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# Algebra and Geometry of Gelfand-Tsetlin Patterns

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# Abstract

The problem of studying special bases in irreducible representations of Lie groups has already attracted a lot of attention. Algebraic and geometric structures arising from these bases are of great importance due to numerous applications in various areas of modern mathematics and physics.

This thesis is devoted to detailed study of Gelfand-Tsetlin patterns appearing in explicit formulas for certain special functions on classical groups. Originally Gelfand-Tsetlin patterns appeared in parametrization of basis of finite-dimensional representations. Remarkably the same patterns (in the form of Givental-GLO graphs) appear in integral representations of Whittaker functions related to principal series (infinite-dimensional) representations of Lie groups. We study basic properties of Gelfand-Tsetlin patterns appearing both for finite-dimensional and infinite-dimensional representations and uncover relations between them.

# CONTENTS

# NOTATIONS

Lie algebras and root systems:

- $\mathfrak{g}$ – complex semisimple or reductive Lie algebra;
- $\mathfrak{h} \subset \mathfrak{g}$ – Cartan subalgebra;
- $\delta_{i,j}$ – Kronecker delta: $\delta_{i,j} = 1$ if $i = j$ and $\delta_{i,j} = 0$ if $i \neq j$.
- $\langle \cdot, \cdot \rangle$ – pairing between vectors and dual vectors of in $\mathfrak{h}$ and $\mathfrak{h}^*$;
- $(\cdot, \cdot)$ – inner product in $\mathfrak{h}^*$;
- $\Phi_n$ – root system of rank $n$ in $\mathfrak{h}^*$;
- $\Phi_n^+$ – subset of positive roots in $\Phi_n$;
- $\alpha_i \in \Phi_n^+$ – simple roots of $\Phi_n$;
- $\Phi_n^\vee$ – coroot system dual to $\Phi_n$;
- $\alpha^\vee \in \Phi_n^\vee$ – coroots;
- $\alpha_i^\vee$ – simple coroots in $\Phi_n^\vee$;
- $\mathcal{W}^{\Phi_n}$ – Weyl group of the root system $\Phi_n$;
- $s_\alpha \in \mathcal{W}^{\Phi_n}$ – reflection corresponding to the root $\alpha$;
- $s_i$ – simple reflection;
- $\omega_i$ – fundamental weights;
- $M^{\Phi_n}$ – Cartan matrix of the root system $\Phi_n$;
- $h_i, e_i, f_i$ – a set of Chevalley generators of Lie algebra;

- $\rho$ – half-sum of the positive roots or sum of fundamental weights;
- $\mathfrak{S}_n$ – a group of permutations of $n$ elements;

Patterns and polytopes:

- $\mathcal{P}_\lambda^{A_n}$ – set of Gelfand-Tsetlin patterns for the weight $\lambda$;
- $\mathcal{P}_\lambda^{B_n}$ – set of Gelfand-Tsetlin orthogonal patterns for the weight $\lambda$(odd rank);
- $\mathcal{P}_\lambda^{C_n}$ – set of Zhelobenko patterns for the weight $\lambda$;
- $\mathcal{P}_\lambda^{2n-split}$ – set of $2n$-split orthogonal patterns for the weight $\lambda$;
- $\mathcal{P}_\lambda^{D_n}$ – set of Gelfand-Tsetlin orthogonal patterns for the weight $\lambda$(even rank);
- $\mathcal{P}_\lambda^{BZ_n}$ – set of Berenstein-Zelevinsky patterns for the weight $\lambda$;
- $\theta(x)$ – Heaviside step-function, such that $\theta(x) = 1$ for $x \geq 0$ and $\theta(x) = 0$ for $x < 0$;
- $\Theta^{A_n}(p)$ – a characteristic function for polytope defined by Gelfand-Tsetlin patterns;
- $\Theta^{B_n}(p)$ – a characteristic function for polytope defined by $2n$-split orthogonal patterns;
- $\Theta^{C_n}(p)$ – a characteristic function for polytope defined by Zhelobenko patterns;
- $\Theta^{D_n}(p)$ – a characteristic function for polytope defined by Berenstein-Zelevinsky orthogonal patterns;

Functions on the maximal torus:

- $S_\lambda^{\Phi_n}(x)$ – character of the Lie algebra defined by the root system $\Phi_n$ for the highest weight $\lambda$;

- $Z^{\Phi_n}(\lambda_1, \ldots, \lambda_n | x_1, \ldots, x_n)$ – equivariant volume of $G/P$, $x \in \mathfrak{h}$, $P$ is a minimal parabolic subgroup;
- $\Psi_\lambda^{A_n}(x)$ – $\mathfrak{gl}_{n+1}$-Whittaker function, $x \in \mathfrak{h}$;
- $\Psi_\lambda^{B_n}(x)$ – $\mathfrak{so}_{2n+1}$-Whittaker function, $x \in \mathfrak{h}$;
- $\Psi_\lambda^{C_n}(x)$ – $\mathfrak{sp}_{2n}$-Whittaker function, $x \in \mathfrak{h}$;
- $\Psi_\lambda^{D_n}(x)$ – $\mathfrak{so}_{2n}$-Whittaker function, $x \in \mathfrak{h}$;

# 1

# INTRODUCTION

## 1.1 OVERVIEW

The notion of classical groups was introduced by Weyl [48]. These groups play important role in various areas of mathematics and physics.

Finite dimensional representations of classical groups have several remarkable properties. One of them was discovered by Gelfand and Tsetlin in their works [17, 16]. They provided a way to construct a canonical basis in irreducible representations using restriction of it on a subgroup of a lower rank. It appears that the basis can be described in terms of sets of (half-)integer numbers which we now call Gelfand-Tsetlin patterns. However, they only considered the case of general linear and orthogonal groups. The case of symplectic groups was a bit more special and was treated later by Zhelobenko in [49]. Zhelobenko described patterns which are called symplectic, or Zhelobenko patterns. Later Proctor in [39] showed that for each classical group one can write down a character of finite dimensional irreducible representation as a sum over appropriate patterns. Proctor, and independently Berenstein and Zelevinsky in [3], introduced alternative patterns for orthogonal groups which were more convenient to use in character formula. Proctor also described a correspondence among different realisations of patterns and Young tableaux. For more details on patterns and character formulas see [2, 14, 15, 16, 17, 18, 34, 37, 27, 38, 39, 45, 49, 50].

The construction of Gelfand-Tsetlin basis stimulated progress in studying infinite-dimensional representations of Lie groups done by Cherednik[13], Vinberg[47], Molev[35], Gerasimov et al[20]. In particular, new integral representations of Archimedean Whittaker functions were proposed in [19] and [20], which use Gelfand-Tsetlin patterns with real-valued entries. There is an important problem of uncovering geometrical meaning of the Gelfand-Tsetlin construction and its relations to the Kashiwara-Lusztig theory of crystal bases.

In the paper [21] Gerasimov and Lebedev proposed an original way to study geometry of Gelfand-Tsetlin patterns using special functions called equivariant volumes $Z(\lambda|x)$ of flag manifolds $G/P$. Equivariant volumes are simpler objects than other special functions like characters or Whittaker functions. In particular, they have simpler geometrical meaning, see [1, 4, 32, 46].

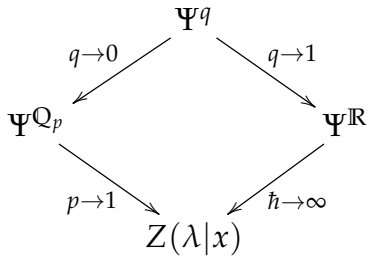
It was stated in [21] that equivariant volumes can be obtained from non-Archemedean Whittaker functions $\Psi^{\mathbb{Q}_p}$ by taking a formal limit $p \to 1$. The Whittaker functions of Lie groups over $\mathbb{Q}_p$ are identified with characters of irreducible (finite-dimensional) representations of Langlands dual complex groups due to Shintani-Casselman-Shalika formula which was introduced in [43] and [12]. This formula provides an explicit formulation of the Langlands duality. More on special functions on $p$-adic groups and Langlands duality can be found in [7, 8, 9, 10, 11, 25, 26, 33, 41, 42, 40].

The relation between characters of finite dimensional representations and equivariant volumes has deep meaning. According to Kirillov in [28, 31](see also [29, 30]), characters of finite dimensional representations of Lie groups have representations as integrals over coajoint orbits. These integral representations allow a natural limit to equivariant volumes of flag varieties.

It was also stated in [21] that the equivariant volumes of $G/P$ are degenerate versions of Whittaker functions over $\mathbb{R}$. One can identify the $G(\mathbb{R})$-Whittaker functions with equivariant volumes of infinite-dimensional space of holomorphic maps of a two-dimensional disk into $G/P$, see [23]. By

taking a limit where the disc degenerates to a point we get an equivariant volume of the finite-dimensional space itself. For details on Archimedean Whittaker functions and integral representations related to Gelfand-Tsetlin patterns see [19, 23, 21, 22, 36].

It all can be visualised by the following diagram




(1)

where $\Psi^q$ is the q-deformed Whittaker function which interpolates between the Archimedean and non-Archimedean types of Whittaker functions, see [21, 22].

It is evident that equivariant volumes of $G/P$ incorporate both combinatorial structure from roots systems and analytic properties from the Whittaker functions over $\mathbb{R}$. In this thesis we are interested in the lower half of the diagram (1).

## 1.2 MAIN RESULTS

The main goal of this work is to study several fundamental properties of Schur functions originated from Gelfand-Tsetlin construction. Namely, we derive and study the analogous properties of equivariant volumes and identify with the corresponding properties of Archimedean Whittaker functions. We expect that our results will allow to discover new deep properties of $G(\mathbb{R})$-Whittaker functions for classical groups.

RECURSION. Gelfand-Tsetlin patterns provide a recurrent structure for equivariant volumes of $G/P$ where $G$ is a classical Lie group. Namely, we have

**Theorem.** *The following relation holds*

$$Z^{\Phi_n}(\lambda|x) = \int\limits_{\mathbb{R}^{|\Phi_n^+|}} Dp\, \Theta^{\Phi_n}(\lambda, p) e^{F(x,\lambda,p)}. \tag{2}$$

Here $\Phi_n$ is one of the root systems $A_n$, $B_n$, $C_n$, $D_n$. As a corollary we have a recurrent relation:

$$Z^{\Phi_n}(\lambda|x) = \int\limits_{\mathbb{R}^r} Dp\, \Theta^{\Phi_n}_{\Phi_{n-1}}(\lambda, p) Z^{\Phi_{n-1}}(p|x). \tag{3}$$

SURGERY OF GIVENTAL-GLO GRAPHS. We show explicitly how equivariant volumes are obtained from the Whittaker functions over $\mathbb{R}$ for each classical Lie algebra. Integral representations of Whittaker functions can be described in terms of Givental graphs as it was shown in [19]. We show how they correspond to the respective Gelfand-Tsetlin polytopes of equivariant volumes of $G/P$ when $P \subset G$ is a minimal parabolic subgroup.

The case when $P$ is not minimal is treated separately. We show how the polytopes look in this case and how the Givental graphs should be transformed according to the correspondence.

OKADA FORMULAS. We show that due to connection between characters and equivariant volumes there are relations analogous to Okada formulas for equivariant volumes:

$$Z_J^{A_{n+m-1}}(t^n, 0^m|x, y) = \int\limits_{\mathbb{R}^m} \prod_{i=1}^{m} d\lambda_i\, \Theta_m(\lambda, t) \cdot Z_{J_1}^{A_{n-1}}(t^{n-m}, \lambda|x) Z^{A_{m-1}}(t - \lambda'|y), \tag{4}$$

$$Z_J^{C_{n+m}}(t^{n+m}|x, y) = \int\limits_{\mathbb{R}^m} \prod_{i=1}^{m} d\lambda_i\, \Theta_m(\lambda, t) \cdot Z_{J_1}^{C_n}(t^{n-m}, \lambda|x) Z^{C_m}(\lambda|y), \tag{5}$$

$$Z_J^{D_{n+m+1}}(t^{n+m}, \pm t|x, y, 0) = \int\limits_{\mathbb{R}^m} \prod_{i=1}^{m} d\lambda_i\, \Theta_m(\lambda, t) \cdot Z_{J_1}^{B_n}(t^{n-m}, \lambda|x) Z^{B_m}(\lambda|y), \tag{6}$$

$$Z_{J_1}^{A_{m+n-1}}(s^m,0^n|x)\cdot Z_{J_2}^{A_{m+n-1}}(t^n,0^m|x)=$$
$$=\int\limits_{\mathbb{R}^m}\prod_{i=1}^{m}d\lambda_i\,\Theta_m(\lambda,s,t)\,Z_J^{A_{m+n-1}}(\lambda,t^{n-m},s+t-\lambda'|x) \tag{7}$$

$$Z_J^{B_n}(s^n|x)\cdot Z_J^{B_n}(t^n|x)=\int\limits_{\mathbb{R}^n}\prod_{i=1}^{n}d\lambda_i\,\Theta_m(\lambda,2s)\cdot Z^{B_n}(\lambda+t-s|x). \tag{8}$$

$$Z_J^{C_n}(s^n|x)\cdot Z_J^{C_n}(t^n|x)=2^{-n}\cdot\int\limits_{\mathbb{R}^n}\prod_{i=1}^{n}d\lambda_i\cdot\Theta_m(\lambda,2s)\cdot Z^{C_n}(\lambda+t-s|x). \tag{9}$$

## 1.3 PLAN OF THESIS

The plan of this text is as follows.

In Chapter 2 we describe the original Gelfand-Tsetlin patterns for general linear and orthogonal groups, as well as patterns for symplectic groups discovered by Zhelobenko and alternative patterns for orthogonal groups provided by Proctor and Berenstein and Zelevinsky. In each case there is a character formula as a sum over the corresponding patterns. We also show that these patterns define a certain polytopes called Gelfand-Tsetlin polytopes. Then we take a limit of characters and show that the patterns provide representations of equivariant volumes of $G/P$ as integrals over polytopes. It follows that equivariant volumes inherit the recurrent structure of the respective polytopes.

In Chapter 3 we review the formulas for $G(\mathbb{Q}_p)$-Whittaker functions and their correspondence to the functions on Langlands dual complex groups, see [7, 12, 40]. We see that they are closely related to characters of complex groups and Demazure operators.

In Chapter 4 we prove for the case of $P$ being minimal parabolic subgroup that the limit of $\mathfrak{g}$-Whittaker functions gives equivariant volumes for four classical series. The integral representation of Whittaker functions allows a description via Givental graphs as it was shown in [19]. We show that there

is a correspondence between these graphs and Gelfand-Tsetlin polytopes, equivalently, Gelfand-Tsetlin patterns for each root system.

In Chapter 5 we consider the case when the weight has a non-trivial stabilizer, which refers to the equivariant volumes of $G/P$ when the parabolic subgroup $P$ is not minimal. The Whittaker functions for certain cases of $\mathfrak{gl}_n$ were described in [36], we show that a limit similar to the one in chapter 4 takes place. There is again a correspondence between Givental graphs and Gelfand-Tsetlin polytopes. We also write explicitly the similar cases for the other groups, but for them there is no such description of Whittaker functions yet. In the end we show that Okada formulas, which were given for characters with the highest weight having a stabilizer, can be translated to similar formulas for the equivariant volumes by taking the limit.

In Appendix A we provide some basic information on the root systems and root data for the four classical series $A_n$, $B_n$, $C_n$, $D_n$ corresponding to the Lie algebras $\mathfrak{gl}_{n+1}$, $\mathfrak{so}_{2n+1}$, $\mathfrak{sp}_{2n}$, $\mathfrak{so}_{2n}$.

In Appendix B we provide canonical formulas for characters given by Weyl and Demazure, combinatorial formulas using tableaux and relations between characters including Okada formulas.

# 2

# GELFAND-TSETLIN PATTERNS

## 2.1 GELFAND-TSETLIN METHOD

Gelfand and Tsetlin in their paper [17] described a method to construct a canonical basis for irreducible representations of groups $GL(n+1)$. The method was based on the following key points:

- a finite dimensional irreducible representation is defined by the highest weight $\lambda = (\lambda_1, \ldots, \lambda_{n+1})$, an integer-valued vector;
- a restriction of the representation to the subgroup $GL(n) \subset GL(n+1)$ can be decomposed into a sum of irreducible representations of $GL(n)$:

$$\mathrm{Res}^{GL(n+1)}_{GL(n)} = \bigoplus_{\mu} W^{\mu}, \tag{10}$$

  each irreducible representation in the sum has multiplicity one, and the highest weights $\mu = (\mu_1, \ldots, \mu_n)$ satisfy the conditions $\lambda_i \geq \mu_i \geq \lambda_{i+1}$ for $i = 1, \ldots, n$.

One can continue the process of decomposition for $GL(n)$ and so on down to $GL(1)$, which is an abelian group and all its irreducible representations are of dimension 1. Thus we have a decomposition of the representation of $GL(n+1)$ into a direct sum of one-dimensional spaces. We can choose

one vector in each of these spaces and they form a canonical basis in our representation. They can be labeled by sets of integers

$$P_\lambda = \begin{bmatrix} p_{n+1,1} & & p_{n+1,2} & & \cdots & & p_{n+1,n+1} \\ & p_{n,1} & & p_{n,2} & \cdots & p_{n,n} & \\ & & & \cdots & & & \\ & & & p_{1,1} & & & \end{bmatrix} \tag{11}$$

such that the top row is fixed and equals highest weight of the representation, $p_{n+1,i} = \lambda_i$, and for the other rows the conditions are

$$p_{i+1,j} \geq p_{i,j} \geq p_{i+1,j+1}, \qquad i = 1, \ldots, n, \qquad j = 1, \ldots, i. \tag{12}$$

**Definition 2.1.1.** The set of integers (11) is called a Gelfand-Tsetlin pattern for group $GL(n+1)$.

EXAMPLE. For the case of $GL(3)$ and the highest weight $\lambda = (2, 1, 0)$ the set of patterns is as follows:

$$\begin{bmatrix} 2 & & 1 & & 0 \\ & 2 & & 1 & \\ & & 2 & & \end{bmatrix}, \begin{bmatrix} 2 & & 1 & & 0 \\ & 2 & & 1 & \\ & & 1 & & \end{bmatrix}, \begin{bmatrix} 2 & & 1 & & 0 \\ & 2 & & 0 & \\ & & 2 & & \end{bmatrix}, \begin{bmatrix} 2 & & 1 & & 0 \\ & 2 & & 0 & \\ & & 1 & & \end{bmatrix},$$
$$\begin{bmatrix} 2 & & 1 & & 0 \\ & 2 & & 0 & \\ & & 0 & & \end{bmatrix}, \begin{bmatrix} 2 & & 1 & & 0 \\ & 1 & & 1 & \\ & & 1 & & \end{bmatrix}, \begin{bmatrix} 2 & & 1 & & 0 \\ & 1 & & 0 & \\ & & 1 & & \end{bmatrix}, \begin{bmatrix} 2 & & 1 & & 0 \\ & 1 & & 0 & \\ & & 0 & & \end{bmatrix}. \tag{13}$$

## 2.2 ORTHOGONAL GELFAND-TSETLIN PATTERNS

In paper [16] Gelfand and Tsetlin described similar basis and patterns for groups of orthogonal matrices. Proctor in [39] used these patterns for a character formula of irreducible representations of the group $Spin(N)$ which is a double covering of $SO(N)$. We separate odd and even case of $N$.

**Definition 2.2.1.** $2n$-orthogonal Gelfand-Tsetlin patterns for a weight $\lambda = (\lambda_1, \dots, \lambda_n)$ are sets of numbers

$$P = \begin{bmatrix} p_{2n,1} & & p_{2n,2} & \cdots & & p_{2n,n} & \\ & p_{2n-1,1} & & p_{2n-1,2} & \cdots & & p_{2n-1,n} \\ & & p_{2n-2,1} & & \cdots & p_{2n-2,n-1} & \\ & & & \cdots & & \cdots & \\ & & & & & p_{2,1} & \\ & & & & & & p_{1,1} \end{bmatrix} \tag{14}$$

which are all integers or all half-integers and satisfy the following relations:

$$p_{2n,i} = \lambda_i; \tag{15}$$

$$p_{2i+1,j} \geq p_{2i,j} \geq p_{2i+1,j+1}, \qquad i = 1, \dots n-1, \quad j = 1, \dots, i; \tag{16}$$

$$p_{2i,j} \geq p_{2i-1,j} \geq p_{2i,j+1}, \qquad i = 1, \dots n, \quad j = 1, \dots, i-1; \tag{17}$$

$$p_{2i,i} \geq |p_{2i-1,i}|, \qquad i = 1, \dots, n; \tag{18}$$

$$p_{2i,i} \geq |p_{2i+1,i+1}|, \qquad i = 1, \dots, n-1. \tag{19}$$

**Definition 2.2.2.** $(2n-1)$-orthogonal Gelfand-Tsetlin patterns for a weight $\lambda = (\lambda_1, \dots, \lambda_n)$ are sets of numbers

$$P = \begin{bmatrix} p_{2n-1,1} & & p_{2n-1,2} & & \cdots & & p_{2n-1,n} \\ & p_{2n-2,1} & & p_{2n-2,2} & \cdots & p_{2n-2,n-1} & \\ & & \cdots & & \cdots & & \\ & & & & & p_{2,1} & \\ & & & & & & p_{1,1} \end{bmatrix} \tag{20}$$

which are all integers or all half-integers and satisfy the following relations:

$$p_{2n-1,i} = \lambda_i; \tag{21}$$

$$p_{2i+1,j} \geq p_{2i,j} \geq p_{2i+1,j+1}, \qquad i = 1, \dots n-1, \quad j = 1, \dots, i; \tag{22}$$

$$p_{2i,j} \geq p_{2i-1,j} \geq p_{2i,j+1}, \qquad i = 1, \dots n-1, \quad j = 1, \dots, i-1; \tag{23}$$

$$p_{2i,i} \geq |p_{2i-1,i}|, \qquad i = 1, \dots, n-1; \tag{24}$$

$$p_{2i,i} \geq |p_{2i+1,i+1}|, \qquad i = 1, \dots, n-1. \tag{25}$$

EXAMPLES. For the weight $\lambda = (2,1)$ the 4-orthogonal Gelfand-Tsetlin patterns are:

$$\begin{bmatrix} 2 & & 1 & \\ & 2 & & 1 \\ & & 2 & \\ & & & 2 \end{bmatrix}, \begin{bmatrix} 2 & & 1 & \\ & 2 & & 1 \\ & & 2 & \\ & & & 1 \end{bmatrix}, \begin{bmatrix} 2 & & 1 & \\ & 2 & & 1 \\ & & 2 & \\ & & & 0 \end{bmatrix}, \begin{bmatrix} 2 & & 1 & \\ & 2 & & 1 \\ & & 2 & \\ & & & -1 \end{bmatrix},$$
$$\begin{bmatrix} 2 & & 1 & \\ & 2 & & 1 \\ & & 2 & \\ & & & -2 \end{bmatrix}, \begin{bmatrix} 2 & & 1 & \\ & 2 & & 1 \\ & & 1 & \\ & & & 1 \end{bmatrix}, \begin{bmatrix} 2 & & 1 & \\ & 2 & & 1 \\ & & 1 & \\ & & & 0 \end{bmatrix}, \begin{bmatrix} 2 & & 1 & \\ & 2 & & 1 \\ & & 1 & \\ & & & -1 \end{bmatrix},$$
$$\begin{bmatrix} 2 & & 1 & \\ & 2 & & 0 \\ & & 2 & \\ & & & 2 \end{bmatrix}, \begin{bmatrix} 2 & & 1 & \\ & 2 & & 0 \\ & & 2 & \\ & & & 1 \end{bmatrix}, \begin{bmatrix} 2 & & 1 & \\ & 2 & & 0 \\ & & 2 & \\ & & & 0 \end{bmatrix}, \begin{bmatrix} 2 & & 1 & \\ & 2 & & 0 \\ & & 2 & \\ & & & -1 \end{bmatrix},$$
$$\begin{bmatrix} 2 & & 1 & \\ & 2 & & 0 \\ & & 2 & \\ & & & -2 \end{bmatrix}, \begin{bmatrix} 2 & & 1 & \\ & 2 & & 0 \\ & & 1 & \\ & & & 1 \end{bmatrix}, \begin{bmatrix} 2 & & 1 & \\ & 2 & & 0 \\ & & 1 & \\ & & & 0 \end{bmatrix}, \begin{bmatrix} 2 & & 1 & \\ & 2 & & 0 \\ & & 1 & \\ & & & -1 \end{bmatrix},$$
$$\begin{bmatrix} 2 & & 1 & \\ & 2 & & 0 \\ & & 0 & \\ & & & 0 \end{bmatrix}, \begin{bmatrix} 2 & & 1 & \\ & 2 & & -1 \\ & & 2 & \\ & & & 2 \end{bmatrix}, \begin{bmatrix} 2 & & 1 & \\ & 2 & & -1 \\ & & 2 & \\ & & & 1 \end{bmatrix}, \begin{bmatrix} 2 & & 1 & \\ & 2 & & -1 \\ & & 2 & \\ & & & 0 \end{bmatrix},$$
$$\begin{bmatrix} 2 & & 1 & \\ & 2 & & -1 \\ & & 2 & \\ & & & -1 \end{bmatrix}, \begin{bmatrix} 2 & & 1 & \\ & 2 & & -1 \\ & & 2 & \\ & & & -2 \end{bmatrix}, \begin{bmatrix} 2 & & 1 & \\ & 2 & & -1 \\ & & 1 & \\ & & & 1 \end{bmatrix}, \begin{bmatrix} 2 & & 1 & \\ & 2 & & -1 \\ & & 1 & \\ & & & 0 \end{bmatrix},$$

$$\begin{bmatrix} 2 & & 1 & & \\ & 2 & & -1 & \\ & & 1 & & \\ & & & -1 \end{bmatrix}, \begin{bmatrix} 2 & & 1 & & \\ & 1 & & 1 & \\ & & 1 & & \\ & & & 1 \end{bmatrix}, \begin{bmatrix} 2 & & 1 & & \\ & 1 & & 1 & \\ & & 1 & & \\ & & & 0 \end{bmatrix}, \begin{bmatrix} 2 & & 1 & & \\ & 1 & & 1 & \\ & & 1 & & \\ & & & -1 \end{bmatrix},$$
$$\begin{bmatrix} 2 & & 1 & & \\ & 1 & & 0 & \\ & & 1 & & \\ & & & 1 \end{bmatrix}, \begin{bmatrix} 2 & & 1 & & \\ & 1 & & 0 & \\ & & 1 & & \\ & & & 0 \end{bmatrix}, \begin{bmatrix} 2 & & 1 & & \\ & 1 & & 0 & \\ & & 1 & & \\ & & & -1 \end{bmatrix}, \begin{bmatrix} 2 & & 1 & & \\ & 1 & & 0 & \\ & & 0 & & \\ & & & 0 \end{bmatrix},$$
$$\begin{bmatrix} 2 & & 1 & & \\ & 1 & & -1 & \\ & & 1 & & \\ & & & 1 \end{bmatrix}, \begin{bmatrix} 2 & & 1 & & \\ & 1 & & -1 & \\ & & 1 & & \\ & & & 0 \end{bmatrix}, \begin{bmatrix} 2 & & 1 & & \\ & 1 & & -1 & \\ & & 1 & & \\ & & & -1 \end{bmatrix}. \tag{26}$$

For the weight $\lambda = (3, -1)$ the 3-orthogonal Gelfand-Tsetlin patterns are:

$$\begin{bmatrix} 3 & & -1 & \\ & 3 & & \\ & & 3 \end{bmatrix}, \begin{bmatrix} 3 & & -1 & \\ & 3 & & \\ & & 2 \end{bmatrix}, \begin{bmatrix} 3 & & -1 & \\ & 3 & & \\ & & 1 \end{bmatrix}, \begin{bmatrix} 3 & & -1 & \\ & 3 & & \\ & & 0 \end{bmatrix},$$
$$\begin{bmatrix} 3 & & -1 & \\ & 3 & & \\ & & -1 \end{bmatrix}, \begin{bmatrix} 3 & & -1 & \\ & 3 & & \\ & & -2 \end{bmatrix}, \begin{bmatrix} 3 & & -1 & \\ & 3 & & \\ & & -3 \end{bmatrix}, \begin{bmatrix} 3 & & -1 & \\ & 2 & & \\ & & 2 \end{bmatrix},$$
$$\begin{bmatrix} 3 & & -1 & \\ & 2 & & \\ & & 1 \end{bmatrix}, \begin{bmatrix} 3 & & -1 & \\ & 2 & & \\ & & 0 \end{bmatrix}, \begin{bmatrix} 3 & & -1 & \\ & 2 & & \\ & & -1 \end{bmatrix}, \begin{bmatrix} 3 & & -1 & \\ & 2 & & \\ & & -2 \end{bmatrix},$$
$$\begin{bmatrix} 3 & & -1 & \\ & 1 & & \\ & & 1 \end{bmatrix}, \begin{bmatrix} 3 & & -1 & \\ & 1 & & \\ & & 0 \end{bmatrix}, \begin{bmatrix} 3 & & -1 & \\ & 1 & & \\ & & -1 \end{bmatrix}. \tag{27}$$

## 2.3 SYMPLECTIC PATTERNS

For symplectic group $Sp(2n)$ the patterns and basis were described by Zhelobenko in [49].

**Definition 2.3.1.** Sympletic patterns, or Zhelobenko patterns, for a weight $\lambda = (\lambda_1, \dots, \lambda_n)$ are sets of numbers

$$P = \begin{bmatrix} p_{2n,1} & & p_{2n,2} & \cdots & & p_{2n,n} & \\ & p_{2n-1,1} & & p_{2n-1,2} & \cdots & & p_{2n-1,n} \\ & & p_{2n-2,1} & & \cdots & p_{2n-2,n-1} & \\ & & & \cdots & & \cdots & \\ & & & & & p_{2,1} & \\ & & & & & & p_{1,1} \end{bmatrix} \tag{28}$$

which are all integers and satisfy the following relations:

$$p_{2n,i} = \lambda_i; \tag{29}$$

$$p_{2i+1,j} \geq p_{2i,j} \geq p_{2i+1,j+1}, \qquad i = 1, \dots n-1, \quad j = 1, \dots, i; \tag{30}$$

$$p_{2i,j} \geq p_{2i-1,j} \geq p_{2i,j+1}, \qquad i = 1, \dots n, \quad j = 1, \dots, i-1; \tag{31}$$

$$p_{2i,i} \geq p_{2i-1,i} \geq 0, \qquad i = 1, \dots, n. \tag{32}$$

*Remark* 2.3.1. It was noted in [49] that, unlike general linear and orthogonal cases, here one should restrict the representation of $Sp(2n)$ to $Sp(2n-2)$ of the rank lower by 2. It follows from the fact that the group of rank $2n-1$ between them is not reductive.

EXAMPLE. For the case of $n = 2$ and the weight $\lambda = (2, 1)$ the symplectic patterns are the following set:

$$\begin{bmatrix} 2 & & 1 & \\ & 2 & & 1 \\ & & 2 & \\ & & & 2 \end{bmatrix}, \begin{bmatrix} 2 & & 1 & \\ & 2 & & 1 \\ & & 2 & \\ & & & 1 \end{bmatrix}, \begin{bmatrix} 2 & & 1 & \\ & 2 & & 1 \\ & & 2 & \\ & & & 0 \end{bmatrix}, \begin{bmatrix} 2 & & 1 & \\ & 2 & & 1 \\ & & 1 & \\ & & & 1 \end{bmatrix},$$
$$\begin{bmatrix} 2 & & 1 & \\ & 2 & & 1 \\ & & 1 & \\ & & & 0 \end{bmatrix}, \begin{bmatrix} 2 & & 1 & \\ & 2 & & 0 \\ & & 2 & \\ & & & 2 \end{bmatrix}, \begin{bmatrix} 2 & & 1 & \\ & 2 & & 0 \\ & & 2 & \\ & & & 1 \end{bmatrix}, \begin{bmatrix} 2 & & 1 & \\ & 2 & & 0 \\ & & 2 & \\ & & & 0 \end{bmatrix},$$
$$\begin{bmatrix} 2 & & 1 & \\ & 2 & & 0 \\ & & 1 & \\ & & & 1 \end{bmatrix}, \begin{bmatrix} 2 & & 1 & \\ & 2 & & 0 \\ & & 1 & \\ & & & 0 \end{bmatrix}, \begin{bmatrix} 2 & & 1 & \\ & 2 & & 0 \\ & & 0 & \\ & & & 0 \end{bmatrix}, \begin{bmatrix} 2 & & 1 & \\ & 1 & & 1 \\ & & 1 & \\ & & & 1 \end{bmatrix},$$
$$\begin{bmatrix} 2 & & 1 & \\ & 1 & & 1 \\ & & 1 & \\ & & & 0 \end{bmatrix}, \begin{bmatrix} 2 & & 1 & \\ & 1 & & 0 \\ & & 1 & \\ & & & 1 \end{bmatrix}, \begin{bmatrix} 2 & & 1 & \\ & 1 & & 0 \\ & & 1 & \\ & & & 0 \end{bmatrix}, \begin{bmatrix} 2 & & 1 & \\ & 1 & & 0 \\ & & 0 & \\ & & & 0 \end{bmatrix}. \quad (33)$$

## 2.4 ALTERNATIVE PATTERNS FOR ORTHOGONAL GROUPS

For further purposes it is more convenient to use alternative patterns for orthogonal case. They can be found in [39] and were introduced by Proctor for a character formula of irreducible representations of $Spin(2n + 1)$ and by Berenstein and Zelevinsky for $Spin(2n)$.

**Definition 2.4.1.** A $2n$-split orthogonal pattern for a weight $\lambda = (\lambda_1, \dots, \lambda_n)$ is a set of numbers

$$P = \begin{bmatrix} p_{2n,1} & & p_{2n,2} & \cdots & & p_{2n,n} & \\ & p_{2n-1,1} & & p_{2n-1,2} & \cdots & & p_{2n-1,n} \\ & & p_{2n-2,1} & & \cdots & p_{2n-2,n-1} & \\ & & & \cdots & & \cdots & \\ & & & & & p_{2,1} & \\ & & & & & & p_{1,1} \end{bmatrix} \tag{34}$$

which satisfy the following conditions:

$$p_{2n,i} = \lambda_i; \tag{35}$$

$$p_{2i+1,j} \geq p_{2i,j} \geq p_{2i+1,j+1}, \qquad i = 1, \dots n-1, \quad j = 1, \dots, i; \tag{36}$$

$$p_{2i,j} \geq p_{2i-1,j} \geq p_{2i,j+1}, \qquad i = 1, \dots n, \quad j = 1, \dots, i-1; \tag{37}$$

$$p_{2i,i} \geq p_{2i-1,i} \geq 0, \qquad i = 1, \dots, n. \tag{38}$$

All entries here except $p_{2i-1,i}$ for $i = 1, \dots, n$ are simultaneously integers or half-integers. The entries $p_{2i-1,i}$ for $i = 1, \dots, n$ are integers or half-integers independently.

