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Homological and motivic invariants of torsors

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Abstract

Many interesting objects in algebraic geometry arise as torsors of linear algebraic groups over a field. Some notable examples are provided by vector bundles, quadratic forms, Hermitian forms, octonion algebras, Severi-Brauer varieties and many others. The main aim of this thesis is to investigate torsors from a motivic homotopic perspective, by using Nisnevich classifying spaces and their characteristic classes. In order to do so, we will need a Gysin long exact sequence induced by fibrations with motivically invertible reduced fiber. The leading example is provided by the work of Smirnov and Vishik where they introduce subtle Stiefel-Whitney classes, by computing the motivic cohomology of the Nisnevich classifying spaces of orthogonal groups, with the purpose of studying quadratic forms.

In this work, we will mainly deal with spin groups and unitary groups. In particular, we will give descriptions of the motivic cohomology rings of their Nisnevich classifying spaces. These will provide us with subtle characteristic classes for $Spin$-torsors and for Hermitian forms. As a result, we will obtain information about the kernel invariant of quadratic forms belonging to I^3 on the one hand, and of quadratic forms divisible by a one-fold Pfister form on the other. Moreover, in order to approach the case of Severi-Brauer varieties, we will develop a Serre spectral sequence induced by fibrations with motivically cellular fiber. This could be a successful approach to compute the motivic cohomology of the Nisnevich classifying spaces of projective general linear groups.

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CONTENTS

INTRODUCTION

Motivic homotopy theory finds its roots in the groundbreaking work of Morel and Voevodsky (see [36]). It has been developed in the attempt of studying algebraic varieties with a certain flexibility, similar in some ways to the one of topological spaces. Indeed, due to the development of this theory, many topological techniques have been introduced in the algebro-geometric world, leading to undoubtedly extraordinary results. As a remarkable example of the power of this rather recent theory we can mention of course the proof of the Milnor conjecture (see [64]) and subsequently of the Bloch-Kato conjecture (see [65]) by Voevodsky jointly with Rost. In this new motivic homotopic environment one can find represented different cohomology theories for algebraic varieties, such as motivic cohomology, algebraic K-theory of Quillen (see [44]) and algebraic cobordism of Voevodsky, whose pure part is the algebraic cobordism of Levine and Morel (see [29]), corresponding respectively to ordinary cohomology, topological K-theory and complex cobordism in the classical topological world. Motivic homotopy theory has seen since its birth the development of surprisingly sophisticated techniques in a relatively short period of time. This is certainly a demonstration of the high potential of this theory to shed light on many longstanding problems in algebra, geometry and topology and to exploit and discover the deep interconnections among these mathematical areas, leading to new and interesting directions of research.

In this thesis, we will focus on a tiny but very interesting part of this vast universe, namely the classification of torsors of linear algebraic groups. Many newsworthy objects in algebra and geometry arise as torsors of some algebraic groups, for example vector bundles, quadratic forms, Hermitian forms, octonion algebras, Severi-Brauer varieties and so on. Therefore, having a robust theory to approach the classification of torsors would certainly bring in turn various interesting results about the above mentioned objects and many others.

In topology, the investigation of principal bundles of topological groups is of fundamental importance. Much study has been dedicated to it and several approaches have been implemented. One of these, which has shown incredible advantages, is the study of characteristic classes. In few words, for any topological group G one can construct a topological space BG such that all homotopic maps from the base space X to BG are in one-to-one correspondence with principal G-bundles over X. For this reason, BG deserves the name of classifying space of the group G. Now, in order to produce invariants which are rich enough to detect interesting properties but sufficiently computable to provide an actual advantage, one can apply cohomological funtors to the homotopic maps representing the principal bundles. Essentially, for any cohomological functor and any pricipal G-bundle over X, one has a morphism from the cohomology of BG to the cohomology of X. The images of the cohomology classes of BG in the cohomology of X provide the needed invariants of the principal G-bundles, which are called characteristic classes. Hence, in order to produce these invariants, one first needs to compute the cohomology of the classifying space for a chosen cohomology theory.

Pursuing the classification of torsors of algebraic groups, one could follow the path indicated by topology. The first natural question to ask is: what is the space that deserves the name of classifying space? It turns out that the answer to this question in the motivic world is not univocal. Indeed, we have two candidates, the Nisnevich and the étale classifying spaces, both introduced by Morel and Voevodsky in [36], and in principle many others corresponding to different topologies on the site of smooth schemes over a certain base. These two types of classifying spaces are, in general, different from each other. Indeed, it has been proven that they coincide only for special algebraic groups, namely groups with no non-trivial torsors over the point. In practice, it is particularly interesting to study étale locally trivial torsors. For example, torsors over the point are all étale locally trivial for smooth algebraic groups. Then, the space that actually classifies them is the étale classifying space. So, one can obtain characteristic classes for étale locally trivial torsors by applying some cohomological functor to it.

From this perspective, much attention has been devoted to study étale classifying spaces and their Chow rings. In particular, the Chow rings of general linear groups, special linear groups and symplectic groups over complex numbers have been computed by Totaro in his enlightening work (see [53]). Then, the Chow rings of orthogonal groups over complex numbers appear in works of Totaro and Pandharipande (see [53] and [41]). Pandharipande also computed the Chow rings of odd special orthogonal groups and SO_4 over complex numbers in [41]. Field completed the picture by computing in [13] the Chow rings of all even special orthogonal groups over complex numbers. Several results about the Chow rings of finite groups are provided by works of Guillot, Totaro and Yagita (see [18], [53] and [67]). The projective general linear case is notoriously more difficult. The Chow ring of the first non-trivial case, namely PGL_3, over complex numbers has been computed, almost completely, by Vezzosi in [54]. Subsequently, using his stratification method, Molina and Vistoli computed in [33] the Chow rings of general linear groups, special linear groups and symplectic groups over any field and of orthogonal groups and special orthogonal groups over fields of characteristic different from 2. Going back to the projective general linear case, Vistoli has given in [57] an almost complete description of the Chow ring of PGL_p over complex numbers, for p prime. Regarding the spin groups, Guillot computed almost completely the Chow ring of the first non-trivial case, namely $Spin_7$, together with the one of G_2, over complex numbers in [19]. Next, Molina obtained the almost complete description of the Chow ring of $Spin_8$ over complex numbers in [32].

The situation for motivic cohomology is much more complicated. Very few results are known about the motivic cohomology rings of étale classifying spaces. The complete description is known for general linear groups, special linear groups and symplectic groups over any field from the work of Smirnov and Vishik (see [47]), since these are special algebraic groups and their étale and Nisnevich classifying spaces coincide. Yagita computed in [69] the motivic cohomology of the étale classifying spaces of orthogonal groups over complex numbers. Moreover, the case of the special orthogonal groups has been handled by Harada and Nakada in [20]. Besides, new studies on Chow-Witt theory have lead to interesting results about the Chow-Witt rings of special linear groups and symplectic groups by Hornbostel and Wendt in [21].

On the other hand, a systematic investigation of torsors by using Nisnevich classifying spaces instead of étale ones has been carried out by Smirnov and Vishik in [47]. In particular, with the purpose of classifying quadratic forms, they compute the motivic cohomology rings of the Nisnevich classifying spaces of orthogonal groups over fields of characteristic different from 2. These motivic cohomology rings are shown to be, exactly as in topology, polynomial rings over the cohomology of the point in some generators, called subtle Stiefel-Whitney classes. Unlike characteristic classes from the étale space, for any quadratic form, these subtle invariants take values in the cohomology of the Čech simplicial scheme associated to the respective quadratic form, which is highly non-trivial. The advantage is that in this way subtle Stiefel-Whitney classes see much more than the respective étale characteristic classes. However, they have the drawback to take values in different cohomology rings which makes it difficult to compare different quadratic forms. Anyway, these new invariants have proven to be powerful enough to see the triviality of a quadratic form, more precisely they see the power of the fundamental ideal of the Witt ring a quadratic form belongs to. Moreover, they are surprisingly related to the J-invariant of quadrics defined in [56] and they allow to give a description of the motive of the torsor associated to a quadratic form in terms of simpler pieces, namely motives of Čech simplicial schemes. Nisnevich classifying spaces have also been investigated from a motivic homotopic point of view by Asok, Hoyois and Wendt in [3], [2] and [66], where they study in particular octonion algebras and *Spin*-torsors over low-dimensional schemes.

In this thesis, we will proceed further in the direction indicated by Smirnov and Vishik in [47]. In particular, we will generalise their technique for computing the motivic cohomology rings of the Nisnevich classifying spaces of orthogonal groups in order to produce Gysin long exact sequences for morphisms of simplicial schemes with fibers whose reduced motives are invertible. These will provide us with the major tool we will use throughout this work to prove our main results.

On the one hand, after noticing that the special orthogonal case does not differ much from the orthogonal one, in the sense that the motivic cohomology rings of the Nisnevich classifying spaces of special orthogonal groups are polynomial in all subtle Stiefel-Whitney classes except the first, we will move to the study of spin groups. This case is drastically more challenging since relations appear related to the action of the motivic Steenrod algebra on the second subtle Stiefel-Whitney class. In topology, the singular cohomology of the classifying spaces of spin groups has been completely determined by Quillen in [43]. With his result in mind, we will be able to compute a large part of the motivic cohomology of the Nisnevich classifying spaces of spin groups, which happens to be the same as in topology, and reduce the whole description to checking the regularity of a certain sequence in a polynomial ring over $\mathbb{Z}/2$. Moreover, from this result, we will deduce information on the kernel invariant of quadratic forms with trivial discriminant and Clifford invariant, namely *Spin*-torsors. Furthermore, from the motivic cohomology of the Nisnevich classifying space of $Spin_7$ we will reconstruct the motivic cohomology of the Nisnevich classifying space of G_2.

On the other hand, we will get a complete description of the motivic cohomology rings of the Nisnevich classifying spaces of unitary groups of a quadratic extesion. We will see that these are not polynomial, in contrast to the respective topological counterparts. The main reason relies on the fact that the classifying space of the unitary group is not cellular and the Rost motive of the quadratic extension appears in the picture. This computation provides us with subtle invariants for torsors of the unitary group, namely hermitian forms. We will then compare these invariants to subtle Stiefel-Whitney classes and deduce information about the kernel invariant of quadratic forms divisible by a one-fold Pfister form. Moreover, we will show that, as subtle Stiefel-Whitney classes do for quadratic forms, these new subtle classes see the triviality of a hermitian form. Furthermore, we will express the motive of the torsor associated to a hermitian form in terms of its subtle characteristic classes. We want to notice now that we expect these subtle invariants for hermitian forms to be related to the J-invariant of hermitian forms studied by Fino in [14], in analogy to the orthogonal picture.

If we want to approach the case of projective general linear groups, the Gysin long exact sequence in motivic cohomology is not enough anymore. With this in mind, we generalise the Gysin sequence to some spectral sequence for morphisms of simplicial schemes with fibers which are motivically cellular, in the sense that their motives are sum of Tates. This spectral sequence is similar to the Serre spectral sequence associated to a fibration in topology since it allows to reconstruct the motivic cohomology of the total simplicial scheme from the cohomology of the base and of the fiber. The main difference resides in how the filtrations inducing the spectral sequences are obtaind. In fact, while the Serre spectral sequence in topology is built up by filtering the base, our spectral sequence is instead obtained by filtering the fiber. We will see that this spectral sequence applies to the case of the projective general linear group. However, although we believe it could be a successful tool for getting some information about the motivic cohomology of the Nisnevich classifying spaces of projective general linear groups, we do not perform any of these computations in this thesis, since this research is still at an embryonic stage.

Slightly off the main topic of this work that, as we have discussed, mainly deals with torsors and subtle characteristic classes, at the end we will focus on Rost motives and Čech simplicial schemes of Pfister quadrics. In particular, we will investigate their deep connection with the motivic Steenrod algebra. Rost motives, Pfister quadrics and cohomology operations have shown to be fundamental for the proof of the Milnor conjecture (see [62]). At the end of this thesis, we will describe completely the action of the motivic Steenrod algebra on the motivic cohomology of the Čech simplicial scheme associated to a Pfister quadric. Due to this description, we will be able to present the motivic cohomology of the reduced Rost motive as a quotient of the motivic Steenrod algebra by a left ideal generated by some special cohomology operations. This could be the first step towards the analysis of some stable motivic homotopy groups of objects related to Rost motives.

We would like to finish this introduction by quickly summarising the contents of each chapter of this thesis. In the first chapter, we will just introduce all the homotopic and motivic categories we will need next. More precisely, we will present the category of motivic spaces, the simplicial homotopy category, the unstable and stable motivic homotopy categories and the triangulated category of motives. We will end this chapter by recalling the structure of the motivic Steenrod algebra together with some of its main features. The second chapter is devoted to the introduction of the main techniques and objects we will study in this thesis. We will start by recalling the triangulated category of motives over a simplicial scheme, in order to be able to produce Gysin long exact sequences for morphisms with motivically invertible reduced fibers. Then, we will talk about torsors and classifying spaces, focusing in particular on the Nisnevich classifying space. At the end, we will concentrate on the orthogonal case, introducing subtle Stiefel-Whitney classes, as a leading example for everything we will do in the remaining part of this work. In the third chapter we will perform the computation for spin groups. We will recall Quillen's results in topology which we are inspired by, then prove some results on the action of the motivic Steenrod algebra on the second subtle Stiefel-Whitney class. This will lead us to the main theorem of this chapter, which describes a large part of the motivic cohomology rings of the Nisnevich classifying spaces of spin groups, reducing the complete computation to assessing the regularity of a certain sequence of polynomials. We will finish the chapter by finding some very simple relations for subtle Stiefel-Whitney classes of quadratic forms belonging to the cube of the fundamental ideal of the Witt ring. Moreover, we will describe the motivic cohomology of the Nisnevich classifying space of the exceptional group G_2. The fourth chapter is instead devoted to unitary groups. We will first recall well known results about Pfister quadrics and Rost motives of quadratic extensions and then compute some cohomology rings related to them. We will then move to the main result of the chapter, namely the computation of the motivic cohomology rings of the Nisnevich classifying spaces of unitary groups. Next, we will compare the new subtle classes arising from these cohomology rings to subtle Stiefel-Whitney classes. This will provide us with some information on the kernel invariant of quadratic forms divisible by a one-fold Pfister form. We will conclude with a few applications on hermitian forms. In the fifth chapter, we will generalise the above mentioned Gysin sequence by constructing a spectral sequence in some sense of Serre's type for morphisms with motivically cellular fibers. Then, we will see that this spectral sequence applies in particular to the case of projective general linear groups leading to a possibly helpful tool for computing the motivic cohomology of their Nisnevich classifying spaces, at least in some cases. The sixth chapter is devoted to describe completely the action of the motivic Steenrod algebra on Rost motives and Čech simplicial schemes of Pfister quadrics. In particular, we will present some well known cohomology rings as left-modules over the motivic Steenrod algebra.

1

HOMOTOPIC AND MOTIVIC CATEGORIES

We start in this chapter by providing an introduction of the main categories we will work with. First, we will recall the construction of the category of *motivic spaces* (see [36]), which is the analogous in algebraic geometry of the category of topological spaces. Then, the *simplicial homotopy category* will be introduced, built up from the category of motivic spaces by inverting a special class of morphisms, called simplicial weak equivalences (see [36]). Next, we will deal with the *unstable motivic homotopy category* of Morel and Voevodsky (see [36]), constructed by, roughly speaking, contracting the affine line, in the same fashion as the unit interval is contracted in the topological homotopy category. We will continue on this path in parallel to topology by recalling the *stable motivic homotopy category* (see [25] and [9]), obtained by stabilising with respect to the projective line and, then, the *triangulated category of motives* (see [60]), which is a substitute in the motivic world of the derived category of abelian groups. These are all natural environments to study algebro-geometric problems from a homotopic perspective, since they allow, at different levels of complexity, richness and computability, to deal with algebraic varieties in a sufficiently flexible way, similar in some sense to the one characteristic of topological spaces. Furthermore, at the end, we will recall the structure and main properties of the *motivic Steenrod algebra* (see [22] and [61]), namely the algebra of bistable motivic cohomology operations, which will provide a fundamental tool we shall use frequently throughout this thesis.

Essentially, this chapter will serve the purpose of producing the right background and language for studying torsors of *linear algebraic groups over a field* by motivic homotopic means, such as classifying spaces and characteristic classes. In this sense, it contains well known definitions and results in the field of motivic homotopy theory that we will exploit in the following chapters.

1.1 MOTIVIC SPACES

We want to begin this chapter by defining the category of motivic spaces, which all the other categories we will present later on are constructed from. The general idea is that, in order to do homotopy theory in algebraic geometry, one would like to work in a "nicely behaving" category of spaces which should be deeply related to the category of smooth schemes over a certain base. This category is what we shall call the category of motivic spaces, and it will serve as a motivic counterpart of the category of CW-complexes in topology. We will now recall a construction of this category on the lines of the groundbreaking work of Morel and Voevodsky (see [36]).

Let Y be a Noetherian scheme of finite Krull dimension. The category of smooth schemes of finite type over Y, which we will denote by Sm/Y, is not a good category on its own for homotopical purposes. One of the main reasons is that it is not closed under small colimits, which means in particular that it is not possible to do quotients in it. A standard way to solve this problem is to consider the category of presheaves on Sm/Y, which we will denote by $Pre(Sm/Y)$. We know by Yoneda embedding that Sm/Y sits inside $Pre(Sm/Y)$ by sending a smooth scheme X to the presheaf $U \mapsto Hom_{Sm/Y}(U, X)$. This category is large enough to contain all small colimits (and limits) but it does not pay any attention to any particular topology. For this reason, one would like first to choose a topology, which could be for example Zariski, Nisnevich or étale, and then consider sheaves in the respective site. This way one would add more colimits to the category of presheaves. We will be mainly interested in the Nisnevich topology, therefore we will focus on the category of Nisnevich sheaves over Sm/Y, which we will denote by $Shv_{Nis}(Sm/Y)$. Nisnevich sheaves are essentially presheaves which send elementary distinguished squares to cartesian squares of sets. We now recall the definition of elementary distinguished square.

Definition 1.1.1. *An elementary distinguished square in Sm/Y is a square*

$$\begin{array}{ccc} p^{-1}(U) & \longrightarrow & V \\ \downarrow & & \downarrow{\scriptstyle p} \\ U & \xrightarrow[j]{} & X \end{array}$$

where j is an open embedding, p is an étale morphism and the restriction of p to the complement of U, namely $p^{-1}(X - U) \to X - U$, is an isomorphism for the reduced structure.

Note that elementary distinguished squares are also pushout squares. It is desirable to work with a category of sheaves that send these pushout squares to cartesian squares of sets. This observation leads to the following definition of Nisnevich sheaves.

Definition 1.1.2. *A presheaf F on Sm/Y is a Nisnevich sheaf if:*
1) $F(\emptyset) = pt$;
2) F sends elementary distinguished squares to pullback squares of sets

$$\begin{array}{ccc} F(X) & \longrightarrow & F(V) \\ \downarrow & & \downarrow \\ F(U) & \longrightarrow & F(p^{-1}(U)) \end{array}$$

The category of Nisnevich sheaves is still not big enough to do homotopy. For example, we would like to have a simplicial category to work with. Hence, we enlarge $Shv_{Nis}(Sm/Y)$ by considering the category of simplicial Nisnevich sheaves over Sm/Y.

Definition 1.1.3. *The category of motivic spaces over the base scheme Y is the category $\Delta^{op}Shv_{Nis}(Sm/Y)$ of simplicial Nisnevich sheaves over Sm/Y. We will simply denote this category by $Spc(Y)$.*

This category naturally contains Nisnevich sheaves over Y as constant simplicial Nisnevich sheaves and simplicial sets as simplicial constant Nisnevich sheaves. In particular, it contains algebraic varieties X over Y, which are represented in $Spc(Y)$ by the simplicial Nisnevich sheaves whose simplicial components are all given by $U \mapsto Hom_{Sm/Y}(U, X)$.

Notice that, as we have already mentioned, this construction is possible with other topologies leading to different categories of motivic spaces, and subsequently to different homotopy theories.

Following the parallel with topology, there is a natural way to construct a category of pointed motivic spaces. Indeed, it is enough to consider the category of pointed simplicial Nisnevich sheaves.

Definition 1.1.4. *The category of pointed motivic spaces over the base scheme Y is the category $\Delta^{op}Shv_{Nis}(Sm/Y)_*$ of pointed simplicial Nisnevich sheaves over Sm/Y, i.e. simplicial Nisnevich sheaves F endowed with a morphism $pt \to F$. We will simply denote this category by $Spc_*(Y)$.*

For any unpointed space F, we can consider its associated pointed space $F_+ = F \amalg pt$, obtained just by adding a point. Once we have pointed spaces we can define a smash product, just as in topology. More precisely, for any couple of pointed motivic spaces F and G, the simplicial sheaf $F \wedge G$ is defined as the Nisnevisch sheafification of the presheaf $U \mapsto F(U) \wedge G(U)$. With the smash product just defined, $Spc_*(Y)$ becomes a symmetric monoidal category with unit object $S^0 = Y_+$.

1.2 SIMPLICIAL HOMOTOPY CATEGORY

The next natural step one needs to take in order to do homotopy in algebraic geometry is to provide $Spc(Y)$ (and $Spc_*(Y)$) with a model structure. In principle, there are several model structures available which will give at the end the same homotopy category. What follows is a recollection of the model structure considered in [36].

By a point x of the site Sm/Y endowed with the Nisnevich topology we mean a functor $x^* : Shv_{Nis}(Sm/Y) \to Set$ from the category of Nisnevich sheaves over Y to the category of sets which commutes with finite limits and all colimits. We present now one of the possible definitions of a model structure on the category of motivic spaces.

Definition 1.2.1. *A morphism $f : F \to G$ in $Spc(Y)$ is:*
1) a weak equivalence if $x^*f : x^*F \to x^*G$ is a weak equivalence of simplicial sets for any point x of Sm/Y;
2) a cofibration if it is a monomorphism;
3) a fibration if it has the right lifting property respect to all acyclic cofibrations.

The following key result assures that, with the previous definition, motivic spaces are actually endowed with a model structure.

Theorem 1.2.2. *The classes of weak equivalences, cofibrations and fibrations as defined above give $Spc(Y)$ the structure of a proper model category.*

Proof. See [36, Theorem 1.4], [24, Corollary 2.7] and [23]. □

This model structure on the category of motivic spaces is often called the simplicial model structure.

Definition 1.2.3. *The homotopy category of the simplicial model category, obtained by inverting weak equivalences, is called the simplicial homotopy category over Y. We will denote this category by $\mathcal{H}_s(Y)$ or $\mathcal{H}_s((Sm/Y)_{Nis})$ if we want to stress the topology we are working with.*

Analogously, it is possible to provide $Spc_*(Y)$ with a simplicial model structure, by considering as weak equivalences, fibrations and cofibrations those morphisms of pointed motivic spaces which are respectively weak equivalences, fibrations and cofibrations of motivic spaces without base points. The resulting pointed simplicial homotopy category obtained by inverting weak equivalences will be denoted by $\mathcal{H}_{s,*}(Y)$. The smash product defined in the previous section provides $\mathcal{H}_{s,*}(Y)$ with the structure of a symmetric monoidal category.

For any X in Sm/Y and any point x of X, let us denote by $O^h_{X,x}$ the henselization of the local ring of X at x. Then, for any motivic space F,

$F(Spec(O^h_{X,x}))$ is defined to be the colimit $colim_U F(U)$, where U runs over all the Nisnevich neighborhoods of x in X, i.e. étale morphisms $g : U \to X$ endowed with a point $u \in g^{-1}(x)$ such that the induced map on residue fields $k(x) \to k(u)$ is an isomorphism. The next important result tells us that we can check simplicial weak equivalences by evaluating them on local henselian rings.

Lemma 1.2.4. *A morphism $f : F \to G$ in $Spc(Y)$ is a simplicial weak equivalence if and only if for any X in Sm/Y and any point x of X the induced map $F(Spec(O^h_{X,x})) \to G(Spec(O^h_{X,x}))$ is a weak equivalence of simplicial sets.*

Proof. See [36, Lemma 1.11]. □

At this point we are ready to provide examples of simplicial weak equivalences that involve a very important class of simplicial schemes, notably Čech simplicial schemes.

Definition 1.2.5. *For any smooth scheme X over Y the Čech simplicial scheme of X is the simplicial scheme $\check{C}(X)$ with simplicial components given by $\check{C}(X)_n = X_Y^{n+1}$ and face and degeneracy maps given by partial projections and partial diagonals respectively.*

A fundamental property of Čech simplicial schemes we will often use in this thesis is the following.

Lemma 1.2.6. *If $Hom_{Sm/Y}(V, U) \neq \emptyset$, then $\check{C}(U) \times V \to V$ is a simplicial weak equivalence.*

Proof. See [62, Lemma 9.2]. □

From the previous lemma one obtains the following useful result that holds if the base scheme Y is a point, i.e. $Y = Spec(k)$ for some field k.

Proposition 1.2.7. *For any pair of smooth schemes X and X' over k, we have that the following conditions are equivalent:*
1) $X(E) \neq \emptyset$ if and only if $X'(E) \neq \emptyset$, for each field extension E/k;
2) $\check{C}(X) \cong \check{C}(X')$ in $\mathcal{H}_s(k)$.

Proof. See [47, 2.3.10 and 2.3.11]. □

In particular, the previous proposition assures that all Čech simplicial schemes are projectors in $\mathcal{H}_s(k)$, namely we have an isomorphism in the simplicial homotopy category $\check{C}(X) \times \check{C}(X) \cong \check{C}(X)$ for any X.

1.3 UNSTABLE A^1-HOMOTOPY CATEGORY

In topology, the homotopy category is obtained, roughly speaking, by contracting the interval $[0,1]$. Till now we did not care about contracting any interval in the simplicial homotopy category. The core idea at the base of the A^1-homotopy theory of Morel and Voevodsky is that, pursuing this parallel path to topology, a good and natural candidate to consider as a substitute of the unit interval is the affine line A^1_Y, which for brevity we will denote by A^1. In this section we will recall their construction of the unstable A^1-homotopy category. In order to do so, we need to define an A^1-structure in $\mathcal{H}_s(Y)$. First, we define A^1-local objects following [36].

Definition 1.3.1. *A motivic space G is A^1-local if for any motivic space F we have a bijection*

$$Hom_{\mathcal{H}_s(Y)}(F,G) \to Hom_{\mathcal{H}_s(Y)}(F \times A^1, G)$$

induced by the projection $F \times A^1 \to F$.

We are now ready to introduce A^1-weak equivalences.

Definition 1.3.2. *A morphism $f : F \to F'$ is an A^1-weak equivalence if for any A^1-local space G we have that the induced map*

$$Hom_{\mathcal{H}_s(Y)}(F',G) \to Hom_{\mathcal{H}_s(Y)}(F,G)$$

is a bijection.

With this definition of weak equivalences in mind, it is possible to define a model structure on $Spc(Y)$ which actually takes into account the affine line.

Theorem 1.3.3. *The category $Spc(Y)$ with weak equivalences given by A^1-weak equivalences, cofibrations given by monomorphisms and fibrations given by morphisms with the right lifting property respect to acyclic cofibrations is a proper model category.*

Proof. See [36, Theorem 3.2] □

Definition 1.3.4. *The homotopy category we obtain from this model structure is called the unstable motivic or A^1-homotopy category. We will denote this category by $\mathcal{H}_{A^1}(Y)$.*

As for the simplicial homotopy category, we have an analogous model structure on $Spc_*(Y)$ which gives rise to a pointed A^1-homotopy category denoted by $\mathcal{H}_{A^1,*}(Y)$. The smash product on pointed motivic spaces defines a symmetric monoidal structure on $\mathcal{H}_{A^1,*}(Y)$.

At this stage, we are ready to present the characters that play the role of the spheres in this motivic setting. First, we need to mention that, unlike the topological situation, in the motivic environment there are two types of circles, the simplicial circle S_s^1 and the Tate circle S_t^1. The simplicial circle is just the pointed simplicial set $\Delta^1/\partial\Delta^1$ sitting inside $Spc_*(Y)$ while the Tate circle is the punctured affine line $G_m = A^1 - 0$ pointed by 1. Smashing these two kinds of circles one obtains bigraded spheres $S^{p,q} = S_s^{p-q} \wedge S_t^q$.

On the other hand, let us consider the quotient $A^1/(A^1 - 0)$, which we will shortly denote by $\mathbf{T}$. Namely, $\mathbf{T}$ is defined by the following pushout square of pointed simplicial Nisnevich sheaves

$$\begin{array}{ccc} A^1 - 0 & \longrightarrow & A^1 \\ \downarrow & & \downarrow \\ Spec(k) & \longrightarrow & \mathbf{T} \end{array}$$

A classical result in motivic homotopy theory tells us that the space $\mathbf{T}$ is A^1-homotopy equivalent to the projective line P^1 and to the bigraded sphere $S^{2,1}$. Namely, we have the following proposition.

Proposition 1.3.5. *$S^{2,1} \cong \mathbf{T} \cong P^1$ in $\mathcal{H}_{A^1,*}(Y)$.*

Proof. See [36, Lemma 2.15 and Corollary 2.18]. □

1.4 STABLE A^1-HOMOTOPY CATEGORY

Once one has an unstable homotopy category, the natural following step would be to stabilise it respect to a certain suspension in order to get a triangulated category. With this in mind, we want to achieve a stable motivic homotopy category by stabilising with respect to P^1, which plays the role of the sphere in topology, as we have already noticed in the previous section. A possible approach, following [59] or [25], is to obtain the pursued stabilisation by considering the category of $\mathbf{T}$-spectra.

Definition 1.4.1. *A $\mathbf{T}$-spectrum is a collection of pointed motivic spaces $E = \{E_m\}$ endowed with structure morphisms*

$$\sigma_m : \mathbf{T} \wedge E_m \to E_{m+1}$$

A morphism $f : E \to F$ of $\mathbf{T}$-spectra is a collection of morphisms of pointed motivic spaces $\{f_m : E_m \to F_m\}$ such that, for any m, the following square commutes

$$\begin{array}{ccc} \mathbf{T} \wedge E_m & \xrightarrow{id \wedge f_m} & \mathbf{T} \wedge F_m \\ \sigma_m \downarrow & & \downarrow \sigma_m \\ E_{m+1} & \xrightarrow[f_{m+1}]{} & F_{m+1} \end{array}$$

The category of $\mathbf{T}$-spectra will be denoted by $Spt(Y)$.

For any pointed space F we will denote by $\Sigma^\infty F$ the corresponding suspension $\mathbf{T}$-spectrum, defined by $(\Sigma^\infty F)_m = \mathbf{T}^m \wedge F$.

Definition 1.4.2. *For any* $\mathbf{T}$*-spectrum* E*, the stable homotopy groups of* E *are the presheaves of groups defined by*

$$\pi_{p,q}(E)(U) = colim_n \, Hom_{\mathcal{H}_{A^1}(Y)}(S_s^{p-q+n} \wedge S_t^{q+n}, E_n(U))$$

This definition of stable homotopy groups allows to define a model structure on the category of motivic spectra.

Definition 1.4.3. *A morphism* $f : E \to F$ *of* $\mathbf{T}$*-spectra is:*
1) a stable weak equivalence if the induced morphism on stable homotopy groups $\pi_{p,q}(E) \to \pi_{p,q}(F)$ *is an isomorphism of presheaves of groups for any* p *and* q*;*
2) a cofibration if it is a monomorphism at each level;
3) a fibration if it has the right lifting property with respect to all acyclic cofibrations.

The classes of stable weak equivalences, cofibrations and fibrations just defined actually provide the category of $\mathbf{T}$-speactra $Spt(Y)$ with a stable model structure.

Theorem 1.4.4. *The category of* $\mathbf{T}$*-spectra, together with stable weak equivalences, cofibrations and fibrations defined above, is a proper simplicial model category.*

Proof. See [25, Theorem 2.9 and Lemma 3.7]. □

In a nutshell, by inverting stable weak equivalences we get the stable A^1-homotopy category.

Definition 1.4.5. *The homotopy category associated to the stable model structure on the category of* $\mathbf{T}$*-spectra is called the stable motivic or* A^1*-homotopy category over* Y *and will be denoted by* $\mathcal{SH}_{A^1}(Y)$*.*

From the fact S_s^1 is a cogroup object in $\mathcal{H}_{A^1}(Y)$ one has that hom-sets in $\mathcal{SH}_{A^1}(Y)$ are always abelian groups since each $\mathbf{T}$-spectrum is a two-fold simplicial suspension. It follows that the stable motivic homotopy category is additive and, moreover, triangulated. Indeed, one gets a triangulated structure by considering as shift the simplicial suspension, namely $E[1] = S_s^1 \wedge E$, and as distinguished triangles the triangles isomorphic to the cofiber sequences

$$E \xrightarrow{f} F \to Cone(f) \to S_s^1 \wedge E$$

where $Cone(f)$ is defined in $Spt(Y)$ by the following pushout square

$$\begin{array}{ccc} E & \xrightarrow{f} & F \\ \downarrow & & \downarrow \\ E \wedge \Delta[1] & \longrightarrow & Cone(f) \end{array}$$

Among other advantages, the stable motivic homotopy category is the natural environment to deal with homology and cohomology. The main reason is that, actually, homology and cohomology theories are represented in $\mathcal{SH}_{A^1}(Y)$. More precisely, we have that for any **T**-spectrum E the corresponding homology and cohomology are defined respectively by

$$E_{p,q}(F) = Hom_{\mathcal{SH}_{A^1}(Y)}(S^{p,q}, E \wedge F)$$

and

$$E^{p,q}(F) = Hom_{\mathcal{SH}_{A^1}(Y)}(F, E(q)[p])$$

where by $E(q)[p]$ we mean the **T**-spectrum $S^{p,q} \wedge E$. All homology and cohomology theories arising in this way are called *"large" theories*.

In particular, motivic cohomology, which is the main cohomology theory we will consider throughout this thesis, is represented by the motivic Eilenberg-MacLane spectrum $\mathbf{H}\mathbb{Z}$, defined by

$$(\mathbf{H}\mathbb{Z})_m = L(A^m)/L(A^m - 0) \cong L((P^1)^{\wedge m})$$

where by $L(X)$ we mean the sheaf that sends any smooth scheme U over Y to the free abelian group generated by the closed irreducible subschemes of $U \times_Y X$ which are finite and surjective over a connected component of U, provided that the base scheme Y is regular (we will discuss better this sheaf in the next section). The assembly morphisms are given by the compositions

$$\mathbf{T} \wedge (\mathbf{H}\mathbb{Z})_m \cong P^1 \wedge L((P^1)^{\wedge m}) \to L(P^1) \wedge L((P^1)^{\wedge m})$$
$$\to L((P^1)^{\wedge m+1}) \cong (\mathbf{H}\mathbb{Z})_{m+1}$$

1.5 TRIANGULATED CATEGORY OF MOTIVES

We have till now introduced the motivic counterparts of homotopic categories. At this point, we would like to complete this picture by showing the construction of a triangulated category which replaces somehow in the motivic setting the derived category of abelian groups in topology. As we will see, this provides a natural environment to work with motivic cohomology.

Fix a commutative ring with identity R and a regular Noetherian scheme Y.

Definition 1.5.1. *For any U and V in Sm/Y define $c(U,V)$ by*

$$c(U,V) = \bigoplus R \cdot \{W \mid W \textit{ integral closed subscheme of } U \times_Y V \textit{ finite over } U \textit{ and surjective over a connected component of } U\}$$

Elements of $c(U,V)$ are called finite correspondences from U to V.

Let ϕ be a finite correspondence from U to V and ψ a finite correspondence from V to W. Then, one can define the composition $\psi \circ \phi$ as the finite correspondence from U to W obtained by $(p_U \times p_W)_*((p_U \times p_V)^*(\phi) \cap (p_V \times p_W)^*(\psi))$, where p_U, p_V and p_W are the obvious projections. With this composition one can construct the category of finite correspondences.

Definition 1.5.2. *The category of finite correspondences with R-coefficients is the category whose objects are smooth schemes over Y and morphisms are finite correspondences. We will denote this category, which is naturally R-linear, by $Cor(Y, R)$.*

Note that there exists a motivic functor from Sm/Y to $Cor(Y, R)$ sending each smooth scheme U to itself and each smooth morphism $f : U \to V$ to its graph Γ_f.

Definition 1.5.3. *A presheaf with transfers on Sm/Y is an R-linear contravariant functor from $Cor(Y, R)$ to the category of R-modules. It will be called Nisnevich sheaf with transfers if the underlying presheaf of R-modules on Sm/Y is a sheaf in the Nisnevich topology.*

The category of presheaves with transfers will be denoted by $PST(Y, R)$ while the category of Nisnevich sheaves with transfers will be denoted by $ST_{Nis}(Y, R)$. Every smooth scheme V over Y is represented in $PST(Y, R)$ by the presheaf with transfers $L(V)$ defined by $U \to cor(U, V)$. Indeed, $L(V)$ is a Nisnevich sheaf for any $V \in Sm/Y$.

Theorem 1.5.4. *The category of Nisnevich sheaves with transfers is abelian.*

Proof. See [60, Theorem 3.1.4] and [8, Proposition 10.3.9]. □

Hence, one can construct its derived category of complexes of Nisnevich sheaves with transfers bounded from above, which we will compactly denote by $D^-(ST_{Nis}(Y, R))$. Then, we can consider its smallest thick subcategory $\mathcal{E}(Y)$ containing the morphisms $L(U \times A^1) \to L(U)$ for any smooth scheme U and closed under direct sums. Notice that the quotient category $D^-(ST_{Nis}(Y, R))/\mathcal{E}(Y)$ is the localization $D^-(ST_{Nis}(Y, R))[\mathcal{W}(Y)^{-1}]$, where $\mathcal{W}(Y)$ is the class of morphisms which have cone in $\mathcal{E}(Y)$. Morphisms in $\mathcal{W}(Y)$ are called A^1-weak equivalences.

