Investigations into Local Class Field Theory with General Residue Fields

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To my family

Abstract

Local Fields, complete discrete valuation fields with a finite residue field, are an important mathematical tool in number theory; in particular in the topics of Class Field Theory and Arithmetic Geometry. However, while a lot of work has been done on when the residue field, $\overline{F}$, of the complete discrete valuation field, F, is finite; dealing with less restrictions on the fields is also an incredibly fruitful undertaking. This enterprise, both examining some of the work already done on this topic and also expanding on them, is the focus of the thesis.

In this thesis we look at two important aspects of the theory of complete discrete valuation fields. To begin with we examine local class field theory on complete discrete valuation fields whose residue field is an imperfect field of positive characteristic p. We investigate for which finite abelian totally ramified p-extensions L/F is the map, $\Psi_{L/F}$, an isomorphism; as before we only knew it was an isomorphism when L/F was a finite cyclic extension.

We then move onto abelian varieties, A, over complete discrete valuation fields. The research on this topic focuses on generalising a result by Barry Mazur about the rational points of an abelian variety with good ordinary reduction over a complete discrete valuation field with finite residue field, $A(F)$, to also deal with complete discrete valuation fields whose residue field is perfect and has positive characteristic p.

We finish off the thesis by then bringing up further directions to take both topics in the future. The work here opens up several new avenues in the subjects to explore and plenty of new research opportunities to be investigated later on.

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Finally, I would like to bring up the people who work at Games Workshop. Though perhaps unconventional to mention; these people provided companionship and distraction that probably greatly helped my mental health during the long years of my PhD.

Contents

Chapter 1

Introduction

An important concept in algebra, and in particular algebraic number theory, is the process of localisation. The exact process is not important right now, and is likely known to the reader, but in essence it involves taking a multiplicatively closed set of a ring, often a prime ideal, and using it to create a set of denominators of a different ring. This new ring is a complete discrete valuation ring into which the original ring naturally injects. Readers who need a refresher on the subject can read section 4 of Chapter **II** of Serge Lang's "Algebra" [6].

A simple example of localisation is taking a prime number p from the integers, $\mathbb{Z}$, to form the ring: $\mathbb{Z}_{(p)} = \{\frac{a}{b} : a, b \in \mathbb{Z}, p \nmid b\}$. This ring is now a discrete valuation ring, with valuation $\nu(\frac{a}{b})$ being equal to the highest power of p that divides a, containing the single prime ideal (p).

David Eisenbud described the importance of localisation in his book "Commutative Algebra with a View Toward Algebraic Geometry":

"The technique of localization reduces many problems in commutative algebra to problems about local rings. This often turns out to be extremely useful: Most of the problems with which commutative algebra has been successful are those that can be reduced to the local case." [1]

Of course we can do more than just localise around a point in a field; we can then complete the resulting field with respect to its respective valuation. For example the completion of the above $\mathbb{Z}_{(p)}$ makes the p-adic integers $\mathbb{Z}_p$. This gives the benefit of making every Cauchy sequence convergent, giving rise to properties that were not necessarily there before, such as Hensel's Lemma. Taking the field of fractions of such complete discrete valuation rings gives us complete discrete valuation fields, which are the subject of this thesis.

In the case of this piece of work, we shall only be looking at the subject of local class field theory and abelian varieties. Both of which can be done over both "global fields" and, in the case of this work, "local fields".

Before we go any further we should clarify a piece of terminology. Though one of the topics we are focusing on is called "local class field theory" we shall be calling the objects "complete discrete valuation fields"; this is because the term "local field" is sometimes used to refer exclusively to a complete discrete valuation field with a finite residue field, see for instance "Algebraic Number Theory" by Jürgen Neukirch [10]. As we are going to be extensively talking about complete discrete valuation fields with non-finite residue fields we should head off any confusion by being more exact in our terminology. The one exception to this rule is in chapter titles for the purely aesthetic reason that "complete discrete valuation fields" is a lot more unwieldy when it comes to a title than "local fields".

We shall be assuming that the reader already has a basic knowledge of complete discrete valuation fields, local class field theory and algebraic geometry. Chapter 2, "Literature Review", directs you to where you can find out about those topics while also going over other important definitions and results which you are not expected to already know.

As noted previously, this document focuses on two different topics in the subject of complete discrete valuation fields. Chapter 3, "Local Fields with Imperfect Residue Fields", is on local class field theory and investigating whether a fundamental result still holds when we assume that the base field has an imperfect residue field. Meanwhile Chapter 4, "Abelian Varieties Over Local Fields", is about Arithmetic Algebraic Geometry and generalising a result that is known to hold over a discrete valuation field with a finite residue field to work over fields with perfect residue fields. A more extensive summary of the main results that we investigate of these topics will happen later in this "Introduction".

An example of the importance of there work undertaken here, is the potential applications of the chapter on local class field theory where the residue field has an imperfect residue field has on the subject of higher local fields. The exact definition of such constructs does not matter at this moment, as we do not go over this subject in the document, but basically K is a higher local field if K is a complete discrete valuation field whose residue field $\overline{K}$ is also a complete discrete valuation field. In fact one can form a chain of complete discrete valuation fields, with each field being the residue field of the previous field, as long as the inner-most complete discrete valuation field has a finite residue field [2]. More information can be found in "Geometry and Topology Monographs, Volume 3: An Invitation to Higher Local Fields" [2], however one thing to note here is that if $\overline{K}$ is equal to, for example, $\mathbb{F}_p((X))$, it is imperfect and thus the local class field theory that we talk about in Chapter 3 would be very useful for when discussing K.

1.1 Overview of the Thesis

There are four chapters, not including the "Introduction", in this thesis. Chapter 2, "Literature Review", goes over important background that is required in order to properly understand the rest of this document. It contains little original mathematics, other than some considerations into tying different sections together.

As noted in the previous section it is Chapter 3, "Local Fields with Imperfect Residue Fields", that discusses the main topic of this thesis. Our consideration of local class field theory when the residue field is imperfect produces several important results.

We will now go over the important results of this chapter. While this will be self-contained, with all of the relevant definitions and notations stated, readers are advised to go over Chapter 3, and the relevant sections of the "Literature Review", to get the full picture of the work.

We will let $p > 0$ be a prime number and set F as a complete discrete valuation field whose residue field, denoted by $\overline{F}$, has characteristic p. The principal units of F, those elements of the ring of integers of F reduced to 1 when mapped to $\overline{F}$, will be written as $U_{1,F}$.

Assume that $\overline{F}$ is perfect and let L_1/F and L_2/F be two finite abelian totally ramified p-extensions. It is known, and gone over in Lemma 2.3.4, that if a prime element, π, of F is contained in $N_{L_1/F}(L_1^*) \cap N_{L_2/F}(L_2^*)$ then L_1L_2/F is a totally ramified extension.

Now, we will now longer require that $\overline{F}$ is perfect and we will construct a, not necessarily unique, field extension $\mathcal{F}/F$ such that $\mathcal{F}$ is a complete discrete valuation field with the following properties:

1) The ramification index of $\mathcal{F}/F$, denoted as $e(\mathcal{F} \mid F)$, is equal to 1.

2) We have that $\overline{\mathcal{F}} = \bigcup_{n \geq 1} \overline{F}^{p^{-n}}$, so $\overline{\mathcal{F}}$ is the perfection of $\overline{F}$.

Theorem 1.1.1. *Let $\overline{F}$ be a field, which may be imperfect, of characteristic p and let L_1/F and L_2/F be two finite abelian totally ramified p-extensions and π be a prime element of F. Theorem 3.3.2 states that if $\pi \in N_{L_1\mathcal{F}/\mathcal{F}}((L_1\mathcal{F})^*) \cap N_{L_2\mathcal{F}/\mathcal{F}}((L_2\mathcal{F})^*)$, then L_1L_2/F is a totally ramified extension.*

Still keeping that $\overline{F}$ may be imperfect, let L/F be a finite abelian totally ramified p-extension. Let $\widehat{F}/F$ be the maximal unramified p-extension of F.

Define $\widehat{L}/L$ in the equivalent way, we do that $\widehat{L} = L\widehat{F}$. Denote the group of homomorphisms $\mathrm{Hom}_{\mathbb{Z}_p}(\mathrm{Gal}(\widehat{F}/F), \mathrm{Gal}(L/F))$ by $\mathrm{Gal}(L/F)^{\wedge}$.

If $\overline{F}$ is finite then classical local class field theory tells us there is an isomorphism $\Psi_{L/F} : F^*/N_{L/F}(L^*) \to \mathrm{Gal}(L/F)$. This has an analogue if $\overline{F}$ is perfect as now there is a map $\Psi_{L/F} : U_{1,F}/N_{L/F}(U_{1,L}) \to \mathrm{Gal}(L/F)^{\wedge}$, properly described in Definition 2.3.5. From Theorem 2.3.1 we have that this version of $\Psi_{L/F}$ has also been shown to always be an isomorphism.

Let $\overline{F}$ be imperfect and denote $U_{1,F} \cap N_{\widehat{L}/\widehat{F}}(U_{1,\widehat{L}})$ by $U(L/F)$ and $N_{L/F}(U_{1,L})$ by $N(L/F)$. In this case, as defined by Definition 2.4.5, there is a map $\Psi_{L/F} : U(L/F)/N(L/F) \to \mathrm{Gal}(L/F)^{\wedge}$, which is a generalisation of the map we saw where $\overline{F}$ was assumed to be perfect. It is not currently known whether the version of the map is always an isomorphism.

Theorem 1.1.2. *Let $\overline{F}$ be a field, which may be imperfect, of characteristic p and set L/F as a totally ramified Galois extension whose Galois group is isomorphic to $(\mathbb{Z}/p\mathbb{Z})^2$. Theorem 3.4.1 shows that $\Psi_{L/F}$ must be an isomorphism.*

Keeping $\overline{F}$ as imperfect, let L/F be a finite abelian totally ramified p-extension, with Galois group denoted by G. Denote by G_i, with $i \geq 0$, the Lower Ramificaion Groups of G and say that G has a ramification jump at a if $G_a \neq G_{a+1}$.

Theorem 1.1.3. *Let $\overline{F}$ be a field, which may be imperfect, of characteristic p and set L/F as a finite abelian totally ramified p-extension whose Galois group, G, has a single ramification jump; this means that there exists a non-negative integer a with $G = G_a \neq G_{a+1}$ and $G_{a+1} = 1$. Theorem 3.7.1 tells us that $\Psi_{L/F}$ is an isomorphism.*

It is still an open, and interesting, question about for what abelian finite totally ramified p-extensions, L/F, has $\Psi_{L/F}$ as an isomorphism. While we are not able to determine the nature of $\Psi_{L/F}$ for all such extensions we are able to make some inroads on certain types of extension.

Theorem 1.1.4. *Let $\overline{F}$ be a field, which may be imperfect, of characteristic p and set L/F as a finite abelian totally ramified p-extension whose Galois group, G, has two ramification jumps. This means that there are non-negative integers $b > a$ such that $G = G_a \neq G_{a+1}$ and $G_b \neq G_{b+1}$, with G_b equally G_{a+1} and $G_{b+1} = 1$. Sections 8 and 9 of Chapter 3 is able to reduce the question of* *whether $\Psi_{L/F}$ is an isomorphism or not to whether a certain property holds for the extension.*

The property mentioned in the above theorem is a bit too intricate to be included in this summary and thus sections 8 and 9 should be read for the details.

Chapter 4, titled "Abelian Varieties over Local Fields", deals with the subject of Arithmetic Algebraic Geometry. Again, readers are recommended to study Chapter 4 and the last section of the "Literature Review", to properly understand the following mathematics.

Let $p > 0$ be a prime number and $F/\mathbb{Q}_p$ be a finite extension. Set A as a d-dimensional abelian variety with good ordinary reduction over F, where d is a positive integer. Denote by $\widehat{A}$ the formal group over the rings of integers $\mathcal{O}_F$ and $A(\overline{F})_p$ the p-torsion points of $A(\overline{F})$. Finally, let L/F be a totally ramified $\mathbb{Z}_p$-extension.

Jonathan Lubin and Michael Rosen's paper "The Norm Map for Ordinary Abelian Varieties" [7] shows that we can find a "twist matrix" of $\widehat{A}$, a non-singular $d \times d$ matrix, denoted by u, over $\mathbb{Z}_p$. How to construct u is given in Definition 2.5.3.

They go onto to show that the following exact sequence, described in Theorem 2.5.1, can be constructed:

$$\mathbb{Z}_p^d/((I-u)\mathbb{Z}_p^d) \to A(F)/N_{L/F}(A(L)) \to A(\overline{F})_p \to 1$$

The nature of the map $\mathbb{Z}_p^d/((I-u)\mathbb{Z}_p^d) \to A(F)/N_{L/F}(A(L))$ shall be explained in Lemma 2.5.3 and Theorem 2.5.1.

We should note that the above exact sequence by Rosen and Lubin is not original and [7] instead offers an alternate proof to a result by Barry Mazur, which they call "Mazur's Proposition 4.39", in his paper "Rational Points of Abelian Varieties with Values in Towers of Number Fields" [8]. However, the work in this thesis is building on the mathematics of [7] so that is the source that we will mainly be referencing.

Chapter 4 is exploring the subject of generalising the result of [7] to when $\overline{F}$ is only assumed to be a perfect field of positive characteristic p. While a complete generalisation is not achieved in this thesis the following result is found:

Theorem 1.1.5. *Let F be a complete discrete valuation field such that $\overline{F}$ is perfect and has positive characteristic p. Let $\widehat{F}/F$ be the maximal unramified*

p-extension of F. Next, let A be a d-dimensional abelian variety with good ordinary reduction over F such that the formal goup associated to A has an isomorphism to $\widehat{\mathbb{G}}_m^d$ over $\mathcal{O}_{\widehat{F}}$. Finally, assume that L/F is a totally ramified $\mathbb{Z}_p$-extension.

We can construct the group Q and create the following exact sequence:

$$Q \to A(F)/N_{L/F}(A(L)) \to A(\overline{F})_p$$

While the definition of the group Q is given in Theorem 4.2.2 and Notation 4.2.2, its exact properties are currently unknown, though some analysis is given in the last two sections of Chapter 4.

After Chapter 4 the final chapter, which is called "Conclusion" briefly goes over how to continue the research that we have already done in this thesis. It is there to provide some ideas for mathematicians, either myself or someone else, to explore in the future.

Chapter 2

Literature Review

This chapter shall be briefly going over the background information for local class field theory and arithmetic geometry that I used for my own research. Full proofs and explanations shall not be provided but we will be talking about the relevant documents where the information can be found.

As a reminder we shall be assuming that readers are familiar with the basics of complete discrete valuation fields, local class field theory and algebraic geometry. If the reader is not proficient in such matters they are directed towards the first four chapters of Professors Ivan Fesenko and Sergei Vostokov's book "Local Fields and their Extensions" [5], an invaluable resource on the first two subjects and the course notes on "Algebraic Geometry" by Professor Emeritus James S. Milne, the latest version of the notes can be found on Milne's official website, for the last [9].

2.1 Preliminary Matters

Before we can begin the proper review of other mathematician's work there are a few of notations and definitions that we use in this thesis and must be established. These are all from "Local Fields and their Extensions" but we explain them here in case the reader is not familiar with them.

Notation 2.1.1. Let F be a complete discrete valuation field. We will set $\mathcal{O}_F$ to be the ring of integers of F and $\mathcal{M}_F$ to be the maximal ideal of $\mathcal{O}_F$.

We will next let $\overline{F}$ be the notation for the residue field of F, so $\overline{F} = \mathcal{O}_F/\mathcal{M}_F$. Finally, for $\alpha \in \mathcal{O}_F$, we will let $\overline{\alpha}$ be the image of α under the natural map from $\mathcal{O}_F$ to $\overline{F} = \mathcal{O}_F/\mathcal{M}_F$.

Notation 2.1.2. Let F be a complete discrete valuation ring with fixed prime element π_F. For non-negative integers i define the λ_i-maps as follows:

- $\lambda_0 : U_F \to \overline{F}^*$ with $\lambda_0(\alpha) = \overline{\alpha}$.
- $\lambda_i : U_{i,F} \to \overline{F}$ sends $\lambda_i(1 + \alpha\pi_F^i) = \overline{\alpha}$, for $i \geq 1$.

Definition 2.1.1. Let F be a complete discrete valuation field. We will denote by F^{ur} the maximal unramified extension of F. This makes F^{ur} the minimal extensions of F such that $e(F^{\text{ur}}/F) = 1$ and $\overline{F^{\text{ur}}}$ is the separable closure of $\overline{F}$.

Notation 2.1.3. Let F be a complete discrete valuation field with $\overline{F}$ having positive characteristic p. Let L/F be a totally ramified cyclic extension of degree p, with σ as a generator of $\text{Gal}(L/F)$. Finally let π_L be a prime element of L.

Define $s(L/F)$ as $\nu_L((\sigma(\pi_L)/\pi_L) - 1)$.

Note 2.1.1. For details about the properties of $s(L/F)$, such as how it is a well defined construction, please look at sections (1.4) and (1.5) Chapter **III** of [5].

Notation 2.1.4. Let F be a complete discrete valuation field with characteristic 0 and whose residue field has positive characteristic p. We will define $e(F)$ to be $\nu_F(p)$.

There is also the following basic definition that we should clarify.

Definition 2.1.2. Let F be a complete discrete valuation field. An extension L/F is called abelian if it is a Galois extension and $\text{Gal}(L/F)$ is an abelian group.

With the above necessities out of the way we can start the overview of Local Class Field Theory where the residue fields are not finite.

2.2 Review of Local Class Field Theory with Quasi-Finite Residue Fields

This section shall be focusing on the work done on local class field theory where the base field is assumed to have a quasi-finite residue field. It is the next step up from local class field theory where the base field has a finite residue field.

Most of the information has can be found in George Whaples' series of papers on the subject [11] [12] [13] [14] [15], however there is a summary of Whaples' work in the first three sections Chapter **V** of [5].

We should note that despite the fact that "Local Fields and their Extensions" is not the primary source of this information that is what we are going to be referring to in this section. This is because one of the authors of the book also wrote the papers we discuss in the next two sections. He uses the same terminology and types of mathematics throughout his work on local class field theory, which makes it much easier to relate what is written here to the mathematics of the later sections.

Definition 2.2.1. A field K is quasi-finite if it has the following two properties:

1) K is a perfect field.

2) For each integer $n > 0$, there is precisely one extension N/K of degree n. We also have that N/K is a Galois cyclic extension.

Note 2.2.1. It is clear that if K is a finite field then it is quasi-finite.

Note 2.2.2. Throughout this section we will be assuming that F is a complete discrete valuation field whose residue field, $\overline{F}$, is quasi-finite. We will also be assuming that $\text{char}\,(\overline{F}) = p > 0$ as it more closely relates to the work we will do later. Finally, we will assume that L/F is an arbitrary finite Galois field extension of F.

Note 2.2.3. It should be stated that nearly everything we will be initially looking at here is the same as if $\overline{F}$ was finite. The main difference between local class field theory when $\overline{F}$ is finite and when we assume only that $\overline{F}$ is quasi-finite comes with the Existence Theorem; only when $\overline{F}$ is finite is every open subgroup of F^* with finite index a norm group. We will however be going over the mathematics again in order to refresh any readers on the subject and to showcase the similarity.

Definition 2.2.2. We will let $\mathcal{F}$ be equal to either F^{ur} or its completion, as the situation demands. This option is because F^{ur} is not necessarily a complete discrete valuation field. For a finite Galois extension L/F we will set $\mathcal{L}$ to be $L\mathcal{F}$.

Note 2.2.4. For a finite Galois extension L/F we have that $\mathcal{F} \subseteq \mathcal{L}$ and that $\mathcal{L}/\mathcal{F}$ is Galois with $\text{Gal}\,(L/F) \cong \text{Gal}\,(\mathcal{L}/\mathcal{L}_0)$, here L_0/L is the maximal unramified subextension of L/F.

Definition 2.2.3. Let us fix an isomorphism $t : \mathrm{Gal}\,(\overline{F}^{\mathrm{sep}}/\overline{F}) \to \widehat{\mathbb{Z}}$, which exists as the definition of quasi-finite fields means the absolute Galois group of $\overline{F}$ is isomorphic to $\widehat{\mathbb{Z}}$. We will denote the element of $\mathrm{Gal}\,(\overline{F}^{\mathrm{sep}}/\overline{F})$ mapped to 1 under t by $\overline{\varphi}$. We know that we have $\mathrm{Gal}\,(F^{\mathrm{ur}}/F) \cong \widehat{\mathbb{Z}} \cong \mathrm{Gal}\,(\overline{F}^{\mathrm{sep}}/\overline{F})$, and thus we call the element of $\mathrm{Gal}\,(F^{\mathrm{ur}}/F)$ mapped to $\overline{\varphi}$ the Frobenius automorphism and denote it by φ_F.

Note 2.2.5. There is no canonical unique Frobenius automorphism of F unless $\overline{F}$ is finite. As such we need to fix which Frobenius automorphism of F we are using ahead of time.

Definition 2.2.4. Using the Frobenius automorphism in Definition 2.2.3 define the following set:

$$\mathrm{Frob}\,(L/F) = \{\tilde{\sigma} \in \mathrm{Gal}\,(L^{\mathrm{ur}}/F) : \tilde{\sigma}\,|_{F^{\mathrm{ur}}} = \varphi_F^n, n \in \mathbb{Z}_{>0}\}$$

Note 2.2.6. It should be clear that every element of $\mathrm{Frob}\,(L/F)$ is an extension of a unique element of $\mathrm{Gal}\,(L/F)$. We will denote $\tilde{\sigma}\,|_L$, for $\tilde{\sigma} \in \mathrm{Frob}\,(L/F)$, by σ.

We will now define a proto-Neukirch homomorphism, denoted by $\tilde{\Upsilon}_{L/F}$.

Definition 2.2.5. Let $\tilde{\sigma}$ be an element of $\mathrm{Frob}\,(L/F)$. We have that $\tilde{\sigma} \in \mathrm{Gal}\,(L^{\mathrm{ur}}/F)$ and let Σ be the fixed field of $\tilde{\sigma}$ in L^{ur}. Fix a prime element of Σ, denoted by π_Σ, and define $\tilde{\Upsilon}(\tilde{\sigma})$ as being $N_{\Sigma/F}(\pi_\Sigma) \mod N_{L/F}(L^*)$.

Lemma 2.2.1. *We have that* $\tilde{\Upsilon}_{L/F} : \mathrm{Frob}\,(L/F) \to F^*/N_{L/F}(L^*)$ *is a well defined homomorphism.*

Proof. The proof of this lemma can be found in two chapters, Lemma (2.2) Chapter **IV** and section (1.3) Chapter **V** in [5]. □

Note 2.2.7. Since we are using the mathematics in [5] we are going to be following the methods Professor Ivan Fesenko use. This means that at the moment we will define the Neukirch homomorphism, and show that it is an isomorphism, only in the cases where L/F is a finite unramified extension. We should point out that since $\overline{F}$ is quasi-finite all finite unramified extensions of F are abelian. We shall deal with the other finite Galois extensions of F after we have discussed and defined the map $\Psi_{L/F}$.

Lemma 2.2.2. *Let L/F be a finite unramified extension, we have that $\tilde{\Upsilon}_{L/F}$ is independent of what element of* $\mathrm{Frob}\,(L/F)$ *extends a given element of* $\mathrm{Gal}\,(L/F)$*. As such if $\tilde{\sigma}$ and $\tilde{\theta}$ are two elements of* $\mathrm{Frob}\,(L/F)$ *such that $\tilde{\sigma}\mid_L = \tilde{\theta}\mid_L$ then $\tilde{\Upsilon}_{L/F}(\tilde{\sigma}) = \tilde{\Upsilon}_{L/F}(\tilde{\theta})$.*

Proof. The proof of this lemma can be found in two chapters, Theorem (2.4) Chapter **IV** and section (1.3) Chapter **V** in [5]. □

Definition 2.2.6. From the above lemma we may define the Neukirch Homomorphism for all finite unramified extensions L/F as $\Upsilon_{L/F} : \mathrm{Gal}\,(L/F) \to F^*/N_{L/F}(L^*)$. This is defined by taking an element $\sigma \in \mathrm{Gal}\,(L/F)$ extending it to an element $\tilde{\sigma} \in \mathrm{Frob}\,(L/F)$ and applying $\tilde{\Upsilon}_{L/F}$ to $\tilde{\sigma}$.

Theorem 2.2.1. *For a finite unramified extension L/F, the Neukirch homomorphism:*

$$\Upsilon_{L/F} : \mathrm{Gal}\,(L/F) \to F^*/N_{L/F}(L^*)$$

is an isomorphism.

Proof. The proof of this theorem can be found in two chapters, Theorem (2.4) 2 Chapter **IV** and section (1.3) Chapter **V** in [5]. □

In order to be able to define $\Upsilon_{L/F}$ for all finite abelian extensions of F we also have to establish a few functorial properties of $\tilde{\Upsilon}_{L/F}$.

Proposition 2.2.1. *$\tilde{\Upsilon}_{L/F}$ has the following three functorial properties:*

1) *Fix $\sigma \in \mathrm{Gal}\,(F^{\mathrm{sep}}/F)$ and let L/M be a finite Galois extension, with M/F being a finite seperable extension. Define the map $\sigma^* : \mathrm{Frob}\,(L/M) \to \mathrm{Frob}\,(\sigma L/\sigma M)$ as $\sigma^*(\tilde{\rho}) = \sigma\tilde{\rho}\sigma^{-1}\mid_{L^{\mathrm{ur}}}$ for $\tilde{\rho} \in \mathrm{Frob}\,(L/M)$. The following diagram:*

$$\begin{array}{ccc} \mathrm{Frob}\,(L/M) & \xrightarrow{\tilde{\Upsilon}_{L/M}} & M^*/N_{L/M}(L^*) \\ \downarrow{\scriptstyle \sigma^*} & & \downarrow{\scriptstyle \sigma} \\ \mathrm{Frob}\,(\sigma L/\sigma M) & \xrightarrow{\tilde{\Upsilon}_{\sigma L/\sigma M}} & (\sigma M)^*/N_{\sigma L/\sigma M}((\sigma L)^*) \end{array}$$

is commutative.

2) Let L/F be a finite Galois extension and let M/F and E/L be finite seperable extensions such that E/M is a finite Galois extension. The following diagram:

$$\begin{array}{ccc} \mathrm{Frob}\,(E/M) & \xrightarrow{\tilde{\Upsilon}_{E/M}} & M^*/N_{E/M}(E^{ast}) \\ \downarrow & & \downarrow N^*_{M/F} \\ \mathrm{Frob}\,(L/F) & \xrightarrow{\tilde{\Upsilon}_{L/F}} & F^*/N_{L/F}(L^*) \end{array}$$

is commutative. Here the left vertical homorphism sends $\tilde{\sigma} \in \mathrm{Frob}\,(E/M)$ to $\tilde{\sigma} \mid_{L^{\mathrm{ur}}}$, this is surjective if $M = F$, and the right vertical homomorphism is induced by the norm map $N_{M/F} : M^* \to F^*$.

3) Let L/F be a finite Galois extension with subextension M/F. The following diagram:

$$\begin{array}{ccccccc} \mathrm{Frob}\,(L/M) & \longrightarrow & \mathrm{Frob}\,(L/F) & \longrightarrow & \mathrm{Frob}\,(M/F) & & \\ \downarrow \tilde{\Upsilon}_{L/M} & & \downarrow \tilde{\Upsilon}_{L/F} & & \downarrow \tilde{\Upsilon}_{M/F} & & \\ M^*/N_{L/M}(L^*) & \xrightarrow{N^*_{M/F}} & F^*/N_{L/F}(L^*) & \longrightarrow & F^*/N_{M/F}(M^*) & \longrightarrow & 1 \end{array}$$

is commutative. Here the map $F^*/N_{L/F}(L^*) \to F^*/N_{M/F}(M^*)$ is induced by the identity map of F^*.

Proof. The proof of this lemma can be found in two chapters, the Lemma, Proposition and Corollary of (2.5) Chapter **IV** and section (1.3) Chapter **V** in [5]. □

We will now define the map $\Psi_{L/F} : F^*/N_{L/F}(L^*) \to \mathrm{Gal}\,(L/F)^{\mathrm{ab}}$ in the case where L/F is a finite Galois totally ramified extension. We will use this to deduce that $\Upsilon_{L/F}$ exists in the case where L/F is finite and totally ramified, and is an isomorphism when L/F is abelian.

For the time being we will be assuming that L/F is a finite totally ramified Galois extension.

Definition 2.2.7. Define $U(\mathcal{L}/\mathcal{F})$ as the subgroup of $U_{1,\mathcal{L}}$ generated by the elements $u^{\sigma-1}$. where u runs through every element of $U_{1,\mathcal{L}}$ and σ runs through all elements of $\mathrm{Gal}\,(\mathcal{L}/\mathcal{F})$.

Definition 2.2.8. Let E/F be the maximal abelian subextension of L/F and let φ_L be any fixed Frobenius automorphism of L^{ur}. Next, let φ be an extension of φ_L to $\mathcal{L}$. Finally, let π be an arbitrary prime element of $\mathcal{L}$.

Fix $\alpha \in F^*$, we know that there is a $\beta \in \mathcal{L}^*$ such that $N_{\mathcal{L}/\mathcal{F}}(\beta) = \alpha$. As $N_{\mathcal{L}/\mathcal{F}}(\beta^{\varphi-1}) = \alpha^{\varphi-1} = 1$ we also know that there is a unique $\sigma \in \mathrm{Gal}\,(\mathcal{L}/\mathcal{F})$ such that:

$$\beta^{\varphi-1} \equiv \pi^{\sigma-1} \mod U(\mathcal{L}/\mathcal{F})$$

We have that $\mathrm{Gal}\,(\mathcal{L}/\mathcal{F}) \cong \mathrm{Gal}\,(L/F)$ and thus we may say that σ is contained in $\mathrm{Gal}\,(L/F)$. We will let $\Psi_{L/F}(\alpha)$ be equal to $\sigma\mid_E$, with σ being thought of as an element of $\mathrm{Gal}\,(L/F)$.

Lemma 2.2.3. *For a finite totally ramified extension L/F, the map $\Psi_{L/F}$: $F^*/N_{L/F}(L^*) \to \mathrm{Gal}\,(L/F)^{\mathrm{ab}}$ is a well defined homomorphism.*

Proof. The proof of this lemma can be found in two chapters, Lemma (3.1) Chapter **IV** and section (1.3) Chapter **V** in [5]. □

Theorem 2.2.2. *Let L/F be a finite totally ramified Galois extension then $\Psi_{L/F}$ is an isomorphism. We also have that $\tilde{\Upsilon}_{L/F}$ does not depend on the choice of $\tilde{\sigma}$ for $\sigma \in \mathrm{Gal}\,(L/F)$, so $\Upsilon_{L/F}$ is well defined in this case, and that the inverse of $\Psi_{L/F}$ is equal to $\Upsilon^{\mathrm{ab}}_{L/F}$. Here $\Upsilon^{\mathrm{ab}}_{L/F}$ is the map between $\mathrm{Gal}\,(L/F)^{\mathrm{ab}}$ and $F^*/N_{L/F}(L^*)$ induced by $\Upsilon_{L/F}$.*

Proof. The proof of this theorem can be found in two chapters, Theorem (3.2) Chapter **IV** and section (1.3) Chapter **V** in [5]. □

The following lemma is important for the proof of the next theorem.

Lemma 2.2.4. *Let L/F be a finite abelian extension. We have that there exists a finite unramified extension M/L such that M/F is abelian and $M = M_0K$. Here K/F is a abelian totally ramified extension and M_0/F is an unramified extension.*

For all M that have the above properties we have that:

$$N_{M/F}(M^*) = N_{M_0/F}(M_0^*) \cap N_{K/F}(K^*)$$

Proof. The proof of this lemma can be found in two chapters, Lemma (3.3) Chapter **IV** and section (1.3) Chapter **V** in [5]. □

We can use the above lemma and Proposition 2.2.1 to finish our description of $\Upsilon_{L/F}$ for all finite Galois extensions L/F.

Theorem 2.2.3. *Let L/F be a finite Galois extension. We have that $\tilde{\Upsilon}_{L/F}$ does not depend on the choice of $\tilde{\sigma} \in \mathrm{Frob}\,(L/F)$ for $\sigma \in \mathrm{Gal}\,(L/F)$. This means that we get a well defined Neukirch homomorphism $\Upsilon_{L/F} : \mathrm{Gal}\,(L/F) \to F^*/N_{L/F}(L^*)$.*

We also have that $\Upsilon_{L/F}$ induces an isomorphism, $\Upsilon^{\mathrm{ab}}_{L/F}$, between $\mathrm{Gal}\,(L/F)^{\mathrm{ab}}$ and $F^*/N_{L/F}(L^*)$.

Proof. The proof of this theorem can be found in two chapters, Theorem (3.3) Chapter **IV** and section (1.3) Chapter **V** in [5]. □

Although there is more that can be discussed about $\Upsilon_{L/F}$ and $\Psi_{L/F}$ in the case where $\overline{F}$ is assumed to be quasi-finite it is not necessary to bring it up in this thesis. Those who wish to find out should check out Chapters **IV** and **V** of [5].

2.2.1 Additive Polynomials

This will be an aside looking at additive polynomials over quasi-finite and perfect fields. The part about quasi-finite fields shall be coming from section 2 Chapter **V** of [5]. We will also be referencing Professor Fesenko's paper "Local Class Field Theory: Perfect Residue Case" [3] in this subsection. This is because [5] only talks about additive polynomials with how they relate to local class field theory with quasi-finite residue fields and as such we need to look at the paper about perfect residue fields in order to get the rest of information we want.

We should note that work on perfect fields is not necessary to understand this section, which is only about complete discrete valuation fields with quasi-finite residue fields. That being said it will come up when we look at local class field theory when $\overline{F}$ is assumed to be perfect in the next section. Discussing the topic here will help relate the coming section back to this one.

Definition 2.2.9. Let K be a field. If $f(X) \in K[X]$ has the property that for all α and β in K we have $f(\alpha+\beta) = f(\alpha) + f(\beta)$ then $f(X)$ is called an additive polynomial.

Lemma 2.2.5. *Let K be a field with characteristic 0 then a polynomial $f(X)$ over K is additive if and only if $f(X) = \alpha X$ for some $\alpha \in K$.*

Now let K be a field with positive characteristic p. This time the polynomial $f(X)$ over K is additive if and only if it has the form $\sum_{i=0}^{n} \alpha_i X^{p^i}$, where $\alpha_i \in K$ for all i.

Proof. The proof of this lemma can be found in Lemma (2.1) Chapter **V** in [5]. □

Note 2.2.8. As it is a lot less trivial, and relates to the work we are talking about on local class field theory, from now on we will assume that the field, K, we are working with has positive characteristic p.

Notation 2.2.1. We shall denote the additive polynomial $X^p - X$ as $\wp(X)$. If K is a field then $\wp(K)$ is equal to the image of $X^p - X$ when applied to the elements of K.

Note 2.2.9. We will occasionally refer to the derivative of a polynomial $f(X)$, written as $f'(X)$. In this case we are talking about the formal derivative. Here if $f(X) = \sum_{n\in\mathbb{Z}} \alpha_n X^n$ then $f'(X) = \sum_{n\in\mathbb{Z}} n\alpha_n X^{n-1}$.

We should note that as we are dealing with fields, K, of positive characteristic p and the polynomials we are talking about are of the form $f(X) = \sum_{n\geq 0} \alpha_n X^{p^n}$ we always have that $f'(X) = \sum_{n\geq 0} p^n \alpha_n X^{p^n-1} = \alpha_0$, which is a constant.

Definition 2.2.10. While the sum of two additive polynomials is additive the product does not have to be. As such if $f(X)$ and $g(X)$ are two additive polynomials over K we will define the product of $f(X)$ and $g(X)$, denoted by $(f \circ g)(X)$, as $f(g(X))$.

Lemma 2.2.6. *If $f(X)$ and $g(X)$ are two additive polynomials over K then so is $(f \circ g)(X)$.*

Additive polynomials over K have a ring structure with addition being regular polynomial addition and multiplication being $\circ$. The additive identity of the ring of additive polynomials is the constant polynomial $f(X) = 0$, while the multiplicative identity is $f(X) = X$.

Proof. The proof of this lemma can be found at the beginning of section (2.2) Chapter **V** in [5]. □

Note 2.2.10. General polynomials over K do not have a ring structure, as distribution often fails, the fact that we are working with the restriction that the polynomials must be additive solves that problem. For instance if $f(X)$ is of the form $\Sigma_{i\geq 0}\alpha_i X^{p^i}$, and the same holds for $g(X)$ and $h(X)$, then we will always get that $f \circ (g+h)(X) = (f \circ g + f \circ h)(X)$.

Definition 2.2.11. If $f(X) \in K[x]$ is of the form $(g \circ h)(X) = g(h(X))$ then we say that $g(X)$ is an outer component of $f(X)$ while $h(X)$ is an inner component of $f(X)$.

Definition 2.2.12. An additive polynomial $f(X) \in K[X]$ is called K-decomposable if the kernel of f over K is contained in K. We shall denote the set of K-decomposable polynomials by DP_K.

Lemma 2.2.7. *Let K be a perfect field and set $f(X) \in DP_K$ with $f'(0) \neq 0$. Then there is a set $\{\alpha_1, \alpha_2, ...\alpha_{n+1}\}$ of elements of K such that:*

$$\alpha_i^{-1} \in (\wp(X) \circ \alpha_{i+1}X \circ \wp(X) \circ \alpha_{i+2}X \circ ... \circ \alpha_{n+1}X)(K)$$

for all i and:

$$f(X) = \alpha_1 X \circ \wp(X) \circ \alpha_2 X \circ \wp(X) \circ ... \circ \wp(X) \circ \alpha_{n+1}X$$

Conversely every polynomial of the above form is K-decomposable.