**Definition 2.4.2.** A Berenstein-Zelevinsky orthogonal pattern (BZ-pattern) for a weight $\lambda = (\lambda_1, \dots, \lambda_n)$ is a set of numbers

$$P = \begin{bmatrix} p_{2n-1,1} & & p_{2n-1,2} & & \cdots & & p_{2n-1,n} \\ & p_{2n-2,1} & & p_{2n-2,2} & \cdots & p_{2n-2,n-1} & \\ & & \cdots & & \cdots & & \\ & & & & & p_{2,1} & \\ & & & & & & p_{1,1} \end{bmatrix} \tag{39}$$

which are all integers or all half-integers simultaneously and satisfy the following conditions:

$$p_{2n-1,i} = \lambda_i; \tag{40}$$

$$p_{2i+1,j} \geq p_{2i,j} \geq p_{2i+1,j+1}, \qquad i = 1, \dots n-1, \quad j = 1, \dots, i; \tag{41}$$

$$p_{2i,j} \geq p_{2i-1,j} \geq p_{2i,j+1}, \qquad i = 2, \dots n-1, \quad j = 1, \dots, i-1; \tag{42}$$

$$p_{2i,i} \geq p_{2i-1,i}, \qquad i = 1, \dots, n-1, \tag{43}$$

$$p_{2i+1,j} + p_{2i+1,j+1} + p_{2i-1,j} \geq p_{2i,j}, \qquad i = 1, \dots, n-1, \quad j = 1, \dots, i, \tag{44}$$

$$p_{2i-1,j-1} + p_{2i+1,j+1} + p_{2i-1,j} \geq p_{2i,j}, \qquad i = 2, \dots, n-1, \quad j = 2, \dots, i. \tag{45}$$

*Remark* 2.4.1. The inequalities for BZ-patterns can be simplified. Note that since we have

$$p_{2i+1,j} + p_{2i+1,j+1} + p_{2i-1,j} \geq p_{2i,j}, \qquad p_{2i+1,j} \geq p_{2i,j} \tag{46}$$

$$p_{2i-1,j-1} + p_{2i+1,j+1} + p_{2i-1,j} \geq p_{2i,j}, \qquad p_{2i-1,j-1} \geq p_{2i,j}, \tag{47}$$

the additional conditions (44) and (45) are applied only when $p_{2i-1,j} + p_{2i,j} < 0$. One can also notice that

$$\left.\begin{array}{c} p_{2i-1,j-1} + p_{2i+1,j+1} + p_{2i-1,j} \geq p_{2i,j} \\ p_{2i,j} \geq p_{2i+1,j+1} \\ p_{2i-1,j-1} \geq p_{2i,j} \geq p_{2i-1,j} \end{array}\right\} \Rightarrow p_{2i-1,j-1} \geq |p_{2i-1,j}| \geq 0, \tag{48}$$

which implies that all the entries except the row ends $p_{2k-1,k}$ and $p_{2k-2,k-1}$ are non-negative. It means the additional conditions (44) and (45) only matter for the row ends, so the set of inequalities can be rewritten as

$$p_{2n-1,i} = \lambda_i; \tag{49}$$

$$p_{2i+1,j} \geq p_{2i,j} \geq p_{2i+1,j+1}, \qquad i = 1, \dots n-1, \quad j = 1, \dots, i; \tag{50}$$

$$p_{2i,j} \geq p_{2i-1,j} \geq p_{2i,j+1}, \qquad i = 2, \dots n-1, \quad j = 1, \dots, i-1; \tag{51}$$

$$p_{2i,i} \geq p_{2i-1,i}, \qquad i = 1, \dots, n-1, \tag{52}$$

$$p_{2i+1,i} + p_{2i+1,i+1} + p_{2i-1,i} \geq p_{2i,i}, \qquad i = 1, \dots, n-1, \tag{53}$$

$$p_{2i-1,i-1} + p_{2i+1,i+1} + p_{2i-1,i} \geq p_{2i,i}, \qquad i = 2, \dots, n-1. \tag{54}$$

EXAMPLES. For the weight $\lambda = (2,1)$ one has the following set of 4-split orthogonal patterns:

$$\begin{bmatrix} 2 & & 1 & \\ & 2 & & 1 \\ & & 2 & \\ & & & 2 \end{bmatrix}, \quad \begin{bmatrix} 2 & & 1 & \\ & 2 & & 1 \\ & & 2 & \\ & & & \frac{3}{2} \end{bmatrix}, \quad \begin{bmatrix} 2 & & 1 & \\ & 2 & & 1 \\ & & 2 & \\ & & & 1 \end{bmatrix}, \quad \begin{bmatrix} 2 & & 1 & \\ & 2 & & 1 \\ & & 2 & \\ & & & \frac{1}{2} \end{bmatrix},$$

$$\begin{bmatrix} 2 & & 1 & \\ & 2 & & 1 \\ & & 2 & \\ & & & 0 \end{bmatrix}, \quad \begin{bmatrix} 2 & & 1 & \\ & 2 & & 1 \\ & & 1 & \\ & & & 1 \end{bmatrix}, \quad \begin{bmatrix} 2 & & 1 & \\ & 2 & & 1 \\ & & 1 & \\ & & & \frac{1}{2} \end{bmatrix}, \quad \begin{bmatrix} 2 & & 1 & \\ & 2 & & 1 \\ & & 1 & \\ & & & 0 \end{bmatrix},$$

$$\begin{bmatrix} 2 & & 1 & \\ & 2 & & \frac{1}{2} \\ & & 2 & \\ & & & 2 \end{bmatrix}, \quad \begin{bmatrix} 2 & & 1 & \\ & 2 & & \frac{1}{2} \\ & & 2 & \\ & & & \frac{3}{2} \end{bmatrix}, \quad \begin{bmatrix} 2 & & 1 & \\ & 2 & & \frac{1}{2} \\ & & 2 & \\ & & & 1 \end{bmatrix}, \quad \begin{bmatrix} 2 & & 1 & \\ & 2 & & \frac{1}{2} \\ & & 2 & \\ & & & \frac{1}{2} \end{bmatrix},$$

$$\begin{bmatrix} 2 & & 1 & \\ & 2 & & \frac{1}{2} \\ & & 2 & \\ & & & 0 \end{bmatrix}, \quad \begin{bmatrix} 2 & & 1 & \\ & 2 & & \frac{1}{2} \\ & & 1 & \\ & & & 1 \end{bmatrix}, \quad \begin{bmatrix} 2 & & 1 & \\ & 2 & & \frac{1}{2} \\ & & 1 & \\ & & & \frac{1}{2} \end{bmatrix}, \quad \begin{bmatrix} 2 & & 1 & \\ & 2 & & \frac{1}{2} \\ & & 1 & \\ & & & 0 \end{bmatrix},$$

$$\begin{bmatrix} 2 & & 1 & \\ & 2 & & 0 \\ & & 2 & \\ & & & 2 \end{bmatrix}, \quad \begin{bmatrix} 2 & & 1 & \\ & 2 & & 0 \\ & & 2 & \\ & & & \frac{3}{2} \end{bmatrix}, \quad \begin{bmatrix} 2 & & 1 & \\ & 2 & & 0 \\ & & 2 & \\ & & & 1 \end{bmatrix}, \quad \begin{bmatrix} 2 & & 1 & \\ & 2 & & 0 \\ & & 2 & \\ & & & \frac{1}{2} \end{bmatrix},$$

$$\begin{bmatrix} 2 & & 1 & \\ & 2 & & 0 \\ & & 1 & \\ & & & 1 \end{bmatrix}, \quad \begin{bmatrix} 2 & & 1 & \\ & 2 & & 0 \\ & & 1 & \\ & & & \frac{1}{2} \end{bmatrix}, \quad \begin{bmatrix} 2 & & 1 & \\ & 2 & & 0 \\ & & 1 & \\ & & & 0 \end{bmatrix},$$

$$\begin{bmatrix} 2 & & 1 & \\ & 2 & & 0 \\ & & 0 & \\ & & & 0 \end{bmatrix}, \begin{bmatrix} 2 & & 1 & \\ & 1 & & 1 \\ & & 1 & \\ & & & 1 \end{bmatrix}, \begin{bmatrix} 2 & & 1 & \\ & 1 & & 1 \\ & & 1 & \\ & & & \frac{1}{2} \end{bmatrix}, \begin{bmatrix} 2 & & 1 & \\ & 1 & & 1 \\ & & 1 & \\ & & & 0 \end{bmatrix},$$
$$\begin{bmatrix} 2 & & 1 & \\ & 1 & & \frac{1}{2} \\ & & 1 & \\ & & & 1 \end{bmatrix}, \begin{bmatrix} 2 & & 1 & \\ & 1 & & \frac{1}{2} \\ & & 1 & \\ & & & \frac{1}{2} \end{bmatrix}, \begin{bmatrix} 2 & & 1 & \\ & 1 & & \frac{1}{2} \\ & & 1 & \\ & & & 0 \end{bmatrix}, \begin{bmatrix} 2 & & 1 & \\ & 1 & & 0 \\ & & 1 & \\ & & & 1 \end{bmatrix},$$
$$\begin{bmatrix} 2 & & 1 & \\ & 1 & & 0 \\ & & 1 & \\ & & & \frac{1}{2} \end{bmatrix}, \begin{bmatrix} 2 & & 1 & \\ & 1 & & 0 \\ & & 1 & \\ & & & 0 \end{bmatrix}, \begin{bmatrix} 2 & & 1 & \\ & 1 & & 0 \\ & & 0 & \\ & & & 0 \end{bmatrix}. \tag{55}$$

For the weight $\lambda = (3, -1)$ BZ-patterns form the following set:

$$\begin{bmatrix} 3 & & -1 & \\ & 3 & & \\ & & 3 & \end{bmatrix}, \begin{bmatrix} 3 & & -1 & \\ & 3 & & \\ & & 2 & \end{bmatrix}, \begin{bmatrix} 3 & & -1 & \\ & 3 & & \\ & & 1 & \end{bmatrix}, \begin{bmatrix} 3 & & -1 & \\ & 2 & & \\ & & 2 & \end{bmatrix},$$
$$\begin{bmatrix} 3 & & -1 & \\ & 2 & & \\ & & 1 & \end{bmatrix}, \begin{bmatrix} 3 & & -1 & \\ & 2 & & \\ & & 0 & \end{bmatrix}, \begin{bmatrix} 3 & & -1 & \\ & 1 & & \\ & & 1 & \end{bmatrix}, \begin{bmatrix} 3 & & -1 & \\ & 1 & & \\ & & 0 & \end{bmatrix},$$
$$\begin{bmatrix} 3 & & -1 & \\ & 1 & & \\ & & -1 & \end{bmatrix}, \begin{bmatrix} 3 & & -1 & \\ & 0 & & \\ & & 0 & \end{bmatrix}, \begin{bmatrix} 3 & & -1 & \\ & 0 & & \\ & & -1 & \end{bmatrix}, \begin{bmatrix} 3 & & -1 & \\ & 0 & & \\ & & -2 & \end{bmatrix},$$
$$\begin{bmatrix} 3 & & -1 & \\ & -1 & & \\ & & -1 & \end{bmatrix}, \begin{bmatrix} 3 & & -1 & \\ & -1 & & \\ & & -2 & \end{bmatrix}, \begin{bmatrix} 3 & & -1 & \\ & -1 & & \\ & & -3 & \end{bmatrix}. \tag{56}$$

## 2.5 CHARACTER FORMULA

One of the main applications which is going to be used in this work is the character formula. The Gelfand-Tsetlin method provides a basis for irreducible representation, so naturally the character can be written as a sum over all patterns for a given highest weight. These results can be found in [39].

**Theorem 2.5.1** (Proctor). *The character of a finite dimensional irreducible representation of $GL(n+1)$ of the highest weight $\lambda = (\lambda_1, \ldots, \lambda_{n+1})$ is given by the formula*

$$S_\lambda^{A_n}(x_1, \ldots, x_{n+1}) = \sum_{P \in \mathcal{P}_\lambda^{A_n}} x_1^{p_1} \cdots x_{n+1}^{p_{n+1}}, \tag{57}$$

*where $\mathcal{P}_\lambda$ is a set of all Gelfand-Tsetlin patterns (11) for the highest weight $\lambda$ and*

$$p_i = \sum_{j=1}^{i} p_{i,j} - \sum_{j=1}^{i-1} p_{i-1,j}, \qquad i = 2, \ldots, n+1, \tag{58}$$

$$p_1 = p_{1,1}. \tag{59}$$

**Theorem 2.5.2** (Proctor). *The character of a finite dimensional irreducible representation of $Sp(2n)$ of the highest weight $\lambda = (\lambda_1, \ldots, \lambda_n)$ is given by the formula*

$$S_\lambda^{C_n}(x_1, \ldots, x_n) = \sum_{P \in \mathcal{P}_\lambda^{C_n}} x_1^{p_1} \cdots x_n^{p_n}, \tag{60}$$

*where $\mathcal{P}_\lambda^{C_n}$ is a set of all Gelfand-Tsetlin patterns (28) for the highest weight $\lambda$ and*

$$p_i = \sum_{j=1}^{i} p_{2i,j} - 2\sum_{j=1}^{i} p_{2i-1,j} + \sum_{j=1}^{i-1} p_{2i-2,j}, \qquad i = 2, \ldots, n, \tag{61}$$

$$p_1 = p_{2,1} - 2p_{1,1}. \tag{62}$$

**Theorem 2.5.3** (Proctor). *The character of a finite dimensional irreducible representation of $Spin(2n+1)$ of the highest weight $\lambda = (\lambda_1, \ldots, \lambda_n)$ is given by the formula*

$$S_\lambda^{B_n}(x_1, \ldots, x_n) = \sum_{P \in \mathcal{P}_\lambda^{B_n}} x_1^{p_1} \cdots x_n^{p_n}, \tag{63}$$

*where $\mathcal{P}_\lambda^{B_n}$ is a set of all 2n-orthogonal Gelfand-Tsetlin patterns (14) for the highest weight $\lambda$ and*

$$p_i = \operatorname{sign}(p_{2i-1,i}) \operatorname{sign}(p_{2i-3,i-1}) \left( \sum_{j=1}^{i} |p_{2i-1,j}| - 2 \sum_{j=1}^{i-1} |p_{2i-2,j}| + \sum_{j=1}^{i-1} |p_{2i-3,j}| \right), \tag{64}$$

$$i = 2, \dots, n,$$

$$p_1 = p_{1,1}. \tag{65}$$

**Theorem 2.5.4** (Proctor). *The character of a finite dimensional irreducible representation of $Spin(2n+1)$ of the highest weight $\lambda = (\lambda_1, \dots, \lambda_n)$ is given by the formula*

$$S_\lambda^{B_n}(x_1, \dots, x_n) = \sum_{P \in \mathcal{P}_\lambda^{2n-split}} x_1^{p_1} \cdots x_n^{p_n}, \tag{66}$$

*where $\mathcal{P}_\lambda^{2n-split}$ is a set of all 2n-split orthogonal patterns (34) for the highest weight $\lambda$ and*

$$p_i = \sum_{j=1}^{i} p_{2i,j} - 2 \sum_{j=1}^{i} p_{2i-1,j} + \sum_{j=1}^{i-1} p_{2i-2,j}, \qquad i = 2, \dots, n, \tag{67}$$

$$p_1 = p_{2,1} - 2p_{1,1}. \tag{68}$$

**Theorem 2.5.5** (Proctor). *The character of a finite dimensional irreducible representation of $Spin(2n)$ of the highest weight $\lambda = (\lambda_1, \dots, \lambda_n)$ is given by the formula*

$$S_\lambda^{D_n}(x_1, \dots, x_n) = \sum_{P \in \mathcal{P}_\lambda^{D_n}} x_1^{p_1} \cdots x_n^{p_n}, \tag{69}$$

*where $\mathcal{P}_\lambda^{D_n}$ is a set of all (2n-1)-orthogonal Gelfand-Tsetlin patterns (20) for the highest weight $\lambda$ and*

$$p_i = \operatorname{sign}(p_{2i-1,i}) \operatorname{sign}(p_{2i-3,i-1}) \left( \sum_{j=1}^{i} |p_{2i-1,j}| - 2 \sum_{j=1}^{i-1} |p_{2i-2,j}| + \sum_{j=1}^{i-1} |p_{2i-3,j}| \right), \tag{70}$$

$$i = 2, \dots, n,$$

$$p_1 = p_{1,1}. \tag{71}$$

**Theorem 2.5.6** (Berenstein-Zelevinsky). *The character of a finite dimensional irreducible representation of $Spin(2n)$ of the highest weight $\lambda$ is given by the formula*

$$S_\lambda^{D_n}(x_1,\dots,x_n) = \sum_{P\in\mathcal{P}_\lambda^{BZ_n}} x_1^{p_1}\cdots x_n^{p_n}, \tag{72}$$

*where $\mathcal{P}_\lambda^{BZ_n}$ is a set of all Berenstein-Zelevinsky patterns (39) for the highest weight $\lambda$ and*

$$p_i = \sum_{j=1}^{i} p_{2i-1,j} - 2\sum_{j=1}^{i-1} p_{2i-2,j} + \sum_{j=1}^{i-1} p_{2i-3,j}, i = 2,\dots,n, \tag{73}$$

$$p_1 = p_{1,1}. \tag{74}$$

EXAMPLES. Using the examples of patterns from previous sections one can write down characters for the corresponding cases. For the patterns (13) we have

$$\begin{aligned} S_{(2,1,0)}^{A_2}(x_1,x_2,x_3) = x_1^2x_2 + x_1x_2^2 + x_1^2x_3 + x_1x_2x_3 + \\ + x_2^2x_3 + x_1x_2x_3 + x_1x_3^2 + x_2x_3^2. \end{aligned} \tag{75}$$

For the patterns (33):

$$\begin{aligned} S_{(2,1)}^{C_2}(x_1,x_2) = x_1^{-2}x_2^{-1} + x_2^{-1} + x_1^2x_2^{-1} + x_1^{-1}x_2^{-2} + x_1x_2^{-2} + x_1^{-2}x_2 + \\ + x_2 + x_1^2x_2 + x_1^{-1} + x_1 + x_2^{-1} + x_1^{-1} + x_1 + x_1^{-1}x_2^2 + x_1x_2^2 + x_2. \end{aligned} \tag{76}$$

For the patterns (55):

$$\begin{aligned} S_{(2,1)}^{B_2}(x_1,x_2) = x_1^{-2}x_2^{-1} + x_1^{-1}x_2^{-1} + x_2^{-1} + x_1x_2^{-1} + x_1^2x_2^{-1} + x_1^{-1}x_2^{-2} + \\ + x_2^{-2} + x_1x_2^{-2} + x_1^{-2} + x_1^{-1} + 1 + x_1 + x_1^2 + x_1^{-1}x_2^{-1} + \\ + x_2^{-1} + x_1x_2^{-1} + x_1^{-2}x_2 + x_1^{-1}x_2 + x_2 + x_1x_2 + x_1^2x_2 + \\ + x_1^{-1} + 1 + x_1 + x_2^{-1} + x_1^{-1} + 1 + x_1 + x_1^{-1}x_2 + \\ + x_2 + x_1x_2 + x_1^{-1}x_2^2 + x_2^2 + x_1x_2^2 + x_2. \end{aligned} \tag{77}$$

For the patterns (56):

$$\begin{aligned} S_{(3,-1)}^{D_2}(x_1,x_2) = x_1^3x_2^{-1} + x_1^2x_2^{-2} + x_1x_2^{-3} + x_1^2 + x_1x_2^{-1} + x_2^{-2} + x_1x_2 + \\ + 1 + x_1^{-1}x_2^{-1} + x_2^2 + x_1^{-1}x_2 + x_1^{-2} + x_1^{-1}x_2^3 + x_1^{-2}x_2^2 + x_1^{-3}x_2. \end{aligned} \tag{78}$$

## 2.6 RECURSION OVER THE RANK

Gelfand-Tsetlin method relies on the decomposition of the representation of the group into a sum of representations of its subgroups of lower rank. In terms of characters this means that the character of irreducible representation can be represented as a sum over characters of lower rank. The next statements can be easily derived from the character formulas in the previous section, see also [39].

**Proposition 2.6.1.** *The following relation holds*

$$S_\lambda^{A_n}(x_1,\dots,x_{n+1}) = \sum_{\substack{\lambda_i\geq\mu_i\geq\lambda_{i+1}\\ i=1,\dots,n+1}} x_n^{\sum_{i=1}^{n+1}\lambda_i-\sum_{i=1}^{n}\mu_i} S_\mu^{A_{n-1}}(x_1,\dots,x_n) \tag{79}$$

*where*

$$\lambda_i\in\mathbb{Z},\qquad i=1,\dots,n+1, \tag{80}$$

$$\mu_j\in\mathbb{Z},\qquad j=1,\dots,n. \tag{81}$$

**Proposition 2.6.2.** *The following relation holds*

$$S_\lambda^{C_n}(x_1,\dots,x_n) = \sum_{\mu,\nu} x_n^{\sum_{i=1}^{n}\lambda_i-2\sum_{i=1}^{n}\mu_i+\sum_{j=1}^{n-1}\nu_j} S_\nu^{C_{n-1}}(x_1,\dots,x_{n-1}), \tag{82}$$

*where the sum is taken over $\mu$ and $\nu$ satisfying the following conditions:*

$$\lambda_i\geq\mu_i\geq\lambda_{i+1},\qquad i=1,\dots,n-1, \tag{83}$$

$$\lambda_n\geq\mu_n\geq 0, \tag{84}$$

$$\mu_j\geq\nu_j\geq\mu_{j+1},\qquad j=1,\dots,n-1, \tag{85}$$

*and $\lambda_i$, $\mu_j$ and $\nu_k$ are all integers or half-integers simultaneously for $i,j=1,\dots,n$ and $k=1,\dots,n-1$.*

**Proposition 2.6.3.** *The following relation holds*

$$S_\lambda^{B_n}(x_1,\dots,x_n) = \sum_{\mu,\nu} x_n^{\sum_{i=1}^{n}\lambda_i-2\sum_{i=1}^{n}\mu_i+\sum_{j=1}^{n-1}\nu_j} S_\nu^{B_{n-1}}(x_1,\dots,x_{n-1}), \tag{86}$$

*where the sum is taken over $\mu$ and $\nu$ satisfying the following conditions:*

$$\lambda_i \geq \mu_i \geq \lambda_{i+1}, \qquad i = 1, \ldots, n-1, \tag{87}$$

$$\lambda_n \geq \mu_n \geq 0, \tag{88}$$

$$\mu_j \geq \nu_j \geq \mu_{j+1}, \qquad j = 1, \ldots, n-1, \tag{89}$$

*and $\lambda_i$, $\mu_j$ and $\nu_k$ are all integers or half-integers simultaneously for $i = 1, \ldots, n$ and $j, k = 1, \ldots, n-1$, and $\mu_n \in \mathbb{Z}/2$.*

**Proposition 2.6.4.** *The following relation holds*

$$S_\lambda^{D_n}(x_1, \ldots, x_n) = \sum_{\mu,\nu} x_n^{\sum_{i=1}^{n} \lambda_i - 2\sum_{i=1}^{n-1} \mu_i + \sum_{j=1}^{n-1} \nu_j} S_\nu^{D_{n-1}}(x_1, \ldots, x_{n-1}), \tag{90}$$

*where the sum is taken over $\mu$ and $\nu$ satisfying the following conditions:*

$$\lambda_i \geq \mu_i \geq \lambda_{i+1}, \qquad i = 1, \ldots, n-1, \tag{91}$$

$$\mu_j \geq \nu_j \geq \mu_{j+1}, \qquad j = 1, \ldots, n-2, \tag{92}$$

$$\mu_{n-1} \geq \nu_{n-1}, \tag{93}$$

$$\lambda_{n-1} + \lambda_n + \nu_{n-1} \geq \mu_{n-1}, \tag{94}$$

$$\nu_{n-2} + \lambda_n + \nu_{n-1} \geq \mu_{n-1}, \tag{95}$$

*and $\lambda_i$, $\mu_j$ and $\nu_k$ are all integers or half-integers simultaneously for $i = 1, \ldots, n$ and $j, k = 1, \ldots, n-1$.*

## 2.7 EQUIVARIANT VOLUMES

The characters of a group $G$ can be viewed as functions on its maximal torus $H \subset G$ (see B.1). Here we consider another kind of functions on torus. If $B \subset G$ is a Borel subgroup, then the homogeneous space $G/B$ is called a full flag manifold (see [14, 30]). There is a connection between cohomology theory of flag spaces and representation theory of $G$ due to Borel-Weil-Bott theorem (see [21, 32, 46]). In particular, functions $Z^{\Phi_n}$ called equivariant volumes of $G/B$ [21] have a certain relation to characters of $G$.

Let $\mathfrak{h}$ be a Cartan subalgebra of the Lie algebra $\mathfrak{g}$, $\Phi_n$ corresponding root system, $\alpha_i \in \Phi_i$ simple roots, $s_i \in \mathcal{W}^{\Phi_n}$ corresponding simple reflections and $w_0 \in \mathcal{W}^{\Phi_n}$ the element of maximal length. Let also $\langle \lambda, \alpha_i^\vee \rangle > 0$ for all simple coroots $\alpha_i^\vee$. Then, according to [21], we have the following formula for $Z^{\Phi_n}$:

$$Z^{\Phi_n}(\lambda|x) = \mathcal{A}_{w_0}(e^\lambda)(x). \tag{96}$$

Here $\mathcal{A}_{w_0}$ is a BGG operator for $w_0$ defined by (606). We define $Z^{\Phi_n}$ only for the domain

$$\langle \lambda, \alpha_i^\vee \rangle > 0, \qquad i = 1, \dots, n. \tag{97}$$

The cases where $\langle \lambda, \alpha_i^\vee \rangle = 0$ for some $i$ are going to be considered in Chapter 5.

**Proposition 2.7.1** (see [21]). *Let $\Phi_n$ be a root system and $S_\lambda^{\Phi_n}(x)$ its character for the highest weight $\lambda$ satisfying the condition (97). Then the following relation holds*

$$\lim_{\varepsilon \to 0} \varepsilon^{|\Phi_n^+|} \cdot S^{\Phi_n}_{\frac{\lambda_1}{\varepsilon}, \dots, \frac{\lambda_n}{\varepsilon}}(e^{\varepsilon x_1}, \dots, e^{\varepsilon x_n}) = Z^{\Phi_n}(\lambda_1, \dots, \lambda_n | x_1, \dots, x_n). \tag{98}$$

*Proof.* Apply Demazure character formula (603) on the left-hand side and use Proposition (B.2.1):

$$\lim_{\varepsilon \to 0} \varepsilon^{|\Phi_n^+|} \cdot \mathcal{D}_{w_0}(e^{\frac{\lambda}{\varepsilon}})(\varepsilon x) = \mathcal{A}_{w_0}(e^\lambda)(x) = Z^{\Phi_n}(\lambda|x). \tag{99}$$

□

**Corollary 2.7.1** (see [21]). *The following relation holds*

$$Z^{\Phi_n}(\lambda|x) = \sum_{w \in \mathcal{W}^{\Phi_n}} \frac{(-1)^w e^{\langle w\lambda, x\rangle}}{\prod\limits_{\alpha \in \Phi_n^+} \langle \alpha, x \rangle}. \tag{100}$$

*Proof.* The Weyl character formula (591) can be written as

$$S_\lambda^{\Phi_n}(e^{x_1}, \dots, e^{x_n}) = \frac{\sum\limits_{w \in \mathcal{W}} (-1)^w e^{\langle w\lambda + w\rho, x\rangle}}{e^{\langle \rho, x\rangle} \cdot \prod\limits_{\alpha \in \Phi_n^+} (1 - e^{-\langle \alpha, x\rangle})}, \tag{101}$$

where $\rho$ is a half-sum of positive roots. Apply it to the left-hand side in Proposition (2.7.1):

$$\lim_{\varepsilon\to 0}\varepsilon^{|\Phi_n^+|}\cdot S^{\Phi_n}_{\frac{\lambda_1}{\varepsilon},\dots,\frac{\lambda_n}{\varepsilon}}(e^{\varepsilon x_1},\dots,e^{\varepsilon x_n}) = \lim_{\varepsilon\to 0}\varepsilon^{|\Phi_n^+|}\frac{\sum\limits_{w\in\mathcal{W}}(-1)^w e^{\langle w\lambda+\varepsilon w\rho,x\rangle}}{e^{\varepsilon\langle\rho,x\rangle}\cdot\prod\limits_{\alpha\in\Phi_n^+}(1-e^{-\varepsilon\langle\alpha,x\rangle})} =$$

$$= \sum_{w\in\mathcal{W}^{\Phi_n}}\frac{(-1)^w e^{\langle w\lambda,x\rangle}}{\prod\limits_{\alpha\in\Phi_n^+}\langle\alpha,x\rangle} = Z^{\Phi_n}(\lambda_1,\dots,\lambda_n|x_1,\dots,x_n). \tag{102}$$

□

**Corollary 2.7.2** (see [21]). *Consider the coroot system $\Phi_n^\vee$, then the following relation holds*

$$\prod_{\alpha\in\Phi_n^+}\langle\alpha,x\rangle\, Z^{\Phi_n}(\lambda_1,\dots,\lambda_n|x_1,\dots,x_n) =$$

$$= \prod_{\alpha^\vee\in(\Phi_n^\vee)^+}\left\langle\lambda,\alpha^\vee\right\rangle Z^{\Phi_n^\vee}(x_1,\dots,x_n|\lambda_1,\dots,\lambda_n). \tag{103}$$

*Proof.* Straightforward check using (100). □

For the root systems $A_n$, $B_n$, $C_n$ and $D_n$ the formula (100) gives:

$$Z^{A_n}(\lambda_1,\dots,\lambda_{n+1}|x_1,\dots,x_{n+1}) = \sum_{w\in\mathfrak{S}_{n+1}}\frac{(-1)^w e^{\langle w\lambda,x\rangle}}{\prod\limits_{i<j}(x_i-x_j)}, \tag{104}$$

$$\lambda_1 > \dots > \lambda_{n+1}. \tag{105}$$

$$Z^{B_n}(\lambda_1,\dots,\lambda_n|x_1,\dots,x_n) = \sum_{w\in(\mathbb{Z}/2\mathbb{Z})^n\rtimes\mathfrak{S}_n}\frac{(-1)^w e^{\langle w\lambda,x\rangle}}{\prod\limits_{i<j}(x_i-x_j)(x_i+x_j)\prod\limits_{i=1}^n x_i}, \tag{106}$$

$$\lambda_1 > \dots > \lambda_n > 0. \tag{107}$$

$$Z^{C_n}(\lambda_1,\dots,\lambda_n|x_1,\dots,x_n) = \sum_{w\in(\mathbb{Z}/2\mathbb{Z})^n\rtimes\mathfrak{S}_n}\frac{(-1)^w e^{\langle w\lambda,x\rangle}}{\prod\limits_{i<j}(x_i-x_j)(x_i+x_j)\prod\limits_{i=1}^n 2x_i}, \tag{108}$$

$$\lambda_1 > \dots > \lambda_n > 0. \tag{109}$$

$$Z^{D_n}(\lambda_1,\dots,\lambda_n|x_1,\dots,x_n)=\sum_{w\in(\mathbb{Z}/2\mathbb{Z})^{n-1}\rtimes\mathfrak{S}_n}\frac{(-1)^w e^{\langle w\lambda,x\rangle}}{\prod_{i<j}(x_i-x_j)(x_i+x_j)}, \tag{110}$$

$$\lambda_1>\dots>\lambda_{n-1}>|\lambda_n|. \tag{111}$$

*Remark* 2.7.1. Note that

$$Z^{B_n}(\lambda_1,\dots,\lambda_n|x_1,\dots,x_n)=2^n Z^{C_n}(\lambda_1,\dots,\lambda_n|x_1,\dots,x_n). \tag{112}$$

It follows from the fact that the root systems $B_n$ and $C_n$ are dual and have the same Weyl group $\mathcal{W}^{B_n}=\mathcal{W}^{C_n}=(\mathbb{Z}/2\mathbb{Z})^n\rtimes\mathfrak{S}_n$.

In the previous section, characters of classical groups were represented as sums over corresponding patterns. One can apply the idea of taking the limit in theorem (2.7.1) to these sums, which will give an integral representation of equivariant volumes. This is the continuous analogue of sums over patterns for characters.

**Theorem 2.7.1** (see [21]). *The equivariant volume for the root system $A_n$ has the following integral representation:*

$$Z^{A_n}(\lambda_1,\dots,\lambda_{n+1}|x_1,\dots,x_{n+1})=$$
$$=\int\limits_{\mathbb{R}^{(n+1)n/2}}\prod_{i=1}^{n}\prod_{j=1}^{i}dp_{i,j}\,\Theta^{A_n}(p)\cdot e^{x_1p_{11}+\sum\limits_{i=2}^{n+1}x_i\left(\sum\limits_{j=1}^{i}p_{i,j}-\sum\limits_{j=1}^{i-1}p_{i-1,j}\right)}, \tag{113}$$

*where $p_{n+1,i}=\lambda_i$ for $i=1,\dots,n+1$, and $\Theta^{A_n}(p)$ defines the domain:*

$$\Theta^{A_n}(p)=\prod_{i=1}^{n}\prod_{j=1}^{i}\theta(p_{i+1,j}-p_{i,j})\theta(p_{i,j}-p_{i+1,j+1}). \tag{114}$$

*Here $\theta(x)=1$ if $x\geq 0$ and $\theta(x)=0$ if $x<0$.*

*Proof.* Consider the limit (2.7.1) and use the representation of the character as a sum over patterns (57):

$$\varepsilon^{|\Phi_n^+|}\cdot S^{\Phi_n}_{\frac{\lambda_1}{\varepsilon},\dots,\frac{\lambda_n}{\varepsilon}}(e^{\varepsilon x_1},\dots,e^{\varepsilon x_n})=\varepsilon^{|\Phi_n^+|}\cdot\sum_{p_{i,j}}e^{x_1p_{11}+\sum\limits_{i=2}^{n+1}x_i\left(\sum\limits_{j=1}^{i}p_{i,j}-\sum\limits_{j=1}^{i-1}p_{i-1,j}\right)}, \tag{115}$$

where the sum is taken over all sets of $p_{i,j}$ satisfying the following conditions:

$$p_{i+1,j} \geq p_{i,j} \geq p_{i+1,j+1}, \qquad i = 1, \dots, n, \quad j = 1, \dots, i \tag{116}$$

$$p_{i,j} \in \varepsilon\mathbb{Z}. \tag{117}$$

Taking $\Delta p_{i,j} = \varepsilon$ one can rewrite it as

$$\sum_{p_{i,j}} \prod_{i=1}^{n} \prod_{j=1}^{i} \Delta p_{i,j} e^{x_1 p_{11} + \sum\limits_{i=2}^{n+1} x_i \left( \sum\limits_{j=1}^{i} p_{i,j} - \sum\limits_{j=1}^{i-1} p_{i-1,j} \right)}. \tag{118}$$

Then the limit gives

$$\begin{aligned} &\lim_{\varepsilon \to 0} \varepsilon^{|\Phi_n^+|} \cdot S^{\Phi_n}_{\frac{\lambda_1}{\varepsilon}, \dots, \frac{\lambda_n}{\varepsilon}}(e^{\varepsilon x_1}, \dots, e^{\varepsilon x_n}) = \\ &= \lim_{\varepsilon \to 0} \sum_{p_{i,j}} \prod_{i=1}^{n} \prod_{j=1}^{i} \Delta p_{i,j} e^{x_1 p_{11} + \sum\limits_{i=2}^{n+1} x_i \left( \sum\limits_{j=1}^{i} p_{i,j} - \sum\limits_{j=1}^{i-1} p_{i-1,j} \right)} = \\ &= \int_{\mathcal{C}} \prod_{i=1}^{n} \prod_{j=1}^{i} dp_{i,j} \cdot e^{x_1 p_{11} + \sum\limits_{i=2}^{n+1} x_i \left( \sum\limits_{j=1}^{i} p_{i,j} - \sum\limits_{j=1}^{i-1} p_{i-1,j} \right)}. \end{aligned} \tag{119}$$

The domain $\mathcal{C}$ is defined by the inequalities for Gelfand-Tsetlin patterns (12). This is identical to defining it with (114). □

**Theorem 2.7.2.** *The equivariant volume for the root system $B_n$ has the following integral representation:*

$$\begin{aligned} Z^{B_n}(\lambda_1, \dots, \lambda_n | x_1, \dots, x_n) = 2^n \int\limits_{\mathbb{R}^{n^2}} \prod_{i=1}^{n} \prod_{j=1}^{i} dp_{2i-1,j} \prod_{i=1}^{n-1} \prod_{j=1}^{i} dp_{2i,j} \, \Theta(p)^{B_n} \times \\ \times e^{x_1(p_{21} - 2p_{11}) + \sum\limits_{i=2}^{n} x_i \left( \sum\limits_{j=1}^{i} p_{2i,j} - 2 \sum\limits_{j=1}^{i} p_{2i-1,j} + \sum\limits_{j=1}^{i-1} p_{2i-2,j} \right)}, \end{aligned} \tag{120}$$

*where $p_{2n,i} = \lambda_i$ for $i = 1, \dots, n$, and $\Theta^{B_n}(p)$ defines the domain:*

$$\begin{aligned} \Theta^{B_n}(p) = \prod_{i=1}^{n-1} \prod_{j=1}^{i} \theta(p_{2i+1,j} - p_{2i,j}) \theta(p_{2i,j} - p_{2i+1,j+1}) \times \\ \times \prod_{i=2}^{n} \prod_{j=1}^{i-1} \theta(p_{2i,j} - p_{2i-1,j}) \theta(p_{2i-1,j} - p_{2i,j+1}) \prod_{i=1}^{n} \theta(p_{2i-1,i}) \theta(p_{2i,i} - p_{2i-1,i}). \end{aligned} \tag{121}$$

*Here $\theta(x) = 1$ if $x \geq 0$ and $\theta(x) = 0$ if $x < 0$.*

**Theorem 2.7.3.** *The equivariant volume for the root system $C_n$ has the following integral representation:*

$$Z^{C_n}(\lambda_1,\dots,\lambda_n|x_1,\dots,x_n) = \int\limits_{\mathbb{R}^{n^2}} \prod_{i=1}^{n}\prod_{j=1}^{i} dp_{2i-1,j} \prod_{i=1}^{n-1}\prod_{j=1}^{i} dp_{2i,j}\, \Theta^{C_n}(p)\times \\ \times e^{x_1(p_{21}-2p_{11})+\sum\limits_{i=2}^{n} x_i\left(\sum\limits_{j=1}^{i} p_{2i,j}-2\sum\limits_{j=1}^{i} p_{2i-1,j}+\sum\limits_{j=1}^{i-1} p_{2i-2,j}\right)}, \tag{122}$$

*where $p_{2n,i} = \lambda_i$ for $i = 1,\dots,n$, and $\Theta^{C_n}(p)$ defines the domain:*

$$\Theta^{C_n}(p) = \prod_{i=1}^{n-1}\prod_{j=1}^{i} \theta(p_{2i+1,j}-p_{2i,j})\theta(p_{2i,j}-p_{2i+1,j+1})\times \\ \times \prod_{i=2}^{n}\prod_{j=1}^{i-1} \theta(p_{2i,j}-p_{2i-1,j})\theta(p_{2i-1,j}-p_{2i,j+1}) \prod_{i=1}^{n} \theta(p_{2i-1,i})\theta(p_{2i,i}-p_{2i-1,i}). \tag{123}$$

*Here $\theta(x) = 1$ if $x \geq 0$ and $\theta(x) = 0$ if $x < 0$.*

**Theorem 2.7.4.** *The equivariant volume for the root system $D_n$ has the following integral representation:*

$$Z^{D_n}(\lambda_1,\dots,\lambda_n|x_1,\dots,x_n) = \int\limits_{\mathbb{R}^{n(n-1)}} \prod_{i=1}^{n-1}\prod_{j=1}^{i} dp_{2i-1,j} \prod_{i=1}^{n-1}\prod_{j=1}^{i} dp_{2i,j}\, \Theta^{D_n}(p)\times \\ \times e^{x_1 p_{11}+\sum\limits_{i=2}^{n} x_i\left(\sum\limits_{j=1}^{i} p_{2i-1,j}-2\sum\limits_{j=1}^{i} p_{2i-2,j}+\sum\limits_{j=1}^{i-1} p_{2i-3,j}\right)}, \tag{124}$$

*where $p_{2n-1,i} = \lambda_i$ for $i = 1,\dots,n$, and $\Theta^{D_n}(p)$ defines the domain:*

$$\Theta^{D_n}(p) = \prod_{i=1}^{n-1}\prod_{j=1}^{i} \theta(p_{2i+1,j}-p_{2i,j})\theta(p_{2i,j}-p_{2i+1,j+1})\times \\ \times \prod_{i=2}^{n-1}\prod_{j=1}^{i-1} \theta(p_{2i,j}-p_{2i-1,j})\theta(p_{2i-1,j}-p_{2i,j+1}) \prod_{i=1}^{n-1} \theta(p_{2i,i}-p_{2i-1,i})\times \\ \times \prod_{i=1}^{n-1} \theta(p_{2i+1,i}+p_{2i+1,i+1}+p_{2i-1,i}-p_{2i,i})\times \\ \times \prod_{i=2}^{n-1} \theta(p_{2i-1,i-1}+p_{2i+1,i+1}+p_{2i-1,i}-p_{2i,i}). \tag{125}$$

*Here $\theta(x) = 1$ if $x \geq 0$ and $\theta(x) = 0$ if $x < 0$.*

The proofs for the cases $B_n$, $C_n$, $D_n$ are similar to $A_n$.