Definition 1.5.5. *The localization $D^-(ST_{Nis}(Y, R))[\mathcal{W}(Y)^{-1}]$ is denoted simply by $\mathcal{DM}^-_{eff}(Y, R)$ and is called the triangulated category of motives over Y with R-coefficients.*

Let $\Delta^\bullet$ be the standard cosimplicial object defined by

$$\Delta^n = Y \times_{Spec(\mathbb{Z})} Spec(\mathbb{Z}[t_0, \ldots, t_n]/\sum_{i=0}^{n} t_i - 1)$$

Then, if F is a presheaf with transfers we can consider its Suslin complex $C_*(F)$ defined by $C_n(F)(U) = F(U \times \Delta^n)$ endowed with differentials given by $\sum_i (-1)^i \delta_i^*$, where the δ_i^* correspond to the boundary maps of $\Delta^\bullet$. If F is a Nisnevich sheaf with transfers, then one can easily see that its Suslin complex is a complex of Nisnevich sheaves with transfers.

Definition 1.5.6. *For any smooth scheme X over Y, the motive of X is the object $M(X)$ in $\mathcal{DM}^-_{eff}(Y,R)$ corresponding to the Suslin complex $C_*(L(X))$. Indeed, M defines a functor from Sm/Y to $\mathcal{DM}^-_{eff}(Y,R)$.*

The category of motives is moreover tensor triangulated (see [60] and [8]) with a tensor product which will be denoted by $\otimes$.

Definition 1.5.7. *The thick subcategory of $\mathcal{DM}^-_{eff}(Y,R)$ generated by motives of smooth schemes is denoted by $\mathcal{DM}^-_{gm}(Y,R)$ and is called the category of geometric motives.*

We will now recall the main properties of the triangulated category of motives (see [60] and [8]):
1) (A^1-homotopy invariance) the projection $U \times A^1 \to U$ induces an isomorphism $M(U \times A^1) \cong M(U)$;
2) (Kunneth formula) $M(U \times V) \cong M(U) \otimes M(V)$;
3) (Mayer-Vietoris triangle) for any open cover $\{U, V\}$ of a smooth scheme X there is a distinguished triangle

$$M(U \cap V) \to M(U) \oplus M(V) \to M(X) \to M(U \cap V)[1]$$

4) (vector bundle) for any vector bundle $E \to X$ the induced map on motives $M(E) \to M(X)$ is an isomorphism;
5) (projective bundle) for any projective bundle $P \to X$ of rank n there is an isomorphism

$$M(P) \cong \bigoplus_{i=0}^{n} M(X)(i)[2i]$$

6) (Gysin triangle) for any closed immersion $Z \to X$ of pure codimension c between smooth separated schemes of finite type there is a distinguished triangle

$$M(X - Z) \to M(X) \to M(Z)(c)[2c] \to M(X - Z)[1]$$

7) (cancellation) If Y is a perfect field, then for any M and N in $\mathcal{DM}^-_{eff}(Y,R)$ the natural map

$$Hom_{\mathcal{DM}^-_{eff}(Y,R)}(M, N) \to Hom_{\mathcal{DM}^-_{eff}(Y,R)}(M(1), N(1))$$

is an isomorphism;
8) (Chow motives) for any pair of smooth projective schemes U and V, where U has pure relative dimension d over Y, there is an isomorphism

$$Hom_{\mathcal{DM}^-_{eff}(Y,R)}(M(U), M(V)) \cong CH^d(U \times V)$$

Definition 1.5.8. *The motive of Y, considered over itself with the identity morphism, will be denoted by T and will be called the trivial Tate motive.*

For any smooth scheme X over Y, we have a morphism $M(X) \to T$ in $\mathcal{DM}^-_{eff}(Y, R)$. We define the reduced motive of X as $Cone(M(X) \to T)[-1]$ and we will denote it by $\widetilde{M}(X)$.

As a particular case, we have a distinguished triangle

$$M(G_m) \to T \to \widetilde{M}(G_m)[1] \to M(G_m)[1]$$

which is split since there is a morphism from Y to G_m. Then, we will define the Tate motive $T(1)$ to be $\widetilde{M}(G_m)[-1]$. Hence, $M(G_m) = T \oplus T(1)[1]$. This leads to the definition of Tate motives with various shifts simply by imposing $T(q) = T(1)^{\otimes q}$.

The power of the triangulated category of motives is, for example, stressed by the fact that motivic cohomology is represented in $\mathcal{DM}^-_{eff}(Y, R)$. Indeed, we have that

$$H^{p,q}(X, R) = Hom_{\mathcal{DM}^-_{eff}(Y,R)}(M(X), T(q)[p])$$

The following fundamental result computes the motivic cohomology with $\mathbb{Z}/2$-coefficients of the point, which we will compactly denote by H for the rest of this thesis.

Theorem 1.5.9. *We have that $H = K^M(k)/2[\tau]$, where $K^M(k)$ is the Milnor K-theory of k and τ is the non-trivial cohomology class in bidegree $(1)[0]$.*

Proof. See [62]. □

We end this section with a result that claims the triviality in certain regions of the motivic cohomology of a smooth simplicial scheme.

Proposition 1.5.10. *Let $X_\bullet$ be a smooth simplicial scheme over k. Then, the motivic cohomology group $H^{p,q}(X_\bullet, \mathbb{Z})$ is zero in the following cases:*
1) $q < 0$;
2) $q = 0$ and $p < 0$;
3) $q = 1$ and $p \leq 0$.

Proof. See [58, Corollary 2.2]. □

1.6 THE MOTIVIC STEENROD ALGEBRA

Cohomology operations are pivotal tools for the study of algebro-geometric problems from a motivic homotopic point of view. As in topology, they enrich the structure of motivic cohomology rings, which are naturally left modules over the algebra of stable cohomology operations. This algebra is called motivic Steenrod algebra and shares some of its main features, which we will recall in this section, with its classical topological counterpart. Since in this thesis we will mainly consider motivic cohomology rings with $\mathbb{Z}/2$-coefficients, this section will report only results about the mod 2 motivic Steenrod algebra.

We fix for the rest of this section the base scheme Y to be a point $Spec(k)$ for some field k of characteristic different from 2. We recall that in [61] there is a construction of Steenrod squares Sq^i for fields of characteristic 0 which is generalised to fields of any characteristic different from 2 in [22].

Definition 1.6.1. *The motivic Steenrod algebra that we will denote by $\mathcal{A}$ is the subalgebra of the algebra of bistable operations in mod 2 motivic cohomology generated over H by the Steenrod squares Sq^i of bidegree $([i/2])[i]$ for any $i \geq 0$.*

The following important result tells us that indeed $\mathcal{A}$ contains all bistable cohomology operations.

Theorem 1.6.2. *$\mathcal{A}$ is the algebra of bistable cohomology operations in mod 2 motivic cohomology.*

Proof. See [22, Theorem 1.1] and [64, Theorem 3.49]. □

Notice that, unlike in topology, the action of the motivic Steenrod algebra on the cohomology of the point is non-trivial. In fact, we have that $Sq^1(\tau) = \rho$, where ρ is the class of -1 in the first Milnor K-group of k modulo 2.

We have the following important result which describes essentially all relations between Steenrod squares.

Theorem 1.6.3. *(Adem relations) In $\mathcal{A}$ we have the following relations for any $a < 2b$:*
1) if a and b are even

$$Sq^a Sq^b = \sum_{c=0}^{[a/2]} \tau^{c \; mod2} \binom{b-c-1}{a-2c} Sq^{a+b-c} Sq^c$$

2) if a is even and b is odd

$$Sq^a Sq^b = \sum_{c=0}^{[a/2]} \binom{b-c-1}{a-2c} Sq^{a+b-c} Sq^c + \sum_{c=0, c\; odd}^{[a/2]} \rho \binom{b-c-1}{a-2c} Sq^{a+b-c-1} Sq^c$$

3) if a is odd and b is even

$$Sq^a Sq^b = \sum_{c=0}^{[a/2]} \binom{b-c-1}{a-2c} Sq^{a+b-c} Sq^c + \sum_{c=0, c\; odd}^{[a/2]} \rho \binom{b-c-1}{a-2c-1} Sq^{a+b-c-1} Sq^c$$

4) if a and b are odd

$$Sq^a Sq^b = \sum_{c=0}^{[a/2]} \binom{b-c-1}{a-2c} Sq^{a+b-c} Sq^c$$

Proof. See [22, Theorem 5.1] and [61, Theorem 10.2]. □

As in topology, it follows from Adem relations the following useful corollary that provides a set of generators of $\mathcal{A}$ as an H-algebra.

Corollary 1.6.4. *$\mathcal{A}$ is generated as an algebra over H by the Steenrod squares Sq^{2^i} for any $i \geq 0$.*

The following proposition describes the behaviour of Steenrod squares respect to the cup product in motivic cohomology.

Proposition 1.6.5. *(Cartan formula)*

$$Sq^{2n}(xy) = \sum_i \tau^{i \, mod 2} Sq^i(x) Sq^{2n-i}(y)$$

$$Sq^1(xy) = Sq^1(x)y + xSq^1(y)$$

Proof. See [61, Proposition 9.7]. □

Other important properties that we are going to use in this thesis are summarised in the following two results.

Lemma 1.6.6. *For any x in bidegree $(n)[2n]$ one has that $Sq^{2n}(x) = x^2$.*

Proof. See [61, Lemma 9.8]. □

Lemma 1.6.7. *For any x in bidegree $(q)[p]$ one has that $Sq^{2n}(x) = 0$ for any $n > p - q$ and $n \geq q$.*

Proof. See [61, Lemma 9.9]. □

Now, we recall the structure of the dual of the motivic Steenrod algebra $\mathcal{A}_{*,*}$.

Theorem 1.6.8. *The dual of the motivic Steenrod algebra $\mathcal{A}_{*,*}$ is isomorphic to the graded commutative H-algebra generated by elements τ_i, for $i \geq 0$, in bidegree $(2^i - 1)[2^{i+1} - 1]$ and ξ_i, for $i > 0$, in bidegree $(2^i - 1)[2^{i+1} - 2]$ subject to relations $\tau_i^2 = \tau\xi_{i+1} + \rho\tau_{i+1} + \rho\tau_0\xi_{i+1}$ for any $i \geq 0$.*

Proof. See [22, Theorem 5.6] and [61, Theorem 12.6]. □

In the Steenrod algebra there are very special operations, namely the Milnor operations Q_j, which have shown to be invaluable tools for the proof of the Milnor conjecture (see [62]) and of the Bloch-Kato conjecture (see [65]).

Definition 1.6.9. *Q_j is the element in $\mathcal{A}$ dual with respect to the standard skew-polynomial basis to the element τ_j in $\mathcal{A}_{*,*}$.*

Following [28, Remark 5], they can also be defined inductively by

$$Q_0 = Sq^1$$

and

$$Q_j = Sq^{2^j} Q_{j-1} + Q_{j-1} Sq^{2^j} + \rho Sq^{2^{j-1}} Q_{j-1} Q_{j-2}$$

The following result recapitulates the main features of Milnor operations.

Proposition 1.6.10. *$Q_j^2 = 0$ and $[Q_i, Q_j] = 0$, i.e. Milnor operations generate an exterior subalgebra in $\mathcal{A}$.*

Proof. See [61, Proposition 13.4]. □

We finish this section by noticing that, when -1 is a square in the base field, i.e. $\rho = 0$ in H, the behaviour of the motivic Steenrod algebra is particularly similar to the topological one's. For example, when ρ is zero, the action of the motivic Steenrod algebra on the cohomology of the point is indeed trivial and the Milnor operations can be just defined by the usual recursive formula

$$Q_j = [Sq^{2^j}, Q_{j-1}]$$

well known from topology.

2

CLASSIFYING SPACES AND CHARACTERISTIC CLASSES FOR TORSORS

The principal aim of this thesis is to make a few steps forward in the investigation of torsors from a motivic homotopic point of view. In a nutshell, we will compute the motivic cohomology rings of some classifying spaces where characteristic classes for torsors come from.

In order to do so, we first need to recall definitions and main properties of motivic categories over a simplicial base (see [63]). These triangulated categories of motives constitute the right environment to work with fibrations of simplicial schemes with reduced fibers which are motivically invertible. This study will lead us to the construction in good generality of long exact sequences in motivic cohomology of the same type of *Gysin long exact sequences* for spherical fibrations in topology.

After having dealt with the necessary general results that will enable further computations, we will present the main characters of this thesis, i.e. *torsors of linear algebraic groups*. Then, we will introduce *Nisnevich* and *étale classifying spaces* (see [36]). In particular, we will highlight their deep connection and prove some of their main features, which will be used often in the next chapters. We will see that some specific morphisms of classifying spaces meet exactly the conditions required to produce the above mentioned Gysin sequences. This will constitute the main technique we are going to use in order to obtain the description of the motivic cohomology rings of certain Nisnevich classifying spaces, which will provide us with *subtle invariants* for certain types of torsors.

At the end of this chapter, we will recall, as a leading example, the case of orthogonal groups, whose torsors over the point are *quadratic forms*, studied by Smirnov and Vishik in [47]. Our future results are modelled on and find their roots in the just mentioned orthogonal case.

2.1 MOTIVES OVER A SIMPLICIAL BASE

The main purpose of this section is to recall some key definitions and results regarding the triangulated category of motives over a simplicial base, which will be a pivotal tool for our computations.

Let us fix a smooth simplicial scheme $Y_\bullet$ over k and a commutative ring with identity R. Following [63], we will denote by $Sm/Y_\bullet$ the category whose objects are given by pairs (U, j), where j is a non-negative integer and U is a smooth scheme over Y_j, and whose morphisms from (U, j) to (V, i) are given by pairs (f, θ), where $\theta : [i] \to [j]$ is a simplicial map and $f : U \to V$ is a morphism of schemes, such that the following diagram is commutative

$$\begin{array}{ccc} U & \xrightarrow{f} & V \\ \downarrow & & \downarrow \\ Y_j & \xrightarrow[Y_\theta]{} & Y_i \end{array}$$

The definition of motivic spaces over simplicial schemes then is essentially the same of spaces over schemes given in section 1.

Definition 2.1.1. *We will denote by $Spc(Y_\bullet) = \Delta^{op} Shv_{Nis}(Sm/Y_\bullet)$ the category of motivic spaces over $Y_\bullet$ and by $Spc_*(Y_\bullet)$ its pointed counterpart, consisting respectively of unpointed and pointed simplicial Nisnevich sheaves over $Sm/Y_\bullet$.*

For any morphism $f : F \to G$ in $Spc_*(Y_\bullet)$ there is a cofiber sequence

$$F \to G \to Cone(f) \to S^1_s \wedge F$$

where $Cone(f)$ is defined by the following pushout diagram in $Spc_*(Y_\bullet)$

$$\begin{array}{ccc} F & \xrightarrow{f} & G \\ \downarrow & & \downarrow \\ F \wedge \Delta[1] & \longrightarrow & Cone(f) \end{array}$$

Definition 2.1.2. *A presheaf with transfers on the simplicial scheme $Y_\bullet$ is given by a collection $\{F_i\}_{i \geq 0}$ of presheaves with transfers on Sm/Y_i respectively together with a morphism of presheaves with transfers $F_\theta : Y_\theta^*(F_i) \to F_j$ for any simplicial map $\theta : [i] \to [j]$, such that $F_{id} = id$ and $F_{\phi\psi} : Y_{\phi\psi}^*(F_i) \to F_k$ is equal to the composition of $Y_\phi^* F_\psi : Y_\phi^* Y_\psi^*(F_i) \to Y_\phi^*(F_j)$ and $F_\phi : Y_\phi^*(F_j) \to F_k$, where $\phi : [j] \to [k]$ and $\psi : [i] \to [j]$ are simplicial maps.*

For any (V, i) in $Sm/Y_\bullet$, let $L(V, i)$ be the presheaf with transfers on $Y_\bullet$ whose j-th component is given by

$$L(V, i)_j = \bigoplus_{\theta:[i]\to[j]} L(V \times_\theta Y_j)$$

where $V \times_\theta Y_j$ is defined via the pullback square

$$\begin{array}{ccc} V \times_\theta Y_j & \longrightarrow & Y_j \\ \downarrow & & \downarrow Y_\theta \\ V & \longrightarrow & Y_i \end{array}$$

for any simplicial map $\theta : [i] \to [j]$.

We will denote by $PST(Y_\bullet, R)$ the category of presheaves with transfers over $Y_\bullet$ with R-coefficients and by $Cor(Y_\bullet, R)$ its full subcategory generated by all possible direct sums of objects of the type $L(V, i)$.

Lemma 2.1.3. *The category $PST(Y_\bullet, R)$ is naturally equivalent to the category of R-linear contravariant functors from $Cor(Y_\bullet, R)$ to the category of R-modules.*

Proof. See [63, Lemma 2.3]. □

For any i consider the functor $r_i : Cor(Y_i, R) \to Cor(Y_\bullet, R)$ which sends V to $L(V, i)$. These functors r_i induce in the standard way a pair of adjoint functors

$$\begin{array}{c} PST(Y_\bullet, R) \\ r_{i,\#} \uparrow \downarrow r_i^* \\ PST(Y_i, R) \end{array}$$

Sheafifying in the Nisnevich topology we get the category of Nisnevich sheaves with transfers over $Y_\bullet$, which we will denote by $ST_{Nis}(Y_\bullet, R)$. Let $\mathcal{E}_i(Y_\bullet)$ be the class in $PST(Y_\bullet)$ obtained as $r_{i,\#}(\mathcal{E}(Y_i))$, where $\mathcal{E}(Y_i)$ has been defined in Section 1.5, and $\mathcal{E}(Y_\bullet)$ be the smallest thick subcategory of $PST(Y_\bullet)$ containing all $\mathcal{E}_i(Y_\bullet)$ and closed under direct sums. A morphism in $D^-(ST_{Nis}(Y_\bullet, R))$ is called an A^1-weak equivalence if its cone lives in $\mathcal{E}(Y_\bullet)$. Let us denote by $\mathcal{W}(Y_\bullet)$ the class of A^1-weak equivalences.

Definition 2.1.4. *The triangulated category of motives over the simplicial scheme $Y_\bullet$ is the localization of $D^-(ST_{Nis}(Y_\bullet, R))$ with respect to A^1-weak equivalences, i.e. $D^-(ST_{Nis}(Y_\bullet, R))[\mathcal{W}(Y_\bullet)^{-1}]$. We will denote this category by $\mathcal{DM}^-_{eff}(Y_\bullet, R)$.*

This category of motives over a simplicial base is a tensor triangulated category. One of the main properties of this category is the following.

Proposition 2.1.5.

$$H^{p,q}(Y_\bullet, R) = Hom_{\mathcal{DM}^-_{eff}(Y_\bullet, R)}(T, T(q)[p])$$

Proof. See [63, Proposition 5.3]. □

We notice that every cofiber sequence in $Spc_*(Y_\bullet)$ induces a distinguished triangle in $\mathcal{DM}^-_{eff}(Y_\bullet, R)$. Besides, attached to this category there is a sequence of restriction functors

$$r_i^* : \mathcal{DM}^-_{eff}(Y_\bullet, R) \to \mathcal{DM}^-_{eff}(Y_i, R)$$

induced by the funtors above defined on the categories of presheaves with transfers. The image of a motive $N \in \mathcal{DM}^-_{eff}(Y_\bullet, R)$ under r_i^* will be simply denoted by N_i. Furthermore, we have the following adjunction for any morphism $p : Y_\bullet \to Y'_\bullet$ of smooth simplicial schemes

$$\begin{array}{c} \mathcal{DM}^-_{eff}(Y_\bullet, R) \\ Lp^* \uparrow \downarrow Rp_* \\ \mathcal{DM}^-_{eff}(Y'_\bullet, R) \end{array}$$

In the case that p is smooth, together with the previous one, there is also the following adjunction

$$\begin{array}{c} \mathcal{DM}^-_{eff}(Y_\bullet, R) \\ Lp_\# \downarrow \uparrow p^* \\ \mathcal{DM}^-_{eff}(Y'_\bullet, R) \end{array}$$

We finish this section by noticing that, for any smooth simplicial scheme $Y_\bullet$ over k, we have a pair of adjoint functors

$$\begin{array}{c} \mathcal{DM}^-_{eff}(Y_\bullet, R) \\ Lc_\# \downarrow \uparrow c^* \\ \mathcal{DM}^-_{eff}(k, R) \end{array}$$

where $c : Y_\bullet \to Spec(k)$ is the projection to the base. Then, $\mathcal{DM}^-_{eff}(Y_\bullet, R)$ contains Tate objects which are defined by $T(q)[p] = c^*(T(q)[p])$. In general, for any motive M in $\mathcal{DM}^-_{eff}(k, R)$ we will also denote by M its image $c^*(M)$ in $\mathcal{DM}^-_{eff}(Y_\bullet, R)$.

2.2 GYSIN LONG EXACT SEQUENCES IN MOTIVIC COHOMOLOGY

We report below some results about the category of motives over a simplicial base which will be central throughout this thesis in order to deal with fibrations with motivically invertible reduced fibers.

First of all, we recall some facts about coherence taken from [47].

Definition 2.2.1. *A smooth coherent morphism is a smooth morphism $\pi : X_\bullet \to Y_\bullet$ such that there is a cartesian diagram*

$$\begin{array}{ccc} X_j & \xrightarrow{\pi_j} & Y_j \\ {\scriptstyle X_\theta}\downarrow & & \downarrow{\scriptstyle Y_\theta} \\ X_i & \xrightarrow[\pi_i]{} & Y_i \end{array}$$

for any simplicial map $\theta : [i] \to [j]$.

Definition 2.2.2. *A motive N in $\mathcal{DM}^-_{eff}(Y_\bullet, R)$ is said to be coherent if all simplicial morphisms $\theta : [i] \to [j]$ induce structural isomorphisms $N_\theta : LY^*_\theta(N_i) \to N_j$.*

The full subcategory of $\mathcal{DM}^-_{eff}(Y_\bullet, R)$ whose objects are coherent motives will be denoted by $\mathcal{DM}^-_{coh}(Y_\bullet, R)$. The fact that LY^*_θ is a triangulated functor implies that $\mathcal{DM}^-_{coh}(Y_\bullet, R)$ is closed under taking cones and arbitrary direct sums. On the other hand, we have that $L\pi_\#$ maps coherent objects to coherent ones for any smooth coherent morphism π. Hence, $M(X_\bullet \xrightarrow{\pi} Y_\bullet)$ is a coherent motive, where by $M(X_\bullet \xrightarrow{\pi} Y_\bullet)$ we mean the image $L\pi_\#(T)$ of the trivial Tate motive.

We are now ready to report some technical propositions which will enable us to work quite confidently with fibrations of simplicial schemes with motivically invertible reduced fibers. Before stating the results, we define the simplicial set $CC(Y_\bullet)$ by applying to $Y_\bullet$ the functor CC which sends each connected component of each Y_i to the point and commutes with colimits.

Proposition 2.2.3. *Suppose that $H^1(CC(Y_\bullet), R^\times) = 0$. Let T be the trivial Tate motive and $N \in \mathcal{DM}^-_{coh}(Y_\bullet, R)$ be such a motive that its graded components $N_i \in \mathcal{DM}^-_{eff}(Y_i, R)$ are isomorphic to T. Then N is isomorphic to T.*

Proof. See [47, Proposition 3.1.5]. □

From the previous proposition we immediately deduce the following corollary which is a generalisation for all invertible motives.

Corollary 2.2.4. *Suppose that $H^1(CC(Y_\bullet), R^\times) = 0$. Let M be an invertible motive in $\mathcal{DM}^-_{eff}(k, R)$ and $N \in \mathcal{DM}^-_{coh}(Y_\bullet, R)$ be such a motive that its graded components $N_i \in \mathcal{DM}^-_{eff}(Y_i, R)$ are isomorphic to M. Then N is isomorphic to M.*

Proof. Consider the motive $N \otimes M^{-1}$ in $\mathcal{DM}^-_{eff}(Y_\bullet, R)$. We notice that

$$(N \otimes M^{-1})_i \cong N_i \otimes M^{-1} \cong M \otimes M^{-1} \cong T$$

and, for any simplicial map $\theta : [i] \to [j]$, the morphisms

$$LY^*_\theta((N \otimes M^{-1})_i) \to (N \otimes M^{-1})_j$$

are nothing else but the isomorphisms $(LY_\theta^*(N_i) \to N_j) \otimes M^{-1}$. Then, it follows from Proposition 2.2.3 that $N \otimes M^{-1} \cong T$, which completes the proof. □

Notice that the condition $H^1(CC(Y_\bullet), R^\times) = 0$ is automatically satisfied if $R = \mathbb{Z}/2$, which is the case we will be mainly interested in.

Recall that, in topology, for a spherical fibration

$$S^{n-1} \to E \to B$$

there exists a long exact sequence of cohomology groups

$$\cdots \to H^{*-1}(E) \to H^{*-n}(B) \to H^*(B) \to H^*(E) \to \dots$$

called *Gysin sequence*, where the middle morphism is the multiplication by a certain class in the cohomology of the base space (see for example [50, Section 15.30]).

Now, we want to present the core technique inspired by [47] we will use in this thesis. This result allows to generate long exact sequences in motivic cohomology associated to fibrations with reduced fibers which are motivically invertible, of the same nature of Gysin sequences for sphere bundles in topology.

Proposition 2.2.5. *Let $\pi : X_\bullet \to Y_\bullet$ be a smooth coherent morphism of smooth simplicial schemes over k and A a smooth k-scheme such that:*
1) over the 0th simplicial component π is isomorphic to the projection $Y_0 \times A \to Y_0$;
2) $H^1(CC(Y_\bullet), R^\times) = 0$;
3) $\widetilde{M}(A)$ is an invertible motive in $\mathcal{DM}^-_{eff}(k, R)$.
Then, $M(Cone(\pi)) \cong \widetilde{M}(A)[1] \in \mathcal{DM}^-_{eff}(Y_\bullet, R)$ where $Cone(\pi)$ is the cone of π in $Spc_*(Y_\bullet)$.

Proof. In $Spc_*(Y_\bullet)$ we have a cofiber sequence

$$X_\bullet \xrightarrow{\pi} Y_\bullet \to Cone(\pi) \to S^1_s \wedge X_\bullet$$

which induces a distinguished triangle

$$M(X_\bullet \xrightarrow{\pi} Y_\bullet) \to T \to M(Cone(\pi)) \to M(X_\bullet \xrightarrow{\pi} Y_\bullet)[1]$$

in the motivic category $\mathcal{DM}^-_{eff}(Y_\bullet, R)$. Since π is smooth coherent we have by 1) that it is the projection over any simplicial component. It immediately follows that $\pi_i : Y_i \times A \cong X_i \to Y_i$ induces the morphism $M(A) \to T$ in $\mathcal{DM}^-_{eff}(Y_i, R)$ for any i, from which we get that $M(Cone(\pi))_i \cong \widetilde{M}(A)[1]$ in $\mathcal{DM}^-_{eff}(Y_i, R)$. Moreover, we point out that $M(Cone(\pi))$ is in $\mathcal{DM}^-_{coh}(Y_\bullet, R)$, since both $M(X_\bullet \xrightarrow{\pi} Y_\bullet)$ and T are coherent objects. Hence, Proposition 2.2.3 implies that $M(Cone(\pi)) \cong \widetilde{M}(A)[1]$ in $\mathcal{DM}^-_{eff}(Y_\bullet, R)$, and the proof is complete. □

Moreover, we get a *Thom isomorphism* of $H(Y_\bullet, R)$-modules

$$H^{*-s,*'-r}(Y_\bullet, R) \to H^{*,*'}(Cone(\pi), R)$$

in the case that the reduced motive of A is the Tate motive $T(r)[s-1]$.

Definition 2.2.6. *The image of* 1 *under the Thom isomorphism will be called Thom class and it will be denoted by* α.

In order to understand exactly why the previous proposition generates sequences of Gysin type we need to find the right substitute for the topological sphere S^{n-1} in the motivic setting. The character that plays the role of the sphere in our case is, indeed, the split affine quadric A_{q_n} defined by the equation $q_n = 1$, where q_n is the standard split n-dimensional quadratic form. As we will see better lately, the reduced motive of A_{q_n} is the Tate motive $T([n/2])[n-1]$. Hence, by the Thom isomorphism, we get a long exact sequence of motivic cohomology groups

$$\cdots \to H^{*-1,*'}(X_\bullet, R) \to H^{*-n,*'-[n/2]}(Y_\bullet, R) \to$$
$$H^{*,*'}(Y_\bullet, R) \to H^{*,*'}(X_\bullet, R) \to \dots$$

which completes the analogy with Gysin sequences.

Later, we will also need the following result about functoriality of the isomorphism found in the previous proposition.

Proposition 2.2.7. *Let* $\pi : X_\bullet \to Y_\bullet$ *and* $\pi' : X'_\bullet \to Y'_\bullet$ *be smooth coherent morphisms of smooth simplicial schemes over* k *and* A *a smooth* k*-scheme that satisfies all conditions from the previous proposition with respect to* π' *and such that the following diagram is cartesian with all morphisms smooth*

$$\begin{array}{ccc} X_\bullet & \xrightarrow{\pi} & Y_\bullet \\ \downarrow{\scriptstyle p_X} & & \downarrow{\scriptstyle p_Y} \\ X'_\bullet & \xrightarrow{\pi'} & Y'_\bullet \end{array}$$

Then, the induced square of motives in the category $\mathcal{DM}^-_{eff}(Y'_\bullet, R)$ *extends uniquely to a morphism of triangles where* $Lp_{Y\#}M(Cone(\pi)) \to M(Cone(\pi'))$ *is given by* $M(p_Y) \otimes id_{\widetilde{M}(A)[1]}$.

Proof. We start by noticing that in $Spc_*(Y'_\bullet)$ we can complete our commutative diagram to a morphism of cofiber sequences

$$\begin{array}{ccccccc} X_\bullet & \xrightarrow{\pi} & Y_\bullet & \longrightarrow & Cone(\pi) & \longrightarrow & S^1_s \wedge X_\bullet \\ \downarrow{\scriptstyle p_X} & & \downarrow{\scriptstyle p_Y} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle id \wedge p_X} \\ X'_\bullet & \xrightarrow{\pi'} & Y'_\bullet & \longrightarrow & Cone(\pi') & \longrightarrow & S^1_s \wedge X'_\bullet \end{array}$$

which induces a morphism of distinguished triangles in $\mathcal{DM}^{-}_{eff}(Y'_\bullet, R)$

$$\begin{array}{ccccccc}
Lp_{Y\#}M(X_\bullet \xrightarrow{\pi} Y_\bullet) & \longrightarrow & Lp_{Y\#}T & \longrightarrow & Lp_{Y\#}Cone(\pi) \cong Lp_{Y\#}\widetilde{M}(A)[1] & \longrightarrow & Lp_{Y\#}M(X_\bullet \xrightarrow{\pi} Y_\bullet)[1] \\
\downarrow{\scriptstyle M(p_X)} & & \downarrow{\scriptstyle M(p_Y)} & & \downarrow{\scriptstyle M(p)} & & \downarrow{\scriptstyle M(p_X)[1]} \\
M(X'_\bullet \xrightarrow{\pi'} Y'_\bullet) & \longrightarrow & T & \longrightarrow & Cone(\pi') \cong \widetilde{M}(A)[1] & \longrightarrow & M(X'_\bullet \xrightarrow{\pi'} Y'_\bullet)[1]
\end{array}$$

where the isomorphisms in the third column follow by Proposition 2.2.5. If we restrict our previous diagrams to the 0th simplicial component we obtain in $Spc_*(Y'_0)$

$$\begin{array}{ccccccc}
Y_0 \times A & \xrightarrow{\pi_0} & Y_0 & \longrightarrow & Cone(\pi_0) & \longrightarrow & S^1_s \wedge (Y_0 \times A) \\
\downarrow{\scriptstyle p_{Y_0}\times id} & & \downarrow{\scriptstyle p_{Y_0}} & & \downarrow{\scriptstyle p_0} & & \downarrow{\scriptstyle id \wedge (p_{Y_0}\times id)} \\
Y'_0 \times A & \xrightarrow{\pi'_0} & Y'_0 & \longrightarrow & Cone(\pi'_0) & \longrightarrow & S^1_s \wedge (Y'_0 \times A)
\end{array}$$

and in $\mathcal{DM}^{-}_{eff}(Y'_0, R)$

$$\begin{array}{ccccccc}
Lp_{Y_0\#}M(A) & \longrightarrow & Lp_{Y_0\#}T & \longrightarrow & Lp_{Y_0\#}\widetilde{M}(A)[1] & \longrightarrow & Lp_{Y_0\#}M(A)[1] \\
\downarrow{\scriptstyle M(p_{Y_0})\otimes id_{M(A)}} & & \downarrow{\scriptstyle M(p_{Y_0})} & & \downarrow{\scriptstyle M(p_0)} & & \downarrow{\scriptstyle M(p_{Y_0})\otimes id_{M(A)}[1]} \\
M(A) & \longrightarrow & T & \longrightarrow & \widetilde{M}(A)[1] & \longrightarrow & M(A)[1]
\end{array}$$

Note that

$$Hom_{\mathcal{DM}^{-}_{eff}(Y'_0,R)}(Lp_{Y_0\#}\widetilde{M}(A)[1], T) \cong Hom_{\mathcal{DM}^{-}_{eff}(Y_0,R)}(\widetilde{M}(A)[1], p^*_{Y_0}T) \cong$$

$$Hom_{\mathcal{DM}^{-}_{eff}(Y_0,R)}(\widetilde{M}(A)[1], T) \cong Hom_{\mathcal{DM}^{-}_{eff}(k,R)}(\widetilde{M}(Y_0 \times A)[1], T) \cong 0$$

since $Y_0 \times A$ is a smooth scheme over k, so has no cohomology in bidegree $(0)[-1]$ by Proposition 2.4.8, and $H^{0,0}(Spec(k), R) \to H^{0,0}(Y_0 \times A, R)$ is injective. From this we deduce that $M(p_0)$ must be $M(p_{Y_0}) \otimes id_{\widetilde{M}(A)[1]}$, as $M(p_{Y_0}) \otimes id_{\widetilde{M}(A)[1]}$ is the only possible map extending the commutative square on the left.

At this point we notice that both $M(p)$ and $M(p_Y) \otimes id_{\widetilde{M}(A)[1]}$ belong to

$$Hom_{\mathcal{DM}^{-}_{eff}(Y'_\bullet,R)}(Lp_{Y\#}\widetilde{M}(A)[1], \widetilde{M}(A)[1]) \cong$$

$$Hom_{\mathcal{DM}^{-}_{eff}(Y_\bullet,R)}(\widetilde{M}(A), p^*_Y\widetilde{M}(A)) \cong$$

$$Hom_{\mathcal{DM}^{-}_{eff}(Y_\bullet,R)}(\widetilde{M}(A), \widetilde{M}(A)) \cong H^{0,0}(Y_\bullet, R)$$

since $\widetilde{M}(A)$ is an invertible motive. Similarly $M(p_0) = M(p_{Y_0}) \otimes id_{\widetilde{M}(A)[1]}$ belongs to

$$Hom_{\mathcal{DM}^-_{eff}(Y'_0,R)}(Lp_{Y_0\#}\widetilde{M}(A)[1], \widetilde{M}(A)[1]) \cong$$

$$Hom_{\mathcal{DM}^-_{eff}(Y_0,R)}(\widetilde{M}(A), p^*_{Y_0}\widetilde{M}(A)) \cong$$

$$Hom_{\mathcal{DM}^-_{eff}(Y_0,R)}(\widetilde{M}(A), \widetilde{M}(A)) \cong H^{0,0}(Y_0, R)$$

Now, recall that $H^{0,0}(Y_\bullet, R)$ is the free R-module with rank equal to the number of connected components of $Y_\bullet$ and $H^{0,0}(Y_0, R)$ is the free R-module with rank equal to the number of connected components of Y_0. The set of connected components of $Y_\bullet$ is obtained from the set of connected components of Y_0 by identifying all the couples of components of Y_0 linked by a connected component of Y_1 via the face maps. In other words, $H^{0,0}(Y_\bullet, R)$ is just the kernel of the morphism $H^{0,0}(Y_0, R) \to H^{0,0}(Y_1, R)$ induced by the simplicial data. It follows that the restriction

$$r_0^* : H^{0,0}(Y_\bullet, R) \to H^{0,0}(Y_0, R)$$

is injective, hence $M(p) = M(p_Y) \otimes id_{\widetilde{M}(A)[1]}$, which is what we aimed to prove. □

In particular, from the previous proposition it immediately follows the next corollary about functoriality of Thom classes.

Corollary 2.2.8. *Under the hypothesis of Proposition 2.2.7 with $\widetilde{M}(A)$ being a Tate motive, the homomorphism of $H(Y'_\bullet, R)$-modules*

$$p^* : H^{*,*'}(Cone(\pi'), R) \to H^{*,*'}(Cone(\pi), R)$$

sends α' to α, where α' and α are the respective Thom classes.

2.3 TORSORS AND CLASSIFYING SPACES

We introduce in this section the objects of investigation of this thesis which is essentially devoted to the analysis of torsors from a motivic homotopic point of view.

Let $X_\bullet$ be a motivic space (with Nisnevich or étale topology) over k and G a linear algebraic group over k. A right action of G on $X_\bullet$ is a morphism $a : X_\bullet \times G \to X_\bullet$. The action is free if the morphism $X_\bullet \times G \to X_\bullet \times X_\bullet$ defined by $(x, g) \mapsto (a(x, g), x)$ is a monomorphism. For any right G-action on $X_\bullet$ one can define the quotient $X_\bullet/G$ as the coequilizer of the projection $X_\bullet \times G \to X_\bullet$ and the action.

