optimizers.Adamax(learning_rate\
    =learning_rate[i3])
elif optimiser == 'Adadelta':
```

```
    opt = tf.keras.optimizers.Adadelta(learning_rate\
    =learning_rate[i3])
elif optimiser == 'Adagrad':
    opt = tf.keras.optimizers.Adadelta(learning_rate\
    =learning_rate[i3])
elif optimiser == 'RMSprop':
    opt = tf.keras.optimizers.Adadelta(learning_rate\
    =learning_rate[i3])
model.compile(optimizer=opt,
                                    loss=tf.keras.losses.SparseCategoricalCrossentropy(\
                                    from_logits=True),
                                    metrics=['accuracy'])
model.fit(train_images, train_labels, epochs=num_epochs)
test_loss, test_acc = model.evaluate(test_images, test_labels, \
verbose=2)
print('\nTest accuracy:', test_acc)
probability_model = tf.keras.Sequential([model,
                                    tf.keras.layers.Softmax()])
predictions = probability_model.predict(test_images)
predictions[0]
np.argmax(predictions[0])
test_labels[0]
class_names[test_labels[0]]
```


## A. 5 Using the trained model to make a prediction about a single image

If we wanted to use this for magnetic induction tomography of the heart in a clinical setting (Marmugi and Renzoni, 2016; Jensen et al., 2019; Deans et al., 2020), for example, you could take a set of test data from a patient, then input that image into this programme, and the programme would be able to use its trained model to try and determine the optimum operating procedure for a surgeon by finding the position of the defect in the patient's heart. A test data set with a defect was selected (easy to observe in Fig. A.4 by eye) and the computer could say with $100 \%$ certainty that there is a defect. This is a trivial example of how to use machine learning to detect a defect in an object. In the future, this could be developed further with more complex training data and challenging test data to test the limitations of the programme.

```
# Testing a single image
test_data_of_choice = 0
def plot_value_array(i, predictions_array, true_label):
    true_label = true_label[i]
    plt.grid(False)
```

```
    plt.xticks(range(len(class_names)))
    plt.yticks([])
    thisplot = plt.bar(range(len(class_names)), predictions_array, \
    color="#777777")
    plt.ylim([0, 1])
    predicted_label = np.argmax(predictions_array)
    thisplot[predicted_label].set_color('red')
    thisplot[true_label].set_color('blue')
def plot_image(i, predictions_array, true_label, img):
    true_label, img = true_label[i], img[i]
    plt.grid(False)
    plt.xticks([])
    plt.yticks([])
    plt.imshow(img, cmap=plt.cm.binary)
    predicted_label = np.argmax(predictions_array)
    if predicted_label == true_label:
        color = 'blue'
    else:
        color = 'red'
    plt.xlabel("{} {:2.0f}% ({})".format(class_names[predicted_label],
                                    100*np.max(predictions_array),
                                    class_names[true_label]),
                                    color=color)
# Here the 22nd test data set is selected.
i = 22
plt.figure(figsize = (6,3))
plt.subplot(1,2,1)
plot_image(i, predictions[i], test_labels, test_images)
plt.subplot(1,2,2)
plot_value_array(i, predictions[i], test_labels)
plt.show()
```



Figure A.4: A test data set with a defect was picked, input to the computer programme and the machine predicted with $100 \%$ accuracy that this object had a defect.

## Appendix B

## Saturated absorption spectroscopy

On the D1 line, the spectral lines are resolved from one another, because the $F^{\prime}=3$ and $F^{\prime}=4$ excited states are separated by 1.167 GHz (Steck, 2022), whereas the Doppler broadening at, for example, $45^{\circ} \mathrm{C}$ is 371 MHz . In the case of the D 2 line,


Figure B.1: Saturated absorption spectroscopy on the D2 line. (a) Unnormalised saturated absorption spectroscopy. (b) The data in (a) is subtracted from a normal absorption spectrum without the pump laser (see Fig. 2.6a). (c) The $x$-axis is converted to frequency and the individual hyperfine transitions can be observed.
however, the spectral lines cannot be individually resolved because the excited states are closer to each other than the Doppler broadening. It can be useful to perform saturated absorption spectroscopy (Preston, 1996), or Doppler-free spectroscopy, for optimal locking of a D2 laser to a specific atomic transition, for example to the $F=4 \rightarrow F^{\prime}=5$ transition. A high power pump beam propagates in the opposite direction to a low power probe beam in the vapour cell. Both lasers have the same frequencies. Consider the pump beam to be red-detuned from $F=4 \rightarrow F^{\prime}=5$. Photons from the pump beam will be absorbed by atoms travelling away from the propagation direction of this beam with a specific velocity. This same atom will not absorb a photon from the probe beam as this beam will appear blue-detuned to the atom. However, when the atom has no velocity component along the propagation direction of the pump and probe beams, then both pump and probe beams will appear on resonance. Due to the high power of the pump beam many of the atoms will be in the excited state and so the weak probe beam can cause stimulated emission, thus leading to an increased power being measured on the photodiode. This leads to the kinks observed in Fig. B.1. None of the data taken in this thesis was obtained with laser locking using saturated absorption spectroscopy, but we often refer to these graphs to understand the transitions. Most notably, the $F=$ $3 \rightarrow F^{\prime}=2$ transition has a characteristic kink that goes downwards, in contrast to the $F=4 \rightarrow F^{\prime}=5$ transition where the kink goes upwards.

