Annals of Pure and Applied Logic 164 (2013) 566–576

Contents lists available at SciVerse ScienceDirect

Annals of Pure and Applied Logic

www.elsevier.com/locate/apal

On the Kolmogorov complexity of continuous real functions

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ARTICLE INFO

Article history:
Received 19 December 2011
Received in revised form 28 October 2012
Available online 27 November 2012

MSC:
68Q30
03D32
28C20
06B35

Keywords:
Kolmogorov complexity
Algorithmic randomness
Computable analysis
Domain theory
Measure theory
Prevalence

ABSTRACT

Kolmogorov complexity was originally defined for finitely-representable objects. Later, the definition was extended to real numbers based on the asymptotic behaviour of the sequence of the Kolmogorov complexities of the finitely-representable objects—such as rational numbers—used to approximate them.

This idea will be taken further here by extending the definition to *continuous functions* over real numbers, based on the fact that every continuous real function can be represented as the limit of a sequence of finitely-representable enclosures, such as polynomials with rational coefficients.

Based on this definition, we will prove that for any growth rate imaginable, there are real functions whose Kolmogorov complexities have higher growth rates. In fact, using the concept of *prevalence*, we will prove that 'almost every' continuous real function has such a high-growth Kolmogorov complexity. An asymptotic bound on the Kolmogorov complexities of total single-valued computable real functions will be presented as well.

1. Introduction

In computer science, Kolmogorov complexity of an object provides a measure of the complexity of its description, hence the alternative name descriptive complexity. Although commonly known as Kolmogorov complexity, Solomonoff [21,22] was the first to develop the concept, for the purpose of studying the complexity of *finite* objects, such as the finite sequences of binary digits.

This concept of complexity has indeed been extended and studied over non-finite objects such as real numbers, regarding which, there are at least two main approaches in the literature: one based on the real Turing machine (RTM) of Blum, Shub, and Smale [3]; and the other based on an effective setting, such as the Type-2 Theory of Effectivity (TTE) of Weihrauch [24].

A real Turing machine is conceptually very similar to an ordinary Turing machine, except that each of its registers is capable of holding the exact value of a real number at any time, and the machine is capable of carrying out arithmetic operations on real numbers in unit time. Montaña and Pardo [17] and Ziegler and Koolen [25] have studied Kolmogorov complexity over sequences of real numbers based on the theory of real Turing machines. This approach is very elegant but too abstract to address the issue of effective representation of real numbers.

Type-2 theory of effectivity is another framework for studying computability over real numbers, within which the issue of effective representation is addressed. Hence, it provides a theoretical foundation for exact real computation [7–9,24]. In this framework, each ideal object (such as a non-finitely-representable real number, function, manifold, etc.) is represented as the limit of a sequence of finitely-representable approximations.

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0168-0072/$ – see front matter
http://dx.doi.org/10.1016/j.apal.2012.11.003

As an example, consider the case of real numbers. Each real number can be represented as the limit of a sequence of rational numbers. Another choice for a basis of representation of real numbers is the set of arbitrary precision floating point numbers [10,16,19]. In fact, it is in this setting that the Kolmogorov complexity of real numbers has been studied by Cai and Hartmanis [4] and Staiger [23].

In a sound and complete exact framework, the correct result of any computation must be obtainable to within any accuracy that is demanded. For instance, in one viable protocol one could represent accuracy using integers, and interpret an accuracy n as 'being within the radius 2^{-n} of the exact result'. In general, the concrete indicator of accuracy is arbitrary. Nonetheless, the expectation is that with 'higher' accuracy demanded, one needs more computational resources to provide a satisfying answer to the querying party.

The inspiration for the framework of this paper comes mainly from the work of Cai and Hartmanis [4]. Every real number $x \in [0,1]$ can be approximated by a sequence $\langle x_i \rangle_{i\in\mathbb{N}}$ of arbitrary precision binary floating point numbers in $[0,1]$, in such a way that:

1. Each x_n has a binary representation of the form $r_{x_n} = 0.d_1^{x_n} d_2^{x_n} \ldots d_n^{x_n}$ where $\forall i \in \mathbb{N}$: $d_i^{x_n} \in \{0,1\}$. In other words, each x_n has at least one representation of maximum length $2+n$. If we ignore the common leading "0." prefix, then we can say that each x_n has a representation of maximum length n.
2. $\forall n \in \mathbb{N}$: $|x - x_n| < 2^{-n}$.

In a fixed language, for each $n \in \mathbb{N}$, there may be a description of x_n with a string of characters which is shorter than its binary expansion. Therefore, for each real number $x \in [0,1]$ and each $n \in \mathbb{N}$, there exists a rational number $x_n \in [0,1]$ which has a description of length at most n and satisfies $|x - x_n| < 2^{-n}$. If we let $K(x_n)$ denote the length of the shortest possible description of any such x_n then $K(x_n) \leqslant n$. For the real number x, the Kolmogorov complexity $K_{\mathbb{R}}(x)$ can be defined as:

$$K_{\mathbb{R}}(x) := \frac{1}{2}\left(\liminf_{n\to\infty} \frac{K(x_n)}{n} + \limsup_{n\to\infty} \frac{K(x_n)}{n}\right)$$

It should be clear that $\forall x \in [0,1]$: $0 \leqslant K_{\mathbb{R}}(x) \leqslant 1$. Cai and Hartmanis [4] prove that:

(i) For Lebesgue-almost every x in $[0,1]$: $K_{\mathbb{R}}(x) = 1$.
(ii) For every $t \in [0,1]$, the set $K_{\mathbb{R}}^{-1}(t)$ is uncountable and has Hausdorff dimension t.
(iii) The graph of $K_{\mathbb{R}}$ is a fractal.

We try to address the cost of representation of functions by extending the definition of Kolmogorov complexity to the set $C[0,1]$ of continuous real functions from $[0,1]$ to $\mathbb{R}$. *The main result of this paper states that no matter what rate of growth one considers, 'almost all' functions in the Banach space $C[0,1]$ have Kolmogorov complexities with higher growth rates.* As such, this result can be regarded as an extension of item (i) to the case of the function space $C[0,1]$.

The extension of the concept of Kolmogorov complexity to the function spaces such as $C[0,1]$ or $C(2^{\mathbb{N}})$ has already been studied in contexts different to that of ours. Barmpalias et al. [2] have studied the concept over the function space $C(2^{\mathbb{N}})$ in the context of descriptive set theory. Fouché [11,12] has extensively studied this subject in the context of Brownian motion.

2. Kolmogorov complexity of a continuous real function

In what follows, by representation of the functions in $C[0,1]$ we mean the representation of each such element as the limit of a (not necessarily computable) sequence of finitely-representable objects. Domain theory [1,6,8,13] provides a suitable setting for this purpose. Of course, the actual structure that we introduce will not be a domain, but concepts such as approximation will be developed in accordance with domain theory. To start, we use enclosures to approximate functions:

Definition 1 *($[f,g]$: function enclosure).*

1. For $f, g \in \mathbb{R}^{[0,1]}$ we define the function enclosure $[f,g]$ by
$$[f,g] := \left\{h \in \mathbb{R}^{[0,1]} \;\middle|\; \forall x \in [0,1]:\ f(x) \leqslant h(x) \leqslant g(x)\right\}$$
(Obviously if $\exists x_0 \in [0,1]$: $g(x_0) < f(x_0)$ then $[f,g]$ will be empty.)
2. The function f (respectively g) is called the *lower boundary* (respectively *upper boundary*) of the enclosure $[f,g]$.
3. Enclosures H_1 and H_2 are said to be *consistent* if $H_1 \cap H_2 \neq \emptyset$. Otherwise, they are said to be *inconsistent*.