Proof. The proof of this lemma can be found in Lemma 2.2 of [3]. □

Lemma 2.2.8. *The ring of additive polynomials over K is a left euclidean principal ideal ring, if K is a perfect field then it is also a right euclidean principal ideal ring.*

This means that if $f(X)$ and $g(X)$ are additive polynomials over K then there exists additive polynomials $h_1(X)$ and $q_1(X)$, with the degree of $q_1(X)$ being less than the degree of $g(X)$, such that $f(X) = (h_1 \circ g)(X) + q_1(X)$. If K is perfect then there are also polynomials $h_2(X)$ and $q_2(X)$, with the degree of $q_2(X)$ being less than the degree of $g(X)$, such that $f(X) = (g \circ h_2)(X) + q_2(X)$.

Proof. The proof of this lemma can be found in Lemma (2.2) and Proposition (2.2) Chapter **V** in [5]. □

Note 2.2.11. Lemma 2.2.8 means that when talking about additive polynomials over K, where K is a perfect field, the least common outer (inner) multiple and the greatest common outer (inner) divisor of two additive polynomials is a well defined thing we can discuss.

Corollary 2.2.1. *Let K be a perfect field and let $f_1(X)$ and $f_2(X)$ be two additive polynomials over K. Denote by $f_3(X)$ the greatest common outer divisor of $f_1(X)$ and $f_2(X)$. This means that $f_3(X)$ is the unique polynomial, up to multiplication by a constant, with the maximal degree such that there exists additive polynomials $h_1(X)$ and $h_2(X)$ over K with $f_1(X) = f_3(X) \circ h_1(X)$ and $f_2(X) = f_3(X) \circ h_2(X)$. We have the equality $f_3(K) = f_1(K) + f_2(K)$.*

Next denote the least common outer multiple of $f_1(X)$ and $f_2(X)$ by $f_4(X)$. As such $f_4(X)$ is the unique polynomial, up to multiplication by a constant, with the minimal degree such that there exists additive polynomials $h_1(X)$ and $h_2(X)$ over K with $f_4(X) = f_1(X) \circ h_1(X)$ and $f_4(X) = f_2(X) \circ h_2(X)$. We have that $f_3(K) = f_1(K) \cap f_2(K)$.

Proof. The proof of this corollary can be found in Corollary (2.2) Chapter **V** in [5]. □

Proposition 2.2.2. *Let $H \subseteq K$ be a finite additive subgroup, so H is a subgroup of the additive group over K. Then there is a unique additive polynomial, up to multiplication by a constant, $f(X)$ over K such that H is the set of roots of $f(X)$ and the degree of $f(X)$ is equal to $|H|$.*

Proof. The proof of this proposition can be found in Proposition (2.3) Chapter **V** in [5]. □

Corollary 2.2.2. *Let H be a finite additive subgroup of K^{sep} with the property that for all $\sigma \in \text{Gal}\,(K^{\text{sep}}/K)$ we have $\sigma(H) = H$, then H is the set of roots of some additive polynomial $f(X)$ over K.*

Proof. The proof of this corollary can be found in Corollary 1 of section (2.3) Chapter **V** in [5]. □

Corollary 2.2.3. *Considering K as a $\mathbb{F}_p$-vector space, let $\{\alpha_1, \alpha_2, ...\alpha_n\}$ be a set of elements of K that are linearly independent over $\mathbb{F}_p$. Next, let $\{\beta_1, \beta_2, ...\beta_n\}$ be a set of arbitrary elements of K. We have that there exists an additive polynomial, $f(X)$, over K with degree at most p^n such that for all i we have $f(\alpha_i) = \beta_i$.*

Proof. The proof of this corollary can be found in Corollary 2 of section (2.3) Chapter **V** in [5]. □

Proposition 2.2.3. *Let K be a perfect field and let $f(X)$ be a K-decomposable additive polynomial. Then there is an isomorphism:*

$$\lambda : K/f(K) \to \mathrm{Hom}_{\mathbb{Z}_p}\left(G_K^{\mathrm{abp}}, \ker(f(X))\right)$$

Here G_K^{abp} represents the Galois group $\mathrm{Gal}\left(K^{\mathrm{abp}}/K\right)$, with K^{abp} being the maximal abelian p-extension of K, and, if $f(b) = a$, we have $\lambda(a)(\phi) = \phi(b) - b$.

Proof. The proof of this proposition can be found in Proposition 2.3 of [3]. □

Corollary 2.2.4. *Let K be a perfect field, let $f(X) \in DP_K$ be such that $f'(0) \neq 0$ and finally let $g(X)$ be an additive polynomial over K. We have that $f(X)$ is an outer component of $g(X)$ if and only if $g(K) \subseteq f(K)$.*

Proof. The proof of this corollary can be found in Corollary 2.3 of [3]. □

We can change the result of Corollary 2.2.4 to talk about inner components if we expand the field we are over.

Lemma 2.2.9. *Let K be a perfect field and let $f(X)$ and $g(X)$ be additive polynomials over K with $f'(0) \neq 0$. If we expand the domain of $f(X)$ and $g(X)$ to all of K^{sep} then we have that $\ker(f(X)) \subseteq \ker(g(X))$ if and only if $f(X)$ is an inner component of $g(X)$.*

Proof. The proof of this lemma can be found in Proposition (2.5) Chapter **V** in [5]. □

Before we state the next result we need a few new definitions.

Definition 2.2.13. Let K be a field. We can create the notion of generalized additive polynomials over K. These are equations of the form $g(X) = \sum \beta_m X p^m$, with $m \in \mathbb{Z}$ and $\beta_m \in K$ for all m. Like with regular additive polynomials over K, we call a generalized additive polynomial over K, $g(X)$, K-decomposable if the kernel of $g(X)$ belongs to K. We again denote this by $g(X) \in DP_K$.

Let $f(X)$ be an additive polynomial over the field K. We can write $f(X)$ as $\sum_{n\geq 0} \alpha_n X^{p^n}$ and we will denote by $f^*(X)$ the generalized additive polynomial $\sum_{n\geq 0} \alpha_n^{p^{-n}} X^{p^{-n}}$.

Proposition 2.2.4. *Let K be a perfect field and let $f(X)$ be an additive polynomial over K. Then $f(X) \in DP_K$ if and only if $f^*(X) \in DP_K$.*

Proof. The proof of this proposition can be found in Proposition 2.4 of [3]. □

Corollary 2.2.5. *If we let K be a perfect field and set $f(X) \in DP_K$ then there is a set of cardinality $|\ker(f(X))|$ consisting of $\alpha_i \in K$ such that $f(K) = \cap_i \alpha_i \wp(K)$.*

Proof. The proof of this corollary can be found in Corollary 2.4 of [3]. □

Proposition 2.2.5. *Let K be a perfect field and let $f_1(X)$ and $f_2(X)$ both be elements of DP_K. The following statements are true:*

1) *The least common inner (outer) multiples and greatest common inner (outer) divisors of $f_1(X)$ and $f_2(X)$ are all K-decomposable.*

2) *Let $f_3(X)$ be the least common outer multiple of $f_1(X)$ and $f_2(X)$, then $f_3(K) = f_1(K) \cap f_2(K)$.*

3) *The set $\{\alpha \in K : f_1(\alpha) \in f_2(K)\}$ is equal to $h(K)$, where $h(X) \in DP_K$.*

Proof. The proof of this proposition can be found in Proposition 2.5 of [3]. □

Proposition 2.2.6. *Let K be a quasi-finite field and let $f(X)$ be an additive polynomial over K. Then there exists $g(X) \in DP_K$ such that $g'(0) \neq 0$ and $g(K) = f(K)$. We also have that $g(X)$ is an outer component of $f(X)$, so $f(X) = g(X) \circ h(X)$ for some additive polynomial, $h(X)$, over K.*

Proof. The proof of this proposition can be found in Proposition (2.6) Chapter **V** in [5]. □

Corollary 2.2.6. *Let K be a quasi-finite field and let $f(X)$ be an additive polynomial over K. The number of roots of $f(X)$ in K coincides with the index of $f(K)$ in K.*

Proof. This result falls out when you combine Proposition 2.2.6 with Proposition 2.2.3, taking into account that:

$$\mathrm{Hom}_{\mathbb{Z}_p}(G_K^{\mathrm{abp}}, \ker(f(X))) \cong \ker(f(X))$$

since K is quasi-finite and hence $G_K^{\mathrm{abp}} \cong \mathbb{Z}_p$. □

Corollary 2.2.7. *Let K to be a quasi-finite field and let $f(X)$ be an additive polynomial over K, then there is a finite set of $\alpha_i \in K$ such that $f(K) = \cap_i \alpha_i \wp(K)$.*

Proof. This result comes about from combining the result of Corollary 2.2.5 with the result of Proposition 2.2.6. □

Corollary 2.2.8. *Let K be a quasi-finite field and set $f(X)$ to be a non-zero additive polynomial over K. Then the following statements are equivalent:*

1) *$f(K) \neq K$.*

2) *$f(X)$ has a non-zero root in K.*

3) *There exists an $\alpha \in K^*$ such that $\alpha\wp(X) = \alpha X^p - \alpha X$ is an outer component of $f(X)$.*

4) *There exists an $\beta \in K^*$ such that $\wp(\beta X) = \beta^p X^p - \beta X$ is an inner component of $f(X)$.*

Proof. The proof of this corollary can be found in Corollary (2.6) Chapter **V** in [5]. □

Proposition 2.2.7. *Let K be a quasi-finite field and set $f(X) \in K[X]$ to be an additive polynomial. We know that $S = f(K)$ is a subgroup of finite index in the additive group K. Treating K/S as a finite dimensional $\mathbb{F}_p$-vector space; the endomorphisms of K/S are induced by additive polynomials.*

Proof. The proof of this proposition can be found in the beginning of section (2.7) and Proposition (2.7) Chapter **V** in [5]. □

Corollary 2.2.9. *Let K be a quasi-finite field and let $f(X)$ be a non-zero additive polynomial over K. Taking K as an additive group, with $f(K)$ as a subgroup, every intermediate subgroup between $f(K)$ and K is equal to $g(K)$ for some is an additive polynomial, $g(X)$, over K.*

Proof. The proof of this corollary can be found in Corollary 1 of section (2.7) Chapter **V** in [5]. □

Corollary 2.2.10. *Let K be a quasi-finite field and set $f_1(X)$ and $f_2(X)$ to be two additive polynomials over K. The homomorphisms from $K/f_1(K)$ to $K/f_2(K)$ are induced by additive polynomials.*

Proof. The proof of this corollary can be found in Corollary 2 of section (2.7) Chapter **V** in [5]. □

We will now briefly discuss the idea of additive polynomials forming a topology on a quasi-finite field.

Definition 2.2.14. Let K be a quasi-finite field; the results of Proposition 2.2.7 and its attendant corollaries show that the set of all $f(K)$, where $f(X) \in K[X]$ runs through all the additive polynomials over K, form a basis of neighbourhoods of a linear topology over K. We call this topology an additive topology over K and any neighbourhood of 0 can be written as $T = f(K)$ for some additive polynomial $f(X)$ over K.

Proposition 2.2.8. *Let K be a quasi-finite field and let us consider it having the additive topology defined in the above definition. In this case additive polynomials over K define continuous endomorphisms of K. Also if we consider the ring of all continuous endomorphisms of K then the subring defined by additive polynomials is dense.*

Proof. The proof of this proposition can be found in Proposition (2.8) Chapter **V** in [5]. □

Lemma 2.2.10. *Let K be a perfect field and set $f(X) \in K[X]$ to be non-zero and have the property $f(0) = 0$. Also we will let $g(X)$ be any non-zero additive polynomial element of DP_K. There exists finite sequences of polynomials $h_i(X)$ and $q_i(X)$ such that $\sum_i q_i(X)$ is a non-zero element of DP_K and both $\sum_i f(h_i(X))$ and $\sum_i f(q_i(X))$ are additive polynomials. We next have that $g(X)$ is an outer component of both $\sum_i f(h_i(X))$ and $\sum_i f(q_i(X))$ and finally that $\sum_i f(h_i(X)) \neq 0$.*

Proof. The proof of this lemma can be found in Lemma 2.6 of [3]. □

Note 2.2.12. This is a stronger version of a similar result, namely Proposition (2.9) Chapter **V** in [5]. For example this result talks about perfect fields rather than just quasi-finite fields. As such there would be no point in going over the result from [5] here, but there are a couple of corollaries brought up after that proposition, one of which shows that there is a even a third version of Lemma 2.2.10 that applies only to quasi-finite fields.

Corollary 2.2.11. *Let K be a quasi-finite field and set $f(X) \in K[X]$ to be non-zero and have the property $f(0) = 0$; likewise let $g(X)$ be any non-zero additive polynomial over K. There exists finite sequences of polynomials $h_i(X)$ and $q_i(X)$ such that $\sum_i q_i(X)$ is non-zero and both $\sum_i f(h_i(X))$ and $\sum_i f(q_i(X))$ are additive polynomials. We also have that $g(X)$ is an outer component of both $\sum_i f(h_i(X))$ and $\sum_i f(q_i(X))$ and finally that $\sum_i f(h_i(X)) \neq 0$.*

Proof. The proof of this corollary can be found in Corollary 1 of section (2.9) Chapter **V** in [5]. □

Note 2.2.13. The difference between Corollary 2.2.11 and Lemma 2.2.10 is that the requirement for $g(X)$ to be K-decomposable is dropped. However the fact that we only know that $\sum_i q_i(X)$ is a non-zero additive polynomial, and not necessarily an element of DP_K, means that Corollary 2.2.11 is only a variation on Lemma 2.2.10 and not a stronger version.

Corollary 2.2.12. *Let K be a quasi-finite field and look at the additive topology on K, as described in Definition 2.2.14. A neighbourhood of 0 in that topology can be considered as a vector subspace over $\mathbb{F}_p$ that contains the set of values of a non-zero element $f(X) \in K[X]$ with the property $f(0) = 0$.*

Proof. The proof of this corollary can be found in Corollary 2 of section (2.9) Chapter **V** in [5]. □

2.2.2 Back to Local Class Field Theory

We will now move onto a brief look at the modified Existence Theorem that occurs with complete discrete valuation fields with quasi-finite residue fields. This is the first place that any substantial difference between this topic and regular local class field theory appears.

Note 2.2.14. We will be going back to the assumptions that we made in Note 2.2.2. These are that F is a complete discrete valuation field whose residue field is both quasi-finite and has positive characteristic p. We will also have that L/F is an arbitrary finite Galois field extensions of F.

Definition 2.2.15. Let π_F be any prime element of F. An open subgroup N of F^* is called Normic if for all $i > 0$ we have that there exists a polynomial $f_i(X) \in \mathcal{O}_F[X]$ such that the residue polynomial $\overline{f}_i(X) \in \mathcal{O}_{\overline{F}}[X]$ is non-constant and $1 + f_i(\mathcal{O}_F)\pi_F^i \subseteq N$.

Note 2.2.15. The definition of Normic subgroups is independent of the choice of prime element of F. If π and π' are two prime elements of F then we know that $\pi' = \alpha\pi$ for some $\alpha \in U_F$. As such if $N \subseteq F^*$ is a Normic subgroup such that for a given $i > 0$ we have that $1 + f_i(\mathcal{O}_F)(\pi')^i \subseteq N$ where $\overline{f}_i(X)$ is non-constant then we see that $1 + \alpha f_i(\mathcal{O}_F)\pi^i \subseteq N$. We know that $\overline{\alpha} \neq 0$, as $\alpha \in U_F$, and thus $\overline{\alpha f}_i(X)$ is also a non-constant polynomial.

Note 2.2.16. Section (3.1) Chapter **V** in [5] shows that when dealing with any Normic subgroup we can always choose a family of $f_i(X)$ such that for each $i > 0$ we have $\overline{f}_i(X)$ is a non-zero additive polynomial over $\overline{F}$ with $f_i(0) = 0$.

Note 2.2.17. If $\overline{F}$ is finite, and thus we are dealing with regular local class field theory, then open and Normic subgroups of F^* have the same definition. It is trivial from the definition that Normic subgroups are open so we must show that open subgroups are Normic. Assume that $\overline{F}$ has cardinality $q \in \mathbb{Z}_{>0}$, where q is some positive power of prime number p, and let N be an open subgroup of F^*. We know that, since N is an open subgroup of F^*, that there exists a $t \in \mathbb{Z}_{>0}$ such that $U_{t,F} \subseteq N$. Let $\alpha \in \mathcal{O}_F$, we know that $\overline{\alpha}^q = \overline{\alpha}$ and thus $(\alpha^q - \alpha) \in \pi_F\mathcal{O}_F$, where π_F is a prime element of F. This means that $(\alpha^q - \alpha)^{p^t} \in \pi_F^t\mathcal{O}_F$. If, for all $i > 0$, we set $f_i(X)$ to equal $(X^q - X)^{p^t}$ we have that $\overline{f}_i(X) = (X^q - X)^{p^t}$, which is not a constant, and $1 + f_i(\mathcal{O}_F) \subseteq U_{t,F} \subseteq N$ and thus $1 + f_i(\mathcal{O}_F)\pi_F^i \subseteq N$ for all positive i. This is all that is required to show that N is a Normic subgroup of F^*. This result was taken from section (3.1) Chapter **V** in [5].

Lemma 2.2.11. *Let N_1 and N_2 be two Normic subgroups of F^*, then $N_1 \cap N_2$ and N_1N_2 are also Normic subgroups of F^*.*

Proof. The proof of this lemma can be found in the proof of Proposition (3.3) Chapter **V** of [5] and the fact that an open subgroup of F^* that contains a Normic subgroup clearly has the requirements for a Normic subgroup. □

Note 2.2.18. Lemma 2.2.11 means the Normic subgroups of F^* form a lattice with respect to product, N_1N_2, and intersection, $N_1 \cap N_2$. This is in fact a sublattice of the lattice that all open subgroups of F^* already naturally creates.

Lemma 2.2.12. *Let L/F be a finite abelian extension, then $N_{L/F}(L^*)$ is a Normic subgroup of finite index in F^*.*

Proof. The proof of this lemma can be found in Proposition (3.2) Chapter **V** of [5]. □

We are now ready to state the modified Existence Theorem that occurs when we assume that $\overline{F}$ is quasi-finite.

Theorem 2.2.4. *The map $L/F \mapsto N_{L/F}(L^*)$ is an order reversing bijection between the lattice of finite abelian extensions of F and the lattice of Normic subgroups of finite index in F^*.*

Proof. The proof of this theorem can be found in Theorem (3.4) Chapter **V** of [5]. □

Note 2.2.19. We have shown, in Note 2.2.17, that if $\overline{F}$ is finite, then open and Normic subgroups of F^* are equivalent. Likewise, Note 2.2.18 showcases that the lattice of Normic subgroups of F^* is a sublattice of the lattice of open subgroups of F^*. Therefore, it is clear that the above Existence Theorem is just a generalisation of the one we have when $\overline{F}$ is finite.

There is a lot more about local class field theory discussed, both in George Whaples' papers and Professors Fesenko and Vostokov's book, however it is not obvious how it would help the reader understand this thesis so we will not discuss the topic further here. Those who wish to understand more are recommended to look at the both of them.

2.3 Review of Local Class Field Theory with Perfect Residue Fields

In the following section we shall be discussing local class field theory when the only requirements on the base field, F, is that $\overline{F}$ is perfect and has positive characteristic. As such for the remainder of this section we shall assume that F is a complete discrete valuation field with a perfect residue field of characteristic $p > 0$.

We shall be getting most of our results from Professor Ivan Fesenko's paper "Local Class Field Theory: Perfect Residue Case" [3]. However, it should be noted that there is a mistake in this paper which was corrected when the summary of the subject was written in section 4 Chapter **V** of [5]. This change shall be explained when we get to it.

Note 2.3.1. In this section, and in the following section where we deal with imperfect residue fields, we only deal with totally ramified p-extensions. This is because it is there that things change from what we have already seen. No matter the properties of $\overline{F}$, if L/F is an unramified extension of degree n then $N_{L/F}(L^*) = \langle \pi_F^n \rangle \cdot U_F$, where π_F is a prime element of F. Meanwhile, if L/F is a totally tamely ramified extension of degree n, let T be the subgroup of U_F such that if $\alpha \in T$ then $\overline{\alpha} \in \overline{F}^n$ and for every $\theta \in \overline{F}$ there is a unique $\alpha \in T$ such that $\overline{\alpha} = \theta$. In this case $N_{L/F}(L^*) = \langle \pi \rangle \cdot T \cdot U_{1,F}$.

Since the rest is already known to us the only thing that needs to be investigated is when L/F is a finite abelian totally ramified p-extension.

Note 2.3.2. In the case when we are dealing with perfect residue fields we only deal with $U_{1,F}/N_{L/F}(U_{1,L})$ and not the full $F^*/N_{L/F}(L^*)$. This is okay as we are only dealing with finite abelian totally ramified p-extensions, L/F, and we know that, for all totally ramified fields, $N_{L/F}(L^*) \cap U_{1,F} = N_{L/F}(U_{1,L})$. We also know, in the cases where $\overline{F}$ is perfect, that both a prime element, π_F, of F is contained in $N_{L/F}(L^*)$ and that $\lambda_0(N_{L/F}(U_L)) = \overline{F}$, the λ_0-map is taken from Notation 2.1.2. This gives us that the group $N_{L/F}(L^*)$ is uniquely determined by the subgroup $N_{L/F}(U_{1,L})$.

Definition 2.3.1. We will define $\widehat{F}$ as the maximal unramified p-extension of F. We also construct $\widehat{L}$ to be the equivalent for L.

Note 2.3.3. If L/F is a finite abelian totally ramified p-extension then we have that $\widehat{L} = L\widehat{F}$, this means that $\widehat{F} \subseteq \widehat{L}$.

Note 2.3.4. This is where [3] contains the error that was mentioned earlier. It was originally thought that we only needed to define $\widehat{F}$, which was in the paper denoted by $\tilde{F}$, to be the maximal abelian unramified p-extension of F, and not the maximal unramified p-extension of F [3]. Luckily by the time of the later publication, [5], it had been realised what properties $\widehat{F}/F$ needed to posses, as seen in section 4 Chapter **V** in [5].

Note 2.3.5. From now on when referring to [3] we shall be assuming, except when explicitly noted, that every instance of $\tilde{F}$ is replaced by $\widehat{F}$.

Note 2.3.6. The original [3] deals with generic field extension L/F, where it is a finite totally ramified p-extension [3]. As such a lot of statements and proofs of that paper involve the abelianisation of $\mathrm{Gal}\,(L/F)$; this will not be necessary for our work as we will always be assuming that L/F is an abelian extension. This means, as it is not necessary, that we will be missing out the abelianisation of the Galois groups when going over the mathematics of the paper.

Definition 2.3.2. Let L/F be a finite abelian totally ramified p-extension. We shall denote the homomorphism group $\mathrm{Hom}_{\mathbb{Z}_p}(\mathrm{Gal}\,(\widehat{F}/F), \mathrm{Gal}\,(L/F))$ as $\mathrm{Gal}\,(L/F)^{\wedge}$.

Note 2.3.7. The above definition replaces the $\mathrm{Gal}\,(L/F)^*$ found in [3], that construct is not used as it uses $\mathrm{Gal}\,(\tilde{F}/F)$ rather than $\mathrm{Gal}\,(\widehat{F}/F)$.

Note 2.3.8. Since the extensions we are dealing with are abelian we do have that $\mathrm{Gal}\,(L/F)^{\wedge} = \mathrm{Gal}\,(L/F)^*$. However it is still important that we deal with $\widehat{F}/F$, as the mathematics does not work if we are only dealing with the maximal abelian unramified p-extension of F, even if in this summary of definitions and results it is not obvious that is the case.

Note 2.3.9. Let L/F be a finite abelian totally ramified p-extension but assume that $\overline{F}$ is quasi-finite. By the nature of quasi-finite fields we know that $\mathrm{Gal}\,(\widehat{F}/F) \cong \mathbb{Z}_p$. This means that:

$$\begin{aligned}\mathrm{Gal}\,(L/F)^{\wedge} = \mathrm{Hom}_{\mathbb{Z}_p}(\mathrm{Gal}\,(\widehat{F}/F), \mathrm{Gal}\,(L/F)) &\cong \mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Z}_p, \mathrm{Gal}\,(L/F)) \\ &\cong \mathrm{Gal}\,(L/F)\end{aligned}$$

Definition 2.3.3. Let L/F be a finite abelian totally ramified p-extension. We want to create a Neukirch homomorphism:

$$\Upsilon_{L/F} : \mathrm{Gal}\,(L/F)^{\wedge} \to U_{1,F}/N_{L/F}(U_{1,F})$$

that is a generalisation of the Neukirch homomorphism we have seen when there are more restrictions on F.

Let χ be an element of $\mathrm{Gal}\,(L/F)^{\wedge}$. Next, let Σ_χ be the field in $\widehat{L}$ fixed by the elements $\tau_\varphi \in \mathrm{Gal}\,(\widehat{L}/F)$ such that $\tau_\varphi\mid_{\widehat{F}} = \varphi$ and $\tau_\varphi\mid_L = \chi(\varphi)$ while φ varies through all the elements of $\mathrm{Gal}\,(\widehat{F}/F)$. We have that $\widehat{L}/\Sigma_\chi$ is unramified while Σ_χ/F is a finite totally ramified p-extension.

Let π_L be a prime element of L and π_χ be a prime element of Σ_χ, then define $\Upsilon_{L/F}(\chi)$ to be:

$$N_{\Sigma_\chi/F}(\pi_\chi)N_{L/F}(\pi_L^{-1}) \mod N_{L/F}(U_{1,L})$$

Note 2.3.10. In the above definition we do not need φ to vary through every element of $\mathrm{Gal}\,(\widehat{F}/F)$ but instead just through a topological basis of $\mathrm{Gal}\,(\widehat{F}/F)$. By the definition of a basis it will lead to the same fixed field Σ_χ.

Lemma 2.3.1. *Let L/F be a finite abelian totally ramified p-extension, then the Neukirch map $\Upsilon_{L/F} : \mathrm{Gal}\,(L/F)^{\wedge} \to U_{1,F}/N_{L/F}(U_{1,F})$ is a well defined homomorphism.*

Proof. The proof of this lemma can be found in Lemma 1.2 of [3]. □

Note 2.3.11. Suppose we assume that L/F is a finite abelian totally ramified p-extension but that $\overline{F}$ is quasi-finite. We know that in this case $\mathrm{Gal}\,(L/F)^{\wedge} \cong \mathrm{Gal}\,(L/F)$, because of Note 2.3.9. Thus we now have a map:

$$\Upsilon_{L/F} : \mathrm{Gal}\,(L/F) \to U_{1,F}/N_{L/F}(U_{1,L})$$

Now fix $\sigma \in \mathrm{Gal}\,(L/F)$ and choose $\tau \in \mathrm{Gal}\,(\widehat{L}/F)$ such that $\tau\mid_L = \sigma$ and $\tau\mid_{\widehat{F}} = 1_{\widehat{F}}$, which is a topological generator of the group $\mathrm{Gal}\,(\widehat{F}/F) \cong \mathbb{Z}_p$. Let Σ_σ be a the fixed field of τ in $\widehat{F}$, then from the definition of $\Upsilon_{L/F}$ we can see that:

$$\Upsilon_{L/F}(\sigma) = N_{\Sigma_\sigma/F}(\pi_\sigma)N_{L/F}(\pi_L^{-1}) \mod N_{L/F}(U_{1,L})$$

where π_L is a prime element of L and π_σ is a prime element of Σ_σ.

However, F is a totally ramified field with quasi-finite residue field, and looking at the definition of the Neukirch homomorphism in that case, via Definition 2.2.6, we can see that τ is an element of $\mathrm{Frob}\,(L/F)$ that extends σ. So it is easy to see that the quasi-finite residue field version of the Neukirch homomorphism maps σ to the same value in $U_{1,F}/N_{L/F}(U_{1,L})$ as the perfect field case. Finally, we can then send it to the value over $N_{L/F}(L^*)$ by the map $U_{1,F}/N_{L/F}(U_{1,L}) \to F^*/N_{L/F}(L^*)$.

This shows that the Neukirch homomorphism that has just been defined is a generalisation of what we have seen before.

For L/F, a finite abelian totally ramified p-extension, we wish to create a new map $\Psi_{L/F} : U_{1,F}/N_{L/F}(U_{1,L}) \to \mathrm{Gal}\,(L/F)^\wedge$ which is an inverse of the map $\Upsilon_{L/F}$ and a generalisation of the map, also denoted by $\Psi_{L/F}$, found in the case where we have more restrictions on $\overline{F}$.

Notation 2.3.1. We have that if L/F is a finite abelian totally ramified p-extension then $\mathrm{Gal}\,(\widehat{L}/L) \cong \mathrm{Gal}\,(\widehat{F}/F)$. As such if φ is an element of $\mathrm{Gal}\,(\widehat{F}/F)$, we shall denote its extension to $\mathrm{Gal}\,(\widehat{L}/L)$ as φ as well. For an in-depth proof of why $\mathrm{Gal}\,(\widehat{L}/L) \cong \mathrm{Gal}\,(\widehat{F}/F)$, in both the case where $\overline{F}$ is perfect and the case where it is imperfect, you can read section 2 of Chapter 3 of this thesis, and in particular Corollary 3.2.2.

Definition 2.3.4. Let L/F be a finite abelian totally ramified p-extension. We shall denote the subgroup generated by:

$$\{\alpha^{\sigma-1} : \alpha \in U_{1,\widehat{L}}, \sigma \in \mathrm{Gal}\,(\widehat{L}/\widehat{F})\}$$

with the symbol $V(L \mid F)$.

Note 2.3.12. Remember that if $\overline{F}$ is quasi-finite then we have $U(\mathcal{L}/\mathcal{F})$, which is equal to the subgroup generated by:

$$\{\alpha^{\sigma-1} : \alpha \in U_{1,\mathcal{L}}, \sigma \in \mathrm{Gal}\,(\mathcal{L}/\mathcal{F})\}$$

Now $\mathcal{F}$ is a complete discrete valuation field with residue field equal to the algebraic closure of $\overline{F}$. This means that we can take $\widehat{F} \subseteq \mathcal{F}$, from the definition of $\widehat{F}$, and we also have $\widehat{L} \subseteq \mathcal{L}$. Since both $\mathrm{Gal}\,(\widehat{L}/\widehat{F})$ and $\mathrm{Gal}\,(\mathcal{L}/\mathcal{F})$ are isomorphic to $\mathrm{Gal}\,(L/F)$, and the restriction homomorphism between $\mathrm{Gal}\,(\widehat{L}/\widehat{F})$ and $\mathrm{Gal}\,(\mathcal{L}/\mathcal{F})$ is an isomorphism, we have that $V(L \mid F) \subseteq U(\mathcal{L}/\mathcal{F})$.

With the above definition we have the following exact sequence:

$$1 \to \mathrm{Gal}\,(\widehat{L}/\widehat{F}) \to U_{1,\widehat{L}}/V(L \mid F) \to U_{1,\widehat{F}} \to 1$$

where the map $U_{1,\widehat{L}}/V(L \mid F) \to U_{1,\widehat{F}}$ is induced by $N_{\widehat{L}/\widehat{F}}$.

Lemma 2.3.2. *Let L/F be a finite abelian totally ramified p-extension. Let φ_j, $j \in J$, be $\mathbb{Z}_p$-linearly independent elements of* $\mathrm{Gal}(\widehat{F}/F)$. *Let σ be an element of* $\mathrm{Gal}(\widehat{L}/F)$ *such that $\sigma|_{\widehat{F}} \in \mathrm{Gal}(\widehat{F}/F)$ is $\mathbb{Z}_p$-linearly independent of every φ_j. Then set T to be the fixed subfield of $\widehat{L}$ corresponding to $\{\varphi_j\}_{j\in J}$ and finally have β be an element of T. Then there exists a $\gamma \in U_{1,T}$ such that $\gamma^{\sigma-1} = \beta$.*

Proof. The proof of this lemma can be found in Lemma 1.4 of [3]. □

We are now ready to define the map $\Psi_{L/F}$.

Definition 2.3.5. Let L/F be a finite abelian totally ramified p-extension. Let $\varepsilon \in U_{1,F}$, $\varphi \in \mathrm{Gal}(\widehat{F}/F)$ and fix a prime element, π_L, of L . By Lemma 2.3.1 we have that there exists an $\alpha \in U_{1,\widehat{L}}$ such that $N_{\widehat{L}/\widehat{F}}(\alpha) = \varepsilon$. Since φ commutes with the elements of $\mathrm{Gal}(\widehat{L}/\widehat{F})$ we have that $N_{\widehat{L}/\widehat{F}}(\alpha^{-1}\varphi(\alpha)) = 1$. We can now use the exact sequence that we have previously seen in order to get that there is a $\sigma \in \mathrm{Gal}(\widehat{L}/\widehat{F})$ such that $\alpha^{-1}\varphi(\alpha) \equiv \pi_L\sigma(\pi_L^{-1}) \mod V(L \mid F)$. We next define a map $\chi : \mathrm{Gal}(\widehat{F}/F) \to \mathrm{Gal}(L/F)$ by setting $\chi(\varphi) = \sigma|_L$. We have that χ is in fact a member of $\mathrm{Gal}(L/F)^\wedge$, and we can define $\Psi_{L/F} : U_{1,F}/N_{L/F}(U_{1,L}) \to \mathrm{Gal}(L/F)^\wedge$ by having $\Psi_{L/F}(\varepsilon)$ being equal to χ.

Note 2.3.13. Let us now assume that $\overline{F}$ has a quasi-finite residue field and that L/F is a finite abelian totally ramified p-extension; finally set t' as an element of F^*.

Like in Note 2.3.11 we can simplify $\Psi_{L/F}$ to being:

$$\Psi_{L/F} : U_{1,F}/N_{L/F}(U_{1,L}) \to \mathrm{Gal}(L/F)$$

Since both a prime element of F is in $N_{L/F}(L^*)$ and $\lambda_0(N_{L/F}(U_L)) = \overline{F}$ we have that for all $\alpha \in F^*$ there is a $\beta \in U_{1,F}$ such that $\alpha \equiv \beta \mod N_{L/F}(L^*)$. As $N_{L/F}(U_{1,L}) \subseteq N_{L/F}(L^*)$ we can now make $\Psi_{L/F}$ a map between $F^*/N_{L/F}(L^*)$ and $\mathrm{Gal}(L/F)$.

Let $t \in U_{1,F}$ be such that $t' \equiv t \mod N_{L/F}(L^*)$. From the definition of $\Psi_{L/F}$ we know that there is an $\eta \in U_{1,\widehat{L}}$ such that $N_{\widehat{L}/\widehat{F}}(\eta) = t$ and $N_{\widehat{L}/\widehat{F}}(\eta^{-1}\varphi(\eta)) = 1$, where φ is a topological generator of $\mathrm{Gal}(\widehat{F}/F) \cong \mathbb{Z}_p$. This means that $N_{\widehat{L}/\widehat{F}}(\eta^{-1}\varphi'(\eta)) = 1$, where φ' is any element of $\mathrm{Gal}(\widehat{F}/F)$.

So, if we let π_L be a prime element of L we see that $\eta^{\varphi-1} \equiv \pi_L^{\sigma-1} \mod V(L \mid F)$ for some $\sigma \in \mathrm{Gal}(L/F)$. By the isomorphism between $\mathrm{Gal}(L/F)^\wedge$ and $\mathrm{Gal}(L/F)$ that sends χ to $\chi(\varphi)$, we may count $\Psi_{L/F}(t)$ as σ.

However, we have already seen in Note 2.3.12 that when $\overline{F}$ is quasi-finite $V(L \mid F) \subseteq U(\mathcal{L}/\mathcal{F})$. This means that we have that $N_{\mathcal{L}/\mathcal{F}}(\eta) = t$ and $\eta^{\varphi-1} \equiv \pi_L^{\sigma-1} \mod U(\mathcal{L}/\mathcal{F})$. So if we take the map, $\Psi_{L/F}$, that was defined for quasi-finite residue fields and apply it to t we get the same value of $\mathrm{Gal}\,(L/F)$ as we did when we used the version for perfect residue fields.

This shows that $\Psi_{L/F}$ we have just defined is a generalisation of the map with the same notation that we have seen for quasi-finite residue fields.

Theorem 2.3.1. *Let L/F be a finite abelien totally ramified p-extension, then the map $\Psi_{L/F} : U_{1,F}/N_{L/F}(U_{1,L}) \to \mathrm{Gal}\,(L/F)^\wedge$ is well defined and an isomorphism. We also get that the Neukirch map:*

$$\Upsilon_{L/F} : \mathrm{Gal}\,(L/F)^\wedge \to U_{1,F}/N_{L/F}(U_{1,L})$$

is the inverse of $\Psi_{L/F}$.

Proof. The proof of this theorem can be found in Lemma 1.5, Proposition 1.6 and Theorem 1.7 of [3]. □

Lemma 2.3.3. *The map $\Psi_{L/F}$, has the following functorial properties:*

1) *Let L/F and L'/F' be finite abelian totally ramified p-extensions, with F'/F and L'/L being finite totally ramified extensions. The following diagram is commutative:*

$$\begin{array}{ccc} U_{1,F'}/N_{L'/F'}(U_{1,L'}) & \xrightarrow{\Psi_{L'/F'}} & \mathrm{Gal}\,(L'/F')^\wedge \\ \downarrow{\scriptstyle N_{F'/F}} & & \downarrow \\ U_{1,F}/N_{L/F}(U_{1,L}) & \xrightarrow{\Psi_{L/F}} & \mathrm{Gal}\,(L/F)^\wedge \end{array}$$

Here the right vertical homomorphism is induced by the restriction $\mathrm{Gal}\,(L'/F') \to \mathrm{Gal}\,(L/F)$ *and the isomorphism between* $\mathrm{Gal}\,(\widehat{F}/F)$ *and* $\mathrm{Gal}\,(\widehat{F'}/F')$.