Definition 2.3.1. *A G-torsor over a motivic space $Y_\bullet$ is a morphism $X_\bullet \to Y_\bullet$ endowed with a free right action of G on $X_\bullet$ over $Y_\bullet$ such that $X_\bullet/G \to Y_\bullet$ is an isomorphism.*

We will denote by $P(Y_\bullet, G)_{Nis\, or\, ét}$ the set of isomorphism classes of G-torsors over $Y_\bullet$ in the Nisnevich or étale topology.

In this thesis we will focus in particular on Nisnevich classifying spaces of linear algebraic groups over $Spec(k)$. In fact, their motivic cohomology is used to produce invariants of torsors which should supply an efficient tool for their classification. In this section we recall some of their properties and relations with étale classifying spaces.

Given a linear algebraic group G over k, let us denote by EG the simplicial scheme defined on simplicial components by $(EG)_n = G^{n+1}$ with partial projections and partial diagonals as face and degeneracy maps respectively. In few words, EG is the Čech simplicial scheme of the algebraic group G. The operation in G induces a natural action on EG.

Definition 2.3.2. *The Nisnevich classifying space BG is obtained by taking the quotient of EG respect to the natural right G-action, i.e. $BG = EG/G$.*

Moreover, from the morphism of sites $\pi : (Sm/k)_{ét} \to (Sm/k)_{Nis}$ we obtain the following adjunction

$$\begin{array}{c} \mathcal{H}_s((Sm/k)_{ét}) \\ \pi^* \uparrow \downarrow R\pi_* \\ \mathcal{H}_s((Sm/k)_{Nis}) \end{array}$$

where π_* is the restriction to Nisnevich topology and π^* is étale sheafification.

Definition 2.3.3. *The étale classifying space of G is defined by $B_{ét}G = R\pi_*\pi^*BG$.*

Although this definition presents étale classifying spaces as objects of $\mathcal{H}_s((Sm/k)_{Nis})$, there exists a geometric construction for their A^1-homotopy type (see [36]) obtained from a faithful representation $\rho : G \hookrightarrow GL(V)$ by taking the quotient respect to the diagonal action of G on an open subscheme of an infinite-dimensional affine space $\oplus_{i=1}^{\infty} V$ where G acts freely.

The Nisnevich and étale classifying spaces are generally different from each other. The following result gives a sufficient and necessary condition for them to become equivalent.

Lemma 2.3.4. *The canonical morphism $BG \to B_{ét}G$ is an isomorphism in $\mathcal{H}_s(k)$ if and only if G is a sheaf in the étale topology and for any smooth scheme U over k one has $H^1_{Nis}(U, G) = H^1_{ét}(U, G)$.*

Proof. See [36, Lemma 1.18]. □

Now, let H be an algebraic subgroup of G. Then, we can define two simplicial objects related to BH.

Definition 2.3.5. *By $\widetilde{B}H$ we denote the bisimplicial scheme $(EH \times EG)/H$ and by $\widehat{B}H$ the simplicial scheme EG/H, where the quotients are understood as étale quotients.*

We highlight that the obvious morphism of simplicial schemes $\pi : \widehat{B}H \to BG$ is trivial over each simplicial component with G/H-fibers. At this point, let us call by $\phi : \widetilde{B}H \to BH$ and $\psi : \widetilde{B}H \to \widehat{B}H$ the two natural projections. Notice that ϕ is always trivial over each simplicial component with contractible fiber EG, therefore an isomorphism in $\mathcal{H}_s(k)$. The behaviour of ψ is somewhat different. Indeed, we need to impose a precise condition in order to make it an isomorphism.

Proposition 2.3.6. *If the map*

$$Hom_{\mathcal{H}_s(k)}(Spec(R), B_{\acute{e}t}H) \to Hom_{\mathcal{H}_s(k)}(Spec(R), B_{\acute{e}t}G)$$

is injective for any henselian local ring R over k, then ψ is an isomorphism in $\mathcal{H}_s(k)$. In particular, $BH \cong \widehat{B}H$ in $\mathcal{H}_s(k)$.

Proof. We start by noticing that the restriction of ψ over any simplicial component is given by the morphism $(EH \times G^{n+1})/H \to G^{n+1}/H$. The simplicial scheme $(EH \times G^{n+1})/H$ is nothing else but the Čech simplicial scheme $\check{C}(G^{n+1} \to G^{n+1}/H)$ associated to the H-torsor $G^{n+1} \to G^{n+1}/H$ which becomes split once extended to G. In order to check that

$$\check{C}(G^{n+1} \to G^{n+1}/H) \to G^{n+1}/H$$

is a simplicial weak equivalence it is enough, by Lemma 1.2.4, to evaluate on henselian local rings. Therefore, we need to look at the morphism of simplicial sets

$$\check{C}(G^{n+1}(R) \to G^{n+1}/H(R)) \to G^{n+1}/H(R)$$

for any henselian local ring R over k. Now, the fiber of $G^{n+1} \to G^{n+1}/H$ over any point $Spec(R)$ of G^{n+1}/H is given by a H-torsor $P \to Spec(R)$ whose extension to G is split, so split itself by hypothesis. In other words, this fiber is nothing else but the split H-torsor $H \times Spec(R) \to Spec(R)$. In this way we have found a splitting of $G^{n+1}(R) \to G^{n+1}/H(R)$ which proves that $\check{C}(G^{n+1}(R) \to G^{n+1}/H(R)) \to G^{n+1}/H(R)$ is a weak equivalence of simplicial sets, for any henselian local ring R. This implies that ψ is a weak equivalence over any simplicial component, hence an isomorphism in $\mathcal{H}_s(k)$. □

In practice, due to results from [12], [38], [39] and [42], in most cases we can check if ψ is a simplicial weak equivalence just by looking at field extensions of k. In fact, we have the following theorems which we are going to use later.

Theorem 2.3.7. *Let R be a local ring of a smooth variety over a field k of characterisitic different from 2 and K the field of fractions of R. Let q be a quadratic space over R. If q_K is hyperbolic, then q is hyperbolic.*

Proof. See [39, Theorem 5.1]. □

Theorem 2.3.8. *Let R be a local ring containing a field k of characterisitic different from 2 and K the field of fractions of R. Let (A, σ) be an Azumaya algebra with involution over R and h a hermitian space over (A, σ). If h_K is hyperbolic, then h is hyperbolic.*

Proof. See [39, Theorem 9.2]. □

More generally, one has the following result.

Theorem 2.3.9. *Let R be a regular semi-local domain containing a field, and let K be its field of fractions. Let G be a reductive group scheme over R. Then the map*

$$H^1_{ét}(R, G) \to H^1_{ét}(K, G)$$

induced by the inclusion of R into K has a trivial kernel.

Proof. See [12, Theorem 1] and [42, Theorem 1.1]. □

In a nutshell, rationally trivial torsors of the othogonal group, the unitary group and, in general, any other reductive group are locally trivial.

The natural embedding of algebraic groups $H \hookrightarrow G$ induces two morphisms $j : BH \to \widehat{B}H$ and $g : BH \to BG$. The following result tells us that, under the hypothesis of the previous proposition, j identifies BH and $\widehat{B}H$ in $\mathcal{H}_s(k)$.

Proposition 2.3.10. *Under the hypothesis of Proposition 2.3.6, j is an isomorphism in $\mathcal{H}_s(k)$.*

Proof. We already know that in this case the morphisms of bisimplicial schemes ϕ and ψ become weak equivalences once restricted to simplicial components. It follows that the morphisms they induce on the respective diagonal simplicial objects, namely $\phi : \Delta(\widetilde{B}H) \to BH$ and $\psi : \Delta(\widetilde{B}H) \to \widehat{B}H$, are weak equivalences. So, in order to get the result, it is enough to provide a simplicial homotopy $F_i^{(n)} : (H^{n+1} \times G^{n+1})/H \to G^{n+2}/H$ between $j\phi$ and ψ. One is given by

$$F_i^{(n)}(h_0, \dots, h_n, g_0, \dots, g_n) = (h_0, \dots, h_i, g_i, \dots, g_n)$$

for any n and any $0 \leq i \leq n$. □

Moreover, from the fact that $g = \pi j$, we obtain that $j^* : H(\widehat{B}H) \to H(BH)$ is an isomorphism of $H(BG)$-modules.

We want to point out at this stage that the main reason why we would like to work with $\pi : \widehat{B}H \to BG$ instead of $g : BH \to BG$ is that π is a smooth coherent morphism which is trivial over the 0th simplicial component with fiber G/H. This is particularly desirable since, provided the reduced G/H is motivically invertible, we are entitled to use Proposition 2.2.5 in order to get information on $H(BG)$ out of $H(BH)$. This is essentially how we will be able to reconstruct in some cases the cohomology of the Nisnevich classifying space of an algebraic group inductively by looking at some natural filtration of it.

The following crucial result assures us that actually classifying spaces classify torsors in the respective topology.

Proposition 2.3.11. *There exist bijections*

$$Hom_{\mathcal{H}_s(k)}(Y_\bullet, BG) = P(Y_\bullet, G)_{Nis}$$

and

$$Hom_{\mathcal{H}_s(k)}(Y_\bullet, B_{\acute{e}t}G) = P(Y_\bullet, G)_{\acute{e}t}$$

Proof. See [36, Proposition 1.15]. □

We note that the torsors which are often worthy to be studied are the ones in the étale topology. As an example, étale locally trivial torsors of the orthogonal group over the point are all quadratic forms while there is only one Nisnevich locally trivial torsor. For this reason, for the rest of this thesis, by torsor we will automatically mean torsor in the étale topology.

Although from the previous proposition it is clear that torsors are classified by étale classifying spaces anyway it is convenient in certain situations to investigate Nisnevich classifying spaces. The reason is clarified by the following proposition.

Proposition 2.3.12. *There exists a commutative square in $\mathcal{H}_s(k)$*

$$\begin{array}{ccc} \check{C}(X) & \longrightarrow & BG \\ \downarrow & & \downarrow \\ Spec(k) & \xrightarrow{X} & B_{\acute{e}t}G \end{array}$$

for any torsor X over the point.

Proof. See [47, 2.5.3]. □

If we apply the motivic cohomology functor to the previous square we obtain

$$\begin{array}{ccc} H(\check{C}(X)) & \longleftarrow & H(BG) \\ \uparrow & & \uparrow \\ H & \longleftarrow & H(B_{\acute{e}t}G) \end{array}$$

Then, it is clear that characteristic classes coming from the étale classifying spaces take value in a much simpler object, namely H, than the one where the characteristic classes coming from the Nisnevich classifying spaces land, namely $H(\check{C}(X))$, which is infinite-dimensional. As a result, we deduce that these invariants coming from $H(BG)$, also called *subtle characteristic classes*, may carry more information in principle about G-torsors than the corresponding étale ones. Moreover, since the cohomology of the point is trivial over the 0th diagonal we have that all subtle characteristic classes coming from the étale classifying space in the above 0th diagonal part trivialise in $H(\check{C}(X))$. Unfortunately, as a drawback, we have that subtle charateristic classes of different torsors take values in principle in different places making it more difficult to compare them. On the other hand, following [47], one can define the kernel invariant of a torsor which can be used to make helpful comparisons.

Definition 2.3.13. *For any G-torsor X over $Spec(k)$, the kernel invariant of X is defined as*

$$Ker(X) = Ker(H(BG) \to H(\check{C}(X)))$$

where $H(\check{C}(X)) \to H(BG)$ is the morphism induced by $\check{C}(X) \to BG$ from Proposition 2.3.12.

In this thesis, from the computation of the cohomology of certain classifying spaces, we will be able to deduce some information on the kernel invariant for some classes of quadratic forms.

We finish this section by recalling an important result which will be useful in the next chapters. First, we need to define torsor triples which are triples (G, X, G') where G and G' are linear algebraic groups over a field k and X is both a left G-torsor and a right G'-torsor. Then, we have the following proposition.

Proposition 2.3.14. *For any torsor triple (G, X, G') there is a natural isomorphism in $\mathcal{H}_s(k)$*

$$\begin{array}{ccccc} \check{C}(X) \times BG' & & \xleftrightarrow{\cong} & & BG \times \check{C}(X) \\ & \searrow & & \swarrow & \\ & & \check{C}(X) & & \end{array}$$

Proof. See [47, Proposition 2.6.1]. □

Moreover, we notice that, by looking at the proof of the previous proposition, it is clear that the claimed isomorphism is functorial, i.e. for any morphism of torsor triples $(H, Z, H') \to (G, X, G')$ there exists a commutative diagram in $\mathcal{H}_s(k)$

$$\begin{array}{ccc} \check{C}(Z) \times BH' & \overset{\cong}{\longleftrightarrow} & BH \times \check{C}(Z) \\ \downarrow & & \downarrow \\ \check{C}(X) \times BG' & \overset{\cong}{\longleftrightarrow} & BG \times \check{C}(X) \end{array}$$

2.4 SUBTLE STIEFEL-WHITNEY CLASSES

We now move our attention to one particular case which is of main interest for the purposes of this thesis, i.e. the case of *orthogonal groups*. This case has been deeply studied by Smirnov and Vishik in [47]. In this section we will report their main results, which will be fundamental throughout the rest of this work. In particular, we will recall what subtle Stiefel-Whitney classes are and summarise some of their main applications.

First, note that, since O_n-torsors correspond to quadratic forms, for which Witt cancellation theorem holds over any field, O_{n-1}-torsors inject in O_n-torsors via the map $q \mapsto q \perp \langle (-1)^{n-1} \rangle$. Then, due to Theorem 2.3.7, it is possible to apply Propositions 2.3.6 and 2.3.10 to the case that G and H are respectively O_n and O_{n-1}. Moreover, we recall that

$$A_{q_n} \cong O_n / O_{n-1}$$

where A_{q_n} is the affine quadric defined by the equation $q_n = 1$, where q_n is the standard split quadratic form $\perp_{i=1}^{n} \langle (-1)^{i-1} \rangle$. The motive of this split affine quadric is provided by the following proposition.

Proposition 2.4.1. *In $\mathcal{DM}^{-}_{eff}(k)$ we have that*

$$M(A_{q_n}) = T \oplus T([n/2])[n-1]$$

Proof. See [47, Proposition 3.1.3]. □

Hence, we can apply Proposition 2.2.5 to the morphism $\widehat{B}O_{n-1} \to BO_n$, which we have already noticed to be coherent and trivial over simplicial components with fiber O_n/O_{n-1}.

Indeed, by exploiting the results above mentioned and by an induction argument starting from the fact that $O_1 \cong \mu_2$, the following description of the motivic cohomology rings of the Nisnevich classifying spaces of orthogonal groups can be obtained.

Theorem 2.4.2. *Let k be a field of characteristic different from 2. Then, there is a unique set $u_1, \dots, u_n$ of classes in the motivic $\mathbb{Z}/2$-cohomology of BO_n such that $deg(u_i) = ([i/2])[i]$, u_i vanishes when restricted to $H(BO_{i-1})$ for any $2 \leq i \leq n$ and*

$$H(BO_n) = H[u_1, \dots, u_n]$$

Proof. See [47, Theorem 3.1.1]. □

Moreover, we have the following generalisation of the previous theorem.

Proposition 2.4.3. *For any simplicial scheme $X_\bullet$*

$$H(X_\bullet \times BO_n) = H(X_\bullet)[u_1, \dots, u_n]$$

Proof. See [47, Proposition 3.2.4]. □

Definition 2.4.4. *The generators u_i are called subtle Stiefel-Whitney classes. For any quadratic form q, their images in $H(\check{C}(X_q))$ will be denoted by $u_i(q)$, where X_q is the torsor associated to q, i.e. $X_q = Iso(q \leftrightarrow q_n)$.*

As shown in [47], these classes provide very interesting and informative invariants for quadratic forms. In particular, as we have already noticed in the previous section, they see more than their étale counterparts since they take value in the motivic cohomology of a Čech simplicial scheme which is highly non-trivial respect to the cohomology of the point. Most importantly, the étale characteristic classes are expressible through subtle classes. Furthermore, it is possible to describe the motive of X_q in terms of subtle Stiefel-Whitney classes. More precisely, we have the following two results.

Proposition 2.4.5. *In $\mathcal{DM}^-_{eff}(BO_n, \mathbb{Z}/2)$ we have that*

$$M(EO_n \to BO_n) = \bigotimes_{1 \leq i \leq n} Cone[-1](T \xrightarrow{u_i} T([i/2])[i])$$

Proof. See [47, Proposition 3.1.11]. □

Theorem 2.4.6. *For any n-dimensional quadratic form q we have that*

$$M(X_q) = \bigotimes_{1 \leq i \leq n} Cone[-1](\mathfrak{X}_q \xrightarrow{u_i(q)} \mathfrak{X}_q([i/2])[i])$$

in $\mathcal{DM}^-_{eff}(k, \mathbb{Z}/2)$, where $\mathfrak{X}_q$ is the motive of $\check{C}(X_q)$.

Proof. See [47, Theorem 3.2.2]. □

Moreover, subtle Stiefel-Whitney classes are able to see the power of the fundamental ideal I of the Witt ring $W(k)$ where a quadratic form belongs. Indeed, we have the following theorem.

Theorem 2.4.7. *For any quadratic form q, we have that $q \in I^n$ if and only if $u_{2^r}(q) = 0$ for any $0 \leq r \leq n-2$.*

Proof. See [47, Theorem 3.2.27]. □

From the previous theorem, one gets the following result which states essentially that subtle Stiefel-Whitney classes see the triviality of quadratic forms. More precisely, the following corollary holds.

Corollary 2.4.8. *$q \cong q_n$ if and only if $u_{2^r}(q) = 0$ for any r.*

Proof. See [47, Corollary 3.2.32]. □

By the very same arguments exploited for the orthogonal groups it is possible to get the same description for $H(BSO_n)$ with the only difference given by the fact that $u_1 = 0$. More precisely, from the short exact sequence of algebraic groups

$$1 \to SO_n \to O_n \to \mu_2 \to 1$$

we get that

$$A_{q_n} \cong SO_n / SO_{n-1}$$

and by recalling that SO_n-torsors correspond to n-dimensional quadratic forms with trivial discriminant and SO_1 is the point we obtain the following theorem.

Theorem 2.4.9. *Let k be a field of characteristic different from 2. Then, the motivic cohomology ring of BSO_n is described by*

$$H(BSO_n) = H[u_2, \ldots, u_n]$$

In the following chapters, we will also use the action of the motivic Steenrod algebra on subtle classes which is given by the following Wu formula as in the classical case at least when $\rho = 0$.

Proposition 2.4.10. *(Wu formula) Suppose $\rho = 0$. Then, the following formula holds*

$$Sq^k u_m = \begin{cases} \sum_{j=0}^{k} \binom{m+j-k-1}{j} u_{k-j} u_{m+j}, & 0 \leq k < m \\ u_m^2, & k = m \\ 0, & k > m \end{cases}$$

Proof. See [47, Proposition 3.1.12]. □

From the previous result we immediately deduce the following corollary which will be helpful in the next chapters.

Corollary 2.4.11. *Suppose $\rho = 0$ and let w be a monomial of bidegree $([\frac{m}{2}])[m]$ in the polynomial ring $\mathbb{Z}/2[\tau, u_2, \ldots, u_{n+1}]$. Then, $Sq^m w = w^2$ and $Sq^j w = 0$ for any $j > m$.*

Proof. If m is even, by Lemmas 1.6.6 and 1.6.7 and by noticing that $Sq^{2h+1} = Sq^1Sq^{2h}$, there is nothing to prove since w is on the slope 2 diagonal. Consider m odd, then $w = \tau^{\frac{r-1}{2}} x u_{h_1} \cdots u_{h_r}$ where x is a monomial in even subtle classes and u_{h_i} are odd subtle classes (notice that r must be odd by degree reason). Therefore, by Cartan formula we have that

$$Sq^m w = Sq^m(\tau^{\frac{r-1}{2}} x u_{h_1} \cdots u_{h_r}) = Sq^{m-h_r}(\tau^{\frac{r-1}{2}} x u_{h_1} \cdots u_{h_{r-1}}) Sq^{h_r} u_{h_r} =$$

$$(\tau^{\frac{r-1}{2}} x u_{h_1} \cdots u_{h_{r-1}})^2 u_{h_r}^2 = w^2$$

since the monomial $\tau^{\frac{r-1}{2}} x u_{h_1} \cdots u_{h_{r-1}}$ is on the slope 2 diagonal in bidegree $(\frac{m-h_r}{2})[m-h_r]$, so $Sq^j(\tau^{\frac{r-1}{2}} x u_{h_1} \cdots u_{h_{r-1}}) = 0$ for $j > m - h_r$, and $Sq^k u_{h_r} = 0$ for $k > h_r$ by Wu formula. Moreover, $Sq^j w = 0$ for $j > m$ for the same reason. □

3

SUBTLE CHARACTERISTIC CLASSES FOR SPIN-TORSORS

In this chapter we will compute a large part of the motivic cohomology ring of the Nisnevich classifying space of $Spin_n$, the spin group of the standard split quadratic form q_n. This will provide us with *subtle characteristic classes* for $Spin$-torsors or, which is the same, *quadratic forms from* I^3, where I is the fundamental ideal of the Witt ring.

The topological counterpart of this computation has been obtained by Quillen in [43]. In the first section of this chapter we will recall Quillen's main result which relies on two key tools: 1) the regularity of a certain sequence in a polynomial ring; 2) the Serre spectral sequence associated to a fibration. With reference to the first one, unfortunately we do not have a proof of the regularity of the corresponding motivic sequence. Anyway, using the regularity of Quillen's sequence in topology we will still be able to obtain a large part of the motivic cohomology of $BSpin_n$. Regarding the second point, we may have a sort of Serre spectral sequence available for our case (see Section 5 for more details), but, as we will show, techniques reported in previous chapters, as Gysin sequences, will be sufficient and, in fact, more suitable to deal with this case.

All in all, we will describe completely the motivic cohomology of $BSpin_n$ in bidegrees $(*')[*]$ satisfying the condition $* \leq 2*'+1$ and reduce the complete computation to the question whether the motivic sequence is regular or not. Moreover, in the last sections we will get very nice and simple relations among subtle Stiefel-Whitney classes of *quadratic forms with trivial discriminant and Clifford invariant*. These relations will give information on their kernel invariant and highlight the deep connection with the J-invariant.

On another side, we will apply our main result to compute the motivic cohomology ring of BG_2, the Nisnevich classifying space of the split exceptional algebraic group G_2, providing *subtle invariants* for *octonion algebras*.

3.1 THE COHOMOLOGY OF $BSpin_n$ IN TOPOLOGY

In this section we will present Quillen's main results on the computation of the singular cohomology of the classifying space of the spin group associated to the real euclidean quadratic form. These results will lead us in the aim of finding a similar description in the motivic world. More precisely, in [43] the following two theorems are proved.

Theorem 3.1.1. *The sequence*

$$w_2, Sq^1 w_2, \dots, Sq^{2^{k-2}} Sq^{2^{k-3}} \cdots Sq^4 Sq^2 Sq^1 w_2$$

is a regular sequence in $H^*(BSO_n) = \mathbb{Z}/2[w_2, \dots, w_n]$, where k depends on n as in the following table

n	*k*
8l+1	*4l*
8l+2	*4l+1*
8l+3	*4l+2*
8l+4	*4l+2*
8l+5	*4l+3*
8l+6	*4l+3*
8l+7	*4l+3*
8l+8	*4l+3*

and w_i is the i-th Stiefel-Whitney class.

Proof. See [43, Theorem 6.3]. □

Moreover, we recall that the values written in the previous table are related to the dimension of spin representations of $Spin_n$.

Theorem 3.1.2. *Let I_k be the ideal in $H^*(BSO_n)$ generated by the previous regular sequence and Δ be a spin representation of $Spin_n$. Then the canonical homomorphism*

$$H^*(BSO_n)/I_k \otimes \mathbb{Z}/2[w_{2^k}(\Delta)] \to H^*(BSpin_n)$$

is an isomorphism.

Proof. See [43, Theorem 6.5]. □

From Theorem 3.1.1 and Theorem 3.1.2 it follows that

$$k(n+1) = \begin{cases} k(n), & Sq^{2^{k(n)-1}} \cdots Sq^1 w_2 \in I_{k(n)} \\ k(n)+1, & Sq^{2^{k(n)-1}} \cdots Sq^1 w_2 \notin I_{k(n)} \end{cases}$$

where here by $I_{k(n)}$ we mean the ideal in $H(BSO_{n+1}) = \mathbb{Z}/2[w_2, \dots, w_{n+1}]$ generated by the elements $w_2, Sq^1 w_2, \dots, Sq^{2^{k(n)-2}} \cdots Sq^1 w_2$.

Furthermore, we notice that Theorem 3.1.2 relies on the Serre spectral sequence for the fibration $B\mathbb{Z}/2 \to BSpin_n \to BSO_n$. We will use instead techniques developed in the previous chapters.

3.2 THE FIBRATION $BSpin_n \to BSO_n$

We have already noticed that the special orthogonal case does not differ much from the orthogonal one, at least from the cohomological perspective, in the sense that their motivic cohomology rings are both polynomial over the cohomology of the point in subtle Stiefel-Whitney classes. This is not true anymore for spin groups. The main reason is that in this case there are much more complicated relations among subtle classes given by the action of the motivic Steenrod algebra on u_2 which make the cohomology rings not polynomial in subtle Stiefel-Whitney classes anymore (precisely for $n > 9$) and, moreover, new classes appear. For this reason, in order to get our main result, together with an inductive argument we will need to consider the fibration $BSpin_n \to BSO_n$. More precisely, in order to investigate the motivic cohomology of $BSpin_n$, we will need to consider for any $n \geq 2$ the cartesian square

$$\begin{array}{ccc} \widehat{B}Spin_n & \xrightarrow{\widehat{a}_n} & \widehat{B}SO_n \\ {\scriptstyle \widetilde{\pi}}\downarrow & & \downarrow{\scriptstyle \pi} \\ BSpin_{n+1} & \xrightarrow{a_{n+1}} & BSO_{n+1} \end{array}$$

where π and $\widetilde{\pi}$ are smooth coherent morphisms, trivial over simplicial components, with fiber isomorphic to the affine quadric $A_{q_{n+1}}$ defined by the equation $q_{n+1} = 1$.

In $Spc_*(BSO_{n+1})$ we can complete the previous diagram to the following one, which is commutative up to a sign in the right bottom square, where each row and each column is a cofiber sequence

$$\begin{array}{ccccccc} \widehat{B}Spin_n & \xrightarrow{\widehat{a}_n} & \widehat{B}SO_n & \xrightarrow{\widehat{b}_n} & Cone(\widehat{a}_n) & \xrightarrow{\widehat{c}_n} & S^1_s \wedge \widehat{B}Spin_n \\ {\scriptstyle \widetilde{\pi}}\downarrow & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \overline{\pi}} & & \downarrow \\ BSpin_{n+1} & \xrightarrow{a_{n+1}} & BSO_{n+1} & \xrightarrow{b_{n+1}} & Cone(a_{n+1}) & \xrightarrow{c_{n+1}} & S^1_s \wedge BSpin_{n+1} \\ {\scriptstyle \widetilde{f}}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \overline{f}} & & \downarrow \\ Cone(\widetilde{\pi}) & \longrightarrow & Cone(\pi) & \longrightarrow & Cone(\overline{\pi}) & \longrightarrow & S^1_s \wedge Cone(\widetilde{\pi}) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ S^1_s \wedge \widehat{B}Spin_n & \longrightarrow & S^1_s \wedge \widehat{B}SO_n & \longrightarrow & S^1_s \wedge Cone(\widehat{a}_n) & \longrightarrow & S^2_s \wedge \widehat{B}Spin_n \end{array}$$

The previous induces, in turn, a commutative diagram of long exact sequences in motivic cohomology with $\mathbb{Z}/2$-coefficients, where all the homomorphisms are compatible with Steenrod operations and respect the

$H(BSO_{n+1})$-module structure. This remark comes from the fact that the following diagram of categories

$$\begin{array}{ccc} Spc_*(BSO_{n+1}) & \longrightarrow & \mathcal{H}_{A^1,*}(k) \\ \downarrow & & \downarrow \\ \mathcal{DM}^-_{eff}(BSO_{n+1}, \mathbb{Z}/2) & \longrightarrow & \mathcal{DM}^-_{eff}(k, \mathbb{Z}/2) \end{array}$$

is commutative up to a natural equivalence and both functors in the right bottom corner have adjoints from the right, so we have the action of Steenrod operations on the motivic cohomology of objects belonging to the image of $Spc_*(BSO_{n+1})$ in $\mathcal{DM}^-_{eff}(BSO_{n+1}, \mathbb{Z}/2)$ pulled from $\mathcal{H}_{A^1,*}(k)$.

All in all, using Propositions 2.3.6 and 2.3.10 and Theorem 2.3.7, which one is allowed to use since $Spin$-torsors are quadratic forms from I^3 and for quadratic forms we have Witt cancellation, we have the following infinite grid of long exact sequences (#)

$$\begin{array}{ccccccccccc}
 & & \vdots & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & H^{p-2,q}(BSpin_n) & \xrightarrow{c_n^*} & H^{p-1,q}(Cone(a_n)) & \xrightarrow{b_n^*} & H^{p-1,q}(BSO_n) & \xrightarrow{a_n^*} & H^{p-1,q}(BSpin_n) & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle \widetilde{h}^*} & & \downarrow{\scriptstyle \overline{h}^*} & & \downarrow{\scriptstyle h^*} & & \downarrow & & \\
\cdots & \longrightarrow & H^{p-1,q}(Cone(\widetilde{\pi})) & \longrightarrow & H^{p,q}(Cone(\overline{\pi})) & \longrightarrow & H^{p,q}(Cone(\pi)) & \longrightarrow & H^{p,q}(Cone(\widetilde{\pi})) & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle \widetilde{f}^*} & & \downarrow{\scriptstyle \overline{f}^*} & & \downarrow{\scriptstyle f^*} & & \downarrow & & \\
\cdots & \longrightarrow & H^{p-1,q}(BSpin_{n+1}) & \xrightarrow{c_{n+1}^*} & H^{p,q}(Cone(a_{n+1})) & \xrightarrow{b_{n+1}^*} & H^{p,q}(BSO_{n+1}) & \xrightarrow{a_{n+1}^*} & H^{p,q}(BSpin_{n+1}) & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle \widetilde{g}^*} & & \downarrow{\scriptstyle \overline{g}^*} & & \downarrow{\scriptstyle g^*} & & \downarrow & & \\
\cdots & \longrightarrow & H^{p-1,q}(BSpin_n) & \longrightarrow & H^{p,q}(Cone(a_n)) & \longrightarrow & H^{p,q}(BSO_n) & \longrightarrow & H^{p,q}(BSpin_n) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & & \vdots & &
\end{array}$$

where all the homomorphisms are compatible with Steenrod operations and respect the $H(BSO_{n+1})$-module structure.

We recall that, by applying Proposition 2.2.5 to the smooth coherent morphism $\pi : \widehat{B}SO_n \to BSO_{n+1}$, which has fiber isomorphic to $A_{q_{n+1}}$ whose reduced motive is Tate, there is a Thom isomorphism

$$H^{p-n-1,q-[\frac{n+1}{2}]}(BSO_{n+1}) \to H^{p,q}(Cone(\pi))$$

which sends 1 to the Thom class α from Definition 2.2.6. By Theorem 2.4.2, modulo this isomorphism f^* is just the multiplication by the subtle Stiefel-Whitney class u_{n+1}, since it is the only class of its bidegree vanishing in

$H(BO_n)$. Since $Spin_{n+1}/Spin_n \cong A_{q_{n+1}}$, Proposition 2.2.5 applies also to the smooth coherent morphism $\widetilde{\pi} : \widehat{B}Spin_n \to BSpin_{n+1}$. Therefore, we have a Thom isomorphism

$$H^{p-n-1,q-[\frac{n+1}{2}]}(BSpin_{n+1}) \to H^{p,q}(Cone(\widetilde{\pi}))$$

and a Thom class $\widetilde{\alpha} \in H^{n+1,[\frac{n+1}{2}]}(Cone(\widetilde{\pi}))$. We notice that, by Corollary 2.2.8, $\widetilde{\alpha}$ is nothing else but the restriction of α from $H^{n+1,[\frac{n+1}{2}]}(Cone(\pi))$ to $H^{n+1,[\frac{n+1}{2}]}(Cone(\widetilde{\pi}))$. Hence, modulo the Thom isomorphism, $\widetilde{f}^*$ is multiplication by u_{n+1}. Moreover, from Propositions 2.2.7 we have that

$$M(Cone(\overline{\pi})) \cong M(Cone(a_{n+1}))([(n+1)/2])[n+1]$$

in $\mathcal{DM}^-_{eff}(BSO_{n+1}, \mathbb{Z}/2)$ which induces an isomorphism

$$H^{p-n-1,q-[\frac{n+1}{2}]}(Cone(a_{n+1})) \to H^{p,q}(Cone(\overline{\pi}))$$

Note that, from Theorem 2.4.2, h^* is always the 0 homomorphism, which means at the same time that g^* is surjective and f^* is injective. From these remarks we obtain the next proposition.

Proposition 3.2.1. *$Sq^m\alpha = u_m\alpha$ for any $m \leq n+1$ and 0 otherwise. The same holds for $\widetilde{\alpha}$.*

Proof. We just notice that $f^*(Sq^m\alpha) = Sq^m f^*(\alpha) = Sq^m u_{n+1} = u_m u_{n+1} = u_m f^*(\alpha) = f^*(u_m\alpha)$. The result follows by injectivity of f^*. □

3.3 THE ACTION OF SOME STEENROD SQUARES ON u_2

Our first aim is to prove a result similar to Theorem 3.1.1. Our proof will basically consist in deducing the motivic case from the topological one. However, this method does not provide a proof of the regularity of Quillen's sequence in the motivic case. We notice that the main result we get in this section will be enough to compute a large part of the motivic cohomology of $BSpin_n$ but not the whole, which, as we will see, would be possible by exactly the same methods of the next section if we knew the regularity of Quillen's sequence. For the rest of this section, the base field k will always be of characteristic different from 2 containing $\sqrt{-1}$.

We start by defining the elements θ_j in $H(BSO_n)$ inductively by the following formulas:

$$\theta_0 = u_2$$
$$\theta_{j+1} = Sq^{2^j}\theta_j$$

Similarly, define the elements ρ_j in $H_{top}(BSO_n) = \mathbb{Z}/2[w_2, \ldots, w_n]$ starting from w_2.

At this point, let us consider three homomorphisms $i : H_{top}(BSO_n) \to H(BSO_n)$, $h : H_{top}(BSO_n) \to H(BSO_n)$ and $t : H(BSO_n) \to H_{top}(BSO_n)$, where i is defined by imposing $i(w_i) = u_i$ and extending to a ring homomorphism, h by imposing, for any monomial x, $h(x) = \tau^{[\frac{p_{i(x)}}{2} - q_{i(x)}]} i(x)$, where $(q_{i(x)})[p_{i(x)}]$ is the bidegree of $i(x)$, and extending linearly and t by imposing $t(u_i) = w_i$, $t(\tau) = 1$ and $t(K_r^M(k)/2) = 0$ for any $r > 0$ and extending to a ring homomorphism.

We start by describing some properties of these homomorphisms. First of all, i and h are graded with respect to the usual grading in $H_{top}(BSO_n)$ and the square grading in $H(BSO_n)$. Besides, by the very definition of h, $h(x)$ has bidegree $([\frac{p_{i(x)}}{2}])[p_{i(x)}]$ for any homogeneous polynomial x. On the other hand, we notice that h is not a ring homomorphism. Anyway, we have the following lemmas.

Lemma 3.3.1. *For any homogeneous polynomials x and y in $H_{top}(BSO_n)$, we have that $h(xy) = \tau^{\epsilon} h(x)h(y)$, where ϵ is 1 if $p_{i(x)}p_{i(y)}$ is odd and 0 otherwise.*

Proof. At first consider two monomials x and y. Then, we get

$$h(xy) = \tau^{[\frac{p_{i(x)}+p_{i(y)}}{2} - q_{i(x)} - q_{i(y)}]} i(xy) =$$

$$\tau^{\epsilon + [\frac{p_{i(x)}}{2} - q_{i(x)}] + [\frac{p_{i(y)}}{2} - q_{i(y)}]} i(x)i(y) = \tau^{\epsilon} h(x)h(y)$$

where ϵ is 1 if $p_{i(x)}p_{i(y)}$ is odd and 0 otherwise. For homogeneous polynomials $x = \sum_{j=0}^{l} x_j$ and $y = \sum_{k=0}^{m} y_k$, where x_j and y_k are monomials, we have

$$h(xy) = h(\sum_{j=0}^{l}\sum_{k=0}^{m} x_j y_k) = \sum_{j=0}^{l}\sum_{k=0}^{m} h(x_j y_k) = \sum_{j=0}^{l}\sum_{k=0}^{m} \tau^{\epsilon_{jk}} h(x_j)h(y_k)$$

where ϵ_{jk} is 1 if $p_{i(x_j)}p_{i(y_k)}$ is odd and 0 otherwise. Now, we recall that $p_{i(x_j)} = p_{i(x)}$ and $p_{i(y_k)} = p_{i(y)}$ for any j and k, from which it immediately follows that $h(xy) = \tau^{\epsilon} \sum_{j=0}^{l}\sum_{k=0}^{m} h(x_j)h(y_k) = \tau^{\epsilon} h(x)h(y)$, where ϵ is 1 if $p_{i(x)}p_{i(y)}$ is odd and 0 otherwise. □

Lemma 3.3.2. *For any homogeneous (respect to bidegree) $z \in \mathbb{Z}/2[\tau, u_2, \dots, u_n]$, we have that $ht(z) = \tau^{[\frac{p_z}{2} - q_z]} z$ (where $[\frac{p_z}{2} - q_z]$ can possibly be negative).*

Proof. Write z as $\sum_{j=0}^{m} z_j$, where z_j are monomials in $\mathbb{Z}/2[\tau, u_2, \dots, u_n]$. Then, $ht(z) = \sum_{j=0}^{m} ht(z_j) = \sum_{j=0}^{m} \tau^{[\frac{p_{it(z_j)}}{2} - q_{it(z_j)}]} it(z_j)$. Notice that $z_j = \tau^{n_j} x_j$, for some monomials x_j in $\mathbb{Z}/2[u_2, \dots, u_n]$. By the very definition of i and t we get that $it(z_j) = x_j$. Thus,

$$ht(z) = \sum_{j=0}^{m} \tau^{[\frac{p_{x_j}}{2} - q_{x_j}]} x_j = \sum_{j=0}^{m} \tau^{[\frac{p_{x_j}}{2} - q_{x_j} - n_j]} z_j = \tau^{[\frac{p_z}{2} - q_z]} z$$

since $p_{x_j} = p_{z_j} = p_z$ and $q_{x_j} + n_j = q_{z_j} = q_z$. □

Lemma 3.3.3. *For any j, $t(\theta_j) = \rho_j$ and $h(\rho_j) = \theta_j$.*

Proof. Since a Wu formula holds even in the motivic case by 2.4.10, we get that $t(\theta_j) = \rho_j$ by the very definition of t. Then, $h(\rho_j) = ht(\theta_j) = \theta_j$ by Lemma 3.3.2 and by recalling that θ_j is in bidegree $(2^{j-1})[2^j + 1]$. □

At this point, denote by I_j the ideal in $H(BSO_n)$ generated by $\theta_0, \ldots, \theta_{j-1}$ and by I_j^{top} the ideal in $H_{top}(BSO_n)$ generated by $\rho_0, \ldots, \rho_{j-1}$. We are now ready to prove the main result of this section.