Furthermore, tighter enclosures provide better approximations of a function. This is expressed using the width of an enclosure:

Definition 2 *(w: width operator).* Consider $f, g \in C[0,1]$. For the enclosure $[f,g]$ the width is defined as $\mathrm{w}([f,g]) := \max\{g(t) - f(t) \mid t \in [0,1]\}$.

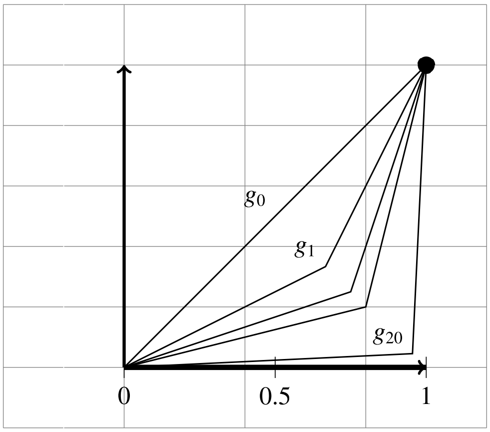

Fig. 1. $\mathbb{FE}$ is not a complete partial order: each g_i is continuous, but the limit is not.

Note that $\mathrm{w}([f, g])$ is well defined as $[0, 1]$ is compact and both f and g are assumed to be continuous.

Definition 3 *($\Gamma(h)$: graph of a function enclosure).* Let $h = [f, g]$ be an enclosure. By the *graph of* h we mean the set of points in $[0, 1] \times \mathbb{R}$ lying between the graphs of its lower and upper boundaries. Set theoretically, this is just the union of all functions in the enclosure, i.e. $\Gamma(h) := \bigcup h$.

The set of continuous-function enclosures under the reverse inclusion forms a poset:

Definition 4 *($\mathbb{FE}$).* We denote the set of non-empty continuous-function enclosures by $\mathbb{FE}$, i.e. $\mathbb{FE} := \{[f, g] \mid f, g \in C[0, 1],\ \forall t \in [0, 1]\colon\ f(t) \leqslant g(t)\}$. We define the order $\sqsubseteq$ over this set as follows: $\forall h_1, h_2 \in \mathbb{FE} : h_1 \sqsubseteq h_2 \Leftrightarrow h_2 \subseteq h_1$. The pair $(\mathbb{FE}, \sqsubseteq)$ is a partial order which we simply denote by $\mathbb{FE}$.

An element $h \in \mathbb{FE}$ is maximal if and only if $\mathrm{w}(h) = 0$, in which case $h = [f, f]$, for some $f \in C[0, 1]$. A sequence $\langle [f_i, g_i] \rangle_{i\in\mathbb{N}}$ of enclosures is called a *chain* if $\forall i \in \mathbb{N}$: $[f_i, g_i] \sqsubseteq [f_{i+1}, g_{i+1}]$. The enclosure $[f, g]$ is said to be the limit of such a chain if $f = \lim_{n\to\infty} f_n$ and $g = \lim_{n\to\infty} g_n$, where the limits are taken with respect to the supremum norm on $C[0, 1]$.

Let $\langle [f_i, g_i] \rangle_{i\in\mathbb{N}}$ be an arbitrary chain of enclosures. Then, for each $x \in [0, 1]$, the sequence $\langle f_i(x) \rangle_{i\in\mathbb{N}}$ is a non-decreasing sequence which is bounded from above by $g_0(x)$. By a dual argument, $\langle g_i(x) \rangle_{i\in\mathbb{N}}$ is a non-increasing sequence which is bounded from below by $f_0(x)$. Therefore, for each $x \in [0, 1]$ these sequences are convergent. This enables us to define the pointwise limits of the sequences of functions $\langle f_i \rangle_{i\in\mathbb{N}}$ and $\langle g_i \rangle_{i\in\mathbb{N}}$, which we name ϕ and ψ, respectively, as follows:

$$\forall x \in [0, 1]\colon \begin{cases} \phi(x) := \lim_{i\to\infty} f_i(x) \\ \psi(x) := \lim_{i\to\infty} g_i(x) \end{cases} \tag{1}$$

The functions $\phi, \psi : [0, 1] \to \mathbb{R}$ are well-defined but not necessarily continuous. As an example, take the chain $\{[f_i, g_i] \mid i \in \mathbb{N}\}$ defined (for all $i \in \mathbb{N}$ and $x \in [0, 1]$) by $f_i(x) = 0$ and

$$g_i(x) = \begin{cases} x/(i+1) & \text{if } 0 \leqslant x \leqslant 1 - 1/(i+2) \\ (i+1)x - i & \text{if } 1 - 1/(i+2) \leqslant x \leqslant 1 \end{cases}$$

Fig. 1 depicts this chain which is made up of piecewise linear enclosures.[1] For an example with polynomial enclosures, one can consider $g_i(x) = x^i$. It should be clear that in both cases $\forall i \in \mathbb{N}$: $[f_i, g_i] \sqsubseteq [f_{i+1}, g_{i+1}]$, but the limit of $\langle g_i \rangle_{i\in\mathbb{N}}$ is the non-continuous function

$$\psi(x) = \begin{cases} 0 & \text{if } x \in [0, 1) \\ 1 & \text{if } x = 1 \end{cases}$$

Nonetheless, if $\{h_i \mid i \in \mathbb{N}\}$ is a chain such that $\lim_{i\to\infty} \mathrm{w}(h_i) = 0$, then the chain has a limit in $\mathbb{FE}$. To see this, let us assume that for each $i \in \mathbb{N}$, $h_i = [f_i, g_i]$ and define ϕ and ψ as in (1) above. The assumption $\lim_{i\to\infty} \mathrm{w}(h_i) = 0$ implies that $\phi = \psi$. Thus, we only need to show that (say) ψ is continuous, which we will prove by contradiction.

[1] Strictly speaking, each boundary is a piecewise *affine* function consisting of affine segments. Nonetheless, to make the material more accessible, we will keep using the terms 'piecewise linear' functions and 'linear' segments, as they are well-established terms in literature.