2) *Let L/F be a finite abelian totally ramified p-extension and let σ be an automorphism of* $\mathrm{Gal}\,(F^{\mathrm{sep}}/F)$. *Define the map* $\sigma^\wedge\, \mathrm{Gal}\,(L/F)^\wedge \to \mathrm{Gal}\,(\sigma L/\sigma F)^\wedge$ *as:*

$$(\sigma^\wedge(\chi))(\sigma\varphi\sigma^{-1}) = \sigma\chi(\varphi)\sigma^{-1}$$

where $\chi \in \mathrm{Gal}\,(L/F)^\wedge$ *and* $\varphi \in \mathrm{Gal}\,(\widehat{F}/F)$*. Then the following diagram is commutative:*

$$\begin{array}{ccc} U_{1,F}/N_{L/F}(U_{1,L}) & \xrightarrow{\Psi_{L/F}} & \mathrm{Gal}\,(L/F)^\wedge \\ \downarrow & & \downarrow \sigma^\wedge \\ U_{1,\sigma F}/N_{\sigma L/\sigma F}(U_{1,\sigma L}) & \xrightarrow{\Psi_{\sigma L/\sigma F}} & \mathrm{Gal}\,(\sigma L/\sigma F)^\wedge \end{array}$$

3) *Let* L/F *be a finite abelian totally ramified* p*-extension with subextension* M/F*. Let the map* $\mathrm{Ver}^\wedge : \mathrm{Gal}\,(L/F)^\wedge \to \mathrm{Gal}\,(L/M)^\wedge$ *be induced by* $\mathrm{Ver} : \mathrm{Gal}\,(L/F) \to \mathrm{Gal}\,(L/M)$*. Then the following diagram is commutative:*

$$\begin{array}{ccc} U_{1,F}/N_{L/F}(U_{1,L}) & \xrightarrow{\Psi_{L/F}} & \mathrm{Gal}\,(L/F)^\wedge \\ \downarrow & & \downarrow \mathrm{Ver}^\wedge \\ U_{1,M}/N_{L/M}(U_{1,L}) & \xrightarrow{\Psi_{L/M}} & \mathrm{Gal}\,(L/M)^\wedge \end{array}$$

Proof. The proof of this lemma can be found in Proposition 1.8 of [3]. □

The next result is in neither [3] nor [5], however it follows quite simply from the above lemma and we will be using it in Chapter 3.

Corollary 2.3.1. *Let* L/F *be a finite abelian totally ramified* p*-extension with* M/F *being a subextension. We have that the kernel of the map* $N_{M/F} : U_{1,M} \to U_{1,L}$ *is contained in the image of the map* $N_{L/M} : U_{1,L} \to U_{1,M}$*.*

Proof. Let us use part 1) of Lemma 2.3.3 with $L' = L$ and $F' = M$. This will give us the following commutative diagram:

$$\begin{array}{ccc} U_{1,M}/N_{L/M}(U_{1,L}) & \xrightarrow{\Psi_{L/M}} & \mathrm{Gal}\,(L/M)^\wedge \\ \downarrow N_{M/F} & & \downarrow \\ U_{1,F}/N_{L/F}(U_{1,L}) & \xrightarrow{\Psi_{L/F}} & \mathrm{Gal}\,(L/F)^\wedge \end{array}$$

The map from $\mathrm{Gal}\,(L/M)^\wedge$ to $\mathrm{Gal}\,(L/F)^\wedge$ is induced by the inclusion from $\mathrm{Gal}\,(L/M)$ to $\mathrm{Gal}\,(L/F)$ and is therefore injective. We also have that both $\Psi_{L/M}$ and $\Psi_{L/F}$ are isomorphisms which gives us that the map:

$$N_{M/F} : U_{1,M}/N_{L/M}(U_{1,L}) \to U_{1,F}/N_{L/F}(U_{1,L})$$

is injective.

This gives us that if $a \in U_{1,M}$ is such that $N_{M/F}(a) = 1$, which is contained in $N_{L/F}(U_{1,L})$, then $a \in N_{L/M}(U_{1,L})$. This is precisely what we were looking for. □

Corollary 2.3.2. *Let L_1/F and L_2/F be finite abelian totally ramified p-extensions such that L_3/F, where $L_3 = L_1L_2$, is also totally ramified. Denote $L_1 \cap L_2$ as L_4. We have the following equalities:*

1) $N_{L_3/F}(U_{1,L_3}) = N_{L_1/F}(U_{1,L_1}) \cap N_{L_2/F}(U_{1,L_2})$

2) $N_{L_4/F}(U_{1,L_4}) = N_{L_1/F}(U_{1,L_1})N_{L_2/F}(U_{1,L_2})$

We also have that $L_1 \subseteq L_2$ if and only if $N_{L_1/F}(U_{1,L_1}) \supseteq N_{L_2/F}(U_{1,L_2})$.

Proof. The proof of this corollary can be found in Corollary 1.8 of [3]. □

We are next going to talk about the Existence Theorem for when we only assume $\overline{F}$ is perfect. We will be taking information from subsection 1 of section 2 in the this chapter, in which we took a look at additive polynomials in the following work.

Definition 2.3.6. Let K be a field. Taking K as an additive group, a subgroup N of K is called polynomial if $N = f(K)$ for some non-zero K-decomposable polynomial $f(X)$.

Definition 2.3.7. Let N be a subgroup of $U_{1,F}$. We call N a Normic subgroup if the following three criteria are met:

1) N is an open subgroup of $U_{1,F}$.

2) For all $i > 0$, there exists a polynomial $f_i(X) \in F[x]$ such that $\overline{f}_i(X)$ is a non-zero $\overline{F}$-decomposable polynomial and we have that $1 + f_i(\mathcal{O}_F)\pi_F^i \subseteq N$, here π_F is any prime element of F.

3) Let i be any positive integer, then the image of the subgroup $(N \cap U_{i,F})/U_{i+1,F}$ under the λ_i-map is polynomial.

Note 2.3.14. The name "Normic subgroup" is the same name we used in the previous section, in Definition 2.2.15, when talking about quasi-finite residue fields.

It is in fact possible to show that this version of Normic subgroup is in fact a generalisation, of a sort, of the previous one we saw. However, rather than prove it here it would be simpler if we did it a bit later on, in Note 2.3.19, once we have got a few more results.

Proposition 2.3.1. *Let N be a Normic subgroup of $U_{1,F}$, by definition 2.3.7, and let L/F be an abelian totally ramified p-extension. Then $N_{L/F}^{-1}(N)$ is a Normic subgroup of $U_{1,L}$, again as in Definition 2.3.7.*

Proof. The proof of this proposition can be found in Proposition 3.2 of [3]. □

Definition 2.3.8. Let π be a fixed prime element of F. We shall denote by $\mathfrak{F}_\pi$ the set of all finite abelian totally ramified p-extensions of F, L/F, such that $\pi \in N_{L/F}(L^*)$.

We will denote the composition of all elements of $\mathfrak{F}_\pi$ by F_π. Thus F_π/F is the composition of L_i/F, $i \in I$, for all $L_i/F \in \mathfrak{F}_\pi$.

Note 2.3.15. The reason $\mathfrak{F}_\pi$ needs to be constructed is because of the fact that we are only dealing with totally ramified extensions. If L_1/F and L_2/F are two finite abelian totally ramified p-extensions then L_1L_2/F is a finite abelian p-extension but not necessarily totally ramified. $\mathfrak{F}_\pi$ helps deal with this, as the next lemma shows.

Lemma 2.3.4. *If we let π be a fixed prime element of F then $\mathfrak{F}_\pi$ is closed under intersection and composition. As such if both L_1/F and L_2/F are in $\mathfrak{F}_\pi$, and thus $\pi \in N_{L_1/F}(L_1^*) \cap N_{L_2/F}(L_2^*)$, then so is L_1L_2/F and $(L_1 \cap L_2)/F$.*

Proof. The proof of this lemma can be found in section 3.3 of [3]. □

Note 2.3.16. Lemma 2.3.4 means that $\mathfrak{F}_\pi$ is a lattice with respect to composition and intersection. Also, as $\mathfrak{F}_\pi$ is closed under composition any finite subextension of F_π/F is an element of $\mathfrak{F}_\pi$.

Lemma 2.3.5. *If we let π be a fixed prime element of F then there is a bijection between extensions in $\mathfrak{F}_\pi$ and Normic subgroups of $U_{1,F}$ that is created by $L/F \mapsto N_{L/F}(U_{1,L})$.*

Proof. The proof of this lemma can be found in Proposition 3.4 and Theorem 3.5 of [3]. □

Note 2.3.17. Lemmas 2.3.4 and 2.3.5, combined with Corollary 2.3.2, show that the Normic subgroups of $U_{1,F}$ form a lattice with respect to multiplication and intersection.

Summarizing the work we have done gives us the Existence Theorem as it pertains to local class field theory when we only assume that $\overline{F}$ is perfect and has positive characteristic p.

Theorem 2.3.2. *Let π be a fixed prime element of F. We have that the map $L/F \mapsto N_{L/F}(U_{1,L})$ is an order reversing bijection between the lattice consisting of elements of $\mathfrak{F}_\pi$ and the lattice of Normic subgroups of $U_{1,F}$.*

Proof. The proof of this theorem is a combination of Lemma 2.3.5 and Corollary 2.3.2 of this section. □

Corollary 2.3.3. *As a reminder, from Note 2.3.4, we have that $\tilde{F}/F$ is the maximal unramified abelian p-extension of F. Let π be a fixed prime element of F, then we have that F_π/F is totally ramified and $F_\pi\tilde{F}/F$ is the maximal abelian p-extension of F, which we denote by F^{abp}.*

This means that $\text{Gal}\,(F^{\text{abp}}/F) \cong \text{Gal}\,(\tilde{F}/F) \times \text{Gal}\,(F_\pi/F)$.

Proof. The proof of this corollary can be found in Corollary 3.4 of [3]. □

Note 2.3.18. Let T/F be the maximal abelian extension of F. As $\text{Gal}\,(T/F)$ is an abelian group we know that $\text{Gal}\,(T/F) \cong \text{Gal}\,(A/F) \times \text{Gal}\,(F^{\text{abp}}/F)$, where A/F is the composition of all finite abelian extensions of F whose degrees are coprime to p.

From Corollary 2.3.3 above we have that:

$$\text{Gal}\,(F^{\text{abp}}/F) \cong \text{Gal}\,(\tilde{F}/F) \times \text{Gal}\,(F_\pi/F)$$

As such $\text{Gal}\,(T/F) \cong \text{Gal}\,(A'/F) \times \text{Gal}\,(F_\pi/F)$, where A'/F is the composition of A/F and $\tilde{F}/F$, and is thus the composition of all abelian unramified and tamely ramified extensions of F.

Note 2.3.19. Let F be a complete discrete valuation field with quasi-finite residue field. We know that $\overline{F}$ is perfect so the results from this section will apply to F.

Let N be a Normic subgroup of F^*, by Definition 2.2.15, and set $N' = N \cap U_{1,F}$. Let $Q = \langle \pi_F \rangle \cdot B \cdot N'$, where π_F is a prime element of F and B is a group of multiplicative representatives of $\overline{F}$. We have that N fits the requirements of being a Normic subgroup and $N \subseteq Q$ so, trivially, Q is a Normic subgroup of F^*.

By Theorem 2.2.4 there is a finite abelian extension L/F such that $N_{L/F}(L^*) = Q$. Since Q is of the form $\langle \pi_F \rangle \cdot B \cdot N'$ we have that L/F is a finite abelian totally ramified p-extension, by section 1 Chapter **III** of [5], and thus $N_{L/F}(U_{1,L})$ is a Normic subgroup of $U_{1,F}$, by Definition 2.3.7. We know that:

$$N' = Q \cap U_{1,F} = N_{L/F}(L^*) \cap U_{1,F} = N_{L/F}(U_{1,L})$$

and that $N' = N \cap U_{1,F}$, with N being a generic Normic subgroup of F^*. This gives us that for any Normic subgroup N of F^* we have that $N \cap U_{1,F}$ is a Normic subgroup of $U_{1,F}$.

Let N' be a Normic subgroup of $U_{1,F}$, by Definition 2.3.7. We have that $F^* = \langle \pi_F \rangle \cdot B \cdot U_{1,F}$ and set $N = \langle \pi_F \rangle \cdot B \cdot N'$, thus $N \cap U_{1,F} = N'$. We know that N' is an open subgroup of $U_{1,F}$ and thus N is an open subgroup of F^*. Also as N' fits the second property of Normic subgroups of $U_{1,F}$ we trivially get that N is a Normic subgroup of F^*, by Definition 2.2.15. Thus we get that every Normic subgroup of $U_{1,F}$ is the intersection of a Normic subgroup of F^* and $U_{1,F}$.

The above results show that the Normic subgroups described in Definition 2.3.7 are indeed a generalisation of the Normic subgroups described in Definition 2.2.15. This proves the result that Note 2.3.14 promised would be dealt with.

Note 2.3.20. It is relatively easy to show that the Existence Theorem we saw in Theorem 2.3.2 is a generalisation, of a sort, of the Existence Theorem of Theorem 2.2.4 which with deals quasi-finite residue fields.

Let $\overline{F}$ be quasi-finite, as such $\overline{F}$ is perfect and thus all results of this section apply to F, and fix π as a prime element of F. We will be looking at $L/F \in \mathfrak{F}_\pi$ and let B be the group of multiplicative representatives of $\overline{F}$. By definition we have $\langle \pi \rangle \subseteq N_{L/F}(L^*)$. Likewise, from the fact that $\mathcal{F} = \cap_{n \geq 1} (F^*)^{p^n}$, from Proposition (7.1) Chapter **I** of [5], and $N_{L/F}(F^*) = (F^*)^{p^t}$, where p^t is the degree of L/F, we have that $B \subseteq N_{L/F}(L^*)$. So, from the above and Theorem 2.2.4 we have that $N_{L/F}(L^*)$ is equal to a unique Normic subgroup of F^* of the form $\langle \pi \rangle \cdot B \cdot N$, where N is a Normic subgroup, by Note 2.3.19, of $U_{1,F}$.

Keeping π and B from above we can see that any finite abelian extension, L/F, such that $\langle\pi\rangle \cdot B \subseteq N_{L/F}(L^*)$ is an element of $\mathfrak{F}_\pi$, the description shows that L/F is a totally ramified p-extension and that $\pi \in N_{L/F}(L^*)$. Let N be a Normic subgroup of $U_{1,F}$; then we can extend N' to be a subgroup of F^* by setting $N' = \langle\pi\rangle \cdot B \cdot N$, a group that is a Normic subgroup of F^* by Note 2.3.19. By Theorem 2.2.4 there is a unique abelian extension L/F such that $N_{L/F}(L^*) = N'$. We know that $L/F \in \mathfrak{F}_\pi$ and thus, by Theorem 2.3.2, we have that L/F is the unique element of $\mathfrak{F}_\pi$ such that $N_{L/F}(U_{1,L}) = N_{L/F}(L^*) \cap U_{1,F} = N$.

By combining the above two paragraphs together we can see that the Existence Theorem of Theorem 2.3.2 is a generalisation of the Existence Theorem of Theorem 2.2.4. However, there are the caveats that we have to restrict ourselves to elements of $\mathfrak{F}_\pi$, for a fixed prime element of F, rather than deal with all finite abelian extensions of F and that we nnow look at $N_{L/F}(U_{1,L})$ rather than $N_{L/F}(L^*)$.

Though there has been a lot more discussion on this subject, in particular in Professor Fesenko's paper, it is not relevant for the work we do here and thus we will not go over it at this juncture.

2.4 Review of Local Class Field Theory with Imperfect Residue Fields

The first main parts of independent research of the thesis, and the whole of Chapter 3, is on trying to answer open problems in local class field theory with arbitrary residue fields of positive characteristic, so the residue fields can be imperfect.

We build on the work done by Professor Ivan Fesenko in the paper "On General Local Reciprocity Maps" [4], in particular the first part. Please note that the original paper, published in 1996 in "Journal für die reine und angewandte Mathematik", has a few errors in the first section so we will be working from a corrected version that can be found online.

Throughout this section we will be assuming that F is a complete discrete valuation field with the only restriction on its residue field, $\overline{F}$, is that it has positive characteristic p. This means that $\overline{F}$ may be imperfect.

Notation 2.4.1. We again will let $\widehat{F}/F$ be the maximal unramified p-extension of F. Likewise, for finite abelian totally ramified p-extensions we will again denote the group $\mathrm{Hom}_{\mathbb{Z}_p}(\mathrm{Gal}\,(\widehat{F}/F), \mathrm{Gal}\,(L/F))$ by $\mathrm{Gal}\,(L/F)^{\wedge}$.

If L/F is a finite abelian totally ramified p-extension, we have that $\widehat{L} = L\widehat{F}$, giving us that $\widehat{F} \subseteq \widehat{L}$.

Unlike when we require that $\overline{F}$ be perfect we unfortunately do not always have that $N_{\widehat{L}/\widehat{F}}(U_{1,\widehat{L}}) = U_{1,\widehat{F}}$. As such we require further notation.

Definition 2.4.1. Let L/F be a finite abelian totally ramified field extension, then define $U(L/F)$ as $U_{1,F} \cap N_{\widehat{L}/\widehat{F}}(U_{1,\widehat{L}})$.

Note 2.4.1. If $\overline{F}$ is perfect we get that $U(L/F) = U_{1,F} \cap N_{\widehat{L}/\widehat{F}}(U_{1,\widehat{L}}) = U_{1,F}$. This tracks with how we deal with the case where $\overline{F}$ is perfect.

Notation 2.4.2. Let L/F be a finite abelian totally ramified p-extension, we shall denote $N_{L/F}(U_{1,L})$ as $N(L/F)$.

We can now define the Neukirch homomorphism: $\Upsilon_{L/F} : \mathrm{Gal}\,(L/F)^{\wedge} \to U(L/F)/N(L/F)$.

Definition 2.4.2. Let L/F be a finite abelian totally ramified p-extension and set χ as a an element of $\mathrm{Gal}\,(L/F)^{\wedge}$. Denote by $\Sigma_\chi \subseteq \widehat{L}$ the fixed field of the elements $\tau_\varphi \in \mathrm{Gal}\,(\widehat{L}/F)$, where $\tau_\varphi\,|_L = \chi(\varphi)$ and $\tau_\varphi\,|_{\widehat{F}} = \varphi$ and φ runs through every element of $\mathrm{Gal}\,(\widehat{F}/F)$. We have that L/Σ_χ is unramified while Σ_χ/F is a totally ramified p-extension. Let π_χ be a prime element of Σ_χ and π_L be a prime element of L. Set:

$$\Upsilon_{L/F}(\chi) = N_{\Sigma_\chi/F}(\pi_\chi)N_{L/F}(\pi_L^{-1}) \mod N(L/F)$$

Note 2.4.2. Like in the previous section we do not need φ to vary through every element of $\mathrm{Gal}\,(\widehat{F}/F)$ but instead just through a topological basis of $\mathrm{Gal}\,(\widehat{F}/F)$. By the definition of a basis it will yield the same fixed field Σ_χ.

Lemma 2.4.1. *If L/F is a finite abelian totally ramified p-extension, then the Neukirch map $\Upsilon_{L/F}$ is a well defined homomorphism.*

Proof. The proof of this lemma can be found in Lemma 1.2 of [4]. □

Note 2.4.3. If we have that $\overline{F}$ is a perfect field and that L/F is a finite abelian totally ramified p-extension then we know that $U(L/F) = U_{1,F}$ and thus $\Upsilon_{L/F}$ becomes a map between $\mathrm{Gal}\,(L/F)^\wedge$ and $U_{1,F}/N(L/F)$. It is also clear that $\Upsilon_{L/F}$ is the same map that we defined in the perfect residue field case meaning that the Neukirch map that we have just defined is a generalisation of what we have already seen.

Definition 2.4.3. Let $\mathfrak{F}/\widehat{F}$ be a field extension such that $\mathfrak{F}$ is a complete discrete valuation field with the following properties:

1) The ramification index of $\mathfrak{F}/\widehat{F}$, so $e(\mathfrak{F} \mid \widehat{F})$, is equal to 1.

2) We have that $\overline{\mathfrak{F}} = \bigcup_{n\geq 0} \overline{\widehat{F}}^{p^{-n}}$. This means that $\overline{\mathfrak{F}}$ is the perfection of $\overline{\widehat{F}}$.

If L/F is a finite abelian totally ramified p-extension define $\mathfrak{L}$ as being $L\mathfrak{F}$. This gives $\mathfrak{L}$ the same properties as $\mathfrak{F}$, but substituting L for F, while also making sure that $\mathfrak{F} \subseteq \mathfrak{L}$.

Note 2.4.4. If $\overline{F}$ is perfect then $\mathfrak{F}$ is unique and is the completion of $\widehat{F}$; the uniqueness of $\mathfrak{F}$ cannot be assumed if we do not require $\overline{F}$ to be perfect.

Definition 2.4.4. Let L/F be a finite abelian totally ramified p-extension. We will define $I(L \mid F)$ as a subgroup of $U_{1,\widehat{L}}$ equal to the intersection of $U_{1,\widehat{L}}$ and the subgroup of $U_{1,\mathfrak{L}}$ generated by the elements $\varepsilon^{-1}\sigma(\varepsilon)$, where $\varepsilon \in U_{1,\mathfrak{L}}$ and $\sigma \in \mathrm{Gal}\,(L/F)$.

Definition 2.4.5. We can now define the map, $\Psi_{L/F}$, which is the left-hand inverse of $\Upsilon_{L/F}$, for a finite abelian totally ramified p-extension L/F.

Define the map $c : \mathrm{Gal}\,(L/F) \to U_{1,\widehat{L}}/I(L \mid F)$ as follows:

$$c(\sigma) = \pi_L^{-1}\sigma(\pi_L) \mod I(L \mid F)$$

where $\sigma \in \mathrm{Gal}\,(L/F)$ and π_L is any prime element of L.

We now have the exact sequence:

$$1 \longrightarrow \mathrm{Gal}\,(L/F) \xrightarrow{c} U_{1,\widehat{L}}/I(L \mid F) \xrightarrow{N_{\widehat{L}/\widehat{F}}} N_{\widehat{L}/\widehat{F}}(U_{1,\widehat{L}}) \longrightarrow 1$$

For reasons why this sequence is exact please check section 1.4 of [4].

Next we can define $\Psi_{L/F}$:

Let $\alpha \in U(L/F)$ and $\varphi \in \mathrm{Gal}\,(\widehat{F}/F)$, with $\widehat{\varphi} \in \mathrm{Gal}\,(\widehat{L}/F)$ being a fixed extension of φ. Choose an element $\beta \in U_{1,\widehat{L}}$ such that $N_{\widehat{L}/\widehat{F}}(\beta) = \alpha$. We know that $N_{\widehat{L}/\widehat{F}}(\beta^{-1}\widehat{\varphi}(\beta)) = 1$, so from the above exact sequence we get $\beta^{-1}\widehat{\varphi}(\beta) \equiv$

$c(\sigma^{-1}) \mod I(L \mid F)$, for an element $\sigma \in \mathrm{Gal}\,(L/F)$. Define $\Psi_{L/F}(\alpha)$ as the map in $\mathrm{Gal}\,(L/F)^{\wedge}$ that sends φ to σ as φ varies through elements of $\mathrm{Gal}\,(\widehat{F}/F)$.

Lemma 2.4.2. *Let L/F be a finite abelian totally ramified p-extension. Then $\Psi_{L/F}$ is a well defined homomorphism and $\Psi_{L/F} \circ \Upsilon_{L/F}$ is the identity map on $\mathrm{Gal}\,(L/F)^{\wedge}$, thus making $\Upsilon_{L/F}$ injective and $\Psi_{L/F}$ surjective.*

Proof. The proof of this lemma can be found in Lemma 1.4. and Proposition 1.5 of [4]. □

Definition 2.4.6. Let $\mathcal{F}/F$ be a field extension such that $\mathcal{F}$ is a complete discrete valuation field with the following properties:

1) The ramification index of $\mathcal{F}/F$, so $e(\mathcal{F} \mid F)$, is equal to 1.

2) $\overline{\mathcal{F}} = \bigcup_{n \geq 0} \overline{F}^{p^{-n}}$. This means that $\overline{\mathcal{F}}$ is the perfection of $\overline{F}$.

If L/F is a finite abelian totally ramified p-extension. Fix a field to be $\mathcal{F}$ and define $\mathcal{L}$ as being $L\mathcal{F}$.

This definition of $\mathcal{L}$ fits the requirements we have shown for $\mathcal{F}$ while also having the property that $\mathcal{F} \subseteq \mathcal{L}$.

Note 2.4.5. Like with $\mathfrak{F}$, there are possibly multiple fields that fit the requirements of $\mathcal{F}$. Likewise, if $\overline{F}$ is perfect then $\mathcal{F}$ is unique but in this case is equal to F.

Lemma 2.4.3. *Let L/F be a finite abelian totally ramified p-extension. Denote the map $U(L/F)/N(L/F) \to U_{1,\mathcal{F}}/N_{\mathcal{L}/\mathcal{F}}(U_{1,\mathcal{L}})$ induced by inclusion with $\lambda_{L/F}$. Then the following two diagrams are commutative:*

$$\begin{array}{ccc} \mathrm{Gal}\,(L/F)^{\wedge} & \xrightarrow{\Upsilon_{L/F}} & U(L/F)/N(L/F) \\ \downarrow & & \downarrow \lambda_{L/F} \\ \mathrm{Gal}\,(\mathcal{L}/\mathcal{F})^{\wedge} & \xrightarrow{\Upsilon_{\mathcal{L}/\mathcal{F}}} & U_{1,\mathcal{F}}/N_{\mathcal{L}/\mathcal{F}}(U_{1,\mathcal{L}}) \end{array}$$

and:

$$\begin{array}{ccc} U(L/F)/N(L/F) & \xrightarrow{\Psi_{L/F}} & \mathrm{Gal}\,(L/F)^{\wedge} \\ \downarrow \lambda_{L/F} & & \downarrow \\ U_{1,\mathcal{F}}/N_{\mathcal{L}/\mathcal{F}}(U_{1,\mathcal{L}}) & \xrightarrow{\Psi_{\mathcal{L}/\mathcal{F}}} & \mathrm{Gal}\,(\mathcal{L}/\mathcal{F})^{\wedge} \end{array}$$

Proof. The proof of this lemma can be found in section 1.6 of [4]. □

We know from previous work that since $\overline{\mathcal{F}}$ is a perfect field then $\Psi_{\mathcal{L}/\mathcal{F}}$ is an injective map. This gives us that $\lambda_{L/F}$ is surjective and that the kernel of $\Psi_{L/F}$ is the kernel of $\lambda_{L/F}$, which is equal to $(U_{1,F} \cap N_{\widehat{L}/\widehat{F}}(U_{1,\widehat{L}}) \cap N_{\mathcal{L}/\mathcal{F}}(U_{1,\mathcal{L}}))/N(L/F)$.

Notation 2.4.3. Denote $U_{1,F} \cap N_{\widehat{L}/\widehat{F}}(U_{1,\widehat{L}}) \cap N_{\mathcal{L}/\mathcal{F}}(U_{1,\mathcal{L}})$ by the notation $N_\star(L/F)$.

We now have that $\Psi_{L/F}$ induces an isomorphism between $U(L/F)/N_\star(L/F)$ and $\mathrm{Gal}\,(L/F)^\wedge$.

Notation 2.4.4. In order to differentiate between when we are dealing with the map from $U(L/F)/N_*(L/F)$ and the map from $U(L/F)/N_{L/F}(U_{1,L})$, we shall denote the map from $U(L/F)/N_{L/F}(U_{1,L})$ to $\mathrm{Gal}\,(L/F)^\wedge$ as $\Psi_{L/F}$ and the induced isomorphism from $U(L/F)/N_*(L/F)$ to $\mathrm{Gal}\,(L/F)^\wedge$ as $\Psi^*_{L/F}$.

Note 2.4.6. If $\overline{F}$ is perfect we have already seen that $U(L/F) = U_{1,F}$ and it is easy to see from the definition that $N_\star(L/F) = N(L/F)$. This gives us that $\Psi_{L/F}$ is an isomorphism between $U_{1,F}/N(L/F)$ and $\mathrm{Gal}\,(L/F)^\wedge$ like we should expect.

Let L/F be a finite abelian totally ramified p-extension and set M/F as a subextension, then from the working in section 1.7 of [4] we have the following properties:

1) $\lambda^{-1}_{L/F}(N(\mathcal{M}/\mathcal{F})) = N_{M/F}(U(L/M)N_\star(L/F))$
2) $\ker(N_{\mathcal{M}/\mathcal{F}}) \subseteq N(\mathcal{L}/\mathcal{M})$
3) $U(L/F) \cap N_\star(M/F) = N_{M/F}(U(L/M)N_\star(L/F))$

We can use the above in the proof of the following proposition.

Proposition 2.4.1. *Let L/F be a finite abelian totally ramified p-extension. Then the following properties are equivalent:*

1) *Both $\Upsilon_{L/F}$ and $\Psi_{L/F}$ are isomorphisms.*
2) *$\Psi_{L/F}$ is a monomorphism.*
3) *$\Upsilon_{L/F}$ is a surjective map.*
4) *$N_\star(L/F) = N(L/F)$.*
5) *Fix $\varepsilon \in U_{1,\widehat{L}}$, then if for every $\sigma \in \mathrm{Gal}\,(\widehat{L}/L)$ we have $\varepsilon^{\sigma-1} \in I(L \mid F)$, then we also have that $\varepsilon^{\sigma-1} \in I(L \mid F)^{\sigma-1}$ for all $\sigma \in \mathrm{Gal}\,(\widehat{L}/L)$.*
6) *$U(L/F)$ is equal to the set of elements which have the form:*

$$N_{\Sigma_\chi/F}(\pi_\chi)/N_{L/F}(\pi_L)$$

where Σ_χ*,* π_χ *and* π_L *are as in the definition of* $\Upsilon_{L/F}$*, see definition 2.4.2, and* χ *runs through all of the elements of* $\mathrm{Gal}\,(L/F)^\wedge$*.*

Proof. The proof of this proposition can be found in Proposition 1.8 of [4]. □

Note 2.4.7. The above proposition means that if L/F is a finite abelian totally ramified p-extension we only need to prove one of the conditions in order to show that $\Psi_{L/F}$ is an isomorphism.

From the previous proposition we get the following theorem:

Theorem 2.4.1. *Let* L/F *be a finite cyclic totally ramified* p*-extension; then* $\Psi_{L/F}$ *is an isomorphism with* $\Upsilon_{L/F}$ *being its inverse.*

Proof. The proof of this theorem can be found in Theorem 1.9 of [4]. □

Note 2.4.8. The above theorem means that if L/F is a finite totally ramified cyclic p-extension then:

$$\mathrm{Gal}\,(L/F)^\wedge \cong U(L/F)/N(L/F) = (U_{1,F} \cap N_{\widehat{L}/\widehat{F}}(U_{1,\widehat{L}}))/N_{L/F}(U_{1,F})$$

Theorem 2.4.2. *Let* L_1/F *and* L_2/F *be finite abelian totally ramified* p*-extensions such that* L_1L_2/F *is totally ramified. Then* $L_1 \subseteq L_2$ *if and only if* $N(L_2/F) \subseteq N(L_1/F)$*.*

Proof. The proof of this theorem can be found in Theorem 1.10 of [4]. □

Note 2.4.9. The above theorem is an analogue to one of the results in Corollary 2.3.2, which applies when we assume that $\overline{F}$ is perfect.

Proposition 2.4.2. *Let* F *be a complete discrete valuation field with no restrictions on* $\overline{F}$ *other than that* $\mathrm{char}\,(\overline{F}) = p > 0$*. Next let* L_1/F *and* L_2/F *be a pair of finite abelian totally ramified* p*-extensions such that there is a prime element,* π*, of* F*, with* $\pi \in N_{L_1/F}(L_1^*) \cap N_{L_2/F}(L_2^*)$*. Then we have that* L_1L_2/F *is also a totally ramified extension.*

Proof. The proof of this proposition can be found in Proposition 2.1 of [4]. □

Note 2.4.10. Unlike the previous two sections, we will not be discussing the Existence Theorem in this case. It is not necessary for the work in this document so there is no point in going over it here.

We will note that like with $\Upsilon_{L/F}$ and $\Psi_{L/F}$, where we can only prove they are an isomorphism in limited cases, the current Existence Theorem is a lot less complete than in cases with more restrictions on the residue field of F. For instance [4] only describes the one-to-one correspondence between certain subgroups of $U_{1,F}$ and finite cyclic totally ramified p-extensions of F; rather than all finite abelian totally ramified p-extensions of F.

Like with the previous parts of this chapter there is a lot more in Professor Fesenko's paper about local class field theory when dealing with imperfect residue fields, however it is not relevant to this work so we will not be discussing it here.

2.5 Review of the Norm Map for Ordinary Abelian Varieties

In this section we shall have a brief look at the paper "The Norm Map for Ordinary Abelian Varieties" by Jonathan Lubin and Michael Rosen [7]. This paper offers an alternate proof of a result that is in Barry Mazur's paper "Rational Points of Abelian Varieties with Values in Towers of Number Fields" [8]; we shall not be looking at that paper however as the result and method we are interested in is all contained in [7].

Notation 2.5.1. Let p be a prime number and let $F/\mathbb{Q}_p$ be a finite field extension. This means that F is a complete discrete valuation field with finite residue field. We will denote the cardinality of $\overline{F}$ by $q \in \mathbb{Z}$.

We will be assuming, for the rest of this section, that d is a positive integer.

Let A be a d-dimensional abelian variety over F with good ordinary reduction.

Finally, fix L as a totally ramified $\mathbb{Z}_p$-extension of F.

Notation 2.5.2. Let E/F be either an algebraic extension of F or the completion of such an extension. We shall denote the ring of integers of E by $\mathcal{O}_E$ and the maximal ideal of E by $\mathcal{M}_E$.

Notation 2.5.3. We will let $\widehat{F}$ denote the completion of the maximal unramified extension of F. Since F has a finite residue field there exists an unique F-Frobenius automorphism of $\widehat{F}$ which we shall denote by ϕ.

Note 2.5.1. We said in Notation 2.5.1 that q is the cardinality of $\overline{F}$. This means that for all $a \in \mathcal{O}_{\widehat{F}}$ we have that $\phi(a) \equiv a^q \mod \mathcal{M}_{\widehat{F})}$.

Definition 2.5.1. Let n be a positive integer, denote by L_n the fields such that $F \subseteq L_n \subseteq L$ and $[L_n : K] = p^n$. Denote by $N_{L/F}(A(L))$ the intersection:

$$\bigcap_{n \geq 1} N_{L_n/F}(A(L_n))$$

Definition 2.5.2. Let K be a d-dimensional formal group over $\mathcal{O}_F$. We call K toroidal if we have $K \cong \widehat{\mathbb{G}}_m^d$, over $\mathcal{O}_{\widehat{F}}$.

Definition 2.5.3. Let H be a d-dimensional toroidal formal group over $\mathcal{O}_F$ and let $k : H \to \widehat{\mathbb{G}}_m^d$ be an isomorphism over $\mathcal{O}_{\widehat{F}}$.

We can represent k as d power series and we can form another d power series, denoted by k^ϕ, by applying the Frobenius automorphism ϕ to the coefficients of k.

$k^\phi \circ k^{-1}$ is an automorphism on $\widehat{\mathbb{G}}_m^d$. This automorphism corresponds to a non-singular $d \times d$ matrix over $\mathbb{Z}_p$, which we shall denote by u. A matrix of the form u is called a twist matrix of H.

Note 2.5.2. If H is a d-dimensional toroidal formal group over $\mathcal{O}_F$ then any two twist matrices of H will by similar, so it does not matter which twist matrix we use in our later work.

Definition 2.5.4. Let H be a a d-dimensional toroidal formal group over $\mathcal{O}_F$ and set u as a twist matrix of H. Next, let L/F be a totally ramified Galois extension and extend the F-Frobenius automorphism, ϕ to $L\widehat{F}$ by having it fix elements of L. We define the following set:

$$V(L) = V_u(L) = \{\alpha \in U_{L\widehat{F}}^d : \alpha^\phi = \alpha^u\}$$

Lemma 2.5.1. *Keeping the notation of Definition 2.5.4; we have that $V_u(L)$ is a* $\mathrm{Gal}\,(L/F)$*-module and is isomorphic to $H(\mathcal{O}_L)$, as* $\mathrm{Gal}\,(L/F)$*-modules.*

Proof. The proof of this lemma can be found in, and slightly before, the first lemma of section 1 of [7]. □

Lemma 2.5.2. *Let A be a d-dimensional abelian variety over F with good ordinary reduction. Denote by $\widehat{A}$ the formal group over $\mathcal{O}_F$ that corresponds to A. Then $\widehat{A}$ is a toroidal formal group over $\mathcal{O}_{\widehat{F}}$.*

Proof. The proof of this lemma can be found in Lemma 4.27 in [8]. □

Lemma 2.5.3. *Let A be a d-dimensional abelian variety over F with good ordinary reduction. Denote by $\widehat{A}$ the formal group over $\mathcal{O}_F$ that relates to A. From Lemma 2.5.2 we know that $\widehat{A}$ is toroidal and let u be a twist matrix of $\widehat{A}$. Finally, fix L_n/F as a finite totally ramified Galois p-extension and denote by G_n the Galois group of L_n/F, with $V_u(F)$ and $V_u(L_n)$ being as described in Definition 2.5.4.*

We then have the following isomorphism:

$$V_u(F)/N_{L_n/F}(V_u(L_n) \cong (G_n^{ab})^d/((I-u)(G_n^{ab})^d)$$

Proof. The proof of this lemma can be found in Theorem 1 of [7]. □

From the above lemma we now come to the main theorem of the paper, which the paper calls "Mazur's Proposition 4.39" [7]:

Theorem 2.5.1. *Keep the notation that we have in the above lemma and now let L/F be a totally ramified $\mathbb{Z}_p$-extension, we have that the following is an exact sequence:*

$$\mathbb{Z}_p^d/((I-u)\mathbb{Z}_p^d) \to A(F)/N_{L/F}(A(L)) \to A(\overline{F})_p \to 1$$

Here the map $\mathbb{Z}_p^d/((I-u)\mathbb{Z}_p^d) \to A(F)/N_{L/F}(A(L))$ is constructed using the isomorphism in Lemma 2.5.3 and the natural map:

$$\widehat{A}(\mathcal{O}_F)/N_{L/F}(\widehat{A}(\mathcal{O}_L)) \to A(F)/N_{L/F}(A(L))$$

Proof. The proof of this theorem is gone over in the first two sections of [7]. □

Though there is more in the paper the above is all that we will need for this document. This makes the rest of the paper irrelevant for us at the time being.

