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School of Mathematical Sciences

A Survey of the Number Theoretic Properties of Real Analytic Modular Forms

Joshua Drewitt

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"Be your own person. Forget the expectations of others.

True success is doing what you love."

– David Orobosa Omoregie

This thesis is dedicated to my parents,
James and Gillian Drewitt.

And to Daniel Hunt.

Abstract

The purpose of this thesis is to study the number theoretic properties of real analytic modular forms, which were recently introduced by F. Brown. We begin with a brief review of the classical theory of modular forms, focusing on mathematical objects such as Hecke operators, L-functions and period polynomials.

In Chapter 2, we introduce and discuss real analytic modular forms. We define L-functions for the entirety of this space and establish their main properties. We then define the analogue of the period polynomial for modular iterated integrals of length one.

In Chapter 3, we review the action of Hecke operators on the period polynomials of standard modular forms, with the aim of extending this theory to real analytic modular forms. We achieve this for a certain type of length one modular iterated integral called a real analytic cusp form.

In the final chapter, we present a theorem for producing new and interesting length three modular iterated integrals. This can be viewed as an extension of the length two case given by Brown, a review of which is included in this chapter. We discuss how modular iterated integrals could help understand modular graph functions, which arise in string perturbation theory.

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Statement of originality

I declare that this thesis is a result of my own work which has been mainly undertaken during my period of registration for this degree at The University of Nottingham.

Related Publications

- Chapter 2 of this thesis closely follows my paper "Period functions associated to real analytic modular forms", co-authored with Nikolaos Diamantis [1].
- Chapter 4 closely follows my preprint "Triple equivariant Eisenstein integrals" [2].

CONTENTS

0 INTRODUCTION 1

1 THE CLASSICAL THEORY OF MODULAR FORMS 7

- 1.1 Defining a Modular Form . . . 7
- 1.2 Examples . . . 9
- 1.3 Hecke Operators . . . 13
- 1.4 L-Functions . . . 17
- 1.5 Period Polynomials . . . 21
- 1.6 Period Functions associated to Maass Wave Cusp Forms . . . 25
- 1.7 Weakly Holomorphic Modular Forms . . . 29

2 L-FUNCTIONS AND PERIOD FUNCTIONS ASSOCIATED TO REAL ANALYTIC MODULAR FORMS 33

- 2.1 An Introduction to Real Analytic Modular Forms . . . 33
- 2.2 Eigenforms for the Laplacian . . . 36
- 2.3 The Real Analytic Eisenstein Series . . . 38
- 2.4 Modular Iterated Integrals of Length One . . . 40
- 2.5 L-Functions associated to Real Analytic Modular Forms . . . 42
  - 2.5.1 L-Functions in $\mathcal{HM}^!(\lambda)$ and in $\mathcal{MI}_1^!$. . . 45
  - 2.5.2 L-Functions in $\mathcal{M}_{r,s}$. . . 50
- 2.6 Period Functions associated to Real Analytic Modular Forms 52
  - 2.6.1 Maass-Selberg Forms associated to Length One Modular Iterated Integrals . . . 53
  - 2.6.2 Cocycles associated to Modular Iterated Integrals of Length One. . . . 55
  - 2.6.3 An Application to Algebraicity . . . 64

3 THE ACTION OF HECKE OPERATORS ON EICHLER INTEGRALS OF REAL ANALYTIC FORMS 71
3.1 Hecke Operators acting on Eichler Integrals of Modular Forms 72
3.2 An Analogous Result for Length One Modular Iterated Integrals 75
3.2.1 Real Analytic Cusp Forms 77
3.2.2 Hecke Operators acting on Eichler Integrals of Modular Iterated Integrals 81

4 LAPLACE-EIGENVALUE EQUATIONS FOR MODULAR ITERATED INTEGRALS 93
4.1 The Space of Modular Iterated Integrals revisited 93
4.2 Laplace Equations for Length Two Iterated Integrals 96
4.2.1 Double Eisenstein Integrals 98
4.2.2 Equivariant Versions of Double Eisenstein Integrals . . 101
4.2.3 Example 104
4.3 Modular Graph Functions 105
4.4 Laplace Equations for Length Three Iterated Integrals 108
4.4.1 Equivariant Versions of Triple Integrals 111
4.4.2 Examples 114

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INTRODUCTION

Modular forms are some of the most fascinating and significant mathematical objects. They played a crucial role in Andrew Wiles' proof of Fermat's Last Theorem and can be associated with cryptography, string theory, representation theory, the Langlands program and much more.

Modular forms began to appear naturally in the nineteenth century during the study of elliptic functions and were investigated by mathematicians such as Eisenstein. They also appeared, as theta functions, in the work of Jacobi during the 1820s. However, it is believed that the term modular form (or Modulform in German) was not introduced until around the 1890s, by the German mathematician Klein. The first use of this term in publication is believed to have been in [3].

In the twentieth century, modular forms began to play an increasingly significant role in mathematics and appeared in the work of many mathematicians, including Ramanujan. Hecke began studying modular forms in the 1920s and shortly introduced a family of operators, now known as the Hecke operators. These operators played a key role in capturing and showcasing the important arithmetic information that modular forms have to offer. As a result of this, modular forms became securely rooted in their position at the forefront of mathematics.

Today, modular forms continue to be a subject of enormous interest, constantly producing beautiful and surprising results. They are closely related to a range of other mathematical objects, including elliptic curves and Maass wave forms. Indeed, the term modular form itself can now relate to a variety of modular objects, such as weakly holomorphic modular forms, mock modular forms and, the recently defined, real analytic modular forms.

Chapter 1 of this thesis illustrates some of the fascinating theory of modular forms we alluded to above. We do, however, try to discuss only what is relevant to the thesis. We will see that a modular form f, of weight k, obeys the condition

$$f(\gamma z) = (cz+d)^k f(z), \qquad \forall\, \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{Z}), \tag{0.1}$$

and has a Fourier expansion of the form

$$f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z}. \tag{0.2}$$

In Section 1.2, some examples of modular forms are given. This will help demonstrate how the Fourier coefficients of these forms can contain interesting and important information. In the section after this, we discuss how the Hecke operators play a key role in capturing this information.

This leads to the definition of an L-function for a modular form, given in Section 1.4. The study of L-functions is not restricted to modular forms and they are an interesting mathematical object in their own right. Indeed, they can be used as a tool to associate various arithmetic objects to each other, such as modular forms with elliptic curves. They also provide us with important number theoretic results, such as information about the distribution of prime numbers.

To help us understand L-functions, we define the period polynomial of a modular form in Section 1.5. These polynomials are closely related to L-functions and have been used to prove many important results, such as Manin's Periods Theorem (Theorem 1.13).

We also study the space of Maass wave cusp forms, which are modular objects similar to modular forms. However, despite this similarity, developing an analogue of the period polynomial for Maass wave cusp forms proved difficult. In Section 1.6, we present a review of how this was eventually achieved in the form of a period function. Chapter 1 ends with an investigation of weakly holomorphic modular forms.

In Chapter 2, we introduce a space which contains or intersects all the spaces studied in Chapter 1. This is the space of real analytic modular forms, which was recently introduced by Brown in [4–6]. These forms can be viewed as a unifying tool for the spaces appearing in Chapter 1.

The aim of this thesis is to study the classical number theoretic properties of real analytic modular forms. We want to produce a theory for these forms that is an extension of the theory for standard modular forms, as presented in Chapter 1.

We will see that a real analytic modular form of weights (r, s) must obey the following two conditions:

(i) $f(\gamma z) = (cz+d)^r (c\bar{z}+d)^s f(z), \quad \forall\, \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{Z}).$

(ii) For some $M, N \in \mathbb{N}$, $a_{m,n}^{(j)} \in \mathbb{C}$

$$f(z) = \sum_{|j| \leq M} y^j \left(\sum_{m,n \geq -N} a_{m,n}^{(j)} e^{2\pi i m z} e^{-2\pi i n \bar{z}} \right).$$

These two conditions can be compared to equations (0.1) and (0.2). We denote the space of all real analytic modular forms of weights (r, s) by $\mathcal{M}^!_{r,s}$ and set $\mathcal{M}^! = \bigoplus_{r,s} \mathcal{M}^!_{r,s}$.

Chapter 2 begins with the background of these real analytic forms and sets the groundwork needed for the rest of the thesis. We look at an example of a real analytic modular form, called the real analytic Eisenstein series, in Section 2.3.

In Section 2.5, we define L-functions for the entirety of $\mathcal{M}^!$ and establish their main properties. When restricted to specific subspaces, this matches the L-functions given in Chapter 1.

It was not immediately obvious, however, how to define suitable period polynomials for all of $\mathcal{M}^!$. This is understandable though, since the period polynomial usually reflects arithmeticity and the space $\mathcal{M}^!$ is too large to be of arithmetic nature in its entirety. We show that, in a certain subspace, it is possible to define such polynomials. This subspace is the space of modular iterated integrals of length one and is of great interest to us in this thesis. We will see, in Section 2.6, how the construction of the period function from the theory of Maass wave forms (Section 1.6) will form the basis of these period polynomials.

Evidence why our period polynomial is the analogue of the period polynomial of a classical modular form is also given. Additionally, we show how our period polynomial is closely related to our L-function (Theorem 2.23).

In the final section of this chapter, it is shown how our L-function can be used to prove that an analogue of Manin's Periods Theorem holds for weakly holomorphic modular forms (Theorem 2.29).

In Chapter 3, we return to the topic of Hecke operators. The eigenvalues of these operators are closely related to the Fourier coefficients of standard modular forms, and therefore, learning about the former will uncover important arithmetic information about the latter. These eigenvalues, in turn, can be recovered from the Hecke operators' trace. Therefore, formulas describing the trace of a Hecke operator are of great importance.

Such a formula for standard modular forms was first introduced by Selberg in [7]. This trace formula, however, was rather complicated and since then simpler versions of this formula have been given. These new versions are dependent on the Hecke operators being compatible with the period polynomials of certain standard modular forms. We study this compatibility in Section 3.1.

It would be useful if we could extend this compatibility to real analytic modular forms. In Section 3.2, a certain type of length one modular iterated integral called a real analytic cusp form is introduced. We discuss some of the interesting properties that belong to this function. We then show that, because of these properties, the Hecke operators are compatible with the period polynomials of real analytic cusp forms.

Chapter 4, the final chapter, is an investigation of the modular iterated integrals of length higher than one. One of the original motivations for defining real analytic modular forms was due to the close relationship between modular iterated integrals and modular graph functions, which arise in string perturbation theory. In fact, there is good evidence that all modular graph functions are contained within the space of modular iterated integrals. Therefore, studying the space of modular iterated integrals will help with the long-standing problem of giving a complete description of the modular graph functions.

In Section 4.2, we provide a review of the research presented in Chapter 9 of [4]. In this chapter, Brown gives a theorem for producing new and interesting length two modular iterated integrals. Each of these functions has an associated Laplace-eigenvalue equation, for example, the function $F^{(1)}_{1,1}$ satisfies

$$(\Delta + 2)F^{(1)}_{1,1} = -4\mathbb{L}^3\mathbb{G}_4\bar{\mathbb{G}}_4, \tag{0.3}$$

where $\mathbb{G}_4$ is an Eisenstein series (1.7), Δ is the Laplacian (2.7) and $\mathbb{L} = i\pi(z - \bar{z})$.

In Section 4.3, we give a brief overview of the modular graph functions of interest to us in this thesis. We also explain how Brown has provided an important use for the above Laplace equation. Indeed, he has shown that we can use this equation to give, for the first time, an explicit expression of an important modular graph function.

If other modular graph functions can be expressed using Laplace equations associated to modular iterated integrals, then these equations can help to give a complete description of the modular graph functions.

In Section 4.4, we extend the theory presented in Chapter 9 of [4] from length two modular iterated integrals to length three. We are able to produce Laplace equations for new length three modular iterated integrals.

1

THE CLASSICAL THEORY OF MODULAR FORMS

In this chapter, we provide a brief overview of the classical theory of modular forms, setting the groundwork from which we will build upon in the later chapters. Since this is a vast and rich area of mathematics, we will try to only discuss what is relevant to this thesis.

We hope, nonetheless, to still give the reader enough insight to appreciate how fascinating this theory can be. This, in turn, should help justify why we would want to extend the theory of modular forms to other functions, such as the real analytic modular forms. In Sections 1.1–1.5, we closely follow [8–10].

1.1 DEFINING A MODULAR FORM

We begin by defining what it means for a function to be a modular form, we will also see some of the immediate consequences of this. Before we can proceed, however, we must first set some basic notation.

The special linear group $SL(2,\mathbb{R})$ acts on the upper half plane $\mathfrak{H}$ via Möbius transformations. In other words, if $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{R})$ then, for all $z \in \mathfrak{H}$, we have

$$\gamma(z) = \frac{az+b}{cz+d}.$$

A modular form can be defined over any discrete subgroup of $SL(2,\mathbb{R})$. For this thesis, however, we will concentrate on the full modular group $SL(2,\mathbb{Z})$, which we denote by Γ_1:

$$\Gamma_1 := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,:\, ad - bc = 1 \text{ and } a,b,c,d \in \mathbb{Z} \right\}.$$

The matrices

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

generate this group.

For an integer k, a function $f\colon \mathfrak{H} \to \mathbb{C}$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL(2,\mathbb{R})$ with $det(\gamma) > 0$, we define the (single) slash operator $f|_k$ by

$$(f|_k\gamma)(z) := det(\gamma)^{k/2}(cz+d)^{-k}f(\gamma(z)). \tag{1.1}$$

This is a group action, and therefore, we have

$$f|_k(\gamma\mu) = (f|_k\gamma)|_k\,\mu\,, \quad \forall\ \gamma,\mu \in GL(2,\mathbb{R}). \tag{1.2}$$

We are now ready to define a modular form.

Definition 1.1. Let k be a non-negative integer. A function $f\colon \mathfrak{H} \to \mathbb{C}$ is a modular form of weight k with respect to Γ_1 if it obeys the following three conditions:

$$(i) \quad f \text{ is holomorphic on } \mathfrak{H}. \tag{1.3}$$

$$(ii) \quad (f|_k\gamma)(z) = f(z), \quad \forall\ \gamma \in \Gamma_1. \tag{1.4}$$

$$(iii) \quad f \text{ is holomorphic as } z \to i\infty. \tag{1.5}$$

Remark 1.2. We call property (ii) the transformation law, this can also be written as $f(\gamma z) = (cz+d)^k f(z)$, $\forall\ \gamma \in \Gamma_1$.

We will now discuss some of the immediate consequences of the above conditions. Firstly, since $T \in \Gamma_1$, the second condition implies that any modular form f must obey the equation $f(z) = f(z+1)$, for all $z \in \mathfrak{H}$.

This, combined with the fact that f is holomorphic, means it has a Fourier expansion at infinity of the form

$$f(z) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi i n z}.$$

The third condition is equivalent to a_n vanishing for every $n < 0$. Therefore, any modular form f can be expressed as

$$f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z}. \tag{1.6}$$

Finally, the second condition implies that, as $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \in \Gamma_1$, if f is of odd weight then

$$f(z) = (-1)^k f(z),$$

and therefore, f must be the zero function.

We denote the space of all modular forms of weight k with respect to Γ_1 by M_k and set $M = \bigoplus_k M_k$. We should write $M_k(\Gamma_1)$ but, as we are focused solely on modular forms defined over Γ_1, we write M_k for convenience.

A modular form that vanishes at infinity is called a cusp form. When f is written in the form of equation (1.6), this is equivalent to a_0 vanishing. We denote the space of all cusp forms of weight k with respect to Γ_1 by S_k and set $S = \bigoplus_k S_k$. Clearly, for a fixed k, each S_k is a subspace of M_k and, in turn, S is a subspace of M.

1.2 EXAMPLES

Having defined what a modular form is, it is natural to take a look at some examples. This will actually be useful for two reasons:

Firstly, it will provide us with an insight into how interesting the theory of modular forms can be. We will see, for example, how the Fourier coefficients of a modular form can contain interesting arithmetic information. Secondly, investigating these functions will provide us with a variety of useful facts about the space of modular forms, which we will use in later chapters. We start by looking at the Eisenstein series.

Definition 1.3. For an integer $k \geq 4$ and even, we define the Eisenstein series of weight k by

$$G_k(z) := \frac{(k-1)!}{2(2\pi i)^k} \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \frac{1}{(mz+n)^k}. \tag{1.7}$$

This is a modular form as it obeys the three conditions of Definition 1.1: Firstly, as $k \geq 4$, we have absolute convergence to a holomorphic function on $\mathfrak{H}$. (We have absolute convergence for $k > 2$ since the number of pairs (m, n) such that $N \leq |mz + n| < N + 1$ is $O(N)$, and therefore, the above series converges like $\sum_{N=1}^{\infty} N/N^k$, see [8] for more information.) Secondly, G_k obeys the transformation law as, for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1$, we have

$$\begin{aligned} G_k(\gamma z) &= \frac{(k-1)!}{2(2\pi i)^k} \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \left(\frac{cz+d}{m(az+b)+n(cz+d)}\right)^k \\ &= \frac{(k-1)!}{2(2\pi i)^k} \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \left(\frac{cz+d}{(ma+nc)z+(mb+nd)}\right)^k \\ &= (cz+d)^k \, G_k(z). \end{aligned}$$

The validity of the argument above relies on the absolute convergence of the sum. Finally, the Fourier expansion of G_k can be calculated to be

$$G_k(z) = \frac{-B_k}{2k} + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n, \tag{1.8}$$

where $q = e^{2\pi i z}$, B_k is the k-th Bernoulli number and $\sigma_x(n)$ is the divisor function, given by $\sum_{d|n} d^x$. We can then deduce that G_k satisfies equation (1.5) and is therefore a modular form. The first few examples of G_k are given by

$$\begin{aligned} G_4(z) &= \frac{1}{240} + q + 9q^2 + 28q^3 + 73q^4 + 126q^5 + \cdots \\ G_6(z) &= -\frac{1}{504} + q + 33q^2 + 244q^3 + 1057q^4 + \cdots \\ G_8(z) &= \frac{1}{480} + q + 129q^2 + 2188q^3 + \cdots \end{aligned}$$

We can also use equation (1.8) to define the Eisenstein series when $k = 2$:

$$G_2(z) = -\frac{1}{24} + q + 3q^2 + 4q^3 + 7q^4 + 6q^5 + 12q^6 + \cdots$$

but, since we no longer have absolute convergence, this does not obey the transformation law. Therefore, G_2 is not a modular form. This Fourier expansion can be adjusted, however, to give a modular form of weight two:

$$G_2^* := G_2 + \frac{1}{8\pi y}.$$

We can see from equation (1.8) that the Fourier expansions of the Eisenstein series contain important arithmetic information relating to the divisor function. For example, the following identities can be derived from the Fourier coefficients of the Eisenstein series of weights ≤ 14 (see [8] for more information).

$$\sum_{m=1}^{n-1} \sigma_3(m)\sigma_3(n-m) = \frac{\sigma_7(n) - \sigma_3(n)}{120},$$

$$\sum_{m=1}^{n-1} \sigma_3(m)\sigma_9(n-m) = \frac{\sigma_{13}(n) - 11\sigma_9(n) + 10\sigma_3(n)}{2640}.$$

In fact, the Fourier coefficients of modular forms, in general, are a remarkable source of arithmetic information and are of great interest to number theorists (and many others). In the next section, we will see how studying a family of operators, called the Hecke operators, can lead to uncovering some of this arithmetic information.

The Eisenstein series G_k can be normalised so that its Fourier expansion has slightly different coefficients. We multiply G_k by a constant so that the zeroth term in its Fourier expansion is equal to 1. We denote this new normalisation by E_k, for example:

$$E_4 = 240G_4 = 1 + 240q + 2160q^2 + 6720q^3 + 17520q^4 + \ldots$$

$$E_6 = -504G_6 = 1 - 504q - 16632q^2 - 122976q^3 - \ldots$$

This normalisation leads nicely to the following definition of a certain cusp form.

Definition 1.4. Let E_4 and E_6 be as above, then the modular discriminant function is given by

$$\Delta(z) := \frac{1}{1728} \cdot \left(E_4^3(z) - E_6^2(z)\right).$$

This function is a modular form of weight 12. Furthermore, as the constant parts of the Fourier series of E_4^3 and E_6^2 cancel out, the modular discriminant function is a cusp form. We can also express this function as

$$\begin{aligned}\Delta(z) &= e^{2\pi iz} \prod_{n=1}^{\infty} \left(1 - e^{2\pi inz}\right)^{24} \\ &= q - 24q^2 + 252q^3 - 1472q^4 + 4830q^5 - \dots\end{aligned}$$

The modular discriminant is an incredibly useful function as it can be used to provide a variety of important results involving the space of modular forms. We state some of these results below without proof (more information on this topic can be found in [8] or [10]).

We have already seen that if f is a modular form of odd weight, then f must be the zero function. The modular discriminant can be used to show that this also holds for any f of negative weight. (Therefore, the inclusion of k being non-negative in Definition 1.1 is not needed.) Furthermore, if $f \in M_0$ (resp. $f \in M_2$) then f must be a constant function (resp. the zero function).

Now, for an even integer $k > 2$ we have the decomposition $M_k = \langle G_k \rangle \oplus S_k$ and we can deduce that

$$dim(M_k) = dim(S_k) + 1.$$

The idea behind this is that, since the Eisenstein series G_k has a Fourier series with a non-zero constant term, we can subtract a multiple of G_k from any $f \in M_k$ to give a weight k cusp form.

The space of weight k cusp forms is isomorphic to the space of weight $k - 12$ modular forms and hence

$$dim(S_k) = dim(M_{k-12}).$$

To see where this identity comes from, we note that, for $f \in S_k$, the quotient $f/\Delta \in M_{k-12}$.

The above facts give us enough information to determine the dimensions of any space M_k or S_k. For example, we know $dim(M_k) = 0$ for $k = -8, -6, -4, -2$ and this implies that $dim(S_k) = 0$ for $k = 4, 6, 8, 10$. This, in turn, implies that $dim(M_k) = 1$ for these values of k and, using this information, we can deduce that $dim(S_k) = 1$ for $k = 16, 18, 20, 22$, and so on.

The dimensions of all M_k and S_k are summarised in the following theorem.

Theorem 1.5. *Let k be an odd or negative integer, then $dim(M_k) = dim(S_k) = 0$. For an even integer $k \geq 0$, the dimensions of M_k and S_k are given by*

k	0	2	4	6	8	10	12	14	$\cdots$	k	$\cdots$	$k+12$	$\cdots$
$dim(M_k)$	1	0	1	1	1	1	2	1		n		$n+1$	
$dim(S_k)$	0	0	0	0	0	0	1	0		$n-1$		n	

We can immediately deduce from this theorem that there are no cusp forms of weight $k < 12$ or $k = 14$. This fact will be of great importance to us in Chapter 4.

1.3 HECKE OPERATORS

In the 1930s Hecke introduced a family of operators that act on the space of modular forms, building upon the work of Mordell. This section serves as an introduction to these operators and will show us many of their interesting properties. For example, we will see how they play a significant role in capturing the important arithmetic information held by the Fourier coefficients of modular forms. First, however, we must define these operators.

Definition 1.6. Let $m > 0$ be an integer, then we define T_m to be the m-th Hecke operator, which acts on a modular form f of weight k in the following way:

$$T_m f(z) := m^{k-1} \sum_{\mu \in \Gamma_1 \backslash \mathcal{M}_m} (cz+d)^{-k} f(\mu(z)).$$

Here $\mu = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, $\mathcal{M}_m$ is the set of all 2×2 matrices with integral entries and determinant m, and $\Gamma_1 \backslash \mathcal{M}_m$ is the (finite) set of cosets.

Remark 1.7. As this definition is dependent on k, we should write $T_m^k f(z)$ but we write $T_m f(z)$ for convenience and consider the weight fixed.

When T_m acts on a modular form of weight k not only is the result still a modular form, it remains a modular form of weight k. In other words, T_m maps the space M_k of weight k modular forms onto itself. Indeed, we can see from the definition that $T_m f$ is still holomorphic and hence obeys equation (1.3). To see how $T_m f$ obeys equation (1.4), we express it in terms of the slash operator defined by equation (1.1). Therefore, the definition of the Hecke operator becomes

$$T_m f(z) = m^{k/2-1} \sum_{\mu \in \Gamma_1 \backslash \mathcal{M}_m} f|_k \mu. \tag{1.9}$$

We can now see that this operator obeys equation (1.4), as equation (1.2) gives $(f|_k\mu)|_k\gamma = f|_k(\mu\gamma)$ and $\{\mu\gamma : \mu \in \Gamma_1 \backslash \mathcal{M}_m\}$, with $\gamma \in \Gamma_1$, is just another set of representatives for $\Gamma_1 \backslash \mathcal{M}_m$. The slash operator also allows us to see that $T_m f$ is well defined as changing the representative for the coset from μ to $\gamma\mu$, with $\gamma \in \Gamma_1$, does not effect the summation, since $f|_k(\gamma\mu) = (f|_k\gamma)|_k\mu = f|_k\mu$.

Finally, to see that $T_m f$ obeys equation (1.5), we will express it in terms of its Fourier series. In order to achieve this, we first note that every equivalence class in $\Gamma_1 \backslash \mathcal{M}_m$ contains exactly one representative $A = \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$ with $d > b \geq 0$ and $a > 0$. Therefore, as the determinant of $A = m$, we have

$$T_m f(z) = m^{k-1} \sum_{\substack{ad=m \\ a,d>0}} d^{-k} \sum_{b=0}^{d-1} f\left(\frac{az+b}{d}\right) \tag{1.10}$$

and for $f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi inz}$ this gives

$$T_m f(z) = m^{k-1} \sum_{\substack{ad=m \\ a,d>0}} d^{-k} \sum_{b=0}^{d-1} \sum_{n=0}^{\infty} a_n e^{2\pi inaz/d} e^{2\pi inb/d},$$

Now, using the fact that

$$\sum_{b=0}^{d-1} e^{2\pi inb/d} = \begin{cases} d & \text{if } d \mid n \\ 0 & \text{otherwise} \end{cases}$$

and setting $n = gd$, we can deduce that

$$T_m f(z) = m^{k-1} \sum_{\substack{ad=m \\ a,d>0}} \sum_{g=0}^{\infty} d^{-k+1} a_{gd} e^{2\pi igaz}. \tag{1.11}$$

Finally, setting $h = ag$, and noting that as $ad = m$ we have $a \mid h$ and $a \mid m$, we get

$$\begin{aligned} T_m f(z) &= \sum_{h=0}^{\infty} \sum_{\substack{a|(h,m) \\ ad=m \\ a,d>0}} (m/d)^{k-1} a_{hd/a} e^{2\pi ihz} \\ &= \sum_{h=0}^{\infty} \Bigg(\sum_{\substack{a|(h,m) \\ a>0}} a^{k-1} a_{hm/a^2} \Bigg) q^h, \end{aligned}$$

where $q = e^{2\pi iz}$. Therefore, we conclude that

$$T_m f(z) = \sum_{h=0}^{\infty} a'_h q^h, \tag{1.12}$$

with

$$a'_h = \sum_{\substack{r|(h,m) \\ r>0}} r^{k-1} a_{hm/r^2}. \tag{1.13}$$

This shows that $T_m f$ obeys equation (1.5), and therefore, it does indeed map the space M_k onto itself. It also follows that if f is a cusp form then so is $T_m f$, as the constant term of $T_m f$ is $\sigma_{k-1}(m) a_0$ and is therefore 0 if $a_0 = 0$.

We also have as a consequence of equation (1.13):

$$T_l T_m = \sum_{r|(l,m)} r^{k-1} T_{lm/r^2} \tag{1.14}$$

and hence

$$T_l T_m = T_m T_l. \tag{1.15}$$

Furthermore if l and m are co-prime then

$$T_l T_m = T_{lm}. \tag{1.16}$$

If we wish to calculate T_m for any m then we only need to know the T_p for the primes p dividing m. To see this, we let $m > 1$ be divisible by a prime p. If $p^2 \mid m$ then equation (1.14) gives

$$T_{m/p} T_p - p^{k-1} T_{m/p^2} = T_m.$$

If $p^2 \nmid m$ then equation (1.16) gives $T_m = T_{m/p} T_p$.

To calculate T_p for p prime we use equation (1.10) to give

$$\begin{aligned} T_p f(z) &= p^{k-1} f(pz) + p^{-1} \sum_{b=0}^{p-1} f\left(\frac{z+b}{p}\right) \\ &= p^{k-1} \sum_{n=0}^{\infty} a_n q^{np} + \sum_{n=0}^{\infty} a_{np} q^n. \end{aligned}$$

For the rest of this section, we will focus on modular forms that are eigenvectors of T_m for every m. If $f(z) = \sum_{n=0}^{\infty} a_n q^n \in M_k$, then such a form satisfies

$$T_m f = \lambda_m f, \tag{1.17}$$

for all $m > 0$ and some complex numbers λ_m. Therefore, by equations (1.13) and (1.17), we have

$$\sum_{n=0}^{\infty} \Bigg(\sum_{\substack{r \mid (n,m) \\ r>0}} r^{k-1} a_{nm/r^2} \Bigg) q^n = \lambda_m \sum_{n=0}^{\infty} a_n q^n \tag{1.18}$$

and, by comparing the coefficients of q^n, this becomes

$$\sum_{\substack{r \mid (n,m) \\ r>0}} r^{k-1} a_{nm/r^2} = \lambda_m a_n. \tag{1.19}$$

Setting $n = 1$, we conclude that

$$a_m = \lambda_m a_1, \quad \forall\ m > 0. \tag{1.20}$$

Definition 1.8. Let $f(z) = \sum_{n=0}^{\infty} a_n q^n$ be a modular form, then f is a Hecke eigenform if $T_m f = \lambda_m f$, for every $m > 0$, and $a_1 = 1$.

If f is a Hecke eigenform then, by equation (1.20), $a_m = \lambda_m$ for all $m > 0$. This shows how closely related the Fourier coefficients of modular forms can be to the eigenvalues of the Hecke operators. Therefore, learning about these eigenvalues will allow us to uncover important arithmetic information held by the Fourier coefficients of modular forms. This idea is examined further in Chapter 3.

An example of a Hecke eigenform is the Eisenstein series G_k (when $k \geq 4$), we have $T_m G_k = a_m G_k = \sigma_{k-1}(m) G_k$.

As seen above, the theory of Hecke operators is closely interwoven with the theory of modular forms. Further proof of this is provided by the fact that these operators play a fundamental role in the internal structure of the space of modular forms. We will not explore this topic in detail here, instead we state the theorem below to provide a brief example of this and refer the reader to [10] for more information.

Theorem 1.9. *The Hecke eigenforms in M_k form a basis of M_k for every k.*

Hecke operators and Hecke eigenforms also serve as useful tools for the introduction and definition of L-functions of modular forms.

1.4 *L*-FUNCTIONS

The theory of L-functions is vast and not at all restricted to modular forms. In fact, they can be used as a useful tool in associating various arithmetic objects to each other, for example, modular forms with elliptic curves. The information contained within the location of its zeroes and poles can also be used as a way to study arithmetic objects. Furthermore, they provide us with information relating to the distribution of primes and appear in the Birch and Swinnerton-Dyer conjecture.

In this section, we will explore some of the properties that belong to an L-function of a modular form, such as analytic continuation and satisfying a functional equation. We return to the topic of Hecke eigenforms to help us define the L-function of a modular form.

If $f(z) = \sum_{n=0}^{\infty} a_n q^n$ is a Hecke eigenform, then $T_m f = \lambda_m f$ with $a_m = \lambda_m$. Inserting this into equation (1.19) gives

$$\sum_{\substack{r|(n,m) \\ r>0}} r^{k-1} a_{nm/r^2} = a_m a_n. \tag{1.21}$$

Therefore, when n and m are co-prime we have $a_{nm} = a_m a_n$. Combining this with the fact that $a_1 = 1$, we see that the sequence $\{a_n\}$ is multiplicative and, as is often the case when presented with such a sequence, we define a Dirichlet series. Specifically, we define the L-Function (or L-series) of a Hecke eigenform f by

$$L_f(s) := \sum_{n=1}^{\infty} \frac{a_n}{n^s}.$$

As the coefficients a_n are multiplicative, we have

$$\begin{aligned} L_f(s) &= \prod_{p \text{ prime}} \left(1 + a_p p^{-s} + a_{p^2} p^{-2s} + \cdots\right) \\ &= \prod_{p \text{ prime}} A_p(p^{-s}), \end{aligned} \tag{1.22}$$

where $A_p(x) = \sum_{t=0}^{\infty} a_{p^t} x^t$. Now inserting $m = p$ and $n = p^t$ into equation (1.21) gives

$$a_{p^{t+1}} = a_p a_{p^t} - p^{k-1} a_{p^{t-1}},$$

which we will use to rewrite $A_p(x)$.

$$\begin{aligned} A_p(x) &= 1 + \sum_{t=0}^{\infty} a_{p^{t+1}} x^{t+1} \\ &= 1 + \sum_{t=0}^{\infty} \left(a_p a_{p^t} - p^{k-1} a_{p^{t-1}}\right) x^{t+1} \\ &= 1 + \sum_{t=0}^{\infty} a_p a_{p^t} x^{t+1} - \sum_{t=1}^{\infty} p^{k-1} a_{p^{t-1}} x^{t+1} \\ &= 1 + a_p x A_p(x) - p^{k-1} x^2 A_p(x) \end{aligned}$$

and hence

$$A_p(x) = \frac{1}{1 - a_p x + p^{k-1}x^2}.$$

This allows us to rewrite equation (1.22) as

$$L_f(s) = \prod_{p \text{ prime}} \frac{1}{1 - a_P p^{-s} + p^{k-1}p^{-2s}}.$$

When f is given by the Eisenstein series G_k, with $k \geq 4$, we have

$$L_{G_k}(s) = \prod_{p \text{ prime}} \frac{1}{1 - \sigma_{k-1}(p)p^{-s} + p^{k-1-2s}}$$

and, as

$$\sigma_{k-1}(p) = \sum_{d|p} d^{k-1} = 1 + p^{k-1},$$

we conclude that

$$\begin{aligned} L_{G_k}(s) &= \prod_{p \text{ prime}} \frac{1}{1 - p^{-s} - p^{k-1-s} + p^{k-1-2s}} \\ &= \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}} \cdot \frac{1}{1 - p^{k-1-s}}. \end{aligned}$$

We will now remove our restriction that f has to be a Hecke eigenform and extend our definition of an L-function to all modular forms.

Definition 1.10. Let $f(z) = \sum_{n=0}^{\infty} a_n q^n$ be any modular form (not necessarily a Hecke eigenform), then we define the L-function (or L-series) of f by

$$L_f(s) := \sum_{n=1}^{\infty} \frac{a_n}{n^s}.$$

Furthermore, we define the completed L-function of f by

$$L_f^*(s) := \frac{\Gamma(s)}{(2\pi)^s} L_f(s) = \frac{\Gamma(s)}{(2\pi)^s} \sum_{n=1}^{\infty} \frac{a_n}{n^s},$$

where

$$\Gamma(s) = \int_0^{\infty} e^{-t} t^{s-1}\, dt$$

is the Gamma function.

For the rest of this section, we assume f to be a cusp form for simplicity. The methods used below can also be applied to non-cusp forms but more work is required. This is because, for example, we have problems with convergence if f is not a cusp form.

The L-function of a cusp form of weight k converges absolutely in the half plane $Re(s) > 1 + k/2$, we will see how the completed L-function extends this to an entire function of s. We consider the Mellin transform of f, which is given by

$$\{\mathcal{M}f\}(s) := \int_0^\infty f(it)t^{s-1}\,dt. \tag{1.23}$$

After inserting $f(z) = \sum_{n=1}^\infty a_n q^n$ and then using the substitution $2\pi nt \to t$, we have

$$\begin{aligned}\int_0^\infty f(it)t^{s-1}dt &= \int_0^\infty \sum_{n=1}^\infty a_n e^{-2\pi nt} t^{s-1}\,dt \\ &= \int_0^\infty \sum_{n=1}^\infty \frac{a_n e^{-t} t^{s-1}}{(2\pi n)^s}\,dt \\ &= \frac{1}{(2\pi)^s}\int_0^\infty e^{-t}t^{s-1}\,dt \sum_{n=1}^\infty \frac{a_n}{n^s} \\ &= \frac{\Gamma(s)}{(2\pi)^s}\sum_{n=1}^\infty \frac{a_n}{n^s}\end{aligned}$$

(we can move the summation and integration around since we have absolute convergence). Therefore, we have

$$L_f^*(s) = \int_0^\infty f(it)t^{s-1}\,dt.$$

Since f is a cusp form, we can deduce that $f(it)$ tends to zero exponentially as t tends to infinity. Furthermore, $f(it)$ also tends to zero exponentially as t tends to zero, as the transformation law for f (Remark 1.2) when $\gamma = S$ gives $f(-1/z) = z^k f(z)$. This implies that the integral above converges absolutely for all $s \in \mathbb{C}$, and therefore, $L_f^*(s)$ extends holomorphically to the entire complex plane. This representation of $L_f^*(s)$ also allows us to easily see that it obeys the functional equation

$$L_f^*(k-s) = i^k L_f^*(s),$$

as

$$L_f^*(k-s) = \int_0^\infty f(it)t^{k-s-1}dt$$

and the substitution $t \to 1/t$ gives

$$L_f^*(k-s) = \int_0^\infty f(-1/it)t^{s-k-1}\,dt.$$

Now, again by the transformation law of f, we have $f(-1/it) = (it)^k f(it)$ and hence

$$L_f^*(k-s) = \int_0^\infty (it)^k f(it)t^{s-k-1}\,dt = i^k L_f^*(s).$$

1.5 PERIOD POLYNOMIALS

Period polynomials are fundamental objects associated to modular cusp forms, which help reveal crucial arithmetic information about L-functions. The period polynomial of a weight k cusp form is a combination of the periods of f. We define these periods to be the $k-1$ numbers

$$r_n(f) := \int_0^{i\infty} f(\tau)\tau^n d\tau, \tag{1.24}$$

where $0 \le n \le k-2$. We can see how closely this relates to our definition of an L-function:

$$r_n(f) = \int_0^{i\infty} f(\tau)\tau^n d\tau = \int_0^\infty f(it)(it)^n\, i\,dt = \; i^{n+1}L_f^*(n+1).$$

The period polynomial of f is defined by combining these periods.