Assume that ψ is discontinuous at some point $z \in [0,1]$. Without loss of generality, we assume that for some $\epsilon > 0$, there exists a sequence of points $\langle z_i \rangle_{i \in N}$ such that[2]:

$$\Big(\lim_{i\to\infty} z_i = z\Big) \wedge \big(\forall i \in \mathbb{N}\colon\ \psi(z_i) - \psi(z) > \epsilon\big) \tag{2}$$

As $\psi(z) = \lim_{i\to\infty} g_i(z)$, we can find an $n_0 \in \mathbb{N}$ such that $\forall n \geqslant n_0\colon\ |g_n(z) - \psi(z)| < \epsilon/4$. Because $\langle g_i(z)\rangle_{i\in\mathbb{N}}$ is a non-increasing sequence, we have $\forall n \geqslant n_0\colon\ g_n(z) - \psi(z) < \epsilon/4$. Let us fix n at n_0 for now and use the continuity of g_{n_0} to derive:

$$\begin{aligned}
& & & g_{n_0}(z) - \psi(z) < \epsilon/4 \\
&(\text{continuity of } g_{n_0} \text{ at } z) & \Rightarrow\ & \lim_{i\to\infty} g_{n_0}(z_i) - \psi(z) < \epsilon/4 \\
&(\text{property of limit}) & \Rightarrow\ & \exists i_0 \in \mathbb{N}\colon\ \forall i \geqslant i_0\colon\ g_{n_0}(z_i) < \psi(z) + \epsilon/4 \\
&\big(\text{by (2) and fixing } i \text{ at } i_0\big) & \Rightarrow\ & g_{n_0}(z_{i_0}) < \psi(z_{i_0}) - \epsilon + \epsilon/4 < \psi(z_{i_0}) - 3\epsilon/4 \\
&(\text{as } \epsilon > 0) & \Rightarrow\ & g_{n_0}(z_{i_0}) < \psi(z_{i_0})
\end{aligned}$$

which is a contradiction as by definition, at every point $x \in [0,1]$ the value of each g_i must be greater than or equal to that of ψ, i.e. $g_{n_0}(z_{i_0}) \geqslant \psi(z_{i_0})$.

We summarise these results in the following proposition:

Proposition 5. *The partial order $\mathbb{FE}$ is not complete, i.e. it is not closed under the limit operator over chains. However, if $\{h_i \mid i \in \mathbb{N}\}$ is a chain such that $\lim_{i\to\infty} \mathrm{w}(h_i) = 0$, then the chain has a limit in $\mathbb{FE}$.*

The condition $\lim_{i\to\infty} \mathrm{w}(h_i) = 0$ is sufficient but not necessary for convergence in $\mathbb{FE}$. For instance, the sequence $\langle [f_i, g_i] \rangle_{i\in\mathbb{N}}$ in which

$$\forall i \in \mathbb{N}\colon\ \forall x \in [0,1]\colon\ \begin{cases} f_i(x) = 0 \\ g_i(x) = x(i+2)/(i+1) \end{cases}$$

is a chain which converges to the enclosure $[\lambda x.0, \lambda x.x]$ whose width is 1.

Basic concepts in function enclosure arithmetic [5,20] will be used in this paper. The intuition behind this arithmetic is very simple as operators on functions mimic operators of ordinary interval arithmetic [18]. For instance, addition can be easily overloaded with function enclosures as in $[f_1, g_1] + [f_2, g_2] := [f_1 + f_2, g_1 + g_2]$. With other operations (such as $\times$, etc.), a bit of tweaking is needed, similar to the case of ordinary interval arithmetic.

It is possible to be more practically minded and restrict oneself to a set of finitely-representable function enclosures, and still be able to approximate every function in $C[0,1]$. For instance, one may consider the set of enclosures $[f, g]$ such that f and g are polynomials with rational coefficients.

Of course objects are finitely-representable only relative to some specific language $L \subseteq \Sigma^*$, for some alphabet Σ. To make the presentation easier, we assume that our alphabet is rich enough to include:

- 0 and 1;
- some appropriate symbols for arithmetic operators;
- some appropriate symbols for forming lists, pairs and lambda expressions.

All of these can be encoded by appropriate Turing machines, and as we study asymptotic behaviour, the inclusion or exclusion of these elements in Σ will not affect our results. Nonetheless, it is crucial that we keep the alphabet finite, as we will see from the proof of Proposition 10.

We say that an enclosure h is *approximated* by another enclosure g if g is finitely-representable and $g \sqsubseteq h$.[3] A countable set $\mathcal{B}$ of finitely-representable enclosures is called a *basis* for $\mathbb{FE}$ if each $h \in \mathbb{FE}$ is the limit of a chain $\langle h_i \rangle_{i\in\mathbb{N}}$ of elements in $\mathcal{B}$. Examples of bases are:

- the set of polynomials with rational coefficients;
- the set of continuous piecewise-linear functions, where the end-points of each linear segment have rational coordinates.

Now let $\mathcal{B}$ be a basis for $\mathbb{FE}$. We call $\rho : \mathbb{N} \to \mathcal{B}$ a binary representation of $f \in C[0,1]$ if $\forall n \in \mathbb{N}\colon\ (\rho(n) \sqsubseteq [f, f]) \wedge (\mathrm{w}(\rho(n)) < 2^{-n})$.

[2] The case of $(\forall i \in \mathbb{N}\colon\ \psi(z) - \psi(z_i) > \epsilon)$ can be dealt with in a similar manner.

[3] Note that our definition of approximation is different from the way-below relation used in domain theory [1, Definition 2.2.1]. In particular, in our framework any finitely-representable enclosure g approximates itself.

Remark 6. All of our results can be easily generalised to any base. Thus, to save space, we will refer to binary representations simply as representations.

Notation 1. We reserve the notation $K(x)$ to denote the Kolmogorov complexity of any finitely-representable object x. This includes objects such as integer numbers, finite strings over a finite alphabet, arbitrary-precision floating-point numbers, finitely-representable function enclosures, and the like.

Proposition 7. *For every function $f \in C[0,1]$ there exists a representation $\hat{f}$ of f of minimal Kolmogorov complexity, i.e. for any other representation of f such as $\rho : \mathbb{N} \to \mathcal{B}$:*

$$\forall n \in \mathbb{N}\colon\ K\big(\hat{f}(n)\big) \leqslant K\big(\rho(n)\big)$$

Proof. Assume that $\mathcal{B}$ is enumerated as $\langle h_i \rangle_{i\in\mathbb{N}}$. In order to define $\hat{f}$ over a certain $n \in \mathbb{N}$, first let X_n be those elements of $\mathcal{B}$ that approximate f and have width smaller than 2^{-n}, i.e. $X_n := \{h \in \mathcal{B} \mid h \sqsubseteq [f, f], \mathrm{w}(h) < 2^{-n}\}$ and consider the set $K(X_n) = \{K(h) \mid h \in X_n\}$ which is the image of X_n under K. This is a subset of $\mathbb{N}$, hence it has a least element n_0. All that remains is to assign to $\hat{f}(n)$ one of the elements in $K^{-1}(n_0)$, for instance, the one with the smallest index in the enumeration $\langle h_i \rangle_{i\in\mathbb{N}}$ of $\mathcal{B}$. □

Definition 8 *(Optimal representation).* The representation $\hat{f}$ as in Proposition 7 is called an optimal representation of the function f.

Definition 9 *($K_C(f)$).* Let $\hat{f} : \mathbb{N} \to \mathcal{B}$ be an optimal representation of $f \in C[0,1]$. The *Kolmogorov complexity function* of f is defined as:

$$\begin{aligned} K_C(f) &: \mathbb{N} \to \mathbb{N} \\ &\quad n \mapsto K\big(\hat{f}(n)\big) \end{aligned}$$

We will consider the asymptotic growth of $K_C(f)$ for functions in $C[0,1]$. This way we can study the set of functions f for which $K_C(f)$ is bounded in some way, e.g. by some polynomial, or some exponential function.