Chapter 3

Local Fields with Imperfect Residue Fields

This chapter is about furthering our understanding of local class field theory. In particular, it advances the knowledge imparted by Professor Ivan Fesenko's "On General Local Reciprocity Maps" [4], which is on the subject of local class field theory when the residue field is not necessarily perfect. We shall be using the aforementioned paper as well as Ivan Fesenko's paper "Local Class Field Theory: Perfect Residue Field Case" [3] and the book he co-authored, "Local Fields and their Extensions" [5], to explore several different avenues on the topic.

[4], [5] and [3] have already been discussed in the "Literature Review". [4] deals with local class field theory with imperfect residue fields, which will of course be useful for this topic. Meanwhile, [3] deals with local class field theory when the residue field is required to be perfect, an area of mathematics with a lot more concrete results than when we let the residue field be imperfect. There are a few results in that paper that are useful in the study of the following mathematics.

In [5], Chapter **III** is our main resource as it deals with the nature of norm maps and the Hasse-Herbrand function, $h_{L/F}$.

Over the course of the chapter, unless explicitly stated otherwise, we will be letting F be a complete discrete valuation field with no restrictions on $\overline{F}$ other than it having positive characteristic p, and will be looking at field extensions, L/F, which are finite abelian totally ramified p-extensions.

We shall be briefly looking at the structure of the norm group $N_{L/F}$, as well as how the N_* groups, defined in section 1.7 of [4], and in section 3 of the "Literature Review", intersect and compose with one another.

The main topic of consideration however is the map $\Psi_{L/F}$ and when it is an isomorphism. We shall prove that $\Psi_{L/F}$ is an isomorphism if $\mathrm{Gal}\,(L/F) \cong (\mathbb{Z}/p\mathbb{Z})^2$, or if L/F has a single ramification jump.

We shall also look at what happens if L/F has two ramification jumps. Unfortunately, the problem of establishing the isomorphism property of $\Psi_{L/F}$ remains open. We will, however, explain what we can prove and show where the problem presents itself. This is then followed by a failed idea of trying to resolve the problem and an idea of where to go in the future.

The final sections of this chapter deals with extensions created by adjoining the p'th-root of elements to F. We deal with both the basic nature of such extensions and look at if we can relate them back to the mathematics we have already done on $\Psi_{L/F}$.

3.1 Notation

Throughout this chapter we will be using the same notation and definitions that Professor Fesenko uses in the first section of [4]. This has already been gone over in section 3 of the "Literature Review", but as a refresher we will go back over them here.

Notation/Definition 3.1.1. Let F and L/F be as described in the previous section. We let $\widehat{F}$ be the unramified extension of F corresponding to the maximal separable
p-extension of $\overline{F}$ and define $\widehat{L}$ to be equal to $L\widehat{F}$.

We next define $\mathcal{F}$ to be a complete discrete valuation field which is an extension of F with the following two properties. Firstly, we have that
$e(\mathcal{F} \mid F) = 1$ and secondly $\overline{\mathcal{F}} = \bigcup_{n\geq 1} \overline{F}^{p^{-n}}$, this means that $\overline{\mathcal{F}}$ is the perfection of $\overline{F}$. We again let $\mathcal{L}$ equal $L\mathcal{F}$.

We finally define $N_*(L/F)$ to be equal to $U_{1,F} \cap N_{\widehat{L}/\widehat{F}}(U_{1,\widehat{L}}) \cap N_{\mathcal{L}/\mathcal{F}}(U_{1,\mathcal{L}})$ and write $N_{L/F}(U_{1,L})$ as $N(L/F)$ and let $\Psi^*_{L/F} : U(L/F)/N_*(L/F) \to \mathrm{Gal}\,(L/F)^\wedge$ be the isomorphism induced by $\Psi_{L/F} : U(L/F)/N(L/F) \to \mathrm{Gal}\,(L/F)^\wedge$.

Proposition 2.4.1, and by extension Proposition 1.8 of [4], also tells us that $\Psi_{L/F}$ is an isomorphism if and only if $N_*(L/F) = N(L/F)$.

3.2 An Important Exact Sequence

The first thing we shall do is show that an important exact sequence in local class field theory holds when the residue field of a complete discrete valuation field F is assumed to be perfect, and then we shall show that the sequence still holds when the residue field is imperfect.

Proposition 3.2.1. *Let F be an arbitrary complete discrete valuation field. There is an order and degree preserving bijection between finite separable unramified extensions of F and finite separable extensions of $\overline{F}$.*

If L/F is sent to $T/\overline{F}$ by the bijection then L/F is Galois if and only if $T/\overline{F}$ is Galois, in that case we have $\mathrm{Gal}(L/F) \cong \mathrm{Gal}(T/\overline{F})$.

Proof. As a reminder a finite extension of L/F is unramified if $\overline{L}/\overline{F}$ is a separable extension with the same degree as L/F, taken from section (3.1) Chapter **II** in [5].

If L/F is a finite unramified Galois extension then, by Proposition (3.3) Chapter **II** in [5], we have that $\overline{L}/\overline{F}$ is Galois and that $\mathrm{Gal}(L/F) \cong \mathrm{Gal}(\overline{L}/\overline{F})$. This isomorphism takes $\sigma \in \mathrm{Gal}(L/F)$ to $\overline{\sigma} \in \mathrm{Gal}(\overline{L}/\overline{F})$, where for α in L we define $\overline{\sigma}(\overline{\alpha})$ as $\overline{\sigma(\alpha)}$.

Let L_1/F, L_2/F and L/F be finite separable unramified extensions of F with $L_1 \subseteq L_2 \subseteq L$ and L/F being Galois. We have that $\mathrm{Gal}(L/F) \cong \mathrm{Gal}(\overline{L}/\overline{F})$, and that $\mathrm{Gal}(L/L_1) \cong \mathrm{Gal}(\overline{L}/\overline{L_1})$ and $\mathrm{Gal}(L/L_2) \cong \mathrm{Gal}(\overline{L}/\overline{L_2})$. From Proposition (3.2) Chapter **II** in [5] we may choose α_1 and α_2 be such that $L_1 = F(\alpha_1)$ and $L_2 = F(\alpha_2)$, with both α_1 and α_2 are contained in $\mathcal{O}_F$ with $\overline{L_1} = \overline{F}(\overline{\alpha_1})$ and $\overline{L_2} = \overline{F}(\overline{\alpha_2})$.

Now $L_1 \subseteq L_2$, which means that $\alpha_1 \in L_2$. This gives us that $\overline{\alpha_1} \in \overline{L_2}$ and thus $\overline{L_1} \subseteq \overline{L_2}$. Since we have that $\mathrm{Gal}(L/L_1) \cong \mathrm{Gal}(\overline{L}/\overline{L_1})$ and $\mathrm{Gal}(L/L_2) \cong \mathrm{Gal}(\overline{L}/\overline{L_2})$ we have that the degree of L_2/L_1 is equal to the degree of $\overline{L_2}/\overline{L_1}$.

Next suppose $L_1 \not\subseteq L_2$ but both are still contained in L with L/F being a finite unramified Galois extension. We thus have that L_1L_2/L_2 has an degree greater than 1. By Corollary (3.2) Chapter **II** in [5] we have that $\overline{L_1L_1}/\overline{F} = \overline{L_1}\,\overline{L_2}/\overline{F}$ is unramified. Since L contained both L_1 and L_2 we get that it contains L_1L_2 and by the previous paragraph we have that $\overline{L_2} \subseteq \overline{L_1}\,\overline{L_2}$ and that the degree of $\overline{L_1}\,\overline{L_2}/\overline{L_2}$ is greater than 1. From this we get that $\overline{L_1} \not\subseteq \overline{L_2}$ and thus the natural map from finite unramified extensions of F to finite seperable extensions of $\overline{F}$ is both order and degree preserving and injective.

Conversely, let $T/\overline{F}$ be a finite separable extension with $T = \overline{F}(\theta)$ and $g(X) \in \overline{F}[X]$ being the monic seperable irreducible polynomial which has θ as a root. Next, let $f(X) \in \mathcal{O}_F[X]$ be a monic polynomial such that $\overline{f}(X) = g(X)$. Since $g(X)$ is both monic and separable then, by Proposition (3.2) Chapter **II** in [5], we have that $F(\beta)$, where β is any root of $f(X)$, is an unramified extension of F.

Let M/F be the splitting field of $f(X)$ over F. Then, by Corollary (3.2) Chapter **II** in [5], we get that M/F is unramified and Galois. We also have, by Proposition (3.3) Chapter **II** in [5], that $\overline{M}/\overline{F}$ is Galois with $\operatorname{Gal}(M/F) \cong \operatorname{Gal}(\overline{M}/\overline{F})$. Letting β_i be the root of $f(X)$ such that $\overline{\beta_i} = \theta$ we have that $F(\beta_i)/F$ is an unramified extension with the degree of $F(\beta_i)/F$ being equal to the degree of $T/\overline{F}$. Since $\theta \in \overline{F(\beta_i)}$ we have that $T \subseteq \overline{F(\beta_i)}$ and thus $T = \overline{F(\beta_i)}$. This gives us that the natural map from finite unramified extensions of F to finite separable extensions of $\overline{F}$ is surjective.

Summing up what we have worked out, we have that the map $L/F \mapsto \overline{L}/\overline{F}$ is an order and degree preserving bijection from finite unramified extensions of F and separable extensions of $\overline{F}$.

Finally, if the $T/\overline{F}$ is Galois then, by Proposition (3.3) Chapter **II** of [5], we see, as $\overline{F(\beta_i)} = T$, that $F(\beta_i)/F$ is also a Galois extension. This combined with what we did earlier in the proof shows that the bijection induces a bijection between finite Galois extensions of F and finite Galois extensions of $\overline{F}$ and that the bijection preserves Galois groups. □

Corollary 3.2.1. *Let F be an arbitrary complete discrete valuation field with residue field of characteristic p. Remembering that $\widehat{F}/F$ is the maximal unramified p-extension of F, we have that it is Galois and $\overline{\widehat{F}}/\overline{F}$ is the maximal p-extension of $\overline{F}$ with* $\operatorname{Gal}(\widehat{F}/F) \cong \operatorname{Gal}(\overline{\widehat{F}}/\overline{F})$.

Proof. Let α be an arbitrary element of $\widehat{F}$. This means that $\overline{\alpha}$ is contained in a separable p-extension of $\overline{F}$. Let $f(X) \in \overline{F}[X]$ be the minimal polynomial of $\overline{\alpha}$. As $\overline{\alpha} \in \overline{\widehat{F}}$ we have that the degree of $f(X)$ must be of the form p^n for some non-negative integer n.

Let $\overline{\sigma}$ be a field automorphism of $\overline{F}^{\text{sep}}/\overline{F}$. We have that $\overline{\sigma}(\overline{\alpha})$ must also be a root of the irreducible polynomial $f(X)$. This means that $\overline{F}(\overline{\sigma}(\overline{\alpha}))/\overline{F}$ must have degree p^n. If L/F is an unramified extension such that $\overline{L} = \overline{F}(\overline{\sigma}(\overline{\alpha}))$ then the degree of L/F is also equal to p^n and thus, by definition, $L \subseteq \widehat{F}$. This gives us that $\overline{\sigma}(\overline{\alpha})$ is contained in $\overline{\widehat{F}}$, so the field extension $\overline{\widehat{F}}/\overline{F}$ contains the Galois closure of $\overline{F}(\overline{\alpha})/\overline{F}$.

By Proposition (3.2) Chapter **II** in [5] we get that $\widehat{F}$ contains the Galois closure of $F(\alpha)/F$. Since α was an arbitrary element of $\widehat{F}$ we see that $\widehat{F}$ contains the Galois closure of any finite subextension of $\widehat{F}/F$ and thus, as $\widehat{F}/F$ is the limit of all of its finite subextensions, $\widehat{F}/F$ is a Galois extension.

Likewise, if we let γ be an element of the separable p-closure of $\overline{F}$ then using similar mathematics to the above, and Proposition (3.2) again, we get that there exists a $\theta \in \widehat{F}$ such that $\overline{\theta} = \gamma$. This means that $\overline{\widehat{F}}$ contains the separable p-closure of $\overline{F}$. Next, if σ is not in the separable p-closure of $\overline{F}$ then $\overline{F}(\gamma)/\overline{F}$ has degree equal to mp^t, where m is an integer greater than 1 and coprime to p. Therefore, if L/F is a separable extension that contains an element θ such that $\overline{\theta} = \gamma$ we have that $F(\gamma)/F$ is also a separable extension and it has degree mp^t; which means $L \not\subseteq \widehat{F}$. So $\overline{\widehat{F}}$ is contained in the separable p-closure of $\overline{F}$ and thus is the separable p-closure of $\overline{F}$.

Now, the Galois group of $\widehat{F}/F$ is equal to the limit of the Galois groups of all of its finite subextensions and the same holds for the Galois group of $\overline{\widehat{F}}/\overline{F}$. From Proposition 3.2.1 we see that the map $L/F \mapsto \overline{L}/\overline{F}$ induces a bijection between finite Galois unramified extensions of F and finite Galois extensions of $\overline{F}$, and this bijection preserves Galois groups and order. From the bijection we see that, by taking the limits, $\mathrm{Gal}\,(\widehat{F}/F) \cong \mathrm{Gal}\,(\overline{\widehat{F}}/\overline{F})$. □

Corollary 3.2.2. *Let F be an arbitrary complete discrete valuation ring whose residue field has characteristic p. Next, let L/F be a finite totally ramified extension. We have that* $\mathrm{Gal}\,(\widehat{F}/F) \cong \mathrm{Gal}\,(\widehat{L}/L)$.

Proof. As L/F is totally ramified we have that $\overline{L} = \overline{F}$ and thus by Corollary 3.2.1 we get that $\overline{\widehat{F}} = \overline{\widehat{L}}$, since both are equal to the maximal p-extension of $\overline{F}$, and:

$$\mathrm{Gal}\,(\widehat{F}/F) \cong \mathrm{Gal}\,(\overline{\widehat{F}}/\overline{F}) = \mathrm{Gal}\,(\overline{\widehat{L}}/\overline{L}) \cong \mathrm{Gal}\,(\widehat{L}/L)$$

□

Note 3.2.1. The above corollary was already assumed to be true when were writing up about the generalisation of classic local class field theory in the "Literature Review", see Notation 2.3.1 for us explicitly assuming that Corollary 3.2.2 held. However, none of the maths we have used in this section so far relies on local class field theory. Therefore, there should be no issue with us proving the corollary here; in fact Notation 2.3.1 directs readers here to get a proof of the fact that $\mathrm{Gal}\,(\widehat{F}/F) \cong \mathrm{Gal}\,(\widehat{L}/L)$, for a finite abelian totally ramified p-extension L/F.

With the groundwork out of the way we can start on the exact sequence. This will be done in two parts, first when $\overline{F}$ is perfect and then when it is imperfect.

Lemma 3.2.1. *Let F be an arbitrary complete discrete valuation field with perfect residue field of characteristic p. Let L/F be a finite abelian totally ramified p-extension with M/F being a subextension of L/F. There is the following exact sequence:*

$$1 \to \mathrm{Gal}\,(L/M)^{\wedge} \to \mathrm{Gal}\,(L/F)^{\wedge} \to \mathrm{Gal}\,(M/F)^{\wedge} \to 1$$

here the two middle maps exist from the result of Corollary 3.2.2 that identifies $\mathrm{Gal}\,(\widehat{F}/F)$ *with* $\mathrm{Gal}\,(\widehat{L}/L)$ *and* $\mathrm{Gal}\,(\widehat{M}/M)$. *The map from* $\mathrm{Gal}\,(L/M)^{\wedge}$ *to* $\mathrm{Gal}\,(L/F)^{\wedge}$ *is induced by the natural inclusion from* $\mathrm{Gal}\,(L/M)$ *to* $\mathrm{Gal}\,(L/F)$ *and the map from* $\mathrm{Gal}\,(L/F)^{\wedge}$ *to* $\mathrm{Gal}\,(M/F)^{\wedge}$ *arises from the restriction of* $\mathrm{Gal}\,(L/F)$ *to* $\mathrm{Gal}\,(M/F)$.

Proof. It is clear that the map from $\mathrm{Gal}\,(L/M)^{\wedge}$ to $\mathrm{Gal}\,(L/F)^{\wedge}$ is injective and the kernel of the map to $\mathrm{Gal}\,(M/F)^{\wedge}$ is the image in $\mathrm{Gal}\,(L/F)^{\wedge}$ of $\mathrm{Gal}\,(L/M)^{\wedge}$. The only thing left to do is to show that the map to $\mathrm{Gal}\,(M/F)^{\wedge}$ is surjective.

As we have that $\overline{F}$ is a perfect field we can use the mathematics of section 3 of the "Literature Review".

We shall use the first diagram of Lemma 2.3.3 to get the following:

$$\begin{array}{ccc} U_{1,F}/N_{L/F}(U_{1,L}) & \xrightarrow{\Psi_{L/F}} & \mathrm{Gal}\,(L/F)^{\wedge} \\ \downarrow & & \downarrow \\ U_{1,F}/N_{M/F}(U_{1,M}) & \xrightarrow{\Psi_{M/F}} & \mathrm{Gal}\,(M/F)^{\wedge} \end{array}$$

with the left downward arrow being induced by the identity on $U_{1,F}$ and the right downward arrow being induced by the restriction map from $\mathrm{Gal}\,(L/F)$ to $\mathrm{Gal}\,(M/F)$. We also have that both $\Psi_{L/F}$ and $\Psi_{M/F}$ are isomorphisms.

We can use Corollary 2.3.2 to get that, as $M \subseteq L$, we have $N_{L/F}(U_{1,L}) \subseteq N_{M/F}(U_{1,M})$. This combined with the mathematics from earlier in the proof shows that the map from $\mathrm{Gal}\,(L/F)^{\wedge}$ to $\mathrm{Gal}\,(M/F)^{\wedge}$ is surjective, like we wanted.

□

Theorem 3.2.1. *We shall keep the assumptions of Lemma 3.2.1 other than we will now assume that $\overline{F}$ is imperfect. The exact sequence described, with the same maps, in Lemma 3.2.1 still holds.*

Proof. Like with Lemma 3.2.1 the only thing that is not clear is that the map $\mathrm{Gal}(L/F)^\wedge \to \mathrm{Gal}(M/F)^\wedge$ is surjective. Fix a $\mathcal{F}$, as in Notation/Definition 3.1.1, and let $\mathcal{L} = L\mathcal{F}$ and $\mathcal{M} = M\mathcal{F}$. As the ramification index of $\mathcal{F}/F$ is equal to 1, and the same holds for $\mathcal{L}/L$ and $\mathcal{M}/M$, with L/F being totally ramified we have that $\mathrm{Gal}(\mathcal{L}/\mathcal{F}) \cong \mathrm{Gal}(L/F)$, with the map being the restriction map of automorphisms of $\mathcal{L}$ to L. There are similar results for $\mathrm{Gal}(\mathcal{L}/\mathcal{M})$ and $\mathrm{Gal}(\mathcal{M}/\mathcal{F})$. We also see that the isomorphism from $\mathrm{Gal}(\mathcal{L}/\mathcal{F})$ to $\mathrm{Gal}(L/F)$ induces the isomorphisms between $\mathrm{Gal}(\mathcal{L}/\mathcal{M})$ and $\mathrm{Gal}(L/M)$ and between $\mathrm{Gal}(\mathcal{M}/\mathcal{F})$ and $\mathrm{Gal}(M/F)$.

Now, let $T/\overline{\mathcal{F}}$ be a finite Galois extension. As $\overline{\mathcal{F}}/\overline{F}$ is a purely inseparable extension, by definition, we have that there exists a Galois extension $T'/\overline{F}$ such that $T'\overline{\mathcal{F}} = T$ and $\mathrm{Gal}(T/\overline{\mathcal{F}}) \cong \mathrm{Gal}(T'/\overline{F})$. Likewise for all finite Galois extensions $S/\overline{F}$ we have that $S\overline{\mathcal{F}}/\overline{\mathcal{F}}$ is also Galois and has a Galois group isomorphic to the Galois group of $S/\overline{F}$.

Finally, it is clear that if $S/\overline{F}$ is a Galois extension, with $S'/\overline{F}$ being a subextension, then $S'\overline{\mathcal{F}}/\overline{\mathcal{F}}$ is a subextension of the Galois extension $S\overline{\mathcal{F}}/\overline{\mathcal{F}}$. We also have that $\mathrm{Gal}(S/S') \cong \mathrm{Gal}(S\overline{\mathcal{F}}/S'\overline{\mathcal{F}})$. The same holds when going from Galois extensions of $\overline{\mathcal{F}}$ to Galois extensions of $\overline{F}$.

The above all combines to mean that there is an order, and Galois group, preserving bijection between finite Galois extensions of $\overline{F}$ and finite Galois extensions of $\overline{\mathcal{F}}$. Since we can take infinite Galois extensions of $\overline{F}$ to be the composition of finite Galois extensions we get a bijection from infinite Galois extensions of $\overline{F}$ and $\overline{\mathcal{F}}$.

Now, we know that $\overline{\widehat{F}}$ is the maximal p-extension of $\overline{F}$, with $\overline{\widehat{F}}/\overline{F}$ being Galois, and that the same holds for $\overline{\widehat{\mathcal{F}}}$ and $\overline{\mathcal{F}}$. So then, from the previous paragraph and Corollary 3.2.1, we have that $\mathrm{Gal}(\widehat{\mathcal{F}}/\mathcal{F})$ is isomorphic to $\mathrm{Gal}(\widehat{F}/F)$. The extension $\widehat{F}/F$ is Galois, so letting elements of $\mathrm{Gal}(\widehat{\mathcal{F}}/\mathcal{F})$ act on $\widehat{F}$ creates elements of $\mathrm{Gal}(\widehat{F}/F)$. Likewise, we have that the restriction map induces an isomorphism between $\mathrm{Gal}(\overline{\widehat{\mathcal{F}}}/\overline{\mathcal{F}})$ and $\mathrm{Gal}(\overline{\widehat{F}}/\overline{F})$ and thus the restriction map induces an isomorphism between $\mathrm{Gal}(\widehat{\mathcal{F}}/\mathcal{F})$ and $\mathrm{Gal}(\widehat{F}/F)$.

Combining the previous paragraph with the isomorphism from $\mathrm{Gal}(L/F)$ to $\mathrm{Gal}(\mathcal{L}/\mathcal{F})$, we have that $\mathrm{Gal}(L/F)^\wedge \cong \mathrm{Gal}(\mathcal{L}/\mathcal{F})^\wedge$. Using the same methods we also get that $\mathrm{Gal}(L/M)^\wedge \cong \mathrm{Gal}(\mathcal{L}/\mathcal{M})^\wedge$ and $\mathrm{Gal}(M/F)^\wedge \cong \mathrm{Gal}(\mathcal{M}/\mathcal{F})^\wedge$.

Now we need to show that as the map from $\mathrm{Gal}(L/F)^\wedge$ to $\mathrm{Gal}(M/F)^\wedge$, which is induced by the restriction map from $\mathrm{Gal}(L/F)$ to $\mathrm{Gal}(M/F)$, is surjective. Let Φ be an element of $\mathrm{Gal}(M/F)^\wedge$, and let Φ' denote the element of $\mathrm{Gal}(\mathcal{M}/\mathcal{F})^\wedge$ the isomorphism between $\mathrm{Gal}(M/F)^\wedge$ and $\mathrm{Gal}(\mathcal{M}/\mathcal{F})^\wedge$ sends Φ to. We know, from Lemma 3.2.1, that the map from $\mathrm{Gal}(\mathcal{L}/\mathcal{F})^\wedge$ to

$\mathrm{Gal}\,(\mathcal{M}/\mathcal{F})^\wedge$, induced by the restriction map, is surjective; so let Ψ' be an element of $\mathrm{Gal}\,(\mathcal{L}/\mathcal{F})^\wedge$ that is mapped to Φ' and let Ψ be the image of Ψ' under the isomorphism between $\mathrm{Gal}\,(\mathcal{L}/\mathcal{F})^\wedge$ and $\mathrm{Gal}\,(L/F)^\wedge$.

We have that the domain of all of these homomorphism groups are isomorphic to each other, so for the sake of simplicity we shall have them all map from $\mathrm{Gal}\,(\widehat{F}/F)$. Let α be an element of $\mathrm{Gal}\,(\widehat{F}/F)$ then we have that $\Phi'(\alpha) = \beta'$, with β' being an element of $\mathrm{Gal}\,(\mathcal{M}/\mathcal{F})$. As the map $\mathrm{Gal}\,(\mathcal{L}/\mathcal{F})^\wedge \to \mathrm{Gal}\,(\mathcal{M}/\mathcal{F})^\wedge$ is induced by the restriction map from $\mathrm{Gal}\,(\mathcal{L}/\mathcal{F})$ to $\mathrm{Gal}\,(\mathcal{M}/\mathcal{F})$ there is a $t' \in \mathrm{Gal}\,(\mathcal{L}/\mathcal{F})$ such that $\Psi'(\alpha) = t'$. We have $t'\,\mathrm{Gal}\,(\mathcal{L}/\mathcal{M}) \mapsto \beta'$ under the natural isomorphism from $\mathrm{Gal}\,(\mathcal{L}/\mathcal{F})/\,\mathrm{Gal}\,(\mathcal{L}/\mathcal{M})$ to $\mathrm{Gal}\,(\mathcal{M}/\mathcal{F})$.

Let $\Phi(\alpha) = \beta \in \mathrm{Gal}\,(M/F)$ and let $\Psi(\alpha) = t \in \mathrm{Gal}\,(L/F)$. We know that the isomorphism from $\mathrm{Gal}\,(\mathcal{L}/\mathcal{F})$ to $\mathrm{Gal}\,(L/F)$ induces the isomorphisms between $\mathrm{Gal}\,(\mathcal{L}/\mathcal{M})$ and $\mathrm{Gal}\,(L/M)$ and between $\mathrm{Gal}\,(\mathcal{M}/\mathcal{F})$ and $\mathrm{Gal}\,(M/F)$. This gives us that $t\,\mathrm{Gal}\,(L/F) \mapsto \beta$ under the natural isomorphism from $\mathrm{Gal}\,(L/F)$ to $\mathrm{Gal}\,(M/F)$. As α is an arbitrary element of $\widehat{F}/F$, this is precisely what we wanted to show that Ψ maps to Φ and thus that map $\mathrm{Gal}\,(L/F)^\wedge \to \mathrm{Gal}\,(M/F)^\wedge$ is surjective. □

3.3 Intersection and Composition of Extensions

Theorem 3.3.1. *Let L_1/F, L_2/F be finite abelian totally ramified p-extension such that L_1L_2/F is a finite abelian totally ramified p-extension. Then we have $N_*(L_1L_2/F) = N_*(L_1/F) \cap N_*(L_2/F)$ and $N_*((L_1 \cap L_2)/F) = N_*(L_1/F)N_*(L_2/F)$.*

We also have that $L_1 \subseteq L_2$ if and only if $N_*(L_2/F) \subseteq N_*(L_1/F)$.

Proof. Let L/F be a finite abelian totally ramified p-extension with subextension M/F. We have that both:

$$\Psi^*_{L/F} : U(L/F)/N_*(L/F) \to \mathrm{Gal}\,(L/F)^\wedge$$

and:

$$\Psi^*_{M/F} : U(M/F)/N_*(M/F) \to \mathrm{Gal}\,(M/F)^\wedge$$

are isomorphisms. We also have, from Theorem 3.2.1, a natural surjective map from $\mathrm{Gal}\,(L/F)^\wedge$ to $\mathrm{Gal}\,(M/F)^\wedge$, whose kernel is $\mathrm{Gal}\,(L/M)^\wedge$, and which commutes with:

$$U(L/F)/N_*(L/F) \to U(M/F)/N_*(M/F)$$

So, if we allow $L = L_1L_2$, then we have that $\mathrm{Gal}\,(L/L_1) \cap \mathrm{Gal}\,(L/L_2) = 1$. This gives us that:

$$N_*(L/F)/N_*(L/F) = (\Psi^*_{L/F})^{-1}(\mathrm{Gal}\,(L/L_1)^\wedge \cap \mathrm{Gal}\,(L/L_2)^\wedge\,)$$

This is then equal to, as $\Psi^*_{L/F}$ is an isomorphism, $(\Psi^*_{L/F})^{-1}(\mathrm{Gal}\,(L/L_1)^\wedge\,) \cap (\Psi^*_{L/F})^{-1}(\mathrm{Gal}\,(L/L_2)^\wedge\,)$, which equals $(N_*(L_1/F) \cap N_*(L_2/F))/N_*(L/F)$. We therefore get the result we want of $N_*(L_1/F) \cap N_*(L_2/F) = N_*(L/F)$.

For $N_*((L_1 \cap L_2)/F)$, remember that $\Psi^*_{L/F}$ is an isomorphism and $\mathrm{Gal}\,(L/(L_1 \cap L_2)) = \mathrm{Gal}\,(L/L_1) \times \mathrm{Gal}\,(L/L_2)$, we get:

$$N_*((L_1 \cap L_2)/F)/N_*(L/F) \cong (\Psi^*_{L/F})^{-1}(\mathrm{Gal}\,(L/(L_1 \cap L_2))^\wedge\,)$$
$$\cong (\Psi^*_{L/F})^{-1}(\mathrm{Gal}\,(L/L_1)^\wedge \times \mathrm{Gal}\,(L/L_2)^\wedge\,) \cong (N_*(L_1/F)N_*(L_2/F))/N_*(L/F)$$

This then gives us $N_*((L_1 \cap L_2)/F) = N_*(L_1/F)N_*(L_2/F)$..

We obviously have that if $L_2 \subseteq L_1$, then $N_*(L_1/F) \subseteq N_*(L_2/F)$. Meanwhile, if $N_*(L_1/F) \subseteq N_*(L_2/F)$, then $N_*(L_1/F) = N_*(L/F)$. As $\Psi^*_{L/F}$ is an isomorphism this would mean that $\mathrm{Gal}\,(L/F)^\wedge \cong \mathrm{Gal}\,(L_1/F)^\wedge$ and therefore that $L_1 = L$, thus $L_2 \subseteq L_1$. □

Note 3.3.1. The above theorem is a separate result from the result proved in theorem 1.10 of [4], which says that $L_1 \subseteq L_2$ if and only if $N(L_2/F) \subseteq N(L_1/F)$, as seen in Theorem 2.4.2.

Corollary 3.3.1. *If L/F is a finite abelian totally ramified p-extension such that $L = L_1...L_n/F$, where each L_i/F is a cyclic subextension, then $N_*(L/F) = \cap_{i=1}^n N_*(L_i/F) = \cap_{i=1}^n N(L_i/F)$.*

Proof. Suppose that for $i \neq j$, we have that $L_i \cap L_j = F$. We then get that $\mathrm{Gal}\,(L/F) \cong \oplus_{i=1}^n M_i$ where $M_i = \mathrm{Gal}\,(L_i/F)$. From Theorem 3.3.1 we get that $N_*(L/F) = \bigcap_{i=1}^n N_*(L_i/F) = \bigcap_{i=1}^n N(L_i/F)$, since $N_*(L_i/F) = N(L_i/F)$ when L_i/F is a cyclic extension, by Theorem 2.4.1. □

Theorem 3.3.2. *Let F be a complete discrete valuation field with imperfect residue field of positive characteristic, and fix L_1/F and L_2/F as two finite abelian totally ramified p-extensions. Next .let $\mathcal{F}/F$ be as defined in Notation/Definition 3.1.1 and be chosen such that $L_1\mathcal{F} \cap L_2\mathcal{F} = (L_1 \cap L_2)\mathcal{F}$. Suppose there is a prime element π of F such that $\pi \in N_{L_1\mathcal{F}/\mathcal{F}}((L_1\mathcal{F})^*) \cap N_{L_2\mathcal{F}/\mathcal{F}}((L_2\mathcal{F})^*)$. Then L_1L_2/F is a totally ramified extension.*

Proof. We have that π is a prime element of $\mathcal{F}$ and that $\pi \in N_{\mathcal{L}_1/\mathcal{F}}(\mathcal{L}_1^*) \cap N_{\mathcal{L}_2/\mathcal{F}}(\mathcal{L}_2^*)$. We can now use local class field theory on $\mathcal{F}$, which has a perfect residue field, to get that $\mathcal{L}_1\mathcal{L}_2/\mathcal{F}$ is a totally ramified extension, with degree equal to that of L_1L_2/F and with $\pi \in N_{\mathcal{L}_1\mathcal{L}_2/\mathcal{F}}((\mathcal{L}_1\mathcal{L}_2)^*)$, see Lemma 2.3.4. This gives us that L_1L_2/F is a totally ramified extension. □

Note 3.3.2. As $N_{L_1/F}(L_1^*) \subseteq N_{L_1\mathcal{F}/\mathcal{F}}((L_1\mathcal{F})^*) \cap F^*$ and the same thing holds for L_2, the above result in Theorem 3.3.2 is a stronger version of Proposition 2.1 from [4].

We shall now prove an analogue to a result in section (3.3) of [3]; however only in the limited case where $\operatorname{char}(F) = \operatorname{char}(\overline{F}) = p$.

Proposition 3.3.1. *Let F have characteristic $p > 0$. Then we can construct a fixed $\mathcal{F}$ such that for all finite abelian totally ramified p-extensions L_1/F and L_2/F we have $L_1\mathcal{F} \cap L_2\mathcal{F} = (L_1 \cap L_2)\mathcal{F}$.*

Proof. Now $\operatorname{char}(F) = \operatorname{char}(\overline{F}) = p$ so by Proposition (5.4) Chapter **II** of [5] there is a field of coefficients M contained in F such that under the natural map between F and $\overline{F}$ we have that M is mapped isomorphically onto $\overline{F}$.

We can let the field F' to be equal to $F(\bigcup_{i\geq 0} M^{p^{-i}})$. As $\operatorname{char}(F) = p$ we have that F'/F is a purely inseparable extension.

Let α be such that $\alpha^p \in M$ but $\alpha \notin M$ and look at the polynomial $X^p\alpha^p$. Since the natural map between F and $\overline{F}$ isomorphically sends M to $\overline{F}$ we get that, as $\alpha \notin M$, then $\overline{\alpha} \notin \overline{F}$. This means that $F(\alpha)/F$ contains a non-trivial extension of the residue fields. Since $\operatorname{char}(F) = p$ we have that the only root of $X^p - \alpha^p$ is α and thus $X^p - \alpha^p$ is irreducible over F; which means $F(\alpha)/F$ is an extension of degree p. By Proposition (2.4) Chapter **II** of [5], since $F(\alpha)/F$ has a non-trivial extension of residue fields and has degree p we get that $\overline{F}(\overline{\alpha})/\overline{F}$ has degree p while the ramification index of $F(\alpha)/F$ is 1.

We can then iterate the above, extending F to a field that is a finite extension of F when necessary, to get that $F'/F = F(\bigcup_{i\geq 0} M^{p^{-i}})/F$ has a ramification index of 1. The extension $\overline{F'}/\overline{F}$ is generated by the roots of polynomials of the form $X^{p^t} - \overline{\beta}$. As $\operatorname{char}(\overline{F}) = p$ we get that $\overline{F'}/F$ is a purely inseparable extension. $\overline{F'}/\overline{F}$ is purely inseparable, so we have that $\overline{F'} \subseteq \bigcup_{i\geq 0} \overline{F}^{p^{-i}}$; the latter field is the perfection of $\overline{F}$.

Let $\overline{\gamma} \in \overline{F}$. As the characteristic of $\overline{F}$ if p we have that $\overline{\gamma}$ has a single p^t-th root for all $t \geq 0$. By the isomorphism from M to $\overline{F}$ we see that there exists an element of M that is mapped to $\overline{\gamma}$ by the natural map from F to $\overline{F}$; we shall call this element γ. As F also has characteristic p we have that γ also has a single p^t-th root, that we shall labal γ'. From how we constructed F' we have that $\gamma' \in F'$ and we find out that $\overline{\gamma'}$ is the unique p^t-th root of $\overline{\gamma}$.

Since $\overline{\gamma}$ was an arbitrary element of $\overline{F}$ and t was an arbitrary non-negative integer, we have $\bigcup_{i\geq 0} \overline{F}^{p^{-i}} \subseteq \overline{F'}$. Combining this with what we have seen earlier we get that $\bigcup_{i\geq 0} \overline{F}^{p^{-i}} = \overline{F'}$, and thus $\overline{F'}$ is the perfection of $\overline{F}$.