Definition 1.11. Let f be a modular cusp form, then the period polynomial of f is given by

$$P_f(z) := \sum_{n=0}^{k-2} \binom{k-2}{n} (-1)^n\, r_n(f) z^{k-2-n} = \sum_{n=0}^{k-2} \binom{k-2}{n} i^{1-n} L_f^*(n+1) z^{k-2-n}. \tag{1.25}$$

Let P_{k-2} denote the space of polynomials with degree less than or equal to $k-2$ in one variable, this is acted upon by Γ_1 via $|_{2-k}$. For example

$$z^2|_{2-k}\,\gamma = (cz+d)^{k-2}(\gamma z)^2,$$

where $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1$. We can see that the period polynomial of a modular cusp form is an element of P_{k-2}. By using equation (1.24) and applying the binomial formula, P_f can also be expressed as

$$P_f(z) = \int_0^{i\infty} f(\tau)(\tau - z)^{k-2} d\tau. \tag{1.26}$$

Period polynomials have been interpreted in many different ways, one such way is from the perspective of Eichler cohomology. A good introduction to Eichler cohomology was given in both [11] and [12], we will give a summary of these below.

We start with some general cohomological theory which we will then specialise for Eichler cohomology. We let M be a right Γ_1-module and for an integer $i \geq 0$ we call a map from Γ_1^i to M an i-cochain for Γ_1 with coefficients in M. The group formed by such maps is denoted by $C^i(\Gamma_1, M)$. The differential $d^i \colon C^i(\Gamma_1, M) \to C^{i+1}(\Gamma_1, M)$ is given by

$$\begin{aligned}(d^i\sigma)(g_1, ..., g_{i+1}) :=\ & \sigma(g_2, ..., g_{i+1}).g_1 \\ & + \sum_{j=1}^{i} (-1)^j \sigma(g_1, ..., g_{j+1}g_j, ..., g_{i+1}) + (-1)^{i+1}\sigma(g_1, .., g_i).\end{aligned} \tag{1.27}$$

Definition 1.12. With the above notation, we define the group of i-cocycles to be the kernel of the map d^i

$$Z^i(\Gamma_1, M) := \operatorname{Ker}(d^i)$$

and the group of i-coboundaries to be the image of d^{i-1}

$$B^i(\Gamma_1, M) := \operatorname{Im}(d^{i-1}).$$

We set $B^0(\Gamma_1, M) := 0$. Finally, we define the i-th cohomology group to be the quotient group

$$H^i(\Gamma_1, M) := Z^i(\Gamma_1, M)/B^i(\Gamma_1, M).$$

In Eichler cohomology the module M is P_{k-2} and, in this thesis, we will focus on the 1st cohomology group $H^1(\Gamma_1, P_{k-2})$. A 0-cochain $\phi \in C^0(\Gamma_1, P_{k-2})$

is an element $P \in P_{k-2}$. For $i = 0$, we can then deduce that equation (1.27) becomes

$$(d^0\phi)(\gamma) = P|_{k-2}\gamma - P, \quad \gamma \in \Gamma_1.$$

A 1-coboundary is then a map from Γ_1 to P_{k-2} of the form

$$\gamma \mapsto P|_{2-k}\gamma - P, \quad \gamma \in \Gamma_1,$$

for some fixed $P \in P_{k-2}$. We now take a closer look at 1-cocycles, we first note that

$$(d^1\phi)(\gamma_2, \gamma_1) = \phi(\gamma_1)|_{2-k}\gamma_2 - \phi(\gamma_1\gamma_2) + \phi(\gamma_2), \quad \forall\, \gamma_1, \gamma_2 \in \Gamma_1.$$

Setting $(d^1\phi)(\gamma_2, \gamma_1) = 0$, we see that a 1-cocycle is a map $\phi\colon \Gamma_1 \to P_{k-2}$ such that

$$\phi(\gamma_1\gamma_2) = \phi(\gamma_1)|_{2-k}\gamma_2 + \phi(\gamma_2), \quad \forall\, \gamma_1, \gamma_2 \in \Gamma_1.$$

The identity

$$P|_{2-k}\gamma_1\gamma_2 - P = (P|_{2-k}\gamma_1 - P)|_{2-k}\gamma_2 + (P|_{2-k}\gamma_2 - P)$$

allows us to see that $B^1(\Gamma_1, P_{k-2})$ is a subspace of $Z^1(\Gamma_1, P_{k-2})$, and therefore, we can confirm that the quotient group $H^1(\Gamma_1, P_{k-2})$ does indeed make sense. We define the Eichler integral of f by

$$v_f(z) := \int_{i\infty}^{z} f(\tau)(\tau - z)^{k-2} d\tau.$$

If we apply the slash operator to v_f, we have, for any $\gamma \in \Gamma_1$, the equation

$$(v_f|_{2-k}\gamma)(z) = \int_{i\infty}^{\gamma z} f(\tau)(\tau - \gamma z)^{k-2}(cz + d)^{k-2} d\tau$$

and the substitution $\tau \to \gamma\tau$ gives

$$\begin{aligned}(v_f|_{2-k}\gamma)(z) &= \int_{\gamma^{-1}(i\infty)}^{z} f(\gamma\tau)(\gamma\tau - \gamma z)^{k-2}(cz + d)^{k-2}(c\tau + d)^{-2} d\tau \\ &= \int_{\gamma^{-1}(i\infty)}^{z} f(\tau)(\gamma\tau - \gamma z)^{k-2}(cz + d)^{k-2}(c\tau + d)^{k-2} d\tau \\ &= \int_{\gamma^{-1}(i\infty)}^{z} f(\tau)(\tau - z)^{k-2} d\tau,\end{aligned}$$

where we used the identity $(\gamma\tau - \gamma z)(cz+d)(c\tau+d) = (\tau - z)$. Therefore, the period polynomial can be viewed in terms of the Eichler integral:

$$P_f(z) = (v_f|_{2-k}S)(z) - v_f(z), \tag{1.28}$$

where $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. This can also be written in a slightly more concise manner:

$$P_f(z) = (v_f|_{2-k}(S-1))(z). \tag{1.29}$$

(We note that we are still performing the calculation the same way as in equation (1.28), but we write it this way for convenience.) Looking back at the general case, for any $\gamma \in \Gamma_1$ we can define a 1-cocycle by

$$\sigma_f(\gamma) := d^0 v_f = v_f|_{2-k}(\gamma) - v_f = v_f|_{2-k}(\gamma - 1),$$

since σ_f is defined as the differential of a 0-cochain it will be a 1-coboundary, and therefore, it will satisfy the 1-cocycle relation. Furthermore, it is a 1-cocycle in P_{k-2}. To see how σ_f is an element of P_{k-2}, we can use the same procedure as above to write it as

$$\sigma_f(\gamma) = \int_{\gamma^{-1}(i\infty)}^{i\infty} f(\tau)(\tau - z)^{k-2} d\tau, \quad \gamma \in \Gamma_1.$$

This is an essential object in the theory of modular forms. An application of this cocycle can be used to give an isomorphism between the Eichler cohomology group and the space of modular forms. Specifically, the Eichler-Shimura Theorem tells us that

$$\sigma \colon \bar{S}_k \oplus M_k \xrightarrow{\sim} H^1(\Gamma_1, P_{k-2}), \tag{1.30}$$

where $\bar{S}_k = \{\bar{f} : f \in S_k\}$, and the isomorphism σ is induced by the assignment of $f \in M_k$ and $\bar{g} \in \bar{S}_k$ to σ_f and $\sigma_{\bar{g}}$, respectively. In some of the literature the map

$$\phi_f(\gamma) := \int_i^{\gamma^{-1}i} f(\tau)(\tau - z)^{k-2} d\tau, \quad \gamma \in \Gamma_1, \tag{1.31}$$

is used instead of $\sigma_f(\gamma)$, this difference is unimportant though since these cocycles belong to the same cohomology class. Indeed $v_f(z)$ can be written as

$$v_f(z) = \int_{i\infty}^{i} f(\tau)(\tau - z)^{k-2} d\tau + \int_i^z f(\tau)(\tau - z)^{k-2} d\tau$$

and we can deduce that

$$\sigma_f(z) = \left(\int_{i\infty}^{i} f(\tau)(\tau - z)^{k-2} d\tau \right) \Big|_{2-k} (\gamma - 1) + \int_{\gamma^{-1}i}^{i} f(\tau)(\tau - z)^{k-2} d\tau.$$

Since the first integral belongs to P_{k-2}, then $\sigma_f(\gamma)$ and $\phi_f(\gamma)$ differ by a 1-coboundary, and therefore, they belong to the same cohomology class.

An example of the importance of the above isomorphism, and of period polynomials in general, is given by its significant role in the proving the following theorem.

Theorem 1.13 (Manin's Periods Theorem [13]). *Let $f \in S_k$ be a Hecke eigenform and K_f denote the field generated by the Fourier coefficients of f. Then there exist $\omega^{\pm}(f) \in \mathbb{C}$ such that*

$$L_f^*(j) \in \omega^+(f) K_f, \quad \textit{for odd } j \in \{1, \ldots, k-1\}$$

and

$$L_f^*(j) \in \omega^-(f) K_f, \quad \textit{for even } j \in \{1, \ldots, k-1\}.$$

This theorem allows us to see how the period polynomial can be used to reveal crucial arithmetic information about the L-function, in this case at the arguments $1, 2, \ldots, k-1$.

1.6 PERIOD FUNCTIONS ASSOCIATED TO MAASS WAVE CUSP FORMS

In this section, we will see how the theory of period polynomials for modular cusp forms can be extended to Maass wave cusp forms (usually abbreviated to Maass forms). The analogue of the period polynomial for a Maass form is the period function. This function was first introduced for Maass functions, a subset of Maass forms.

Definition 1.14. A Maass wave cusp function on Γ_1 of eigenvalue $\lambda \in \mathbb{C}$ is a smooth function $u\colon \mathfrak{H} \to \mathbb{C}$ such that

(*i*) $u(\gamma z) = u(z), \quad \forall\, \gamma \in \Gamma_1,$

(*ii*) $\forall\, r \in \mathbb{R}$, u satisfies the growth condition $u(z) = \mathcal{O}(y^r)$ as $y \to \infty$ (where $z = x + iy$),

(*iii*) There exists a $\lambda \in \mathbb{C}$ such that $\Omega_0 u = \lambda u$,

where Ω_0 is given by $-y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$. Furthermore, u is an even Maass function if $u(z) = u(-\bar{z})$, $\forall\, z \in \mathfrak{H}$.

In the late 1990s, Lewis [14] proved there was a 1-1 correspondence between the space of even Maass functions that have eigenvalue $\lambda = s(1-s)$ and the space of holomorphic functions $\psi\colon \mathbb{C}/(-\infty, 0] \to \mathbb{C}$ such that

$$\psi(z) := \psi(z+1) + z^{-2s}\,\psi\left(\frac{z+1}{z}\right).$$

Lewis gave this correspondence via an integral transform. This method, however, did not make the relation between ψ and the period polynomial of a classical modular form obvious.

Shortly after, Lewis and Zagier [15] gave a different description of ψ and extended the correspondence between ψ and even Maass functions to include odd Maass functions, they did this by making use of L-functions. The authors also showed how the function ψ corresponding to a Maass function is the analogue of the period polynomial for a modular form, and hence, why we will call ψ the period function corresponding to u. Mühlenbruch [16] extended the concept of period functions for Maass functions to the larger space of Maass forms.

Definition 1.15. Let $k \in 2\mathbb{Z}$, we then define a Maass wave cusp form on Γ_1 of weight k and eigenvalue $\lambda \in \mathbb{C}$ as a smooth function $u\colon \mathfrak{H} \to \mathbb{C}$ such that

$$(i) \quad u(\gamma z) = \left(\frac{cz+d}{|cz+d|}\right)^k u(z), \quad \forall\, \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1,$$

(ii) $\forall\, r \in \mathbb{R}$, u satisfies the growth condition $u(z) = \mathcal{O}(y^r)$ as $y \to \infty$ (where $z = x + iy$),

(iii) There exists a $\lambda \in \mathbb{C}$ such that $\Omega_k u = \lambda u$, (1.32)

where Ω_k is the Casimir operator given by

$$-y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + iky\frac{\partial}{\partial x}. \tag{1.33}$$

We denote the space of all Maass forms of weight k and eigenvalue λ by $S(k,\lambda)$. We can see that the space of Maass functions with eigenvalue λ is a subset of $S(k,\lambda)$, denoted by $S(0,\lambda)$. In Chapter 2.3 of [16], Mühlenbruch demonstrated how to associate a period function to a Maass form of weight k. We will summarise how he did this below:

We start with some definitions, we let $k \in \mathbb{R}$ and define the Maass operators E_k^+ and E_k^- by

$$E_k^+ := 2iy\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y} + k \quad \text{and} \quad E_k^- := -2iy\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y} - k. \tag{1.34}$$

These act on real-analytic functions and can also be written as

$$E_k^+ = 4iy\frac{\partial}{\partial z} + k \quad \text{and} \quad E_k^- = -4iy\frac{\partial}{\partial \bar{z}} - k.$$

Ideally, the period function we find will have a form that can be represented as an integral, like equation (1.26). Furthermore, the integrand of this integral will be closed. Why the closure of the integrand of a period polynomial is important is not fully explored in this thesis, but an idea of this importance can be seen in allowing us to express the period polynomial in terms of its Eichler Integral. Without this closure, the equality

$$\int_{\gamma^{-1}(i\infty)}^{z} f(\tau)(\tau-z)^{k-2}d\tau + \int_{z}^{i\infty} f(\tau)(\tau-z)^{k-2}d\tau = \int_{\gamma^{-1}(i\infty)}^{i\infty} f(\tau)(\tau-z)^{k-2}d\tau$$

used to give equation (1.28) would not be certain. We did not have to worry about closure when dealing with standard modular forms since these are holomorphic, unlike Maass wave forms.

To help us construct a closed form, we define the function $R_{k,\nu}(z,\zeta)$ as follows:

$$R_{k,\nu}(z,\zeta) := \left(\frac{\zeta-\bar{z}}{\zeta-z}\right)^{\frac{k}{2}} \left(\frac{\mathrm{Im}(z)}{(\zeta-z)(\zeta-\bar{z})}\right)^{1/2-\nu}$$

for $z \in \mathfrak{H}$, $\nu \in \mathbb{C}$ and $\zeta \in \mathbb{R}$. In the next proposition, we give some useful equations involving this function.

Proposition 1.16 (Prop. 36 of [16])**.** *Let $z \in \mathfrak{H}$, $\nu \in \mathbb{C}$ and $\zeta, k \in \mathbb{R}$. The function $R_{k,\nu}$ satisfies the following equations:*

$$i) \quad \Omega_k R_{k,\nu}(\cdot,\zeta) = (1/4-\nu^2) R_{k,\nu}(\cdot,\zeta). \tag{1.35}$$

$$ii) \quad E_k^{\pm} R_{k,\nu}(\cdot,\zeta) = (1-2\nu \pm k) R_{k\pm 2,\nu}(\cdot,\zeta).$$

Now let f and g be smooth functions in an open subset U of $\mathfrak{H} \cup \bar{\mathfrak{H}}$. For $z = x+iy$, we define

$$\{f,g\}^+(z) := f(z)g(z)\frac{dz}{y} \quad \text{and} \quad \{f,g\}^-(z) := f(z)g(z)\frac{d\bar{z}}{y}. \tag{1.36}$$

We are now ready to define the Maass-Selberg form, on which the definition of the period function is based.

Definition 1.17. Let $k \in \mathbb{R}$, the Maass-Selberg form $\tilde{\eta}_k$ is defined by

$$\tilde{\eta}_k(f,g) := \{E_k^+ f, g\}^+ - \{f, E_{-k}^- g\}^-.$$

The next lemma provides the key property of the Maass-Selberg form that we are in search of.

Lemma 1.18. *(Lemma 39 of [16]) Suppose that, for some $\lambda \in \mathbb{R}$, we have $\Omega_k f = \lambda f$ and $\Omega_{-k} g = \lambda g$. Then $\tilde{\eta}_k(f,g)$ is closed.*

Finally, we can now give the definition of the period function of a Maass wave form.

Definition 1.19. Let $\zeta \in (0, \infty)$, $\nu \in \mathbb{C}$, $k \in 2\mathbb{Z}$ and let $u \in S(k, 1/4 - \nu^2)$. We then associate the period function $P_{k,\nu}(\zeta)$ to u by

$$P_{k,\nu}(\zeta) := \int_0^{i\infty} \tilde{\eta}_k(u, R_{-k,\nu}(\cdot, \zeta)).$$

Remark 1.20. We know that $\tilde{\eta}_k(u, R_{-k,\nu}(\cdot, \zeta))$ is closed, since equation (1.32) tells us that $\Omega_k u = (1/4 - \nu^2)u$ and equation (1.35) gives

$$\Omega_{-k} R_{-k,\nu}(\cdot, \zeta) = (1/4 - \nu^2) R_{-k,\nu}(\cdot, \zeta).$$

We will briefly give some justification why this period function is consistent with the period polynomial of a modular cusp form. (More information on this consistency is given in [16].) We let $k \geq 2$ be an even integer and $f \in S_k$, then we have the map $\sigma\colon S_k \to S(k, k/2 \cdot (1 - k/2))$ given by

$$\sigma(f)(z) := \operatorname{Im}(z)^{k/2} f(z).$$

The period polynomial $P_{k,(k-1)/2}$ associated to $\sigma(f)$ is given in the proof of Proposition 49 of [16] to be

$$P_{k,\frac{k-1}{2}}(\zeta) = (2k - 2) \int_0^{i\infty} f(z)(z - \zeta)^{k-2} dz.$$

By comparison, the period polynomial of f, given by equation (1.26), is

$$P_f(\zeta) = \int_0^{i\infty} f(z)(z - \zeta)^{k-2} dz.$$

1.7 WEAKLY HOLOMORPHIC MODULAR FORMS

Since weakly holomorphic modular forms will begin to appear throughout this thesis, we will provide a brief description of these forms in this section. A weakly holomorphic modular form is similar to a standard modular form except it is no longer required to be holomorphic as $z \to i\infty$.

Definition 1.21. Let k be an integer. A function $f\colon \mathfrak{H} \to \mathbb{C}$ is a weakly holomorphic modular form of weight k with respect to Γ_1 if it obeys the following conditions:

(i) f is holomorphic on $\mathfrak{H}$ (and meromorphic as $z \to i\infty$).

(ii) $(f|_k\gamma)(z) = f(z), \quad \forall\, \gamma \in \Gamma_1.$

As with standard modular forms, any weakly holomorphic modular form f has a Fourier expansion. In this case any such f can be written in the form

$$f(z) = \sum_{n=-N}^{\infty} a_n e^{2\pi i n z}, \tag{1.37}$$

where $N \in \mathbb{N}$. We denote the space of all weakly holomorphic modular forms of weight k with respect to Γ_1 by $M_k^!(\Gamma_1)$ or, since we are only focusing on forms defined over Γ_1, by $M_k^!$. We set $M^! = \bigoplus_k M_k^!$.

Unlike with standard modular forms, there exist weakly holomorphic forms of negative weight. We can also have non-constant weight 0 weakly holomorphic forms.

A weakly holomorphic modular form is called a weakly holomorphic cusp form if its Fourier expansion has no constant term ($a_0 = 0$). We denote the space of all weakly holomorphic cusp forms of weight k by $S_k^!$ and set $S^! = \bigoplus_k S_k^!$, clearly $S_k^!$ is a subspace of $M_k^!$. Furthermore, the space of standard holomorphic modular forms (resp. standard holomorphic cusp forms) of weight k, denoted by M_k (resp. S_k), is a subspace of $M_k^!$ (resp. $S_k^!$).

The usual definition of an L-function for a standard modular form, given by

$$L_f^*(s) = \int_0^\infty f(it)t^{s-1}dt, \tag{1.38}$$

is no longer well defined for $f \in S_k^!$ since $f(it)$ has a possible pole at $t = \infty$.

We shall only briefly discuss L-functions and period polynomials for weakly holomorphic modular forms but more information on this topic can be found in [17]. It is shown in [17] that we can define the L-function of a weakly holomorphic cusp form f of weight k by

$$L_f^*(s) := \sum_{n=-N}^{\infty} \frac{a_n \Gamma(s, 2\pi n)}{(2\pi n)^s} + i^k \sum_{n=-N}^{\infty} \frac{a_n \Gamma(k-s, 2\pi n)}{(2\pi n)^{k-s}}, \tag{1.39}$$

where $\Gamma(s,z) = \int_z^\infty e^{-t} t^{s-1} dt$ is the incomplete Gamma function. The absolute convergence of $L_f^*(s)$ is guaranteed, see the remarks below equation (1.5) of [17] for more information. When f is a standard modular form this definition can be re-written as the original definition of an L-function, given by (1.38).

The period polynomial of a weakly holomorphic cusp form f of weight k is given by

$$P_f(z) := \sum_{n=0}^{k-2} \binom{k-2}{n} i^{1-n} L_f^*(n+1) z^{k-2-n},$$

which matches one of our original expressions for the period polynomial (1.25).

Hecke theory for weakly holomorphic modular forms is also different compared to the standard holomorphic forms case. We will see that, generally speaking, there are no Hecke eigenforms in $M_k^!$ (apart from those included in M_k). Nevertheless, it was shown in [18] that we can still have meaningful Hecke theory for $M_k^!$, involving operators induced by the standard Hecke operators. We closely follow the work of [18] and [19] for the rest of this section.

First, we note that, as in the holomorphic case, T_m preserves $M_k^!$ and $S_k^!$:

$$T_m(M_k^!) \subseteq M_k^! \qquad \text{and} \qquad T_m(S_k^!) \subseteq S_k^!.$$

However, when we compare the Fourier coefficients for $T_m f$ and $T_m g$, where $f = \sum_{n=0}^{\infty} a_n q^n \in M_k$ and $g = \sum_{n=-N}^{\infty} b_n q^n \in M_k^!$, we begin to see some differences. We showed in Section 1.3, specifically equation (1.12), that

$$T_m f(z) = \sum_{h=0}^{\infty} a'_h q^h.$$

However, using a similar process to the one we used to obtain the equation above but for g, we have

$$T_m g(z) = \sum_{h=-mN}^{\infty} b'_h q^h$$

and therefore $T_m g$ can have a possible pole at $i\infty$ of order mN. Comparing this with g, which has possible pole at $i\infty$ of maximum order N, we see the difficulty with finding weakly holomorphic Hecke eigenforms.

In order to solve this problem, we consider the operator $D = \frac{1}{2\pi i}\frac{d}{dz}$. Generally, this operator does not preserve modularity, however, in [20] it is shown that

$$D^{k-1}(M^!_{2-k}) \subseteq M^!_k.$$

It is then shown in [18] that the Hecke operators commute with D in the following way:

$$T_m D^{k-1} g = m^{k-1} D^{k-1} T_m g, \tag{1.40}$$

where $g \in M^!_{2-k}$. Combining this with the fact that T_m preserves $M^!_{2-k}$ implies that $D^{k-1}(M^!_{2-k})$ is preserved under T_m:

$$T_m \left(D^{k-1}(M^!_{2-k}) \right) \subseteq D^{k-1}(M^!_{2-k}).$$

A weakly holomorphic Hecke eigenform is then defined to be any $f \in M^!_k \backslash D^{k-1}(M^!_{2-k})$ such that, for all $m > 0$,

$$(T_m - \lambda_m) f = h_m, \tag{1.41}$$

where $h_m \in D^{k-1}(M^!_{2-k})$ and $\lambda_m \in \mathbb{C}$. This definition can indeed be used to give new Hecke eigenforms and to provide a meaningful Hecke theory for $M^!_k$. For example, in [19] the authors consider the elements $[f] := f + D^{2k-1}(M^!_{2-k})$ of the quotient space $M^!_k / D^{k-1}(M^!_{2-k})$. A function f that obeys equation (1.41) can be considered an eigenvector $[f]$ of the Hecke operators in that factor space. We will see an application of this particular Hecke theory in Section 2.6.3.

2

L-FUNCTIONS AND PERIOD FUNCTIONS ASSOCIATED TO REAL ANALYTIC MODULAR FORMS

The space of real analytic modular forms, denoted by $\mathcal{M}^!$, was recently introduced by Brown [4–6]. This space is of great interest to number theorists (and others) as it contains or intersects all the spaces we studied in the previous chapter. Indeed, these real analytic forms can be viewed as a unifying tool for such spaces.

In this chapter, we will define L-functions for the entirety of $\mathcal{M}^!$ which, when restricted to specific subspaces, will match the L-functions we gave in the previous chapter.

We are not, however, able to define period functions for all of $\mathcal{M}^!$. This is understandable though, since the period polynomial usually reflects arithmeticity and the space $\mathcal{M}^!$ is too large to be of arithmetic nature in its entirety. Nevertheless, there exists a space of modular iterated integrals, denoted by $\mathcal{MI}^!$, which sits inside the class of real analytic modular forms. We show that it is possible to define period functions and, indeed, period polynomials for modular iterated integrals of length one.

2.1 AN INTRODUCTION TO REAL ANALYTIC MODULAR FORMS

We will shortly define the space of real analytic modular forms and then explore its underlying theory. We must first, however, define a new type of slash operator.

For integers r, s, a function $f\colon \mathfrak{H} \to \mathbb{C}$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL(2, \mathbb{R})$ with $det(\gamma) > 0$, we define the double slash operator $f||_{r,s}$ by

$$(f||_{r,s}\gamma)(z) := det(\gamma)^{\frac{r+s}{2}}(cz+d)^{-r}(c\bar{z}+d)^{-s}f(\gamma z), \ \ \forall\, z \in \mathfrak{H}.$$

It is easy to see that $f||_{r,s}$ is reduced to the single slash operator (1.1) when $s = 0$. As with the single slash operator, this is a group operation:

$$f||_{r,s}(\gamma\mu) = (f||_{r,s}\gamma)||_{r,s}\,\mu\,, \quad \forall\, \gamma, \mu \in \Gamma_1.$$

Definition 2.1. Let r and s be integers such that $r+s \in 2\mathbb{Z}$. A real analytic function $f\colon \mathfrak{H} \to \mathbb{C}$ is a real analytic modular form of weights (r, s) with respect to Γ_1 if

(i) $(f||_{r,s}\gamma)(z) = f(z), \quad \forall\, \gamma \in \Gamma_1$ and $z \in \mathfrak{H}$. (2.1)

(ii) For some $C > 0$, $f(z) = \mathcal{O}(e^{Cy})$ as $y \to \infty$, uniformly in x (where $z = x + iy$). Furthermore,

$$f(z) = \sum_{|j| \le M} y^j \left(\sum_{m,n \ge -N} a_{m,n}^{(j)} e^{2\pi i m z} e^{-2\pi i n \bar{z}} \right) \tag{2.2}$$

for some $M, N \in \mathbb{N}$, $a_{m,n}^{(j)} \in \mathbb{C}$.

Remark 2.2. We call (i) the transformation law. This can also be written as $f(\gamma z) = (cz+d)^r(c\bar{z}+d)^s f(z)$, $\forall\, \gamma \in \Gamma_1$.

We denote the space of all real analytic modular forms of weights (r, s) with respect to Γ_1 by $\mathcal{M}^!_{r,s}$ and set $\mathcal{M}^! = \bigoplus_{r,s} \mathcal{M}^!_{r,s}$. As mentioned above, the space $\mathcal{M}^!$ contains or intersects previously studied classes of important modular objects; the idea is that we should be able to use real analytic modular forms as a unifying tool for such objects.

For example, when $s = 0$ a holomorphic function f of $\mathcal{M}^!_{r,0}$ is a weakly holomorphic form of weight r, we have seen that the space of such forms is denoted by $M^!_r$. When $r = 0$, we obtain a similar space of weakly anti holomorphic modular forms which is donated by $\overline{M}^!_s$.

If we restrict to the condition that $a^{(j)}_{m,n} = 0$ if either m or n are negative, then we have a new space denoted by $\mathcal{M}_{r,s}$. This space $\mathcal{M}_{r,s}$ was defined and studied thoroughly in [4]. A simple example of a function in $\mathcal{M}_{r,s}$, and specifically $\mathcal{M}_{-1,-1}$, is given by

$$\mathbb{L} := i\pi(z - \bar{z}) = -2\pi y, \quad \text{where } z = x + iy,$$

this function will begin to appear throughout this thesis. Of course, the function y by itself also belongs to $\mathcal{M}_{-1,-1}$ and we have that, for any $k \in \mathbb{Z}$, $\mathbb{L}^k$ and y^k exist in $\mathcal{M}_{-k,-k}$.

This space $\mathcal{M}^!$ is also connected to the space $S(k, \lambda)$ of Maass forms that we saw in the previous chapter. However, as we are no longer limited to forms that are eigenvalues of the Casimir operator (instead we are more restricted by the Fourier expansion than Maass forms are), this overlap is limited.

Having said this, the overlap is large enough for us to exploit it and define a period function. In particular, we will see that the Maass-Selberg form, defined in the previous chapter, plays a crucial role in our construction for period functions of a certain subspace of $\mathcal{M}^!$.

The Fourier expansion in equation (2.2) can be decomposed uniquely into the sum of an exponentially decaying part $\tilde{f}$, a "constant term" f^0 and a "principle part" $\mathring{f}$. Specifically,

$$\tilde{f}(z) := \sum_{|j|\leq M} y^j \sum_{\substack{m,n\geq -N \\ m+n>0}} a^{(j)}_{m,n} q^m \bar{q}^n, \tag{2.3}$$

$$f^0(z) := \sum_{|j|\leq M} y^j \sum_{\substack{m,n\geq -N \\ m+n=0}} a^{(j)}_{m,n} q^m \bar{q}^n, \tag{2.4}$$

$$\mathring{f}(z) := \sum_{|j|\leq M} y^j \sum_{\substack{m,n\geq -N \\ m+n<0}} a^{(j)}_{m,n} q^m \bar{q}^n \tag{2.5}$$

such that

$$f(z) = \tilde{f}(z) + f^0(z) + \mathring{f}(z),$$

where $q = e^{2\pi i z}$.

2.2 EIGENFORMS FOR THE LAPLACIAN

The space $\mathcal{M}^!$ is equipped with the operators

$$\partial_r \colon \mathcal{M}^!_{r,s} \to \mathcal{M}^!_{r+1,s-1} \quad \text{and} \quad \bar{\partial}_s \colon \mathcal{M}^!_{r,s} \to \mathcal{M}^!_{r-1,s+1}$$

given by

$$\partial_r := 2iy\frac{\partial}{\partial z} + r \quad \text{and} \quad \bar{\partial}_s := -2iy\frac{\partial}{\partial \bar{z}} + s,$$

where $z = x + iy$. They induce bigraded operators $\partial, \bar{\partial} \colon \mathcal{M}^! \to \mathcal{M}^!$ of bidegree $(+1, -1)$ and $(-1, +1)$, respectively. If $f \in \mathcal{M}^!_{r,s}$, then ∂ acts on f via ∂_r and similarly $\bar{\partial}$ acts via $\bar{\partial}_s$. We can see that the operators ∂_r and $\bar{\partial}_s$ are closely related to the operators E^+_k and E^-_k from the previous chapter:

$$2\partial_k = E^+_{2k} \quad \text{and} \quad 2\bar{\partial}_{-k} = E^-_{2k}. \tag{2.6}$$

The Laplacian $\Delta_{r,s} \colon \mathcal{M}^!_{r,s} \to \mathcal{M}^!_{r,s}$ is then defined by

$$\Delta_{r,s} := -\bar{\partial}_{s-1}\partial_r + r(s-1) = -\partial_{r-1}\bar{\partial}_s + s(r-1). \tag{2.7}$$

This induces a bigraded operator Δ of bidegree $(0, 0)$ on $\mathcal{M}^!$. If $f \in \mathcal{M}^!_{r,s}$, then Δ acts on f via $\Delta_{r,s}$. The Laplacian can also be written as

$$\begin{aligned}\Delta_{r,s} &= -4y^2\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}} + 2iyr\frac{\partial}{\partial \bar{z}} - 2iys\frac{\partial}{\partial z} \\ &= -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + i(r-s)y\frac{\partial}{\partial x} - (r+s)y\frac{\partial}{\partial y}.\end{aligned}$$

We can easily see from this expression that the Laplacian is closely related to the Casimir operator (1.33):

$$\Delta_{r,s} = \Omega_{r-s} - (r+s)y\frac{\partial}{\partial y}. \tag{2.8}$$

The following lemma will be useful when moving between the Δ and Ω formalism.

Lemma 2.3. *Let r and s be integers such that $r+s \in 2\mathbb{Z}$. Now suppose F is an element of $\mathcal{M}^!_{r,s}$ such that $\Delta_{r,s}F = \lambda F$ for some $\lambda \in \mathbb{R}$, then $F_1 := y^{\frac{r+s}{2}}F$ satisfies the following:*

$$(i) \quad F_1||_{\frac{r-s}{2},\frac{s-r}{2}}\gamma = F_1, \quad \forall\, \gamma \in \Gamma_1.$$

$$(ii) \quad \Omega_{r-s}\, F_1 = \left(\lambda + \left(\frac{r+s}{2}\right)\left(1 - \frac{r+s}{2}\right)\right) F_1.$$

Proof. (i) Observing that

$$\mathrm{Im}(\gamma z) = \mathrm{Im}(z)(cz+d)^{-1}(c\bar{z}+d)^{-1}, \quad \forall\, \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1$$

gives

$$F_1||_{\frac{r-s}{2},\frac{s-r}{2}}\gamma \;=\; y^{\frac{r+s}{2}}F(\gamma z)(cz+d)^{-r}(c\bar{z}+d)^{-s} \;=\; y^{\frac{r+s}{2}}F(z).$$

(ii) We have

$$\begin{aligned}\Delta_{r,s}F_1 =& y^{\frac{r+s}{2}}(\Delta_{r,s}F) - y^2\left(\frac{\partial}{\partial y^2}y^{\frac{r+s}{2}}\right)F - (r+s)y^{\frac{r+s+2}{2}}\frac{\partial F}{\partial y} \\ &- (r+s)y\left(\frac{\partial}{\partial y}y^{\frac{r+s}{2}}\right)F\end{aligned}$$

and

$$(r+s)y\frac{\partial F_1}{\partial y} = (r+s)y\left(\frac{\partial}{\partial y}y^{\frac{r+s}{2}}\right)F + (r+s)y^{\frac{r+s+2}{2}}\frac{\partial F}{\partial y}.$$

Therefore, using equation (2.8) we get the required result. □

We define $\mathcal{HM}^!$ as the space of functions in $\mathcal{M}^!$ that are eigenfunctions of the Laplacian and for $\lambda \in \mathbb{C}$ we let

$$\mathcal{HM}^!(\lambda) := \mathrm{Ker}(\Delta - \lambda : \mathcal{M}^! \to \mathcal{M}^!).$$

The following lemma summarises some of the special features of the Fourier expansion of a function $f \in \mathcal{HM}^!(\lambda)$.