We will first prove in Proposition 10 that there are functions in $C[0,1]$ with arbitrarily fast-growing Kolmogorov complexity functions. In the proof we seem to go to some length to manufacture 'one' such function. But, then—as is usually the case in mathematics—in Section 5 we will show that in fact 'almost all' functions in $C[0,1]$ have this property.

Proposition 10. *For any given $\theta : \mathbb{N} \to \mathbb{N}$, there exists a function f in $C[0,1]$ whose Kolmogorov complexity is above θ over infinitely many points. In other words*:

$$\forall m \in \mathbb{N}\colon\ \exists n \geqslant m\colon\ K_C(f)(n) \geqslant \theta(n)$$

Proof. See Appendix A. □

For instance, in Proposition 10, by taking $\theta(n)$ to be:

1. 2^n, one can show that there exists a real function $f \in C[0,1]$ whose Kolmogorov complexity $K_C(f)$ is not dominated by any polynomial.
2. $n!$, one can show that there exists a real function $f \in C[0,1]$ whose Kolmogorov complexity $K_C(f)$ is not dominated by any exponential function.

3. Invariant ideals

Consider the poset $(\mathbb{N}^{\mathbb{N}}, \preccurlyeq)$ in which $\preccurlyeq$ is the pointwise ordering on functions: $f \preccurlyeq g \Leftrightarrow \forall n \in \mathbb{N}\colon\ f(n) \leqslant g(n)$, and define the operators $\vee, \wedge : \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ by $(f \vee g)(n) = \max(f(n), g(n))$ and $(f \wedge g)(n) = \min(f(n), g(n))$. This way one obtains a lattice, in which certain ideals (and their complements) are of interest to our discussion. For instance, the ideal of all functions smaller than some polynomial (or some exponential) function. Remember that for each $f \in \mathbb{N}^{\mathbb{N}}$ the principal ideal $\downarrow f$ is defined as $\downarrow f := \{g \in \mathbb{N}^{\mathbb{N}} \mid g \preccurlyeq f\}$.

Definition 11 *(Invariant ideal).* We call a set $\mathcal{U} \subseteq \mathbb{N}^{\mathbb{N}}$ a translation-invariant proper ideal—invariant ideal, to be brief—of $\mathbb{N}^{\mathbb{N}}$ if

1. $\mathcal{U}$ contains the identity function: $\lambda n.n \in \mathcal{U}$.

2. $\mathcal{U}$ is a lower set: $\forall f \in \mathcal{U}, g \in \mathbb{N}^{\mathbb{N}}$: $g \preccurlyeq f \Rightarrow g \in \mathcal{U}$.
3. $\mathcal{U}$ is closed under addition: $\forall f, g \in \mathcal{U}$: $\lambda n.f(n) + g(n) \in \mathcal{U}$.
4. $\mathcal{U}$ is closed under translation: $\forall f \in \mathcal{U}, k \in \mathbb{N}$: $\lambda n.f(n+k) \in \mathcal{U}$.
5. There exists a countable set of functions B such that $\mathcal{U} = \bigcup\{\downarrow f \mid f \in B\}$. Such a set B will be referred to as a *basis* for $\mathcal{U}$.

Examples of bases that 'generate' invariant ideals are the set of polynomials with integer coefficients, or the set of exponential functions $\lambda n.a2^n + b$, where $a, b \in \mathbb{N}$.

Proposition 12. *For any invariant ideal $\mathcal{U}$, the following are true*:

(i) $\mathcal{U} \neq \emptyset \wedge \mathcal{U} \neq \mathbb{N}^{\mathbb{N}}$.
(ii) $\mathcal{U}$ *includes every constant function $\lambda n.p$ and every affine function $\lambda n.pn + q$, for every $p, q \in \mathbb{N}$.*

Proof.

(i) $\mathcal{U} \neq \emptyset$ because it contains the identity function. Now consider a countable basis $B = \{f_0, f_1, f_2, \ldots\}$ for $\mathcal{U}$ and define the function $h : \mathbb{N} \to \mathbb{N}$ by $h(n) = f_n(n) + 1$. Then $h \notin \mathcal{U}$.
(ii) Let p and q be natural numbers. The invariant ideal $\mathcal{U}$ contains the identity and is closed under translation, therefore $\lambda n.n + p \in \mathcal{U}$. Moreover, $\mathcal{U}$ is a lower set, thus $\lambda n.p \in \mathcal{U}$.
As $\mathcal{U}$ is closed under addition, adding identity p times to itself would give $\lambda n.pn \in \mathcal{U}$, and by adding the constant function $\lambda n.q$, we obtain $\lambda n.pn + q \in \mathcal{U}$. □

From now on we reserve the symbol $\mathcal{U}$ to denote an invariant ideal. Note that for each invariant ideal $\mathcal{U}$, the set $K_C^{-1}(\mathcal{U})$ consists of those functions in $C[0,1]$ whose Kolmogorov complexity functions are members of $\mathcal{U}$, i.e.

$$K_C^{-1}(\mathcal{U}) = \{f \in C[0,1] \mid K_C(f) \in \mathcal{U}\}$$

Proposition 13. *For any invariant ideal $\mathcal{U}$ the following are true*:

(a) *The set $K_C^{-1}(\mathcal{U})$ is closed under arithmetic operations.* (*Note that for division $h = f/g$, we require that g be nowhere zero, i.e. we do not allow division by zero.*)
(b) *The set $K_C^{-1}(\mathcal{U})$ is an F_σ set, i.e. it is the union of a countable family of closed sets.*
(c) *The set $K_C^{-1}(\mathcal{U})$ is Borel.*
(d) *Let $\mathcal{U}^c$ denote the complement of $\mathcal{U}$, then*:

$$\forall f \in K_C^{-1}(\mathcal{U}^c),\ g \in K_C^{-1}(\mathcal{U}),\ \tau \in \mathbb{R} \setminus \{0\}: \quad (\tau f + g) \in K_C^{-1}(\mathcal{U}^c)$$

Proof. See Appendix B. □

4. Prevalence and shyness

Consider a topological real vector space V and let Φ be a predicate defined over V with support $S \subseteq V$, i.e. $\forall s \in V$: $(\Phi(s) \Leftrightarrow s \in S)$. In the case $\dim(V) = k < \infty$, one may use the k-dimensional Lebesgue measure Λ_k in order to express statements such as '*Φ holds almost everywhere in V*', which would mean $\Lambda_k(V \setminus S) = 0$.

Now let V be an infinite-dimensional, separable[4] Banach space. Over such a space, we do not have any measure with properties similar to that of the Lebesgue measure over Euclidean spaces. To be more precise: "any translation-invariant measure μ over V which is not identically zero has the property that all open sets have infinite measure" [15, p. 2].

A translation-invariant alternative for 'almost every' in such infinite-dimensional spaces is 'prevalence', as introduced by Hunt et al. [15]. In other words, when V is infinite-dimensional, the statement '*S is a prevalent subset of V*' gives us the same quality of information as the statement 'Φ holds almost everywhere in V' would give, were V finite-dimensional.

Here we do not need the full generality of the original definition of prevalence, for which the reader is referred to [15]. Instead, we use a simpler condition for prevalence which is based on the concept of a probe:

Definition 14 *(Probe).* Let $S \subseteq C[0,1]$. A finite-dimensional subspace P of $C[0,1]$ is said to be a probe for S if for all $f \in C[0,1]$, Lebesgue-almost every point in the hyperplane $f + P$ belongs to S.