This means that we have that $e(F' \mid F) = 1$ and that $\overline{F'} = \bigcup_{i\geq 0} \overline{F}^{p^{-i}}$ and thus, by definition, we may make F' be the fixed $\mathcal{F}$ that we use. Naturally, for the rest of this proof we will refer to F' with $\mathcal{F}$.

Now, let L_1/F and L_2/F be two finite abelian totally ramified p-extensions. Now L_1/F and L_2/F are both separable extensions while $\mathcal{F}/F$ is a purely inseparable, this means that we have that $L_1\mathcal{F} \cap L_2\mathcal{F} = (L_1 \cap L_2)\mathcal{F}$.

Since L_1/F and L_2/F were arbitrary finite abelian totally ramified p-extensions we get that the $\mathcal{F}$ that we constructed has the property we want. □

Definition 3.3.1. Suppose $\mathrm{char}\,(F) > 0$ and let L/F be a finite abelian totally ramified p-extension. Let us fix $\mathcal{F}$ for all such extensions and define $N^*(L/F)$ as $N_{L\mathcal{F}/\mathcal{F}}((L\mathcal{F})^*) \cap F^*$.

Corollary 3.3.2. *Let us keep the requirement that* $\mathrm{char}\,(F) > 0$. *Fix a prime element* π *of* F. *Let* F_π *be the family of all finite abelian totally ramified* p*-extensions* L/F *such that* $\pi \in N^*(L/F)$. *This family is closed under composition and intersection.*

Proof. Let L_1/F and L_2/F be members of F_π so that we have that π is contained in $N^*(L_1/F) \cap N^*(L_2/F)$. From Theorem, 3.3.2 we know that L_1L_2/F is a finite abelian totally ramified p-extension and, as:

$$N_{L_1L_2\mathcal{F}/\mathcal{F}}((L_1L_2\mathcal{F})^*) = N_{L_1\mathcal{F}/\mathcal{F}}((L_1\mathcal{F})^*) \cap N_{L_2\mathcal{F}/\mathcal{F}}((L_2\mathcal{F})^*)$$

we have that $\pi \in N^*(L_1L_2/F) = N^*(L_1/F) \cap N^*(L_2/F)$.

Likewise, we already know that $(L_1 \cap L_2)/F$ is a finite abelian totally ramified p-extension and, as:

$$N_{(L_1\cap L_2)\mathcal{F}/\mathcal{F}}(((L_1 \cap L_2)\mathcal{F})^*) = N_{L_1\mathcal{F}/\mathcal{F}}((L_1\mathcal{F})^*)N_{L_2\mathcal{F}/\mathcal{F}}((L_2\mathcal{F})^*)$$

we have that $\pi \in N_{(L_1\cap L_2)\mathcal{F}/\mathcal{F}}(((L_1 \cap L_2)\mathcal{F})^*)$. This gives us that $\pi \in N^*((L_1 \cap L_2)/F)$.

This means that F_π is closed under intersection and composition. □

Unfortunately, although the above corollary is an extension of the results of section 3.3. in the paper on the perfect residue fields [3], we talked about them in Definition 2.3.8, it should be reiterated that the result has only been shown for when char $(F) > 0$.

3.4 $(\mathbb{Z}/p\mathbb{Z})^2$-Extensions

Let L/F be a abelian totally ramified extension with Galois group isomorphic to $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$. We want to know what properties L/F is required to have for $\Psi_{L/F}$ to be an isomorphism.

Let M/F be a subextension of L/F of degree p, so L/M also has degree p, and are thus cyclic. We know, from Theorem 2.4.1, that both $\Psi_{L/M}$ and $\Psi_{M/F}$ are isomorphisms. This means that $N_*(L/M) = N(L/M)$ and $N_*(M/F) = N(M/F)$. What does it therefore mean for:

$$N_*(L/F) = U_{1,F} \cap N_{\widehat{L}/\widehat{F}}(U_{1,\widehat{L}}) \cap N_{\mathcal{L}/\mathcal{F}}(U_{1,\mathcal{L}}) \neq N_{L/F}(U_{1,L}) = N(L/F)?$$

Lemma 3.4.1. *Keeping the assumptions and notation outlined above, we have that $N_*(L/F) \neq N(L/F)$ if and only if there exists an α contained in $N_*(L/F) \subseteq N_{M/F}(U_{1,M})$ such that $N_{M/F}^{-1}(\alpha) \cap N_{\widehat{L}/\widehat{M}}(U_{1,\widehat{L}}) = \emptyset$.*

Proof. $N(L/F) \subseteq N_*(L/F)$, so if the two groups are not equal it always means that there is an element of the latter that is not in the former. Therefore, let us suppose $N_*(L/F) \neq N(L/F)$ by saying there is an α in $N_*(L/F)$ but not in $N(L/F)$. Now ,$\alpha \in N_*(L/F)$ means that there exists a $t_1 \in U_{1,\widehat{L}}$ and $t_2 \in U_{1,\mathcal{L}}$ such that $N_{\widehat{L}/\widehat{F}}(t_1) = N_{\mathcal{L}/\mathcal{F}}(t_2) = \alpha$.

$N_*(L/F) \subseteq N_*(M/F) = N(M/F)$, and thus there is a $\beta \in U_{1,M}$ such that $N_{M/F}(\beta) = \alpha$. We also have, from Corollary 2.3.1, that the kernel of $N_{\mathcal{M}/\mathcal{F}}$ is contained in the image of $N_{\mathcal{L}/\mathcal{M}}$. Next, as $N_{\mathcal{M}/\mathcal{F}}(N_{\mathcal{L}/\mathcal{M}}(t_2)) = \alpha$, we have that if $r \in U_{1,M}$ is such that $N_{M/F}(r) = \alpha$, then $r \in N_{\mathcal{L}/\mathcal{M}}(U_{1,\mathcal{L}})$. Hence $U_{1,M} \subseteq U_{1,\mathcal{M}}$, this means that $\beta \in N_{\mathcal{L}/\mathcal{M}}(U_{1,\mathcal{L}})$.

Now if $\beta \in N_{\widehat{L}/\widehat{F}}(U_{1,\widehat{L}})$, then:

$$\beta \in U_{1,M} \cap N_{\widehat{L}/\widehat{M}}(U_{1,\widehat{L}}) \cap N_{\mathcal{L}/\mathcal{M}}(U_{1,\mathcal{L}}) = N_*(L/M)$$

This latter group is equal to $N(L/M)$, which means that there is a $\gamma \in U_{1,L}$ such that $N_{L/M}(\gamma) = \beta$. Obviously we therefore have that $N_{L/F}(\gamma) = \alpha$, which contradicts the fact that $\alpha \notin N(L/F)$. This means that $\beta \notin N_{\widehat{L}/\widehat{M}}(U_{1,\widehat{L}})$. As β was an arbitrary element of $N_{M/F}^{-1}(\alpha)$ we get that $N_{M/F}^{-1}(\alpha) \cap N_{\widehat{L}/\widehat{M}}(U_{1,\widehat{L}}) = \emptyset$; this is the "if" part of the statement of this lemma.

If $N_*(L/F) = N(L/F)$, then every $\alpha \in N_*(L/F)$ is contained in $N(L/F)$ and thus there exists a $\beta \in N(L/M)$ such that $N_{M/F}(\beta) = \alpha$. We trivially have that $\beta \in N_{\widehat{L}/\widehat{M}}(U_{1,\widehat{L}})$, which gets us the "only if" part of the statement. □

Lemma 3.4.2. *Keeping the same notation as the previous lemma, now let* $s_1 = s(M/F)$ *and let* $s_2 = s(L/M)$*, as described in Notation 2.1.3. Then* $\Psi_{L/F}$ *is an isomorphism if* $s_1 \geq s_2$*.*

Proof. We shall let σ be a generator of Gal (M/F), and therefore of Gal $(\widehat{M}/\widehat{F})$ and Gal $(\mathcal{M}/\mathcal{F})$. The following proof will be by contradiction and therefore we shall start off by assuming that $\Psi_{L/F}$ is not an isomorphism.

So, from the above lemma, we have that there exists an $\alpha \in N_*(L/F)$ such that $N_{M/F}^{-1}(\alpha) \cap N_{\widehat{L}/\widehat{M}}^{-1}(U_{1,\widehat{L}}) = \emptyset$. Let $\beta \in U_{1,M}$ be such that $N_{M/F}(\beta) = \alpha$.

Looking at the diagrams in Proposition (1.5) Chapter **III** of [5], and remembering that $\widehat{\overline{L}}$ is separably p-closed, we see that $U_{s_2,\widehat{M}} \subseteq N_{\widehat{L}/\widehat{M}}(U_{1,\widehat{L}})$ and that $\ker(N_{\widehat{M}/\widehat{F}}) \subseteq U_{s_1,\widehat{M}}$. We have that $s_1 \geq s_2$, so then the kernel of $N_{\widehat{M}/\widehat{F}}$ would be contained in the image of $N_{\widehat{L}/\widehat{M}}$.

The previous paragraph means that $\beta \notin U_{s_2,M} \subseteq U_{s_2,\widehat{M}}$. It also means that we could not find $\gamma \in U_{1,\widehat{M}}$, such that $\beta(\gamma^{\sigma-1}) \in \text{Im}\,(N_{\widehat{L}/\widehat{M}})$. This is because $\gamma^{\sigma-1}$ is contained in the kernel of $N_{\widehat{M}/\widehat{F}}$, and β is not in the image of $N_{\widehat{L}/\widehat{M}}$.

Now $\alpha \in N_*(L/F) = U_{1,F} \cap N_{\widehat{L/F}}(U_{1,\widehat{L}}) \cap N_{\mathcal{L}/\mathcal{F}}(U_{1,\mathcal{L}})$ and therefore $\alpha \in N_{\widehat{L}/\widehat{F}}(U_{1,\widehat{L}})$. This means that there is a t_1' contained in $U_{1,\widehat{L}}$ such that $N_{\widehat{L}/\widehat{F}}(t_1') = \alpha$. let $t_1'' = N_{\widehat{L}/\widehat{F}}(t_1')$. Now $N_{\widehat{M}/\widehat{F}}(t_1'') = \alpha$ and by Corollary (4.1) Chapter **III** of [5] we have that t_1'' is of the form $\beta(\gamma^{\sigma-1})$, for $\gamma \in U_{1,\widehat{M}}$. This contradicts what we concluded in the previous paragraph. This means that such a β cannot exist and therefore by contradiction we have that $N_*(L/F) = N(L/F)$, if $s_1 \geq s_2$. □

There is one final result we must establish before we can prove the main theorem of this section. This is Exercise 2 part b) of section 3 of Chapter **III** of [5]. Since this is not proved in the book we will state and prove the result in the next lemma.

Lemma 3.4.3. *Let* F *still be a complete discrete valuation field where the only restriction on* $\overline{F}$ *is that it has positive characteristic* p*. Let* M_1/F *and* M_2/F

be two totally ramified Galois extensions of degree p *with* $M_1 \cap M_2 = F$*. Let* $L = M_1M_2$ *be such that* L/F *is also totally ramified and set* $s_1 = s(L \mid M_2)$ *and* $s_2 = s(L \mid M_1)$*.*

We have that $s_1 \equiv s_2 \mod p$*. Also, if* $s(M_2 \mid F) > s(M_1 \mid F)$*, then* $s_1 = s(M_1 \mid F)$ *and* $s_2 = ps(M_2 \mid F) - (p-1)s_1$*. Meanwhile, if* $s = s(M_2 \mid F) = s(M_1 \mid F)$*, then* $s_1 = s_2 \leq s$*.*

Proof. Let σ be a generator of $\mathrm{Gal}\,(L/M_1)$ and ϕ be a generator of $\mathrm{Gal}\,(L/M_2)$; finally set G as $\mathrm{Gal}\,(L/F)$. We have that L/F is a totally ramified Galois extension and thus, in terms of lower ramification groups of G, we have that there is an $i, j \geq 1$ such that $\sigma \in G_i$ but not in G_{i+1} and $\phi \in G_j$ but not in G_{j+1}. We can use knowledge of lower ramification groups to help us.

Let π_L be a prime element of L. From how ramification groups are calculated we know that there are units, α and β, of L such that $\sigma(\pi_L)/\pi_L = 1 + \alpha\pi_L^i$ and $\phi(\pi_L)/\pi_L = 1 + \beta\pi_L^j$. Looking at Notation 2.1.2, we see that, by definition, $s_1 = i$ and $s_2 = j$. From the definition of i and j in the previous paragraph they are two ramification jumps of G, it has been established that in this case $i \equiv j \mod p$. This gives us the first statement of the lemma that $s_1 \equiv s_2 \mod p$.

The rest of this proof shall be relying heavily on the properties of the Hasse-Herbrand function, $h_{L/F}$, which are described in detail in section (3) of Chapter **III** of [3]. We are going to be using the following properties:

- $h_{L/F} = h_{L/M_1} \circ h_{M_1/F} = h_{L/M_2} \circ h_{M_2/F}$.
- As L/M_1 is an extension of degree p we have that h_{L/M_1} is formulated as follows:

$$h_{L/M_1}(x) = \begin{cases} x & \text{if } x \leq s_2 \\ px + s_2(1-p) & \text{if } x \geq s_2 \end{cases}$$

  We have similar properties for h_{L/M_2}, $h_{M_1/F}$ and $h_{M_2/F}$.
- All Hasse-Herbrand functions are bijections that maps, though not necessarily surjectively, the set of non-negative numbers into itself.
- If E/P is a finite Galois extension with Q/P being a Galois subextension, then for every $x \geq 0$ we have that the image of $\mathrm{Gal}\,(E/P)_{h_{E/Q}(x)}$ in $\mathrm{Gal}\,(Q/P)$ is equal to $\mathrm{Gal}\,(Q/P)_x$. This is Theorem (3.5) Chapter **III** in [5].

From $h_{L/F} = h_{L/M_1} \circ h_{M_1/F} = h_{L/M_2} \circ h_{M_2/F}$ we have that $h_{L/F}$ has one or two points where its derivative does not exist. These points are at $s(M_2 \mid F)$ and when $h_{M_1/F}(x) = s_1$, the latter is at $x = s_1$ if $s_1 \leq s(M_2 \mid F)$ or at $x = (s_1 + (p-1)s(M_2 \mid F))/p$ if $s_1 > s(M_2 \mid F)$. The same property also holds when we decompose $h_{L/F}$ into $h_{L/M_1} \circ h_{M_1/F}$. So the points where the derivative does not exist must also be at $s(M_1 \mid F)$ and either s_2 or $(s_2 + (p-1)s(M_1 \mid F))/p$.

Now suppose that $s(M_2 \mid F) > s(M_1 \mid F)$. From the previous paragraph we have that $h_{L/F}$ has up to two points where the derivative does not exist; one at $s(M_2 \mid F)$ and one, which may be the same as the previous point, at $s(M_1 \mid F)$. As $s(M_2 \mid F) \neq s(M_1 \mid F)$ we see that $h_{L/F}$ has two points where the derivative does not exist; $x = s(M_2 \mid F)$ and $x = s(M_1 \mid F)$ are the two separate points.

From the work earlier in this proof we therefore get that $s(M_1 \mid F)$ is equal to s_1, if $s_1 \leq s(M_2 \mid F)$, or equal to $(s_1 + (p-1)s(M_2 \mid F))/p$, if $s_1 > s(M_2 \mid F)$. If $s_1 > s(M_2 \mid F)$, then $s_1 + (p-1)s(M_2 \mid F) > ps(M_2 \mid F)$ and thus $s(M_1 \mid F) > s(M_2 \mid F)$. This contradicts what we know which gives us that $s_1 \leq s(M_2 \mid F)$ and $s_1 = s(M_1 \mid F)$.

We can see that the other point where the derivative does not exist is $s(M_2 \mid F)$, which is equal to $(s_2 + (p-1)s(M_1 \mid F))/p$, we get a similar contradiction as before if $s_2 \leq s(M_1 \mid F)$. By rearranging this gives us that:

$$s_2 = ps(M_2 \mid F) - (p-1)s(M_1 \mid F) = ps(M_2 \mid F) - (p-1)s_1$$

as required.

Let us now suppose that $s(M_2 \mid F) = s(M_1 \mid F) = s$. By construction we end up with $h_{M_1/F} = h_{M_2/F}$. A Hasse-Herbrand function is a bijection between the set of non-negative numbers to itself and thus inverses exist. Since we have that $h_{L/F} = h_{L/M_1} \circ h_{M_1/F} = h_{L/M_2} \circ h_{M_2/F}$ we can use the inverse of $h_{M_1/F}$ to get that $h_{L/M_1} = h_{L/M_2}$. We therefore get that $s_1 = s_2 = t$, for some positive integer t.

Suppose that $t > s$. Since $s + 1 \leq t$ we have, by the nature of h_{L/M_1} and h_{L/M_2}, that $h_{L/M_1}(s+1) = h_{L/M_2}(s+1) = s+1$. We can now use the aforementioned Theorem (3.5). This tells us that the image of $\mathrm{Gal}\,(L/F)_{s+1}$ in $\mathrm{Gal}\,(M_1/F)$ is equal to $\mathrm{Gal}\,(M_1/F)_{s+1}$. The last group is equal to 1 as $s(M_1 \mid F) = s$, and from how lower ramification groups are worked out. Likewise we have that the image of $\mathrm{Gal}\,(L/F)_{s+1}$ in $\mathrm{Gal}\,(M_2/F)$ equals 1.

From earlier in this proof we got that σ and ϕ must both be in $\mathrm{Gal}\,(L/F)_{s+1}$, since they both help generate s_1 and s_2 respectively; which are both greater than or equal to $s+1$. We also cannot have $\phi = \sigma^n$, for some n, as otherwise ϕ and σ would have the same fixed field in L and thus $M_1 = M_2$; making

$L = M_1M_2 = M_1$. Since $\mathrm{Gal}(L/F) \cong (\mathbb{Z}/p\mathbb{Z})^2$ and σ and ϕ are independent elements of that group we see that they generate $\mathrm{Gal}(L/F)$. This gives us, since $\phi, \sigma \in \mathrm{Gal}(L/F)_{s+1}$, that $\mathrm{Gal}(L/F)_{s+1} = \mathrm{Gal}(L/F)$.

Going back to what we had previously. We worked out that the image of $\mathrm{Gal}(L/F)_{s+1} = \mathrm{Gal}(L/F)$ in $\mathrm{Gal}(M_1/F)$ is equal to 1. This of course cannot be the case as the image of $\mathrm{Gal}(L/F)$ in $\mathrm{Gal}(M_1/F)$ is $\mathrm{Gal}(M_1/F)$. From this contradiction we get that $s_1 = s_2 = t \leq s = s(M_1 \mid F) = s(M_2 \mid F)$, just like we wanted to show.

□

Theorem 3.4.1. *$\Psi_{L/F}$ is always an isomorphism if L/F is a totally ramified extension whose Galois group is isomorphic to $(\mathbb{Z}/p\mathbb{Z})^2$.*

Proof. Lemma 3.4.3 gives us a few results for when L_1/F and L_2/F are two totally ramified extensions of degree p such that L/F, where $L = L_1L_2$, is totally ramified. The results include that if $s(L_1/F) > s(L_2/F)$, then $s(L_2/F) = s(L/L_1)$, and that if $s(L_1/F) = s(L_2/F)$ we have that $s(L/L_1) = s(L/L_2) \leq s(L_2/F)$. This is important, as it means that we have $s(L_1/F) \geq s(L/L_1)$ or $s(L_2/F) \geq s(L/L_2)$ for every possible L/F that we could have.

Combining this with the result of Lemma 3.4.2 tells us that if L/F is Galois a totally ramified extensions whose Galois groups is isomorphic to $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$, then $\Psi_{L/F}$ is always an isomorphism. □

3.5 General Abelian Extensions

Lemma 3.5.1. *Let F be a complete discrete valuation field such that $\overline{F}$ is not separably p-closed, where $p > 0$ is the characteristic of $\overline{F}$. We will still allow $\overline{F}$ to be imperfect. Fix L/F as a finite abelian totally ramified p-extension such that $G = \mathrm{Gal}(L/F)$ and G_n is the n'th ramification group. Assume that we have that there exists an $a > 0$ such that $G_a = G$ and $G_{a+1} = 1$. In other words G has one ramification jump.*

If $F = L_0 \subset L_1 \subset ... \subset L_n = L$ is a chain of subextensions such that L_{i+1}/L_i has degree p for all i, then $s(L_{i+1}/L_i) = a$ for all i.

Proof. We shall use Proposition (3.6) Chapter **III** of [5]. We know that if the degree of L/F equals p then, by definition, $s(L/F) = a$. Let us inductively suppose that for $[L : F] \leq p^n$, for some positive integer n, then if $F = L_0 \subset$

$L_1 \subset ... \subset L_n = L$, with L_{i+1}/L_i being a degree p extension, we have that $s(L_{i+1}/L_i) = a$ for all $0 \leq i \leq n-1$.

Let L/F now have degree p^{n+1} and let M/F be a subextension of degree p, we know one exists as L/F is abelian. Let $G' = \text{Gal}\,(L/M)$. From how ramification groups are calculated, and the fact that G' is a subgroup of G, we know that $G'_a = G'$ and $G'_{a+1} = 1$, just like it is for G. This means, from Proposition (3.6). that for $m \leq a$ we have both $h_{L/M}(m) = m$ and $h_{L/F}(m) = m$. Now $h_{L/F} = h_{L/M} \circ h_{M/F}$, and this means that for $m \geq a$ we must have that $h_{M/F}(m) = m$. Since M/F has degree p we know, from

$$h_{M/F} = \begin{cases} x & \text{if } x \leq s(M/F) \\ px + s(M/F)(1-p) & \text{if } x \geq s(M/F) \end{cases}$$

that $s(M/F) \geq a$.

Now let us suppose that $s(M/F) = s > a$. From Proposition (3.6) Chapter **III** in [5] we have that, as $s > a$, then $N_{L/M}$ induces a bijection between $U_{h_{L/F}(s),L}/U_{h_{L/F}(s)+1,L}$ and $U_{s,M}/U_{s+1,M}$. Likewise, $N_{L/F}$ induces a bijection between $U_{h_{L/F}(s),L}/U_{h_{L/F}(s)+1,L}$ and $U_{s,F}/U_{s+1,F}$. Now, $N_{L/F} = N_{M/F} \circ N_{L/M}$ and $h_{M/F}(s) = s$, so the fact that $h_{L/F} = h_{L/M} \circ h_{M/F}$ means we have that $N_{M/F}$ induces a bijection between $U_{s,M}/U_{s+1,M}$ and $U_{s,F}/U_{s+1,F}$. This can not be true, since $s = s(M/F)$ and M/F has degree p, which means that the induced map from $U_{s,M}/U_{s+1,M}$ to $U_{s,F}/U_{s+1,F}$ is not injective. From this we get that $s(M/F) \leq a$, and thus $s(M/F) = a$. This is what we need to prove the lemma by induction. □

Theorem 3.5.1. *Let L/F be a finite abelian totally ramified p-extension. Set $G = \text{Gal}\,(L/F)$. Let a_i, $i \in I$, be the ramification jumps. This gives us a chain:*

$$G = G_{a_1} > G_{a_2} > ... > G_{a_n} > G_{a_n+1} = 1$$

Let L_t be the fixed field of G_t in L.

Then if $L_t = M_0 \subset M_1 \subset ... \subset M_{k_t} = L_{t+1}$ is a chain of subextensions such that M_{r+1}/M_r has degree p for all r, then $s(M_{r+1}/M_r) = a_t$ for all r.

Proof. Let us inductively suppose that for all $n \leq m-1$ and $1 \leq t \leq n$, we have that $s(M_{r+1}/M_r) = a_t$.

Let us set $n = m$. Now L_{a_2} is the fixed field of G_{a_2} in G and we have that the extension L/L_{a_2} is Galois, as L/F is an abelian extension, and has Galois group G_{a_2}. We also have, from how the groups are computed, that the lower ramification groups of G_{a_2} are precisely G_{a_t}, for $2 \leq t \leq m$. So by the inductive assumption we have the result we want holds for $2 \leq t \leq m$.

The only thing left to show is that if:

$$F = L_{a_1} = M_0 \subseteq M_1 \subseteq ... \subseteq M_k = L_{a_2} = L_{a_1+1}$$

is a chain of field extensions such that M_{r+1}/M_r has degree p for all $0 \leq r < k$, then $s(M_{r+1}/M_r) = a_1$. By Proposition (3.6) Chapter **III** in [5], $\mathrm{Gal}\,(L_{a_2}/F)$ is a Galois extension such that $\mathrm{Gal}\,(L_{a_2}/F)_{a_1} = \mathrm{Gal}\,(L_{a_2}/F)$. Therefore, we can get the result that we want, namely that for $0 \leq r < k_1$ we have that $s(M_{r+1}/M_r) = a_1$, by applying the same methods as used in the proof of Lemma 3.5.1. □

3.6 Structure of Norm Groups when there is One Ramification Jump

Theorem 3.6.1. *Let L/F be a abelian totally ramified extension of degree p^n, here $n \geq 1$, with one ramification jump. This means there exists an $a > 0$ such that if $G = \mathrm{Gal}\,(L/F)$ we have that $G_a = G$, while $G_{a+1} = 1$.*

The norm group $N_{L/F}(U_{1,L})$ has the following structure:

- *If $i < a$ then $\lambda_i(N_{L/F}(U_{1,L}) \cap U_{i,F})) = \overline{F}^{p^n}$.*
- *If $i = a$ we have that $\lambda_i(N_{L/F}(U_{1,L}) \cap U_{a,F})) = g(\overline{F})$. Here $g(X) \in \overline{F}[X]$ is equal to $(X^p + \overline{\eta_1}X) \circ ... \circ (X^p + \overline{\eta_n}X)$ for some non-zero $\overline{\eta_1}, \overline{\eta_2}, ...\overline{\eta_n}$ contained in $\overline{F}$.*
- *If $i > a$ then we have $\lambda_i(N_{L/F}(U_{1,L}) \cap U_{i,F})) = \overline{F}$.*

Proof. From Lemma 3.5.1 we know that if $F = L_0 \subseteq L_1 ... \subseteq L_n = L$ is a chain of field extensions such that $[L_{t+1} : L_t] = p$, then $s(L_{t+1}/L_t) = a$ for all $0 \leq t < n$.

From Proposition (1.5) Chapter **III** of [5] we have that for $1 \leq t < n$ and for $i \geq 1$ we get N_{L_{t+1}/L_t} induces the following map :

$$U_{h_{L_{t+1}/L_t}(i),L_{t+1}}/U_{h_{L_{t+1}/L_t}(i)+1,L_{t+1}} \to U_{i,L_t}/U_{i+1,L_t}$$

as follows:

- If $i < a$ then $\overline{\theta}$ is mapped to $\overline{\theta}^p$, and $h_{L_{t+1}/L_t}(i) = i$.

- If $i = a$ then $\overline{\theta}$ is mapped to $\overline{\theta}^p - \overline{\mu_{t+1}}^{p-1}\overline{\theta}$ for some fixed non-zero $\overline{\mu_{t+1}} \in \overline{F}$, and $h_{L_{t+1}/L_t}(i) = i$.

- If $i > a$ then $\overline{\theta}$ is mapped to $-\overline{\mu_{t+1}}^{p-1}\overline{\theta}$ for the same fixed $\overline{\mu_{t+1}} \in \overline{F}$ as the $i = a$ case. In this case $h_{L_{t+1}/L_t}(i) = a(1-p) + pi$.

We also have that if $i > 0$ and p does not divide i then:

$$N_{L_{t+1}/L_t}(U_{p+i,L_{t+1}}) = N_{L_{t+1}/L_t}(U_{p+i+1,L_{t+1}})$$

For simplicity we shall replace $-\overline{\mu_{t+1}}^{p-1}$ with $\overline{\eta_{t+1}}$, and will do the same in similar cases, as it is still a fixed non-zero element of $\overline{F}$.

The fact that $s(L_{t+1}/L_t) = a$ for all $0 \leq t < n$ means that we can iterate the above n times to get the map $N_{L/F}$ induces from $U_{h_{L/F}(i),L}/U_{h_{L/F}(i)+1,L}$ to $U_{i,F}/U_{i+1,F}$:

- If $i < a$ then $\overline{\theta}$ is mapped to $\overline{\theta}^{p^n}$, and $h_{L/F}(i) = i$.

- If $i = a$ then there exists non-zero $\overline{\eta_1}, \overline{\eta_2}, ...\overline{\eta_n} \in \overline{F}$ such that $\overline{\theta}$ is mapped to $g(\overline{\theta})$. Here $g \in \overline{F}[X]$ is equal to $(X^p + \overline{\eta_1}X) \circ ... \circ (X^p + \overline{\eta_n}X)$. Here we also have that $h_{L/F}(i) = i$.

- Finally if $i > a$ then, using the same non-zero $\overline{\eta_1}, \overline{\eta_2}, ...\overline{\eta_n} \in \overline{F}$ as above, $\overline{\theta}$ is mapped to $\overline{\eta_1}\overline{\eta_2}...\overline{\eta_n}\overline{\theta}$. We shall let $\overline{\eta}$ denote $\overline{\eta_1}\overline{\eta_2}...\overline{\eta_n}$, and note that it is also a non-zero element of $\overline{F}$. Here we have that:

$$h_{L/F}(i) = a(1-p)+p(a(1-p)+p(...p(a(1-p)+pi))...)) = a(1-p^n)+p^n i$$

From working out all the cases of i we can finally get the structure of the norm group that we wanted:

- If $i < a$ then $\lambda_i(N_{L/F}(U_{1,L}) \cap U_{iF})) = \overline{F}^{p^n}$.

- If $i = a$ we have that $\lambda_i(N_{L/F}(U_{1,L}) \cap U_{a,F})) = g(\overline{F})$, where $g(X) \in \overline{F}[X]$ is defined earlier in this proof.

- If $i > a$ then, as $\overline{\eta} \neq 0$, we have $\lambda_i(N_{L/F}(U_{1,L}) \cap U_{i,F})) = \overline{F}$.

□

3.6.1 An Important Aside about $g(X)$

Lemma 3.6.1. *$g(X) \in \overline{F}[X]$, which was defined in the proof of Theorem 3.6.1 to be equal to $(X^p + \overline{\eta_1}X) \circ ... \circ (X^p + \overline{\eta_n}X)$, is a $\overline{F}$-decomposable polynomial, as explained in Definition 2.2.12.*

Proof. By definition, $g(X)$ is the map $N_{L/F}$ induces from $U_{a,L}/U_{a+1,L}$ to $U_{a,F}/U_{a+1,F}$. Therefore, by statement (3) of section 3.1 from [3], if $\overline{F}$ is a perfect field we have that $g(X)$ is $\overline{F}$-decomposable. As a reminder, $g(X)$ being $\overline{F}$-decomposable means that it is an additive polynomial over $\overline{F}$ and every root of $g(X)$ is contained in $\overline{F}$.

Suppose that $\overline{F}$ is imperfect and let $\mathcal{F}$ and $\mathcal{L} = L\mathcal{F}$ be the complete discrete valuation fields with perfect residue fields that can be constructed as per Notation/Definition 3.1.1. Likewise, for $0 \le t \le n$, fix $\mathcal{L}_t$ as $L_t\mathcal{F}$. Since $L_t \subseteq L_{t+1}$, and L_{t+1}/L_t is a non-trivial totally ramified extension, we see that $\mathcal{L}_{t+1}/\mathcal{L}_t$ is a non-trivial totally ramified extension. We also have that $[L_t : F] = [\mathcal{L}_t : \mathcal{F}]$, and from this we get that $\mathcal{F} = \mathcal{L}_0 \subseteq \mathcal{L}_1 \subseteq ... \subseteq \mathcal{L}_n = \mathcal{L}$ is a chain of totally ramified extensions such that for $0 \le t < n$ we have $[\mathcal{L}_{t+1} : \mathcal{L}_t] = p$.

Let π_{t+1} be a prime element of L_{t+1}, for some $0 \le t < n$, then π_{t+1} is a prime element of $\mathcal{L}_{t+1}$. We also have that $\mathrm{Gal}\,(\mathcal{L}_{t+1}/\mathcal{L}_t) \cong \mathrm{Gal}\,(L_{t+1}/L_t)$, via the restriction map. This means that if σ_t is a generator of $\mathrm{Gal}\,(L_{t+1}/L_t)$ then there is a generator, σ'_t, of $\mathrm{Gal}\,(\mathcal{L}_{t+1}/\mathcal{L}_t)$ such that $\sigma'_t(\pi_{t+1}) = \sigma_t(\pi_{t+1})$.

This gives us that $\sigma'_t(\pi_{t+1})/\pi_{t+1} = \sigma_t(\pi_{t+1})/\pi_{t+1}$, and thus $s(\mathcal{L}_{t+1}/\mathcal{L}_t) = s(L_{t+1}/L_t) = a$. We also get, from how it is computed, that the map from $U_{a,\mathcal{L}_{t+1}}/U_{a+1,\mathcal{L}_{t+1}}$ to $U_{a,\mathcal{L}_t}/U_{a+1,\mathcal{L}_t}$, induced by $N_{\mathcal{L}_{t+1}/\mathcal{L}_t}$ is the same polynomial as the map from $U_{a,L_{t+1}}/U_{a+1,L_{t+1}}$ to $U_{a,L_t}/U_{a+1,L_t}$. This means that the map from $U_{a,\mathcal{L}}/U_{a+1,\mathcal{L}}$ to $U_{a,\mathcal{F}}/U_{a+1,\mathcal{F}}$, induced by $N_{\mathcal{L}/\mathcal{F}}$, is also $g(X)$. Note that these above results hold, for the same reasons, if we instead look at $N_{\widehat{L}/\widehat{F}}$, instead of $N_{\mathcal{L}/\mathcal{F}}$.

$\overline{\mathcal{F}}$ is a perfect field, and we are assuming that $\overline{F}$ is an infinite field as well; it is imperfect so $\overline{F}$ must have infinite cardinality. Therefore, we have that $g(X)$ is a $\overline{\mathcal{F}}$-decomposable polynomial. This means that $g(X) = \sum_{i=0}^{n} \alpha_i X^{p^i}$ and thus the formal derivative of $g(X)$ is the constant α_0, since $\overline{\mathcal{F}}$ has characteristic p. Since $g(X) = (X^p + \overline{\eta_1}X) \circ ... \circ (X^p + \overline{\eta_n}X)$, we can see that in this case $g'(X) = \eta_1 \cdot \eta_2 \cdot ... \cdot \eta_n = \eta \neq 0$.

This means that $g(X)$ has no repeated roots and therefore its roots are contained in $\overline{F}^{\mathrm{sep}}$, since $g(X) \in \overline{F}[X]$. We also know that the roots of $g(X)$ are contained in $\overline{\mathcal{F}}$ and that, by definition, $\overline{\mathcal{F}}/\overline{F}$ is a purely inseparable extension. This means that the roots of $g(X)$ are contained in $\overline{F}$ and we have that $g(X)$

is an additive polynomial when counted over $\overline{\mathcal{F}}$ and thus it is additive when regarded over $\overline{F}$. Therefore $g(X)$ is an $\overline{F}$-decomposable polynomial. □

3.7 Extensions with One Ramification Jump

Theorem 3.7.1. *Let L/F be a finite totally ramified p-extension with one ramification jump, so there is an integer $a > 0$ such that, if we set $\mathrm{Gal}\,(L/F)$ as G, we have $G = G_a$ and $G_{a+1} = 1$. In this scenario $\Psi_{L/F}$ is an isomorphism.*

Proof. We will let $g(X) \in \overline{F}[X]$ be as described in the previous section.

Now, $\Psi_{L/F}$ is an isomorphism if and only if:

$$U_{1,F} \cap N_{\mathcal{L}/\mathcal{F}}(U_{1,\mathcal{L}}) \cap N_{\widehat{L}/\widehat{F}}(U_{1,\widehat{L}}) = N_{L/F}(U_{1,L})$$

This is equivalent to saying that for all $i > 0$ we have that:

$$\lambda_i(U_{i,F} \cap N_{L/F}(U_{1,L})) = \lambda_i(U_{i,F} \cap N_{\mathcal{L}/\mathcal{F}}(U_{1,\mathcal{L}})) \cap \lambda_i(U_{i,F} \cap N_{\widehat{L}/\widehat{F}}(U_{1,\widehat{L}}))$$

The above is true because we know that there is a $b > 0$ such that:

$$U_{b,F} \cap N_{L/F}(U_{1,L}) = U_{b,F} \cap N_{\mathcal{L}/\mathcal{F}}(U_{1,\mathcal{L}}) = U_{b,F} \cap N_{\widehat{L}/\widehat{F}}(U_{1,\widehat{L}}) = U_{b,F}$$

Note, we already know $U_{1,F} \cap N_{\mathcal{L}/\mathcal{F}}(U_{1,\mathcal{L}}) \cap N_{\widehat{L}/\widehat{F}}(U_{1,\widehat{L}}) \supseteq N_{L/F}(U_{1,L})$. This means all we need to show is that if:

$$\alpha \in \lambda_i(U_{i,F} \cap N_{\mathcal{L}/\mathcal{F}}(U_{1,\mathcal{L}})) \cap \lambda_i(U_{i,F} \cap N_{\widehat{L}/\widehat{F}}(U_{1,\widehat{L}}))$$

then α is in $\lambda_i(U_{i,F} \cap N_{L/F}(U_{1,L}))$.

First let $i > a$. We have $N_{L/F}(U_{a+1,L}) = U_{a+1,F}$, and the same holds of $\widehat{F}$ and $\mathcal{F}$. This means that if $i > a$ we get:

$$\lambda_i(U_{i,F} \cap N_{L/F}(U_{1,L})) = \overline{F}$$

and likewise both $\lambda_i(U_{i,F} \cap N_{\mathcal{L}/\mathcal{F}}(U_{1,\mathcal{L}}))$ and $\lambda_i(U_{i,F} \cap N_{\widehat{L}/\widehat{F}}(U_{1,\widehat{L}}))$ are equal to $\overline{F}$. This makes this matter trivial.

