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Low Dimensional Adelic Geometry

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Abstract

Adelic (and idelic) structures can be associated to algebraic and arithmetic varieties, and an adelic geometry can be developed as a bridge between algebraic geometry and arithmetic geometry. We study in detail adelic geometry in dimension one and two. In particular, such a theory can be seen as a generalisation of the theory of algebraic and arithmetic line bundles, so the result is a novel approach to intersection theory. The construction process of adelic objects is "from local to global" and it endows such objects with natural topologies. One of the main richnesses of adelic geometry is given by the topological interactions between adelic structures, and a deep study of them in the case of arithmetic surfaces might be crucial to the solution to higher number theory open problems.

Acknowledgements

Writing this Ph.D. thesis was for me a quite unique process which made me face my own deepest feelings. Even if the outcome bears only my name, it is important to recall that I wasn't alone, in fact a few people, directly or indirectly, have been of great support. With my great pleasure, I will thank them.

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At last but not least I would like to thank *Andrea de Luca* for his enlightening opinions and my *old friends from Catanzaro* who are a second family for me.

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[1] Keep It Mello.

A different image came to me a few weeks ago. The unknown thing to be known appeared to me as some stretch of earth or hard marl, resisting penetration... the sea advances insensibly in silence, nothing seems to happen, nothing moves, the water is so far off you hardly hear it... yet it finally surrounds the resistant substance.

Alexander Grothendieck[2]

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[2] English translation by Colin McLarty in "The Rising Sea: Grothendieck on simplicity and generality I". The original French quote can be found in " Grothendieck A. - Récoltes et Semailles, Université des Sciences et Techniques du Languedoc, Montpellier (pag. 552-553)".

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Chapter 0

Introduction

0.1 Adelic theory as a geometric theory

If we want to solve an analytic problem on $(\mathbb{Q}, |\cdot|)$ (where $|\cdot|$ is the usual euclidean absolute value), it is advisable to embed $\mathbb{Q}$ in its completion $\mathbb{R}$ with respect to $|\cdot|$. In this way we can take advantage of the completeness properties of $\mathbb{R}$ and the discreteness of $\mathbb{Q}$ inside $\mathbb{R}$. Any other absolute value on $\mathbb{Q}$ is given by the p-adic absolute value $|\cdot|_p$ for any prime p, so it makes sense to embed $\mathbb{Q}$ discretely in its completion $\mathbb{Q}_p$ with respect to $|\cdot|_p$. The above argument can be easily generalized for any number field K, and adeles were introduced in 1930s by Chevalley in order to "globalize" the analytic theory on K. In other words, the aim was to consider simultaneously all the completions of K with respect all possible places[1]. It is not very useful to study simply the product $\prod_{\mathfrak{p}} K_{\mathfrak{p}}$ of all completions because the resulting space is "too big" and it doesn't have the correct topological properties. For any place $\mathfrak{p}$ let $\mathcal{O}_{\mathfrak{p}}$ be the closed unit ball in $K_{\mathfrak{p}}$, then the ring of adeles is defined as a subset of $\prod_{\mathfrak{p}} K_{\mathfrak{p}}$, namely:

$$\mathbf{A}_K := \prod_{\mathfrak{p}}{}' K_{\mathfrak{p}}$$

where $\prod'$ is the restricted products with respect to the family of additive subgroups:

$$\{\mathcal{O}_{\mathfrak{p}} : \mathfrak{p} \text{ is non-archimedean}\}.$$

Note that for archimedean places, the unit ball is not an additive subgroup. The most important features of $\mathbf{A}_K$ were well described in [60] and consist

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[1] a place $\mathfrak{p}$ is an equivalence class of absolute values on K where two absolute values are declared equivalent if they generate the same topology.

mainly in the fact that $\mathbf{A}_K$ is a locally compact additive group (so it admits a Haar measure), K is discrete in $\mathbf{A}_K$, and the quotient $\mathbf{A}_K/K$ is compact. Moreover the Pontryagin dual $\mathbf{A}_K^\vee$ has a very simple description and $\mathbf{A}_K \cong \mathbf{A}_K^\vee$.

Remark 0.1. Note the analogies and also the differences between the properties mentioned above with respect to the corresponding "local" properties related to the embeddings $K \subset K_\mathfrak{p}$.

The multiplicative version of the adelic theory is the idelic theory, and the group of ideles attached to K is defined as:

$$\mathbf{A}_K^\times = \prod_{\mathfrak{p}}{}' K_\mathfrak{p}^\times$$

where the restricted product is taken with respect to the subgroups

$$\mathcal{O}_\mathfrak{p}^\times := \{x \in \mathcal{O}_\mathfrak{p} \colon |x|_\mathfrak{p} = 1\}\,.$$

In this case $\mathcal{O}_\mathfrak{p}^\times$ has a group structure also in the archimedean case.

But a number field K can be seen as the function field of the nonsingular arithmetic curve $B = \operatorname{Spec} O_K$, and we know that there is a bijection between points of the completed curve $\widehat{B}$ in the sense of Arakelov geometry and places of K. Therefore the adelic ring attached to K can be described in a more geometric way related to $\widehat{B}$:

$$\mathbf{A}_{\widehat{B}} := \prod_{b \in \widehat{B}}{}' K_b = \mathbf{A}_K$$

where K_b is still the local field attached to the point b. So, classical adelic theory can be deduced from 1-dimensional arithmetic geometry. We can adopt a similar approach but starting from 1-dimensional algebraic geometry: fix a nonsingular algebraic projective curve X over a perfect field k with function field denoted by $k(X)$; then to each point $x \in X$ we can associate a non-archimedean local field K_x with its valuation ring denoted by $\mathcal{O}_x$. The adelic ring associated to X is then:

$$\mathbf{A}_X := \prod_{x \in X}{}' K_x$$

but in this case it is not a locally compact additive group. Each K_x is a locally linearly compact k-vector space (or a 1-Tate space), therefore $\mathbf{A}_X$ is again locally linearly compact and one can show similarly to the arithmetic case that: $k(X)$ is discrete in $\mathbf{A}_X$, the quotient $\mathbf{A}_X/k(X)$ is a linearly compact

k-vector space and $\mathbf{A}_X$ is self dual. In other words, from a topological point of view, the passage from arithmetic theory to algebraic theory implies that we substitute the theory of compactness of groups with the theory of linear compactness of vector spaces. In both arithmetic and geometric 1-dimensional case, adelic and idelic theory give a generalization of the intersection theory (i.e. the theory of degree of divisors):

- Ideles can be easily seen as a generalization of line bundles (resp. Arakelov line bundles), so it is natural to give an extension of the theory of divisors (resp. Arakelov divisors) from an idelic point of view.

- For an algebraic curve X and any divisor $D \in \mathrm{Div}(X)$ we can define an adelic subspace $\mathbf{A}_X(D) \subset \mathbf{A}_X$ and an adelic complex $\mathcal{A}_X(D)$. The cohomology of $\mathcal{A}_X(D)$ is equal to the usual Zariski cohomology $H^i(D)$, therefore we can give an interpretation of $\deg(D)$ in terms of the characteristic of $\mathcal{A}_X(D)$ which will be called the adelic characteristic. For an arithmetic curve $\widehat{B} = \operatorname{Spec} O_K$ we cannot define a complex $\mathcal{A}_{\widehat{B}}(\widehat{D})$ associated to an Arakelov divisor $\widehat{D}$, since for archimedean points closed unit balls are not additive groups. However, one can recover the Arakelov degree of $\widehat{D}$ as the product of volumes (with respect to scaled Haar measures) of certain closed balls in K_b.

The above theory remains valid also in the case of singular curves, because by normalization we can always reduce to the nonsingular case.

The first attempt to construct a 2-dimensional adelic/idelic theory from 2-dimensional geometry was partially made in [49], but only in the algebraic case, in positive characteristic, and with several mistakes. Let's fix a nonsingular, projective surface $(X, \mathscr{O}_X)$ over a perfect field k, then to each "flag" $x \in y$ made of a closed point x inside an integral curve $y \subset X$ we can associate the ring $K_{x,y}$ which will be a 2-dimensional local field if y is nonsingular at x, or a finite product of 2-dimensional local fields if we have a singularity. Note how the geometric dimension of X matches the "dimension" of the ring $K_{x,y}$, and this happens roughly speaking because for a flag $x \in y$ (assuming that x is a nonsingular point of y) we have two distinct levels of discrete valuations: there is the discrete valuation associated to the containment $x \in y$ and the discrete valuation associated to $y \subset X$. In any case $K_{x,y}$ is obtained through a process of successive localisations and completions starting with $\mathscr{O}_{X,x}$. By the symbol $\mathcal{O}_{x,y}$ we denote the product of valuation rings inside $K_{x,y}$. It is possible to perform a "double restricted product": first over all points ranging on a fixed curve and then over all curves in X, in order to obtain the 2-dimensional adelic ring:

$$\mathbf{A}_X := \prod_{\substack{x\in y\\ y\subset X}}'' K_{x,y} \subset \prod_{\substack{x\in y\\ y\subset X}} K_{x,y}\,.$$

The topology on $K_{x,y}$ can be defined canonically thanks to the construction by completions and localisations, and by starting with the standard $\mathfrak{m}_x$-adic topology on $\mathscr{O}_{X,x}$. But here it is important to recall that abstract 2-dimensional local fields don't have a standard topology in general. Then the topology on $\mathbf{A}_X$ can be defined canonically. Moreover, it turns out (see remark 3.21) that $\mathbf{A}_X$ is the restricted product of 2-Tate spaces (i.e. k-vector spaces of the form $V((t))$ where V is a 1-Tate space). For 2-dimensional local fields with the same structure of $K_{x,y}$ there is a well known theory of differential forms and residues (see [63]); one can globalise the constructions in order to obtain a k-character $\xi^\omega : \mathbf{A}_X \to k$ associated to a rational differential form $\omega \in \Omega^1_{k(X)|k}$ and the differential pairing:

$$\begin{array}{rcl} d_\omega : \mathbf{A}_X \times \mathbf{A}_X & \to & k \\ (\alpha, \beta) & \mapsto & \xi^\omega(\alpha\beta)\,. \end{array}$$

In [19] it is shown that: ξ^ω induces the self duality of $\mathbf{A}_X$, the subspace $\mathbf{A}_X/k(X)^\perp$ is linearly compact (orthogonal spaces are calculated with respect to d_ω) and $k(X)$ is discrete in $\mathbf{A}_X$. One can define important subrings of $K_{x,y}$ in the following way:

$$\begin{tikzcd}
& \mathcal{O}_{x,y} & & & K_{x,y} & \\
\mathcal{O}_x \arrow[ur, hook] & & \mathcal{O}_y \arrow[ul, hook] & K_x \arrow[ur, hook] & & K_y \arrow[ul, hook] \\
& & & & k(X) \arrow[ul, hook] \arrow[ur, hook] & \\
& k \arrow[uul, hook] \arrow[uur, hook] & & & k \arrow[u, hook] &
\end{tikzcd}$$

where:

- $\mathcal{O}_x := \widehat{\mathscr{O}_{X,x}}$. It is a Noetherian, complete, regular, local, domain of dimension 2 with maximal ideal $\widehat{\mathfrak{m}_x}$.

- $K'_x := \operatorname{Frac} \mathcal{O}_x$.

- $K_x := k(X)\mathcal{O}_x \subseteq K'_x$.
- $\mathcal{O}_y := \widehat{\mathscr{O}_{X,y}}$. It is a complete DVR with maximal ideal $\widehat{\mathfrak{m}_y}$.
- $K_y := \operatorname{Frac}\mathcal{O}_y$. It is a complete valuation field with valuation ring $\mathcal{O}_y$.

Again such constructions can be globalised in order to get important subspaces of $\mathbf{A}_X$:

$$A_{012} := \mathbf{A}_X\,; \quad A_{12} := \mathbf{A}_X \cap \prod_{\substack{x\in y\\ y\subset X}} \mathcal{O}_{x,y};$$

$$A_{02} := \mathbf{A}_X \cap \prod_{x\in X} K_x\,; \quad A_2 := \mathbf{A}_X \cap \prod_{x\in X} \mathcal{O}_x\,; \quad A_{01} := \mathbf{A}_X \cap \prod_{y\subset X} K_y\,;$$

$$A_1 := \mathbf{A}_X \cap \prod_{y\subset X} \mathcal{O}_y\,; \quad A_0 := k(X)$$

whose containments relations are shown below:

$$\begin{tikzcd}
 & A_0 \arrow[ld, hook] \arrow[d, hook] \arrow[rd, hook] & \\
A_{01} \arrow[r, hook] & A_{012} & A_{02} \arrow[l, hook'] \\
A_1 \arrow[u, hook] \arrow[ru, hook] \arrow[r, hook] & A_{12} \arrow[u, hook] & A_2\,. \arrow[u, hook] \arrow[lu, hook] \arrow[l, hook']
\end{tikzcd}$$

By using all subspaces A_* it is possible to define an idelic complex $\mathcal{A}_X^\times$ and an adelic complex $\mathcal{A}_X(D)$ associated to a divisor $D \in \operatorname{Div}(X)$. The cohomology of such complexes can be calculated by geometric methods thanks to the following important results:

$$H^i(\mathcal{A}_X^\times) \cong H^i(X, \mathscr{O}_X^\times)\,, \tag{0.1}$$

$$H^i(\mathcal{A}_X(D)) \cong H^i(D)\,. \tag{0.2}$$

Again, both idelic and adelic theory give an extension of the intersection theory on X:

- The idelic complex assumes the following detailed form:

$$\mathcal{A}_X^\times : \qquad A_0^\times \oplus A_1^\times \oplus A_2^\times \xrightarrow{d^0_\times} A_{01}^\times \oplus A_{02}^\times \oplus A_{12}^\times \xrightarrow{d^1_\times} A_{012}^\times$$

and it can be shown that the space $\ker(d^1_\times)$ is a generalization of the group $\mathrm{Div}(X)$ since there is a surjective map $\ker(d^1_\times) \to \mathrm{Div}(X)$. Moreover the intersection pairing on $\mathrm{Div}(X)$ can be extended to a pairing on $\ker(d^1_\times)$.

- The characteristic of the complex $\mathcal{A}_X(D)$ can be used to redefine the intersection pairing between two divisors in terms of adeles, even without using isomorphism (0.2). Such a theory gives an alternative approach to the Riemann-Roch theorem for algebraic surfaces.

The adelic theory associated to an arithmetic surface $X \to B = \operatorname{Spec} O_K$ (K is a number field and $K(X)$ is the function field of X) is more complicated and less developed. Locally, the rings $K_{x,y}$ have a completely different strucure as 2-dimensional local fields depending whether y is horizontal or vertical. Moreover, there was the global issue of interpreting the archimedean data of the completed surface $\widehat{X} := X \cup \bigcup_\sigma X_\sigma$ in an adelic and idelic way. The correct definition of the "full" (or completed) ring of adeles $\mathbf{A}_{\widehat{X}}$ has been given only recently in [18]. Roughly speaking it involves the adelic rings $\mathbf{A}_{X_\sigma}$ of the fibres at infinity X_σ, but counted twice:

$$\mathbf{A}_{\widehat{X}} = \mathbf{A}_X \oplus \prod_{\sigma \in B_\sigma} (\mathbf{A}_{X_\sigma} \oplus \mathbf{A}_{X_\sigma})$$

The topology of $\mathbf{A}_{\widehat{X}}$ is easily inherited from the canonical topologies imposed on $K_{x,y}$, but in this case, we cannot say anything about the "compactness theory" involved, so far. The following schematization clarifies the issue:

1-dimensional theory:

- *Algebraic geometry:* $\mathbf{A}_X$ is the restricted product of 1-Tate spaces (and also a 1-Tate space).
- *Arithmetic geometry:* $\mathbf{A}_{\widehat{B}}$ is locally compact.

2-dimensional theory:

- *Algebraic geometry:* $\mathbf{A}_X$ is the restricted product of 2-Tate spaces.
- *Arithmetic geometry:* ?? .

The problem is given mainly by the rings $K_{x,y}$ associated to vertical curves, because they are products of mixed characteristic 2-dimensional local fields.

So, even if we have a well developed local theory of differential forms and residues related to $K_{x,y}$ (see [44]) and a global differential pairing, we are not able to prove that $\mathbf{A}_{\widehat{X}}$ is self dual and that $K(X)$ is discrete in $\mathbf{A}_{\widehat{X}}$.

Thanks to the projection $\ker(d^1_\times) \to \mathrm{Div}(X)$ it is possible to extend the Deligne pairing

$$[[\,,\,]] : \mathrm{Div}(X) \times \mathrm{Div}(X) \to \mathrm{Pic}(B)$$

to an idelic pairing

$$\langle\,,\,\rangle_i : \ker(d^1_\times) \times \ker(d^1_\times) \to \mathrm{Pic}(B)\,.$$

Moreover there is a surjective map:

$$(\mathbf{A}^\times_{X_\sigma} \oplus \mathbf{A}^\times_{X_\sigma}) \supseteq S \to \mathbb{Z}G(X_\sigma)$$

where S is an adequately defined subset of $\mathbf{A}^\times_{X_\sigma} \oplus \mathbf{A}^\times_{X_\sigma}$ and $\mathbb{Z}G(X_\sigma)$ is the vector space of Green functions on X_σ with integer orders. Through such a map we can give an extension of the $*$-product between Green functions to an idelic $*$-product denoted by $*_i$. By putting together $\langle\,,\,\rangle_i$ and $*_i$ it is possible to define an idelic extension of the Arakelov intersection number on $\widehat{X}$. On the other hand, an adelic extension of the Arakelov intersection pairing has still to be found. One of the key features of adelic approach to Arakelov intersection is to get a more symmetric vision of the completed surface $\widehat{X}$: in the current Arakelov geometry there is big discrepancy in the treatment of archimedean and non-archimedean data, whereas at the level of complete adelic objects there are (and we expect) more symmetries such as full reciprocity laws, adelic self-duality etc etc...

It is fundamental to recall that in the framework of 2-dimensional adelic theory it is possible to define also another “smaller” adelic ring with different adelic structure, called the ring of *analytic adeles* (see [18] or [11]). Such a ring, together with all its substructures, is closely related to the theory of 0-cycles, moreover there is an integration theory for it and it is involved the study of the 2-dimensional ζ-function. The ongoing work [16] (see also [11] for a nice summary) uses the interplay between $\mathbf{A}_{\widehat{X}}$ and analytic adeles to attack BSD-conjecture. A key point for a positive outcome of this approach will be to find a proof for the discreteness of $K(X)$ in $\mathbf{A}_{\widehat{X}}$.

Remark 0.2. In [4] Beilinson shortly described how to attach a n-dimensional adelic theory to any n-dimensional Noetherian scheme in a very abstract functorial way. Reworks and clarifications of this approach are [27] and [45, 8]. In particular in [45, 8.4 and 8.5] it is proved that for dimensions 1 and 2 our explicit theory agrees with Beilinson theory of adeles.

0.2 Main results and future work

This thesis gives a detailed description of the adelic theory summarized in section 0.1. The new main results are the followings:

(a) The definition of the idelic Deligne pairing

$$\langle\, ,\, \rangle_i : \ker(d^1_\times) \times \ker(d^1_\times) \to \mathrm{Pic}(B)\,.$$

extending the Deligne pairing (see subsection 3.3.4). The key point is to globalise Kato's local symbol for 2-dimensional local fields containing a local field (see [30] or [36]), which is the generalisation of the usual tame symbol for valuation fields. In the geometric framework given by an arithmetic surface $\varphi : X \to B$, Kato's symbol translates into a skew symmetric, bilinear map:

$$(\, ,\,)_{x,y} : K^\times_{x,y} \times K^\times_{x,y} \to K^\times_b$$

where $\varphi(x) = b \in B$ and $x \in y \subset X$. Roughly speaking, by composing it with the complete valuation v_b on K_b and by summing over all flags $x \in y$ such that $\varphi(x) = b$, we show that we obtain a well defined integer n_b. By repeating the argument for each $b \in B$ we obtain a divisor $\sum_{b\in B} n_b[b]$. At this point we prove that such a pairing descends to the Deligne pairing thanks to the projection $\ker(d^1_\times) \to \mathrm{Div}(X)$.

(b) The idelic description of Green functions, the product $*_i$ which is the idelic extension of the $*$-product and the idelic interpretation of the Arakelov intersection number (see subsection 3.3.5). The key point here is to understand that in the product $\mathbf{A}^\times_{X_\sigma} \oplus \mathbf{A}^\times_{X_\sigma}$ one factor "represents" meromorphic sections of line bundles on the Riemann surface X_σ. The other factor is used to define hermitian products on such line bundles. Then it is enough to recall that any Green function with integer orders can be obtained from hermitian line bundles (see subsection D.1.3). The product $*_i$ is obtained by lifting properties of Green functions to ideles. Finally, we put together $*_i$ and the idelic pairing $\langle\, ,\, \rangle_i$ defined in (a) in order to obtain the idelic version of the Arakelov intersection number.

(c) New functorial properties of adeles with respect to: morphisms between algebraic surfaces, fibred surfaces over a curve, embedding of a curve in a surface (see section 3.2.6). In particular we show how adeles and related structures "pullback" with respect to morphisms.

(d) In subsection 3.3.2 we define the full ring of adeles $\mathbf{A}_{\widehat{X}}$ for arithmetic surfaces without using the lifting maps mentioned in [18]. Moreover, in subsection 3.3.3 we give the reciprocity law around archimedean points (see theorem 3.60(2)).

(e) The idelic interpretation of the intersection pairing on algebraic surfaces given in subsection 3.2.7 is different from that defined in [49]. In particular we relate it to the idelic complex $\mathcal{A}_X^\times$.

Moreover appendices C and D form a detailed introductory course[2] to Arakelov geometry in dimension 1 and 2. In particular D contains an explicit presentation of the Deligne pairing, fixing the mistakes contained in [43].

Some open questions were listed in [19, 5]:

> 1. Study functorial properties of the adelic complex with respect to morphisms of surfaces and their applications. Extend the argument in this paper to the case of a quasi-coherent sheaf (...) Find an adelic proof of the Noether formula and the Hodge index theorem.
>
> (...)
>
> 3. *On arithmetic extension.* Let $S \to \operatorname{Spec} O_k$ be a regular proper scheme of relative dimension one, k a number field (...) It is a fundamental problem to find an analogue $\mathcal{A}_S$ of the adelic complex in the arithmetic case and an extension of the adelic Euler characteristic, which gives in particular an adelic description of the Arakelov intersection index and another proof of the Faltings Euler characteristic theorem (...)

Note that (c) partially answers to 1, moreover (a) and (b) partially solve the "multiplicative version" of problem 3.

Future work will focus on a deeper study of the adelic theory for an arithmetic surface $X \to \operatorname{Spec} O_K$; in particular:

(i) We still don't know how to properly define the subspaces $\widehat{A_*}$ of $\mathbf{A}_{\widehat{X}} = \widehat{A}_{012}$. These should be the arithmetic versions of $A_* \subset A_{012}$ and they should lead to the arithmetic adelic and idelic complexes $\mathcal{A}_{\widehat{X}}$ and $\mathcal{A}_{\widehat{X}}^\times$. The schematic part of each $\widehat{A_*}$ will be obviously A_*, but it is not clear what to do with the archimedean bits.

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[2] Those appendices are the result of some lectures that the author of this thesis gave in 2017/2018 at the university of Nottingham.

(*ii*) Prove that $\mathbf{A}_{\widehat{X}}$ is self dual and that $K(X)$ is discrete in $\mathcal{A}_{\widehat{X}}$ (conjectures 3.62 and 3.63).

(*iii*) Find an adelic interpretation of the Arakelov intersection pairing. If we think in complete analogy to the arithmetic 1-dimensional case, it is unlikely to find a complex of the type $\mathcal{A}_{\widehat{X}}(\widehat{D})$ associated to an Arakelov divisor. On the other hand we will need some "analytic tool" (for arithmetic curves we employed measure theory) which will be the "arithmetic version" of the geometric adelic characteristic:

1-dimensional adelic intersection theory:

- *Algebraic geometry:* We define the complex $\mathcal{A}_X(D)$ and we use its characteristic $\chi\left(A_X(D)\right)$.
- *Arithmetic geometry:* The Arakelov degree of $\widehat{D}$ is the "volume" of a subset of $\mathbf{A}_{\widehat{B}}$

2-dimensional adelic intersection theory:

- *Algebraic geometry:* We define the complex $\mathcal{A}_X(D)$ and we use its characteristic $\chi\left(A_X(D)\right)$.
- *Arithmetic geometry:* ?? .

0.3 How to read this thesis

Each chapter contains its own introduction and a list of the main references. The thesis contains a big appendices sections which should be used as *prerequisite material*.

Proofs of *known results* are in general referred with precision to the appropriate references but an exception to this policy is made in three cases: when the proofs are somehow constructive, so the reader can benefit from a detailed exposition; when we think that the existing proofs are not clear; when proofs are omitted in the literature (folklore results). Another exception is made for appendices C and D where most of the known proofs are presented in great detail.

Theorems, propositions, definitions, lemmas, examples, remarks and conjectures are numbered with the same increasing counter by chapters. Figures and equations have different increasing counters also numbered by chapters. In particular when we want to refer to some equation, we put the referring number between round brackets.

Basic notations. All rings are considered commutative and unitary. When we pick a point x in a scheme X we generally mean a *closed point* if not otherwise specified, also all sums $\sum_{x \in X}$ are meant to be "over all closed points of X". The cardinality of a set T is denoted as $\#(T)$. If F is a field, then $\overline{F}$ doesn't denote the algebraic closure. A Dedekind scheme S is a normal locally Noetherian scheme of dimension 0 or 1. For a morphisms of schemes $f : X \to S$, the schematic preimage of $s \in S$ is X_s. Sheaves are denoted with the "mathscr" lateχ font; particular the structure sheaf of a scheme X is $\mathscr{O}_X$ (note the difference with the font $\mathcal{O}$). If X is a variety over a field k, then the function field of X is denoted by $k(X)$. If K is a number field and $X \to \operatorname{Spec} O_K$ is an integral scheme over the ring of integers O_K, then the function field of X is denoted by $K(X)$. Finally it is important to point out that the letter K will denote different mathematical objects in this thesis (and in different contexts), so the reader should check at the beginning of each section its specific meaning from time to time.

Chapter 1

Local theory

The theory of adeles is a globalisation of the theory of higher local fields. Subsection 1.1 provides a brief introduction to abstract higher local fields just from an algebraic point of view. In section 1.2 we focus on the case when all residue fields have the same characteristic, in any dimension[1]. We will see that higher local fields of this kind come from algebraic geometry of varieties over perfect fields. In section 1.3 we study the 2-dimensional mixed characteristic case in a particular occurrence that we denote as "arithmetic" (this terminology is not standard). Such a restriction is not coincidental, but it allows us to deal exactly with the 2-dimensional local fields arising from 2-dimensional arithmetic geometry.

Main references. An elementary introduction to higher local fields can be found in [45]; for more specific results see [20]. Section 1.2 is inspired to [63] and results of section 1.3 can be found in [44, 2], [36] and [30].

1.1 Higher local fields

Let's recall the definition of local field:

Definition 1.1. A *local field* (or a *1-dimensional local field*) F, is one of the fields listed below:

(1) $F = \mathbb{R}$ endowed with the usual real absolute value$|\cdot|$.

(2) $F = \mathbb{C}$ endowed with the usual complex absolute value $||\cdot||$.

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[1] We are interested just in dimension ≤ 2, but in this case case the theory can be quickly presented for a generic dimension n with no harm.

(3) F is a complete discrete valuation field (the valuation is surjective) such that the residue field $\overline{F}$ is a perfect field. The valuation ring of F is denoted as $\mathcal{O}_F$ and its maximal ideal is $\mathfrak{p}_F$. Moreover if v is the valuation on F, then the absolute value is given by $|x|_v := q^{-v(x)}$, where $q = \#(\overline{F})$ if $\overline{F}$ is a finite field, and $q = e := \exp(1)$ otherwise.

If F is of type (1) or (2), it is an *archimedean local field* otherwise it is a *non-archimedean local field*. A local field is topologized with the topology induced by the absolute value. A morphism between local fields is a continuous field homomorphism.

Remark 1.2. According to our definition, a non-archimedean local field endowed with its natural topology is in general not locally compact.

Remember that if F is a non-archimedean local field there exists only one surjective complete valuation on it (see [45, Theorem 1.4]).

A higher local field is a simple generalization of definition 1.1: given a complete discrete valuation field F, it might happen that the residue field $F^{(1)} := \overline{F}$ is again a complete discrete valuation field; by taking one more time the residue field we have the field $F^{(2)}$. In other words, a complete discrete valuation field might originate a potentially infinite sequence of fields $\{F^{(i)}\}_{i\geq 0}$ such that $F^{(0)} = F$ and $F^{(i+1)} = \overline{F^{(i)}}$. Each $F^{(i)}$ is called the *i-th residue field*.

Definition 1.3. A *n-dimensional local field*, for $n \geq 2$, is a complete discrete valuation field F admitting sequence of residue fields $\{F^{(i)}\}_{i>0}$ such that $F^{(n-1)}$ is a local field. If $F^{(n-1)}$ is an archimedean local field, then F is called *archimedean*, otherwise we say that F is *non-archimedean*. F has *mixed characteristic* if $\mathrm{char}(F) \neq \mathrm{char}(\overline{F})$.

Example 1.4. The simplest n-dimensional local field is the field of iterated Laurent series over a perfect field K:

$$F = K((t_1))\dots((t_n))\,.$$

If $f = \sum a_j t_n^j \in F$, with $a_i \in K((t_1))\dots((t_{n-1}))$, we have the complete discrete valuation defined by $v(f) = \min\{j \colon a_j \neq 0\}$. The valuation ring is $\mathcal{O}_F = K((t_1))\dots((t_{n-1}))[[t_n]]$ and the residue field is $F^{(1)} = K((t_1))\dots((t_{n-1}))$. Clearly $F^{(n)} = K$. When $n = 2$, the elements of $K((t_1))((t_2))$ are the formal power series $\sum_{i,j} a_{i,j} t_1^i t_2^j$ such that $a_{i,j} = 0$ when the indexes i and j are chosen in the following way: let's plot the couples (j, i) as a lattice on the plane, then we select a semiplane like the one which is not coloured in figure 1.1. The coordinate j is bounded from right, whereas the coordinate i is bounded from above by a descending staircase line.