For our purposes, prevalence of a set is implied by its having a Borel subset with a probe:

[4] A topological space is called separable if it contains a countable dense subset.

Proposition 15 *(Prevalent set). Let $S \subseteq C[0,1]$:*

1. *If S is a Borel set, then S is prevalent if it has a probe.*
2. *If S is not a Borel set, then S is prevalent if it has a Borel subset $S' \subseteq S$ that has a probe.*

Proof. See [14,15]. □

Remark 16. The condition of having a probe is sufficient *but not necessary* for a Borel set to be prevalent. In fact, according to the general definition given in [15], a Borel set may be prevalent without having a probe.

Definition 17 *(Shy set).* A set $S \subseteq C[0,1]$ is said to be shy if its complement in $C[0,1]$ is prevalent.

5. Main theorem

Consider an invariant ideal $\mathcal{U}$, with $B = \{\phi_i \mid i \in \mathbb{N}\}$ as its basis. Define the function $\theta : \mathbb{N} \to \mathbb{N}$ over each $n \in \mathbb{N}$ as follows:

$$\theta(n) := \begin{cases} 1 + \phi_0(0) & \text{if } n = 0 \\ 1 + \max(\{\theta(j) \mid 0 \leqslant j \leqslant n-1\} \cup \{\phi_k(n) \mid 0 \leqslant k \leqslant n\}) & \text{if } n \geqslant 1 \end{cases} \tag{3}$$

This way we get:

$$\forall m, n \in \mathbb{N}\colon\ n \geqslant m \quad \Rightarrow \quad \phi_m(n) < \theta(n) \tag{4}$$

Now consider the function f obtained by applying Proposition 10 over this θ. The Kolmogorov complexity of f dominates θ, and by implication—using (4)—is strictly greater than each ϕ_m, over infinitely many points. This means that f belongs to $K_C^{-1}(\mathcal{U}^c)$, and therefore:

Lemma 18. *$K_C^{-1}(\mathcal{U}^c) \neq \emptyset$.*

In fact, we will prove that for any invariant ideal $\mathcal{U}$, 'almost every' function in $C[0,1]$ belongs to $K_C^{-1}(\mathcal{U}^c)$, where $\mathcal{U}^c$ is the set complement of $\mathcal{U}$ in $\mathbb{N}^{\mathbb{N}}$:

Theorem 1. *For any invariant ideal $\mathcal{U}$, the set $K_C^{-1}(\mathcal{U}^c)$ is a prevalent subset of $C[0,1]$.*

Proof. Using Lemma 18, we can pick a function $f \in K_C^{-1}(\mathcal{U}^c)$. The one-dimensional hyperplane $\mathcal{P} := \{\tau f \mid \tau \in \mathbb{R}\}$ is a probe for $K_C^{-1}(\mathcal{U}^c)$. To see this, let g be any function in $C[0,1]$. We show that for at most one $\tau \in \mathbb{R}$, the function $\tau f + g$ may belong to $K_C^{-1}(\mathcal{U})$. Assume that this is not true. Then, there must be two different real numbers $\tau_1 \neq \tau_2$ such that:

$$\begin{cases} h_1 := \tau_1 f + g \in K_C^{-1}(\mathcal{U}) \\ h_2 := \tau_2 f + g \in K_C^{-1}(\mathcal{U}) \end{cases}$$

But then by item (a) of Proposition 13, we must have:

$$f = \frac{h_1 - h_2}{\tau_1 - \tau_2} \in K_C^{-1}(\mathcal{U})$$

which is a contradiction. In other words, for each $g \in C[0,1]$, Lebesgue-almost every point in $\mathcal{P} + g$ belongs to $K_C^{-1}(\mathcal{U}^c)$.

In part (c) of Proposition 13, we proved that $K_C^{-1}(\mathcal{U})$ is Borel, which implies that $K_C^{-1}(\mathcal{U}^c)$ is Borel too. This completes the proof. □

6. An asymptotic bound on the Kolmogorov complexities of the computable functions

Let us denote the set of all computable functions in $C[0,1]$ by $\tilde{C}[0,1]$ and assume that this set is enumerated as a sequence $\langle f_i \rangle_{i \in \mathbb{N}}$, with their respective Kolmogorov complexities enumerated as $\langle \phi_i \rangle_{i \in \mathbb{N}}$. We define θ as in (3) and (using Proposition 10) obtain a function f whose Kolmogorov complexity strictly dominates the Kolmogorov complexity of every computable f_i, over infinitely many points. In fact, we can go further: let $\mathcal{U}_\theta$ be the smallest invariant ideal that includes θ. It can be proved in the usual way that such an invariant ideal does exist. The set $K_C^{-1}(\mathcal{U}_\theta)$ is a shy subset of $C[0,1]$ which includes $\tilde{C}[0,1]$.

Just by using the closure properties as demanded by the definition of an invariant ideal, one can show that for every $f \in K_C^{-1}({\mathcal{U}_\theta}^c)$ and every $f_i \in \tilde{C}[0,1]$, there exists an infinite set $J \subseteq \mathbb{N}$ such that $\forall j \in J\colon\ \phi_i(j) < K_C(f)(j)$, hence:

Theorem 2. *The Kolmogorov complexity function of every* $f \in K_C^{-1}(\mathcal{U}_\theta{}^c)$ *is an asymptotic upper bound for the Kolmogorov complexity function of any computable function* g *in* $\tilde{C}[0,1]$, *i.e. there exists an infinite set* $J \subseteq \mathbb{N}$ *such that* $\forall j \in J\colon K_C(g)(j) < K_C(f)(j)$.

7. Summary and discussion

We have defined a notion of Kolmogorov complexity for functions in $C[0,1]$ by drawing inspiration from the work of Cai and Hartmanis [4]. Essentially, we have taken into account the representation of an infinite object—in our case a continuous real function—as the limit of a sequence of finite approximations, and then have based the definition of its Kolmogorov complexity on the growth rate of the descriptive complexities of those finite approximations.

Even though the material is about a subject almost exclusive to computer science, the space $C[0,1]$ over which we studied the concept of Kolmogorov complexity includes both computable and non-computable functions. Therefore, it is not possible to interpret this result in a purely Turing-computable framework. Yet, it touches upon the issue of representation, and in this respect, Theorem 1 can be interpreted as stating that 'almost all' continuous real functions 'are expensive to represent via finitely-representable approximations'. In light of this interpretation, even if all of $C[0,1]$ is supplied to us by an oracle, computations that require information regarding the values of a function over all the points of its domain—think of integration for example—are infeasible over 'almost all' continuous real functions, as long as we are restricted by finitely-representable approximations.

We focused on some specific subsets of $\mathbb{N}^{\mathbb{N}}$, which we called the invariant ideals, and demonstrated that their inverse images under the Kolmogorov complexity function have got some certain closure, topological and measure-theoretic properties (Proposition 13). Through these invariant ideals we obtain a hierarchy within $C[0,1]$, which we will be studying further in our future research.