*Remark* 2.7.2. Note that the integral representations for $Z^{B_n}$ and $Z^{C_n}$ are essentially the same, which is not a surprise since Zhelobenko patterns and 2n-split orthogonal patterns are very similar. The only difference is that the ends of the odd rows for Zhelobenko patterns are integers, while for orthogonal patterns they are half-integers. This difference gives the factor $2^n$ in the formula for $Z^{B_n}$, because in the proof one should take $\Delta p_{2i-1,i} = \varepsilon/2$.

The variables of integration can be separated in each case. This means that the equivariant volumes of rank $n$ can be represented as integrals of the volumes of rank $n-1$, so we have recurrent relations just like we had for characters. Note that the polytopes can be represented as follows:

$$\Theta^{\Phi_n} = \prod_{k=1}^{n} \Theta^{\Phi_k}_{\Phi_{k-1}}, \tag{126}$$

where $\Theta^{\Phi_1}_{\Phi_0} := \Theta^{\Phi_1}$ and $\Phi_n$ is one of the root systems of classical type. Thus we have the following

**Corollary 2.7.3** (Recurrent formulas)**.** *For $n \geq 1$ the following relation holds*

$$\begin{aligned} Z^{A_n}(\lambda_1, \dots, \lambda_{n+1} | x_1, \dots, x_{n+1}) = \\ = \int\limits_{\mathbb{R}^n} \prod_{j=1}^{n} dp_{n,j}\, \Theta^{A_n}_{A_{n-1}}(p)\, e^{x_{n+1}\left(\sum\limits_{j=1}^{n+1} p_{n+1,j} - \sum\limits_{j=1}^{n} p_{n,j}\right)} \times \\ \times Z^{A_{n-1}}(p_{n,1}, \dots, p_{n,n} | x_1, \dots, x_n), \end{aligned} \tag{127}$$

*where*

$$\Theta^{A_n}_{A_{n-1}}(p) = \prod_{j=1}^{n} \theta(p_{n+1,j} - p_{n,j})\theta(p_{n,j} - p_{n+1,j+1}), \tag{128}$$

$$Z^{A_0}(\lambda_1 | x_1) = e^{x_1 \lambda_1}, \tag{129}$$

$$p_{n+1,i} = \lambda_i, \qquad i = 1, \dots, n+1. \tag{130}$$

*For $n \geq 2$ the following relation holds*

$$Z^{B_n}(\lambda_1,\dots,\lambda_n|x_1,\dots,x_n) = 2 \int\limits_{\mathbb{R}^{2n-1}} \prod_{j=1}^{n} dp_{2n-1,j} \prod_{j=1}^{n-1} p_{2n-2,j} \times$$
$$\times \Theta^{B_n}_{B_{n-1}}(p)\, e^{x_n\left(\sum\limits_{j=1}^{n} p_{2n,j} - 2\sum\limits_{j=1}^{n} p_{2n-1,j} + \sum\limits_{j=1}^{n-1} p_{2n-2,j}\right)} \times$$
$$\times Z^{B_{n-1}}(\lambda_1,\dots,\lambda_{n-1}|x_1,\dots,x_{n-1}), \tag{131}$$

*where*

$$\Theta^{B_n}_{B_{n-1}}(p) = \theta(p_{2n-1,n})\theta(p_{2n,n} - p_{2n-1,n}) \times$$
$$\times \prod_{j=1}^{n-1} \theta(p_{2n-1,j} - p_{2n-2,j})\theta(p_{2n-2,j} - p_{2n-1,j+1}) \times$$
$$\times \prod_{j=1}^{n-1} \theta(p_{2n,j} - p_{2n-1,j})\theta(p_{2n-1,j} - p_{2n,j+1}), \tag{132}$$

$$p_{2n,i} = \lambda_i, \qquad i = 1,\dots,n. \tag{133}$$

*For $n \geq 2$ the following relation holds*

$$Z^{C_n}(\lambda_1,\dots,\lambda_n|x_1,\dots,x_n) = \int\limits_{\mathbb{R}^{2n-1}} \prod_{j=1}^{n} dp_{2n-1,j} \prod_{j=1}^{n-1} p_{2n-2,j} \times$$
$$\times \Theta^{C_n}_{C_{n-1}}(p)\, e^{x_n\left(\sum\limits_{j=1}^{n} p_{2n,j} - 2\sum\limits_{j=1}^{n} p_{2n-1,j} + \sum\limits_{j=1}^{n-1} p_{2n-2,j}\right)} \times$$
$$\times Z^{C_{n-1}}(\lambda_1,\dots,\lambda_{n-1}|x_1,\dots,x_{n-1}), \tag{134}$$

*where*

$$\Theta^{C_n}_{C_{n-1}}(p) = \theta(p_{2n-1,n})\theta(p_{2n,n} - p_{2n-1,n}) \times$$
$$\times \prod_{j=1}^{n-1} \theta(p_{2n-1,j} - p_{2n-2,j})\theta(p_{2n-2,j} - p_{2n-1,j+1}) \times$$
$$\times \prod_{j=1}^{n-1} \theta(p_{2n,j} - p_{2n-1,j})\theta(p_{2n-1,j} - p_{2n,j+1}), \tag{135}$$

$$p_{2n,i} = \lambda_i, \qquad i = 1,\dots,n. \tag{136}$$

*For $n \geq 2$ the following relation holds*

$$Z^{D_n}(\lambda_1,\dots,\lambda_n|x_1,\dots,x_n) = \int\limits_{\mathbb{R}^{2n-2}} \prod_{j=1}^{n-1} dp_{2n-3,j}dp_{2n-2,j}\Theta^{D_n}_{D_{n-1}}(p)\times$$
$$\times e^{x_n\left(\sum\limits_{j=1}^{n} p_{2n-1,j} - 2\sum\limits_{j=1}^{n} p_{2n-2,j} + \sum\limits_{j=1}^{n-1} p_{2n-3,j}\right)}\times$$
$$\times Z^{D_{n-1}}(p_{2n-3,1},\dots,p_{2n-3,n-1}|x_1,\dots,x_{n-1}), \tag{137}$$

*where*

$$\Theta^{D_n}_{D_{n-1}}(p) = \prod_{j=1}^{n-1}\theta(p_{2n-1,j}-p_{2n-2,j})\theta(p_{2n-2,j}-p_{2n-1,j+1})\times$$
$$\times\prod_{j=1}^{n-2}\theta(p_{2n-2,j}-p_{2n-3,j})\theta(p_{2n-3,j}-p_{2n-2,j+1})\times$$
$$\times\theta(p_{2n-1,n-1}+p_{2n-1,n}+p_{2n-3,n-1}-p_{2n-2,n-1})\times$$
$$\times\theta(p_{2n-3,n-2}+p_{2n-1,n}+p_{2n-3,n-1}-p_{2n-2,n-1})\times$$
$$\times\prod_{i=1}^{n-1}\theta(p_{2n-2,n-1}-p_{2n-3,n-1}), \tag{138}$$
$$p_{2n-1,i} = \lambda_i, \qquad i = 1,\dots,n. \tag{139}$$

## 2.8 GELFAND-TSETLIN POLYTOPES

The characteristic functions $\Theta^{\Phi_n}(p)$ in the previous section define a finite region of integration in each case by restricting the values with inequalities. They can also be used in the character formula to define the summation region, so the character of $GL(n+1)$ would look as follows:

$$S^{A_n}_{\lambda}(x_1,\dots,x_{n+1}) = \sum_{\substack{p_{i,j}\in\mathbb{Z}\\ 1\leq j\leq i<n+1}} x_1^{p_1}\cdots x_{n+1}^{p_{n+1}}\Theta^{A_n}(p), \tag{140}$$

where

$$p_{n+1,i} = \lambda_i, \qquad i = 1,\dots,n+1, \tag{141}$$
$$p_i = \sum_{j=1}^{i} p_{i,j} - \sum_{j=1}^{i-1} p_{i-1,j}, \qquad i = 2,\dots,n+1, \tag{142}$$
$$p_1 = p_{1,1}. \tag{143}$$

For the group $Sp(2n)$ one has

$$S_\lambda^{C_n}(x_1,\dots,x_n) = \sum_{p_{2i-1,j},p_{2i,j}\in\mathbb{Z}} \Theta^{C_n}(p) x_1^{p_1}\cdots x_n^{p_n}, \tag{144}$$

where

$$p_{2n,i} = \lambda_i, \qquad i = 1,\dots,n, \tag{145}$$

$$p_i = \sum_{j=1}^{i} p_{2i,j} - 2\sum_{j=1}^{i} p_{2i-1,j} + \sum_{j=1}^{i-1} p_{2i-2,j}, \qquad i = 2,\dots,n, \tag{146}$$

$$p_1 = p_{2,1} - 2p_{1,1}; \tag{147}$$

For the group $Spin(2n+1)$ and $\lambda_i \in \mathbb{Z}$ one has

$$S_\lambda^{B_n}(x_1,\dots,x_n) = \sum_{\substack{p_{2i-1,j},p_{2i,j}\in\mathbb{Z}\\ p_{2i-1,i}\in\mathbb{Z}/2}} \Theta^{B_n}(p) x_1^{p_1}\cdots x_n^{p_n}, \tag{148}$$

where

$$p_{2n,i} = \lambda_i, \qquad i = 1,\dots,n, \tag{149}$$

$$p_i = \sum_{j=1}^{i} p_{2i,j} - 2\sum_{j=1}^{i} p_{2i-1,j} + \sum_{j=1}^{i-1} p_{2i-2,j}, \qquad i = 2,\dots,n, \tag{150}$$

$$p_1 = p_{2,1} - 2p_{1,1}. \tag{151}$$

If for the weight $\lambda_i \in \mathbb{Z} + \frac{1}{2}$ then

$$S_\lambda^{B_n}(x_1,\dots,x_n) = \sum_{\substack{p_{2i-1,j},p_{2i,j}\in\mathbb{Z}+\frac{1}{2}\\ p_{2i-1,i}\in\mathbb{Z}/2}} \Theta^{B_n}(p) x_1^{p_1}\cdots x_n^{p_n}, \tag{152}$$

where

$$p_{2n,i} = \lambda_i, \qquad i = 1,\dots,n, \tag{153}$$

$$p_i = \sum_{j=1}^{i} p_{2i,j} - 2\sum_{j=1}^{i} p_{2i-1,j} + \sum_{j=1}^{i-1} p_{2i-2,j}, \qquad i = 2,\dots,n, \tag{154}$$

$$p_1 = p_{2,1} - 2p_{1,1}. \tag{155}$$

For the group $Spin(2n)$ with $\lambda_i \in \mathbb{Z}$ one has

$$S_\lambda^{D_n}(x_1,\dots,x_n) = \sum_{p_{2i-1,j},p_{2i,j}\in\mathbb{Z}} \Theta^{D_n}(p) x_1^{p_1}\cdots x_n^{p_n}, \tag{156}$$

where

$$p_{2n-1,i} = \lambda_i, \qquad i = 1, \dots, n, \tag{157}$$

$$p_i = \sum_{j=1}^{i} p_{2i-1,j} - 2\sum_{j=1}^{i-1} p_{2i-2,j} + \sum_{j=1}^{i-1} p_{2i-3,j}, i = 2, \dots, n, \tag{158}$$

$$p_1 = p_{1,1}. \tag{159}$$

If for the highest weight $\lambda_i \in \mathbb{Z} + \frac{1}{2}$ then

$$S_\lambda^{D_n}(x_1, \dots, x_n) = \sum_{p_{2i-1,j}, p_{2i,j} \in \mathbb{Z}+\frac{1}{2}} \Theta^{D_n}(p) x_1^{p_1} \cdots x_n^{p_n}, \tag{160}$$

where

$$p_{2n-1,i} = \lambda_i, \qquad i = 1, \dots, n, \tag{161}$$

$$p_i = \sum_{j=1}^{i} p_{2i-1,j} - 2\sum_{j=1}^{i-1} p_{2i-2,j} + \sum_{j=1}^{i-1} p_{2i-3,j}, i = 2, \dots, n, \tag{162}$$

$$p_1 = p_{1,1}. \tag{163}$$

The functions $\Theta^{\Phi_n}(p)$ restrict the summation to the finite number of points on the polytopes in $\mathbb{R}^{|\Phi_n^+|}$, defined by the respective inequalities. These polytopes provide a basic structure for the functions on maximal torus and have the following properties:

RECURRENT STRUCTURE. For each characteristic function one can write

$$\Theta^{\Phi_n}(p) = \Theta^{\Phi_n}_{\Phi_{n-1}}(p)\Theta^{\Phi_{n-1}}(p). \tag{164}$$

Here $\Theta^{\Phi_n}_{\Phi_{n-1}}(p)$ is defined by the inequalities between the first two rows in case of $A_n$ and the first three rows in other cases. It follows that the functions $S^{\Phi_n}$ and $Z^{\Phi_n}$ have recurrent formulas described in previous sections.

INVARIANCE UNDER THE LIMIT. The limit (2.7.1) provides connection between functions on torus. Both are defined by the same polytope, and the limit preserves it up to scaling.

SPECIALIZATION TO IDENTITY. Consider an element $(e^{x_1}, \ldots, e^{x_n})$ of maximal torus $H \subset G$. One can take $x_i \to 0$, in which case the character gives

$$S_\lambda^{\Phi_n}(1, \ldots, 1) = \sum_p \Theta^{\Phi_n}(p). \tag{165}$$

On the left-hand side we have the dimension of the space of representation $V_\lambda$. On the right-hand side one has the total number of points satisfying $\Theta^{\Phi_n}(p)$ and belonging to the polytope, which is also the total number of patterns for the weight $\lambda$:

$$\dim V_\lambda = \sum_p \Theta^{\Phi_n}(p) = |\mathcal{P}_\lambda^{\Phi_n}|. \tag{166}$$

Now take $x_i \to 0$ in the integral formula for the equivariant volume. It gives

$$Z^{\Phi_n}(\lambda|0) = \int\limits_{\mathbb{R}^{|\Phi_n^+|}} \prod_{i,k} dp_{i,k}\, \Theta^{\Phi_n}(p). \tag{167}$$

So $Z^{\Phi_n}(\lambda|0)$ is actually a volume of the polytope with the characteristic function $\Theta^{\Phi_n}(p)$.

# 3

# NON-ARCHIMEDEAN WHITTAKER FUNCTIONS

In this chapter we review $G(\mathbb{Q}_p)$-Whittaker functions and their connection to characters of Langlands dual groups following [7, 12, 40].

## 3.1 WHITTAKER MODELS

Consider the field of $p$-adic rational numbers $\mathbb{Q}_p$. Throughout the chapter let

- $G$ be an algebraic connected reductive group over $\mathbb{Q}_p$;
- $B$ be a minimal parabolic subgroup of $G$;
- $K = G(\mathbb{Z}_p)$ be a maximal compact subgroup in $G$;
- $J$ be an Iwahori subgroup – the inverse image of $B(\mathbb{Z}_p/p\mathbb{Z}_p)$ in $K$;
- $T$ be a maximal split torus in $G$, so $T \subset B \subset G$;
- $X(T)$ be a group of characters of $T$;
- $\Phi_n \subset X(T)$ be the root system of $G$ with respect to torus $T$;
- $N = [B, B]$ be the unipotent radical in $B$ corresponding to the subset of positive roots $\Phi_n^+ \subset \Phi_n$;
- $N_\alpha$ be a subgroup in $N$ generated by $\alpha \in \Phi_n$;

- $P_\Delta$ be the parabolic subgroup corresponding to the subset of simple roots $\Delta$ in $\Phi_n$;
- $\delta$ be a modulus character of $B$ such that for every element $b = mn$, $m \in M$, $n \in N$, it is defined as $|\det \mathrm{Ad}_{\mathfrak{n}}(m)|$, where $\mathfrak{n}$ is a Lie algebra of $N$ and $\mathrm{Ad}_{\mathfrak{n}}(m)$ is an adjoint action of $m$ on $\mathfrak{n}$;
- $C_c^\infty(G)$ be the space of locally constant functions with compact support on $G$.

**Definition 3.1.1.** For each simple root $\alpha_i$ define a smooth complex character $\psi_i$ of $N_{\alpha_i}$, which is not trivial for all $i$. Then the product

$$\psi = \prod_{i=1}^{n} \psi_i \tag{168}$$

defines a character of $N$. Such character is called principal.

For each simple root $\alpha_i$ there is a subgroup $G_{\alpha_i}$ isomorphic to $SL_2$, and $N_{\alpha_i}$ is its unipotent subgroup generated by $\alpha_i$. Then an element of $N_{\alpha_i}$ can be represented as a corresponding unipotent matrix in $SL_2$:

$$\begin{pmatrix} 1 & x_i \\ & 1 \end{pmatrix} \in N_{\alpha_i}, \qquad x_i \in \mathbb{Q}_p. \tag{169}$$

**Definition 3.1.2.** If $\psi_i$ are trivial for $x_i \in \mathbb{Z}_p$ but not trivial for $x_i \in p^{-1}\mathbb{Z}_p$ then the charater $\psi$ is called unramified.

**Definition 3.1.3.** Let $(\rho, V)$ be a smooth representation of $N$. Define the quotient space

$$V_{\psi,N} = V/\{\rho(n)v - \psi(n)v \,|\, n \in N,\, v \in V\}. \tag{170}$$

Then $N$ acts on $V_{\psi,N}$ by $\psi$. The space $V_{\psi,N}$ is called a Jacquet space of the twisted representation $\pi \otimes \psi^{-1}$.

Let $\mathrm{Ind}_N^G(\psi)$ be the induced representation of this character, so that $\mathrm{Ind}_N^G(\psi)$ is a space of functions $f\colon G \to \mathbb{C}$ such that

$$f(ng) = \psi(n)f(g), \qquad \forall n \in N,\, g \in G, \tag{171}$$

and there exists an open subgroup $K$ in $G$ such that

$$f(gk) = f(g), \qquad \forall k \in K,\, g \in G. \tag{172}$$

**Definition 3.1.4.** Let $(\pi, V)$ be an admissible representation of $G$. Then the $G$-embedding of $V$ into $\mathrm{Ind}_N^G(\psi)$ is called a Whittaker model of $V$.

**Proposition 3.1.1** (see [12]). *By Frobenius reciprocity there is an isomorphism:*

$$\mathrm{Hom}_G(V, \mathrm{Ind}_N^G(\psi)) \simeq \mathrm{Hom}_{\mathbb{C}}(V_{\psi,N}, \mathbb{C}). \tag{173}$$

The isomorphism is induced by the map $Q\colon \mathrm{Ind}_N^G(\psi) \to \mathbb{C}$ such that

$$Q\colon f \mapsto f(1). \tag{174}$$

## 3.2 SHINTANI-CASSELMAN-SHALIKA FORMULA

**Definition 3.2.1.** Let $\chi$ be a complex character on a maximal torus $T$. It is called unramified if it is trivial on $T \cap K$.

**Definition 3.2.2.** One can extend the character on a torus $\chi$ to a complex character on $B$. The induced representation $\mathrm{Ind}_B^G(\chi)$ is called a principal series representation.

If $\delta$ is a modulus character on $B$ then we have

$$f(bg) = \chi(b)\delta^{1/2}(b)f(g), \qquad f \in \mathrm{Ind}_B^G(\chi). \tag{175}$$

For an unramified character $\chi$ consider the induced representation $\mathrm{Ind}_B^G(\chi)$. Let $\mathcal{P}_\chi$ be a projection from $C_c^\infty(G)$ to $\mathrm{Ind}_B^G(\chi)$ defined by

$$\mathcal{P}_\chi f(g) = \int_B \chi^{-1}(b)\delta^{1/2}(b)f(bg)db, \tag{176}$$

where the measure is a left Haar measure on $B$. Let $\psi$ be an unramified principal character on $N$. Then Proposition (3.1.1) tells us that there is a map $\mathrm{Ind}_B^G(\chi) \to V_{\psi,N} \to \mathbb{C}$ defined by the following

**Proposition 3.2.1** (see [12]). *Let $w_0$ be the element of maximal length in $W$ and $x$ be its representative in $N_G(T)$. The functional*

$$Q_\chi(\mathcal{P}_\chi f) = \int_N \mathcal{P}_\chi f(xn)\psi^{-1}(n)\,dn \tag{177}$$

*is defined for all $f \in C_c^\infty(Bw_0B)$ and is a unique $N$-morphism from $\mathrm{Ind}_B^G(\chi)$ to $\mathbb{C}$ up to multiplication by a scalar.*

**Proposition 3.2.2** (see [12]). *The function $Q_\chi(\mathcal{P}_\chi f)$ is holomorphic with respect to $\chi$ for all $f \in C_c^\infty(G)$.*

Let $(\mathrm{Ind}_B^G(\chi))^K$ be a subspace of $K$-fixed vectors. It is spanned by the characteristic function $\phi_K = \mathcal{P}_\chi(\mathrm{ch}\,K)$ of the compact subgroup $K$. Due to proposition (3.1.1) we have the $G$-embedding $F\colon \mathrm{Ind}_B^G(\chi) \to \mathrm{Ind}_N^G(\psi)$ corresponding to the functional $Q_\chi$ such that

$$F(f)(g) = Q_\chi(R_g\mathcal{P}_\chi f), \qquad f \in \mathrm{Ind}_B^G(\chi), \tag{178}$$

where $R_g$ is the right action of the group $G$.

**Definition 3.2.3.** The unramified Whittaker function corresponding to the character $\chi$ is the image of $\phi_K$ under the embedding map

$$W_\chi(g) = Q_\chi(R_g\phi_K), \tag{179}$$

which satisfies the condition

$$W_\chi(ngk) = \psi(n)W_\chi(g), \qquad \forall n \in N,\, k \in K,\, g \in G. \tag{180}$$

**Proposition 3.2.3** (see [7, 12, 43]).

1. *The function $W_\chi$ is defined by its values on $T$.*
2. *If for some simple root $\alpha$ we have $\|\alpha(t)\|_p > 1$ then $W_\chi(t) = 0$ for $t \in T$.*

*Proof.* The first statement follows from the fact that the group $G$ has the decomposition $G = NTK$.

For the second statement write

$$W_\chi(t) = W_\chi(tn) = W_\chi(tnt \cdot t^{-1}), \qquad n \in N \cap K. \tag{181}$$

Since $\|\alpha(t)\|_p > 1$ for some simple root $\alpha$, then we have $\psi_\alpha(tnt^{-1}) \neq 1$ for some $n$ in $N_\alpha \cap K$, so $W_\chi(t) = 0$. □

Let $\Phi_n^\vee$ be the dual root system and $G^\vee$ and $T^\vee \subset G^\vee$ be the Langlands dual complex group and its maximal torus. Let $\Lambda_W$ be the weight lattice of $G^\vee$. Then we have the isomorphism

$$\Lambda_W \simeq T/(T \cap K). \tag{182}$$

Since the torus $T$ is split, it is isomorphic to $(\mathbb{Q}_p^\times)^n$. So for each weight $\lambda$ we have the representative of coset $T/(T \cap K)$ in torus:

$$p^\lambda = \begin{pmatrix} p^{\lambda_1} & & \\ & \ddots & \\ & & p^{\lambda_n} \end{pmatrix}. \tag{183}$$

Due to proposition (3.2.3) the Whittaker function $W_\chi$ is defined by its values on $T/(T \cap K)$. Then property 2 in proposition (3.2.3) means that $W_\chi(p^\lambda) = 0$ if $\lambda$ is not dominant.

**Theorem 3.2.1** (Shintani-Casselman-Shalaika formula, see [12])**.** *The following relation holds*

$$W_\chi(p^\lambda) = \zeta(\chi) \cdot \delta^{1/2}(p^\lambda) \cdot \frac{\sum\limits_{w \in W} (-1)^w \cdot \prod\limits_{\substack{\alpha \in \Phi_n^+ \\ w\alpha \in \Phi_n^-}} \chi(p^{\alpha^\vee})^{-1} \cdot w\chi(p^\lambda)}{\prod\limits_{\alpha \in \Phi_n^+} 1 - \chi(p^{\alpha^\vee})^{-1}}. \tag{184}$$

*Here*

$$W_\chi(1) = \zeta(\chi) = \prod_{\alpha \in \Phi_n^+} (1 - p^{-1}\chi(p^{\alpha^\vee})). \tag{185}$$

Due to duality one may put $\chi(p^\lambda) = e^{\langle x, \lambda \rangle} \in T^\vee(\mathbb{C})$. So on the right hand side we have the Weyl character formula (591).

*Remark* 3.2.1. Taking the formal limit $p \to 1$ in this formula is equivalent to the limit in proposition (2.7.1).

As an example, let us look at the case $G = GL(n)$. The formula for $GL(n)$ was originally proved by Shintani in [43] for a bit different normalization:

**Theorem 3.2.2** (Shintani). *For $G = GL(n)$ the following relation holds*

$$W_\chi(p^\lambda) = p^{\sum_{i=1}^{n}(i-n)\lambda_i} \cdot \frac{\det ||x_j^{\lambda_i+n-i}||}{\prod_{i<j}(x_i - x_j)}, \qquad \lambda_1 \geq \cdots \geq \lambda_n. \tag{186}$$

*For $\lambda_1 = \cdots = \lambda_n = 0$ we have $W_\chi(p^\lambda) = 1$.*

## 3.3 CASSELMAN BASIS

Let $J$ be an Iwahori subgroup, defined as inverse image of $B(\mathbb{Z}_p/p\mathbb{Z}_p)$ in $G(\mathbb{Z}_p)$. Let $I(\chi) = \mathrm{Ind}_B^G(\chi)$ and let $I(\chi)^J$ be the subspace of Iwahori-fixed vectors. The dimension of this space is $\dim I(\chi)^J = |\mathcal{W}^{\Phi_n}|$. For the group $G$ we have the decomposition (see [10, 11])

$$G = \bigsqcup_{w \in \mathcal{W}^{\Phi_n}} BwJ, \tag{187}$$

where on the right-hand side there is a disjoint union of double cosets. Thus there is a standard basis of characteristic functions $\phi_w = \mathcal{P}_\chi(\mathrm{ch}\, JwJ)$ in $I(\chi)^J$. More explicitly

$$\phi_w(bw'x) = \begin{cases} \delta^{1/2}(b)\chi(b), & w = w' \\ 0, & w \neq w', \end{cases} \tag{188}$$

where $w, w' \in \mathcal{W}^{\Phi_n}$, $b \in B$, $x \in J$. For each $w \in \mathcal{W}^{\Phi_n}$ there is a unique intertwining operator $\mathcal{T}_w \colon I(\chi) \to I(w\chi)$ such that

$$\mathcal{T}_w f(1) = \int_{(wNw^{-1} \cap N)\backslash N} f(x^{-1}n)dn, \tag{189}$$

where $x \in K$ is a representative of $w$. This operator takes $I(\chi)^J$ to $I(w\chi)^J$ and $I(\chi)^K$ to $I(w\chi)^w$. There are simple formulas for the action of the intertwining operator on the bases. Define rational functions $c_\alpha(\chi)$ as follows for each root:

$$c_\alpha(\chi) = \frac{1 - p^{-1}\chi(p^{\alpha^\vee})}{1 - \chi(p^{\alpha^\vee})}. \tag{190}$$

For each $w \in \mathcal{W}^{\Phi_n}$ put

$$c_w(\chi) = \prod_{\substack{\alpha \in \Phi_n^+ \\ w\alpha \in \Phi_n^-}} c_\alpha(\chi). \tag{191}$$

**Theorem 3.3.1** (Casselman, see [11]). *The following relation holds*

$$\mathcal{T}_w(\phi_K) = c_w(\chi)\phi_K. \tag{192}$$

For the standard basis there is

**Theorem 3.3.2** (Casselman, see [11]). *For a simple root $\alpha_i \in \Phi_n$ and $w$ such that $\ell(w_i w) > \ell(w)$ we have*

$$\mathcal{T}_{w_i}(\phi_w) = (c_\alpha(\chi) - 1)\phi_w + p_{-1}\phi_{w_i w}, \tag{193}$$

$$\mathcal{T}_{w_i}(\phi_{w_i w}) = \phi_w + (c_\alpha(\chi) - p^{-1})\phi_{w_i w}. \tag{194}$$

Let $L_w$ be functionals on $I(\chi)^J$ defined as follows:

$$L_w(\phi) = T_w(\phi)(1). \tag{195}$$

These functionals form a basis in the dual space of $I(\chi)^J$.

**Definition 3.3.1.** The basis $f_w$ in $I(\chi)^J$ dual to $L_w$, in other words,

$$L_w(f_{w'}) = 1, \qquad w = w', \tag{196}$$

$$L_w(f_{w'}) = 0, \qquad w \neq w', \tag{197}$$

is called Casselman basis of Iwahori-fixed vectors.

**Proposition 3.3.1** (Casselman, see [11]). *The following relation holds*

$$\phi_K = \sum_{w \in \mathcal{W}^{\Phi_n}} c_w(\chi) f_w. \tag{198}$$

Reeder in [42] provided an algorithm which recursively calculates the coefficients of the Casselmann basis in the standard one.

**Theorem 3.3.3** (Reeder). *We have*

$$f_w = \sum_{s \in \mathcal{W}^{\Phi_n}} a_{s,w}(\chi)\phi_w. \tag{199}$$

*The coefficients $a_{s,w}$ are defined by the following conditions:*

$$a_{w,w} = 1, \tag{200}$$

$$a_{s,w} = 0, \qquad \ell(s) \leq \ell(w), \tag{201}$$

$$a_{s,s_\alpha w} + (c_\alpha - 1)a_{s,w} = c_\alpha c_{-\alpha} s_\alpha a_{s_\alpha s,w}, \qquad \ell(s_\alpha s) > \ell(s), \tag{202}$$

$$a_{s,s_\alpha w} + (c_\alpha - 1)a_{s,w} = s_\alpha a_{s_\alpha s,w}, \qquad \ell(s_\alpha s) < \ell(s), \tag{203}$$

*where $\alpha \in \Phi_n$.*

Let $W^{Cass}_{\chi,w}$ be the images of $f_w$ under the embedding map $I(\chi) \to \mathrm{Ind}_N^G(\psi)$, so that

$$W^{Cass}_{\chi,w}(g) = Q_\chi(R_g f_w). \tag{204}$$

**Proposition 3.3.2** (Reeder, see [40]). *The following relation holds*

$$W^{Cass}_{\chi,w}(p^\lambda) = \delta^{1/2}(p^\lambda) \prod_{\substack{\alpha\in\Phi_n^+ \\ w\alpha\in\Phi_n^+}} \frac{1-p^{-1}\chi(p^{\alpha^\vee})}{1-\chi(p^{\alpha^\vee})^{-1}} \cdot w\chi(p^\lambda). \tag{205}$$

Let us introduce one more basis in $I(\chi)^J$. Take the following vectors:

$$\tilde{\phi}_w = \sum_{\substack{s\in W^{\Phi_n} \\ \ell(s)\geq\ell(w)}} \phi_s. \tag{206}$$

Explicitly

$$\tilde{\phi}_w(bw'x) = \begin{cases} \delta^{1/2}(b)\chi(b), & \ell(w') \geq \ell(w) \\ 0, & \ell(w') < \ell(w). \end{cases} \tag{207}$$

Note that for $w = 1$ we have $\tilde{\phi}_w = \phi_K$. Take the image of these functions under the embedding map $I(\chi) \to \mathrm{Ind}_N^G(psi)$:

$$W_{\chi,w}(g) = Q_\chi(R_g\tilde{\phi}_w). \tag{208}$$

**Theorem 3.3.4** (see [7]). *For a dominant weight $\lambda$ the following relation holds*

$$W_{\chi,w}(p^\lambda) = \delta^{1/2}(p^\lambda)P_{\lambda,w}(e^x, p^{-1}). \tag{209}$$

*Here*

- $e^x = \mathrm{diag}(e^{x_1}, \dots, e^{x_n}) \in T^\vee$ *corresponding to* $\chi$*;*
- $P_{\lambda,w}$ *is a polynomial in* $p^{-1}$ *and a rational function of* $e^x$*;*
- $P_{\lambda,w}(e^x, 0) = \mathcal{D}'_w e^{\langle wx,\lambda\rangle}$ *where*

$$\mathcal{D}'_{\alpha_i^\vee} f(e^x) = \frac{f(e^x) - e^{\langle x,\alpha_i^\vee\rangle} f(e^{wx})}{1 - e^{\langle x,\alpha_i^\vee\rangle}}, \tag{210}$$

  *and* $\mathcal{D}'_w = \mathcal{D}'_{\alpha^\vee_{i_1}} \cdot \mathcal{D}'_{\alpha^\vee_{i_k}}$ *if* $w = s_1 \dots s_k$*;*
- *for the element of maximal length* $w_0$ *we have*

$$W_{\chi,w_0}(p^\lambda) = \delta^{1/2}(p^\lambda)e^{\langle w_0 x,\lambda\rangle}. \tag{211}$$

Note that the operators (210) are Demazure operators (600) for the roots $-\alpha_i$.

# 4

# ARCHIMEDEAN WHITTAKER FUNCTIONS

In this chapter we prove that the limit of classical Whittaker functions are equivariant volumes of flag manifolds. In the process we see how the Givental and GLO graphs (see [19, 24]) and Gelfand-Tsetlin polytopes are related to each case.

## 4.1 $\mathfrak{gl}_{n+1}$-WHITTAKER FUNCTION

**Theorem 4.1.1** (see [19]). *The $\mathfrak{gl}_{n+1}$-Whittaker function has the following integral representation*

$$\Psi^{A_n}_{x_1,\dots,x_{n+1}}(\lambda_1,\dots,\lambda_{n+1}) = \int_{\mathcal{C}} \prod_{i=1}^{n}\prod_{k=1}^{i} d\lambda_{i,k}\, e^{\mathcal{F}(x,\lambda)}, \tag{212}$$

*where*

$$\begin{aligned}\mathcal{F}(x,\lambda) = \imath \sum_{i=1}^{n+1} x_i \left(\sum_{k=1}^{i}\lambda_{i,k} - \sum_{k=1}^{i-1}\lambda_{i-1,k}\right) - \\ - \sum_{i=1}^{n}\sum_{k=1}^{i}\left(e^{\lambda_{i,k}-\lambda_{i+1,k}} + e^{\lambda_{i+1,k+1}-\lambda_{i,k}}\right),\end{aligned} \tag{213}$$

$$\lambda_{n+1,i} = \lambda_i, \qquad i = 1,\dots,n+1 \tag{214}$$

*and the domain $\mathcal{C}$ is a slight deformation of $\mathbb{R}^{(n+1)n/2}$ providing the convergence of the integral.*

The part

$$e^{-\sum_{i=1}^{n}\sum_{k=1}^{i}\left(e^{\lambda_{i,k}-\lambda_{i+1,k}}+e^{\lambda_{i+1,k+1}-\lambda_{i,k}}\right)} \tag{215}$$

visually can be represented as the following Givental diagram

$$\begin{array}{ccccccccc}
\lambda_{n+1,1} & & & & & & & & \\
\downarrow & & & & & & & & \\
\lambda_{n,1} & \longrightarrow & \lambda_{n+1,2} & & & & & & \\
\downarrow & & \downarrow & & & & & & \\
\cdots & & \cdots & & \cdots & & & & \\
\downarrow & & \downarrow & & & & & & \\
\lambda_{2,1} & \longrightarrow & \lambda_{n+1,2} & \longrightarrow & \cdots & \longrightarrow & \lambda_{n+1,n} & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\lambda_{1,1} & \longrightarrow & \lambda_{2,2} & \longrightarrow & \cdots & \longrightarrow & \lambda_{n,n} & \longrightarrow & \lambda_{n+1,n+1}
\end{array} \tag{216}$$

where each arrow $x \longrightarrow y$ corresponds to the factor $e^{-e^{y-x}}$ and the result is the product of these factors.