Proposition 3.3.4. *For any $j \leq k-1$ and any homogeneous (respect to bidegree) $z \in \mathbb{Z}/2[\tau, u_2, \ldots, u_n]$, $z\theta_j \in I_j$ implies $\tau^{[\frac{p_z}{2} - q_z]} z \in I_j$. Moreover, $\theta_k \in I_k$, where k depends on n as in the table of Theorem 3.1.1.*

Proof. From $z\theta_j \in I_j$ we deduce that $t(z)\rho_j \in I_j^{top}$. Therefore, by Theorem 3.1.1 we have that $t(z) = \sum_{l=0}^{j-1} \psi_l \rho_l$ for some homogeneous $\psi_l \in H_{top}(BSO_n)$ and after applying h we obtain $\tau^{[\frac{p_z}{2} - q_z]} z = \sum_{l=0}^{j-1} \tau^{\epsilon_l} h(\psi_l)\theta_l$ by Lemmas 3.3.1, 3.3.2 and 3.3.3. In order to finish the proof we only need to notice that, since $\rho_k \in I_k^{top}$ by Theorem 3.1.2, then $\theta_k = h(\rho_k) \in h(I_k^{top}) \subset I_k$. □

3.4 THE MOTIVIC COHOMOLOGY RING OF $BSpin_n$

In this section we will prove a motivic version of Theorem 3.1.2 which unfortunately does not give the whole description of $H(BSpin_n)$ since, as we have already pointed out in the previous section, we do not have a proof of the regularity of Quillen's sequence in the motivic case. Anyway, even without the regularity of this sequence, just by using Proposition 3.3.4 and the infinite diagram (#), it is possible to get the complete description of the motivic cohomology of $BSpin_n$ over quadratically closed fields in bidegrees $(*')[*]$ satisfying the condition $* \leq 2*' + 1$.

For now, let k be any field of characteristic different from 2. We start by showing that, as in topology, the second subtle Stiefel-Whitney class is trivial in the motivic cohomology ring $H(BSpin_n)$.

Lemma 3.4.1. *For any $n \geq 2$, u_2 is trivial in $H(BSpin_n)$. Moreover, there exists a unique element x_0 in $H(Cone(a_n))$ such that $b_n^*(x_0) = u_2$.*

Proof. Recall that $SO_2 \cong Spin_2 \cong G_m$, where G_m is the multiplicative group, and the morphism from $Spin_2$ to SO_2 is the double cover $G_m \xrightarrow{(\cdot)^2} G_m$. Then, for $n = 2$ the homomorphism

$$a_2^* : H(BG_m) = H[u_2] \to H(BG_m) = H[v_2]$$

sends u_2 to $2v_2$, hence $u_2 = 0$ in $H(BSpin_2)$.

Now, suppose $u_2 = 0$ in $H(BSpin_n)$, then u_2 should be divisible by u_{n+1} in $H(BSpin_{n+1})$, which forces u_2 to be trivial by degree reasons. Therefore, by induction, $u_2 = 0$ in $H(BSpin_n)$ for any n. It immediately follows that there exists x_0 in $H(Cone(a_n))$ such that $b_n^*(x_0) = u_2$ for any $n \geq 2$. We will prove its uniqueness by showing that b_n^* is a monomorphism in bidegree $(1)[2]$. First of all we notice that, for any $n \geq 2$, $H^{1,1}(BSpin_n) = K_1^M(k)/2$ by induction on n and by observing that $\widetilde{g}^*$ is an isomorphism in bidegree $(1)[1]$. Hence, $c_n^* : H^{1,1}(BSpin_n) \to H^{2,1}(Cone(a_n))$ is the zero homomorphism, since the composition $H^{1,1} \to H^{1,1}(BSO_n) \to H^{1,1}(BSpin_n)$ is surjective and, therefore, so is the second map. It follows that $b_n^* : H^{2,1}(Cone(a_n)) \to H^{2,1}(BSO_n)$ is a monomorphism, as we aimed to show. □

From the previous lemma, for any $n \geq 2$, we have a canonical set of elements x_j in $H(Cone(a_n))$ defined by $x_j = Sq^{2^{j-1}} \cdots Sq^1 x_0$ for any $j > 0$. Denote by $< x_0, \ldots, x_{j-1} >$ the $H(BSO_n)$-submodule of $H(Cone(a_n))$ generated by $x_0, \ldots, x_{j-1}$. Before proceeding we need the following lemma.

Lemma 3.4.2. *$x_j \notin < x_0, \ldots, x_{j-1} >$ in $H(Cone(a_2))$, and consequently in any $H(Cone(a_n))$, for any j.*

Proof. We start by considering the Bockstein homomorphism β associated to the short exact sequence $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/2 \to 0$. The homomorphism a_2^* on cohomology with integer coefficients sends u_2 to $2v_2$ where v_2 is the generator of $H(BSpin_2) = H(BG_m)$ and so is injective, hence b_2^* is the 0 homomorphism on cohomology with integer coefficients, from which it follows that x_0 cannot come from any integral cohomology class. Thus, $y = \beta(x_0) \neq 0$. Moreover, since u_2 comes from an integral cohomology class, we have $b_2^*(y) = 0$, so $y = mc_2^*(v_2)$ for some integer m. At this point we notice that mv_2 is covered by a_2^* for any even m, so m must be odd, which implies that y is not divisible by 2, since $v_2 \, mod(2)$ is not covered by a_2^*. This is enough to conclude that

$$x_1 = Sq^1 x_0 = \beta(x_0) \, mod(2) = y \, mod(2) \neq 0$$

Hence, $x_1 = c_2^*(v_2)$ from which we deduce that

$$x_j = Sq^{2^{j-1}} \ldots Sq^2 x_1 = c_2^*(Sq^{2^{j-1}} \ldots Sq^2 v_2) = c_2^*(v_2^{2^{j-1}})$$

Now, suppose that $x_j \in < x_0, \ldots, x_{j-1} >$, in other words $x_j = \sum_{i=0}^{j-1} \phi_i x_i$ for some $\phi_i \in H[u_2]$. Then, we would have that

$$\phi_0 u_2 = b_2^*(x_j + \sum_{i=0}^{j-1} \phi_i x_i) = 0$$

which implies $\phi_0 = 0$. Moreover, since positive powers of u_2 act trivially on $H[v_2]$ (with $\mathbb{Z}/2$-coefficients), we have that

$$c_2^*(v_2^{2^{j-1}}) = c_2^*(v_2^{2^{j-1}} + \sum_{i=1}^{j-1} \phi_i v_2^{2^{i-1}}) = 0$$

that is impossible since c_2^* is injective on the slope 2 line (above zero), which comes from the fact that $H(BSO_2) = H[u_2]$ and $a_2^*(u_2) = 0$. □

At this point, we are ready to prove our main result which provides the complete description of the $* \leq 2*' + 1$ part of the motivic cohomology of $BSpin_n$ over quadratically closed fields.

Theorem 3.4.3. *Let k be a quadratically closed field of characteristic different from 2. Then, for any $n \geq 2$, there exists a cohomology class v_{2^k} of bidegree $(2^{k-1})[2^k]$ such that the natural homomorphism of H-algebras*

$$H(BSO_n)/I_k \otimes_H H[v_{2^k}] \to H(BSpin_n)$$

is an isomorphism in bidegrees $(*')[*]$ for any $* \leq 2*' + 1$ and a monomorphism for $* = 2*' + 2$, where I_k is the ideal generated by $\theta_0, \ldots, \theta_{k-1}$ and k depends on n as in the table of Theorem 3.1.1.

Proof. Note that, since k is quadratically closed, $K^M(k)/2$ is just $\mathbb{Z}/2$. Hence, $H = \mathbb{Z}/2[\tau]$. This remark will be used only in Lemma 3.4.7 in order to apply Proposition 3.3.4.

Our proof will go by induction on n, starting from $n = 2$.

Base case: For $n = 2$, $H(BSpin_2) = H(BG_m) = H[v_2]$ provides our induction base.

Inductive step: We will denote by θ'_j and θ_j the class $Sq^{2^{j-1}} \cdots Sq^1 u_2$ in $H(BSO_n)$ and $H(BSO_{n+1})$ respectively, by I'_k the ideal generated by the elements $u_2, \theta'_1, \ldots, \theta'_{k-1}$, by I_k the ideal generated by $u_2, \theta_1, \ldots, \theta_{k-1}$, by x'_0 and x_0 the unique lifts of u_2 to $H(Cone(a_n))$ and $H(Cone(a_{n+1}))$ respectively, by x'_j the class $Sq^{2^{j-1}} \cdots Sq^1 x'_0$ and by x_j the class $Sq^{2^{j-1}} \cdots Sq^1 x_0$.

Now, suppose by induction hypothesis that we have a homomorphism

$$H(BSO_n)/I'_k \otimes_H H[v_{2^k}] \to H(BSpin_n)$$

which is an isomorphism in the $* \leq 2*' + 1$ part and a monomorphism for $* = 2*' + 2$, where $k = k(n)$ according to the table of Theorem 3.1.1. From now on we will always consider, unless otherwise specified, bidegrees such that $* \leq 2*' + 1$.

Looking at the long exact sequence

$$\cdots \to H^{*-1,*'}(BSpin_n, R) \to H^{*-n-1,*'-[(n+1)/2]}(BSpin_{n+1}, R) \xrightarrow{\cdot u_{n+1}}$$

$$H^{*,*'}(BSpin_{n+1}, R) \to H^{*,*'}(BSpin_n, R) \to \dots$$

from the diagram (#) in Section 3.2 and by induction on degree we know that, in square degrees less than 2^k, in $H(BSpin_{n+1})$ there are only subtle Stiefel-Whitney classes, i.e. the homomorphism $a^*_{n+1} : H(BSO_{n+1}) \to H(BSpin_{n+1})$ is surjective in these degrees. Let w be a cohomology class in $H^{2^k-n, 2^{k-1}-[\frac{n+1}{2}]}(BSO_{n+1})$ such that $a^*_{n+1}(w)\widetilde{\alpha} = \widetilde{h}^*(v_{2^k})$, where $\widetilde{\alpha}$ is the Thom class of the morphism $\widetilde{g}$. We point out that

$$a^*_{n+1}(u_{n+1}w) = u_{n+1}a^*_{n+1}(w) = \widetilde{f}^*\widetilde{h}^*(v_{2^k}) = 0$$

The following result enables to complete the induction step. It is indeed the core proposition that permits to conduct the proof of our main theorem.

Proposition 3.4.4. *Suppose we have a commutative diagram*

$$\begin{array}{ccc} H(BSO_{n+1}) \otimes_H H[v] & \xrightarrow{g^* \otimes l} & H(BSO_n) \otimes_H H[c] \\ {\scriptstyle p_{n+1}} \downarrow & & \downarrow {\scriptstyle p_n} \\ H(BSpin_{n+1}) & \xrightarrow{\widetilde{g}^*} & H(BSpin_n) \end{array}$$

such that v is a lift from $H(BSpin_n)$ to $H(BSpin_{n+1})$ of a monic homogeneous polynomial c in v_{2^k} with coefficients in $H(BSO_n)$, $l(v) = c$ and $\widetilde{h}^*(c) = 0$.

If $Im(\widetilde{h}^*) = Im(p_{n+1}) \cdot \widetilde{h}^*(v_{2^k})$ in the $* \leq 2*' + 2$ part, then $ker(p_{n+1}) = J_k + (u_{n+1}w)$ in the $* \leq 2*' + 2$ part, where J_k is $I_k \otimes_H H[v]$.

If moreover $ker(\widetilde{h}^*) = Im(\widetilde{g}^* p_{n+1})$ in the $* \leq 2*' + 1$ part, we get that the homomorphism

$$H(BSO_{n+1})/(I_k + (u_{n+1}w)) \otimes_H H[v] \to H(BSpin_{n+1})$$

is an isomorphism in bidegrees $(*')[*]$ such that $* \leq 2*' + 1$ and a monomorphism in bidegrees $(*')[*]$ such that $* = 2*' + 2$.

Proof. We want to prove that $p_{n+1}(x) = 0$ implies $x \in J_k + (u_{n+1}w)$ for any x in bidegrees satisfying the condition $* \leq 2*' + 2$. We proceed by induction on the square degree of x. The induction base is guaranteed by the fact that the degree 2 part of the kernel is generated by u_2 and $u_2 \in I_k$. Now, suppose that the claim is true for square degrees less than the square degree of x. We can write x as $\sum_{j=0}^m \phi_j v^j$ for some $\phi_j \in H(BSO_{n+1})$ in bidegrees satisfying $* \leq 2*' + 2$. Notice that $p_n(g^* \otimes l)(x) = \widetilde{g}^* p_{n+1}(x) = 0$, therefore $\sum_{j=0}^m p_n g^*(\phi_j) c^j = 0$. From this we deduce that $p_n g^*(\phi_j) = 0$ for any j since by hypothesis c is a monic polynomial in v_{2^k} in $H(BSpin_n)$, so $g^*(\phi_j) \in I'_k$. Then, $\phi_j \in I_k + (u_{n+1})$ since $\phi_j + ig^*(\phi_j) \in (u_{n+1})$ and $i(I'_k) \subset I_k + (u_{n+1})$, where i is the inclusion of $H(BSO_n)$ in $H(BSO_{n+1})$ sending u_l to u_l. Hence, there are $\psi_j \in H(BSO_{n+1})$ such that $\phi_j + u_{n+1}\psi_j \in I_k$, from which it follows that $x + u_{n+1}z \in J_k$ where $z = \sum_{j=0}^{m} \psi_j v^j$. Hence, $u_{n+1}p_{n+1}(z) = 0$ which implies that

$$p_{n+1}(z)\widetilde{\alpha} \in Im(\widetilde{h}^*) = Im(p_{n+1}) \cdot \widetilde{h}^*(v_{2^k}) = Im(p_{n+1}) \cdot p_{n+1}(w)\widetilde{\alpha}$$

from which we deduce that there exists an element y in $H(BSO_{n+1}) \otimes_H H[v]$ such that $p_{n+1}(z) = p_{n+1}(yw)$. Therefore, $z + yw \in J_k + (u_{n+1}w)$ by induction hypothesis. It follows that $z \in J_k + (w)$ and $x \in J_k + (u_{n+1}w)$.

In order to prove the last part of the proposition we will show by induction on degree that, if $ker(\widetilde{h}^*) = Im(\widetilde{g}^* p_{n+1})$, then p_{n+1} is surjective in the $* \leq 2*' + 1$ part. The induction basis comes from the fact that, in square degree ≤ 2, $H(BSpin_{n+1})$ is the same as the cohomology of the point. Take an element x and suppose that p_{n+1} is surjective in square degrees less than the square degree of x. From $\widetilde{g}^*(x) \in ker(\widetilde{h}^*) = Im(\widetilde{g}^* p_{n+1})$ it follows that there is an element χ in $H(BSO_{n+1}) \otimes_H H[v]$ such that $\widetilde{g}^*(x) = \widetilde{g}^* p_{n+1}(\chi)$. Therefore, $x + p_{n+1}(\chi) = u_{n+1}z$ for some $z \in H(BSpin_{n+1})$. By induction hypothesis $z = p_{n+1}(\zeta)$ for some element $\zeta \in H(BSO_{n+1}) \otimes_H H[v]$, hence $x = p_{n+1}(\chi + u_{n+1}\zeta)$, which is what we aimed to show. □

So, in order to finalise the proof we only need to find a cohomology class v which does the job. There are two possible cases: 1) $\widetilde{h}^*(v_{2^k}) = 0$; 2) $\widetilde{h}^*(v_{2^k}) \neq 0$.

Case 1: In this case v_{2^k} can be lifted to $H(BSpin_{n+1})$ so $w = 0$ and we can choose $c = v_{2^k}$. It follows that $Im(\widetilde{h}^*) = 0 = Im(p_{n+1}) \cdot \widetilde{h}^*(v_{2^k})$ in the $* \leq 2*' + 2$ part and $ker(\widetilde{h}^*) = H(BSpin_n) = Im(p_n) = Im(p_n(g^* \otimes l)) = Im(\widetilde{g}^* p_{n+1})$ in the $* \leq 2*' + 1$ part, since in this case p_n and $g^* \otimes l$ are surjective in the respective bidegrees. So, by Proposition 3.4.4, we have that the homomorphism

$$H(BSO_{n+1})/I_k \otimes_H H[v_{2^k}] \to H(BSpin_{n+1})$$

is an isomorphism in bidegrees $(*')[*]$ such that $* \leq 2*' + 1$ and a monomorphism for $* = 2*' + 2$. Furthermore, we observe that $k(n+1) = k(n) = k$ is the value predicted by the table of Theorem 3.1.1 since $\theta_k \in I_k$ as it is zero in $H(BSpin_{n+1})$ (because u_2 is). Note that the bidegree of θ_k satisfies $* = 2*' + 1$. This completes the first case.

Case 2: In this case we notice that the element w such that $a^*_{n+1}(w)\widetilde{\alpha} = \widetilde{h}^*(v_{2^k})$ must be different from 0. Moreover, since $H(BSpin_n)$ is generated by $v_{2^k}^i$ as a $H(BSO_n)$-module (and, so, as a $H(BSO_{n+1})$-module) in the $* \leq 2*' + 1$ part by induction hypothesis, we have that $Im(\widetilde{h}^*)$ is generated by $\widetilde{h}^*(v_{2^k}^i)$ as a $H(BSO_{n+1})$-module in the $* \leq 2*' + 2$ part. At this point, we need the following lemmas.

Lemma 3.4.5. *$Sq^m a_{n+1}^*(w) \in < a_{n+1}^*(w) >$ for any m, where $< a_{n+1}^*(w) >$ is the $H(BSO_{n+1})$-submodule of $H(BSpin_{n+1})$ generated by $a_{n+1}^*(w)$.*

Proof. We proceed by induction on m. For $m = 0$ there is nothing to prove and for $m > 2^k - n$ we have that $Sq^m w = 0$ by Corollary 2.4.11. Suppose the statement is true for integers less than $m \leq 2^k - n$. Then,

$$Sq^m(u_{n+1}w) = \sum_{j=0}^{m} \tau^{j \, mod 2} Sq^j u_{n+1} Sq^{m-j} w = \sum_{j=0}^{m} \tau^{j \, mod 2} u_j u_{n+1} Sq^{m-j} w$$

from which it follows by applying a_{n+1}^*, by induction (on m) hypothesis and by recalling that $u_{n+1} a_{n+1}^*(w) = 0$ that

$$0 = Sq^m(u_{n+1} a_{n+1}^*(w)) = \sum_{j=0}^{m} \tau^{j \, mod 2} u_j u_{n+1} Sq^{m-j} a_{n+1}^*(w) = u_{n+1} Sq^m a_{n+1}^*(w)$$

Hence, $\widetilde{f}^*(Sq^m a_{n+1}^*(w)\widetilde{\alpha}) = 0$, from which it follows that $Sq^m a_{n+1}^*(w)\widetilde{\alpha} \in Im(\widetilde{h}^*)$. Now, note that $Sq^m a_{n+1}^*(w)\widetilde{\alpha}$ lies in the $* \leq 2*' + 2$ part. Therefore, by the remark just before this lemma, we obtain that $Sq^m a_{n+1}^*(w)\widetilde{\alpha} = \sum_{i \geq 1} \phi_i \widetilde{h}^*(v_{2^k}^i)$ for some $\phi_i \in H(BSO_{n+1})$. But, for any $i > 1$, the square degree of $\widetilde{h}^*(v_{2^k}^i)$ is greater than that of $Sq^m a_{n+1}^*(w)\widetilde{\alpha}$. We deduce that $Sq^m a_{n+1}^*(w)\widetilde{\alpha} = \phi_1 \widetilde{h}^*(v_{2^k})$, from which it follows that

$$Sq^m a_{n+1}^*(w) = \phi_1 a_{n+1}^*(w) \in < a_{n+1}^*(w) >$$

which is what we aimed to prove. □

Lemma 3.4.6. *For any $m > 1$ there exist elements λ_m and μ_m in $H(BSpin_{n+1})$ such that $\widetilde{h}^*(v_{2^k}^m) = \lambda_m \widetilde{h}^*(v_{2^k})$, $\widetilde{g}^*(\mu_m) = v_{2^k}^m + \widetilde{g}^*(\lambda_m) v_{2^k}$, λ_m and μ_m are in the image of $H(BSO_{n+1}) \otimes_H H[\mu_2]$ and μ_m is divisible by μ_2.*

Proof. We notice that by Proposition 3.2.1 and Corollary 2.4.11

$$\widetilde{h}^*(v_{2^k}^2) = \widetilde{h}^*(Sq^{2^k} v_{2^k}) = Sq^{2^k}(a_{n+1}^*(w)\widetilde{\alpha}) =$$

$$(\tau^{n \, mod 2} Sq^{2^k - n} a_{n+1}^*(w) u_n + \tau^{(n+1) \, mod 2} Sq^{2^k - n - 1} a_{n+1}^*(w) u_{n+1})\widetilde{\alpha}$$

which belongs to $< \widetilde{h}^*(v_{2^k}) >$ by Lemma 3.4.5. In other words, there is an element λ_2 in $H(BSO_{n+1})$ such that $\widetilde{h}^*(v_{2^k}^2) = \lambda_2 \widetilde{h}^*(v_{2^k})$. Indeed, recalling that $a_{n+1}^*(w) u_{n+1} = 0$, the element λ_2 can be obtained in this way: first, note that $Sq^{2^k - n} a_{n+1}^*(w) = r a_{n+1}^*(w)$ by previous lemma, then $\lambda_2 = \tau^{n \, mod 2} r u_n$.

Denote by μ_2 a lift of $v_{2^k}^2 + \widetilde{g}^*(\lambda_2)v_{2^k}$ to $H(BSpin_{n+1})$. Suppose the statement is true for m, so, taking into account that $\widetilde{h}^*$ is $H(BSpin_{n+1})$-linear, we have

$$\widetilde{h}^*(v_{2^k}^{m+1}) = \widetilde{h}^*((v_{2^k}^m + \widetilde{g}^*(\lambda_m)v_{2^k})v_{2^k} + \widetilde{g}^*(\lambda_m)v_{2^k}^2) = \mu_m\widetilde{h}^*(v_{2^k}) + \lambda_m\lambda_2\widetilde{h}^*(v_{2^k})$$

Denote by λ_{m+1} the element $\mu_m + \lambda_m\lambda_2$ and by μ_{m+1} the element $\lambda_m\mu_2$. Then,

$$\widetilde{g}^*(\mu_{m+1}) = \widetilde{g}^*(\lambda_m\mu_2) = \widetilde{g}^*(\lambda_m)(v_{2^k}^2 + \widetilde{g}^*(\lambda_2)v_{2^k}) = \widetilde{g}^*(\lambda_m)v_{2^k}^2 +$$
$$\widetilde{g}^*(\lambda_{m+1} + \mu_m)v_{2^k} = (\widetilde{g}^*(\lambda_m)v_{2^k} + \widetilde{g}^*(\lambda_{m+1}) + v_{2^k}^m + \widetilde{g}^*(\lambda_m)v_{2^k})v_{2^k} =$$
$$v_{2^k}^{m+1} + \widetilde{g}^*(\lambda_{m+1})v_{2^k}$$

and the proof is complete. □

Now consider the following commutative diagram

$$\begin{array}{ccc} H(BSO_{n+1}) \otimes_H H[\mu_2] & \xrightarrow{g^*\otimes l} & H(BSO_n) \otimes_H H[v_{2^k}^2 + \widetilde{g}^*(\lambda_2)v_{2^k}] \\ \downarrow{\scriptstyle p_{n+1}} & & \downarrow{\scriptstyle p_n} \\ H(BSpin_{n+1}) & \xrightarrow{\widetilde{g}^*} & H(BSpin_n) \end{array}$$

From the previous lemma and from the remark before Lemma 3.4.5 we get that $Im(\widetilde{h}^*) = Im(p_{n+1}) \cdot \widetilde{h}^*(v_{2^k})$ in the $* \leq 2*' + 2$ part. Then, by Proposition 3.4.4, we obtain that $ker(p_{n+1}) = J_k + (u_{n+1}w)$ in the $* \leq 2*' + 2$ part.

Recall that, by looking at diagram (#) and by induction on degree, the cohomology group $H^{2^k,2^{k-1}}(BSpin_{n+1})$ consists only of subtle Stiefel-Whitney classes, since we are studying the case that v_{2^k} is not covered by $\widetilde{g}^*$. Hence,

$$c_{n+1}^* : H^{2^k,2^{k-1}}(BSpin_{n+1}) \to H^{2^k+1,2^{k-1}}(Cone(a_{n+1}))$$

is the zero homomorphism and b_{n+1}^* is injective in the bidegree of x_k, from which we deduce that $\theta_k \notin I_k$ since $x_k \notin < x_0, \dots, x_{k-1} >$ by Lemma 3.4.2. Therefore, by observing that $ker(p_{n+1}) = J_k + (u_{n+1}w)$ in the $* \leq 2*' + 2$ part and $p_{n+1}(\theta_k) = 0$ we get that $\theta_k + u_{n+1}w \in I_k$ which implies that $ker(p_{n+1}) = J_{k+1}$ in the same bidegrees.

In order to finish, we need the following lemma.

Lemma 3.4.7. *$ker(\widetilde{h}^*) = Im(\widetilde{g}^* p_{n+1})$ in the $* \leq 2*' + 1$ part.*

Proof. Let us set $\mu_1 = \lambda_0 = 0$ and $\mu_0 = \lambda_1 = 1$. Let x be an element of the kernel. We can write x as $\sum_{j=0}^m \gamma_j v_{2^k}^j$ with $\gamma_j \in H(BSO_{n+1})$. Then, by Lemma 3.4.6,

$$x = \sum_{j=0}^m \gamma_j(\widetilde{g}^*(\mu_j) + \widetilde{g}^*(\lambda_j)v_{2^k})$$

from which it follows by applying $\widetilde{h}^*$ that $\sum_{j=0}^{m} \gamma_j \lambda_j \widetilde{h}^*(v_{2^k}) = 0$. Denote by σ the element $\sum_{j=0}^{m} \gamma_j \lambda_j$ in $H(BSO_{n+1}) \otimes_H H[\mu_2]$. From

$$p_{n+1}(\sigma w)\widetilde{\alpha} = p_{n+1}(\sigma) a_{n+1}^*(w)\widetilde{\alpha} = p_{n+1}(\sigma)\widetilde{h}^*(v_{2^k}) = 0$$

we get $\sigma w \in J_{k+1}$, since $ker(p_{n+1}) = J_{k+1}$ in the $* \leq 2*' + 2$ part. Thus, $\sigma w = \sum_{j=0}^{k} \sigma_j \theta_j$ for some $\sigma_j \in H(BSO_{n+1}) \otimes_H H[\mu_2]$ and, multiplying by u_{n+1}, we obtain that $u_{n+1}\sigma w + u_{n+1}\sigma_k \theta_k \in J_k$. On the other hand, $\theta_k + u_{n+1} w \in I_k$, from which it follows by multiplying by σ that $\sigma\theta_k + u_{n+1}\sigma w \in J_k$. Hence, $(\sigma + u_{n+1}\sigma_k)\theta_k \in J_k$. By Proposition 3.3.4 we deduce that $\tau^m(\sigma + u_{n+1}\sigma_k) \in J_k$ for some m which in this case is non positive since σ is in the $* \leq 2*' + 1$ part of $H(BSO_{n+1}) \otimes_H H[\mu_2]$, from which it follows that $\sigma \in J_k + (u_{n+1})$. Therefore, $\widetilde{g}^* p_{n+1}(\sigma) = 0$ in $H(BSpin_n)$ and

$$x = \sum_{j=0}^{m} \gamma_j \widetilde{g}^*(\mu_j) \in Im(\widetilde{g}^* p_{n+1})$$

as we aimed to show. □

Denote by $v_{2^{k+1}}$ the class μ_2, then by Proposition 3.4.4 we get that the homomorphism

$$H(BSO_{n+1})/I_{k+1} \otimes_H H[v_{2^{k+1}}] \to H(BSpin_{n+1})$$

is an isomorphism in bidegrees $(*')[*]$ such that $* \leq 2*' + 1$ and a monomorphism for $* = 2*' + 2$. Moreover, since $\theta_k \notin I_k$ we have that $\rho_k \notin I_k^{top}$ from which it follows that $k(n+1) = k(n) + 1 = k + 1$ by the remark just after Theorem 3.1.2. This completes the proof of the second case. □

Notice that Proposition 3.4.4 works completely fine without any restriction on degrees. This means that the only obstruction in the proof of the previous theorem to conclude that the homomorphism

$$H(BSO_n)/I_k \otimes_H H[v_{2^k}] \to H(BSpin_n)$$

is an isomorphism everywhere lies in Lemma 3.4.7 which is the only place where the restriction to bidegrees that satisfy the condition $* \leq 2*' + 1$ is really needed. Indeed, this restriction is necessary in the above mentioned lemma since we do not have a proof of the regularity of the sequence considered by Quillen. Anyway, it is reasonable to formulate the following question.

Question 3.4.8. *Is the sequence $u_2, \theta_1, \ldots, \theta_{k-1}$ a regular sequence in the polynomial ring $\mathbb{Z}/2[\tau, u_2, \ldots, u_n]$, where k depends on n as in the table of Theorem 3.1.1?*

The answer to the previous question is easily proved to be positive for $n \leq 12$. Moreover, we notice that the regularity of Quillen's sequence in topology implies the regularity of the sequence $1+\tau, u_2, \theta_1, \dots, \theta_{k-1}$ in $H(BSO_n)$, which unfortunately does not imply the regularity of our sequence since $1+\tau$ is not a homogeneous polynomial of positive degree. For this reason, one would rather work with the sequence $\tau, u_2, \theta_1, \dots, \theta_{k-1}$ which is made of homogeneous elements, but this happens to be not regular already for $n = 11$. Besides, we would like to mention that Quillen's method to prove Theorem 3.1.1 does not immediately apply to our situation since the homomorphism he considered $H(BSO_n) \hookrightarrow H(BO_1^{\times n})$ is not faithfully flat in the motivic case, so it does not in principle preserve regular sequences.

Nevertheless, we would like to conclude this section by highlighting that, by exactly the same arguments of Theorem 3.4.3 (just by substituting Proposition 3.3.4 with the regularity of Quillen's sequence in the motivic case in the proof of Lemma 3.4.7), the following conditional result holds.

Theorem 3.4.9. *Let k be a field of characteristic different from 2 containing $\sqrt{-1}$. If the answer to Question 3.4.8 is affirmative, then we have an isomorphism*

$$H(BSpin_n) \cong H(BSO_n)/I_k \otimes_H H[v_{2^k}]$$

where k depends on n as in the table of Theorem 3.1.1.

As we have pointed out, at the end, everything is reduced to answer Question 3.4.8. Once one has the regularity of the needed sequence in $\mathbb{Z}/2[\tau, u_2, \dots, u_n]$ it is possible to obtain the complete description of the motivic cohomology ring of $BSpin_n$ over any field of characteristic different from 2 containing $\sqrt{-1}$. In fact, in the previous theorem, we do not require k to be quadratically closed anymore, since the regularity of the motivic sequence in $\mathbb{Z}/2[\tau, u_2, \dots, u_n]$ would imply the regularity of the same sequence in $H[u_2, \dots, u_n]$. Moreover, since one can easily check the regularity of the sequence for $n \leq 12$, our main theorem provides the whole computation of $H(BSpin_n)$ over any field of characteristic different from 2 containing $\sqrt{-1}$ for any $n \leq 12$.

3.5 RELATIONS AMONG SUBTLE CLASSES FOR $Spin_n$-TORSORS

In this section we deduce, just from the triviality of u_2 in the motivic cohomology of $BSpin_n$, some very simple relations among subtle classes in the motivic cohomology of the Čech simplicial scheme associated to a $Spin_n$-torsor. Here, we work over a base field k of characteristic different from 2 containing $\sqrt{-1}$.

We start by recalling that $Spin_n$-torsors over the point correspond to n-dimensional quadratic forms from I^3, where I is the fundamental ideal in the Witt ring. Moreover, we have the following commutative diagram

$$\begin{array}{ccc} \check{C}(X_q) & \longrightarrow & BSpin_n \\ \downarrow & & \downarrow \\ Spec(k) & \xrightarrow{q} & B_{\acute{e}t}Spin_n \end{array}$$

for any n-dimensional $q \in I^3$ and all above-diagonal classes in $H(BSpin_n)$ coming from étale classifying space trivialize in $H(\check{C}(X_q))$, since the above-diagonal cohomology of a point is zero. Here $\check{C}(X_q)$ is the Čech simplicial scheme associated to the torsor q. In particular Chern classes $c_i(q) = \tau^{i \, mod 2} u_i(q)^2$ are zero, as these are coming from the étale space .

From previous remarks we obtain the following proposition, which provides us with relations among subtle characteristic classes for quadratic forms from I^3.

Proposition 3.5.1. *For any n-dimensional $q \in I^3$, the following relations hold in $H(\check{C}(X_q))$*

$$\sum_{h=0}^{2^j} u_{2^j-h}(q) u_{2^j+1+h}(q) = 0$$

for any j satisfying $2^j + 1 \leq n$.

Proof. We will actually prove that

$$\theta_{j+1}(q) = \sum_{h=0}^{2^j} u_{2^j-h}(q) u_{2^j+1+h}(q)$$

and the result will follow by recalling that $u_2(q) = 0$. For $j = 0$ and $j = 1$ we have respectively $\theta_1(q) = u_3(q)$ and $\theta_2(q) = u_2(q)u_3(q) + u_5(q)$, which provide our induction basis. Suppose the statement holds for $\theta_j(q)$ with $j \geq 2$, then by Cartan formula and Proposition 2.4.10 we have that

$$\theta_{j+1}(q) = Sq^{2^j} \theta_j(q) = Sq^{2^j} \sum_{h=0}^{2^{j-1}} u_{2^{j-1}-h}(q) u_{2^{j-1}+1+h}(q)$$

$$= \sum_{h=0}^{2^{j-1}-1} (\tau^{h \, mod 2} u_{2^{j-1}-h}(q)^2 Sq^{2^{j-1}+h} u_{2^{j-1}+1+h}(q)$$

$$+ \tau^{(h+1) \, mod 2} Sq^{2^{j-1}-h-1} u_{2^{j-1}-h}(q) u_{2^{j-1}+1+h}(q)^2) + Sq^{2^j} u_{2^j+1}(q)$$

$$= \sum_{h=0}^{2^{j-1}-1} (c_{2^{j-1}-h}(q) Sq^{2^{j-1}+h} u_{2^{j-1}+1+h}(q) + Sq^{2^{j-1}-h-1} u_{2^{j-1}-h}(q) c_{2^{j-1}+1+h}(q))$$

$$+ \sum_{h=0}^{2^j} u_{2^j-h}(q) u_{2^j+1+h}(q) = \sum_{h=0}^{2^j} u_{2^j-h}(q) u_{2^j+1+h}(q)$$

□

In other words, we obtain that

$$u_{2^j+1}(q) = \sum_{h=0}^{2^{j-1}-1} u_{2^{j-1}-h}(q)u_{2^{j-1}+1+h}(q)$$

for any j satisfying $2^j + 1 \leq n$.

In [47], Smirnov and Vishik have highlighted the deep relation between subtle Stiefel-Whitney classes and the J-invariant of quadrics defined in [56]. More precisely, they proved the following result.

Theorem 3.5.2. *Let q be an n-dimensional quadratic form, $p = q$, for even n, and $p = q \perp \langle det_{\pm}(q)\rangle$, for odd n. Then,*

$$u_{2j+1}(p) \in (u_{2l+1}(p) | 0 \leq l < j) \Rightarrow j \in J(q)$$

Proof. See [47, Corollary 3.2.22]. □

From the previous theorem and from Proposition 3.5.1 we immediately deduce the following well known corollary.