## Appendix C

## Optical pumping with circularly-polarised light

The 16 rate equations for the 16 ground state magnetic sublevels are given when circularly-polarised light is resonant on the $F=3 \rightarrow F^{\prime}=4$ transition on the D1 line in a paraffin-coated Cs cell in the absence of a quenching gas:

$$
\begin{align*}
\frac{d p_{3,-3}}{d t}= & R_{p}\left[-p_{3,-3}\left(c_{3,-3 \leftrightarrow 4^{\prime},-2^{\prime}}\right)+p_{3,-3}\left(c_{3,-3 \leftrightarrow 4^{\prime},-2^{\prime}} c_{3,-3 \leftrightarrow 4^{\prime},-2^{\prime}}\right)\right]  \tag{C.1}\\
& -\Gamma_{1} p_{3,-3}+\frac{\Gamma_{1}}{16}, \\
\frac{d p_{3,-2}}{d t}= & R_{p}\left[-p_{3,-2}\left(c_{3,-2 \leftrightarrow 4^{\prime},-1^{\prime}}\right)+p_{3,-2}\left(c_{3,-2 \leftrightarrow 4^{\prime},-1^{\prime}} c_{3,-2 \leftrightarrow 4^{\prime},-1^{\prime}}\right)\right. \\
& \left.+p_{3,-3}\left(c_{3,-3 \leftrightarrow 4^{\prime},-2^{\prime}} c_{3,-2 \leftrightarrow 4^{\prime},-2^{\prime}}\right)\right]-\Gamma_{1} p_{3,-2}+\frac{\Gamma_{1}}{16},  \tag{C.2}\\
\frac{d p_{3,-1}}{d t}= & R_{p}\left[-p_{3,-1}\left(c_{3,-1 \leftrightarrow 4^{\prime}, 0^{\prime}}\right)+p_{3,-1}\left(c_{3,-1 \leftrightarrow 4^{\prime}, 0^{\prime}} c_{3,-1 \leftrightarrow 4^{\prime}, 0^{\prime}}\right)\right. \\
& +p_{3,-2}\left(c_{3,-2 \leftrightarrow 4^{\prime},-1^{\prime}} c_{3,-1 \leftrightarrow 4^{\prime},-1^{\prime}}\right)+p_{3,-3}\left(c_{\left.\left.3,-3 \leftrightarrow 4^{\prime},-2^{\prime} c_{3,-1 \leftrightarrow 4^{\prime},-2^{\prime}}\right)\right]}\right.  \tag{C.3}\\
& -\Gamma_{1} p_{3,-1}+\frac{\Gamma_{1}}{16}, \\
\frac{d p_{3,0}}{d t}= & R_{p}\left[-p_{3,0}\left(c_{3,0 \leftrightarrow 4^{\prime},-1^{\prime}}\right)+p_{3,0}\left(c_{3,0 \leftrightarrow 4^{\prime},-1^{\prime}} c_{3,0 \leftrightarrow 4^{\prime},-1^{\prime}}\right)\right. \\
& \left.+p_{3,-1}\left(c_{3,-1 \leftrightarrow 4^{\prime}, 0^{\prime}} c_{3,0 \leftrightarrow 4^{\prime}, 0^{\prime}}\right)+p_{3,-2}\left(c_{3,-2 \leftrightarrow 4^{\prime},-1^{\prime}} c_{3,0 \leftrightarrow 4^{\prime},-1^{\prime}}\right)\right]  \tag{C.4}\\
& -\Gamma_{1} p_{3,0}+\frac{\Gamma_{1}}{16}, \\
\frac{d p_{3,1}}{d t}= & R_{p}\left[-p_{3,1}\left(c_{3,1 \leftrightarrow 4^{\prime}, 2^{\prime}}\right)+p_{3,0}\left(c_{3,0 \leftrightarrow 4^{\prime},-1^{\prime}} c_{3, \leftrightarrow \leftrightarrow 4^{\prime}, 1^{\prime}}\right)\right.  \tag{C.5}\\
& \left.+p_{3,1}\left(c_{3,1 \leftrightarrow 4^{\prime}, 2^{\prime}} c_{3,1 \leftrightarrow 4^{\prime}, 2^{\prime}}\right)+p_{3,-1}\left(c_{3,-1 \leftrightarrow 4^{\prime}, 0^{\prime}} c_{3,1 \leftrightarrow 4^{\prime}, 0^{\prime}}\right)\right]-\Gamma_{1} p_{3,1}+\frac{\Gamma_{1}}{16}, \\
\frac{d p_{3,2}}{d t}= & R_{p}\left[-p_{3,2}\left(c_{3,2 \leftrightarrow 4^{\prime}, 3^{\prime}}\right)+p_{3,1}\left(c_{3,1 \leftrightarrow 4^{\prime}, 2^{\prime}} c_{3,2 \leftrightarrow 4^{\prime}, 2^{\prime}}\right)\right.  \tag{C.6}\\
& \left.+p_{3,2}\left(c_{3,2 \leftrightarrow 4^{\prime}, 3^{\prime}} c_{3,2 \leftrightarrow 4^{\prime}, 3^{\prime}}\right)+p_{3,0}\left(c_{3,0 \leftrightarrow 4^{\prime},-1^{\prime}} c_{3,2 \leftrightarrow 4^{\prime},,^{\prime}}\right)\right]-\Gamma_{1} p_{3,2}+\frac{\Gamma_{1}}{16},
\end{align*}
$$

$$
\begin{align*}
\frac{d p_{3,3}}{d t}= & R_{p}\left[-p_{3,3}\left(c_{3,3 \leftrightarrow 4^{\prime}, 4^{\prime}}\right)+p_{3,3}\left(c_{3,3 \leftrightarrow 4^{\prime}, 4^{\prime}} c_{3,3 \leftrightarrow 4^{\prime}, 4^{\prime}}\right)\right.  \tag{C.7}\\
& \left.+p_{3,2}\left(c_{3,2 \leftrightarrow 4^{\prime}, 3^{\prime}} c_{3,3 \leftrightarrow 4^{\prime}, 3^{\prime}}\right)+p_{3,1}\left(c_{3,1 \leftrightarrow 4^{\prime}, 2^{\prime}} c_{3,3 \leftrightarrow 4^{\prime}, 2^{\prime}}\right)\right]-\Gamma_{1} p_{3,3}+\frac{\Gamma_{1}}{16} \\