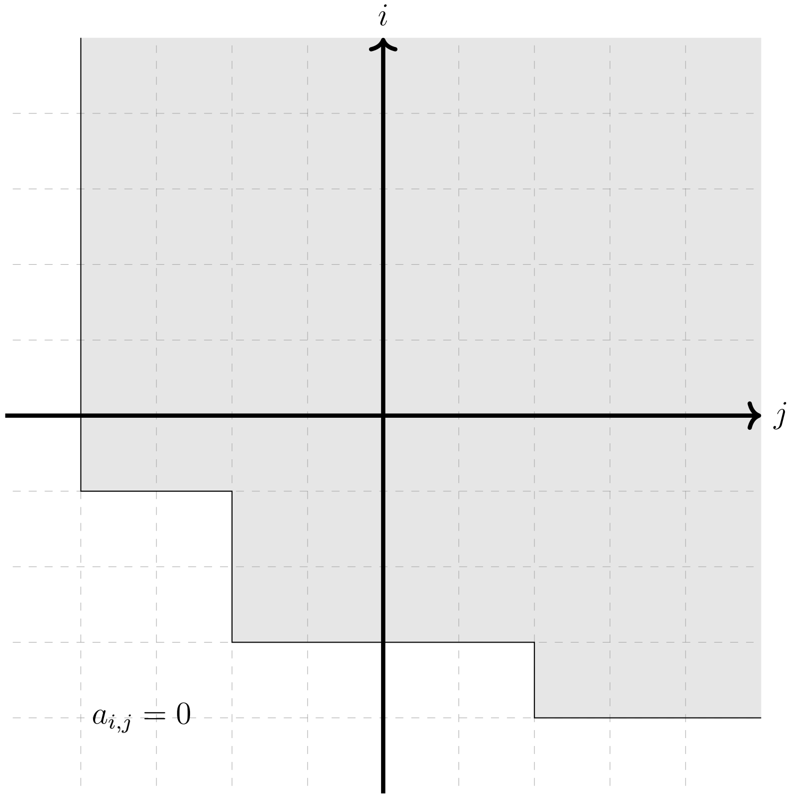

Figure 1.1: A cartesian diagram showing the lattice of couples (j, i) corresponding to the coefficients $a_{i,j}$ of a power series in $\sum_{i,j} a_{i,j} t_1^i t_2^j \in K((t_1))((t_2))$.

Remark 1.5. The above definition of dimension for a n-dimensional local field might seem quite counter-intuitive, indeed a n-dimensional local field can also be a m-dimensional local field for $m \neq n$. For instance $F = K((t_1)) \dots ((t_n))$ is m-dimensional for any $m = 1, \dots, n$. For our purposes it will be clear from the context which dimension we want to take in account. Often it is convenient to consider the maximum amongst all possible dimensions (when it exists).

Remark 1.6. Note in the case of archimedean n-dimensional local fields the n-th residue field doesn't exist.

Let's give a less trivial example of higher local field:

Example 1.7. Let (K, v_K) be a non-archimedean local field and consider the following set of (double) formal series:

$$K\{\{t\}\} := \left\{ \sum_{j=-\infty}^{\infty} a_j t^j \colon a_j \in K,\ \inf_j v_K(a_j) > -\infty,\ \lim_{j \to -\infty} a_j = 0 \right\}$$

Addition and multiplication in $K\{\{t\}\}$ are defined in the following way:

$$\sum_{j=-\infty}^{\infty} a_j t^j + \sum_{j=-\infty}^{\infty} b_j t^j = \sum_{j=-\infty}^{\infty} (a_j + b_j) t^j \tag{1.1}$$

$$\sum_{j=-\infty}^{\infty} a_j t^j \cdot \sum_{j=-\infty}^{\infty} b_j t^j = \sum_{i=-\infty}^{\infty} \left(\sum_{j=-\infty}^{\infty} (a_j b_{i-j}) t^j \right) \tag{1.2}$$

and $K\{\{t\}\}$ become a field. We can also define the following discrete valuation v on $K\{\{t\}\}$:

$$v\left(\sum_{j=-\infty}^{\infty} a_j t^j \right) := \inf_j v_K(a_j) \tag{1.3}$$

It is not difficult to verify that v is a well defined valuation, and moreover that equations (1.1) and (1.3) are not just formal expressions but truly convergent series in $K\{\{t\}\}$ with the topology generated by v. Let's now analyse the structure of $F = K\{\{t\}\}$ as valuation field:

$$\mathcal{O}_F = \left\{ \sum_{j=-\infty}^{\infty} a_j t^j \in K\{\{t\}\} \colon a_j \in \mathcal{O}_K \right\}$$

$$\mathfrak{p}_F = \left\{ \sum_{j=-\infty}^{\infty} a_j t^j \in K\{\{t\}\} \colon a_j \in \mathfrak{p}_K \right\}$$

Consider the surjective homomorphism:

$$\begin{array}{rcl} \pi : \ \mathcal{O}_F & \to & \overline{K}((t)) \\ \sum a_j t^j & \mapsto & \sum \overline{a_j} t_j \end{array}$$

where clearly $\overline{a_j}$ is the natural image of a_j in $\overline{K}$. Now it is evident that π induces an isomorphism $\overline{F} \cong \overline{K}((t))$. In other words F has a structure of 2-dimensional local field such that $F^{(1)} = \overline{K}((t))$ and $F^{(2)} = \overline{K}$. Clearly such a construction can be iterated several times to get the field:

$$K\{\{t_1\}\} \ldots \{\{t_n\}\} .$$

For example if $K = \mathbb{Q}_p$, then $K\{\{t\}\}$ is a 2-dimensional local field of mixed characteristic.

Remember that we have the following classical classification theorem for local fields:

Theorem 1.8 (Classification theorem for local fields)**.** *Let F be a local field:*

(1) *When F is archimedean, then $F = \mathbb{R}$ or $F = \mathbb{C}$.*

(2) *When F is not archimedean there are two cases:*

  (2*a*) *If* $\mathrm{char}\, F = \mathrm{char}\, \overline{F}$, *then* $F \cong \overline{F}((t))$.

  (2*b*) *If* $\mathrm{char}\, F \neq \mathrm{char}\, \overline{F} = p$, *then F is isomorphic to K_p which denotes a finite extension of $\mathbb{Q}_p$.*

Proof. (1) is true just by definition. For (2) see for example [21, II.5]. □

Such a classification can be extended for higher local fields, in particular any n-dimensional local field can be obtained by "combining" the higher local fields presented in examples 1.4 and 1.7:

Theorem 1.9 (Classification theorem for n-dimensional local fields)**.** *Let F be a n-dimensional local field with $n \geq 2$.*

(1) *If* $\mathrm{char}\, F = \mathrm{char}\, F^{(1)} = \ldots = \mathrm{char}\, F^{(n-1)}$, *then*

$$F \cong F^{(n-1)}((t_1))\ldots((t_{n-1}))$$

and $F^{(n-1)}$ is isomorphic to one of the four fields listed in theorem 1.8.

(2) *If $r \in \{2, 3, \ldots n\}$ is the unique number such that* $\mathrm{char}\, F^{(n-r)} \neq \mathrm{char}\, F^{(n-r+1)} =$ *p, then:*

  (2*a*) *When $r \neq n$, F is isomorphic to a finite extension of:*

$$K_p\{\{t_1\}\}\ldots\{\{t_{r-1}\}\}((t_r))\ldots((t_{n-1}))\,.$$

  (2*b*) *When $r = n$ (i.e. in the mixed characteristic case), F is isomorphic to a finite extension of:*

$$K_p\{\{t_1\}\}\ldots\{\{t_{n-1}\}\}\,.$$

Proof. See [45, Theorem 2.18]. □

In this text we will focus mainly on 2-dimensional local fields, so let's give a table with all possible 2-dimensional local fields by using the classification theorem:

2-dimensional local fields			
Geometric	Arithmetic		Archimedean
$(0,0,0)$, (p,p,p)	$(0,p,p)$	$(0,0,p)$	$\mathbb{C}((t))$ or $\mathbb{R}((t))$
$K((t_1))((t_2))$ with K perfect	finite extension of $K_p\{\{t\}\}$	$K_p((t))$	

(1.4)

For a non-archimedean local field (F, v), we have the notion of *local parameter* ϖ which is any generator of the maximal ideal $\mathfrak{p}_F$, or equivalently any element such that $v(\varpi) = 1$. Clearly we have the (recursive) generalization for n-dimensional local fields.

Definition 1.10. Let F be a non-archimedean n-dimensional local field, then a *sequence of local parameters for* F is a n-tuple $(\varpi_1, \ldots, \varpi_n) \in F \times \ldots \times F$ satisfying the following properties:

- ϖ_n is a local parameter for F.
- $(\varpi_1, \ldots, \varpi_{n-1}) \in \mathcal{O}_F \times \ldots \times \mathcal{O}_F$ and the sequence of natural projections $(\overline{\varpi}_1, \ldots, \overline{\varpi}_{n-1})$ is a sequence of local parameters for the residue field $\overline{F}$.

One can obtain a sequence of local parameters, by applying the following algorithm: choose any local parameter for $F^{(n-1)}$, then pick any of its liftings in F, this will be ϖ_1. Choose choose any local parameter for $F^{(n-2)}$, then pick any of its liftings in F, this will be ϖ_2, etc.

Let's give another definition that will be useful later:

Definition 1.11. Let F be a n-dimensional local field and put $\mathcal{O}_F^{(0)} := F$, then we define recursively the *j-th valuation ring* (for $j \geq 1$):

$$\mathcal{O}_F^{(j)} := \left\{ x \in \mathcal{O}_F : \overline{x} \in \mathcal{O}_{\overline{F}}^{(j-1)} \right\}$$

It is clear that $\mathcal{O}_F^{(1)} = \mathcal{O}_F$. For the algebraic properties of $\mathcal{O}_F^{(j)}$ the reader can check [45, 3].

1.2 $F \cong K((t_1))\dots((t_n))$ over k

In this section we will heavily use the theory of semi-topological rings and their related structures. All the preliminary material is covered in appendix E.

1.2.1 Topology

The topology of higher dimensional fields is a delicate matter, especially in the equal characteristic case (see [8] or [45, 4] for a summary), because in general there is no canonical way to define it.
The first problem consists in defining a reasonable topology on higher local fields of the form $K((t_1))\dots((t_n))$ where K is a generic field. Consider for example $\mathbb{R}((t))$ and suppose that $\mathbb{R}$ is endowed with a topology of ST ring, then we have an ind-pro description in **STring**

$$\mathbb{R}((t)) := \varinjlim_r t^r\mathbb{R}[[t]] = \varinjlim_r \varprojlim_m \frac{t^r\mathbb{R}[[t]]}{t^{r+m}\mathbb{R}[[t]]}$$

which gives a structure of ST ring on $\mathbb{R}((t))$ as described in appendix E. The crucial point in the above process is the choice of the topology on $\mathbb{R}$, indeed if we endow $\mathbb{R}$ with the discrete topology, we get the discrete valuation topology on $\mathbb{R}((t))$, on the other hand if we start with the Euclidean topology on $\mathbb{R}$ we obtain a completely different topological structure on $\mathbb{R}((t))$. In the former case we are considering $\mathbb{R}((t))$ as a 1-dimensional local field, in the latter situation $\mathbb{R}((t))$ has a 2-dimensional structure. Both possibilities are legitimate, and in general "the geometry" will guide us. By the way, we decide to make a temporary choice; the following convention will always hold *if not otherwise specified*:

$K((t))$ is a ST ring endowed with the ind-pro topology where the starting topology on K is discrete.

Once that we have fixed a topology on $K((t))$, we can iterate the ind-pro procedure in order to get a ST ring structure on $K((t_1))\dots((t_n))$. That's not the end of the story; by the classification theorem, for any non-archimedean n-dimensional local field F, such that $\operatorname{char} F = \operatorname{char} F^{(1)} = \ldots = \operatorname{char} F^{(n)}$, we have an algebraic isomorphism

$$F \cong F^{(n)}((t_1))\dots((t_n)),$$

and one might think about transferring the topology described above from $F^{(n)}((t_1))\dots((t_n))$ to F. That is a bad idea, in fact Yekutieli in [63] showed that the resulting topology on F would depend on the choice of the isomorphism. We want to put this issue of the choice of the isomorphism under the carpet, and by following [63] we slightly restrict the category of higher local fields. Since higher local fields of the type $K((t_1))\dots((t_n))$ arise naturally from algebraic varieties over a field, we can fix a field k (trivially topologized) and carry it in any definition.

Definition 1.12. A *n-dimensional local field over a field k* is a n-dimensional local field F satisfying the following properties:

1. There is a ring homomorphism $k \hookrightarrow \mathcal{O}_F^{(n)}$.
2. The field extension $k \hookrightarrow F^{(n)}$ induced by the ring homomorphism of property (1) is finite.

A morphism of n-dimensional local fields over k is a homomorphism of k-algebras $\varphi : F \to L$ which is also a homomorphism of n-dimensional local fields. The category of n-dimensional local fields over k is denoted by $\mathbf{LF}^n(k)$.

Definition 1.13. A *topological n-dimensional local field over a field k* is a n-dimensional local field F over k satisfying the following properties:

(T) F has a structure of ST ring.

(P) There exists a bijection

$$F \cong F^{(n)}((t_1))\dots((t_n))$$

which is an isomorphism of: k-algebras, higher valuation fields and ST rings. This map is called *a parametrization* of F.

A morphism of topological n-dimensional local fields over k is a continuous homomorphism of k-algebras $\varphi : F \to L$ which is also a homomorphism of n-dimensional local fields. The category of topological n-dimensional local fields over k is denoted by $\mathbf{TLF}^n(k)$.

Remark 1.14. In the above definition (T) stands for topology and (P) for parametrization. In particular (P) requires particular attention, indeed note that we only ask for the existence of a parameterization of F, we are not fixing it.

If we assume that the base field k is perfect, then for any object in $F \in \mathbf{TLF}^n(k)$ it is possible to find parametrizations in a very simple way. This phenomenon is a consequence of the following well known commutative algebra result which is a version of the Cohen structure theorem:

Theorem 1.15. *Fix a perfect field k. Let A be a complete local ring which is also a k-algebra and let L be the residue field of A, then there exists a morphism of k-algebras $\sigma : L \to A$ that is a section of the canonical projection $\pi : A \to L$. Moreover, if $k \hookrightarrow L$ is a finite extension, then such a section σ is unique.*

Proof. See [64, Theorem 1.1]. $\square$

Corollary 1.16. *Let k be a perfect field and let F be a n-dimensional local field over k. Then there exists a unique k-homorphism $\sigma_F : F^{(n)} \hookrightarrow \mathcal{O}_F^{(n)}$ which is a section of the canonical projection $\mathcal{O}_F^{(n)} \twoheadrightarrow F^{(n)}$.*

Proof. Let's proceed by induction. For $n = 1$ we apply directly theorem 1.15. By simplicity put $\overline{F} = F^{(1)}$ and suppose that the thesis is true for the $(n-1)$-dimensional local field $\overline{F}$, which means that there exists the unique section:

$$\sigma_{\overline{F}} : \overline{F}^{(n-1)} = F^{(n)} \hookrightarrow \mathcal{O}_{\overline{F}}^{(n-1)} .$$

Then we have the following commutative diagram in the category of k-algebras:

$$\begin{tikzcd}
 & & \mathcal{O}_F \arrow[d, two heads] \\
F^{(n)} \arrow[r, hook, "\sigma_{\overline{F}}"] & \mathcal{O}_{\overline{F}}^{(n-1)} \arrow[r, phantom, "\subset"] & \overline{F} \arrow[u, bend right, "\sigma"']
\end{tikzcd}$$

where σ is the unique section of $\mathcal{O}_F \twoheadrightarrow \overline{F}$ obtained thanks to theorem 1.15. Let's put $\sigma_F := \sigma \circ \sigma_{\overline{F}}$; clearly $\sigma_F(F^{(n)}) \subset \mathcal{O}_F^{(n)}$ and moreover it is easy to verify that σ_F is the unique section of $\mathcal{O}_F^{(n)} \twoheadrightarrow F^{(n)}$. $\square$

Definition 1.17. Let k be a perfect field and consider $F \in \mathbf{LF}^n(k)$. The k-homomorpism $\sigma_F : F^{(n)} \hookrightarrow \mathcal{O}_F^{(n)}$ of corollary 1.16 is called *the canonical lifting of $F^{(n)}$*.

Theorem 1.18. *Let k be a perfect field and consider $F \in \mathbf{TLF}^n(k)$. For any sequence of uniformizer parameters $(\varpi_1 \dots, \varpi_n)$ on F, the canonical lifting $\sigma_F : F^{(n)} \hookrightarrow \mathcal{O}_F^{(n)}$ extends uniquely to a parameterization $\psi : F^{(n)}((t_1)) \dots ((t_n)) \to F$ such that $\psi(t_i) = \varpi_i$ for any $i = 1 \dots, n$.*

Proof. It is a consequence of [63, Corollary 2.1.19]. $\square$

Proposition 1.19. *Let $F \in \mathbf{TLF}^n(k)$ with k perfect. Then the parametrizations of F are in bijection with all possible sequences of uniformizer parameters for F.*

Proof. Given a sequence of uniformizer parameters $(\varpi_1 \ldots, \varpi_n)$, we can associate a parametrization in a unique way thanks to theorem 1.18. Viceversa given a parametrization $p : F^{(n)}((t_1))\ldots((t_n)) \to F$, it is evident that $(p(t_1), \ldots, p(t_n))$ is the unique sequence of local parameters giving p. □

Consider the functor **unt** : $\mathbf{TLF}^n(k) \to \mathbf{LF}^n(k)$ which maps an object on itself but forgetting the topology, then the next theorem summarizes the issue of the non-unicity of the topology on a higher local field of equal characteristic:

Theorem 1.20. *In general* **unt** : $\mathbf{TLF}^n(k) \to \mathbf{LF}^n(k)$ *is not injective on objects, but it is an equivalence of categories when* $n = 1$ *or when* $\mathrm{char}(k) > 0$.

Proof. To show that the functor is not injective on objects one can use [64, Example 3.13]. The assertion about $n = 1$ is a well know fact, whereas for the case $\mathrm{char}(k) > 0$ see [63, Proposition 2.1.21]. □

1.2.2 Differential forms and residues

Fix a perfect field k, fix an element $F \in \mathbf{TLF}^n(k)$ and remember the definition of the module of m-differential forms $\Omega^{m,\mathrm{sep}}_{F|k}$ introduced in appendix E.2. For 1-dimensional complete valuation fields, Tate in [59] defined an abstract notion of residue; here we present the generalization of this notion of residue for fields isomorphic to $K((t_1))\ldots((t_n))$ (for a more comprehensive and general approach see [63]).

Let's fix $L = K((t_1))\ldots((t_n)) \in \mathbf{TLF}^n(k)$ (in other words $K|k$ is a finite extension and the parametrization is the identity), then we have a nice explicit expression of $\Omega^{1,\mathrm{sep}}_{L|k}$.

Proposition 1.21. *There is a unique isomorphism*

$$\Omega^{1,\mathrm{sep}}_{L|k} \cong L dt_1 + \ldots + L dt_n\,.$$

In other words $\Omega^{1,\mathrm{sep}}_{L|k}$ *is a* n*-dimensional* L*-vector space. As immediate consequence we have that:*

$$\Omega^{j,\mathrm{sep}}_{L|k} \cong \bigoplus_{1\le i_1 \le \ldots \le i_j \le n} L dt_{i_1} \wedge \ldots \wedge dt_{i_j}$$

Proof. Let's consider the continuous k-derivation:

$$\begin{aligned} \frac{\partial}{\partial t_n} : L &\to L \\ \sum_{j\ge m} a_j t_n^j &\mapsto \sum_{j \ge m} j a_j t_n^{j-1} \end{aligned}$$

with $a_j \in L^{(1)}$. A usual we put $\frac{\partial f}{\partial t_n} = \frac{\partial}{\partial t_n}(f)$. Then for any $0 < i < n$ we define inductively:

$$\begin{aligned} \frac{\partial}{\partial t_i} : L &\rightarrow L \\ \sum_{j \geq m} a_j t_n^j &\mapsto \sum_{j \geq m} \frac{\partial a_j}{\partial t_i} t_n^j \end{aligned}$$

Clearly the map

$$\begin{aligned} d : L &\rightarrow Ldt_1 + \ldots + Ldt_n \\ f &\mapsto \frac{\partial f}{\partial t_1} dt_1 + \ldots + \frac{\partial f}{\partial t_n} dt_n \end{aligned}$$

is a continuous k-derivation. By the basic properties of derivations, any other continuous k-derivation $d' : L \rightarrow M$ (where M is a $T2$ semi-topological module) should be of the form:

$$d' f = \frac{\partial f}{\partial t_1} d' t_1 + \ldots + \frac{\partial f}{\partial t_n} d' t_n \, .$$

It means that $Ldt_1 + \ldots + Ldt_n$ satisfies the universal property of the module of separated differential forms. $\square$

By propositions 1.21 and 1.19, for any sequence of uniformizer parameters $(\varpi_1, \ldots, \varpi_n)$ and any $\omega \in \Omega^{n,\mathrm{sep}}_{F|k}$, we can write $\omega = a d\varpi_1 \wedge \ldots \wedge d\varpi_n$ for

$$a = \sum_{i_1, \ldots, i_n} a_{i_1, \ldots, i_n} \varpi_1^{i_1} \ldots \varpi_n^{i_n} \in F \, . \tag{1.5}$$

The following theorem introduces the higher residue in $\mathbf{TLF}^n(k)$:

Theorem 1.22. *Let $F \in \mathbf{TLF}^n(k)$, then there exists a k-linear homomorphism*

$$\mathrm{res}_{F|k} : \Omega^{n,\mathrm{sep}}_{F|k} \rightarrow k$$

satisfying the following properties:

(1) *It is continuous.*

(2) *For any sequence of uniformizer parameters $(\varpi_1, \ldots, \varpi_n)$ and any $\alpha \in F^{(n)}$:*

$$\mathrm{res}_{F|k}\left(\alpha \varpi_1^{i_1} \ldots \varpi_n^{i_n}\right) = \begin{cases} \mathrm{Tr}_{F^{(n)}|k}(\alpha) & \textit{if } i_1 = i_2 = \ldots = i_n = -1 \\ 0 & \textit{otherwise} \, . \end{cases}$$

(3) If $F \hookrightarrow L$ is a finite extension of fields in $\mathbf{TLF}^n(k)$, then:

$$\mathrm{res}_{L|k} = \mathrm{res}_{F|k} \circ \mathrm{Tr}_{L|F} \ .$$

(4) For any $b \in F$ and $\omega \in \Omega^{n,\mathrm{sep}}_{F|k}$, define $\langle b, \omega \rangle_{\mathrm{res}} := \mathrm{res}_{F|k}(b\omega)$, then the maps

$$\begin{array}{rcl} \Omega^{n,\mathrm{sep}}_{F|k} & \to & F^\vee \\ \omega & \mapsto & \langle \cdot, \omega \rangle_{\mathrm{res}} \end{array}$$

$$\begin{array}{rcl} F & \to & \left(\Omega^{n,\mathrm{sep}}_{F|k}\right)^\vee \\ b & \mapsto & \langle b, \cdot \rangle_{\mathrm{res}} \end{array}$$

are algebraic and topological isomorphisms.

Moreover the map $\mathrm{res}_{F|k}$ is uniquely defined by properties (1) and (2).

Proof. It is a particular occurence of the theory developed in [63, 2.4]. See also [64, Remark 5.5]. □

It is evident from properties (1) and (2) of theorem 1.22 that $\mathrm{res}_{F|k}$ is the map:

$$\begin{array}{rcl} \mathrm{res}_{F|k} : \Omega^{n,\mathrm{sep}}_{L|k} & \to & k \\ \omega & \mapsto & \mathrm{Tr}_{F^{(n)}|k}(a_{-1,-1,\dots,-1}) \, . \end{array}$$

where $\omega = a d\varpi_1 \wedge \ldots \wedge d\varpi_n$ and the power series a is expressed as in equation 1.5.

Fix a nonzero differential form $\omega \in \Omega^{n,\mathrm{sep}}_{F|k}$; the homomorphism $\rho_\omega : F \to \Omega^{n,\mathrm{sep}}_{F|k}$ given by $b \mapsto b\omega$ is an isomorphism of ST vector spaces, so the dual map

$$\begin{array}{rcl} \rho_\omega^\vee : \left(\Omega^{n,\mathrm{sep}}_{F|k}\right)^\vee & \to & F^\vee \\ f & \mapsto & f(\cdot\, \omega) \end{array}$$

is again in isomorphism of ST vector spaces. It follows that the composition:

$$F \to \left(\Omega^{n,\mathrm{sep}}_{F|k}\right)^\vee \xrightarrow{\rho_\omega^\vee} F^\vee$$

which is given explicitely by $b \mapsto \mathrm{res}_{F|k}(\cdot\, b\omega)$, induces a self duality for the k-vector space F. This means that for any choice of a nonzero differential form ω, the k-character:

$$\begin{array}{rcl} \psi_\omega : F & \to & k \\ v & \mapsto & \mathrm{res}_{F|k}(v\omega) \end{array}$$

is a standard character for F (see definition F.8).

1.3 Arithmetic 2-dimensional local fields over a local field

In the previous section we analysed the structure of higher local fields of the type $K((t_1))\dots((t_n))$ (over a base field k), so by the classification theorem we know that the only remaining case to be studied consists of higher local fields of the type $F \cong K_p\{\{t_1\}\}\dots\{\{t_r\}\}((t_{r+1}))((t_m))$. In this section, by simplicity and for geometric purposes, we focus just on a particular case. We assume that F is a 2-dimensional local field such that:

- $\mathrm{char}\, F = 0$ and $\mathrm{char}\, F^{(2)} = p$.
- F is endowed with a ST-ring topology and there exists a mixed characteristic local field K with a fixed embedding $K \hookrightarrow F$ of ST-rings (K has the discrete valuation ring topology).

In this case we say that F is an *arithmetic* 2*-dimensional local field over* K (look at table 1.4). The presence of the local field K inside F comes from the theory of arithmetic surfaces and it will be explained in section 3.3. In this section we will also introduce the *Kato symbol*:

$$(\ ,\ )_{F|K} : F^\times \times F^\times \to K^\times$$

which is the generalization of the usual tame symbol (see definition G.1).

1.3.1 Equal characteristic

Let F be an arithmetic 2-dimensional local field such that $\mathrm{char}\, \overline{F} = 0$.

Definition 1.23. The *coefficient field of* F *(with respect to* K*)* is the algebraic closure of K inside F and it is denoted as k_F.

The coefficient field k_F is a finite extension of K and moreover $\overline{F} = k_F$. In particular $F \cong k_F((t))$. The local field K carries the usual discrete valuation topology and k_F is endowed with the finite product topology, so we can put on $k_F((t))$ the ind/pro-topology or equivalently the P-topology (see proposition E.11). However, this is not enough to define a canonical topology on F, because, as we noticed in the previous section, the result will depend on the choice of the isomorphism $F \cong k_F((t))$. We again have the concept of parametrization

Definition 1.24. A *parametrization* for F is a fixed isomorphism $F \cong k_F((t))$ of higher valuation fields and ST-rings.

Let's put $L := k_F((t))$, it is a semi-topological K-algebra, so we can define the module of separated differential forms $\Omega^{1,\text{sep}}_{L|K}$ and by using the same proof of proposition 1.21, we can show that $\Omega^{1,\text{sep}}_{L|K} = Ldt$. We define the residue map for L relative to K to be the K-linear homomorphism:

$$\begin{aligned} \operatorname{res}_{L|K} : \Omega^{1,\text{sep}}_{L|K} &\to K \\ \omega &\mapsto \operatorname{Tr}_{k_F|K}(a_{-1}) \end{aligned}$$

where $\omega = adt$ and $a = \sum_{j \geq m} a_j t^j \in L$. It is straightforward to see that $\operatorname{res}_{L|K}$ is continuous.

Definition 1.25. If $p : F \to L$ is a parametrization for F and $\tilde{p} : \Omega^{1,\text{sep}}_{F|K} \to \Omega^{1,\text{sep}}_{L|K}$ is the induced homomorphism, then we define:

$$\operatorname{res}_{F|K} := \operatorname{res}_{L|K} \circ \tilde{p} : \Omega^{1,\text{sep}}_{F|K} \to K\,.$$

Proposition 1.26. *The map* $\operatorname{res}_{F|K}$ *doesn't depend on the choice of the parametrization.*

Proof. It is an easy calculation. See [44, pages 14-15] for details. ☐

Remark 1.27. It is important to emphasize how the theory developed for F so far resembles exactly the theory of a local field of the type $\overline{F}((t))$ introduced in section 1.2. The crucial difference is that here K and $\overline{F}$ carry a non-trivial topology, which is the reason why F is considered as 2-dimensional local field.

The valuation field F is naturally endowed with the usual tame symbol $(\ ,\ )_F : F^\times \times F^\times \to k_F^\times$, so we can obtain the Kato symbol (or two dimensional tame symbol) by simply composing it with the field norm map:

Definition 1.28. The *Kato symbol* for F (with respect to K) is given by:

$$(\ ,\ )_{F|K} : N_{k_F|K} \circ (\ ,\ )_F : F^\times \times F^\times \to K^\times\,.$$

1.3.2 Mixed characteristic

Now we assume that F is an arithmetic 2-dimensional local field of mixed characteristic. By the classification theorem $\mathbb{Q}_p$ is contained in F and we have the notion of constant field of F which replaces the one of coefficient field:

Definition 1.29. The *constant field of* F is the algebraic closure of $\mathbb{Q}_p$ in F, and it is denoted by k_F.

Remark 1.30. Note that the definition of the constant field doesn't depend on K so it makes sense for any 2-dimensional local field of mixed characteristic. Of course it might happen that $K = \mathbb{Q}_p$.

Since K is a finite extension of $\mathbb{Q}_p$ (by the 1-dimensional classification theorem), we know that k_F is an intermediate field between K and F. The constant field k_F is a finite extension of $\mathbb{Q}_p$ (so also a finite extension of K).