Corollary 3.5.3. *For any n-dimensional $q \in I^3$, $2^{j-1} \in J(q)$ for any j satisfying $2^j + 1 \leq n$.*

3.6 THE MOTIVIC COHOMOLOGY OF BG_2

In this last section, we use the main result of this chapter, namely Theorem 3.4.3, to compute the motivic cohomology ring of the Nisnevich classifying space of G_2. This enables us to obtain motivic invariants for G_2-torsors, i.e. octonion algebras.

We start by noticing that there is a fiber sequence

$$A_{q_8} \to BG_2 \to BSpin_7$$

(see for example [3]). We can exploit this sequence and previous results to compute the motivic cohomology ring of BG_2. Before proceeding, note that by the remark just after Theorem 3.4.9 we know the complete description of $H(BSpin_7)$ over any field of characteristic different from 2 containing $\sqrt{-1}$.

Theorem 3.6.1. *Let k be a field of characteristic different from 2 containing $\sqrt{-1}$. Then, the motivic cohomology ring of BG_2 is completely described by*

$$H(BG_2) = H[u_4, u_6, u_7]$$

Proof. Since $Spin_7/G_2 \cong A_{q_8}$, by applying Proposition 2.2.5 to the smooth coherent morphism $\widehat{B}G_2 \to BSpin_7$, we get a Gysin long exact sequence of $H(BSpin_7)$-modules in motivic cohomology

$$\cdots \to H^{p-8,q-4}(BSpin_7) \to H^{p,q}(BSpin_7) \to$$

$$H^{p,q}(BG_2) \to H^{p-7,q-4}(BSpin_7) \to \dots$$

Hence, in order to be able to describe $H(BG_2)$ we only need to understand where 1 is sent under the morphism $H^{p-8,q-4}(BSpin_7) \to H^{p,q}(BSpin_7)$. Recall that from Theorem 3.4.3 we have that $H(BSpin_7) = H[u_4, u_6, u_7, v_8]$.

Note that there is a commutative diagram

$$\begin{array}{ccccc} BSL_2 & \xrightarrow{\Delta} & BSL_2 \times BSL_2 & \longrightarrow & BSL_4 \\ \cong\updownarrow & & \cong\updownarrow & & \updownarrow\cong \\ BSpin_3 & \longrightarrow & BSpin_4 & \longrightarrow & BSpin_6 \end{array}$$

where all the vertical maps are induced by the sporadic isomorphisms $SL_2 \cong Spin_3$, $SL_2 \times SL_2 \cong Spin_4$ and $SL_4 \cong Spin_6$. It induces a commutative diagram of motivic cohomology rings

$$\begin{array}{ccccc} H[c] & \xleftarrow{\Delta^*} & H[c', c''] & \longleftarrow & H[c_2, c_3, c_4] \\ \cong\updownarrow & & \cong\updownarrow & & \updownarrow\cong \\ H[v_4] & \longleftarrow & H[u_4, v_4] & \longleftarrow & H[u_4, u_6, v_8] \end{array}$$

where the first vertical arrow identifies c with v_4, the last vertical arrow identifies c_2 with u_4 and c_3 with u_6, c' and c'' are sent both to c and c_4 maps to $c'c''$. Now, note that $H(BSpin_6) \to H(BSpin_4)$ factors through $H(BSpin_5) = H[u_4, v_8]$. Since $\widetilde{h}^* : H(BSpin_4) \to H(BSpin_5)$ is nontrivial, the class w from Theorem 3.4.3 is equal to 1 and, so, by Lemma 3.4.6 we know that $\lambda_2 = u_4$. Hence, v_8 maps to $v_4^2 + u_4 v_4$. Moreover, notice that the second vertical arrow identifies c' with v_4 and c'' with $v_4 + u_4$. It follows that $c'c''$ is identified with $v_4^2 + u_4 v_4$. Therefore, c_4 is identified with v_8 since they are the only classes in their degrees that restrict to the same element.

Moreover, we can notice that there is a cartesian square of simplicial schemes given by

$$\begin{array}{ccc} \widehat{B}SL_3 & \longrightarrow & BSL_4 \cong BSpin_6 \\ \downarrow & & \downarrow \\ \widehat{B}G_2 & \longrightarrow & BSpin_7 \end{array}$$

Recall that $H(BSL_3) = H[c_2, c_3]$, $H(BSL_4) = H[c_2, c_3, c_4]$ and $H(BSpin_6) = H[u_4, u_6, v_8]$ with the identifications $c_2 = u_4$, $c_3 = u_6$ and $c_4 = v_8$ discussed above. Hence, by Corollary 2.2.8 we easily deduce that the morphism $H^{p-8,q-4}(BSpin_7) \to H^{p,q}(BSpin_7)$ is multiplication by $v_8 + \{a\}u_7$, from which it immediately follows that $H(BG_2) = H[u_4, u_6, u_7]$, which is what we aimed to prove. □

4

SUBTLE CHARACTERISTIC CLASSES AND HERMITIAN FORMS

In this chapter we will focus instead on the unitary group $U_n(E/k)$ associated to the standard split hermitian form of a quadratic extension E/k. In particular, we will compute the motivic cohomology with $\mathbb{Z}/2$-coefficients of its Nisnevich classifying space. As in the orthogonal case, this will provide us with *subtle characteristic classes* which allow to approach the classification of $U_n(E/k)$-torsors over the point, which are nothing else but n-dimensional *hermitian forms* of E/k, which are in one-to-one correspondence with $2n$-dimensional quadratic forms over k divisible by the norm form of the quadratic extension considered.

In [47], the computation of the motivic cohomology of BO_n is conducted inductively by using fibrations with motivically Tate fibers. In our situation, new features will appear. In particular, the fibrations in the unitary case, similar to those considered in the orthogonal one, will have reduced fibers which (depending on parity) are not motivically Tate but, anyway, invertible, which will still allow the computation. These invertible motives are, not surprisingly, closely related to the *Rost motive* of our quadratic extension. As a consequence, we obtain that, unlike the orthogonal case, the classifying space of the unitary group is not cellular, but it becomes one once tensored with the Čech simplicial scheme of the Pfister form of the quadratic extension. Related to this, we observe an interesting interaction between invertible objects and idempotents in Voevodsky category. It is manifested, in particular, by the fact that the cohomology of the tensor product of $BU_n(E/k)$ with the Čech simplicial scheme above mentioned happens to be a direct limit of the cohomology of $BU_n(E/k)$ tensored with powers of an invertible motive.

We also note that, although studying hermitian forms is the same as studying quadratic forms divisible by a Pfister form, the understanding of the unitary case allows to trace back information from the hermitian world to the quadratic one. In particular, from the computation of the motivic cohomology of $BU_n(E/k)$ we get relations among subtle Stiefel-Whitney classes in the cohomology of the Čech simplicial scheme of the respective quadratic form divisible by a binary Pfister form. These relations supply information about the kernel invariant of this particular class of quadratic forms. In this sense, for quadratic forms associated to hermitian forms, the cohomology of $BU_n(E/k)$ is much closer to one of the Čech simplicial scheme of the torsor than the cohomology of BO_{2n}.

We will finish the chapter by showing that these subtle invariants see the triviality of hermitian forms, in the same way as subtle Stiefel-Whitney classes do for quadratic forms. Moreover, we will get a description of the motive of the torsor associated to a hermitian form in terms of its subtle classes.

4.1 SOME GENERALITIES ON HERMITIAN FORMS

We will start this section by recalling a few general facts and fixing some conventions about hermitian forms and unitary groups. Everything can be found in some standard reference such as [46].

Fix a base field k of characteristic different from 2. Given a quadratic extension $E = k(\sqrt{\alpha})$ and an n-dimensional E-vector space V, an n-dimensional hermitian form is a map $h : V \times V \to E$ which is E-linear in the first factor and such that $h(v, w) = \sigma(h(w, v))$ (where σ is the generator of $Gal(E/k)$). It follows immediately from the definition that the diagonal part of a hermitian form takes values in k and is a quadratic form. We will denote by $\widetilde{h}$ this $2n$-dimensional quadratic form over k defined by $\widetilde{h}(v) = h(v, v)$ for any $v \in V$ considered as a $2n$-dimensional k-vector space. Moreover, notice that the quadratic form $\widetilde{h}$ just defined is divisible by $\langle\langle \alpha \rangle\rangle$, the 1-fold Pfister form associated to α. Indeed, more is true, namely any quadratic form over k divisible by $\langle\langle \alpha \rangle\rangle$ is associated to some hermitian form, and the correspondence is bijective. In fact, given two n-dimensional hermitian forms h and h', we have that $h \cong h'$ if and only if $\widetilde{h} \cong \widetilde{h'}$ ([26, Corollary 9.2]).

As before, we will express by q_n the standard split quadratic form $\perp_{i=1}^{n} \langle (-1)^{i-1} \rangle$ and by $\mathbb{H}$ the hyperbolic form $\langle 1, -1 \rangle$. Similarly, we will denote by h_n the standard split hermitian form $\perp_{i=1}^{n} \langle (-1)^{i-1} \rangle$. Notice, in particular, that $\widetilde{h}_n = \langle\langle \alpha \rangle\rangle \otimes q_n$. By $U_n(E/k)$ we will mean the unitary group of invertible $n \times n$-matrices over E that preserve the standard split hermitian form h_n. Notice that this is a linear algebraic group over k.

4.2 Čech simplicial scheme and rost motive of a quadratic extension

Let $E = k(\sqrt{\alpha})$ be a quadratic extension of k. Then, the motive of $Spec(E)$ in $\mathcal{DM}^-_{eff}(k, \mathbb{Z}/2)$ is the Rost motive M_α of the Pfister form $\langle\langle\alpha\rangle\rangle$. It is proven in [45] (see 6.1.4 for more details) that this motive comes endowed with two morphisms $M_\alpha \to T$ and $T \to M_\alpha$ such that the composition $T \to M_\alpha \to T$ is the 0 morphism and becomes a split distinguished triangle in $\mathcal{DM}^-_{eff}(E, \mathbb{Z}/2)$.

Moreover, in [62] (see 6.1.6 for more details) it is shown that M_α can be presented as an extension of two motives of Čech simplicial schemes. More precisely, in $\mathcal{DM}^-_{eff}(k, \mathbb{Z}/2)$ there is the following distinguished triangle

$$M_\alpha \to \mathfrak{X}_\alpha \to \mathfrak{X}_\alpha[1] \to M_\alpha[1] \qquad (*)$$

where $\mathfrak{X}_\alpha$ is the motive of the Čech simplicial scheme of the Pfister quadric associated to the Pfister form $\langle\langle\alpha\rangle\rangle$.

Let $N_>$ be $Cone(T \to M_\alpha)$ and $N_<$ be $Cone(M_\alpha \to T)[-1]$. Since we have that $Hom(T, T[j]) = 0$ for $j \neq 0$ and we are working with $\mathbb{Z}/2$-coefficients, the morphism $T \to M_\alpha$ is uniquely liftable to $N_<$ while the morphism $M_\alpha \to T$ is uniquely extendable to $N_>$. It immediately follows from the octahedron axiom that $Cone(N_> \to T)[-1] \cong Cone(T \to N_<)$. We will denote this motive by $\widehat{M_\alpha}$.

In this section, we will study the above mentioned motives and their motivic cohomology. We start by establishing relations among them.

Proposition 4.2.1. *The following isomorphisms hold in $\mathcal{DM}^-_{eff}(k, \mathbb{Z}/2)$:*
1) $M_\alpha \otimes \mathfrak{X}_\alpha \cong M_\alpha$ via $M_\alpha \otimes (\mathfrak{X}_\alpha \to T)$;
2) $N_> \otimes \mathfrak{X}_\alpha \cong \mathfrak{X}_\alpha$ via $(N_> \to T) \otimes \mathfrak{X}_\alpha$;
3) $M_\alpha \otimes N_> \cong M_\alpha$ via $M_\alpha \otimes (N_> \to T)$;
4) $N_< \otimes N_> \cong T$;
5) $\widehat{M_\alpha} \otimes N_> \cong \widehat{M_\alpha}[1]$ via $\widehat{M_\alpha} \otimes (N_> \to T[1])$.

Proof. 1) Since $\mathfrak{X}_\alpha$ is a projector in $\mathcal{DM}^-_{eff}(k, \mathbb{Z}/2)$ we have that $\mathfrak{X}_\alpha \otimes \mathfrak{X}_\alpha \cong \mathfrak{X}_\alpha$. Hence, by tensoring with $\mathfrak{X}_\alpha$ the distinguished triangle

$$M_\alpha \to \mathfrak{X}_\alpha \to \mathfrak{X}_\alpha[1] \to M_\alpha[1]$$

we obtain that $M_\alpha \otimes \mathfrak{X}_\alpha \cong M_\alpha$.

2) Therefore, by tensoring with $\mathfrak{X}_\alpha$ the distinguished triangle

$$T \to M_\alpha \to N_> \to T[1]$$

and by recalling that the morphism $\mathfrak{X}_\alpha \to M_\alpha$ from $(*)$ factors through T we get that $N_> \otimes \mathfrak{X}_\alpha \cong \mathfrak{X}_\alpha$.

3) It follows formally from 1) and 2).

4) On the other hand, by tensoring with $N_>$ the distinguished triangle

$$N_< \to M_\alpha \to T \to N_<[1]$$

and by noticing that $(M_\alpha \to T) \otimes N_>$ coincides with $M_\alpha \to N_>$ we obtain that $N_< \otimes N_> \cong T$.

5) Finally, by tensoring with $N_>$ the distinguished triangle

$$T \to N_< \to \widehat{M_\alpha} \to T[1]$$

and by noticing that $(T \to N_<) \otimes N_>$ coincides with $N_> \to T$ we have that $\widehat{M_\alpha} \otimes N_> \cong \widehat{M_\alpha}[1]$. □

From the previous proposition we immediately deduce the following lemma.

Lemma 4.2.2. *In $\mathcal{DM}^-_{eff}(k, \mathbb{Z}/2)$ for any $n \in \mathbb{N}$ there are the following distinguished triangles:*
1) $N_>^{\otimes n-1} \to M_\alpha \to N_>^{\otimes n} \to N_>^{\otimes n-1}[1]$;
2) $\widehat{M_\alpha}[n-1] \to N_>^{\otimes n} \to N_>^{\otimes n-1} \to \widehat{M_\alpha}[n]$.
Here, $M_\alpha \to N_>^{\otimes n}$ and $N_>^{\otimes n} \to N_>^{\otimes n-1}$ are the unique non-zero morphisms between the respective objects.

Proof. 1) It follows immediately from 3) of Proposition 4.2.1 by tensoring the distinguished triangle

$$T \to M_\alpha \to N_> \to T[1]$$

with the appropriate power of $N_>$.

2) It follows immediately from 5) of Proposition 4.2.1 by tensoring the distinguished triangle

$$\widehat{M_\alpha} \to N_> \to T \to \widehat{M_\alpha}[1]$$

with the appropriate power of $N_>$. □

At this point, we present the motivic cohomology of $\widehat{M_\alpha}$, which will be used in the main result of this section, namely the computation of the motivic cohomology of tensor powers of $N_>$.

Lemma 4.2.3. *There exists a cohomology class μ of bidegree $(0)[1]$ such that the motivic cohomology of $\widehat{M_\alpha}$ is given by*

$$H(\widehat{M_\alpha}) = \frac{K^M(k)/2}{Ann(\{\alpha\})} \cdot \mu$$

So, the motivic cohomology of $\widehat{M_\alpha}$ is concentrated on a single diagonal.

Proof. After applying the octahedron axiom twice to the distinguished triangle

$$M_\alpha \to \mathfrak{X}_\alpha \to \mathfrak{X}_\alpha[1] \to M_\alpha[1]$$

we get the distinguished triangle

$$\widehat{M_\alpha}[-1] \to \widetilde{\mathfrak{X}_\alpha}[1] \to \widetilde{\mathfrak{X}_\alpha} \to \widehat{M_\alpha}$$

where $\widetilde{\mathfrak{X}_\alpha}$ is $Cone(\mathfrak{X}_\alpha \to T)[-1]$.

The motivic cohomology of $\widetilde{\mathfrak{X}_\alpha}$ has been computed in the original version of [40] and [68] (see Theorem 6.1.3 for more details). It is described by

$$H(\widetilde{\mathfrak{X}_\alpha}) = \mathbb{Z}/2[\mu] \cdot \mu \otimes \frac{K^M(k)/2}{Ann(\{\alpha\})}$$

Therefore, by the long exact sequence in motivic cohomology induced by the previous distinguished triangle and by recalling that the homomorphism $H^{*,*'}(\widetilde{\mathfrak{X}_\alpha}) \to H^{*-1,*'}(\widetilde{\mathfrak{X}_\alpha})$ sends μ^j to μ^{j-1} since $H(\widehat{M_\alpha})$ is trivial above the 1-st diagonal just by definition, we get the description of $H(\widehat{M_\alpha})$. □

We are now ready to compute the motivic cohomology of any tensor power of $N_>$. This result will be essential in the next section for the proof of the main result.

Proposition 4.2.4. *For any $n \in \mathbb{N}$ there exist cohomology classes μ_i of bidegree $(0)[i]$ for $1 \le i \le n$ such that the motivic cohomology of the n-th tensor power of $N_>$ as an H-module is given by*

$$H(N_>^{\otimes n}) = H \oplus \bigoplus_{i=1}^{n} \frac{K^M(k)/2}{Ann(\{\alpha\})} \cdot \mu_i$$

where the H-module structure is described by the relations $\tau\mu_i = \{\alpha\}\mu_{i-1}$ ($\mu_0 = 1$ by convention).

Proof. We will proceed by induction on n. For $n = 1$ the distinguished triangle

$$\widehat{M_\alpha} \to N_> \to T \to \widehat{M_\alpha}[1]$$

induces the following long exact sequence in motivic cohomology

$$\cdots \to H^{*-1,*'}(\widehat{M_\alpha}) \to H^{*,*'} \to H^{*,*'}(N_>) \to H^{*,*'}(\widehat{M_\alpha}) \to \ldots$$

From Lemma 4.2.3 and the fact that the motivic cohomology of a point is trivial above the main diagonal it follows that

$$H^{*,*'}(N_>) = \begin{cases} H^{*,*'}(\widehat{M_\alpha}), & * > *' \\ H^{*,*'}, & * \leq *' \end{cases}$$

which implies that

$$H(N_>) = H \oplus \frac{K^M(k)/2}{Ann(\{\alpha\})} \cdot \mu$$

On the other hand, after tensoring with $\mathfrak{X}_\alpha$ the distinguished triangle

$$T \to M_\alpha \to N_> \to T[1]$$

we get a morphism of long exact sequences in motivic cohomology

$$\begin{array}{ccccccccc} \ldots \to & H^{*-1,*'}(M_\alpha) & \to & H^{*-1,*'} & \to & H^{*,*'}(N_>) & \to & H^{*,*'}(M_\alpha) & \to \ldots \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \ldots \to & H^{*-1,*'}(M_\alpha) & \to & H^{*-1,*'}(\mathfrak{X}_\alpha) & \to & H^{*,*'}(\mathfrak{X}_\alpha) & \to & H^{*,*'}(M_\alpha) & \to \ldots \end{array}$$

By a four lemma argument, since the second vertical map is injective (by Beilinson-Lichtenbaum "conjecture"), we deduce that $H(N_>) \to H(\mathfrak{X}_\alpha)$ is injective. Therefore, $\tau\mu = \{\alpha\}$ in $H(N_>)$, since the same relation holds in $H(\mathfrak{X}_\alpha)$. That completes the induction basis.

Now, suppose the statement holds for $n-1$. Then, by 2) of Lemma 4.2.2 we have the following long exact sequence in motivic cohomology

$$\cdots \to H^{*-n,*'}(\widehat{M_\alpha}) \to H^{*,*'}(N_>^{\otimes n-1}) \to H^{*,*'}(N_>^{\otimes n}) \to H^{*-n+1,*'}(\widehat{M_\alpha}) \to \ldots$$

From Lemma 4.2.3 and by induction hypothesis we have

$$H^{*,*'}(N_>^{\otimes n}) = \begin{cases} H^{*-n+1,*'}(\widehat{M_\alpha}), & * > *' + n - 1 \\ H^{*,*'}(N_>^{\otimes n-1}), & * \leq *' + n - 1 \end{cases}$$

which implies that there exists μ_n in bidegree $(0)[n]$ such that

$$H(N_>^{\otimes n}) = H(N_>^{\otimes n-1}) \oplus \frac{K^M(k)/2}{Ann(\{\alpha\})} \cdot \mu_n = H \oplus \bigoplus_{i=1}^{n} \frac{K^M(k)/2}{Ann(\{\alpha\})} \cdot \mu_i$$

From 1) of Lemma 4.2.2 we have the following long exact sequence in motivic cohomology

$$\cdots \to H^{*-1,*'}(N_>^{\otimes n-1}) \to H^{*,*'}(N_>^{\otimes n}) \to H^{*,*'}(M_\alpha) \to H^{*,*'}(N_>^{\otimes n-1}) \to \ldots$$

that maps $\mu_{i-1} \in H^{i-1,0}(N_>^{\otimes n-1})$ to $\mu_i \in H^{i,0}(N_>^{\otimes n})$ since $H^{i,0}(M_\alpha) = 0$ for $i > 0$. Hence, by induction hypothesis, $\tau\mu_i = \{\alpha\}\mu_{i-1}$ in $H(N_>^{\otimes n})$ and the proof is complete. □

By 2) of Lemma 4.2.1 there is a chain of morphisms

$$\mathfrak{X}_\alpha \to \cdots \to N_>^{\otimes n} \to N_>^{\otimes n-1} \to \cdots \to N_> \to T$$

that induces in cohomology the chain of homomorphisms

$$H \to H(N_>) \to \cdots \to H(N_>^{\otimes n-1}) \to H(N_>^{\otimes n}) \to \cdots \to H(\mathfrak{X}_\alpha)$$

which sends $\mu \in H(N_>)$ to $\mu \in H(\mathfrak{X}_\alpha)$.

We now highlight an interesting relation between the invertible motive $N_>$ and the projector $\mathfrak{X}_\alpha$.

Proposition 4.2.5. *The homomorphisms $H(N_>^{\otimes n}) \to H(\mathfrak{X}_\alpha)$ are injective for all $n \in \mathbb{N}$. Moreover, $H(\mathfrak{X}_\alpha) = \varinjlim H(N_>^{\otimes n})$.*

Proof. We have already noticed that $H(N_>) \to H(\mathfrak{X}_\alpha)$ is injective and maps $\mu = \mu_1$ to μ. Now, suppose by induction hypothesis that the homomorphism $H(N_>^{\otimes n-1}) \to H(\mathfrak{X}_\alpha)$ is injective. Notice that there is a commutative diagram

$$\begin{array}{ccc} H(N_>^{\otimes n-1}) \otimes H(N_>) & \longrightarrow & H(N_>^{\otimes n}) \\ \downarrow & & \downarrow \\ H(\mathfrak{X}_\alpha) \otimes H(\mathfrak{X}_\alpha) & \xrightarrow{\smile} & H(\mathfrak{X}_\alpha) \end{array}$$

where the bottom horizontal map is the usual cup product in $H(\mathfrak{X}_\alpha)$. It follows that the right vertical map sends μ_i to μ^i for any $i \leq n$. This completes the proof. □

Later on we will need also the following description of the motivic cohomology of $N_<$.

Lemma 4.2.6. *The motivic cohomology of $N_<$ is given by*

$$H(N_<) = Ann(\{\alpha\}) \oplus H \cdot \tau$$

Proof. After applying the octahedron axiom to the distinguished triangle

$$M_\alpha \to \mathfrak{X}_\alpha \to \mathfrak{X}_\alpha[1] \to M_\alpha[1]$$

we obtain

$$\mathfrak{X}_\alpha \to N_< \to \widetilde{\mathfrak{X}_\alpha} \to \mathfrak{X}_\alpha[1]$$

which induces in motivic cohomology the following long exact sequence

$$\cdots \to H^{*-1,*'}(\mathfrak{X}_\alpha) \to H^{*,*'}(\widetilde{\mathfrak{X}_\alpha}) \to H^{*,*'}(N_<) \to H^{*,*'}(\mathfrak{X}_\alpha) \to \dots$$

Hence, the result follows by noticing that $H^{*,*'}(\widetilde{\mathfrak{X}_\alpha})$ is the $* > *'$ part of $H^{*,*'}(\mathfrak{X}_\alpha)$, while $H^{*,*'}$ is the $* \leq *'$ part of it, and that $H^{*-1,*'}(\mathfrak{X}_\alpha) \to H^{*,*'}(\widetilde{\mathfrak{X}_\alpha})$ sends μ^{i-1} to μ^i. So, the latter map is surjective everywhere, injective for $* \geq *' + 2$ and zero for $* \leq *'$. □

4.3 THE MOTIVIC COHOMOLOGY RING OF $BU_n(E/k)$

Our goal in this section is to compute by using the techniques presented in previous chapters the motivic cohomology of the Nisnevich classifying space of $U_n(E/k)$, the unitary group associated to the standard split hermitian form h_n of the extension E/k.

At first, let us show some preliminary results which will be useful in the proof of the main theorem.

Proposition 4.3.1. *The homogeneous variety $U_n(E/k)/U_{n-1}(E/k)$ is isomorphic to the affine quadric A_{h_n} defined by the equation $\widetilde{h}_n = 1$.*

Proof. Let V be an n-dimensional E-vector space and let A_{h_n} be the subset of V defined by the equation $h_n = 1$. Then, V can be considered as a $2n$-dimensional k-vector space in which A_{h_n} is the affine quadric defined by the equation $\widetilde{h}_n = 1$. The action of $U_n(E/k)$ on A_{h_n} is transitive (see for example [46]) and, moreover, the isotropy group of the vector $(1, 0, \ldots, 0)$ is isomorphic to $U_{n-1}(E/k)$. This implies the desired result. □

At this point, in order to apply Proposition 2.2.5 to the unitary case, we need to study the motive of the affine quadric A_{h_n}.

Proposition 4.3.2. *The motive in $\mathcal{DM}^-_{eff}(k, \mathbb{Z}/2)$ of the affine quadric A_{h_n} is given by*

$$M(A_{h_n}) = \begin{cases} T \oplus N_>(n)[2n-1], & n \text{ odd} \\ T \oplus T(n)[2n-1], & n \text{ even} \end{cases}$$

Proof. We start by noticing that the quadratic form

$$\widetilde{h}_n = \begin{cases} \langle\langle \alpha \rangle\rangle \perp (n-1)\,\mathbb{H}, & n \text{ odd} \\ n\,\mathbb{H}, & n \text{ even} \end{cases}$$

For a quadratic form q let us denote by Q the projective quadric defined by $q = 0$, by Q' the projective quadric defined by $q = z^2$ and by A the affine quadric defined by $q = 1$. Then, we have in $\mathcal{DM}^-_{eff}(k, \mathbb{Z}/2)$ the following Gysin triangle

$$M(A) \to M(Q') \to M(Q)(1)[2] \to M(A)[1]$$

In the case $q = n\,\mathbb{H}$ the previous triangle becomes

$$M(A) \to \bigoplus_{i=0}^{2n-1} T(i)[2i] \to \bigoplus_{i=1}^{2n-1} T(i)[2i] \oplus T(n)[2n] \to M(A)[1]$$

which implies that, for n even, $M(A_{h_n}) = M(A) = T \oplus T(n)[2n-1]$.

In the case $q = \langle\langle \alpha \rangle\rangle$ we have

$$M(A) \to T \oplus T(1)[2] \to M_\alpha(1)[2] \to M(A)[1]$$

from which it follows that $M(A_{h_1}) = M(A) = T \oplus N_>(1)[1]$.

The general case n odd follows from [5, Lemma 34]. Namely, we have

$$\widetilde{M}(A_{h_n}) = \widetilde{M}(A_{h_1})(n-1)[2n-2] = N_>(n)[2n-1]$$

that implies $M(A_{h_n}) = T \oplus N_>(n)[2n-1]$. □

Before going ahead with the main theorem of this section, we notice that $U_n(E/k)$-torsors over $Spec(k)$ are in one-to-one correspondence with hermitian forms associated to the quadratic extension E/k or, which is the same, with quadratic forms over k divisible by $\langle\langle \alpha \rangle\rangle$. Since Witt cancellation holds for quadratic forms, the previous remark assures that $U_{n-1}(E/k)$-torsors inject in $U_n(E/k)$-torsors over any field extension of k, which allows us to use Propositions 2.3.6 and 2.3.10 in the unitary case taking into account Theorem 2.3.8.

Theorem 4.3.3. *For any $m, n \in \mathbb{Z}_{\geq 0}$ there exist cohomology classes c_i of bidegree $(i)[2i]$ for $1 \leq i \leq n$ such that the motivic cohomology of $\mathfrak{X}_\alpha \otimes BU_n(E/k)$ and $N_>^{\otimes m} \otimes BU_n(E/k)$ is described respectively by*

$$H(\mathfrak{X}_\alpha \otimes BU_n(E/k)) = H(\mathfrak{X}_\alpha)[c_1, \dots, c_n]$$

and

$$H(N_>^{\otimes m} \otimes BU_n(E/k)) = \bigoplus_{i_1,\dots,i_n \in \mathbb{Z}_{\geq 0}} H(N_>^{\otimes m + \sum_{l \text{ odd}} i_l}) \cdot c_1^{i_1} \cdots c_n^{i_n}$$

where the obvious homomorphisms of $H(BU_n(E/k))$-modules

$$H(N_>^{\otimes m} \otimes BU_n(E/k)) \to H(\mathfrak{X}_\alpha \otimes BU_n(E/k))$$

are injective. Moreover, $H(\mathfrak{X}_\alpha \otimes BU_n(E/k)) = \varinjlim H(N_>^{\otimes m} \otimes BU_n(E/k))$.

Proof. We will proceed by induction on n. The induction basis follows immediately from the fact that $BU_0(E/k) \cong Spec(k)$ and by Proposition 4.2.5.

Now, suppose the result holds for $n-1$. Then, since $N_> \otimes \mathfrak{X}_\alpha \cong \mathfrak{X}_\alpha$ by 2) of Proposition 4.2.1 and applying Propositions 2.2.5, 2.3.6, 2.3.10 and 4.3.2 to the coherent morphism $\mathfrak{X}_\alpha \otimes \widehat{B}U_{n-1}(E/k) \to \mathfrak{X}_\alpha \otimes BU_n(E/k)$, we obtain the following long exact sequence in motivic cohomology

$$\dots \to H^{*-1,*'}(\mathfrak{X}_\alpha \otimes BU_{n-1}(E/k)) \xrightarrow{h^*} H^{*-2n,*'-n}(\mathfrak{X}_\alpha \otimes BU_n(E/k)) \xrightarrow{f^*}$$

$$H^{*,*'}(\mathfrak{X}_\alpha \otimes BU_n(E/k)) \xrightarrow{g^*} H^{*,*'}(\mathfrak{X}_\alpha \otimes BU_{n-1}(E/k)) \to \dots$$

Note that, even after having replaced $H(\mathfrak{X}_\alpha \otimes \widehat{B}U_{n-1}(E/k))$ with $H(\mathfrak{X}_\alpha \otimes BU_{n-1}(E/k))$, this stays a sequence of $H(\mathfrak{X}_\alpha \otimes BU_n(E/k))$-modules by the remark just after Proposition 2.3.10. By induction hypothesis, the cohomology ring $H(\mathfrak{X}_\alpha \otimes BU_{n-1}(E/k))$ is freely generated as an $H(\mathfrak{X}_\alpha)$-algebra by $c_1, \dots, c_{n-1}$ which are all uniquely liftable to $H(\mathfrak{X}_\alpha \otimes BU_n(E/k))$, since $\mathfrak{X}_\alpha \otimes BU_n(E/k)$ is the motive of a smooth simplicial scheme and, so, has no cohomology in negative round degrees. Hence, g^* is an epimorphism as it is a ring homomorphism, h^* is trivial and f^* is a monomorphism. Denoting by c_n the element $f^*(1)$ we obtain the result

$$H(\mathfrak{X}_\alpha \otimes BU_n(E/k)) = H(\mathfrak{X}_\alpha)[c_1, \dots, c_n]$$

For the rest of the induction step we will consider separately two cases.

1) *n even*: for any $m \in \mathbb{N}$ we have the following long exact sequence in motivic cohomology of $H(BU_n(E/k))$-modules

$$\dots \to H^{*-1,*'}(N_>^{\otimes m} \otimes BU_{n-1}(E/k)) \xrightarrow{h^*} H^{*-2n,*'-n}(N_>^{\otimes m} \otimes BU_n(E/k)) \xrightarrow{f^*}$$

$$H^{*,*'}(N_>^{\otimes m} \otimes BU_n(E/k)) \xrightarrow{g^*} H^{*,*'}(N_>^{\otimes m} \otimes BU_{n-1}(E/k)) \to \dots$$

For $m = 0$, by induction hypothesis, $H(BU_{n-1}(E/k))$ is generated as an H-algebra by $c_1, \dots, c_{n-1}$ and μc_l for any odd $l < n$. By degree reasons these cohomology classes are all uniquely liftable to $H(BU_n(E/k))$. Therefore, g^* is an epimorphism since it is a ring homomorphism. This assures that, for any m, $H(N_>^{\otimes m} \otimes BU_{n-1}(E/k))$ is generated as an $H(BU_n(E/k))$-module by μ_i for all $i \leq m$. By degree reasons the μ_i are all uniquely liftable to $H(N_>^{\otimes m} \otimes BU_n(E/k))$. Now g^* happens to be surjective since it is a homomorphism of $H(BU_n(E/k))$-modules. Hence, h^* is the 0 homomorphism and f^* is a monomorphism. Then, denoting by c_n the cohomology class $f^*(1)$ we have, for any m, the following morphism of short exact sequences of $H(BU_n(E/k))$-modules

$$\begin{array}{ccccccccc}
0 & \to & H^{*-2n,*'-n}(N_>^{\otimes m} \otimes BU_n(E/k)) & \xrightarrow{\cdot c_n} & H^{*,*'}(N_>^{\otimes m} \otimes BU_n(E/k)) & \to & H^{*,*'}(N_>^{\otimes m} \otimes BU_{n-1}(E/k)) & \to & 0 \\
 & & \downarrow & & \downarrow & & \hookdownarrow & & \\
0 & \to & H^{*-2n,*'-n}(\mathfrak{X}_\alpha \otimes BU_n(E/k)) & \xrightarrow{\cdot c_n} & H^{*,*'}(\mathfrak{X}_\alpha \otimes BU_n(E/k)) & \to & H^{*,*'}(\mathfrak{X}_\alpha \otimes BU_{n-1}(E/k)) & \to & 0
\end{array}$$

By induction on square degree and by a standard four lemma argument, the central vertical morphism is injective. Moreover, by an induction argument on square degree and looking at the previous upper short exact sequence we get that

$$H(N_>^{\otimes m} \otimes BU_n(E/k)) = \bigoplus_{i \in \mathbb{Z}_{\geq 0}} H(N_>^{\otimes m} \otimes BU_{n-1}(E/k)) \cdot c_n^i$$

$$= \bigoplus_{i_1,\ldots,i_n \in \mathbb{Z}_{\geq 0}} H(N_>^{\otimes m+\sum_{l\,odd} i_l}) \cdot c_1^{i_1} \cdots c_n^{i_n}$$

as an $H(BU_n(E/k))$-submodule of $H(\mathfrak{X}_\alpha \otimes BU_n(E/k))$.

2) *n odd*: as before for any m we have the following long exact sequence in motivic cohomology of $H(BU_n(E/k))$-modules

$$\cdots \to H^{*-1,*'}(N_>^{\otimes m} \otimes BU_{n-1}(E/k)) \xrightarrow{h^*} H^{*-2n,*'-n}(N_>^{\otimes m+1} \otimes BU_n(E/k)) \xrightarrow{f^*}$$

$$H^{*,*'}(N_>^{\otimes m} \otimes BU_n(E/k)) \xrightarrow{g^*} H^{*,*'}(N_>^{\otimes m} \otimes BU_{n-1}(E/k)) \to \ldots$$

As in the previous case, for $m = 0$ the induction hypothesis implies that $H(BU_{n-1}(E/k))$ is generated as an H-algebra by $c_1, \ldots, c_{n-1}$ and μc_l for any odd $l < n$. By the same degree reasons they are all uniquely liftable to $H(BU_n(E/k))$. Thus, g^* is an epimorphism since it is a ring homomorphism. This is enough to show that, for any m, $H(N_>^{\otimes m} \otimes BU_{n-1}(E/k))$ is generated as an $H(BU_n(E/k))$-module by μ_i for all $i \leq m$. Again the μ_i are uniquely liftable to $H(N_>^{\otimes m} \otimes BU_n(E/k))$. It follows that g^* is surjective, h^* is trivial and f^* is injective. Then, denoting by c_n the cohomology class $f^*(1)$ we have, for any m, the following morphism of short exact sequences of $H(BU_n(E/k))$-modules

$$\begin{array}{ccccccccc}
0 & \to & H^{*-2n,*'-n}(N_>^{\otimes m+1} \otimes BU_n(E/k)) & \xrightarrow{\cdot c_n} & H^{*,*'}(N_>^{\otimes m} \otimes BU_n(E/k)) & \to & H^{*,*'}(N_>^{\otimes m} \otimes BU_{n-1}(E/k)) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & H^{*-2n,*'-n}(\mathfrak{X}_\alpha \otimes BU_n(E/k)) & \xrightarrow{\cdot c_n} & H^{*,*'}(\mathfrak{X}_\alpha \otimes BU_n(E/k)) & \to & H^{*,*'}(\mathfrak{X}_\alpha \otimes BU_{n-1}(E/k)) & \to & 0
\end{array}$$

By the very same arguments of the previous case, the central vertical morphism is injective and

$$H(N_>^{\otimes m} \otimes BU_n(E/k)) = \bigoplus_{i \in \mathbb{Z}_{\geq 0}} H(N_>^{\otimes m+i} \otimes BU_{n-1}(E/k)) \cdot c_n^i$$

$$= \bigoplus_{i_1,\ldots,i_n \in \mathbb{Z}_{\geq 0}} H(N_>^{\otimes m+\sum_{l\,odd} i_l}) \cdot c_1^{i_1} \cdots c_n^{i_n}$$

as an $H(BU_n(E/k))$-submodule of $H(\mathfrak{X}_\alpha \otimes BU_n(E/k))$, which completes the proof. □

As a corollary of the previous theorem we obtain the description of the motivic cohomology ring of $BU_n(E/k)$ as an H-algebra.