\frac{d p_{4,-4}}{d t}= & 0  \tag{C.8}\\
\frac{d p_{4,-3}}{d t}= & R_{p} p_{3,-3}\left(c_{3,-3 \leftrightarrow 4^{\prime},-2^{\prime}} c_{4,-3 \leftrightarrow 4^{\prime},-2^{\prime}}\right)-\Gamma_{1} p_{4,-3}+\frac{\Gamma_{1}}{16},  \tag{C.9}\\
\frac{d p_{4,-2}}{d t}= & R_{p}\left[p _ { 3 , - 2 } \left(c_{\left.\left.3,-2 \leftrightarrow 4^{\prime},-1^{\prime} c_{4,-2 \leftrightarrow 4^{\prime},-1^{\prime}}\right)+p_{3,-3}\left(c_{3,-3 \leftrightarrow 4^{\prime},-2^{\prime}} c_{4,-2 \leftrightarrow 4^{\prime},-2^{\prime}}\right)\right]}\right.\right.  \tag{C.10}\\
& -\Gamma_{1} p_{4,-2}+\frac{\Gamma_{1}}{16}, \\
\frac{d p_{4,-1}}{d t}= & R_{p}\left[p_{3,-1}\left(c_{3,-1 \leftrightarrow 4^{\prime}, 0^{\prime}} c_{4,-1 \leftrightarrow 4^{\prime}, 0^{\prime}}\right)+p_{3,-2}\left(c_{\left.3,-2 \leftrightarrow 4^{\prime},-1^{\prime} c_{4,-1 \leftrightarrow 4^{\prime},-1^{\prime}}\right)}\right.\right.  \tag{C.11}\\
& \left.+p_{3,-3}\left(c_{3,-3 \leftrightarrow 4^{\prime},-2^{\prime}} c_{4,-1 \leftrightarrow 4^{\prime},-2^{\prime}}\right)\right]-\Gamma_{1} p_{4,-1}+\frac{\Gamma_{1}}{16}, \\
\frac{d p_{4,0}}{d t}= & R_{p}\left[p _ { 3 , 0 } \left(c_{\left.3,0 \leftrightarrow 4^{\prime},-1^{\prime} c_{4,0 \leftrightarrow 4^{\prime}, 1^{\prime}}\right)+p_{3,-1}\left(c_{3,-1 \leftrightarrow 4^{\prime}, 0^{\prime}} c_{4,0 \leftrightarrow 4^{\prime}, 0^{\prime}}\right)}\right.\right.  \tag{C.12}\\
& \left.+p_{3,-2}\left(c_{3,-2 \leftrightarrow 4^{\prime},-1^{\prime}} c_{4,0 \leftrightarrow 4^{\prime},-1^{\prime}}\right)\right]-\Gamma_{1} p_{4,0}+\frac{\Gamma_{1}}{16}, \\
\frac{d p_{4,1}}{d t}= & R_{p}\left[p_{3,0}\left(c_{3,0 \leftrightarrow 4^{\prime},-1^{\prime}} c_{4,1 \leftrightarrow 4^{\prime}, 1^{\prime}}\right)+p_{3,1}\left(c_{3,1 \leftrightarrow 4^{\prime}, 2^{\prime}} c_{4,1 \leftrightarrow 4^{\prime}, 2^{\prime}}\right)\right. \\
& \left.+p_{3,-1}\left(c_{3,-1 \leftrightarrow 4^{\prime}, 0^{\prime}} c_{4,1 \leftrightarrow 4^{\prime}, 0^{\prime}}\right)\right]-\Gamma_{1} p_{4,1}+\frac{\Gamma_{1}}{16},  \tag{C.13}\\
\frac{d p_{4,2}}{d t}= & R_{p}\left[p_{3,1}\left(c_{3,1 \leftrightarrow 4^{\prime}, 2^{\prime}} c_{4,2 \leftrightarrow 4^{\prime}, 2^{\prime}}\right)+p_{3,2}\left(c_{3,2 \leftrightarrow 4^{\prime}, 3^{\prime}} c_{4,2 \leftrightarrow 4^{\prime}, 3^{\prime}}\right)\right.  \tag{C.14}\\
& \left.+p_{3,0}\left(c_{3,0 \leftrightarrow 4^{\prime},-1^{\prime}} c_{4,2 \leftrightarrow 4^{\prime}, 1^{\prime}}\right)\right]-\Gamma_{1} p_{4,2}+\frac{\Gamma_{1}}{16}, \\
\frac{d p_{4,3}}{d t}= & R_{p}\left[p_{3,3}\left(c_{3,3 \leftrightarrow 4^{\prime}, 4^{\prime}} c_{4,3 \leftrightarrow 4^{\prime}, 4^{\prime}}\right)+p_{3,2}\left(c_{3,2 \leftrightarrow 4^{\prime}, 3^{\prime}} c_{4,3 \leftrightarrow 4^{\prime}, 3^{\prime}}\right)\right. \\
& +p_{3,1}\left(c_{\left.\left.3,1 \leftrightarrow 4^{\prime}, 2^{\prime} c_{4,3 \leftrightarrow 4^{\prime}, 2^{\prime}}\right)\right]-\Gamma_{1} p_{4,3}+\frac{\Gamma_{1}}{16}}^{d t}=\right.  \tag{C.15}\\
\frac{d p_{4,4}}{d t}= & R_{p}\left[p_{3,3}\left(c_{3,3 \leftrightarrow 4^{\prime}, 4^{\prime}} c_{4,4 \leftrightarrow 4^{\prime}, 4^{\prime}}\right)+p_{3,2}\left(c_{3,2 \leftrightarrow 4^{\prime}, 3^{\prime}} c_{4,4 \leftrightarrow 4^{\prime}, 3^{\prime}}\right)\right]-\Gamma_{1} p_{4,4}+\frac{\Gamma_{1}}{16}, \tag{C.16}
\end{align*}
$$

where $R_{p}$ is the optical pumping rate, $p_{3,3}=p_{3,3}(t)$ is the population of the $F=$ $3, m=3$ sublevel, $c_{3,3 \rightarrow 4,4^{\prime}}$ is the Clebsch-Gordon coefficient squared for the $\sigma^{+}$ transition from the $F=3, m=3$ sublevel to the $F=4, m=4$ sublevel, and $\Gamma_{1}$ is the longitudinal relaxation rate of the Cs sample.