Definition 1.31. We say that an arithmetic 2-dimensional local field of mixed characteristic F is *standard* if there is a k_F isomorpshism $F \cong k_F\{\{t\}\}$. When an isomorphism is given, we say that we have fixed a *parametrization* of F.

We will study standard fields first and extend any result for a generic F thanks to the following result:

Proposition 1.32. *There exists a standard field L contained in F such that: $[F : L] < \infty$, $k_F = k_L$ and $\overline{F} = \overline{L}$.*

Proof. See [44, Lemma 2.14] □

So, from now on in this subsection we fix L to be a standard field contained in F with the properties described in proposition 1.32. Clearly we have the following field extensions that need to be kept always in mind (we mark the finite extensions with the superscript $\mathfrak{f}$):

$$\mathbb{Q}_p \subseteq^{\mathfrak{f}} K \subseteq^{\mathfrak{f}} k_L = k_F \subseteq L \cong k_L\{\{t\}\} \subseteq^{\mathfrak{f}} F\,. \tag{1.6}$$

Now we define a topology on $k_L\{\{t\}\}$; in [38] it is shown that we can transfer such a topology to L without any dependence on the choice of the parametrization. Then we endow F with the finite product topology which will be called the *canonical topology of* F.

Remark 1.33. We recall one more time that in characteristic 0, elements of $\mathbf{TLF}^n(k)$ don't have a canonical topology in general (theorem 1.20).

An open basis at 0 for $k_L\{\{t\}\}$ is given by subsets of the form:

$$\sum_{i\in\mathbb{Z}} U_i t_i := \left\{\sum a_i t_i \in k_L\{\{t\}\} \colon a_i \in U_i\right\}$$

where $\{U_i\}_{i\in\mathbb{Z}}$ is a family of open neighborhoods of 0 in k_L satisfying the following properties:

(i) For any integer r, there exists an index j such that $\mathfrak{p}_{k_L}^r \subseteq U_i$ for any $i \geq j$.

(ii) $\bigcap_{i\in\mathbb{Z}} U_i$ contains a power of $\mathfrak{p}_{k_L}$.

We need a notion of residue map for F and its construction will be similar to the equal characteristic case, in particular we will go through the following steps:

- Define the residue map for $L \cong k_L\{\{t\}\}$.
- See that it is independent from any parametrization of a standard field.
- Extend the residue map for any arithmetic 2-dimensional local field F.
- See that the construction doesn't depend on the choice of the standard field L.

First of all we need to study the module of separated differential forms $\Omega^{1,\mathrm{sep}}_{k_L\{\{t\}\}|K}$:

Proposition 1.34. *$\Omega^{1,\mathrm{sep}}_{k_L\{\{t\}\}|K} = k_L\{\{t\}\}dt$.*

Proof. See [44, Lemma 2.18]. □

Now the residue map for $k_L\{\{t\}\}$ can be easily defined in the following way:

$$\begin{aligned} \mathrm{res}_{k_L\{\{t\}\}|K} : \Omega^{1,\mathrm{sep}}_{k_L\{\{t\}\}|K} &\to K \\ \omega &\mapsto -\mathrm{Tr}_{k_L|K}(a_{-1}) \end{aligned}$$

where $\omega = adt$ and $a = \sum a_j t^j \in L$. It is straightforward to check that the residue is K-linear and continuous.

Remark 1.35. The presence of the minus sign in the definition of $\mathrm{res}_{k_L\{\{t\}\}|K}$ has a geometric meaning. We will see in section 3.3 that 2-dimensional local fields of equal characteristic arise on horizontal curves on arithmetic surfaces, whereas 2-dimensional local fields of mixed characteristic are related to vertical curves. So, intuitively the minus sign in the residue map remembers of the "change of orientation" of such curves.

Definition 1.36. Let L be any standard field. If $p : L \to k_L\{\{t\}\}$ is a parametrization for L and $\tilde{p} : \Omega^{1,\mathrm{sep}}_{L|K} \to \Omega^{1,\mathrm{sep}}_{k_L\{\{t\}\}|K}$ is the induced homomorphism, then we define:

$$\mathrm{res}_{L|K} := \mathrm{res}_{k_L\{\{t\}\}|K} \circ \tilde{p} : \Omega^{1,\mathrm{sep}}_{L|K} \to K\,.$$

Proposition 1.37. *Let L be any standard field, then the map $\mathrm{res}_{L|K}$ doesn't depend on the parametrization of L.*

Proof. See [44, Proposition 2.19]. ☐

Finally we can use the trace map to extend the residue to F:

Definition 1.38. Let F be any arithmetic 2-dimensional local field of mixed characteristic, then:

$$\mathrm{res}_{F|K} := \mathrm{res}_{L|K} \circ \mathrm{Tr}_{F|L} : F \to K$$

where L is any standard field contained in F with the properties described by proposition 1.32.

Proposition 1.39. *The residue map on F doesn't depend on the choice of the standard field L inside F.*

Proof. See [44, Lemma 2.21]. ☐

Kato symbol. Finally, we want to define the Kato symbol for F and the strategy is the usual one: we start from $k_L\{\{t\}\}$ and we extend our arguments to F by checking that everything is independent from parametrizations and from the choice of the standard fields. We will heavily use some K-theoretic notions developed in appendix G.1.

Fix just for the moment $L = k_L\{\{t\}\}$, then we define:

$$(\ ,\ )_{L|K} : L^\times \times L^\times \xrightarrow{\{\,,\}} K_2(L) \longrightarrow \widehat{K}_2(L) \xrightarrow{-\mathrm{res}^{(2)}_L} \widehat{K}_1(k_L) = k_L^\times \xrightarrow{N_{k_L|K}} K^\times \tag{1.7}$$

where:

- $\{\,,\}$ is the natural projection arising from the definition of $K_2(L)$ (see proposition G.2).
- The morphism $K_2(L) \to \widehat{K}_2(L)$ is the map given by the construction of $\widehat{K}_2(L)$ as projective limit (see equation (G.3)).

- $\mathrm{res}_L^{(2)}$ is the higher Kato residue map constructed in theorem G.11. Note that $\widehat{K}_1(k_L) = k_L^\times$ because k_L is already complete.

Moreover by simplicity we use the following notation:

$$\partial_L : K_2(L) \longrightarrow \widehat{K}_2(L) \xrightarrow{-\mathrm{res}_L^{(2)}} k_L^\times . \tag{1.8}$$

Remark 1.40. [36] gives an explicit description of $\mathrm{res}_L^{(2)}$ which involves winding numbers.

Definition 1.41. Let L be a generic standard field and fix a parametrization: $p : L \to k_L\{\{t\}\}$ then we define:

$$(\,,\,)_{L|K} : L^\times \times L^\times \xrightarrow{\{\,,\}} K_2(L) \xrightarrow{K_2(p)} K_2(k_L\{\{t\}\}) \xrightarrow{\partial_{k_L\{\{t\}\}}} k_L^\times \xrightarrow{N_{k_L|K}} K^\times$$

and we put $\partial_L := \partial_{k_L\{\{t\}\}} \circ K_2(p)$.

Proposition 1.42. *Let L be a standard field, then the definition of $(\,,\,)_{L|K}$ doesn't depend on the parametrization of L.*

Proof. See [36, Corollary 3.7]. □

At this point we are ready to give the general definition of the Kato symbol:

Definition 1.43. Let F an arithmetic 2-dimensional local field and let L be a standard field contained in F, then the *Kato symbol for F* (with respect to K) is given by:

$$(\,,\,)_{F|K} : F^\times \times F^\times \xrightarrow{\{\,,\}} K_2(F) \xrightarrow{K_2(N_{F|L})} K_2(L) \xrightarrow{\partial_L} k_L^\times \xrightarrow{N_{k_L|K}} K^\times \tag{1.9}$$

Proposition 1.44. *The definition of $(\,,\,)_{F|K}$ doesn't depend on the choice of L inside F.*

Proof. See [30, Proposition 3]. □

Chapter 2

One-dimensional adelic geometry

In this chapter we give a detailed account of one dimensional adelic theory in relation with geometry. The material is essentially known, but here it will presented such in way that it will be a guideline for the 2-dimensional theory. It means that the path we will take in order to develop the 2-dimensional theory will be, as much as possible, the direct generalization of what is written here.

Subsection 2.1.1 relates the local theory of chapter 1 to one-dimensional schemes. We will fix a field K to be either an algebraic function field of one variable over a perfect field k or a number field and in subsection 2.1.2 we introduce the abstract concept of adelic line bundle associated to K. Here the important point is that we don't distinguish the two cases when K is a function field (i.e. algebraic geometry) or a number field (i.e. arithmetic geometry). Such an approach doesn't have a direct 2-dimensional extension. Afterwords, we globalize the local theory, but in sections 2.2 and 2.3 we separate respectively the algebraic theory from the arithmetic theory. For algebraic curves every piece of information we need is already in the geometric picture; on the other hand for arithmetic curves (i.e. spectra of orders) we have to deal inevitably with archimedean absolute values and Arakelov geometry. This modus operandi will be adopted also later in the 2-dimensional case, where again we will need to split the global theory.

Main references. A classical introduction to adelic theory is given in [52], which is basically a revisitation of Tate's thesis [60]. The concept of adelic line bundles appears in [9, 2] for other purposes. Moreover [19, 0] gives a very quick overview of the adelic theory in relation to algebraic curves.

2.1 General definitions

2.1.1 Local data on schemes of dimension one

Let $(X, \mathscr{O}_X)$ be a Noetherian and integral scheme of dimension 1 with function field K. In this short section we explain how to attach the objects K_x and $\mathcal{O}_x$ to each closed points of X: the former will be a finite direct product of discrete valuation fields and the latter a finite direct product of discrete valuation rings.

Fix a closed point $x \in X$, then the completion $\widehat{\mathscr{O}_{X,x}}$ with respect its unique maximal ideal $\mathfrak{p}_x$ is a Noetherian local ring (not necessarily a domain) of dimension 1 with maximal ideal $\widehat{\mathfrak{p}_x} := \mathfrak{p}_x\widehat{\mathscr{O}_{X,x}}$. Let's look at the properties of the homomorphism:

$$\phi : \operatorname{Spec}\widehat{\mathscr{O}_{X,x}} \to \operatorname{Spec}\mathscr{O}_{X,x}$$

induced by the inclusion $\mathscr{O}_{X,x} \hookrightarrow \widehat{\mathscr{O}_{X,x}}$.

Proposition 2.1. $\phi^{-1}(0) := \{\mathfrak{q} \in \widehat{\mathscr{O}_{X,x}} : \phi(\mathfrak{q}) = \mathfrak{q} \cap \mathscr{O}_{X,x} = 0\}$ *is the set of all minimal prime ideals of* $\widehat{\mathscr{O}_{X,x}}$. *This in particular implies that* $\phi^{-1}(0)$ *has finite cardinality.*

Proof. It is equivalent to show that $\phi^{-1}(0)$ is the set of all prime ideals of height 0. To do this we will use the going-down property for the ring extension $\mathscr{O}_{X,x} \subset \widehat{\mathscr{O}_{X,x}}$ which in particular says that if $\mathfrak{q} \in \widehat{\mathscr{O}_{X,x}}$ is a prime such that $\mathfrak{q} \cap \mathscr{O}_{X,x} = \mathfrak{p}_x$, then $\operatorname{ht}(\mathfrak{q}) \geq \operatorname{ht}(\mathfrak{p}_x)$.

If $\mathfrak{q}$ is a prime of height 0, then $\operatorname{ht}(\phi(\mathfrak{q})) = 0$, but this means that $\phi(\mathfrak{q}) = 0$ since $\mathscr{O}_{X,x}$ is a domain. Vice versa assume that $\mathfrak{q} \in \phi^{-1}(0)$, if $\operatorname{ht}(\mathfrak{q}) = 1$, then $\mathfrak{q}$ would be the maximal ideal of $\widehat{\mathscr{O}_{X,x}}$ implying that $\phi(\mathfrak{q}) \neq 0$. Therefore it has to be $\operatorname{ht}(\mathfrak{q}) = 0$. □

From the previous proposition it follows that for any $\mathfrak{q} \in \phi^{-1}(0)$

$$K_{x,\mathfrak{q}} := \left(\widehat{\mathscr{O}_{X,x}}\right)_{\mathfrak{q}} = \operatorname{Frac}\left(\widehat{\mathscr{O}_{X,x}}/\mathfrak{q}\right).$$

But there is more:

Proposition 2.2. $K_{x,\mathfrak{q}}$ *is a complete discrete valuation field whose ring of integers is given by* $\mathcal{O}_{x,\mathfrak{q}} := \left(\widehat{\mathscr{O}_{X,x}}/\mathfrak{q}\right)^{\sim}$ *(here by* $\sim$ *we denote the normalization). Furthermore the residue field* $\overline{K}_{x,\mathfrak{q}}$ *is a finite extension of* $k(x)$.

Proof. $\widehat{\mathscr{O}_{X,x}}/\mathfrak{q}$ is a Noetherian local domain such that

$$\dim\left(\widehat{\mathscr{O}_{X,x}}/\mathfrak{q}\right)=\dim\left(\widehat{\mathscr{O}_{X,x}}\right)-\mathrm{ht}(\mathfrak{q})=1\,.$$

Moreover it is also complete (with respect the topology induced by its maximal ideal). It follows that the integral closure $\mathcal{O}_{x,\mathfrak{q}}$ is a complete DVR with fraction field $K_{x,\mathfrak{q}}$. By Nagata theorem ([7, Ch. IX, 4, n.2, theorem 2]) we know that $\widehat{\mathscr{O}_{X,x}}/\mathfrak{q}$ is a Japanese ring ([7, Ch.IX, 4, n.2, def.1]), so in particular $\mathcal{O}_{x,\mathfrak{q}}$ is a finite $\widehat{\mathscr{O}_{X,x}}/\mathfrak{q}$-module. This implies that $\overline{K}_{x,\mathfrak{q}}$ is a finite extension of:

$$\left(\widehat{\mathscr{O}_{X,x}}/\mathfrak{q}\right)/\left(\widehat{\mathfrak{p}_x}/\mathfrak{q}\right)\cong\widehat{\mathscr{O}_{X,x}}/\widehat{\mathfrak{p}_x}\cong\mathscr{O}_{X,x}/\mathfrak{p}_x=k(x)\,.$$

□

Each $K_{x,\mathfrak{q}}$ is endowed with the natural topology induced by the complete discrete valuation and remember that such topology can be defined also by using the constructions (C) and (L) described in example E.17. Now we can define the following rings attached to x:

$$K_x:=\prod_{\mathfrak{q}\in\phi^{-1}(0)}K_{x,\mathfrak{q}}\,. \tag{2.1}$$

$$\mathcal{O}_x:=\prod_{\mathfrak{q}\in\phi^{-1}(0)}\mathcal{O}_{x,\mathfrak{q}}\,. \tag{2.2}$$

Clearly we put the product topology on K_x and the subspace topology on $\mathcal{O}_x$.

When $\widehat{\mathscr{O}_{X,x}}$ is a domain we say that X is *unbranched* at x, in particular X is unbranched at its non-singular points since the property of being a regular local ring is stable under completions ([57, 15.39.4]) and regular local rings are domains ([57, 10.105.2]). By the way, it may happen that X is unbranched at singular points, and the example is given by cusps. If X is unbranched at x, then by proposition 2.1 we can conclude that direct products of equations (2.1) and (2.2) are actually made of a single factor; so K_x is a discrete valuation field and $\mathcal{O}_x$ is its valuation ring. Moreover in this case $\overline{K}_x=k(x)$.

Remark 2.3. Let $\mathcal{N}:\widetilde{X}\to X$ be the normalization of X such that $\mathcal{N}^{-1}(x)=\{\widetilde{x}_1,\dots\widetilde{x}_r\}$, then with some commutative algebra arguments one can show that:

$$\mathcal{O}_x=\prod_{\mathfrak{q}\in\phi^{-1}(0)}\left(\widehat{\mathscr{O}_{X,x}}/\mathfrak{q}\right)^{\sim}\cong\prod_{i=1}^{r}\widehat{\mathscr{O}_{\widetilde{X},\widetilde{x}_i}}\,.$$

2.1.2 Adelic line bundles

The field K here is either an algebraic function field of one variable over a perfect field k or a number field. A *place on* K is an equivalence relation of absolute values on K, and remember that two absolute values are equivalent if they generate the same topology. With the symbol $\mathcal{P}(K)$ we indicate all places on K which are trivial on $k^\times$ (when k is present, otherwise $\mathcal{P}(K)$ is the set of all places)[1]. The completion of K with respect to a place $b \in \mathcal{P}(K)$ is denoted as K_b. We assume that all discrete valuations on K_b are normalized i.e. surjective onto $\mathbb{Z}$. Let's fix a canonical representative for each place $b \in \mathcal{P}(K)$ in the following way:

- If b is non-archimedean and v_b is the unique complete discrete valuation on K_b we choose the following absolute values:

  $$|\cdot|_b := c_b^{-v_b(\cdot)}$$

  where c_b is a constant defined in the following way:

  $$c_b = \begin{cases} \#(\overline{K}) & \text{if } \#(\overline{K}) < \infty \\ e = \exp(1) & \text{otherwise} \end{cases} \tag{2.3}$$

- If b is archimedean, then K is a number field and we make the choice explained in appendix C.1.

Let's recall the notion of normed vector space over a generic field $(F, |\cdot|)$ endowed with an absolute value:

Definition 2.4. Let $(F, |\cdot|)$ be a field endowed with an absolute value and let V be a F-vector space. A norm on V is a function $||\cdot|| : V \to \mathbb{R}_+$ satisfying the following properties:

- $||x|| = 0$ if and only if $x = 0$.
- $||ax|| = |a|||x||$ for any $a \in F$ and $x \in V$.
- $||x + y|| \leq ||x|| + ||y||$ for any $x, y \in V$.

Definition 2.5. An *adelic vector space* over K, is a couple

$$E_A := \big(E, \{||\cdot||_b\}_{b\in\mathcal{P}(K)}\big)$$

such that E is a K-vector space and $||\cdot||_b$ is a norm on the vector space $E_b := E \otimes_K K_b$ over $(K_b, |\cdot|_b)$, satisfying the following properties:

---

[1] The property of being trivial on k is shared by all the absolute values representing the same place.

- If b is non-archimedean, then $||\cdot||_b$ satisfies the ultrametric triangle inequality, namely

$$||x+y||_b \leq \max\{||x||_b + ||x||_b\} \quad \forall x, y \in E_b\,.$$

- For any $x \in E \setminus \{0\}$, $||x \otimes 1||_b = 1$ for all but finitely many $b \in \mathcal{P}(K)$.

By simplicity, if $x \in E$, we put $x = x \otimes 1 \in E_b$ when it is clear from the context and we consider $E \subseteq E_b$.

A morphism between two adelic vector spaces $\psi : E_A \to E_A$ is given by a homomorphism of vector spaces $\psi : E \to E'$ and the collection of induced homomorphisms $\psi_b : E_b \to E'_b$ given by $x \otimes 1 \mapsto \psi(x) \otimes 1$ for any $x \in E$.

Definition 2.6. $\psi : E_A \to E'_A$ is an *isometry of adelic vector spaces* if the underlying homomorphism between K-vector spaces $\psi : E \to E'$ is an isomorphism and moreover $\psi_b : E_b \to E'_b$ is an isometry for any $b \in \mathcal{P}(K)$. The isometry relation between adelic bundles is indicated simply by the symbol $\cong$.

In order to define the concept of adelic line bundle we need the general notion of purity for a normed vector space. Let $(V, ||\cdot||)$ be a normed vector space over a field $(F, |\cdot|)$, then we can define another norm on V in the following way:

$$||x||_\circ := \inf\{|t|\colon t \in F,\ ||x|| \leq |t|\}\,. \tag{2.4}$$

Clearly it holds that $||x|| \leq ||x||_\circ$ for any $x \in V$.

Definition 2.7. A normed vector space $(V, ||\cdot||)$ over $(F, |\cdot|)$ is said *pure* if $||x|| = ||x||_\circ$ for any $x \in V$.

Definition 2.8. An *adelic line bundle over K* is an adelic K-vector space $L_A = (L, \{||\cdot||_b\})$ of dimension 1 such that $(L_b, ||\cdot||_b)$ is pure for all $b \in \mathcal{P}(K)$.

Remark 2.9. In general an adelic vector bundle of rank r is more than an adelic vector E_A space of dimension r with the purity condition at each place, indeed we also require that E_b has an orthonormal basis for all but finitely many b.

Definition 2.10. Let L_A be an adelic line bundle, then we can define the *adelic realization of L_A* as

$$\mathbf{A}_K(L_A) := \prod_b{}' L_b = \prod_b{}'(L \otimes_K K_b)$$

where the restricted product is taken over the unit balls

$$\mathcal{B}(L_b, 1) := \{y \in L_b\colon ||y||_b \leq 1\}\,.$$

Given two adelic line bundles $L_A = (L, \{|| \cdot ||_b\}_b)$ and $L'_A = (L', \{|| \cdot ||'_b\}_b)$ we can construct the adelic line bundle

$$L_A \otimes L'_A := (L \otimes_F L', \{|| \cdot ||_b \otimes || \cdot ||'_b\}_b)$$

where $|| \cdot ||_b \otimes || \cdot ||'_b$ is the norm on $L_b \otimes L'_b \cong (L \otimes_K L')_b$ defined by

$$x \otimes y \mapsto ||x||_b ||y||'_b \,.$$

It is easy to see that adelic line bundles form a group with the identity element given by $K_A = (K, \{| \cdot |_b\})$.

Definition 2.11. The group of adelic line bundles modulo the isometry relation is called the *adelic Picard group* and it is denoted by $\mathrm{APic}(K)$.

Let L_A be an adelic line bundle on K, for any nonzero $s \in L$ we want to define the order of s at $b \in \mathcal{P}(K)$. At any place b, by the purity of L_b we can find[2] an element $\omega_b \in L_b$ such that $||\omega_b||_b = 1$. We can write in a unique way $s = \omega_b \alpha_b$ with $\alpha_b \in K_b$. Define

$$\mathrm{ord}_b(s) := v_b(\alpha_b)$$

it has two important properties:

- For all but finitely many $b \in \mathcal{P}(K)$ we have that $\mathrm{ord}_b(s) = 0$. This is true because for all but finitely places $||s||_b = 1$, hence $|\alpha_b|_b = 1$.
- $\mathrm{ord}_b(s)$ is independent from the choice of ω_b. Indeed let ω'_b be another element such that $||\omega'_b||_b = 1$, then $\omega_b = \mu_b \omega'_b$ for $\mu_b \in K_b$ and $|\mu|_b = 1$. It follows that $s = \omega'_b \mu_b \alpha_b$ and $v_b(\mu_b \alpha_b) = v_b(\alpha_b)$.

2.2 Algebraic curves

We have defined the local data attached to any point of a one dimensional scheme, now it's time to "glue" this data and get the adelic ring.

Notation. In this section we fix a perfect field k and a k-scheme X which is Noetherian, geometrically integral and of dimension 1. The function field of X is $K = k(X)$ and of course it is an algebraic function field of one variable over k. We will call all the schemes, which have the same properties of X, simply *algebraic curves over k*.

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[2] Note that since archimedean places of F are discrete, the "inf" of equation 2.4 is a actually a "min", for any place b.

2.2.1 Adeles

First of all note that we have two inclusions: $k \hookrightarrow \overline{K}_{x,\mathfrak{q}}$ and $k \hookrightarrow \mathscr{O}_{X,x} \subset K_{x,\mathfrak{q}}$, so $K_{x,\mathfrak{q}}$ is a discrete valuation field of equal characteristic and $\operatorname{char} K_{x,\mathfrak{q}} = \operatorname{char} \overline{K}_{x,\mathfrak{q}} = \operatorname{char} k$. The ring K_x, considered with its natural topology of finite product of discrete valuation fields; it is locally compact only when $k = \mathbb{F}_q$. For a generic k we just know that K_x is a locally linearly compact k-vector space (and also a topological ring), such that $\mathcal{O}_x$ a linearly locally compact open subspace (see example F.6 and proposition F.4(5)).

Definition 2.12. The adelic ring of X is defined as:

$$\mathbf{A}_X := \prod_{x \in X}{}' K_x$$

where $\prod'$ is the restricted product, in the category of locally linearly compact vector spaces, with respect to the subspaces $\mathcal{O}_x$.

If X is singular and $\widetilde{X}$ is its normalization, then by remark 2.3 we can deduce that $\mathbf{A}_X \cong \mathbf{A}_{\widetilde{X}}$. This means that we can basically work just with regular curves, on which we have the big advantage that $K_x = \operatorname{Frac} \widehat{\mathscr{O}_{X,x}}$ and $\mathcal{O}_x = \widehat{\mathscr{O}_{X,x}}$. In this case, for a choice of local parameter ϖ we have an isomorphism $K_x \cong k(x)((\varpi))$.

The following theorem is the key link between classical theory of adeles and geometric theory of adeles:

Theorem 2.13. *Assume that X is regular. There is a bijection between the closed points of X and $\mathcal{P}(K)$.*

Proof. For each closed point $x \in X'$, the ring $\mathscr{O}_{X,x}$ is a DVR since x is nonsingular; hence we have a place on $K = \operatorname{Frac} \mathscr{O}_{X,x}$. Let $x, y \in X'$ two distinct points, by the Riemann-Roch theorem it is possible to find an element $f \in K^\times$ such that $v_x(f) \geq 0$ and $v_y(f) < 0$; this implies that x and y give different places on the field K.
It remains to show the surjectivity of the map. Since K is an algebraic function field over k, each place $\mathfrak{p}$ on K which is trivial on k is non-archimedean (See [21, I, 1.2]), so it arises from a discrete valuation $v : K \to \mathbb{Z}$ such that $v(k^\times) = 0$. Let $\mathcal{O}_v$ be the valuation ring of v, since $k^\times \subset \mathcal{O}_v$, we have the following commutative diagram of schemes where the construction of the

arrows is straightforward:

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[r, "\iota"] \arrow[d] & X \arrow[d] \\
\operatorname{Spec} \mathcal{O}_v \arrow[r] & \operatorname{Spec} F\ .
\end{tikzcd} \tag{2.5}$$

Now we apply the valuative criterion of properness ([37, pag. 106]) to deduce that the morphism ι of diagram (2.5) extends to a morphism $\bar{\iota} : \operatorname{Spec} \mathcal{O}_v \to X$. Moreover $\bar{\iota}$ induces a local and ring embedding:

$$\mathscr{O}_{X,x} \hookrightarrow (\mathcal{O}_v)_{\mathfrak{p}_v} = \mathcal{O}_v \ ,$$

so by a property of valuation rings (See [37, lemma 3.24]) which says that valuation rings in K are maximal under the relation of domination amongst local rings contained in K, we have $\mathscr{O}_{X,x} \cong \mathcal{O}_v$. We can conclude that v_x and v define the same topology on K. □

Let's assume the following convention if not otherwise specified:

X is a regular curve, otherwise we can consider its normalization $\widetilde{X}$. Moreover we always identify the set of closed points of X and $\mathcal{P}(K)$.

When $k = \mathbb{F}_q$, then K is a global field of finite characteristic and all places are trivial on $\mathbb{F}_q$. It follows that $\mathbf{A}_X$ is exactly the classical ring of adeles usually denoted by $\mathbf{A}_K$. So, the geometric approach to adeles described here extends the classical theory from global fields of finite characteristic, to function fields of one variable over any perfect field k.

Now we use the theory of differential forms developed in section 1.2.2 applied to the valuation fields K_x. For any $x \in X$ let's put by commodity:

$$\Omega^1_x := \Omega^{1,\mathrm{sep}}_{K_x|k} \,,$$

$$\operatorname{res}_x := \operatorname{res}_{K_x|k} : \Omega^1_x \to k \,.$$

Clearly we have a (nontrivial) composition of maps

$$\Omega^1_{K|k} \to \Omega^1_{K_x|k} \to \Omega^1_x \tag{2.6}$$

where $\Omega^1_{K|k}$ is the usual K-vector space of rational differential forms on X. But by [37, Def. 6.1.14 and Prop. 6.1.15] we know that $\Omega^1_{K|k}$ has dimension 1, hence we can conclude that equation 2.6 gives an embedding $\Omega^1_{K|k} \hookrightarrow \Omega^1_x$. For any $\eta = f d\varpi \in \Omega^1_x$ we can define the valuation $v_x(\eta) := v_x(f)$, where $v_x(f)$ is the usual complete discrete valuation in K_x. It is straightforward to check that the value $v_x(\eta)$ is well defined independently from the choice of the uniformizer ϖ.

The following theorem is a classical result, often presented for curves over algebraically closed fields:

Theorem 2.14 (1D reciprocity law)**.** *Let $\omega \in \Omega^1_{K|k}$, then*

$$\sum_{x \in X} \mathrm{res}_x(\omega) = 0\,.$$

Proof. See [59, Corollary of theorem 3]. $\square$

Lemma 2.15. *Fix a nonzero rational differential form $\omega \in \Omega^1_{K|k}$, then for any $x \in X$ we have that*[3]

$$\mathcal{O}_x^{\vartriangle} = \{\mathrm{res}_x(\cdot\, b\omega) \colon v_x(b) \geq -v_x(\omega)\}\,.$$

Proof. In chapter 1 we showed that the map $\psi := \mathrm{res}_x(\cdot\,\omega) : K_x \to k$ is a standard character, so in particular any other k-character of K_x is of the form $\psi_b := \mathrm{res}_x(\cdot\, b\omega)$ for $b \in K_x$.
Let $b \in K_x$ such that $v_x(b) \geq v_x(\omega)$, then for any $a \in \mathcal{O}_x$: $\psi_b(a) = \mathrm{res}_x(ab\omega)$, but:

$$v_x(ab\omega) = v_x(a) + v_x(b) + v_x(\omega) \geq v_x(a) - v_x(\omega) + v_x(\omega) \geq 0\,;$$

hence $\psi_b(a) = 0$ and one inclusion is proved.
Vice versa assume that b is such that $v_x(b) < -v_x(\omega)$ then $v_x(b\omega) = r < 0$ if $a = \varpi^{-r-1} \in \mathcal{O}_x$, then $\psi_b(a) \neq 0$. $\square$

The following theorem is the generalization of the classical duality result for adeles.