Theorem 4.3.4. *For any $n \in \mathbb{Z}_{\geq 0}$ there exist cohomology classes c_i of bidegree $(i)[2i]$ for $1 \leq i \leq n$ and d_j of bidegree $(j)[2j+1]$ for $1 \leq j$ odd $\leq n$ such that the motivic cohomology ring of $BU_n(E/k)$ is given by*

$$H(BU_n(E/k)) = \frac{H[c_i, d_j]_{1 \leq i \leq n, 1 \leq j\,odd \leq n}}{R}$$

where R is the ideal generated by $\tau d_j + \{\alpha\}c_j$, $Ann(\{\alpha\}) \cdot d_j$ and $c_{j'}d_j + c_jd_{j'}$ for any $1 \leq j, j'$ odd $\leq n$.

Proof. By Theorem 4.3.3 we have a monomorphism of rings

$$H(BU_n(E/k)) = \bigoplus_{i_1,\dots,i_n \in \mathbb{Z}_{\geq 0}} H(N_>^{\otimes \sum_{l\, odd} i_l}) \cdot c_1^{i_1} \cdots c_n^{i_n} \to H(\mathfrak{X}_\alpha)[c_1, \dots, c_n]$$

from which we deduce that $H(BU_n(E/k))$ is generated as an H-algebra by the c_i and the μc_j for j odd. Let us denote by d_j these elements. Then, the relations among c_i and d_j that generate R follow immediately by Proposition 4.2.4 and by noticing that $\mu c_j \cdot c_{j'} = c_j \cdot \mu c_{j'}$ for any $1 \leq j, j'$ *odd* $\leq n$. This amounts to say that there is an epimorphism

$$p : \frac{H[c_i, d_j]_{1 \leq i \leq n, 1 \leq j\, odd \leq n}}{R} \to H(BU_n(E/k))$$

We can check its injectivity by looking separately at each restriction

$$p : p^{-1}(H(N_>^{\otimes \sum_{l\, odd} i_l}) \cdot c_1^{i_1} \cdots c_n^{i_n}) \to H(N_>^{\otimes \sum_{l\, odd} i_l}) \cdot c_1^{i_1} \cdots c_n^{i_n}$$

Notice that $H(N_>^{\otimes \sum_{l\, odd} i_l}) \cdot c_1^{i_1} \cdots c_n^{i_n}$ is generated as a $K^M(k)/2$-module by $\mu^m c_1^{i_1} \cdots c_n^{i_n}$ for any $0 < m \leq \sum_{l\, odd} i_l$ and $\tau^{m'} c_1^{i_1} \cdots c_n^{i_n}$ for any $m' \geq 0$. Moreover, the elements in $p^{-1}(H(N_>^{\otimes \sum_{l\, odd} i_l}) \cdot c_1^{i_1} \cdots c_n^{i_n})$ that map to these generators through p are unique. Then, injectivity follows by looking at the restriction of p on each diagonal of $p^{-1}(H(N_>^{\otimes \sum_{l\, odd} i_l}) \cdot c_1^{i_1} \cdots c_n^{i_n})$ which is an isomorphism to $\frac{K^M(k)/2}{Ann(\{\alpha\})}$ on positive diagonals and to $K^M(k)/2$ on the others. □

4.4 COMPARISON BETWEEN $BU_n(E/k)$ AND $BO(\widetilde{h}_n)$

Since there exists an obvious homomorphism of groups $U_n(E/k) \to O(\widetilde{h}_n)$, it is reasonable to compare the classifying spaces $BU_n(E/k)$ and $BO(\widetilde{h}_n)$ and, in particular, the characteristic classes arising from both.

Before proceeding, we highlight that, given a quadratic form q, there is the following isomorphism

$$O(q \perp \langle b \rangle)/O(q) \cong A_{q \perp \langle b \rangle = b}$$

where by $A_{q \perp \langle b \rangle = b}$ we mean the affine quadric defined by the equation $q \perp \langle b \rangle = b$.

For sake of simplicity, we will express by p_n the quadratic form $\langle\langle \alpha \rangle\rangle \perp (n-1)\,\mathbb{H}$ and by $p_{n-\frac{1}{2}}$ the quadratic form $\langle -\alpha \rangle \perp (n-1)\,\mathbb{H}$.

In the following theorem we compute the motivic cohomology ring of $BO(p_n)$.

Theorem 4.4.1. *For any $n \in \mathbb{Z}_{\geq 0}$ there exist cohomology classes u_i of bidegree $([i/2])[i]$ for $1 \leq i \leq 2n$ and a class v_{2n+1} of bidegree $(n)[2n+1]$ such that the motivic cohomology ring of $BO(p_n)$ is given by*

$$H(BO(p_n)) = \frac{H[u_1, \dots, u_{2n}, v_{2n+1}]}{(\tau v_{2n+1} + \{\alpha\}u_{2n}, Ann(\{\alpha\}) \cdot v_{2n+1})}$$

Proof. We start by noticing that

$$O(p_n)/O(p_{n-\frac{1}{2}}) \cong A_{p_n=1}$$

From the fact that $O(p_{n-\frac{1}{2}}) \cong O_{2n-1}$ we obtain by Theorem 2.4.2 that

$$H(BO(p_{n-\frac{1}{2}})) = H[u_1, \dots, u_{2n-1}]$$

Then, from [47, Proposition 3.2.4] it follows that

$$H(N_>^{\otimes m} \otimes BO(p_{n-\frac{1}{2}})) = H(N_>^{\otimes m}) \otimes_H H[u_1, \dots, u_{2n-1}]$$

and

$$H(\mathfrak{X}_\alpha \otimes BO(p_{n-\frac{1}{2}})) = H(\mathfrak{X}_\alpha)[u_1, \dots, u_{2n-1}]$$

Now, by recalling that $M(A_{p_n=1}) = T \oplus N_>(n)[2n-1]$ and $N_> \otimes \mathfrak{X}_\alpha \cong \mathfrak{X}_\alpha$ and using Propositions 2.2.5, 2.3.6 and 2.3.10, we obtain a long exact sequence in motivic cohomology

$$\dots \to H^{*-1,*'}(\mathfrak{X}_\alpha \otimes BO(p_{n-\frac{1}{2}})) \xrightarrow{h^*} H^{*-2n,*'-n}(\mathfrak{X}_\alpha \otimes BO(p_n)) \xrightarrow{f^*}$$

$$H^{*,*'}(\mathfrak{X}_\alpha \otimes BO(p_n)) \xrightarrow{g^*} H^{*,*'}(\mathfrak{X}_\alpha \otimes BO(p_{n-\frac{1}{2}})) \to \dots$$

Hence, by the same arguments of Theorem 4.3.3 and denoting by u_{2n} the class $f^*(1)$ we get that

$$H(\mathfrak{X}_\alpha \otimes BO(p_n)) = H(\mathfrak{X}_\alpha)[u_1, \dots, u_{2n}]$$

As in the odd case of Theorem 4.3.3, for any m we get a long exact sequence of $H(BO(p_n))$-modules

$$\dots \to H^{*-1,*'}(N_>^{\otimes m} \otimes BO(p_{n-\frac{1}{2}})) \xrightarrow{h^*} H^{*-2n,*'-n}(N_>^{\otimes m+1} \otimes BO(p_n)) \xrightarrow{f^*}$$

$$H^{*,*'}(N_>^{\otimes m} \otimes BO(p_n)) \xrightarrow{g^*} H^{*,*'}(N_>^{\otimes m} \otimes BO(p_{n-\frac{1}{2}})) \to \dots$$

Hence, by exactly the same arguments of Theorem 4.3.3 and denoting by u_{2n} the class $f^*(1)$ we obtain, for any m, a morphism of short exact sequences of $H(BO(p_n))$-modules

$$\begin{array}{ccccccccc} 0 & \to & H^{*-2n,*'-n}(N_>^{\otimes m+1}\otimes BO(p_n)) & \xrightarrow{\cdot u_{2n}} & H^{*,*'}(N_>^{\otimes m}\otimes BO(p_n)) & \to & H^{*,*'}(N_>^{\otimes m}\otimes BO(p_{n-\frac{1}{2}})) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & H^{*-2n,*'-n}(\mathfrak{X}_\alpha\otimes BO(p_n)) & \xrightarrow{\cdot u_{2n}} & H^{*,*'}(\mathfrak{X}_\alpha\otimes BO(p_n)) & \to & H^{*,*'}(\mathfrak{X}_\alpha\otimes BO(p_{n-\frac{1}{2}})) & \to & 0 \end{array}$$

From this it follows that

$$H(N_>^{\otimes m}\otimes BO(p_n)) = \bigoplus_{i\in\mathbb{Z}_{\geq 0}} H(N_>^{\otimes m+i}\otimes BO(p_{n-\frac{1}{2}}))\cdot u_{2n}^i$$

$$= \bigoplus_{i_1,\dots,i_{2n}\in\mathbb{Z}_{\geq 0}} H(N_>^{\otimes m+i_{2n}})\cdot u_1^{i_1}\cdots u_{2n}^{i_{2n}}$$

and, setting $m = 0$, we obtain

$$H(BO(p_n)) = \bigoplus_{i_1,\dots,i_{2n}\in\mathbb{Z}_{\geq 0}} H(N_>^{\otimes i_{2n}})\cdot u_1^{i_1}\cdots u_{2n}^{i_{2n}}$$

Moreover, we have a monomorphism of H-algebras

$$H(BO(p_n)) \to H(\mathfrak{X}_\alpha\otimes BO(p_n))$$

from which we deduce, as in Theorem 4.3.4, that

$$H(BO(p_n)) = \frac{H[u_1,\dots,u_{2n},v_{2n+1}]}{(\tau v_{2n+1}+\{\alpha\}u_{2n}, Ann(\{\alpha\})\cdot v_{2n+1})}$$

where v_{2n+1} is nothing else but the element that maps to μu_{2n} under the monomorphism $H(BO(p_n)) \to H(\mathfrak{X}_\alpha\otimes BO(p_n))$. □

At this point, we recall that $\mathfrak{X}_\alpha\otimes BO(p_n) \cong BO_{2n}\otimes\mathfrak{X}_\alpha$ by Proposition 2.3.14. In the following proposition we describe the isomorphism induced on motivic cohomology by this map.

Proposition 4.4.2. *The isomorphism in motivic cohomology*

$$H(\mathfrak{X}_\alpha\otimes BO(p_n)) \longleftrightarrow H(BO_{2n}\otimes\mathfrak{X}_\alpha)$$

induced by the isomorphism $\mathfrak{X}_\alpha\otimes BO(p_n)\cong BO_{2n}\otimes\mathfrak{X}_\alpha$ maps u_{2i} to u_{2i} and u_{2i-1} to $u_{2i-1}+\mu u_{2i-2}$ for any $1\leq i\leq n$.

Proof. We proceed by induction on n. For $n = 1$ we have the following commutative diagram

$$\begin{array}{ccc} H(\mathfrak{X}_\alpha\otimes BO(\langle\langle\alpha\rangle\rangle)) & \longleftrightarrow & H(BO_2\otimes\mathfrak{X}_\alpha) \\ \downarrow & & \downarrow \\ H(\mathfrak{X}_\alpha\otimes BO(\langle-\alpha\rangle)) & \longleftrightarrow & H(BO(\langle-1\rangle)\otimes\mathfrak{X}_\alpha) \end{array}$$

where the bottom horizontal isomorphism maps u_1 to $u_1 + \mu$ (that is pronounced in [47] just after Lemma 3.2.6). Then, the result follows from the fact that u_1 and u_2 are uniquely determined both in $H(\mathfrak{X}_\alpha \otimes BO(\langle\langle\alpha\rangle\rangle))$ and in $H(BO_2 \otimes \mathfrak{X}_\alpha)$ by the fact that u_1 restricts to u_1 and u_2 vanishes respectively in $H(\mathfrak{X}_\alpha \otimes BO(\langle -\alpha\rangle))$ and in $H(BO(\langle -1\rangle) \otimes \mathfrak{X}_\alpha)$.

Now, suppose the statement is true for $n-1$. Then, we have the following commutative diagram

$$\begin{array}{ccc} H(\mathfrak{X}_\alpha \otimes BO(p_n)) & \longleftrightarrow & H(BO_{2n} \otimes \mathfrak{X}_\alpha) \\ \downarrow & & \downarrow \\ H(\mathfrak{X}_\alpha \otimes BO(p_{n-\frac{1}{2}})) & \longleftrightarrow & H(BO(-q_{2n-1}) \otimes \mathfrak{X}_\alpha) \\ \downarrow & & \downarrow \\ H(\mathfrak{X}_\alpha \otimes BO(p_{n-1})) & \longleftrightarrow & H(BO_{2n-2} \otimes \mathfrak{X}_\alpha) \end{array}$$

In this case we need to understand first the homomorphism

$$H(BO(p_{n-\frac{1}{2}})) \to H(BO(p_{n-1}))$$

In order to do so, we notice that

$$O(p_{n-\frac{1}{2}})/O(p_{n-1}) \cong A_{p_{n-\frac{1}{2}}=-1} \cong A_{\langle\alpha\rangle\perp(n-1)\mathbb{H}=1}$$

From $\widetilde{M}(A_{\alpha x^2=1}) = N_<$ we deduce that

$$\widetilde{M}(A_{\langle\alpha\rangle\perp(n-1)\mathbb{H}=1}) = N_<(n-1)[2n-2]$$

Therefore, by Proposition 2.2.5 we have a long exact sequence in motivic cohomology

$$\cdots \to H^{*-1,*'}(BO(p_{n-1})) \xrightarrow{h^*} H^{*-2n+1,*'-n+1}(N_< \otimes BO(p_{n-\frac{1}{2}})) \xrightarrow{f^*}$$

$$H^{*,*'}(BO(p_{n-\frac{1}{2}})) \xrightarrow{g^*} H^{*,*'}(BO(p_{n-1})) \to \dots$$

At this point, notice that $H(N_< \otimes BO(p_{n-\frac{1}{2}})) = H(N_<) \otimes_H H[u_1, \dots, u_{2n-1}]$ which implies that u_i and v_{2n-1} are all uniquely liftable to $H(BO(p_{n-\frac{1}{2}}))$ by degree reasons, since $H^{*,*'}(N_< \otimes BO(p_{n-\frac{1}{2}}))$ is 0 for $*' < 0$ and for $(*')[*] = (0)[0]$ and $(0)[1]$, and by Lemma 4.2.6. Hence, g^* is an epimorphism since it is a ring homomorphism. Moreover, $g^*(u_i) = u_i$ for $i \leq 2n-2$ since the natural restriction $H(BO(p_{n-\frac{1}{2}})) \to H(BO(p_{n-\frac{3}{2}}))$ factors through $H(BO(p_{n-1}))$ and the classes u_i are uniquely determined, both in $H(BO(p_{n-\frac{1}{2}}))$ and in $H(BO(p_{n-1}))$, by the fact that they restrict to the respective u_i or vanish for $i = 2n-2$ in $H(BO(p_{n-\frac{3}{2}}))$. For the same reason, since v_{2n-1} vanishes in $H(BO(p_{n-\frac{3}{2}}))$, the element that covers v_{2n-1} through g^* has the shape $u_{2n-1}+\epsilon u_1 u_{2n-2}$, where ϵ is 0 or 1. Suppose $\epsilon = 1$, then by Wu formula we have that $Sq^1(u_{2n-1}+u_1u_{2n-2}) = Sq^1Sq^1u_{2n-2} = 0$, so $Sq^1 v_{2n-1} = 0$ as well. But, $Sq^1 v_{2n-1}$ maps to $Sq^1(\mu u_{2n-2})$ in $H(\mathfrak{X}_\alpha \otimes BO(p_{n-1}))$, which again maps to $Sq^1(\mu u_{2n-2}) = \mu^2 u_{2n-2} + \mu u_1 u_{2n-2} \neq 0$ in $H(BO_{2n-2} \otimes \mathfrak{X}_\alpha) = H(\mathfrak{X}_\alpha)[u_1, \ldots, u_{2n-2}]$ by induction hypothesis, and we get a contradiction. Hence, ϵ must be 0 and $g^*(u_{2n-1}) = v_{2n-1}$.

Therefore, we have that the isomorphism

$$H(\mathfrak{X}_\alpha \otimes BO(p_{n-\frac{1}{2}})) \leftrightarrow H(BO(-q_{2n-1}) \otimes \mathfrak{X}_\alpha)$$

maps u_{2i} to u_{2i} and u_{2i-1} to $u_{2i-1} + \mu u_{2i-2}$ for any $1 \leq i \leq 2n-2$. Moreover, since $H(\mathfrak{X}_\alpha \otimes BO(p_{n-\frac{1}{2}})) \to H(\mathfrak{X}_\alpha \otimes BO(p_{n-1}))$ maps $u_{2n-1} + \mu u_{2n-2}$ to 0, we have that $H(\mathfrak{X}_\alpha \otimes BO(p_{n-\frac{1}{2}})) \leftrightarrow H(BO(-q_{2n-1}) \otimes \mathfrak{X}_\alpha)$ maps u_{2n-1} to $u_{2n-1} + \mu u_{2n-2}$.

Now, the result follows from the fact that the u_i are uniquely determined both in $H(\mathfrak{X}_\alpha \otimes BO(p_n))$ and in $H(BO_{2n} \otimes \mathfrak{X}_\alpha)$ by the fact that they restrict to u_i for $i \leq 2n-1$ and vanish for $i = 2n$ respectively in $H(\mathfrak{X}_\alpha \otimes BO(p_{n-\frac{1}{2}}))$ and in $H(BO(-q_{2n-1}) \otimes \mathfrak{X}_\alpha)$. □

From Theorem 4.4.1 we get immediately the following result which provides the motivic cohomology ring of $BO(\widetilde{h}_n)$.

Theorem 4.4.3. *For any $n \in \mathbb{Z}_{\geq 0}$ there exist cohomology classes u_i of bidegree $([i/2])[i]$ for $1 \leq i \leq 2n$ and a class v_{2n+1} of bidegree $(n)[2n+1]$ only for n odd such that the motivic cohomology ring of $BO(\widetilde{h}_n)$ is given by*

$$H(BO(\widetilde{h}_n)) = \begin{cases} \frac{H[u_1, \ldots, u_{2n}, v_{2n+1}]}{(\tau v_{2n+1} + \{\alpha\} u_{2n}, Ann(\{\alpha\}) \cdot v_{2n+1})}, & n \text{ odd} \\ H[u_1, \ldots, u_{2n}], & n \text{ even} \end{cases}$$

Proof. It follows from the fact that $\widetilde{h}_n$ is split for n even and is isomorphic to p_n for n odd. □

Once we know both the motivic cohomology of $BO(\widetilde{h}_n)$ and $BU_n(E/k)$, we can relate the subtle classes arising from the orthogonal group and those arising from the unitary group. In particular, we have the following result.

Proposition 4.4.4. *For any $n \in \mathbb{Z}_{\geq 0}$ the natural embedding $U_n(E/k) \hookrightarrow O(\widetilde{h}_n)$ induces an epimorphism*

$$H(BO(\widetilde{h}_n)) \to H(BU_n(E/k))$$

sending u_{2i} to c_i for any $1 \leq i \leq n$, u_{2l+1} to 0 for any $0 \leq l$ even $< n$, u_{2j+1} to d_j for any $1 \leq j$ odd $< n$ and v_{2n+1} to d_n only for n odd.

Proof. We will proceed by induction. Notice that the induction basis is provided by the isomorphism $U_0 \cong O_0 \cong Spec(k)$.

For n odd we have the following commutative diagrams

$$\begin{array}{ccc} U_{n-1}(E/k) & \longrightarrow & U_n(E/k) \\ \downarrow & & \\ O_{2n-2} & & \downarrow \\ \downarrow & & \\ O(\widetilde{h}_{n-\frac{1}{2}}) & \longrightarrow & O(\widetilde{h}_n) \end{array} \qquad \begin{array}{ccc} H(BU_{n-1}(E/k)) & \longleftarrow & H(BU_n(E/k)) \\ \uparrow & & \\ H(BO_{2n-2}) & & \uparrow \\ \uparrow & & \\ H(BO(\widetilde{h}_{n-\frac{1}{2}})) & \longleftarrow & H(BO(\widetilde{h}_n)) \end{array}$$

where by $\widetilde{h}_{n-\frac{1}{2}}$ here we mean the quadratic form $\langle -\alpha \rangle \perp (n-1)\,\mathbb{H}$.

By induction hypothesis we have that u_{2i} goes to c_i for any $1 \leq i \leq n-1$, u_{2l+1} to 0 for any $0 \leq l\, even < n-1$ and u_{2j+1} to d_j for any $1 \leq j\, odd < n-1$. The class u_{2n-1} goes to 0 via the map $H(BO(\widetilde{h}_n)) \to H(BU_{n-1}(E/k))$ since this factors through $H(BO_{2n-2})$. Hence, u_{2n-1} maps to 0 in $H(BU_n(E/k))$ since the morphism $H(BU_n(E/k)) \to H(BU_{n-1}(E/k))$ is injective in bidegree $(n-1)[2n-1]$. Moreover, noticing that

$$U_n(E/k)/U_{n-1}(E/k) \cong O(\widetilde{h}_n)/O(\widetilde{h}_{n-\frac{1}{2}})$$

and by Proposition 2.2.7, we obtain that u_{2n} goes to c_n and v_{2n+1} goes to d_n.

For n even we have similar commutative diagrams

$$\begin{array}{ccc} U_{n-1}(E/k) & \longrightarrow & U_n(E/k) \\ \downarrow & & \\ O(\widetilde{h}_{n-1}) & & \downarrow \\ \downarrow & & \\ O_{2n-1} & \longrightarrow & O_{2n} \end{array} \qquad \begin{array}{ccc} H(BU_{n-1}(E/k)) & \longleftarrow & H(BU_n(E/k)) \\ \uparrow & & \\ H(BO(\widetilde{h}_{n-1})) & & \uparrow \\ \uparrow & & \\ H(BO_{2n-1}) & \longleftarrow & H(BO_{2n}) \end{array}$$

In this case we need to study the homomorphism

$$H(BO_{2n-1}) \to H(BO(\widetilde{h}_{n-1}))$$

In order to do so, we notice that

$$O_{2n-1}/O(\widetilde{h}_{n-1}) \cong A_{\widetilde{h}_{n-1}\perp\langle\alpha\rangle=\alpha} \cong A_{\alpha^{-1}\widetilde{h}_{n-1}\perp\langle 1\rangle=1}$$

From $\widetilde{M}(A_{x^2=\alpha}) = N_<$ and since $\alpha^{-1}\widetilde{h}_{n-1} \perp \langle 1 \rangle$ is isomorphic to $\langle \alpha^{-1} \rangle \perp (n-1)\,\mathbb{H}$, we deduce that

$$\widetilde{M}(A_{\alpha^{-1}\widetilde{h}_{n-1}\perp\langle 1\rangle=1}) = N_<(n-1)[2n-2]$$

Hence, by Proposition 2.2.5 we have a long exact sequence in motivic cohomology

$$\cdots \to H^{*-1,*'}(BO(\widetilde{h}_{n-1})) \xrightarrow{h^*} H^{*-2n+1,*'-n+1}(N_< \otimes BO_{2n-1}) \xrightarrow{f^*}$$

$$H^{*,*'}(BO_{2n-1}) \xrightarrow{g^*} H^{*,*'}(BO(\widetilde{h}_{n-1})) \to \dots$$

Then, by repeating exactly the same arguments that appear in Proposition 4.4.2 we get that $g^*(u_i) = u_i$ for $i \leq 2n-2$ and $g^*(u_{2n-1}) = v_{2n-1}$.

Therefore, by induction hypothesis we have that u_{2i} goes to c_i for any $1 \leq i \leq n-1$, u_{2l+1} to 0 for any $0 \leq l\ even \leq n-1$ and u_{2j+1} to d_j for any $1 \leq j\ odd \leq n-1$. Moreover, recalling that

$$U_n(E/k)/U_{n-1}(E/k) \cong O_{2n}/O_{2n-1}$$

and by Proposition 2.2.7, we obtain that u_{2n} goes to c_n, as we aimed to show. □

As a corollary of the previous proposition and of Theorem 4.3.4 we get a description of $H(BU_n(E/k))$ as a quotient of $H(BO(\widetilde{h}_n))$.

Corollary 4.4.5. *For any $n \in \mathbb{Z}_{\geq 0}$ there is an isomorphism*

$$H(BU_n(E/k)) \cong \frac{H(BO(\widetilde{h}_n))}{R}$$

where R is the ideal in $H(BO(\widetilde{h}_n))$ generated by u_{4j+1}, $u_{4i+3}u_{4j+2} + u_{4j+3}u_{4i+2}$, $\tau u_{4j+3} + \{\alpha\}u_{4j+2}$ and $Ann(\{\alpha\}) \cdot u_{4j+3}$ for any $0 \leq i, j \leq [\frac{n-1}{2}]$, where u_{2n+1} is substituted by v_{2n+1} for n odd.

4.5 APPLICATIONS TO HERMITIAN FORMS

Throughout this section we exploit previous results to study the kernel invariant of quadratic forms divisible by $\langle\langle\alpha\rangle\rangle$. The general idea is that $H(BU_n(E/k))$ is closer to the cohomology of the Čech simplicial scheme of a quadratic form associated to a hermitian form than $H(BO_{2n})$.

We start by noticing that for every hermitian form h of the quadratic extension E/k there exists a commutative diagram

$$\begin{array}{ccccc} \check{C}(X_h) & \longrightarrow & BU_n(E/k) & \longrightarrow & BO(\widetilde{h}_n) \\ \downarrow & & \downarrow & & \downarrow \\ Spec(k) & \xrightarrow{h} & B_{\acute{e}t}U_n(E/k) & \longrightarrow & B_{\acute{e}t}O(\widetilde{h}_n) \end{array}$$

where $\check{C}(X_h)$ is the Čech simplicial scheme of the torsor $X_h = Iso(h \leftrightarrow h_n)$. Hence, the computation of the motivic cohomology of $BU_n(E/k)$ provides us with subtle characteristic classes for hermitian forms and relations among them. More precisely, we have the following proposition.

Proposition 4.5.1. *For any n-dimensional hermitian form h, in $H(\check{C}(X_h))$ the following relations hold for any $1 \leq j, j'$ odd $\leq n$:*
1) $c_{j'}(h)d_j(h) + c_j(h)d_{j'}(h) = 0$;
2) $\tau d_j(h) + \{\alpha\}c_j(h) = 0$;
3) $Ann(\{\alpha\}) \cdot d_j(h) = 0$.

Proof. It follows immediately from Theorem 4.3.4. □

We now move to consider quadratic forms associated to hermitian ones and their subtle Stiefel-Whitney classes.

Recall that two hermitian forms are isomorphic if and only if the corresponding quadratic forms over k are isomorphic. In particular, for even dimensional hermitian forms we have that they split if and only if the respective quadratic forms split. It follows that $\check{C}(X_h) \cong \check{C}(X_{\widetilde{h}})$, for even dimensional hermitian forms.

Proposition 4.5.2. *For n even, in $H(\check{C}(X_{\widetilde{h}}))$ the following relations hold for any $0 \leq i, j \leq \frac{n}{2} - 1$:*
1) $u_{4j+1}(\widetilde{h}) = 0$;
2) $u_{4i+3}(\widetilde{h})u_{4j+2}(\widetilde{h}) = u_{4j+3}(\widetilde{h})u_{4i+2}(\widetilde{h})$;
3) $\tau u_{4j+3}(\widetilde{h}) = \{\alpha\}u_{4j+2}(\widetilde{h})$;
4) $Ann(\{\alpha\}) \cdot u_{4j+3}(\widetilde{h}) = 0$.

Proof. It follows immediately from Corollary 4.4.5. □

On the other hand, if q is an odd dimensional quadratic form, then $\langle\langle\alpha\rangle\rangle \otimes q$ is split over a field extension of k if and only if $\langle\langle\alpha\rangle\rangle$ is split over the same field extension. It follows from this remark that, for odd dimensional hermitian forms, $\check{C}(X_{\widetilde{h}}) \cong \check{C}(X_\alpha)$, where $\check{C}(X_\alpha)$ stands for the Čech simplicial scheme associated to the Pfister form $\langle\langle\alpha\rangle\rangle$.

Proposition 4.5.3. *For n odd, in $H(\check{C}(X_{\widetilde{h}})) = H(\check{C}(X_\alpha))$ the following relations hold for any $0 \leq j \leq \frac{n-1}{2}$:*
1) $u_{4j+1}(\widetilde{h}) = \mu u_{4j}(\widetilde{h})$;
2) $u_{4j-1}(\widetilde{h}) = 0$.

Proof. Together with the commutative diagram at the beginning of this section, we have the following one

$$\begin{array}{ccc} \check{C}(X_\alpha) & \longrightarrow & BO_{2n} \\ \downarrow & & \downarrow \\ Spec(k) & \xrightarrow{\widetilde{h}} & B_{\acute{e}t}O_{2n} \end{array}$$

By Proposition 2.3.14, we know that after tensoring both with $\check{C}(X_\alpha)$ they coincide. Therefore, our restriction morphism $H(BO_{2n}) \to H(\check{C}(X_\alpha))$ factors as

$$H(BO_{2n}) \to H(BO_{2n} \times \check{C}(X_\alpha)) \leftrightarrow H(\check{C}(X_\alpha) \times BO(\widetilde{h}_n)) \to$$
$$H(\check{C}(X_\alpha) \times BU_n(E/k)) \to H(\check{C}(X_\alpha))$$

which implies the result by Theorem 4.3.3, Proposition 4.4.2 and Proposition 4.4.4. □

We now show that the subtle classes arising in the unitary case see the triviality of the torsor of a hermitian form in the same way subtle Stiefel-Whitney classes do for quadratic forms (see Corollary 2.4.8).

Proposition 4.5.4. *$h \cong h_n$ if and only if $c_{2^r}(h) = 0$ for any r.*

Proof. Let us start from the case n even. Then, we have already noticed that h splits if and only if $\widetilde{h}$ splits. By Corollary 2.4.8, this is equivalent to say that $u_{2^{r+1}}(\widetilde{h})$ vanishes in $H(\check{C}(X_{\widetilde{h}}))$ for any r, which is the same of vanishing of $c_{2^r}(h)$ in $H(\check{C}(X_h))$, since in this case $\check{C}(X_h) \cong \check{C}(X_{\widetilde{h}})$ and by Proposition 4.4.4.

For n odd, we have that h splits if and only if $h \perp \langle -1 \rangle$ (which is even dimensional) splits. This amounts to say that $c_{2^r}(h \perp \langle -1 \rangle) = 0$ in $H(\check{C}(X_{h \perp \langle -1 \rangle}))$ for any r, which is equivalent to say that $c_{2^r}(h) = 0$ in $H(\check{C}(X_h))$ for any r. □

We conclude by presenting an expression of the motive of the torsor associated to a hermitian form. Indeed, by the very same arguments of Proposition 2.4.5 and Theorem 2.4.6 one obtains the description of the motive of the torsor X_h in terms of motives of Čech simplicial schemes and subtle characteristic classes, where h is any hermitian form.

Before stating the results, let us denote by $\widetilde{c}_j$ a morphism $T \to N_>(j)[2j]$ in $\mathcal{DM}^-_{eff}(BU_n(E/k))$ which composed with the only non-zero morphism $N_>(j)[2j] \to T(j)[2j]$ gives c_j for any j odd. It is actually the unique cohomology class in $H(N_< \otimes BU_n(E/k))$ that maps to c_j under the homomorphism induced by the only non-zero morphism $T \to N_<$. Then, we have the following two propositions.

Proposition 4.5.5. *In $\mathcal{DM}^-_{eff}(BU_n(E/k), \mathbb{Z}/2)$ we have that*

$$M(EU_n(E/k) \to BU_n(E/k)) = \bigotimes_{1 \leq i \text{ even} \leq n} Cone[-1](T \xrightarrow{c_i} T(i)[2i]) \otimes$$

$$\bigotimes_{1 \leq j \text{ odd} \leq n} Cone[-1](T \xrightarrow{\widetilde{c}_j} N_>(j)[2j])$$

Proposition 4.5.6. *In $\mathcal{DM}^{-}_{eff}(k, \mathbb{Z}/2)$ we have that*

$$M(X_h) = \bigotimes_{1 \leq i \, even \leq n} Cone[-1](\mathfrak{X}_h \xrightarrow{c_i(h)} \mathfrak{X}_h(i)[2i]) \otimes$$

$$\bigotimes_{1 \leq j \, odd \leq n} Cone[-1](\mathfrak{X}_h \xrightarrow{\widetilde{c}_j(h)} N_{>} \otimes \mathfrak{X}_h(j)[2j])$$

5

A SPECTRAL SEQUENCE FOR MORPHISMS WITH MOTIVICALLY CELLULAR FIBER

In this chapter, we want to generalise the Gysin long exact sequence for fibrations with motivically Tate reduced fiber to a spectral sequence for fibrations with motivically cellular fiber, i.e. with fiber whose motive is an extension of Tate motives.

First, we will recall basic definitions about Postnikov systems in triangulated categories, and how they naturally generate spectral sequences. Moreover, we will report a few facts about convergence issues of a spectral sequence (see [7]).

Then, we will move to the generalisation of Propositions 2.2.3 and 2.2.5. Indeed, this will provide the aimed spectral sequence for coherent morphisms with motivically cellular fibers. In some sense, this spectral sequence is analogous to the *Serre spectral sequence* associated to a fibration in topology. The main reason is that, as in topology, our spectral sequence reconstructs the cohomology of the total simplicial scheme from the cohomology of the base simplicial scheme and the fiber. However, the filtration inducing the spectral sequence in the motivic case is on the fiber instead of the base.

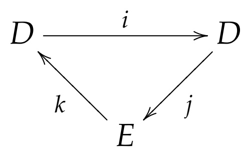
As an example, the spectral sequence investigated in this chapter could be used, in principle, to study PGL_n-torsors, which over the point correspond to *Severi-Brauer varieties*. Indeed, by exploiting a fiber sequence involving PGL_n and GL_n one could try to get information about the motivic cohomology of the Nisnevich classifying space $BPGL_n$, which would in turn provide some subtle classes for Severi-Brauer varieties.

5.1 SOME GENERAL FACTS ABOUT SPECTRAL SEQUENCES

Let us start in this section by recalling some well known facts about Postnikov systems in triangulated categories, spectral sequences associated to them and convergence issues.

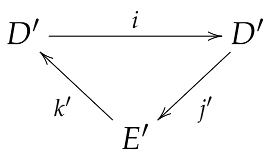
Throughout this section, $\mathcal{C}$ will indicate a triangulated category, $\mathcal{A}$ an abelian category and $H : \mathcal{C} \to \mathcal{A}$ a cohomological functor.

Definition 5.1.1. *An unrolled exact couple in $\mathcal{A}$ is a triangle*



which is exact at each vertex.

First, notice that the morphism d defined as the composition jk is a differential, i.e. $d^2 = 0$. Set $E' = Ker(d)/Im(d)$ and $D' = Im(i)$. By some standard arguments there exist morphisms $j' : D' \to E'$ and $k' : E' \to D'$, defined respectively by $j'(i(x)) = j(x)$ and $k'([y]) = k(y)$ for any $x \in D$ and $y \in Ker(d)$, such that the following triangle



is again an exact couple, called the derived couple. Reiterating this construction one gets a sequence of objects E_r endowed with differentials d_r each of which is the homology of the previous one. More precisely, we can give the following definition.

Definition 5.1.2. *The sequence $\{(E_r, d_r)\}$ constructed inductively by*

$$E_r = Ker(d_{r-1})/Im(d_{r-1})$$

is called the spectral sequence associated to the derived couple.

In practical situations, one often encounters bigraded exact couples, which naturally give rise to bigraded spectral sequences $\{(E_r^{s,t}, d_r^{s,t})\}$. In the next definition we remind what is the limit page of a bigraded spectral sequence.

Definition 5.1.3. *Let $\{(E_r^{s,t}, d_r^{s,t})\}$ be a bigraded spectral sequence and suppose for any couple (s,t) there is an integer $r(s,t)$ such that $E_r^{s,t} \cong E_{r(s,t)}^{s,t}$ for any $r \geq r(s,t)$, then we say that the spectral sequence abuts to $E_\infty^{s,t} = E_{r(s,t)}^{s,t}$.*

We resume now some notions about filtrations taken from [7] which are advantageous to deal with convergence of spectral sequences.

An increasing filtration of an object G in $\mathcal{A}$ is a diagram of the following shape

$$\cdots \hookrightarrow F^1 \hookrightarrow F^2 \hookrightarrow \cdots \hookrightarrow F^m \hookrightarrow F^{m+1} \hookrightarrow \cdots \hookrightarrow G$$

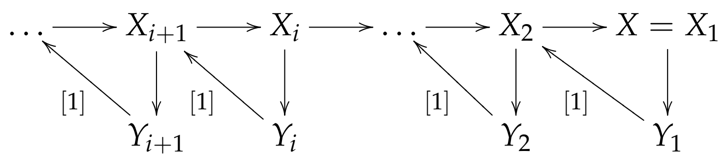
Definition 5.1.4. *The increasing filtration $\{F^m\}_{m\in\mathbb{Z}}$ of G is said to be:*
1) exhaustive if $G = \varinjlim_m F^m$;
2) Hausdorff if $\varprojlim_m F^m = 0$;
3) complete $\varprojlim^1_m F^m = 0$.