## Appendix D

## Optical pumping with linearly-polarised light

The 16 rate equations for the 16 ground state magnetic sublevels are given when linearly-polarised light is resonant on the $F=4 \rightarrow F^{\prime}=3$ transition on the D1 line in a paraffin-coated Cs cell in the absence of a quenching gas:

$$
\begin{align*}
\frac{d p_{4,4}}{d t}= & R_{p}\left(p_{4,3} c_{4,3 \leftrightarrow 3^{\prime}, 3^{\prime}} c_{4,4 \leftrightarrow 3^{\prime}, 3^{\prime}}\right)-\Gamma_{1} p_{4,4}+\frac{\Gamma_{1}}{16},  \tag{D.1}\\
\frac{d p_{4,3}}{d t}= & R_{p}\left[-p_{4,3}\left(c_{4,3 \leftrightarrow 3^{\prime}, 3^{\prime}}\right)+p_{4,3}\left(c_{4,3 \leftrightarrow 3^{\prime}, 3^{\prime}} c_{4,3 \leftrightarrow 3^{\prime}, 3^{\prime}}\right)\right. \\
& \left.+p_{4,2}\left(c_{4,2 \leftrightarrow 3^{\prime}, 2^{\prime}} c_{4,3 \leftrightarrow 3^{\prime}, 2^{\prime}}\right)\right]-\Gamma_{1} p_{4,3}+\frac{\Gamma_{1}}{16},  \tag{D.2}\\
\frac{d p_{4,2}}{d t}= & R_{p}\left[-p_{4,2}\left(c_{4,2 \leftrightarrow 3^{\prime}, 2^{\prime}}\right)+p_{4,3}\left(c_{4,3 \leftrightarrow 3^{\prime}, 3^{\prime}} c_{4,2 \leftrightarrow 3^{\prime}, 3^{\prime}}\right)+p_{4,1}\left(c_{4,1 \leftrightarrow 3^{\prime}, 1^{\prime}} c_{4,2 \leftrightarrow 3^{\prime}, 1^{\prime}}\right)\right.  \tag{D.3}\\
& \left.+p_{4,2}\left(c_{4,2 \leftrightarrow 3^{\prime}, 2^{\prime}} c_{4,2 \leftrightarrow 3^{\prime}, 2^{\prime}}\right)\right]-\Gamma_{1} p_{4,2}+\frac{\Gamma_{1}}{16}, \\
\frac{d p_{4,1}}{d t}= & R_{p}\left[-p_{4,1}\left(c_{4,1 \leftrightarrow 3^{\prime}, 1^{\prime}}\right)+p_{4,2}\left(c_{4,2 \leftrightarrow 3^{\prime}, 2^{\prime}} c_{4,1 \leftrightarrow 3^{\prime}, 2^{\prime}}\right)+p_{4,0}\left(c_{4,0 \leftrightarrow 3^{\prime}, 0^{\prime}} c_{4,1 \leftrightarrow 3^{\prime}, 0^{\prime}}\right)\right. \\
& \left.+p_{4,1}\left(c_{4,1 \leftrightarrow 3^{\prime}, 1^{\prime}} c_{4,1 \leftrightarrow 3^{\prime}, 1^{\prime}}\right)\right]-\Gamma_{1} p_{4,1}+\frac{\Gamma_{1}}{16},  \tag{D.4}\\
\frac{d p_{4,0}}{d t}= & R_{p}\left[-p_{4,0}\left(c_{4,0 \leftrightarrow 3^{\prime}, 0^{\prime}}\right)+p_{4,1}\left(c_{4,1 \leftrightarrow 3^{\prime}, 1^{\prime}} c_{4,0 \leftrightarrow 3^{\prime}, 1^{\prime}}\right)+\right. \\
& \left.+p_{4,-1}\left(c_{4,-1 \leftrightarrow 3^{\prime},-1^{\prime}} c_{4,0 \leftrightarrow 3^{\prime},-1^{\prime}}\right)+p_{4,0}\left(c_{4,0 \leftrightarrow 3^{\prime}, 0^{\prime}} c_{4,0 \leftrightarrow 3^{\prime}, 0^{\prime}}\right)\right]-\Gamma_{1} p_{4,0}  \tag{D.5}\\
& +\frac{\Gamma_{1}}{16}, \\
\frac{d p_{4,-1}}{d t}= & R_{p}\left[-p_{4,-1}\left(c_{4,-1 \leftrightarrow 3^{\prime},-1^{\prime}}\right)+p_{4,0}\left(c_{\left.4,0 \leftrightarrow 3^{\prime}, 0^{\prime} c_{4,-1 \leftrightarrow 3^{\prime}, 0^{\prime}}\right)+}\right.\right. \\
& \left.+p_{4,-2}\left(c_{4,-2 \leftrightarrow 3^{\prime},-2^{\prime}} c_{4,-1 \leftrightarrow 3^{\prime},-2^{\prime}}\right)+p_{4,-1}\left(c_{4,-1 \leftrightarrow 3^{\prime},-1^{\prime}} c_{4,-1 \leftrightarrow 3^{\prime},-1^{\prime}}\right)\right]  \tag{D.6}\\
& -\Gamma_{1} p_{4,-1}+\frac{\Gamma_{1}}{16},
\end{align*}
$$

$$
\begin{align*}
& \frac{d p_{4,-2}}{d t}=R_{p}\left[-p_{4,-2}\left(c_{4,-2 \leftrightarrow 3^{\prime},-2^{\prime}}\right)+p_{4,-1}\left(c_{4,-1 \leftrightarrow 3^{\prime},-1^{\prime}} c_{4,-2 \leftrightarrow 3^{\prime},-1^{\prime}}\right)\right. \\
& \left.+p_{4,-3}\left(c_{4,-3 \leftrightarrow 3^{\prime},-3^{\prime}} c_{4,-2 \leftrightarrow 3^{\prime},-3^{\prime}}\right)+p_{4,-2}\left(c_{4,-2 \leftrightarrow 3^{\prime},-2^{\prime}} c_{4,-2 \leftrightarrow 3^{\prime},-2^{\prime}}\right)\right]  \tag{D.7}\\
& -\Gamma_{1} p_{4,-2}+\frac{\Gamma_{1}}{16}, \\
& \frac{d p_{4,-3}}{d t}=R_{p}\left[-p_{4,-3}\left(c_{4,-3 \leftrightarrow 3^{\prime},-3^{\prime}}\right)+p_{4,-2}\left(c_{4,-2 \leftrightarrow 3^{\prime},-2^{\prime}} c_{4,-3 \leftrightarrow 3^{\prime},-2^{\prime}}\right)\right. \\