It should also be noted that whereas the Kolmogorov complexity of real functions has been studied in some other contexts, such as descriptive set theory [2], Brownian motion [11] and abstract recursion theory [12], we are mainly interested in the broader abstract complexity theory over Banach spaces, of which $C[0,1]$ is an instance. Therefore, we have presented the results of this paper in order to shed some more light on the complexity theoretic properties of real function spaces, and we will be taking this framework further and incorporating it into a more mainstream computational complexity framework in our future research.

Acknowledgements

I am very grateful to Jan Duracz for his proof-reading of the earlier manuscript of this paper and his helpful suggestions. I would also like to thank Douglas Cenzer and Willem Fouché for informing me of their related work on the subject of algorithmic randomness over function spaces.

Appendix A. Proof of Proposition 10

First fix a basis $\mathcal{B}$ for $\mathbb{FE}$. We show that there are sequences $\langle n_i \rangle_{i\in\mathbb{N}}$ of natural numbers and $\langle f_i \rangle_{i\in\mathbb{N}}$ of function enclosures in $\mathcal{B}$ such that:

(i) $\forall i \in \mathbb{N}\colon n_i < n_{i+1}$.
(ii) $\forall i \in \mathbb{N}\colon f_i \sqsubseteq f_{i+1}$.
(iii) $\forall i \in \mathbb{N}\colon K(f_i) \geqslant \theta(n_i)$.
(iv) $\forall i \in \mathbb{N}\colon 0 < \mathrm{w}(f_i) < 2^{-n_i}$. As a result there exists a unique function $f \in C[0,1]$ which is enclosed by every f_i. This will be the function f as required in the statement of the proposition.
(v) For any $g \in \mathcal{B}$ enclosing f: $\mathrm{w}(g) \leqslant 2^{-n_i} \Rightarrow K(g) \geqslant \theta(n_i)$.

This will prove the proposition. The sequences $\langle n_i \rangle_{i\in\mathbb{N}}$ and $\langle f_i \rangle_{i\in\mathbb{N}}$ are both constructed inductively, and to ensure the crucial item (v), we define a countable set $\tilde{P}$ of points in $[0,1]\times\mathbb{R}$ through which the function f has to pass.

Initially, we take $\tilde{P}$ to be the empty set and at each stage we add finitely many points $\tilde{P}_i$ to $\tilde{P}$. Any function that passes through these points avoids being enclosed by 'low' Kolmogorov complexity enclosures. Here is the inductive procedure:

The base case: Let $n_0 = 0$ and enumerate all elements of $\mathcal{B}$ of width less than 1 $(= 2^{-n_0})$ in a sequence such as $\langle g_0, g_1, \ldots, g_k, \ldots \rangle$ in non-descending Kolmogorov complexity order, that is $\forall i \in \mathbb{N}\colon K(g_i) \leqslant K(g_{i+1})$. Note that this is possible as the alphabet in our language is finite and therefore, for any natural number $t \in \mathbb{N}$, there are only finitely many enclosures g in $\mathcal{B}$ such that $K(g) = t$. Now, find the smallest $i_0 \in \mathbb{N}$ such that $K(g_{i_0}) \geqslant \theta(0)$ and $\mathrm{w}(g_{i_0}) \neq 0$. Define $f_0 := g_{i_0}$ and $\tilde{P}_0 := \emptyset$.

The induction step: Assume that we have got finite sequences $\langle n_i \rangle_{0\leqslant i\leqslant m}$ and $\langle f_i \rangle_{0\leqslant i\leqslant m}$ that satisfy the necessary conditions. As f_m is an enclosure we assume $f_m = [f_m^{\min}, f_m^{\max}]$, and as $\mathrm{w}(f_m) > 0$ then $\exists z \in (0,1)\colon f_m^{\min}(z) < f_m^{\max}(z)$.

Both $f_m^{\min}$ and $f_m^{\max}$ are continuous, therefore for some compact interval $D = [\ell, r]$ of non-zero width (i.e. $\ell < r$) and some $\rho > 0$ we get $\forall x \in D\colon f_m^{\max}(x) - f_m^{\min}(x) \geqslant \rho$. Choose $n_{m+1} \in \mathbb{N}$ in such a way that:

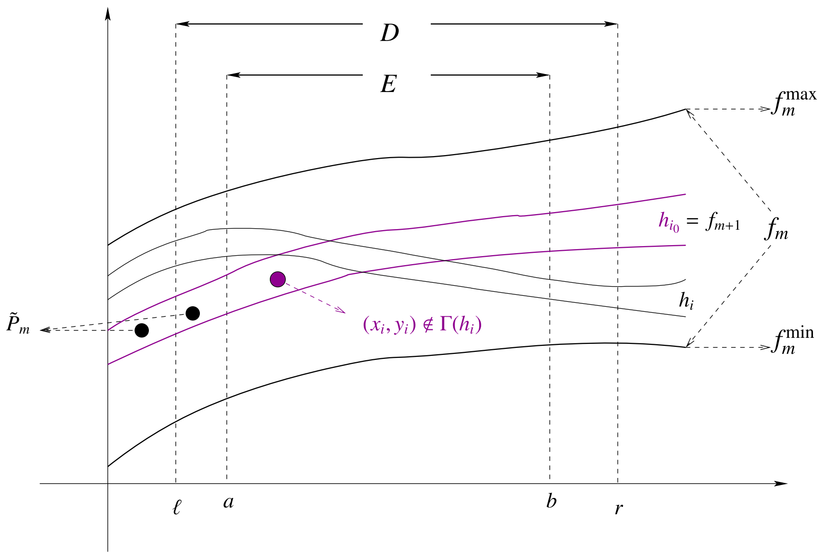

Fig. 2. (x_i, y_i) guarantees that $f = \bigsqcup_{m\in\mathbb{N}} f_m$ is not enclosed by the 'low' Kolmogorov complexity enclosure h_i.

$$(n_{m+1} > n_m) \wedge \left(2^{-n_{m+1}} < \frac{\rho}{2}\right)$$

Let $\langle h_0, h_1, \ldots, h_k, \ldots\rangle$ be an enumeration of all enclosures in $\mathcal{B}$ with widths smaller than $2^{-n_{m+1}}$ arranged in non-descending Kolmogorov complexity order. Initially, we assume that $\tilde{P}_{m+1} = \tilde{P}_m$ and search for the smallest $i_0 \in \mathbb{N}$ with the following properties:

(a) $0 < \mathrm{w}(h_{i_0}) < 2^{-n_{m+1}}$.

(b) $f_m \sqsubseteq h_{i_0}$.

(c) $K(h_{i_0}) \geqslant \theta(n_{m+1})$.

(d) $\tilde{P}_{m+1} \subseteq \Gamma(h_{i_0})$, and if h is any enclosure in $\mathcal{B}$ with $K(h) < \theta(n_{m+1})$, then $\tilde{P}_{m+1} \nsubseteq \Gamma(h)$.

If $i_0 = 0$ satisfies all these properties, then we are done. Otherwise, we go through an inductive process. On the way, we modify $\tilde{P}_{m+1}$ and make sure that condition (d) is guaranteed (see Fig. 2).

First let $\langle h_0, h_1, \ldots, h_q\rangle$ be the initial segment of the sequence $\langle h_i\rangle_{i\in\mathbb{N}}$ such that:

1. $\forall i \leqslant q$: $K(h_i) < \theta(n_{m+1})$.
2. $\forall i > q$: $K(h_i) \geqslant \theta(n_{m+1})$.