**Proposition 4.1.1** (see [21]). *Let $\lambda_1 > \cdots > \lambda_{n+1}$. The following relation holds*

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \varepsilon^{n(n+1)/2} \cdot \Psi^{A_n}_{\varepsilon x_1,\dots,\varepsilon x_{n+1}}(\varepsilon^{-1}\lambda_1,\dots,\varepsilon^{-1}\lambda_{n+1}) = \\
&\qquad = Z^{A_n}(\lambda_1,\dots,\lambda_n|\imath x_1,\dots,\imath x_n).
\end{aligned} \tag{217}$$

*Proof.* Note that

$$\lim_{\varepsilon\to 0} e^{-e^{-\varepsilon^{-1}x}} = \theta(x). \tag{218}$$

Then we have

$$e^{-\sum\limits_{i=1}^{n}\sum\limits_{k=1}^{i}\left(e^{\lambda_{i,k}-\lambda_{i+1,k}}+e^{\lambda_{i+1,k+1}-\lambda_{i,k}}\right)} \to \prod_{i=1}^{n}\prod_{k=1}^{i}\theta(\lambda_{i+1,k}-\lambda_{i,k})\theta(\lambda_{i,k}-\lambda_{i+1,k+1}). \tag{219}$$

The function on the right-hand side is $\Theta^{A_n}(\lambda)$, so the limit takes the following form:

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \varepsilon^{n(n+1)/2} \cdot \Psi^{A_n}_{\varepsilon x_1,\dots,\varepsilon x_{n+1}}(\varepsilon^{-1}\lambda_1,\dots,\varepsilon^{-1}\lambda_{n+1}) = \\
&= \int\limits_{\mathbb{R}^{(n+1)n/2}} \prod_{i=1}^{n}\prod_{k=1}^{i} d\lambda_{i,k}\, \Theta^{A_n}(\lambda) \cdot e^{\imath\sum\limits_{i=1}^{n+1} x_i\left(\sum\limits_{k=1}^{i}\lambda_{i,k}-\sum\limits_{k=1}^{i-1}\lambda_{i-1,k}\right)}.
\end{aligned} \tag{220}$$

We got the integral representation (113) of $Z^{A_n}(\lambda_1,\dots,\lambda_n|\imath x_1,\dots,\imath x_n)$. □

## 4.2 $\mathfrak{so}_{2n+1}$-WHITTAKER FUNCTION

**Theorem 4.2.1** (see [19]). *$\mathfrak{so}_{2n+1}$-Whittaker function has the following integral representation*

$$\Psi^{B_n}_{x_1,\dots,x_n}(\lambda_{n,1},\dots,\lambda_{n,n}) = \int\limits_{\mathcal{C}} \prod_{i=1}^{n-1}\prod_{k=1}^{i} d\lambda_{i,k} \prod_{i=1}^{n}\prod_{k=1}^{i} d\mu_{i,k}\, e^{\mathcal{F}(\lambda,\mu,x)}, \tag{221}$$

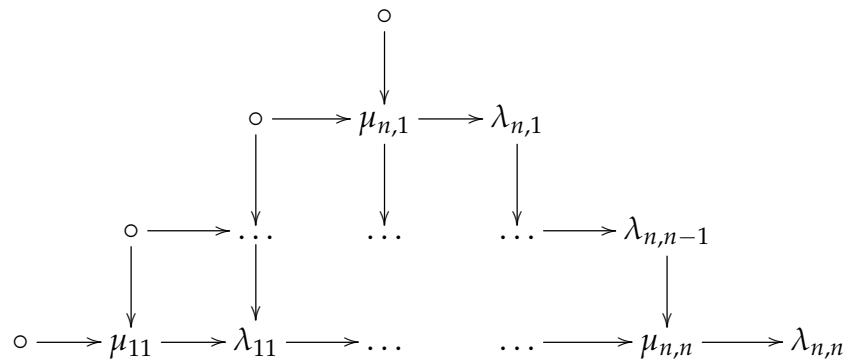
*where*

$$\begin{aligned}\mathcal{F}(\lambda,\mu,x) = -\imath x_1(2\mu_{11}-\lambda_{11}) - \\ -\imath\sum_{k=2}^{n} x_k\left(\sum_{i=1}^{k}\lambda_{k,i} + 2\mu_{k,1} - 2\sum_{i=2}^{k}\mu_{k,i} + \sum_{i=1}^{k-1}\lambda_{k-1,i} - 2\ln(e^{\lambda_{k,1}}+e^{\lambda_{k-1,1}})\right) - \\ -\sum_{k=1}^{n} e^{\mu_{k,1}} - \sum_{i=1}^{n-1}\sum_{k=i+1}^{n}(e^{\lambda_{k-1,i}-\mu_{k,i}} + e^{\lambda_{k,i}-\mu_{k,i}}) - \\ -\sum_{i=2}^{n}\sum_{k=i}^{n}(e^{\mu_{k,i}-\lambda_{k-1,i-1}} + e^{\mu_{k,i}-\lambda_{k,i-1}}) - \sum_{k=1}^{n} e^{\lambda_{k,k}-\mu_{k,k}},\end{aligned} \tag{222}$$

*and the domain $\mathcal{C}$ is a slight deformation of $\mathbb{R}^{n^2}$ providing the convergence of the integral.*

Like for $\mathfrak{gl}_{n+1}$-Whittaker function, here we can introduce the Givental diagram

(223)




where each $x \longrightarrow y$ corresponds to the factor $e^{-e^{y-x}}$ and each $\circ \longrightarrow x$ corresponds to $e^{-\frac{1}{2}e^{x}}$.

**Theorem 4.2.2.** *Let $\lambda_1 > \dots > \lambda_n > 0$. The following relation holds*

$$\begin{aligned}\lim_{\varepsilon\to 0}(\varepsilon)^{n^2}\Psi^{B_n}_{\varepsilon x_1,\dots,\varepsilon x_n}(-\varepsilon^{-1}\lambda_n,\dots,-\varepsilon^{-1}\lambda_1) = \\ = Z^{C_n}(\lambda_1,\dots,\lambda_n|\imath x_1,\dots,\imath x_n).\end{aligned} \tag{224}$$

*Proof.* Take the function on the left-hand side:

$$(\varepsilon)^{n^2}\cdot\Psi^{B_n}_{\varepsilon x_1,\dots,\varepsilon x_n}(-\varepsilon^{-1}\lambda_{n,1},\dots,-\varepsilon^{-1}\lambda_{n,n})=$$
$$=\int\limits_{\mathcal{C}}\prod_{i=1}^{n-1}\prod_{k=1}^{i}d\lambda_{i,k}\prod_{i=1}^{n}\prod_{k=1}^{i}d\mu_{i,k}\,e^{\mathcal{F}(-\varepsilon^{-1}\lambda,-\varepsilon^{-1}\mu,\varepsilon x)}. \tag{225}$$

After taking the limit using (218) for the exponents we have

$$e^{\mathcal{F}(-\varepsilon^{-1}\lambda,-\varepsilon^{-1}\mu,\varepsilon x)}\to\Theta(\lambda,\mu)\cdot\exp\{\imath x_1(2\mu_{11}-\lambda_{11})+$$
$$+\imath\sum_{k=2}^{n}x_k\left(\sum_{i=1}^{k}\lambda_{k,i}+2\mu_{k,1}-2\sum_{i=2}^{k}\mu_{k,i}+\sum_{i=1}^{k-1}\lambda_{k-1,i}-2\min(\lambda_{k,1},\lambda_{k-1,1})\right)\}, \tag{226}$$

where

$$\Theta(\lambda,\mu)=\prod_{k=1}^{n}\theta(\mu_{k,1})\prod_{i=1}^{n-1}\prod_{k=i+1}^{n}\theta(\lambda_{k-1,i}-\mu_{k,i})\theta(\lambda_{k,i}-\mu_{k,i})\times$$
$$\times\prod_{i=2}^{n}\prod_{k=i}^{n}\theta(\mu_{k,i}-\lambda_{k-1,i-1})\theta(\mu_{k,i}-\lambda_{k,i-1})\prod_{k=1}^{n}\theta(\lambda_{k,k}-\mu_{k,k}). \tag{227}$$

The function $\Theta(\lambda,\mu)$ defines the domain such that, in particular,

$$\min(\lambda_{k,1},\lambda_{k-1,1})\geq\mu_{k,1}\geq 0,\qquad k=2,\dots,n, \tag{228}$$
$$\lambda_{11}\geq\mu_{11}\geq 0. \tag{229}$$

So we can change the variables the following way:

$$\mu_{k,1}\to\min(\lambda_{k,1},\lambda_{k-1,1})-\mu_{k,1},\qquad k=2,\dots,n, \tag{230}$$
$$\mu_{11}\to\lambda_{11}-\mu_{11}.$$

After the changing the domain, and so the function $\Theta(\lambda,\mu)$, remains the same. So in the end we have

$$\lim_{\varepsilon\to 0}(\varepsilon)^{n^2}\Psi^{B_n}_{\varepsilon x_1,\dots,\varepsilon x_n}(-\varepsilon^{-1}\lambda_{n,1},\dots,-\varepsilon^{-1}\lambda_{n,n})=$$
$$=\int_{\mathbb{R}^{n^2}}\prod_{i=1}^{n-1}\prod_{k=1}^{i}d\lambda_{i,k}\prod_{i=1}^{n}\prod_{k=1}^{i}d\mu_{i,k}\,\Theta(\lambda,\mu)\times$$
$$\times e^{\imath x_1(\lambda_{11}-2\mu_{11})+\imath\sum\limits_{k=2}^{n}x_k\left(\sum\limits_{i=1}^{k}\lambda_{k,i}-2\sum\limits_{i=1}^{k}\mu_{k,i}+\sum\limits_{i=1}^{k-1}\lambda_{k-1,i}\right)}. \tag{231}$$

Now put

$$\lambda_{n,i} = \lambda_{n+1-i} = p_{2n,n+1-i}, \qquad i = 1, \dots, n,$$
$$\lambda_{k,i} = p_{2k,k+1-i}, \qquad k = 1, \dots, n-1, \quad i = 1, \dots, k,$$
$$\mu_{k,i} = p_{2k-1,k+1-i}, \qquad k = 1, \dots, n, \quad i = 1, \dots, k,$$

so that $\Theta(\lambda, \mu) = \Theta^{C_n}(p)$ which is given by (123). The right-hand side is then the integral representation (122) of $Z^{C_n}(\lambda_1, \dots, \lambda_n | \imath x_1, \dots, \imath x_n)$. □

*Remark* 4.2.1. One can use the change of variables (230) directly in the integral representation of the Whittaker function. It takes the following form there:

$$\mu_{k,1} \to \ln(e^{\lambda_{k,1}} + e^{\lambda_{k-1,1}}) - \mu_{k,1}, \qquad k = 2, \dots, n, \tag{232}$$
$$\mu_{11} \to \lambda_{11} - \mu_{11}.$$

Then one has

$$e^{\mu_{k,1}} \to e^{\lambda_{k,1} - \mu_{k,1}} + e^{\lambda_{k-1,1} - \mu_{k,1}}, \tag{233}$$
$$e^{\lambda_{k,1} - \mu_{k,1}} + e^{\lambda_{k-1,1} - \mu_{k,1}} \to e^{\mu_{k,1}}, \tag{234}$$
$$e^{\mu_{11}} \to e^{\lambda_{11} - \mu_{11}}, \tag{235}$$
$$e^{\lambda_{11} - \mu_{11}} \to e^{\mu_{11}}, \tag{236}$$

so the Givental diagram remains the same, which is in agreement with the invariance of polytop $\Theta(\lambda, \mu)$. Thus we get the following representation of the Whittaker function:

$$\Psi^{B_n}_{x_1,\dots,x_n}(\lambda_{n,1}, \dots, \lambda_{n,n}) = \int_{\mathcal{C}} \prod_{i=1}^{n-1} \prod_{k=1}^{i} d\lambda_{i,k} \prod_{i=1}^{n} \prod_{k=1}^{i} d\mu_{i,k}\, e^{\mathcal{F}'(\lambda,\mu,x)}, \tag{237}$$

where

$$\begin{aligned}
\mathcal{F}'(\lambda, \mu, x) = &-\imath x_1(\lambda_{11} - 2\mu_{11}) - \\
&-\imath \sum_{k=2}^{n} x_k \left( \sum_{i=1}^{k} \lambda_{k,i} - 2 \sum_{i=1}^{k} \mu_{k,i} + \sum_{i=1}^{k-1} \lambda_{k-1,i} \right) - \\
&- \sum_{k=1}^{n} e^{\mu_{k,1}} - \sum_{i=1}^{n-1} \sum_{k=i+1}^{n} \left( e^{\lambda_{k-1,i} - \mu_{k,i}} + e^{\lambda_{k,i} - \mu_{k,i}} \right) - \\
&- \sum_{i=2}^{n} \sum_{k=i}^{n} (e^{\mu_{k,i} - \lambda_{k-1,i-1}} + e^{\mu_{k,i} - \lambda_{k,i-1}}) - \sum_{k=1}^{n} e^{\lambda_{k,k} - \mu_{k,k}}.
\end{aligned} \tag{238}$$

## 4.3 $\mathfrak{sp}_{2n}$-WHITTAKER FUNCTION

**Theorem 4.3.1** (see [19]). *$\mathfrak{sp}_{2n}$-Whittaker function has the following integral representation*

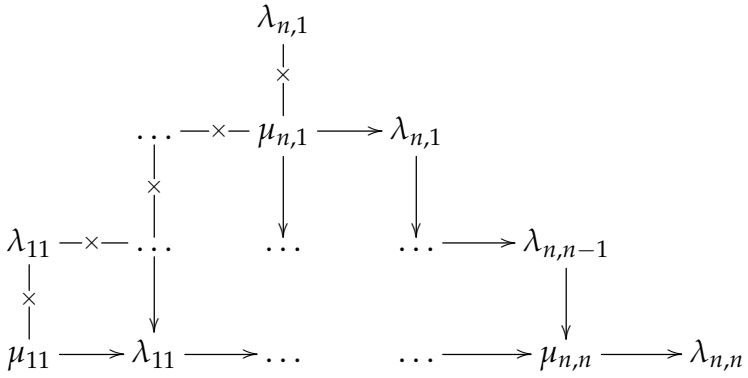
$$\Psi^{C_n}_{x_1,\dots,x_n}(\lambda_{n,1},\dots,\lambda_{n,n}) = \int\limits_{\mathcal{C}} \prod_{i=1}^{n-1}\prod_{k=1}^{i} d\lambda_{i,k} \prod_{i=1}^{n}\prod_{k=1}^{i} d\mu_{i,k}\, e^{\mathcal{F}(\lambda,\mu,x)}, \tag{239}$$

*where*

$$\begin{aligned}
\mathcal{F}(\lambda,\mu,x) &= \imath x_1\mu_{11} - \\
-\imath\sum_{k=2}^{n} x_k \left(\sum_{i=1}^{k}\lambda_{k,i} - \mu_{k,1} - 2\sum_{i=2}^{k}\mu_{k,i} + \sum_{i=1}^{k-1}\lambda_{k-1,i} - \ln(e^{\lambda_{k,1}} + e^{\lambda_{k-1,1}})\right) - \\
- \sum_{i=1}^{n} e^{\lambda_{i,i}-\mu_{i,i}} - \sum_{i=1}^{n-1}\sum_{k=i+1}^{n}(e^{\lambda_{k-1,i}-\mu_{k,i}} + e^{\lambda_{k,i}-\mu_{k,i}}) - e^{\mu_{11}+\lambda_{11}} - \\
- \sum_{k=2}^{n}(e^{\lambda_{k-1,1}+\mu_{k,1}} + e^{\lambda_{k,1}+\mu_{k,1}}) - \sum_{i=2}^{n}\sum_{k=i}^{n}(e^{\mu_{k,i}-\lambda_{k-1,i-1}} + e^{\mu_{k,i}-\lambda_{k,i-1}}),
\end{aligned} \tag{240}$$

*and the domain $\mathcal{C}$ is a slight deformation of $\mathbb{R}^{n^2}$ providing the convergence of the integral.*

The diagram for this case is as follows:



(241)

where each $x \longrightarrow y$ corresponds to the factor $e^{-e^{y-x}}$ and each $x - \times - y$ corresponds to $e^{-e^{x+y}}$.

**Theorem 4.3.2.** *Let $\lambda_1 > \cdots > \lambda_n > 0$. The following relation holds*

$$\begin{aligned}
&\lim_{\varepsilon\to 0}(\varepsilon)^{n^2}\Psi^{C_n}_{\varepsilon x_1,\dots,\varepsilon x_n}(-\varepsilon^{-1}\lambda_n,\dots,-\varepsilon^{-1}\lambda_1) = \\
&\qquad = Z^{B_n}(\lambda_1,\dots,\lambda_n|\imath x_1,\dots,\imath x_n).
\end{aligned} \tag{242}$$

*Proof.* Take the function on the left-hand side:

$$(\varepsilon)^{n^2}\Psi^{C_n}_{\varepsilon x_1,\ldots,\varepsilon x_n}(-\varepsilon^{-1}\lambda_{n,1},\ldots,-\varepsilon^{-1}\lambda_{n,n}) = \\ = \int\limits_{\mathcal{C}} \prod_{i=1}^{n-1}\prod_{k=1}^{i} d\lambda_{i,k} \prod_{i=1}^{n}\prod_{k=1}^{i} d\mu_{i,k}\, e^{\mathcal{F}(-\varepsilon^{-1}\lambda,-\varepsilon^{-1}\mu,\varepsilon x)}. \tag{243}$$

After taking the limit using (218) for the exponents we have

$$e^{\mathcal{F}(-\varepsilon^{-1}\lambda,-\varepsilon^{-1}\mu,\varepsilon x)} \to \Theta(\lambda,\mu)\cdot\exp\{\imath x_1\mu_{11}+ \\ +\imath\sum_{k=2}^{n} x_k\left(\sum_{i=1}^{k}\lambda_{k,i}-\mu_{k,1}-2\sum_{i=2}^{k}\mu_{k,i}+\sum_{i=1}^{k-1}\lambda_{k-1,i}-\min(\lambda_{k,1},\lambda_{k-1,1})\right)\} \tag{244}$$

where

$$\Theta(\lambda,\mu) = \theta(\mu_{11}+\lambda_{11})\prod_{k=2}^{n}\theta(\lambda_{k,1}+\mu_{k,1})\theta(\lambda_{k-1,1}+\mu_{k,1})\times \\ \times\prod_{k=1}^{n}\theta(\lambda_{k,k}-\mu_{k,k})\prod_{i=1}^{n-1}\prod_{k=i+1}^{n}\theta(\lambda_{k-1,i}-\mu_{k,i})\theta(\lambda_{k,i}-\mu_{k,i})\times \\ \times\prod_{i=2}^{n}\prod_{k=i}^{n}\theta(\mu_{k,i}-\lambda_{k-1,i-1})\theta(\mu_{k,i}-\lambda_{k,i-1}). \tag{245}$$

The function $\Theta(\lambda,\mu)$ defines a domain such that

$$\min(\lambda_{k,1},\lambda_{k-1,1}) \geq |\mu_{k,1}|, \qquad k=2,\ldots,n, \tag{246}$$

$$\lambda_{11} \geq |\mu_{11}|, \tag{247}$$

so we can use the following change of variables:

$$\mu_{k,1} \to 2\mu_{k,1} - \min(\lambda_{k,1},\lambda_{k-1,1}), \qquad k=2,\ldots,n, \tag{248}$$

$$\mu_{11} \to \lambda_{11} - 2\mu_{11}.$$

For the domain then we have

$$\theta(\mu_{11}+\lambda_{11})\theta(\lambda_{11}-\mu_{11}) \to \theta(\mu_{11})\theta(\lambda_{11}-\mu_{11}), \tag{249}$$

$$\theta(\lambda_{k,1}+\mu_{k,1})\theta(\lambda_{k-1,1}+\mu_{k,1})\theta(\lambda_{k,1}-\mu_{k,1})\theta(\lambda_{k-1,1}-\mu_{k,1}) \to \\ \to \theta(\mu_{k,1})\theta(\lambda_{k,1}-\mu_{k,1})\theta(\lambda_{k-1,1}-\mu_{k,1}), \qquad k=2,\ldots,n, \tag{250}$$

$$\Theta(\lambda,\mu)\to\Theta'(\lambda,\mu)=\prod_{k=1}^{n}\theta(\mu_{k,1})\prod_{k=1}^{n}\theta(\lambda_{k,k}-\mu_{k,k})\times$$
$$\times\prod_{i=1}^{n-1}\prod_{k=i+1}^{n}\theta(\lambda_{k-1,i}-\mu_{k,i})\theta(\lambda_{k,i}-\mu_{k,i})\times$$
$$\times\prod_{i=2}^{n}\prod_{k=i}^{n}\theta(\mu_{k,i}-\lambda_{k-1,i-1})\theta(\mu_{k,i}-\lambda_{k,i-1}), \tag{251}$$

and the integral gets additional factor $(2^n)$ from the measure. Gathering everything together:

$$\lim_{\varepsilon\to 0}(\varepsilon)^{n^2}\Psi^{C_n}_{\varepsilon x_1,\dots,\varepsilon x_n}(-\varepsilon^{-1}\lambda_{n,1},\dots,-\varepsilon^{-1}\lambda_{n,n})=$$
$$=2^n\int\limits_{\mathbb{R}^{n^2}}\prod_{i=1}^{n-1}\prod_{k=1}^{i}d\lambda_{i,k}\prod_{i=1}^{n}\prod_{k=1}^{i}d\mu_{i,k}\,\Theta'(\lambda,\mu)\times$$
$$\times e^{\imath x_1(\lambda_{11}-2\mu_{11})+\imath\sum\limits_{k=2}^{n}x_k\left(\sum\limits_{i=1}^{k}\lambda_{k,i}-2\sum\limits_{i=1}^{k}\mu_{k,i}+\sum\limits_{i=1}^{k-1}\lambda_{k-1,i}\right)}. \tag{252}$$

Now put

$$\lambda_{n,i}=\lambda_{n+1-i}=p_{2n,n+1-i},\qquad i=1,\dots,n,$$
$$\lambda_{k,i}=p_{2k,k+1-i},\qquad k=1,\dots,n-1,\quad i=1,\dots,k,$$
$$\mu_{k,i}=p_{2k-1,k+1-i},\qquad k=1,\dots,n,\quad i=1,\dots,k,$$

so that $\Theta(\lambda,\mu)=\Theta^{C_n}(p)=\Theta^{B_n}(p)$ which is given by (123). The right-hand side is then the integral representation (120) of $Z^{B_n}(\lambda_1,\dots,\lambda_n|\imath x_1,\dots,\imath x_n)$.

□

*Remark* 4.3.1. The variables change (248) can be applied to the Whittaker function itself in the following form:

$$\mu_{k,1}\to 2\mu_{k,1}-\ln(e^{\lambda_{k,1}}+e^{\lambda_{k-1,1}}),\qquad k=2,\dots,n, \tag{253}$$
$$\mu_{11}\to\lambda_{11}-2\mu_{11}. \tag{254}$$

Then one has

$$e^{\lambda_{k-1,1}-\mu_{k,1}}+e^{\lambda_{k,1}-\mu_{k,1}}+e^{\lambda_{k-1,1}+\mu_{k,1}}+e^{\lambda_{k,1}+\mu_{k,1}}\to \tag{255}$$
$$\to(e^{\lambda_{k-1,1}-\mu_{k,1}}+e^{\lambda_{k,1}-\mu_{k,1}})^2+(e^{\mu_{k,1}})^2,$$
$$e^{\lambda_{11}-\mu_{11}}+e^{\lambda_{11}+\mu_{11}}\to(e^{\lambda_{11}-\mu_{11}})^2+(e^{\mu_{11}})^2.$$

The integral representation of the Whittaker function takes the following form:

$$\Psi^{C_n}_{x_1,\dots,x_n}(\lambda_{n,1},\dots,\lambda_{n,n}) = 2^n \int\limits_{\mathcal{C}} \prod_{i=1}^{n-1}\prod_{k=1}^{i} d\lambda_{i,k} \prod_{i=1}^{n}\prod_{k=1}^{i} d\mu_{i,k}\, e^{\mathcal{F}'(\lambda,\mu,x)}, \tag{256}$$

where

$$\begin{aligned}\mathcal{F}'(\lambda,\mu,x) = \imath x_1(\lambda_{11}-2\mu_{11}) - \imath\sum_{k=2}^{n} x_k\left(\sum_{i=1}^{k}\lambda_{k,i} - 2\sum_{i=1}^{k}\mu_{k,i} + \sum_{i=1}^{k-1}\lambda_{k-1,i}\right) - \\ - \sum_{i=2}^{n} e^{\lambda_{i,i}-\mu_{i,i}} - \sum_{i=2}^{n-1}\sum_{k=i+1}^{n}(e^{\lambda_{k-1,i}-\mu_{k,i}} + e^{\lambda_{k,i}-\mu_{k,i}}) - (e^{\lambda_{11}-\mu_{11}})^2 - (e^{\mu_{11}})^2 - \\ - \sum_{k=2}^{n}((e^{\lambda_{k-1,1}-\mu_{k,1}} + e^{\lambda_{k,1}-\mu_{k,1}})^2 + (e^{\mu_{k,1}})^2) - \sum_{i=2}^{n}\sum_{k=i}^{n}(e^{\mu_{k,i}-\lambda_{k-1,i-1}} + e^{\mu_{k,i}-\lambda_{k,i-1}}).\end{aligned} \tag{257}$$

As we can see, this form looks very different from the one that allows visual representation via the Givental diagram. Nevertheless, it does not pose any problem when taking the limit because

$$\lim_{\varepsilon\to 0} e^{-(e^{-\varepsilon^{-1}x})^2} = \lim_{\varepsilon\to 0} e^{-e^{-\varepsilon^{-1}x}} = \theta(x). \tag{258}$$

## 4.4 $\mathfrak{so}_{2n}$-WHITTAKER FUNCTION

**Theorem 4.4.1** (see [19]). *$\mathfrak{so}_{2n}$-Whittaker function has the following integral representation*

$$\Psi^{D_n}_{x_1,\dots,x_n}(\mu_{n,1},\dots,\mu_{n,n}) = \int\limits_{\mathcal{C}} \prod_{i=1}^{n-1}\prod_{k=1}^{i} d\lambda_{i,k} \prod_{i=1}^{n-1}\prod_{k=1}^{i} d\mu_{i,k}\, e^{\mathcal{F}(\lambda,\mu,x)}, \tag{259}$$

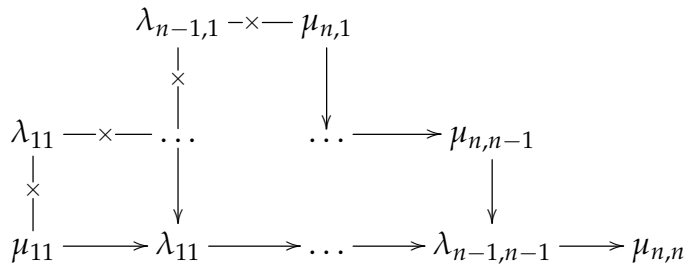
*where*

$$\begin{aligned}\mathcal{F}(\lambda,\mu,x) = \imath x_1\mu_{11} - \\ -\imath\sum_{k=2}^{n} x_k\left(\sum_{i=1}^{k}\mu_{k,i} - 2\sum_{i=1}^{k-1}\lambda_{k-1,i} + \sum_{i=1}^{k-1}\mu_{k-1,i} - 2\ln(e^{\mu_{k,1}} + e^{\mu_{k-1,1}})\right) - \\ - \sum_{k=1}^{n-1} e^{\mu_{k+1,k+1}-\lambda_{k,k}} - \sum_{k=1}^{n-2}\sum_{i=k+1}^{n-1}(e^{\mu_{i,k+1}-\lambda_{i,k}} + e^{\mu_{i+1,k+1}-\lambda_{i,k}}) -\end{aligned}$$

$$-\sum_{i=1}^{n-1}(e^{\lambda_{i,1}-\mu_{i,1}}+e^{\lambda_{i,1}-\mu_{i+1,1}}+e^{\lambda_{i,1}+\mu_{i,1}}+e^{\lambda_{i,1}+\mu_{i+1,1}})- \\ -\sum_{k=2}^{n-1}\sum_{i=k}^{n-1}(e^{\lambda_{i,k}-\mu_{i,k}}+e^{\lambda_{i,k}-\mu_{i+1,k}}), \tag{260}$$

*and the domain $\mathcal{C}$ is a slight deformation of $\mathbb{R}^{n(n-1)}$ providing the convergence of the integral.*

The diagram is similar to $C_n$ case and looks as follows:




(261)

where each $x \longrightarrow y$ corresponds to the factor $e^{-e^{y-x}}$ and each $x - \times - y$ corresponds to $e^{-e^{x+y}}$.

To prove the limit of the Whittaker function is an equivariant volume one should consider the cases of even and odd $n$ separately.

**Theorem 4.4.2.** *Let $\mu_1 > \cdots > \mu_{n-1} > |\mu_n|$ and $n = 2N$. Then the following relation holds*

$$\lim_{\varepsilon\to 0}(\varepsilon)^{n(n-1)}\Psi^{D_n}_{\varepsilon x_1,\ldots,\varepsilon x_n}(-\varepsilon^{-1}\mu_n,\ldots,-\varepsilon^{-1}\mu_1) = \\ = Z^{D_n}(\mu_1,\ldots,\mu_n|\imath x_1,\ldots,\imath x_n). \tag{262}$$

*Proof.* Take the function on the left-hand side:

$$(\varepsilon)^{2N(2N-1)}\Psi^{D_{2N}}_{\varepsilon x_1,\ldots,\varepsilon x_{2N}}(-\varepsilon^{-1}\mu_{2N,1},\ldots,-\varepsilon^{-1}\mu_{2N,2N}) = \\ = \int_{\mathcal{C}}\prod_{i=1}^{2N-1}\prod_{k=1}^{i}d\lambda_{i,k}\prod_{i=1}^{2N-1}\prod_{k=1}^{i}d\mu_{i,k}\,e^{\mathcal{F}(-\varepsilon^{-1}\lambda,-\varepsilon^{-1}\mu,\varepsilon x)}. \tag{263}$$

After taking the limit the exponent in the integral should look like this:

$$e^{\mathcal{F}(-\varepsilon^{-1}\lambda,-\varepsilon^{-1}\mu,\varepsilon x)} \to \Theta(\lambda,\mu)\mathcal{E}(\lambda,\mu,x) \tag{264}$$

where

$$\Theta(\lambda,\mu) = \prod_{k=1}^{2N-1} \theta(\mu_{k+1,k+1} - \lambda_{k,k}) \times$$
$$\times \prod_{k=1}^{2N-2} \prod_{i=k+1}^{2N-1} \theta(\mu_{i,k+1} - \lambda_{i,k}) \theta(\mu_{i+1,k+1} - \lambda_{i,k}) \times$$
$$\times \prod_{i=1}^{2N-1} \theta(\lambda_{i,1} - \mu_{i,1}) \theta(\lambda_{i,1} - \mu_{i+1,1}) \theta(\lambda_{i,1} + \mu_{i,1}) \theta(\lambda_{i,1} + \mu_{i+1,1}) \times$$
$$\times \prod_{k=2}^{2N-1} \prod_{i=k}^{2N-1} \theta(\lambda_{i,k} - \mu_{i,k}) \theta(\lambda_{i,k} - \mu_{i+1,k}), \tag{265}$$

$$\mathcal{E}(\lambda,\mu,x) = \exp\{-\imath x_1 \mu_{11} +$$
$$+\imath \sum_{k=2}^{2N} x_k \left( \sum_{i=1}^{k} \mu_{k,i} - 2\sum_{i=1}^{k-1} \lambda_{k-1,i} + \sum_{i=1}^{k-1} \mu_{k-1,i} - 2\min(\mu_{k,1}, \mu_{k-1,1}) \right)\}. \tag{266}$$

Let us separate the even and odd parts in the exponent, since they are going to behave differently:

$$\mathcal{E}(\lambda,\mu,x) = \exp\{-\imath x_1 \mu_{11} +$$
$$+\imath \sum_{k=1}^{N} x_{2k} \left( \sum_{i=1}^{2k} \mu_{2k,i} - 2\sum_{i=1}^{2k-1} \lambda_{2k-1,i} + \sum_{i=1}^{2k-1} \mu_{2k-1,i} - 2\min(\mu_{2k,1}, \mu_{2k-1,1}) \right) +$$
$$+\imath \sum_{k=1}^{N-1} x_{2k+1} \left( \sum_{i=1}^{2k+1} \mu_{2k+1,i} - 2\sum_{i=1}^{2k} \lambda_{2k,i} + \sum_{i=1}^{2k} \mu_{2k,i} - 2\min(\mu_{2k+1,1}, \mu_{2k,1}) \right)\} \tag{267}$$

Take the following change of variables:

$$\lambda_{2k-1,1} \to \lambda_{2k-1,1} + \mu_{2k-1,1} - \min(\mu_{2k,1}, \mu_{2k-1,1}), \qquad k = 1, \dots, N, \tag{268}$$
$$\lambda_{2k,1} \to \lambda_{2k,1} + \mu_{2k+1,1} - \min(\mu_{2k+1,1}, \mu_{2k,1}), \qquad k = 1, \dots, N-1. \tag{269}$$

For the exponential part then we have the following:

$$\mathcal{E}(\lambda,\mu,x) \to \mathcal{E}'(\lambda,\mu,x) = \exp\{-\imath x_1 \mu_{11} +$$
$$+\imath \sum_{k=1}^{N} x_{2k} \left( \sum_{i=1}^{2k} \mu_{2k,i} - 2\sum_{i=1}^{2k-1} \lambda_{2k-1,i} + \sum_{i=2}^{2k-1} \mu_{2k-1,i} - \mu_{2k-1,1} \right) +$$
$$+\imath \sum_{k=1}^{N-1} x_{2k+1} \left( \sum_{i=2}^{2k+1} \mu_{2k+1,i} - 2\sum_{i=1}^{2k} \lambda_{2k,i} + \sum_{i=1}^{2k} \mu_{2k,i} - \mu_{2k+1,1} \right)\} \tag{270}$$

For the step-functions of $\lambda_{2k-1,1}$ we have

$$\theta(\mu_{2k,2}-\lambda_{2k-1,1}) \to \theta(\mu_{2k,2}-\lambda_{2k-1,1}-\mu_{2k-1,1}+\min(\mu_{2k,1},\mu_{2k-1,1})) = \\ = \theta(\mu_{2k,2}-\lambda_{2k-1,1})\theta(\mu_{2k,2}-\lambda_{2k-1,1}-\mu_{2k-1,1}+\mu_{2k,1}), \tag{271}$$

$$\theta(\mu_{2k-1,2}-\lambda_{2k-1,1}) \to \theta(\mu_{2k-1,2}-\lambda_{2k-1,1}-\mu_{2k-1,1}+\min(\mu_{2k,1},\mu_{2k-1,1})) = \\ = \theta(\mu_{2k-1,2}-\lambda_{2k-1,1})\theta(\mu_{2k-1,2}-\lambda_{2k-1,1}-\mu_{2k-1,1}+\mu_{2k,1}), \tag{272}$$

$$\theta(\lambda_{2k-1,1}-\mu_{2k-1,1})\theta(\lambda_{2k-1,1}-\mu_{2k,1}) \to \\ \to \theta(\lambda_{2k-1,1}-\min(\mu_{2k,1},\mu_{2k-1,1}))\times \\ \times\theta(\lambda_{2k-1,1}+\mu_{2k-1,1}-\min(\mu_{2k,1},\mu_{2k-1,1})-\mu_{2k,1}) = \\ = \theta(\lambda_{2k-1,1}-\mu_{2k,1}) \tag{273}$$

$$\theta(\lambda_{2k-1,1}+\mu_{2k-1,1})\theta(\lambda_{2k-1,1}+\mu_{2k,1}) \to \\ \to \theta(\lambda_{2k-1,1}+2\mu_{2k-1,1}-\min(\mu_{2k,1},\mu_{2k-1,1}))\times \\ \times\theta(\lambda_{2k-1,1}+\mu_{2k-1,1}-\min(\mu_{2k,1},\mu_{2k-1,1})+\mu_{2k,1}) = \\ = \theta(\lambda_{2k-1,1}+\mu_{2k-1,1}). \tag{274}$$

For the step-functions of $\lambda_{2k,1}$ we have

$$\theta(\mu_{2k+1,2}-\lambda_{2k,1}) \to \theta(\mu_{2k+1,2}-\lambda_{2k,1}-\mu_{2k+1,1}+\min(\mu_{2k+1,1},\mu_{2k,1})) = \\ \theta(\mu_{2k+1,2}-\lambda_{2k,1})\theta(\mu_{2k+1,2}-\lambda_{2k,1}-\mu_{2k+1,1}+\mu_{2k,1}), \tag{275}$$

$$\theta(\mu_{2k,2}-\lambda_{2k,1}) \to \theta(\mu_{2k,2}-\lambda_{2k,1}-\mu_{2k+1,1}+\min(\mu_{2k+1,1},\mu_{2k,1})) = \\ = \theta(\mu_{2k,2}-\lambda_{2k,1})\theta(\mu_{2k,2}-\lambda_{2k,1}-\mu_{2k+1,1}+\mu_{2k,1}), \tag{276}$$

$$\theta(\lambda_{2k,1}-\mu_{2k,1})\theta(\lambda_{2k,1}-\mu_{2k+1,1}) \to \\ \to \theta(\lambda_{2k,1}+\mu_{2k+1,1}-\min(\mu_{2k+1,1},\mu_{2k,1})-\mu_{2k,1})\times \\ \times\theta(\lambda_{2k,1}-\min(\mu_{2k+1,1},\mu_{2k,1})) = \\ = \theta(\lambda_{2k,1}-\mu_{2k,1}), \tag{277}$$

$$\theta(\lambda_{2k,1}+\mu_{2k,1})\theta(\lambda_{2k,1}+\mu_{2k+1,1}) \to$$
$$\to \theta(\lambda_{2k,1}+\mu_{2k+1,1}-\min(\mu_{2k+1,1},\mu_{2k,1})+\mu_{2k,1})\times$$
$$\times\theta(\lambda_{2k,1}+2\mu_{2k+1,1}-\min(\mu_{2k+1,1},\mu_{2k,1}))=$$
$$=\theta(\lambda_{2k,1}+\mu_{2k+1,1}). \tag{278}$$