& \left.+p_{4,-3}\left(c_{4,-3 \leftrightarrow 3^{\prime},-3^{\prime}} c_{4,-3 \leftrightarrow 3^{\prime},-3^{\prime}}\right)\right]-\Gamma_{1} p_{4,-3}+\frac{\Gamma_{1}}{16},  \tag{D.8}\\
& \frac{d p_{4,-4}}{d t}=R_{p}\left[p_{4,-3}\left(c_{4,-3 \leftrightarrow 3^{\prime},-3^{\prime}} c_{4,-4 \leftrightarrow 3^{\prime},-3^{\prime}}\right)\right]-\Gamma_{1} p_{4,-4}+\frac{\Gamma_{1}}{16},  \tag{D.9}\\
& \frac{d p_{3,3}}{d t}=R_{p}\left[p_{4,3}\left(c_{4,3 \leftrightarrow 3^{\prime}, 3^{\prime}} c_{3,3 \leftrightarrow 3^{\prime}, 3^{\prime}}\right)+p_{4,2}\left(c_{4,2 \leftrightarrow 3^{\prime}, 2^{\prime}} c_{3,3 \leftrightarrow 3^{\prime}, 2^{\prime}}\right)\right]-\Gamma_{1} p_{3,3}  \tag{D.10}\\
& +\frac{\Gamma_{1}}{16}, \\
& \frac{d p_{3,2}}{d t}=R_{p}\left[p_{4,3}\left(c_{4,3 \leftrightarrow 3^{\prime}, 3^{\prime}} c_{3,2 \leftrightarrow 3^{\prime}, 3^{\prime}}\right)+p_{4,2}\left(c_{4,2 \leftrightarrow 3^{\prime}, 2^{\prime}} c_{3,2 \leftrightarrow 3^{\prime}, 2^{\prime}}\right)\right.  \tag{D.11}\\
& \left.+p_{4,1}\left(c_{4,1 \leftrightarrow 3^{\prime}, 1^{\prime}} c_{3,2 \leftrightarrow 3^{\prime}, 1^{\prime}}\right)\right]-\Gamma_{1} p_{3,2}+\frac{\Gamma_{1}}{16}, \\
& \frac{d p_{3,1}}{d t}=R_{p}\left[p_{4,2}\left(c_{4,2 \leftrightarrow 3^{\prime}, 2^{\prime}} c_{3,1 \leftrightarrow 3^{\prime}, 2^{\prime}}\right)+p_{4,1}\left(c_{4,1 \leftrightarrow 3^{\prime}, 1^{\prime}} c_{3,1 \leftrightarrow 3^{\prime}, 1^{\prime}}\right)\right. \\
& \left.+p_{4,0}\left(c_{4,0 \leftrightarrow 3^{\prime}, 0^{\prime}} c_{3,1 \leftrightarrow 3^{\prime}, 0^{\prime}}\right)\right]-\Gamma_{1} p_{3,1}+\frac{\Gamma_{1}}{16},  \tag{D.12}\\
& \frac{d p_{3,0}}{d t}=R_{p}\left[p_{4,1}\left(c_{4,1 \leftrightarrow 3^{\prime}, 1^{\prime}} c_{3,0 \leftrightarrow 3^{\prime}, 1^{\prime}}\right)+p_{4,0}\left(c_{4,0 \leftrightarrow 3^{\prime}, 0^{\prime}} c_{3,0 \leftrightarrow 3^{\prime}, 0^{\prime}}\right)\right. \\
& \left.+p_{4,-1}\left(c_{4,-1 \leftrightarrow 3^{\prime},-1^{\prime}} c_{3,0 \leftrightarrow 3^{\prime},-1^{\prime}}\right)\right]-\Gamma_{1} p_{3,0}+\frac{\Gamma_{1}}{16},  \tag{D.13}\\
& \frac{d p_{3,-1}}{d t}=R_{p}\left[p_{4,0}\left(c_{4,0 \leftrightarrow 3^{\prime}, 0^{\prime}} c_{3,-1 \leftrightarrow 3^{\prime}, 0^{\prime}}\right)+p_{4,-1}\left(c_{4,-1 \leftrightarrow 3^{\prime},-1^{\prime}} c_{3,-1 \leftrightarrow 3^{\prime},-1^{\prime}}\right)\right.  \tag{D.14}\\
& \left.+p_{4,-2}\left(c_{4,-2 \leftrightarrow 3^{\prime},-2^{\prime}} c_{3,-1 \leftrightarrow 3^{\prime},-2^{\prime}}\right)\right]-\Gamma_{1} p_{3,-1}+\frac{\Gamma_{1}}{16}, \\
& \frac{d p_{3,-2}}{d t}=R_{p}\left[p_{4,-1}\left(c_{4,-1 \leftrightarrow 3^{\prime},-1^{\prime}} c_{3,-2 \leftrightarrow 3^{\prime},-1^{\prime}}\right)+p_{4,-2}\left(c_{4,-2 \leftrightarrow 3^{\prime},-2^{\prime}} c_{3,-2 \leftrightarrow 3^{\prime},-2^{\prime}}\right)\right. \\
& \left.+p_{4,-3}\left(c_{4,-3 \leftrightarrow 3^{\prime},-3^{\prime}} c_{3,-2 \leftrightarrow 3^{\prime},-3^{\prime}}\right)\right]-\Gamma_{1} p_{3,-2}+\frac{\Gamma_{1}}{16},  \tag{D.15}\\
& \frac{d p_{3,-3}}{d t}=R_{p}\left[p_{4,-2}\left(c_{4,-2 \leftrightarrow 3^{\prime},-2^{\prime}} c_{3,-3 \leftrightarrow 3^{\prime},-2^{\prime}}\right)+p_{4,-3}\left(c_{4,-3 \leftrightarrow 3^{\prime},-3^{\prime}} c_{3,-3 \leftrightarrow 3^{\prime},-3^{\prime}}\right)\right] \\
& -\Gamma_{1} p_{3,-3}+\frac{\Gamma_{1}}{16}, \tag{D.16}
\end{align*}
$$

where $R_{p}$ is the optical pumping rate, $p_{4,-4}=p_{4,-4}(t)$ is the population of the $F=$ $4, m=-4$ sublevel, $c_{4,-1 \rightarrow 3^{\prime},-1}$, is the Clebsch-Gordon coefficient squared for the $\pi$ transition from the $F=4, m=-1$ sublevel to the $F=3, m=-1$ sublevel, and $\Gamma_{1}$ is the longitudinal relaxation rate of the Cs sample.