As $\tilde{P}_m$ is a finite set, there exists a compact interval $E = [a, b]$ of non-zero width (i.e. $a < b$) such that $E \subseteq D \setminus \pi_1(\tilde{P}_m)$, where π_1 is the projection over the first coordinate. In other words, E is a compact subinterval of D that does not include the x-coordinate of any of the points in $\tilde{P}_m$.

Let $P_{m+1} = \{(x_0, y_0), (x_1, y_1), \ldots, (x_q, y_q)\} \subseteq E \times \mathbb{R}$ be a set of points such that:

1. No two points have the same x-coordinate: $\forall i < j \leqslant q$: $x_i \neq x_j$. This is possible because the compact interval E is not a singleton.
2. $\forall i \leqslant q$: $(x_i, y_i) \in \Gamma(f_m) \setminus (\Gamma(h_i) \cup \tilde{P}_m)$. This is possible because for each $i \in \mathbb{N}$ we have $\mathrm{w}(h_i) \leqslant \rho/2$.

The points in P_{m+1} are added to $\tilde{P}_m$ to get the set $\tilde{P}_{m+1}$. Thus $\tilde{P}_{m+1} := \tilde{P}_m \cup P_{m+1}$. Now we can start searching from $(q+1)$ upwards and find the smallest $i_0 > q$ such that:

1. $f_m \sqsubseteq h_{i_0}$.
2. $\tilde{P}_{m+1} \subseteq \Gamma(h_{i_0})$.
3. $0 < \mathrm{w}(h_{i_0})$.

It is straightforward to see that there exists such an i_0. Properties (c) and (d) follow from the way we constructed our sequences.

Now define $f_{m+1} := h_{i_0}$ and continue with the next iteration of the inductive process.

Finally, let $\tilde{P} = \bigcup\{\tilde{P}_m \mid m \in \mathbb{N}\}$. It should be obvious that $\langle n_i\rangle_{i\in\mathbb{N}}$ and $\langle f_i\rangle_{i\in\mathbb{N}}$ satisfy (i)–(iv) on page 573. To see how condition (v) is satisfied, note that if $g \in \mathcal{B}$ is any enclosure of f of width less than 2^{-n_m} (for any $m \in \mathbb{N}$), then $\tilde{P}_{m+1} \subseteq \Gamma(g)$. This means that g cannot be any of the enclosures $h_0, h_1, \ldots, h_q$ of Kolmogorov complexities less than $\theta(n_m)$. □

Appendix B. Proof of Proposition 13

(a) Consider $f, g \in K_C^{-1}(\mathcal{U})$ with optimal representations $\hat{f}$ and $\hat{g}$, respectively. Let us go through the proof for each of the arithmetic operators one by one:

addition: let $h := f + g$. To obtain a function enclosure h_n of h of width smaller than 2^{-n}, all we need is to take h_n to be $\hat{f}(n+1) + \hat{g}(n+1)$. Addition over finitely-representable approximations $\hat{f}(n+1)$ and $\hat{g}(n+1)$ adds only a constant C^+ to the Kolmogorov complexity, in other words:

$$K(h_n) \leqslant K\big(\hat{f}(n+1)\big) + K\big(\hat{g}(n+1)\big) + C^+$$

Remember that $\mathcal{U}$ is closed under addition and translation and it contains the constant function $\lambda n.C^+$. Thus $K_C(h) \in \mathcal{U}$.

subtraction: similar to the case of addition.

multiplication: let $h := f \times g$. As g is continuous, it must be bounded by a constant M_g, i.e. $\forall x \in [0, 1]$: $|g(x)| < M_g$. Take N_g to be a natural number such that $2^{N_g} > M_g$. To obtain a function enclosure h_n of h of width smaller than 2^{-n}, all we need is to take h_n to be $\hat{f}(n + N_g) \times \hat{g}(n + N_g)$. As $\times$ adds only a constant to the sum of the Kolmogorov complexities of $\hat{f}(n + N_g)$ and $\hat{g}(n + N_g)$, the rest of the proof for this case is similar to the case of addition.

division: similar to the case of multiplication: let $h := f/g$. Note that division by zero is ruled out. Hence, there exists a natural number M_g such that

$$\forall x \in [0, 1]\colon\ \big|g(x)\big| > \frac{1}{M_g}$$

Thus, if we pick some $N_g \in \mathbb{N}$ large enough so that $2^{N_g} > M_g$, then $\forall n \geqslant N_g$: $\forall x \in [0, 1]$: $(x, 0) \notin \Gamma(\hat{g}(n))$. In other words, for n greater than N_g, no enclosure $\hat{g}(n)$ crosses the x-axis. The rest of the proof is similar to the case of multiplication.

(b) Let $B = \{\theta_i \mid i \in \mathbb{N}\}$ be a basis for $\mathcal{U}$, which means that $\mathcal{U} = \bigcup\{\downarrow\theta_i \mid i \in \mathbb{N}\}$. We write $\mathrm{cl}(A)$ to denote the closure of a set A and we define:

$$\mathrm{UCL} := \bigcup\big\{\mathrm{cl}\big(K_C^{-1}(\downarrow\theta_i)\big)\ \big|\ i \in \mathbb{N}\big\}$$

and prove that

$$K_C^{-1}(\mathcal{U}) = \mathrm{UCL}$$

Obviously $K_C^{-1}(\mathcal{U}) \subseteq \mathrm{UCL}$. So, it will suffice to prove that:

$$\forall i \in \mathbb{N}\colon\ \mathrm{cl}\big(K_C^{-1}(\downarrow\theta_i)\big) \subseteq K_C^{-1}(\mathcal{U})$$

Fix a natural number $p \in \mathbb{N}$ and take an arbitrary function $f \in \mathrm{cl}(K_C^{-1}(\downarrow\theta_p))$. This means that there exists a sequence $\langle h_0, h_1, \ldots, h_n, \ldots\rangle$ of functions in $K_C^{-1}(\downarrow\theta_p)$ such that $f = \lim_{i\to\infty} h_i$, where this limit is taken with regards to the supremum norm. There exists a Cauchy subsequence $\langle h_{i_0}, h_{i_1}, \ldots, h_{i_n}, \ldots\rangle$ of this sequence which converges to f, therefore:

$$\forall j \in \mathbb{N}\colon\ \|f - h_{i_j}\|_{\mathrm{sup}} < 2^{-j}$$

Define the sequence $\langle f_0, f_1, \ldots, f_n, \ldots\rangle$ by letting $f_j := h_{i_j}$ for all $j \in \mathbb{N}$. Each f_i has an optimal representation $\hat{f}_i$ such that $K(\hat{f}_i) \in \downarrow\theta_p$.