Lemma 2.4. *[6] Let $f \in \mathcal{HM}^!(\lambda) \cap \mathcal{M}^!_{r,s}$, then $\lambda \in \mathbb{Z}$ (or $f = 0$):*

$$\mathcal{HM}^! = \bigoplus_{n\in\mathbb{Z}} \mathcal{HM}^!(n).$$

Furthermore, there exists a $k_0 \in \mathbb{Z}$ *such that* $k_0 < 1-r-s-k_0$ *and* $\lambda = k_0(1-r-s-k_0)$. *There is also a unique decomposition of* $f = f^h + f^a + f^0$ *given by*

$$f^h(z) := \sum_{j=k_0}^{-s} y^j \sum_{\substack{m\geq -N \\ m\neq 0}} a_m^{(j)} q^m, \tag{2.9}$$

$$f^a(z) := \sum_{j=k_0}^{-r} y^j \sum_{\substack{m\geq -N' \\ m\neq 0}} b_m^{(j)} \bar{q}^m, \tag{2.10}$$

$$f^0(z) := ay^{k_0} + by^{1-r-s-k_0},$$

for some $a, b, a_m^{(j)}, b_m^{(j)} \in \mathbb{C}$ *and* $N, N' \in \mathbb{N}$.

Finally f^h, f^a, y^{k_0} *and* $y^{1-r-s-k_0}$ *are eigenfunctions of* Δ *with eigenvalue* λ.

Proof. Lemma 4.3 of [6] together with the remarks following it. □

Therefore, when $f \in \mathcal{HM}^!(\lambda) \cap \mathcal{M}^!_{r,s}$, this lemma allows us to rewrite $\tilde{f}$ (2.3) and $\mathring{f}$ (2.5) in the following way:

$$\tilde{f} = \sum_{j=k_0}^{-s} y^j \sum_{m>0} a_m^{(j)} q^m + \sum_{j=k_0}^{-r} y^j \sum_{m>0} b_m^{(j)} \bar{q}^m, \tag{2.11}$$

$$\mathring{f} = \sum_{j=k_0}^{-s} y^j \sum_{-N\leq m<0} a_m^{(j)} q^m + \sum_{j=k_0}^{-r} y^j \sum_{-N'\leq m<0} b_m^{(j)} \bar{q}^m. \tag{2.12}$$

2.3 THE REAL ANALYTIC EISENSTEIN SERIES

An example of an element of $\mathcal{M}^!_{r,s}$, and also of $\mathcal{M}_{r,s}$, which is an eigenform of the Laplacian is the non-holomorphic Eisenstein series. Using a normalisation and notation that is more appropriate for the theory of real analytic modular forms, we give the following definition.

Definition 2.5. If $r, s \geq 0$ and $r + s = w \geq 1$, then the real analytic Eisenstein series of weights (r, s) is given by

$$\mathcal{E}_{r,s} := \frac{w!}{2\cdot(2\pi i)^{w+2}} \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \frac{\mathbb{L}}{(mz+n)^{r+1}(m\bar{z}+n)^{s+1}}.$$

This series converges absolutely and is an element of $\mathcal{M}_{r,s}$. The following lemma provides us with some properties of the real analytic Eisenstein series.

Lemma 2.6. *(Prop. 4.1 and Cor. 4.2 of [4]) Let* $r, s \geq 0$ *and* $r + s = w \geq 1$*, then* $\mathcal{E}_{r,s}$ *satisfies the following equations:*

$$\partial \mathcal{E}_{w,0} = \mathbb{L}\mathbb{G}_{w+2},$$
$$\partial \mathcal{E}_{r,s} - (r+1)\mathcal{E}_{r+1,s-1} = 0, \qquad \text{for } s \geq 1.$$

$$\bar{\partial} \mathcal{E}_{0,w} = \mathbb{L}\bar{\mathbb{G}}_{w+2},$$
$$\bar{\partial} \mathcal{E}_{r,s} - (s+1)\mathcal{E}_{r-1,s+1} = 0, \qquad \text{for } r \geq 1.$$

$$\Delta_{r,s}\mathcal{E}_{r,s} = (-r-s)\mathcal{E}_{r,s}.$$

We recall that $\mathbb{G}_{w+2}$ is the Eisenstein series we defined in equation (1.7). The Fourier expansion of the real analytic Eisenstein series has been well studied; it has been explicitly computed in various literature (see Theorem 3.1. of [21], Lemma 5.2.15 of [22] or Section 4.2 of [4]). We give the Fourier expansion here to see how it fits with Lemma 2.4. With the notation of that lemma, we have

$$\mathcal{E}_{r,s}(z) = \frac{\pi B_{w+2}}{(w+1)(w+2)}\, y + \frac{(-1)^r w!\, \zeta(w+1)}{2^{2w+1}\pi^w}\binom{w}{r} y^{-r-s}$$
$$+ \sum_{j=-r-s}^{-s} y^j \sum_{m>0} (-1)^r (2\pi)^j \binom{2w}{r}\binom{r}{-j-s}(-j)!\, \frac{\sigma_{2w+1}(m)}{(2m)^{1-j}}\, q^m$$
$$+ \sum_{j=-r-s}^{-r} y^j \sum_{m>0} (-1)^s (2\pi)^j \binom{2w}{s}\binom{s}{-j-r}(-j)!\, \frac{\sigma_{2w+1}(m)}{(2m)^{1-j}}\, \bar{q}^m,$$

where B_x is the x-th Bernoulli number and $\sigma_x(n) = \sum_{d|n} d^x$ is the divisor function. Therefore

$$a_m^{(j)} = (-1)^r (2\pi)^j \binom{2w}{r}\binom{r}{-j-s}(-j)!\, \frac{\sigma_{2w+1}(m)}{(2m)^{1-j}} \tag{2.13}$$

$$b_m^{(j)} = (-1)^s (2\pi)^j \binom{2w}{s}\binom{s}{-j-r}(-j)!\, \frac{\sigma_{2w+1}(m)}{(2m)^{1-j}} \tag{2.14}$$

and $k_0 = -r-s$, which is consistent with Lemma 2.4

2.4 MODULAR ITERATED INTEGRALS OF LENGTH ONE

In this section, we look at the space of modular iterated integrals, introduced by Brown in [6]. This space sits naturally inside the space $\mathcal{M}^!$ of real analytic modular forms.

Definition 2.7. We let $\mathcal{MI}^!_{-1} = 0$. For any $k \geq 0 \in \mathbb{Z}$ we define the space of modular iterated integrals of length k, $\mathcal{MI}^!_k$, to be the largest subspace of $\bigoplus_{r,s\geq 0} \mathcal{M}^!_{r,s}$ which satisfies

$$\partial \mathcal{MI}^!_k \subset \mathcal{MI}^!_k + M^![\mathbb{L}] \times \mathcal{MI}^!_{k-1},$$
$$\bar{\partial} \mathcal{MI}^!_k \subset \mathcal{MI}^!_k + \overline{M}^![\mathbb{L}] \times \mathcal{MI}^!_{k-1},$$

where $M^!$ is the space of weakly holomorphic forms.

We have the following description of $\mathcal{MI}_0$:

Proposition 2.8. *(Prop. 5.2 of of [6]) The space of length zero modular iterated integrals is given by*

$$\mathcal{MI}^!_0 = \mathbb{C}[\mathbb{L}^{-1}]. \tag{2.15}$$

For the rest of this chapter we will only be interested in the special case of length one modular iterated integrals. Using the above proposition, this is simply the largest subspace $\mathcal{MI}^!_1$ of $\bigoplus_{r,s\geq 0} \mathcal{M}^!_{r,s}$ which satisfies

$$\partial \mathcal{MI}^!_1 \subset \mathcal{MI}^!_1 + M^![\mathbb{L}^{\pm}],$$
$$\bar{\partial} \mathcal{MI}^!_1 \subset \mathcal{MI}^!_1 + \overline{M}^![\mathbb{L}^{\pm}].$$

We note that $M^![\mathbb{L}^{\pm}] = M^![\mathbb{L}, \mathbb{L}^{-1}]$. A characterisation of$\mathcal{MI}^!_1$ is provided in [6].

Proposition 2.9. *(Prop. 5.8 of [6]) Any element F of $\mathcal{MI}^!_1$ of weights (r,s) can be written uniquely as*

$$F = \sum_{k=0}^{min(r,s)} F_k$$

for some elements of $\mathcal{MI}_1^!$ *of weights* (r,s) *such that* $\Delta_{r,s}F_k = \lambda_k F_k$ *where*

$$\lambda_k = (k-1)(r+s-k).$$

This, in particular, implies that the value of the invariant k_0 *(see Lemma 2.4) for* F_k *is*

$$k_0 = k - r - s.$$

We will interpret the functions F_k of the last proposition in the setting of Section 2.2.

Proposition 2.10. *Let* F *be an element of* $\mathcal{MI}_1^!$ *with weights* (r,s) *and let* F_k *(*$k \in \{0,\ldots,min(r,s)\}$*) be as in Proposition 2.9. Then* $y^{\frac{r+s}{2}}F_k(z)$ *is* Γ_1*-invariant under the action of* $||_{\frac{r-s}{2},\frac{s-r}{2}}$ *and is an eigenfunction of* Ω_{r-s} *with eigenvalue*

$$\frac{1}{4} - \mu_k^2 \qquad\qquad \text{where } \mu_k = -k + \frac{r+s+1}{2}.$$

For $g = \tilde{F}_k, \mathring{F}_k, y^{k-r-s}, y^{1-k}$ *(in the notation of* (2.3), (2.5)*) we have*

$$\Omega_{r-s}\left(y^{\frac{r+s}{2}}g\right) = \left(\frac{1}{4} - \mu_k^2\right) y^{\frac{r+s}{2}}g. \tag{2.16}$$

Proof. By Lemma 2.3 and Proposition 2.9, $y^{\frac{r+s}{2}}F_k(z)$ is Γ_1-invariant under the action of $||_{\frac{r-s}{2},\frac{s-r}{2}}$ and is an eigenfunction of Ω_{r-s} with eigenvalue

$$\begin{aligned}\lambda_k + \left(\frac{r+s}{2}\right)\left(1-\frac{r+s}{2}\right) &= (k-1)(r+s-k) + \left(\frac{r+s}{2}\right)\left(1-\frac{r+s}{2}\right)\\ &= \frac{1}{4} - \left(k - \frac{r+s+1}{2}\right)^2.\end{aligned}$$

Now we just need to establish the eigen-properties given by equation (2.16). We start by applying the last statement of Lemma 2.4 to $F_k \in \mathcal{HM}^!(\lambda_k)$. Then, since $\Delta_{r,s}F_k^h = \lambda_k F_k^h$, we have

$$(\Delta_{r,s} - \lambda_k)\left(\sum_{j=k_0}^{-s} y^j \sum_{-N\le m<0} a_m^{(j)} q^m\right) = -\left(\Delta_{r,s} - \lambda_k\right)\left(\sum_{j=k_0}^{-s} y^j \sum_{m>0} a_m^{(j)} q^m\right). \tag{2.17}$$

Now, let $\mathcal{P}$ be the space of polynomials of y over $\mathbb{C}$. By equation (2.22) of [4] we have

$$\Delta_{r,s}(\mathcal{P}\cdot q^m\bar{q}^n) \subset \mathcal{P}\cdot q^m\bar{q}^n \tag{2.18}$$

for each $m, n \in \mathbb{Z}$. Therefore, the LHS of (2.17) will be a polynomial in q^{-1} with coefficients in $\mathcal{P}$. This implies that the LHS of (2.17) either has exponential growth as $y \to \infty$ or is identically 0. This former, however, is impossible because, by (2.18), the RHS of (2.17) decays exponentially as $y \to \infty$. Therefore the LHS vanishes and

$$\Delta_{r,s}\left(\sum_{j=k_0}^{-s} y^j \sum_{-N\leq m<0} a_m^{(j)} q^m\right) = \lambda_k \left(\sum_{j=k_0}^{-s} y^j \sum_{-N\leq m<0} a_m^{(j)} q^m\right).$$

Using a similar method we see that $\sum_{j=k_0}^{-r} y^j \sum_{-N'\leq m<0} b_m^{(j)} \bar{q}^m$ is an eigenfunction of $\Delta_{r,s}$ with eigenvalue λ_k. Thus, using equation (2.12), $\mathring{F}_k$ is an eigenfunction of $\Delta_{r,s}$ with eigenvalue λ_k.

By Lemma 2.4, y^{k-r-s} and y^{1-k} are also eigenfunctions of $\Delta_{r,s}$ with eigenvalue λ_k, we can use Lemma 2.3 to deduce the desired eigenproperties of $y^{\frac{r+s}{2}} y^{k-r-s}$ and $y^{\frac{r+s}{2}} y^{1-k}$. The eigenproperties of $y^{\frac{r+s}{2}} \mathring{F}_k$, $y^{\frac{r+s}{2}} y^{k-r-s}$ and $y^{\frac{r+s}{2}} y^{1-k}$, which we just proved, together with the eigenproperty of $y^{\frac{r+s}{2}} F_k$ then imply the eigenproperty of $y^{\frac{r+s}{2}} \tilde{F}_k$. □

2.5 L-FUNCTIONS ASSOCIATED TO REAL ANALYTIC MODULAR FORMS

The difficulty with extending the definition of an L-function from standard modular forms to all real analytic modular forms is twofold. Firstly, we have to handle the potentially exponential growth of these real analytic forms and, secondly, we have to tackle their lack of holomorphicity. To overcome the first problem, we give a definition that is based on the expression of standard L-functions via a Mellin transform, as in (1.23). In fact, this will also solve our second problem and allow us to define L-functions on the entire space $\mathcal{M}^!$ of real analytic modular forms.

Definition 2.11. Let $f \in \mathcal{M}^!_{r,s}$ with an expansion of the form given by equation (2.2). We let the implied logarithm take the principal branch of the logarithm and we set, for $v \neq -j, r+s+j$ ($|j| \leq M$),

$$L_f^*(v) := \left(\int_1^\infty \tilde{f}(it)t^{v-1}dt + \int_1^{-\infty} \mathring{f}(it)t^{v-1}dt - \sum_{|j|\leq M} \sum_{\substack{m,n\geq -N \\ m+n=0}} \frac{a^{(j)}_{m,n}}{v+j} \right)$$
$$+ i^{r-s}\left(\int_1^\infty \tilde{f}(it)t^{r+s-v-1}dt + \int_1^{-\infty} \mathring{f}(it)t^{r+s-v-1}dt \right.$$
$$\left. - \sum_{|j|\leq M} \sum_{\substack{m,n\geq -N \\ m+n=0}} \frac{a^{(j)}_{m,n}}{r+s-v+j} \right). \quad (2.19)$$

The rigorous meaning of the first integral from 1 to $-\infty$ is

$$\sum_{|j|\leq M} \sum_{\substack{m,n\geq -N \\ m+n<0}} \frac{a^{(j)}_{m,n}}{(2\pi(m+n))^{j+v}} \Gamma(j+v, 2\pi(m+n)) \quad (2.20)$$

where $\Gamma(r,z)$ denotes the incomplete Gamma function given by

$$\Gamma(r,z) := \int_z^\infty e^{-t}t^r \frac{dt}{t}.$$

For $z \neq 0$, this has an analytic continuation to the entire r-plane, and therefore, as $m+n \neq 0$ for $\mathring{f}(z)$, equation (2.20) is well-defined for all values of v by the analytic continuation of the incomplete Gamma function. We interpret the second integral from 1 to $-\infty$ in (2.19) in a similar way. The reason we have chosen to write these terms formally as integrals was to stress the symmetry with the other terms and to hint at the origin of the definition in a "regularisation" introduced in [23].

Since we also have the exponential decay of $\tilde{f}$ at infinity, all integrals in (2.19) are well-defined. As mentioned previously, the above construction was inspired by the "regularisation" introduced in Section 4 of [23]. (For another application of this idea, see [24].)

We will shortly see that, when we restrict to specific subspaces, this definition will match the L-functions we gave in the previous chapter. Indeed, the requirement to match these various L-functions heavily influenced our definition of L_f^*.

The above definition immediately gives the following proposition:

Proposition 2.12. *Let $f \in \mathcal{M}^!_{r,s}$ (with $r \equiv s \mod 2$). The L-function of f is meromorphic with finitely many poles and satisfies*

$$L^*_f(v) = i^{r-s} L^*_f(r+s-v)$$

for all v away from the poles.

By allowing such generality, the definition we have given is rather formal and would be unlikely to uncover any important arithmetic information involving the entirety of $\mathcal{M}^!$. Therefore, in order to obtain more refined information, we will restrict our point of view to specific subspaces of $\mathcal{M}^!$.

Before doing so, we mention a few important points. Firstly, for a function $f \in \mathcal{M}^!_{r,s}$ with moderate growth at infinity, $f \in \mathcal{M}_{r,s}$, our definition matches that of Brown's given in Section 9.4 of [4]. We will return to this in more detail in Section 2.5.2. See also, Section 13 of [5] where Brown's construction is applied to the important subclass of $\mathcal{M}_{r,s}$ consisting of modular analogues of the single-valued polylogarithms.

Secondly, by taking the antiderivative of f^0 for large enough v, we note that equation (2.19) can also be written as

$$\begin{aligned} L^*_f(v) = &\int_1^\infty \tilde{f}(it) t^{v-1} dt + \int_1^{-\infty} \mathring{f}(it) t^{v-1} dt \\ &+ i^{r-s} \left(\int_1^\infty \tilde{f}(it) t^{r+s-v-1} dt + \int_1^{-\infty} \mathring{f}(it) t^{r+s-v-1} dt \right) \\ &+ \int_0^1 \left(i^{r-s} f^0(i/t) t^{-r-s} - f^0(it) \right) t^v \frac{dt}{t}, \end{aligned} \tag{2.21}$$

which will be useful when performing computations later in this chapter.

Finally, if we let $F \in \mathcal{MI}_1^!$ be of weights (r, s) and F_k be as in Proposition 2.9, then the constant part of F_k is given by $F_k^0 = ay^{k-r-s} + by^{1-k}$. This allows us to write the L-function of F_k as

$$\begin{aligned} L^*_{F_k}(v) = &\int_1^\infty \tilde{F}_k(it)(t^{v-1} + i^{r-s}t^{r+s-v-1})dt - \frac{a}{v+k-r-s} - \frac{i^{r-s}a}{k-v} \\ &+ \int_1^{-\infty} \mathring{F}_k(it)(t^{v-1} + i^{r-s}t^{r+s-v-1})dt - \frac{b}{v-k+1} - \frac{i^{r-s}b}{r+s-v-k+1}. \end{aligned} \tag{2.22}$$

2.5.1 *L-Functions in $\mathcal{HM}^!(\lambda)$ and in $\mathcal{MI}_1^!$.*

Let $f \in \mathcal{HM}^!(\lambda) \cap \mathcal{M}^!_{r,s}$. Using the Fourier expansion of f provided by Lemma 2.4, the general definition of $L_f^*(v)$ we gave in Definition 2.11 leads to an expression as a series. This is more natural because it is reminiscent of our original definition of an L-function of a standard modular form (see Definition 1.10). Furthermore, in the case of weakly holomorphic modular forms, it matches the L-functions already associated with such forms (See Section 1.7 or [25] and references therein).

We need to ensure that the series we will eventually obtain converges absolutely. To do this we will need an analogue of the "trivial bound" about the Fourier coefficients. As in the case of weakly holomorphic forms, see Lemma 3.2 of [26], the growth is, in general, exponential. Although the proof parallels that of [26], the presence of two weights and of the powers of y give rise to new complications, and therefore, we will provide a full proof.

Proposition 2.13. *Let $f \in \mathcal{HM}^!(\lambda)$. With the notation of Lemma 2.4, for each $j \in \{k_0, \dots, -s\}$ (resp. $j \in \{k_0, \dots, -r\}$), there is a $C > 0$ such that*

$$a_n^{(j)} \ll e^{C\sqrt{n}} \qquad \left(\text{resp. } b_n^{(j)} \ll e^{C\sqrt{n}}\right) \qquad \text{as } n \to \infty.$$

Proof. Set N_0 =max(N, N') and let $n > N_0$. Then we have

$$\begin{aligned}\int_0^1 f(z)e^{-2\pi inz}dx &= \int_0^1 (f^0 + f^h + f^a)e^{-2\pi inz}dx \\ &= \int_0^1 f^0(z)e^{-2\pi inz}dx + \sum_{j=k_0}^{-s} y^j \sum_{\substack{m\geq -N \\ m\neq 0}} a_m^{(j)} \int_0^1 e^{2\pi i(m-n)z}dx \\ &\quad + \sum_{j=k_0}^{-r} y^j \sum_{\substack{m\geq -N' \\ m\neq 0}} b_m^{(j)} e^{2\pi(n-m)y} \int_0^1 e^{-2\pi i(m+n)x}dx \\ &= \sum_{j=k_0}^{-s} y^j a_n^{(j)} \end{aligned} \tag{2.23}$$

since, for $n > N_0$ and $m \geq -N'$, $m + n > 0$. Likewise,

$$\int_0^1 f(z)e^{2\pi inz}dx = \sum_{j=k_0}^{-r} y^j b_n^{(j)} e^{-4\pi ny}. \tag{2.24}$$

Suppose that $-s > k_0$ (and hence $-s - k_0 > 0$). Then (2.23) implies

$$e^{2\pi ny}y^{-k_0} \int_0^1 f(z)e^{-2\pi inx}dx = \sum_{j=0}^{-s-k_0} y^j a_n^{(j+k_0)} \tag{2.25}$$

and therefore

$$\frac{\partial^{-s-k_0}}{\partial y^{-s-k_0}} \left(e^{2\pi ny}y^{-k_0} \int_0^1 f(z)e^{-2\pi inx}dx \right) = (-s - k_0)!a_n^{(-s)}. \tag{2.26}$$

The LHS will be a sum of products of $e^{2\pi ny}$, polynomials in y and n and

$$\int_0^1 \frac{\partial^j f(z)}{\partial y^j} e^{-2\pi inx}dx \tag{2.27}$$

for $j \in \{0, \ldots, -s - k_0\}$. Now, we note that, for all k, l,

$$\frac{\partial}{\partial y} = \frac{1}{2y} \left(\partial_k + \bar{\partial}_l - k - l \right) \tag{2.28}$$

and, since $y^{-1} \in \mathcal{M}^!_{1,1}$, we have $\frac{\partial}{\partial y}\left(\mathcal{M}^!\right) \subset \mathcal{M}^!$. Therefore, we have

$$\frac{\partial^j}{\partial y^j}\left(\mathcal{M}^!\right) \subset \mathcal{M}^!. \tag{2.29}$$

We will use this to bound (2.27) with the help the following lemma:

Lemma 2.14. *For each $f \in \mathcal{M}^!_{r,s}$, there exists a constant $C > 0$ such that $f(z) = O(e^{C/y}y^{\frac{-(r+s)}{2}})$ as $y \to 0$, uniformly in x.*

Proof. In this lemma we introduce the function known as the j -invariant

$$j(z) := e^{-2\pi iz} + 744 + 19688e^{2\pi iz} + 21493760e^{4\pi iz} + ...$$

This function is modular invariant, $j(\gamma z) = j(z)$ for every $\gamma \in \Gamma_1$, and does not vanish in the interior of the fundamental domain $\mathcal{F}$ of Γ_1. (See [27], for example, for more information on this function.)

Therefore, if there is a $c > 0$ such that $f(z) = O(e^{cy})$ as $y \to \infty$ (uniformly in x), then $F(z) = |f(z)y^{\frac{r+s}{2}}j(z)^{-c-\epsilon}|$ is bounded in the standard fundamental domain $\mathcal{F}$ of Γ_1. We can then rearrange $F(z)$ to get, as $y \to \infty$,

$$|f(z)y^{\frac{r+s}{2}}| = O(e^{Cy})$$

for some $C > 0$. On the other hand, $|f(z)y^{\frac{r+s}{2}}|$ is invariant under any $\gamma \in \Gamma_1$ and thus under $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. This implies that there is a $C > 0$, such that, as $y \to 0$,

$$|f(z)y^{\frac{r+s}{2}}| = |f(-1/z)\mathrm{Im}(-1/z)^{\frac{r+s}{2}}| = O(e^{C\mathrm{Im}(-1/z)}) = O(e^{C/y})$$

uniformly in x, which gives the result. □

With this lemma and (2.29) we see that, for $y \to 0$,

$$\frac{\partial^j f}{\partial y^j} = O(e^{C/y}y^M)$$

for some constant depending on r and s. From (2.26), we then conclude $a_n^{(-s)} \ll e^{2\pi ny}e^{C/y}y^{M_1}N^{M_2}$, where C, M_1, M_2 and the implied constant depend only on r, s and k_0. For $y = 1/\sqrt{n}$ this implies the bound of the proposition in the case $j = -s$.

Next, by rearranging equation (2.25) we have

$$e^{2\pi ny}y^{-k_0}\int_0^1 f(z)e^{-2\pi inx}dx - y^{-s-k_0}a_n^{(-s)} = \sum_{j=0}^{-s-k_0-1} y^j a_n^{(j+k_0)}$$

and then we differentiate both sides by y to get

$$\frac{\partial^{-s-k_0-1}}{\partial y^{-s-k_0-1}}\left(e^{2\pi ny}y^{-k_0}\int_0^1 f(z)e^{-2\pi inx}dx\right) - (-s-k_0)!ya_n^{(-s)}$$
$$= (-s-k_0-1)!a_n^{(-s-1)}.$$

Arguing as above for the second term, and using the bound for $a_n^{(-s)}$ we proved above, we deduce the bound for $j = -s-1$. Continuing in this way, we deduce the result for all j.

It is clear from the argument (essentially by interchanging the roles of s and k_0), that it remains valid when $-s \le k_0$.

To prove the bound for $b_n^{(j)}$, we work in the same way but based on (2.24), instead of (2.23). □

With the assumption that $f \in \mathcal{HM}^!(\lambda)$, we are now ready to use the Fourier expansion given in Lemma 2.4 to express the L-function of f as a series. The first integral in (2.19) becomes

$$\int_1^\infty \tilde{f}(it)t^{v-1}dt = \int_1^\infty \left(\sum_{j=k_0}^{-s} t^j \sum_{m>0} a_m^{(j)} e^{-2\pi mt} + \sum_{j=k_0}^{-r} t^j \sum_{m>0} b_m^{(j)} e^{-2\pi mt} \right) t^v \frac{dt}{t}.$$

For compactness of notation, we set, for each $j \in \mathbb{Z}$, $c_m^{(j)} = a_m^{(j)} + b_m^{(j)}$, where $a_m^{(j)}$ (resp. $b_m^{(j)}$) are taken to be 0 if j or m is outside the range of j- or m-summation in (2.9) (resp. (2.10)). Therefore, the above equation becomes

$$\begin{aligned}\int_1^\infty \tilde{f}(it)t^{v-1}dt &= \sum_{j\in\mathbb{Z}} \sum_{m>0} \int_1^\infty c_m^{(j)} e^{-2\pi mt} t^{v+j} \frac{dt}{t} \\ &= \sum_{j\in\mathbb{Z}} \sum_{m>0} \int_{2\pi m}^\infty \frac{c_m^{(j)} e^{-t}\, t^{j+v}}{(2\pi m)^{v+j}} \frac{dt}{t} \\ &= \sum_{j\in\mathbb{Z}} \sum_{m>0} \frac{c_m^{(j)}\Gamma(v+j, 2\pi m)}{(2\pi m)^{v+j}},\end{aligned}$$

where we used the change of variables $2\pi mt \to t$. Using equation (2.20), the second integral in (2.19) becomes

$$\int_1^{-\infty} \mathring{f}(it)t^{v-1}dt = \sum_{j\in\mathbb{Z}} \sum_{m<0} \frac{c_m^{(j)}\Gamma(v+j, 2\pi m)}{(2\pi m)^{v+j}}.$$

Applying similar methods for the other integrals in (2.19), we deduce, for $v \neq -k_0, -k_0+1, k_0+r+s-1, k_0+r+s$,

$$\begin{aligned}L_f^*(v) &= \sum_{j\in\mathbb{Z}} \sum_{m\neq 0} \frac{c_m^{(j)}\Gamma(v+j, 2\pi m)}{(2\pi m)^{v+j}} + i^{r-s} \sum_{j\in\mathbb{Z}} \sum_{m\neq 0} \frac{c_m^{(j)}\Gamma(s+r+j-v, 2\pi m)}{(2\pi m)^{s+r+j-v}} \\ &\quad + \frac{i^{r-s}a}{v-k_0-r-s} + \frac{i^{r-s}b}{v+k_0-1} - \frac{a}{v+k_0} - \frac{b}{v-k_0-r-s+1}. \qquad (2.30)\end{aligned}$$

Because of Proposition 2.13 and the asymptotics $\Gamma(r,x) \sim e^{-x}x^{r-1}$ as $x \to \infty$ we see that this series converges absolutely for all $v \in \mathbb{C}$. Equation 2.30 can be seen to be more reminiscent our original definition of the completed L-function for a standard modular cusp form, given by Definition 1.10.

Example: L-function of a weakly holomorphic modular form

When f is a weakly holomorphic cusp form of weight k, equation (2.19) can be written as

$$\begin{aligned} L_f^*(v) = &\int_1^{\infty} \tilde{f}(it)t^{v-1}dt + \int_1^{-\infty} \mathring{f}(it)t^{v-1}dt \\ &+ i^k \left(\int_1^{\infty} \tilde{f}(it)t^{k-v-1}dt + \int_1^{-\infty} \mathring{f}(it)t^{k-v-1}dt \right). \end{aligned} \tag{2.31}$$

We will see that our L-function matches the earlier definition of an L-function we gave for such forms (equation (1.39)). In fact, our L-function will match the L-function for all weakly holomorphic modular forms, not just those of cusp forms, and so we will focus on the general case. (The main reason for giving (2.31) was so we could recall it easily in Section 2.6.3).

We let $f \in M_k^!$ be a weakly holomorphic modular form. Any such f can be considered as an element of $\mathcal{HM}^!(0) \cap \mathcal{M}_{k,0}^!$ with, using the notation of Lemma 2.4, $k_0 = 1-k, a = b_m^{(j)} = 0$ for all j, m, and $a_m^{(j)} = 0$ for $j \neq 0$ or $m < m_0$ for some $m_0 \in \mathbb{Z}$. Then, equation (2.30) becomes

$$L_f^*(v) = \sum_{\substack{m \geq m_0 \\ m \neq 0}} \frac{a_m^{(0)}\Gamma(v, 2\pi m)}{(2\pi m)^v} + i^k \sum_{\substack{m \geq m_0 \\ m \neq 0}} \frac{a_m^{(0)}\Gamma(k-v, 2\pi m)}{(2\pi m)^{k-v}} - b\left(\frac{1}{v} + \frac{i^k}{k-v}\right)$$

which matches, say, (6.1) of [25] or, when f is a cusp form and hence $b = 0$, equation (1.39).

L-functions of modular iterated integrals of length one.

Using the decomposition given in Proposition 2.9, we can now express the L-function of a function F in the broader class $\mathcal{MI}_1^!$ of modular iterated integrals of length 1, in terms of the L-function in $\mathcal{HM}^!(\lambda)$ for varying λ.

We let F be an element of $\mathcal{MI}_1^!$ of weights (r,s), then, for each $v \in \mathbb{C}$, we have

$$L_F^*(v) = \sum_{k=0}^{\min(r,s)} L_{F_k}^*(v),$$

where F_k are elements of $\mathcal{HM}^!$ of weights (r,s) such that $F = F_0 + \cdots + F_{\min(r,s)}$ as in Proposition 2.9.

2.5.2 *L-Functions in $\mathcal{M}_{r,s}$.*

We now consider the case that f is of polynomial growth at the cusps so that $a_{m,n}^{(j)}$ vanish when m or n are negative, i.e $f \in \mathcal{M}_{r,s}$. This means that $\mathring{f} = 0$ and hence $\tilde{f} = f - f^0$. Furthermore, for $Re(v) \gg 0$, the integral $\int_0^\infty \tilde{f}(it)t^{v-1}dt$ converges which, in turn, allows us to make the change of variables $t \to 1/t$ in the third integral of (2.21) to derive

$$\begin{aligned}
L_f^*(v) &= \int_1^\infty \Big(f(it) - f^0(it)\Big)t^v \frac{dt}{t} + i^{r-s}\int_0^1 \Big(f(i/t) - f^0(i/t)\Big)t^{-r-s+v}\frac{dt}{t} \\
&\qquad + \int_0^1 \Big(i^{r-s}f^0(i/t))t^{-r-s} - f^0(it)\Big)t^v\frac{dt}{t} \\
&= \int_1^\infty \Big(f(it) - f^0(it)\Big)t^v \frac{dt}{t} + \int_0^1 \Big(i^{r-s}f(i/t)t^{-r-s} - f^0(it)\Big)t^v\frac{dt}{t}.
\end{aligned}$$

Now using the transformation law given by equation (2.1) for $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, we have

$$\begin{aligned}
L_f^*(v) &= \int_1^\infty \Big(f(it) - f^0(it)\Big)t^v \frac{dt}{t} + \int_0^1 \Big(f(it) - f^0(it)\Big)t^v\frac{dt}{t} \\
&= \int_0^\infty \Big(f(it) - f^0(it)\Big)t^v \frac{dt}{t}. \qquad (2.32)
\end{aligned}$$

This matches Brown's definition of an L-function for $f \in \mathcal{M}_{r,s}$ given in Section 9.4 of [4]. There, up to a different normalisation, the L-function is defined, for $Re(v) \gg 0$, by

$$L_f^*(v) = \sum_{|j|\le M} (2\pi)^{-j-v}\Gamma(v+j)L_f^{(j)}(v+j) \qquad (2.33)$$

where, with the notation of (2.2),

$$L_f^{(j)}(v) := \sum_{N\ge 1}\frac{1}{N^v}\left(\sum_{m+n=N} a_{m,n}^{(j)}\right).$$

The equivalence of this with our definition is established in the proof of Theorem 9.7 of [4].

L-functions in $\mathcal{HM}^!(\lambda) \cap \mathcal{M}_{r,s}$

When $f \in \mathcal{M}_{r,s}$ and is also an eigen-function of the Laplacian, then $L_f^*(v)$ obtains a more familiar form, which is valid for $Re(v) \gg 0$. We let $f \in \mathcal{HM}^!(\lambda) \cap \mathcal{M}_{r,s}$, then, by Lemma 2.4, there is a $k_0 \in \mathbb{Z}$ such that $f = f^0 + f^h + f^a$ with

$$f^0 = ay^{k_0} + by^{1-r-s-k_0}, \quad f^h = \sum_{j=k_0}^{-s} y^j \sum_{m>0} a_m^{(j)} q^m \text{ and } f^a = \sum_{j=k_0}^{-r} y^j \sum_{m>0} b_m^{(j)} \bar{q}^m,$$

where a, b, $a_m^{(j)}$ and $b_m^{(j)} \in \mathbb{C}$. Then for $Re(v) \gg 0$, (2.32) (or, directly, (2.19)) becomes

$$\begin{aligned} L_f^*(v) &= \int_0^\infty \left(\sum_{j=k_0}^{-s} t^j \sum_{m>0} a_m^{(j)} e^{-2\pi m t} + \sum_{j=k_0}^{-r} t^j \sum_{m>0} b_m^{(j)} e^{-2\pi m t} \right) t^v \frac{dt}{t} \\ &= \sum_{j=k_0}^{-s} \frac{\Gamma(j+v)}{(2\pi)^{j+v}} \sum_{m>0} \frac{a_m^{(j)}}{m^{j+v}} + \sum_{j=k_0}^{-r} \frac{\Gamma(j+v)}{(2\pi)^{j+v}} \sum_{m>0} \frac{b_m^{(j)}}{m^{j+v}}, \end{aligned}$$

where we used a similar method as we did to obtain equation (2.30). As with equation (2.30), we can see that the above equation resembles the completed L-function for a standard cusp form given by Definition 1.10.

Example: L-function of the real analytic Eisenstein series

We can use the above representation of $L_f^*(v)$ and equations (2.13) and (2.14) to explicitly compute the L-function of the real analytic Eisenstein series $\mathcal{E}_{r,s}$. For $Re(v) \gg 0$, we have

$$L^*_{\mathcal{E}_{r,s}}(v) = \sum_{j=-r-s}^{-s} \frac{\Gamma(j+v)}{(2\pi)^{j+v}} \sum_{m>0} (-1)^r (2\pi)^j \binom{2w}{r} \binom{r}{-j-s} (-j)! \, \frac{\sigma_{2w+1}(m)}{2^{1-j} m^{v+1}}$$
$$+ \sum_{j=-r-s}^{-r} \frac{\Gamma(j+v)}{(2\pi)^{j+v}} \sum_{m>0} (-1)^s (2\pi)^j \binom{2w}{s} \binom{s}{-j-r} (-j)! \, \frac{\sigma_{2w+1}(m)}{2^{1-j} m^{v+1}}$$

$$= \frac{\zeta(v+1)\zeta(v-2w)}{2\cdot(2\pi)^v} \Bigg((-1)^r \binom{2w}{r} \sum_{j=-r-s}^{-s} 2^j\, \Gamma(j+v) \binom{r}{-j-s} (-j)!$$
$$+ (-1)^s \binom{2w}{s} \sum_{j=-r-s}^{-r} 2^j\, \Gamma(j+v) \binom{s}{-j-r} (-j)! \Bigg),$$

where we have used $\sum_{m>0} \frac{\sigma_x(m)}{m^y} = \zeta(y)\zeta(y-x)$. The last expression also gives the meromorphic continuation to the entire v-plane.