Theorem 2.16. *Fix a nonzero rational differential form $\Omega^1_{K|k}$ and consider the map:*

$$\begin{array}{rcl} \psi^\omega : \mathbf{A}_X & \to & k \\ (\alpha_x)_x & \mapsto & \displaystyle\sum_{x \in X} \mathrm{res}_x(\alpha_x\omega)\,. \end{array}$$

Then ψ^ω is a standard character for $\mathbf{A}_X$. In particular $\mathbf{A}_X$ is self dual.

---

[3] For the notation "$\vartriangle$" see appendix F.

Proof. We show that any k-character of $\mathbf{A}_X$ is of the form $\psi^\omega(\cdot\,\beta)$ for a uniquely determined element $\beta \in \mathbf{A}_X$, the rest of the proof will follow easily by using theorem F.13 which in particular says that

$$\mathbf{A}_X^\vee = \left\{ \sum_{x\in X'} \psi_x \colon \psi_x \in K_x^\vee \text{ and } \psi_x \in \mathcal{O}_x^\vartriangle \text{ for all but finitely many } x \right\} .$$

We know that $\psi_x = \mathrm{res}_x(\cdot\, b_x\omega)$ for a unique $b_x \in K_x$ and for all but finitely many x $v_x(b_x) \geq -v_x(\omega)$ (see lemma 2.15). The classical product formula says that $v_x(\omega) = 0$ for all but finitely many x, hence $b_x \in \mathcal{O}_x$ for all but finitely many x and we conclude that $\beta = (b_x)_x \in \mathbf{A}_X$. □

Definition 2.17. Fix a nonzero rational differential form $\omega \in \Omega^1_{K|k}$, then the standard character ψ^ω of theorem 2.16 gives a so called *global differential pairing* (associated to ω) on $\mathbf{A}_X$:

$$\begin{aligned} d_\omega : \mathbf{A}_X \times \mathbf{A}_X &\to k \\ (\alpha, \beta) &\mapsto \psi^\omega(\alpha\beta) = \sum_{x\in X} \mathrm{res}_x(\alpha_x\beta_x\omega)\,. \end{aligned}$$

It is straightforward to notice that d_ω is bilinear, continuous and symmetric.

2.2.2 Adelic complexes

We want to show that invertible sheaves (up to isomorphism) on X correspond exactly to adelic line bundles on K (up to isometry).

Let $\mathscr{L}$ be a locally free sheaf of rank r over X (since X is noetherian $\mathscr{L}$ is coherent), if η is the generic point of X then $\mathscr{L}_\eta$ is a r-dimensional K-vector space. First of all, for any $x \in X$ put:

$$K_x(\mathscr{L}) := \mathscr{L}_\eta \otimes_K K_x = (\mathscr{L}_x \otimes_{\mathscr{O}_{X,x}} K) \otimes_K K_x = \mathscr{L}_x \otimes_{\mathscr{O}_{X,x}} K_x\,,$$

$$\mathcal{O}_x(\mathscr{L}) := \mathscr{L}_x \otimes_{\mathscr{O}_{X,x}} \widehat{\mathscr{O}_{X,x}}\,.$$

We are mostly interested in the case when $\mathscr{L}$ is an invertible sheaf, therefore from now on we assume that $r = 1$. If e_x is a local $\mathscr{O}_{X,x}$-basis at x for $\mathscr{L}_x$, then any nonzero element $t \in K_x(\mathscr{L})$ can be written as $t = e_x a_x$, with $a_x \in K_x$, and its norm can be defined in the following way:

$$||t||_x := |a_x|_x\,.$$

Since two basis at x of $\mathscr{L}_x$ differ by an element in $\mathscr{O}_{X,x}^\times$, we deduce that $||\cdot||_x$ is well defined independently from the choice of e_x. Fix a nonzero rational section $s \in \mathscr{L}_\eta$ then we can find an element $a \in K^\times$ and a collection of local $\mathscr{O}_{X,x}$-basis $\{l_x\}_{x\in X}$ of $\mathscr{L}_x$ such that $s = l_x a$ (note that now a is the same at each x). By [37, Lemma 7.2.5] we know that $a \in \mathscr{O}_{X,x}^\times$ for all but finitely many x, therefore $\mathscr{L}_\eta$ is an adelic vector space. Finally, the purity condition is trivially true, so $\mathscr{L}_\eta$ is an adelic line bundle. With the above metric structure given on $K_x(\mathscr{L})$ we have that

$$\mathcal{O}_x(\mathscr{L}) = \{t \in K_x(\mathscr{L}) \colon ||t||_x \leq 1\} \, .$$

It is important to understand in detail what is the local metric structure associated the invertible sheaf $\mathscr{O}_X(D)$ for a divisor $D = \sum_{x\in X} n_x[x] \in \mathrm{Div}(X)$. Consider D as a Cartier divisor and assume that f_x is a local equation of D at x, then

$$K_x(\mathscr{O}_X(D)) = f_x^{-1}\mathscr{O}_{X,x} \otimes_{\mathscr{O}_{X,x}} K_x = K_x \, .$$

Pick the element f_x^{-1} as $\mathscr{O}_{X,x}$-basis of $f_x^{-1}\mathscr{O}_{X,x}$, the any $t \in K_x(\mathscr{O}_X(D))$ can be written as $t = f_x^{-1}(f_x t)$, and we conclude that $||t||_x = |f_x|_x \, |t|_x$. Now remember the relation, $n_x = v_x(f_x^{-1})$, then we get the following explicit expression for the norm on $K_x(\mathscr{O}_X(D))$:

$$|| \cdot ||_x = c_x^{-n_x} | \cdot |_x \, . \tag{2.7}$$

where the constant c_x was defined in equation (2.3).

The following theorem gives the equivalence between invertible sheaves and adelic line bundles:

Theorem 2.18. $\mathrm{Pic}(X) \cong \mathrm{APic}(K)$.

Proof. To any line bundle $\mathscr{L}$ we associate the adelic line bundle $(\mathscr{L}_\eta, || \, ||_x)$ as explained above. Let's find the inverse of such a map. Let $(L, || \, ||_x)$ be an adelic line bundle, then for any open set $U \subseteq X$ we define:

$$\mathscr{L}_L(U) = \{t \in L \colon ||t||_x \leq 1, \ \forall x \in U\} \, .$$

Clearly $\mathscr{L}_L$ is a sheaf of $\mathscr{O}_X$-modules. Choose any $s \in L$, then on $V := X \setminus \{x_1, \dots x_r\}$ by the general properties of adelic vector spaces we have that $||s||_x = 1$; it means that for $t \in \mathscr{L}_L(V)$ we can write $t = \alpha s$ for $\alpha \in K$ and such that $||t||_x = |\alpha|_x \leq 1$ for any $x \in V$. The last condition actually implies that $\alpha \in \mathscr{O}_X(V)$. We can clearly cover X with subspaces constructed like V, and this shows that $\mathscr{L}_L$ is a line bundle. Finally it is straightforward to check that we have constructed the inverse map we were looking for. $\square$

The adelic realization (see definition 2.10) of the adelic line bundle $\mathscr{L}_n$ associated to an invertible sheaf $\mathscr{L}$ is:

$$\mathbf{A}_X(\mathscr{L}) := \prod_{x\in X}{}' K_x(\mathscr{L})$$

where the restricted product is taken with respect to the rings $\mathcal{O}_x(\mathscr{L})$. For $\mathscr{L} = \mathscr{O}_X$ we obtain $\mathbf{A}_X(\mathscr{O}_X) = \mathbf{A}_X$, so we have recovered the usual adelic ring attached to X. The vector space $\mathscr{L}_\eta$ is diagonally embedded in $\mathbf{A}_X(\mathscr{L})$ with the identity morphism, moreover let's define the space

$$\mathbf{A}_X(\mathscr{L})(0) := \prod_{x\in X} \mathcal{O}_x(\mathscr{L})\,,$$

then we get following adelic complex:

$$\mathcal{A}_X(\mathscr{L}): \quad 0 \to \mathscr{L}_\eta \oplus \mathbf{A}_X(\mathscr{L})(0) \to \mathbf{A}_X(\mathscr{L}) \to 0 \tag{2.8}$$

with the main map given by

$$(s, (\alpha_x)_x) \mapsto (s - \alpha_x)_x\,.$$

Let $D = \sum_{x\in X} n_x[x] \in \mathrm{Div}(X)$ and let $\mathscr{L} = \mathscr{O}_X(D)$. Consider an expression $D = (f_i, U_i)$ as Cartier divisor, we obtain the idelic point $\alpha_D = (f_{i,x}^{-1})_x \in \mathbf{A}_X^\times$, where for each $x \in X'$ we can choose any (f_i, U_i) such that $x \in U_i$. Then it is evident that

$$\mathbf{A}_X(\mathscr{O}_X(D)) = \alpha_D \mathbf{A}_X = \mathbf{A}_X\,.$$

Furthermore thanks to equation (2.7):

$$\mathbf{A}_X(D) := \mathbf{A}_X(\mathscr{O}_X(D))(0) = \{(t_x)_x \colon \|t_x\|_x \leq 1\} = \{(t_x)_x \colon v_x(t_x)_x \geq -n_x\}\,.$$

Proposition 2.19. *Fix a nonzero rational differential form $\omega \in \Omega^1_{K|k}$, and denote with $(\omega) \in \mathrm{Div}(X)$ the divisor associated to ω. Then for any divisor $D \in \mathrm{Div}(X)$ we have that*

$$\mathbf{A}_X(D)^\perp \cong \mathbf{A}_X((\omega) - D)$$

Proof. This is an easy generalization of lemma 2.15. □

In other words the adelic complex $\mathcal{A}_X(\mathscr{O}_X(D))$ is exactly the adelic complex

$$\mathcal{A}_X(D): \quad 0 \to K \oplus \mathbf{A}_X(D) \to \mathbf{A}_X \to 0 \tag{2.9}$$

defined in [19]. When $D = 0$ we put simply $\mathcal{A}_X := \mathcal{A}_X(0)$, and since $\mathbf{A}_X(0) = \prod_{x\in X} \mathcal{O}_x$, we obtain

$$\mathcal{A}_X : \quad 0 \to K \oplus \prod_{x\in X} \mathcal{O}_x \to \mathbf{A}_X \to 0 \tag{2.10}$$

The cohomology groups of the complex $\mathcal{A}_X(D)$ are the following ones:

$$H^0(\mathcal{A}_X(D)) = K \cap \mathbf{A}_X(D); \quad H^1(\mathcal{A}_X(D)) = \mathbf{A}_X/(K + \mathbf{A}_X(D))\,.$$

We want to prove that the cohomology of $\mathcal{A}_X(D)$ is equal to the cohomology associated to the sheaf $\mathscr{O}_X(D)$.

Lemma 2.20. *Assume that* $[K : k(t)] = m$, *then we have the following algebraic and topological isomorphism:*

$$\mathbf{A}_X \cong \mathbf{A}_{\mathbb{P}^1_k} \otimes_{k(t)} K \cong \bigoplus_{i=1}^{m} \mathbf{A}_{\mathbb{P}^1_k}$$

Proof. See [10, Ch II, 14]. □

The following theorem is fundamental and it is the generalization of the classical version proved in Tate's thesis in the case when K is a global field. Here we give the proof in the case of the function field K and the locally linearly compact space $\mathbf{A}_X$.

Theorem 2.21. *Consider* $K \subset \mathbf{A}_X$ *with the diagonal embedding, then the following statements are true:*

(1) *K is discrete in $\mathbf{A}_X$.*

(2) *The quotient $\mathbf{A}_X/K$ is a linearly compact k-vector space.*

(3) *$K = K^{\perp}$ with respect to the differential pairing of definition 2.17.*

Proof. Thanks to 2.20 we can assume that $X = \mathbb{P}^1_k$ and $K = k(t)$, then the general result will follow easily.
(1) Let ∞ be the point at infinity of $\mathbb{P}^1_k$ and consider the divisor $-D_\infty = -[\infty]$. Let's check that

$$K \cap \mathbf{A}_X(-D_\infty) = 0\,, \tag{2.11}$$

this would be enough to conclude that K is discrete in $\mathbf{A}_X$. Suppose that $f \in K \cap \mathbf{A}_X(-D_\infty)$ and $f \neq 0$, then:

- $v_x(f) \geq 0$ i.e. $|f|_x \leq 1$ when $x \neq \infty$

- $v_\infty(f) \geq 1$ i.e. $|f|_\infty \leq \frac{1}{c_\infty} < 1$ where the constant c_∞ was defined in equation (2.3).

The above items and the product formula imply the following contradiction:

$$1 = \prod_{x \in X} |f|_x < 1\,.$$

Thus we can conclude that equation (2.11) holds.
(2) To prove this point we need two substeps:

(2a) $\mathbf{A}_X(-D_\infty)$ is linearly compact.

(2b) $\mathbf{A}_X = K \oplus \mathbf{A}_X(-D_\infty)$.

(2a) Note that $\mathbf{A}_X(-D_\infty) = \alpha \mathbf{A}_X(0)$ where $\alpha = (\alpha_x)$ is any element such that $v_x(\alpha_x) = 0$ for $x \neq \infty$, and $v_\infty(\alpha_\infty) = 1$. Therefore we have a surjective continuous homomorphism:

$$\begin{array}{rcl} \lambda_\alpha : \mathbf{A}_X(0) & \to & \mathbf{A}_X(-D_\infty) \\ \beta & \mapsto & \alpha\beta \end{array}$$

which implies (thanks to proposition F.4(2)) that $\mathbf{A}_X(-D_\infty)$ is linearly compact.
(2b) The prime ideals of K are in bijective correspondence with a set of irreducible polynomials in $k(t)$, therefore just by simplicity we will write $p(t) \in \operatorname{Spec} K$ if $p(t)$ is a chosen generator for a prime ideal $\mathfrak{p} \in \operatorname{Spec} K$. Thus any adelic element $\alpha = (\alpha_x)_x \in \mathbf{A}_X$ can be written as:

$$\alpha = \big((\alpha_{p(t)})_{p(t) \in \operatorname{Spec} K},\, \alpha_\infty\big)\,;$$

moreover remember how the valuations on each $K_{p(t)}$ are defined:

$$\begin{cases} v_{p(t)}\left(\sum_{i \geq m} \lambda_{i,p(t)} p(t)^i\right) = m \\ v_\infty\left(\sum_{i \geq m} \lambda_{i,\infty} t^i\right) = -m \end{cases}$$

for $m \in \mathbb{Z}$. The strategy is the following: for any $\alpha \in \mathbf{A}_X$ we find an element $\gamma \in K$ such that $\alpha - \gamma \in \mathbf{A}_X(-D_\infty)$; and this γ will be the sum of two elements δ and δ' that we are going to construct. For any $p(t) \in \operatorname{Spec} K$ consider the following two objects

$$\alpha'_{p(t)} := \sum_{i \leq -1} \lambda_{i,p(t)} p(t)^i\,,$$

$$\delta := \sum_{p(t) \in \operatorname{Spec} K} \,. \tag{2.12}$$

Note that the sum of equation (2.12) is well defined thanks to the adelic condition. For any irreducible polynomial $q(t) \in \operatorname{Spec} K$ we have:

$$v_{q(t)}(\alpha_q(t) - \delta) = v_{q(t)} \left((\alpha_{q(t)} - \alpha'_{q(t)}) - \sum_{p(t) \neq q(t)} \alpha'_{p(t)} \right) \geq 0 \,,$$

hence for any point $x \in X' \setminus \infty$, $\alpha_x - \delta \in \mathcal{O}_x$. Now suppose that

$$\alpha_\infty - \delta = \sum_{i \geq m} \lambda_{i,\infty} t^i \,,$$

then consider $c = \sum_{i \geq 0} \lambda_{i,\infty} t^i \in k[[t]]$, then $v_\infty(\alpha_\infty - \delta - c) \geq 1$. Since $k[t]$ is dense in $k[[t]]$, there exists $\delta' \in k[t]$ such that:

$$v_\infty(\alpha_\infty - \delta - \delta') \geq v_\infty(\alpha_\infty - \delta - c) \,.$$

With such a choice of δ' we conclude that $\gamma = \delta + \delta' \in K$ and moreover:

- $v_x(\alpha_x - \delta - \delta') \geq 0$ when $x \neq \infty$.
- $v_\infty(\alpha_\infty - \delta - \delta') \geq 1$.

This means that $\alpha - \gamma \in \mathbf{A}_X(-D_\infty)$ as we wanted.
Finally we use use (2b) to ensure that $\mathbf{A}_X/K \cong \mathbf{A}_X(-D_\infty)$, furthermore (2$a$) implies that $\mathbf{A}_X/K$ is linearly compact.
(3) By the residue formula it follows immediately that $K^\perp \supseteq K$. Moreover $K^\perp \cong (\mathbf{A}_X/K)^\vee$ is discrete since by (2) $\mathbf{A}_X/K$ is linearly compact (see proposition F.9); therefore also the quotient $K^\perp/K$ turns out to be discrete. On the other hand:

$$(\mathbf{A}_X/K)/(K^\perp/K) \cong \mathbf{A}_X/K^\perp$$

is a T2 space because $K^\perp$ is closed[4], hence $K^\perp/K$ is closed in the linearly compact space $\mathbf{A}_X/K$ (we are using (2)). By proposition F.4(1) we conclude that $K^\perp/K$ is linearly compact and discrete, so it is a finite dimensional k-vector space by proposition F.4(3). Suppose that $\dim_k K^\perp/K = m$, then $\dim_K K^\perp/K = l \leq m$ and we have:

$$K^\perp/K \cong \prod_{i=1}^{l} K \cong \prod_{i=1}^{\infty} k$$

---

[4] Remember that the quotient G/H of two topological groups is $T2$ is and only if H is closed in G

thus the only possibility to obtain $\dim_k K^\perp/K < \infty$ is $l = 0$ which means $K^\perp = K$. $\square$

At this point we are ready to prove the main result regarding the adelic complex:

Theorem 2.22. *For any divisor* $D = \sum_{x\in X} n_x[x] \in \mathrm{Div}(X)$ *and any* $i \geq 0$ *we have the following isomorphism of* k*-vector spaces:*

$$H^i(D) \cong H^i(\mathcal{A}_X(D))$$

where $H^i(D) := H^i(X, \mathscr{O}_X(D))$ *and* $H^i(\mathcal{A}_X(D))$ *is the* i*-th cohomology group of the adelic complex (2.9).*

Proof. For $i \geq 2$ it is straightforward to notice that $H^i(D) = H^i(\mathcal{A}_X(D)) = 0$. The case $i = 0$ is described in the following equation:

$$H^0(\mathcal{A}_X(D)) = K \cap \mathcal{A}_X(D) = \{f \in K \colon v_x(f) \geq -n_x\} \cong H^0(D)\,.$$

So it remains to prove only the case $i = 1$. Remember that by Serre's duality we have $H^1(D) \cong H^0((\omega) - D)^*$, where $*$ is the operation of taking the algebraic dual of a k-vector space, then by using the already proven case of $i = 0$ we get:

$$H^1(D) \cong H^0((\omega) - D)^* \cong H^0(\mathcal{A}_X((\omega) - D)) = (K \cap \mathbf{A}_X((\omega) - D))^*\,.$$

It follows that $(K \cap \mathbf{A}_X((\omega) - D))^*$ is finite dimensional over k, so

$$(K \cap \mathbf{A}_X((\omega) - D))^* = (K \cap \mathbf{A}_X((\omega) - D))^\vee\,.$$

Now thanks to proposition 2.19, proposition F.11 and theorem 2.21 we obtain:

$$H^1(D) \cong \mathbf{A}_X((\omega) - D))^\vee = (K \cap \mathbf{A}_X(D)^\perp)^\vee \cong \mathbf{A}_X/(K \cap \mathbf{A}_X(D)^\perp)^\perp \cong$$

$$\cong \mathbf{A}_X\Big/\overline{(K^\perp + \mathbf{A}_X(D)^{\perp\perp})} \cong \mathbf{A}_X\Big/\overline{(K + \mathbf{A}_X(D))}\,.$$

Finally if we show that $K + \mathbf{A}_X(D)$ is closed in $\mathbf{A}_X$, we obtain the required isomorphism $H^1(D) \cong H^1(\mathcal{A}_X(D))$. Consider $K \cap \mathbf{A}_X(D)$, it is of finite dimension over k and discrete, so it is linearly compact by proposition F.4(4). This means that $K \cap \mathbf{A}_X(D)$ is closed in $\mathbf{A}_X(D)$ (because the latter is linearly compact too). The isomorphism:

$$(K + \mathbf{A}_X(D))/K \cong \mathbf{A}_X(D)/(K \cap \mathbf{A}_X(D))$$

implies that $(K+\mathbf{A}_X(D))/K$ is linearly compact, hence it is closed in $\mathbf{A}_X/K$ (which is linearly compact). Therefore

$$(\mathbf{A}_X/K)/((K+\mathbf{A}_X(D))/K) \cong \mathbf{A}_X/(K+\mathbf{A}_X(D))$$

is a T2 topological space and we can conclude that $K+\mathbf{A}_X(D)$ is closed in $\mathbf{A}_X$. □

Remark 2.23. A first immediate consequence of theorem 2.22 is the adelic interpretation of the genus of X:

$$g(X) = \dim_k(H^1(0)) = \dim_k(H^1(\mathbf{A}_X(0))) = \dim_k(\mathbf{A}_X/(K+\mathbf{A}_X(0)))$$

One can also define the *idelic complex* $\mathcal{A}_X^\times$ attached to X (in this case there is no dependence on the chosen divisor):

$$\begin{array}{rrcl} \mathcal{A}_X^\times : & 0 \to K^\times \oplus \mathbf{A}_X(0)^\times & \to & \mathbf{A}_X^\times \to 0 \\ & (f,(\alpha_x)_x) & \mapsto & (f\alpha_x^{-1})_x \end{array}$$

Remark 2.24. Note that the idelic complex is well defined because $\mathcal{O}_X$ is a multiplicative group for any $x \in X'$, therefore $\mathbf{A}_X(0)^\times$ is a subgroup of $\mathbf{A}_X^\times$.

The cohomology of $\mathcal{A}_X^\times$ is given by

$$H^0(\mathcal{A}_X^\times) = K^\times \cap \mathbf{A}_X(0)^\times; \quad H^1(\mathcal{A}_X^\times) = \mathbf{A}_X^\times/K^\times\mathbf{A}_X(0)^\times\,.$$

Theorem 2.25. *For any $i \geq 0$ we have the following isomorphism of groups:*

$$H^i(X,\mathscr{O}_X^\times) \cong H^i(\mathcal{A}_X^\times)\,.$$

In particular $\mathbf{A}_X^\times/\mathbf{A}_X(0)^\times \cong \mathrm{Div}(X)$ and $H^1(\mathcal{A}_X^\times) \cong \mathrm{Cl}(X)$.

Proof. Clearly we have to check just the cases $i=0$ and $i=1$. The former one is trivial:

$$H^0(\mathcal{A}_X^\times) = K^\times \cap \mathbf{A}_X(0)^\times = \bigcap_{x\in X} \mathscr{O}_{X,x}^\times = H^0(X,\mathscr{O}_X^\times) = k^\times\,.$$

Now construct the map:

$$\begin{aligned} p : \mathbf{A}_X^\times &\to \mathrm{Div}(X) \\ (\alpha_x)_x &\mapsto \sum_{x\in X} v_x(\alpha_x)[x] \end{aligned} \tag{2.13}$$

which is: well defined, evidently surjective and with kernel given by

$$\mathbf{A}_X(0)^\times = \prod_{x\in X'} \mathcal{O}_x^\times\,.$$

It follows that $\mathbf{A}_X^\times/\mathbf{A}_X(0)^\times \cong \mathrm{Div}(X)$ and $H^1(\mathcal{A}_X^\times) \cong \mathrm{Cl}(X) \cong H^1(X,\mathscr{O}_X^\times)$. □

2.2.3 Functoriality: adeles and morphisms of curves

Let's fix a surjective morphism $\varphi : X \to Y$ of algebraic curves over k, not necessarily regular, then we want to study how the adelic rings of X and Y are related through the map φ. The usual properties of morphisms of schemes will guide us, indeed we know that for any fixed closed point $\bar{y} = \varphi(\bar{x})$ there is a local injective homomorphism of rings $\varphi^{\#}_{\bar{x}} : \mathscr{O}_{Y,\bar{y}} \to \mathscr{O}_{X,\bar{x}}$. We now want to show how the local maps $\varphi^{\#}_{\bar{x}}$ "glue together" to a morphisms of ST-rings

$$\varphi^a : \mathbf{A}_Y \to \mathbf{A}_X \, .$$

The local homomorphism $\varphi^{\#}_{\bar{x}}$ induces a commutative diagram (with continuous maps with respect to the obvious topologies):

$$\begin{tikzcd}
\widehat{\mathscr{O}_{Y,\bar{y}}} \arrow[r, "\psi"] & \widehat{\mathscr{O}_{X,\bar{x}}} \\
\mathscr{O}_{Y,\bar{y}} \arrow[u, hook] \arrow[r, "\varphi^{\#}_{\bar{x}}"] & \mathscr{O}_{X,\bar{x}} \arrow[u, hook]
\end{tikzcd} .$$

Now let $\mathfrak{q}$ any minimal prime ideal of $\widehat{\mathscr{O}_{X,\bar{x}}}$ and let's put $\varphi(\mathfrak{q}) := \psi^{-1}(\mathfrak{q})$, then by localisation we obtain in a canonical way a continuous homomorphism:

$$(\widehat{\mathscr{O}_{Y,\bar{y}}})_{\varphi(\mathfrak{q})} \to (\widehat{\mathscr{O}_{X,\bar{x}}})_{\mathfrak{q}}$$

which, with the notation of subsection 2.1.1 is a homomorphism:

$$\varphi_{\bar{x},\mathfrak{q}} : K_{\bar{y},\varphi(\mathfrak{q})} \to K_{\bar{x},\mathfrak{q}} \, .$$

such that $\varphi_{\bar{x},\mathfrak{q}}(k(Y)) \subseteq k(X)$.

By using the notation of proposition 2.1 for the map $\phi_{\bar{y}} : \operatorname{Spec} \widehat{\mathscr{O}_{Y,\bar{y}}} \to \operatorname{Spec} \mathscr{O}_{Y,\bar{y}}$ let's put

$$\begin{array}{rcl} \varphi'_{\bar{x},\mathfrak{q}} : K_{\bar{y}} = \displaystyle\prod_{\mathfrak{p} \in \phi_{\bar{y}}^{-1}(0)} K_{\bar{y},\mathfrak{p}} & \to & K_{\bar{x},\mathfrak{q}} \\ (\beta_{\mathfrak{p}})_{\mathfrak{p}} & \mapsto & \varphi_{\bar{x},\mathfrak{q}}(\beta_{\varphi(\mathfrak{q})}) \end{array}$$

so we can define:

$$\begin{array}{rcl} \varphi_{\bar{x},\bar{y}} : K_{\bar{y}} & \to & K_{\bar{x}} \\ \beta & \mapsto & (\varphi'_{\bar{x},\mathfrak{q}}(\beta))_{\mathfrak{q} \in \phi_{\bar{x}}^{-1}(0)} \end{array}$$

Moreover we have

$$\begin{aligned} \varphi'_{\bar{x},\bar{y}} : \prod_{y\in Y} K_y &\to K_{\bar{x}} \\ (\alpha_y)_{y\in Y} &\mapsto \varphi_{\bar{x},\bar{y}}(\alpha_{\bar{y}}) \end{aligned}$$

and finally we obtain

$$\begin{aligned} \varphi^a : \prod_{y\in Y} K_y &\to \prod_{x\in X} K_x \\ \alpha &\mapsto (\varphi_{x,y}(\alpha))_{x\in X} \,. \end{aligned}$$

The homomorphism φ^a clearly induces by restriction a continuous homomorphism:

$$\varphi^a : \mathbf{A}_Y \to \mathbf{A}_X$$

as expected. When φ is also flat, the pullback of divisors $\varphi^* : \mathrm{Div}(Y) \to \mathrm{Div}(X)$ is well defined and we have the following important proposition:

Proposition 2.26. *Assume that φ is flat and fix a divisor $D \in \mathrm{Div}(Y)$. Let $\varphi^* D \in \mathrm{Div}(X)$ be the pullback of D, then the map $\varphi^a : \mathbf{A}_Y \to \mathbf{A}_X$ induces by restriction a morphism $\varphi^a(D) : \mathbf{A}_Y(D) \to \mathbf{A}_X(\varphi^* D)$.*

Proof. For any closed point $x \in X$ let's put $y := \varphi(x)$ and let $D = \sum_{y\in Y} n_y[y]$. By definition of the pullback of a divisor we have:

$$\varphi^* D = \sum_{x\in X} e_{x|y} n_y [x] \,,$$

where if t_y is a local parameter of $\mathscr{O}_{Y,y}$, then $e_{x|y} := v_x(\varphi^\#(t_y)) \in \mathscr{O}_{X,x}$ is the ramification index at x. Now let $\alpha_y \in K_y$ such that $m := v_y(\alpha_y) \geq -n_y$, then we can write $\alpha_y = u t_y^m$ where $u \in \mathcal{O}_y^\times$ and we have:

$$\varphi^\#(\alpha_y) = \varphi^\#(u t_y^m) = \varphi^\#(u)\varphi^\#(t_y)^m \,.$$

It follows that $v_x(\varphi^\#(\alpha_y)) = e_{x|y} m \geq -e_{x|y} n_y$. The above argument implies that if $(\alpha_y)_y \in \mathbf{A}_Y(D)$, then $\varphi^a((\alpha_y)_y) \in \mathbf{A}_X(\varphi^* D)$. ☐

Under the hypotheses of proposition 2.26 we can easily conclude that there is the following homomorphism of complexes:

$$\begin{tikzcd}
\mathcal{A}_Y(D)\colon & 0 \arrow[r] & k(Y) \oplus \mathbf{A}_Y(D) \arrow[r] \arrow[d, "\varphi^a \oplus \varphi^a"] & \mathbf{A}_Y \arrow[r] \arrow[d, "\varphi^a"] & 0 \\
\mathcal{A}_X(\varphi^* D)\colon & 0 \arrow[r] & k(X) \oplus \mathbf{A}_X(\varphi^* D) \arrow[r] & \mathbf{A}_X \arrow[r] & 0 \,.
\end{tikzcd}$$

2.2.4 Idelic and adelic interpretation of the degree of a divisor

For algebraic curves the “intersection theory” is related simply to the concept of degree of a divisor; in this subsection we will see how to re-interpret it by means of ideles and adeles. As an application of the adelic point of view we will derive the classical Riemann-Roch theorem.