Next, let $i = a$. This first thing to note is that if $j < a$, then the map from $U_{j,L}/U_{j+1,L}$ to $U_{j,F}/U_{j+1,F}$ is an injective map, and the same thing holds for $N_{\mathcal{L}/\mathcal{F}}$ and $N_{\widehat{L}/\widehat{F}}$. This means that if $\gamma \in U_{j,L}$, for $j < a$, then $N_{L/F}(\gamma) \in U_{j,F}$. So $U_{a,F} \cap N_{L/F}(U_{1,L})) = U_{a,F} \cap N_{L/F}(U_{a,L})$ and $\lambda_a(U_{a,F} \cap N_{L/F}(U_{1,L})) =$

$\lambda_a(U_{a,F} \cap N_{L/F}(U_{a,L})) = g(\overline{F})$. Meanwhile, by the same logic, we also have that $\lambda_a(U_{a,F} \cap N_{\mathcal{L}/\mathcal{F}}(U_{1,\mathcal{L}})) = \overline{F} \cap g(\overline{\mathcal{F}})$ and $\lambda_a(U_{a,F} \cap N_{\widehat{L}/\widehat{F}}(U_{1,\widehat{L}})) = \overline{F} \cap g(\overline{\widehat{F}})$.

By Corollary 3.2.1 we have $\overline{\widehat{F}}$ is the separable p-closure of $\overline{F}$ and thus, since $g(X)$ is a separable polynomial over $\overline{F}$, we have that $g(\overline{\widehat{F}}) = \overline{\widehat{F}}$. This means that $\overline{F} \cap g(\overline{\widehat{F}}) = \overline{F}$. Since $\overline{\mathcal{F}}/\overline{F}$ is a purely inseparable extension and $g(X)$ is separable over $\overline{F}$, we get $\overline{F} \cap g(\overline{\mathcal{F}}) = g(\overline{F})$. These two facts tell us that we have that result that we wanted, since $\overline{F} \cap g(\overline{F}) = g(\overline{F})$.

Finally, let $i < a$. Again we have that, if $j < i$, then the map from $U_{j,L}/U_{j+1,L}$ to $U_{j,F}/U_{j+1,F}$ is an injective map, and the same thing holds for $N_{\mathcal{L}/\mathcal{F}}$ and $N_{\widehat{L}/\widehat{F}}$. This means that $\lambda_i(U_{i,F} \cap N_{L/F}(U_{1,L})) = \overline{F}^{p^n}$, where $p^n = [L : F]$. We also have that $\lambda_i(U_{i,F} \cap N_{\mathcal{L}/\mathcal{F}}(U_{1,\mathcal{L}})) = \overline{\mathcal{F}}^{p^n} \cap \overline{F}$ and $\lambda_i(U_{i,F} \cap N_{\widehat{L}/\widehat{F}}(U_{1,\widehat{L}})) = \overline{\widehat{F}}^{p^n} \cap \overline{F}$.

$\overline{\mathcal{F}}$ is a perfect field and thus $\overline{\mathcal{F}}^{p^n} = \overline{\mathcal{F}}$, and therefore $\overline{\mathcal{F}}^{p^n} \cap \overline{F} = \overline{F}$. Meanwhile, $\overline{\widehat{F}}/\overline{F}$ is a separable extension; as such if $\overline{\beta} \in \overline{\widehat{F}}$ is such that $\overline{\beta}^p \in \overline{F}$ then $\overline{\beta} \in \overline{F}$. This tells us that $\overline{\widehat{F}}^{p^n} \cap \overline{F} = \overline{F}^{p^n}$. Since we can conclude that $\overline{F} \cap \overline{F}^{p^n} = \overline{F}^{p^n}$, we again have the result that we wanted.

Putting the three above cases together tells us that:

$$U_{1,F} \cap N_{\mathcal{L}/\mathcal{F}}(U_{1,\mathcal{L}}) \cap N_{\widehat{L}/\widehat{F}}(U_{1,\widehat{L}}) \subseteq N_{L/F}(U_{1,L})$$

and we already know:

$$U_{1,F} \cap N_{\mathcal{L}/\mathcal{F}}(U_{1,\mathcal{L}}) \cap N_{\widehat{L}/\widehat{F}}(U_{1,\widehat{L}}) \supseteq N_{L/F}(U_{1,L})$$

This leads us to the conclusion:

$$U_{1,F} \cap N_{\widehat{L}/\widehat{F}}(U_{1,\widehat{L}}) \cap N_{\mathcal{L}/\mathcal{F}}(U_{1,\mathcal{L}}) = N_{L/F}(U_{1,L})$$

This is enough to show us that $\Psi_{L/F}$ is always an isomorphism in this case.

As L/F was an arbitrary extension we have that the map $\Psi_{L/F}$ is an isomorphism whenever L/F is a finite abelian totally ramified p-extension with one ramification jump. □

3.8 Extensions with Two Ramification Jumps

Suppose that L/F is a finite abelian totally ramified$p$-extension that has two ramification jumps. This means that there exists $b > a > 0$ such that, if $G = \mathrm{Gal}\,(L/F)$, we have $G = G_a \neq G_{a+1}$ and $G_b \neq 1$, while $G_{b+1} = 1$. Let M be the fixed field of G_b and let $[M : F] = p^{n_1}$, while $[L : M] = p^{n_2}$; for non-negative integers n_1 and n_2. We can use Theorem 3.6.1 to get the structure of $N_{L/M}(U_{1,L})$ and $N_{M/F}(U_{1,M})$. Finally, let $g_1(X)$ be the $\overline{F}$-decomposable polynomial that is the map from $U_{a,M}/U_{a+1,M}$ to $U_{a,F}/U_{a+1,F}$ induced by $N_{M/F}$, while $g_2(X)$ is the $\overline{F}$-decomposable polynomial that is the map from $U_{b,L}/U_{b+1,L}$ to $U_{b,M}/U_{b+1,M}$ induced by $N_{L/M}$. We get a and b from the aforementioned Theorem 3.6.1.

All of the notation in the previous paragraph will be fixed throughout our talk about extensions with two ramification jumps.

We will be using the fact that $N_{L/F} = N_{M/F} \circ N_{L/M}$, the fact that both L/M and M/F have only a single ramification jump and that we have $h_{L/F} = h_{L/M} \circ h_{M/F}$. This means we can use the mathematics we have already done, and the fact that $a < b$ to work out what $\lambda_i(N_{L/F}(U_{1,L}) \cap U_{i,F})$ is for any $i > 0$; we will also be able to do the same for $\mathcal{L}/\mathcal{F}$ and $\widehat{L}/\widehat{F}$.

Aim 3.8.1. *We want to find out for which be $i \geq 1$ we have the equality:*

$$\lambda_i(U_{i,F} \cap N_{L/F}(U_{1,L})) = \lambda_i(U_{i,F} \cap N_{\mathcal{L}/\mathcal{F}}(U_{1,\mathcal{L}})) \cap \lambda_i(U_{i,F} \cap N_{\widehat{L}/\widehat{F}}(U_{1,\widehat{L}})) \ (*)$$

Towards the Aim: We should first note that if $i > b$ then $N_{L/F}(U_{1,L}) \cap U_{i,F} = U_{i,F}$, and the same result holds for $\mathcal{L}/\mathcal{F}$ and $\widehat{L}/\widehat{F}$. we therefore trivially get what we want.

This means we just have to check what happens when $i \leq b$. We should remember that $h_{M/F}(i) = i$, for $i \leq a$, and $h_{M/F}(i) = a(1 - p^{n_1}) + p^{n_1}i$, for $i \geq a$. We get a similar formula for $h_{L/M}$.

Lemma 3.8.1. *If $i < a$, then $\lambda_i(U_{i,F} \cap N_{L/F}(U_{1,L}))$ satisfies the equality $(*)$.*

Proof. The first thing to say is that, because $i + 1 \leq a < b$, we have $h_{L/F}(i + 1) = h_{L/M} \circ h_{M/F}(i + 1) = i + 1$. This means, by the definition of $h_{L/F}$, that $N_{L/F}(U_{i+1,L}) \subseteq U_{i+1,F}$; thus $\lambda_i(N_{L/F}(U_{1,F}) \cap U_{i,F})$ depends only on $N_{L/F}(\alpha)$ for $\alpha \notin U_{i+1,L}$.

If $j \leq i$ then, as $j < a < b$, we know that the map from $U_{j,L}/U_{j+1,L}$ to $U_{j,M}/U_{j+1,M}$, induced by $N_{L/M}$, is $\overline{\theta} \mapsto \overline{\theta}^{p^{n_2}}$. Likewise, the map from $U_{j,M}/U_{j+1,M}$ to $U_{j,F}/U_{j+1,F}$, induced by $N_{M/F}$, is $\overline{\theta} \mapsto \overline{\theta}^{p^{n_1}}$. This makes the map from $U_{j,L}/U_{j+1,L}$ to $U_{j,F}/U_{j+1,F}$, induced by $N_{L/F}$, send $\overline{\theta}$ to $\overline{\theta}^{p^{n_1+n_2}}$, which is injective.

So, we see that if $\alpha \in U_{1,L}$, but not in $U_{j+1,L}$, then $N_{L/F}(\alpha) \notin U_{j+1,F}$. Therefore, we have that $\lambda_i(N_{L/F}(U_{1,L}) \cap U_{i,F}) = \overline{F}^{p^{n_1+n_2}}$. The same result comes about with $\mathcal{L}/\mathcal{F}$ and $\widehat{L}/\widehat{F}$. We can use the same mathematics that we used in the proof of Theorem 3.7.1 to get the equality that we want. □

Lemma 3.8.2. *If $i = a$, then $\lambda_i(U_{i,F} \cap N_{L/F}(U_{1,L}))$ satisfies the equality $(*)$.*

Proof. From the mathematics in the proof of Lemma 3.8.1 we know that $N_{L/F}(U_{1,L}) \cap U_{a,F} = N_{L/F}(U_{a,L})$. We also have that $h_{M/F}(a) = a$ and $h_{L/M}(a) = a$ which means that $h_{L/F}(a) = a$. This gives us that $\lambda_i(N_{L/F}(U_{1,L}) \cap U_{a,F})$ is isomorphic to the image of the map from $U_{a,L}/U_{a+1,L}$ to $U_{a,F}/U_{a+1,F}$, induced by $N_{L/F}$.

We next have that $N_{L/M}(U_{1,L}) \cap U_{a,M} = N_{L/M}(U_{a,L})$ and $N_{M/F}(U_{1,M}) \cap U_{a,F} = N_{M/F}(U_{a,M})$. Since $h_{L/M}(a) = h_{M/F}(a) = a$, we get that the map from $U_{a,L}/U_{a+1,L}$ to $U_{a,F}/U_{a+1,F}$, in the previous paragraph, is the map from $U_{a,L}/U_{a+1,L}$ to $U_{a,M}/U_{a+1,M}$, induced by $N_{L/M}$, followed by the map from $U_{a,M}/U_{a+1,M}$ to $U_{a,F}/U_{a+1,F}$, induced by $N_{M/F}$. The former map is $\overline{\theta} \mapsto \overline{\theta}^{p^{n_2}}$; while the latter is $\overline{\theta} \mapsto g_1(\overline{\theta})$. This gives us that the map from $U_{a,L}/U_{a+1,L}$ to $U_{a,F}/U_{a+1,F}$ is $\overline{\theta} \to g_1(\overline{\theta}^{p^{n_2}})$. Please note that we get the same map for the homomorphism from $U_{a,\mathcal{L}}/U_{a+1,\mathcal{L}}$ to $U_{a,\mathcal{F}}/U_{a+1,\mathcal{F}}$ and the homomorphism from $U_{a,\widehat{L}}/U_{a+1,\widehat{L}} \to U_{a,\widehat{F}}/U_{a+1,\widehat{F}}$.

This means that we are left trying to prove that $g_1(\overline{\mathcal{F}}^{p^{n_2}}) \cap g_1(\overline{\widehat{F}}^{p^{n_2}}) \cap \overline{F} = g_1(\overline{F}^{p^{n_2}})$. Obviously, we have that $g_1(\overline{F}^{p^{n_2}}) \subseteq g_1(\overline{\mathcal{F}}^{p^{n_2}}) \cap g_1(\overline{\widehat{F}}^{p^{n_2}}) \cap \overline{F}$, and thus we are trying to prove the opposite inclusion.

Now, the first thing to note is that $\overline{\mathcal{F}}$ is perfect and thus $\overline{\mathcal{F}}^{p^{n_2}} = \overline{\mathcal{F}}$, giving us that $g_1(\overline{\mathcal{F}}^{p^{n_2}}) = g_1(\overline{\mathcal{F}})$. We also know that $g_1(X)$ is a $\overline{F}$-decomposable polynomial. This means that it is a separable polynomial and, as $\overline{\mathcal{F}}/\overline{F}$ is a purely inseparable extension, we have that $g_1(\overline{\mathcal{F}}) \cap \overline{F} = g_1(\overline{F})$.

So, let γ be in $g_1(\overline{\mathcal{F}}^{p^{n_2}}) \cap g_1(\overline{\widehat{F}}^{p^{n_2}}) \cap \overline{F}$, which equals $g_1(\overline{F}) \cap g_1(\overline{\widehat{F}}^{p^{n_2}})$. So there is a $\rho_1 \in \overline{F}$ and a $\rho_2 \in \overline{\widehat{F}}$ such that $g_1(\rho_1) = g_1(\rho_2^{p^{n_2}}) = \gamma$. Now $g_1(X)$ is an additive polynomial and thus we have that $\rho_1 - \rho_2^{p^{n_2}}$ is a root of $g_1(X)$. However, $g_1(X)$ is $\overline{F}$-decomposable and thus $\rho_1 - \rho_2^{p^{n_2}} \in \overline{F}$. Since ρ_1 is also in $\overline{F}$, we have that $\rho_2^{p^{n_2}} \in \overline{F}$. It is also true that $\rho_2 \in \overline{\widehat{F}}$ and $\overline{\widehat{F}}/\overline{F}$ is a separable extension, and thus we have that $\rho_2 \in \overline{F}$. As γ was arbitrary, this shows us that:

$$g_1(\overline{\mathcal{F}}^{p^{n_2}}) \cap g_1(\overline{\widehat{F}}^{p^{n_2}}) \cap \overline{F} \subseteq g_1(\overline{F}^{p^{n_2}})$$

Which is precisely the inclusion that we wanted in order to finish the proof. □

3.9 The Final Values of i

We have reduced our work on Aim 3.8.1 to case when we are dealing with $\lambda_i(U_{i,F} \cap N_{L/F}(U_{1,L}))$ and $a < i \leq b$.

If $|M : F| = p^{n_1}$, then we know, from Proposition (5.7) Chapter **I** of [5], that, for $r > 0$, we have that the map induced from $N_{M/F}$ sends $U_{a+rp^{n_1},M}/U_{a+rp^{n_1}+1.M}$ surjectively onto $U_{a+r,F}/U_{a+r+1,F}$. The induced map from $U_{a+rp^t,M}/U_{a+rp^t+1,M}$ to $U_{a+r,F}/U_{a+r+1,F}$ sends $\bar{t}$ to $\eta\bar{t}$ for some non-zero fixed $\eta \in \overline{F}$. We also have that, if $p^{n_1} \nmid r$, then $N_{M/F}(U_{a+r,M}) = N_{M/F}(U_{a+r+1,M})$. Note, that this also holds if we replace the F and M by $\widehat{F}$ and $\widehat{M}$, or by $\mathcal{F}$ and $\mathcal{M}$, respectively.

Meanwhile, if $|L : M| = p^{n_2}$, for $r < h_{M/F}(b) - a$ we have that $N_{L/M}$ induces the injective map from $U_{a+r,L}/U_{a+r+1,L}$ to $U_{a+r,M}/U_{a+r+1,M}$ that sends $\bar{t}$ to $\bar{t}^{p^{n_2}}$. We also have that $N_{L/M}$ induces a map from $U_{h_{M/F}(b),L}/U_{h_{M/F}(b)+1,L}$ to $U_{h_{M/F}(b),M}/U_{h_{M/F}(b)+1,M}$ via the map that sends $\bar{t}$ to $g_2(\bar{t})$, for some $\overline{F}$-decomposable polynomial $g_2(X)$. Note, that this also holds if we replace the M and L by $\widehat{M}$ and $\widehat{L}$, or by $\mathcal{M}$ and $\mathcal{L}$, respectively.

We should point out that, by definition, $h_{M/F}(b) = a + (b-a)p^{n_1}$, which means that $N_{M/F}$ induces a map from $U_{h_{M/F}(b),M}/U_{a+rh_{M/F}(b)+1,M}$ to $U_{b,F}/U_{b+1,F}$ which sends $\bar{t}$ to $\eta\bar{t}$. We also have that $h_{L/M}(h_{M/F}(b)) = h_{M/F}(b)$ and thus $h_{L/F}(b) = h_{M/F}(b)$.

Lemma 3.9.1. *Let j be such that $a < j < h_{M/F}(b)$, and let $\bar{t}$ be in $\overline{\widehat{F}}^{p^{n_2}}$, and not be equal to 0. Finally, pick $\alpha \in \ker(N_{\widehat{M}/\widehat{F}} : U_{1,\widehat{M}} \to U_{1,\widehat{F}})$, such that $\alpha \in U_{j,\widehat{M}}$ and $\lambda_j(\alpha) = \bar{t}$.*

Now suppose we have the following property: For all j between a and $h_{M/F}(b)$, and all choices of $\bar{t}$ and α derived from j, there is $\beta \in N_{\widehat{L}/\widehat{M}}(U_{1,\widehat{L}}) \cap U_{j,\widehat{M}}$, such that $\lambda_j(\beta) = \bar{t}$ and $\beta \in \ker(N_{\widehat{M}/\widehat{F}} : U_{1,\widehat{M}} \to U_{1,\widehat{F}})$.

If the above property holds then we get the equality $(*)$ for $\lambda_i(N_{L/F}(U_{1,L}) \cap U_{i,F})$, with all i such that $a < i \leq b$. This gives us that in this case $\Psi_{L/F}$ is an isomorphism.

Proof. Suppose the property is true and let $p^{n_1} \nmid r$, with $0 < r \leq h_{L/F}(b) - a$. As $h_{L/F}(b) = h_{M/F} = a + (b-a)p^{n_1}$ we have that r is strictly less than $h_{L/F}(b) - a$. As $p^{n_1} \nmid r$, we have $N_{\widehat{M}/\widehat{F}}(U_{a+r,\widehat{M}}) = N_{\widehat{M}/\widehat{F}}(U_{a+r+1,\widehat{M}})$.

As such for all $\bar{t}$ in $\overline{\widehat{F}}^{p^{n_2}}$, not equal to 0, there is an $\alpha \in \ker(N_{\widehat{M}/\widehat{F}} : U_{1,\widehat{M}} \to U_{1,\widehat{F}})$, such that $\alpha \in U_{a+r,\widehat{M}}$ and $\lambda_{a+r}(\alpha) = \bar{t}$. From the statement of the lemma. we must have that there is a $\beta \in N_{\widehat{L}/\widehat{M}}(U_{1,\widehat{L}}) \cap U_{a+r,\widehat{M}}$, such that $\lambda_{a+r}(\beta) = \bar{t}$ and $\beta \in \ker(N_{\widehat{M}/\widehat{F}} : U_{1,\widehat{M}} \to U_{1,\widehat{F}})$.

Now, we know that $N_{\widehat{L}/\widehat{M}}(U_{1,\widehat{L}}) \cap U_{a+r,\widehat{M}} = N_{\widehat{L}/\widehat{M}}(U_{1,\widehat{M}})$ and $N_{\widehat{L}/\widehat{M}}$ induces the injective map from $U_{a+r,\widehat{L}}/U_{a+r+1,\widehat{L}}$ to $U_{a+r,\widehat{M}}/U_{a+r+1,\widehat{M}}$ that sends $\overline{t'}$ to $\overline{t'}^{p^{n_2}}$. Let γ' be in $U_{a+r,\widehat{L}}$, but not in $U_{a+r+1,\widehat{L}}$; as such that there is $\overline{t} \in \overline{\widehat{F}}^{p^{n_2}}$, not equal to 0, such that $\lambda_{a+r}(N_{\widehat{L}/\widehat{M}}(\gamma')) = \overline{t}$. Denote $N_{\widehat{L}/\widehat{M}}(\gamma')$ by γ.

The considerations that we had previously in this proof means that there is a $\beta \in N_{\widehat{L}/\widehat{M}}(U_{1,\widehat{L}}) \cap U_{a+r,\widehat{M}}$, such that $\lambda_{a+r}(\beta) = \overline{t}$ and $\beta \in \ker(N_{\widehat{M}/\widehat{F}} : U_{1,\widehat{M}} \to U_{1,\widehat{F}})$. Let $\beta' \in U_{1,\widehat{L}}$ be such that $N_{\widehat{L}/\widehat{M}}(\beta') = \beta$. We have, since $\beta \in \ker(N_{\widehat{M}/\widehat{F}} : U_{1,\widehat{M}} \to U_{1,\widehat{F}})$, that:

$$N_{\widehat{M}/\widehat{F}}(\gamma\beta^{-1}) = N_{\widehat{M}/\widehat{F}}(\gamma)N_{\widehat{M}/\widehat{F}}(\beta^{-1}) = N_{\widehat{M}/\widehat{F}}(\gamma)$$

We also get that $\gamma\beta^{-1} \in U_{a+r+1,\widehat{M}}$ and thus, by the fact that the map $N_{\widehat{L}/\widehat{M}}$ induced on $U_{a+r,\widehat{L}}/U_{a+r+1,\widehat{L}}$ is injective, $\gamma'(\beta')^{-1} \in U_{a+r+1,\widehat{L}}$. So, by the arbitrariness of the values chosen, then we get that $N_{\widehat{L}/\widehat{F}}(U_{a+r,\widehat{L}}) = N_{\widehat{L}/\widehat{F}}(U_{a+r+1,\widehat{L}})$ for all r such that $p^{n_1} \nmid r$ and $r \leq h_{L/F}(b) - a$.

If it is necessary we can iterate the above. After only a finite number of iterations we are left with only needing to look at $N_{\widehat{L}/\widehat{F}}(U_{h_{L/F}(b),\widehat{L}})$ and $N_{\widehat{L}/\widehat{F}}(U_{a+rp^{n_1},\widehat{L}})$, the latter when $0 < rp^{n_1} < h_{L/F}(b) - a$.

Please note that $\mathcal{F}$ has perfect residue field, and thus $\overline{\mathcal{F}}^{p^{n_2}} = \overline{\mathcal{F}}$. This means that the property of the lemma automatically holds if we replace $\widehat{F}$, $\widehat{M}$ and $\widehat{L}$ by $\mathcal{F}$, $\mathcal{M}$ and $\mathcal{L}$ respectively. We, therefore, get that we only need to look at groups of the same sort as the above when looking at the norm maps between these fields; it should be emphasised that the result of this paragraph requires no assumptions, beyond the basics of $\overline{F}$ having characteristic p and L/F being a finite abelian totally ramified p-extension with two ramification jumps, as it will always hold.

With $N_{\widehat{L}/\widehat{F}}(U_{a+rp^{n_1},\widehat{L}})$, we get that:

$$\lambda_{a+r}(N_{\widehat{L}/\widehat{F}}(U_{a+rp^{n_1},\widehat{L}})) = \lambda_{a+r}(N_{\widehat{L}/\widehat{F}}(U_{1,\widehat{L}}) \cap U_{a+r,\widehat{F}}) = \eta\overline{\widehat{F}}^{p^{n_2}}$$

for the $\eta \in \overline{\widehat{F}}$ mentioned at the beginning of this section.
Because $\eta \in \overline{F}$ and $\overline{\widehat{F}}^{p^{n_2}} \cap \overline{F} = \overline{F}^{p^{n_2}}$, the above gives us that:

$$\lambda_{a+r}(N_{\widehat{L}/\widehat{F}}(U_{1,\widehat{L}}) \cap U_{a+r,F}) = \eta\overline{F}^{p^{n_2}}$$

The previous group is equal to $\lambda_{a+r}(N_{L/F}(U_{1,L}) \cap U_{a+r,F})$.

We also have that:

$$\lambda_{a+r}(N_{\mathcal{L}/\mathcal{F}}(U_{a+rp^{n_1},\mathcal{L}})) = \eta\overline{\mathcal{F}}^{p^{n_2}} = \overline{\mathcal{F}}$$

as $\overline{\mathcal{F}}$ is perfect, so $\overline{\mathcal{F}}^{p^{n_2}} = \overline{\mathcal{F}}$. So then $\lambda_{a+r}(N_{\mathcal{L}/\mathcal{F}}(U_{a+rp^{n_1},\mathcal{L}}) \cap U_{a+r,F}) = \overline{F}$. This result gives us that:

$$\lambda_{a+r}(N_{\widehat{L}/\widehat{F}}(U_{1,\widehat{L}}) \cap N_{\mathcal{L}/\mathcal{F}}(U_{1,\mathcal{L}}) \cap U_{a+r,F}) = \lambda_{a+r}(N_{\widehat{L}/\widehat{F}}(U_{1,\widehat{L}}) \cap U_{a+r,F})$$

From the induced maps we have $\lambda_{a+r}(N_{L/F}(U_{1,L}) \cap U_{a+r,F}) = \eta\overline{F}^{p^{n_2}}$.

Combining the results obtained above yields:

$$\lambda_{a+r}(N_{L/F}(U_{1,L}) \cap U_{a+r,F}) = \lambda_{a+r}(N_{\widehat{L}/\widehat{F}}(U_{1,\widehat{L}}) \cap N_{\mathcal{L}/\mathcal{F}}(U_{1,\mathcal{L}}) \cap U_{a+r,F})$$

which is what we need to get the equality $(*)$.

Next, let us look at $N_{\widehat{L}/\widehat{F}}(U_{h_{L/F}(b),\widehat{L}})$. This gives us:

$$\lambda_b(N_{\widehat{L}/\widehat{F}}(U_{h_{L/F}(b),\widehat{L}})) = \lambda_b(N_{\widehat{L}/\widehat{F}}(U_{1,\widehat{L}}) \cap U_{b,\widehat{F}}) = \eta g_2(\overline{\widehat{F}})$$

for the $\overline{F}$-decomposable polynomial $g_2(X)$ and η we have seen before. We also have that:

$$\lambda_b(N_{\mathcal{L}/\mathcal{F}}(U_{h_{L/F}(b),\mathcal{L}})) = \eta g_2(\overline{\mathcal{F}})$$

and

$$\lambda_b(N_{L/F}(U_{h_{L/F}(b),L})) = \eta g_2(\overline{F})$$

Consider, since $\eta \in \overline{F}$ is non-zero, $(\eta g_2)(X)$ as its own $\overline{F}$-decomposable polynomial. We can use the same mathematics as in the proof of Theorem 3.7.1 to get the equality $(*)$ in this case.

Combining the cases we considered gives the equality $(*)$ for $\lambda_i(N_{L/F}(U_{1,L}) \cap U_{i,F})$, with all i such that $a < i \leq b$

Using the information of this proof, and the previous considerations of the other values of i we see that, in the case where L/F has two ramification jumps and has the property of the lemma, $\Psi_{L/F}$ is an isomorphism. □

Note 3.9.1. If we can verify the hypothesis of Lemma 3.9.1 then I think it is just a matter of tweaking them and applying that repeatedly to get that $\Psi_{L/F}$ is an isomorphism for all finite abelian totally ramified p-extensions; though this is just speculation at this point.

3.10 An Alternate Proof of Theorem 3.4.1

In this section we are going to prove something that was already shown to be true in section 3.3. Namely Theorem 3.4.1, which shows that $\Psi_{L/F}$ is always an isomorphism if $\mathrm{Gal}\,(L/F) \cong (\mathbb{Z}/p\mathbb{Z})^2$. This time, however, we will be using a different method which I hope can be expanded to more general extensions.

However, first we must establish a new result that does hold in general.

Lemma 3.10.1. *Let M/F be a totally ramified extension of degree p, and set $s = s(M/F)$. Let π_M be a fixed prime element of M and σ be a generator of* $\mathrm{Gal}\,(M/F)$. *Now, we know from section* (1.4) *Chapter **III** of [5], that $\pi_M^{\sigma-1} = 1 + \eta\pi_M^s$, for some $\eta \in U_M$ such that $\overline{\eta}$ is a non-zero element of $\overline{M} = \overline{F}$.*

Let L/M be an extension such that L/F is a finite abelian totally ramified p-extension. We shall assume that all of the ramification jumps of L/M are greater than s. Finally, set $|L : M|$ equal to p^t. We have that $\overline{\eta} \in \overline{F}^{p^t}$.

Proof. Since $p \geq 2$, we have $\lambda_{s+1}(N_{L/M}(U_{1,L}) \cap U_{s+1,M}) = \overline{F}^{p^t}$. This statement is because all the ramification jumps of L/M are of the form $s + pa$ for some $a > 0$. Therefore $s + 1$ is less than all the ramification jumps of L/M, which means that we get that $\lambda_{s+1}(N_{L/M}(U_{1,L}) \cap U_{s+1,M}) = \overline{F}^{p^t}$ from the work we did on the structure of norm groups in the proof of Theorem 3.6.1.

Let $\alpha \in U_{1,M} \cap N_{L/M}(U_{1,L})$ be such that $\alpha \notin U_{2,M}$. This means that $\lambda_1(\alpha) = \overline{\gamma} \in \overline{F}^{p^t}$, for $\overline{\gamma} \neq 0$.

We can extend σ to $\mathrm{Gal}\,(L/F)$, we call the extension σ as well, and in doing so we can see that since $\alpha \in N_{L/M}(U_{1,L})$, then so is $\alpha^{\sigma-1}$. Now, $p \nmid 1$ so $\alpha^{\sigma-1} \equiv 1 + \eta\gamma\pi_M^{s+1} \mod \pi_M^{s+2}$. Here γ is some element of U_F such that $\overline{\gamma}$ is the same as before.

Now, both $\overline{\eta}$ and $\overline{\gamma}$ are non-zero elements of $\overline{F}$ and thus so is $\overline{\eta\gamma}$. Therefore $\alpha^{\sigma-1} \notin U_{s+2,M}$ and, since $\alpha^{\sigma-1} \in N_{L/M}(U_{1,L})$, we have that $\overline{\eta\gamma} \in \overline{F}^{p^t}$. We have already established that $\overline{\gamma} \in \overline{F}^{p^t}$, and therefore $\overline{\eta} \in \overline{F}^{p^t}$ as well. □

Now it is time to prove, by a different method, Theorem 3.4.1 again.

Theorem 3.10.1. *Let L/F be a abelian totally ramified extension whose Galois group is isomorphic to $(\mathbb{Z}/p\mathbb{Z})^2$. Then $\Psi_{L/F}$ is an isomorphism.*

Proof. Since $\mathrm{Gal}\,(L/F) \cong (\mathbb{Z}/p\mathbb{Z})^2$ it has either one or two ramification jumps. If L/F has one ramification jump, then Theorem 3.7.1 tells us that $\Psi_{L/F}$ is an isomorphism so let us assume that L/F has two ramification jumps.

By the notation that we have using in the last few sections we know there exists a subextension of L/F, that we can call M/F, such that both L/M and M/F have degree p. We also have that $s(L/M) = b$ while $s(M/F) = a$, with $b > a$. The work of section 3.7 tells us that the only things we need to investigate are $\lambda_i(N_{L/F}(U_{h_{L/F}(i),L}))$ for i such that $a < i \leq b$.

Let σ be a generator of $\mathrm{Gal}\,(M/F)$ and denote an extension of σ to $\mathrm{Gal}\,(L/F)$ as σ as well. Finally, let π_M be a prime element of M, with $\pi_M^{\sigma-1} = 1 + \eta\pi_M^a$, for some $\eta \in U_M$.

We know from Proposition (4.1) Chapter **III** of [5] that, since M/F is cyclic, the kernel of the norm map $N_{\widehat{M}/\widehat{F}} : U_{1,\widehat{M}} \to U_{1,\widehat{F}}$ is equal to $\varepsilon^{\sigma-1}$ for $\varepsilon \in U_{1,\widehat{M}}$. We also have, by the diagrams of Proposition (1.5) Chapter **III** of [5], that $\lambda_{a+k} : N_{\widehat{M}/\widehat{F}}(U_{a+pk,\widehat{M}}) \to \overline{\widehat{F}}$ is injective for all positive integers k. This injectivity means that we have trivially fulfilled the property of Lemma 3.9.1 for the case of $a + pk$.

Now, fix $\overline{\gamma} \in \overline{\widehat{F}}^p$, with $\overline{\gamma}$ not being equal to 0. Next, let j be an integer such that $p \nmid j$ and $0 < j < h_{M/F}(b) - a$. We have $j < h_{M/F}(b)$, and thus there exists an $\alpha \in N_{\widehat{L}/\widehat{M}}(U_{1,\widehat{L}}) \cap U_{j,\widehat{M}}$ such that $\alpha \equiv \overline{\gamma}(\overline{\eta}j)^{-1} \mod U_{j+1,\widehat{M}}$. This holds because we established, in Lemma 3.10.1, that in this case $\overline{\eta} \in \overline{F}^p \subseteq \overline{\widehat{F}}^p$, so $\overline{\gamma}(\overline{\eta}j)^{-1} \in \overline{\widehat{F}}^p$.

As $\alpha \in N_{\widehat{L}/\widehat{M}}(U_{1,\widehat{L}})$, then so is $\alpha^{\sigma-1}$; we shall now label $\alpha^{\sigma-1}$ as β. We have that β is in the kernel of $N_{\widehat{M}/\widehat{F}} : U_{1,\widehat{M}} \to U_{1,\widehat{F}}$ and is in $U_{a+j,\widehat{M}}$, with $\beta \equiv \overline{\gamma}(\overline{\eta}j)^{-1}(\overline{\eta}j) \equiv \overline{\gamma} \mod U_{a+j+1,\widehat{M}}$. We should remember that j was an arbitrary positive integer less than $h_{M/F}(b) - a$, so $a < a + j < h_{M/F}(b)$, and j is not divisible by p. We also made $\overline{\gamma}$ an arbitrary element of $\overline{\widehat{F}}^p$.

What we have therefore shown is that for all such $0 < j < h_{M/F}(b) - a$ such that $\lambda_{a+j}(\ker(N_{\widehat{M}/\widehat{F}}(U_{1,\widehat{M}} \to U_{1,\widehat{F}}) \cap U_{a+j.\widehat{M}}) \neq 0$ and for all $\overline{\gamma} \in \overline{\widehat{F}}^p$ there exists a $\beta \in N_{\widehat{L}/\widehat{M}}(U_{1,\widehat{L}}) \cap U_{a+j,\widehat{M}}$ such that $\lambda_{a+j}(\beta) = \overline{\gamma}$ and $\beta \in \ker(N_{\widehat{M}/\widehat{F}} : U_{1,\widehat{M}} \to U_{1,\widehat{F}})$. This gives us the required property of Lemma 3.9.1 for $a+j$, with $p \nmid j$ and $0 < j < h_{M/F}(b) - a$. This combined with the work on $a+pk$ means that we have the property for all $a+j$, with $0 < j < h_{M/F}(b) - a$.

This means that the equality $(*)$ for $\lambda_i(N_{L/F}(U_{1,L} \cap U_{i,F})$ for all $a < i \leq b$. This was the last thing we needed to prove to get that $\Psi_{L/F}$ is an isomorphism.

□

3.10.1 This Method for Greater Extensions

The following subsections will be talking about applying the method used in Theorem 3.10.1 to the case of finite abelian totally ramified extensions L/F of degree p^3. We will also be assuming that there exists a subextension M/F of degree p^2, with M/F having a single ramification jump at $a \geq 1$ and L/M having a ramification jump at b, with $b > a$.

We want to know whether the method of the previous subsection can be extended in order to prove $\Psi_{L/F}$ is an isomorphism.

Let $M = M_2$ and make M_1/F a subextension of M_2/F such that M_2/M_1 and M_1/F both have degree p. Let σ_1 and σ_2 be generators of $\mathrm{Gal}\,(M_1/F)$ and $\mathrm{Gal}\,(M_2/M_1)$ respectively, and we will use the same notation to denote any extensions of those automorphisms.

We need to recall that:

$$s(M_1/F) = s(M_2/M_1) = a$$

and that $N_{\widehat{M_2}/\widehat{M_1}}(U_{a+pj,\widehat{M_2}}) = U_{a+j,\widehat{M_1}}$ and $N_{\widehat{M_2}/\widehat{M_1}}(U_{a+pj+1,\widehat{M_2}}) = U_{a+j+1,\widehat{M_1}}$ for any non-negative integer j.

If $p \nmid j$, then we know:

$$\lambda_{a+j}(\ker(N_{\widehat{M_1}/\widehat{F}}) \cap U_{a+j,\widehat{M_1}}) = \overline{\widehat{F}} = \lambda_{a+j}(\ker(N_{\widehat{M_2}/\widehat{M_1}}) \cap U_{a+j,\widehat{M_2}})$$

As such, since $N_{\widehat{M_2}/\widehat{F}} = N_{\widehat{M_2}/\widehat{F}} \circ N_{\widehat{M_2}/\widehat{M_1}}$, we get that:

$$\lambda_{a+pj}(\ker(N_{\widehat{M_2}/\widehat{F}}) \cap U_{a+pj,\widehat{M_2}}) = \overline{\widehat{F}} = \lambda_{a+j}(\ker(N_{\widehat{M_2}/\widehat{F}}) \cap U_{a+j,\widehat{M_2}})$$

as well.