2.6 PERIOD FUNCTIONS ASSOCIATED TO REAL ANALYTIC MODULAR FORMS

It is not obvious how to define appropriate period functions for the entirety of $\mathcal{M}^!$. As mentioned in the beginning of this chapter, this is not surprising since the period function usually reflects arithmeticity and the space $\mathcal{M}^!$ is too large to be of arithmetic nature in its entirety.

In this section, we will show that, in the subspace $\mathcal{MI}_1^!$ of modular iterated integrals of length one, it is possible to define period functions and, indeed, period polynomials. We will see how constructions from the theory of Maass forms, given in Section 1.6, will form the basis for these period functions.

2.6.1 *Maass-Selberg Forms associated to Length One Modular Iterated Integrals*

We start by noting that, as we are dealing with modular iterated integrals of length one, the weights r, s are always ≥ 0. We now recall the Maass-Selberg form that we defined in Section 1.6:

$$\eta_k(f,g) = \{\partial_{k/2} f, g\}^+ - \{f, \bar{\partial}_{k/2} g\}^-,$$

where f and g are smooth functions in an open subset U of $\mathfrak{H} \cup \bar{\mathfrak{H}}$ and $\{f,g\}^{\pm}$ are defined in equation (1.36). We have normalised slightly differently form Definition 1.17 because we now use Brown's operators instead of E_k^+ and E_k^-. The relationship between these sets of operators was given in (2.6).

The next lemma gives a property of the Maass-Selberg form we have not yet seen but will need later in this chapter.

Lemma 2.15. *(Lemma 39 of [16]) Let $\gamma \in \Gamma_1$ and let $\gamma^*(\eta_k(f,g))$ denote the pull-back of the differential form $\eta_k(f,g)$ by the map $z \mapsto \gamma z$ ($z \in \mathfrak{H} \cup \bar{\mathfrak{H}}$), then*

$$\gamma^*(\eta_k(f,g)) = \eta_k\left(f||_{\frac{k}{2},\frac{-k}{2}}\gamma,\, g||_{\frac{-k}{2},\frac{k}{2}}\gamma\right).$$

To define the Maass-Selberg form that we will associate to modular iterated integrals of length one we will also need the function $R_{n,v}$, which we defined to be

$$R_{n,\nu}(z,\zeta) := \left(\frac{\zeta - \bar{z}}{\zeta - z}\right)^{\frac{n}{2}} \left(\frac{\mathrm{Im}(z)}{(\zeta - z)(\zeta - \bar{z})}\right)^{\frac{1}{2}-\nu}.$$

However, we now extend this to $z \in \mathfrak{H} \cup \bar{\mathfrak{H}}$ and a complex $\zeta \notin \{z, \bar{z}\}$ (rather than $z \in \mathfrak{H}$ and $\zeta \in \mathbb{R}$). For each $\zeta \in \mathbb{C}$, this gives a well-defined real-analytic function of z if we restrict z to the complement in $\mathfrak{H}$ of some path joining ζ and $\bar{\zeta}$ and then choose an appropriate branch for the implied logarithm. Likewise, for a suitable subset of $\bar{\mathfrak{H}}$.

Furthermore, for the specific values of n, ν which we will use for the function $R_{n,\nu}$, it can be defined for all $\zeta \in \mathbb{C}$ and $z \in \mathfrak{H} \cup \bar{\mathfrak{H}}$. Specifically, for

$n = s - r$ and $\nu = \mu_k = -k + (r+s+1)/2$ with k as in Proposition 2.9 we have

$$R_{s-r,\mu_k}(z,\zeta) = \left(\frac{\zeta - \bar{z}}{\zeta - z}\right)^{\frac{s-r}{2}} \left(\frac{\operatorname{Im}(z)}{(\zeta - z)(\zeta - \bar{z})}\right)^{k-\frac{r+s}{2}}$$
$$= \operatorname{Im}(z)^{k-\frac{r+s}{2}}(\zeta - z)^{r-k}(\zeta - \bar{z})^{s-k}. \tag{2.34}$$

Since $k \leq r, s$, this can be defined for all $\zeta \in \mathbb{C}$ and $z \in \mathfrak{H} \cup \bar{\mathfrak{H}}$.

The function R_{n,μ_k} has some very useful properties, which we will be able to make use of later on. These properties are summarised in our next lemma.

Lemma 2.16. *Set $n = s - r$ and $\mu_k = -k + (r+s+1)/2$ with $k \leq r, s$, then we have the following:*

i) For each $\zeta \in \mathbb{C}$ we have

$$\bar{\partial}_{-\frac{n}{2}} R_{n,\mu_k}(\cdot,\zeta) = \frac{1}{2}(1 - 2\mu_k - n) R_{n-2,\mu_k}(\cdot,\zeta),$$
$$\Omega_n R_{n,\mu_k}(\cdot,\zeta) = \left(\frac{1}{4} - \mu_k^2\right) R_{n,\mu_k}(\cdot,\zeta).$$

ii) For each $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1$, $z \in \mathfrak{H} \cup \mathfrak{H}$ and $\zeta \in \mathbb{C}$ we have

$$R_{n,\mu_k}(\gamma z, \gamma\zeta) = (c\zeta + d)^{1-2\mu_k} \left(\frac{cz+d}{c\bar{z}+d}\right)^{n/2} R_{n,\mu_k}(z,\zeta).$$

Proof. This is essentially Proposition 36 of [16] but there it is proved with the restriction that $\zeta \in \mathbb{R}$ and $(c\zeta + d) > 0$ due to the more general μ and n to which the proposition applies. □

We now associate Maass-Selberg forms to each F_k, with F_k as given in Proposition 2.9. This will be the basis for our construction of the period function for all elements of $\mathcal{MI}_1^!$.

Proposition 2.17. *Let $F \in \mathcal{MI}_1^! \cap \mathcal{M}_{r,s}$ and, for $k \in \{0, \dots, min(r,s)\}$, let F_k be the k-th term in the decomposition of F in eigenfunctions of $\Delta_{r,s}$, as in Proposition*

2.9. *Then, for* $\mu_k = -k + (r+s+1)/2$ *and each* $\zeta \in \mathbb{C}$*, the following forms are closed:*

$$\begin{aligned} i)\quad & \eta_{r-s}\left(y^{\frac{r+s}{2}}\left(\tilde{F}_k + ay^{k-r-s}\right), R_{s-r,\mu_k}(\cdot,\zeta)\right), \\ ii)\quad & \eta_{r-s}\left(y^{\frac{r+s}{2}}\mathring{F}_k, R_{s-r,\mu_k}(\cdot,\zeta)\right), \\ iii)\quad & \eta_{r-s}\left(y^{\frac{r+s}{2}}\left(by^{1-k}\right), R_{s-r,\mu_k}(\cdot,\zeta)\right). \end{aligned}$$

Proof. By Proposition 2.10, $y^{\frac{r+s}{2}}\left(\tilde{F}_k + ay^{k-r-s}\right)$, $y^{\frac{r+s}{2}}\mathring{F}_k$ and $y^{\frac{r+s}{2}}\left(by^{1-k}\right)$ are eigenfunctions of Ω_{r-s} and, by Lemma 2.16, $R_{s-r,\mu_k}(\cdot,\zeta)$ is an eigenfunction of Ω_{s-r}. They all have eigenvalue $\frac{1}{4} - \mu_k^2$. Therefore, by Lemma 1.18, we deduce the assertion. □

2.6.2 *Cocycles associated to Modular Iterated Integrals of Length One.*

We will now associate to the F_k's of the last section a 1-cocycle in the Γ_1-module $P_{r+s-2k}(\mathbb{C})$. We define it as the coboundary of a 0-cochain in a larger module than $P_{r+s-2k}(\mathbb{C})$. The construction follows the definition of the "integral at a tangential base point at infinity" of Section 4 of [23].

For convenience of notation, we set $\eta_{r-s}(f;\zeta) := \eta_{r-s}(y^{\frac{r+s}{2}}f, R_{s-r,\mu_k}(\cdot,\zeta))$.

Proposition 2.18. *Let* $F \in \mathcal{MI}_1^! \cap \mathcal{M}_{r,s}$*,* F_k *be as in Proposition 2.9 and* $\mu_k = -k + (r+s+1)/2$*. Then the function* $v_F^{(k)} : \mathfrak{H} \to \mathbb{C}$ *given by*

$$v_F^{(k)}(\zeta) := \int_\zeta^{i\infty} \eta_{r-s}(\tilde{F}_k + ay^{k-r-s};\zeta) + \int_\zeta^0 \eta_{r-s}(by^{1-k};\zeta) + \int_\zeta^{-i\infty} \eta_{r-s}(\mathring{F}_k;\zeta),$$

where the line of integration in the last integral includes the origin, is well-defined. The differential forms to be integrated in $v_F^{(k)}$ *can be written more explicitly in the form*

$$\begin{aligned} & y^{k-1}\,\partial_r\left(f(z)\right)(\zeta - z)^{r-k}(\zeta - \bar{z})^{s-k}\,dz \\ & \qquad + (s-k)\,y^{k-1}f(z)(\zeta - z)^{r-k+1}(\zeta - \bar{z})^{s-k-1}\,d\bar{z} \end{aligned} \tag{2.35}$$

for each smooth function $f : \mathfrak{H} \cup \bar{\mathfrak{H}} \to \mathbb{C}$.

Proof. We will show the second assertion first. Using the definition of η_{r-s} and Lemma 2.16 we have

$$\begin{aligned}\eta_{r-s}(f;\zeta) &= \left(2iy\frac{\partial}{\partial z}\left(y^{\frac{r+s}{2}}f(z)\right) + \frac{r-s}{2}y^{\frac{r+s}{2}}f(z)\right)R_{s-r,\mu_k}(\cdot,\zeta)\frac{dz}{y}\\ &\quad - y^{\frac{r+s}{2}}f(z)\left(\frac{1-2\mu_k-s+r}{2}\right)R_{s-r-2,\mu_k}(\cdot,\zeta)\frac{d\bar{z}}{y}\\ &= \left(ry^{\frac{r+s}{2}}f(z) + 2iy^{\frac{r+s}{2}+1}\frac{\partial\left(f(z)\right)}{\partial z}\right)R_{s-r,\mu_k}(\cdot,\zeta)\frac{dz}{y}\\ &\quad - y^{\frac{r+s}{2}}f(z)\left(\frac{1-2\mu_k-s+r}{2}\right)R_{s-r-2,\mu_k}(\cdot,\zeta)\frac{d\bar{z}}{y}.\end{aligned}$$

Now, using equation (2.34), we have

$$\begin{aligned}\eta_{r-s}(f;\zeta) &= y^{\frac{r+s}{2}}\partial_r\left(f(z)\right)y^{k-\frac{r+s}{2}}(\zeta-z)^{r-k}(\zeta-\bar{z})^{s-k}\frac{dz}{y}\\ &\quad - y^{\frac{r+s}{2}}f(z)\left(\frac{1-2\mu_k-s+r}{2}\right)y^{k-\frac{r+s}{2}}(\zeta-z)^{r-k+1}(\zeta-\bar{z})^{s-k-1}\frac{d\bar{z}}{y}.\end{aligned}$$

Substituting the value for μ_k we get (2.35) as required. From this we deduce that, if f and $\partial f/\partial z$ decay exponentially as $y \to \infty$, the same holds for $\eta_{r-s}(y^{\frac{r+s}{2}}f, R_{s-r,\mu_k}(\cdot,\zeta))$. This condition holds for $f = \tilde{F}_k$ and it also holds for $f = \mathring{F}_k$ as $y \to -\infty$.

The term corresponding to ay^{k-r-s} in the first integral is $O(y^{-2})$ as $y \to \infty$, which assures convergence. (Note that each of the two summands in (2.35) individually has a term of order y^{-1} but they cancel each other out on the upper imaginary axis).

Since the second integral in the definition of $v_F^{(k)}$ is convergent too, we can see that the integrals are all convergent.

Furthermore, using Proposition 2.17, we can see that $\eta_{r-s}(\tilde{F}_k + ay^{k-r-s};\zeta)$, $\eta_{r-s}(by^{1-k};\zeta)$ and $\eta_{r-s}(\mathring{F}_k,\zeta)$ are closed in $\mathfrak{H}$. The last form is also closed in $\bar{\mathfrak{H}}$. Indeed, for each fixed $\zeta \in \mathbb{C}$, by (2.35), we have that $d(\eta_{r-s}(\mathring{F}_k,\zeta)) = P(e^{-2\pi iz})dz \wedge d\bar{z}$, where P is a polynomial whose coefficients are polynomials in $z, \bar{z}$. Since $\eta_{r-s}(\mathring{F}_k,\zeta)$ is closed in $\mathfrak{H}$, each of those coefficients are identically zero in $\mathfrak{H}$, and therefore, they vanish in $\bar{\mathfrak{H}}$ too. □

Equation (2.35) will be used often in both this chapter and the next. Indeed, we will use it to prove our next lemma, which will be of use to us later:

Lemma 2.19. *The integrand $\eta_{r-s}(y^{k-r-s};0)$ vanishes along the positive imaginary axis.*

Proof. Using equation (2.35), we have

$$\eta_{r-s}(y^{k-r-s};0) = y^{k-1}2iy\frac{\partial}{\partial z}(y^{k-r-s})(-z)^{r-k}(-\bar{z})^{s-k}dz \qquad (2.36)$$
$$+ ry^{2k-r-s-1}(-z)^{r-k}(-\bar{z})^{s-k}dz + (s-k)y^{2k-r-s-1}(-z)^{r-k+1}(-\bar{z})^{s-k-1}d\bar{z}.$$

Since the first summand on the RHS of this equation is equal to

$$(k-r-s)y^{2k-r-s-1}(-z^{r-k})(-\bar{z})^{s-k}dz, \qquad (2.37)$$

equation (2.36) can be written as

$$\eta_{r-s}(y^{k-r-s};0) = (k-s)y^{2k-r-s-1}(-z)^{r-k}(-\bar{z})^{s-k}dz$$
$$+ (s-k)y^{2k-r-s-1}(-z)^{r-k+1}(-\bar{z})^{s-k-1}d\bar{z}.$$

Integrating along the positive imaginary axis yields

$$i\left[(k-s)y^{2k-r-s-1}(-iy)^{r-k}(iy)^{s-k} - (s-k)y^{2k-r-s-1}(-iy)^{r-k+1}(iy)^{s-k-1}\right]dy$$
$$= i\left[(k-s)y^{2k-r-s-1}(-iy)^{r-k}(iy)^{s-k} + (s-k)y^{2k-r-s-1}(-iy)^{r-k}(iy)^{s-k}\right]dy$$
$$= 0.$$
□

We now define the 1-coboundary $\sigma_F^{(k)}$ by

$$\sigma_F^{(k)}(\gamma) := d^0 v_F^{(k)} = v_F^{(k)}||_{2k-r-s,0}(\gamma - 1).$$

We will show that, although $v_F^{(k)}$ does not belong to $P_{r+s-2k}(\mathbb{C})$, its differential does and, in fact, it belongs to a cohomology class reminiscent to that of (1.31) in the classical Eichler cohomology.

Proposition 2.20. *Let $F \in \mathcal{MI}_1^! \cap \mathcal{M}_{r,s}$. For $k \in \{0,\dots,min(r,s)\}$, let F_k be the k-th term in the decomposition of F in eigenfunctions of $\Delta_{r,s}$, as in Proposition 2.9. Then we have the following:*

i) The map $\sigma_F^{(k)}$ induces a 1-cocycle in $P_{r+s-2k}(\mathbb{C})$.

ii) Let $\tilde{\sigma}_F^{(k)} : \Gamma_1 \to P_{r+s-2k}(\mathbb{C})$ be the map given by

$$\tilde{\sigma}_F^{(k)}(\gamma)(\zeta) := \int_i^{\gamma^{-1}i} \eta_{r-s}\left(y^{\frac{r+s}{2}}F_k, R_{s-r,\mu_k}(\cdot,\zeta)\right),$$

where $\gamma \in \Gamma_1$. This gives a 1-cocycle which belongs to the same cohomology class as $\sigma_F^{(k)}$.

Proof. Throughout this proof we use again the abbreviation $\eta_{r-s}(g;\zeta) := \eta_{r-s}(y^{\frac{r+s}{2}}g, R_{s-r,\mu_k}(\cdot,\zeta))$. The main challenge of proving part (i) is showing that $\sigma_F^{(k)} \in P_{r+s-2k}(\mathbb{C})$, the proof that it is a 1-cocycle in $P_{r+s-2k}(\mathbb{C})$ is then trivial since every 1-coboundary is automatically a 1-cocycle. Since $\eta_{r-s}(g;\zeta)$ are closed for $g = \tilde{F}_k$, ay^{k-r-s}, by^{1-k} and $\mathring{F}_k$, we have

$$\begin{aligned} v_F^{(k)}(\zeta) &= \int_\zeta^i \eta_{r-s}\left(F_k - by^{1-k} - \mathring{F}_k;\zeta\right) + \int_i^{i\infty} \eta_{r-s}\left(\tilde{F}_k + ay^{k-r-s};\zeta\right) \\ &\quad + \int_\zeta^0 \eta_{r-s}\left(by^{1-k};\zeta\right) + \int_\zeta^{-i\infty} \eta_{r-s}(\mathring{F}_k;\zeta) \\ &= \int_\zeta^i \eta_{r-s}(F_k;\zeta) + \int_i^{i\infty} \eta_{r-s}\left(\tilde{F}_k + ay^{k-r-s};\zeta\right) \\ &\quad + \int_i^0 \eta_{r-s}\left(by^{1-k};\zeta\right) + \int_i^{-i\infty} \eta_{r-s}(\mathring{F}_k;\zeta), \end{aligned} \tag{2.38}$$

where the last integral is also taken to be over a path that includes the origin. By (2.35), the last three terms of (2.38) are in $P_{r+s-2k}(\mathbb{C})$. (However, we note that to reach this conclusion, we must first fix a specific path of integration before expanding the integrand in ζ.) Thus, the image of those integrals under the action by $||_{2k-r-s,0}(\gamma-1)$ is in $P_{r+s-2k}(\mathbb{C})$ too.

To show

$$\int_\zeta^i \eta_{r-s}(F_k;\zeta)||_{2k-r-s,0}(\gamma-1) \in P_{r+s-2k}(\mathbb{C}),$$

we observe that, by Lemma 2.15, we have

$$\gamma^* \left(\eta_{r-s}\left(y^{\frac{r+s}{2}}F_k, R_{s-r,\mu_k}(\cdot,\gamma\zeta)\right)\right) = \\ \eta_{r-s}\left((y^{\frac{r+s}{2}}F_k)||_{\frac{r-s}{2},\frac{s-r}{2}}\gamma,\ R_{s-r,\mu_k}(\cdot,\gamma\zeta)||_{\frac{s-r}{2},\frac{r-s}{2}}\gamma\right)$$

where $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1$ and the action $||_{\frac{s-r}{2},\frac{r-s}{2}}$ refers to the implied variable z. We can deduce, using Lemma 2.3 and part (ii) of Lemma 2.16, that the RHS of the above equation is equal to

$$\eta_{r-s}\left(y^{\frac{r+s}{2}}F_k, R_{s-r,\mu_k}(\cdot,\zeta)\right)(c\zeta+d)^{1-2\mu_k}.$$

Therefore, noting that $1-2\mu_k = 2k-r-s$, we have

$$\begin{aligned}
&\left(\int_\zeta^i \eta_{r-s}\left(y^{\frac{r+s}{2}}F_k, R_{s-r,\mu_k}(\cdot,\zeta)\right)\right)\Big|\Big|_{1-2\mu_k,0}\gamma \\
&\quad= \left(\int_{\gamma\zeta}^i \eta_{r-s}\left(y^{\frac{r+s}{2}}F_k, R_{s-r,\mu_k}(\cdot,\gamma\zeta)\right)\right)(c\zeta+d)^{2\mu_k-1} \\
&\quad= \int_\zeta^{\gamma^{-1}i} \gamma^*\left(\eta_{r-s}\left(y^{\frac{r+s}{2}}F_k, R_{s-r,\mu_k}(\cdot,\gamma\zeta)\right)\right)(c\zeta+d)^{2\mu_k-1} \\
&\quad= \int_\zeta^{\gamma^{-1}i} \eta_{r-s}\left(y^{\frac{r+s}{2}}F_k, R_{s-r,\mu_k}(\cdot,\zeta)\right).
\end{aligned}$$

This implies that

$$\begin{aligned}
&\left(\int_\zeta^i \eta_{r-s}\left(y^{\frac{r+s}{2}}F_k, R_{s-r,\mu_k}(\cdot,\zeta)\right)\right)\Big|\Big|_{1-2\mu_k,0}(\gamma-1) \\
&\qquad\qquad= \int_i^{\gamma^{-1}i} \eta_{r-s}\left(y^{\frac{r+s}{2}}F_k, R_{s-r,\mu_k}(\cdot,\zeta)\right)
\end{aligned} \tag{2.39}$$

which is in $P_{r+s-2k}(\mathbb{C})$. We conclude that $\sigma_F^{(k)}(\gamma) \in P_{r+s-2k}(\mathbb{C})$, and therefore, it is a 1-coboundary in $P_{r+s-2k}(\mathbb{C})$. It then follows that it must be a 1-cocycle in $P_{r+s-2k}(\mathbb{C})$.

We now prove the second part. We see with (2.38) and (2.39) that, for all $\gamma \in \Gamma_1$,

$$\begin{aligned}
&\sigma_F^{(k)}(\gamma)(\zeta) = \tilde{\sigma}_F^{(k)}(\gamma)(\zeta) + \\
&\left(\int_i^{i\infty}\eta_{r-s}(\tilde{F}_k + ay^{k-r-s};\zeta) + \int_i^{-i\infty}\eta_{r-s}(\mathring{F}_k;\zeta) + \int_i^0 \eta_{r-s}(by^{1-k};\zeta)\right)\Big|\Big|_{2k-r-s,0}(\gamma-1).
\end{aligned} \tag{2.40}$$

Since the integrals in the bottom line belong to $P_{r+s-2k}(\mathbb{C})$, then $\tilde{\sigma}_F^{(k)}$ is a 1-cocycle with coefficients in $P_{r+s-2k}(\mathbb{C})$. Furthermore, $\sigma_F^{(k)}$ and $\tilde{\sigma}_F^{(k)}$ differ by a 1-coboundary in $P_{r+s-2k}(\mathbb{C})$, and therefore, they belong to the same cohomology class. □

This proposition allows us to see how $\sigma_F^{(k)}$ belongs to a cohomology class reminiscent to that of (1.31). We can also now give the definition of a period function associated to an element of $\mathcal{MI}_1^!$.

Definition 2.21. Let $F \in \mathcal{M}_{r,s}^!$ be a modular iterated integral of length one, then the period function of F is given by

$$P_F(\zeta) := \sum_{k=0}^{\min(r,s)} \sigma_F^{(k)}(S)(\zeta) = \sum_{k=0}^{\min(r,s)} \left(v_F^{(k)} ||_{2k-r-s,0} (S-1) \right)(\zeta),$$

where $v_F^{(k)}$ is as given in Proposition 2.18.

Furthermore, we call $v_F^{(k)}$ the k-th Eichler integral of F.

Remark 2.22. We can compare the above expression to how we express the period polynomial of a modular form using standard Eichler integrals (1.29).

As with the case of the classical period polynomial, the value of the cocycle $\sigma_F^{(k)}$ at the involution S encapsulates the critical values of the L-functions of F_k. However, in the general case, its leading and constant terms must be "truncated".

Theorem 2.23. *Assume that $r \equiv s \mod 4$. Then,*

$$\sigma_F^{(k)}(S)(\zeta) - \sigma_F^c(S)(\zeta) = i \sum_{l=1}^{s+r-2k-1} \left(\sum_{n=0}^{l} \alpha_{n,l} \right) L_{F_k}^*(k+l)\zeta^l,$$

where $\sigma_F^c(S)(\zeta)$ denotes the sum of the leading and constant term of $\sigma_F^{(k)}(S)(\zeta)$ and

$$\alpha_{n,l} = i^{-l-2n} \left((r-k)\binom{s-k+1}{l-n}\binom{r-k+1}{n} - (s-k)\binom{s-k-1}{l-n}\binom{r-k+1}{n} \right).$$

Proof. Using equation (2.40), and noting that $\tilde{\sigma}_F^{(k)}(S)(\zeta) = 0$, we see that $\sigma_F^{(k)}(S)(\zeta)$ is equal to

$$\Big(\int_i^{i\infty} \eta_{r-s}(\tilde{F}_k + ay^{k-r-s};\zeta) + \int_i^{-i\infty} \eta_{r-s}(\mathring{F}_k;\zeta) + \int_i^0 \eta_{r-s}(by^{1-k};\zeta)\Big) \Big|\Big|_{2k-r-s,0} (S-1). \tag{2.41}$$

Now, for each smooth function $h\colon \mathfrak{H} \cup \bar{\mathfrak{H}} \to \mathbb{C}$, we deduce from equation (2.35) that

$$\begin{aligned}\eta_{r-s}(h;\zeta) = &2i\frac{\partial}{\partial z}\Big(y^k h(z)(\zeta-z)^{r-k}(\zeta-\bar{z})^{s-k}\Big)\,dz \\ &+ (r-k)y^{k-1}h(z)(\zeta-z)^{r-k-1}(\zeta-\bar{z})^{s-k+1}dz \\ &+ (s-k)y^{k-1}h(z)(\zeta-z)^{r-k+1}(\zeta-\bar{z})^{s-k-1}d\bar{z}.\end{aligned} \tag{2.42}$$

We will still need to express the above equation for $h = \tilde{F}_k$, $\mathring{F}_k$, y^{k-r-s}, y^{1-k} in a way that is more convenient for us to use in this proof. We look at the more complicated case of $h = y^{k-r-s}$ first.

Using Lemma 2.19, we have

$$\begin{aligned}\int_i^{i\infty} \eta_{r-s}(y^{k-r-s};\zeta) &= \int_i^{i\infty} \eta_{r-s}(y^{k-r-s};\zeta) - \int_i^{i\infty} \eta_{r-s}(y^{k-r-s};0) \\ = \int_i^{i\infty} \Bigg[&2i\frac{\partial}{\partial z}\Big(y^{2k-r-s}\Big((\zeta-z)^{r-k}(\zeta-\bar{z})^{s-k} - (-z)^{r-k}(-\bar{z})^{s-k}\Big)\Big)\,dz \\ &+ (r-k)y^{2k-r-s-1}\Big((\zeta-z)^{r-k-1}(\zeta-\bar{z})^{s-k+1} - (-z)^{r-k-1}(-\bar{z})^{s-k+1}\Big)\,dz \\ &+ (s-k)y^{2k-r-s-1}\Big((\zeta-z)^{r-k+1}(\zeta-\bar{z})^{s-k-1} - (-z)^{r-k+1}(-\bar{z})^{s-k-1}\Big)\,d\bar{z}\Bigg].\end{aligned}$$

The addition of $\int_i^{i\infty} \eta_{r-s}(y^{k-r-s};0)$ ensures that each of the polynomials in z have degree $\leq s+r-2k-1$. Hence, the integral $\int_i^{i\infty} \eta_{r-s}(y^{k-r-s};\zeta)$ converges and, by integrating along the positive imaginary axis, we conclude that it equals

$$\begin{aligned}&(r-k)i\int_1^{\infty} y^{2k-r-s-1}\Big((\zeta-iy)^{r-k-1}(\zeta+iy)^{s-k+1} - (-iy)^{r-k-1}(iy)^{s-k+1}\Big)\,dy \\ &- (s-k)i\int_1^{\infty} y^{2k-r-s-1}\Big((\zeta-iy)^{r-k+1}(\zeta+iy)^{s-k-1} - (-iy)^{r-k+1}(iy)^{s-k-1}\Big)\,dy \\ &\quad - 2i((\zeta-i)^{r-k}(\zeta+i)^{s-k} - 1),\end{aligned}$$

where we used the fact that $r \equiv s \mod 4$ to get -1 in the last summand. We can use equation (2.42) directly for $h = \tilde{F}_k, \mathring{F}_k, by^{1-k}$. For example, when $h = \tilde{F}_k$ we have

$$\begin{aligned}\int_i^{i\infty} \eta_{r-s}(\tilde{F}_k;\zeta) = (r-k)i\int_1^\infty y^{k-1}\tilde{F}_k(iy)(\zeta-iy)^{r-k-1}(\zeta+iy)^{s-k+1}dy \\ - (s-k)i\int_1^\infty y^{k-1}\tilde{F}_k(iy)(\zeta-iy)^{r-k+1}(\zeta+iy)^{s-k-1}dy \\ - 2i\tilde{F}_k(i)(\zeta-i)^{r-k}(\zeta+i)^{s-k},\end{aligned}$$

with a similar result when $h = \mathring{F}_k, by^{1-k}$.

Now, we note that

$$\begin{aligned}&(r-k)i\int_1^\infty y^{k-1}\tilde{F}_k(iy)(\zeta-iy)^{r-k-1}(\zeta+iy)^{s-k+1}dy \underset{2k-r-s,0}{||} (S-1) \\ &= (r-k)i\int_1^\infty y^{k-1}\tilde{F}_k(iy)\Big[(-1/\zeta-iy)^{r-k-1}(-1/\zeta+iy)^{s-k+1}\zeta^{r+s-2k} \\ &\qquad - (\zeta-iy)^{r-k-1}(\zeta+iy)^{s-k+1}\Big]dy \\ &= (r-k)i\int_1^\infty y^{k-1}\tilde{F}_k(iy)\Big[(1+iy\zeta)^{r-k-1}(1-iy\zeta)^{s-k+1} \\ &\qquad - (\zeta-iy)^{r-k-1}(\zeta+iy)^{s-k+1}\Big]dy,\end{aligned}$$

with an analogous equation for

$$(s-k)i\int_1^\infty y^{k-1}\tilde{F}_k(iy)(\zeta-iy)^{r-k+1}(\zeta+iy)^{s-k-1}dy \underset{2k-r-s,0}{||} (S-1).$$

Furthermore, $2i\tilde{F}_k(i)(\zeta-i)^{r-k}(\zeta+i)^{s-k}||_{2k-r-s,0}(S-1)$ vanishes because $r \equiv s \mod 4$. Therefore, we have

$$\eta_{r-s}(\tilde{F}_k;\zeta) \underset{2k-r-s,0}{||} (S-1) = i\int_1^\infty y^{k-1}\tilde{F}_k(iy)R(y,\zeta)\,dy,$$

where

$$\begin{aligned}R(y,\zeta) := (r-k)\left((1+iy\zeta)^{r-k-1}(1-iy\zeta)^{s-k+1} - (\zeta-iy)^{r-k-1}(\zeta+iy)^{s-k+1}\right) \\ - (s-k)\left((1+iy\zeta)^{r-k+1}(1-iy\zeta)^{s-k-1} - (\zeta-iy)^{r-k+1}(\zeta+iy)^{s-k-1}\right).\end{aligned}$$

We get a similar result for $h = \mathring{F}_k, y^{k-r-s}, y^{1-k}$. Combining this with equation (2.41), we can deduce that $\sigma_F^{(k)}(S)(\zeta) - \sigma_F^c(S)(\zeta)$ equals

$$i\int_1^\infty y^{k-1}(\tilde{F}_k(iy) + ay^{k-r-s})(R(y,\zeta) - R_0(y,\zeta))\,dy \qquad (2.43)$$
$$+ i\int_1^{-\infty} y^{k-1}(\mathring{F}_k(iy))(R(y,\zeta) - R_0(y,\zeta))\,dy$$
$$+ i\int_1^0 y^{k-1}(by^{1-k})(R(y,\zeta) - R_0(y,\zeta))\,dy,$$

where R_0 is the sum of the constant and leading term of the expansion of $R(y,\zeta)$ in ζ. Using the binomial expansion and the notation from the statement of the theorem, we can write (2.43) as

$$i\sum_{l=1}^{s+r-2k-1}\left(\sum_{n=0}^{l}\alpha_{n,l}\right)\left(\int_1^\infty \left(\tilde{F}_k(iy) + ay^{k-r-s}\right)\left(y^{k+l} + i^{s-r}y^{s+r-k-l}\right)\frac{dy}{y}\right.$$
$$\left. + \int_1^{-\infty}\mathring{F}_k(iy)\left(y^{k+l} + i^{s-r}y^{s+r-k-l}\right)\frac{dy}{y} + \int_1^0 by^{1-k}\left(y^{k+l} + i^{s-r}y^{s+r-k-l}\right)\frac{dy}{y}\right)\zeta^l,$$

where we also used the identity

$$\left(\sum_{i=0}^{m} a_i x^i\right)\left(\sum_{j=0}^{n} b_j x^j\right) = \sum_{l=0}^{m+n}\left(\sum_{n=0}^{l} a_n b_{l-n}\right)x^l.$$

Using the definition of the L-function for F_k (say, equation (2.22)) we deduce the theorem. □

From Proposition 2.20 we obtain a map from the space of modular iterated integrals of length one to a direct sum of copies of the space of classical modular (resp. cusp) forms.

Theorem 2.24. *The maps $\sigma_F^{(k)}$ ($k \in \{0,\ldots,min(r,s)\}$) defined in Proposition 2.20 induce a map*

$$\mathcal{MI}_1^! \cap \mathcal{M}_{r,s}^! \to \bigoplus_{k=0}^{min(r,s)} H^1(\Gamma_1, P_{r+s-2k}(\mathbb{C})) \cong \bigoplus_{k=0}^{min(r,s)} \left(\bar{S}_{r+s-2k+2} \oplus M_{r+s-2k+2}\right).$$

Proof. Proposition 2.20 induces a map sending each $F \in \mathcal{MI}_1^! \cap \mathcal{M}_{r,s}^!$ to

$$\left(\left[\sigma_F^{(0)}\right],\ldots,\left[\sigma_F^{(\min(r,s))}\right]\right) \in \bigoplus_{k=0}^{\min(r,s)} H^1(\Gamma_1, P_{r+s-2k}(\mathbb{C})).$$

Here $[\sigma_F^{(k)}]$ stands for the cohomology class of the 1-cocycle defined in that proposition. The last isomorphism of this theorem follows from the Eichler-Shimura isomorphism (1.30). □

Corollary 2.25. *Let F be a modular iterated integral of length one and weights (r,s) and let $F = F_0 + \cdots + F_{min(r,s)}$ be its decomposition into eigenfunctions of the Laplacian. Then, for each $k \in \{0,\ldots,min(r,s)\}$, there is a $p_k(\zeta) \in P_{r+s-2k}(\mathbb{C})$ and unique $f_k \in S_{r+s-2k+2}$, $g_k \in M_{r+s-2k+2}$ such that, for all $\gamma \in \Gamma_1$,*

$$\int_i^{\gamma^{-1}i} \eta_{r-s}(y^{\frac{r+s}{2}} F_k, R_{s-r,\mu_k}(\cdot,\zeta)) = \overline{\int_i^{\gamma^{-1}i} f(z)(z-\bar{\zeta})^{r+s-2k}dz} + \int_i^{\gamma^{-1}i} g(z)(z-\zeta)^{r+s-2k}dz + p_k\Big|\Big|_{2k-r-s,0}(\gamma-1).$$

2.6.3 *An Application to Algebraicity*

In [28] an Eichler-Shimura isomorphism is proved for weakly holomorphic modular forms. As a proof of concept for the "correctness" of our definition of the L-function in Section 2.5, we will use the results of [28] to show an analogue of Manin's Periods Theorem (Theorem 1.13) holds for weakly holomorphic forms. It should be mentioned that K. Bringmann has shown an alternative way, based on results of [29], to establish a statement that implies the same result.

Before stating and proving our result, we first summarise the setup of [28] and then show that it is compatible with the explicit expressions for the cocycles given in the last section.