In equation 2.13 we defined the projection map $p : \mathbf{A}_X^\times \to \mathrm{Div}(X)$, so basically we can see ideles as a generalization of divisors on a curve.

Definition 2.27. The *idelic degree* is the map:

$$\begin{aligned} \mathrm{ideg} : \mathbf{A}_X^\times &\to \mathbb{Z} \\ (\alpha_x)_x &\mapsto \sum_{x \in X'} v_x(\alpha_x)[k(x) : k] \end{aligned}$$

which descends to a map $\mathrm{ideg} : H^1(\mathcal{A}_X^\times) \to \mathbb{Z}$.

It is easy to check that it the correct definition since we have the following commutative diagram:

$$\begin{tikzcd} \mathbf{A}_X^\times \arrow[d, "p"] \arrow[dr, "\mathrm{ideg}"] & \\ \mathrm{Div}(X) \arrow[r, "\deg"] & \mathbb{Z} \end{tikzcd}$$

which in turns induces the following commutative diagram:

$$\begin{tikzcd} H^1(\mathcal{A}_X^\times) \arrow[d, "\cong"] \arrow[dr, "\mathrm{ideg}"] & \\ \mathrm{Pic}(X) \arrow[r, "\deg"] & \mathbb{Z} \end{tikzcd} .$$

This completes the idelic interpretation of the degree of a divisor; on the other hand the adelic interpretation is a bit more involved. Let’s start with a very general proposition involving locally linearly compact vector spaces:

Proposition 2.28. *Let V be a locally linearly compact k-vector space and let $\mathrm{Lo}(V)$ be the set of open locally compact subspaces. Then there exists a unique map:*

$$\begin{aligned} w_V : \mathrm{Lo}(V) \times \mathrm{Lo}(V) &\to \mathbb{Z} \\ (A, B) &\mapsto w_V(A, B) \end{aligned}$$

satisfying the following conditions:

(1) *$w_V(A,B) + w_V(B,C) = w_V(A,C)$ for any $A, B, C \in \mathrm{Lo}(V)$.*

(2) *If $A \supset B$, then $w_V(A,B) = [A : B]_k$ where we use the following notation $[A : B]_k := \dim_k A/B$.*

Proof. Let's define:

$$w_V(A,B) := [A : A \cap B]_k - [B : A \cap B]_k\,;$$

it is easy to check that this map verifies conditions (1) and (2), so let's check the unicity. Let w' be another map satisfying (1) and (2), then for any $C \in A \cap B$ and $C \in \mathrm{Lo}(V)$ we have:

$$w'(A,B) = w'(A,C) - w'(B,C) = [A : C]_k - [B : C]_k =$$

$$= [A : A \cap B]_k + [A \cap B : C]_k - [B : A \cap B]_k - [A \cap B : C]_k = w_V(A,B)\,.$$

□

We can give an alternative expression of the map w_V if we fix a closed subspace $T \subset V$. For any $A, B \in \mathrm{Lo}(V)$ put:

$$w'_T(A,B) := w_{V/T}((A+T)/T, (B+T)/T)$$

$$w''_T(A,B) := w_T(T \cap A, T \cap B)$$

(remember that V/T and T are again locally linearly compact) then the map:

$$\widetilde{w}(A,B) := w'_T(A,B) + w''_T(A,B)$$

satisfies the two conditions of theorem 2.28 on V and we can conclude that $\widetilde{w} = w_V$.

Definition 2.29. Let $D \in \mathrm{Div}(X)$, then the *adelic degree* of D is:

$$\mathrm{adeg}(D) := w_V(\mathcal{A}_X(D), \mathcal{A}_X(0))$$

The characteristic of the adelic complex $\mathcal{A}_X(D)$ is defined in the following way (it is the usual definition of characteristic of a complex in commutative algebra):

$$\chi(\mathcal{A}_X(D)) := \sum_{i \geq 0} (-1)^i h^i(\mathcal{A}_X(D)) = h^0(\mathcal{A}_X(D)) - h^1(\mathcal{A}_X(D))\,.$$

where $h^i(\mathcal{A}_X(D)) = \dim_k(H^i(\mathcal{A}_X(D))$. The adelic Riemann-Roch theorem is a formula connecting the values of $\mathrm{adeg}(D)$ and $\chi(\mathcal{A}_X(D))$:

Theorem 2.30 (Adelic Riemann-Roch)**.** *For any divisor D on X we have:*

$$\mathrm{adeg}(D) = \chi(\mathcal{A}_X(D)) - \chi(\mathcal{A}_X(0))$$

Proof. Consider the closed k-subspace $K \subset \mathbf{A}_X$, then:

$$\mathrm{adeg}(D) = w'_K(\mathbf{A}_X(D), \mathbf{A}_X(0)) + w''_K(\mathbf{A}_X(D), \mathbf{A}_X(0)) =$$

$$= -w_{\mathbf{A}_X/K}\big(\mathbf{A}_X/K, (\mathbf{A}_X(D) + K)/K\big) + w_{\mathbf{A}_X/K}\big(\mathbf{A}_X/K, (\mathbf{A}_X(0) + K)/K\big) +$$
$$+ w_K(\mathbf{A}_X(D) \cap K, 0) - w_K(\mathbf{A}_X(0) \cap K, 0)\,.$$

But:

$$(\mathbf{A}_K/K)\big/((\mathbf{A}_X(D) + K)/K) \cong \mathbf{A}_K/(\mathbf{A}_X(D) + K) = H^1(\mathcal{A}_X(D))\,,$$

$$\mathbf{A}_X(D) \cap K = H^1(\mathcal{A}_X(D))\,.$$

In other words the above expression of $\mathrm{adeg}(D)$ becomes:

$$\mathrm{adeg}(D) = -h^1(\mathbf{A}_X(D)) + h^1(\mathbf{A}_X(0)) + h^0(\mathbf{A}_X(D)) - h^0(\mathbf{A}_X(0)) =$$

$$= \chi(\mathcal{A}_X(D)) - \chi(\mathcal{A}_X(0))\,.$$

□

Finally we want to show that the adelic degree coincides with the usual degree of a divisor, and this will give a proof of the Riemann-Roch theorem based just on the adelic theory .

Lemma 2.31. *If $D, D' \in Div(X)$ and $D \geq D'$, then:*

$$[\mathbf{A}_X(D) : \mathbf{A}_X(D')]_k = \deg(D) - \deg(D')\,.$$

Proof. Suppose that $D = \sum_{x \in X} n_x[x]$, we work by induction on $\deg(D) - \deg(D')$. When it is 0, then everything is trivial, so let's prove the basis of induction in the case $\deg(D) - \deg(D') = 1$. It means that $D = D' + [z]$ for a point $z \in X$; then let's define a map:

$$\begin{array}{rcl} \psi\colon \mathbf{A}_X(D' + [z]) & \to & k(z) \\ (\alpha_x)_x & \mapsto & \alpha_{z, -n_z - 1} \end{array}$$

where

$$\alpha_z = \sum_{i \geq m} \alpha_{z,i} \varpi^i \in k(z)((\varpi))\,.$$

Note that we are fixing a uniformizer parameter ϖ on K_z. It is easy to verify that ψ is surjective and moreover its kernel is $\mathbf{A}_X(D')$, thus we get an isomorphism $\mathbf{A}_X(D)/\mathbf{A}_X(D') \cong k(z)$ which implies the following equalities:

$$[\mathbf{A}_X(D) : \mathbf{A}_X(D')]_k = [k(z) : k] = \deg([z]) = \deg(D) - \deg(D')\,.$$

Suppose now that the lemma is true for $\deg(D) - \deg(D') = n$, then if $\deg(D) - \deg(D') = n + 1$ we can always find a divisor D'' such that:

- $D' \leq D'' \leq D$.
- $\deg(D'') - \deg(D') = n$.
- $\deg(D) - \deg(D'') = 1$.

Then we obtain:

$$\deg(D) - \deg(D') = \deg(D) = \deg(D'') + \deg(D'') - \deg(D') =$$

$$[\mathbf{A}_X(D) : \mathbf{A}_X(D'')]_k + [\mathbf{A}_X(D'') : \mathbf{A}_X(D')]_k = [\mathbf{A}_X(D) : \mathbf{A}_X(D')]_k\,.$$

□

Proposition 2.32. *Let* $D \in \mathrm{Div}(X)$, *then* $\mathrm{adeg}(D) = \deg(D)$.

Proof. First of let's write denote with D^- the negative part of the divisor D, and note that $D \geq 0 \geq D^-$. Then we have

$$\mathrm{adeg}(D) = w_V(\mathbf{A}_X(D), \mathbf{A}_X(0)) =$$

$$= w_V(\mathbf{A}_X(D), \mathbf{A}_X(D^-)) - w_V(\mathbf{A}_X(0), \mathbf{A}_X(D^-)) =$$

$$= [\mathbf{A}_X(D) : \mathbf{A}_X(D^-)]_k - [\mathbf{A}_X(0) : \mathbf{A}_X(D^-)]_k =$$

$$=^{(\text{lem. } 2.31)} \deg(D) - \deg(D^-) - \deg(0) + \deg(D^-) = \deg(D)\,.$$

□

By the adelic Riemann-Roch theorem and proposition 2.32, for any divisor D on X we obtain the formula:

$$\deg(D) = \chi(\mathcal{A}_X(D)) - \chi(\mathcal{A}_X(0))\,.$$

2.3 Arithmetic curves

For a number field K, the classical adelic ring $\mathbf{A}_K$ can be interpreted as an object attached to the arithmetic scheme $X = \operatorname{Spec} O_K$. But there is substantial difference with respect to the geometric case, in fact the result of theorem 2.13, is no longer true in this setting. The archimedean places on K don't correspond to any geometric data on the scheme X, so in order to restore the correspondence between the arithmetic of K and the geometry of X, we have to work in the setting of Arakelov geometry, where we complete X to $\widehat{X}$ by including the so called "archimedean points".

Notations. In this section we fix a number field K whose ring of integers is denoted by O_K. Our prototype of (possibly singular) "arithmetic curve" will be $B = \operatorname{Spec} A$, where $A \subseteq O_K$ is an order of K. The normalization of X is clearly $\operatorname{Spec} O_K$ and as we did in the previous section, we will see that by the purpose of adelic theory it is always enough to work with $\operatorname{Spec} O_K$. Appendix C is a prerequisite for this section.

2.3.1 Adeles

For each closed point b of $B = \operatorname{Spec} A$ we consider the ring K_b defined in section 1 (a sum of complete valuation fields). Moreover for any embedding $\sigma : A \hookrightarrow \mathbb{C}$ up to conjugation, we define K_σ to be $\mathbb{R}$ or $\mathbb{C}$ (depending on whether σ is real or complex) and the choice of the absolute value $|\cdot|_\sigma$ on K_σ is made like in appendix C. As usual we denote with B_∞ the set of archimedean points of B i.e. the set of complex embeddings of A up to the conjugation relation and moreover we set $\widehat{B} = B \cup B_\infty$. Clearly (0) is the unique non-closed point of B and with the notation $b \in \widehat{B}$ we allow x to be either a point of the scheme B or an archimedean point. Note that for any $b \in \widehat{B} \setminus (0)$ the additive group K_b is linearly compact.

Definition 2.33. The adelic ring of $\widehat{B}$ is defined as:

$$\mathbf{A}_{\widehat{B}} := \prod_{x \in \widehat{B}}{}' K_b$$

where $\prod'$ is the restricted product, in the category of locally compact groups, with respect to the subgroups $\mathcal{O}_b$.

Remark 2.34. By simplicity we will often denote $\mathbf{A}_{\widehat{B}}$ as $\mathbf{A}_K$. Moreover for obvious reasons we identify the set of places $\mathcal{P}(K)$ with $\widehat{B} \setminus (0)$.

By definition, A contains an integral basis of $K|\mathbb{Q}$, then all the complex embeddings of O_K arise from the complex embedding of A, so by using remark 2.3, it is evident that $\mathbf{A}_{\widehat{B}} = \mathbf{A}_{\widehat{\operatorname{Spec} O_K}}$. For this reason from now on, if not otherwise specified we will consider $B = \operatorname{Spec} O_K$. Under this assumption K_b is a complete discrete valuation field and for b non-archimedean $\mathcal{O}_b$ is its valuation ring.

The theory of duality and pairings on $\mathbf{A}_{\widehat{B}}$ is simpler than the geometric theory, since we don't need to deal with differential forms and local linear compactness. It is enough to use the classical Pontryagin duality theory for locally compact groups; each field K_b is self dual with respect to a standard character $\psi_b : K_b \to \mathbb{C}^\times$ which can be defined in the following way (see [52, Exercise 7.1]):

- If b is archimedean, then $\psi_b(a) = e^{-2\pi i \operatorname{Tr}_{K_b|\mathbb{R}}(a)}$ for any $a \in K_b$.
- If b is non-archimedean then K_b is a finite extension of $\mathbb{Q}_p$ for a prime p. If for any $a \in K_b$ can write $\operatorname{Tr}_{K_b|\mathbb{Q}_p}(a) = \sum_{j\geq m} a_j p^j$, then let's put:
$$\psi_b(a) = e^{2\pi i \sum_{j=m}^{-1} a_j p^j}$$

Theorem 2.35. *The map:*

$$\begin{array}{rcl} \psi^0 : \mathbf{A}_{\widehat{B}} & \to & \mathbb{C}^\times \\ (\alpha_b)_b & \mapsto & \displaystyle\sum_{b\in\mathcal{P}(K)} \psi_b(\alpha_b)\,. \end{array}$$

is a standard character for the locally compact group $\mathbf{A}_{\widehat{B}}$. In particular $\mathbf{A}_{\widehat{B}}$ is self dual.

Proof. See [52, Lemma 5.3 and Theorem 5.4]. ☐

Definition 2.36. The standard character ψ^0 of theorem 2.35 gives a so called *global pairing* on $\mathbf{A}_{\widehat{B}}$:

$$\begin{array}{rcl} d : \mathbf{A}_{\widehat{B}} \times \mathbf{A}_{\widehat{B}} & \to & \mathbb{C}^\times \\ (\alpha, \beta) & \mapsto & \psi^0(\alpha\beta) = \displaystyle\sum_{b\in\mathcal{P}(K)} \psi_b(\alpha_b\beta_b)\,. \end{array}$$

It is straightforward to notice that d is continuous and symmetric.

The topological interplay between K (with its diagonal embedding) and $\mathbf{A}_{\widehat{B}}$ is similar to geometric case (see theorem 2.21). Linear compactness is just replaced by usual compactness of groups:

Theorem 2.37. *Consider $K \subset \mathbf{A}_{\widehat{B}}$ with the diagonal embedding, then the following statements are true:*

(1) *K is discrete in $\mathbf{A}_{\widehat{B}}$.*

(2) *The quotient $\mathbf{A}_{\widehat{B}}/K$ is compact k-vector space.*

(3) *$K = K^{\perp}$ with respect to the pairing of definition 2.36.*

Proof. See [52, Theorem 5.11] for (1) and (2). The proof of (3) is similar to 2.21(3). □

Note that unit discs in $\mathbb{R}$ and $\mathbb{C}$ are not groups, so we don't have any adelic complex. By the way unit circles are multiplicative groups, so:

$$\mathbf{A}_{\widehat{B}}(0)^{\times} := \prod_{b \in \widehat{B}} \mathcal{O}_b^{\times}$$

is a subgroup of $\mathbf{A}_{\widehat{B}}^{\times}$ and we have a well defined arithmetic idelic complex

$$\begin{array}{rrcl} \mathcal{A}_{\widehat{B}}^{\times}: & 0 \to K^{\times} \oplus \mathbf{A}_{\widehat{B}}(0)^{\times} & \to & \mathbf{A}_{\widehat{B}}^{\times} \to 0 \\ & (f, (\alpha_b)_b) & \mapsto & (f\alpha_b^{-1})_b \,. \end{array}$$

2.3.2 Idelic and adelic interpretation of the Arakelov degree

As we did for algebraic curves, we can give an interpretation of the Arakelov degree of a divisors in terms of ideles and adeles. The idelic version will be very simple and natural, whereas for the adelic version there is a bit of work to do, and in this case measure theory is involved.

First of all we define surjective morphism:

$$\begin{array}{rcl} \widehat{p} : \mathbf{A}_{\widehat{B}}^{\times} & \to & \mathrm{Div}_{\mathrm{Ar}}(B) \\ (\alpha_{\mathfrak{p}})_{\mathfrak{p}} \times (\alpha_{\sigma})_{\sigma} & \mapsto & \displaystyle\sum_{\mathfrak{p} \in B} v_{\mathfrak{p}}(\alpha_{\mathfrak{p}})[\mathfrak{p}] + \sum_{\sigma \in B_{\infty}} 2 v_{\sigma}(\alpha_{\sigma})[\sigma] \end{array}$$

then we have the idelic version of the degree:

Definition 2.38. The *Arakelov idelic degree* is the map:

$$\begin{array}{rcl} \mathrm{ideg}_{\mathrm{Ar}} : \quad \mathbf{A}_{\widehat{B}}^{\times} & \to & \mathbb{R} \\ (\alpha_{\mathfrak{p}})_{\mathfrak{p}} \times (\alpha_{\sigma})_{\sigma} & \mapsto & \displaystyle\sum_{\mathfrak{p} \in B'} v_{\mathfrak{p}}(\alpha_{\mathfrak{p}}) \log \mathfrak{N}(\mathfrak{p}) + \frac{1}{2} \sum_{\sigma \in B_{\infty}} \varepsilon_{\sigma} v_{\sigma}(\alpha_{\sigma}) \end{array}$$

which descends to a map $\mathrm{ideg}_{\mathrm{Ar}} : H^1(\mathcal{A}_{\widehat{B}}^{\times}) \to \mathbb{R}$.

The idelic description of the Arakelov degree is now complete, since we have the following commutative diagram:

$$\begin{tikzcd}
\mathbf{A}_{\widehat{B}}^{\times} \arrow[d, "\widehat{p}"] \arrow[dr, "\mathrm{ideg}_{\mathrm{Ar}}"] & \\
\mathrm{Div}_{\mathrm{Ar}}(B) \arrow[r, "\deg_{\mathrm{Ar}}"] & \mathbb{R}
\end{tikzcd}$$

which in turns induces:

$$\begin{tikzcd}
H^1(\mathcal{A}_{\widehat{B}}^{\times}) \arrow[d, "\cong"] \arrow[dr, "\mathrm{ideg}"] & \\
\mathrm{Pic}_{\mathrm{Ar}}(B) \arrow[r, "\deg_{\mathrm{Ar}}"] & \mathbb{R}
\end{tikzcd}$$

In order to describe the adelic theory, we have to study adelic line bundles on the number field K. On algebraic curves, invertible sheaves correspond to adelic line bundles, now we expect the arithmetic version of this correspondence, that should involve hermitian invertible O_K-modules (see C.2) and adelic line bundles. First of all we show how to define an adelic line bundle on K by starting from an hermitian invertible O_K-module $(L, \{h_\sigma\}_\sigma)$. The underline K-vector space is $L \otimes_{O_K} K$, moreover for any $b \in \mathcal{P}(K)$ the norms on $K_b(L) := L \otimes_{O_K} K_b$ are defined in the following way:

- If b is non-archimedean, consider a O_K-basis ω of L, then any element of $t \in K_b(L)$ can be written uniquely as $t = \omega a$ with $a \in K_b$. Then we put
$$||t||_b := |a|_b \,,$$
and of course this definition is independent from the choice of ω.
- If $b = \sigma$ is archimedean, then on $K_\sigma(L) = L \otimes_{O_K}^{\sigma} \mathbb{C}$ we consider the norm induced by the hermitian product h_σ.

By the classical product formula it is easy to see that for any nonzero $v \in L \otimes_{O_K} K$, $||v_b|| = 1$ for all but finitely many $b \in \mathcal{P}(K)$; moreover the purity condition obvious. By commodity we give a name to the unit ball in $K_b(L)$:

$$\mathcal{O}_b(L) := \{t \in K_b(L) \colon ||t||_b \leq 1\} \,.$$

As we did in the case of geometric curves, we want to find explicitly the metrics on the hermitian invertible O_K-module $(\mathscr{O}_B(D), \{e^{\alpha_\sigma} h_{0_\sigma}\}_\sigma)$ arising from the Arakelov divisor $\widehat{D} = \sum_{b \in \mathcal{P}(K)} n_b[b]$. The very easy calculation follows directly from the definition of $\mathscr{O}_B(D)$ (like in the geometric case), indeed $K_b(\mathscr{O}_B(D)) = K_b$ and we get:

$$||\cdot||_b = c_b^{-n_b}|\cdot|_b\,.$$

Theorem 2.39. $\mathrm{Pic}_{\mathrm{Ar}}(B) \cong \mathrm{APic}(B)$.

Proof. It is an easy modification of the proof of theorem 2.18. $\square$

So far the theory seems completely parallel to the adelic theory on algebraic curves, but this is exactly the point where we can find a big conceptual difference. There is no straightforward way to define an adelic subspace $\mathbf{A}_{\widehat{B}}(\widehat{D})$ for a fixed Arakelov divisor $\widehat{D}$ and this happens because of the presence of archimedean points $\sigma \in B_\infty$. In fact the balls $\mathcal{O}_\sigma(L) \subset K_\sigma(L)$ are not additive subgroups as we mentioned earlier. So, if we decided to proceed like in the geometric case and we tried to define a "naive version" of $\mathbf{A}_{\widehat{B}}(\widehat{D})$, we would simply get a subset of $\mathbf{A}_{\widehat{B}}$; but this is clearly not enough to get an adelic complex and an adelic cohomology.
Even without the adelic complex, we can obtain and adelic interpretation of the degree of an Arakelov divisor by using the Haar measures on K_b. First of all, for any place b let's fix a Haar measure μ_b on K_b:

- If $b = \mathfrak{p} \in B$, $\mu_\mathfrak{p}$ is the Haar measure such that $\mu_\mathfrak{p}(\mathcal{O}_\mathfrak{p}) = 1$.
- If $b = \sigma \in B_\infty$ we put $\mu_\sigma = \frac{\lambda}{\pi}$ where λ is the complex Lebesgue measure.

The product $\mu := \prod_b \mu_b$ is a Haar measure on the adelic ring $\mathbf{A}_{\widehat{B}}$, but here we consider the function

$$\widehat{\mu} := \prod_{\mathfrak{p}} \mu_\mathfrak{p} \cdot \prod_\sigma \mu_\sigma^{\frac{\epsilon_\sigma}{2}}$$

acting on each measurable set $\prod_b U_b \subseteq \mathbf{A}_{\widehat{B}}$ in the following way:

$$\widehat{\mu}\left(\prod_b U_b\right) = \prod_{\mathfrak{p}} \mu_\mathfrak{p}(U_\mathfrak{p}) \cdot \prod_\sigma \mu_\sigma(U_\sigma)^{\frac{\epsilon_\sigma}{2}}$$

Let L_A be an adelic line bundle and fix a local basis ω_b such that $||\omega_b||_b = 1$ for any vector space L_b; this choice gives an isomorphism $L_b \cong K_b$ which induces a function $\tilde{\mu}$ on L_A. Then we define the degree of L_A to be:

$$\deg(L_A) := \log \widehat{\mu}\left(\prod_{b\in\widehat{B}} \mathcal{O}_b(L)\right)\,.$$

It is straightforward to verify that $\deg(L_A)$ doesn't depend from the choice of $(\omega_b)_b$ and moreover that it is constant on isometric classes of adelic line bundles. In other words we have a well defined map $\deg : \mathrm{APic} \to \mathbb{R}$. Finally let's check that if $\widehat{D} = \sum_{\mathfrak{p}} n_{\mathfrak{p}}[\mathfrak{p}] + \sum_{\sigma} \alpha_{\sigma}[\sigma]$ is an Arakelov divisor, the Arakelov degree of $\widehat{D}$ coincides with the degree of the adelic line bundle $L_A(\widehat{D})$ that we define as the adelic line bundle associated to the hermitian invertible O_K-module $(\mathscr{O}_B(D), \{e^{\alpha_\sigma} h_{0_\sigma}\}_\sigma)$:

$$\deg L_A(\widehat{D}) = \log \widehat{\mu}\left(\prod_{b\in\widehat{B}} \mathcal{O}_b(L)\right) =$$

$$= \log \widehat{\mu}\left(\prod_{\mathfrak{p}} \{x \in K_{\mathfrak{p}} \colon \mathfrak{N}(\mathfrak{p})^{-n_{\mathfrak{p}}}|x|_{\mathfrak{p}} \leq 1\} \cdot \prod_{\sigma} \{z \in \mathbb{C} \colon e^{-\alpha_\sigma}|z| \leq 1\}\right) =$$

$$= \log\left(\prod_{\mathfrak{p}} \mu_{\mathfrak{p}}(\mathfrak{p}^{-n_{\mathfrak{p}}}\mathcal{O}_{\mathfrak{p}}) \cdot \prod_{\sigma} \mu_{\sigma}(\{z \in \mathbb{C} \colon |z| \leq e^{\alpha_\sigma}\})^{\frac{\epsilon_\sigma}{2}}\right) =$$

$$\log\left(\prod_{\mathfrak{p}} \mathfrak{N}(\mathfrak{p})^{n_{\mathfrak{p}}} \cdot \prod_{\sigma} e^{\epsilon_\sigma \alpha_\sigma}\right) = \sum_{\mathfrak{p}} n_{\mathfrak{p}} \log \mathfrak{N}(\mathfrak{p}) + \sum_{\sigma} \epsilon_\sigma \alpha_\sigma = \deg(\widehat{D})\,.$$

Chapter 3

Two-dimensional adelic geometry

In this chapter adelic theory is generalized to the 2-dimensional case. Section 3.1 relates 2-dimensional local theory to schemes of dimension 2. Then similarly to chapter 2 we consider two distinct cases:

- *Adeles on algebraic surfaces.* We describe in detail the structure of the adelic ring and of its subspaces in subsections 3.2.1 and 3.2.2. Duality results of subsection 3.2.3 follow basically from the explicit relationship between the 2-dimensional adelic ring and the adelic rings associated to curves on a surface (see proposition 3.17). After introducing adelic complexes (subsection 3.2.4) and explaining the link between 0-cicles and adeles (subsection 3.2.5), we deal with the property of adeles with respect to morphisms (subsection 3.2.6). In particular we will see that adeles "pullback". The last two subsections extend intersection theory respectively in a idelic and adelic way.

- *Adeles on arithmetic surfaces.* A detailed description of the local data is given in subsection 3.3.1. Here, the important point is that the nature of the local data changes dramatically between vertical curves and horizontal curves. We will see that for vertical curves we will have less explicit constructions because 2-dimensional local fields of mixed characteristic are involved. The definition of the adelic ring and residues will involve archimedean local and global data, in particular we will appeal to the 1-dimensional adelic ring of Riemann surfaces (the fibres at infinity). Idelic extensions of Deligne pairing and $*$-product between Green functions are developed and merged together to obtain the idelic Arakelov intersection pairing (subsections 3.3.4 and 3.3.5).

Main references. The theory of 2-dimensional adeles on algebraic surfaces has been partially introduced by Parshin in [49], but his global theory deals with non-completed adeles. The link between ideles and classical intersection theory is partially studied in [50]. A complete text for adeles on algebraic surfaces is [19]. The only text containing the correct concept of (full) adeles on arithmetic surfaces is [18].

3.1 Local data on schemes of dimension two

Let $(X, \mathscr{O}_X)$ be a Noetherian, integral and regular scheme of dimension 2 and let K be the function field of X. As we did in section 2.1.1, we construct the local data necessary for the definition of the adelic ring of X. In the 1-dimensional case we attached a complete valuation field to each nonsingular point, here on the other hand we work with 2-dimensional valuation fields. However, the situation on X is more complicated (even if we are considering just the case of a regular scheme), since for each closed point x the inclusion $x \in Y$ in any integral curve $Y \subset X$ "carries" a one dimensional valuation field that should be taken in account. So, instead of taking the closed points of X as "set of parameters", we consider the set of all possible *flags* $x \in Y \subset X$ where x is a closed point of X contained in an integral curve Y.

From now on a curve Y on X will always be an integral curve and its unique generic point will be denoted with the letter y. By simplicity we will often identify Y with its generic point y, which means that by an abuse of language and notation we will use sentences like "let $y \subset X$ be a curve on X..." or "let $x \in y \subset X$ be a flag on X...". In other words y is considered as a scheme or as a point depending on the context.

Definition 3.1. Fix a closed point $x \in X$, then:

- $\mathcal{O}_x := \widehat{\mathscr{O}_{X,x}}$. It is a Noetherian, complete, regular, local, domain of dimension 2 with maximal ideal $\widehat{\mathfrak{m}_x}$.
- $K'_x := \operatorname{Frac} \mathcal{O}_x$.
- $K_x := K\mathcal{O}_x \subseteq K'_x$. Notice that this is not a field.

For a curve $y \subset X$ we put:

- $\mathcal{O}_y := \widehat{\mathscr{O}_{X,y}}$. It is a complete DVR with maximal ideal $\widehat{\mathfrak{m}_y}$.
- $K_y := \operatorname{Frac} \mathcal{O}_y$. It is a complete valuation field with valuation ring $\mathcal{O}_y$.

Fix a flag $x \in y \subset X$, then we have a surjective local homomorphism $\mathscr{O}_{X,x} \to \mathscr{O}_{y,x}$ with kernel $\mathfrak{p}_{y,x}$ induced by the closed embedding $y \subset X$ (note that $\mathfrak{p}_{y,x}$ is a prime ideal of height 1).