Gathering them all up we have get the following characteristic function of the domain:

$$\begin{aligned}\Theta(\lambda,\mu)\to\Theta'(\lambda,\mu)=&\prod_{k=1}^{2N-1}\theta(\mu_{k+1,k+1}-\lambda_{k,k})\times\\
\times&\prod_{k=1}^{2N-2}\prod_{i=k+1}^{2N-1}\theta(\mu_{i,k+1}-\lambda_{i,k})\theta(\mu_{i+1,k+1}-\lambda_{i,k})\times\\
\times&\prod_{k=1}^{N-1}\theta(\mu_{2k+1,2}-\lambda_{2k,1}-\mu_{2k+1,1}+\mu_{2k,1})\times\\
\times&\prod_{k=1}^{N-1}\theta(\mu_{2k,2}-\lambda_{2k,1}-\mu_{2k+1,1}+\mu_{2k,1})\times\\
\times&\prod_{k=1}^{N}\theta(\mu_{2k,2}-\lambda_{2k-1,1}-\mu_{2k-1,1}+\mu_{2k,1})\times\\
\times&\prod_{k=2}^{N}\theta(\mu_{2k-1,2}-\lambda_{2k-1,1}-\mu_{2k-1,1}+\mu_{2k,1})\times\\
\times&\prod_{k=1}^{N-1}\theta(\lambda_{2k,1}-\mu_{2k,1})\theta(\lambda_{2k,1}+\mu_{2k+1,1})\times\\
\times&\prod_{k=1}^{N}\theta(\lambda_{2k-1,1}-\mu_{2k,1})\theta(\lambda_{2k-1,1}+\mu_{2k-1,1})\times\\
\times&\prod_{k=2}^{2N-1}\prod_{i=k}^{2N-1}\theta(\lambda_{i,k}-\mu_{i,k})\theta(\lambda_{i,k}-\mu_{i+1,k}).\end{aligned} \tag{279}$$

Now use another change of variables:

$$\mu_{2k-1,1}\to-\mu_{2k-1,1},\qquad k=1,\dots,N. \tag{280}$$

Then for the exponent we have

$$\mathcal{E}'(\lambda,\mu,x)\to\mathcal{E}''(\lambda,\mu,x)=\exp\{\imath x_1\mu_{11}+$$
$$+\imath\sum_{k=1}^{N}x_{2k}\left(\sum_{i=1}^{2k}\mu_{2k,i}-2\sum_{i=1}^{2k-1}\lambda_{2k-1,i}+\sum_{i=1}^{2k-1}\mu_{2k-1,i}\right)+$$
$$+\imath\sum_{k=1}^{N-1}x_{2k+1}\left(\sum_{i=1}^{2k+1}\mu_{2k+1,i}-2\sum_{i=1}^{2k}\lambda_{2k,i}+\sum_{i=1}^{2k}\mu_{2k,i}\right)\}=$$
$$=e^{\imath x_1\mu_{11}+\imath\sum\limits_{k=1}^{2N}x_k\left(\sum\limits_{i=1}^{k}\mu_{k,i}-2\sum\limits_{i=1}^{k-1}\lambda_{k-1,i}+\sum\limits_{i=1}^{k-1}\mu_{k-1,i}\right)}. \tag{281}$$

And for the characteristic function:

$$\Theta'(\lambda,\mu)\to\Theta''(\lambda,\mu)=\prod_{k=1}^{2N-1}\theta(\mu_{k+1,k+1}-\lambda_{k,k})\times$$
$$\times\prod_{k=1}^{2N-2}\prod_{i=k+1}^{2N-1}\theta(\mu_{i,k+1}-\lambda_{i,k})\theta(\mu_{i+1,k+1}-\lambda_{i,k})\times$$
$$\times\prod_{k=1}^{2N-1}\theta(\mu_{k+1,2}-\lambda_{k,1}+\mu_{k+1,1}+\mu_{k,1})\times$$
$$\times\prod_{k=2}^{2N-1}\theta(\mu_{k,2}-\lambda_{k,1}+\mu_{k+1,1}+\mu_{k,1})\times$$
$$\times\prod_{k=1}^{2N-1}\prod_{i=k}^{2N-1}\theta(\lambda_{i,k}-\mu_{i,k})\theta(\lambda_{i,k}-\mu_{i+1,k}). \tag{282}$$

Gathering everything together we get the following result of taking the limit:

$$\lim_{\varepsilon\to0}(\varepsilon)^{n(n-1)}\Psi^{D_{2N}}_{\varepsilon x_1,\dots,\varepsilon x_n}(-\varepsilon^{-1}\mu_{n,1},\dots,-\varepsilon^{-1}\mu_{n,n})=$$
$$=\int\limits_{\mathbb{R}^{2N(2N-1)}}\prod_{i=1}^{2N-1}\prod_{k=1}^{i}d\lambda_{i,k}\prod_{i=1}^{2N-1}\prod_{k=1}^{i}d\mu_{i,k}\,\Theta''(\lambda,\mu)\times$$
$$\times e^{\imath x_1\mu_{11}+\imath\sum\limits_{k=1}^{2N}x_k\left(\sum\limits_{i=1}^{k}\mu_{k,i}-2\sum\limits_{i=1}^{k-1}\lambda_{k-1,i}+\sum\limits_{i=1}^{k-1}\mu_{k-1,i}\right)}. \tag{283}$$

To see what we need one can put

$$\mu_{n,i}=p_{2n-1,n+1-i}=\mu_{n+1-i},\qquad i=1,\dots,n, \tag{284}$$
$$\mu_{k,i}=p_{2k-1,k+1-i},\qquad k=1,\dots,n-1,\quad i=1,\dots,k, \tag{285}$$
$$\lambda_{k,i}=p_{2k,k+1-i},\qquad k=1,\dots,n-1,\quad i=1,\dots,k, \tag{286}$$

so that the function on the right-hand side looks exactly like the integral representation (124) of the equivariant volume $Z^{D_{2N}}(\mu_1,\dots,\mu_n|\imath x_1,\dots,\imath x_n)$ and the function $\Theta''(\lambda,\mu)$ is actually the BZ-polytope characteristic function (125). □

**Theorem 4.4.3.** *Let $\mu_1 > \cdots > \mu_{n-1} > |\mu_n|$ and $n = 2N+1$. Then the following relation holds*

$$\lim_{\varepsilon\to 0}(\varepsilon)^{n(n-1)}\Psi^{D_n}_{\varepsilon x_1,\dots,\varepsilon x_n}(\varepsilon^{-1}\mu_n,-\varepsilon^{-1}\mu_{n-1},\dots,-\varepsilon^{-1}\mu_1) = \\ = Z^{D_n}(\mu_1,\dots,\mu_n|\imath x_1,\dots,\imath x_n). \tag{287}$$

*Proof.* The proof follows the same steps as for the case $n = 2N$. □

## 4.5 CONNECTION TO GELFAND-TSETLIN POLYTOPES

The limit (218) translates the exponential factor $e^{-e^{-x}}$ in the integral representation of the Whittaker function to step-function, which is a part of characteristic function. On the other hand, there is a visual representation of the exponential factors in a form of Givental diagram, or Givental graph. Naturally, the graph has connection to the patterns and polytopes.

TYPE $A_n$. We can see that the Gelfand-Tsetlin patterns, defining the polytop $\Theta^{A_n}$, and the Givental diagram are in very close relation. Namely, each arrow $x \to y$ in the Givental diagram (216) corresponds to inequality $x \geq y$ from (12) or to the step-function $\theta(x-y)$ in the characteristic function of the Gelfand-Tsetlin polytope.

TYPE $B_n$. The graph here consists of two types of arrows. Each $x \longrightarrow y$ in the graph (223) after taking the limit corresponds to the inequality $y \geq x$. Each $\circ \longrightarrow y$ originally gave the factor $e^{-\frac{1}{2}e^x}$, but for each such node $x$ one has two arrows with $\circ$, so each pair gives $e^{-e^x}$. After taking the limit it gives the step-functions $\theta(x)$, or equally the inequalities $x \geq 0$. So there is

again a one-to-one correspondence between the Givental graph and $2n$-split patterns and the polytope $\Theta^{B_n}(p)$.

TYPE $C_n$. The graph here consists again of two types of arrows. Again, each arrow $x \longrightarrow y$ in the graph (223) after taking the limit corresponds to the inequality $y \geq x$ and the step-function $\theta(y-x)$. On the other hand $x -\times- y$ gives the inequalities $y \geq -x$, which with the normal arrows gives $y \geq |x|$. Note that in the proof we had to change the variables like (248). One can check that they transform the inequalities the following way

$$y \geq |x| \longrightarrow y \geq x \geq 0. \tag{288}$$

Which gives us the proper symplectic patterns and the corresponding polytope $\Theta^{C_n}(p)$.

TYPE $D_n$. The graph of type $D_n$ is a graph of type $C_n$ without the top row. So we have the same initial translation:

$$(\ x \longrightarrow y\ ) \longrightarrow \qquad y \geq x, \quad \theta(y-x), \tag{289}$$

$$(\ x -\times- y\ ) \longrightarrow \qquad y \geq -x, \quad \theta(y+x). \tag{290}$$

After that one needs to take a specific change of variables (268) which transforms the polytope given by (265) into the polytope $\Theta^{D_n}(p)$ corresponding to the BZ-patterns (39).

# 5

# EQUIVARIANT VOLUMES OF $G/P_J$

## 5.1 STABILIZER OF $\lambda$

Let $\mathfrak{h}$ be a Cartan subalgebra of the Lie algebra $\mathfrak{g}$, $\Phi_n$ corresponding root system, $\alpha_i \in \Phi_i$ simple roots, $s_i \in \mathcal{W}^{\Phi_n}$ corresponding simple reflections and $w_0 \in \mathcal{W}^{\Phi_n}$ the element of maximal length. Consider the case when condition (97) is not satisfied, so for some $\alpha_{i_k}^\vee \in \Phi_n^\vee$, $k = 1, \dots, m$ we have $\left\langle \lambda, \alpha_{i_k}^\vee \right\rangle = 0$. This means $\lambda \in \mathfrak{h}^*$ has non-trivial stabilizer in $\mathcal{W}^{\Phi_n}$ and

$$\mathcal{A}_{i_k}(e^\lambda)(x) = 0, \tag{291}$$

where $\mathcal{A}_{i_k}$ is defined by (604). Denote the stabilizer $\mathrm{Stab}_{\mathcal{W}}(\lambda)$. It is a subgroup in $\mathcal{W}$ and is generated by simple reflections $s_j \in \mathcal{W}$ where $j \in \bar{J} = \{i_1, \dots, i_m\}$, $\bar{J} \subset \{1, \dots, n\}$, $J = \{1, \dots, n\} \setminus \bar{J}$. Then for the element of maximal length in $\mathcal{W}$ one can write a decomposition

$$w_0 = w_0^J \cdot w_0^{\bar{J}} \quad , \tag{292}$$

where $w_0^{\bar{J}}$ is the element of maximal length in $\mathrm{Stab}_{\mathcal{W}}(\lambda)$ and $w_0^J$ is the longest element in $\mathcal{W}/\mathrm{Stab}_{\mathcal{W}}(\lambda)$. According to [4], this corresponds to the case of degenerate flag manifold $G/P_J$, where $P_J \subset G$ is a parabolic subgroup generated by Lie algebra

$$\mathfrak{p}_J = \mathfrak{h} \bigoplus_\alpha \mathfrak{g}_\alpha, \quad \alpha \in \Phi_n^+ \cup w_0^{\bar{J}} \Phi_n^+, \tag{293}$$

and formula for the equivariant volume of $G/P_J$ holds

$$Z_J^{\Phi_n}(\lambda | x) = \mathcal{A}_{w_0^J}(e^\lambda)(x). \tag{294}$$

Define the subset of positive roots

$$\Phi_J^+ = \{\alpha \in \Phi_n^+ \mid w_0^J \alpha \in \Phi_n^-\}. \tag{295}$$

**Proposition 5.1.1.** *If $\lambda$ has a non-trivial stabilizer, the following relation holds*

$$\lim_{\varepsilon \to 0} \varepsilon^{|\Phi_J^+|} \cdot S^{\Phi_n}_{\frac{\lambda_1}{\varepsilon}, \ldots, \frac{\lambda_n}{\varepsilon}}(e^{\varepsilon x_1}, \ldots, e^{\varepsilon x_n}) = Z_J^{\Phi_n}(\lambda_1, \ldots, \lambda_n | x_1, \ldots, x_n). \tag{296}$$

*Proof.* The Demazure character formula (603) gives

$$S^{\Phi_n}_{\lambda}(e^{x_1}, \ldots, e^{x_n}) = \mathcal{D}_{w_0^J} \cdot \mathcal{D}_{w_0^{\bar{J}}}(e^{\lambda})(x) = \mathcal{D}_{w_0^J}(e^{\lambda})(x), \tag{297}$$

since $\mathcal{D}_i(e^{\lambda}) = e^{\lambda}$ if $s_i \lambda = \lambda$. The rest of the proof is similar to the proof of (2.7.1). □

**Corollary 5.1.1.** *The following relation holds*

$$Z_J^{\Phi_n}(\lambda | x) = \sum_{w \in \mathcal{W}^{\Phi_n} / \operatorname{Stab}_{\mathcal{W}^{\Phi_n}}} \frac{e^{\langle w\lambda, x \rangle}}{\prod\limits_{\alpha \in \Phi_J^+} \langle w\alpha, x \rangle}. \tag{298}$$

*Proof.* The proof is similar to the proof of Corollary (2.7.1). □

## 5.2 ROOT SYSTEM $A_n$

Let $\lambda$ have a non-trivial stabilizer, which means that $\lambda_j = \lambda_{j+1}$ for $j \in \bar{J}$. Consider Gelfand-Tsetlin patterns (11) with such top row. Then certain parts of patterns are fixed due to inequalities (12), and effectively we have lower number of variables. The polytope defined by characteristic function (114) is degenerate and has lower dimension. Let us take a look at certain particular cases.

### 5.2.1 *Case* $\lambda = (\lambda_1, 0, \ldots, 0)$

The stabilizer here is

$$\operatorname{Stab}_{\mathfrak{S}_{n+1}}(\lambda) = \langle s_2, \ldots, s_n \rangle = \mathfrak{S}_n. \tag{299}$$

The reduced word then

$$w_0^J = s_n s_{n-1} \dots s_1 \tag{300}$$

and the equivariant volume can be calculated using BGG operators

$$Z_J^{A_n}(\lambda|x) = \mathcal{A}_n \mathcal{A}_{n-1} \dots \mathcal{A}_1 e^{\langle \lambda, x \rangle}. \tag{301}$$

The explicit formula looks as follows

$$Z_J^{A_n}(\lambda|x) = \sum_{i=1}^{n+1} \frac{e^{\lambda_1 x_i}}{\prod\limits_{j \neq i} (x_i - x_j)}. \tag{302}$$

Gelfand-Tsetlin patterns are reduced to the following string of inequalities:

$$\lambda_1 = p_{n+1,1} \geq p_{n,1} \geq \dots \geq p_{2,1} \geq p_{1,1} \geq 0. \tag{303}$$

This corresponds to the polytope with the characteristic function

$$\Theta_J^{A_n}(p) = \theta(p_{1,1}) \prod_{i=1}^{n} \theta(p_{i+1,1} - p_{i,1}). \tag{304}$$

The character formula gives

$$S_\lambda^{\Phi_n}(x) = \sum_{\mathbb{Z}^n} \Theta_J^{A_n}(p) \cdot x_1^{p_{1,1}} \prod_{i=2}^{n+1} x_i^{p_{i,1} - p_{i-1,1}}. \tag{305}$$

Using the proposition (5.1.1) we obtain the following integral representation

$$Z_J^{A_n}(\lambda|x) = \int\limits_{\mathbb{R}^n} \prod_{i=1}^{n} dp_{i,1}\, \Theta_J^{A_n}(p) \cdot e^{x_1 p_{1,1} + \sum\limits_{i=2}^{n+1} x_i (p_{i,1} - p_{i-1,1})}. \tag{306}$$

The polytope here corresponds to the following Givental diagram for Whittaker functions

$$\lambda_{n+1,1} \longrightarrow \lambda_{n,1} \longrightarrow \dots \longrightarrow \lambda_{2,1} \longrightarrow \lambda_{1,1} \longrightarrow 0 \tag{307}$$

It corresponds to the case of the Grassmannian $(1, n+1)$ in [23, 36], so we have the integral representation for the Whittaker function:

**Theorem 5.2.1** (see [23, 36]). *For the* $\mathrm{Gr}_{1,n+1}$*-Whittaker function the following relation holds*

$$\Psi^{(1,n+1)}_{x_1,\dots,x_{n+1}}(\lambda_1, 0, \dots, 0) = \int\limits_{\mathcal{C}} \prod_{i=1}^{n} d\lambda_{i,1}\, e^{\mathcal{F}_{(1,n+1)}(\lambda_1, x)} \tag{308}$$

*where*

$$\mathcal{F}_{(1,n+1)}(\lambda_1, x) = \imath x_1 \lambda_{1,1} + \imath \sum_{i=2}^{n+1} x_i(\lambda_{i,1} - \lambda_{i-1,1}) - e^{-\lambda_{1,1}} - \sum_{i=1}^{n} e^{\lambda_{i,1} - \lambda_{i+1,1}} \tag{309}$$

*and the domain $\mathcal{C}$ is a slight deformation of $\mathbb{R}^n$ in $\mathbb{C}^n$ providing the convergence of the integral.*

**Proposition 5.2.1** (see [21]). *The following relation holds*

$$\lim_{\varepsilon \to 0} \varepsilon^n \cdot \Psi^{(1,n+1)}_{\varepsilon x_1, \ldots, \varepsilon x_{n+1}}(\varepsilon^{-1}\lambda_1, 0, \ldots, 0) = Z_J^{A_n}(\lambda_1, 0, \ldots, 0 | \imath x_1, \ldots, \imath x_{n+1}). \tag{310}$$

*Proof.* Straighforward calculation using (218). □

### 5.2.2 *Case* $\lambda = (\lambda_1^k, 0^{n+1-k})$

For convenience let us put $k \leq n + 1 - k$, because the case $k > n + 1 - k$ is very similar. The stabilizer for this case is generated by every simple reflection except $s_k$:

$$\mathrm{Stab}_{\mathfrak{S}_{n+1}}(\lambda) = \langle s_1, \ldots, s_{k-1} \rangle \times \langle s_{k+1}, \ldots, s_n \rangle = \mathfrak{S}_k \times \mathfrak{S}_{n+1-k}. \tag{311}$$

This is the most degenerate case of $G/P_J$, when $P_J$ is a maximal proper parabolic subgroup in $G$, and $G/P_J$ is actually a Grassmannian $\mathrm{Gr}_{k,n+1}$ (see [30]). The element of the maximal length in stabilizer is a product of the respective elements of $\mathfrak{S}_k$ and $\mathfrak{S}_{n+1-k}$:

$$w_0^{\bar{J}} = s_1 s_2 s_1 s_3 s_2 s_1 \ldots s_{k-1} s_{k-2} \ldots s_1 \cdot s_n s_{n-1} s_n \ldots s_{k+1} s_{k+2} \ldots s_n. \tag{312}$$

Then one has

$$w_0^J = w_0 \cdot w_0^{\bar{J}} = (s_n \ldots s_1)^k. \tag{313}$$

The reduced decomposition for this word looks as follows:

$$w_0^J = (s_{n-k+1} \ldots s_n) \cdot (s_{n-k} \ldots s_{n-1}) \cdots (s_1 \ldots s_k). \tag{314}$$

Using BGG operators one can write

$$Z_J^{A_n}(\lambda | x) = (\mathcal{A}_{n-k+1} \ldots \mathcal{A}_n) \cdots (\mathcal{A}_1 \ldots \mathcal{A}_k) e^{\langle \lambda, x \rangle}. \tag{315}$$

The explicit formula from Corollary (5.1.1) gives

$$Z_J^{A_n}(\lambda|x) = \sum_{w\in\mathfrak{S}_{n+1}/\,\mathrm{Stab}_{\mathfrak{S}_{n+1}}(\lambda)} \frac{e^{\lambda_1 \sum\limits_{i=1}^{k} x_{w(i)}}}{\prod\limits_{i=1}^{k}\prod\limits_{j=k+1}^{n+1}(x_{w(i)}-x_{w(j)})}. \tag{316}$$

Let us see what happens to the characteristic function (114) and what the integral representation of $Z_J^{A_n}$ looks like. The Gelfand-Tsetlin patterns (11) with a top row $\lambda$ are described by the following set of inequalities

$$\lambda_1 = p_{n+1,1} \geq p_{n+1-k,1}, \tag{317}$$

$$p_{k,k} \geq 0, \tag{318}$$

$$p_{i+j-k,j} \geq p_{i+j-k-1,j}, \qquad i = k+1,\dots,n, \quad j = 1,\dots,k, \tag{319}$$

$$p_{i+j-k,j} \geq p_{i+j-k+1,j+1} \qquad i = k,\dots,n, \quad j = 1,\dots,k-1 \tag{320}$$

The total number of variables is $k(n+1-k)$ forming a rectangular pattern of dimensions $k\times(n+1-k)$. The polytope characteristic function then can be written as follows:

$$\Theta_J^{A_n}(p) = \theta(p_{n+1,1}-p_{n+1-k,1})\theta(p_{k,k})\prod_{i=1}^{n-k}\prod_{j=1}^{k}\theta(p_{i+j,j}-p_{i+j-1,j})\times$$
$$\times\prod_{i=1}^{n+1-k}\prod_{j=1}^{k-1}\theta(p_{i+j-1,j}-p_{i+j,j+1}). \tag{321}$$

The character formula gives

$$S_\lambda^{\Phi_n}(x) = \sum_{\mathbb{Z}^{k(n+1-k)}} \Theta_J^{A_n}(p)\cdot x_1^{p_{11}}\prod_{i=1}^{k} x_i^{\sum\limits_{j=1}^{i}p_{i,j}-\sum\limits_{j=1}^{i-1}p_{i-1,j}}\prod_{i=k+1}^{n+1-k} x_i^{\sum\limits_{j=1}^{k}(p_{i,j}-p_{i-1,j})}\times$$
$$\times\prod_{i=n+2-k}^{n} x_i^{\lambda_1+\sum\limits_{j=i-n+k}^{k}p_{i,j}-\sum\limits_{j=i-n+k-1}^{k}p_{i-1,j}}\cdot x_{n+1}^{\lambda_1-p_{n,k}}. \tag{322}$$

Using Proposition (5.1.1) we obtain the following integral representation for equivariant volume

$$Z_J^{A_n}(\lambda|x) = \int\limits_{\mathbb{R}^{k(n+1-k)}} \prod_{j=1}^{k} \prod_{i=j}^{n+j-k} dp_{i,j}\, \Theta_J^{A_n}(p) \times$$
$$\times \exp\left\{ x_1 p_{1,1} + \lambda_1 \sum_{i=n+2-k}^{n} x_i + \sum_{i=1}^{k} x_i \left( \sum_{j=1}^{i} p_{i,j} - \sum_{j=1}^{i-1} p_{i-1,j} \right) + \right.$$
$$+ \sum_{i=k+1}^{n+1-k} x_i \sum_{j=1}^{k} (p_{i,j} - p_{i-1,j}) +$$
$$\left. + \sum_{i=n+2-k}^{n} x_i \left( \sum_{j=i-n+k}^{k} p_{i,j} - \sum_{j=i-n+k-1}^{k} p_{i-1,j} \right) - x_{n+1} p_{n,k} \right\}. \tag{323}$$

The polytope here corresponds to the following Givental diagram for Whittaker functions

$$\begin{array}{ccccc} \lambda_{n+1,1} & & & & \\ \downarrow & & & & \\ \lambda_{n+1-k,1} & \longrightarrow & \cdots & \longrightarrow & \lambda_{n,k} \\ \downarrow & & & & \downarrow \\ \vdots & & \ddots & & \vdots \\ \downarrow & & & & \downarrow \\ \lambda_{1,1} & \longrightarrow & \cdots & \longrightarrow & \lambda_{k,k} \longrightarrow 0 \end{array} \tag{324}$$

This is exactly the diagram for $\mathrm{Gr}_{k,n+1}$-Whittaker function in [36] for $k \leq n+1-k$. One can take the integral representation of this function and show that its limit is the equivariant volume $Z_J^{A_n}(\lambda|x)$.

**Theorem 5.2.2** (Oblezin, see [36]). *Let $k \leq n+1-k$. For the $\mathrm{Gr}_{k,n+1}$-Whittaker function the following relation holds*

$$\Psi^{(k,n+1)}_{x_1,\ldots,x_{n+1}}(\lambda_{n+1,1}, 0, \ldots, 0) = \int\limits_{\mathcal{C}} \prod_{j=1}^{k} \prod_{i=j}^{n+j-k} d\lambda_{i,j}\, e^{\mathcal{F}_{(k,n+1)}(\lambda,x)} \tag{325}$$

*where*

$$\mathcal{F}_{(k,n+1)}(\lambda,x)=\imath\sum_{i=1}^{k}x_{n+1-k+i}\lambda_{n+1,1}+\imath\sum_{i=1}^{n+1-k}(x_i-x_{i+1})\sum_{j=1}^{\min(i,k)}\lambda_{i,j}+$$
$$+\imath\sum_{i=1}^{k-1}(x_{n+1-k+i}-x_{n+2-k+i})\sum_{j=i+1}^{k}\lambda_{n+1-k+i,j}-e^{-\lambda_{k,k}}-e^{\lambda_{n+1-k,1}-\lambda_{n+1,1}}-$$
$$-\sum_{i=1}^{k}\sum_{j=1}^{n-k}e^{\lambda_{i+j-1,i}-\lambda_{i+j,i}}-\sum_{i=1}^{n+1-k}\sum_{j=1}^{k-1}e^{\lambda_{i+j,j+1}-\lambda_{i+j-1,j}} \tag{326}$$

*and the domain $\mathcal{C}$ is a slight deformation of $\mathbb{R}^{k(n+1-k)}$ in $\mathbb{C}^{k(n+1-k)}$ providing the convergence of the integral.*

**Theorem 5.2.3.** *The following relation holds*

$$\lim_{\varepsilon\to 0}\varepsilon^{k(n+1-k)}\cdot\Psi^{(k,n+1)}_{\varepsilon x_1,\dots,\varepsilon x_{n+1}}(\varepsilon^{-1}\lambda_1,0,\dots,0)=$$
$$=Z_J^{A_n}(\underbrace{\lambda_1,\dots,\lambda_1}_{k},0,\dots,0|x_1,\dots,x_{n+1}). \tag{327}$$

*Proof.* The proof is similar to the proof of (217). □

## 5.3 ROOT SYSTEM $C_n$

The highest weight $\lambda=(\lambda_1,\dots,\lambda_n)$ satisfies the relations

$$\lambda_1\geq\lambda_2\geq\dots\geq\lambda_n\geq 0. \tag{328}$$

For the general rank it is hard to cover all cases of stabilizers. Thus we are going to look into the case when

$$\lambda=(\lambda_1^k,0^{n-k})=(\underbrace{\lambda_1,\dots,\lambda_1}_{k},\underbrace{0,\dots,0}_{n-k}). \tag{329}$$

Just like in previous section, it corresponds to the most degenerate case of $G/P_J$, when $P_J$ is a maximal proper parabolic subgroup in $G$. The Weyl group $\mathcal{W}^{C_n}$ is generated by $\{s_1,\dots s_{n-1},\varepsilon_n\}$ and the longest element there is

$$w_0=\varepsilon_1\dots\varepsilon_n, \tag{330}$$

where $\varepsilon_i = s_i \dots s_{n-1} \varepsilon_n s_{n-1} \dots s_i$. The stabilizer of $\lambda$ is a product of two groups:

$$\mathrm{Stab}_{\mathcal{W}^{C_n}}(\lambda) = \langle s_1, \dots, s_{k-1} \rangle \times \langle s_{k+1}, \dots, s_{n-1}, \varepsilon_n \rangle = \mathfrak{S}_k \times \mathcal{W}^{C_{n-k}}. \tag{331}$$

The element of the maximal length in stabilizer is a product of respective elements of the factors:

$$w_0^{\bar{J}} = s_1 (s_2 s_1)(s_3 s_2 s_1) \dots (s_{k-1} \dots s_1) \cdot \varepsilon_{k+1} \dots \varepsilon_n. \tag{332}$$

The positive roots from $\Phi_J^+ \subset \Phi_n^+$ are those which are mapped by $w_0^J$ to negative roots, so we have

$$\Phi_J^+ = \{\epsilon_i \pm \epsilon_j\}_{i=1,\dots,k}^{j=k+1,\dots,n} \cup \{\epsilon_i + \epsilon_j\}_{1 \le i \le j \le k}, \tag{333}$$

$$|\Phi_J^+| = 2k(n-k) + \frac{k(k+1)}{2}. \tag{334}$$

The equivariant volume is given by the formula

$$Z_J^{C_n}(\lambda, x) = \\ = \sum_{w \in W} \frac{e^{\lambda_1 \sum_{i=1}^k x_{w(i)}}}{\prod\limits_{i=1}^{k} \prod\limits_{j=k+1}^{n} (x_{w(i)} - x_{w(j)})(x_{w(i)} + x_{w(j)}) \prod\limits_{1 \le i \le j \le k} (x_{w(i)} + x_{w(j)})} \tag{335}$$

where $W = \mathcal{W}^{C_n} / \mathrm{Stab}_{\mathcal{W}^{C_n}}(\lambda)$.

The symplectic patterns (28) for the weight $\lambda$ have a certain number of variables with fixed values and are defined by the following inequalities:

$$\lambda_1 \ge p_{2n-k,1}, \tag{336}$$

$$p_{i+j-1,j} \ge p_{i+j,j+1}, \qquad i = k, \dots, 2n-k, \quad j = 1, \dots, k-1, \tag{337}$$

$$p_{i+j-1,j} \ge p_{i+j-2,j}, \qquad i = k+1, \dots, 2n-k, \quad j = 1, \dots, k, \tag{338}$$

$$p_{i+j-1,j} \ge p_{i+j,j+1}, \qquad i = 2, \dots, k-1, \quad j = 1, \dots, i-1, \tag{339}$$

$$p_{i+j-1,j} \ge p_{i+j-2,j}, \qquad i = 2, \dots, k, \quad j = 1, \dots, i-1, \tag{340}$$

$$p_{2i-1,i} \ge 0, \qquad i = 1, \dots, k. \tag{341}$$

The effective number of variables is

$$k(2n-2k+1) + \frac{k(k-1)}{2} = 2k(n-k) + \frac{k(k+1)}{2} = |\Phi_J^+|. \tag{342}$$

The characteristic function for the polytope defined by these inequalities can be written as

$$\Theta_J^{C_n}(p) = \theta(\lambda_1 - p_{2n-k,1}) \prod_{i=k}^{2n-k} \prod_{j=1}^{k-1} \theta(p_{i+j-1,j} - p_{i+j,j+1}) \times$$
$$\times \prod_{i=k+1}^{2n-k} \prod_{j=1}^{k} \theta(p_{i+j-1,j} - p_{i+j-2,j}) \prod_{i=2}^{k-1} \prod_{j=1}^{i-1} \theta(p_{i+j-1,j} - p_{i+j,j+1}) \times$$
$$\times \prod_{i=2}^{k} \prod_{j=1}^{i-1} \theta(p_{i+j-1,j} - p_{i+j-2,j}) \prod_{i=1}^{k} \theta(p_{2i-1,i}). \tag{343}$$

The character formula provides the following if $k$ is odd:

$$S_\lambda^{C_n}(x) = \sum_{\mathbb{Z}^{|\Phi_J^+|}} \Theta_J^{C_n}(p)\, x_1^{p_{2,1}-2p_{1,1}} \prod_{i=2}^{k} x_i^{\sum_{j=1}^{i} p_{2i,j} - 2\sum_{j=1}^{i} p_{2i-1,j} + \sum_{j=1}^{i-1} p_{2i-2,j}} \times$$
$$\times \prod_{i=k}^{(2n-k-1)/2} x_i^{\sum_{j=1}^{k} (p_{2i,j} - 2p_{2i-1,j} + p_{2i-2,j})} \times$$
$$\times \left( x_{(2n-k+1)/2} \right)^{\lambda_1 + \sum_{j=2}^{k} p_{2i,j} - 2\sum_{j=1}^{k} p_{2i-1,j} + \sum_{j=1}^{k} p_{2i-2,j}} \times$$
$$\times \prod_{i=\frac{2n-k+3}{2}}^{n} x_i^{\sum_{j=2i-2n+k+1}^{k} p_{2i,j} - \sum_{j=2i-2n+k}^{k} p_{2i-1,j} + \sum_{j=2i-2n+k-1}^{k} p_{2i-2,j}}. \tag{344}$$

If $k$ is even then

$$S_\lambda^{C_n}(x) = \sum_{\mathbb{Z}^{|\Phi_J^+|}} \Theta_J^{C_n}(p)\, x_1^{p_{2,1}-2p_{1,1}} \prod_{i=2}^{k} x_i^{\sum_{j=1}^{i} p_{2i,j} - 2\sum_{j=1}^{i} p_{2i-1,j} + \sum_{j=1}^{i-1} p_{2i-2,j}} \times$$
$$\times \prod_{i=k}^{(2n-k)/2} x_i^{\sum_{j=1}^{k} (p_{2i,j} - 2p_{2i-1,j} + p_{2i-2,j})} \times$$
$$\times \prod_{i=\frac{2n-k+2}{2}}^{n} x_i^{\sum_{j=2i-2n+k+1}^{k} p_{2i,j} - \sum_{j=2i-2n+k}^{k} p_{2i-1,j} + \sum_{j=2i-2n+k-1}^{k} p_{2i-2,j}}. \tag{345}$$

Using the proposition (5.1.1) we can get the integral representation of the equivariant volume:

$$Z_J^{C_n}(\lambda|x) = \int_{\mathbb{R}^{|\Phi_J^+|}} \prod_{i=k}^{2n-k} \prod_{j=1}^{k} dp_{i+j-1,j} \prod_{i=1}^{k-1} \prod_{j=1}^{i} dp_{i+j-1,j}\, \Theta_J^{C_n}(p) e^{\mathcal{F}_k(p,\lambda,x)} \tag{346}$$

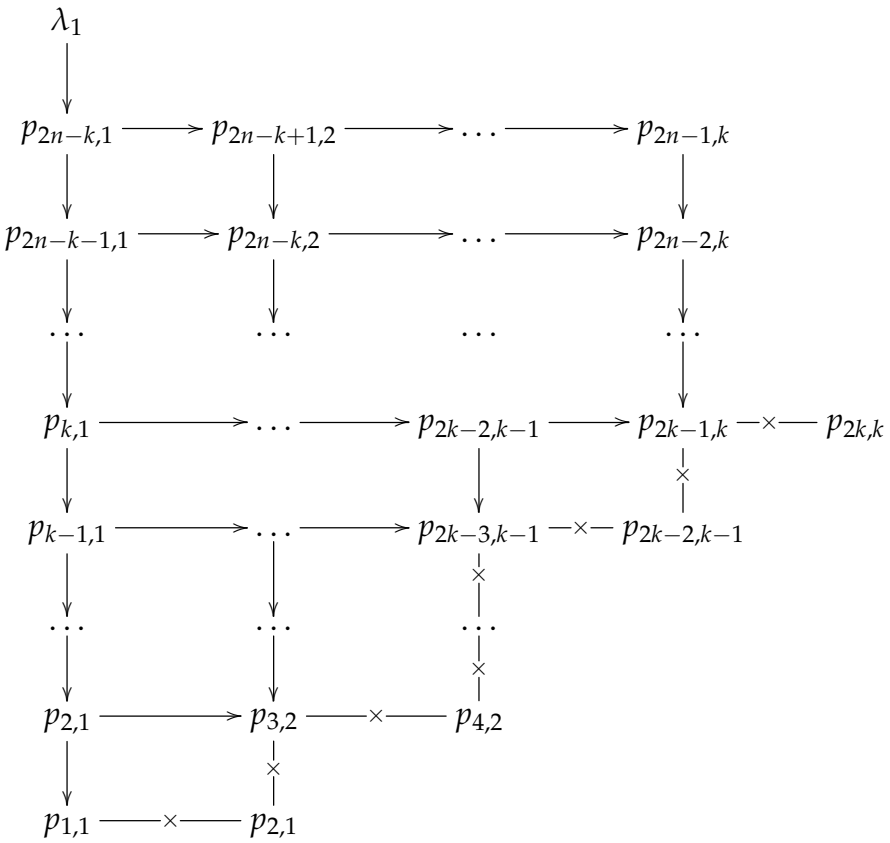
where for odd $k$

$$\begin{aligned}
\mathcal{F}_k(p,\lambda,x) &= x_1(p_{2,1}-2p_{1,1})+\\
&+\sum_{i=2}^{k} x_i\left(\sum_{j=1}^{i} p_{2i,j}-2\sum_{j=1}^{i} p_{2i-1,j}+\sum_{j=1}^{i-1} p_{2i-2,j}\right)+\\
&+\sum_{i=k}^{(2n-k-1)/2} x_i\sum_{j=1}^{k}(p_{2i,j}-2p_{2i-1,j}+p_{2i-2,j})+\\
&+x_{(2n-k+1)/2}\left(\lambda_1+\sum_{j=2}^{k} p_{2i,j}-2\sum_{j=1}^{k} p_{2i-1,j}+\sum_{j=1}^{k} p_{2i-2,j}\right)+\\
&+\sum_{i=\frac{2n-k+3}{2}}^{n} x_i\left(\sum_{j=2i-2n+k+1}^{k} p_{2i,j}-2\sum_{j=2i-2n+k}^{k} p_{2i-1,j}+\sum_{j=2i-2n+k-1}^{k} p_{2i-2,j}\right)
\end{aligned} \tag{347}$$

and for even $k$

$$\begin{aligned}
\mathcal{F}_k(p,\lambda,x) &= x_1\left(p_{2,1}-2p_{1,1}\right)+\\
&+\sum_{i=2}^{k} x_i\left(\sum_{j=1}^{i} p_{2i,j}-2\sum_{j=1}^{i} p_{2i-1,j}+\sum_{j=1}^{i-1} p_{2i-2,j}\right)+\\
&+\sum_{i=k}^{(2n-k)/2} x_i\sum_{j=1}^{k}(p_{2i,j}-2p_{2i-1,j}+p_{2i-2,j})+\\
&+\sum_{i=\frac{2n-k+2}{2}}^{n} x_i\left(\sum_{j=2i-2n+k+1}^{k} p_{2i,j}-2\sum_{j=2i-2n+k}^{k} p_{2i-1,j}+\sum_{j=2i-2n+k-1}^{k} p_{2i-2,j}\right).
\end{aligned} \tag{348}$$

The polytope $\Theta_J^{C_n}(p)$ corresponds to the following Givental graph of type $C_n$ in variables $p_{i,j}$:




(349)

In the next sections we are going to look into the low rank cases $n = 2, 3$. It is easy to derive formulas for equivariant volumes for each $k$ explicitly and there is a connection to $Z_J^{A_n}$.