We let $M^!_{k,\mathbb{Q}}$ denote the $\mathbb{Q}$-vector space of weight k weakly holomorphic modular forms for Γ_1, with rational Fourier coefficients. Similarly, we let $S^!_{k,\mathbb{Q}}$ denote the same space but with weakly holomorphic modular cusp forms instead. In Section 1.7, we showed that, although there are generally no Hecke eigenforms in $M^!_k$, there are well-defined operators on $M^!_k/D^{k-1}M^!_{2-k}$ induced by the standard Hecke operators. Using this terminology and notation, we have the following theorem:

Theorem 2.26. *(Cor. 1.3 of [28]) Let ϕ denote the map that assigns to each $f \in M_k^!$ the function* (1.31)*, then ϕ induces a Hecke invariant isomorphism*

$$[\phi] : M^!_{k,\mathbb{Q}}/D^{k-1}M^!_{2-k,\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} H^1(\Gamma_1, P_{k-2}(\mathbb{Q})) \otimes_{\mathbb{Q}} \mathbb{C}.$$

The image of $S^!_{k,\mathbb{Q}}/D^{k-1}M^!_{2-k,\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C}$ under $[\phi]$ is the parabolic cohomology group defined by

$$H^1_{\text{par}}(\Gamma_1, P_{k-2}(\mathbb{C})) := Z^1_{\text{par}}(\Gamma_1, P_{k-2}(\mathbb{C}))/B^1_{\text{par}}(\Gamma_1, P_{k-2}(\mathbb{C}))$$

where, with $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$,

$$Z^1_{\text{par}}(\Gamma_1, P_{k-2}(\mathbb{C})) := \{\psi \in Z^1(\Gamma_1, P_{k-2}(\mathbb{C})); \psi(T) = 0\}$$

is the space of parabolic cocycles and

$$B^1_{\text{par}}(\Gamma_1, P_{k-2}(\mathbb{C})) := \{\psi \in B^1(\Gamma_1, P_{k-2}(\mathbb{C})); \psi(T) = 0\}$$

is the space of parabolic coboundaries. $B^1_{\text{par}}(\Gamma_1, P_{k-2}(\mathbb{C}))$ is generated by ψ_0 such that

$$\psi_0(\gamma) = 1||_{2-k,0}(\gamma - 1). \tag{2.44}$$

Furthermore, we let F_∞ be the "real Frobenius" induced by the map sending $\sigma \in Z^1_{\text{par}}(\Gamma_1, P_{k-2}(\mathbb{C}))$ to $F_\infty\sigma \in Z^1_{\text{par}}(\Gamma_1, P_{k-2}(\mathbb{C}))$ such that

$$F_\infty\sigma\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)(z) = \sigma\left(\begin{smallmatrix} a & -b \\ -c & d \end{smallmatrix}\right)(-z).$$

Then $H^1_{\text{par}}(\Gamma_1, P_{k-2}(\mathbb{C}))$ is decomposed into F_∞-eigenspaces as follows:

$$H^1_{\text{par}}(\Gamma_1, P_{k-2}(\mathbb{C})) = H^{1,+}_{\text{par}}(\Gamma_1, P_{k-2}(\mathbb{C})) \oplus H^{1,-}_{\text{par}}(\Gamma_1, P_{k-2}(\mathbb{C})).$$

Each class of $H^{1,+}_{\text{par}}(\Gamma_1, P_{k-2}(\mathbb{C}))$ (resp. $H^{1,-}_{\text{par}}(\Gamma_1, P_{k-2}(\mathbb{C}))$ is represented by a cocycle σ such that $\sigma(S)$ is an even (resp. odd) polynomial.

Now let ϕ' be the map that assigns to each $f \in S^!_{k,\mathbb{Q}}$ the cocycle $\phi'_f = d^0 v'_f = v'_f||_{2-k,0}(\gamma - 1)$, where $v'_f \colon \mathfrak{H} \to P_{k-2}(\mathbb{C})$ is given by

$$v'_f(z) := \int_{i\infty}^{z} \tilde{f}(w)(w - z)^{k-2}dw + \int_{-i\infty}^{z} \mathring{f}(w)(w - z)^{k-2}dw,$$

where $\tilde{f}$ and $\mathring{f}$ are defined by (2.3) and (2.5).

We will now prove the following proposition:

Proposition 2.27. *The map ϕ' induces a map $[\phi']$ from $S^!_k/D^{k-1}M^!_{2-k}\otimes_{\mathbb{Q}}\mathbb{C}$ to $H^1_{par}(\Gamma_1, P_{k-2}(\mathbb{Q}))\otimes_{\mathbb{Q}}\mathbb{C}$. The resulting diagram*

$$\begin{array}{ccc} M^!_k/D^{k-1}M^!_{2-k}\otimes_{\mathbb{Q}}\mathbb{C} & \xrightarrow{[\phi]} & H^1(\Gamma_1, P_{k-2}(\mathbb{Q}))\otimes_{\mathbb{Q}}\mathbb{C} \\ i\uparrow & & j\uparrow \\ S^!_k/D^{k-1}M^!_{2-k}\otimes_{\mathbb{Q}}\mathbb{C} & \xrightarrow{[\phi']} & H^1_{par}(\Gamma_1, P_{k-2}(\mathbb{Q}))\otimes_{\mathbb{Q}}\mathbb{C} \end{array} \tag{2.45}$$

(where i, j are natural injections and ϕ is as in Theorem 2.26) is commutative.

Proof. Let $f \in S^!_{k,\mathbb{Q}}$, then

$$\begin{aligned} v'_f||_{2-k,0}(T) &= \int_{i\infty}^{z+1}\tilde{f}(w)(w-(z+1))^{k-2}dw + \int_{-i\infty}^{z+1}\mathring{f}(w)(w-(z+1))^{k-2}dw \\ &= \int_{i\infty}^{z}\tilde{f}(w+1)(w-z)^{k-2}dw + \int_{-i\infty}^{z}\mathring{f}(w+1)(w-z)^{k-2}dw \\ &= v'_f, \end{aligned}$$

where we used the change of variables $w \to w+1$ and then the fact that both $\tilde{f}$ and $\mathring{f}$ are periodic with period 1. Therefore,

$$\phi'_f(T) = v'_f||_{2-k,0}(T-1)(z) = 0.$$

In addition, ϕ'_f is a cocycle by construction, and therefore, it is a parabolic cocycle. This proves the first assertion.

Now, using $f = \tilde{f} + \mathring{f}$, we have

$$\begin{aligned} v'_f(z) &= \int_{i\infty}^{i}\tilde{f}(w)(w-z)^{k-2}dw + \int_{i}^{z}\tilde{f}(w)(w-z)^{k-2}dw \\ &\qquad + \int_{-i\infty}^{i}\mathring{f}(w)(w-z)^{k-2}dw + \int_{i}^{z}\mathring{f}(w)(w-z)^{k-2}dw \\ &= \int_{i}^{z} f(w)(w-z)^{k-2}dw + \int_{i\infty}^{i}\tilde{f}(w)(w-z)^{k-2}dw + \int_{-i\infty}^{i}\mathring{f}(w)(w-z)^{k-2}dw. \end{aligned}$$

We then observe that

$$d(\gamma w) = (cw+d)^{-2}dw \quad \text{and} \quad (\gamma w - \gamma z) = \frac{w-z}{(cw+d)(cz+d)},$$

for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1$. Therefore,

$$\begin{aligned} \int_{i}^{z} f(w)(w-z)^{k-2}dw||_{2-k,0}\gamma &= \int_{i}^{\gamma z} f(w)(w-\gamma z)^{k-2}dw(cz+d)^{k-2} \\ &= \int_{\gamma^{-1}i}^{z} f(\gamma w)(\gamma w-\gamma z)^{k-2}(cz+d)^{k-2}d(\gamma w) \\ &= \int_{\gamma^{-1}i}^{z} f(w)(w-z)^{k-2}dw. \end{aligned}$$

This implies that

$$\int_i^z f(w)(w-z)^{k-2}dw||_{2-k,0}(\gamma-1) = \int_{\gamma^{-1}i}^i f(w)(w-z)^{k-2}dw$$

and hence we can write

$$\begin{aligned}\phi'_f(\gamma) = &\int_{\gamma^{-1}i}^i f(w)(w-z)^{k-2}dw \\ &+ \left[\int_{i\infty}^i \tilde{f}(w)(w-z)^{k-2}dw + \int_{-i\infty}^i \mathring{f}(w)(w-z)^{k-2}dw\right]\Big|\Big|_{2-k,0}(\gamma-1).\end{aligned} \tag{2.46}$$

Since the term inside the square brackets belongs to $P_{k-2}(\mathbb{C})$, the cohomology class $j([\phi'](f))$ of this cocycle coincides with $[\phi](f)$. □

Remark 2.28. In [29], Theorem 1.2 proves that $[\phi']$ is an isomorphism.

We are now ready to prove our analogous version of Manin's Periods Theorem.

Theorem 2.29. *Suppose that the class of $f \in S^!_k$ in $S^!_k/D^{k-1}M^!_{2-k}$ is an eigenclass of the Hecke operators. Let K_f denote the field generated by the Fourier coefficients of f, then there exist $\omega^\pm(f) \in \mathbb{C}$ such that*

$$L^*_f(j) \in \omega^+(f)K_f, \quad \text{for odd } j \in \{2,\dots,k-2\}$$

and

$$L^*_f(j) \in \omega^-(f)K_f, \quad \text{for even } j \in \{2,\dots,k-2\}.$$

Proof. We first note that in [18] it is proven that the eigenspace of the class of f in $S^!_{k,\mathbb{Q}}/D^{k-1}M^!_{2-k,\mathbb{Q}} \otimes_\mathbb{Q} \mathbb{C}$ is two-dimensional. We let

$$V_f^{\mathrm{deR}} \subset S^!_{k,\mathbb{Q}}/D^{k-1}M^!_{2-k,\mathbb{Q}} \otimes_\mathbb{Q} K_f$$

denote the Hecke eigenspace generated by f and then let V_f^B be the corresponding eigenspace in $H^1(\Gamma_1, P_{k-2}(\mathbb{Q})) \otimes_\mathbb{Q} K_f$. We follow the notation of [28] to indicate the de Rham-cohomological and Betti-cohomological origin of those eigenspaces.

The space V_f^B is two dimensional as a K_f-vectorspace, and therefore, so is the corresponding eigenspace V_f^W in $H^1_{\text{par}}(\Gamma_1, P_{k-2}(\mathbb{Q})) \otimes_{\mathbb{Q}} K_f$. It decomposes into invariant and anti-invariant eigenspaces with respect to the real Frobenius:

$$V_f^W = V_f^{W,+} \oplus V_f^{W,-}.$$

Furthermore, by Theorem 2.26, the map

$$[\phi'] : V_f^{\text{deR}} \otimes_{K_f} \mathbb{C} \xrightarrow{\sim} V_f^W \otimes_{K_f} \mathbb{C}$$

is a canonical isomorphism. Therefore, for some $\omega^\pm(f) \in \mathbb{C}$ and some $\phi^\pm(f) \in V_f^{W,\pm} \subset H^{1,\pm}_{\text{par}}(\Gamma_1, P_{k-2}(K_f))$, we have

$$[\phi'(f)] = \omega^+(f)\phi^+(f) + \omega^-(f)\phi^-(f).$$

Thus, for an even $P^+ \in Z^1_{\text{par}}(\Gamma_1, P_{k-2}(K_f))$, an odd $P^- \in Z^1_{\text{par}}(\Gamma_1, P_{k-2}(K_f))$ and a $c_f \in \mathbb{C}$,

$$\phi'(f) = \omega^+(f)P^+ + \omega^-(f)P^- + c_f\psi_0,$$

where we recall that ψ_0 is defined by (2.44) and generates $B^1_{\text{par}}(\Gamma_1, P_{k-2}(\mathbb{C}))$. This gives

$$\phi'(f)(S) = \omega^+(f)P^+(S) + \omega^-(f)P^-(S) + c_f||_{2-k,0}(S-1). \tag{2.47}$$

On the other hand, using (2.46), we have

$$\phi'_f(S)(z) = \left[\int_{i\infty}^{i} \tilde{f}(w)(w-z)^{k-2}dw + \int_{-i\infty}^{i} \mathring{f}(w)(w-z)^{k-2}dw\right]\Bigg|\Bigg|_{2-k,0}(S-1)$$

and, with an application of the binomial formula, this gives

$$\begin{aligned}\phi'_f(S)(z) = \sum_{a=0}^{k-2}\binom{k-2}{a} i^{a+1} z^a \cdot \Bigg[\int_{\infty}^{1} \tilde{f}(it)t^a dt + \int_{-\infty}^{1} \mathring{f}(it)t^a dt \\ + i^k\left(\int_{\infty}^{1} \tilde{f}(it)t^{k-2-a}dt + \int_{-\infty}^{1} \mathring{f}(it)t^{k-2-a}dt\right)\Bigg].\end{aligned}$$

Comparing this to equation (2.31), we can see that the coefficient of z^a for $a = j-1$ in $\phi'(f)(S)$ is a multiple of $L_f^*(j)$ by an element of $\mathbb{Q}[i]$. By comparing, in (2.47), the coefficients of z^{j-1} for j odd (other than $j = k-1$ and $j = 1$, since $c_f||_{2-k,0}(S-1)$ contributes terms with powers in z of $k-2$ and 0) and j even , we deduce the assertion. □

We note that, unlike Manin's Periods Theorem, this theorem does not say anything about the L-function at the arguments $k-1$ and 1. As confirmed by numerical experiments carried out by F. Strömberg, the lack of K_f-proportionality of $L_f^*(k-1)$ and $L_f^*(1)$ seems to be genuine and not due to any incompleteness of our proof.

3

THE ACTION OF HECKE OPERATORS ON EICHLER INTEGRALS OF REAL ANALYTIC FORMS

The Fourier coefficients of modular forms are a source of remarkable and extremely useful arithmetic information, as demonstrated in Section 1.2. Furthermore, the eigenvalues of Hecke operators are closely related to the Fourier coefficients (see Section 1.3), and therefore, learning about the former will uncover important arithmetic information about the latter. In general, information about the eigenvalues of a linear map $T : V \to V$ from a vector space to itself can be recovered from its trace, since this value is equal to the sum of the eigenvalues (including multiplicities).

We are able to define the trace of T by first choosing a basis for V, we then describe T as a matrix relative to this basis and take the trace of this square matrix. This value does not depend on the basis we have chosen.

Formulas, therefore, that describe the trace of Hecke operators are of great importance. Such a formula, now known as the Eichler-Selberg trace formula, was introduced by Selberg in 1956 [7]. This original formula, however, was rather complicated and involved calculating the Hurwitz-Kronecker class number; this number is related to the number of Γ_1 equivalence classes of certain quadratic forms. Since then, equivalent but simpler variations of this formula have been given using purely algebraic approaches (see [30], [31] or [32], for example). We will see one such formula in Theorem 3.2.

The key idea behind this new algebraic approach involves looking at how Hecke operators act on the Eichler integrals of standard modular cusp forms. We begin this chapter by discussing this part of the theory in detail. Later, we extend this theory to real analytic modular forms by looking at how Hecke operators act on Eichler integrals of length one modular iterated integrals.

3.1 HECKE OPERATORS ACTING ON EICHLER INTEGRALS OF MODULAR FORMS

Let $f(z) = \sum_{m>0} a_m q^m$ be a modular cusp form of weight r, we previously defined an Eichler integral for such a form by

$$v_f(z) := \int_{i\infty}^{z} f(\tau)(\tau - z)^{r-2} d\tau.$$

Using the change of variables $\tau \to \tau + z$, this becomes

$$v_f(z) = \int_{i\infty}^{0} f(\tau + z)\tau^{r-2} d\tau = -\int_0^{i\infty} \sum_{m>0} a_m e^{2\pi i m\tau} e^{2\pi i m z} \tau^{r-2} d\tau$$

and another change of variables $\tau \to it$ followed by $2\pi m t \to t$ gives

$$\begin{aligned} v_f(z) &= -i \int_0^{\infty} \sum_{m>0} a_m e^{-2\pi m t} e^{2\pi i m z} (it)^{r-2} dt \\ &= -i^{r-1} \int_0^{\infty} \sum_{m>0} a_m e^{-t} e^{2\pi i m z} (t/2\pi m)^{r-2} (2\pi m)^{-1} dt \\ &= -i^{r-1} \left(\int_0^{\infty} e^{-t} t^{r-2} dt \right) \sum_{m>0} a_m e^{2\pi i m z} (2\pi m)^{1-r} \\ &= \frac{\Gamma(r-1)}{(2\pi i)^{r-1}} \sum_{m>0} \frac{a_m}{m^{r-1}} e^{2\pi i m z}. \end{aligned}$$

Therefore, for a modular cusp form $f(z) = \sum_{m>0} a_m q^m$, we have

$$v_f(z) = A \cdot \sum_{m>0} \frac{a_m}{m^{r-1}} e^{2\pi i m z} \tag{3.1}$$

where

$$A := \frac{\Gamma(r-1)}{(2\pi i)^{r-1}}.$$

Usually, the constant A is dropped in the literature that focuses solely on the action of Hecke operators on Eichler integrals. We have seen that the Eichler integral of a modular cusp form is closely related to its period polynomial P_f:

$$P_f(z) = v_f(z)|_{2-r}(S-1), \quad \text{where } S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right). \tag{3.2}$$

A well known relation between the Hecke operators and the Eichler integral is given by the following equation:

$$n^{\frac{2-r}{2}} \cdot v_{T_n f}(z) = v_f(z)|_{2-r} T_n^\infty, \tag{3.3}$$

where

$$T_n^\infty := \sum_{\alpha \in M_n^\infty} \alpha$$

and M_n^∞ is the set of matrices $\alpha = \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$ with $d > b \geq 0$, $a > 0$ and $ad = n$.

In order to see how equation (3.3) holds we first look at the RHS, using equations (3.1) and (1.1) we have

$$\begin{aligned} v_f(z)|_{2-r} T_n^\infty &= \left(A \cdot \sum_{m>0} \frac{a_m}{m^{r-1}} \, e^{2\pi i m z} \right) \Big|_{2-r} T_n^\infty \\ &= n^{\frac{2-r}{2}} A \cdot \sum_{\alpha \in M_n^\infty} d^{r-2} \sum_{m>0} \frac{a_m}{m^{r-1}} \, e^{2\pi i m \alpha z}. \end{aligned} \tag{3.4}$$

We now focus on the LHS of (3.3) and note that, as in Section 1.3, every equivalence class in $\Gamma_1 \backslash \mathcal{M}_n$ contains exactly one representative $\alpha = \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$ with $d > b \geq 0$ and $a > 0$. Therefore, the action of T_n on f can be written as

$$\begin{aligned} T_n f(z) &= \sum_{\alpha \in M_n^\infty} n^{r-1} d^{-r} f(\alpha z) \\ &= \sum_{\alpha \in M_n^\infty} n^{r-1} d^{-r} \sum_{m>0} a_m e^{2\pi i m a z/d} e^{2\pi i m b/d} \\ &= \sum_{m>0} \sum_{\alpha \in M_n^\infty} n^{r-1} d^{-r} a_m e^{2\pi i m b/d} \, q^{ma/d}, \end{aligned}$$

which means, using equation (3.1),

$$\begin{aligned} v_{T_n f}(z) &= A \cdot \sum_{m>0} \sum_{\alpha \in M_n^\infty} \frac{n^{r-1} d^{-r} a_m e^{2\pi i m b/d}}{(ma/d)^{r-1}} e^{2\pi i m a z/d} \\ &= A \cdot \sum_{\alpha \in M_n^\infty} d^{-1} n^{r-1} a^{1-r} \sum_{m>0} \frac{a_m}{m^{r-1}} e^{2\pi i m \alpha z}. \end{aligned}$$

Now, since $a = n/d$, this becomes

$$v_{T_n f}(z) = A \cdot \sum_{\alpha \in M_n^\infty} d^{r-2} \sum_{m>0} \frac{a_m}{m^{r-1}} e^{2\pi i m \alpha z}. \tag{3.5}$$

Comparing equations (3.4) and (3.5), we can see that the identity given by equation (3.3) does indeed hold. We will now look at how this identity plays a key role in defining a formula for the trace of Hecke operators.

First, we need to check that the Eichler integral is T-invariant:

$$v_f(z)|_{2-r}(T-1) = 0, \qquad \text{where } T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right). \tag{3.6}$$

We have

$$v_f(z)|_{2-r}T = A \cdot \sum_{m>0} \frac{a_m}{m^{r-1}} e^{2\pi i m T(z)}$$

and, since $e^{2\pi i m z}$ is invariant under $z \to z+1$, we conclude that

$$v_f(z)|_{2-r}T = A \cdot \sum_{m>0} \frac{a_m}{m^{r-1}} e^{2\pi i m z} = v_f(z).$$

Therefore, equation (3.6) holds. Combining this identity with the next theorem, we can derive information regarding the Hecke operator's compatibility with the period polynomial of a cusp form.

Theorem 3.1 (Thm. 1 of [31]). *Let $n > 0$ be an integer and M_n be the set of all 2×2 matrices with integral entries and determinant n, then there exists a $\tilde{T}_n \in \mathbb{Q}[M_n]$ which satisfies the following:*

a) $(S-1)\tilde{T}_n = T_n^\infty(S-1) + (T-1)Y_n$, *for some* $Y_n \in \mathbb{Q}[M_n]$. (3.7)

b) $(U-1)\tilde{T}_n(S+1) = 0$, *where* $U = \left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. (3.8)

c) $(S-1)\tilde{T}_n(U^2+U+1) = 0$. (3.9)

Applying equation (3.7) from this theorem to the Eichler integral of f and using equation (3.6) gives

$$v_f|_{2-r}(S-1)|_{2-r}\tilde{T}_n = v_f|_{2-r}T_n^\infty|_{2-r}(S-1)$$

and, using equations (3.2) and (3.3), we deduce that

$$P_f|_{2-r}\tilde{T}_n = n^{\frac{2-r}{2}} \cdot v_{T_n f}|_{2-r}(S-1).$$

A second application of equation (3.2) shows that the Hecke action is compatible with the period polynomials of modular cusp forms. In particular, we have the equation

$$P_f|_{2-r}\tilde{T}_n = n^{\frac{2-r}{2}} \cdot P_{T_n f}. \tag{3.10}$$

In [31], it is shown how we can use this identity, along with equations (3.8) and (3.9), to prove the next theorem. This theorem provides us with a trace formula for Hecke operators.

Theorem 3.2 (Thm. 1 of [31]). *Suppose* $\tilde{T}_n = \sum_{M \in M_n} c(M)[M]$ *satisfies equations* (3.7), (3.8) *and* (3.9), *then*

$$tr(T_n, S_k) + tr(T_n, M_k) = \sum_{M \in M_n} c(M) p_{k-2}(tr(M), n),$$

where $k > 2$ *and*

$$p_v(t,n) = \sum_{r=0}^{v/2} (-1)^r \binom{v-r}{r} t^{v-r} n^r$$

is the coefficient of x^v *in* $(1 - tx + nx^2)^{-1}$.

3.2 AN ANALOGOUS RESULT FOR LENGTH ONE MODULAR ITERATED INTEGRALS

In this section, we will look at an analogous version of equation (3.3) but for Eichler integrals of length one modular iterated integrals. First, however, we will need to extend the definition of a Hecke operator to include real analytic modular forms, this extension was first explicitly defined by Brown in [6].

Definition 3.3. Let $n > 0$ be an integer, then we define T_n to be the n-th Hecke operator which acts on a real analytic modular form F of weights (r,s) in the following way:

$$T_n f(z) := n^{r+s-1} \sum_{\mu \in \Gamma_1 \backslash \mathcal{M}_n} (cz+d)^{-r} (c\bar{z}+d)^{-s} f(\mu(z)),$$

where $\mu = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, $\mathcal{M}_n$ is the set of all 2×2 matrices with integral entries and determinant n, and $\Gamma_1 \backslash \mathcal{M}_m$ is the (finite) set of cosets.

Remark 3.4. As this definition is dependent on the weights (r, s), we should write $T_n^{r,s} f(z)$ but for convenience we write $T_n f(z)$ and consider the weights fixed.

Remark 3.5. In the case when F is a standard modular form, this definition reduces to the original definition of a Hecke operator we gave in Section 1.3.

An important relation between the Hecke and Laplace operators, which we will need to make use of later, is given by the lemma below.

Lemma 3.6. *(Lemma 6.2 of [6]) Let F be a real analytic modular form, then*

$$T_n(\Delta F) = \Delta(T_n F).$$

In the previous section we did not work with Eichler integrals of all modular forms but just those of modular cusp forms, we will follow a similar approach for real analytic modular forms. This is, however, complicated by the fact that there exist two different spaces that we can consider as the space of cusp forms of real analytic modular forms. Interestingly, our analogous version of equation (3.3) for real analytic modular forms (given by Theorem 3.14) holds for both types of cusp forms. This means we will not have to worry too much about the distinction between the two. We will, however, give an explanation of both types of cusp forms below.

Classical modular cusp forms can be seen as the modular forms $f = \sum_{m=0}^{\infty} a_m q^m$ whose constant part a_0 vanishes. The "constant" part of a real analytic modular form F is given by

$$F^0 = \sum_{|j| \leq M} y^j a_{0,0}^{(j)}.$$

Although this is not a constant function on the upper half plane, we call it the constant part as each $y^j a_{0,0}^{(j)}$ is constant in terms of the operators ∂_{-j} and $\bar{\partial}_{-j}$.

In [6], a function $F \in \mathcal{M}_{r,s}^!$ is defined to be a cusp form if F^0 vanishes. The subspace of such forms is denoted by $S_{r,s}^!$ with $\mathcal{S}^! = \bigoplus_{r,s} \mathcal{S}_{r,s}^!$. This is one of the spaces of cusp forms we consider. As mentioned above, we will see that our analogous version of equation (3.3) does indeed hold for such cusp forms. However, the space $\mathcal{MI}_1^!$ is a relatively new space and we do not yet know if there is any overlap between $\mathcal{S}^!$ and $\mathcal{MI}_1^!$. It may not be so useful, then, to show that a version of (3.3) holds for a potentially empty space.

The second type of cusp forms we consider are called "real analytic cusp forms", their space is denoted by $\mathcal{H}(f)$. Not only will we see that our version of equation (3.3) holds for these types of cusp forms as well, we will also see that every real analytic cusp form is an element of $\mathcal{MI}_1^!$. Therefore, as long as $\mathcal{H}(f)$ is non-empty, the space $\mathcal{H}(f) \cap \mathcal{MI}_1^!$ is also non-empty. We will take a closer look at these forms in the following section.

3.2.1 *Real Analytic Cusp Forms*

Real analytic cusp forms were first introduced by Brown in [6], to define such forms we will first return to the definition of $\mathcal{MI}_1^!$. We have seen that $\mathcal{MI}_1^!$ is the largest subspace of $\bigoplus_{r,s\geq 0} \mathcal{M}_{r,s}^!$ such that

$$\partial \mathcal{MI}_1^! \subset \mathcal{MI}_1^! + M^![\mathbb{L}^{\pm}],$$
$$\bar{\partial} \mathcal{MI}_1^! \subset \mathcal{MI}_1^! + \overline{M}^![\mathbb{L}^{\pm}].$$

Since the operator ∂ is of bidegree $(+1,-1)$, if $F \in \mathcal{M}_{r,s}$ then $\partial F \in \mathcal{M}_{r+1,s-1}$. Now let $F \in \mathcal{M}_{r,0} \cap \mathcal{MI}_1^!$, it follows that $\partial F \in M^![\mathbb{L}^{\pm}]$ and, since $\mathbb{L}$ can be viewed as an element of $\mathcal{MI}^!$ of weights $(-1,-1)$, then

$$\partial F = \mathbb{L} \cdot f$$

for some $f \in M_{r+2}^!$. We call such an F a modular primitive of $\mathbb{L}f$. We are now prepared to look at a certain class of functions, denoted by $X_{r,s}$, which contains the real analytic cusp forms.

Proposition 3.7 (Prop. 5.4 of [6]). *Let $w \geq 0$ and $X_{w,0} \in \mathcal{M}^!$ be a modular primitive of $\mathbb{L}f$:*

$$\partial X_{w,0} = \mathbb{L}f,$$

where $f \in M^!_{w+2}$. *Then there exists a family of unique elements* $X_{r,s} \in \mathcal{M}^!_{r,s}$, *with* $r \geq 0$, $s > 0$ *and* $r + s = w$, *such that*

$$\partial X_{r,s} = (r+1)X_{r+1,s-1}, \qquad \text{for } s \geq 1,$$

and

$$\bar{\partial} X_{r,s} = (s+1)X_{r-1,s+1}, \qquad \text{for } r \geq 1,$$
$$\bar{\partial} X_{0,w} = \mathbb{L}\bar{g},$$

for some $g \in M^!_{w+2}$. *The functions* $X_{r,s}$ *are also eigenfunctions of the Laplacian:*

$$\Delta X_{r,s} = -(r+s)X_{r,s}. \tag{3.11}$$

We want to study how Hecke operators act on the Eichler integrals and, in turn, period polynomials of these functions. Before we can do this, however, we must first check that we are even able to associate Eichler integrals to any such $X_{r,s}$. To solve this problem, we must simply show that these functions are elements of $\mathcal{MI}^!_1$. The next lemma will prove that this is indeed the case. This lemma is a generalisation of proposition 4.1 of [4].

Lemma 3.8. *Let* $r, s \geq 0$ *and set* $w = r + s$. *Suppose* $F_{r,s} \in \mathcal{M}^!_{r,s}$ *are a family of elements such that*

$$\partial F_{w,0} \in M^![\mathbb{L}] \times \mathcal{MI}^!_{n-1}, \tag{3.12}$$
$$\partial F_{r,s} - (r+1)F_{r+1,s-1} \in M^![\mathbb{L}] \times \mathcal{MI}^!_{n-1}, \qquad \text{if } s \geq 1, \tag{3.13}$$

and

$$\bar{\partial} F_{0,w} \in \overline{M}^![\mathbb{L}] \times \mathcal{MI}^!_{n-1}, \tag{3.14}$$
$$\bar{\partial} F_{r,s} - (s+1)F_{r-1,s+1} \in \overline{M}^![\mathbb{L}] \times \mathcal{MI}^!_{n-1}, \qquad \text{if } r \geq 1. \tag{3.15}$$

Then the $F_{r,s} \in \mathcal{MI}^!_n$.

Proof. By the definition of $\mathcal{MI}^!_n$ (Definition 2.7), any space $B \subset \bigoplus_{r,s \geq 0} \mathcal{M}^!_{r,s}$ which satisfies

$$\partial B \subseteq B + M^![\mathbb{L}] \times \mathcal{MI}^!_{n-1},$$
$$\bar{\partial} B \subseteq B + \overline{M}^![\mathbb{L}] \times \mathcal{MI}^!_{n-1}$$

must be contained within $\mathcal{MI}^!_n$. Let B denote the space of the family of elements $F_{r,s}$ then, by equations (3.12) and (3.13),

$$\partial F_{w,0} \subseteq M^![\mathbb{L}] \times \mathcal{MI}^!_{n-1},$$
$$\partial F_{r,s} \subseteq B + M^![\mathbb{L}] \times \mathcal{MI}^!_{n-1}, \qquad \text{if } s \geq 1,$$

and hence $\partial B \subseteq B + M^![\mathbb{L}] \times \mathcal{MI}^!_{n-1}$. Similarly, by equations (3.14) and (3.15), we have $\bar{\partial} B \subseteq B + \overline{M}^![\mathbb{L}] \times \mathcal{MI}^!_{n-1}$. Due to the maximality of $\mathcal{MI}^!_n$, we conclude that B must be contained in this space, and therefore, the $F_{r,s} \in \mathcal{MI}^!_n$. □

We can now see that each $X_{r,s}$ is an element of $\mathcal{MI}^!_1$. Therefore, by Proposition 2.9, it can be decomposed uniquely into eigenfunctions of the Laplacian:

$$X_{r,s} = \sum_{k=0}^{\min(r,s)} F_k, \tag{3.16}$$

such that $\Delta F_k = \lambda_k F_k$ with $\lambda_k = (k-1)(r+s-k)$. Furthermore, each $F_k \in \mathcal{MI}^!_1$ and is of weights (r,s). We recall that, for each $k \in \{0, \ldots, \min(r,s)\}$, we can associate a k-th Eichler integral, $v^{(k)}_{X_{r,s}}$, to $X_{r,s}$. We will show that $v^{(0)}_{X_{r,s}}$ is the only non-zero Eichler integral.

Indeed, in the decomposition above, we have $F_k = 0$ for any $k \neq 0$. Furthermore, when $k = 0$, F_0 is equal to $X_{r,s}$. This (unique) decomposition of (3.16) holds since, using equation (3.11), we have

$$\begin{aligned} \Delta F_0 =& \Delta X_{r,s} = -(r+s)X_{r,s} = (k-1)(r+s-k)X_{r,s} \\ =& (k-1)(r+s-k)F_0, \end{aligned} \tag{3.17}$$

when $k = 0$.

Now, as each $v^{(k)}_F$ is defined using the F_k from the decomposition (3.16), we can see all but $v^{(0)}_F$ vanish. This, in turn, implies that the period polynomial of $X_{r,s}$ is given by

$$P_{X_{r,s}} = \sum_{k=0}^{\min(r,s)} \left(v^{(k)}_{X_{r,s}} ||_{2k-r-s,0} (S-1) \right) = v^{(0)}_{X_{r,s}} ||_{-r-s,0} (S-1). \tag{3.18}$$

The next step is to find an explicit expression for any $X_{r,s}$, we will define the following functions in order to achieve this.

Definition 3.9. For a weakly holomorphic modular form

$$f = \sum_{m=-N}^{\infty} a_m q^m$$

we define

$$f_{(k)} := \sum_{\substack{m \geq -N \\ m \neq 0}} \frac{a_m}{(2m)^k} q^m$$

and (with $r + s = w$)

$$R_{r,s}(f) := (-1)^r \binom{w}{r} \sum_{k=s}^{w} k! \binom{r}{k-s} \frac{f_{(k+1)}}{(2\pi y)^k}.$$

Proposition 3.10 (Cor. 5.12 of [6]). *Let $X_{w,0}$ be a modular primitive of $\mathbb{L}f$, where $f \in M^!_{w+2}$. Now let $X_{r,s}$ and $g \in M^!_{w+2}$ be as determined in Proposition 3.7. Then, for all $r, s \geq 0$ and $r + s = w$,*

$$X_{r,s} = \alpha(-1)^r \binom{w}{r} y^{-r-s} + \frac{-2\pi\beta}{w+1} y + R_{r,s}(f) + \overline{R_{s,r}(g)},$$

for some $\alpha \in \mathbb{C}$ and

$$\beta = a_0(f) = \overline{a_0(g)}.$$

Therefore, if f is a weakly holomorphic cusp form, then the associated $X_{r,s}$ can be written as

$$X_{r,s} = \alpha(-1)^r \binom{w}{r} y^{-r-s} + R_{r,s}(f) + \overline{R_{s,r}(g)}. \tag{3.19}$$

Such functions are called real analytic cusp forms, we will denote these forms by $\mathcal{H}(f)_{r,s}$. We denote the space of all real analytic cusp forms by $\mathcal{H}(f)$. Since these $\mathcal{H}(f)_{r,s}$ are eigenfunctions of the Laplacian (see (3.17)) then, by Lemma 2.4 and Proposition 2.9, we have the decomposition

$$\mathcal{H}(f)_{r,s} = F^h + F^a + ay^{-r-s} + by.$$

Comparing this with equation (3.19), we observe that $b = 0$, and therefore, $\mathcal{H}(f)_{r,s}$ has a decomposition of the form

$$\mathcal{H}(f)_{r,s} = F^h + F^a + ay^{-r-s}.$$

This can also be written slightly differently as

$$\mathcal{H}(f)_{r,s} = \tilde{F} + \mathring{F} + ay^{-r-s}.$$

In the next section, we will look at how Hecke operators act on the Eichler integrals of real analytic forms that share the above decomposition.

3.2.2 *Hecke Operators acting on Eichler Integrals of Modular Iterated Integrals*

An identity describing the action of Hecke operators on the Eichler integrals of classical modular cusp forms is given by equation (3.3). We have shown how this can be applied to prove that the Hecke action is compatible with the period polynomials of standard modular cusp forms (equation (3.10)). In this section, we provide analogous versions of these equations but for modular iterated integrals of length one.

We begin by recalling certain decompositions of length one modular iterated integrals and looking at how Hecke operators affect these decompositions. Let F be an element of $\mathcal{M}^!_{r,s} \cap \mathcal{MI}^!_1$, then we know that F can be decomposed uniquely as

$$F = \tilde{F} + \mathring{F} + F^0,$$

where

$$F^0 = ay^{k_0} + by^{1-r-s-k_0} \tag{3.20}$$

for some $k_0 \in \mathbb{Z}$ and $a, b \in \mathbb{C}$. For most of this section, we will be focusing on the case when $b = 0$:

$$F = \tilde{F} + \mathring{F} + F^0 = \tilde{F} + \mathring{F} + ay^{k_0}. \tag{3.21}$$

Using equations (2.11) and (2.12), we can write $\tilde{F}$ and $\mathring{F}$ as

$$\tilde{F} = \sum_{m>0} \sum_{j=k_0}^{-s} y^j a_m^{(j)} q^m + \sum_{m>0} \sum_{j=k_0}^{-r} y^j b_m^{(j)} \bar{q}^m, \tag{3.22}$$

$$\mathring{F} = \sum_{m<0} \sum_{j=k_0}^{-s} y^j a_m^{(j)} q^m + \sum_{m<0} \sum_{j=k_0}^{-r} y^j b_m^{(j)} \bar{q}^m. \tag{3.23}$$

We also observe that

$$T_n(F) = T_n(\tilde{F} + \mathring{F} + F^0) = T_n(\tilde{F}) + T_n(\mathring{F}) + T_n(F^0),$$

where we assume T_n is fixed as if it is acting on a modular form of weights (r, s). We will need to check that the following proposition involving this decomposition holds.