In practice, one often studies filtrations of graded objects. At this point, we are ready to mention the types of convergence for spectral sequences, which are in turn related to the different types of filtrations of the target object.

Definition 5.1.5. *A spectral sequence associated to an exact couple is called:*
1) weakly convergent to G if there exists an increasing filtration of G which is exhaustive and such that $E_\infty^{s,t} \cong F^{s,t}/F^{s-1,t}$ for any s, where s is the parameter of the filtration;
2) convergent to G if it is weakly convergent and the filtration of G is Hausdorff;
3) strongly convergent to G if it is weakly convergent and the filtration of G is complete Hausdorff.

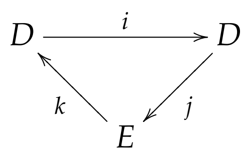
We now recall what Postnikov systems in triangulated categories are. Indeed, there are left and right Postnikov systems, but since we will use only one type, we will not put any adjective and simply call Postnikov system the type we need.

Definition 5.1.6. *A Postnikov system for an object X in $\mathcal{C}$ is a diagram*



where all the triangles are distinguished triangles in $\mathcal{C}$.

Associated to a Postnikov system in a triangulated category one can always construct an exact couple by applying a cohomological functor H. More precisely, we have the following unrolled bigraded exact couple



where $D^{s,t} = H(X_s[-t])$, $E^{s,t} = H(Y_s[-t])$ and the morphisms $i : D^{s,t} \to D^{s+1,t}$, $j : D^{s,t} \to E^{s-1,t+1}$ and $k : E^{s,t} \to D^{s,t}$ are induced by the morphisms in the Postnikov system.

Notice that we can obtain an increasing filtration of the object $H(X)$ in $\mathcal{A}$ given by

$$F^1 \hookrightarrow F^2 \hookrightarrow \cdots \hookrightarrow F^m \hookrightarrow F^{m+1} \hookrightarrow \cdots \hookrightarrow H(X)$$

where $F^m = Ker(H(X) \to H(X_m))$ and the morphism $H(X) \to H(X_m)$ is the one induced by the Postnikov system. Moreover, observe that the filtration $\{F^m\}$ just introduced is obviously complete Hausdorff, since it is bounded from below, but not necessarily exhaustive. Anyway, we have the following result which guarantees the strong convergence of the spectral sequence provided a certain condition is met.

Theorem 5.1.7. *If $\varinjlim_m H(X_m) = 0$, then the spectral sequence associated to the above Postnikov system is strongly convergent to $H(X)$.*

Proof. See [7, Theorem 6.1]. □

5.2 A SERRE SPECTRAL SEQUENCE FOR MOTIVIC COHOMOLOGY

We begin this section by recalling a result from [47] about coherent motives that we will use later on.

Proposition 5.2.1. *For any motive N in $\mathcal{DM}^-_{coh}(Y_\bullet, R)$ there exists a functorial increasing filtration*

$$(N)_{\leq 0} \to (N)_{\leq 1} \to \cdots \to (N)_{\leq n-1} \to (N)_{\leq n} \to \cdots \to N$$

with graded pieces $(N)_n = Cone((N)_{\leq n-1} \to (N)_{\leq n}) \cong Lr_{n,\#}r_n^*(N)[n]$ which converges in the sense that

$$\bigoplus_n (N)_{\leq n} \xrightarrow{id-sh} \bigoplus_n (N)_{\leq n} \to N$$

extends to a distinguished triangle, where $sh : (N)_{\leq n-1} \to (N)_{\leq n}$ is the map from the filtration.

Proof. See [47, Proposition 3.1.8]. □

The next proposition is a generalisation of Proposition 2.2.3. Indeed, it allows to construct Postnikov systems for coherent motives with simplicial components which are direct sums of Tate motives satisfying some specific conditions. The proof is essentially the same of Proposition 2.2.3 with some minor adjustments, but we will report it for completeness. Before proceeding, we need to define an order on the bidegrees $(q)[p]$.

Definition 5.2.2. *We set $(q)[p] \prec (q')[p']$ if and only if one of the following two conditions is satisfied:*
1) $q < q'$;
2) $q = q'$ and $p < p'$.
This is clearly a strict order relation.

For any $j \geq 1$, let T_j be the possibly infinite direct sum $\bigoplus_{I_j} T(q_j)[p_j]$ in $\mathcal{DM}^-_{eff}(k,R)$ such that $(q_j)[p_j] \prec (q_{j+1})[p_{j+1}]$ and let $N \in \mathcal{DM}^-_{coh}(Y_\bullet, R)$ be such a motive that its simplicial components $N_i \in \mathcal{DM}^-_{eff}(Y_i, R)$ are isomorphic to the direct sum $\bigoplus_{j\geq 1} T_j$.

Note that in $\mathcal{DM}^-_{eff}(Y_1, R)$ the automorphism group $Aut(\bigoplus_{j\geq 1} T_j)$ consists of invertible upper triangular matrices, since

$$Hom_{\mathcal{DM}^-_{eff}(Y_1,R)}(T, T(q)[p]) = H^{p,q}(Y_1, R) = 0$$

for any $(q)[p] \prec (0)[0]$.

Recall that, since N is coherent, for any simplicial map $\theta : [i] \to [j]$, the structural map $N_\theta : LY^*_\theta(N_i) \to N_j$ is an isomorphism. Then, for any connected component of Y_1, fix the automorphism of $\bigoplus_{j\geq 1} T_j$ given by

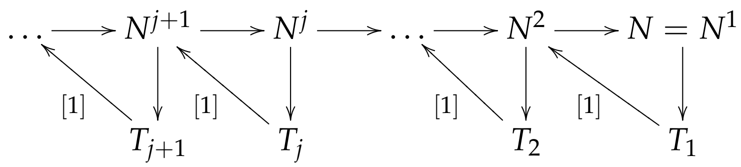
$$N_{\partial_0} : LY^*_{\partial_0}(N_0) \cong \bigoplus_{j\geq 1} T_j \to N_1 \cong \bigoplus_{j\geq 1} T_j$$

to be the identity 1. This way, to each connected component of Y_1 one can associate the automorphism of $\bigoplus_{j\geq 1} T_j$ induced by the other face map, namely

$$N_{\partial_1} : LY^*_{\partial_1}(N_0) \cong \bigoplus_{j\geq 1} T_j \to N_1 \cong \bigoplus_{j\geq 1} T_j$$

Hence, we have produced a morphism of groupoids ω^N from $\pi_1(CC(Y_\bullet))$ to $Aut(\bigoplus_{j\geq 1} T_j)$. Now, for any $j \geq 1$, we can compose ω^N with the morphism $Aut(\bigoplus_{j\geq 1} T_j) \to Aut(T_j)$ that sends each invertible upper triangular matrix to its j-th diagonal entry. We will denote these morphisms by ω^N_j.

Proposition 5.2.3. *Using the notations just introduced, if ω^N_j is trivial for any $j \geq 1$, then there exists a Postnikov system in $\mathcal{DM}^-_{eff}(Y_\bullet, R)$*



such that the simplicial components N^j_i are isomorphic to the direct sum $\bigoplus_{k\geq j} T_k$ and the morphisms $r^*_i(N^j \to T_j)$ are the natural projections $\bigoplus_{k\geq j} T_k \to T_j$ in $\mathcal{DM}^-_{eff}(Y_i, R)$.

Proof. To construct the aimed Postnikov system we just need to produce morphisms $N^j \to T_j$ where each N^j is defined as the cone of the previous morphism, namely $N^j = Cone(N^{j-1} \to T_{j-1})[-1]$. We will do so by induction.

Notice that each simplicial component of N is isomorphic to $\bigoplus_{j\geq 1} T_j$ and T_1 is the direct sum of possibly infinite $T(q_1)[p_1]$ such that $(q_1)[p_1] \prec (q_j)[p_j]$ for any $j \geq 2$ by hypothesis. By applying the triangulated functor $Lc_\#$ introduced at the end of Section 2.1 to the filtration of the previous proposition one gets a filtration $(Lc_\# N)_{\leq n}$ for $Lc_\# N$ with graded pieces $(Lc_\# N)_n \cong \bigoplus_{j\geq 1} \bigoplus_{I_j} M(Y_n)(q_j)[p_j + n]$. Following the lines of the proof of Proposition 2.2.3, we will denote by $(Lc_\# N)_{>n}$ the cone $Cone((Lc_\# N)_{\leq n} \to (Lc_\# N)_n)$ and by $(Lc_\# N)_{m\geq *>n}$ the cone $Cone((Lc_\# N)_{\leq n} \to (Lc_\# N)_{\leq m})$ for any $m > n$. Now, note that

$$(Lc_\# N)_{>n} \cong Cone(\bigoplus_{m>n} (Lc_\# N)_{m\geq *>n} \xrightarrow{id-sh} (Lc_\# N)_{m\geq *>n})$$

and moreover $(Lc_\# N)_{m\geq *>n}$ is an extension of $(Lc_\# N)_k$ for $n < k \leq m$. Therefore, we have that

$$Hom_{\mathcal{DM}^-_{eff}(k,R)}((Lc_\# N)_{>0}, T_1) = 0$$

$$Hom_{\mathcal{DM}^-_{eff}(k,R)}((Lc_\# N)_{>1}, T_1) = 0$$

$$Hom_{\mathcal{DM}^-_{eff}(k,R)}((Lc_\# N)_{>1}, T_1[1]) = 0$$

since $Hom_{\mathcal{DM}^-_{eff}(k,R)}(M(Y_n), T(q)[p]) = 0$ for any $n \geq 0$ and any $(q)[p] \prec (0)[0]$. We deduce from these remarks and by applying the cohomological functor $Hom_{\mathcal{DM}^-_{eff}(k,R)}(-, T_1)$ to the distinguished triangle

$$(Lc_\# N)_0 \to (Lc_\# N) \to (Lc_\# N)_{>0} \to (Lc_\# N)_0[1]$$

that there exists an exact sequence

$$0 \to Hom_{\mathcal{DM}^-_{eff}(k,R)}(Lc_\#(N), T_1) \to Hom_{\mathcal{DM}^-_{eff}(k,R)}(Lc_{\#,0}(N_0), T_1)$$
$$\to Hom_{\mathcal{DM}^-_{eff}(k,R)}(Lc_{\#,1}(N_1), T_1)$$

Repeating the same arguments for T_1 in $\mathcal{DM}^-_{coh}(Y_\bullet, R)$ one gets a similar sequence

$$0 \to Hom_{\mathcal{DM}^-_{eff}(k,R)}(Lc_\#(T_1), T_1) \to Hom_{\mathcal{DM}^-_{eff}(k,R)}(Lc_{\#,0}(T_1), T_1)$$
$$\to Hom_{\mathcal{DM}^-_{eff}(k,R)}(Lc_{\#,1}(T_1), T_1)$$

In order to produce an isomorphism between $Hom_{\mathcal{DM}^-_{eff}(k,R)}(Lc_\#(N), T_1)$ and $Hom_{\mathcal{DM}^-_{eff}(k,R)}(Lc_\#(T_1), T_1)$ we need to identify the last morphisms of the two exact sequences. Since in the exact sequences only the 0th and the 1st simplicial components appear, it is enough to get a compatibility between the coherent system (N_i, N_θ) and the one of T_1 for $i = 0, 1$ and simplicial maps $\theta : [0] \to [1]$, where N_θ is the structural isomorphism $LY_\theta^*(N_0) \to N_1$. In other words, we want a commutative diagram

$$\begin{array}{ccc} LY^*_{\partial_1}(N_0) \cong \bigoplus_{j\geq 1} T_j & \xrightarrow{N_{\partial_1}} & N_1 \cong \bigoplus_{j\geq 1} T_j \\ \downarrow & & \downarrow \\ LY^*_{\partial_1}(T_1) \cong T_1 & \xrightarrow[id]{} & T_1 \end{array}$$

having fixed the automorphism of $Aut(\bigoplus_{j\geq 1} T_j)$ given by N_{∂_0} to be the identity 1 for any connected component of Y_1. Indeed, we have such a commutative diagram since by hypothesis ω_1^N is trivial. Hence, the two exact sequences above coincide. Then, we have that

$$Hom_{\mathcal{DM}^-_{eff}(Y_\bullet,R)}(N, T_1) \cong Hom_{\mathcal{DM}^-_{eff}(k,R)}(Lc_\#(N), T_1) \cong$$

$$Hom_{\mathcal{DM}^-_{eff}(k,R)}(Lc_\#(T_1), T_1) \cong Hom_{\mathcal{DM}^-_{eff}(Y_\bullet,R)}(T_1, T_1)$$

by adjunctions and the identity of T_1 provides the pursued morphism $N \to T_1$ whose restriction on each simplicial component is given by the natural projection $\bigoplus_{k\geq 1} T_k \to T_1$. It follows that N_i^2 is isomorphic to $\bigoplus_{k\geq 2} T_k$ for any i. This proves the induction basis.

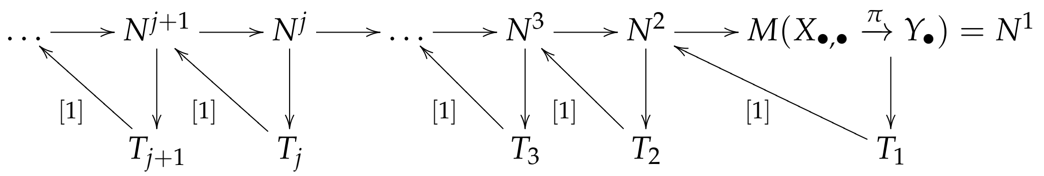
Now, suppose we have a morphism from N^k to T_k for any $1 \leq k \leq j-1$, where each N^k is defined as $Cone(N^{k-1} \to T_{k-1})[-1]$. We denote by N^j the cone $Cone(N^{j-1} \to T_{j-1})[-1]$. Notice that the simplicial components of N^j are all isomorphic to $\bigoplus_{l\geq j} T_l$ and T_j is the direct sum of possibly infinite $T(q_j)[p_j]$ such that $(q_j)[p_j] \prec (q_l)[p_l]$ for any $l \geq j+1$ by hypothesis. Therefore, by applying the same arguments of the induction basis, using the fact that $\omega_1^{N^j} = \omega_j^N$ is trivial by hypothesis, there exists a morphism $N^j \to T_j$. This completes the proof. □

We just want to point out that the proof of the previous proposition works in the same way if each simplicial component of N is a finite direct sum of Tates. In this case, we would obtain a finite Postnikov system for N in $\mathcal{DM}^-_{eff}(Y_\bullet, R)$.

We next apply the previous proposition to produce a spectral sequence for fibrations which have motivically cellular fibers, i.e. fibers whose motives are sum of Tates satisfying certain conditions.

Proposition 5.2.4. *Let $\pi : X_{\bullet,\bullet} \to Y_\bullet$ be a smooth coherent morphism from a smooth bisimplicial scheme to a smooth simplicial scheme over k, $A_\bullet$ a smooth simplicial k-scheme and, for any $j \geq 1$, T_j the possibly infinite direct sum of Tate*

motives $\bigoplus_{I_j} T(q_j)[p_j]$ *in* $\mathcal{DM}^-_{eff}(k,R)$ *such that* $(q_j)[p_j] \prec (q_{j+1})[p_{j+1}]$*. Moreover, suppose the following conditions hold:*
1) over the 0*th simplicial component* π *is isomorphic to the projection* $Y_0 \times A_\bullet \to Y_0$*;*
2) $\omega_j^{M(X_{\bullet,\bullet}\to Y_\bullet)}$ *is trivial for any* $j \geq 1$*;*
3) $M(A_\bullet) = \bigoplus_{j\geq 1} T_j \in \mathcal{DM}^-_{eff}(k,R)$*.*
Then, there exists a Postnikov system in $\mathcal{DM}^-_{eff}(Y_\bullet, R)$



such that the simplicial components N_i^j *are isomorphic to the direct sum* $\bigoplus_{k\geq j} T_k$ *and the morphisms* $r_i^*(N^j \to T_j)$ *are the natural projections* $\bigoplus_{k\geq j} T_k \to T_j$ *in* $\mathcal{DM}^-_{eff}(Y_i, R)$*.*

Proof. This follows essentially the steps of the proof of Proposition 2.2.5. Indeed, by coherence of π we have that $\pi_i : Y_i \times A_\bullet \cong X_{i,\bullet} \to Y_i$ is the projection onto the first factor for any i. It follows that the coherent motive N^1 has simplicial components given by $N_i^1 \cong M(A_\bullet)$ in $\mathcal{DM}^-_{eff}(Y_i,R)$ for any i. Therefore, Proposition 5.2.4 implies the existence of the aimed Postnikov system in $\mathcal{DM}^-_{eff}(Y_\bullet,R)$, and the proof is complete. □

As reported in the previous section, once constructed a Postnikov system in a triangulated category and considered a suitable cohomological functor, one can obtain a spectral sequence which may converge if some extra requirements are met. The following theorem just states the existence of a strongly convergent spectral sequence related to the Postnikov system of the previous proposition.

Theorem 5.2.5. *Let* $\pi : X_{\bullet,\bullet} \to Y_\bullet$ *be a smooth coherent morphism from a smooth bisimplicial scheme to a smooth simplicial scheme over k and* $A_\bullet$ *a smooth simplicial k-scheme satisfying all conditions of the previous proposition. Moreover, for any bidegree* $(q)[p]$*, suppose there is an integer l such that* $(q)[p] \prec (q_l)[p_l]$*. Then, there exists a strongly convergent spectral sequence*

$$E_1^{p,q,s} = \bigoplus_{I_s} H^{p-p_s,q-q_s}(Y_\bullet,R) \Rightarrow H^{p,q}(X_{\bullet,\bullet},R)$$

Proof. We start by applying the construction of the exact couple associated to a Postnikov system of the previous section to the cohomological functor $Hom_{\mathcal{DM}^-_{eff}(Y_\bullet,R)}(-,T(q))$, for any q. This way, we get a spectral sequence with E_1-page given by

$$E_1^{p,q,s} = Hom_{\mathcal{DM}^-_{eff}(Y_\bullet,R)}(T_s, T(q)[p]) = \bigoplus_{I_s} H^{p-p_s,q-q_s}(Y_\bullet,R)$$

The filtration we are considering is defined by $F^m = ker(H(X_{\bullet,\bullet}) \to H(N^m))$. We have already noticed that this filtration is complete Hausdorff. In order to get the strong convergence we need to study $\varinjlim_m H(N^m) = 0$. Since all the N^m are coherent motives, by Proposition 5.2.1 we have filtrations $(N^m)_{\leq n}$ with graded pieces $(N^m)_n \cong Lr_{n,\#}r_n^*(N^m)[n]$. Hence, we have filtrations $(Lc_\# N^m)_{\leq n}$ with graded pieces

$$(Lc_\# N^m)_n \cong \bigoplus_{k\geq m} \bigoplus_{I_k} M(Y_n)(q_k)[p_k + n]$$

Now, fix a bidegree $(q)[p]$, then by hypothesis there exists an integer l such that $(q)[p] \prec (q_l)[p_l]$, from which it follows that

$$Hom_{\mathcal{DM}^-_{eff}(k,R)}((Lc_\# N^l)_n, T(q)[p]) = 0$$

for any n. Therefore,

$$Hom_{\mathcal{DM}^-_{eff}(k,R)}(Lc_\#(N^l), T(q)[p]) = 0$$

from which we deduce by adjunction that $H^{p,q}(N^l) = 0$ that implies, in particular, the triviality of $\varinjlim_m H(N^m)$. Hence, by Theorem 5.1.7 we obtain the result. $\square$

In the same fashion of Proposition 2.2.7, the next result assures that the spectral sequence just constructed is indeed functorial.

Proposition 5.2.6. *Let $\pi : X_{\bullet,\bullet} \to Y_\bullet$ and $\pi' : X'_{\bullet,\bullet} \to Y'_\bullet$ be smooth coherent morphisms from smooth bisimplicial schemes to smooth simplicial schemes over k and $A_\bullet$ a smooth simplicial k-scheme that satisfies all conditions from Proposition 5.2.4 with respect to π' and such that the following diagram is cartesian with all morphisms smooth*

$$\begin{array}{ccc} X_{\bullet,\bullet} & \xrightarrow{\pi} & Y_\bullet \\ {\scriptstyle p_X}\downarrow & & \downarrow{\scriptstyle p_Y} \\ X'_{\bullet,\bullet} & \xrightarrow[\pi']{} & Y'_\bullet \end{array}$$

Then, the induced square of motives in the category $\mathcal{DM}^-_{eff}(Y'_\bullet, R)$ extends uniquely to a morphism of Postnikov systems where, for any $j \geq 1$, $Lp_{Y\#}T_j \to T_j$ is given by $\bigoplus_{I_j} M(p_Y)(q_j)[p_j]$.

Proof. The proof follows the lines of Proposition 2.2.7. We will denote by N^j the objects from the Postnikov system of π and by N'^j the ones from the Postnikov system of π'.

First, recall that by Proposition 5.2.1 there is a filtration of $Lc_\# N^j$ with graded pieces

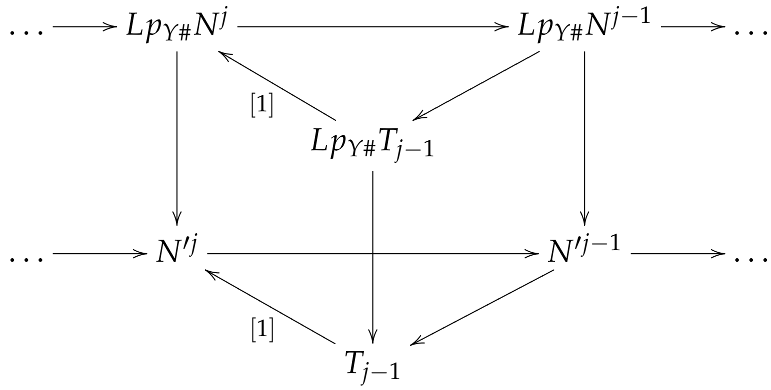
$$(Lc_\# N^j)_n \cong \bigoplus_{k\geq j} \bigoplus_{I_k} M(Y_n)(q_k)[p_k + n]$$

It follows that $Hom_{\mathcal{DM}^-_{eff}(k,R)}((Lc_\# N^j)_n, T_{j-1}) = 0$ for any n since, for any $k \geq j$, we have that $(q_{j-1})[p_{j-1}] \prec (q_k)[p_k]$ by hypothesis. Therefore,

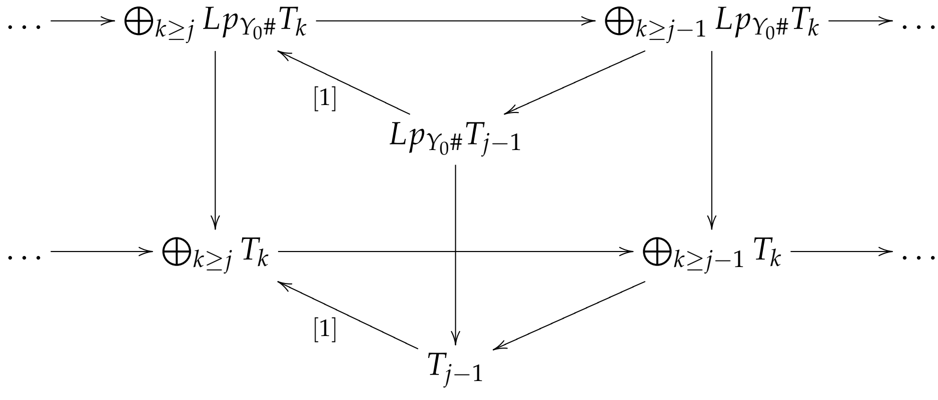
$$Hom_{\mathcal{DM}^-_{eff}(Y'_\bullet,R)}(Lp_{Y\#}N^j, T_{j-1}) \cong Hom_{\mathcal{DM}^-_{eff}(Y_\bullet,R)}(N^j, T_{j-1}) \cong$$

$$Hom_{\mathcal{DM}^-_{eff}(k,R)}(Lc_\# N^j, T_{j-1}) = 0$$

from which we deduce that there are no non-trivial maps from $Lp_{Y\#}N^j$ to T_{j-1}. It follows that there exist unique morphisms $Lp_{Y\#}N^j \to N'^j$ fitting into a morphism of Postnikov systems in $\mathcal{DM}^-_{eff}(Y'_\bullet, R)$



If we restrict our previous diagram to the 0th simplicial component we obtain in $\mathcal{DM}^-_{eff}(Y'_0, R)$ the following Postnikov system



where each triangle is split. By hypothesis, the morphism $Lp_{Y_0\#}T_{j-1} \to T_{j-1}$ in the previous diagram is basically given by $\bigoplus_{I_{j-1}} M(p_{Y_0})(q_{j-1})[p_{j-1}]$ while the map $Lp_{Y_0\#}N_0^{j-1} \to N_0'^{j-1}$ is given by $\bigoplus_{k\geq j-1}\bigoplus_{I_k} M(p_{Y_0})(q_k)[p_k]$.

Now, recall that the morphisms $Lp_{Y\#}T_j \to T_j$ and $\bigoplus_{I_j} M(p_Y)(q_j)[p_j]$ are both in

$$Hom_{\mathcal{DM}^-_{eff}(Y'_\bullet,R)}(Lp_{Y\#}T_j, T_j) \cong Hom_{\mathcal{DM}^-_{eff}(Y_\bullet,R)}(T_j, p_Y^* T_j) \cong$$

$$Hom_{\mathcal{DM}^-_{eff}(Y_\bullet,R)}(T_j,T_j) \cong \bigoplus H^{0,0}(Y_\bullet,R)$$

and, for the same reason, $(Lp_{Y\#}T_j \to T_j) = \bigoplus_{I_j} M(p_{Y_0})(q_j)[p_j]$ is in

$$Hom_{\mathcal{DM}^-_{eff}(Y_0',R)}(Lp_{Y_0\#}T_j,T_j) \cong Hom_{\mathcal{DM}^-_{eff}(Y_0,R)}(T_j,p^*_{Y_0}T_j) \cong$$

$$Hom_{\mathcal{DM}^-_{eff}(Y_0,R)}(T_j,T_j) \cong \bigoplus H^{0,0}(Y_0,R)$$

Recall that $H^{0,0}(Y_\bullet,R)$ is the free R-module with rank equal to the number of connected components of $Y_\bullet$ and, analogously, $H^{0,0}(Y_0,R)$ is the free R-module with rank equal to the number of connected components of Y_0. Since, as we have already noticed at the end of Proposition 2.2.7, the homomorphism

$$r_0^* : \bigoplus H^{0,0}(Y_\bullet,R) \to \bigoplus H^{0,0}(Y_0,R)$$

is injective, we deduce that $Lp_{Y\#}T_j \to T_j$ and $\bigoplus_{I_j} M(p_Y)(q_j)[p_j]$ are identified, which completes the proof. □

We would like to finish this section by establishing a comparison between this spectral sequence and the Serre spectral sequence associated to a fiber bundle in topology. Recall that in topology for a fibre sequence

$$F \to E \to B$$

with $\pi_1(B)$ acting trivially on $H(F)$ one has a spectral sequence converging to $H^*(E)$

$$E_2^{s,t} = H^s(B,H^t(F)) \Rightarrow H^*(E)$$

called *Serre spectral sequence* (see for example [50, Theorem 15.27]).

Analogously, the spectral sequence of our last theorem allows to reconstruct somehow the cohomology of the total simplicial scheme from the cohomology of the base and of the fiber, provided that the fiber is motivically cellular, which shows a certain similarity with the topological Serre spectral sequence. Moreover, the triviality condition on the morphisms

$$\omega_j^{M(X_{\bullet,\bullet}\to Y_\bullet)} : \pi_1(CC(Y_\bullet)) \to Aut(T_j)$$

for any $j \geq 1$ is analogous to the topological condition on the triviality of the action of $\pi_1(B)$ on $H(F)$. On the other hand, the main difference resides in how the spectral sequences are obtained. In fact, while the topological Serre spectral sequence is achieved by filtering the base, our spectral sequence is instead realised by filtering the fiber.

5.3 THE CASE OF PGL_n

The spectral sequence in the previous section has been constructed in the attempt of finding a possible way to study subtle characteristic classes for Severi-Brauer varieties, which are in one-to-one correspondence with PGL_n-torsors over the point.

From the short exact sequence of algebraic groups

$$1 \to G_m \to GL_n \to PGL_n \to 1$$

and by noticing that all GL_n-torsors over a field are trivial we get the following fiber sequence (see [4, Lemma 3.4])

$$BG_m \to BGL_n \to BPGL_n$$

Indeed, we can consider $\widetilde{B}GL_n = (EGL_n \times EPGL_n)/GL_n$ as a model for BGL_n. Then, over any simplicial component, the map $\widetilde{B}GL_n \to BPGL_n$ looks like

$$(EGL_n \times PGL_n^{m+1})/GL_n \to PGL_n^{m+1}/PGL_n$$

which is equivalent to the projection

$$(EGL_n \times PGL_n)/GL_n \times PGL_n^m \to PGL_n^m$$

By recalling that $PGL_n \cong GL_n/G_m$ via a central extension, we obtain that $(EGL_n \times PGL_n)/GL_n \cong EGL_n/G_m = \widehat{B}G_m \cong BG_m$, since G_m is a special algebraic group. It follows that the morphism $\widetilde{B}GL_n \to BPGL_n$ is smooth coherent and, over each simplicial component, is isomorphic to the projection $BG_m \times PGL_n^m \to PGL_n^m$.

Now, the main point is that the cohomology of BGL_n is known to be the polynomial algebra over H generated by Chern classes, i.e.

$$H(BGL_n) = H[c_1, \dots, c_n]$$

and $BG_m \cong P^\infty$ whose motive is cellular. Indeed, we have that $M(P^\infty) = \bigoplus_{j=0}^{\infty} T(j)[2j]$. Therefore, our spectral sequence can be constructed for this particular case, which leads to the following result.

Corollary 5.3.1. *There exists a strongly convergent spectral sequence*

$$E_1^{p,q,s} = H^{p-2s,q-s}(BPGL_n, R) \Rightarrow H^{p,q}(BGL_n, R)$$

Proof. Just apply Theorem 5.2.5 to the coherent morphism $\widetilde{B}GL_n \to BPGL_n$. □

The hope is to get in future some computational advantages from this spectral sequence in order to produce some invariants for Severi-Brauer varieties. The natural path to follow would be first to enrich the structure of this spectral sequence, for example with multiplicative structure or compatibility with Steenrod operations, then to produce computations for the motivic cohomology of $BPGL_n$, at least in some cases. This part of the research is still at an early stage and will not be performed in this thesis.

6

THE ACTION OF THE MOTIVIC STEENROD ALGEBRA ON ROST MOTIVES

As we have already pointed out, a really interesting and important part of motivic homotopy theory is the study of cohomology operations. As an example, the proof of the *Milnor conjecture* requires a considerable use of the *motivic Steenrod algebra*, namely the algebra of bistable cohomology operations in motivic cohomology, and in particular of the *Milnor operations* and of the *Margolis homology*. The mod p motivic Steenrod algebra has been constructed in [61] and has been shown to be the algebra of bistable cohomology operations for fields of characteristic zero in [64] by Voevodsky and for all fields of characteristic different from p in [22] by Hoyois, Kelly and Østvær. In [31], Milnor constructed Milnor operations and proved some very useful inductive formulas for them in the classical Steenrod algebra, which are the same in the motivic case when -1 is a square in the base field. These operations have been studied in the general case by Kylling in [28]. We will use these results throughout this chapter in order to understand the action of the motivic Steenrod algebra on the motivic cohomology of Čech simplicial schemes of norm quadrics.

Norm quadrics, which are quadrics associated to Pfister quadratic forms, are the mod 2 case of norm varieties. These extremely interesting objects and their motives have been deeply studied by several authors and constitute one of the pillars of the proof of the Milnor conjecture (Bloch-Kato in the general case). In particular, Rost has proved in [45] that motives of Pfister quadrics are direct sums of particular motives, called *Rost motives*, suitably shifted. A Rost motive has the property that it splits into two Tate motives over the algebraic closure. Moreover, a Rost motive can be represented in a suitable category as the extension of two simpler objects, namely two copies of the Čech simplicial scheme associated to the corresponding quadric appropriately shifted.

The aim of this chapter is to go a bit further in the understanding of Čech simplicial schemes of norm quadrics and, in particular, in their relations with the motivic Steenrod algebra. More precisely, we will give a complete description of the action of the motivic Steenrod algebra over the motivic cohomology of the reduced Čech simplicial scheme of a Pfister quadric in the case -1 is a square in k. This result will be used to describe the cyclic left module over the motivic Steenrod algebra generated by a certain element in the motivic cohomology of the Čech simplicial scheme of the Pfister quadric. We will show that this is the quotient of the motivic Steenrod algebra by its left ideal generated by the operations $Sq^{2^r}Sq^{2^{r-1}}\cdots Sq^2Sq^1$, Sq^{2^i} for any $i > 0$, τ and $Ann(a)$. Moreover, we will relate this module to the motivic cohomology of the reduced Rost motive, presenting it in turn as a quotient of the Steenrod algebra.

6.1 PFISTER QUADRICS AND ROST MOTIVES

We will start with a recollection on Pfister quadrics and Rost motives.

Definition 6.1.1. *We denote by Q_a the small Pfister quadric over the field k associated to the pure symbol $a = \{a_1, \ldots, a_r\}$ in $K_r^M(k)/2$, i.e. the quadric described by the equation $\langle\langle a_1, \ldots, a_{r-1}\rangle\rangle = a_r t^2$, and by P_a the big Pfister quadric ssociated to a, i.e. the quadric defined by $\langle\langle a_1, \ldots, a_r\rangle\rangle = 0$.*

We know from [62] that Q_a is a 2-splitting variety, which means that the symbol a is 0 in the Milnor K-theory of $k(Q_a)$ modulo 2.

Proposition 6.1.2. *The symbol $a = \{a_1, \ldots, a_r\}$ is divisible by 2 in $K_r^M(k(Q_a))$.*

Proof. See [62, Proposition 4.1]. □

Essentially, Q_a splits over a field extension E of k if and only if a is zero in $K_r^M(E)/2$.

Let $\check{C}(Q_a)$ be the Čech simplicial scheme associated to Q_a and $\mathfrak{X}_a$ its motive in $\mathcal{DM}^-_{eff}(k, \mathbb{Z}/2)$. We denote by $\widetilde{\mathfrak{X}_a}$ the reduced motive of $\check{C}(Q_a)$, i.e. the object in $\mathcal{DM}^-_{eff}(k, \mathbb{Z}/2)$ defined by the distinguished triangle

$$\widetilde{\mathfrak{X}_a} \to \mathfrak{X}_a \to T \to \widetilde{\mathfrak{X}_a}[1]$$

In the preliminary version of [40] and later in [68], it has been computed the motivic cohomology of $\mathfrak{X}_a$ and $\widetilde{\mathfrak{X}_a}$.

Theorem 6.1.3. *There exists an isomorphism of H-modules*

$$H(\mathfrak{X}_a) \cong H \oplus \left(\mathbb{Z}/2[\mu] \otimes \Lambda(Q_0, \ldots, Q_{r-2})(\gamma) \otimes \frac{K^M(k)/2}{Ann(a)}\right)$$

where Q_j are the Milnor operations, γ is the only element in bidegree $(r-1)[r]$ which maps to the symbol a once multiplied by τ and μ is the element in bidegree $(2^{r-1}-1)[2^r-1]$ corresponding to $Q_{r-2}Q_{r-3}\cdots Q_1Q_0\gamma$.

Moreover, there exists an isomorphism of H-modules

$$H(\widetilde{\mathfrak{X}_a}) \cong \mathbb{Z}/2[\mu] \otimes \Lambda(Q_0, \ldots, Q_{r-2})(\gamma) \otimes \frac{K^M(k)/2}{Ann(a)}$$

Proof. See [68, Theorem 5.8]. □

Rost has computed the motive of the Pfister quadric P_a that is reported in the following result.

Theorem 6.1.4. *There exists an object M_a in $\mathcal{DM}^-_{eff}(k, \mathbb{Z}/2)$ such that*

$$M(P_a) \cong M_a \otimes M(P^{2^{r-1}-1})$$

Moreover, M_a is endowed with two morphisms $T(2^{r-1}-1)[2^r-2] \to M_a$ and $M_a \to T$ such that the sequence

$$T(2^{r-1}-1)[2^r-2] \to M_a \to T$$

is a split distinguished triangle in $\mathcal{DM}^-_{eff}(E, \mathbb{Z}/2)$, for any field extension E/k where Q_a has a point.

Proof. See [45, Theorem 17] and [62, Theorem 4.3]. □

The previous result allows to define Rost motives.

Definition 6.1.5. *The motive M_a appearing in the previous theorem is called Rost motive of the pure symbol a.*

Besides, we denote by $\widehat{M_a}$ the reduced Rost motive obtained from M_a by removing both the cells $M_a \to T$ and $T(2^{r-1}-1)[2^r-2] \to M_a$. The Rost motive M_a is the extension of two simpler objects, in fact the following theorem holds.

Theorem 6.1.6. *There exists a distinguished triangle in $\mathcal{DM}^-_{eff}(k, \mathbb{Z}/2)$*

$$M_a \to \mathfrak{X}_a \to \mathfrak{X}_a(2^{r-1}-1)[2^r-1] \to M_a[1]$$

where the second arrow represents μ.