Remark 3.2. Obviously $\mathscr{O}_{X,x}/\mathfrak{p}_{y,x} \cong \mathscr{O}_{y,x}$, and by working with open affine neighbourhoods of x one can show that $\left(\mathscr{O}_{X,x}\right)_{\mathfrak{p}_{y,x}} \cong \mathscr{O}_{X,y}$. Moreover if K_Y is the function field of Y, we have the following isomorphisms:

$$K_Y = \operatorname{Frac} \mathscr{O}_{y,x} \cong \operatorname{Frac}(\mathscr{O}_{X,x}/\mathfrak{p}_{y,x}) \cong$$

$$\cong \left(\mathscr{O}_{X,x}\right)_{\mathfrak{p}_{y,x}} / \mathfrak{p}_{y,x} \left(\mathscr{O}_{X,x}\right)_{\mathfrak{p}_{y,x}} \cong \mathscr{O}_{X,y}/\mathfrak{m}_y = k(y)\,.$$

So the residue field at the point y is identified with the function field of the curve Y.

The inclusion $\mathscr{O}_{X,x} \subset \mathcal{O}_x$ induces a morphism of schemes $\varphi : \operatorname{Spec} \mathcal{O}_x \to \operatorname{Spec} \mathscr{O}_{X,x}$ and we define the *local branches of* y *at* x as the elements of the set

$$y(x) := \varphi^{-1}(\mathfrak{p}_{y,x}) = \{\, \mathfrak{z} \in \operatorname{Spec} \mathcal{O}_x \colon \mathfrak{z} \cap \mathscr{O}_{X,x} = \mathfrak{p}_{y,x} \,\}\ .$$

If $y(x)$ contains only an element, we say that y is unbranched at x.

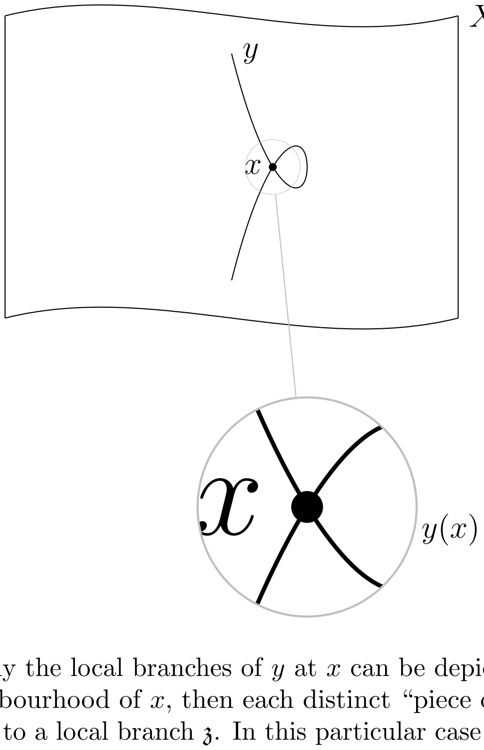

Figure 3.1: Informally the local branches of y at x can be depicted in the following way: consider a small neighbourhood of x, then each distinct “piece of y” that we see passing through x corresponds to a local branch $\mathfrak{z}$. In this particular case y has a simple node at x, so 2 local branches at x.

Remark 3.3. If x is a cusp point on y, one can show that y unbranched at x.

Definition 3.4. Let $\mathfrak{z} \in y(x)$ be a local branch of a curve y at point x, then let's define the field

$$K_{x,\mathfrak{z}} := \mathrm{Frac}\left(\widehat{(\mathcal{O}_x)_{\mathfrak{z}}}\right) .$$

in other words: we localise $\mathcal{O}_x$ at the prime ideal $\mathfrak{z}$, then we complete it at its maximal ideal and finally we take the fraction field. By convenience we put $\mathcal{O}_{x,\mathfrak{z}} := \widehat{(\mathcal{O}_x)_{\mathfrak{z}}}$.

The proof of the following proposition relies on some basic commutative algebra results about localisations, completions and normalisations. It is the "2-dimensional version" of proposition 2.2.

Proposition 3.5. *Let $x \in y \subset X$ be a flag and let $\mathfrak{z} \in y(x)$. Then $K_{x,\mathfrak{z}}$ is a 2-dimensional valuation field such that $\mathcal{O}_{K_{x,\mathfrak{z}}} = \mathcal{O}_{x,\mathfrak{z}}$ and $K^{(2)}_{x,\mathfrak{z}}$ is a finite extension of $k(x)$.*

Proof. First of all $\mathrm{ht}\,\mathfrak{z} \geq \mathrm{ht}\,\mathfrak{p}_{y,x} = 1$, but if $\mathrm{ht}\,\mathfrak{z} = 2$ then $\mathfrak{z}$ is the maximal ideal of $\mathcal{O}_x$ and we have that $\mathfrak{z} \cap \mathscr{O}_{X,x} = \mathfrak{m}_x$, a contradiction. Therefore $\mathrm{ht}\,\mathfrak{z} = 1$ and $\dim (\mathcal{O}_x)_{\mathfrak{z}} = 1$. It follows that $\widehat{(\mathcal{O}_x)_{\mathfrak{z}}}$ is a Noetherian, complete, local, domain of dimension 1, i.e. a complete DVR which is the valuation ring of the complete discrete valuation field $K_{x,\mathfrak{z}}$. The residue field of $K_{x,\mathfrak{z}}$ is by definition:

$$K^{(1)}_{x,\mathfrak{z}} := (\mathcal{O}_x)_{\mathfrak{z}} / \mathfrak{z} (\mathcal{O}_x)_{\mathfrak{z}} = \mathrm{Frac}\left(\mathcal{O}_x/\mathfrak{z}\right) .$$

Note that $\mathcal{O}_x/\mathfrak{z}$ is a Noetherian, complete, local domain of dimension 1 (in general we may lose the regularity by passing to the quotient). Consider the normalisation $\widetilde{\mathcal{O}_x/\mathfrak{z}}$ of $\mathcal{O}_x/\mathfrak{z}$; the domain $\widetilde{\mathcal{O}_x/\mathfrak{z}}$ is obviously normal and again Noetherian and complete. Moreover by Nagata theorem (see [7, Ch. IX, 4, no 2, Theorem 2]) $\mathcal{O}_x/\mathfrak{z}$ is a Japanese ring, therefore in particular $\widetilde{\mathcal{O}_x/\mathfrak{z}}$ is a finite $\mathcal{O}_x/\mathfrak{z}$-module. Now [15, Corollary 7.6] implies that $\widetilde{\mathcal{O}_x/\mathfrak{z}}$ is also local, and by summing up all the listed property we can conclude that $\widetilde{\mathcal{O}_x/\mathfrak{z}}$ is a complete DVR with fraction field $\mathrm{Frac}\left(\mathcal{O}_x/\mathfrak{z}\right)$. This proves that $K^{(1)}_{x,\mathfrak{z}}$ is a complete valuation field.

It remains to show only that the second residue field $K^{(2)}_{x,\mathfrak{z}}$ is a finite extension of $k(x)$. By definition $K^{(2)}_{x,\mathfrak{z}}$ is the residue field of the local ring $\widetilde{\mathcal{O}_x/\mathfrak{z}}$, but we already know that $\widetilde{\mathcal{O}_x/\mathfrak{z}}$ is a finite $\mathcal{O}_x/\mathfrak{z}$-module, so $K^{(2)}_{x,\mathfrak{z}}$ is a finite extension of:

$$\left(\mathcal{O}_x/\mathfrak{z}\right) \big/ \left(\widehat{\mathfrak{m}_x}/\mathfrak{z}\right) \cong \mathcal{O}_x/\widehat{\mathfrak{m}_x} \cong \mathscr{O}_{X,x}/\mathfrak{m}_x = k(x) .$$

$\square$

Definition 3.6. Let $x \in y \subset X$ be a flag and let $\mathfrak{z} \in y(x)$, then we put $E_{x,\mathfrak{z}} := K^{(1)}_{x,\mathfrak{z}}$ and $k_{\mathfrak{z}}(x) := K^{(2)}_{x,\mathfrak{z}}$.

$$\begin{tikzcd}
K_{x,\mathfrak{z}} \arrow[r, phantom, "\supset"] \arrow[dr, dash, dashed, bend right] & \mathcal{O}_{x,\mathfrak{z}} := \mathcal{O}_{K_{x,\mathfrak{z}}} \arrow[r, phantom, "\supset"] \arrow[d] & \mathcal{O}^{(2)}_{K_{x,\mathfrak{z}}} \arrow[d] \\
 & E_{x,\mathfrak{z}} := K^{(1)}_{x,\mathfrak{z}} \arrow[r, phantom, "\supset"] \arrow[dr, dash, dashed, bend right] & \mathcal{O}_{E_{x,\mathfrak{z}}} \arrow[d] \\
 & & k_{\mathfrak{z}}(x) := K^{(2)}_{x,\mathfrak{z}}
\end{tikzcd}$$

Moreover:

$$K_{x,y} := \prod_{\mathfrak{z} \in y(x)} K_{x,\mathfrak{z}}\,, \quad \mathcal{O}_{x,y} := \prod_{\mathfrak{z} \in y(x)} \mathcal{O}_{x,\mathfrak{z}}\,,$$

$$E_{x,y} := \prod_{\mathfrak{z} \in y(x)} E_{x,\mathfrak{z}}\,, \quad k_y(x) := \prod_{\mathfrak{z} \in y(x)} k_{\mathfrak{z}}(x)\,.$$

So, in perfect analogy with the one dimensional case, to any complete flag $x \in y \subset X$ we have associated a ring $K_{x,y}$ which is a product of 2-dimensional valuation field.

Remark 3.7. If y is nonsingular at x, then $K_{x,y}$ is a 2-dimensional valuation field and $k_{\mathfrak{z}}(x) = k(x)$.

At this stage we don't know much about the structure of $K_{x,\mathfrak{z}}$, since it depends heavily on the nature of X, in particular whether X is an algebraic surface or an arithmetic surface. Despite this lack of information, we can still define a canonical topology on $K_{x,\mathfrak{z}}$ by using the theory developed in appendix E. But keep in mind that this is a very peculiar occurrence, in fact remember from chapter 1 that the topology on 2-dimensional local fields is in general not canonical. Let's endow $\mathscr{O}_{X,x}$ with the $\mathfrak{m}_x$-adic topology with respect to its maximal ideal, then we construct the canonical topology with the following steps explained at the end of appendix E.1:

$$\mathscr{O}_{X,x} \overset{(C)}{\rightsquigarrow} \mathcal{O}_x = \widehat{\mathscr{O}_{X,x}} \overset{(L)}{\rightsquigarrow} (\mathcal{O}_x)_{\mathfrak{z}} \overset{(C)}{\rightsquigarrow} \widehat{(\mathcal{O}_x)_{\mathfrak{z}}} \overset{(L)}{\rightsquigarrow} K_{x,\mathfrak{z}} = \operatorname{Frac}\left(\widehat{(\mathcal{O}_x)_{\mathfrak{z}}}\right)\,. \tag{3.1}$$

Then $K_{x,y}$ is endowed with the product topology and it is a ST-ring.

3.2 Algebraic surfaces

In this section we fix a perfect field k (endowed with the discrete topology) and a 2-dimensional k-scheme X which is: of finite type (over k), Noetherian, geometrically integral and regular. We say that X is a *(nonsingular) surface over* k and moreover we denote its function field by $k(X)$.

3.2.1 Structure of $K_{x,\mathfrak{z}}$

Let's keep the general notations introduced in section 3.1, then for any flag $x \in y \subset X$ we have a ST ring $K_{x,y}$ which is a sum of 2-dimensional local fields $K_{x,\mathfrak{z}}$. Let's take a closer look to their structure.

The field $K_{x,\mathfrak{z}}$ is endowed with a natural k-algebra structure given by the canonical embedding $\iota : k \hookrightarrow \mathcal{O}_{x,\mathfrak{z}} \subseteq K_{x,\mathfrak{z}}$ induced by the morphism $X \to \operatorname{Spec} k$. It means that $K_{x,\mathfrak{z}}$ is a two dimensional local field of equal characteristic.

$$\begin{tikzcd}
 & \mathcal{O}_{x,y} & \\
 & \mathcal{O}_{x,\mathfrak{z}} \arrow[u, hook] & \\
\mathcal{O}_x \arrow[ur, hook] & & \mathcal{O}_y \arrow[ul, hook] \\
 & k \arrow[ul, hook] \arrow[ur, hook] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & K_{x,y} & \\
 & K_{x,\mathfrak{z}} \arrow[u, hook] & \\
K_x \arrow[ur, hook] & & K_y \arrow[ul, hook] \\
 & k(X) \arrow[ul, hook] \arrow[ur, hook] & \\
 & k \arrow[u, hook] &
\end{tikzcd}$$

Now we show that the image of the morphism $k \hookrightarrow \mathcal{O}_{x,\mathfrak{z}}$ is actually contained in the rank 2 valuation ring $\mathcal{O}^{(2)}_{x,\mathfrak{z}}$: there is a proper inclusion $\overline{\iota(k)} \subsetneq E_{x,\mathfrak{z}}$, but $E_{x,\mathfrak{z}}$ is the fraction field of $\mathcal{O}_{E_{x,\mathfrak{z}}}$ which implies that $\overline{\iota(k)} \subset \mathcal{O}_{E_{x,\mathfrak{z}}}$. In other words $\iota(k) \subset \mathcal{O}^{(2)}_{x,\mathfrak{z}}$.

So far we know that $K_{x,\mathfrak{z}}$ is an object of $\mathbf{LF}^2(k)$ satisfying the condition (T) of definition 1.13, but it is not immediately evident that $K_{x,\mathfrak{z}} \in \mathbf{TLF}^2(k)$ since we need the condition (P).

Proposition 3.8. *$K_{x,\mathfrak{z}}$ is a two dimensional topological local field over k.*

Proof. See [63, Proposition 3.3.6], but keep in mind that adeles in [63] are introduced as functors. □

Let's choose a sequence of local parameters (ϖ_1, ϖ_2) for $K_{x,\mathfrak{z}}$, by theorem 1.18 this implies that we are fixing a parametrization:

$$K_{x,\mathfrak{z}} \cong k_{\mathfrak{z}}(x)((t_1))((t_2)) \cong E_{x,\mathfrak{z}}((t_2)) \tag{3.2}$$

such that $\varpi_1 \mapsto t_1$ and $\varpi_2 \mapsto t_2$. Now we show that the choice of ϖ_2 (the uniformizer parameter for the valuation on $K_{x,\mathfrak{z}}$) can be made in a nice way.

Remark 3.9. Remember from commutative algebra the following chain of implications:

$$(A \text{ regular local}) \Rightarrow (A \text{ a UFD}) \Rightarrow (\text{Any prime } \mathfrak{p} \text{ s.t. } \operatorname{ht}(\mathfrak{p}) = 1 \text{ is principal})\,.$$

So, $\mathscr{O}_{X,x}$ is a UFD and $\mathfrak{p}_{y,x}$ is principal, but also $\mathcal{O}_x$ is a UFD and $\mathfrak{z}$ is principal.

Proposition 3.10. *Let $\mathfrak{p}_{x,y} = (\varpi_y)$ for $\varpi_y \in \mathscr{O}_{X,x}$, then we can choose the uniformizer parameter for $K_{x,\mathfrak{z}}$ to be ϖ_y.*

Proof. We show that ϖ_y generates the maximal ideal of $\mathcal{O}_{x,\mathfrak{z}}$. First of all we notice that the ring $\mathscr{O}_{X,x}/\varpi_y\mathscr{O}_{X,x} \cong \mathscr{O}_{Y,x}$ is reduced, and this implies that $\widehat{\mathscr{O}_{Y,x}} = \mathcal{O}_x/\varpi_y\mathcal{O}_x$ is reduced too. By remark 3.9 ϖ_y has a unique factorization $\varpi_y = p_1 \dots p_m$ in $\mathcal{O}_x$, and all the $p_i' s$ are distinct prime elements thanks to the fact that $\mathcal{O}_x/\varpi_y\mathcal{O}_x$ is reduced. Again remark 3.9 implies that $\mathfrak{z} = (p_j)$ for some index j. Any element of $\mathfrak{z}(\mathcal{O}_x)_{\mathfrak{z}}$ can be written as $\frac{p_j a}{b}$ with $b \notin \mathfrak{z}$ but:

$$\frac{p_j a}{b} = \frac{p_1 \dots p_m a}{p_1 \dots p_{j-1} p_{j+1} \dots p_m b} = \frac{\varpi_y a}{p_1 \dots p_{j-1} p_{j+1} \dots p_m b}$$

Since $p_1 \dots p_{j-1} p_{j+1} \dots p_m b \notin \mathfrak{z}$, we can conclude that ϖ_y generates the prime ideal $\mathfrak{z}(\mathcal{O}_x)_{\mathfrak{z}}$ of $(\mathcal{O}_x)_{\mathfrak{z}}$. ☐

Corollary 3.11. *If ϖ_y is a uniformizer parameter for the complete valuation field K_y, then it is a uniformizer parameter for $K_{x,\mathfrak{z}}$.*

Proof. It follows from proposition 3.10 and the fact that $(\mathscr{O}_{X,x})_{\mathfrak{p}_{x,y}} \cong \mathscr{O}_{X,y}$. ☐

Remark 3.12. Fix a flag $x \in y \subset X$. The local homomorphism $\mathscr{O}_{X,x} \to \mathscr{O}_{y,x}$ induces a local homomorphism $\mathcal{O}_x \to \widehat{\mathscr{O}_{y,x}}$ which gives a bijective correspondence between the ideals in $y(x)$ and the minimal prime ideals of $\widehat{\mathscr{O}_{y,x}}$. Moreover $E_{x,y} = \prod_{\mathfrak{z} \in y(x)} E_{x,\mathfrak{z}}$ is exactly the one dimensional object associated to the inclusion $x \in y$ that we have defined in the previous chapter. Hence the adelic ring of the curve y is recovered in the following way:

$$\mathbf{A}_y = \prod_{x \in y}{}' E_{x,y}\,.$$

Proposition 3.13. *Let's denote with $v_{x,\mathfrak{z}}$ the valuation of $K_{x,\mathfrak{z}}$ and with v_y the valuation of K_y. Then the restriction of $v_{x,\mathfrak{z}}$ to K_y is equal to v_y.*

Proof. By remark 3.12 we deduce that $E_{x,\mathfrak{z}}$ contains $k(y)$, which is in turns the residue field of K_y, so the claims follows directly from corollary 3.11. $\square$

3.2.2 Adeles

The adelic ring $\mathbf{A}_X$ will be the result of a "glueing" of the local data $\{K_{x,y}\}_{x\in y\subset X}$ where the couple (x, y) runs amongst all flags in X. The glueing procedure will be described precisely, but roughly speaking we will define the k-vector space $\mathbf{A}_X$ inside the big product of rings

$$\mathbf{A}_X \subset \prod_{\substack{x\in y,\\ y\subset X}} K_{x,y}$$

as a sort of "double restricted product".

First "restricted product": the adelic spaces $\mathbb{A}_y^{(r)}$ and $\mathbb{A}_y$. In this paragraph we fix a curve $y \subset X$, and denote with $\mathfrak{J}_{x,y}$ the Jacobson radical of $\mathcal{O}_{x,y}$.

Definition 3.14. Let's put:

$$\mathbb{A}_y^{(0)} = \begin{Bmatrix} (\alpha_{x,y})_{x\in y} \in \prod_{x\in y} \mathcal{O}_{x,y} \colon \forall s > 0,\ \alpha_{x,y} \in \mathcal{O}_x + \mathfrak{J}_{x,y}^s \\ \text{for all but finitely many } x \in y. \end{Bmatrix}$$

then for any $r \in \mathbb{Z}$

$$\mathbb{A}_y^{(r)} := \widehat{\mathfrak{m}}_y^r \mathbb{A}_y^{(0)} \subset \prod_{x\in y} K_{x,y}\,.$$

Clearly $\mathbb{A}_y^{(r)} \supseteq \mathbb{A}_y^{(r+1)}$ and $\bigcap_{r\in\mathbb{Z}} \mathbb{A}_y^{(r)} = 0$. Moreover we define

$$\mathbb{A}_y := \bigcup_{r\in\mathbb{Z}} \mathbb{A}_y^{(r)}\,.$$

Remark 3.15. We have the inclusion $\mathbb{A}_y \subset \prod_{x\in y} K_{x,y}$, therefore we can interpret $\mathbb{A}_y$ as a "restricted product" of the rings $K_{x,y}$ for y fixed and $x \in y$. Thus we can write:

$$\mathbb{A}_y = \prod_{x\in y}{}' K_{x,y}$$

where $\prod'$ here is just a piece of notation without any formal meaning.

Let's associate a notational symbol to the following assumption:

$$\diamondsuit = \left\{ \begin{array}{l} \text{For every } x \in y \text{ and } \mathfrak{z} \in y(x) \text{ we fix a sequence of local} \\ \text{parameters } (\varpi_1, \varpi_2) \text{ on } K_{x,\mathfrak{z}} \ (\varpi_2 \text{ is a uniformizer of} \\ K_{x,\mathfrak{z}}). \text{ Furthemore, thanks to corollary 3.11 we choose} \\ \varpi_2 = \varpi_y \text{ where } \varpi_y \text{ is a local parameter for } K_y. \end{array} \right\}$$

Remember that assumption $\diamondsuit$ induces the identifications $K_{x,\mathfrak{z}} = k(x)_{\mathfrak{z}}((t_1))((t_2))$ and $E_{x,y} = k_{\mathfrak{z}}(x)((t_1))$. Moreover when y is regular at x, then $\mathfrak{J}_{x,y}$ is the maximal ideal of $\mathcal{O}_{x,y}$. Let's give a nice explicit description of $\mathbb{A}_y^{(r)}$ and $\mathbb{A}_y$ under the conditions imposed by $\diamondsuit$ and the hypothesis that y is a regular curve:

Lemma 3.16. *Let y be a regular curve and assume that the condition $\diamondsuit$ is valid. For any $r \in \mathbb{Z}$, $\mathbb{A}_y^{(r)}$ is equal to the following ring:*

$$\Xi_y^{(r)} := \left\{ (\alpha_{x,y})_{x\in y} \in \prod_{x \in y} K_{x,y} \colon \alpha_{x,y} \textit{ satisfies the following conditions } (*) \textit{ and } (**) \right\}$$

(∗) $\alpha_{x,y} \in t_2^r E_{x,y}[[t_2]]$.

(∗∗) *Assume that:*

$$\alpha_{x,y} = t_2^r \sum_{i \geq 0} \Gamma_{x,i} t_2^i \qquad \textit{with } \ \Gamma_{x,i} \in E_{x,y} \,,$$

then for any fixed index i the sequence $(\Gamma_{x,i})_{x\in y} \in \mathbf{A}_y$ (here $\mathbf{A}_y$ is the adelic ring of the curve y defined in chapter 2). In other words for all but finitely many $x \in y$ we have that $\Gamma_{x,i} \in \mathcal{O}_{E_{x,y}}$.

Proof. Inclusion $\mathbb{A}_y^{(r)} \subseteq \Xi_y^{(r)}$. Let's start with $r = 0$, the general case will follow trivially. Consider an element $(\alpha_{x,y})_{x\in y}$, then clearly (∗) is true because $\mathcal{O}_{x,y} = E_{x,y}[[t_2]]$. Suppose that $\alpha_{x,y} = \sum_{i\geq 0} \Gamma_{x,i} t_2^i$, then there exists a unique decomposition:

$$\alpha_{x,y} = \sum_{i\geq 0} \Theta_{x,i} t_2^i + \sum_{i \geq 0} \Lambda_{x,i} t_2^i \in \mathcal{O}_x + \mathcal{O}_{x,y}$$

where $\Theta_{x,i} \in k(x)[[t_1]] = \mathcal{O}_{E_{x,y}}$, $\Lambda_{x,i} \in t_1^{-1} k(x)[t_1^{-1}] \subset E_{x,y} \setminus \mathcal{O}_{E_{x,y}}$, and $\Gamma_{x,i} = \Theta_{x,i} + \Lambda_{x,i}$. Now fix an index $h \geq 0$, then the set

$$S_h := \{x \in y \colon \Lambda_{x,h} \neq 0\}$$

is finite, indeed note that $\mathcal{O}_x + \mathfrak{J}^s_{x,y} = \mathcal{O}_{E_{x,y}}[[t_2]] + t_2^s E_{x,y}[[t_2]]$, thus if $\Lambda_{x,h} \neq 0$, then $\alpha_{x,y} \notin \mathcal{O}_x + \mathfrak{J}^{h+1}_{x,y}$. In other words if for infinitely many $x \in y$ we had that $\Lambda_{x,h} \neq 0$, then for the same points $\alpha_{x,y} \notin \mathcal{O}_x + \mathfrak{J}^{h+1}_{x,y}$ against the definition of $\mathbb{A}^{(0)}_y$. We have shown that for all but finitely many $x \in y$, $\Gamma_{x,i} = \Theta_{x,i} \in \mathcal{O}_{E_{x,y}}$ which is equivalent to say that $(\Gamma_{x,i})_{x \in y} \in \mathbf{A}_y$.
The case when $r \neq 0$ follows easily from the fact that $\widehat{\mathfrak{m}}^r_y \Xi^{(0)}_y = \Xi^{(r)}_y$.

Inclusion $\Xi^{(r)}_y \subseteq \mathbb{A}^{(r)}_y$. As above it is enough to write the proof for $r = 0$. Let $(\alpha_{x,y})_{x \in y} \in \Xi^{(0)}_y$, then for any index $i \geq 0$ define:

$$T_i := \left\{ x \in y \colon \Gamma_{x,i} \notin \mathcal{O}_{E_{x,y}}[[t_2]] \right\} ;$$

by the property $(**)$ T_i is a finite set. Now fix an index $h > 0$ then for all $x \in y \setminus \cup_{i=1}^{h-1} T_i$, (i.e. for all but finitely many $x \in y$) it holds that $\Gamma_{x,i} = \Theta_{x,i}$ when $i < h$, which means that

$$\alpha_{x,y} = \sum_{i \geq 0} \Theta_{x,i} t_2^i + \sum_{i \geq h} \Lambda_{x,i} t_2^i \in \mathcal{O}_x + \mathfrak{J}^h_{x,y} .$$

□

Proposition 3.17. *Let y be a regular curve and assume that the condition $\diamondsuit$ is valid. For any $r \in \mathbb{Z}$, $\mathbb{A}^{(r)}_y \cong t^r \mathbf{A}_y[[t]]$ as rings (here t is simply a variable). In particular $\mathbb{A}_y \cong \mathbf{A}_y((t))$ and $\mathbb{A}^{(0)}_y \cong \mathbf{A}_y[[t]]$.*

Proof. By lemma 3.16 we have the equality $\mathbf{A}^{(r)}_y = \Xi^r_y$ and the map $\Xi^r_y \to t^r \mathbf{A}_y[[t]]$ is given in the following way and it is well defined:

$$(\alpha_{x,y})_{x \in y} = \left(t_2^r \sum_{i \geq 0} \Gamma_{x,i} t_2^i \right)_{x \in y} \mapsto t^r \sum_{i \geq 0} (\Gamma_{x,i})_{x \in y} \, t^i .$$

It is routine check to show that is is a ring isomorphism. □

Remark 3.18. Proposition 3.17 is true also when y is a singular curve. The proof is based on a slightly modified version of lemma 3.16; the only difference consists in the fact that if $x \in y$ is singular then $K_{x,y} = \prod_{\mathfrak{z} \in y(x)} K_{x,\mathfrak{z}}$ is a sum of 2-dimensional valuation fields and $\mathfrak{J}_{x,y}$ is the sum of the maximal ideals of $K_{x,\mathfrak{z}}$. Here we restricted the proof to the case of nonsingular curves just by simplicity of notations.

On the product $\prod_{x \in y} K_{x,y}$ we take the product topology, thus we can endow each $\mathbb{A}^{(r)}_y$ and $\mathbb{A}_y$ with the subspace topology which gives a structure of ST k-algebras. An equivalent way to topologize these spaces consists in giving the ind-pro topology to $\mathbf{A}_y((t))$ (remember that the ring of one dimensional adeles is a ST ring), and then using proposition 3.17.

Remark 3.19. In other words the algebraic isomorphism of proposition 3.17 becomes an isomorphism of ST k-algebras.

Second "restricted product": the adelic space $\mathbf{A}_X$. The construction of $\mathbb{A}_y$ can be seen as a way to take the restricted product of $\prod_{x\in y} K_{x,y}$. The final step in order to construct the ring of adeles $\mathbf{A}_X$ is to take the restricted product of the groups $\mathbb{A}_y$ over all the curves in X with respect to the subgroups $\mathbb{A}_y^{(0)}$.

Definition 3.20.

$$\mathbf{A}_X := \left\{ (\beta_y)_{y\subset X} \in \prod_{y\subset X} \mathbb{A}_y \colon \beta_y \in \mathbb{A}_y^{(0)} \text{ for all but finitely many } y \right\} \subset \prod_{\substack{x\in y,\\ y\subset X}} K_{x,y}\,.$$

In a more suggestive way, we write by commodity

$$\mathbf{A}_X = \prod_{\substack{x\in y\\ y\subset X}}'' K_{x,y}$$

where the symbol "$\prod''$" is just a piece of notation which remembers that we are taking a "double restricted product".

In order to topologize $\mathbf{A}_X$ we need to recall the description of the restricted product, by means of categorical limits, for linearly topologized groups. Let $\{G_i\}_{i\in I}$ a set of linearly topologized groups and for any i let $H_i \subset G_i$ be a closed subgroup[1] endowed with the subspace topology. We denote the family of finite subsets of I as $\mathcal{P}_f(I)$; it forms a directed set with the relation $J \subseteq J'$. For any $J \in \mathcal{P}_f(I)$ define

$$G_J := \prod_{i\in J} G_i \times \prod_{i\notin J} H_i\,,$$

if $J \subseteq J'$ the identity in each factor induces an embedding $G_J \hookrightarrow G_{J'}$, thus we have a direct system $\{G_J\}_J$ and it is easy to see that

$$\prod_i{}' G_i = \varinjlim_J G_J\,,$$

where $\prod'_i G_i$ is the usual restricted product of the G_i with respect to the subgroups H_i. At this point, on each G_J we put the product topology and $\prod'_i G_i$ is endowed with the linear direct limit topology.

---

[1] Recall that for linear topologies open subgroups are closed.

By definition $\mathbf{A}_X$ is the restricted product of the groups $\mathbb{A}_y$ with respect to the subgroups $\mathbb{A}_y^{(0)}$ for any $y \subset X$. Therefore we endow $\mathbf{A}_X$ with the topology described above. It follows that $\mathbf{A}_X$ is again a ST k-algebra.