Now look at $a+i$, with $0 < i < h_{M_2/F}(b) - a$ and $p \nmid i$. Let π_{M_2} be a prime element of M_2, and thus a prime element of $\widehat{M_2}$. We can use Lemma 3.10.1 on L/M_1, since $s(L/M_2) = b > a = s(M_2/M_1)$, to get that $\pi_{M_2}^{\sigma_2 - 1} = 1 + \eta_2 \pi_{M_2}^a$ for some $\eta_2 \in U_{M_2}$ such that $\overline{\eta_2} \in \overline{M}^p = \overline{F}^p$.

We can now use the same method in the proof of Theorem 3.10.1 on L/M_2, since it is an extension whose Galois group is isomorphic to $(\mathbb{Z}/p\mathbb{Z})^2$. We end up getting that for all $\overline{\gamma} \in \overline{\widehat{F}}^p$ there exists a $\beta \in N_{\widehat{L}/\widehat{M_2}}(U_{1,\widehat{L}}) \cap U_{a+i,\widehat{M_2}}$, such that $\lambda_{a+i}(\beta) = \overline{\gamma}$ and β is in the kernel of $N_{\widehat{M_2}/\widehat{M_1}}$. Now, $N_{\widehat{M_2}/\widehat{F}} = N_{\widehat{M_2}/\widehat{F}} \circ N_{\widehat{M_2}/\widehat{M_1}}$, so β is also in the kernel of $N_{\widehat{M_2}/\widehat{F}}$.

We have therefore shown the property of Lemma 3.9.1, with $U_{a+i,\widehat{M_2}}$ for $0 < i < h_{M_2/F}(b) - a$ and $p \nmid i$.

We do not need to worry about when the case of where $a+i$ is of the form $a+p^2j$, for some non-negative integer j such that $0 < p^2j < h_{M_2/F}(b)-a$, since we have that:

$$\lambda_{a+p^2j}(\ker(N_{\widehat{M_2}/\widehat{F}}) \cap U_{a+p^2j,\widehat{M_2}}) = 0$$

This means that we are left with trying to show the property of Lemma 3.9.1, with $a+i$ for $0 < i < h_{M_2/F}(b)-a$, where $p^2 \nmid i$ but $p \mid i$.

With that in mind, let $p \nmid j$ and let $\alpha_1 \in U_{j,\widehat{M_1}} \cap N_{\widehat{L}/\widehat{M_1}}(U_{1,\widehat{L}})$ be such that $\alpha_1 \notin U_{j+1,\widehat{M_1}}$. Let $\beta \in U_{1,\widehat{L}}$, be such that $N_{\widehat{L}/\widehat{M_1}}(\beta) = \alpha_1$ and let $\alpha_2 \in U_{1,\widehat{M_2}}$ equal $N_{\widehat{L}/\widehat{M_2}}(\beta)$.

Next, we know that $\alpha_1^{\sigma_1-1} \in U_{a+j,\widehat{M_1}}$ and that $\alpha_1^{\sigma_1-1} = N_{\widehat{L}/\widehat{M_1}}(\beta^{\sigma_1-1})$. We also see that $N_{\widehat{M_2}/\widehat{M_1}}(\alpha_2^{\sigma_1-1}) = \alpha_1^{\sigma_1-1}$ and $N_{\widehat{L}/\widehat{M_2}}(\beta^{\sigma_1-1}) = \alpha_2^{\sigma_1-1}$. Since we have $\alpha_1^{\sigma_1-1} \in U_{a+j,\widehat{M_1}}$ and not in $U_{a+j+1,\widehat{M_1}}$, as we know that $\alpha_1 \notin U_{j+1,\widehat{M_1}}$. we get that there exists a $0 \leq t < p$ such that $\alpha_2^{\sigma_1-1} \in U_{a+pj-t,\widehat{M_2}}$, while $\alpha_2^{\sigma_1-1}$ is not in $U_{a+pj-t+1,\widehat{M_2}}$.

If $t=0$, and thus we have that $\alpha_2^{\sigma_1-1} \in U_{a+pj,\widehat{M_2}}$, we can skip the next part. If, however $t > 0$ then we need to do a bit more work.

Since $0 < t < p$, we have that $p \nmid (pj-t)$. Likewise, since $\alpha_2^{\sigma-1} \in N_{\widehat{L}/\widehat{M_2}}(U_{1,\widehat{L}}) \cap U_{a+pj-t,\widehat{M_2}}$, we know that there is a $\gamma \in N_{\widehat{L}/\widehat{M_2}}(U_{1,\widehat{L}}) \cap U_{a+pj-t,\widehat{M_2}}$, such that $\lambda_{a+pj-t}(\gamma) = \lambda_{a+pj-t}(\alpha_2^{\sigma_1-1})$. We also have that $\gamma \in \ker(N_{\widehat{M_2}/\widehat{M_1}})$.

The aforementioned properties shows us that:

$$\alpha_2^{\sigma-1}\gamma^{-1} \in N_{\widehat{L}/\widehat{M_2}}(U_{1,\widehat{L}}) \cap U_{a+pj-t+1,\widehat{M_2}}$$

Likewise, the fact that γ is contained in the kernel of $N_{\widehat{M_2}/\widehat{M_1}}$ means that $N_{\widehat{M_2}/\widehat{M_1}}(\alpha_2^{\sigma_1-1}\gamma^{-1}) = \alpha_1^{\sigma_1-1}$. This gives us that $\alpha_2^{\sigma-1}\gamma^{-1} \notin U_{a+pj+1,\widehat{M_2}}$.

If $\alpha_2^{\sigma-1}\gamma^{-1}$ is still not in $U_{a+pj,\widehat{M_2}}$ then we can iterate the method we have just done with $\alpha_2^{\sigma-1}\gamma^{-1}$. After only a finite number of iterations we will end up with an $\alpha \in U_{a+pj,\widehat{M_2}} \cap N_{\widehat{L}/\widehat{M_2}}(U_{1,\widehat{L}})$ such that $N_{\widehat{M_2}/\widehat{M_1}}(\alpha) = \alpha_1^{\sigma_1-1}$.

Now, $N_{\widehat{M_1}/\widehat{F}}(\alpha_1^{\sigma_1-1}) = 1$, and thus $\alpha \in \mathrm{Im}\,(N_{\widehat{L}/\widehat{M_2}}) \cap \ker(N_{\widehat{M_2}\widehat{F}})$. We also have that $\alpha \in U_{a+pj,\widehat{M_2}}$ while also having $\alpha \notin U_{a+pj+1,\widehat{M_2}}$, which is precisely what we wanted.

Unfortunately, the mathematics that we have just done has a problem with the value of α; which means that it does not showcase the property of Lemma 3.9.1, like we wanted.

Let π_{M_1} be a fixed prime element of M_1 and $\eta_1 \in U_F$ be the element such that $\pi_{M_1}^{\sigma_1-1} = 1+\eta_1\pi_{M_1}^a$ Next, fix a prime element, π_{M_2}, of M_2 and let $\eta_2 \in U_F$ be the element such that $\pi_{M_2}^{\sigma_2-1} = 1+\eta_2\pi_{M_2}^a$.

By looking at L/M_2, we know that $\overline{\eta_2} \in \overline{F}^p \subseteq \overline{\widehat{F}}^p$. We also know that the map induced by $N_{\widehat{M_2}/\widehat{M_1}}$ on $U_{a+pj,\widehat{M_2}}/U_{a+pj+1,\widehat{M_2}}$ to $U_{a+j,\widehat{M_1}}/U_{a+j+1,\widehat{M_1}}$ is $\overline{\theta} \mapsto -\overline{\eta_2}^{p-1}\overline{\theta}$.

If we also have $\lambda_j(\alpha_1) = \overline{c} \in \overline{\widehat{F}}$, then $\lambda_{a+j}(\alpha_1^{\sigma_1-1}) = j\overline{a\eta_1}$. This means if $\alpha \in U_{a+pj,\overline{M_2}}$ is such that $N_{\overline{M_2}/\overline{M_1}}(\alpha) = \alpha_1^{\sigma_1-1}$, then $\lambda_{a+pj}(\alpha) = -j\overline{\eta_1}(\overline{\eta_2}^{1-p})\overline{c}$.

Let $j < a$, therefore, as $N_{\widehat{L}/\widehat{M_1}} = N_{\widehat{M_2}/\widehat{M_1}} \circ N_{\widehat{L}/\widehat{M_1}}$, we have $\overline{c} \in \overline{\widehat{F}}^{p^2}$. Now $-j\overline{\eta_1}(\overline{\eta_2}^{1-p})\overline{c} \in \overline{\widehat{F}}^p$ and thus $-j\overline{\eta_1}(\overline{\eta_2}^{1-p}) \in \overline{\widehat{F}}^p$. This problem is that $\overline{c} \in \overline{\widehat{F}}^{p^2}$ and $-j\overline{\eta_1}(\overline{\eta_2}^{1-p})\overline{\widehat{F}}^{p^2} \neq \overline{\widehat{F}}^p$.

This means that we fail to have the property we desire when dealing with $a + i = a + pj$, for $0 < j < a$ and $p \nmid j$. This is because we need to be able to construct such an α the correlates to every value over $\overline{\widehat{F}}^p$ in order fit the requirements of the property of Lemma 3.9.1.

The problem that we have identified comes about from requiring that $\alpha_2 \in N_{\widehat{L}/\widehat{M_2}}(U_{1,\widehat{L}})$. If $j < a$, this forces for us to have that $\lambda_j(\alpha_2) \in \overline{\widehat{F}}^p$. Since $N_{\widehat{M_2}/\widehat{M_1}}(\alpha_2) = \alpha_1$ and remembering the definition of $\overline{c}$, we have end up with $\overline{c} \in \overline{\widehat{F}}^{p^2}$.

3.10.2 A Possible Way Forward

There is a possible result, that if we could prove would deal with the problem of the method of the previous subsection.

Proposition 3.10.1. *Keeping the notation and mathematics of the last subsection. Suppose we could allow α_2 to be any element of $N^{-1}_{\widehat{M_2}/\widehat{M_1}}(\alpha_1)$, for $\alpha_1 \in U_{j,\widehat{M_1}}$ and $j < a$; this is instead of requiring that α_2 must be in $N_{\widehat{L}/\widehat{M_2}}(U_{1,\widehat{L}})$. Then we would get equality $(*)$ for all cases like we wanted.*

Proof. If we could let α_2 be any element of $N^{-1}_{\widehat{M_2}/\widehat{M_1}}(\alpha_1)$ we would have that $\overline{c}$ could be any value in $\overline{\widehat{F}}^p$. Keeping with the mathematics of the previous subsection would therefore mean that we could work over all of $-j\overline{\eta_1}(\overline{\eta_2}^{1-p})\overline{\widehat{F}}^p$, which does equal $\overline{\widehat{F}}^p$, rather than $-j\overline{\eta_1}(\overline{\eta_2}^{1-p})\overline{\widehat{F}}^{p^2}$. This will let us fulfil the property of Lemma 3.9.1 for the final values of $a + i$, and thus get the equality $(*)$ like we wanted. □

Unfortunately, it is not obvious how the acquire the requirements of the above lemma at the moment and it is probably time to move on to a slightly different topic.

3.11 Kummer Extensions of Degree p

In the following sections we will be looking at the next simplest type of abelian p-extensions after those of degree $(\mathbb{Z}/p\mathbb{Z})^2$. We will suppose F contains a primitive p'th-root of unity and let $L = F(\alpha_1^{1/p}, \alpha_2^{1/p}, \alpha_3^{1/p})$. Here the α_i are three members of F that are linearly independent in the $\mathbb{Z}/p\mathbb{Z}$ space F^*/F^{*p}. This will mean that L/F is a abelian totally ramified p-extension with Galois group isomorphic to $(\mathbb{Z}/p\mathbb{Z})^3$.

However, while we will eventually be combining Kummer Extensions we will do so only in the next section. For the rest of this section we shall be only looking at $L = F(\alpha^{1/p})$, for $\alpha \in F^*$, such that L/F is a totally ramified Galois extension of degree p.

We will note that several of the upcoming results, namely Lemmas 3.11.2, 3.11.3 and 3.11.5, are also proved in Professor Emeritus Bostwick F. Wyman's paper "Wildly Ramified Gamma Extensions" [16]. However, that paper makes the general assumption that we are only dealing with complete discrete valuation fields with perfect residue fields. As we are dealing with imperfect residue fields we are doing different proofs, for very similar results, here.

3.11.1 Basics of Such Extensions

Lemma 3.11.1. *Let F and the α be as described at the beginning of the section. We get that $L = F(\alpha^{1/p})$ is also equal to $F(\alpha'^{1/p})$, where $\alpha' = \beta'\pi_F^{n'}$; here $\beta' \in U_F$, the integer n' is equal to 0 or 1 and π_F is a prime element of F.*

Proof. Assume that $\alpha = \beta\pi_F^n$, where $\beta \in U_F$ and $n \in \mathbb{Z}$. By multiplying by powers of π_F^p, we can also assume that $\alpha \mod F^{*p} \equiv \beta\pi_F^{n'}$, where $\beta \in U_F$ and $0 \le n' \le p-1$. We therefore get that $L = F((\beta\pi_F^n)^{1/p})$.

Let $n' > 0$ and assume that $\beta \notin U_F^{n'}$. Since $0 < n' < p$, then there exists a $t > 0$ such that $n' \mid tp+1$. This gives us $F((\beta\pi_F^{n'})^{1/p}) = F((\beta^{tp+1}\pi_F^{n'})^{1/p})$, and the same holds if F is replaced by an extension of F, and thus we may substitute β with $\beta' \in U_F^{n'}$.

So then $\beta' = \gamma^{n'}$ and thus $L = L(((\gamma\pi_F)^{n'})^{1/p})$. We have $(\beta'\pi_F^{n'})^{1/p} \in F((\gamma\pi_F)^{1/p})$ and as such $F((\beta'\pi_F^{n'})^{1/p}) \subseteq F((\gamma\pi_F)^{1/p})$. We have that the field extension $F((\beta'\pi_F^{n'})^{1/p})/F$ has degree p, while $F((\gamma\pi_F)^{1/p})/F$ has degree at most p. From that we get that $F((\beta'\pi_F^{n'})^{1/p}) = F((\gamma\pi_F)^{1/p})$.

If we let $n' = 0$ we cover the other option brought up in the lemma's statement.

□

From now on we will be assuming that α is of the form $\beta\pi_F^n$ with $\beta \in U_F$ and n equal to 0 or 1.

Note 3.11.1. For any extension L'/F, as long as $(\beta\pi_F^n)^{1/p} \notin L'$ and $n > 0$, we have that $L'((\beta\pi_F^n)^{1/p}) = L'((\gamma)\pi_F)^{1/p})$. This means that we can keep making the assumption of the above lemma when we get to the composition of Kummer Extensions in the next section.

3.11.2 p'th-root of Prime Element Extensions

Lemma 3.11.2. *Let n, as defined in the previous subsection, be equal to 1. This means that α is a prime element of F. Then if we let $L = F(\alpha^{1/p})$ we have that $s(L/F) = ep/(p-1)$, here $e = e(F) = \nu_F(p)$.*

Proof. From the restrictions we have that $\alpha = \beta\pi_F$ is a prime element of F. So if we let $L = F(\alpha^{1/p})$ we have that $\alpha^{1/p} = \pi_L$ is a prime element of L.

If σ is a generator of $\mathrm{Gal}\,(L/F)$, we know that there is a primitive p'th-root of unity ε such that $\sigma(\pi_L) = \varepsilon\pi_L$. This gives us that $s = s(L/F)$ is equal to the valuation in L of $\varepsilon\pi_L/\pi_L = \varepsilon$. Now $\nu_F(\varepsilon - 1) = e/(p-1)$, where $e = e(F)$, so, as L/F is a totally ramified extension of degree p, we have that $\nu_L(\varepsilon - 1) = ep/(p-1)$.

This means that if $n = 1$ we have that $s = ep/(p-1)$. This is the result we were after. □

3.11.3 p'th-root of Unit Extensions

Let F, α, L and n be the same as in the rest of this section.

Lemma 3.11.3. *Suppose that $n = 0$, this means that $\alpha \in U_F$; we then may assume that $\alpha \in U_{1,F}$.*

Proof. Assume that $\alpha \notin U_{1,F}$, then $\overline{\alpha} \neq 1$ in $\overline{F}$. We have $\overline{\alpha} \in \overline{F}^p$, as otherwise the field $L = F(\alpha^{1/p})$ would have a larger residue field than F which would contradict that L/F is totally ramified. Let $\gamma \in F$ be such that $\overline{\gamma} = \overline{\alpha}^{1/p}$, then $\overline{\gamma^p} = \overline{\alpha}$. This means that $\alpha/\gamma^p \in U_{1,F}$ and as $\gamma^p \in F^{*p}$ we have that $F(\alpha^{1/p}) = F((\alpha/\gamma^p)^{1/p})$. This means that we may relabel α/γ^p as α and state that $\alpha \in U_{1,F}$. □

Lemma 3.11.4. *Keeping the notation of the previous lemma, let* $L = F(\alpha^{1/p})$*. We can always find a* $\gamma \in L$ *such that* $F(\gamma) = L$*, with* $\gamma^p \in U_{1,F}$ *and* $p \nmid \nu_L(\gamma - 1)$*.*

Proof. Let $\gamma = 1 + t'\pi_L^{n'}$ with $\gamma^p = \alpha$, here π_L is a prime element of L and t is in U_L. Suppose that $p \mid n'$.

As $\nu_L(\pi_F) = p$, we can write γ as $1 + t'\pi_F^m$, with $m \geq 1$ and t' being a unit of L. We have that L/F is totally ramified so there is a $k \in F$ such that $\overline{k} = \overline{t'}$. We then get that $\gamma = (1 + k\pi_F^m)(1 + k'\pi_L^{m'})$, with $k' \in U_L$ and $m' > n'$. Then we may take γ to be $1 + k'\pi_L^{m'}$, instead, and iterate if we still have that $p \mid m'$. Eventually we will end up with $L = F(\gamma')$, with $\gamma' \in U_{m'',L}$ but not in $U_{m''+1,L}$ and $p \nmid m''$, if we do not then as F is a complete field, then by taking the limit, we will get that the original γ was in F giving us that $L = F$ which contradicts the fact that L/F has degree p.

What the above means is that we may take $\gamma = 1 + tpi_L^{n'}$ and assume $p \nmid n'$ from now on without losing anything.

□

Lemma 3.11.5. *Still keeping the notation that we have been using, let* $L = F(\alpha^{1/p})$*, here* $\alpha^{1/p} = \gamma = 1 + t\pi_L^{n'}$*, with* $t \in U_L$*, and* $p \nmid n'$*. Let* σ *be a generator of* Gal (L/F) *and* $e = e(F)$*. Then we have that* $n' < ep/(p-1)$ *and* $s(L/F) = ep/(p-1) - n'$*.*

Proof. As $\gamma^p = \alpha$ we have that $\sigma(\gamma) = \varepsilon\gamma$, here ε is a primitive p'th-root of 1.

$n' = \nu_L(\gamma - 1)$ and $p \nmid n'$ and so, by section (1.4) Chapter **III** in [5], we have that:

$$s(L/F) = \nu_L\left(\frac{\sigma(\gamma - 1)}{\gamma - 1} - 1\right)$$

We know that $\nu_L(\varepsilon - 1) = ep/(p-1)$, with $\nu_L(\gamma) = 1$, and that:

$$s(L/F) = \nu_L\left(\frac{\sigma(\gamma - 1)}{\gamma - 1} - 1\right) = \nu_L\left(\frac{\gamma \cdot (\varepsilon - 1)}{\gamma - 1}\right) = \nu_L(\varepsilon - 1) - \nu_L(\gamma - 1)$$

since $\nu_L(a \cdot b) = \nu_L(a) + \nu_L(b)$. This gives us that:

$$s(L/F) = ep/(p-1) - \nu_L(\gamma - 1) = ep/(p-1) - n'$$

We have that $s(L/F) \geq 1$, and thus $n' < ep/(p-1)$. □

Theorem 3.11.1. *Keeping the notation of the previous lemma we can also assume that the* α*, such that* $L = F(\alpha^{1/p})$*, has the property that* $\nu_F(\alpha - 1) = n'$*.*

Proof. We are keeping ε as a primitive p'th-root of unity. This means that ε^i, for $1 \leq i < p$, is also a primitive p'th-root of unity and thus $\nu_L(\varepsilon^i - 1) = \nu_L(\varepsilon - 1) = ep/(p-1)$. This gives us that $n' = \nu_L(\gamma - 1) < \nu(\varepsilon^i - 1)$ for $1 \leq i < p$. This means that for all $1 \leq i < p$ we have, from how the valuation on L works, that:

$$\nu_L(\gamma - \varepsilon^i) = \nu_L((\gamma - 1) - (\varepsilon^i - 1)) = \nu_L(\gamma - 1)$$

As a reminder, we have that $\nu_L(a \cdot b) = \nu_L(a) + \nu_L(b)$ and, that for all $\beta \in F$, we get $\nu_L(\beta) = p\nu_F(\beta)$. We also have that the polynomial $X^p - 1 \in L[X]$ is equal to $\prod_{i=1}^p (X - \varepsilon^i)$. Taking all this in mind we get that:

$$\nu_F(\alpha - 1) = \frac{1}{p}\nu_L(\gamma^p - 1)$$

$$= \nu_L\left(\prod_{i=1}^{p} \gamma - \varepsilon^i\right) = \frac{1}{p}\sum_{i=1}^{p} \nu_L(\gamma - \varepsilon^i) = \frac{1}{p} \cdot p \cdot \nu_L(\gamma - 1) = n'$$

just like we wanted. □

3.12 Combining p'th-root Extensions

Use e to denote $e(F)$ again, let α_1 and α_2 be in F. Makes $L_i = F(\alpha_i^{1/p})$ a totally ramified extension of F of degree p for $i = 1$ and 2. Finally, set L/F to be a abelian totally ramified extension of degree p^2 where $L = L_1L_2$. Let $\alpha_1 = \pi_F$, a prime element of F. From the work we did in the previous section we may assume that either $\alpha_2 = k\pi_F$, for some $k \in U_F$, or that $\alpha_2 \in U_{n,F}$ for some $n < ep/(p-1)$, where $p \nmid n$.

Please note, that we do not need the two options for α_2. If $\alpha_1 = \pi_F$ and $\alpha_2 = k \in U_{n,F}$, for some $n < ep/(p-1)$ and where $p \nmid n$, then we can see that $F((k\pi_F)^{1/p})/F$ is an extension of degree p that is a subextension of $L_1L_2/F = L/F$. Assuming that L/F is totally ramified then so is $F((k\pi_F)^{1/p})/F$ and since $k^{1/p} \in L(\pi_F^{1/p})F((k\pi_F)^{1/p})$, we can see that $L_1F((k\pi_F)^{1/p}) = L$ as well. This means that we can relabel $k\pi_F$ as α_2 and $F((k\alpha_F)^{1/p})$ as L_2 to get that both α_1 and α_2 are prime elements of F.

Theorem 3.12.1. *Assume that the α_1 and α_2 above are both prime elements of F. Then we have that $s(L/L_1) = ep^2/(p-1) - (n+ep)$, here n is a positive integer less than $ep/(p-1)$ which is not divisible by p.*

Proof. Note that $L = L_1(\alpha_2^{1/p})$; so if $\alpha_2 = k\pi_F$, then $L = L_1(k^{1/p}\pi_F^{1/p})$. This means that $L = L_1(k^{1/p})$, and $\pi_F^{1/p} \in L_1$. Let $L_3 = F(k^{1/p})$, here L_3/F is a subextension of L/F and is thus totally ramified. $k \in U_F$ and thus L_3/F has degree 1 or p. However as $L = L_1L_3$ and $L \neq L_1$ we conclude that L_3/F is a non-trivial extension and thus a totally ramified extension of degree p. Since $L = L_1L_3$ and $L_3 = F(k^{1/p})$, we may therefore replace $\alpha_2 = k\pi_F$ by k and relabel k as α_2 and L_3 as L_2 without changing L. This means, using the results of the previous subsection, that without changing L_2 we can always assume that $\nu_F(\alpha_2 - 1) = n$ for some $n < ep/(p-1)$ and that $p \nmid n$.

Using Lemmas 3.11.2 and 3.11.5, we have that $s_1 = s(L_1/F) = ep/(p-1)$ and $s_2 = s(L_2/F) = ep/(p-1) - n$. We have that $s_1 > s_2$ and that L/F is a totally ramified Galois Extension. Since L_1/F and L_2/F has degree p while L_1L_2/F has degree p^2, we also have that $L_1 \cap L_2 = F$. Using Lemma 3.4.3, we have $s(L/L_1) = s_2 = ep/(p-1) - n$. Now $L = L_1(\alpha_2^{1/p})$ and L_1/F is a totally ramified extension of degree p, which gives us that $e(L_1) = pe(F) = pe$. Since $\alpha_2 \in U_{n,F}$, we have $\alpha_2 \in U_{pn,L_1}$. As $p \mid pn$, we know there is a $\beta_2 \in U_{m,L_1}$ but not in U_{m+1,L_1}, where $m < ep^2/(p-1)$ and $p \nmid m$. This β has the gives us $L = L_1(\beta_2^{1/p})$. We have $s(L/L_1)$ is equal to $ep^2/(p-1) - m$ but is also equal to $s_2 = ep/(p-1) - n$. This gives us that:

$$m = n + ep^2/(p-1) - ep/(p-1) = n + ep$$

Note that as $p \nmid n$ we have that $p \nmid (n+ep)$ as well. □

3.12.1 Applying these Two Sections to the Map $\Psi_{L/F}$

There is a problem with using the mathematics of these last two sections to deal with the issue of the map $\Psi_{L/F}$. Please note that since we do not actually prove anything concrete the following mathematics shall be in the form of a note, rather than a lemma or something like that.

Note 3.12.1. Let L/F be a totally ramified extension with Galois group isomorphic to $(\mathbb{Z}/p\mathbb{Z})^3$, such that $L = L_1L_2L_3$. Here $L_i = F((k_i\pi_F)^{1/p})$, where $k_i \in U_F$ and π_F is a prime element of F. This means that L is F extended by three p'th-roots of prime elements of F.

We know, from Lemma 3.11.2, that $s_i = s(L_i/F) = e(F)p/(p-1)$, for all i. Let σ_i be a generator of $\mathrm{Gal}\,(L_i/F)$ and label σ_i extension to $\mathrm{Gal}\,(L/F)$ with σ_i'. Label the fixed field of σ_i' in L as L_i'. Let π_L be a prime element of L. The work we have done on whether $\Psi_{L/F}$ is isomorphic or not has depended on $s_i' = s(L/L_i')$, this is dependent on $\sigma_i'(\pi_L)/\pi_L$. We have shown that if the s_i' are all equal then $\Psi_{L/F}$ is an isomorphism.

However, despite the fact that the s_i are all equal and they are dependent on $\sigma_i(\pi_{L_i})/\pi_{L_i} = \sigma_i'(\pi_{L_i})/\pi_{L_i}$, where π_{L_i} is a prime element of L_i, this tells us nothing about the nature of the s_i' and whether they are equal or not.

We can relabel $k_1\pi_F$ as π_F and assume that $L_2 = F((k_2\pi_F)^{1/p})$, with $k_2 \in U_{n_3,F}$ and $n_3 < e(F)p/(p-1)$. From the mathematics we have done before, and Lemma 3.4.3, we can look at $L_1L_2 = L_3'$. What we have, since $s(L_1/F) = s(L_2/F) == e(F)p/(p-1)$, is that:

$$s(L_3'/L_1) = s(L_3'/L_2) = e(F)p/(p-1) - n_3$$

By the same reasoning that exists positive integers n_1 and n_2, less than $e(F)p/(p-1)$, such that:

$$s(L_1'/L_2) = s(L_1'/L_3) = e(F)p/(p-1) - n_1$$

and:

$$s(L_2'/L_1) = s(L_2'/L_3) = e(F)p/(p-1) - n_2$$

Now, L/L_i' is a totally ramified extension of degree p, and so is L_i'/L_j for $j \neq i$. We have $L_2'L_3' = L$, so to work out s_2' and s_3' we need to apply Lemma 3.4.3 to the extension L/L_1. We have $s(L_2'/L_1) = e(F)p/(p-1) - n_2$ and $s(L_3'/L_1) = e(F)p/(p-1) - n_3$.

However, despite the fact that we started with three prime elements of F, and thus the $s(L_i/F)$ were all equal to each other, we will notice that that tells us nothing about the values of the n_i. We do not know anything, other than the most generic facts, about $e(F)p/(p-1) - n_2$ and $e(F)p/(p-1) - n_3$ or how they relate to each other. This means that we cannot use them and Lemma 3.4.3 to work out the value of $s(L/L_2')$ or $s(L/L_3')$. Likewise, since we do not know enough information pertaining to $e(F)p/(p-1) - n_1$ we cannot use it to work out $s(L/L_1')$.

Since Theorem 3.7.1 tells us that $\Psi_{L/F}$ is an isomorphism if $s(L/L_1') = s(L/L_2') = s(L/L_3')$, and we have failed to conclude on the veracity of the statement otherwise, we can see that with what we have found out we cannot tell whether $\Psi_{L/F}$ is an isomorphism or not in this case.

It is not clear how working only with such extensions can be applied to the mathematics that we have done early in this chapter. This means that we probably cannot use it to expand the collection of extensions L/F for which we know whether $\Psi_{L/F}$ is an isomorphism, or even to tell us more about the nature of $\Psi_{L/F}$.

This is all we are going to discuss about this topic at the moment. Please check the chapter named "Conclusion" for further directions one could take in their study of whether $\Psi_{L/F}$ is an isomorphism in the case when you are dealing with extensions over complete discrete valuation fields with imperfect residue fields.

Chapter 4

Abelian Varieties over Local Fields

This chapter is about the norm map for abelian varieties with ordinary good reduction over complete discrete valuation fields. It is related to and is an attempt to further extend Jonathan Lubin and Michael I. Rosen's 1977 paper "The Norm Map for Ordinary Abelian Varieties" [7]. We will endeavour to generalise the main result of that paper, which is Lubin and Rosen reproving Mazur's Proposition 4.39, from the paper "Rational Points of Abelian Varieties with Values in Towers of Number Fields" [8], to a wider collection of complete discrete valuation fields.

We have briefly gone over the results of [7] in section 4 of the "Literature Review". We are attempting to prove a related result that does not build directly on the results of that paper, instead it uses a very similar proof, so it is not actually necessary to be familiar with [7] or its results to understand this chapter.

In the original paper Lubin and Rosen did not reference local class field theory; this is despite the fact that the topic involves Galois extensions of complete discrete valuation fields, and one of the results they get can be seen as an analogue to a major result of local class field theory, as we mention in the "Conclusion". We, however, shall be burrowing ideas from the topic, in particular from Professors Fesenko and Vostokov's book "Local Fields and their Extensions"; in which the first three parts of Chapter 5 has a brief introduction to local class field theory when the complete discrete valuation field has a quasi-finite residue field [5], as described in section 1 of the "Literature Review". We will also be utilising Fesenko's paper "Local Class Field Theory: Perfect Residue Field Case", which we discussed in section 2 of the "Literature Review".

As a reminder, this is the result that we want to generalise [7]:

Theorem 4.0.1. *Fix a prime number p and let F be a finite extension of $\mathbb{Q}_p$. Next, let A be a d-dimensional abelian variety with good ordinary reduction over F and let u be a twist matrix of A, as explained in Definition* 2.5.3. *Denote by $A(\overline{F})_p$ the group of p-torsion points of $A(\overline{F})$, so the elements of $A(\overline{F})$ whose order is a finite power of p. Meanwhile, $A(\overline{F})$ is the group of the $\overline{F}$-valued points of the reduced abelian variety $\overline{A}$, which is defined over $\overline{F}$.*

If L/F is a totally ramified $\mathbb{Z}_p$-extension of F with:

$$N_{L/F}(A(L)) = \bigcap N_{L_n/F}(A(L_n))$$

where $L/L_n/F$ and $[L_n : F] = p^n$, then the following exact sequence can be constructed:

$$\mathbb{Z}_p^d/((I-u)\mathbb{Z}_p^d) \to A(F)/N_{L/F}(A(L)) \to A(\overline{F})_p \to 1$$

Now that we have established the result that we want to generalise we shall being by note that we will from now on be talking F being a complete discrete valuation field of characteristic 0, such that $\overline{F}$ is a perfect field of characteristic $p > 0$. This is instead of assuming that $F/\mathbb{Q}_p$ is a finite extension, which would force $\overline{F}$ to be finite.

Note 4.0.1. We should observe that as we are trying to generalise the main result of [7] we shall be following the steps, and trying to adapt each of them, that that paper uses on the way to its principle conclusion.

4.1 Notation and Definitions

Before we can start generalising Theorem 4.0.1 we first must define more of the notation, and a couple of definitions, that we will be using. Despite this being connected to [7], we will not necessarily be using the same notation that Lubin and Rosen uses, since they unfortunately differ from the symbols we have been using in this regard and it is better for this document to be internally consistent.

Notation 4.1.1. As mentioned previously, we will let F be a complete discrete valuation field with a perfect residue field of positive characteristic p. Like in the previous chapters we will define $\widehat{F}/F$ be the maximal unramified p-extension of F. Next, let A be a d-dimensional abelian variety with good ordinary reduction over F such that the formal goup associated to A, denoted by $\widehat{A}$, has an isomorphism to $\widehat{\mathbb{G}}_m^d$ over $\mathcal{O}_{\widehat{F}}$.

Similar to when we were dealing with local class field theory over fields with perfect residue fields, instead of using $\mathrm{Gal}\,(L/F)$ we shall be using $\mathrm{Gal}\,(L/F)^{\wedge} = \mathrm{Hom}_{\mathbb{Z}_p}(\mathrm{Gal}\,(\widehat{F}/F), \mathrm{Gal}\,(L/F))$.

Next, we will let E/F be either an arbitrary algebraic extension of F or the completion of one, either way E is a discrete valuation ring.

Next, let H be a d-dimensional formal group over $\mathcal{O}_F$ and suppose there exists an isomorphism over $\mathcal{O}_{\widehat{F}}$ written as $k : H \to \widehat{\mathbb{G}}_m^d$, as explained in Note 4.1.1.

Finally, if L/F is a totally ramified Galois extension then we may extend any of the $\phi \in \mathrm{Gal}\,(\widehat{F}/F)$, to $L\widehat{F}$ by having ϕ act trivially on elements of L. This extension shall also be labelled ϕ.

The use of $\mathrm{Gal}\,(L/F)^{\wedge}$, mentioned above, leads to our first major problem when it comes to generalising Theorem 4.0.1.

Note 4.1.1. We should point out that Lemma 4.27 from [8], which Theorem 4.0.1 relies on, states that there is definitely an isomorphism from $\widehat{A}$ to $\widehat{\mathbb{G}}_m^d$ only if we are over $\mathcal{O}_{F^{\mathrm{ur}}}$, where F^{ur} is the maximal unramified extension of F. We are only dealing with $\widehat{F}$ and $\overline{\widehat{F}}$ may not be algebraically closed and thus Lemma 4.27 cannot be used. This unfortunately leads us to have to restrict the type of abelian varieties over F that we can talk about rather than all of them that have good ordinary reduction, like Lubin and Rosen could do.

Definition 4.1.1. Keep the notation that we have defined above and choose a $\phi \in \mathrm{Gal}\,(\widehat{F}/F)$. We have that k can be described by d power series and by applying ϕ to the coefficients of all the power series we end up with the new isomorphism of formal groups $k^{\phi} : H \to \widehat{\mathbb{G}}_m^d$. The automoprhism group of $\widehat{\mathbb{G}}_m^d$ is isomorphic to $GL_d(\mathbb{Z}_p)$ and so fix an isomorphism between the two groups. Next, Let u_ϕ be the invertible $d \times d$ matrix over $\mathbb{Z}_p$ that corresponds to the automorphism of $\widehat{\mathbb{G}}_m^d$, which is $k^{\phi} \circ k^{-1}$. The u_ϕ is called a twist matrix of H.

4.1.1 The Module $V(L)$

The following set is an adaptation of the set, $V_u(L)$, we saw in Definition 2.5.4.

Definition 4.1.2. Keeping the notation and assumptions of Definition 4.1.1, let L/F be a totally ramified Galois extension, and denote $\mathrm{Gal}\,(L/F)$ by G. By having the elements of G fix the elements of $\widehat{F}$ we may extend them to $L\widehat{F}$ and see that $G \cong \mathrm{Gal}\,(L\widehat{F}/\widehat{F})$, we will also refer to the latter group as G. Next, extend the $\phi \in \mathrm{Gal}\,(\widehat{F}/F)$ also to to $L\widehat{F}$, by having them fix elements of L. We may define the group:

$$V(L) = V_u(L) = \{\alpha \in U^d_{1,L\widehat{F}} : \alpha^\phi = \alpha^{u_\phi}, \forall \phi \in \mathrm{Gal}\,(\widehat{F}/F)\}$$

For the sake of notation, we shall denote $V_u(L)$ as $V(L)$ when the u_ϕ we are using is obvious.

We have that ϕ acts diagonally on α and u_ϕ acts in the obvious way.

Lemma 4.1.1. *Keeping the notation of the above definition we have that $V(L)$ is a G-module. What is more $H(\mathcal{O}_L) \cong V(L)$ as G-modules.*

Proof. The actions of the ϕ commute with the elements of G, which gives us that $V(L)$ is a G-module.