## 5.4 EXAMPLE: ROOT SYSTEM $C_2$

The root system $C_2$ has only two simple roots:

$$\alpha_1 = \epsilon_1 - \epsilon_2, \qquad \alpha_2 = 2\epsilon_2, \tag{350}$$

with the corresponding generators of the Weyl group $s_1 \in \mathfrak{S}_2$ and $\varepsilon_2 \colon \epsilon_2 \to -\epsilon_2$. The element of the maximal length here is $w_0 = s_1\varepsilon_2 s_1\varepsilon_2 = \varepsilon_2 s_1\varepsilon_2 s_1$. The highest weight $\lambda = (\lambda_1, \lambda_2)$ satisfies the condition $\lambda_1 \geq \lambda_2 \geq 0$. There are two cases when the stabilizer is not trivial:

1. $\lambda_1 \neq 0$ and $\lambda_2 = 0$;

2. $\lambda_1 = \lambda_2 \neq 0$.

### 5.4.1 *Case* $\lambda = (\lambda_1, 0)$

The stabilizer of $\lambda$ in the Weyl group is generated by $\varepsilon_2$, so it is easy to see that

$$w_0^J = s_1 \varepsilon_2 s_1. \tag{351}$$

The equivariant volume is given by the expression

$$\begin{gathered} Z_J^{C_2}(\lambda|x) = \mathcal{A}_1\mathcal{A}_2\mathcal{A}_1 e^{\langle\lambda,x\rangle} = \\ = \frac{e^{\lambda_1 x_1}}{2x_1(x_1-x_2)(x_1+x_2)} + \frac{e^{\lambda_1 x_2}}{2x_2(x_2-x_1)(x_1+x_2)} + \\ + \frac{e^{-\lambda_1 x_2}}{2x_2(x_1+x_2)(x_1-x_2)} + \frac{e^{-\lambda_1 x_1}}{2x_1(x_1+x_2)(x_2-x_1)}. \end{gathered} \tag{352}$$

The symplectic patterns are reduced to the string of inequalities:

$$\lambda_1 = p_{4,1} \geq p_{3,1} \geq p_{2,1} \geq p_{1,1} \geq 0. \tag{353}$$

They define the polytope with the characteristic function

$$\Theta_J^{C_2}(p) = \theta(p_{4,1} - p_{3,1})\theta(p_{3,1} - p_{2,1})\theta(p_{2,1} - p_{1,1})\theta(p_{1,1}). \tag{354}$$

Using it one can write the character formula as well as the integral representation of the equivariant volume:

$$S_\lambda^{C_2}(x) = \sum_{\mathbb{Z}^3} \Theta_J^{C_2}(p) x_1^{p_{2,1}-2p_{1,1}} x_2^{p_{4,1}-2p_{3,1}+p_{2,1}}, \tag{355}$$

$$Z_J^{C_2}(\lambda|x) = \int\limits_{\mathbb{R}^3} dp_{3,1}dp_{2,1}dp_{1,1}\, \Theta_J^{C_2}(p) e^{x_1(p_{2,1}-2p_{1,1})+x_2(\lambda_1-2p_{3,1}+p_{2,1})} \tag{356}$$

The Givental diagram of type $C_2$, corresponding to this polytope:

$$\lambda_{2,2} \longrightarrow \mu_{2,2} \longrightarrow \lambda_{1,1} \longrightarrow \mu_{1,1} \text{—}\times\text{—} \lambda_{1,1} \tag{357}$$

Note that it looks very similar to the case of $\lambda = (\lambda_1, 0, 0, 0)$ for $A_3$. Take the corresponding equivariant volume:

$$Z^{A_3}_{J'}(\lambda_1, 0, 0, 0|y_1, y_2, y_3, y_4) = \frac{e^{\lambda_1 y_1}}{(y_1 - y_2)(y_1 - y_3)(y_1 - y_4)} + \\ + \frac{e^{\lambda_1 y_2}}{(y_2 - y_1)(y_2 - y_3)(y_2 - y_4)} + \frac{e^{\lambda_1 y_3}}{(y_3 - y_1)(y_3 - y_2)(y_3 - y_4)} + \\ + \frac{e^{\lambda_1 y_4}}{(y_4 - y_1)(y_4 - y_2)(y_4 - y_3)}. \tag{358}$$

Now change the variables the following way:

$$y_1 = x_1, \qquad y_2 = -x_1, \qquad , y_3 = x_2, \qquad y_4 = -x_2. \tag{359}$$

Then

$$Z^{A_3}_{J'}(\lambda_1, 0, 0, 0|x_1, -x_1, x_2, -x_2) = \\ = \frac{e^{\lambda_1 x_1}}{2x_1(x_1 - x_2)(x_1 + x_2)} + \frac{e^{-\lambda_1 x_1}}{2x_1(x_1 + x_2)(x_2 - x_1)} + \\ + \frac{e^{\lambda_1 x_2}}{2x_2(x_2 - x_1)(x_1 + x_2)} + \frac{e^{-\lambda_1 x_2}}{2x_2(x_1 + x_2)(x_1 - x_2)} = \\ = Z^{C_2}_J(\lambda_1, 0|x_1, x_2). \tag{360}$$

### 5.4.2 *Case* $\lambda = (\lambda_1, \lambda_1)$

The stabilizer of $\lambda$ in the Weyl group is generated by $s_1$, so we have

$$w_0^J = \varepsilon_2 s_1 \varepsilon_2. \tag{361}$$

The equivariant volume is given by the expression

$$Z^{C_2}_J(\lambda|x) = \mathcal{A}_2 \mathcal{A}_1 \mathcal{A}_2 e^{\langle \lambda, x \rangle} = \\ = \frac{e^{\lambda_1(x_1 + x_2)}}{4x_1 x_2 (x_1 + x_2)} + \frac{e^{\lambda_1(x_1 - x_2)}}{4x_1 x_2 (x_2 - x_1)} + \frac{e^{\lambda_1(x_2 - x_1)}}{4x_1 x_2 (x_1 - x_2)} - \frac{e^{\lambda_1(-x_1 - x_2)}}{4x_1 x_2 (x_1 + x_2)}. \tag{362}$$

The symplectic patterns are reduced to the following set of inequalities:

$$\lambda_1 = p_{4,1} \geq p_{2,1}, \tag{363}$$

$$p_{2,1} \geq p_{3,2} \geq 0, \tag{364}$$

$$p_{2,1} \geq p_{1,1} \geq 0. \tag{365}$$

They define the polytope with the characteristic function

$$\Theta_J^{C_2}(p) = \theta(p_{4,1} - p_{2,1})\theta(p_{2,1} - p_{3,2})\theta(p_{2,1} - p_{1,1})\theta(p_{1,1})\theta(p_{3,2}). \tag{366}$$

Using it one can write the character formula as well as the integral representation of the equivariant volume:

$$S_\lambda^{C_2}(x) = \sum_{\mathbb{Z}^3} \Theta_J^{C_2}(p) x_1^{p_{2,1}-2p_{1,1}} x_2^{p_{4,1}-2p_{3,2}+p_{2,1}}, \tag{367}$$

$$Z_J^{C_2}(\lambda|x) = \int_{\mathbb{R}^3} dp_{3,2}dp_{2,1}dp_{1,1}\, \Theta_J^{C_2}(p) e^{x_1(p_{2,1}-2p_{1,1})+x_2(\lambda_1-2p_{3,2}+p_{2,1})} \tag{368}$$

The Givental diagram of type $C_2$ corresponding to the polytope:

$$\begin{array}{ccccc} \lambda_{2,2} & \longrightarrow & \lambda_{1,1} & \longrightarrow & \mu_{1,1} \\ & & \downarrow & & \times \\ & & \mu_{2,1} & -\times- & \lambda_{1,1} \end{array} \tag{369}$$

Take again the equivariant volume (5.4.1) and substitute the following:

$$y_1 = x_1 + x_2, \qquad y_2 = -x_1 - x_2, \qquad , y_3 = x_1 - x_2, \qquad y_4 = x_2 - x_1. \tag{370}$$

Then

$$\begin{gathered} Z_{J'}^{A_3}(\lambda_1, 0, 0, 0|x_1 + x_2, -x_1 - x_2, x_1 - x_2, x_2 - x_1) = \\ = \frac{e^{\lambda_1(x_1+x_2)}}{8x_1x_2(x_1+x_2)} - \frac{e^{\lambda_1(-x_1-x_2)}}{8x_1x_2(x_1+x_2)} + \frac{e^{\lambda_1(x_1-x_2)}}{8x_1x_2(x_2-x_1)} + \frac{e^{\lambda_1(x_2-x_1)}}{8x_1x_2(x_1-x_2)} = \\ = \frac{1}{2} Z_J^{C_2}(\lambda_1, \lambda_1|x_1, x_2). \end{gathered} \tag{371}$$

## 5.5 EXAMPLE: ROOT SYSTEM $C_3$

The root system of type $C_3$ has three simple roots:

$$\alpha_1 = \epsilon_1 - \epsilon_2, \qquad \alpha_2 = \epsilon_2 - \epsilon_3, \qquad \alpha_3 = 2\epsilon_3. \tag{372}$$

The corresponding generators of the Weyl group and the element of maximal length are:

$$s_1 \colon \epsilon_1 \leftrightarrow \epsilon_2, \tag{373}$$

$$s_2 \colon \epsilon_2 \leftrightarrow \epsilon_3, \tag{374}$$

$$\varepsilon_3 \colon \epsilon_3 \to -\epsilon_3, \tag{375}$$

$$w_0 = s_1 s_2 \varepsilon_3 s_2 s_1 s_2 \varepsilon_3 s_2 \varepsilon_3. \tag{376}$$

The highest weight $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ satisfies the condition $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 0$, so there are three cases with non-trivial stabilizer:

1. $\lambda_1 \neq 0$, $\lambda_2 = \lambda_3 = 0$;
2. $\lambda_1 = \lambda_2 \neq 0$, $\lambda_3 = 0$;
3. $\lambda_1 = \lambda_2 = \lambda_3 \neq 0$.

### 5.5.1 *Case* $\lambda = (\lambda_1, 0, 0)$

The stabilizer $\mathrm{Stab}_{\mathcal{W}^{C_3}}(\lambda)$ here is a subgroup $\mathcal{W}^{C_2}$ generated by $s_2$ and $\varepsilon_3$. The longest element in the stabilizer is $w_0^{\bar{I}} = s_2 \varepsilon_3 s_2 \varepsilon_3$, so for the longest element in the factor $\mathcal{W}^{C_3} / \mathrm{Stab}_{\mathcal{W}^{C_3}}(\lambda)$ one has

$$w_0^I = s_1 s_2 \varepsilon_3 s_2 s_1. \tag{377}$$

The equivariant volume is then given by the following expression:

$$Z_I^{C_3}(\lambda | x) = \mathcal{A}_1 \mathcal{A}_2 \mathcal{A}_3 \mathcal{A}_2 \mathcal{A}_1 e^{\langle \lambda, x \rangle} =$$

$$= \frac{e^{\lambda_1 x_1}}{2x_1(x_1 - x_2)(x_1 - x_3)(x_1 + x_2)(x_1 + x_3)} +$$

$$+ \frac{e^{\lambda_1 x_2}}{2x_2(x_2 - x_1)(x_2 - x_3)(x_1 + x_2)(x_2 + x_3)} +$$

$$+\frac{e^{\lambda_1 x_3}}{2x_3(x_3-x_1)(x_3-x_2)(x_1+x_3)(x_2+x_3)}-$$
$$-\frac{e^{-\lambda_1 x_3}}{2x_3(x_1+x_3)(x_2+x_3)(x_1-x_3)(x_2-x_3)}-$$
$$-\frac{e^{-\lambda_1 x_2}}{2x_2(x_1+x_2)(x_2+x_3)(x_1-x_2)(x_3-x_2)}-$$
$$-\frac{e^{-\lambda_1 x_1}}{2x_1(x_1+x_2)(x_1+x_3)(x_1-x_3)(x_1-x_3)}. \tag{378}$$

The symplectic patterns are reduced to the following string of inequalities:

$$\lambda_1 = p_{6,1} \geq p_{5,1} \geq p_{4,1} \geq p_{3,1} \geq p_{2,1} \geq p_{1,1} \geq 0. \tag{379}$$

The corresponding characteristic function of the polytope is

$$\Theta_J^{C_3}(p) = \theta(p_{6,1}-p_{5,1})\theta(p_{5,1}-p_{4,1})\times$$
$$\times\theta(p_{4,1}-p_{3,1})\theta(p_{3,1}-p_{2,1})\theta(p_{2,1}-p_{1,1})\theta(p_{1,1}). \tag{380}$$

Then for the character one has

$$S_\lambda^{C^3}(x) = \sum_{(p_{1,1},p_{2,1},p_{3,1},p_{4,1},p_{5,1})\in\mathbb{Z}^5} \Theta_J^{C_3}(p) x_1^{p_{2,1}-p_{1,1}} x_2^{p_{4,1}-2p_{3,1}+p_{2,1}} x_3^{\lambda_1-2p_{5,1}+p_{4,1}}. \tag{381}$$

If now one use the (5.1.1) to take the limits, it should give the following representation for the equivariant volume:

$$Z_J^{C_3}(\lambda|x) = \int_{\mathbb{R}^5} dp_{1,1}dp_{2,1}dp_{3,1}dp_{4,1}dp_{5,1}\Theta_J^{C_3}(p)\times$$
$$\times e^{x_1(p_{2,1}-p_{1,1})+x_2(p_{4,1}-2p_{3,1}+p_{2,1})+x_3(\lambda_1-2p_{5,1}+p_{4,1})}. \tag{382}$$

The Givental diagram of type $C_3$, corresponding to this polytope:

$$\lambda_{3,3} \longrightarrow \mu_{3,3} \longrightarrow \lambda_{2,2} \longrightarrow \mu_{2,2} \longrightarrow \lambda_{1,1} \longrightarrow \mu_{1,1} -\times- \lambda_{1,1} \tag{383}$$

Note that just like in the case $(\lambda_1, 0)$ for $C_2$ the following relation holds

$$Z_{J'}^{A_5}(\lambda_1,0,0,0,0,0|x_1,-x_1,x_2,-x_2,x_3,-x_3) = Z_J^{C_3}(\lambda_1,0,0|x_1,x_2,x_3). \tag{384}$$

### 5.5.2 *Case* $\lambda = (\lambda_1, \lambda_1, 0)$

The stabilizer of $\lambda$ in $\mathcal{W}^{C_3}$ is generated by $s_1$ and $\varepsilon_3$, which commute, so one has the following reduced word:

$$w_0^J = s_2\varepsilon_3 s_2 s_1 s_2 \varepsilon_3 s_2. \tag{385}$$

The equivariant volume then is given by the following expression:

$$\begin{aligned}
&Z_J^{C_3}(\lambda|x) = \mathcal{A}_2\mathcal{A}_3\mathcal{A}_2\mathcal{A}_1\mathcal{A}_2\mathcal{A}_3\mathcal{A}_2 e^{\langle\lambda,x\rangle} = \\
&= \frac{e^{\lambda_1(x_1+x_2)}}{4x_1x_2(x_1-x_3)(x_2-x_3)(x_1+x_2)(x_1+x_3)(x_2+x_3)} + \\
&+ \frac{e^{\lambda_1(x_1+x_3)}}{4x_1x_3(x_1-x_2)(x_3-x_2)(x_1+x_2)(x_1+x_3)(x_2+x_3)} + \\
&+ \frac{e^{\lambda_1(x_1-x_3)}}{4x_1x_3(x_1-x_2)(x_1-x_3)(x_2-x_3)(x_2+x_3)(x_1+x_2)} + \\
&+ \frac{e^{\lambda_1(x_2-x_3)}}{4x_2x_3(x_2-x_1)(x_2-x_3)(x_1-x_3)(x_1+x_3)(x_1+x_2)} + \\
&+ \frac{e^{\lambda_1(x_2+x_3)}}{4x_2x_3(x_2-x_1)(x_3-x_1)(x_2+x_3)(x_1+x_3)(x_1+x_2)} + \\
&+ \frac{e^{\lambda_1(x_3-x_2)}}{4x_2x_3(x_3-x_1)(x_3-x_2)(x_1-x_2)(x_1+x_3)(x_1+x_2)} + \\
&+ \frac{e^{\lambda_1(x_3-x_1)}}{4x_1x_3(x_3-x_2)(x_3-x_1)(x_2-x_1)(x_2+x_3)(x_1+x_2)} + \\
&+ \frac{e^{\lambda_1(-x_1-x_3)}}{4x_1x_3(x_1-x_2)(x_2-x_3)(x_1+x_2)(x_2+x_3)(x_1+x_3)} + \\
&+ \frac{e^{\lambda_1(-x_2-x_3)}}{4x_2x_3(x_2-x_1)(x_1-x_3)(x_1+x_2)(x_1+x_3)(x_2+x_3)} + \\
&+ \frac{e^{\lambda_1(-x_1-x_2)}}{4x_1x_2(x_1-x_3)(x_3-x_2)(x_1+x_2)(x_2+x_3)(x_1+x_3)} + \\
&+ \frac{e^{\lambda_1(x_1-x_2)}}{4x_1x_2(x_1-x_3)(x_1-x_2)(x_3-x_2)(x_2+x_3)(x_1+x_3)} + \\
&+ \frac{e^{\lambda_1(x_2-x_1)}}{4x_1x_2(x_2-x_3)(x_1-x_3)(x_1-x_2)(x_2+x_3)(x_1+x_3)}.
\end{aligned} \tag{386}$$

The symplectic patterns are defined by the following relations:

$$\lambda_1 \geq p_{4,1}, \quad p_{4,1} \geq p_{5,2}, \quad p_{4,1} \geq p_{3,1}, \tag{387}$$

$$p_{5,2} \geq p_{4,2}, \quad p_{3,1} \geq p_{4,2}, \quad p_{4,2} \geq p_{3,2} \tag{388}$$

$$p_{3,1} \geq p_{2,1} \geq p_{3,2}, \quad p_{3,2} \geq 0, \tag{389}$$

$$p_{2,1} \geq p_{1,1} \geq 0. \tag{390}$$

Then characteristic function is as follows:

$$\begin{aligned}\Theta_J^{C_3}(p) = \theta(\lambda_1 - p_{4,1})\theta(p_{4,1} - p_{5,2})\theta(p_{4,1} - p_{3,1})\times \\ \times\theta(p_{5,2} - p_{4,2})\theta(p_{3,1} - p_{4,2})\theta(p_{4,2} - p_{3,2})\theta(p_{3,1} - p_{2,1})\times \\ \times\theta(p_{2,1} - p_{3,2})\theta(p_{2,1} - p_{1,1})\theta(p_{3,2})\theta(p_{1,1}).\end{aligned} \tag{391}$$

The character formula using this polytope gives

$$S_\lambda^{C_3}(x) = \sum_{\mathbb{Z}^7} \Theta_J^{C_3}(p)\, x_1^{p_{2,1}-p_{1,1}} x_2^{p_{4,1}+p_{4,2}-2p_{3,1}-2p_{3,2}+p_{2,1}} x_3^{p_{4,1}+p_{4,2}-2p_{5,2}}. \tag{392}$$

Taking the limit (5.1.1) of this character gives the following equivariant volume:

$$\begin{aligned}Z_J^{C_3}(\lambda|x) = \int_{\mathbb{R}^7} dp_{1,1}dp_{2,1}dp_{3,1}dp_{3,2}dp_{4,1}dp_{4,2}dp_{5,2}\,\Theta_J^{C_3}(p)\times \\ \times e^{x_1(p_{2,1}-p_{1,1})+x_2(p_{4,1}+p_{4,2}-2p_{3,1}-2p_{3,2}+p_{2,1})+x_3(p_{4,1}+p_{4,2}-2p_{5,2})}.\end{aligned} \tag{393}$$

The Givental diagram of type $C_3$ corresponding to this polytope:

$$\begin{array}{ccccccccc} \lambda_{3,3} & \longrightarrow & \lambda_{2,2} & \longrightarrow & \mu_{2,2} & \longrightarrow & \lambda_{1,1} & \longrightarrow & \mu_{1,1} \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow\times \\ & & \mu_{3,2} & \longrightarrow & \lambda_{2,1} & \longrightarrow & \mu_{2,1} & -\times- & \lambda_{1,1} \\ & & & & & & \downarrow\times & & \\ & & & & & & \lambda_{2,1} & & \end{array} \tag{394}$$

### 5.5.3 *Case* $\lambda = (\lambda_1, \lambda_1, \lambda_1)$

The stabilizer in this case is $\mathfrak{S}_3$ generated by $s_1$ and $s_2$. The longest word in stabilizer is $w_0^{\bar{J}} = s_1 s_2 s_1$ so we have

$$w_0^J = \varepsilon_3 s_2 s_1 \varepsilon_3 s_2 \varepsilon_3. \tag{395}$$

For the equivariant volume we have the following:

$$Z_J^{C_3}(\lambda|x) = \mathcal{A}_3\mathcal{A}_2\mathcal{A}_1\mathcal{A}_3\mathcal{A}_2\mathcal{A}_3 e^{\langle\lambda,x\rangle} =$$
$$= \frac{e^{\lambda_1(x_1+x_2+x_3)}}{8x_1x_2x_3(x_1+x_2)(x_1+x_3)(x_2+x_3)} + \frac{e^{\lambda_1(x_1+x_2-x_3)}}{8x_1x_2x_3(x_1+x_2)(x_1-x_3)(x_3-x_2)} +$$
$$+ \frac{e^{\lambda_1(x_1+x_3-x_2)}}{8x_1x_2x_3(x_1+x_3)(x_1-x_2)(x_2-x_3)} + \frac{e^{\lambda_1(x_1-x_3-x_2)}}{8x_1x_2x_3(x_1-x_3)(x_2-x_1)(x_2+x_3)} +$$
$$+ \frac{e^{\lambda_1(x_2-x_3-x_1)}}{8x_1x_2x_3(x_2-x_3)(x_1-x_2)(x_1+x_3)} + \frac{e^{\lambda_1(x_3-x_2-x_1)}}{8x_1x_2x_3(x_3-x_2)(x_1-x_3)(x_1+x_2)} +$$
$$+ \frac{e^{\lambda_1(-x_3-x_2-x_1)}}{8x_1x_2x_3(x_2+x_3)(x_1+x_3)(x_1+x_2)} + \frac{e^{\lambda_1(x_2+x_3-x_1)}}{8x_1x_2x_3(x_2+x_3)(x_2-x_1)(x_1-x_3)}. \tag{396}$$

The symplectic patterns are defined by the following relations:

$$\lambda_1 \geq p_{3,1}, \quad p_{3,1} \geq p_{4,2}, \quad p_{3,1} \geq p_{2,1}, \tag{397}$$
$$p_{4,2} \geq p_{5,3} \geq 0, \quad p_{4,2} \geq p_{3,2} \geq 0, \tag{398}$$
$$p_{2,1} \geq p_{3,2}, \quad p_{2,1} \geq p_{1,1} \geq 0. \tag{399}$$

The characteristic function of the polytope:

$$\Theta_J^{C_3}(p) = \theta(\lambda_1 - p_{3,1})\theta(p_{3,1} - p_{4,2})\theta(p_{3,1} - p_{2,1})\times$$
$$\times\theta(p_{4,2} - p_{5,3})\theta(p_{4,2} - p_{3,2})\theta(p_{2,1} - p_{3,2})\theta(p_{2,1} - p_{1,1})\theta(p_{5,3})\theta(p_{3,2})\theta(p_{1,1}). \tag{400}$$

The character formula using the polytope gives

$$S_\lambda^{C_3}(x) = \sum_{\mathbb{Z}^6} \Theta_J^{C_3}(p)\, x_1^{p_{2,1}-2p_{1,1}} x_2^{\lambda_1+p_{4,2}-2p_{3,1}-2p_{3,2}+p_{2,1}} x_3^{p_{4,2}-2p_{5,3}}. \tag{401}$$

Taking the limit (5.1.1) gives

$$Z_J^{C_3}(\lambda|x) = \int_{\mathbb{R}^6} dp_{1,1}dp_{2,1}dp_{3,1}dp_{3,2}dp_{4,2}dp_{5,3}\, \Theta_J^{C_3}(p)\times$$
$$\times e^{x_1(p_{2,1}-2p_{1,1})+x_2(\lambda_1+p_{4,2}-2p_{3,1}-2p_{3,2}+p_{2,1})+x_3(p_{4,2}-2p_{5,3})}. \tag{402}$$

The Givental diagram of type $C_3$ corresponding to the polytope looks as follows:

$$\begin{array}{ccccccc}
\lambda_{3,3} & \longrightarrow & \mu_{2,2} & \longrightarrow & \lambda_{1,1} & \longrightarrow & \mu_{1,1} \\
 & & \downarrow & & \downarrow & & \times\downarrow \\
 & & \lambda_{2,1} & \longrightarrow & \mu_{2,1} & -\times- & \lambda_{1,1} \\
 & & \downarrow & & \times\downarrow & & \\
 & & \mu_{3,1} & -\times- & \lambda_{2,1} & &
\end{array} \tag{403}$$

## 5.6 ROOT SYSTEM $D_n$

The Weyl group $\mathcal{W}^{D_n}$ is generated by $\{s_1, \dots, s_{n-1}, \bar{\varepsilon}_n\}$. The element of the maximal length is

$$w_0 = \varepsilon_{1,n} \dots \varepsilon_{n-1,n}, \tag{404}$$

$$\varepsilon_{i,n} = s_i \dots s_{n-1} \bar{\varepsilon}_n s_{n-2} \dots s_i, \tag{405}$$

$$\ell(w_0) = n(n-1). \tag{406}$$

The highest weight $\lambda$ satisfies the relations

$$\lambda_1 \geq \dots \geq \lambda_{n-1} \geq |\lambda_n|. \tag{407}$$

We are interested in the following case of non-trivial stabilizer of the weight:

$$\lambda = (\underbrace{\lambda_1, \dots, \lambda_1}_{k}, \underbrace{0, \dots, 0}_{n-k}), \tag{408}$$

which again corresponds to the most degenerate case of the flag space $G/P_J$. The stabilizer of $\lambda$ is a product of two groups:

$$\mathrm{Stab}_{\mathcal{W}^{D_n}}(\lambda) = \langle s_1, \dots, s_{k-1} \rangle \times \langle s_{k+1}, \dots, s_{n-1}, \bar{\varepsilon}_n \rangle = \mathfrak{S}_k \times \mathcal{W}^{D_{n-k}}. \tag{409}$$

The element of the maximal length here is a product of the respective elements of the factors:

$$w_0^{\bar{J}} = s_1(s_2 s_1) \dots (s_{k-1} s_1) \cdot \varepsilon_{k+1,n} \dots \varepsilon_{n-1,n}. \tag{410}$$

The subset of positive roots $\Phi_J^+ \subset \Phi_n^+$:

$$\Phi_J^+ = \{\epsilon_i \pm \epsilon_j\}_{i=1,\dots,k}^{j=k+1,\dots,n} \cup \{\epsilon_i + \epsilon_j\}_{1\le i<j\le k}, \tag{411}$$

$$|\Phi_J^+| = 2k(n-k) + \frac{k(k-1)}{2}. \tag{412}$$

Then one has the following formula for the equivariant volume:

$$Z_J^{D_n}(\lambda, x) = \\ = \sum_{w\in W} \frac{e^{\lambda_1 \sum_{i=1}^{k} x_{w(i)}}}{\prod_{i=1}^{k}\prod_{j=k+1}^{n}(x_{w(i)} - x_{w(j)})(x_{w(i)} + x_{w(j)}) \prod_{1\le i<j\le k}(x_{w(i)} + x_{w(j)})} \tag{413}$$

where $W = \mathcal{W}^{D_n} / \operatorname{Stab}_{W^{D_n}}(\lambda)$.

The BZ-patterns (39) for the weight $\lambda$ have some of the variables fixed, so the effective number of variables is lower. The patterns are defined by the inequalities:

$$\lambda_1 \ge p_{2n-1-k,1}, \tag{414}$$

$$p_{i+j-1,j} \ge p_{i+j,j+1}, \qquad i = k,\dots,2n-1-k, \quad j = 1,\dots,k-1, \tag{415}$$

$$p_{i+j-1,j} \ge p_{i+j-2,j}, \qquad i = k+1,\dots,2n-1-k, \quad j = 1,\dots,k, \tag{416}$$

$$p_{i+j-1,j} \ge p_{i+j,j+1}, \qquad i = 2,\dots,k-1, \quad j = 1,\dots,i-1, \tag{417}$$

$$p_{i+j-1,j} \ge p_{i+j-2,j}, \qquad i = 2,\dots,k, \quad j = 1,\dots,i-1, \tag{418}$$

$$p_{2i-1,i} + p_{2i+1,i+1} + p_{2i+1,i} \ge p_{2i,i}, \qquad i = 1,\dots,k-1, \tag{419}$$

$$p_{2i-1,i} + p_{2i+1,i+1} + p_{2i-1,i-1} \ge p_{2i,i}, \qquad i = 2,\dots,k-1, \tag{420}$$

$$p_{2k-1,k} + p_{2k+1,k} \ge p_{2k,k}, \tag{421}$$

$$p_{2k-1,k} + p_{2k-1,k-1} \ge p_{2k,k}. \tag{422}$$

The effective number of variables is

$$k(2n-2k) + \frac{k(k-1)}{2} = 2k(n-k) + \frac{k(k-1)}{2} = |\Phi_J^+|. \tag{423}$$

The characteristic function of the polytope defined by the inequalities above is

$$\Theta_J^{D_n}(p) = \theta(\lambda_1 - p_{2n-1-k,1}) \cdot \prod_{i=k}^{2n-1-k} \prod_{j=1}^{k-1} \theta(p_{i+j-1,j} - p_{i+j,j+1}) \times$$
$$\times \prod_{i=k+1}^{2n-1-k} \prod_{j=1}^{k} \theta(p_{i+j-1,j} - p_{i+j-2,j}) \prod_{i=2}^{k-1} \prod_{j=1}^{i-1} \theta(p_{i+j-1,j} - p_{i+j,j+1}) \times$$
$$\times \prod_{i=2}^{k} \prod_{j=1}^{i-1} \theta(p_{i+j-1,j} - p_{i+j-2,j}) \times$$
$$\times \prod_{i=2}^{k-1} \theta(p_{2i-1,i} + p_{2i+1,i+1} + p_{2i+1,i} - p_{2i,i}) \theta(p_{2i-1,i} + p_{2i+1,i+1} + p_{2i-1,i-1} - p_{2i,i}) \times$$
$$\times \theta(p_{2i-1,i} + p_{2i+1,i+1} + p_{2i+1,i} - p_{2i,i}) \times$$
$$\times \theta(p_{2k-1,k} + p_{2k+1,k} - p_{2k,k}) \theta(p_{2k-1,k} + p_{2k-1,k-1} - p_{2k,k}). \qquad (424)$$

Just like for the type $C_n$, the character formula is a bit different for the cases of even and odd $k$. For odd $k$ we have

$$S_\lambda^{D_n}(x) = \sum_{\mathcal{C}} \Theta_J^{D_n}(p) x_1^{p_{1,1}} \prod_{i=2}^{k} x_i^{\sum_{j=1}^{i} p_{2i-1,j} - 2\sum_{j=1}^{i-1} p_{2i-2,j} + \sum_{j=1}^{i-1} p_{2i-3,j}} \times$$
$$\times \prod_{i=k}^{(2n-k-1)/2} x_i^{\sum_{j=1}^{k} (p_{2i-1,j} - 2p_{2i-2,j} + p_{2i-3,j})} \times$$
$$\times \left( x_{(2n-k+1)/2} \right)^{\lambda_1 + \sum_{j=2}^{k} p_{2n-k,j} - 2\sum_{j=1}^{k} p_{2n-k-1,j} + \sum_{j=1}^{k} p_{2n-k-2,j}} \times$$
$$\times \prod_{i=\frac{2n-k+3}{2}}^{n} x_i^{\sum_{j=2i-2n+k+1}^{k} p_{2i-1,j} - 2\sum_{j=2i-2n+k}^{k} p_{2i-2,j} + \sum_{j=2i-2n+k-1}^{k} p_{2i-3,j}}. \qquad (425)$$

For even $k$ we have

$$S_\lambda^{D_n}(x) = \sum_{\mathcal{C}} \Theta_J^{D_n}(p) x_1^{p_{1,1}} \prod_{i=2}^{k} x_i^{\sum_{j=1}^{i} p_{2i-1,j} - 2\sum_{j=1}^{i-1} p_{2i-2,j} + \sum_{j=1}^{i-1} p_{2i-3,j}} \times$$
$$\times \prod_{i=k}^{(2n-k)/2} x_i^{\sum_{j=1}^{k} (p_{2i-1,j} - 2p_{2i-2,j} + p_{2i-3,j})} \times$$
$$\times \prod_{i=\frac{2n-k+2}{2}}^{n} x_i^{\sum_{j=2i-2n+k+1}^{k} p_{2i-1,j} - 2\sum_{j=2i-2n+k}^{k} p_{2i-2,j} + \sum_{j=2i-2n+k-1}^{k} p_{2i-3,j}}. \qquad (426)$$

In both cases $\mathcal{C} = \mathbb{Z}^{|\Phi_J^+|}$ if $\lambda_i \in \mathbb{Z}$ and $\mathcal{C} = \left(\mathbb{Z}+\frac{1}{2}\right)^{|\Phi_J^+|}$ if $\lambda_i \in \mathbb{Z}+\frac{1}{2}$. Applying the proposition (5.1.1) we get the integral representation of the equivariant volume:

$$Z_J^{D_n}(\lambda|x) = \int\limits_{\mathbb{R}^{|\Phi_J^+|}} \prod_{i=k}^{2n-1-k} \prod_{j=1}^{k} dp_{i+j-1,j} \prod_{i=1}^{k-1} \prod_{j=1}^{i} dp_{i+j-1,j} \Theta_J^{D_n}(p) e^{\mathcal{F}_k(p,\lambda,x)} \quad (427)$$

where for odd $k$ we have

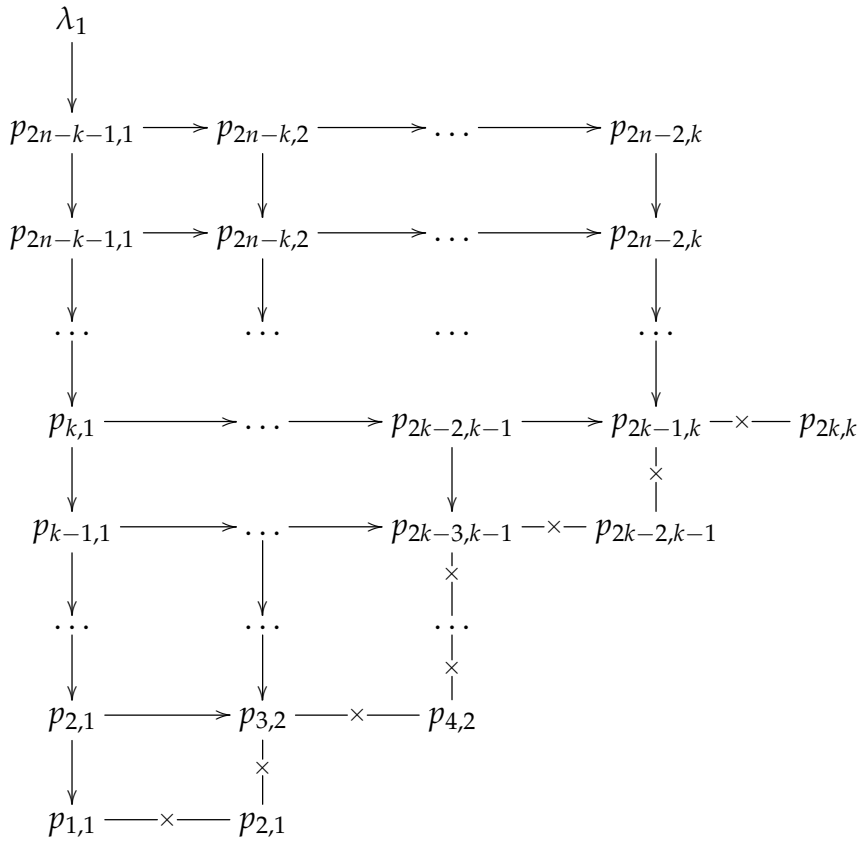
$$\begin{gathered}
\mathcal{F}_k(p,\lambda,x) = x_{1,1}p_{1,1} + \\
+ \sum_{i=2}^{k} x_i \left( \sum_{j=1}^{i} p_{2i-1,j} - 2\sum_{j=1}^{i-1} p_{2i-2,j} + \sum_{j=1}^{i-1} p_{2i-3,j} \right) + \\
+ \sum_{i=k}^{(2n-k-1)/2} x_i \sum_{j=1}^{k} (p_{2i-1,j} - 2p_{2i-2,j} + p_{2i-3,j}) + \\
+ x_{(2n-k+1)/2} \left( \lambda_1 + \sum_{j=2}^{k} p_{2n-k,j} - 2\sum_{j=1}^{k} p_{2n-k-1,j} + \sum_{j=1}^{k} p_{2n-k-2,j} \right) + \\
+ \sum_{i=\frac{2n-k+3}{2}}^{n} x_i \left( \sum_{j=2i-2n+k+1}^{k} p_{2i-1,j} - 2 \sum_{j=2i-2n+k}^{k} p_{2i-2,j} + \sum_{j=2i-2n+k-1}^{k} p_{2i-3,j} \right)
\end{gathered} \quad (428)$$

and for even $k$ we have

$$\begin{gathered}
\mathcal{F}_k(p,\lambda,x) = x_{1,1}p_{1,1} + \\
+ \sum_{i=2}^{k} x_i \left( \sum_{j=1}^{i} p_{2i-1,j} - 2\sum_{j=1}^{i-1} p_{2i-2,j} + \sum_{j=1}^{i-1} p_{2i-3,j} \right) + \\
+ \sum_{i=k}^{(2n-k)/2} x_i \sum_{j=1}^{k} (p_{2i-1,j} - 2p_{2i-2,j} + p_{2i-3,j}) + \\
+ \sum_{i=\frac{2n-k+2}{2}}^{n} x_i \left( \sum_{j=2i-2n+k+1}^{k} p_{2i-1,j} - 2 \sum_{j=2i-2n+k}^{k} p_{2i-2,j} + \sum_{j=2i-2n+k-1}^{k} p_{2i-3,j} \right).
\end{gathered} \quad (429)$$

Using the correspondence between the BZ-patterns and the Givental graphs of type $D_n$ from the previous chapter one can derive the conjectural graph

for $\Theta_J^{D_n}(p)$ in variables $p_{i,j}$. It is the same graph as for $C_n$ but without one top row:




(430)

Like for $C_n$ before, we are going to look into the low rank cases $n = 2, 3$ to show explicit formulas for each $k$. There is an additional relation to $A_n$ due to isomorphism $A_3 \simeq D_3$ described in A.6.