Remark 3.21. By proposition 3.17 and remark 3.18 $\mathbf{A}_X$ is simply the restricted product of the rings $\mathbf{A}_y((t))$ with respect to $\mathbf{A}_y[[t]]$, for y ranging over all the curves in X.

We can embed diagonally $k(X)$ inside the big product $\prod_{\substack{x\in y\\ y\subset X}} K_{x,y}$; but for any $y \subset X$ we have the inclusion $k(X) \subset K_y$ and moreover if $f \in k(X)$, then $f \in \mathscr{O}_{X,y}$ for all but finitely many curves. These facts imply that $k(X) \subset \mathbf{A}_X$.

Remark 3.22. It is fundamental to notice that the abstract definition of $\mathbf{A}_X$ makes sense even if X is just a Noetherian, integral and regular scheme of dimension 2 and not a surface over k.

3.2.3 Residues

Fix a flag $x \in y \subset X$ and let $\mathfrak{z}$ be a local branch of y at x. Let's put by simplicity:

$$\Omega^1_{x,\mathfrak{z}} := \Omega^{1,\mathrm{sep}}_{K_{x,\mathfrak{z}}|k}, \quad \Omega^2_{x,\mathfrak{z}} := \Omega^1_{x,\mathfrak{z}} \wedge \Omega^1_{x,\mathfrak{z}}, \quad \Omega^j_{x,y} = \oplus_{\mathfrak{z}\in y(x)} \Omega^j_{x,\mathfrak{z}},$$

$$\mathrm{res}_{x,\mathfrak{z}} := \mathrm{res}_{K_{x,\mathfrak{z}}|k} : \Omega^2_{x,\mathfrak{z}} \to k,$$

$$\mathrm{res}_{x,y} := \sum_{x\in y(x)} \mathrm{res}_{x,\mathfrak{z}} : \Omega^2_{x,y} \to k.$$

It is easy to see that $\Omega^2_{k(X)|k} \hookrightarrow \Omega^2_{x,\mathfrak{z}} \subseteq \Omega^2_{x,y}$. Since the residue map is independent from parametrization, for any $K_{x,\mathfrak{z}}$, in this subsection we can fix a parametrization $K_{x,\mathfrak{z}} \cong k_{\mathfrak{z}}(x)((t_1))((t_2))$; so we can consider the identifications:

$$\Omega^2_{x,\mathfrak{z}} = K_{x,\mathfrak{z}} dt_1 \wedge dt_2, \quad \Omega^2_{x,y} = K_{x,y} dt_1 \wedge dt_2.$$

Residues satisfy some local properties ([49, Proposition 2]), but the most important global feature is given by the 2-dimensional geometric reciprocity laws. It is the two dimensional version of theorem 2.14.

Theorem 3.23 (2D geometric reciprocity laws)**.** *Let* $\omega \in \Omega^2_{k(X)|k}$*, then:*

(1) *For any fixed curve* $y \subset X$*,* $\sum_{x\in y} \mathrm{res}_{x,y}(\omega) = 0$.

(2) *For any fixed closed point* $x \in X$*,* $\sum_{y\ni x} \mathrm{res}_{x,y}(\omega) = 0$*, where the sum is over all curves containing* x.

Proof. See [49, prop. 7]. □

For a fixed nonzero rational 2-form $\omega \in \Omega^2_{k(X)|k}$, consider the map:

$$\begin{array}{rcl} \xi^\omega : \mathbf{A}_X & \to & k \\ (\alpha_{x,y})_{\substack{x\in y\\ y\subset X}} & \mapsto & \displaystyle\sum_{\substack{x\in y\\ y\subset X}} \operatorname{res}_{x,y}(\omega\alpha_{x,y})\,. \end{array}$$

First of all let's check that the sum in the definition of ξ^ω is finite. Assume $\omega = f dt_1 \wedge dt_2$; for all but finitely many curves $y \subset X$ we have that $f \in \mathscr{O}_{X,y} \subset \mathcal{O}_{x,y}$ (for all $x \in y$) and also $(\alpha_{x,y})_{x\in y} \in \mathbb{A}_y^{(0)}$, therefore we don't have to worry about the summation of residues over curves. It remains to show that for a fixed curve $y \subset X$, the sum $\sum_{x\in y} \operatorname{res}_{x,y}(\omega\alpha_{x,y})$ is actually finite. By simplicity assume that y is nonsingular, then we have:

$$\alpha_{x,y} = \sum_{i\geq r} \Gamma_{x,i} t_2^i \quad \text{with } \Gamma_{x,-1} \in k(x)[[t_1]] \text{ for all but finitely many } x \in y$$

$$f = \sum_{j\geq m} b_j t_2^j \quad \text{with } b_j \in k(y) \text{ and } b \in k(x)[[t_1]] \text{ for all but finitely many } x \in y$$

It follows that the coefficient of degree $(-1,-1)$ of $\omega\alpha_{x,y}$ is 0 for all but finitely many $x \in y$.

At this point we want to show that ξ^ω is a standard k-character for the ST k-algebra $\mathbf{A}_X$ (i.e. it is continuous and it induces the self duality of the adelic ring. See definition F.8). In order to do this we need some preliminary notions and lemmas extending results, which are true for a linearly locally compact vector space V, to $V((t))$.

Remark 3.24. Fix a locally linearly compact vector space V over k, in literature is it often called a 1-*Tate space.* A 2-*Tate space* is a vector space isomorphic to $V((t))$; a concrete example is $\mathbb{A}_y \cong \mathbf{A}_y((t))$. Of course, by induction one can give the notion of a n-Tate space and it is natural to conjecture that the n-dimensional adelic ring on a scheme over k is the restricted product of n-Tate spaces.

Definition 3.25. Let V be a ST k-algebra and let $\xi \in V((t))^\vee$ be a nontrivial character. The *conductor of* ξ is

$$c_\xi := \min\left\{ i \in \mathbb{Z} \colon \xi \in \left(t^i V[[t]]\right)^\triangle \right\}$$

(See appendix F for the definition of the symbol $^\triangle$).

Lemma 3.26. *Let V be a ST k-algebra endowed with a standard k-character. Then $V((t))$ has a standard k-character with conductor equal to* 0.

Proof. Let ξ be a standard character of V. First of all let's find explicitly a nontrivial k-character of $V((t))$ which has conductor equal to 0. Consider:

$$\begin{array}{rcl} \psi^0 : V((t)) & \to & k \\ \sum_{i \geq m} a_i t_i & \mapsto & \xi(a_{-1}) \end{array}$$

It is obviously k-linear and continuous.
Let $\psi \in V((t))^{\vee}$, we want to show that that there exists a uniquely determined $\alpha \in V((t))$ such that $\psi = \psi^0_{\alpha}$. Assume that $c_{\psi} = i$, for any $b \in V$ the map $b \mapsto \psi(bt^{i-1})$ defines a k-character on V that by hypothesis is equal to ξ_{a_0} for a uniquely determined $a_0 \in V$. So consider the k-character:

$$\psi^1(x) := \psi(x) - \psi^0(xa_0t^{-i}) \quad \text{for } x \in V((t))\,,$$

it is easy to verify that $\psi^1\left(t^{i-1}V[[t]]\right) = 0$. Iterating the above argument, for any $j \geq 1$ one finds a uniquely determined $a_j \in V$ such that

$$\psi^{j+1}(x) := \psi^j(x) - \psi_0(xa_jt^{-i+j}) = \psi(x) - \psi^0\left(x\sum_{h=0}^{j} a_h t^{-i+h}\right)$$

is a k-character trivial on $t^{i-1-j}V[[t]]$. By taking the limit for $j \to \infty$ we obtain:

$$0 = \lim_{j\to\infty} \psi^j(x) = \psi(x) - \psi^0\left(x\sum_{h\geq 0} a_h t^{-i+h}\right) .$$

So we put $\alpha := \sum_{h\geq 0} a_h t^{-i+h}$ and it follows that $\psi(x) = \psi^0(x\alpha)$.
We prove that the map $\alpha \mapsto \psi^0_{\alpha}$ is closed. To do this, it is enough to show that for any $s \in \mathbb{Z}$ there exists an integer $r \leq s$ such that:

$$\psi^0_{t^sV[[t]]} := \{\psi^0_{\alpha} \in V((t))^{\vee} \colon \alpha \in t^sV[[t]]\} \supset (t^rV[[t]])^{\triangle}$$

Suppose by contradiction that for any $r \leq s$ we can find an k-character $\varphi_r = \psi^0_{\alpha_r} \in (t^rV[[t]])^{\triangle}$ such that $\varphi_r \notin \psi^0_{t^sV[[t]]}$ i.e. $\alpha_r \notin t^sV[[t]]$. Then

$$\lim_{r\to-\infty} \varphi_r = 0$$

which means that $\lim_{r\to-\infty} \alpha_r = 0$, against the construction of the sequence $\{\alpha_r\}_r$. $\square$

Lemma 3.27. *Let $\{V_i\}_{i\in I}$ be a family of locally linearly compact vector spaces which are also ST k-algebras. Assume that any V_i has a standard character, then the following map is an isomorphism:*

$$\begin{aligned}\prod_{i\in I}{}' V_i((t))^\vee &\to \left(\prod_{i\in I}{}' V_i((t))\right)^\vee\\ (\psi_i) &\mapsto \sum_i \psi_i\end{aligned}$$

Where:

- *The restricted product on the left is taken with respect to the closed additive subgroups $V_i[[t]]^\triangle$.*
- *The restricted product on the right is taken with respect to the closed additive subgroups $V_i[[t]]$.*

Proof. It follows from lemma 3.26 and a modified version of theorem F.13. □

Theorem 3.28. *Fix a nonzero rational 2-form $\omega \in \Omega^2_{k(X)|k}$ and a curve $y \subset X$, then the map*

$$\begin{aligned}\xi^\omega_y : \mathbb{A}_y &\to k\\ (\alpha_{x,y})_{x\in y} &\mapsto \sum_{x\in y} \mathrm{res}_{x,y}(\omega\alpha_{x,y})\,,\end{aligned}$$

is a standard k-character for $\mathbb{A}_y$. In particular it follows that $\mathbb{A}_y$ is self dual.

Proof. Suppose that $\omega = f dt_1 \wedge dt_2$ and let's consider the k-vector space $\mathbf{A}_y((t)))$. Moreover let $dt \in \Omega^1_{k(y)|k}$ be a rational differential form on y, then by theorem 2.16 we know that $\psi^{dt} : \mathbf{A}_y \to k$ is a standard character on the one-dimensional ring of adeles. By lemma 3.26 and proposition 3.17 (see also remark 3.19) the following composition of maps from $\mathbb{A}_y$ to k gives a standard k-character:

$$\mathbb{A}_y \ni \alpha_{x,y} = \left(\sum_{i\geq m} \Gamma_{x,i} t_2^i\right)_{x\in y} \mapsto \alpha_{x,y} f = \left(\sum_{i\geq m} \Gamma'_{x,i} t_2^i\right)_{x\in y} \mapsto \sum_{i\geq 0} (\Gamma'_{x,i})_{x\in y}\, t^i \mapsto$$

$$\mapsto \psi^{dt}(\Gamma'_{x,-1}) = \sum_{x\in y} \mathrm{res}_x(\Gamma'_{x,-1} dt) \in k\,,$$

where res_x is the one dimensional residue map on $E_{x,y}$. On the other hand it is evident that the above map is equal to ξ^ω_y since in each component roughly speaking takes the monomial coefficient of order $(-1,-1)$ of the 2-form $\omega\alpha_{x,y}$ (remember that there is also the trace in the explicit definition of $\mathrm{res}_{x,y}$). □

Theorem 3.29. *Fix a nonzero rational* 2*-form* $\omega \in \Omega^2_{k(X)|k}$*, then the map*

$$\begin{aligned} \xi^\omega : \mathbf{A}_X &\to k \\ (\alpha_{x,y})_{\substack{x\in y \\ y\subset X}} &\mapsto \sum_{\substack{x\in y \\ y\subset X}} \operatorname{res}_{x,y}(\omega\alpha_{x,y}) \,, \end{aligned}$$

is a standard k-character for $\mathbf{A}_X$. In particular it follows that $\mathbf{A}_X$ is self dual.

Proof. The following chain of algebraic and topological isomorphisms hold thanks to lemma 3.27 and theorem 3.28:

$$\mathbf{A}_X^\vee = \left(\prod_{y\subset X}{}' \mathbf{A}_y((t))\right)^\vee \cong \prod_{y\subset X}{}' \mathbf{A}_y((t))^\vee \cong \prod_{y\subset X}{}' \mathbf{A}_y((t)) = \mathbf{A}_X \,.$$

□

Definition 3.30. Fix a nonzero rational differential 2-form $\omega \in \Omega^2_{k(X)|k}$, then we have the *global differential pairing* (associated to ω) on $\mathbf{A}_X$:

$$\begin{aligned} d_\omega : \mathbf{A}_X \times \mathbf{A}_X &\to k \\ (\alpha,\beta) &\mapsto \xi^\omega(\alpha\beta) = \sum_{\substack{x\in y \\ y\subset X}} \operatorname{res}_{x,y}(\alpha\beta\omega) \,. \end{aligned}$$

It is starightfoward to notice that d_ω is bilinear, continuous and symmetric.

3.2.4 Adelic complexes

We now introduce some important subspaces in order to construct the adelic complexes associated to the surface X. Here the definitions are made "by hands" , but such subspaces can bee recovered as a particular case of the general theory of Beilinson adeles (see [45, 8]). First of all let's consider the following diagonal embeddings:

$$K_x \subset \prod_{y\ni x} K_{x,y}, \quad K_y \subset \prod_{x\in y} K_{x,y} \,,$$

so we can consider:

$$\prod_{x\in X} K_x \subset \prod_{\substack{x\in y \\ y\subset X}} K_{x,y}, \quad \prod_{y\subset X} K_y \subset \prod_{\substack{x\in y \\ y\subset X}} K_{x,y} \,.$$

Let's define:

$$A_{012} := \mathbf{A}_X\,; \quad A_{12} := \mathbf{A}_X \cap \prod_{\substack{x\in y\\ y\subset X}} \mathcal{O}_{x,y} = \prod_{y\subset X} \mathbb{A}_y^{(0)} \cong \prod_{y\subset X} \mathbf{A}_y[[t]];$$

$$A_{02} := \mathbf{A}_X \cap \prod_{x\in X} K_x\,; \quad A_2 := \mathbf{A}_X \cap \prod_{x\in X} \mathcal{O}_x\,; \quad A_{01} := \mathbf{A}_X \cap \prod_{y\subset X} K_y\,;$$

$$A_1 := \mathbf{A}_X \cap \prod_{y\subset X} \mathcal{O}_y\,; \quad A_0 := k(X)$$

The containment relations are depicted in the following diagram:

$$\begin{tikzcd}
 & & A_0 \arrow[dl, hook] \arrow[d, hook] \arrow[dr, hook] & & \\
 & A_{01} \arrow[r, hook] & A_{012} & A_{02} \arrow[l, hook'] & \\
A_1 \arrow[ur, hook] \arrow[urr, hook] \arrow[rr, hook] & & A_{12} \arrow[u, hook] & & A_2 \arrow[ul, hook'] \arrow[ull, hook'] \arrow[ll, hook']
\end{tikzcd}$$

And we have the adelic complex of k-vector spaces:

$$\begin{tikzcd}
\mathcal{A}_X & A_0 \oplus A_1 \oplus A_2 \arrow[r, "d^0"] & A_{01} \oplus A_{02} \oplus A_{12} \arrow[r, "d^1"] & A_{012} \\
 & (a_0, a_1, a_2) \arrow[r, mapsto] & (a_0 - a_1, a_2 - a_0, a_1 - a_2) & \\
 & & (a_{01}, a_{02}, a_{12}) \arrow[r, mapsto] & a_{01} + a_{02} + a_{12}\,.
\end{tikzcd} \tag{3.3}$$

If $D = \sum_{y\subset X} n_y[y]$ is a divisor of X we can define the vector subspace

$$\mathbf{A}_X(D) := \prod_{y\subset X} \mathbb{A}_y^{(-n_y)}\,.$$

Note that $\mathbf{A}_X(D)$ is a well defined subspace of $\mathbf{A}_X$ because $n_y = 0$ for all but finitely many y.

Proposition 3.31. *For any $D = \sum_{y\subset X} n_y[y] \in \mathrm{Div}(X)$, we have the equality:*

$$\mathbf{A}_X(D) = \left\{ (\alpha_{x,\mathfrak{z}})_{\substack{x\in y\\ \mathfrak{z}\in y(x)}} \in \mathbf{A}_X \colon v_{x,\mathfrak{z}}(\alpha_{x,\mathfrak{z}}) \geq -n_y \right\}$$

where $v_{x,\mathfrak{z}}$ is the valuation on $K_{x,\mathfrak{z}}$.

Proof. It is enough to use property $(*)$ of lemma 3.16 (in the case of possibly singular curves). $\square$

Let's define the adelic subspaces

$$A_{12}(D) := A_{012} \cap \mathbf{A}_X(D) = \mathbf{A}_X(D)\,.$$

$$A_1(D) := A_{01} \cap \mathbf{A}_X(D); \quad A_2(D) := A_{02} \cap \mathbf{A}_X(D);$$

in order to get the complex

$$\mathcal{A}_X(D): \quad A_0 \oplus A_1(D) \oplus A_2(D) \xrightarrow{d_D^0} A_{01} \oplus A_{02} \oplus A_{12}(D) \xrightarrow{d_D^1} A_{012} \tag{3.4}$$

such that the maps are the same of those in equation (3.3). Furthermore note that $\mathcal{A}_X = \mathcal{A}_X(0)$.

The following proposition describes the behaviour of the subspaces $A_*(D)$ with respect to the pairing d_ω.

Proposition 3.32. *Fix a nonzero rational* 2*-form* $\omega \in \Omega^2_{k(X)|k}$ *and a divisor* $D \in \mathrm{Div}(X)$*:*

(1) $A_{12}(D)^\perp = A_{12}((\omega) - D)$, $A_{01}^\perp = A_{01}$, $A_{02}^\perp = A_{02}$. *Moreover* A_{01}, A_{02} *and* A_{12} *are closed in* A_{012}.

(2) $A_i = A_{ij} \cap A_{ik}$ *for* $0 \leq i \leq 2$, $i < j$ *and* $i < k$.

(3) *Each of* A_* *and* $A_*(D)$ *and any of their sums is closed in* A_{012}. *Moreover:*

$$A_0^\perp = A_{01} + A_{02}\,,$$

$$A_1(D)^\perp = A_{01} + A_{12}((\omega) - D)\,, \quad A_2(D)^\perp = A_{02} + A_{12}((\omega) - D)\,.$$

Proof. See [19, 2, Theorem (4),(5),(6)]. $\square$

The following theorem is the 2-dimensional version of theorem 2.21:

Theorem 3.33. $k(X) = A_0$ *is discrete in* $\mathbf{A}_X$ *and the quotient* $\mathbf{A}_X/A_0^\perp = \mathbf{A}_X/(A_{01} + A_{02})$ *is a linearly compact* k*-vector space.*

Proof. See [19, 3, Theorem (2)]. $\square$

The main property of the adelic complex $\mathcal{A}_X(D)$ is described by the following crucial theorem which says that the adelic cohomology computes the usual sheaf cohomology on X relative to $\mathscr{O}_X(D)$.

Theorem 3.34. *Let* $D \in \mathrm{Div}(X)$*, then* $H^j(D) \cong H^j(\mathcal{A}_X(D))$ *for any* $j \geq 0$.

Proof. This is a special case of [27, 4.2], which proves that adelic cohomology coincides with Zariski cohomology in any dimension. □

There is also the idelic version of complex (3.3):

$$\begin{tikzcd}
\mathcal{A}_X^\times : & A_0^\times \oplus A_1^\times \oplus A_2^\times \arrow[r, "d^0_\times"] & A_{01}^\times \oplus A_{02}^\times \oplus A_{12}^\times \arrow[r, "d^1_\times"] & A_{012}^\times \\
& (a_0, a_1, a_2) \arrow[r, mapsto] & (a_0a_1^{-1}, a_2a_0^{-1}, a_1a_2^{-1}) & \\
& & (a_{01}, a_{02}, a_{12}) \arrow[r, mapsto] & a_{01}a_{02}a_{12}
\end{tikzcd} \tag{3.5}$$

whose cohomology is closely related to the geometry of X as well.

Theorem 3.35. *$H^j(X, \mathscr{O}_X^\times) \cong H^j(\mathcal{A}_X^\times)$ for any $j \geq 0$. In particular we have the isomorphism* $\mathrm{Pic}(X) \cong H^1(\mathcal{A}_X^\times)$.

Proof. We have to show the following equalities:

(*i*) $H^0(\mathcal{A}_X^\times) = k^\times$.

(*ii*) $H^1(\mathcal{A}_X^\times) = \mathrm{Pic}(X)$.

(*iii*) $H^2(\mathcal{A}_X^\times) = 0$.

(*i*) Clearly $H^0(\mathcal{A}_X^\times) = \ker d^0_\times = k^\times$.
(*iii*) Note that

$$H^2(\mathcal{A}_X^\times) = \frac{A_{012}^\times}{A_{01}^\times A_{02}^\times A_{12}^\times}\,.$$

For any $a_{x,\mathfrak{z}} \in K_{x,\mathfrak{z}}$ we can write $a_{x,\mathfrak{z}} = \varpi_y^m b_{x,\mathfrak{z}}$ where ϖ_y is the uniformizer parameter of K_y (see corollary 3.11) and $b_{x,\mathfrak{z}} \in \mathcal{O}_{x,\mathfrak{z}}^\times$. Therefore $A_{012}^\times = A_{01}^\times A_{12}^\times$ and $H^2(\mathcal{A}_X^\times) = 0$.

(ii) Let's consider the following diagram with exact rows:

$$\begin{tikzcd}
 & & (a_{01}, a_{02}, a_{12}) \arrow[r, mapsto, "\pi_1"] & a_{01} & \\
 & a \arrow[r, mapsto] & (1, a, a) & & \\
0 \arrow[r] & A_{02}^{\times} \cap A_{12}^{\times} \arrow[r] & \ker(d_{\times}^{1}) \arrow[r, "\pi_1"] & A_{01}^{\times} \arrow[r] & 0 \\
0 \arrow[r] & A_{2}^{\times} \arrow[r] \arrow[u, hook, "\mathrm{id}"] & \mathrm{im}(d_{\times}^{0}) \arrow[r] \arrow[u, hook] & A_{0}^{\times} A_{1}^{\times} \arrow[r] \arrow[u, hook] & 0
\end{tikzcd} \tag{3.6}$$

By applying the snake lemma we get the exact sequence:

$$\begin{tikzcd}
0 \arrow[r] & H^{1}(\mathcal{A}_{X}^{\times}) \arrow[r, "\cong"] & \frac{A_{01}^{\times}}{A_{0}^{\times} A_{1}^{\times}} \arrow[r] & 0
\end{tikzcd}$$

Now define a map:

$$\begin{array}{rcl}
p_{01} : A_{01}^{\times} & \twoheadrightarrow & \mathrm{Div}(X) \\
(a_{x,y})_{x,y} & \mapsto & \sum\limits_{y \subset X} v_y(a_{x,y})[y]
\end{array}$$

and by composing it with π_1 we obtain the following commutative diagram which completes the proof:

$$\begin{tikzcd}
\ker(d_{\times}^{1}) \arrow[r, "\pi_1"] \arrow[d, two heads] & A_{01}^{\times} \arrow[r, "p_{01}"] \arrow[d, two heads] & \mathrm{Div}(X) \arrow[d, two heads] \\
H^{1}(\mathcal{A}_{X}^{\times}) \arrow[r, "\cong"] & \frac{A_{01}^{\times}}{A_{0}^{\times} A_{1}^{\times}} \arrow[r, "\cong"] & \mathrm{Pic}(X) \, .
\end{tikzcd} \tag{3.7}$$

□

3.2.5 K_2-adeles

In appendix G.1 we introduced some notions of K-theory for fields and local rings. In particular, for any 2-dimensional local field $K_{x,\mathfrak{z}}$ we have the r-th K-group $K_r(K_{x,\mathfrak{z}})$. It is possible to "glue together" all this data to get an adelic object. The only non trivial and interesting case for our 2-dimensional investigation is the one related to $K_2(K_{x,\mathfrak{z}})$, so we will define the concept of K_2-adeles.

We will try to make definitions as similar as possible to those in subsection 3.2.2. First of all, for any flag $x \in y \subset X$ let's put:

$$K_2(K_{x,y}) := \prod_{\mathfrak{z} \in y(x)} K_2(K_{x,\mathfrak{z}})\,; \quad K_2(\mathcal{O}_{x,y}) := \prod_{\mathfrak{z} \in y(x)} K_2(\mathcal{O}_{x,\mathfrak{z}})\,.$$

Moreover fix a curve $y \subset X$; remember that $K_2(\mathcal{O}_y) \subset K_2(K_y)$ and that we have a map $\iota_y : K_2(K_y) \to K_2(K_{x,y})$ induced by the embedding $K_y \hookrightarrow K_{x,y}$. Let's define

$$K_2(\mathbb{A}_y) := \left\{ \begin{aligned} &(a_{x,y})_{x,y} \in \prod_{x \in y} K_2(K_{x,y})\colon \forall s > 0 \; a_{x,y} \in \iota_y(K_2(\mathcal{O}_y)) \cdot K_2(\mathcal{O}_{x,y}, \mathfrak{J}^s_{x,y}) \\ &\text{for all but finitely many nonsingular points } x \in y \end{aligned} \right\}$$

$$K_2(\mathbb{A}_y^{(0)}) := \prod_{x \in y} K_2(\mathcal{O}_{x,y})\,.$$

Definition 3.36. The group of K_2-*adeles* is defined as:

$$K_2(\mathbf{A}_X) := \left\{ (b_y)_{y \subset X} \in \prod_{y \subset X} K_2(\mathbb{A}_y)\colon b_y \in K_2(\mathbb{A}_y^{(0)}) \text{ for all but finitely many } y \right\}$$

Clearly also the embedding $K_x \hookrightarrow K_{x,y}$ induces a map at the level of K_2 groups $\iota_x : K_2(K_x) \to K_2(K_{x,y})$. We consider the diagonal embeddings:

$$\iota_x(K_2(K_x)) \subset \prod_{x \in y} K_2(K_{x,y})\,, \quad \iota_y(K_2(K_y)) \subset \prod_{y \ni x} K_2(K_{x,y})$$

which in turns give the inclusions:

$$\prod_{x \in X} \iota_x(K_2(K_x)) \subset \prod_{\substack{x \in y \\ y \subset X}} K_2(K_{x,y})\,, \quad \prod_{y \subset X} \iota_y(K_2(K_y)) \subset \prod_{\substack{x \in y \\ y \subset X}} K_2(K_{x,y})\,.$$

We can repeat the same constructions to get the maps $\iota_x^0 : K_2(\mathcal{O}_x) \to K_2(K_{x,y})$ and $\iota_y^0 : K_2(\mathcal{O}_y) \to K_2(K_{x,y})$ with the relative diagonal embeddings. At this point we can define the K_2-adelic subspaces.

$$K_2(A_{012}) := K_2(\mathbf{A}_X)\,; \quad K_2(A_{12}) := K_2(\mathbf{A}_X) \cap \prod_{\substack{x \in y \\ y \subset X}} K_2(\mathcal{O}_{x,y});$$

$$K_2(A_{02}) := K_2(\mathbf{A}_X) \cap \prod_{x \in X} \iota_x(K_2(K_x))\,; \quad K_2(A_2) := K_2(\mathbf{A}_X) \cap \prod_{x \in X} \iota_x^0(K_2(\mathcal{O}_x))\,;$$

$$K_2(A_{01}) := K_2(\mathbf{A}_X) \cap \prod_{y \subset X} \iota_y(K_2(K_y))\,; \quad K_2(A_1) := K_2(\mathbf{A}_X) \cap \prod_{y \subset X} \iota^0_y(K_2(\mathcal{O}_y))\,;$$
$$K_2(A_0) := K_2(K)\,.$$

We obtain the K_2 adelic complex.

$$K_2(\mathcal{A}_X):$$

$$\begin{tikzcd}
K_2(A_0) \oplus K_2(A_1) \oplus K_2(A_2) \arrow[r, "\delta^0"] & K_2(A_{01}) \oplus K_2(A_{02}) \oplus K_2(A_{12}) \arrow[r, "\delta^1_\times"] & K_2(A_{012}) \\
(a_0, a_1, a_2) \arrow[r, mapsto] & (a_0 a_1^{-1}, a_2 a_0^{-1}, a_1 a_2^{-1}) & \\
& (a_{01}, a_{02}, a_{12}) \arrow[r, mapsto] & a_{01} a_{02} a_{12}
\end{tikzcd}$$

Remark 3.37. Note that if in the above constructions we use K_1 instead of K_2 we obtain exactly the idelic theory.

Ideles (i.e. K_1-adeles) are closely related to divisors on the surface X, in particular $H^1(\mathcal{A}_X^\times) \cong \mathrm{Pic}(X)$. Now we are going to explain how K_2-adeles are closely related to 0-cycles on X in a very similar way.