Proof. See [62, Theorem 4.4 and Lemma 4.7]. □

As a corollary to the previous result, we get that even $\widehat{M_a}$ is defined by a distinguished triangle involving Čech simplicial schemes. More precisely, we have the following result.

Corollary 6.1.7. *There exists a distinguished triangle in $\mathcal{DM}^-_{eff}(k, \mathbb{Z}/2)$*

$$\widetilde{\mathfrak{X}_a}(2^{r-1}-1)[2^r-1] \to \widetilde{\mathfrak{X}_a} \to \widehat{M_a} \to \widetilde{\mathfrak{X}_a}(2^{r-1}-1)[2^r]$$

Proof. It follows immediately from the previous theorem by applying the octahedron axiom. ☐

Then, we deduce that the motivic cohomology of $\widehat{M_a}$ consists of the diagonals from γ to μ. Indeed, one can easily get the following corollary.

Corollary 6.1.8. *The motivic cohomology of the reduced Rost motive is given by*

$$H(\widehat{M_a}) = \Lambda(Q_0, \dots, Q_{r-2})(\gamma) \otimes \frac{K^M(k)/2}{Ann(a)}$$

Proof. The proof is basically the same as Lemma 4.2.3. ☐

6.2 THE ACTION OF STEENROD OPERATIONS ON γ

We begin this section by showing how Steenrod squares corresponding to powers of 2 act on the element γ.

Lemma 6.2.1. $Sq^2\gamma = \rho Q_0 \gamma$ *and* $Sq^{2^j}\gamma = 0$ *for any* $1 < j < r$.

Proof. First, recall that Milnor operations generate an exterior subalgebra of the motivic Steenrod algebra and that Steenrod operations respect the $K^M(k)/2$-module structure. From [28, Remark 5] we know that

$$Q_j = Sq^{2^j} Q_{j-1} + Q_{j-1} Sq^{2^j} + \rho Sq^{2^{j-1}} Q_{j-1} Q_{j-2}$$

First, we notice that by degree reasons $Sq^{2^j}\gamma = xQ_{j-1}\gamma$ with $x \in K_1^M(k)/2$. Hence, evaluating the previous recursive formula in $Sq^{2^j}\gamma$, by Adem relations, we obtain that

$$Q_j Sq^{2^j}\gamma = Q_{j-1} Sq^{2^j} Sq^{2^j}\gamma =$$

$$Q_{j-1}(Sq^{2^{j+1}-2^{j-1}} Sq^{2^{j-1}}\gamma + \cdots + Sq^{2^{j+1}-2} Sq^2\gamma + \tau Sq^{2^{j+1}-1} Sq^1\gamma)$$

At this point, we notice that for $j = 1$

$$Q_1 Sq^2\gamma = Q_0(\tau Sq^3 Sq^1\gamma) = \rho Sq^3 Sq^1\gamma = \rho Q_0 Sq^2 Q_0\gamma = \rho Q_1 Q_0 \gamma$$

Therefore, by injectivity of Q_1 on the target, we get that $Sq^2\gamma = \rho Q_0 \gamma$. In order to prove that $Sq^{2^j}\gamma = 0$ for any $j > 1$ we proceed by induction. For $j = 2$, we obtain from the previous formula that $Q_2 Sq^4\gamma = Q_1(Sq^6 Sq^2\gamma + \tau Sq^7 Sq^1\gamma)$. Hence,

$$Q_0 Q_2 Sq^4\gamma = Q_0 Q_1(Sq^6 Sq^2\gamma + \tau Sq^7 Sq^1\gamma) = Q_1(\rho Sq^7 Q_0\gamma + \rho Sq^7 Q_0\gamma) = 0$$

On the other hand, we already know that $Sq^4\gamma = xQ_1\gamma$, so $xQ_0Q_1Q_2\gamma = 0$ which implies that $x = 0$ and $Sq^4\gamma = 0$. Now, suppose that $Sq^{2^i}\gamma = 0$

for any $1 < i < j$. Then, $Q_j Sq^{2^j}\gamma = Q_{j-1}(Sq^{2^{j+1}-2}Sq^2\gamma + \tau Sq^{2^{j+1}-1}Sq^1\gamma)$. Applying Q_0 we obtain, as before, that

$$Q_0 Q_j Sq^{2^j}\gamma = Q_0 Q_{j-1}(Sq^{2^{j+1}-2}Sq^2\gamma + \tau Sq^{2^{j+1}-1}Sq^1\gamma) =$$

$$Q_{j-1}(\rho Sq^{2^{j+1}-1}Q_0\gamma + \rho Sq^{2^{j+1}-1}Q_0\gamma) = 0$$

But we already know that $Sq^{2^j}\gamma = xQ_{j-1}\gamma$, so $xQ_0Q_{j-1}Q_j\gamma = 0$ which implies that $x = 0$ and $Sq^{2^j}\gamma = 0$, which is what we wanted to prove. □

From the previous lemma we get the description of the action of any Steenrod square on γ.

Proposition 6.2.2. *$Sq^1\gamma = Q_0\gamma$, $Sq^2\gamma = \rho Q_0\gamma$ and $Sq^m\gamma = 0$ for any $m > 2$.*

Proof. We need to prove only the last part of the statement. First, we notice that $Sq^3\gamma = Sq^1Sq^2\gamma = 0$. If $r = 1$, 2 or 3 we know from Lemma 1.6.7 that $Sq^m\gamma = 0$ for any $m \geq 4$, so we can suppose from now on that $r > 3$. We proceed by induction and restrict to the case m even, since the odd case immediately follows from the relation $Sq^{2n+1} = Sq^1Sq^{2n}$. Suppose now that $Sq^t\gamma = 0$ for any $2 < t < m$ with $2^j < m < 2^{j+1}$ and $j < r$. At first, let us consider the case $m \equiv 0 \pmod 4$. Then, we have the following Adem relations

$$Sq^{m-2^j}Sq^{2^j} = Sq^m + \sum_{t=1}^{\frac{m}{2}-2^{j-1}} \binom{2^j-1-t}{m-2^j-2t}\tau^{t \mod 2}Sq^{m-t}Sq^t$$

since $\binom{2^j-1}{m-2^j}$ is odd. Evaluating in γ we get

$$Sq^m\gamma = \binom{2^j-2}{m-2^j-2}\tau Sq^{m-1}Sq^1\gamma + \binom{2^j-3}{m-2^j-4}Sq^{m-2}Sq^2\gamma$$

At this point, we notice that

$$\binom{2^j-2}{m-2^j-2} = \binom{2^j-3}{m-2^j-4}\frac{(2^j-2)(2^{j+1}-m+1)}{(m-2^j-2)(m-2^j-3)}$$

Since $m \equiv 0 \pmod 4$, then $\frac{2^j-2}{m-2^j-2}$ is odd and $\binom{2^j-2}{m-2^j-2} \equiv \binom{2^j-3}{m-2^j-4} \pmod 2$. If they are both even, then we immediately get that $Sq^m\gamma = 0$. Otherwise, we have $Sq^m\gamma = \tau Sq^{m-1}Sq^1\gamma + Sq^{m-2}Sq^2\gamma$. Applying Q_0 we obtain $Q_0Sq^m\gamma = \rho Sq^{m-1}Q_0\gamma + \rho Sq^{m-1}Q_0\gamma = 0$. But Q_0 is a monomorphism on odd diagonals, hence $Sq^m\gamma = 0$. It remains to consider the case $m \equiv 2 \pmod 4$. In this case we can use the Adem relations

$$Sq^2Sq^{m-2} = Sq^m + \tau Sq^{m-1}Sq^1$$

Evaluating in γ we get

$$Sq^m\gamma = \tau Sq^{m-1}Sq^1\gamma = \tau Q_0 Sq^{m-2}Q_0\gamma = 0$$

since Q_0 is the 0 homomorphism on even diagonals. If $m \geq 2^r$, from Lemma 1.6.7 we have that $Sq^m\gamma = 0$ since $2^{r-1} \geq r-1$, which completes the proof. □

The following result shows how Milnor operations act on γ.

Corollary 6.2.3. *$Q_j\gamma = Sq^{2^j}Sq^{2^{j-1}}\cdots Sq^2Sq^1\gamma$ for any j.*

Proof. We proceed by induction. Notice that the equalities $Q_0\gamma = Sq^1\gamma$ and $Q_1\gamma = Sq^2Q_0\gamma + Q_0Sq^2\gamma = Sq^2Sq^1\gamma$ provide us with the induction basis. Now, suppose the statement is true for values less than $j > 1$. Then, we have from the recursive formulas (see [28, Corollary 4]) that

$$Q_j\gamma = Sq^{2^j}Q_{j-1}\gamma + Q_{j-1}Sq^{2^j}\gamma + \rho Q_{j-1}Sq^{2^{j-1}}Q_{j-2}\gamma = Sq^{2^j}Q_{j-1}\gamma$$

since $Sq^{2^{j-1}}Q_{j-2}\gamma = Q_{j-1}\gamma$ by induction hypothesis. Therefore, the statement holds for any j. □

At this point we recall that μ denotes the element $Q_{r-2}Q_{r-3}\cdots Q_1Q_0\gamma$ in bidegree $(2^{r-1}-1)[2^r-1]$.

Lemma 6.2.4. *$Sq^{2^{r+k}}(\mu^{2^k}) = \rho^{2^k}\mu^{2^{k+1}}$ for any $k \geq 0$.*

Proof. We proceed by induction. So, let us at first examine the case $k = 0$. In order to simplify the notation, we call ν the element $Q_{r-2}Q_{r-3}\cdots Q_2Q_1\gamma$ of bidegree $(2^{r-1}-2)[2^r-1]$. The element ν belongs to the slope 2 line and we know from Lemmas 1.6.6 and 1.6.7 that $Sq^{2^r-2}\nu = \nu^2$ and $Sq^m\nu = 0$ for $m > 2^r-2$. From an easy computation and from the formula $[Q_0, Sq^{2^r}] = Q_1Sq^{2^r-2}$, we deduce that

$$\begin{aligned} Sq^{2^r}\mu &= Sq^{2^r}Q_{r-2}Q_{r-3}\cdots Q_1Q_0\gamma \\ &= Q_0Sq^{2^r}Q_{r-2}Q_{r-3}\cdots Q_2Q_1\gamma + Q_1Sq^{2^r-2}Q_{r-2}Q_{r-3}\cdots Q_2Q_1\gamma \\ &= Sq^{2^r+1}\nu + Q_1Sq^{2^r-2}\nu = Q_1(\nu^2) = Sq^1Sq^2(\nu^2) + Sq^2Sq^1(\nu^2) \\ &= Sq^1(\tau Sq^1\nu Sq^1\nu) = \rho\mu^2 \end{aligned}$$

which provides the induction basis. Now, suppose the statement is true for k. Then, from Cartan formula we get

$$\begin{aligned} Sq^{2^{r+k+1}}(\mu^{2^{k+1}}) &= \sum_{s=0}^{2^{r+k}} (Sq^{2s}(\mu^{2^k})Sq^{2^{r+k+1}-2s}(\mu^{2^k}) \\ &\quad + \tau Sq^{2s+1}(\mu^{2^k})Sq^{2^{r+k+1}-2s-1}(\mu^{2^k})) \\ &= Sq^{2^{r+k}}(\mu^{2^k})Sq^{2^{r+k}}(\mu^{2^k}) = (\rho^{2^k}\mu^{2^{k+1}})(\rho^{2^k}\mu^{2^{k+1}}) = \rho^{2^{k+1}}\mu^{2^{k+2}} \end{aligned}$$

that is exactly what we aimed to prove. □

Proposition 6.2.5. *$Q_{r+k}\gamma = \rho^{2^{k+1}-1}\mu^{2^{k+1}}\gamma$ for any $k \geq -1$.*

Proof. We proceed by induction. First, notice that $Q_{r-1}\gamma = \mu\gamma$ which provides the induction basis for $k = -1$. Suppose the statement holds for k. Then,

$$\begin{aligned} Q_{r+k+1}\gamma &= Sq^{2^{r+k+1}}Q_{r+k}\gamma = Sq^{2^{r+k+1}}(\rho^{2^{k+1}-1}\mu^{2^{k+1}}\gamma) \\ &= \rho^{2^{k+1}-1}(Sq^{2^{r+k+1}}(\mu^{2^{k+1}})\gamma + \tau Sq^{2^{r+k+1}-1}(\mu^{2^{k+1}})Sq^1\gamma \\ &+ Sq^{2^{r+k+1}-2}(\mu^{2^{k+1}})Sq^2\gamma) = \rho^{2^{k+1}-1}\rho^{2^{k+1}}\mu^{2^{k+2}}\gamma = \rho^{2^{k+2}-1}\mu^{2^{k+2}}\gamma \end{aligned}$$

since Q_0 is a monomorphism on odd diagonals and

$$\begin{aligned} &Q_0(\tau Sq^{2^{r+k+1}-1}(\mu^{2^{k+1}})Sq^1\gamma + Sq^{2^{r+k+1}-2}(\mu^{2^{k+1}})Sq^2\gamma) = \\ &Q_0(\tau Sq^{2^{r+k+1}-1}(\mu^{2^{k+1}})Q_0\gamma + \rho Sq^{2^{r+k+1}-2}(\mu^{2^{k+1}})Q_0\gamma) = \\ &\quad \rho Sq^{2^{r+k+1}-1}(\mu^{2^{k+1}})Q_0\gamma + \rho Sq^{2^{r+k+1}-1}(\mu^{2^{k+1}})Q_0\gamma = 0 \end{aligned}$$

This completes the proof. □

The previous proposition tells us that $Q_j\gamma = \rho^{2^{j-r+1}-1}\mu^{2^{j-r+1}}\gamma$ when $j \geq r-1$, which implies immediately the following corollary.

Corollary 6.2.6. *When $\rho = 0$, i.e. -1 is a square in k, $Q_j\gamma = 0$ for any $j > r-1$.*

Indeed, we notice that the previous corollary holds also for the more general case $\rho \in Ann(a)$.

6.3 THE ACTION OF THE MOTIVIC STEENROD ALGEBRA

Here, we compute the complete action of the motivic Steenrod algebra on Čech simplicial schemes of Pfister quadrics and, therefore, on Rost motives.

We start by proving some relations in the motivic Steenrod algebra involving Milnor operations. It is proven in [28, Remark 5] that $[Q_j, Sq^{2^i}] = 0$ for any $i < j$. In the following lemma we extend this result.

Lemma 6.3.1. *$[Q_j, Sq^m] = 0$ for any $m < 2^j$.*

Proof. Our intention is to proceed by induction. We clearly have the induction basis for $m = 1, 2, 3, 4$. Now, consider an even number m such that $2^i < m < 2^{i+1}$ with $i < j$ and suppose that the statement is true for any $t < m$. Then, we have the following Adem relations

$$Sq^{m-2^i}Sq^{2^i} = Sq^m + \sum_{t=1}^{\frac{m}{2}-2^{i-1}} \binom{2^i-1-t}{m-2^i-2t}\tau^{t \, mod 2}Sq^{m-t}Sq^t$$

from which it follows that

$$[Q_j, Sq^{m-2^i} Sq^{2^i}] = [Q_j, Sq^m] + \sum_{t=1}^{\frac{m}{2}-2^{i-1}} \binom{2^i - 1 - t}{m - 2^i - 2t} \tau^{t \, mod 2} [Q_j, Sq^{m-t} Sq^t]$$

Hence, $[Q_j, Sq^m] = 0$ since all the other commutators in the previous formula are 0 by induction hypothesis, and the lemma is proved. □

Also, from [28, Remark 5], we know that $[Q_j, Sq^{2^j}] = \rho Q_j Q_{j-1}$. Therefore, when $\rho = 0$ the previous commutator is 0. This remark and the fact that Steenrod operations are expressible in terms of Sq^{2^i} allow us to extend the previous lemma in the following way.

Lemma 6.3.2. *When $\rho = 0$, $[Q_j, Sq^m] = 0$ for any $m < 2^{j+1}$.*

From now on, we consider only the case $\rho = 0$. In order to simplify the notation we write γ_j for the generic nonzero element $Q_{i_1} \cdots Q_{i_n} \gamma$ with $0 \leq i_n < \cdots < i_1 < r$ and $j = 2^{i_1} + \cdots + 2^{i_n} + 1$. From this notation, we have naturally that $2 \leq j \leq 2^r$ and we agree that $\gamma = \gamma_1$. We notice that clearly $\gamma_j = Q_{r-1}\gamma_{j-2^{r-1}} = \mu\gamma_{j-2^{r-1}}$ for any $j > 2^{r-1}$. Our aim is to prove that every Steenrod square that, acting on some γ_j, hits something above the diagonal of μ^2 acts indeed trivially. We start with the following lemma.

Lemma 6.3.3. *When $\rho = 0$, $[Q_{r-1}, Sq^m]\gamma_j = 0$ for any $m < 2^{r+1} + 2$ and any $j \leq 2^{r-1}$.*

Proof. From the previous lemma, we already know that the statement holds for any $m < 2^r$. Since $[Q_{r-1}, Sq^{2^r}] = Q_r$ and $Q_r\gamma = 0$, we get $[Q_{r-1}, Sq^{2^r}]\gamma_j = Q_r\gamma_j = 0$. Let m be an even integer such that $2^r < m < 2^{r+1}$ and suppose the statement is true for $t < m$. As before, we have the Adem relations

$$Sq^{m-2^r} Sq^{2^r} = Sq^m + \sum_{t=1}^{\frac{m}{2}-2^{r-1}} \binom{2^r - 1 - t}{m - 2^r - 2t} \tau^{t \, mod 2} Sq^{m-t} Sq^t$$

from which it follows that

$$[Q_{r-1}, Sq^{m-2^r} Sq^{2^r}]\gamma_j =$$

$$[Q_{r-1}, Sq^m]\gamma_j + \sum_{t=1}^{\frac{m}{2}-2^{r-1}} \binom{2^r - 1 - t}{m - 2^r - 2t} \tau^{t \, mod 2} [Q_{r-1}, Sq^{m-t} Sq^t]\gamma_j$$

First, we notice that $[Q_{r-1}, Sq^{m-2^r}] = 0$ since $m - 2^r < 2^r$. Hence, we get that $[Q_{r-1}, Sq^{m-2^r} Sq^{2^r}]\gamma_j = 0$. Similarly, since $t \leq \frac{m}{2} - 2^{r-1} < 2^{r-1}$, we get that $Sq^{m-t}Sq^t Q_{r-1}\gamma_j = Sq^{m-t}Q_{r-1}Sq^t\gamma_j$. Now, let us examine two cases: $Sq^t\gamma_j$ is an element below the diagonal of $\mu\gamma$; $Sq^t\gamma_j$ is an element above the diagonal of μ. In the first case, $Sq^t\gamma_j$ is a generic element of the form $x\gamma_h$ where x in an element of the Milnor K-theory and $h \leq 2^{r-1}$, so $Sq^{m-t}Q_{r-1}Sq^t\gamma_j = Q_{r-1}Sq^{m-t}Sq^t\gamma_j$ by induction hypothesis. In the second case, $Sq^t\gamma_j = x\mu\gamma_h = xQ_{r-1}\gamma_h$ where x is an element of the Milnor K-theory and $h \leq 2^{r-1}$. Hence,

$$Sq^{m-t}Q_{r-1}Sq^t\gamma_j = xSq^{m-t}Q_{r-1}Q_{r-1}\gamma_h = 0$$

and

$$Q_{r-1}Sq^{m-t}Sq^t\gamma_j = xQ_{r-1}Sq^{m-t}Q_{r-1}\gamma_h = xQ_{r-1}Q_{r-1}Sq^{m-t}\gamma_h = 0$$

by induction hypothesis. In any case, we get that $[Q_{r-1}, Sq^m]\gamma_j = 0$. We finish the proof by noticing that the formula $[Q_{r-1}, Sq^{2^{r+1}}] = Q_rSq^{2^r}$ from [28] implies

$$[Q_{r-1}, Sq^{2^{r+1}}]\gamma_j = Q_rSq^{2^r}\gamma_j = Sq^{2^r}Q_r\gamma_j = 0$$

since $Q_r\gamma = 0$. □

Proposition 6.3.4. *Considering the case $\rho = 0$ and using the same notation as before, we have that $Sq^m\gamma_j = 0$ when $\frac{m}{2} + j > 2^r$.*

Proof. We write $(q_j)[p_j]$ for the bidegree of γ_j. From Lemma 1.6.7 we know that $Sq^m\gamma_j = 0$ for any $m \geq 2\max\{j+1, q_j\}$. Clearly, the last inequality is satisfied when $m \geq 2^{r+1} + 2$, so we can restrict to the case $m < 2^{r+1} + 2$. First, we consider the case $j \leq 2^{r-1}$ and $q_j \leq 2^{r-1} - 1$. Then, the condition $\frac{m}{2} + j \geq 2^r + 1$ implies that

$$m \geq 2^{r+1} + 2 - 2j \geq 2^{r+1} + 2 - 2^r = 2^r + 2 \geq 2\max\{j+1, q_j\}$$

At this point, let us study the case $j > 2^{r-1}$. Therefore, we have the following equality

$$Sq^m\gamma_j = Sq^mQ_{r-1}\gamma_{j-2^{r-1}} = Q_{r-1}Sq^m\gamma_{j-2^{r-1}}.$$

When $\frac{m}{2} + j - 2^{r-1} > 2^r$, from the first case we know that $Sq^m\gamma_{j-2^{r-1}} = 0$ which implies that $Sq^m\gamma_j = 0$. Hence, we can restrict again to the case $2^{r-1} < \frac{m}{2} + j - 2^{r-1} \leq 2^r$. In this case, $Sq^m\gamma_{j-2^{r-1}} = xQ_{r-1}\gamma_h$, from which it follows that

$$Sq^m\gamma_j = Q_{r-1}Sq^m\gamma_{j-2^{r-1}} = xQ_{r-1}Q_{r-1}\gamma_h = 0$$

And the proof is complete. □

We already know that the relations given by $Sq^{2^r}Sq^{2^{r-1}}\cdots Sq^2Sq^1\gamma = 0$ and $Sq^{2^i}\gamma = 0$ for $i > 0$ hold when $\rho = 0$. Moreover, we notice that $\tau\gamma$ is 0 since there is nothing below the diagonal of γ in the motivic cohomology of $\widetilde{\mathfrak{X}_a}$ and $x\gamma = 0$ for any $x \in Ann(a)$ since multiplication by τ is a monomorphism from the 1st to the 0th diagonal of the motivic cohomology of $\mathfrak{X}_a$.

On the other hand, we can consider the epimorphism $\phi : \mathcal{A} \to \mathcal{M}$ defined by $\phi(\theta) = \theta\gamma$, where $\mathcal{A}$ is the motivic Steenrod algebra and $\mathcal{M}$ is the left $\mathcal{A}$-module generated by γ. Let $\mathcal{I}$ be the left ideal of $\mathcal{A}$ corresponding to the kernel of the morphism ϕ, i.e. the left ideal such that $\mathcal{M} = \frac{\mathcal{A}}{\mathcal{I}}$. Our aim is to prove that $\mathcal{I}$ is generated by the previous relations. Before proceeding, we recall that, by Corollary 1.6.4, $\mathcal{A}$ is generated as an algebra over the motivic cohomology of H by the operations Sq^{2^i} for $i \geq 0$.

In the following preliminary result we will denote by $\overline{\mathcal{A}}$ the subring of $\mathcal{A}$ generated by τ and Sq^{2^i} for $i \geq 0$.

Lemma 6.3.5. *When $\rho = 0$, for every $\theta \in \overline{\mathcal{A}}$, $\theta\gamma$ is either 0 or γ_j for some j.*

Proof. We want to proceed by induction on the square degree of θ. The induction basis has already been proven, as we know that $Sq^1\gamma = \gamma_2$ and $\tau^m Sq^1\gamma = 0$ for any $m > 0$ since τ commutes with Steenrod operations. Now, fix a square degree l and suppose the statement is true for cohomology operations of square degree less than l. Note that, by the remark just before this lemma, it is enough to prove the statement for monomials $Sq^{2^{t_1}} \cdots Sq^{2^{t_m}}$ of square degree l since τ commutes with every operation. Hence, by induction hypothesis, we know that $Sq^{2^{t_1}} \cdots Sq^{2^{t_m}}\gamma = Sq^{2^{t_1}}\gamma_h$, where $\gamma_h = Q_{i_1} \cdots Q_{i_n}\gamma$, whether it is non-trivial. From well known recursive formulas and by induction hypothesis, we deduce the following:
if $t_1 \leq i_1$, then

$$Sq^{2^{t_1}}\gamma_h = Q_{i_1} Sq^{2^{t_1}}\gamma_{h-2^{i_1}} = Q_{i_1}\gamma_{j-2^{i_1}} = \gamma_j$$

whether it is not 0;
if $t_1 > i_1$, then

$$Sq^{2^{t_1}}\gamma_h = Q_{i_1} Sq^{2^{t_1}}\gamma_{h-2^{i_1}} + Q_{i_1+1} Sq^{2^{t_1}-2^{i_1+1}}\gamma_{h-2^{i_1}} = \gamma_j$$

whether it is not 0. This finishes the proof. □

Now we describe the complete action of $\mathcal{A}$ on the motivic cohomology of the reduced Čech simplicial scheme of a Pfister quadric.

Proposition 6.3.6. *When $\rho = 0$, $Sq^1\gamma_j = j\gamma_{j+1}$ and*

$$Sq^{2^k}\gamma_j = \binom{2j-2}{2^k}\binom{j+2^k}{2^k}\gamma_{j+2^{k-1}}$$

for any $k > 0$ and $j + 2^{k-1} \leq 2^r$.

Proof. The first formula is obvious once noticed that $Sq^1 = Q_0$ acts trivially from even to odd diagonals. In order to prove the second formula, we use an induction argument on j. Clearly, $Sq^{2^k}\gamma_1 = Sq^{2^k}\gamma = 0$ which agrees with the fact that $\binom{0}{2^k}$ is always zero. Now suppose the claim holds for any $h < j = 2^{i_1} + \cdots + 2^{i_n} + 1$. If $k \leq i_1$, then

$$Sq^{2^k}\gamma_j = Q_{i_1} Sq^{2^k}\gamma_{j-2^{i_1}} =$$

$$Q_{i_1}\binom{2j-2^{i_1+1}-2}{2^k}\binom{j-2^{i_1}+2^k}{2^k}\gamma_{j-2^{i_1}+2^{k-1}} = \binom{2j-2}{2^k}\binom{j+2^k}{2^k}\gamma_{j+2^{k-1}}$$

In fact, $\binom{2j-2^{i_1+1}-2}{2^k} \equiv \binom{2j-2}{2^k} \pmod 2$ and, when $k < i_1$, $\binom{j-2^{i_1}+2^k}{2^k} \equiv \binom{j+2^k}{2^k} \pmod 2$. On the other hand, we have either $\binom{2j-2}{2^{i_1}} \equiv 0 \pmod 2$ or $i_1 = i_2 + 1$. In the second case, $\gamma_{j-2^{i_1-1}}$ contains Q_{i_1} which implies that the element $Q_{i_1}\binom{2j-2^{i_1+1}-2}{2^{i_1}}\binom{j}{2^{i_1}}\gamma_{j-2^{i_1-1}}$ is zero as $\binom{2j-2}{2^{i_1}}\binom{j+2^{i_1}}{2^{i_1}}\gamma_{j+2^{i_1-1}}$. In the case $k = i_1 + 1$, we get that

$$\begin{aligned} Sq^{2^{i_1+1}}\gamma_j &= Q_{i_1} Sq^{2^{i_1+1}}\gamma_{j-2^{i_1}} + \gamma_{j+2^{i_1}} \\ &= Q_{i_1}\binom{2j-2^{i_1+1}-2}{2^{i_1+1}}\binom{j+2^{i_1}}{2^{i_1+1}}\gamma_j + \gamma_{j+2^{i_1}} \\ &= \gamma_{j+2^{i_1}} = \binom{2j-2}{2^{i_1+1}}\binom{j+2^{i_1+1}}{2^{i_1+1}}\gamma_{j+2^{i_1}} \end{aligned}$$

If $k > i_1 + 1$, then we notice that $Sq^{2^k}\gamma_j$ is on the diagonal of $\gamma_{j+2^{k-1}}$ but does not hit it. Hence, by the previous lemma we deduce that $Sq^{2^k}\gamma_j$ is zero as $\binom{2j-2}{2^k}\binom{j+2^k}{2^k}\gamma_{j+2^{k-1}}$ since $2^k \geq 2^{i_1+2} \geq 2j$. And the claim is proved. □

Let us call by $\mathcal{J}$ the left ideal of $\mathcal{A}$ generated by $Sq^{2^r}Sq^{2^{r-1}}\cdots Sq^2Sq^1$, Sq^{2^i} for any $i > 0$, τ and $Ann(a)$. We already know that $\mathcal{J} \subseteq \mathcal{I}$. Our aim is to prove the other side inclusion. Before starting the proof of the main theorem, we prove the following lemma.

Lemma 6.3.7. *When $\rho = 0$, $Q_r \in \mathcal{J}$.*

Proof. What we want to prove is that it is possible to write Q_j as the sum of $Sq^{2^j}Sq^{2^{j-1}}\cdots Sq^2Sq^1$ and θ_j, where θ_j belongs to the left ideal generated only by the Steenrod squares Sq^{2^k} for any $k > 0$. At fist we notice that $Q_0 = Sq^1$ satisfies the previous condition. Therefore, it provides us with an induction basis. Now, suppose the condition we are considering is satisfied by Q_{j-1}. Then,

$$Q_j = Sq^{2^j}Q_{j-1} + Q_{j-1}Sq^{2^j} = Sq^{2^j}Sq^{2^{j-1}}\cdots Sq^2Sq^1 + Sq^{2^j}\theta_{j-1} + Q_{j-1}Sq^{2^j}$$

$$= Sq^{2^j}Sq^{2^{j-1}}\cdots Sq^2Sq^1 + \theta_j$$

where $\theta_j = Sq^{2^j}\theta_{j-1} + Q_{j-1}Sq^{2^j}$. So, the statement is true and for $j = r$ we get that $Q_r \in \mathcal{J}$. □

At this point we are ready to demonstrate the main result.

Theorem 6.3.8. *When $\rho = 0$, $\mathcal{I} = \mathcal{J}$, i.e. $\mathcal{I}$ is generated as a left $\mathcal{A}$-module by $Sq^{2^r}Sq^{2^{r-1}} \cdots Sq^2Sq^1$, Sq^{2^i} for any $i > 0$, τ and $Ann(a)$.*

Proof. Before starting, we notice that it is enough to consider only monomials $Sq^{2^{t_1}} \cdots Sq^{2^{t_m}}$ since τ and elements in the Milnor K-theory commute with all Steenrod operations. We proceed by induction on the number of the hit diagonal. The induction basis is provided by the fact that the only non zero operation which hits the second diagonal acting trivially on γ is Sq^2 that clearly belongs to $\mathcal{J}$. Now, suppose the statement is true for each operation that hits a diagonal below $j + 2^{k-1}$. Besides, suppose that $Sq^{2^{t_1}} \cdots Sq^{2^{t_m}}\gamma = \gamma_j$ and $Sq^{2^k}Sq^{2^{t_1}} \cdots Sq^{2^{t_m}}\gamma = 0$. We want to prove that $Sq^{2^k}Sq^{2^{t_1}} \cdots Sq^{2^{t_m}}$ belongs to $\mathcal{J}$. From our assumptions, we deduce that $Sq^{2^{t_1}} \cdots Sq^{2^{t_m}} + Q_{i_1} \cdots Q_{i_n} \in \mathcal{I}$ which implies by induction hypothesis that $Sq^{2^{t_1}} \cdots Sq^{2^{t_m}} + Q_{i_1} \cdots Q_{i_n} \in \mathcal{J}$. Therefore,

$$Sq^{2^k}Sq^{2^{t_1}} \cdots Sq^{2^{t_m}} + Sq^{2^k}Q_{i_1} \cdots Q_{i_n} \in \mathcal{J}$$

So, it is enough to prove that $Sq^{2^k}Q_{i_1} \cdots Q_{i_n} \in \mathcal{J}$. For $j = 1$, there is nothing to prove. Hence, suppose from now on that $j > 1$. If $k \leq i_1$, then

$$Sq^{2^k}Q_{i_1} \cdots Q_{i_n} = Q_{i_1}Sq^{2^k}Q_{i_2} \cdots Q_{i_n}$$

At this point, we distinguish two cases. When $k < i_1$ or $i_1 > i_2 + 1$, we get that $Sq^{2^k}\gamma_{j-2^{i_1}} = 0$. Therefore, by induction hypothesis, $Sq^{2^k}Q_{i_2} \cdots Q_{i_n} \in \mathcal{J}$, from which we deduce that $Sq^{2^k}Q_{i_1} \cdots Q_{i_n} \in \mathcal{J}$. On the other hand, the condition $k = i_1 = i_2 + 1$ implies that

$$\begin{aligned} Q_{i_1}Sq^{2^k}Q_{i_2} \cdots Q_{i_n} &= Q_{i_1}Q_{i_1}Q_{i_3} \cdots Q_{i_n} + Q_{i_1}Q_{i_2}Sq^{2^k}Q_{i_3} \cdots Q_{i_n} \\ &= Q_{i_1}Q_{i_2}Sq^{2^k}Q_{i_3} \cdots Q_{i_n} \end{aligned}$$

which belongs to $\mathcal{J}$ by induction hypothesis, since $Sq^{2^k}\gamma_{j-2^{i_1}-2^{i_2}} = 0$. If $k > i_1 + 1$, then we have that

$$Sq^{2^k}Q_{i_1} \cdots Q_{i_n} = Q_{i_1}Sq^{2^k}Q_{i_2} \cdots Q_{i_n} + Q_{i_1+1}Sq^{2^k-2^{i_1+1}}Q_{i_2} \cdots Q_{i_n}$$

Now, we recall that $Sq^{2^k}\gamma_{j-2^{i_1}} = 0$ and $Sq^{2^k-2^{i_1+1}}\gamma_{j-2^{i_1}} = 0$ since they both fail to hit their respective generators. Then, by induction hypothesis, $Sq^{2^k}Q_{i_2} \cdots Q_{i_n} \in \mathcal{J}$ and $Sq^{2^k-2^{i_1+1}}Q_{i_2} \cdots Q_{i_n} \in \mathcal{J}$, from which we deduce that $Sq^{2^k}Q_{i_1} \cdots Q_{i_n} \in \mathcal{J}$. If $k = i_1 + 1$, then $Sq^{2^k}\gamma_j = 0$ implies that $i_1 = r - 1$. In this case the formula

$$Sq^{2^r}Q_{r-1} \cdots Q_{i_n} = Q_{r-1}Sq^{2^r}Q_{i_2} \cdots Q_{i_n} + Q_rQ_{i_2} \cdots Q_{i_n}$$

implies that $Sq^{2^r} Q_{r-1} \cdots Q_{i_n} \in \mathcal{J}$ by the previous lemma and by induction hypothesis, since $Sq^{2^r} \gamma_{j-2^{r-1}} = 0$. At the end, suppose that

$$Sq^{2^k} Sq^{2^{t_1}} \cdots Sq^{2^{t_m}} \gamma + Sq^{2^{k'}} Sq^{2^{t'_1}} \cdots Sq^{2^{t'_{m'}}} \gamma = 0$$

where both addends are non-trivial. Using the same trick as before we can limit ourselves to consider the sum $Sq^{2^k} Q_{i_1} \cdots Q_{i_n} + Sq^{2^{k'}} Q_{i'_1} \cdots Q_{i'_{n'}}$. Then,

$$Sq^{2^k} Q_{i_1} \cdots Q_{k-1} \cdots Q_{i_n} + Sq^{2^{k'}} Q_{i'_1} \cdots Q_{k'-1} \cdots Q_{i'_{n'}} =$$

$$Q_{i_1} \cdots Q_{k-1} Sq^{2^k} \cdots Q_{i_n} + Q_{i_1} \cdots Q_k \cdots Q_{i_n} +$$

$$Q_{i'_1} \cdots Q_{k'-1} Sq^{2^{k'}} \cdots Q_{i'_{n'}} + Q_{i'_1} \cdots Q_{k'} \cdots Q_{i'_{n'}} =$$

$$Q_{i_1} \cdots Q_{k-1} Sq^{2^k} \cdots Q_{i_n} + Q_{i'_1} \cdots Q_{k'-1} Sq^{2^{k'}} \cdots Q_{i'_{n'}}$$

since $Q_{i_1} \cdots Q_k \cdots Q_{i_n} = Q_{i'_1} \cdots Q_{k'} \cdots Q_{i'_{n'}}$. At this point we notice that both the elements $Sq^{2^k} \cdots Q_{i_n} \gamma$ and $Sq^{2^{k'}} \cdots Q_{i'_{n'}} \gamma$ are zero, from which we deduce that $Sq^{2^k} \cdots Q_{i_n}$ and $Sq^{2^{k'}} \cdots Q_{i'_{n'}}$ belong to $\mathcal{J}$, which shows that

$$Sq^{2^k} Q_{i_1} \cdots Q_{i_n} + Sq^{2^{k'}} Q_{i'_1} \cdots Q_{i'_{n'}} \in \mathcal{J}$$

Since the Steenrod squares Sq^{2^k} generate $\mathcal{A}$ as an algebra over the motivic cohomology of the point, the proof is complete. □

At the end, we notice that the module we have studied consists of all the diagonals from γ to μ^2, while the motivic cohomology of $\widehat{M_a}$ consists of the diagonals from γ to μ. Hence, we deduce immediately the following corollary.

Corollary 6.3.9. *When $\rho = 0$, the motivic cohomology of $\widehat{M_a}$ is the quotient of $\mathcal{A}$ by its left ideal generated by the elements $Sq^{2^{r-1}} \cdots Sq^2 Sq^1$, Sq^{2^i} for any $i > 0$, τ and $Ann(a)$.*

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