Let $k : H \to \widehat{\mathbb{G}}^d_m$ be an isomorphism over $\mathcal{O}_{\widehat{F}}$ such that $u_\phi = k^\phi \circ k^{-1}$ for all $\phi \in \mathrm{Gal}\,(\widehat{F}/F)$; we know such an k exists from the definition of the u_ϕ. We have that k induces a group isomorphism between $H(\mathcal{O}_{L\widehat{F}})$ and $U^d_{1,L\widehat{F}}$. The d power series that represent k all have coefficients in $\widehat{F}$ which are, by definition, fixed by the elements of G; this means k is a G-module isomorphism.

Now, let α be in $H(\mathcal{O}_{L\widehat{F}})$ and suppose $k(\alpha) \in V(L)$; we have that for all $\phi \in \mathrm{Gal}\,(\widehat{F}/F)$:

$$k(\alpha)^\phi = k(\alpha)^{u_\phi} = k^\phi \circ k^{-1} \circ k(\alpha) = k^\phi(\alpha)$$

By definition, $k(\alpha)^\phi = k^\phi(\alpha^\phi)$. Since k^ϕ is an isomorphism then the fact we have $k^\phi(\alpha^\phi) = k^\phi(\alpha)$ means that $\alpha^\phi = \alpha$. So $k(\alpha) \in V(L)$ tells us that $\alpha^\phi = \alpha$ for all $\phi \in \mathrm{Gal}\,(\widehat{F}/F)$.

Conversely, if $\alpha \in H(\mathcal{O}_{L\widehat{F}})$ is such that $\alpha^\phi = \alpha$, for all $\phi \in \mathrm{Gal}\,(\widehat{F}/F)$, we have:

$$k(\alpha)^\phi = k^\phi(\alpha)^\phi = k^\phi(\alpha) = k^\phi \circ k^{-1} \circ k(\alpha) = k(\alpha)^{u_\phi}$$

and thus $k(\alpha) \in V(L)$.

To finish the proof we note that $\alpha \in H(\mathcal{O}_L)$ if and only if $\alpha^\phi = \alpha$, for all $\phi \in \mathrm{Gal}\,(\widehat{F}/F)$. We, therefore, have that $k(\alpha) \in V(L)$ if and only if $\alpha \in H(\mathcal{O}_L)$. □

4.2 Generalising "Theorem 1"

Readers who are familiar with [7] will know that the authors of that paper utilise the following theorem, which they call "Theorem 1", as a major stepping stone for the proof of Theorem 4.0.1. This has already been explained in the "Literature Review", as Lemma 2.5.3, but we shall reiterate the result here as it is very important to the work we are about to do.

Theorem 4.2.1. *Let F be a finite extension of $\mathbb{Q}_p$ and let A be a d-dimensional abelian variety over F with good ordinary reduction. Denote by $\widehat{A}$ the formal group over $\mathcal{O}_F$ that relates to A. As per Note 4.1.1 we know that $\widehat{A}$ is toroidal, and let u be a twist matrix of $\widehat{A}$. Finally, fix L_n/F as a finite totally ramified p-extension and denote by G the Galois group of L_n/F, with $V_u(F)$ and $V_u(L_n)$ being as described in Definition 2.5.4, rather than using Definition 4.1.2 above.*

We then have the following isomorphism:

$$V_u(F)/N_{L_n/F}(V_u(L_n) \cong (G_n^{ab})^d/((I-u)(G_n^{ab})^d)$$

As talked about in Note 4.0.1, we will be following the proof that [7] uses to prove its main result. As such the aim of this section will be attempting to generalise Theorem 4.2.1 to the case where $\overline{F}$ is perfect.

We should state that, for the sake of simplicity, L_n/F will not be used. Instead, we will be using L/F, as a finite abelian totally ramified p-extension with Galois group $G = \mathrm{Gal}\,(L/F)$.

If we let $E = L\widehat{F}$, then we get that $\mathrm{Gal}\,(E/\widehat{F}) \cong G$ and $\mathrm{Gal}\,(E/L) \cong \mathrm{Gal}\,(\widehat{F}/F)$, with $\mathrm{Gal}\,(E/L)$ being generated by the modified versions of the ϕ that we saw when discussing $V(L)$ in section 4.1. Finally, it should be noted that, E/L is the completion of the maximal unramified p-extension of L.

To continue the generalisation of Theorem 4.2.1 we first need to deal with a few lemmas.

Lemma 4.2.1. *Keeping the notation that we have defined earlier in this chapter we get that the G-module E^* is cohomologically trivial.*

Proof. Firstly, as E and $\widehat{F}$ have algebraically p-closed residue fields then the norm map $N_{E/\widehat{F}}$ is surjective, from Remark (1.6) Chapter **IV** of [5]. This means that for any subgroup $H \subseteq G$, we have, using Tate cohomology, that $\widehat{H}^0(H, E^*) = 1$. Likewise, by Hilbert's Theorem 90, we have $\widehat{H}^1(H, E^*) = H^1(H, E^*) = 1$. So, the Tate cohomology groups $\widehat{H}^n(H, E^*)$, for any subgroup

H of G, vanish in the successive dimensions $n = 0$ and $n = 1$; thus E^* is a chomologically trivial G-module. □

Before we get to the next lemma there is some more notation that needs to be defined.

Notation 4.2.1. Using the same notation as earlier in this chapter, define I_G as the kernel of the augmentation map from $\mathbb{Z}[G]$ to $\mathbb{Z}$ that sends $\sum_{g \in G} n_g g \in \mathbb{Z}[G]$, with $n_g \in \mathbb{Z}$, to $\sum_{g \in G} n_g \in \mathbb{Z}$.

For the map $N_{E/\widehat{F}} : U_E \to U_{\widehat{F}}$, denote its kernel by ${}_N U_E$.

Lemma 4.2.2. *Using the same notation as before there is an exact sequence:*

$$1 \to ((G^{\text{ab}})^d)^\wedge \to (U^d_{1,E}/(I_G U^d_{1,E}))^\wedge \to (U^d_{1,\widehat{F}})^\wedge$$

Proof. The main part of this proof of to show that we have the exact sequence:

$$1 \to (G^{\text{ab}})^d \to U^d_{1,E}/(I_G U^d_{1,E}) \to U^d_{1,\widehat{F}} \to 1$$

To start proving the above exact sequence exists, use Lemma 4.2.1 and the exact sequence:

$$1 \to U_E \to E^* \to \mathbb{Z} \to 1$$

to get the isomorphisms:

$$G^{\text{ab}} \cong \widehat{H}^{-2}(G, \mathbb{Z}) \cong \widehat{H}^{-1}(G, U_{1,E}) \cong {}_N U_E/(I_G U_E)$$

This gives us, by acting term by term, that:

$$(G^{\text{ab}})^d \cong ({}_N U_E/(I_G U_E))^d \cong ({}_N U_E)^d/(I_G U^d_E)$$

From the surjectivity of the norm map, we have $N_{E/\widehat{F}}(U_E) = U_{\widehat{F}}$. This means, by the above isomorphisms, that $(G^{\text{ab}})^d$ maps isomorphically into the kernel of the map $N_{E/F} : U^d_E/(I_G U^d_E) \to U^d_{\widehat{F}}$. Thus we get the exact sequence:

$$1 \to (G^{\text{ab}})^d \to U^d_E/(I_G U^d_E) \to U^d_{\widehat{F}} \to 1$$

This is nearly the exact sequence that we are after.

$\overline{\widehat{F}}$ is the separable p-closure of $\overline{F}$. As $E = L\widehat{F}$ and L/F is totally ramified so, by Corollary 3.2.1, we get $\overline{E} = \overline{\widehat{F}}$. We also have that $\overline{E}$ has characteristic p, and thus $\overline{E}^*$ is uniquely p-divisible. So for all $m \in \mathbb{Z}$, we get $\widehat{H}^m(G, \overline{E}^*) = 0$. This, when combined with the exact sequence:

$$1 \to U_{1,E} \to U_E \to \overline{E}^* \to 1$$

means that for all $m \in \mathbb{Z}$, we have $\widehat{H}^m(G, U_{1,E}) \cong \widehat{H}^m(G, U_E)$.

Set $m = -1$ to get $U_E/(I_G U_E) \cong U_{1,E}/(I_G U_{1,E})$. Then, if we let $m = 0$, we observe that $\widehat{H}^0(G, U_{1,E}) \cong \widehat{H}^0(G, U_E)$. The latter group is isomorphic to $U_E^G/N_{E/\widehat{F}}(U_E) = 1$, from the surjectivity of the norm map and the fact that the elements of U_E that G acts trivially on are precisely those in $U_{\widehat{F}}$. This, alongside with $U_{1,E}^G = U_{1,\widehat{F}}$ and $N_{E/\widehat{F}}(U_{1,E}) \subseteq U_{1,\widehat{F}}$, shows that the norm map $N_{E/\widehat{F}}$ maps $U_{1,E}$ surjectively onto $U_{1,\widehat{F}}$.

Using the above term by term and combining the previous results with the intermediate exact sequence we got before gives us the exact sequence:

$$1 \to (G^{\mathrm{ab}})^d \to U_{1,E}^d/(I_G U_{1,E}^d) \to U_{1,\widehat{F}}^d \to 1$$

From this we can then take the functor $B \mapsto B^\wedge = \mathrm{Hom}_{\mathrm{cont}}(G, B)$ on the above to get the exact sequence:

$$1 \to ((G^{\mathrm{ab}})^d)^\wedge \to (U_{1,E}^d/(I_G U_{1,E}^d))^\wedge \to (U_{1,\widehat{F}}^d)^\wedge$$

that we were trying to prove. □

Before we begin the next lemma we must first define a pair of maps:

Definition 4.2.1. Let ϕ an element of $\mathrm{Gal}\,(\widehat{F}/F)$, we define $\iota_\alpha(\phi)$ to be equal to $\alpha^{\phi - u_\phi} = \phi(\alpha)/u_\phi(\alpha)$. We can now create a map from $U_{1,\widehat{F}}^d$ to $(U_{1,\widehat{F}}^d)^\wedge$. This will send $\alpha \in U_{1,\widehat{F}}^d$ to the map ι_α.

Define a similar map from $U_{1,E}^d$ to $(U_{1,E}^d)^\wedge$.

Lemma 4.2.3. *Keeping the notation that we have been using previously we have the following exact sequence:*

$$1 \longrightarrow V(F) \longrightarrow U_{1,\widehat{F}}^d \longrightarrow (U_{1,\widehat{F}}^d)^\wedge$$

Here the map from $U_{1,\widehat{F}}^d$ to $(U_{1,\widehat{F}}^d)^\wedge$ sends $\alpha \in U_{1,\widehat{F}}^d$ to $\iota_\alpha \in (U_{1,\widehat{F}}^d)^\wedge$.

Proof. This is obvious from the G-module isomorphism between $V(F)$ and $\widehat{A}(\mathcal{O}_F)$, this was shown in Lemma 4.1.1. □

Note 4.2.1. The workings of the above lemma also gives us the exact sequence:

$$1 \longrightarrow V(L) \longrightarrow U^d_{1,\widehat{E}} \longrightarrow (U^d_{1,\widehat{E}})^\wedge$$

4.2.1 A Commutative Diagram

before we can finish off the generalisation of Theorem 4.2.1, we need to define a couple of maps.

Definition 4.2.2. We will let $\omega : G^{\text{ab}} \to (G^{\text{ab}})^\wedge$ be the map that sends $g \in G^{\text{ab}}$ to the element of $(G^{\text{ab}})^\wedge$ that sends $\phi \in \text{Gal}(\widehat{F}/F)$ to $g^{1-u_\phi} \in G^{\text{ab}}$.

Meanwhile, the map $\chi : U^d_{1,E}/I_G U^d_{1,E} \to (U^d_{1,E}/I_G U^d_{1,E})^\wedge$ will send sends $\alpha \in U^d_{1,E}/I_G U^d_{1,E}$ to ι_α, with ι_α described in Definition 4.2.1.

Theorem 4.2.2. *Using the notation and definitions that we have previously seen in this chapter, there is an isomorphism:*

$$V(F)/N_{L/K}(V(L)) \cong \ker(\text{coker}(\omega) \to \text{coker}(\chi))$$

Proof. Using the previous lemmas we can form the square:

$$\begin{array}{ccc} (G^{\text{ab}})^d & \longrightarrow & U^d_{1,E}/I_G U^d_{1,E} \\ \downarrow \omega & & \downarrow \chi \\ ((G^{\text{ab}})^d)^\wedge & \longrightarrow & (U^d_{1,E}/I_G U^d_{1,E})^\wedge \end{array}$$

which we want to show is commutative. Here the maps in the two rows are derived from the exact sequences created in the statement and proof of Lemma 4.2.2. For the sake of simplicity we will assume that $d = 1$ as the more general case uses nearly the exact same method.

Now, E/L is unramified; so let π be a fixed prime element of L, it is also a prime element of E. We may use this π in the homomorphism between G^{ab} and $U_{1,E}/I_G U_{1,E}$ and the homomorphism between $(G^{\text{ab}})^\wedge$ and $(U_{1,E}/I_G U_{1,E})^\wedge$. The former homomorphism sends $\alpha \in G^{\text{ab}}$ to $\pi^{\alpha-1} I_G U_{1,E}$, while the latter sends $f \in (G^{\text{ab}})^\wedge$ to the element of $(U_{1,E}/I_G U_{1,E})^\wedge$ that sends $\phi \in \text{Gal}(\widehat{F}/F)$ to $\pi^{f(\phi)-1} I_G U_{1,E}$.

Finally, we have that the map from G^{ab} to $U_{1,E}/I_G U_{1,E}$ sends $\alpha \in G^{\text{ab}}$ to $\pi^{\alpha-1} I_G U_{1,E}$. In order to show that the square is commutative we therefore want that, for all $\alpha \in G^{\mathrm{ab}}$, the image of $\omega(\alpha)$ under the map from $(G^{\mathrm{ab}})^\wedge$ to $(U_{1,E}/I_G U_{1,E})^\wedge$ is the same as $\chi(\pi^{\alpha-1} I_G U_{1,E})$.

From how the bottom row of the square acts on $(G^{\mathrm{ab}})^\wedge$ we may pick an arbitrary $\phi \in \mathrm{Gal}\,(\widehat{F}/F)$, and a u_ϕ., and just look at what happens with that. We, therefore, want to show that, for $f \in (G^{\mathrm{ab}})^\wedge$, we have:

$$(\pi^{f(\phi)-1})^{\phi-u_\phi} \equiv \pi^{f(\phi)^{1-u_\phi}-1} \mod I_G U_{1,E}$$

The map from $(G^{\mathrm{ab}})^\wedge$ to $(U_{1,E}/I_G U_{1,E})^\wedge$ is a homomorphism, so for every $n \in \mathbb{Z}$, $\phi \in \mathrm{Gal}\,(\widehat{F}/F)$ and $f \in (G^{\mathrm{ab}})^\wedge$ we have:

$$(\pi^{f(\phi)-1})^n \equiv \pi^{f(\phi)^n-1} \mod I_G U_{1,E}$$

We also have $\pi \in L$, which means $\phi(\pi^{f(\phi)-1}) = \pi^{f(\phi)-1}$; this gives us $(\pi^{f(\phi)-1})^{\phi-u_\phi} = (\pi^{f(\phi)-1})^{1-u_\phi}$.

As $d = 1$, we have u is a unit of $\mathbb{Z}_p$. Now, G^{ab} is a finite abelian p-group and thus u_ϕ acts as the integer u'_ϕ when acting on G^{ab}. The map from $(G^{\mathrm{ab}})^\wedge$ to $(U_{1,E}/I_G U_{1,E})^\wedge$ is a homomorphism, so:

$$(\pi^{f(\phi)-1})^{1-u_\phi} \equiv (\pi^{f(\phi)-1})^{1-u'_\phi} \equiv \pi^{f(\phi)^{1-u'_\phi}-1} \equiv \pi^{f(\phi)^{1-u_\phi}-1} \mod I_G U_{1,E}$$

which is precisely what we wanted to show.

To finish off the proof of the generalisation Theorem 4.2.1 observe the diagram below:

$$
\begin{array}{ccccccccc}
 & & & & 1 & & 1 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & V(L)/(V(L)\cap I_G U_{1,E}) & \xrightarrow{N} & V(F) & \longrightarrow & 1 \\
 & & & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & G^{\mathrm{ab}} & \longrightarrow & U_{1,E}/I_G U_{1,E} & \xrightarrow{N} & U_{1,\widehat{F}} & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle\omega} & & \downarrow{\scriptstyle\chi} & & \downarrow & & \\
1 & \longrightarrow & (G^{\mathrm{ab}})^\wedge & \longrightarrow & (U_{1,E}/I_G U_{1,E})^\wedge & \xrightarrow{N} & (U_{1,\widehat{F}})^\wedge & &
\end{array}
$$

Lemmas 2 and 3 tell us that the bottom two rows and right two columns are exact. Meanwhile, the work we did earlier this section combined with the fact that χ commutes with the elements of G, thus with the norm maps labelled N, tells us the diagram is commutative.

We now use the Snake Lemma to get the exact sequence:

$$V(L)/(V(L) \cap I_G U_{1,E}) \xrightarrow{N} V(F) \longrightarrow \text{coker}\,(\omega) \longrightarrow \text{coker}\,(\chi)$$

This is achieved by taking into account the information shown on the diagram.

We have so far been assuming that $d = 1$. When $d > 1$ the mathematics is formally the same as before. we just have to adjust the maps to take into account that we are now dealing with groups that are the direct product of d copies of the previous groups; for instance u_ϕ is a $d \times d$ invertible matrix over $\mathbb{Z}_p$ rather than a unit of $\mathbb{Z}_p$. Taking into account all values of $d \geq 1$, we get the exact sequence:

$$V(L)/(V(L) \cap I_G U_{1,E}^d) \xrightarrow{N} V(F) \longrightarrow \text{coker}\,(\omega) \longrightarrow \text{coker}\,(\chi)$$

This gives us:

$$V(F)/N_{L/K}(V(L)) \cong \ker\,(\text{coker}\,(\omega) \to \text{coker}\,(\chi))$$

□

This leaves us with trying to work out the kernel of the map from coker (ω) to coker (χ).

Notation 4.2.2. Keep everything from above. We will, for simplicity, denote $\ker\,(\text{coker}\,(\omega) \to \text{coker}\,(\chi))$ as Q.

This gives us that $V(F)/N_{L/F}(V(L)) \cong Q$. This is the generalistion of Theorem 4.2.1 that we were trying to find, at least up to explicit calculation of Q.

Note 4.2.2. By Lemma 4.1.1 we see that both $V(F)$ and $V(L)$ are G-modules and are isomorphic, as G-modules, to $\widehat{A}(\mathcal{O}_F)$ and $\widehat{A}(\mathcal{O}_L)$ respectively.

This means that we can rewrite the above isomorphism as:

$$\widehat{A}(\mathcal{O}_F)/N_{L/F}(\widehat{A}(\mathcal{O}_L)) \cong Q$$

4.3 Finishing the Generalisation of Theorem 4.0.1

This is the point, unfortunately, where we have to stop being concrete with the mathematics we produce on this subject; working out the exact answer to what Q equals is beyond the scope of this thesis. However, we will provide a brief overview of what finishing the generalisation of Theorem 4.0.1 will probably look like.

Notation 4.3.1. Before we get to the sketch of completing the generalisation we need to do some relabelling of the notation that we have been using so far, this is to make our work closer to Theorem 4.0.1. Keeping F the same , we shall now let L/F be a totally ramified $\mathbb{Z}_p$-extension of F such that $L = \bigcup L_n$, where L_n/F is an extension of fields for every positive integer n such that L/L_n and $[L_n : F] = p^n$. We shall let E_n be equal to $L_n\widehat{F}$ and E be equal to $L\widehat{F}$.

From the above we can see that L_n/F is a finite abelian totally ramified p-extension. We also have, using the same abelian variety as before, that $N_{L/F}(A(L)) = \bigcap N_{L_n/F}(A(L_n))$. Next, we shall denote $\mathrm{Gal}\,(L_n/F)$ by G_n, the group $\widehat{A}(\mathcal{O}_F)/N_{L_n/F}(\widehat{A}(\mathcal{O}_{L_n}))$ by Q_n and the ω and χ maps, of Definition 4.2.2, will be ω_n and χ_n respectively. Finally, we now let $\mathrm{Gal}\,(L/F)$ be Γ and label $\widehat{A}(\mathcal{O}_F)/N_{L/F}(\widehat{A}(\mathcal{O}_L))$ as Q.

Note 4.3.1. We should briefly point out that as L/F is a $\mathbb{Z}_p$-extension it is abelian, and thus so are all the L_n/F. This means that we no longer need to bother with abelianisation of the Galois groups.

Proposition 4.3.1. *Keeping the notation outlined in Notation 4.3.1 we have the exact sequence:*

$$Q \to A(F)/N_{L/F}(A(L)) \to A(\overline{F})_p$$

Proof. By the theory of inverse limits, as each Q_n is a G_n-module, we have that $\varprojlim_n Q_n = Q$. The homomorphisms involved in this case are the norm maps between the L_n. We likewise can construct the maps $\omega : \Gamma^d \to (\Gamma^d)^\wedge$ and $\chi : U^d_{1,E}/I_\Gamma U^d_{1,E} \to (U^d_{1,E}/I_\Gamma U^d_{1,E})^\wedge$, as the inverse limits of the maps $\omega_n : G_n \to G_n^\wedge$ and $\chi_n : U^d_{1,E_n}/I_{G_n} U^d_{1,E_n} \to (U^d_{1,E_n}/I_{G_n} U^d_{1,E_n})^\wedge$, respectively.

Taking into account that $A(\mathcal{O}_{L_n}) = A(L_n)$, we have the exact sequence:

$$1 \to \widehat{A}(\mathcal{O}_{L_n}) \to A(L_n) \to A(\overline{F}) \to 1$$

since L_n/F is totally ramified, we have $\overline{L_n} = \overline{F}$.

Using the norm map $N_{L_n/F}$, and noting that the induced map on $A(\overline{F})$ is multiplication by p^n, we get the new exact sequence:

$$\widehat{A}(\mathcal{O}_F)/N_{L/F}(\widehat{A}(\mathcal{O}_{L_n})) \to A(F)/N_{L_n/F}(A(L_n)) \to A(\overline{F})/p^n A(\overline{F}) \to 1$$

This can be simplified to:

$$Q_n \to A(F)/N_{L_n/F}(A(L_n)) \to A(\overline{F})/p^n A(\overline{F}) \to 1$$

Using inverse limits, and taking into account that $\varprojlim_n Q_n = Q$, we therefore get the exact sequence:

$$Q \to A(F)/N_{L/F}(A(L)) \to A(\overline{F})_p$$

like we wanted. □

This is the generalisation of Theorem 4.0.1 that we were looking for, up to computing what the group Q is and whether the last map is surjective.

4.4 Some ideas that we can Explore

While explicitly constructing the Q_n, and thus Q, will not be covered in this thesis; there are some observations and reasonable guesses that we can make about them.

Notation 4.4.1. Let $\widehat{F}/F$ still be as the maximal unramified p-extension of F. Fix a topological basis of $\mathrm{Gal}\,(\widehat{F}/F)$, and denote by J an index such that the set of of $\phi_j \in \mathrm{Gal}\,(\widehat{F}/F)$, with $j \in J$ forms the basis. Label a twist matrix associated with ϕ_j with u_j.

As they are homomorphism groups it is clear, by the definition of a topological basis, that the elements of $(G_n^d)^\wedge$ and $(U_{1,E_n}^d/I_{G_n}U_{1,E_n}^d)^\wedge$ are uniquely determined by how the maps act on the ϕ_j, with $j \in J$. The same holds for elements $(\Gamma^d)^\wedge$ and $(U_{1,E}^d/I_\Gamma U_{1,E}^d)^\wedge$.

Let us assume that $|J|$ is finite. Taking into account the above paragraph we get, for all finite n, that $(G_n^d)^\wedge \cong \oplus_{j\in J} G_n^d$ and $(U_{1,E_n}^d/I_{G_n}U_{1,E_n}^d)^\wedge \cong \oplus_{j\in J}(U_{1,E_n}^d/I_{G_n}U_{1,E_n}^d)$; we also get $(\Gamma^d)^\wedge \cong \oplus_{j\in J}\Gamma^d$ and $(U_{1,E}^d/I_\Gamma U_{1,E}^d)^\wedge \cong \oplus_{j\in J}(U_{1,E}^d/I_\Gamma U_{1,E}^d)$. As we have fixed the topological basis beforehand, we can see that the maps we have defined earlier between the homomorphism groups now act on a term by term basis on the direct sums.

Now, taking the maps from Definition **??** the image of the ω_n is all elements of $\oplus_{j\in J}G_n^d$ of the form $\oplus_{j\in J}g^{1-u_j}$, as g varies in G_n^d. If $|J| > 1$, as G_n is a finite group, we have $\omega_n : G_n \to \oplus_{j\in J}G_n$ cannot be surjective and thus $\operatorname{coker}(\omega_n) \neq 1$.

Now as $|J|$ tends to infinity it is clear that $|\operatorname{coker}(\omega_n)|$ must tend to infinity as well. Let $|J|$ be infinite. We can create subextensions of $\widehat{F}/F$ that are topologically generated by a finite subset of the topological basis of $\operatorname{Gal}(\widehat{F}/F)$. Redefining the groups and homomorphisms to instead be based on such subextensions, and tending the cardinality of the finite subsets to infinity, we can see that if $|J|$ is infinite then so is $|\operatorname{coker}(\omega_n)|$.

Since ω is the inverse limits of the ω_n we get that if $|J| > 1$, then $\operatorname{coker}(\omega)$ is non-trivial. Likewise, if $|J|$ is not finite then neither is $\operatorname{coker}(\omega)$. If $|J| > 1$ but finite then more work is required to see whether $\operatorname{coker}(\omega)$ is finite or not. Meanwhile, χ_n will always map from an infinite group to an infinite group, no matter the cardinality of J. it is therefore difficult to tell, for any cardinality of J, if $\operatorname{coker}(\chi_n)$ is finite or not, or even trivial, and thus difficult to make a definitive statement about the cardinality of $\operatorname{coker}(\chi)$.

While this is all we are going to talk about on the topic of Q at the moment, we will also be briefly going over how to take this topic further in the next chapter, "Conclusion".

4.5 Quasi-Finite Residue Fields

We will finish off this chapter reintroducing the concept of a quasi-finite fields, which were talked about in the first section of the "Literature Review".

As a reminder, a field k is quasi-finite if it is perfect and if its absolute Galois group is isomorphic to $\widehat{\mathbb{Z}}$. In other words, if every finite extension of k is a Galois cyclic extension and for every finite cyclic group, H, we have that k has a unique extension whose Galois group is isomorphic to H. If k is a finite field then it is obviously quasi-finite, but k is also quasi-finite if it is a separable, but not necessarily finite, extension of a finite field. We should note that those two cases does not cover all examples of quasi-finite fields.

We saw that when working with the mathematics of local class field theory, dealing with a complete discrete valuation field F that has a quasi-finite residue field is not much harder than if we required $\overline{F}$ to be finite. It is, however, much simpler than if $\overline{F}$ is only required to be perfect and have positive characteristic. This means that, when generalising classical local class field theory, it is good to first consider the case where $\overline{F}$ is quasi-finite before dealing with arbitrary perfect residue fields.

If we were to restrict $\overline{F}$ to being quasi-finite we would have $\mathrm{Gal}\,(\widehat{F}/F) \cong \mathbb{Z}_p$, we could then use Notation 4.4.1 to get that the set J would consist of a single element. This would mean that we would only need to use one automorphism, which we could denote ϕ, and one corresponding twist matrix, denote this by u. However, we would still be required to use different mathematics than Lubin and Rosen. This is because that by assuming $\overline{F}$ is quasi-finite we are still not requiring $\overline{F}$ is finite, something that seems necessary for the method Lubin and Rosen employ. They make use of the Frobenius automorphism to prove their version Lemma 4.2.3, they call their version "Lemma 3", which by Note 2.2.5 only exists when $\overline{F}$ is finite [7].

However, some simplifications may emerge. As noted before, we have that $\mathrm{Gal}\,(\widehat{F}/F) \cong \mathbb{Z}_p$,; as such for any abelian p-group T we have that $T^\wedge = \mathrm{Hom}_{\mathbb{Z}_l}\,(\mathrm{Gal}\,(\widehat{F}/F), T) \cong T$. As such, keeping the notation that we used in the previous section, we have that ω is now a map between Γ^d and itself, while χ is map between $U^d_{1,E}/(I_G U^d_{1,E})$ and itself.

Likewise, if we look at the finite field extension L_n/F we see that G_n is a finite group and thus ω_n is now a map between a finite group and itself. We should note that we do still have, as we pointed out at the end of the last section, $U^d_{1,E_n}/(I_{G_n} U^d_{1,E_n})$ being infinite, so χ_n does not have the simplicity of ω_n.

This all means that it may be easier to compute $\mathrm{coker}\,(\omega)$ and $\mathrm{coker}\,(\chi)$ in this case. However, it is currently unknown what results we do actually have.

For instance, despite the ω_n being maps from a finite group to itself we should still not expect them to be always surjective. If such maps were always surjective, we would have that $\mathrm{coker}\,(\omega_n) = 1$ and thus so both Q_n and Q must be equal to 1 as well. This result would give us that $A(F)/N_{L/F}(A(L))$ is always isomorphic to $A(\overline{F})_p$; which would lead to a serious issue with our work since it is a generalisation of the main result of [7]. A finite field is quasi-finite, and thus we would have that $A(F)/N_{L/F}(A(L)) \cong A(\overline{F})_p$ for all finite extensions $F/\mathbb{Q}_p$.

Despite the above caveats, quasi-finite residue fields is another avenue that it may be fruitful to explore at a later date.

Chapter 5

Conclusion

We have extensively talked about two topics on the issue of complete discrete valuation fields in this document. Those two being the map $\Psi_{L/F}$ relating to certain extensions of discrete valuation fields with imperfect residue fields and abelian varieties over complete discrete valuation fields. However, there is still a lot more research that could be done on the pair of them in the future.

The first thing we should talk about is the topic of Chapter 3, which is focused on $\Psi_{L/F}$ when dealing with finite abelian totally ramified p-extensions of complete discrete valuation fields with imperfect residue fields. The obvious point to note is that while we have made strides in trying to prove whether $\Psi_{L/F}$ is always an isomorphism; the question unfortunately remains open.

There are a few avenues a person could take to further what we have written about in Chapter 3. The one that appeals to me the most is solving the problem we established in section 8 of Chapter 3, "The Final Values of i".

As a reminder this is showing that the following statement is true:

Let F is a complete discrete valuation field with residue field, that may be imperfect, of characteristic p. We also have that L/F is a abelian totally ramified p-extension with two ramification jumps with M being the fixed field of G_b, where $G_b \neq 1$ and $G_{b+1} = 1$, and we say that $|L : M| = p^{n_2}$ and $|M : F| = p^{n_1}$.

Let j be such that $a < j < h_{M/F}(b)$, and let $\bar{t}$ be in $\overline{\widehat{F}}^{p^{n_2}}$, and not be equal to 0. Finally, pick $\alpha \in \ker(N_{\widehat{M}/\widehat{F}} : U_{1,\widehat{M}} \to U_{1,\widehat{F}})$, such that $\alpha \in U_{j,\widehat{M}}$ and $\lambda_j(\alpha) = \bar{t}$. Then, for all j between a and $h_{M/F}(b)$, and all choices of $\bar{t}$ and α derived from j, there is $\beta \in N_{\widehat{L}/\widehat{M}}(U_{1,\widehat{L}}) \cap U_{j,\widehat{M}}$, such that $\lambda_j(\beta) = \bar{t}$ and $\beta \in \ker(N_{\widehat{M}/\widehat{F}} : U_{1,\widehat{M}} \to U_{1,\widehat{F}})$.

If this problem could be solved positively one would have proved that $\Psi_{L/F}$ is an isomorphism for all finite abelian p-extensions with two or less ramification jumps over all complete discrete valuation fields with, perfect or imperfect, residue field of characteristic p. Since I believe that such a proof would be relatively easy to extend to field extensions with higher number of ramification jumps it could be rather easy to close the entire line of inquiry about whether the maps are an isomorphism.

Of course one may try to be less ambitious and just try to extend the set of abelian p-extensions we know have isomorphic $\Psi_{L/F}$. The obvious first step in this is to try to answer the problem we ended Chapter 3 with. This was trying to see whether the $\Psi_{L/F}$ are isomorphic for certain extensions with Galois group $(\mathbb{Z}/p\mathbb{Z})^3$.

As a reminder from the "Introduction", one idea that we did not have the time to look at here is higher local fields. These are complete discrete valuation rings whose residue field is another complete discrete valuation field, and in fact forms a chain of such fields with the inner most discrete valuation field having a finite residue field. More details can be found in "Geometry and Topology Monographs, Volume 3: An Invitation to Higher Local Fields" [2].

This is important because higher local fields are a well studied type of complete discrete valuation field that may have an imperfect residue field. For instance, a two dimensional local field of characteristic 0 with residue field, which is a complete discrete valuation field with finite residue field, of characteristic p has an understood local class field theory about them that relates abelian extensions to the 2-dimensional Milnor K-group of the field. It also could have an imperfect residue field, as the residue field has characteristic p and infinite cardinality. We could then use this theory to try to answer open questions about extensions of the form $(\mathbb{Z}/p\mathbb{Z})^3$, and then possibly other extensions after that,; at least on this particular type of field.

We should note that a brief introduction to the idea of the Milnor K-groups of a field is gone over in Chapter **IX** of [5].

Of course the other method someone could take, that is not talked about earlier in this thesis, is to find an example of a finite abelian p-extension where $\Psi_{L/F}$ ia not an isomorphism. This is different from the aim in Chapter 3, where we were trying to extend the set of extensions where we know that the $\Psi_{L/F}$ is an isomorphism, but such a counter-example would also give a definitive answer as to whether the $\Psi_{L/F}$ is always an isomorphism.

If such an example could be found the work on the subject would be far from over though. The first line on enquiry could be for what subset of finite abelian p-extensions is $\Psi_{L/F}$ always an isomorphism. Ironically, in this hypothetical scenario, despite the question we were trying to tackle in this document being answered by a different method than the one we used; this thesis, and any further extension to it, could still be useful for dealing with the further problems that would then arise.

The other way a person may take this new hypothetical answer is whether there is a particular property of this complete discrete valuation field that causes it to have such an extension. Is it something in addition to the imperfect residue field? We know that all complete discrete valuation fields with perfect residue fields must not have this hypothetical property. Is there a subset of complete discrete valuation fields with imperfect residue fields in which $\Psi_{L/F}$ is an isomorphism for all finite abelian totally ramified p-extensions, or at the very least this particular counter-example does not work. It could be that this type of extension has non-isomorphic $\Psi_{L/F}$ for all such fields complete discrete valuation fields with imperfect?

The above paragraph may be getting a bit ahead of ourselves. Such an extension has not been found and, has been made abundantly clear, we do not even know whether one exists or not. This was merely to demonstrate that even if such a construct could were to be discovered the topic of $\Psi_{L/F}$ would be far from closed.

Next, we can talk about Chapter 4, "Abelian Varieties over Local Fields", and how we can expand on that topic. Naturally, the first thing we can look at is trying to find more information about the group Q, if not explicity compute Q, and thus get further in the generalisation that we started.

We had some notations that we briefly went over in section 4.4, "Some Ideas that we can Explore", which could be a good place to start. For instance finding out more about $\operatorname{Im}(\chi)$, and thus $\operatorname{coker}(\chi)$, could be very useful. For example, proving that χ is surjective, at least in some cases, would simplify Q to be $\operatorname{coker}(\omega)$.

Section 4.8, "Quasi-finite Residue Fields", brings up a way to simplify the mathematics, while still keeping our assumptions more general that those Lubin and Rosen have. Admittedly, all assuming that $\overline{F}$ is quasi-finite does is make it that $|J| = 1$, though it may yield results which we can then try to extrapolate to cases where the cardinality of J is greater than 1. The best way to tackle this problem would probably be to combine the ideas and both this paragraph and the last.

Other than trying to work out the properties of Q; there is another intriguing thought that may be worth looking at more closely. This is to relate the mathematics of Chapter 4, and [7], back to the topic of local class field theory.

For instance, Theorem 1 of [7] gives a result showing an isomorphism between $\widehat{A}(\mathcal{O}_F)/N_{L/F}(\widehat{A}(\mathcal{O}_L)$ and $\mathrm{Gal}\,(L/F)^{\mathrm{ab}}/((I-u)\,\mathrm{Gal}\,(L/F)^{\mathrm{ab}})$, which seems very similar to the isomorphism between $F^*/N_{L/F}(L^*)$ and $\mathrm{Gal}\,(L/F)^{\mathrm{ab}}$ that is an important result in classical local class field theory.

We should note that interpreting mathematics of [7] through a local class field theory lens was an idea briefly brought up and explored in the last section of Chapter **V** in Professors Ivan Fesenko and Sergei Vostokov's book "Local Fields and their Extensions" [5]. Here the isomorphism of Theorem 1 is labelled the "Twisted Reciprocity Homomorphism". If we want to look at the work of Chapter 4 as a variation of local class field theory a good way to do it is to start with the ideas expressed in [5], and look at the works referenced in that section, and see what can be extrapolated to more general base fields F.

While the above can be done before we get more concrete information on Q it is recommend to do it afterwards; explicitly calculating Q would significantly help us in this matter. This would allow us to go into the topic with more information about the isomorphism between $\widehat{A}(\mathcal{O}_F)/N_{L/F}(\widehat{A}(\mathcal{O}_L))$ and Q, and thus make it easier to see if we can relate it back to the more complicated forms of local class field theory that we saw in the "Literature Review".

Of course those are just a few ideas of how to take the subjects of this document further. There are likely many more, a lot of them using mathematics not touched upon in the previous chapters, but what we have here is at least a place to start further investigations into these mysteries.

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