## 5.7 EXAMPLE: ROOT SYSTEM $D_2$

The root system is generated by two simple roots:

$$\alpha_1 = \epsilon_1 - \epsilon_2, \qquad \alpha_2 = \epsilon_1 + \epsilon_2. \tag{431}$$

These roots are orthogonal, so we have the isomorphism $D_2 \simeq A_1 \times A_1$. The corresponding simple reflections for these roots:

$$s_1 \colon e_1 \leftrightarrow e_2, \tag{432}$$

$$\bar{\varepsilon}_2 = \varepsilon_2 s_1 \varepsilon_2 \colon e_1 \leftrightarrow -e_2. \tag{433}$$

The Weyl group is a direct product of two Weyl groups of $A_1$:

$$\mathcal{W}^{D_2} = \mathfrak{S}_2 \times \mathfrak{S}_2. \tag{434}$$

The longest element is

$$w_0 = \bar{\varepsilon}_2 s_1 = s_1 \bar{\varepsilon}_2 = s_1 \varepsilon_2 s_1 \varepsilon_2 = \varepsilon_2 s_1 \varepsilon_2 s_1. \tag{435}$$

The highest weight $\lambda = (\lambda_1, \lambda_2)$ satisfies the condition $\lambda_1 \geq |\lambda_2|$. Thus we have two following cases of non-trivial stabilizer of the weight:

1. $\lambda_1 = \lambda_2 \neq 0$;
2. $\lambda_1 = -\lambda_2 \neq 0$.

### 5.7.1 *Case* $\lambda = (\lambda_1, \lambda_1)$

The stabilizer $\mathrm{Stab}_{\mathcal{W}^{D_2}}(\lambda)$ is generated by $s_1$, so one has the following longest element in the factor $\mathcal{W}^{D_2} / \mathrm{Stab}_{\mathcal{W}^{D_2}}(\lambda)$:

$$w_0^J = \varepsilon_2 s_1 \varepsilon_2 = \bar{\varepsilon}_2. \tag{436}$$

The equivariant volume is as follows:

$$Z_J^{D_2}(\lambda|x) = \mathcal{A}_2 e^{\langle \lambda, x \rangle} = \frac{e^{\lambda_1(x_1+x_2)}}{x_1+x_2} - \frac{e^{\lambda_1(-x_1-x_2)}}{x_1+x_2} = Z^{A_1}(\lambda_1, -\lambda_1 | x_1, -x_2). \tag{437}$$

The BZ-patterns are defined by the following simple inequalities:

$$\lambda_1 = p_{3,1} \geq p_{1,1} \geq -p_{3,1}. \tag{438}$$

So the polytope has the following characteristic function:

$$\Theta_J^{D_2}(p) = \theta(p_{3,1} - p_{1,1})\theta(p_{3,1} + p_{1,1}). \tag{439}$$

The character and equivariant volume formulas using patterns give simple expressions:

$$S_\lambda^{D_2}(x) = \sum_{p_{1,1} \in \mathbb{Z}} \theta(\lambda_1 - p_{1,1})\theta(\lambda_1 + p_{1,1}) x_1^{p_{1,1}} x_2^{p_{1,1}}, \tag{440}$$

$$Z_J^{D_2}(\lambda|x) = \int_{\mathbb{R}} dp_{1,1}\, \theta(\lambda_1 - p_{1,1})\theta(\lambda_1 + p_{1,1}) e^{(x_1+x_2)p_{1,1}}. \tag{441}$$

The Givental diagram of type $D_2$ corresponding to the polytope:

$$\mu_{2,2} \longrightarrow \mu_{1,1} \text{—}\times\text{—} \mu_{2,2} \tag{442}$$

### 5.7.2 *Case* $\lambda = (\lambda_1, -\lambda_1)$

The stabilizer here is generated by $\bar{\varepsilon}_2$, so one has

$$w_0^J = s_1. \tag{443}$$

The equivariant volume then

$$Z_J^{D_2}(\lambda|x) = \mathcal{A}_1 e^{\langle\lambda,x\rangle} = \frac{e^{\lambda_1(x_1-x_2)}}{x_1-x_2} - \frac{e^{\lambda_1(x_2-x_1)}}{x_1-x_2} = Z^{A_1}(\lambda_1, -\lambda_1|x_1, x_2). \tag{444}$$

The BZ-patterns are defined by the following inequalities:

$$p_{3,1} = \lambda_1 \geq p_{2,1} \geq p_{3,2} = -\lambda_1, \tag{445}$$

$$p_{2,1} \geq p_{1,1} \geq p_{2,1} - p_{3,1} - p_{3,2} = p_{2,1} \Rightarrow p_{1,1} = p_{2,1}. \tag{446}$$

The corresponding polytope is given by the characteristic function

$$\Theta_J^{D_2}(p) = \theta(p_{3,1} - p_{2,1})\theta(p_{3,1} + p_{2,1}). \tag{447}$$

The character of equivariant volume has the similar form as in the previous case:

$$S_\lambda^{D_2}(x) = \sum_{p_{2,1}\in\mathbb{Z}} \theta(\lambda_1 - p_{2,1})\theta(\lambda_1 + p_{2,1}) x_1^{p_{2,1}} x_2^{-p_{2,1}}, \tag{448}$$

$$Z_J^{D_2}(\lambda|x) = \int_{IR} dp_{2,1}\, \theta(\lambda_1 - p_{2,1})\theta(\lambda_1 + p_{2,1}) e^{(x_1-x_2)p_{2,1}}. \tag{449}$$

The corresponding Givental diagram of type $D_2$ is the same as in the previous case:

$$\mu_{2,2} \longrightarrow \mu_{1,1} \text{—}\times\text{—} \mu_{2,2} \tag{450}$$

## 5.8 EXAMPLE: ROOT SYSTEM $D_3$

The root system of type $D_3$ has three simple roots:

$$\alpha_1 = \epsilon_1 - \epsilon_2, \qquad \alpha_2 = \epsilon_2 - \epsilon_3, \qquad \alpha_3 = \epsilon_2 + \epsilon_3. \tag{451}$$

These roots provide the following simple reflections:

$$s_1 \colon \epsilon_1 \leftrightarrow \epsilon_2, \tag{452}$$

$$s_2 \colon \epsilon_2 \leftrightarrow \epsilon_3, \tag{453}$$

$$\bar{\varepsilon}_3 = \varepsilon_3 s_2 \varepsilon_3 \colon \epsilon_2 \leftrightarrow -\epsilon_3. \tag{454}$$

The element of maximal length:

$$w_0 = s_2 s_1 s_2 \bar{\varepsilon}_3 s_1 s_2. \tag{455}$$

The highest weight $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ satisfies the condition $\lambda_1 \geq \lambda_2 \geq |\lambda_3|$ so we have three cases of non-trivial stabilizer:

1. $\lambda_1 \neq 0$, $\lambda_2 = \lambda_3 = 0$;
2. $\lambda_1 = \lambda_2 = \lambda_3$;
3. $\lambda_1 = \lambda_2 = -\lambda_3$.

### 5.8.1 *Case* $\lambda = (\lambda_1, 0, 0)$

The stabilizer of $\lambda$ in $\mathcal{W}^{D_3}$ is generated by commuting simple reflections $s_2$ and $\bar{\varepsilon}_3$. The element of maximal length in the stabilizer is $w_0^J = s_2 \bar{\varepsilon}_3$, so we have

$$w_0^J = s_1 s_2 \bar{\varepsilon}_3 s_1 = s_1 \bar{\varepsilon}_3 s_2 s_1. \tag{456}$$

The equivariant volume is given by

$$Z_J^{D_3}(\lambda, x) = \mathcal{A}_1 \mathcal{A}_2 \mathcal{A}_3 \mathcal{A}_1 e^{\langle \lambda, x \rangle} =$$
$$= \frac{e^{\lambda_1 x_1}}{(x_1 - x_2)(x_1 - x_3)(x_1 + x_2)(x_1 + x_3)} +$$
$$+ \frac{e^{\lambda_1 x_2}}{(x_2 - x_1)(x_2 - x_3)(x_1 + x_2)(x_2 + x_3)} +$$
$$+ \frac{e^{\lambda_1 x_3}}{(x_3 - x_1)(x_3 - x_2)(x_1 + x_3)(x_2 + x_3)} +$$

$$+\frac{e^{-\lambda_1 x_2}}{(x_1+x_2)(x_3-x_2)(x_1-x_2)(x_2+x_3)}+$$
$$+\frac{e^{-\lambda_1 x_1}}{(x_1+x_2)(x_3-x_1)(x_2-x_1)(x_1+x_3)}+$$
$$+\frac{e^{-\lambda_1 x_3}}{(x_1+x_3)(x_2-x_3)(x_1-x_3)(x_2+x_3)}. \tag{457}$$

The BZ-patterns are defined by the following inequalities:

$$\lambda_1 = p_{5,1} \geq p_{4,1} \geq p_{3,1} \geq p_{2,1} \geq p_{1,1}, \tag{458}$$

$$p_{1,1} \geq p_{2,1} - p_{3,1}, \qquad p_{2,1} \geq 0. \tag{459}$$

The corresponding characteristic function of the polytope is

$$\Theta_J^{D_3}(p) = \theta(\lambda_1 - p_{4,1})\theta(p_{4,1} - p_{3,1})\theta(p_{3,1} - p_{2,1})\times$$
$$\times\theta(p_{2,1} - p_{1,1})\theta(p_{2,1})\theta(p_{1,1} + p_{3,1} - p_{2,1}). \tag{460}$$

The character formula gives

$$S_\lambda^{D_3}(x) = \sum_{\mathbb{Z}^4} \Theta_J^{D_3}(p)\, x_1^{p_{1,1}} x_2^{p_{3,1}-2p_{2,1}+p_{1,1}} x_3^{\lambda_1-2p_{4,1}+p_{3,1}}. \tag{461}$$

Taking the limit of this character gives the following representation of the equivariant volume

$$Z_J^{D_3}(\lambda|x) = \int_{\mathbb{R}^4} dp_{1,1}dp_{2,1}dp_{3,1}dp_{4,1}\, \Theta_J^{D_3}(p)\times$$
$$\times e^{x_1 p_{1,1} + x_2(p_{3,1}-2p_{2,1}+p_{1,1}) + x_3(\lambda_1-2p_{4,1}+p_{3,1})}. \tag{462}$$

The Givental graph of type $D_3$ corresponding to the polytope looks as follows:

$$\mu_{3,3} \longrightarrow \lambda_{2,2} \longrightarrow \mu_{2,2} \longrightarrow \lambda_{1,1} \longrightarrow \mu_{1,1} \text{ —}\times\text{— } \lambda_{1,1} \tag{463}$$

One can use the identification of $D_3$ and $A_3$ from the section (A.6) to show that

$$\begin{aligned}
&Z_J^{D_3}(\lambda_1,0,0|x_1,x_2,x_3) = \\
&= \frac{e^{\frac{\lambda_1}{2}(y_1+y_2-y_3-y_4)}}{(y_2-y_3)(y_1-y_3)(y_1-y_4)(y_2-y_4)} + \\
&+ \frac{e^{\frac{\lambda_1}{2}(y_1-y_2+y_3-y_4)}}{(y_3-y_2)(y_1-y_2)(y_1-y_4)(y_3-y_4)} + \\
&+ \frac{e^{\frac{\lambda_1}{2}(-y_1+y_2+y_3-y_4)}}{(y_3-y_1)(y_2-y_1)(y_2-y_4)(y_3-y_4)} + \\
&+ \frac{e^{\frac{\lambda_1}{2}(-y_1+y_2-y_3+y_4)}}{(y_1-y_4)(y_2-y_1)(y_2-y_3)(y_3-y_4)} + \\
&+ \frac{e^{\frac{\lambda_1}{2}(-y_1-y_2+y_3+y_4)}}{(y_1-y_4)(y_3-y_1)(y_3-y_2)(y_2-y_4)} + \\
&+ \frac{e^{\frac{\lambda_1}{2}(y_1-y_2-y_3+y_4)}}{(y_2-y_4)(y_1-y_2)(y_1-y_3)(y_3-y_4)} = \\
&= Z_{J'}^{A_3}(\lambda_1',\lambda_2',\lambda_3',\lambda_4'|y_1,y_2,y_3,y_4),
\end{aligned} \tag{464}$$

where

$$\lambda' = \left(\frac{\lambda_1}{2}, \frac{\lambda_1}{2}, -\frac{\lambda_1}{2}, -\frac{\lambda_1}{2}\right). \tag{465}$$

### 5.8.2 *Case* $\lambda = (\lambda_1, \lambda_1, \lambda_1)$

The stabilizer of $\lambda$ is generated by the subgroup $\mathfrak{S}_3$ generated by $s_1$ and $s_2$. The element of maximal length in this subgroup is $w_0^J = s_1 s_2 s_1$, so we have

$$w_0^J = s_2 s_1 \bar{\varepsilon}_3 \tag{466}$$

and for the equivariant volume the following holds

$$\begin{aligned}
&Z_J^{D_3}(\lambda|x) = \mathcal{A}_2\mathcal{A}_1\mathcal{A}_3 e^{\langle\lambda,x\rangle} = \\
&= \frac{e^{\lambda_1(x_1+x_2+x_3)}}{(x_1+x_2)(x_1+x_3)(x_2+x_3)} + \frac{e^{\lambda_1(x_1-x_2-x_3)}}{(x_1-x_3)(x_2-x_1)(x_2+x_3)} + \\
&+ \frac{e^{\lambda_1(x_2-x_1-x_3)}}{(x_2-x_3)(x_1-x_2)(x_1+x_3)} + \frac{e^{\lambda_1(x_3-x_1-x_2)}}{(x_3-x_2)(x_1-x_3)(x_1+x_2)}.
\end{aligned} \tag{467}$$

The BZ-patterns are defined by the following relations:

$$\lambda_1 \geq p_{2,1} \geq p_{3,2} \geq -\lambda_1, \tag{468}$$

$$p_{2,1} \geq p_{1,1} \geq p_{2,1} - p_{3,2} - \lambda_1. \tag{469}$$

The corresponding characteristic function of the polytope is as follows:

$$\begin{aligned}\Theta_J^{D_3}(p) = \theta(\lambda_1 - p_{2,1})\theta(p_{2,1} - p_{3,2})\theta(p_{3,2} + \lambda_1)\times \\ \times\theta(p_{2,1} - p_{1,1})\theta(p_{1,1} + p_{3,2} + \lambda - p_{2,1}).\end{aligned} \tag{470}$$

The character is given by the sum over finite number of points on polytope:

$$S_\lambda^{D_3}(x) = \sum_{\mathbb{Z}^3} \Theta_J^{D_3}(p)\, x_1^{p_{1,1}} x_2^{\lambda_1 + p_{3,2} - 2p_{2,1} + p_{1,1}} x_3^{p_{3,2}}. \tag{471}$$

Taking the limits (5.1.1) gives the following expression for the equivariant volume:

$$Z_J^{D_3}(\lambda|x) = \int\limits_{\mathbb{R}^3} dp_{1,1}dp_{2,1}dp_{3,2}\Theta_J^{D_3}(p)\, e^{x_1 p_{1,1} + x_2(\lambda_1 + p_{3,2} + p_{1,1} - 2p_{2,1}) + x_3 p_{3,2}}. \tag{472}$$

The Givental graph of type $D_3$ corresponding to the polytope looks as follows:

$$\begin{array}{ccccc} \mu_{3,3} & \longrightarrow & \lambda_{1,1} & \longrightarrow & \mu_{1,1} \\ & & \downarrow & & \vdots \\ & & \mu_{2,1} & -\times- & \lambda_{1,1} \end{array} \tag{473}$$

The identification of $D_3$ and $A_3$ gives the following:

$$\begin{aligned} Z_J^{D_3}(\lambda_1, \lambda_1, \lambda_1|x_1, x_2, x_3) = \\ = \frac{e^{\frac{\lambda_1}{2}(y_1+y_2+y_3-3y_4)}}{(y_1-y_4)(y_2-y_4)(y_3-y_4)} + \frac{e^{\frac{\lambda_1}{2}(y_1+y_2-3y_3+y_4)}}{(y_1-y_3)(y_3-y_2)(y_3-y_4)} + \\ + \frac{e^{\frac{\lambda_1}{2}(y_1+y_3-3y_2+y_4)}}{(y_1-y_2)(y_2-y_3)(y_2-y_4)} + \frac{e^{\frac{\lambda_1}{2}(-3y_1+y_2+y_3+y_4)}}{(y_2-y_1)(y_1-y_3)(y_1-y_4)} = \\ = Z_{J'}^{A_3}(\lambda_1', \lambda_2', \lambda_3', \lambda_4'|y_1, y_2, y_3, y_4), \end{aligned} \tag{474}$$

where

$$\lambda' = \left(\frac{\lambda_1}{2}, \frac{\lambda_1}{2}, \frac{\lambda_1}{2}, -\frac{3\lambda_1}{2}\right). \tag{475}$$

### 5.8.3 *Case* $\lambda = (\lambda_1, \lambda_1, -\lambda_1)$

The stabilizer of $\lambda$ in this case is another subgroup $\mathfrak{S}_3$ in $\mathcal{W}^{D_3}$ generated by $s_1$ and $\bar{\varepsilon}_3$. The element of maximal length is $w_0^J = s_1\bar{\varepsilon}_3 s_1$ and we have

$$w_0^J = \bar{\varepsilon}_3 s_1 s_2. \tag{476}$$

Then the equivariant volume is given by

$$\begin{gathered} Z_J^{D_3}(\lambda|x) = \mathcal{A}_3\mathcal{A}_1\mathcal{A}_2 e^{\langle\lambda,x\rangle} = \\ = \frac{e^{\lambda_1(x_1+x_2-x_3)}}{(x_1+x_2)(x_2-x_3)(x_1-x_3)} + \frac{e^{\lambda_1(x_1+x_3-x_2)}}{(x_1+x_3)(x_3-x_2)(x_1-x_2)} + \\ + \frac{e^{\lambda_1(x_2+x_3-x_1)}}{(x_2+x_3)(x_3-x_1)(x_2-x_1)} - \frac{e^{\lambda_1(-x_1-x_2-x_3)}}{(x_2+x_3)(x_1+x_2)(x_1+x_3)}. \end{gathered} \tag{477}$$

The BZ-patterns are defined by the following relations:

$$\lambda_1 \geq p_{2,1} \geq p_{3,2} \geq -\lambda_1, \tag{478}$$

$$p_{2,1} \geq p_{1,1} \geq p_{2,1} - \lambda_1 - p_{3,2}. \tag{479}$$

The corresponding characteristic function of the polytope:

$$\begin{aligned} \Theta_J^{D_3}(p) = \theta(\lambda_1 - p_{2,1})\theta(p_{2,1} - p_{3,2})\theta(p_{3,2} + \lambda_1)\times \\ \times\theta(p_{2,1} - p_{1,1})\theta(p_{1,1} + \lambda_1 + p_{3,2} - p_{2,1}). \end{aligned} \tag{480}$$

The character formula gives

$$S_\lambda^{D_3}(x) = \sum_{\mathbb{Z}^3} \Theta_J^{D_3}(p)\, x_1^{p_{1,1}} x_2^{\lambda_1 + p_{3,2} - 2p_{2,1} + p_{1,1}} x_3^{p_{3,2}}. \tag{481}$$

Taking the limit (5.1.1) gives the following integral representation for the equivariant volume:

$$Z_J^{D_3}(\lambda|x) = \int\limits_{\mathbb{R}^3} dp_{1,1}dp_{2,1}dp_{3,2}\Theta_J^{D_3}(p)\, e^{x_1p_{1,1} + x_2(\lambda_1 + p_{3,2} - 2p_{2,1} + p_{1,1}) + x_3p_{3,2}}. \tag{482}$$

The Givental graph of type $D_3$ corresponding to the polytope is the same as in previous case:

$$\begin{array}{ccccc} \mu_{3,3} & \longrightarrow & \lambda_{1,1} & \longrightarrow & \mu_{1,1} \\ & & \downarrow & & \times \\ & & \mu_{2,1} & -\times- & \lambda_{1,1} \end{array} \tag{483}$$

The identification of $D_3$ and $A_3$ gives the following:

$$Z_J^{D_3}(\lambda_1,\lambda_1,-\lambda_1|x_1,x_2,x_3) =$$
$$= \frac{e^{\frac{\lambda_1}{2}(3y_1-y_2-y_3-y_4)}}{(y_1-y_4)(y_1-y_2)(y_1-y_3)} + \frac{e^{\frac{\lambda_1}{2}(-y_1+3y_2-y_3-y_4)}}{(y_2-y_4)(y_2-y_1)(y_2-y_3)} +$$
$$+ \frac{e^{\frac{\lambda_1}{2}(-y_1-y_2+3y_3-y_4)}}{(y_3-y_4)(y_3-y_1)(y_3-y_2)} - \frac{e^{\frac{\lambda_1}{2}(-y_1-y_2-y_3+3y_4)}}{(y_3-y_4)(y_1-y_4)(y_2-y_4)} =$$
$$= Z_{J'}^{A_3}(\lambda_1',\lambda_2',\lambda_3',\lambda_4'|y_1,y_2,y_3,y_4), \tag{484}$$

where

$$\lambda' = \left(\frac{3\lambda_1}{2}, -\frac{\lambda_1}{2}, -\frac{\lambda_1}{2}, -\frac{\lambda_1}{2}\right). \tag{485}$$

## 5.9 OKADA FORMULAS

Okada formulas for characters (see Appendix B.6) can be translated to the similar formulas for equivariant volumes with non-trivial stabilizer of $\lambda$.

**Theorem 5.9.1.** *Let $n \geq m$, $n, m \in \mathbb{N}$. The following relations hold*

1.

$$Z_J^{A_{n+m-1}}(\underbrace{t,\dots,t}_{n},\underbrace{0,\dots,0}_{m}|x_1,\dots,x_n,y_1,\dots,y_m) =$$
$$= \int\limits_{\mathbb{R}^m} \prod_{i=1}^{m} d\lambda_i\, \Theta_m(\lambda,t)\cdot Z_{J_1}^{A_{n-1}}(\underbrace{t,\dots,t}_{n-m},\lambda_1,\dots,\lambda_m|x_1,\dots,x_n)\times$$
$$\times Z^{A_{m-1}}(t-\lambda_m,\dots,t-\lambda_1|y_1,\dots,y_m) \tag{486}$$

2.

$$Z_J^{C_{n+m}}(\underbrace{t,\dots,t}_{n+m}|x_1,\dots,x_n,y_1,\dots,y_m) =$$
$$= \int\limits_{\mathbb{R}^m} \prod_{i=1}^{m} d\lambda_i\, \Theta_m(\lambda,t)\cdot Z_{J_1}^{C_n}(\underbrace{t,\dots,t}_{n-m},\lambda_1,\dots,\lambda_m|x_1,\dots,x_n)\times$$
$$\times Z^{C_m}(\lambda_1,\dots,\lambda_m|y_1,\dots,y_m). \tag{487}$$

*3.*

$$Z_J^{D_{n+m+1}}(\underbrace{t,\dots,t}_{n+m},\pm t|x_1,\dots,x_n,y_1,\dots,y_m,0)=$$
$$=\int\limits_{\mathbb{R}^m}\prod_{i=1}^m d\lambda_i\,\Theta_m(\lambda,t)\cdot Z_{J_1}^{B_n}(\underbrace{t,\dots,t}_{n-m},\lambda_1,\dots,\lambda_m|x_1,\dots,x_n)\times$$
$$\times Z^{B_m}(\lambda_1,\dots,\lambda_m|y_1,\dots,y_m). \tag{488}$$

*In all three cases*

$$\Theta_m(\lambda,t)=\theta(t-\lambda_1)\theta(\lambda_m)\cdot\prod_{i=1}^{m-1}\theta(\lambda_i-\lambda_{i+1}). \tag{489}$$

**Theorem 5.9.2.** *1. Let $n\geq m$. Then we have*

$$Z_{J_1}^{A_{m+n-1}}(\underbrace{s,\dots,s}_{m},\underbrace{0,\dots,0}_{n}|x)\cdot Z_{J_2}^{A_{m+n-1}}(\underbrace{t,\dots,t}_{n},\underbrace{0,\dots,0}_{m}|x)=$$
$$=\int\limits_{\mathbb{R}^m}\prod_{i=1}^m d\lambda_i\,\Theta_m(\lambda,s,t)\times$$
$$\times Z_J^{A_{m+n-1}}(\lambda_1,\dots,\lambda_m,\underbrace{t,\dots,t}_{n-m},s+t-\lambda_m,\dots,s+t-\lambda_1|x) \tag{490}$$

*where*

$$\Theta_m(\lambda,s,t)=\theta(s+t-\lambda_1)\theta(\lambda_m-s)\theta(\lambda_m-t)\cdot\prod_{i=1}^{m-1}\theta(\lambda_i-\lambda_{i+1}). \tag{491}$$

*2. For $t\geq s$ we have*

$$Z_J^{B_n}(\underbrace{s,\dots,s}_{n}|x_1,\dots,x_n)\cdot Z_J^{B_n}(\underbrace{t,\dots,t}_{n}|x_1,\dots,x_n)=$$
$$=\int\limits_{\mathbb{R}^n}\prod_{i=1}^n d\lambda_i\,\Theta_m(\lambda,2s)\cdot Z^{B_n}(\lambda_1+t-s,\dots,\lambda_n+t-s|x_1,\dots,x_n).$$

(492)

*3. For $t\geq s$ we have*

$$Z_J^{C_n}(\underbrace{s,\dots,s}_{n}|x_1,\dots,x_n)\cdot Z_J^{C_n}(\underbrace{t,\dots,t}_{n}|x_1,\dots,x_n)=$$
$$=2^{-n}\cdot\int\limits_{\mathbb{R}^n}\prod_{i=1}^n d\lambda_i\cdot\Theta_m(\lambda,2s)\cdot Z^{C_n}(\lambda_1+t-s,\dots,\lambda_n+t-s|x_1,\dots,x_n).$$

(493)

*Proof.* The statements can be proved just by using the proposition (5.1.1) on both sides of the formulas in theorems (B.6.1) and (B.6.2). □

# A

# ROOT SYSTEMS

## A.1 DEFINITIONS

Here we provide the set of definitions and notations related to root systems, which can be found in [6] and [14].

**Definition A.1.1.** Take a real space $\mathfrak{h}^* = \mathbb{R}^n$ equipped with an inner product $(\cdot, \cdot)$. The root system is a finite set of vectors $\Phi_n \in \mathbb{R}^n$ with the following properties:

- $\Phi_n$ spans $\mathfrak{h}^*$;
- if $\alpha \in \Phi_n$ then $-\alpha \in \Phi_n$ and no other multiples of that root are in the system;
- Each root defines a reflection

$$s_\alpha(\beta) = \beta - \frac{2(\beta, \alpha)}{(\alpha, \alpha)}\alpha \tag{494}$$

  with respect to the hyperplane orthogonal to it, and these reflections map $\Phi_n$ to itself;
- for any $\alpha, \beta \in \Phi_n$ the number $\frac{2(\beta,\alpha)}{(\alpha,\alpha)}$ is an integer.

**Definition A.1.2.** For the space $\mathfrak{h}^*$ one can take a dual space $\mathfrak{h}$ with the dual root system $\Phi_n^\vee$ and coroots $\alpha^\vee \in \Phi_n^\vee$ defined as

$$\left\langle \beta, \alpha^\vee \right\rangle := \alpha^\vee(\beta) = \frac{2(\beta, \alpha)}{(\alpha, \alpha)}, \qquad \alpha, \beta \in \Phi_n. \tag{495}$$

The inner product provides then the identification

$$\alpha^\vee \leftrightarrow \frac{2\alpha}{(\alpha,\alpha)} \tag{496}$$

and the reflections can be written as

$$s_\alpha(\beta) = \beta - \left\langle \beta, \alpha^\vee \right\rangle \alpha. \tag{497}$$

**Definition A.1.3.** The reflections of the root system form a group $\mathcal{W}^{\Phi_n}$ called Weyl group.

**Definition A.1.4.** One can choose a direction in $\mathfrak{h}^*$ defined by a vector $\beta \in \mathfrak{h}^*$, so that the set of root can be represented as a disjoint union

$$\Phi_n = \Phi_n^+ \cup \Phi_n^-. \tag{498}$$

where

$$\Phi_n^+ = \{\alpha \in \Phi_n \colon (\alpha,\beta) > 0\}, \tag{499}$$

$$\Phi_n^- = \Phi_n \setminus \Phi_n^+. \tag{500}$$

The roots from the subset $\Phi_n^+$ are called positive roots, the roots from the subset $\Phi_n^-$ are called negative roots.

**Definition A.1.5.** The positive root is called simple if it cannot be represented as a sum of 2 other positive roots. The number of simple roots is called a rank of the root system. We denote them as $\alpha_i \in \Phi_n$ for $i = 1, \ldots, n$. The reflections defined by simple roots are called simple reflections $s_i \in \mathcal{W}^{\Phi_n}$ and are generators of the Weyl group.

**Definition A.1.6.** The fundamental weights are defined as vectors $\omega_i \in \mathfrak{h}^*$ such that

$$\left\langle \omega_i, \alpha_j^\vee \right\rangle = \delta_{i,j}, \qquad i,j = 1, \ldots, n, \tag{501}$$

where $\delta_{i,j} = 1$ if $i = j$ and $\delta_{i,j} = 0$ if $i \neq j$.

**Definition A.1.7.** The fundamental weights $\omega_i$ generate a $\mathbb{Z}$-lattice called weight lattice $\Lambda_W \subset \mathfrak{h}^*$. Each $\omega \in \Lambda_W$ is called a weight.

**Definition A.1.8.** The Cartan matrix for the root system $\Phi_n$ is a $n \times n$ matrix defined as

$$M^{\Phi_n} = \| \left\langle \alpha_i, \alpha_j^\vee \right\rangle \| . \tag{502}$$

**Definition A.1.9** (see also [44])**.** The Chevalley basis for a Lie algebra with the root system $\Phi_n$ is a set of generators $h_i, e_i, f_i$, $i = 1, \ldots, n$, defined by the following relations:

$$h_i = \alpha_i^\vee, \qquad i = 1, \ldots, n; \tag{503}$$

$$[h_i, h_j] = 0, \qquad i, j = 1, \ldots, n; \tag{504}$$

$$[h_i, e_j] = e_j M^{\Phi_n}_{j,i} = e_j \left\langle \alpha_j, \alpha_i^\vee \right\rangle, \qquad i, j = 1, \ldots, n; \tag{505}$$

$$[h_i, f_j] = -f_j M^{\Phi_n}_{j,i} = -f_j \left\langle \alpha_j, \alpha_i^\vee \right\rangle, \qquad i, j = 1, \ldots, n; \tag{506}$$

$$[e_i, f_j] = h_i \delta_{i,j}, \qquad i, j = 1, \ldots, n; \tag{507}$$

$$[e_i, e_j] = [f_i, f_j] = 0, \qquad M^{\Phi_n}_{i,j} = 0; \tag{508}$$

$$[e_i, [e_i, e_j]] = [f_i, [f_i, f_j]] = 0, \qquad M^{\Phi_n}_{i,j} = -1; \tag{509}$$

$$[e_i, [e_i, [e_i, e_j]]] = [f_i, [f_i, [f_i, f_j]]] = 0, \qquad M^{\Phi_n}_{i,j} = -2; \tag{510}$$

$$[e_i, [e_i, [e_i, [e_i, e_j]]]] = [f_i, [f_i, [f_i, [f_i, f_j]]]] = 0, \qquad M^{\Phi_n}_{i,j} = -3. \tag{511}$$

The Chevalley basis is defined by the roots and relations between them and allows us to reconstruct the Lie algebra from its root system. The space $\mathfrak{h}$ spanned by $h_i$ is the Cartan subalgebra of the Lie algebra.

## A.2 $A_n$ DATA

Take the real space $\mathbb{R}^{n+1}$ spanned by its basis vectors $\epsilon_i$, $i = 1, \ldots, n+1$ and the inner product defined by

$$(\epsilon_i, \epsilon_j) = \delta_{i,j}. \tag{512}$$

It is convenient to take the base space as a factor

$$\mathfrak{h}^* = \mathbb{R}^{n+1} / (\epsilon_1 + \cdots + \epsilon_{n+1} = 0). \tag{513}$$

The roots are taken as

$$\alpha = \epsilon_i - \epsilon_j, \qquad i, j = 1, \ldots, n+1, \quad i \neq j. \tag{514}$$

One can see that the coroots from the identification (496) are equal to the roots, so the dual system is $A_n$ system itself. Positive roots can be chosen as follows

$$\alpha = \epsilon_i - \epsilon_j, \qquad i, j = 1, \ldots, n+1, \quad i < j. \tag{515}$$

For the simple roots then we have

$$\alpha_i = \epsilon_i - \epsilon_{i+1}, \qquad i = 1, \ldots, n, \tag{516}$$

$$(\alpha_i, \alpha_{i+1}) = -1, \qquad (\alpha_i, \alpha_i) = 2, \qquad i = 1, \ldots, n. \tag{517}$$

The fundamental weights

$$\omega_i = \epsilon_1 + \cdots + \epsilon_i, \qquad i = 1, \ldots, n. \tag{518}$$

The Weyl group is generated by simple reflections

$$s_i \colon \epsilon_i \leftrightarrow \epsilon_{i+1}, \qquad i = 1, \ldots, n, \tag{519}$$

so we have just a permutation group:

$$\mathcal{W}^{A_n} = \mathfrak{S}_{n+1}. \tag{520}$$

The Cartan matrix for the $A_n$ system takes the following form:

$$M^{A_n} = \begin{pmatrix} 2 & -1 & 0 & \ldots & \\ -1 & 2 & -1 & \ldots & \\ 0 & -1 & 2 & \ldots & \\ \ldots & \ldots & \ldots & \ldots & \\ & \ldots & & 2 & -1 \\ & \ldots & & -1 & 2 \end{pmatrix}. \tag{521}$$

The root system $A_n$ corresponds to the Lie algebra $\mathfrak{sl}_{n+1}$. We prefer to use the algebra $\mathfrak{gl}_{n+1}$ which can be represented as

$$\mathfrak{gl}_{n+1} = \mathfrak{sl}_{n+1} \oplus \mathbb{C}. \tag{522}$$

This means that the root system $A_n$ defines the structure of $\mathfrak{gl}_{n+1}$ as well and all relations above work there except one should add the fundamental weight

$$\omega_{n+1} = \epsilon_1 + \cdots + \epsilon_{n+1}. \tag{523}$$

## A.3 $B_n$ DATA

Let $\mathfrak{h}^*$ be a real space $\mathbb{R}^n$ spanned by its basis vectors $\epsilon_i$, $i = 1, \ldots, n$ and the inner product defined by

$$(\epsilon_i, \epsilon_j) = \delta_{i,j}. \tag{524}$$

The root system $B_n$ is defined as follows

$$\Phi_n = \{\epsilon_i - \epsilon_j\}_{i \neq j} \cup \{\pm\epsilon_i \pm \epsilon_j\}_{i<j} \cup \{\pm\epsilon_i\}_{i=1,\ldots,n}. \tag{525}$$

Using the identification (496) one can check that the dual root system is

$$\Phi_n^\vee = \{\epsilon_i - \epsilon_j\}_{i \neq j} \cup \{\pm\epsilon_i \pm \epsilon_j\}_{i \leq j}. \tag{526}$$

This is the root system $C_n$ which will be defined further, so we have the duality $B_n \leftrightarrow C_n$. The positive roots can be chosen as

$$\Phi_n^+ = \{\epsilon_i - \epsilon_j\}_{i<j} \cup \{\epsilon_i + \epsilon_j\}_{i<j} \cup \{\epsilon_i\}_{i=1,\ldots,n}. \tag{527}$$

Then for the simple roots we have:

$$\alpha_i = \epsilon_i - \epsilon_{i+1}, \qquad i = 1, \ldots, n-1, \tag{528}$$

$$\alpha_n = \epsilon_n. \tag{529}$$

$$(\alpha_i, \alpha_{i+1}) = -1, \qquad (\alpha_i, \alpha_i) = 2, \qquad i = 1, \ldots, n-1, \tag{530}$$

$$(\alpha_n, \alpha_n) = 1. \tag{531}$$

The fundamental weights

$$\omega_i = \epsilon_1 + \cdots + \epsilon_i, \qquad i = 1, \ldots, n-1, \tag{532}$$

$$\omega_n = \frac{1}{2}(\epsilon_1 + \cdots + \epsilon_n). \tag{533}$$

The Weyl group is generated by the following simple reflections:

$$s_i : \epsilon_i \leftrightarrow \epsilon_{i+1}, \qquad i = 1, \ldots, n-1, \tag{534}$$

$$\varepsilon_n : \epsilon_n \to -\epsilon_n, \tag{535}$$

so we have a semidirect product of permutations of basis vectors and inverting axes:

$$\mathcal{W}^{B_n} = (\mathbb{Z}/2\mathbb{Z})^n \rtimes \mathfrak{S}_n. \tag{536}$$

The Cartan matrix for the $B_n$ system takes the following form:

$$M^{B_n} = \begin{pmatrix} 2 & -1 & 0 & \ldots & \\ -1 & 2 & -1 & \ldots & \\ 0 & -1 & 2 & \ldots & \\ \ldots & \ldots & \ldots & \ldots & \\ & \ldots & & 2 & -2 \\ & \ldots & & -1 & 1 \end{pmatrix}. \tag{537}$$

The root system $B_n$ corresponds to the Lie algebra $\mathfrak{so}_{2n+1}$.