## Appendix E

## Faraday rotation calculations

The Hamiltonian for the off-resonant interaction of a probe beam in a two-level system is (Julsgaard, 2003; Sherson, 2006)

$$
\begin{align*}
H= & \hbar \omega\left(a_{+}^{\dagger} a_{+}+a_{-}^{\dagger} a_{-}\right)+\hbar \omega_{0}\left(\rho_{33}+\rho_{44}\right)+\hbar g\left(a_{+}^{\dagger} e^{i \omega t} \rho_{14}+\rho_{41} a_{+} e^{-i \omega t}\right) \\
& +\hbar g\left(a_{-}^{\dagger} e^{i \omega t} \rho_{23}+\rho_{32} a_{-} e^{-i \omega t}\right) . \tag{E.1}
\end{align*}
$$

Following the same methodology from Sec. 3, we calculate $d \rho_{14} / d t, d \rho_{41} / d t, d \rho_{23} / d t$ and $d \rho_{32} / d t$ :

$$
\begin{align*}
\frac{d \rho_{14}}{d t} & =\frac{1}{i \hbar}\left[\rho_{14}, H\right]=\rho_{14} H-H \rho_{14}=-i\left[\omega_{0} \rho_{14}+g a_{+} e^{-i \omega t}\left(\rho_{11}-\rho_{44}\right)\right]  \tag{E.2}\\
\frac{d \rho_{41}}{d t} & =\frac{1}{i \hbar}\left[\rho_{41}, H\right]=\rho_{41} H-H \rho_{41}=-i\left[-\omega_{0} \rho_{41}+g a_{+}^{\dagger} e^{i \omega t}\left(\rho_{44}-\rho_{11}\right)\right],  \tag{E.3}\\
\frac{d \rho_{23}}{d t} & =\frac{1}{i \hbar}\left[\rho_{23}, H\right]=\rho_{23} H-H \rho_{23}=-i\left[\omega_{0} \rho_{23}+g a_{-} e^{-i \omega t}\left(\rho_{22}-\rho_{33}\right)\right]  \tag{E.4}\\
\frac{d \rho_{32}}{d t} & =\frac{1}{i \hbar}\left[\rho_{32}, H\right]=\rho_{32} H-H \rho_{32}=-i\left[-\omega_{0} \rho_{32}+g a_{-}^{\dagger} e^{i \omega t}\left(\rho_{33}-\rho_{22}\right)\right] . \tag{E.5}
\end{align*}
$$

As the probe beam is detuned, we assume that the populations of the excited states $\rho_{44}=\rho_{33}=0$. Going to a rotating frame where $\Delta=\omega_{0}-\omega$ and $\tilde{\rho}_{14}=\rho_{14} e^{i \omega t}$, $\tilde{\rho}_{41}=\rho_{41} e^{-i \omega t}, \tilde{\rho}_{23}=\rho_{23} e^{i \omega t}, \tilde{\rho}_{32}=\rho_{32} e^{-i \omega t}$, we obtain

$$
\begin{align*}
\frac{d \tilde{\rho}_{14}}{d t} & =-i\left[\Delta \tilde{\rho}_{14}+g a_{+} \rho_{11}\right]  \tag{E.6}\\
\frac{d \tilde{\rho}_{41}}{d t} & =-i\left[-\Delta \tilde{\rho}_{41}-g a_{+}^{\dagger} \rho_{11}\right]  \tag{E.7}\\
\frac{d \tilde{\rho}_{23}}{d t} & =-i\left[\Delta \tilde{\rho}_{23}+g a_{-} \rho_{22}\right]  \tag{E.8}\\
\frac{d \tilde{\rho}_{32}}{d t} & =-i\left[-\Delta \tilde{\rho}_{32}-g a_{-}^{\dagger} \rho_{22}\right] \tag{E.9}
\end{align*}
$$

In the steady state, we assume that $d \tilde{\rho}_{14} / d t=d \tilde{\rho}_{41} / d t=d \tilde{\rho}_{23} / d t=d \tilde{\rho}_{32} / d t=0$ due to the assumption of adiabatic elimination. We can then determine

$$
\begin{align*}
& \tilde{\rho}_{14}=-\frac{g a_{+} \rho_{11}}{\Delta},  \tag{E.10}\\
& \tilde{\rho}_{41}=-\frac{g a_{+}^{\dagger} \rho_{11}}{\Delta},  \tag{E.11}\\
& \tilde{\rho}_{23}=-\frac{g a_{-} \rho_{22}}{\Delta},  \tag{E.12}\\
& \tilde{\rho}_{32}=-\frac{g a_{-}^{\dagger} \rho_{22}}{\Delta} . \tag{E.13}
\end{align*}
$$

## Appendix F

## High RF amplitude and transmission data

## F. 1 High RF amplitude data

At larger RF amplitudes for the paraffin-coated alignment-based magnetometer, interesting phenomena appear (Zigdon et al., 2010). Some theoretical examples


Figure F.1: The (a) in-phase ( $X$ ), (b) out-of-phase $(Y)$ and (c) magnitude ( $R$ ) plotted as a function of the detuning $\Delta_{\mathrm{RF}}$ from Eqs. 4.53 and 4.54 for situations when $\gamma<\Omega_{\mathrm{RF}}<\omega_{L}$.
when $\gamma<\Omega_{\mathrm{RF}}<\omega_{L}$, where $\gamma=100 \mathrm{~Hz}$, are shown in Fig. F.1. Note that the effective strength of $B_{\mathrm{RF}}$ "observed" by the atoms depends on the detuning of the RF field. At large detunings $\Delta_{\mathrm{RF}}$ then the "peanut" will be the shape of the atomic polarisation as in Fig. 4.2. However, when the detuning $\Delta_{\mathrm{RF}}=0$ and the effective RF field strength increases, then the atomic polarisation averages around the $x$ axis, leading to a "doughnut" with the centre of the doughnut along the $x$-axis (Rochester, 2023; Zigdon et al., 2010). High RF amplitude experimental data was taken with the paraffin-coated alignment-based magnetometer. This data was fitted to Eqs. 4.53 and 4.54 and can be found in Fig. F.2.


Figure F.2: High RF amplitude magnetic resonance with a paraffin-coated cell in an alignment-based magnetometer. (a) $X$, (b) $Y$ and (c) $R$ are plotted, alongside fits for $X$ and $Y$ using Eqs. 4.53 and 4.54 .