Proposition 3.11. *Let F be a real analytic modular form that can be decomposed as in equation* (3.21)*. We set $A := T_n(F) = T_n(\tilde{F}) + T_n(\mathring{F}) + T_n(F^0)$, then the following equations hold:*

$$\begin{aligned} i)&\quad \tilde{A} = T_n(\tilde{F}), \\ ii)&\quad \mathring{A} = T_n(\mathring{F}), \\ iii)&\quad A^0 = T_n(F^0). \end{aligned}$$

Proof. We will start by looking at $T_n(\tilde{F})$ and noting that, as in Section 1.3, every equivalence class in $\Gamma_1 \backslash \mathcal{M}_n$ contains exactly one representative $\alpha = \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$ with $d > b \geq 0$, $a > 0$ and $ad = n$. Therefore,

$$\begin{aligned} T_n(\tilde{F}) &= n^{r+s-1} \sum_{\substack{ad=n \\ a,d>0}} d^{-r-s} \sum_{b=0}^{d-1} \tilde{F}\left(\frac{az+b}{d}\right) \\ &= n^{r+s-1} \sum_{\substack{ad=n \\ a,d>0}} d^{-r-s} \sum_{b=0}^{d-1} \sum_{m>0} \sum_{j=k_0}^{-s} \left(Im\left(\frac{az+b}{d}\right)\right)^j a_m^{(j)} e^{2\pi i m a z/d} e^{2\pi i m b/d} \\ &+ n^{r+s-1} \sum_{\substack{ad=n \\ a,d>0}} d^{-r-s} \sum_{b=0}^{d-1} \sum_{m>0} \sum_{j=k_0}^{-r} \left(Im\left(\frac{az+b}{d}\right)\right)^j b_m^{(j)} e^{-2\pi i m a \bar{z}/d} e^{2\pi i m b/d}. \end{aligned}$$

Using the fact that

$$\sum_{b=0}^{d-1} \left(Im\left(\frac{az+b}{d}\right)\right)^j e^{2\pi i m b/d} = \begin{cases} \left(\frac{ay}{d}\right)^j d & \text{if } d \mid m \\ 0 & \text{otherwise} \end{cases}$$

and setting $gd = m$, we can deduce that

$$\begin{aligned} T_n(\tilde{F}) = n^{r+s-1} &\sum_{\substack{ad=n \\ a,d>0}} d^{-r-s+1} \sum_{g>0} \sum_{j=k_0}^{-s} \left(\frac{a}{d}\right)^j y^j a_{gd}^{(j)} e^{2\pi i g a z} \\ &+ n^{r+s-1} \sum_{\substack{ad=n \\ a,d>0}} d^{-r-s+1} \sum_{g>0} \sum_{j=k_0}^{-r} \left(\frac{a}{d}\right)^j y^j b_{gd}^{(j)} e^{-2\pi i g a \bar{z}}. \end{aligned}$$

Finally, setting $h = ag$, and noting that as $ad = n$ we have $a \mid h$ and $a \mid n$, we conclude that

$$T_n(\tilde{F}) = \sum_{h>0} n^{r+s-1} \sum_{\substack{a|(h,n) \\ ad=n \\ a,d>0}} d^{-r-s+1} \sum_{j=k_0}^{-s} \left(\frac{a}{d}\right)^j y^j a^{(j)}_{hd/a} q^h + \sum_{h>0} n^{r+s-1} \sum_{\substack{a|(h,n) \\ ad=n \\ a,d>0}} d^{-r-s+1} \sum_{j=k_0}^{-r} \left(\frac{a}{d}\right)^j y^j b^{(j)}_{hd/a} \bar{q}^h.$$

$$= \sum_{h>0} \sum_{j=k_0}^{-s} y^j a'^{(j)}_h q^h + \sum_{h>0} \sum_{j=k_0}^{-r} y^j b'^{(j)}_h \bar{q}^h, \tag{3.24}$$

where $a'^{(j)}_h$ and $b'^{(j)}_h \in \mathbb{C}$. The case for $T_n(\mathring{F})$ is very similar

$$T_n(\mathring{F}) = \sum_{h<0} n^{r+s-1} \sum_{\substack{a|(h,n) \\ ad=n \\ a,d>0}} d^{-r-s+1} \sum_{j=k_0}^{-s} \left(\frac{a}{d}\right)^j y^j a^{(j)}_{hd/a} q^h + \sum_{h<0} n^{r+s-1} \sum_{\substack{a|(h,n) \\ ad=n \\ a,d>0}} d^{-r-s+1} \sum_{j=k_0}^{-r} \left(\frac{a}{d}\right)^j y^j b^{(j)}_{hd/a} \bar{q}^h.$$

$$= \sum_{h<0} \sum_{j=k_0}^{-s} y^j a'^{(j)}_h q^h + \sum_{h<0} \sum_{j=k_0}^{-r} y^j b'^{(j)}_h \bar{q}^h, \tag{3.25}$$

where $a'^{(j)}_h$ and $b'^{(j)}_h \in \mathbb{C}$. To finish this proof, we just have to evaluate $T_n(F^0)$. Using "a_0" in our expression for F^0 to avoid confusion with the the "a" from the matrix $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$ used in this proof, we have

$$T_n(a_0 y^{k_0}) = n^{r+s-1} \sum_{\substack{ad=n \\ a,d>0}} d^{-r-s} \sum_{b=0}^{d-1} a_0 \, Im \left(\frac{az+b}{d}\right)^{k_0}$$

$$= a_0 \sum_{\substack{ad=n \\ a,d>0}} n^{r+s-1} d^{-r-s+1} \left(\frac{a}{d}\right)^{k_0} Im(z)^{k_0}, \tag{3.26}$$

which exists in $\mathbb{C}y^{k_0}$. We can now view A as a combination of equations (3.24), (3.25) and (3.26). Comparing this to equations (3.22), (3.23) and (3.20), we deduce the assertion. □

We recall that the function $F \in \mathcal{MI}^!_1 \cap \mathcal{M}^!_{r,s}$ can be written uniquely as

$$F = \sum_{k=0}^{\min(r,s)} F_k, \tag{3.27}$$

where $F_k \in \mathcal{MI}_1^! \cap \mathcal{M}_{r,s}^!$ and $\Delta_{r,s} F_k = \lambda_k F_k$ with $\lambda_k = (k-1)(r+s-k)$.

Now suppose $\zeta \in \mathfrak{H}$ and there exists a $k \in \{0, \ldots, \min(r,s)\}$ such that F_k can be decomposed as in equation (3.21), then the k-th Eichler integral of F has been defined to be

$$v_F^{(k)}(\zeta) := \int_\zeta^{i\infty} \eta_{r-s} \left(y^{\frac{r+s}{2}} (\tilde{F}_k + a y^{k-r-s}), R_{s-r,\mu_k}(\cdot, \zeta) \right) + \int_\zeta^{-i\infty} \eta_{r-s} \left(y^{\frac{r+s}{2}} \mathring{F}_k, R_{s-r,\mu_k}(\cdot, \zeta) \right), \tag{3.28}$$

with $\mu_k = -k + (r+s+1)/2$. We need to check that $T_n(F)$ also has a unique decomposition of the form given by (3.27), so that $v_{T_nF}^{(k)}$ is well defined. Indeed, if this decomposition was not unique then there is no guarantee that the expression given by (3.28), when F is replaced by $T_n(F)$, would be unique.

Proposition 3.12. *Let $F \in \mathcal{MI}_1^! \cap \mathcal{M}_{r,s}^!$, then $T_n(F)$ has a unique decomposition of the form*

$$T_n(F)(z) = \sum_{k=0}^{min(r,s)} G_k(z),$$

where $G_k := T_n(F_k)$ and $\Delta_{r,s} G_k = \lambda_k G_k$ with $\lambda_k = (k-1)(r+s-k)$.

Proof. Using the decomposition given by (3.27), we have

$$T_n(F) = T_n \left(\sum_{k=0}^{\min(r,s)} F_k \right) = \sum_{k=0}^{\min(r,s)} T_n(F_k)$$

and, using Lemma 3.6, we observe that

$$\Delta(T_n(F_k)) = T_n(\Delta F_k) = T_n(\lambda_k F_k) = \lambda_k (T_n F_k).$$

Therefore, $G_k := T_n F_k$ has eigenvalue λ_k and, since the eigenvalues are distinct for distinct values of k, the the decomposition

$$T_n(F) = \sum_{k=0}^{\min(r,s)} G_k,$$

where $\Delta G_k = \lambda_k G_k$, is unique. □

We can now safely associate k-th Eichler integrals to $T_n F_k$. Before we do this explicitly, however, we first want to express (3.28) in a form that will be more convenient for us in this chapter.

Lemma 3.13. *Let $F \in \mathcal{MI}_1^!$ be of weights (r,s) and suppose there exists a $k \in \{0, \dots, min(r,s)\}$ such that F_k has a decomposition of the from given by (3.21). Then,*

$$v_F^{(k)}(\zeta) = C(\zeta)\left(\tilde{F}_k(\zeta) + \mathring{F}_k(\zeta)\right) + \int_\zeta^{i\infty} y^{k-1}\tilde{F}_k A(\cdot,\zeta) + \int_\zeta^{-i\infty} y^{k-1}\mathring{F}_k A(\cdot,\zeta) + \int_\zeta^{i\infty} F_k^0 B(\cdot,\zeta), \tag{3.29}$$

where

$$\begin{aligned} A(z,\zeta) &:= (r-k)(\zeta-z)^{r-k-1}(\zeta-\bar{z})^{s-k+1}dz \\ &\qquad + (s-k)(\zeta-z)^{r-k+1}(\zeta-\bar{z})^{s-k-1}d\bar{z}, \\ B(z,\zeta) &:= (k-s)y^{k-1}(\zeta-z)^{r-k}(\zeta-\bar{z})^{s-k}dz \\ &\qquad + (s-k)y^{k-1}(\zeta-z)^{r-k+1}(\zeta-\bar{z})^{s-k-1}d\bar{z} \end{aligned}$$

and $C(\zeta) := 0$, unless $k = r$, in which case

$$C(\zeta) := -(2i)^{s-r+1}(Im(\zeta))^s.$$

Proof. For $h = \tilde{F}_k$ we have, by Proposition 2.18,

$$\begin{aligned} \int_\zeta^{i\infty} \eta_{r-s}\left(y^{\frac{r+s}{2}}h, R_{s-r,\mu_k}(\cdot,\zeta)\right) &= 2i\int_\zeta^{i\infty} y^k \frac{\partial\,(h(z))}{\partial z}(\zeta-z)^{r-k}(\zeta-\bar{z})^{s-k}\,dz \\ &\quad + r\int_\zeta^{i\infty} y^{k-1}h(z)(\zeta-z)^{r-k}(\zeta-\bar{z})^{s-k}\,dz \\ &\quad + (s-k)\int_\zeta^{i\infty} y^{k-1}h(z)(\zeta-z)^{r-k+1}(\zeta-\bar{z})^{s-k-1}\,d\bar{z}. \end{aligned} \tag{3.30}$$

Upon integrating by parts the first integral in the RHS, it becomes

$$-k\int_\zeta^{i\infty} y^{k-1}h(z)(\zeta-z)^{r-k}(\zeta-\bar{z})^{s-k} + (r-k)\int_\zeta^{i\infty}(2iy)y^{k-1}h(z)(\zeta-z)^{r-k-1}(\zeta-\bar{z})^{s-k} + C(\zeta)h(\zeta) \tag{3.31}$$

where

$$C(\zeta) = -2iIm(\zeta)^k(\zeta-\zeta)^{r-k}(\zeta-\bar{\zeta})^{s-k}.$$

First, we observe that if $k < r$ then $C(\zeta) = 0$. Looking at the remaining case of $k = r$, we have

$$\begin{aligned} C(\zeta) &= -2iIm(\zeta)^r(\zeta - \bar{\zeta})^{s-r} \\ &= -2iIm(\zeta)^r(2iIm(\zeta))^{s-r} = -(2i)^{s-r+1}Im(\zeta)^s. \end{aligned}$$

Using the identity $2iy = (\zeta - \bar{z}) - (\zeta - z)$ and substituting (3.31) back into (3.30), we conclude that

$$\begin{aligned} \int_\zeta^{i\infty} \eta_{r-s}\Big(y^{\frac{r+s}{2}}h, R_{s-r,\mu_k}(\cdot,\zeta)\Big) &= (r-k)\int_\zeta^{\infty} y^{k-1}h(z)(\zeta-z)^{r-k-1}(\zeta-\bar{z})^{s-k+1}dz \\ &+ (s-k)\int_\zeta^{i\infty} y^{k-1}h(z)(\zeta-z)^{r-k+1}(\zeta-\bar{z})^{s-k-1}\, d\bar{z} + C(\zeta)h(\zeta). \end{aligned}$$

We get a similar result for $h = \mathring{F}_k$. The above gives the first three summands of the RHS of equation (3.29).

To get the last summand we note that, for $h = F_k^0 = ay^{k-r-s}$, the first integral in the RHS of equation (3.30) is equal to

$$(k-r-s)\int_\zeta^{i\infty} ay^{2k-r-s-1}(\zeta-z)^{r-k}(\zeta-\bar{z})^{s-k}\, dz$$

and inserting this back into equation (3.30) gives

$$\begin{aligned} \int_\zeta^{i\infty} \eta_{r-s}\left(y^{\frac{r+s}{2}}h, R_{s-r,\mu_k}(\cdot,\zeta)\right) &= (k-s)\int_\zeta^{i\infty} y^{k-1}h(z)(\zeta-z)^{r-k}(\zeta-\bar{z})^{s-k}\, dz \\ &+ (s-k)\int_\zeta^{i\infty} y^{k-1}h(z)(\zeta-z)^{r-k+1}(\zeta-\bar{z})^{s-k-1}\, d\bar{z} \\ &= \int_\zeta^{i\infty} h(z)B(\cdot,\zeta). \end{aligned}$$

This completes the proof. □

We have now set up all the essential groundwork needed to prove that an analogous version of equation (3.3) holds for certain length one modular iterated integrals.

Theorem 3.14. *Let $F \in \mathcal{MI}_1^! \cap \mathcal{M}_{r,s}^!$ and, for $k \in \{0,\dots,min(r,s)\}$, let F_k be the k-th term in the decomposition of F in eigenfunctions of $\Delta_{r,s}$, as in Proposition 2.9. Suppose further that F_k has a decomposition of the form*

$$F_k = \tilde{F}_k + \mathring{F}_k + F_k^0 = \tilde{F}_k + \mathring{F}_k + a_0y^{k-r-s},$$

where $a_0 \in \mathbb{C}$. Then, we have the following equation:

$$n^{\frac{2-r-s}{2}} \cdot v_{T_nF}^{(k)} = v_F^{(k)}||_{2k-r-s,0}T_n^{\infty}. \tag{3.32}$$

Proof. We will start by evaluating the LHS of equation (3.32). Using $G_k := T_nF_k$ and Lemma 3.13, we have

$$v_{T_nF}^{(k)}(\zeta) = C(\zeta)\left(\tilde{G}_k(\zeta) + \mathring{G}_k(\zeta)\right) + \int_{\zeta}^{i\infty} y^{k-1}\tilde{G}_k(z)A(z,\zeta) + \int_{\zeta}^{-i\infty} y^{k-1}\mathring{G}_k(z)A(z,\zeta) + \int_{\zeta}^{i\infty} G_k^0(z)B(z,\zeta).$$

Using Proposition 3.11, this can be written as

$$\begin{aligned} v_{T_nF}^{(k)}(\zeta) &= C(\zeta)\cdot T_n\left(\tilde{F}_k(\zeta) + \mathring{F}_k(\zeta)\right) + \int_{\zeta}^{i\infty} y^{k-1}(T_n\tilde{F}_k(z))A(z,\zeta) \\ &\quad + \int_{\zeta}^{-i\infty} y^{k-1}(T_n\mathring{F}_k(z))A(z,\zeta) + \int_{\zeta}^{i\infty}(T_nF_k^0(z))B(z,\zeta) \\ &= n^{r+s-1}C(\zeta)\sum_{\alpha\in M_n^{\infty}}(\tilde{F}_k(\alpha\zeta) + \mathring{F}_k(\alpha\zeta))(c\zeta+d)^{-r}(c\bar{\zeta}+d)^{-s} \qquad (3.33) \\ &\quad + n^{r+s-1}\int_{\zeta}^{i\infty} y^{k-1}\sum_{\alpha\in M_n^{\infty}}\tilde{F}_k(\alpha z)A(z,\zeta)(cz+d)^{-r}(c\bar{z}+d)^{-s} \\ &\quad + n^{r+s-1}\int_{\zeta}^{-i\infty} y^{k-1}\sum_{\alpha\in M_n^{\infty}}\mathring{F}_k(\alpha z)A(z,\zeta)(cz+d)^{-r}(c\bar{z}+d)^{-s} \\ &\quad + n^{r+s-1}\int_{\zeta}^{i\infty}\sum_{\alpha\in M_n^{\infty}} a_0 Im(\alpha z)^{k-r-s}B(z,\zeta)(cz+d)^{-r}(c\bar{z}+d)^{-s}. \end{aligned}$$

We will now evaluate the RHS of equation (3.32) and show that it matches equation (3.33). For convenience of notation we set $||T_n^{\infty} := ||_{2k-r-s,0}T_n^{\infty}$,

$$\begin{aligned} v_F^{(k)}(\zeta)||T_n^{\infty} &= C\left(\zeta\right)(\tilde{F}_k(\zeta) + \mathring{F}_k(\zeta))\,||T_n^{\infty} + \left(\int_{\zeta}^{i\infty} y^{k-1}\tilde{F}_k(z)A(z,\zeta)\right)\Bigg|\Bigg|\,T_n^{\infty} \\ &+ \left(\int_{\zeta}^{-i\infty} y^{k-1}\mathring{F}_k(z)A(z,\zeta)\right)\Bigg|\Bigg|\,T_n^{\infty} + \left(\int_{\zeta}^{i\infty} a_0 y^{k-r-s}B(z,\zeta)\right)\Bigg|\Bigg|\,T_n^{\infty}. \end{aligned} \tag{3.34}$$

We will look at each of these summands separately and show that they match their corresponding part in equation (3.33). Taking a look at the first integral, we have

$$\begin{aligned} &\left(\int_{\zeta}^{i\infty} y^{k-1}\tilde{F}_k(z)A(z,\zeta)\right)\Bigg|\Bigg|T_n^{\infty} \\ &\qquad = \sum_{\alpha\in M_n^{\infty}}\int_{\alpha\zeta}^{i\infty} Im(z)^{k-1}\tilde{F}_k(z)A(z,\alpha\zeta)(c\zeta+d)^{r+s-2k}n^{\frac{2k-r-s}{2}} \end{aligned}$$

and, using the change of variables $z \to \alpha z$, this becomes

$$\sum_{\alpha \in M_n^\infty} \int_\zeta^{\alpha^{-1}(i\infty)} Im(\alpha z)^{k-1} \tilde{F}_k(\alpha z) A(\alpha z, \alpha\zeta)(c\zeta + d)^{r+s-2k} n^{\frac{2k-r-s}{2}}. \tag{3.35}$$

Our next step will be to rewrite the above equation but with $A(z,\zeta)$ instead of $A(\alpha z, \alpha\zeta)$. We first note that

$$\begin{aligned} A(\alpha z, \alpha\zeta) = (r-k)(\alpha\zeta - \alpha z)^{r-k-1}(\alpha\zeta - \alpha\bar{z})^{s-k+1} d(\alpha z) \\ + (s-k)(\alpha\zeta - \alpha z)^{r-k+1}(\alpha\zeta - \alpha\bar{z})^{s-k-1} d(\alpha\bar{z}) \end{aligned}$$

and then observe the following:

$i)$ $d(\alpha z) = n \cdot (cz+d)^{-2}\, dz$ and $d(\alpha\bar{z}) = n \cdot (c\bar{z}+d)^{-2}\, d\bar{z}$.

$ii)$ $(\alpha\zeta - \alpha z) = \dfrac{n \cdot (\zeta - z)}{(c\zeta+d)(cz+d)}$ and $(\alpha\zeta - \alpha\bar{z}) = \dfrac{n \cdot (\zeta - \bar{z})}{(c\zeta+d)(c\bar{z}+d)}$.

This allows us to rewrite $A(\alpha z, \alpha\zeta)$ as

$$\begin{aligned} A(\alpha z, \alpha\zeta) =& \frac{(r-k)(\zeta - z)^{r-k-1}(\zeta - \bar{z})^{s-k+1}(cz+d)^{-2} n^{r+s-2k+1}}{(c\zeta+d)^{r-k-1}(cz+d)^{r-k-1}(c\zeta+d)^{s-k+1}(c\bar{z}+d)^{s-k+1}} dz \\ &+ \frac{(s-k)(\zeta - z)^{r-k+1}(\zeta - \bar{z})^{s-k-1}(c\bar{z}+d)^{-2} n^{r+s-2k+1}}{(c\zeta+d)^{r-k+1}(cz+d)^{r-k+1}(c\zeta+d)^{s-k-1}(c\bar{z}+d)^{s-k-1}} d\bar{z}. \end{aligned}$$

Substituting this back into equation (3.35), and using the fact that ∞ is invariant under α^{-1} and $Im(\alpha z) = nIm(z)(cz+d)^{-1}(c\bar{z}+d)^{-1}$, we conclude that

$$\begin{aligned} &\left(\int_\zeta^{i\infty} y^{k-1} \tilde{F}_k(z) A(z,\zeta) \right) \Bigg\| T_n^\infty \\ &\qquad = n^{\frac{r+s}{2}} \int_\zeta^{i\infty} y^{k-1} \sum_{\alpha \in M_n^\infty} \tilde{F}_k(\alpha z) A(z,\zeta)(cz+d)^{-r}(c\bar{z}+d)^{-s}. \end{aligned} \tag{3.36}$$

An analogous result holds for $\mathring{F}_k$

$$\begin{aligned} &\left(\int_\zeta^{-i\infty} y^{k-1} \mathring{F}_k(z) A(z,\zeta) \right) \Bigg\| T_n^\infty \\ &\qquad = n^{\frac{r+s}{2}} \int_\zeta^{-i\infty} y^{k-1} \sum_{\alpha \in M_n^\infty} \mathring{F}_k(\alpha z) A(z,\zeta)(cz+d)^{-r}(c\bar{z}+d)^{-s}. \end{aligned} \tag{3.37}$$

Next, we look at the first summand on the RHS of (3.34). Since $C(\zeta) \neq 0$ only if $r = k$, we have

$$\begin{aligned} &C(\zeta)(\tilde{F}_k(\zeta) + \mathring{F}_k(\zeta))||T_n^\infty \\ &\quad = -(2i)^{s-r+1} \sum_{\alpha \in M_n^\infty} Im(\alpha\zeta)^s(\tilde{F}_k(\alpha\zeta) + \mathring{F}_k(\alpha\zeta))(c\zeta+d)^{s-r} n^{\frac{r-s}{2}} \\ &\quad = -n^{\frac{r+s}{2}}(2i)^{s-r+1} \sum_{\alpha \in M_n^\infty} Im(\zeta)^s(\tilde{F}_k(\alpha\zeta) + \mathring{F}_k(\alpha\zeta))(c\zeta+d)^{-r}(c\bar{\zeta}+d)^{-s} \\ &\quad = n^{\frac{r+s}{2}} C(\zeta) \sum_{\alpha \in M_n^\infty} (\tilde{F}_k(\alpha\zeta) + \mathring{F}_k(\alpha\zeta))(c\zeta+d)^{-r}(c\bar{\zeta}+d)^{-s}. \end{aligned} \tag{3.38}$$

Finally, we just have to look at the last summand on the RHS of (3.34) to finish the proof. Using similar methods to the above, we get

$$\begin{aligned} &\left(\int_\zeta^{i\infty} a_0 y^{k-r-s} B(z,\zeta)\right)\Bigg|\Bigg| T_n^\infty \\ &\qquad = \sum_{\alpha \in M_n^\infty} \int_\zeta^{\alpha^{-1}(i\infty)} a_0 Im(\alpha z)^{k-r-s} B(\alpha z, \alpha\zeta)(c\zeta+d)^{r+s-2k} n^{\frac{2k-r-s}{2}}, \end{aligned} \tag{3.39}$$

with

$$\begin{aligned} B(\alpha z, \alpha\zeta) &= (k-s) Im(\alpha z)^{k-1}(\alpha\zeta - \alpha z)^{r-k}(\alpha\zeta - \alpha\bar{z})^{s-k} d(\alpha z) \\ &\qquad + (s-k) Im(\alpha z)^{k-1}(\alpha\zeta - \alpha z)^{r-k+1}(\alpha\zeta - \alpha\bar{z})^{s-k-1} d(\alpha\bar{z}) \\ &= \frac{(k-s)Im(z)^{k-1}(\zeta - z)^{r-k}(\zeta-\bar{z})^{s-k}(cz+d)^{-2}n^{r+s-k}}{(cz+d)^{r-1}(c\bar{z}+d)^{s-1}(c\zeta+d)^{r+s-2k}} dz \\ &\qquad + \frac{(s-k)Im(z)^{k-1}(\zeta - z)^{r-k+1}(\zeta-\bar{z})^{s-k-1}(c\bar{z}+d)^{-2}n^{r+s-k}}{(cz+d)^{r}(c\bar{z}+d)^{s-2}(c\zeta+d)^{r+s-2k}} d\bar{z}. \end{aligned}$$

A fact which is true throughout this proof but we have not yet needed is that, since $\alpha = \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$, $c = 0$. Therefore,

$$\begin{aligned} B(\alpha z, \alpha\zeta) = n^{r+s-k} d^{2k-2r-2s} \Big[&(k-s) Im(z)^{k-1}(\zeta - z)^{r-k}(\zeta-\bar{z})^{s-k} dz \\ &+ (s-k) Im(z)^{k-1}(\zeta - z)^{r-k+1}(\zeta - \bar{z})^{s-k-1} d\bar{z}\Big]. \end{aligned}$$

substituting this back into equation (3.39), and observing that ∞ is invariant under α^{-1}, we conclude that

$$\left(\int_{\zeta}^{i\infty} a_0 y^{k-r-s} B(z,\zeta)\right)\Big|\Big| T_n^{\infty} = n^{\frac{r+s}{2}} \int_{\zeta}^{i\infty} \sum_{\alpha \in M_n^{\infty}} a_0 Im(\alpha z)^{k-r-s} B(z,\zeta) d^{-r-s}. \tag{3.40}$$

By comparing equations (3.36), (3.37), (3.38) and (3.40) to equation (3.33), we deduce the assertion. □

We can compare the identity from the above theorem with the identity given by equation (3.3) to see the similarities between the two. Surprisingly, the constant $n^{\frac{2-r-s}{2}}$ in equation (3.32) does not depend on the value of k; this leads immediately to the following corollary:

Corollary 3.15. *Let $F \in \mathcal{MI}_1^! \cap \mathcal{M}_{r,s}^!$ and suppose that F_k has a decomposition of the from given by* (3.21) *for all $k \in \{0, \ldots, min(r,s)\}$, then*

$$n^{\frac{2-r-s}{2}} \sum_{k=0}^{min(r,s)} v_{T_n F}^{(k)} = \sum_{k=0}^{min(r,s)} v_F^{(k)} ||_{2k-r-s,0} T_n^{\infty}.$$

Following a similar approach to the classical case, we will use Theorem 3.14 to provide us with an identity relating the action of Hecke operators to the period polynomials of length one modular iterated integrals. This can be seen as an analogous version of the identity given by equation (3.10).

We start by showing that, under our usual conditions, the Eichler integral of a length one modular iterated integral is T-invariant.

Proposition 3.16. *Let $F_k \in \mathcal{MI}_1^!$ be of weights (r,s) and suppose there exists a $k \in \{0, \ldots, min(r,s)\}$ such that F_k has a decomposition of the from given by* (3.21), *then $v_F^{(k)}$ is T-invariant:*

$$v_F^{(k)} ||_{2k-r-s,0}(T-1) = 0, \qquad \text{where } T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right).$$

Proof. Throughout this proof we use the abbreviation $||_{2k-r-s,0}T = ||T$. We start by proving the T-invariance of

$$\int_{\zeta}^{i\infty} \eta_{r-s}\left(y^{\frac{r+s}{2}}(\tilde{F}_k(z) + ay^{k-r-s}), R_{s-r,\mu_k}(z,\zeta)\right). \tag{3.41}$$

To show this, we first observe that

$$\int_{\zeta}^{i\infty} \eta_{r-s} \left(y^{\frac{r+s}{2}} (\tilde{F}_k(z) + ay^{k-r-s}), R_{s-r,\mu_k}(z,\zeta) \right) \Big|\Big| T$$
$$= \int_{T(\zeta)}^{i\infty} \eta_{r-s} \left(y^{\frac{r+s}{2}} (\tilde{F}_k(z) + ay^{k-r-s}), R_{s-r,\mu_k}(z, T(\zeta)) \right)$$
$$= \int_{\zeta}^{T^{-1}(i\infty)} T^* \left(\eta_{r-s} \left(y^{\frac{r+s}{2}} (\tilde{F}_k(z) + ay^{k-r-s}), R_{s-r,\mu_k}(z, T(\zeta)) \right) \right).$$

By Lemma 2.15, this is equal to

$$\int_{\zeta}^{T^{-1}(i\infty)} \eta_{r-s} \left(y^{\frac{r+s}{2}} (\tilde{F}_k(z) + ay^{k-r-s}) ||_{\frac{r-s}{2},\frac{s-r}{2}} T,\ R_{s-r,\mu_k}(z, T(\zeta)) ||_{\frac{s-r}{2},\frac{r-s}{2}} T \right)$$
$$= \int_{\zeta}^{T^{-1}(i\infty)} \eta_{r-s} \left(Im(Tz)^{\frac{r+s}{2}} \left(\tilde{F}_k(Tz) + aIm(Tz)^{k-r-s} \right), R_{s-r,\mu_k}(Tz, T\zeta) \right).$$

Using the fact that $T^{-1}(i\infty) = i\infty$, $Im(z+1) = Im(z)$ and, by part (ii) of Lemma 2.16, $R_{s-r,\mu_k}(Tz, T\zeta) = R_{s-r,\mu_k}(z,\zeta)$, this becomes

$$\int_{\zeta}^{i\infty} \eta_{r-s} \left(y^{\frac{r+s}{2}} (\tilde{F}_k(Tz) + ay^{k-r-s}), R_{s-r,\mu_k}(z,\zeta) \right).$$

Therefore, we can deduce that if $\tilde{F}_k(z)$ is T-invariant then so is the function given by equation (3.41). Indeed, we have

$$\tilde{F}_k(Tz) = \sum_{m>0} \sum_{j=k-r-s}^{-s} Im(Tz)^j\, a_m^{(j)} e^{2\pi i m Tz} + \sum_{m>0} \sum_{j=k-r-s}^{-r} Im(Tz)^j\, b_m^{(j)} e^{-2\pi i m T\bar{z}}$$

and, since both $Im(z)$ and $e^{2\pi i m z}$ are invariant under $z \mapsto z+1$, we conclude that $\tilde{F}_k(Tz) = \tilde{F}_k(z)$. This proves the T-invariance of the function given by (3.41). A similar approach can be used to show the T-invariance of

$$\int_{\zeta}^{-i\infty} \eta_{r-s} \left(y^{\frac{r+s}{2}} \mathring{F}_k(z), R_{s-r,\mu_k}(z,\zeta) \right) \Big|\Big| T.$$

Using the definition of $v_F^{(k)}$ given by equation (3.28), we can deduce the assertion. □

As before, we combine the T-invariance of the Eichler integral with equation (3.7) to give the following equation (here the assumptions of the previous proposition still hold):

$$v_F^{(k)} ||_{2k-r-s,0} (S-1) ||_{2k-r-s,0} \tilde{T}_n = v_F^{(k)} ||_{2k-r-s,0} T_n^{\infty} ||_{2k-r-s,0} (S-1).$$

Using Theorem 3.14, we can rewrite this as

$$v_F^{(k)}||_{2k-r-s,0}(S-1)||_{2k-r-s,0}\tilde{T}_n = n^{\frac{2-r-s}{2}} \cdot v_{T_nF}^{(k)}||_{2k-r-s,0}(S-1).$$

If we suppose further that $F \in \mathcal{MI}_1^!$ such that each F_k has a decomposition of the form (3.21) (for $k \in \{0,\ldots,\min(r,s)\}$), then

$$\sum_{k=0}^{min(r,s)} v_F^{(k)}||_{2k-r-s,0}(S-1)||_{2k-r-s,0}\tilde{T}_n = n^{\frac{2-r-s}{2}} \sum_{k=0}^{min(r,s)} v_{T_nF}^{(k)}||_{2k-r-s,0}(S-1).$$

Using the definition of the period polynomial of a length one modular iterated integral (Definition 2.21) this becomes

$$\sum_{k=0}^{min(r,s)} v_F^{(k)}||_{2k-r-s,0}(S-1)||_{2k-r-s,0}\tilde{T}_n = n^{\frac{2-r-s}{2}} \cdot P_{T_nF}. \tag{3.42}$$

We can compare this with equation (3.10) from the standard modular form case:

$$P_f|_{2-r}\tilde{T}_n = n^{\frac{2-r}{2}} \cdot P_{T_nf}, \tag{3.43}$$

which can be rewritten as

$$v_f|_{2-r}(S-1)|_{2-r}\tilde{T}_n = n^{\frac{2-r}{2}} \cdot P_{T_nf}.$$

We recall that if F is a real analytic cusp form, then every k-th Eichler integral $v_F^{(k)}$ vanishes except for $v_F^{(0)}$. Therefore, in this case, equation (3.42) can be written as

$$v_F^{(0)}||_{-r-s,0}(S-1)||_{-r-s,0}\tilde{T}_n = n^{\frac{2-r-s}{2}} \cdot P_{T_nF}.$$

Finally, using the definition of the period polynomial of a real analytic cusp form (equation (3.18)), we can give an identity in which the period polynomial appears on both sides of the equation:

$$P_F||_{-r-s,0}\tilde{T}_n = n^{\frac{2-r-s}{2}} \cdot P_{T_nF}.$$

This identity, which is of close resemblance to equation (3.43), shows how the Hecke action is compatible with the period polynomial of a real analytic cusp form.

4

LAPLACE-EIGENVALUE EQUATIONS FOR MODULAR ITERATED INTEGRALS

One motivation for defining real analytic modular forms was due to the connection between modular iterated integrals and modular graph functions, which arise in string perturbation theory. These modular graph functions share many of the differential and algebraic properties of modular iterated integrals. Indeed, there is good evidence that all modular graph functions are contained within the space of modular iterated integrals (this is discussed in more detail in [4, 5]).

Studying the space of modular iterated integrals, therefore, will help with the long-standing problem of giving a complete description of the modular graph functions. This problem has been written about extensively in physics literature, see any of [33–42] for example.

In this chapter, we study the spaces of length two and three iterated integrals. We will provide functions that belong to these spaces and show that they satisfy certain Laplace-eigenvalue equations. We will also see the importance of these equations in relation to modular graph functions.

4.1 THE SPACE OF MODULAR ITERATED INTEGRALS REVISITED

In Section 2.4 we introduced and discussed the space $\mathcal{MI}^!$ of modular iterated integrals, which sits naturally inside $\mathcal{M}^!$. In this chapter, we will restrict to the condition that the coefficients $a^{(j)}_{m,n}$ in equation (2.2) vanish if either m or n are negative. This provides us with the space $\mathcal{MI}$ of modular iterated integrals, which sits inside $\mathcal{M}$ and was first studied in [4]. For the rest of this chapter, when we refer to the space of modular iterated integrals we will mean $\mathcal{MI}$ not $\mathcal{MI}^!$. The definition of $\mathcal{MI}$ is very similar to the definition of $\mathcal{MI}^!$ but, for completeness, we state it below.

Definition 4.1. We let $\mathcal{MI}_{-1} = 0$. For any $n \geq 0 \in \mathbb{Z}$ we define the space of modular iterated integrals of length n, $\mathcal{MI}_n$, to be the largest subspace of $\bigoplus_{r,s\geq 0} \mathcal{M}_{r,s}$ which satisfies

$$\partial \mathcal{MI}_n \subset \mathcal{MI}_n + M[\mathbb{L}] \times \mathcal{MI}_{n-1},$$
$$\bar{\partial} \mathcal{MI}_n \subset \mathcal{MI}_n + \overline{M}[\mathbb{L}] \times \mathcal{MI}_{n-1},$$

where M is the space of holomorphic modular forms.