## A.4 $C_n$ DATA

Similar to the case of $B_n$, let $\mathfrak{h}^*$ be a real space $\mathbb{R}^n$ spanned by its basis vectors $\epsilon_i$, $i = 1, \ldots, n$ and the inner product defined by

$$(\epsilon_i, \epsilon_j) = \delta_{i,j}. \tag{538}$$

The root system $C_n$ is defined as follows

$$\Phi_n = \{\epsilon_i - \epsilon_j\}_{i \neq j} \cup \{\pm\epsilon_i \pm \epsilon_j\}_{i \leq j}. \tag{539}$$

Using the identification (496) one can check that the dual root system is

$$\Phi_n^\vee = \{\epsilon_i - \epsilon_j\}_{i \neq j} \cup \{\pm\epsilon_i \pm \epsilon_j\}_{i < j} \cup \{\pm\epsilon_i\}_{i=1,\ldots,n}. \tag{540}$$

As expected, the dual root system is $B_n$ defined in the previous section, so again we have the duality $B_n \leftrightarrow C_n$. The positive roots can be chosen as

$$\Phi_n^+ = \{\epsilon_i - \epsilon_j\}_{i < j} \cup \{\epsilon_i + \epsilon_j\}_{i \leq j}. \tag{541}$$

Then for the simple roots we have:

$$\alpha_i = \epsilon_i - \epsilon_{i+1}, \qquad i = 1, \ldots, n-1, \tag{542}$$

$$\alpha_n = 2\epsilon_n. \tag{543}$$

$$(\alpha_i, \alpha_{i+1}) = -1, \qquad (\alpha_i, \alpha_i) = 2, \qquad i = 1, \ldots, n-1, \tag{544}$$

$$(\alpha_n, \alpha_n) = 4. \tag{545}$$

The fundamental weights

$$\omega_i = \epsilon_1 + \cdots + \epsilon_i, \qquad i = 1, \ldots, n. \tag{546}$$

The Weyl group is generated by the following simple reflections:

$$s_i \colon \epsilon_i \leftrightarrow \epsilon_{i+1}, \qquad i = 1, \ldots, n-1, \tag{547}$$

$$\varepsilon_n \colon \epsilon_n \to -\epsilon_n. \tag{548}$$

This is exactly the same as in case of the system $B_n$, so we have a semidirect product of permutations of basis vectors and inverting axes again:

$$\mathcal{W}^{C_n} = (\mathbb{Z}/2\mathbb{Z})^n \rtimes \mathfrak{S}_n = \mathcal{W}^{B_n}. \tag{549}$$

The Cartan matrix for the $C_n$ system takes the following form:

$$M^{C_n} = \begin{pmatrix} 2 & -1 & 0 & \ldots & \\ -1 & 2 & -1 & \ldots & \\ 0 & -1 & 2 & \ldots & \\ \ldots & \ldots & \ldots & \ldots & \\ & \ldots & & 2 & -1 \\ & \ldots & & -2 & 1 \end{pmatrix}, \tag{550}$$

so it is a transposed $M^{B_n}$. The root system $C_n$ corresponds to the Lie algebra $\mathfrak{sp}_{2n}$.

## A.5 $D_n$ DATA

Again let $\mathfrak{h}^*$ be a real space $\mathbb{R}^n$ spanned by its basis vectors $\epsilon_i$, $i = 1, \ldots, n$ and the inner product defined by

$$(\epsilon_i, \epsilon_j) = \delta_{i,j}. \tag{551}$$

The root system $D_n$ is defined as follows

$$\Phi_n = \{\epsilon_i - \epsilon_j\}_{i \neq j} \cup \{\pm\epsilon_i \pm \epsilon_j\}_{i<j}. \tag{552}$$

Using the identification (496) one can check that the dual root system is the $D_n$ system itself. The positive roots can be chosen as

$$\Phi_n^+ = \{\epsilon_i - \epsilon_j\}_{i<j} \cup \{\epsilon_i + \epsilon_j\}_{i<j}. \tag{553}$$

Then for the simple roots we have:

$$\alpha_i = \epsilon_i - \epsilon_{i+1}, \qquad i = 1, \ldots, n-1, \tag{554}$$

$$\alpha_n = \epsilon_{n-1} + \epsilon_n. \tag{555}$$

$$(\alpha_i, \alpha_{i+1}) = -1, \qquad i = 1, \ldots, n-2, \tag{556}$$

$$(\alpha_{n-1}, \alpha_n) = 0, \tag{557}$$

$$(\alpha_i, \alpha_i) = 2, \qquad i = 1, \ldots, n. \tag{558}$$

The fundamental weights

$$\omega_i = \epsilon_1 + \cdots + \epsilon_i, \qquad i = 1, \ldots, n-2, \tag{559}$$

$$\omega_{n-1} = \frac{1}{2}(\epsilon_1 + \cdots + \epsilon_{n-1} - \epsilon_n), \tag{560}$$

$$\omega_n = \frac{1}{2}(\epsilon_1 + \cdots + \epsilon_{n-1} + \epsilon_n). \tag{561}$$

The Weyl group is generated by the following simple reflections:

$$s_i \colon \epsilon_i \leftrightarrow \epsilon_{i+1}, \qquad i = 1, \ldots, n-1, \tag{562}$$

$$\bar{\varepsilon}_n \colon \epsilon_n \leftrightarrow -\epsilon_{n-1}. \tag{563}$$

The last generator can be replaced with $\varepsilon_n \cdot \varepsilon_{n-1}$ where $\varepsilon_i$ is defined by

$$\varepsilon_i \colon \epsilon_i \to -\epsilon_i. \tag{564}$$

So we have a semidirect product of permutations of basis vectors and inverting an even number of axes:

$$\mathcal{W}^{D_n} = (\mathbb{Z}/2\mathbb{Z})^{n-1} \rtimes \mathfrak{S}_n. \tag{565}$$

The Cartan matrix for the $D_n$ system takes the following form:

$$M^{D_n} = \begin{pmatrix} 2 & -1 & 0 & \dots & & \dots \\ -1 & 2 & -1 & \dots & & \dots \\ 0 & -1 & 2 & \dots & & \dots \\ \dots & \dots & \dots & \dots & & \dots \\ & \dots & & 2 & -1 & -1 \\ & \dots & & -1 & 2 & 0 \\ & \dots & & -1 & 0 & 2 \end{pmatrix}. \tag{566}$$

The root system $D_n$ corresponds to the Lie algebra $\mathfrak{so}_{2n}$.

## A.6 ISOMORPHISM $A_3 \simeq D_3$

Take the space $\mathbb{R}^4$ with the basis $(\epsilon'_1, \dots, \epsilon'_4)$ and the space $\mathbb{R}^3$ with the basis $(\epsilon_1, \dots, \epsilon_3)$. The simple roots for $A_3$ in $\mathbb{R}^4$ are

$$\alpha'_1 = \epsilon'_1 - \epsilon'_2, \quad \alpha'_2 = \epsilon'_2 - \epsilon'_3, \quad \alpha'_3 = \epsilon'_3 - \epsilon'_4, \tag{567}$$

$$(\alpha'_1, \alpha'_2) = (\alpha'_2, \alpha'_3) = -1, \tag{568}$$

$$(\alpha'_1, \alpha'_3) = 0. \tag{569}$$

Note that they span the hyperplane $\mathbb{R}^3 \subset \mathbb{R}^4$. For the system $D_3$ in the space $\mathbb{R}^3$ one has

$$\alpha_1 = \epsilon_1 - \epsilon_2, \quad \alpha_2 = \epsilon_2 - \epsilon_3, \quad \alpha_3 = \epsilon_2 + \epsilon_3, \tag{570}$$

$$(\alpha_1, \alpha_2) = (\alpha_1, \alpha_3) = -1, \tag{571}$$

$$(\alpha_2, \alpha_3) = 0. \tag{572}$$

The Cartan matrices are the same and the root systems are isomorphic. Let us write the isomorphism map explicitly. Define a linear map $\mathbb{R}^3 \to \mathbb{R}^4$ by identifying the roots:

$$\epsilon_2 + \epsilon_3 \to \epsilon'_3 - \epsilon'_4, \tag{573}$$

$$\epsilon_2 - \epsilon_3 \to \epsilon'_1 - \epsilon'_2, \tag{574}$$

$$\epsilon_1 - \epsilon_2 \to \epsilon'_2 - \epsilon'_3. \tag{575}$$

Using linearity we can rewrite it as follows:

$$\epsilon_1 \to \frac{1}{2}(\epsilon'_1 + \epsilon'_2 - \epsilon'_3 - \epsilon'_4), \tag{576}$$

$$\epsilon_2 \to \frac{1}{2}(\epsilon'_1 - \epsilon'_2 + \epsilon'_3 - \epsilon'_4), \tag{577}$$

$$\epsilon_3 \to \frac{1}{2}(-\epsilon'_1 + \epsilon'_2 + \epsilon'_3 - \epsilon'_4). \tag{578}$$

Take the following elements of Cartan subalgebras and dual algebras:

$$\lambda = (\lambda'_1, \lambda'_2, \lambda'_3, \lambda'_4) \in \mathfrak{h}^*_{A_3}, \tag{579}$$

$$\lambda' = (\lambda_1, \lambda_2, \lambda_3) \in \mathfrak{h}^*_{D_3}, \tag{580}$$

$$y = (y_1, y_2, y_3, y_4) \in \mathfrak{h}_{A_3}, \tag{581}$$

$$x = (x_1, x_2, x_3) \in \mathfrak{h}_{D_3}. \tag{582}$$

Then the map gives the following relations between the coordinates of the vectors:

$$\lambda'_1 = \frac{1}{2}(\lambda_1 + \lambda_2 - \lambda_3), \qquad \lambda'_2 = \frac{1}{2}(\lambda_1 - \lambda_2 + \lambda_3), \tag{583}$$

$$\lambda'_3 = \frac{1}{2}(-\lambda_1 + \lambda_2 + \lambda_3), \qquad \lambda'_4 = \frac{1}{2}(-\lambda_1 - \lambda_2 - \lambda_3), \tag{584}$$

$$x_1 = \frac{1}{2}(y_1 + y_2 - y_3 - y_4), \tag{585}$$

$$x_2 = \frac{1}{2}(y_1 - y_2 + y_3 - y_4), \tag{586}$$

$$x_3 = \frac{1}{2}(-y_1 + y_2 + y_3 - y_4). \tag{587}$$

# B

# CHARACTER FORMULAS

## B.1 WEYL CHARACTER FORMULA

The character of a representation of a group $G$ has the following definition.

**Definition B.1.1.** Let $V$ be a representation of $G$. Then the character is a function on $G$ such that

$$S_V(g) = \mathrm{Tr}_V\, g, \tag{588}$$

where on the right-hand side there is a trace of the element $g \in G$ in the space $V$.

Let $\mathfrak{g}$ be Lie algebra of $G$. For its weight lattice $\Lambda$ there is a group ring $\mathbb{Z}[\Lambda]$ with basis elements $e(\lambda)$ corresponding to weights $\lambda$. For the representation ring $R(\mathfrak{g})$ we can define the following ring homomorphism:

$$\mathrm{Char}\colon R(\mathfrak{g}) \to \mathbb{Z}[\Lambda]. \tag{589}$$

Here each class $[V] \in R(\mathfrak{g})$ maps to $\sum_\lambda m_\lambda e(\lambda)$, where $m_\lambda$ is the multiplicity of the weight space $V_\lambda$ in $V$.

**Definition B.1.2.** The image of $[V]$ in $\mathbb{Z}[\Lambda]$ is called a formal character of $V$.

Let $H \subset G$ be a Cartan subgroup with Lie algebra $\mathfrak{h} \subset \mathfrak{g}$.

**Proposition B.1.1** (see [14])**.** *If* $\mathrm{Char}(V) = \sum_\lambda m_\lambda e(\lambda)$ *is the formal character of* $V$ *and* $\exp(x) \in H$, $x \in \mathfrak{h}$, *then*

$$\mathrm{Tr}_V(\exp(x)) = \sum_\lambda m_\lambda e^{\lambda(x)}. \tag{590}$$

This means that the representation $V$ is defined by a character of its restriction to $H$.

Weyl provided the following canonical formula for characters of irreducible representations of reductive groups.

**Theorem B.1.1** (Weyl character formula, see [14]). *Let $\Phi_n$ be roots system, $\mathcal{W}^{\Phi_n}$ its Weyl group and $\rho$ a half-sum of positive roots. The character of the irreducible representation $V_\lambda$ of the corresponding Lie algebra $\mathfrak{g}$ of the highest weight $\lambda$ is given by the following formula:*

$$S_\lambda^{\Phi_n}(e^{x_1},\dots,e^{x_n}) = \frac{\sum\limits_{w\in\mathcal{W}^{\Phi_n}}(-1)^w e^{\langle w(\lambda+\rho),x\rangle}}{\sum\limits_{w\in\mathcal{W}^{\Phi_n}}(-1)^w e^{\langle w\rho,x\rangle}}. \tag{591}$$

*Here $(-1)^w = (-1)^{\ell(w)}$, $\ell(w)$ is the length of the element $w \in \mathcal{W}^{\Phi_n}$.*

*Remark* B.1.1. For the denominator one has

$$\sum_{w\in\mathcal{W}^{\Phi_n}}(-1)^w e^{\langle w\rho,x\rangle} = \prod_{\alpha\in\Phi_n^+}(e^{\langle\alpha,x\rangle/2} - e^{-\langle\alpha,x\rangle/2}) = e^{\langle\rho,x\rangle}\prod_{\alpha\in\Phi_n^+}(1-e^{-\langle\alpha,x\rangle}). \tag{592}$$

**Corollary B.1.1.** *The dimension formula of the irreducible representation of the highest weight $\lambda$ holds*

$$\dim V_\lambda = \prod_{\alpha\in\Phi_n^+}\frac{\langle\lambda+\rho,\alpha\rangle}{\langle\rho,\alpha^\vee\rangle}. \tag{593}$$

**Corollary B.1.2.** *The Weyl character formula can be written explicitly for each classical Lie algebra. The character of $\mathfrak{gl}_n$, for example, is the following determinant ratio:*

$$S_\lambda^{A_{n-1}}(x_1,\dots,x_n) = \frac{\left|x_j^{\lambda_i+n-i}\right|}{\left|x_j^{n-i}\right|}, \tag{594}$$

*which is a well-known definition of Schur symmetric functions, or Schur polynomials. For the character of $\mathfrak{sp}_{2n}$ we have the following ratio:*

$$S_\lambda^{B_n}(x_1,\dots,x_n) = \frac{\left|x_j^{\lambda_i+n-i+1} - x_j^{-(\lambda_i+n-i+1)}\right|}{\left|x_j^{n-i+1} - x_j^{-(n-i+1)}\right|}. \tag{595}$$

*For the character of $\mathfrak{so}_{2n+1}$ the formula is almost similar:*

$$S_\lambda^{C_n}(x_1,\dots,x_n)=\frac{\left|x_j^{\lambda_i+n-i+\frac{1}{2}}-x_j^{-(\lambda_i+n-i+\frac{1}{2})}\right|}{\left|x_j^{n-i+\frac{1}{2}}-x_j^{-(n-i+\frac{1}{2})}\right|}. \tag{596}$$

*For the character of $\mathfrak{so}_{2n}$ the formula is a bit more complicated:*

$$S_\lambda^{D_n}(x_1,\dots,x_n)=\frac{\left|x_j^{\lambda_i+n-i}+x_j^{-(\lambda_i+n-i)}\right|+\left|x_j^{\lambda_i+n-i}-x_j^{-(\lambda_i+n-i)}\right|}{\left|x_j^{n-i}-x_j^{-(n-i)}\right|}. \tag{597}$$

## B.2 DEMAZURE CHARACTER FORMULA

Let $H\subset G$ be a maximal torus of the group $G$. The element of this torus can be represented as

$$(e^{x_1},\dots,e^{x_n})\in H,\qquad (x_1,\dots,x_n)\in\mathfrak{h}, \tag{598}$$

where $\mathfrak{h}$ is a Cartan subalgebra of the Lie algebra $\mathfrak{g}$ of the group $G$. Take the weight lattice $\Lambda_W$ generated by the fundamental weights $\omega_i$. Each element $\lambda\in\Lambda_W$ acts on the torus $H$ as the following homomorphism:

$$\lambda\colon (e^{x_1},\dots,e^{x_n})\to e^{\langle\lambda,x\rangle}\in\mathbb{C}^*. \tag{599}$$

**Definition B.2.1** (see [32]). Let $\Phi_n$ be a root system, $\alpha_i\in\Phi_n$ simple roots, $s_i\in\mathcal{W}^{\Phi_n}$ corresponding generators of the Weyl group. The Demazure operator is a $\mathbb{Z}$-linear operator $\mathcal{D}_i\colon\mathbb{Z}[\Lambda_W]\to\mathbb{Z}[\Lambda_W]$ defined as

$$\mathcal{D}_i(e^\lambda)=\frac{e^\lambda-e^{s_i\lambda-\alpha_i}}{1-e^{-\alpha_i}},\qquad e^\lambda\in\mathbb{Z}[\Lambda_W]. \tag{600}$$

Here $e^\lambda\colon x\mapsto e^{\langle\lambda,x\rangle}$.

Note that

$$\mathcal{D}_i^2=\mathcal{D}_i. \tag{601}$$

For each element $w\in\mathcal{W}^{\Phi_n}$ with a (reduced) decomposition $w=s_{i_1}\cdots s_{i_k}$ define

$$\mathcal{D}_w=\mathcal{D}_{s_{i_1}}\cdots\mathcal{D}_{s_{i_k}}. \tag{602}$$

**Theorem B.2.1** (Demazure character formula, see [32]). *Let $\mathfrak{g}$ be a Lie algebra, $\Phi_n$ - its root system and $\mathcal{W}^{\Phi_n}$ - its Weyl group. Let $w_0 \in \mathcal{W}^{\Phi_n}$ be the element of the maximal length in the Weyl group $\mathcal{W}$. Then the character of the irreducible representation of the highest weight $\lambda$ is given by the following formula:*

$$S_\lambda^{\Phi_n}(e^{x_1}, \dots, e^{x_n}) = \mathcal{D}_{w_0}(e^\lambda)(x). \tag{603}$$

Demazure operators defined as (600) act on the group algebra of characters $\mathbb{Z}[\Lambda_W]$, where $\Lambda_W \in \mathfrak{h}^*$ is a weight lattice, and provide the formula (603) for characters of irreducible representations. There is a connection between these operators and so called BGG operators which were introduced in [4] in the context of cohomologies of flag spaces $G/P$.

**Definition B.2.2** (Bernstein, Gelfand, Gelfand, see [4],[32]). Let $\mathfrak{h}$ be a Cartan subalgebra of the Lie algebra $\mathfrak{g}$, $\Phi_n$ corresponding root system, $\alpha_i \in \Phi_i$ simple roots and $s_i \in \mathcal{W}^{\Phi_n}$ corresponding simple reflections. Take the group algebra $\mathbb{C}[\mathfrak{h}^*]$. The BGG operator is an operator $\mathcal{A}_i \colon \mathbb{C}[\mathfrak{h}^*] \to \mathbb{C}[\mathfrak{h}^*]$ defined by the action

$$\mathcal{A}_i(f)(x) = \frac{f(x) - f(s_i x)}{\langle \alpha_i, x \rangle}, \qquad f \in \mathbb{C}[\mathfrak{h}^*]. \tag{604}$$

Note that

$$\mathcal{A}_i^2 = 0. \tag{605}$$

The definition is given only for simple reflections $s_i$. For every other $w \in \mathcal{W}$ with a reduced decomposition $w = s_{i_1} \cdots s_{i_k}$ define the BGG operators as

$$\mathcal{A}_w = \mathcal{A}_{i_1} \cdots \mathcal{A}_{i_k}. \tag{606}$$

**Proposition B.2.1.** *The following relation holds*

$$\lim_{\varepsilon \to 0} \varepsilon \mathcal{D}_i(e^{\frac{\lambda}{\varepsilon}})(\varepsilon x) = \mathcal{A}_i(e^\lambda)(x). \tag{607}$$

*Proof.* Straightforward calculation gives

$$\varepsilon \mathcal{D}_i(e^{\frac{\lambda}{\varepsilon}})(\varepsilon x) = \varepsilon \frac{e^{\langle \lambda, x \rangle} - e^{\langle s_i \lambda - \varepsilon \alpha_i, x \rangle}}{1 - e^{-\varepsilon \langle \alpha_i, x \rangle}} \to \frac{e^{\langle \lambda, x \rangle} - e^{\langle s_i \lambda, x \rangle}}{\langle \alpha_i, x \rangle} = \mathcal{A}_i(e^\lambda)(x). \tag{608}$$

□

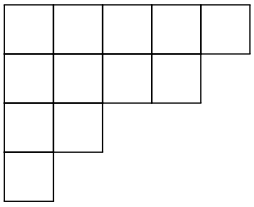
## B.3 YOUNG TABLEAUX

Much like Gelfand-Tsetlin patterns, Young tableaux give a well-known combinatorial representation of characters of classical groups. For more details and tableaux types, see [14, 45, 27, 39, 34].

### B.3.1 *Partitions and Young tableaux*

**Definition B.3.1.** The set of integers $\lambda_1 \geq \ldots \geq \lambda_n \geq 0$ is called a partition of the number $|\lambda| = \lambda_1 + ... + \lambda_n$. The length of partition $l(\lambda)$ is the number of non-zero parts $\lambda_i$.

**Definition B.3.2.** To any partition $\lambda$ there is an associated Young diagram



(609)

where the number of boxes in a row $i$ equals to $\lambda_i$.

**Definition B.3.3.** For each partition $\lambda$ there is a conjugate partition $\lambda^T$ with a Young diagram defined by swapping rows and columns of the diagram $\lambda$.

**Definition B.3.4.** The Young tableau of the shape $\lambda$ is a Young diagram $\lambda$ where boxes are filled with numbers.

**Definition B.3.5.** The standard Young tableau of shape $\lambda$ is a Young diagram $\lambda$ with boxes filled with numbers from 1 to $|\lambda|$ in strict increasing from right to left in rows and from up to down in columns:

| | | | | |
|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 |
| 6 | 7 | 8 | 9 | |
| 10 | 11 | | | |
| 12 | | | | |

(610)

**Definition B.3.6.** The column-strict Young tableau of shape $\lambda$ is a Young diagram $\lambda$ with boxes filled with numbers non-decreasing in rows and strictly increasing in columns:

| | | | | |
|---|---|---|---|---|
| 1 | 2 | 2 | 3 | 3 |
| 2 | 3 | 4 | 4 | |
| 4 | 4 | | | |
| 5 | | | | |

(611)

**Definition B.3.7.** Let $T$ be a standard Young tableau, $\mathfrak{S}_{|\lambda|}$ - the group of permutations of $|\lambda|$ elements which acts on the numbers in an arbitrary tableau of the same shape. The Young symmetrizer is an element of the group algebra $\mathbb{C}\mathfrak{S}_{|\lambda|}$ defined as follows:

$$c_T = \sum_{w \in C_T} \operatorname{sign}(w) w \cdot \sum_{u \in R_T} u, \tag{612}$$

where $C_T$ is a subgroup which preserves columns of $T$, and $R_T$ is a subgroup which preserves rows.

Let $V$ be a standard representation of $\mathrm{GL}_n\mathbb{C}$ and $e_i$ its basis. Fix standard tableau $T$ of shape $\lambda$ and consider the space $V^{\otimes|\lambda|}$. One can identify the basis vector $e_{i_1} \otimes \ldots \otimes e_{i_{|\lambda|}}$ with a Young tableaux $T'$ by putting $i_m$ into the box which has number $m$ in $T$. The symmetric group $\mathfrak{S}_{|\lambda|}$ acts on $V^{\otimes|\lambda|}$ by that identification(by permuting the copies of $V$ in tensor product).

**Definition B.3.8.** The Weyl module $W_\lambda$ is the image of $c_T$ in $V^{\otimes|\lambda|}$.

*Remark* B.3.1. The different choice of $T$ with the same $\lambda$ gives an isomorphic Weyl module.

*Example* 1. For the partition $\lambda = (n)$ we have only one standard tableau:

| | | | |
|---|---|---|---|
| 1 | 2 | ... | $n$ |

(613)

The Young symmetrizer takes the form $c_T = \sum_{w \in R_T} w$, so the resulting Weyl module is $\mathrm{Sym}^n V$.

*Example* 2. For the partition $\lambda = 1, ..., 1$ we also have only one standard tableau:

$$\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline ... \\ \hline n \\ \hline \end{array} \tag{614}$$

The Young symmetrizer takes the form $c_T = \sum_{w \in C_T} \text{sign}(w) w$, so the resulting Weyl module is $\bigwedge^n V$.

**Theorem B.3.1** (Schur)**.** *The basis in the Weyl module $W_\lambda$ of the group $\mathrm{GL}_n\mathbb{C}$ is the set of column-strict tableaux filled with numbers $1, \ldots, n$.*

**Theorem B.3.2.** *Let $p(i, T)$ is the number of entries $i$ in the tableau $T$ and $\mathcal{T}_{\lambda,n}$ is the set of all column-strict Young tableau of shape $\lambda$ and numbers $1, \ldots, n$. The character of irreducible representation $W_\lambda$ of $\mathrm{GL}_n\mathbb{C}$ is given by the formula*

$$S_\lambda^{A_{n-1}}(x_1, ..., x_n) = \sum_{T \in \mathcal{T}_{\lambda,n}} \prod_{i=1}^{n} x^{p(i,T)}. \tag{615}$$

### B.3.2 *Symplectic tableaux*

**Definition B.3.9.** Let $\lambda$ be a partition such that $l(\lambda) \leq n$. The symplectic tableau $T$ is a Young tableau of shape $\lambda$ filled with numbers $1 < \bar{1} < 2 < \bar{2} ... < n < \bar{n}$ with two conditions:

1. $T$ is column-strict;
2. all numbers in $i$-th row are larger or equal to $i$.

*Example* 3. For a partition $\lambda = (4, 3, 2)$ the following tableau is symplectic:

$$\begin{array}{|c|c|c|c|}\hline 1 & \bar{1} & \bar{1} & 2 \\ \hline 2 & \bar{2} & \bar{3} & \\ \cline{1-3} 3 & 3 & & \\ \cline{1-2} \end{array} \tag{616}$$

**Theorem B.3.3** (King, El-Sharkaway, see [45])**.** *Let $p(i, T)$ be the number of entries $i$ in the tableau $T$ and $\mathcal{T}^{Sp}_{\lambda,n}$ is the set of all column-strict Young tableau of*

*shape $\lambda$ in numbers $1 < \bar{1} < 2 < \bar{2} ... < n < \bar{n}$. The character of irreducible representation of $\mathrm{Sp}_{2n}\mathbb{C}$ of the highest weight $\lambda$ is given by the formula:*

$$S_\lambda^{C_n}(x_1, \ldots, x_n) = \sum_{T \in \mathcal{T}_{\lambda,n}^{Sp}} \prod_{i=1}^{n} x_i^{p(i,T)-p(\bar{i},T)}. \tag{617}$$

*Remark* B.3.2. The polynomials $Sp_\lambda(x_1, \ldots, x_n)$ can be referred to as symplectic Schur polynomials.

**Theorem B.3.4** (Berele, see [45]). *The symplectic tableaux of shape $\lambda$ in numbers $1 < \bar{1} < 2 < \bar{2} ... < n < \bar{n}$ form a basis in the irreducible representation of $\mathrm{Sp}_{2n}\mathbb{C}$ of the highest weight $\lambda$.*

### B.3.3 *Orthogonal tableaux*

Take a diagram of shape $\lambda$ and fill it with symbols $1 < \bar{1} < 2 < \bar{2} ... < n < \bar{n} < \infty$ in such way that

1. the tableaux is column-strict;
2. all numbers in $i$-th row are larger of equal to i;
3. if $i$ and $\bar{i}$ are consecutive entries in the same row, then there is $i$ in the previous row above the $\bar{i}$.

Denote the set of such tableaux $\mathcal{T}_\lambda^{SO(2n+1)}$.

**Theorem B.3.5** (King, El-Sharkaway, see [45]). *Let $T$ be the tableaux defined as above and let $m(T)$ be the number of $i$ entries placed above $\bar{i}$ entries in the first column and with $\bar{i}$ in the row $i$ for each $i = 1, \ldots, n$. Let $p(i, T)$ be the number of entries $i$ in the tableaux. Then the character of the irreducible representation of the group $SO(2n+1)$ has the following representation:*

$$S_\lambda^{B_n}(x_1, \ldots, x_n) = \sum_{T \in \mathcal{T}_\lambda^{SO(2n+1)}} 2^{m(T)} \prod_{i=1}^{n} x_i^{p(i,T)-p(\bar{i},T)}. \tag{618}$$

**Theorem B.3.6** (King, El-Sharkaway, see [45]). *In the definition of tableaux $\mathcal{T}_\lambda^{SO(2n+1)}$ add another condition: if $m(T) = 0$, then the number of entries $i$ such that the entry $(i, 1)$ is $\bar{i}$ is even if $i = 1$ and odd if $i = 2$. Denote these modified tableaux as $\mathcal{T}_\lambda^{SO(2n)}$. Then the character of the irreducible representation of the group $SO(2n)$ has the following representation:*

$$S_\lambda^{D_n}(x_1, \dots, x_n) = \sum_{T \in \mathcal{T}_\lambda^{SO(2n)}} 2^{r(T)} \prod_{i=1}^{n} x_i^{p(i,T) - p(\bar{i},T)}, \tag{619}$$

*where $r(T) = m(T) - 1$ if $m(T) > 0$ and $r(T) = 0$ otherwise.*

## B.4 JACOBI-TRUDI IDENTITIES

Jacobi-Trudi identities let us calculate arbitrary Schur functions as determinants of matrices $\det ||a_{i,j}||$ where matrix elements $a_{i,j}$ are elementary symmetric polynomials.

**Theorem B.4.1** (Jacobi-Trudi identities, see [45, 14, 34]). *For Schur polynomials the following relations hold*

$$S_\lambda^{A_n}(x_1, \dots, x_{n+1}) = \det ||S^{A_n}_{(\lambda_i - i + j)}(x_1, \dots, x_{n+1})||_{1 \le i,j \le l(\lambda)}, \tag{620}$$

$$S_\lambda^{A_n}(x_1, \dots, x_{n+1}) = \det ||S^{A_n}_{1^{(\lambda)_i^T - i + j}}(x_1, \dots, x_{n+1})||_{1 \le i,j \le l(\lambda)}. \tag{621}$$

The same kind of identities can be written for the other groups.

**Theorem B.4.2** (Weyl, see [45]). *For the symplectic Schur polynomials one has the following relation:*

$$S_\lambda^{C_n}(x_1, \dots, x_n) =$$
$$= \frac{1}{2} \det ||S^{C_n}_{(\lambda_i - i - j + 2)}(x_1, \dots, x_n) + S^{C_n}_{(\lambda_i - i + j)}(x_1, \dots, x_n)||_{1 \le i,j \le l(\lambda)}. \tag{622}$$

**Theorem B.4.3** (Weyl, see [45]). *For the orthogonal group $SO(2n + 1)$ one has the following relation:*

$$S_\lambda^{B_n}(x_1, \dots, x_n) = \det ||S^{A_{2n+1}}_{(\lambda_i - i - j)} + S^{A_{2n+1}}_{(\lambda_i - i + j)}||_{1 \le i,j \le l(\lambda)}, \tag{623}$$

*where $S_\mu^{A_{2n+1}} = S_\mu^{A_{2n+1}}(x_1, \dots, x_n, x_1^{-1}, \dots, x_n^{-1}, 1)$.*

**Theorem B.4.4** (Weyl, see [45]). *For the orthogonal group $SO(2n)$ one has the following relation:*

$$S_\lambda^{D_n}(x_1,\dots,x_n) = \det||S^{A_{2n}}_{(\lambda_i-i-j)} + S^{A_{2n}}_{(\lambda_i-i+j)}||_{1\le i,j\le l(\lambda)}, \tag{624}$$

*where* $S_\mu^{A_{2n}} = S_\mu^{A_{2n}}(x_1,\dots,x_n,x_1^{-1},\dots,x_n^{-1})$.

## B.5 CAUCHY IDENTITIES

**Theorem B.5.1** (Weyl, see [45]). *The following identities called Cauchy identities hold*

$$\prod_{i=1}^{n}\prod_{j=1}^{m}\frac{1}{1-x_iy_j} = \sum_\lambda S_\lambda^{A_{n-1}}(x_1,\dots,x_n)S_\lambda^{A_{m-1}}(y_1,\dots,y_m), \tag{625}$$

$$\prod_{i=1}^{n-1}\prod_{j=i+1}^{n}(1-x_ix_j)\prod_{k=1}^{n}\frac{1}{1-x_k}\prod_{r=1}^{m}\frac{1}{(1-x_ky_r)(1-x_ky_r^{-1})} = \\ = \sum_\lambda S_\lambda^{A_{n-1}}(x_1,\dots,x_n)S_\lambda^{B_m}(y_1,\dots,y_m) \tag{626}$$

$$\prod_{i=1}^{n-1}\prod_{j=i+1}^{n}(1-x_ix_j)\prod_{k=1}^{n}\prod_{r=1}^{m}\frac{1}{(1-x_ky_r)(1-x_ky_r^{-1})} = \\ = \sum_\lambda S_\lambda^{A_{n-1}}(x_1,\dots,x_n)S_\lambda^{C_m}(y_1,\dots,y_m), \tag{627}$$

$$\prod_{i=1}^{n-1}\prod_{j=i+1}^{n}(1-x_ix_j)\prod_{k=1}^{n}\prod_{r=1}^{m}\frac{1}{(1-x_ky_r)(1-x_ky_r^{-1})} = \\ = \sum_\lambda S_\lambda^{A_{n-1}}(x_1,\dots,x_n)S_\lambda^{D_m}(y_1,\dots,y_m) \tag{628}$$

## B.6 OKADA FORMULAS

In his paper [37] Okada proved several formulas for the characters of the classical groups with a non-trivial stabilizer of the weight. The proof of this formulas using decomposition of Gelfand-Tsetlin patterns was introduced in [5].

**Theorem B.6.1** (Okada). *Let $n \geq m$, $n, m \in \mathbb{N}$. Then the following relations hold*

1.

$$S^{A_{m+n-1}}_{(t^n,0^m)}(x_1,\dots,x_n,y_1,\dots,y_m) = \\ = \sum_{t\geq\lambda_1\geq\cdots\geq\lambda_m\geq 0} S^{A_{n-1}}_{(t^{n-m},\lambda_1,\dots,\lambda_m)}(x)\cdot S^{A_{m-1}}_{(t-\lambda_m,\dots,t-\lambda_1)}(y). \tag{629}$$

2.

$$S^{C_{n+m}}_{(t^{n+m})}(x_1,\dots,x_n,y_1,\dots,y_m) = \\ = \sum_{t\geq\lambda_1\geq\cdots\geq\lambda_m\geq 0} S^{C_n}_{(t^{n-m},\lambda_1,\dots,\lambda_m)}(x)\cdot S^{C_m}_{(\lambda_1,\dots,\lambda_m)}(y). \tag{630}$$

3.

$$S^{D_{m+n+1}}_{(t,\dots,t,\pm t)}(x_1,\dots,x_n,y_1,\dots,y_m,1) = \\ = \sum_{t\geq\lambda_1\geq\cdots\geq\lambda_m\geq 0} S^{B_n}_{(t^{n-m},\lambda_1,\dots,\lambda_m)}(x)\cdot S^{B_m}_{(\lambda_1,\dots,\lambda_m)}(y). \tag{631}$$

**Theorem B.6.2** (Okada). 1. *Let $n \geq m$. Then the following relation holds*

$$S^{A_{m+n-1}}_{(s^m,0^n)}(x)\cdot S^{A_{m+n-1}}_{(t^n,0^m)}(x) = \\ = \sum_{s+t\geq\lambda_1\geq\cdots\geq\lambda_m\geq\max(s,t)} S^{A_{m+n-1}}_{(\lambda_1,\dots,\lambda_m,t^{n-m},s+t-\lambda_m,\dots,s+t-\lambda_1)}(x). \tag{632}$$

2. *For $t \geq s$ the following relation holds*

$$S^{B_n}_{(s^n)}(x)\cdot S^{B(n)}_{(t^n)}(x) = \sum_{2s\geq\lambda_1\geq\cdots\geq\lambda_n\geq 0} S^{B_n}_{(\lambda_1+t-s,\dots,\lambda_n+t-s)}(x). \tag{633}$$

3. *For $t \geq s$ the following relation holds*

$$S^{C_n}_{(s^n)}(x)\cdot S^{C(n)}_{(t^n)}(x) = \sum_{\substack{2s\geq\lambda_1\geq\cdots\geq\lambda_n\geq 0\\ \lambda_i\in 2\mathbb{Z}}} S^{C_n}_{(\lambda_1+t-s,\dots,\lambda_n+t-s)}(x). \tag{634}$$

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