## F. 2 Transmission of the paraffin-coated cell

The power before and after the paraffin-coated vapour cell was measured and the transmission calculated. The transmission is plotted as a function of power before the vapour cell in Fig. F.3. All the paraffin-coated data was taken at room temperature.


Figure F.3: Transmission of the paraffin-coated cell at room temperature ( $\sim 20^{\circ} \mathrm{C}$ ).

## Appendix G

## Eddy current integral calculations

In this appendix the integrals for the case of the on-axis cylinder in Sec. 5.2 are presented.

$$
\begin{equation*}
\Phi=\int_{0}^{\rho^{\prime}} B_{1, z}\left(\rho^{\prime}\right) 2 \pi \rho^{\prime} d \rho^{\prime}=\frac{\mu_{0} m}{2} \int_{0}^{\rho^{\prime}} \frac{\left(2 a^{2} \rho^{\prime}-\rho^{\prime 3}\right)}{\left(a^{2}+\rho^{\prime 2}\right)^{5 / 2}} d \rho^{\prime} \tag{G.1}
\end{equation*}
$$

The integral $I_{1}$

$$
\begin{equation*}
I_{1}=\int_{0}^{\rho^{\prime}} \frac{2 a^{2} \rho^{\prime}}{\left(a^{2}+\rho^{\prime 2}\right)^{5 / 2}} d \rho^{\prime} \tag{G.2}
\end{equation*}
$$

can be solved using integration by substitution. The first set of substitutions $u=\rho^{\prime 2}$ and $d u / d \rho^{\prime}=2 \rho^{\prime}$ give rise to

$$
\begin{equation*}
I_{1}=a^{2} \int_{u=0}^{u=\rho^{\prime 2}} \frac{1}{\left(a^{2}+u\right)^{5 / 2}} d u \tag{G.3}
\end{equation*}
$$

The second set of substitutions are $w=a^{2}+u$ and $d w / d u=1$. The integral then simplifies to

$$
\begin{equation*}
I_{1}=a^{2} \int_{w=a^{2}}^{w=a^{2}+\rho^{\prime 2}} \frac{1}{w^{5 / 2}} d w=\frac{2 a^{2}}{3}\left(\frac{1}{a^{3}}-\frac{1}{\left(a^{2}+\rho^{\prime 2}\right)^{3 / 2}}\right)=\frac{2}{3 a}-\frac{2 a^{2}}{3\left(a^{2}+\rho^{\prime 2}\right)^{3 / 2}} . \tag{G.4}
\end{equation*}
$$

The integral $I_{2}$

$$
\begin{equation*}
I_{2}=\int_{0}^{\rho^{\prime}} \frac{-\rho^{\prime 3}}{\left(a^{2}+\rho^{\prime 2}\right)^{5 / 2}} d \rho^{\prime} \tag{G.5}
\end{equation*}
$$

can also be solved using integration by substitution with $u_{2}=\rho^{\prime 2}$ and $d u_{2} / d \rho^{\prime}=2 \rho^{\prime}$. This gives

$$
\begin{equation*}
I_{2}=-\frac{1}{2} \int_{u_{2}^{\prime}=0}^{u_{2}=\rho^{\prime 2}} \frac{u_{2}}{\left(a^{2}+u_{2}\right)^{5 / 2}} d u_{2} . \tag{G.6}
\end{equation*}
$$

The substitutions $w_{2}=a^{2}+u_{2}$ and $d w_{2} / d u_{2}=1$ are used to simplify the integral to

$$
\begin{equation*}
I_{2}=-\frac{1}{2} \int_{w_{2}=a^{2}}^{w_{2}=a^{2}+\rho^{\prime 2}} \frac{w_{2}-a^{2}}{w_{2}^{5 / 2}} d w_{2}=-\frac{1}{2}{ }_{w_{2}=a^{2}}^{w_{2}=a^{2}+\rho^{\prime 2}}\left(\frac{-2}{w_{2}^{1 / 2}}+\frac{2 a^{2}}{3 w_{2}^{3 / 2}}\right) \tag{G.7}
\end{equation*}
$$

Implementing the limits leads to

$$
\begin{equation*}
I_{2}=\frac{1}{\left(a^{2}+\rho^{\prime 2}\right)^{1 / 2}}-\frac{a^{2}}{3\left(a^{2}+\rho^{\prime 2}\right)^{3 / 2}}-\frac{1}{a}+\frac{1}{3 a}=\frac{2 a^{2}+3 \rho^{\prime 2}}{3\left(a^{2}+\rho^{\prime 2}\right)^{3 / 2}}-\frac{2}{3 a} . \tag{G.8}
\end{equation*}
$$

The integrals are added together to give

$$
\begin{equation*}
I_{1}+I_{2}=\frac{\rho^{\prime 2}}{\left(a^{2}+\rho^{\prime 2}\right)^{3 / 2}} \tag{G.9}
\end{equation*}
$$

and hence

$$
\begin{equation*}
\Phi\left(\rho^{\prime}\right)=\frac{\mu_{0} m}{2} \frac{\rho^{\prime 2}}{\left(a^{2}+\rho^{\prime 2}\right)^{3 / 2}}=\frac{\mu_{0} I_{0} e^{i \omega t^{\prime}} \pi r_{w}^{2}}{2} \frac{\rho^{\prime 2}}{\left(a^{2}+\rho^{\prime 2}\right)^{3 / 2}} . \tag{G.10}
\end{equation*}
$$

## Appendix H

## Orientation-based OPM calculations

## H. 1 Eddy current calculations in a conductive sphere

We now calculate the expected induced magnetic field for a conductive, nonmagnetic solid sphere positioned in between an excitation coil and magnetometer (see inset in Fig. 6.1a). The primary magnetic field from the excitation coil (positioned at $x=0$ ) at the position of the OPM $x=r+r^{\prime}$ is equal to

$$
\begin{equation*}
B_{1}\left(x=r+r^{\prime}\right)=\frac{\mu_{0} m}{2 \pi\left(r+r^{\prime}\right)^{3}}, \tag{H.1}
\end{equation*}
$$

where $m$ is the magnetic moment of the excitation coil, $r$ is the distance from the excitation coil to the sphere with radius $a$ and $r^{\prime}$ is the distance from the sphere to the OPM. Equation H. 1 is the on-axis field for a magnetic dipole and is true when $r+r^{\prime} \gg R_{c}$, where $R_{c}$ is the radius of the excitation coil.