When $n = 0$, we have the following lemma:

Lemma 4.2. *(Lemma 3.10 of [4]) An explicit description of the space of length zero modular iterated integrals can be given by*

$$\mathcal{MI}_0 = \mathbb{C}[\mathbb{L}^{-1}].$$

This matches the description of $\mathcal{MI}_0^!$ (2.15) exactly. We can also give a simple and concise description for $\mathcal{MI}_1$, which is not the same as $\mathcal{MI}_1^!$ (in fact, we are not yet able to give an explicit description for $\mathcal{MI}_1^!$).

Lemma 4.3. *(Cor. 4.4 of [4]) An explicit description of the space of length one modular iterated integrals can be given by*

$$\mathcal{MI}_1 = \mathcal{MI}_0 \otimes_{\mathbb{C}} \bigoplus_{\substack{r,s\geq 0 \\ r+s\geq 2}} \mathbb{C}\, \mathcal{E}_{r,s}.$$

Therefore, we know of all the possible functions that can exist in $\mathcal{MI}_0$ or $\mathcal{MI}_1$. We do not, however, have similar descriptions for $\mathcal{MI}_n$, when $n > 1$. It would be useful if we could implement what we know about $\mathcal{MI}_0$ and $\mathcal{MI}_1$ to determine functions that exist in $\mathcal{MI}_n$.

Proposition 4.4 and Corollary 4.5 will help with this.

Proposition 4.4. *Let $n \geq 0$ be an integer, then*

$$\mathcal{MI}_n \mathcal{MI}_0 \subseteq \mathcal{MI}_n.$$

Proof. By the definition of $\mathcal{MI}_{n+1}$, any space $B \subset \bigoplus_{r,s\geq 0} \mathcal{M}_{r,s}$ which satisfies

$$\partial B \subseteq B + M[\mathbb{L}] \times \mathcal{MI}_n,$$
$$\bar{\partial} B \subseteq B + \overline{M}[\mathbb{L}] \times \mathcal{MI}_n,$$

must be contained within $\mathcal{MI}_{n+1}$. Now, we observe that if $F \in \mathcal{MI}_{n+1}$ and $G \in \mathcal{MI}_0$, then

$$\begin{aligned}\partial(FG) &= (\partial F)G + (\partial G)F \\ &\subseteq (\mathcal{MI}_{n+1} + M[\mathbb{L}] \times \mathcal{MI}_n)\mathcal{MI}_0 + (\mathcal{MI}_0)\mathcal{MI}_{n+1} \\ &\subseteq \mathcal{MI}_{n+1}\mathcal{MI}_0 + M[\mathbb{L}] \times \mathcal{MI}_n\mathcal{MI}_0 + \mathcal{MI}_0\mathcal{MI}_{n+1}.\end{aligned}$$

Therefore, by induction

$$\partial(\mathcal{MI}_{n+1}\mathcal{MI}_0) \subseteq \mathcal{MI}_{n+1}\mathcal{MI}_0 + M[\mathbb{L}] \times \mathcal{MI}_n,$$

with a similar statement for $\bar{\partial}$. Hence $\mathcal{MI}_{n+1}\mathcal{MI}_0$ must be contained in $\mathcal{MI}_{n+1}$ due to the maximality of $\mathcal{MI}_{n+1}$. □

Corollary 4.5. *Let $m, n \geq 0$ be integers, then*

$$\mathcal{MI}_m \mathcal{MI}_n \subseteq \mathcal{MI}_{m+n}.$$ □

This corollary then allows us to see that, as $\mathcal{E}_{r_1,s_1}$ and $\mathcal{E}_{r_2,s_2} \in \mathcal{MI}_1$, we have $\mathcal{E}_{r_1,s_1}\mathcal{E}_{r_2,s_2} \in \mathcal{MI}_2$, where we assume that r_i, $s_i \geq 0$. We can then see that $\mathcal{E}_{r_1,s_1}\mathcal{E}_{r_2,s_2}\mathcal{E}_{r_3,s_3} \in \mathcal{MI}_3$, as $\mathcal{E}_{r_1,s_1}\mathcal{E}_{r_2,s_2} \in \mathcal{MI}_2$ and $\mathcal{E}_{r_3,s_3} \in \mathcal{MI}_1$. Continuing this process we get

$$\mathcal{E}_{r_1,s_1}\mathcal{E}_{r_2,s_2} \cdots \mathcal{E}_{r_n,s_n} \in \mathcal{MI}_n.$$

However, we do not consider these functions to be very interesting as they are just combinations of real analytic Eisenstein series. We do not consider them to be "new" functions. By "new" functions we will therefore mean functions that are not simply combinations of $\mathcal{E}_{r,s}$ or $\mathbb{L}$.

A theorem for producing such "new" functions in $\mathcal{MI}_2$ was discovered by Brown in [4]. We will explain how this was achieved in the next section. Then, in Section 4.3, we will demonstrate how these "new" functions can be applied to the theory of modular graph functions. But first, we will finish this section by providing one more way of checking if a function is in $\mathcal{MI}_n$, for any n. This lemma will be useful when proving the main theorem of Section 4.4.

Lemma 4.6. *Let $r, s \geq 0$ and set $2w = r + s$. Suppose $F_{r,s} \in \mathcal{M}_{r,s}$ are a family of elements such that*

$$\partial F_{2w,0} \in M[\mathbb{L}] \times \mathcal{MI}_{n-1},$$
$$\partial F_{r,s} - (r+1)F_{r+1,s-1} \in M[\mathbb{L}] \times \mathcal{MI}_{n-1}, \qquad \text{if } s \geq 1,$$

and

$$\bar{\partial} F_{0,2w} \in \overline{M}[\mathbb{L}] \times \mathcal{MI}_{n-1},$$
$$\bar{\partial} F_{r,s} - (s+1)F_{r-1,s+1} \in \overline{M}[\mathbb{L}] \times \mathcal{MI}_{n-1}, \qquad \text{if } r \geq 1.$$

Then the $F_{r,s} \in \mathcal{MI}_n$.

Proof. This is essentially Lemma 3.8. □

4.2 LAPLACE EQUATIONS FOR LENGTH TWO ITERATED INTEGRALS

In this section, we examine how Brown produced "new" functions in the space $\mathcal{MI}_2$ of length two modular iterated integrals. This is a review of the work presented by Brown in Section 9 of [4]. We start with some definitions.

Definition 4.7. Let $n \geq 0$, then we define

$$V_{2n} := \bigoplus_{r+s=2n} X^r Y^s \mathbb{Q} \ \subset \ \mathbb{Q}[X, Y],$$

which is equipped with the dagger operator defined by

$$(X,Y)\dagger_\gamma := (aX+bY, cX+dY), \qquad \forall\, \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1.$$

Remark 4.8. We note that we use the notation $\dagger_\gamma$, instead of the $|_\gamma$ used by Brown, to avoid confusion with the single slash operator we defined in equation (1.1).

We have the following identity involving the dagger operator:

$$(X-\gamma(z)Y)\dagger_\gamma = (cz+d)^{-1}(X-zY), \qquad \forall\, \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1.$$

Therefore $g(z) := (X-zY)^r(X-\bar{z}Y)^s$ transforms like a modular object of weights $(-r,-s)$ in the following sense:

$$g(\gamma(z))\dagger_\gamma = (cz+d)^{-r}(c\bar{z}+d)^{-s}g(z). \tag{4.1}$$

We define a real analytic form $f : \mathfrak{H} \to V_{2n} \otimes \mathbb{C}$ to be modular equivariant if

$$f(\gamma(z))\dagger_\gamma = f(z), \qquad \forall\, \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1. \tag{4.2}$$

Using equation (4.1), we can deduce that for any standard modular form f of weight r we can construct a modular equivariant one-form $\underline{\underline{f}}$ by setting

$$\underline{\underline{f}} := 2\pi i f(z)(X-zY)^{r-2}dz.$$

The next lemma provides us with more information about the modular equivariance of real analytic functions.

Lemma 4.9. *(Prop. 7.1 of [4]) Let $f : \mathfrak{H} \to V_{2n} \otimes \mathbb{C}$ be real analytic, then it can be written in the form*

$$f = \sum_{r+s=2n} f_{r,s}(z)(X-zY)^r(X-\bar{z}Y)^s \tag{4.3}$$

such that $(z-\bar{z})^{2n} f_{r,s} : \mathfrak{H} \to \mathbb{C}$ are real analytic. The function f is modular equivariant if and only if every $f_{r,s}$ is modular of weights (r,s) (as defined by equation (2.1)*).*

If any of the above $f_{r,s}$ are also of the form (2.2), then they are real analytic modular forms of weights (r, s). We now give a lemma involving the operators ∂ and $\bar{\partial}$.

Lemma 4.10. *(Prop. 7.2 of [4]) Let $F, A, B\colon \mathfrak{H} \to V_{2n} \otimes \mathbb{C}$ be real analytic, then the equation*

$$\frac{\partial F}{\partial z} = \pi i A(z)$$

(where A is of the form given by (4.3)) is equivalent to the following system of equations (where $r, s \geq 0$ and $r + s = 2n$):

$$\partial F_{2n,0} = \mathbb{L} A_{2n,0},$$
$$\partial F_{r,s} - (r+1) F_{r+1,s-1} = \mathbb{L} A_{r,s}, \qquad \text{for } s \geq 1.$$

Similarly, the equation

$$\frac{\partial F}{\partial \bar{z}} = -\pi i B(z)$$

is equivalent to the following system of equations (where $r, s \geq 0$ and $r + s = 2n$):

$$\bar{\partial} F_{0,2n} = \mathbb{L} B_{0,2n},$$
$$\bar{\partial} F_{r,s} - (s+1) F_{r-1,s+1} = \mathbb{L} B_{r,s}, \qquad \text{for } r \geq 1.$$

4.2.1 *Double Eisenstein Integrals*

Let $a > 0 \in \mathbb{Z}$, then we define the following functions:

$$\mathcal{E}_{2a}(\tau) := \sum_{r+s=2a} \mathcal{E}_{r,s}(\tau)(X - \tau Y)^r (X - \bar{\tau} Y)^s, \tag{4.4}$$

$$E_{2a+2}(\tau) := 2\pi i\, \mathbb{G}_{2a+2}(\tau)(X - \tau Y)^{2a}, \tag{4.5}$$

$$\underline{E}_{2a+2}(\tau) := E_{2a+2}(\tau)\, d\tau.$$

The functions $\mathcal{E}_{2a}(\tau)$ and $\underline{E}_{2a+2}(\tau)$ are modular equivariant and closely linked to one another. By Lemmas 2.6 and 4.10, we have

$$\frac{\partial \mathcal{E}_{2a}(\tau)}{\partial \tau} = \frac{1}{2} E_{2a+2}(\tau) \quad \text{and} \quad \frac{\partial \mathcal{E}_{2a}(\tau)}{\partial \bar{\tau}} = \frac{1}{2} \bar{E}_{2a+2}(\tau). \tag{4.6}$$

For $a, b \geq 2$, we use the above functions to construct the one-forms

$$D_{2a,2b}(\tau) : \mathfrak{H} \to (V_{2a-2} \otimes V_{2b-2}) \otimes (\mathbb{C}\, d\tau \oplus \mathbb{C}\, d\bar{\tau})$$
$$D_{2a,2b}(\tau) := \underline{E}_{2a}(\tau) \otimes \mathcal{E}_{2b-2}(\tau) + \mathcal{E}_{2a-2}(\tau) \otimes \underline{\bar{E}}_{2b}(\tau).$$

These forms are modular equivariant:

$$D_{2a,2b}(\gamma\tau)|_{\gamma} = D_{2a,2b}(\tau), \qquad \forall\, \gamma \in \Gamma_1,$$

and, by Lemma 9.1 of [4], closed:

$$dD_{2a,2b} = 0.$$

We then consider the integral

$$K_{2a,2b}(z) := -\frac{1}{2}\int_z^{\vec{1}_\infty} D_{2a,2b},$$

which, since $D_{2a,2b}$ is closed, only depends on the homotopy class of the chosen path. This integral is regularised as in Section 8 of [4] (see also, Section 4 of [23]). We will now provide a brief explanation on a way to view this regularisation, that will hold for the purpose of this chapter. We first write

$$D_{2a,2b}(\tau) = D^{(0)}_{2a,2b}(\tau) + D^{(\infty)}_{2a,2b}(\tau),$$

such that $D^{(0)}_{2a,2b}(\tau)$ is the part of $D_{2a,2b}(\tau)$ that exponentially tends to zero as τ tends to infinity. The regularisation from above is then defined to be

$$K_{2a,2b}(z) = -\frac{1}{2}\int_z^{\infty} D^{(0)}_{2a,2b}(\tau) + \frac{1}{2}\int_0^{z} D^{(\infty)}_{2a,2b}(\tau).$$

This allows us to see that $K_{2a,2b}(z)$ satisfies the following differential equations:

$$\frac{\partial}{\partial z}K_{2a,2b}(z) = \pi i \mathbb{G}_{2a}(z)(X - zY)^{2a-2} \otimes \mathcal{E}_{2b-2}(z), \tag{4.7}$$
$$\frac{\partial}{\partial \bar{z}}K_{2a,2b}(z) = \mathcal{E}_{2a-2}(z) \otimes \overline{\pi i \mathbb{G}_{2b}}(z)(X - \bar{z}Y)^{2b-2}. \tag{4.8}$$

For $k \geq 0$, there is a Γ_1 equivariant projector given by

$$\delta^k \colon V_{2a} \otimes V_{2b} \to V_{2a+2b-2k}$$
$$\delta^k := m \circ \left(\frac{\partial}{\partial X} \otimes \frac{\partial}{\partial Y} - \frac{\partial}{\partial Y} \otimes \frac{\partial}{\partial X}\right)^k,$$

where $m\colon \mathbb{Q}[X,Y]\otimes\mathbb{Q}[X,Y]\to\mathbb{Q}[X,Y]$ is the multiplication map. We have the following lemma involving this projector:

Lemma 4.11. *(Lemma 7.3 of [4]) Suppose that $D\colon \mathfrak{H}\to V_{2n}\otimes\mathbb{C}$ and set*

$$F(z) := \frac{\delta^k}{(k!)^2}\Big(\mathbb{G}_{2m+2}(z)(X-zY)^{2m}\otimes D(z)\Big).$$

Then, $F\colon \mathfrak{H}\to V_{2m+2n-2k}\otimes\mathbb{C}$ vanishes if either $k>2m$ or $k>2n$. Otherwise, F can be written in the form of equation (4.3) *with*

$$F_{r,s}(z) = (z-\bar{z})^k\mathbb{G}_{2m+2}(z)\binom{2m}{k}\binom{k+s}{k}D_{r-2m+k,s+k}(z),$$

where we set $D_{r,s}=0$ if either r or $s<0$.

Now, for $a,b\geq 1$ and $0\leq k\leq \min\{2a,2b\}$, we define

$$K^{(k)}_{2a+2,2b+2}(z) := (\pi i)^k\frac{\delta^k}{(k!)^2}K_{2a+2,2b+2}(z)$$

and observe that

$$\begin{aligned}\frac{\partial}{\partial z}K^{(k)}_{2a+2,2b+2}(z) &= \frac{\partial}{\partial z}\left((\pi i)^k\frac{\delta^k}{(k!)^2}K_{2a+2,2b+2}(z)\right)\\ &= (\pi i)^k\frac{\delta^k}{(k!)^2}\left(\frac{\partial}{\partial z}K_{2a+2,2b+2}(z)\right).\end{aligned}$$

Then, using equation (4.7) and Lemma 4.11, this becomes

$$\frac{\partial}{\partial z}K^{(k)}_{2a+2,2b+2}(z) \;=\; \pi i J^{(k)}(z), \tag{4.9}$$

where $J^{(k)}\colon \mathfrak{H}\to V_{2a+2b-2k}$ is of the form given by (4.3) with

$$J^{(k)}_{r,s}(z) = \mathbb{L}^k\mathbb{G}_{2a+2}(z)\binom{2a}{k}\binom{k+s}{k}\mathcal{E}_{r-2a+k,s+k}(z), \tag{4.10}$$

and we also used $i\pi(z-\bar{z})=\mathbb{L}$. We get a similar expression for $\partial/\partial\bar{z}$:

$$\frac{\partial}{\partial \bar{z}}K^{(k)}_{2a+2,2b+2}(z) = \pi i H^{(k)}(z), \tag{4.11}$$

where

$$H^{(k)}_{r,s}(z) = \mathbb{L}^k\bar{\mathbb{G}}_{2b+2}(z)\binom{2b}{k}\binom{k+r}{k}\mathcal{E}_{r+k,s-2b+k}(z). \tag{4.12}$$

In the next section, we will be able to see that equations (4.10) and (4.12) form the basis of the equations appearing in Theorem 4.12. This will be the theorem that produces "new" length two modular iterated integrals.

4.2.2 *Equivariant Versions of Double Eisenstein Integrals*

The function $K^{(k)}_{2a+2,2b+2}$ is not quite modular equivariant, however, we can construct a function that is. (We want an equivariant form in order to make use of Lemma 4.9 during the proof of Theorem 4.12. This is covered in more detail in Section 4.4, specifically the proof of Theorem 4.16). Another way to view the equation of modular equivariance (4.2) for $f : \mathfrak{H} \to V_{2a+2b-4-2k} \otimes \mathbb{C}$ is

$$f(\gamma(z))\dagger_\gamma - f(z) = 0, \qquad \forall\ \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1. \tag{4.13}$$

We can also deduce from equations (4.7) and (4.8) that $dK_{2a,2b}$ is modular equivariant. It follows that, $\forall\ \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1$,

$$K_{2a,2b}(\gamma z)\dagger_\gamma - K_{2a,2b}(z) = c_\gamma,$$

for some "constant" c_γ (does not depend on z) which defines a coboundary and hence a cocycle. Combining this with the fact that δ^k is an equivariant projector, we conclude that

$$K^{(k)}_{2a,2b}(\gamma z)\dagger_\gamma - K^{(k)}_{2a,2b}(z) = C_\gamma,$$

for a constant C_γ (does not depend on z) which again defines a cocycle:

$$\gamma \mapsto C_\gamma \in Z^1(\Gamma_1, V_{2a+2b-4-2k} \otimes \mathbb{C}).$$

We can use the Eichler-Shimura Theorem to show that such a cocycle can be written as a linear combination of cocycles of cusp forms, the complex conjugate of said forms, Eisenstein series and a coboundary $c\dagger_{\gamma-\mathrm{id}}$ for a $c \in V_{2a+2b-4-2k} \otimes \mathbb{C}$. We then define a function $M^{(k)}_{2a,2b}$ such that we have modular equivariance and

$$M^{(k)}_{2a,2b} = K^{(k)}_{2a,2b} - c - \frac{1}{2}\int_z^{\overrightarrow{1}\infty} \left(\underline{\underline{f}} + \overline{\underline{\underline{g}}}\right), \tag{4.14}$$

where f (resp. g) is a standard modular form (resp. modular cusp form) of weight $2a+2b-2-2k$. The equation above uniquely determines $M^{(k)}_{2a,2b}, c, f$

and g except for when $2a + 2b - 4 - 2k = 0$, as we could add an arbitrary constant $d \in \mathbb{C}$.

We can use equation (4.14) with equations (4.9) – (4.12), Lemma 4.9 and Lemma 4.10 to produce the next theorem (a full proof of this theorem can be found in [4]).

Theorem 4.12. *(Theorem 9.3 of [4]) Let $a, b \geq 1$, $0 \leq k \leq min\{2a, 2b\}$ and set $w = a + b - k$. There exists a family of elements $(F^{(k)}_{2a+2,2b+2})_{r,s} \in \mathcal{MI}_2 \cap \mathcal{M}_{r,s}$ of total modular weight $2w = r + s$, with $r, s \geq 0$, which satisfy the following:*

$$a)\quad \partial\big(F^{(k)}_{2a+2,2b+2}\big)_{2w,0} = \binom{2a}{k}\mathbb{L}^{k+1}\mathbb{G}_{2a+2}\mathcal{E}_{r-2a+k,k} + \mathbb{L}f, \tag{4.15}$$

$$\partial\big(F^{(k)}_{2a+2,2b+2}\big)_{r,s} - (r+1)\big(F^{(k)}_{2a+2,2b+2}\big)_{r+1,s-1} \tag{4.16}$$

$$= \binom{2a}{k}\binom{k+s}{k}\mathbb{L}^{k+1}\mathbb{G}_{2a+2}\mathcal{E}_{r-2a+k,s+k}, \qquad \text{if } s \geq 1,$$

where f is a unique cusp form of weight $2w + 2$.

$$b)\quad \bar{\partial}\big(F^{(k)}_{2a+2,2b+2}\big)_{0,2w} = \binom{2b}{k}\mathbb{L}^{k+1}\bar{\mathbb{G}}_{2b+2}\mathcal{E}_{k,s-2b+k} + \mathbb{L}\bar{g}, \tag{4.17}$$

$$\bar{\partial}\big(F^{(k)}_{2a+2,2b+2}\big)_{r,s} - (s+1)\big(F^{(k)}_{2a+2,2b+2}\big)_{r-1,s+1} \tag{4.18}$$

$$= \binom{2b}{k}\binom{k+r}{k}\mathbb{L}^{k+1}\bar{\mathbb{G}}_{2b+2}\mathcal{E}_{r+k,s-2b+k}, \qquad \text{if } r \geq 1,$$

where g is a unique cusp form of weight $2w + 2$.

Remark 4.13. To emphasise the importance of the values of a and b, we use the convention $(F^{(k)}_{2a+2,2b+2})_{r,s}$ rather than the $F^{(k)}_{r,s}$ used by Brown.

We can use the family of functions $(F^{(k)}_{2a+2,2b+2})_{r,s}$ from Theorem 4.12 to construct the function

$$F^{(k)}_{2a+2,2b+2}(z) := \sum_{\substack{r+s=\\2a+2b-2k}}\big(F^{(k)}_{2a+2,2b+2}\big)_{r,s}(z)(X - zY)^r(X - \bar{z}Y)^s, \tag{4.19}$$

which is modular equivariant and satisfies the following differential equations:

$$\frac{\partial}{\partial z}F^{(k)}_{2a+2,2b+2}(z) = (i\pi)^k \frac{\delta^{(k)}}{(k!)^2}\Big(\frac{1}{2}E_{2a+2}(z)\otimes\mathcal{E}_{2b}(z)\Big) + \pi i f^{(k)}_{2a+2,2b+2}(z)(X-zY)^{2a+2b-2k}, \tag{4.20}$$

$$\frac{\partial}{\partial \bar{z}}F^{(k)}_{2a+2,2b+2}(z) = (i\pi)^k \frac{\delta^{(k)}}{(k!)^2}\Big(\mathcal{E}_{2a}(z)\otimes\frac{1}{2}\bar{E}_{2b+2}(z)\Big) - \pi i \bar{g}^{(k)}_{2a+2,2b+2}(z)(X-\bar{z}Y)^{2a+2b-2k}, \tag{4.21}$$

where $f^{(k)}_{2a+2,2b+2}$ and $\bar{g}^{(k)}_{2a+2,2b+2}$ are the unique cusp forms determined by Theorem 4.12. These cusp forms are of weight $2a+2b-2k+2$ but, by Theorem 1.5, there exist no cusp forms of weight ≤ 10 or weight 14. Therefore, the differential equations defined above do not include the summands containing f or $\bar{g}$ when $2a+2b-2k \leq 8$ or $2a+2b-2k = 12$.

These differential equations will be crucial in Section 4.4, when we construct an analogous theorem to Theorem 4.12 but for functions in $\mathcal{MI}_3$ rather than $\mathcal{MI}_2$.

In the next section, we will use Theorem 4.12 to generate examples of length two modular iterated integrals. This will include the "new" functions we are looking for.

4.2.3 *Example*

The simplest example is when $2a = 2b = 2$. Since $2a + 2b - 2k \leq 4$, for $k \in \{0, 1, 2\}$, the cusp forms given in equations (4.15) and (4.17) vanish.

When $k = 0$, we have the following set of equations from Theorem 4.12:

$$\partial(F^{(0)}_{4,4})_{4,0} = \mathbb{L}\mathbb{G}_4\mathcal{E}_{2,0} \qquad \overline{\partial}(F^{(0)}_{4,4})_{0,4} = \mathbb{L}\overline{\mathbb{G}_4}\mathcal{E}_{0,2}$$
$$\partial(F^{(0)}_{4,4})_{3,1} - 4(F^{(0)}_{4,4})_{4,0} = \mathbb{L}\mathbb{G}_4\mathcal{E}_{1,1} \qquad \overline{\partial}(F^{(0)}_{4,4})_{1,3} - 4(F^{(0)}_{4,4})_{0,4} = \mathbb{L}\overline{\mathbb{G}_4}\mathcal{E}_{1,1}$$
$$\partial(F^{(0)}_{4,4})_{2,2} - 3(F^{(0)}_{4,4})_{3,1} = \mathbb{L}\mathbb{G}_4\mathcal{E}_{0,2} \qquad \overline{\partial}(F^{(0)}_{4,4})_{2,2} - 3(F^{(0)}_{4,4})_{1,3} = \mathbb{L}\overline{\mathbb{G}_4}\mathcal{E}_{2,0}$$
$$\partial(F^{(0)}_{4,4})_{1,3} - 2(F^{(0)}_{4,4})_{2,2} = 0 \qquad \overline{\partial}(F^{(0)}_{4,4})_{3,1} - 2(F^{(0)}_{4,4})_{2,2} = 0$$
$$\partial(F^{(0)}_{4,4})_{0,4} - (F^{(0)}_{4,4})_{1,3} = 0 \qquad \overline{\partial}(F^{(0)}_{4,4})_{4,0} - (F^{(0)}_{4,4})_{3,1} = 0.$$

When $k = 1$:

$$\partial(F^{(1)}_{4,4})_{2,0} = 2\mathbb{L}^2\mathbb{G}_4\mathcal{E}_{1,1} \qquad \overline{\partial}(F^{(1)}_{4,4})_{0,2} = 2\mathbb{L}^2\overline{\mathbb{G}_4}\mathcal{E}_{1,1}$$
$$\partial(F^{(1)}_{4,4})_{1,1} - 2(F^{(1)}_{4,4})_{2,0} = 4\mathbb{L}^2\mathbb{G}_4\mathcal{E}_{0,2} \qquad \overline{\partial}(F^{(1)}_{4,4})_{1,1} - 2(F^{(1)}_{4,4})_{0,2} = 4\mathbb{L}^2\overline{\mathbb{G}_4}\mathcal{E}_{2,0}$$
$$\partial(F^{(1)}_{4,4})_{0,2} - (F^{(1)}_{4,4})_{1,1} = 0 \qquad \overline{\partial}(F^{(1)}_{4,4})_{2,0} - (F^{(1)}_{4,4})_{1,1} = 0.$$

Finally, when $k = 2$ we have two equations:

$$\partial(F^{(2)}_{4,4})_{0,0} = \mathbb{L}^3\mathbb{G}_4\mathcal{E}_{0,2} \qquad \overline{\partial}(F^{(2)}_{4,4})_{0,0} = \mathbb{L}^3\overline{\mathbb{G}_4}\mathcal{E}_{2,0}.$$

For $k = 0$, the equations above can be solved in terms of real analytic Eisenstein series:

$$(F^{(0)}_{4,4})_{4,0} = \frac{1}{2}\mathcal{E}_{2,0}\mathcal{E}_{2,0}$$
$$(F^{(0)}_{4,4})_{3,1} = \mathcal{E}_{2,0}\mathcal{E}_{1,1} \qquad (F^{(0)}_{4,4})_{1,3} = \mathcal{E}_{1,1}\mathcal{E}_{0,2}$$
$$(F^{(0)}_{4,4})_{2,2} = \mathcal{E}_{2,0}\mathcal{E}_{0,2} + \frac{1}{2}\mathcal{E}_{1,1}\mathcal{E}_{1,1} \qquad (F^{(0)}_{4,4})_{0,4} = \frac{1}{2}\mathcal{E}_{0,2}\mathcal{E}_{0,2}.$$

We can also express $(F^{(2)}_{4,4})_{0,0}$ as a combination of $\mathcal{E}_{r,s}$ and $\mathbb{L}$:

$$(F^{(2)}_{4,4})_{0,0} = \mathbb{L}^2\mathcal{E}_{2,0}\mathcal{E}_{0,2} - \frac{1}{4}\mathbb{L}^2\mathcal{E}_{1,1}\mathcal{E}_{1,1}.$$

However, we cannot do the same for the functions $(F^{(1)}_{4,4})_{r,s}$; they cannot be expressed as a combination of real analytic Eisenstein series and $\mathbb{L}$. These are the "new" functions we have been looking for.

For each of the functions $(F^{(k)}_{4,4})_{r,s}$ above, we have an associated Laplace-eigenvalue equation:

$$\begin{aligned}
(\Delta+4)(F^{(0)}_{4,4})_{4,0} &= -\mathbb{L}\mathbb{G}_4\mathcal{E}_{1,1} & (\Delta+2)(F^{(1)}_{4,4})_{2,0} &= -4\mathbb{L}^2\mathbb{G}_4\mathcal{E}_{0,2}\\
(\Delta+4)(F^{(0)}_{4,4})_{3,1} &= -2\mathbb{L}\mathbb{G}_4\mathcal{E}_{0,2} & (\Delta+2)(F^{(1)}_{4,4})_{1,1} &= -4\mathbb{L}^3\mathbb{G}_4\overline{\mathbb{G}_4}\\
(\Delta+4)(F^{(0)}_{4,4})_{2,2} &= -\mathbb{L}^2\mathbb{G}_4\overline{\mathbb{G}_4} & (\Delta+2)(F^{(1)}_{4,4})_{0,2} &= -4\mathbb{L}^2\overline{\mathbb{G}_4}\mathcal{E}_{2,0}\\
(\Delta+4)(F^{(0)}_{4,4})_{1,3} &= -2\mathbb{L}\overline{\mathbb{G}_4}\mathcal{E}_{2,0} & &\\
(\Delta+4)(F^{(0)}_{4,4})_{0,4} &= -\mathbb{L}\overline{\mathbb{G}_4}\mathcal{E}_{1,1} & \Delta(F^{(2)}_{4,4})_{0,0} &= -\mathbb{L}^4\mathbb{G}_4\overline{\mathbb{G}_4}.
\end{aligned}$$

In the next section, we will see an important application of the Laplace-eigenvalue equations associated to "new" length two modular iterated integrals.

4.3 MODULAR GRAPH FUNCTIONS

Modular graph functions arise in the low-energy expansion of Type II superstring amplitudes at genus one by assigning a lattice sum to a graph. Simply put, they are a type of real analytic function on the upper half plane. These functions have an extremely interesting mathematical structure, provide connections with number theory and particle phenomenology and are of great importance in other areas of physics. They have been discussed in detail in physics literature and we refer the reader to any of [33, 34, 43–45] for more information on their origin.

The type of modular graph functions of interest to us are denoted by $C_{a_1,\dots,a_p}$, where each a_i is an integer greater than zero. Such a function is said to have $p-1$ loops and a total weight of $a_1+\cdots+a_p$. The space of one-loop modular graph functions is well understood. For each total weight there is a unique modular graph function that can be represented in terms of a real analytic Eisenstein series and $\mathbb{L}$:

$$C_{a_1,a_2} = \frac{2^{a+1}}{(2a-2)!}\mathbb{L}^{a-1}\mathcal{E}_{a-1,a-1},$$

where $a = a_1 + a_2$. These one-loop modular graph functions obey the following Laplace-eigenvalue equation:

$$\Delta C_{a_1,a_2} = a(a-1)C_{a_1,a_2}.$$

For example, at weight 4 we have

$$C_{2,2} = C_{1,3} = C_{3,1} := \mathbb{C}_4 = \frac{2}{45}\mathbb{L}^3\mathcal{E}_{3,3}$$

and

$$\Delta\mathbb{C}_4 = 12\mathbb{C}_4.$$

The space of two-loop modular graph functions is also well understood. The authors of [34], building upon the work of [44] and [46], discovered that these functions obey systems of inhomogeneous Laplace-eigenvalue equations. The inhomogeneous parts of these equations are made up of linear and quadratic combinations of real analytic Eisenstein series (and $\mathbb{L}$). Some examples from [34] include

$$\Delta C_{2,2,1} = \frac{4}{315}\mathbb{L}^4\mathcal{E}_{4,4}, \tag{4.22}$$

$$(\Delta+2)C_{2,1,1} = 16\mathbb{L}^2\mathcal{E}_{1,1}\mathcal{E}_{1,1} - \frac{2}{5}\mathbb{L}^3\mathcal{E}_{3,3}, \tag{4.23}$$

$$(\Delta+6)C_{3,1,1} = \frac{32}{3}\mathbb{L}^3\mathcal{E}_{1,1}\mathcal{E}_{2,2} - \frac{8}{315}\mathbb{L}^4\mathcal{E}_{4,4} - 3C_{2,1,1}. \tag{4.24}$$

This first equation was solved easily in [34], giving an explicit expression for $C_{2,2,1}$:

$$C_{2,2,1} = \frac{1}{1575}\mathbb{L}^4\mathcal{E}_{4,4} + \frac{\zeta(5)}{30}.$$

However, finding an explicit expression for the functions $C_{2,1,1}$ or $C_{3,1,1}$ could not be achieved. In [4], it was shown that we should be able to give an explicit expression for $C_{2,1,1}$ with the help of the Laplace-eigenvalue equations from the previous section. To see this, we first observe the following:

$$(\Delta+2)(\mathbb{L}^2\mathcal{E}_{2,0}\mathcal{E}_{0,2}) = -\mathbb{L}^4\mathbb{G}_4\overline{\mathbb{G}_4} - \mathbb{L}^2\mathcal{E}_{1,1}\mathcal{E}_{1,1},$$
$$(\Delta+2)\mathbb{L}^3\mathcal{E}_{3,3} = -10\mathbb{L}^3\mathcal{E}_{3,3}.$$

Therefore, combining this with the equation $(\Delta+2)(F^{(1)}_{4,4})_{1,1} = -4\mathbb{L}^3\mathbb{G}_4\overline{\mathbb{G}_4}$ from the previous section, we have

$$(\Delta+2)\left(4\mathbb{L}(F^{(1)}_{4,4})_{1,1} - 16\mathbb{L}^2\mathcal{E}_{2,0}\mathcal{E}_{0,2} + \frac{1}{25}\mathbb{L}^3\mathcal{E}_{3,3}\right) = 16\mathbb{L}^2\mathcal{E}_{1,1}\mathcal{E}_{1,1} - \frac{2}{5}\mathbb{L}^3\mathcal{E}_{3,3}$$

and hence

$$(\Delta+2)\left(4\mathbb{L}(F^{(1)}_{4,4})_{1,1} - 16\mathbb{L}^2\mathcal{E}_{2,0}\mathcal{E}_{0,2} + \frac{1}{25}\mathbb{L}^3\mathcal{E}_{3,3}\right) = (\Delta+2)C_{2,1,1}. \quad (4.25)$$

We conclude that the modular graph function $C_{2,1,1}$ can be expressed in terms of $\mathbb{L}(F^{(1)}_{4,4})_{1,1}, \mathbb{L}^2\mathcal{E}_{2,0}\mathcal{E}_{0,2}$, $\mathbb{L}^3\mathcal{E}_{3,3}$ and a constant. The coefficients in the expansion (2.2) of $(F^{(1)}_{4,4})_{1,1}$ can be determined using properties of ∂, $\bar{\partial}$ and the above differential equations (see [4] for more details).

If other modular graph functions, such as $C_{3,1,1}$, can be expressed using "new" modular iterated integrals, then these integrals can help with the previously mentioned problem of giving a complete description of the modular graph functions.

We can use Theorem 4.12 to produce more "new" functions and, in turn, to produce more Laplace-eigenvalue equations. By comparing these equations with data on modular graph functions from literature in physics, we should be able to repeat the above result for other modular graph functions.

Of course, it will be useful to have as many Laplace-eigenvalue equations available to use as possible. Indeed, it may be useful, then, to have Laplace-eigenvalue equations for "new" modular iterated integrals that are of a higher length than two. In the next section, we aim to extend the theory presented in Section 4.2 from length two modular iterated integrals to length three. In particular, we wish to produce Laplace-eigenvalue equations for "new" length three modular iterated integrals.

4.4 LAPLACE EQUATIONS FOR LENGTH THREE ITERATED INTEGRALS

The main obstacle with extending the theory from the Section 4.2 to $\mathcal{MI}_3$ is finding a closed modular equivariant function, say $D_{2a,2b,2c}$, which also leads to "new" functions in $\mathcal{MI}_3$. For example, the form

$$\begin{aligned} D_{2a,2b,2c} = \underline{E}_{2a} \otimes \mathcal{E}_{2b-2} \otimes \mathcal{E}_{2c-2} + \mathcal{E}_{2a-2} \otimes \underline{E}_{2b} \otimes \mathcal{E}_{2c-2} + \mathcal{E}_{2a-2} \otimes \mathcal{E}_{2b-2} \otimes \underline{E}_{2c} \\ + \mathcal{E}_{2a-2} \otimes \mathcal{E}_{2b-2} \otimes \overline{\underline{E}}_{2c} + \mathcal{E}_{2a-2} \otimes \overline{\underline{E}}_{2b} \otimes \mathcal{E}_{2c-2} + \overline{\underline{E}}_{2a} \otimes \mathcal{E}_{2b-2} \otimes \mathcal{E}_{2c-2} \end{aligned}$$

may seem like a suitable candidate as it is both closed and modular equivariant. However, when following the same steps from the previous section but with $D_{2a,2b,2c}$ instead of $D_{2a,2b}$, the resulting theorem does not lead to any "new" functions. All the functions produced by this theorem are just combinations of real analytic Eisenstein series and $\mathbb{L}$.