Now define the sequence of function enclosures $\langle \phi_0, \phi_1, \ldots, \phi_n\rangle$ by:

$$\forall n \in \mathbb{N}\colon\ \phi_n = \hat{f}_{n+1}(n+1) + \lambda x.\left[-\frac{1}{2^{n+2}}, \frac{1}{2^{n+2}}\right] \tag{B.1}$$

In other words, the nth enclosure ϕ_n is obtained by taking the enclosure $\hat{f}_{n+1}(n+1)$ (which has width smaller than $2^{-(n+1)}$) and adding the strip $\lambda x.[-\frac{1}{2^{n+2}}, \frac{1}{2^{n+2}}]$ (which has the constant width $2^{-(n+1)}$). It is not difficult to see that the width of ϕ_n is smaller than 2^{-n} and $f \in \phi_n$, because:

(a) $\hat{f}_{n+1}(n+1)$ approximates f_{n+1} to within $2^{-(n+1)}$ accuracy;
(b) f_{n+1} in turn approximates f to within $2^{-(n+1)}$ accuracy;
(c) ϕ_n contains a neighbourhood of f_{n+1} of width $2^{-(n+1)}$.

Thus, $\langle \phi_0, \phi_1, \ldots, \phi_n, \ldots \rangle$ is a valid binary representation of f. On the other hand, the definition of ϕ_n as in (B.1) implies that:

$$\exists C_1, C_2 \in \mathbb{N}\colon\ K(\phi_n) \leqslant K\big(\hat{f}_{n+1}(n+1)\big) + C_1 n + C_2 \tag{B.2}$$

The constant C_2 is added to account for the addition and the lambda term, and C_1 is needed because the length of the binary representation of $1/2^{n+2}$ is linearly proportional to n.

Remember that:

(i) $\forall n \in \mathbb{N}\colon\ \hat{f}_n \in \downarrow \theta_p$.

(ii) $\mathcal{U}$ contains the linear function $\lambda n.C_1 n$, the constant function $\lambda n.C_2$, and is closed under translation and addition.

Therefore, there exists a $q \in \mathbb{N}$ such that:

$$\begin{aligned} & && \lambda n.\theta_p(n) + C_1 n + C_2 \in \downarrow \theta_q \\ &\big(\text{by (B.2)}\big) && \Rightarrow \quad \lambda n.K(\phi_n) \in \downarrow \theta_q \\ &(\theta_q \in B) && \Rightarrow \quad \lambda n.K(\phi_n) \in \mathcal{U} \\ &(\mathcal{U}\text{ is a lower set}) && \Rightarrow \quad K_C(f) \in \mathcal{U} \\ & && \Rightarrow \quad f \in K_C^{-1}(\mathcal{U}). \end{aligned}$$

(c) Follows from (b).

(d) Let $h = \tau f + g$ and assume that $h \in K_C^{-1}(\mathcal{U})$. But then by (a)

$$f = (h - g)/\tau \in K_C^{-1}(\mathcal{U})$$

which is a contradiction. □

References

[1] S. Abramsky, A. Jung, Domain theory, in: S. Abramsky, D.M. Gabbay, T.S.E. Maibaum (Eds.), Handbook of Logic in Computer Science, vol. 3, Clarendon Press, Oxford, 1994, pp. 1–168.

[2] G. Barmpalias, P. Brodhead, D.A. Cenzer, J.B. Remmel, R. Weber, Algorithmic randomness of continuous functions, Arch. Math. Logic 46 (7–8) (2008) 533–546.

[3] L. Blum, M. Shub, S. Smale, On a theory of computation and complexity over the real numbers; NP completeness, recursive functions and universal machines, Bull. Amer. Math. Soc. (N.S.) 21 (1) (1989) 1–46.

[4] J. Cai, J. Hartmanis, On Hausdorff and topological dimensions of the Kolmogorov complexity of the real line, J. Comput. System Sci. 49 (3) (1994) 605–619.

[5] J.A. Duracz, M. Konečný, Polynomial function enclosures and floating point software verification, in: Proceedings of CFV 2008 in Sydney, available from http://www-users.aston.ac.uk/~konecnym/papers/cfv08.pdf, 2008.

[6] A. Edalat, A. Lieutier, Domain theory and differential calculus (functions of one variable), in: Proceedings of 17th Annual IEEE Symposium on Logic in Computer Science (LICS'02), Copenhagen, Denmark, 2002, pp. 277–286.

[7] A. Edalat, P. Sünderhauf, A domain theoretic approach to computability on the real line, Theoret. Comput. Sci. 210 (Jan. 1999) 73–98.

[8] M.H. Escardó, PCF extended with real numbers: a domain theoretic approach to higher order exact real number computation, Ph.D. thesis, Imperial College, 1997.

[9] A. Farjudian, Shrad: A language for sequential real number computation, Theory Comput. Syst. 41 (1) (2007) 49–105.

[10] A. Farjudian, M. Konečný, Time complexity and convergence analysis of domain theoretic Picard method, in: W. Hodges, R. de Queiroz (Eds.), Proceedings of the 15th International Workshop on Logic, Language, Information and Computation, WoLLIC '08, Edinburgh, Scotland, in: Lecture Notes in Artificial Intelligence, vol. 5110, Springer-Verlag, 2008, pp. 149–163.

[11] W.L. Fouché, Arithmetical representations of Brownian motion I, J. Symb. Log. 65 (1) (2000) 421–442.

[12] W.L. Fouché, Dynamics of a generic Brownian motion: Recursive aspects, Theoret. Comput. Sci. 394 (2008) 175–186.

[13] G. Gierz, K.H. Hofmann, K. Keimel, J.D. Lawson, M.W. Mislove, D.S. Scott, Continuous Lattices and Domains, Encyclopedia Math. Appl., vol. 93, Cambridge University Press, 2003.

[14] B.R. Hunt, The prevalence of continuous nowhere differentiable functions, Proc. Amer. Math. Soc. 122 (3) (1994) 711–717.

[15] B.R. Hunt, T. Sauer, J.A. Yorke, Prevalence: A translation-invariant "almost every" on infinite-dimensional spaces, Bull. Amer. Math. Soc. 27 (2) (October 1992) 217–238.

[16] B. Lambov, Reallib: An efficient implementation of exact real arithmetic, Math. Structures Comput. Sci. 17 (1) (2007) 81–98.

[17] J.L. Montaña, L.M. Pardo, On Kolmogorov complexity in the real Turing machine setting, Inform. Process. Lett. 67 (1998) 81–86.

[18] R.E. Moore, Interval Analysis, Prentice Hall, Englewood Cliffs, New Jersey, USA, 1966.

[19] N.T. Müller, The iRRAM: Exact arithmetic in C++, in: Selected Papers from the 4th International Workshop on Computability and Complexity in Analysis (CCA), in: Lecture Notes in Comput. Sci., vol. 2064, Springer-Verlag, 2001, pp. 222–252.

[20] A. Neumaier, Taylor forms – use and limits, Reliab. Comput. 9 (1) (2003) 43–79.

[21] R. Solomonoff, A formal theory of inductive inference, Inf. Control 7 (1) (March 1964) 1–22.

[22] R. Solomonoff, A formal theory of inductive inference, Inf. Control 7 (2) (June 1964) 224–254.

[23] L. Staiger, The Kolmogorov complexity of real numbers, Theoret. Comput. Sci. 284 (2) (2002) 455–466.

[24] K. Weihrauch, Computable Analysis: An Introduction, Springer-Verlag, 2000.

[25] M. Ziegler, W.M. Koolen, Kolmogorov complexity theory over the reals, in: Proceedings of the Fifth International Conference on Computability and Complexity in Analysis, CCA, 2008, in: Electron. Notes Theor. Comput. Sci., vol. 221, 2008, pp. 153–169.