The secondary magnetic field at the position of the OPM $B_{\text {ec }}\left(x=r+r^{\prime}\right)$ is calculated to be

$$
\begin{equation*}
B_{\mathrm{ec}}\left(x=r+r^{\prime}\right)=\frac{\mu_{0} m_{\mathrm{ec}}}{2 \pi r^{\prime 3}}, \tag{H.2}
\end{equation*}
$$

where $m_{\mathrm{ec}}$ is the induced magnetic moment in the sphere and $r^{\prime}$ is the distance from the sphere to the OPM. For a non-magnetic, conductive sphere, $m_{\mathrm{ec}}$ is given by (Bidinosti et al., 2007; Honke and Bidinosti, 2018)

$$
\begin{equation*}
m_{\mathrm{ec}}=\frac{2 \pi a^{3} B_{1}(x=r)}{\mu_{0}} \frac{j_{2}(k a)}{j_{0}(k a)}, \tag{H.3}
\end{equation*}
$$

where

$$
\begin{gather*}
j_{2}(x)=\left(\frac{3}{x^{3}}-\frac{1}{x}\right) \sin x-\frac{3}{x^{2}} \cos x,  \tag{H.4}\\
j_{0}(x)=\frac{\sin x}{x}, \tag{H.5}
\end{gather*}
$$

and

$$
\begin{equation*}
k=\sqrt{\mu \epsilon \omega^{2}+i \mu \sigma \omega}, \tag{H.6}
\end{equation*}
$$

where $k$ is the propagation constant, $\mu=\mu_{0} \mu_{r}$ and $\epsilon=\epsilon_{r} \epsilon_{0}$. The propagation constant can be approximated to be $k \sim \sqrt{i \mu_{0} \sigma \omega}$ for this experiment. In the highfrequency limit where $\delta \ll a$, as is the case in Sec. 6,

$$
\begin{equation*}
\frac{j_{2}(k a)}{j_{0}(k a)} \rightarrow-1 \tag{H.7}
\end{equation*}
$$

and hence the secondary magnetic field at the position of the OPM $B_{\text {ec }}\left(x=r+r^{\prime}\right)$ is calculated to be

$$
\begin{equation*}
B_{\mathrm{ec}}\left(x=r+r^{\prime}\right)=-\frac{a^{3} B_{1}(x=r)}{r^{\prime 3}} . \tag{H.8}
\end{equation*}
$$

The ratio of the induced magnetic field to the primary magnetic field at the position of the OPM is calculated to be

$$
\begin{equation*}
\frac{B_{\mathrm{ec}}\left(x=r+r^{\prime}\right)}{B_{1}\left(x=r+r^{\prime}\right)}=-\frac{a^{3}}{r^{\prime 3}} \frac{B_{1}(x=r)}{B_{1}\left(x=r+r^{\prime}\right)}=-\frac{a^{3}}{r^{\prime 3}} \frac{\left(r+r^{\prime}\right)^{3}}{r^{3}} \tag{H.9}
\end{equation*}
$$

which is chosen as the figure of merit in Sec. 6 for the remote detection of conductive objects.

If the sphere is exactly halfway between the OPM and the excitation coil, i.e., $r=r^{\prime}$, then Eq. H. 9 simplifies to

$$
\begin{equation*}
\frac{B_{\mathrm{ec}}\left(x=r+r^{\prime}\right)}{B_{1}\left(x=r+r^{\prime}\right)}=-\frac{8 a^{3}}{r^{3}} \tag{H.10}
\end{equation*}
$$

## H. 2 Comparison with co-located excitation coil and magnetometer

We now consider the situation where the excitation coil and the magnetometer are co-located. The primary field $B_{1}(x=0)$ at the position of the OPM $x=0$ is in this case given by

$$
\begin{equation*}
B_{1}(x=0)=\frac{\mu_{0} m}{2 R_{c}^{3}}, \tag{H.11}
\end{equation*}
$$

where $m=\pi R_{c}^{2} n I, n$ is the number of windings and $I$ is the current flowing through the coil.

Alternatively, if the OPM is placed on the other side of the object, i.e., a distance $2 r$ away from the primary coil assuming the object is centred between the primary coil and OPM, then $B_{1}(x=2 r)$ will be given by

$$
\begin{equation*}
B_{1}(x=2 r)=\frac{\mu_{0} m}{2\left(R_{c}^{2}+(2 r)^{2}\right)^{3 / 2}} \tag{H.12}
\end{equation*}
$$

In both cases, the induced magnetic field $B_{\text {ec }}$ at the position of the OPM is the same. As previously discussed, it is important to reduce the effect of $B_{1}$ on the OPM. We can compare the primary magnetic field at the OPM position for the two cases

$$
\begin{equation*}
\frac{B_{1}(x=2 r)}{B_{1}(x=0)}=\frac{R_{c}^{3}}{\left(R_{c}^{2}+4 r^{2}\right)^{3 / 2}}=\frac{R_{c}^{3}}{8 r^{3}\left(R_{c}^{2} /\left(4 r^{2}\right)+1\right)^{3 / 2}} \tag{H.13}
\end{equation*}
$$

In the limit where the object is placed far from the excitation coil $\left(r \gg R_{c}\right)$, this expression simplifies to

$$
\begin{equation*}
\frac{B_{1}(x=2 r)}{B_{1}(x=0)}=\frac{R_{c}^{3}}{8 r^{3}} . \tag{H.14}
\end{equation*}
$$

Inserting the relevant numbers for our setup ( $r \sim 25 \mathrm{~cm}$ and $R_{c} \sim 6 \mathrm{~cm}$ ) we calculate $B_{1}(x=2 r) / B_{1}(x=0) \sim 0.002$. By placing the excitation coil and the OPM on opposite sides of the object, the primary magnetic field is orders of magnitude smaller at the OPM position. This configuration will therefore enable much larger detection distances compared to if the excitation coil and the OPM were co-located.

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