We get similar results for any candidate of $D_{2a,2b,2c}$ involving just copies of $\underline{E}_{2w}$ or $\mathcal{E}_{2w-2}$; the form is either not closed, not modular equivariant or, when both these conditions are met, it does not lead to any "new" functions.

In order to solve this problem, we used the functions produced by Theorem 4.12 directly in our definition of $D_{2a,2b,2c}$.

Definition 4.14. For $a, b, c \geq 1$, we define the forms $D_{2a+2,2b+2,2c+2}$ by

$$\begin{aligned} D_{2a+2,2b+2,2c+2} &\colon \mathfrak{H} \to (V_{2a+2b+2c}) \otimes (\mathbb{C}\, d\tau + \mathbb{C}\, d\bar{\tau}) \\ D_{2a+2,2b+2,2c+2}(\tau) &:= E_{2a+2}(\tau) \cdot F^{(0)}_{2b+2,2c+2}(\tau) d\tau + F^{(0)}_{2a+2,2b+2}(\tau) \cdot \bar{E}_{2c+2}(\tau) d\bar{\tau} \\ &\quad + 2\pi i \mathcal{E}_{2c}(\tau) f^{(0)}_{2a+2,2b+2}(\tau)(X - \tau Y)^{2a+2b} d\tau \\ &\quad - 2\pi i \mathcal{E}_{2a}(\tau) \bar{g}^{(0)}_{2b+2,2c+2}(\tau)(X - \bar{\tau} Y)^{2b+2c} d\bar{\tau}. \end{aligned}$$

We will see in Lemma 4.15 that these last two summands are included to deal with the cusp forms that appear in equations (4.20) and (4.21). When $2a + 2b \leq 8$ and $2b + 2c \leq 8$ then, since the cusp forms vanish, these two summands disappear and we are left with a more concise definition:

$$D_{2a+2,2b+2,2c+2} = E_{2a+2}(\tau) \cdot F^{(0)}_{2b+2,2c+2}(\tau)\; d\tau + F^{(0)}_{2a+2,2b+2}(\tau) \cdot \bar{E}_{2c+2}(\tau)\; d\bar{\tau}.$$

This is also true if $2a+2b=12$ and $2b+2c=12$.

We know that $E_{2a+2}(\tau)d\tau$, $\bar{E}_{2c+2}(\tau)d\bar{\tau}$, $\mathcal{E}_{2a}(\tau)$ and $\mathcal{E}_{2c}(\tau)$ are modular equivariant and, by the previous section, $F^{(0)}_{2a+2,2b+2}$ and $F^{(0)}_{2b+2,2c+2}(\tau)$ are also equivariant. Now, as the cusp form $f^{(0)}_{2a+2,2b+2}$ (resp. $g^{(0)}_{2b+2,2c+2}$) is of weight $2a+2b+2$ (resp. $2b+2c+2$), we have the modular equivariance of $D_{2a+2,2b+2,2c+2}$:

$$D_{2a+2,2b+2,2c+2}(\gamma\tau)|_{\gamma} = D_{2a+2,2b+2,2c+2}(\tau), \qquad \forall\, \gamma \in \Gamma_1.$$

Lemma 4.15. *The forms $D_{2a+2,2b+2,2c+2}$ are closed:*

$$dD_{2a+2,2b+2,2c+2} = 0.$$

Proof. Throughout this proof we use the abbreviations $f := f^{(0)}_{2a+2,2b+2}$ and $\bar{g} := \bar{g}^{(0)}_{2b+2,2c+2}$. For a form $\sigma(\tau) := u_1(\tau)\,d\tau + u_2(\tau)\,d\bar{\tau}$, the exterior derivative is given by

$$d\sigma = \left(\frac{\partial u_1}{\partial \bar{\tau}} - \frac{\partial u_2}{\partial \tau}\right) d\bar{\tau} \wedge d\tau.$$

Letting $\sigma(\tau) = D_{2a+2,2b+2,2c+2}(\tau)$, we have

$$\begin{aligned}\frac{\partial u_1}{\partial \bar{\tau}} - \frac{\partial u_2}{\partial \tau} &= \frac{\partial}{\partial \bar{\tau}}\left(E_{2a+2}(\tau)F^{(0)}_{2b+2,2c+2}(\tau) + 2\pi i \mathcal{E}_{2c}(\tau) f(\tau)(X-\tau Y)^{2a+2b}\right)\\ &\quad - \frac{\partial}{\partial \tau}\left(F^{(0)}_{2a+2,2b+2}(\tau)\bar{E}_{2c+2}(\tau) - 2\pi i \mathcal{E}_{2a}(\tau)\bar{g}(\tau)(X-\bar{\tau}Y)^{2b+2c}\right)\\ &= E_{2a+2}(\tau)\frac{\partial}{\partial \bar{\tau}}F^{(0)}_{2b+2,2c+2}(\tau) + 2\pi i\left(\frac{\partial}{\partial \bar{\tau}}\mathcal{E}_{2c}(\tau)\right) f(\tau)(X-\tau Y)^{2a+2b}\\ &\quad - \left(\frac{\partial}{\partial \tau}F^{(0)}_{2a+2,2b+2}(\tau)\right)\bar{E}_{2c+2}(\tau) + 2\pi i\left(\frac{\partial}{\partial \tau}\mathcal{E}_{2a}(\tau)\right)\bar{g}(\tau)(X-\bar{\tau}Y)^{2b+2c}\end{aligned}$$

and, by equation (4.6), this is equal to

$$\begin{aligned}&E_{2a+2}(\tau)\frac{\partial}{\partial \bar{\tau}}F^{(0)}_{2b+2,2c+2}(\tau) + \pi i \bar{E}_{2c+2}(\tau) f(\tau)(X-\tau Y)^{2a+2b}\\ &\quad - \frac{\partial}{\partial \tau}F^{(0)}_{2a+2,2b+2}(\tau)\cdot\bar{E}_{2c+2}(\tau) + \pi i E_{2a+2}(\tau)\bar{g}(\tau)(X-\bar{\tau}Y)^{2b+2c}.\end{aligned} \tag{4.26}$$

Using equations (4.20) and (4.21) for $k=0$, we have

$$\frac{\partial}{\partial \tau}F^{(0)}_{2a+2,2b+2}(\tau) = \frac{1}{2}E_{2a+2}(\tau)\mathcal{E}_{2b}(\tau) + \pi i f(\tau)(X-\tau Y)^{2a+2b}$$

and

$$\frac{\partial}{\partial \bar{\tau}}F^{(0)}_{2b+2,2c+2}(\tau) = \frac{1}{2}\mathcal{E}_{2b}(\tau)\bar{E}_{2c+2}(\tau) - \pi i \bar{g}(\tau)(X-\bar{\tau}Y)^{2b+2c}.$$

Inserting these two equations back into (4.26) gives the required result. □

This lemma leads to the definition of the following function:

$$K_{2a+2,2b+2,2c+2}(z) := -\frac{1}{2}\int_z^{\overrightarrow{1}_\infty} D_{2a+2,2b+2,2c+2}.$$

Since $D_{2a+2,2b+2,2c+2}$ is closed, this integral depends only on the homotopy class of the chosen path and is regularised as in Section 4.2.1. This function also satisfies the following differential equations:

$$\frac{\partial}{\partial z}K_{2a+2,2b+2,2c+2}(z) = \pi i \mathbb{G}_{2a+2}(z)(X-zY)^{2a}\cdot F^{(0)}_{2b+2,2c+2}(z) + \pi i \mathcal{E}_{2c}(z) f^{(0)}_{2a+2,2b+2}(z)(X-zY)^{2a+2b}, \tag{4.27}$$

$$\frac{\partial}{\partial \bar{z}}K_{2a+2,2b+2,2c+2}(z) = F^{(0)}_{2a+2,2b+2}(z)\cdot \overline{\pi i \mathbb{G}_{2c+2}}(z)(X-\bar{z}Y)^{2c} - \pi i \mathcal{E}_{2a}(z)\bar{g}^{(0)}_{2b+2,2c+2}(z)(X-\bar{z}Y)^{2b+2c}. \tag{4.28}$$

We will express equation (4.27) in a way that will be more convenient for us to use. We start by setting

$$J(z) := \mathbb{G}_{2a+2}(z)(X-zY)^{2a}F^{(0)}_{2b+2,2c+2}(z) + \mathcal{E}_{2c}(z)f^{(0)}_{2a+2,2b+2}(z)(X-zY)^{2a+2b},$$

then, by equations (4.4) and (4.19),

$$\begin{aligned} J(z) &= \mathbb{G}_{2a+2}(z)(X-zY)^{2a}\sum_{r+s=2b+2c}\left(F^{(0)}_{2b+2,2c+2}(z)\right)_{r,s}(X-zY)^r(X-\bar{z}Y)^s \\ &\quad + \sum_{r+s=2c}\mathcal{E}_{r,s}(X-zY)^r(X-\bar{z}Y)^s f^{(0)}_{2a+2,2b+2}(X-zY)^{2a+2b} \\ &= \mathbb{G}_{2a+2}(z)\sum_{r+s=2b+2c}\left(F^{(0)}_{2b+2,2c+2}(z)\right)_{r,s}(X-zY)^{r+2a}(X-\bar{z}Y)^s \\ &\quad + f^{(0)}_{2a+2,2b+2}\sum_{r+s=2c}\mathcal{E}_{r,s}(X-zY)^{r+2a+2b}(X-\bar{z}Y)^s. \end{aligned}$$

Using the change of variables $r \mapsto r-2a$ in the first sum and $r \mapsto r-2a-2b$ in the second sum, this is equal to

$$\begin{aligned} J(z) = {} & \mathbb{G}_{2a+2}(z)\sum_{\substack{r+s=\\ 2a+2b+2c}}\left(F^{(0)}_{2b+2,2c+2}\right)_{r-2a,s}(X-zY)^r(X-\bar{z}Y)^s \\ & + f^{(0)}_{2a+2,2b+2}\sum_{\substack{r+s=\\ 2a+2b+2c}}\mathcal{E}_{r-2a-2b,s}(X-zY)^r(X-\bar{z}Y)^s. \end{aligned}$$

Finally, combining these two sums together yields

$$J(z) = \sum_{\substack{r+s=\\2a+2b+2c}} \Bigg[\mathbb{G}_{2a+2}(z) \left(F^{(0)}_{2b+2,2c+2}(z) \right)_{r-2a,s} \\ + f^{(0)}_{2a+2,2b+2} \mathcal{E}_{r-2a-2b,s} \Bigg] (X - zY)^r (X - \bar{z}Y)^s.$$

Therefore, equation (4.27) can be expressed as

$$\frac{\partial}{\partial z} K_{2a+2,2b+2,2c+2}(z) = \pi i \, J(z), \tag{4.29}$$

where $J\colon \mathfrak{H} \to V_{2a+2b+2c}$ is of the form given by (4.3) with

$$J_{r,s}(z) = \mathbb{G}_{2a+2}(z) \left(F^{(0)}_{2b+2,2c+2}(z) \right)_{r-2a,s} + f^{(0)}_{2a+2,2b+2} \mathcal{E}_{r-2a-2b,s}. \tag{4.30}$$

We can also get an analogous equation for $\partial / \partial \bar{z}$:

$$\frac{\partial}{\partial \bar{z}} K_{2a+2,2b+2,2c+2}(z) = \pi i \, J'(z), \tag{4.31}$$

where

$$J'_{r,s}(z) = \bar{\mathbb{G}}_{2c+2}(z) \left(F^{(0)}_{2a+2,2b+2}(z) \right)_{r,s-2c} + \bar{g}^{(0)}_{2b+2,2c+2} \mathcal{E}_{r,s-2b-2c}. \tag{4.32}$$

We will see, in the next section, that equations (4.30) and (4.32) form the basis of the equations appearing in Theorem 4.16. This theorem will be the analogue of Theorem 4.12 and will produce "new" length three modular iterated integrals.

4.4.1 *Equivariant Versions of Triple Integrals*

Using equations (4.27) and (4.28) with the same reasoning as used in Section 4.2.2, we have that, for all $\gamma \in \Gamma_1$,

$$K_{2a+2,2b+2,2c+2}(\gamma z)|_\gamma - K_{2a+2,2b+2,2c+2}(z) = C_\gamma,$$

for some constant C_γ (does not depend on z) which is also a coboundary and hence a cocycle:

$$C_\gamma \in Z^1(\Gamma_1, V_{2a+2b+2c} \otimes \mathbb{C}).$$

As before, we then define a function $M_{2a+2,2b+2,2c+2}$ such that we have modular equivariance and

$$M_{2a+2,2b+2,2c+2} = K_{2a+2,2b+2,2c+2} - c - \frac{1}{2}\int_{z}^{\overrightarrow{1}_\infty} (\underline{\underline{h}} + \overline{\underline{\underline{k}}}),$$

where h (resp. k) is a standard modular form (resp. modular cusp form) of weight $2a+2b+2c+2$. The equation above uniquely determines c, h, k and $M_{2a+2,2b+2,2c+2}$. We now have all the necessary tools to prove the main theorem of this section.

Theorem 4.16. *Let $a, b, c \geq 1$ and set $w = a+b+c$. There exists a family of elements $(G_{2a+2,2b+2,2c+2})_{r,s} \in \mathcal{MI}_3 \cap \mathcal{M}_{r,s}$ of total modular weight $2w = r+s$, with $r, s \geq 0$, which satisfy the following:*

a)
$$\partial(G_{2a+2,2b+2,2c+2})_{2w,0} = \mathbb{L}\mathbb{G}_{2a+2}\left(F^{(0)}_{2b+2,2c+2}\right)_{2b+2c,0} + \mathbb{L}f^{(0)}_{2a+2,2b+2}\mathcal{E}_{2c,0} + \mathbb{L}h, \tag{4.33}$$

$$\partial(G_{2a+2,2b+2,2c+2})_{r,s} - (r+1)(G_{2a+2,2b+2,2c+2})_{r+1,s-1} = \mathbb{L}\mathbb{G}_{2a+2}\left(F^{(0)}_{2b+2,2c+2}\right)_{r-2a,s} + \mathbb{L}f^{(0)}_{2a+2,2b+2}\mathcal{E}_{r-2a-2b,s}, \quad \text{if } s \geq 1, \tag{4.34}$$

where h is a cusp form of weight $2w+2$.

b)
$$\bar{\partial}(G_{2a+2,2b+2,2c+2})_{0,2w} = \mathbb{L}\bar{\mathbb{G}}_{2c+2}\left(F^{(0)}_{2a+2,2b+2}\right)_{0,2a+2b} + \mathbb{L}\bar{g}^{(0)}_{2b+2,2c+2}\mathcal{E}_{0,2a} + \mathbb{L}\bar{k}, \tag{4.35}$$

$$\bar{\partial}(G_{2a+2,2b+2,2c+2})_{r,s} - (s+1)(G_{2a+2,2b+2,2c+2})_{r-1,s+1} = \mathbb{L}\bar{\mathbb{G}}_{2c+2}\left(F^{(0)}_{2a+2,2b+2}\right)_{r,s-2c} + \mathbb{L}\bar{g}^{(0)}_{2b+2,2c+2}\mathcal{E}_{r,s-2b-2c}, \quad \text{if } r \geq 1, \tag{4.36}$$

where k is a cusp form of weight $2w+2$.

Proof. Throughout this proof we use the abbreviation $M_{abc} := M_{2a+2,2b+2,2c+2}$. The function $M_{2a+2,2b+2,2c+2}$ is, by definition, equivariant. Using Lemma 4.9, we obtain its modular components $(M_{2a+2,2b+2,2c+2})_{r,s}$ which are elements of $\mathcal{M}_{r,s}$. Now, looking at $\frac{\partial}{\partial z}M_{2a+2,2b+2,2c+2}$, we have

$$\begin{aligned}\frac{\partial}{\partial z}M_{abc} = \frac{\partial}{\partial z}K_{2a+2,2b+2,2c+2} - \frac{\partial}{\partial z}\int_z^{\vec{1}_\infty} \pi i h(w)(X-wY)^{2w}dw \\ - \frac{\partial}{\partial z}\int_z^{\vec{1}_\infty} \pi i \bar{k}(w)(X-\bar{w}Y)^{2w}d\bar{w} \\ = \frac{\partial}{\partial z}K_{2a+2,2b+2,2c+2} + \pi i h(z)(X-zY)^{2w}.\end{aligned}$$

We write this as

$$\frac{\partial}{\partial z}M_{2a+2,2b+2,2c+2} = \pi i\, H, \tag{4.37}$$

where, using equations (4.29) and (4.30),

$$H_{r,s} = \mathbb{G}_{2a+2}\left(F^{(0)}_{2b+2,2c+2}\right)_{r-2a,s} + f^{(0)}_{2a+2,2b+2}\mathcal{E}_{r-2a-2b,s} + A_{r,s}$$

and $A_{r,s} = 0$ except for $A_{2w,0} = h$. Lemma 4.10 tells us that equation (4.37) is equivalent to the following system of equations, for all $r+s = 2w$ and $r, s \geq 0$:

$$\partial(M_{abc})_{2w,0} = \mathbb{L}H_{2w,0} = \mathbb{L}\left[\mathbb{G}_{2a+2}\left(F^{(0)}_{2b+2,2c+2}\right)_{r-2a,0} + f^{(0)}_{2a+2,2b+2}\mathcal{E}_{r-2a-2b,0} + h\right]$$

$$\begin{aligned}\partial(M_{abc})_{r,s} - (r+1)(M_{abc})_{r+1,s-1} &= \mathbb{L}H_{r,s} \\ &= \mathbb{L}\left[\mathbb{G}_{2a+2}\left(F^{(0)}_{2b+2,2c+2}\right)_{r-2a,s} + f^{(0)}_{2a+2,2b+2}\mathcal{E}_{r-2a-2b,s}\right] \quad \text{, if } s \geq 1.\end{aligned}$$

Modifying $M_{2a+2,2b+2,2c+2}$ by a suitable multiple of $\mathcal{E}_{2w}$, and making use of equation (4.6), allows us to assume that h is a cusp form. We can obtain equation (4.34) directly from the equation above. To acquire equation (4.33), we simply have to note that $r-2a = 2b+2c$ and $r-2a-2b = 2c$ when $s = 0$. Now $\mathbb{G}_{2a+2}$ and $f^{(0)}_{2a+2,2b+2} \in M[\mathbb{L}]$ and each $\left(F^{(0)}_{2b+2,2c+2}\right)_{r-2a,s}$ and $\mathcal{E}_{r-2a-2b,s} \in \mathcal{MI}_2$, therefore the RHS of equations (4.33) and (4.34) exist in $M[\mathbb{L}] \times \mathcal{MI}_2$.

We can obtain equations (4.35) and (4.36) using a similar method, but with (4.31) and (4.32) instead of (4.29) and (4.30). We can also show that the RHS of these equations exist in $\bar{M}[\mathbb{L}] \times \mathcal{MI}_2$.

Then, using Lemma 4.6, we can conclude that the family of functions $(G_{2a+2,2b+2,2c+2})_{r,s} \in \mathcal{MI}_3$. □

As before, since there are no cusp forms of weight ≤ 10, the equations defined in Theorem 4.16, for $2w = 2a + 2b + 2c \leq 8$, do not contain any cusp forms (the cusp forms $f^{(0)}_{2a+2,2b+2}$ and $\bar{g}^{(0)}_{2b+2,2c+2}$ vanish since the above inequality implies that $2a + 2b \leq 8$ and $2b + 2c \leq 8$). Furthermore, since there are also no cusp forms of weight 14, we have no cusp forms appearing if $2a + 2b + 2c = 12$ with $2a + 2b \leq 8$ and $2b + 2c \leq 8$.

In the next section, we will use Theorem 4.16 to produce examples of length three iterated integrals.

4.4.2 *Examples*

Example 1

The first example we look at is when $2a = 2b = 2c = 2$. Theorem 4.16 gives the following set of equations:

$$\partial(G_{4,4,4})_{6,0} = \mathbb{L}\mathbb{G}_4 F(^{(0)}_{4,4})_{4,0} \qquad \bar{\partial}(G_{4,4,4})_{0,6} = \mathbb{L}\overline{\mathbb{G}_4} F(^{(0)}_{4,4})_{0,4}$$

$$\partial(G_{4,4,4})_{5,1} - 6(G_{4,4,4})_{6,0} = \mathbb{L}\mathbb{G}_4 (F^{(0)}_{4,4})_{3,1} \qquad \bar{\partial}(G_{4,4,4})_{1,5} - 6(G_{4,4,4})_{0,6} = \mathbb{L}\overline{\mathbb{G}_4} F(^{(0)}_{4,4})_{1,3}$$

$$\partial(G_{4,4,4})_{4,2} - 5(G_{4,4,4})_{5,1} = \mathbb{L}\mathbb{G}_4 (F^{(0)}_{4,4})_{2,2} \qquad \bar{\partial}(G_{4,4,4})_{2,4} - 5(G_{4,4,4})_{1,5} = \mathbb{L}\overline{\mathbb{G}_4} F(^{(0)}_{4,4})_{2,2}$$

$$\partial(G_{4,4,4})_{3,3} - 4(G_{4,4,4})_{4,2} = \mathbb{L}\mathbb{G}_4 (F^{(0)}_{4,4})_{1,3} \qquad \bar{\partial}(G_{4,4,4})_{3,3} - 4(G_{4,4,4})_{2,4} = \mathbb{L}\overline{\mathbb{G}_4} F(^{(0)}_{4,4})_{3,1}$$

$$\partial(G_{4,4,4})_{2,4} - 3(G_{4,4,4})_{3,3} = \mathbb{L}\mathbb{G}_4 (F^{(0)}_{4,4})_{0,4} \qquad \bar{\partial}(G_{4,4,4})_{4,2} - 3(G_{4,4,4})_{3,3} = \mathbb{L}\overline{\mathbb{G}_4} F(^{(0)}_{4,4})_{4,0}$$

$$\partial(G_{4,4,4})_{1,5} - 2(G_{4,4,4})_{2,4} = 0 \qquad \bar{\partial}(G_{4,4,4})_{5,1} - 2(G_{4,4,4})_{4,2} = 0$$

$$\partial(G_{4,4,4})_{0,6} - (G_{4,4,4})_{1,5} = 0 \qquad \bar{\partial}(G_{4,4,4})_{6,0} - (G_{4,4,4})_{5,1} = 0.$$

Therefore, using the equations from Section 4.2.3, we have:

$$\partial(G_{4,4,4})_{6,0} = \frac{1}{2}\mathbb{L}\mathbb{G}_4\mathcal{E}_{2,0}^2 \qquad \bar{\partial}(G_{4,4,4})_{0,6} = \frac{1}{2}\mathbb{L}\overline{\mathbb{G}_4}\mathcal{E}_{0,2}^2$$

$$\partial(G_{4,4,4})_{5,1} - 6(G_{4,4,4})_{6,0} = \mathbb{L}\mathbb{G}_4\mathcal{E}_{2,0}\mathcal{E}_{1,1} \qquad \bar{\partial}(G_{4,4,4})_{1,5} - 6(G_{4,4,4})_{0,6} = \mathbb{L}\overline{\mathbb{G}_4}\mathcal{E}_{0,2}\mathcal{E}_{1,1}$$

$$\partial(G_{4,4,4})_{4,2} - 5(G_{4,4,4})_{5,1} = \mathbb{L}\mathbb{G}_4\mathcal{E}_{2,0}\mathcal{E}_{0,2} + \frac{1}{2}\mathbb{L}\mathbb{G}_4\mathcal{E}_{1,1}^2 \qquad \bar{\partial}(G_{4,4,4})_{2,4} - 5(G_{4,4,4})_{1,5} = \mathbb{L}\overline{\mathbb{G}_4}\mathcal{E}_{0,2}\mathcal{E}_{2,0} + \frac{1}{2}\mathbb{L}\overline{\mathbb{G}_4}\mathcal{E}_{1,1}^2$$

$$\partial(G_{4,4,4})_{3,3} - 4(G_{4,4,4})_{4,2} = \mathbb{L}\mathbb{G}_4\mathcal{E}_{1,1}\mathcal{E}_{0,2} \qquad \bar{\partial}(G_{4,4,4})_{3,3} - 4(G_{4,4,4})_{2,4} = \mathbb{L}\overline{\mathbb{G}_4}\mathcal{E}_{1,1}\mathcal{E}_{2,0}$$

$$\partial(G_{4,4,4})_{2,4} - 3(G_{4,4,4})_{3,3} = \frac{1}{2}\mathbb{L}\mathbb{G}_4\mathcal{E}_{0,2}^2 \qquad \bar{\partial}(G_{4,4,4})_{4,2} - 3(G_{4,4,4})_{3,3} = \frac{1}{2}\mathbb{L}\overline{\mathbb{G}_4}\mathcal{E}_{2,0}^2$$

$$\partial(G_{4,4,4})_{1,5} - 2(G_{4,4,4})_{2,4} = 0 \qquad \bar{\partial}(G_{4,4,4})_{5,1} - 2(G_{4,4,4})_{4,2} = 0$$

$$\partial(G_{4,4,4})_{0,6} - (G_{4,4,4})_{1,5} = 0 \qquad \bar{\partial}(G_{4,4,4})_{6,0} - (G_{4,4,4})_{5,1} = 0.$$

These equations can then be solved in terms of real analytic Eisenstein series:

$$(G_{4,4,4})_{6,0} = \frac{1}{6}\mathcal{E}_{2,0}\mathcal{E}_{2,0}\mathcal{E}_{2,0}$$

$$(G_{4,4,4})_{5,1} = \frac{1}{2}\mathcal{E}_{2,0}\mathcal{E}_{2,0}\mathcal{E}_{1,1}$$

$$(G_{4,4,4})_{4,2} = \frac{1}{2}\mathcal{E}_{2,0}\mathcal{E}_{2,0}\mathcal{E}_{0,2} + \frac{1}{2}\mathcal{E}_{2,0}\mathcal{E}_{1,1}\mathcal{E}_{1,1}$$

$$(G_{4,4,4})_{3,3} = \mathcal{E}_{2,0}\mathcal{E}_{0,2}\mathcal{E}_{1,1} + \frac{1}{6}\mathcal{E}_{1,1}\mathcal{E}_{1,1}\mathcal{E}_{1,1}$$

$$(G_{4,4,4})_{2,4} = \frac{1}{2}\mathcal{E}_{2,0}\mathcal{E}_{0,2}\mathcal{E}_{0,2} + \frac{1}{2}\mathcal{E}_{1,1}\mathcal{E}_{0,2}\mathcal{E}_{1,1}$$

$$(G_{4,4,4})_{1,5} = \frac{1}{2}\mathcal{E}_{1,1}\mathcal{E}_{0,2}\mathcal{E}_{0,2}$$

$$(G_{4,4,4})_{0,6} = \frac{1}{6}\mathcal{E}_{0,2}\mathcal{E}_{0,2}\mathcal{E}_{0,2}.$$

As with the length two example, each of the above functions has an associated Laplace-eigenvalue equation:

$$(\Delta + 6)(G_{4,4,4})_{6,0} = -\mathbb{L}\mathbb{G}_4\mathcal{E}_{2,0}\mathcal{E}_{1,1}$$

$$(\Delta + 6)(G_{4,4,4})_{5,1} = -\mathbb{L}\mathbb{G}_4\mathcal{E}_{1,1}\mathcal{E}_{1,1} - 2\mathbb{L}\mathbb{G}_4\mathcal{E}_{2,0}\mathcal{E}_{0,2}$$

$$(\Delta + 6)(G_{4,4,4})_{4,2} = -\mathbb{L}^2\mathbb{G}_4\overline{\mathbb{G}_4}\mathcal{E}_{2,0} - 3\mathbb{L}\mathbb{G}_4\mathcal{E}_{1,1}\mathcal{E}_{0,2}$$

$$(\Delta + 6)(G_{4,4,4})_{3,3} = -\mathbb{L}^2\mathbb{G}_4\overline{\mathbb{G}_4}\mathcal{E}_{1,1} - 2\mathbb{L}\mathbb{G}_4\mathcal{E}_{0,2}\mathcal{E}_{0,2} - 2\mathbb{L}\overline{\mathbb{G}_4}\mathcal{E}_{2,0}\mathcal{E}_{2,0}$$

$$(\Delta+6)(G_{4,4,4})_{2,4} = -\mathbb{L}^2\mathbb{G}_4\overline{\mathbb{G}_4}\mathcal{E}_{0,2} - 3\mathbb{L}\overline{\mathbb{G}_4}\mathcal{E}_{1,1}\mathcal{E}_{2,0}$$

$$(\Delta+6)(G_{4,4,4})_{1,5} = -\mathbb{L}\overline{\mathbb{G}_4}\mathcal{E}_{1,1}\mathcal{E}_{1,1} - 2\mathbb{L}\overline{\mathbb{G}_4}\mathcal{E}_{0,2}\mathcal{E}_{2,0}$$

$$(\Delta+6)(G_{4,4,4})_{0,6} = -\mathbb{L}\overline{\mathbb{G}_4}\mathcal{E}_{0,2}\mathcal{E}_{1,1}.$$

However, we would like to have Laplace-eigenvalue equations for "new" length three modular iterated integrals. We will get such equations in the next example.

Example 2

The second example we will look at is when $2a = 4, 2b = 2$ and $2c = 2$. Using Theorem 4.16 we have:

$$\partial(G_{6,4,4})_{8,0} = \mathbb{L}\mathbb{G}_6(F^{(0)}_{4,4})_{4,0} = \frac{1}{2}\mathbb{L}\mathbb{G}_6\mathcal{E}_{2,0}\mathcal{E}_{2,0}$$
$$\partial(G_{6,4,4})_{7,1} - 8(G_{6,4,4})_{8,0} = \mathbb{L}\mathbb{G}_6(F^{(0)}_{4,4})_{3,1} = \mathbb{L}\mathbb{G}_6\mathcal{E}_{2,0}\mathcal{E}_{1,1}$$
$$\partial(G_{6,4,4})_{6,2} - 7(G_{6,4,4})_{7,1} = \mathbb{L}\mathbb{G}_6(F^{(0)}_{4,4})_{2,2} = \mathbb{L}\mathbb{G}_6\mathcal{E}_{2,0}\mathcal{E}_{0,2} + \frac{1}{2}\mathbb{L}\mathbb{G}_6\mathcal{E}_{1,1}\mathcal{E}_{1,1}$$
$$\partial(G_{6,4,4})_{5,3} - 6(G_{6,4,4})_{6,2} = \mathbb{L}\mathbb{G}_6(F^{(0)}_{4,4})_{1,3} = \mathbb{L}\mathbb{G}_6\mathcal{E}_{1,1}\mathcal{E}_{0,2}$$
$$\partial(G_{6,4,4})_{4,4} - 5(G_{6,4,4})_{5,3} = \mathbb{L}\mathbb{G}_6(F^{(0)}_{4,4})_{0,4} = \frac{1}{2}\mathbb{L}\mathbb{G}_6\mathcal{E}_{0,2}\mathcal{E}_{0,2}$$
$$\partial(G_{6,4,4})_{3,5} - 4(G_{6,4,4})_{4,4} = 0$$
$$\partial(G_{6,4,4})_{2,6} - 3(G_{6,4,4})_{3,5} = 0$$
$$\partial(G_{6,4,4})_{1,7} - 2(G_{6,4,4})_{2,6} = 0$$
$$\partial(G_{6,4,4})_{0,8} - (G_{6,4,4})_{1,7} = 0,$$

with similar expressions involving the $\overline{\partial}$ operator. As before, each function has an associated Laplace-eigenvalue equation:

$$(\Delta+8)(G_{6,4,4})_{8,0} = -\mathbb{L}\mathbb{G}_6\mathcal{E}_{2,0}\mathcal{E}_{1,1}$$

$$(\Delta+8)(G_{6,4,4})_{7,1} = -\mathbb{L}\mathbb{G}_6\mathcal{E}_{1,1}\mathcal{E}_{1,1} - 2\mathbb{L}\mathbb{G}_6\mathcal{E}_{2,0}\mathcal{E}_{0,2}$$

$$(\Delta+8)(G_{6,4,4})_{6,2} = -\mathbb{L}^2\mathbb{G}_6\overline{\mathbb{G}_4}\mathcal{E}_{2,0} - 3\mathbb{L}\mathbb{G}_6\mathcal{E}_{1,1}\mathcal{E}_{0,2}$$

$$(\Delta+8)(G_{6,4,4})_{5,3} = -\mathbb{L}^2\mathbb{G}_6\overline{\mathbb{G}_4}\mathcal{E}_{1,1} - 6\mathbb{L}\overline{\mathbb{G}_4}(F^{(0)}_{6,4})_{6,0} - 2\mathbb{L}\mathbb{G}_6\mathcal{E}_{0,2}\mathcal{E}_{0,2}$$

$$(\Delta+8)(G_{6,4,4})_{4,4} = -\mathbb{L}^2\mathbb{G}_6\overline{\mathbb{G}_4}\mathcal{E}_{0,2} - 5\mathbb{L}\overline{\mathbb{G}_4}(F^{(0)}_{6,4})_{5,1}$$

$$(\Delta+8)(G_{6,4,4})_{3,5} = -4\mathbb{L}\overline{\mathbb{G}_4}(F^{(0)}_{6,4})_{4,2}$$

$$(\Delta+8)(G_{6,4,4})_{2,6} = -3\mathbb{L}\overline{\mathbb{G}_4}(F^{(0)}_{6,4})_{3,3}$$

$$(\Delta+8)(G_{6,4,4})_{1,7} = -2\mathbb{L}\overline{\mathbb{G}_4}(F^{(0)}_{6,4})_{2,4}$$

$$(\Delta+8)(G_{6,4,4})_{0,8} = -\mathbb{L}\overline{\mathbb{G}_4}(F^{(0)}_{6,4})_{1,5}.$$

However, we are not able to express any of the functions $(G_{6,4,4})_{r,s}$ as combinations of real analytic Eisenstein series and $\mathbb{L}$. We have, therefore, found Laplace-eigenvalue equations for "new" length three modular iterated integrals, as desired. There are, of course, more values for a, b and c to try. In fact, we will get "new" functions whenever $a \neq b$ or $a \neq c$.

We can use Theorem 4.16 with Theorem 4.12 to create sets of Laplace-eigenvalue equations for length two and three modular iterated integrals. We can then compare this to the data on modular graph functions from literature in physics to repeat the result of (4.25) but with different modular graph functions. This is a relatively new idea and so, as far as the author is aware, it has not yet been achieved on a larger scale than the one equation given by (4.25). There is, however, great interest in implementing such an idea.

We recall that the two-loop modular graph functions satisfy systems of Laplace-eigenvalue equations whose inhomogeneous parts are either linear or quadratic in real analytic Eisenstein series, as in equations (4.22) – (4.24).

However, similar to increasing the length of modular iterated integrals, as we increase the number of loops for the modular graph functions, the space of such functions becomes more obscure. Modular graph functions of three-loops (and higher) are no longer guaranteed to satisfy the linear or quadratic Laplace equations from the one- and two-loop cases. A few relations for low weight three- and four-loop functions have been conjectured and the simplest of these was recently proven in [33]. However, a better understanding of this space is needed.

As we saw in Section 4.2.3, the Laplace equations for length two modular iterated integrals can be seen as quadratic in some form or Eisenstein series ($\mathcal{E}_{r,s}$ or $\mathbb{G}_n$), for example $(\Delta+4)(F^{(0)}_{4,4})_{4,0} = -\mathbb{L}\mathbb{G}_4\mathcal{E}_{1,1}$. However, the Laplace equations in the length three case, in general, involve cubic combinations of some form of Eisenstein series, as seen above.

Therefore, it would not be surprising if the length three Laplace equations were not used in expressing two-loop modular graph functions. Instead, it is more likely that, if they were to be used, it would be with expressing the much less understood three-loop (or higher) modular graph functions. It will also be interesting to see if any modular iterated integrals do not arise in the expression of any modular graph functions, or vice versa.

Of course, producing "new" length three modular iterated integrals will provide crucial insight into $\mathcal{MI}_3$, and $\mathcal{MI}$ as a whole, which is of great importance by itself. Furthermore, this research provides a stepping stone to investigating "new" functions and Laplace-eigenvalue equations for higher length modular iterated integrals.

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