The map $U \mapsto K_2(U)$ for any open set $U \subset X$ is a presheaf of abelian group and we denote with $\mathcal{K}_2(X)$ its sheafification, then we have the following fundamental theorem:

Theorem 3.38. *For $j = 0, 1$ we have the isomorphism $H^j(K_2(\mathcal{A}_X)) \cong H^j(\mathcal{K}_2(X))$ and moreover $H^2(K_2(\mathcal{A}_X)) \cong \mathrm{CH}^2(X)$.*

Proof. Fix a flag $x \in y \subset X$. The function field $k(X)$ is a discrete valuation field with residue field $k(y)$ and the latter is a gain a discrete valuation field with residue field $k_{\mathfrak{z}}(x)$. We can use the Milnor boundary maps and the functoriality of K_2 to obtain the following chain of morphisms:

$$K_2(k(X)) \xrightarrow{\quad \partial_2 \quad} K_1(k(y)) \xrightarrow{\quad \partial_1 \quad} K_0(k_{\mathfrak{z}}(x)) \xrightarrow{K_2(\mathrm{Tr}_{k_{\mathfrak{z}}(x)|k(x)})} K_0(k(x))$$

which leads to the following complex, which is called the Gersten complex on X (or cousin complex):

$$K_2(k(X)) \longrightarrow \prod_{y \subset X} K_1(k(y)) \longrightarrow \prod_{x \in X} K_0(k(x)) \tag{3.8}$$

It is a well known result ([51, 7, Theorems 5.6, 5.11]) that the Gersten complex and the complex $H^j(X, \mathcal{K}_2(X))$ are quasi isomorphic. It means that it is enough to show the result for the Gersten complex. Consider the following big diagram:

$$\begin{tikzcd}
K_2(k(X)) \arrow[r] & \prod_{y\subset X} K_1(k(y)) \arrow[r] & \prod_{x\in X} K_0(k(x)) \\
K_2(k(X)) \times K_2(A_1) \times K_2(A_2) \arrow[r] \arrow[u, "\psi_1"] & K_2(A_{01}) \times K_2(A_{02}) \times K_2(A_{12}) \arrow[r] \arrow[u, "\psi_2"] & K_2(A_{012}) \arrow[u, "\psi_3"] \\
K_2(A_1) \times K_2(A_2) \arrow[r] \arrow[u, "\phi_1"] & K_2(A_1) \times K_2(A_{02}) \times K_2(A_{12}) \arrow[r] \arrow[u, "\phi_2"] & \ker(\psi_3) \arrow[u, "\phi_3"]
\end{tikzcd} \tag{3.9}$$

where maps are defined in the following way:

$$\begin{aligned} \psi_1 : K_2(k(X)) \times K_2(A_1) \times K_2(A_2) &\to K_2(k(X)) \\ (a, b, c) &\mapsto a \end{aligned}$$

$$\begin{aligned} \psi_2 : K_2(A_{01}) \times K_2(A_{02}) \times K_2(A_{12}) &\to \prod_{y\subset X} K_1(k(y)) \\ (a, b, c) &\mapsto \prod_{y\subset X} \partial_2(b) \end{aligned}$$

where $\partial_2 := K_2(K_y) \to K_1(k(y))$ is the Milnor boundary map.

$$\begin{aligned} \psi_3 : K_2(A_{012}) &\to \prod_{x\in X} K_0(k(x)) \\ (\{a_{x,y}, b_{x,y}\})_{\substack{x\in y,\\ y\subset X}} &\mapsto \prod_{x\in X} \left(\sum_{y\ni x} (a_{x,y}, b_{x,y})_{x,y} \right) \end{aligned} \tag{3.10}$$

$$\begin{aligned} \phi_1 : K_2(A_1) \times K_2(A_2) &\to K_2(k(X)) \times K_2(A_1) \times K_2(A_2) \\ (a, b) &\mapsto (1, a, b) \end{aligned}$$

And finally $\phi_2 = \text{id}$ and $\phi_3 = \text{id}$.

Now see [48, Theorem 3] for a detailed analysis of complex 3.9 and for a proof that the two rows on the top are quasi-isomorphic. □

3.2.6 Functoriality

As we did in the one-dimensional case, in this subsection we study the functorial properties of adeles with respect to morphisms. Clearly, due to the local nature of morphism between schemes, we expect a contravariant behaviour of adeles, this means adeles naturally pull back (see the analogy with invertible sheaves). Let's fix our surface X, then we will focus on three cases:

- A surjective morphism $\varphi : X \to Z$ between nonsingular surfaces. This situation was partially studied by Parshin in [49], but he considered only rational adeles and he didn't put much care in the case of curves with singular points.
- A fibred surface $\varphi : X \to C$ over a nonsingular curve. Osipov in [48] considered adeles on a fibered surface, but he was interested mainly in the "pushforward of adelic objects". We'll see that such an approach is very useful for the idelic intersection theory.
- A closed embedding $y \subset X$ of a nonsingular integral curve into a nonsingular surface. In [19] some relations between $\mathbf{A}_{\tilde{y}}/k(\tilde{y})$ and a variation of the adelic complex $\mathcal{A}_X$ are used for purposes of adelic intersection theory.

Morphism between surfaces. Let's fix a surjective morphism $\varphi : X \to Z$ between non singular surfaces over a perfect field k. Fix a flag $\tilde{x} \in \tilde{y} \subset X$; if $\varphi(\tilde{x}) \in \varphi(\tilde{y})$ is not a flag on Z, then we define immediately

$$\begin{aligned} \varphi'_{\tilde{x},\tilde{y}} : \prod_{\substack{z\in w,\\ w\subset Z}} K_{z,w} &\to \quad K_{\tilde{x},\tilde{y}} \\ (a_{z,w})_{\substack{z\in w,\\ w\subset Z}} &\mapsto \quad 0 \end{aligned} \tag{3.11}$$

where clearly w runs over all integral curves in Z. Now let's assume that $\tilde{z} = \varphi(\tilde{x}) \in \tilde{w} = f(\tilde{y}) \subset Z$ is a flag; in other words we have the following situation:

$$\begin{tikzcd} \tilde{x} \arrow[d, mapsto] & \in & \tilde{y} \arrow[r, hook] \arrow[d] & X \arrow[d, "\varphi"] \\ \tilde{z} & \in & \tilde{w} \arrow[r, hook] & Z \end{tikzcd}$$

There is a local homomorphism $\varphi^{\#}_{\tilde{x}} : \mathscr{O}_{Z,\tilde{z}} \to \mathscr{O}_{X,\tilde{x}}$ which is injective thanks to the surjectivity of φ; such a map will be the starting point for the construction of the required homomorphism $\varphi^a : \mathbf{A}_Z \to \mathbf{A}_X$. We will proceed by steps, and the first thing to notice is that, by the property of completions, $\varphi^{\#}_{\tilde{x}}$ induces a continuous homomorphism $\psi_1 : \mathcal{O}_{\tilde{z}} \to \mathcal{O}_{\tilde{x}}$. Now consider the commutative diagram given by the contravariant functor Spec( )

$$\begin{tikzcd}
\operatorname{Spec} \mathcal{O}_{\tilde{x}} \arrow[r, "\operatorname{Spec}(\psi_1)"] \arrow[d] & \operatorname{Spec} \mathcal{O}_{\tilde{z}} \arrow[d] \\
\operatorname{Spec} \mathscr{O}_{X,\tilde{x}} \arrow[r, "\operatorname{Spec}(\varphi^{\#}_{\tilde{x}})"] & \operatorname{Spec} \mathscr{O}_{Z,\tilde{z}}
\end{tikzcd}$$

Pick $\mathfrak{z} \in \tilde{y}(\tilde{x}) \subset \operatorname{Spec} \mathcal{O}_{\tilde{x}}$, then we put by simplicity:

$$\varphi(\mathfrak{z}) := \operatorname{Spec}(\psi_1)(\mathfrak{z}) \in \tilde{w}(\tilde{z}) \subset \operatorname{Spec}(\mathcal{O}_{\tilde{z}})$$

where $\varphi(\mathfrak{z})$ is contained in $\tilde{w}(\tilde{z})$ because of the following commutative diagram:

$$\begin{tikzcd}
\mathcal{O}_{\tilde{x}} & \mathscr{O}_{X,\tilde{x}} \arrow[l, hook'] \arrow[r] & \mathscr{O}_{\tilde{y},\tilde{x}} \\
\mathcal{O}_{\tilde{z}} \arrow[u] & \mathscr{O}_{Z,\tilde{z}} \arrow[l, hook'] \arrow[r] \arrow[u] & \mathscr{O}_{\tilde{w},\tilde{z}} \arrow[u]
\end{tikzcd}$$

We can construct a map

$$\begin{array}{rcl}
\psi_2 : (\mathcal{O}_{\tilde{z}})_{\varphi(\mathfrak{z})} & \to & (\mathcal{O}_{\tilde{x}})_{\mathfrak{z}} \\
\dfrac{a}{b} & \mapsto & \dfrac{\psi_1(a)}{\psi_1(b)}
\end{array}$$

and it is immediate to verify that ψ_2 is well defined and continuous with respect the local ring topologies. By passing again to completions, we have a continuous homomorphism between discrete valuation rings:

$$\psi_3 : \mathcal{O}_{\tilde{z},\varphi(\mathfrak{z})} \to \mathcal{O}_{\tilde{x},\mathfrak{z}} \,.$$

Clearly $\ker(\psi_3)$ is a prime ideal, and since $\mathcal{O}_{z,\varphi(\mathfrak{z})}$ has Krull dimension 1, we can only have $\ker(\psi_3) = 0$ or $\ker(\psi_3)$ is equal to the maximal ideal of $\mathcal{O}_{z,\varphi(\mathfrak{z})}$. But the latter option implies that the image of ψ_3 would be isomorphic to the residue field of $\mathcal{O}_{z,\varphi(\mathfrak{z})}$, and by a moment of reflection about the construction of ψ_3 we can conclude that this is not possible. In other words ψ_3 is injective and it induces a continuous homomorphism:

$$\varphi_{\tilde{x},\mathfrak{z}} : K_{\tilde{z},\varphi(\mathfrak{z})} \to K_{\tilde{x},\mathfrak{z}}$$

The following big diagram summarizes all the steps we needed in order to obtain $\varphi_{\tilde{x},\mathfrak{z}}$, and it is important to notice how it is again a process which alternates "completion" and "localisation" arguments:

$$\begin{tikzcd}
K_{\tilde{z},\varphi(\mathfrak{z})} \arrow[r, "\varphi_{\tilde{x},\mathfrak{z}}"] & K_{\tilde{x},\mathfrak{z}} \\
\mathcal{O}_{\tilde{z},\varphi(\mathfrak{z})} \arrow[r, "\psi_3"] \arrow[u, hook] & \mathcal{O}_{\tilde{x},\mathfrak{z}} \arrow[u, hook] \\
(\mathcal{O}_{\tilde{z}})_{\varphi(\mathfrak{z})} \arrow[r, "\psi_2"] \arrow[u, hook] & (\mathcal{O}_{\tilde{x}})_{\mathfrak{z}} \arrow[u, hook] \\
\mathcal{O}_{\tilde{z}} \arrow[r, "\psi_1"] \arrow[u, hook] & \mathcal{O}_{\tilde{x}} \arrow[u, hook] \\
\mathscr{O}_{Z,\tilde{z}} \arrow[r, "\varphi^{\#}_{\tilde{x}}"] \arrow[u, hook] & \mathscr{O}_{X,\tilde{x}} \arrow[u, hook]
\end{tikzcd}$$

Now we can define

$$\begin{array}{rcl}
\varphi'_{\tilde{x},\mathfrak{z}} : K_{\tilde{z},\tilde{w}} = \displaystyle\prod_{\mathfrak{u}\in\tilde{w}(\tilde{z})} K_{\tilde{z},\mathfrak{u}} & \to & K_{\tilde{x},\mathfrak{z}} \\
(a_{\tilde{z},\mathfrak{u}})_{\mathfrak{u}\in\tilde{w}(\tilde{z})} & \mapsto & \varphi_{\tilde{x},\mathfrak{z}}(a_{\tilde{z},\varphi(\mathfrak{z})})
\end{array}$$

which in turns induces the morphism

$$\begin{array}{rrcl}
\varphi_{\tilde{x},\tilde{y}} : & K_{\tilde{z},\tilde{w}} & \to & K_{\tilde{x},\tilde{y}} \\
& a_{\tilde{z},\tilde{w}} & \mapsto & (\varphi'_{\tilde{x},\mathfrak{z}}(a_{\tilde{z},\tilde{w}}))_{\mathfrak{z}\in\tilde{y}(\tilde{x})}
\end{array}$$

At this point we are ready to (re)define the homomorphism $\varphi'_{\tilde{x},\tilde{y}}$:

$$\begin{array}{c}
\varphi'_{\tilde{x},\tilde{y}} : \displaystyle\prod_{\substack{z\in w,\\ w\subset Z}} K_{z,w} \to K_{\tilde{x},\tilde{y}} \\
(a_{z,w})_{\substack{z\in w,\\ w\subset Z}} \mapsto \varphi_{\tilde{x},\tilde{y}}(a_{\tilde{z},\tilde{w}})
\end{array} \tag{3.12}$$

Note that thanks to equations (3.11) and (3.12) we have a consistent definition of $\varphi'_{\tilde{x},\tilde{y}}$ in all possible cases, so we can put

$$\begin{array}{rcl}
\varphi^a : \displaystyle\prod_{\substack{z\in w,\\ w\subset Z}} K_{z,w} & \to & \displaystyle\prod_{\substack{x\in y,\\ y\subset X}} K_{x,y} \\
(a_{z,w})_{\substack{z\in w,\\ w\subset Z}} & \mapsto & \left(\varphi'_{x,y}\left((a_{z,w})_{\substack{z\in w,\\ w\subset Z}}\right)\right)_{\substack{x\in y,\\ y\subset X}}
\end{array}$$

Example 3.39. Let's assume that $\varphi : X \to Z$ is the blow-up at a point $\tilde{z} \in Z$ with exceptional curve L. Moreover put

$$\prod_{\substack{x\in y,\\ y\subset X}} I_{x,y} = \varphi^a \left(\prod_{\substack{z\in w,\\ w\subset Z}} K_{z,w} \right) \subseteq \prod_{\substack{x\in y,\\ y\subset X}} K_{x,y}$$

Then by the well known properties of blow-ups we have the following equalities:

$$I_{x,y} = \begin{cases} 0 & \text{if } y = L \\ K_{x,y} & \text{if } y \neq L \text{ and } x \notin y \cap L \end{cases}$$

So, the only nontrivial case for $I_{x,y}$ is when $y \neq L$ and $x \in y \cap L$.

By construction (and by the surjectivity of φ), each local map $\varphi_{\tilde{x},\mathfrak{z}}$ has the following properties:

- $\varphi_{\tilde{x},\mathfrak{z}}(k(Z)) \subseteq k(X)$.
- $\varphi_{\tilde{x},\mathfrak{z}}(\mathcal{O}_{\tilde{z}}) \subseteq \mathcal{O}_{\tilde{x}}$ and $\varphi_{\tilde{x},\mathfrak{z}}(K_{\tilde{z}}) \subseteq K_{\tilde{x}}$.
- $\varphi_{\tilde{x},\mathfrak{z}}(\mathcal{O}_{\tilde{w}}) \subseteq \mathcal{O}_{\tilde{y}}$ and $\varphi_{\tilde{x},\mathfrak{z}}(K_{\tilde{w}}) \subseteq K_{\tilde{y}}$. Here it is useful to remember that $\mathscr{O}_{X,\tilde{y}} = (\mathscr{O}_{X,\tilde{x}})_{\mathfrak{p}_{\tilde{y},\tilde{x}}}$ and $\mathscr{O}_{Z,\tilde{w}} = (\mathscr{O}_{Z,\tilde{z}})_{\mathfrak{p}_{\tilde{w},\tilde{z}}}$.
- $\varphi_{\tilde{x},\mathfrak{z}}$ sends the maximal ideal of $\mathcal{O}_{\tilde{z},\varphi(\mathfrak{z})}$ into the maximal ideal of $\mathcal{O}_{\tilde{x},\tilde{z}}$.

Therefore it is not difficult to see that the homomorphism φ^a induces by restriction a map:

$$\varphi^a : \mathbf{A}_Z \to \mathbf{A}_X$$

and a homomorphism of adelic complexes $\varphi^a : \mathcal{A}_Z \to \mathcal{A}_X$. By abuse of notation we have used the symbol φ^a many times to denote different, but closely related, maps, by the way its meaning will be clear from the context.

Remember that when φ is a flat morphism it is possible to pullback divisors, so it is interesting to study the relationship between $\mathbf{A}_X(\varphi^*(D))$ and $\mathbf{A}_Z(D)$ for any $D \in \mathrm{Div}(Z)$.

Proposition 3.40. *Assume that φ is flat and fix a divisor $D \in \mathrm{Div}(Z)$, then the homomorphism $\varphi^a : \mathbf{A}_Z \to \mathbf{A}_X$ induces by restriction a homomorphism $\varphi^a(D) : \mathbf{A}_Z(D) \to \mathbf{A}_X(\varphi^*(D))$.*

Proof. The strategy is very similar to the one-dimensional case because we use proposition 3.31 to express $\mathbf{A}_Z(D)$ and $\mathbf{A}_X(\varphi^*(D))$. Assume that $D = \sum_{w\subset Z} n_w[w]$, then:

$$\varphi^*(D) = \sum_{\substack{y\subset X,\\ y\mapsto w}} e_{y|w} n_w [y]$$

where the sum is over all integral curves $y \subset X$ mapped by φ to integral curves $w \subset Z$. Let $z = \varphi(x) \in w = \varphi(y)$, $\mathfrak{u} = \varphi(\mathfrak{z}) \in w(z)$ and let t_w be a local parameter for $\mathscr{O}_{Z,w}$ then by definition the ramification index at y is $e_{y|w} = v_y(\varphi_y^{\#}(t_w))$. Moreover consider $\alpha_{z,\mathfrak{u}} \in K_{z,\mathfrak{u}}$ such that $m := v_{z,\mathfrak{u}}(\alpha_{z,\mathfrak{u}}) \geq -n_w$, then we can write $\alpha_{z,\mathfrak{u}} = g t_w^m$ with $g \in \mathcal{O}_{z,\mathfrak{u}}^{\times}$, since we can take t_w as local parameter of $K_{z,\mathfrak{u}}$ (see corollary 3.11). Clearly we have:

$$\varphi_{x,\mathfrak{z}}(\alpha_{z,\mathfrak{u}}) = \varphi_{x,\mathfrak{z}}(g)\varphi_{x,\mathfrak{z}}(t_w)^m ,$$

but $v_{z,\mathfrak{u}}$ restricted to K_w is equal to v_w (see proposition 3.13), therefore we can conclude that

$$v_{x,\mathfrak{z}}\left(\varphi_{x,\mathfrak{z}}(\alpha_{z,\mathfrak{u}})\right) = e_{y|w} m \geq -e_{y|w} n_w .$$

The above argument easily implies the claim by looking at products over all flags. □

From proposition 3.40 it follows that φ^a induces a homomorphism of complexes $\varphi^a(D) : \mathcal{A}_Z(D) \to \mathcal{A}_X(\varphi^*(D))$.

Fibred surfaces. Let $\varphi : X \to C$ be a surjective morphism onto a nonsingular projective curve over a perfect field k and as usual fix a flag $\tilde{x} \in \tilde{y} \subset X$ such that $\varphi(\tilde{x}) = \tilde{c} \in C$. Then we have a local and injective homomorphism $\varphi_{\tilde{x}}^{\#} : \mathscr{O}_{C,\tilde{c}} \to \mathscr{O}_{X,\tilde{x}}$ which induces the commutative diagram:

$$\begin{tikzcd}
\mathcal{O}_{\tilde{c}} = \widehat{\mathscr{O}_{C,\tilde{c}}} \arrow[r, "\theta_1"] & \mathcal{O}_{\tilde{x}} \\
\mathscr{O}_{C,\tilde{c}} \arrow[u, hook] \arrow[r, "\varphi_{\tilde{x}}^{\#}"] & \mathscr{O}_{X,\tilde{x}} \arrow[u, hook] .
\end{tikzcd}$$

The map θ_1 is again injective, and to see it one can repeat the same argument used in previous paragraph to show that ψ_3 is injective. It follows that we can induce an homomorphism:

$$\theta_2 : K_{\tilde{c}} \to K'_{\tilde{x}} = \operatorname{Frac} \mathcal{O}_{\tilde{x}} .$$

Let $\mathfrak{z} \in \tilde{y}(\tilde{x})$, then $K'_{\tilde{x}} \subset K_{\tilde{x},\mathfrak{z}}$, so θ_2 gives naturally a homomorphism

$$\varphi_{\tilde{x},\mathfrak{z}} : K_{\tilde{c}} \to K_{\tilde{x},\mathfrak{z}}$$

which induces in turn:

$$\begin{aligned} \varphi_{\tilde{x},\tilde{y}} : K_{\tilde{c}} &\to K_{\tilde{x},\tilde{y}} \\ \alpha_{\tilde{c}} &\mapsto (\varphi_{\tilde{x},\mathfrak{z}}(\alpha_{\tilde{c}}))_{\mathfrak{z} \in \tilde{y}(\tilde{x})} \end{aligned}$$

$$\begin{array}{rcl} \varphi'_{\tilde{x},\tilde{y}} : \prod_{c\in C} K_c & \to & K_{\tilde{x},\tilde{y}} \\ (\alpha_c)_{c\in C} & \mapsto & \varphi_{\tilde{x},\tilde{y}}(\alpha_{\tilde{c}}) \end{array}$$

Finally we get the required homomorphism

$$\begin{array}{rcl} \varphi^a : \prod_{c\in C} K_c & \to & \prod_{\substack{x\in y,\\ y\subset X}} K_{x,y} \\ (\alpha_c)_{c\in C} & \mapsto & (\varphi'_{x,y}(\alpha_c))_{\substack{x\in y,\\ y\subset X}} \end{array}$$

which maps $k(C)$ into $k(X)$ and by restriction gives the required homomorphism of ST- rings $\varphi^a : \mathbf{A}_C \to \mathbf{A}_X$. The one-dimensional adelic complex is shorter than the two-dimensional one, but note that $\varphi_{\tilde{x},\mathfrak{z}}(k(C)\mathcal{O}_{\tilde{c}}) \subseteq k(X)\mathcal{O}_{\tilde{x}} = K_{\tilde{x}}$, therefore we have a well defined homomorphism:

$$\begin{array}{rcl} \theta^a : k(C) \oplus \prod_{c\in C} \mathcal{O}_c & \to & A_{02} = \mathbf{A}_X \cap \prod_{x\in X} K_x \\ (f, (\alpha_c)_c) & \mapsto & \varphi^a(f - (\alpha_c)_c) \end{array}$$

and we have an homomorphism of adelic complexes

$$\begin{tikzcd}
 & & & \mathbf{A}_X \arrow[d, equal] \\
\mathcal{A}_X : & A_0 \oplus A_1 \oplus A_2 \arrow[r] & A_{01} \oplus A_{02} \oplus A_{12} \arrow[r] & A_{012} \\
\mathcal{A}_C : \arrow[u, "\varphi^a"] & 0 \arrow[r] \arrow[u] & k(C) \oplus \prod_{c\in C} \mathcal{O}_c \arrow[r] \arrow[u, "{(0,\theta^a,0)}"'] & \mathbf{A}_C \ . \arrow[u, "\varphi^a"']
\end{tikzcd} \tag{3.13}$$

Again we have a nice behaviour of φ^a with respect to pullback of divisors, when φ^a is flat:

Proposition 3.41. *Assume that* φ *is flat and fix a divisor* $D \in \mathrm{Div}(C)$*, then the homomorphism* $\varphi^a : \mathbf{A}_C \to \mathbf{A}_X$ *induces by restriction a homomorphism* $\varphi^a(D) : \mathbf{A}_C(D) \to \mathbf{A}_X(\varphi^*(D))$.

Proof. Assume $D = \sum_{c\in C} n_c[c]$, then $\varphi^*(D) = \sum_{\substack{y\subset X,\\ y\mapsto c}} e_{y|c} n_c[y]$. Fix a flag $x \in y \subset X$ such that $f(x) = f(y) = c$, and consider the embedding $\iota_{x,y} : \mathscr{O}_{X,x} \hookrightarrow \mathscr{O}_{X,y}$. It is easy to see that $\varphi^{\#}_y = \iota_{x,y} \circ \varphi^{\#}_x$ and this actually implies that $\varphi^{\#}_y$ is the restriction of $\varphi_{x,\mathfrak{z}}$ to $\mathscr{O}_{X,y}$, if $\mathfrak{z} \in y(x)$. At this point we can conclude the proof by using the same strategy used in the proof of proposition 3.40. □

Once again the homomorphism φ^a induces a homomorphism of complexes $\varphi^a(D) : \mathcal{A}_C(D) \to \mathcal{A}_X(\varphi^*(D))$.

Curves inside surfaces. Fix a nonsingular curve $j : \overline{y} \hookrightarrow X$ inside a nonsingular surface X. For any $x \in \tilde{y}$, let $\hat{\mathfrak{p}}_{y,x}$ be the unique local branch of y at x, then we have the canonical residue map:

$$\pi_{x,\tilde{y}} : \mathcal{O}_{x,\tilde{y}} \to E_{x,\tilde{y}}$$

that restricted respectively to $\mathcal{O}_x$ and $\mathcal{O}_{\tilde{y}}$ gives the following canonical homomorphisms:

$$\pi_{x,\tilde{y}}|_{\mathcal{O}_x} : \mathcal{O}_x \to \widehat{\mathscr{O}_{\tilde{y},x}} \tag{3.14}$$

$$\pi_{x,\tilde{y}}|_{\mathcal{O}_{\tilde{y}}} : \mathcal{O}_{\tilde{y}} \to k(\tilde{y}) \tag{3.15}$$

Now we can define the map:

$$\begin{aligned} j^a : \prod_{\substack{x\in y,\\ y\subset X}} \mathcal{O}_{x,y} &\to \prod_{x\in\tilde{y}} E_{x,\tilde{y}} \\ (\alpha_{x,y})_{x,y} &\mapsto (\pi_{x,\tilde{y}}(\alpha_{x,\tilde{y}}))_{x\in\tilde{y}} \end{aligned}$$

that, again, by restriction gives homomorphism of ST-rings:

$$j^a : A_{12} = \mathbf{A}_X \cap \prod_{\substack{x\in y,\\ y\subset X}} \mathcal{O}_{x,y} \to \mathbf{A}_{\tilde{y}}$$

The map j^a together with equations (3.14) and (3.15) induces the following homomorphism of adelic complexes:

$$\begin{tikzcd}
 & & & \mathbf{A}_X \arrow[d, equal] \\
\mathcal{A}_X : \arrow[d, "j^a"] & A_0 \oplus A_1 \oplus A_2 \arrow[r] \arrow[d] & A_{01} \oplus A_{02} \oplus A_{12} \arrow[r] \arrow[d] & A_{012} \arrow[d] \\
\mathcal{A}_{\tilde{y}} : & k(\tilde{y}) \oplus \prod_{x\in\tilde{y}} \widehat{\mathscr{O}_{\tilde{y},x}} \arrow[r] & \mathbf{A}_{\tilde{y}} \arrow[r] & 0 \, .
\end{tikzcd} \tag{3.16}$$

where the nontrivial vertical arrows are defined as:

$$A_0 \oplus A_1 \oplus A_2 \ni (\alpha_0, \alpha_1, \alpha_2) \mapsto (j^a(\alpha_1), j^a(\alpha_2)) \in k(\tilde{y}) \oplus \prod_{x\in\tilde{y}} \widehat{\mathscr{O}_{\tilde{y},x}} \, .$$

$$A_{01} \oplus A_{02} \oplus A_{12} \ni (\alpha_{01}, \alpha_{02}, \alpha_{12}) \mapsto j^a(\alpha_{12}) \in \mathbf{A}_{\tilde{y}} \, .$$

Remark 3.42. Note that both diagrams (3.13) and (3.16) relate the adelic complex a of surface and the adelic complex of a curve. But the latter is shorter than the former, so we need to add a zero on its left or on its right. When the curve is the codomain of the main morphism, we added a zero on the left; when the curve is the domain we added a zero on the right.

3.2.7 Idelic intersection theory

In subsection 2.2.4 we gave a very simple idelic interpretation of the degree of a divisor on a curve C. The fundamental ingredients where the following two maps:

- A surjective homomorphism $p_C : \mathbf{A}_C^\times \to \operatorname{Div}(C)$.
- A notion of "degree" at levels of ideles $\operatorname{ideg} : \mathbf{A}_C^\times \to \mathbb{Z}$.

which induce the following two commutative diagrams:

$$\begin{tikzcd}
\mathbf{A}_C^\times \arrow[d, "p_C"] \arrow[dr, "\operatorname{ideg}"] & \\
\operatorname{Div}(C) \arrow[r, "\deg"] & \mathbb{Z}
\end{tikzcd}
\qquad
\begin{tikzcd}
H^1(\mathcal{A}_C^\times) \arrow[d, "\cong"] \arrow[dr, "\operatorname{ideg}"] & \\
\operatorname{Pic}(C) \arrow[r, "\deg"] & \mathbb{Z}
\end{tikzcd}$$

In the case of a surface X, intersection theory can be described in the following simple way:

$$\begin{tikzcd}
\operatorname{Div}(X) \times \operatorname{Div}(X) \arrow[r, "\cdot"] \arrow[d, two heads] & \mathbb{Z} \\
\operatorname{CH}^1(X) \times \operatorname{CH}^1(X) \arrow[ur] &
\end{tikzcd}$$

The idelic interpretation of the intersection theory consists in constructing a commutative diagram:

$$\begin{tikzcd}
\ker(d^1_\times) \times \ker(d^1_\times) \arrow[r, "?"] \arrow[d, "p \times p"] & K_2(\mathbf{A}_X) \arrow[d, "?"] \\
\operatorname{Div}(X) \times \operatorname{Div}(X) \arrow[r, "\cdot"] \arrow[d, two heads] & \mathbb{Z} \\
\operatorname{CH}^1(X) \times \operatorname{CH}^1(X) \arrow[ur] &
\end{tikzcd} \tag{3.17}$$

where:

- $d^1_\times$ is the map of the idelic complex (3.5).
- The map $p : \ker(d^1_\times) \to \operatorname{Div}(X)$ is defined as:

$$\begin{aligned}
p := p_{01} \circ \pi_1 : \quad \ker(d^1_\times) &\to \operatorname{Div}(X) && (3.18)\\
(\alpha, \beta, \alpha^{-1}\beta^{-1}) &\mapsto \sum_{y \subset X} v_y(\alpha_{x,y})[y] && (3.19)
\end{aligned}$$

> Remember that the maps $p_{01} : A^{\times}_{01} \to \mathrm{Div}(X)$ and $\pi_1 : \ker(d^1_{\times}) \to A^{\times}_{01}$ were defined in the proof of theorem 3.35.

Clearly the main point is to define properly the maps denoted with the question mark.

We start to work locally at a fixed flag $x \in y \subset X$, and then we extend the argument globally. Let $\mathfrak{z} \in y(x)$ a local branch, then we have the following maps:

$$\begin{tikzcd}
K^{\times}_{x,\mathfrak{z}} \times K^{\times}_{x,\mathfrak{z}} \arrow[r, "\{\,,\,\}"] \arrow[rrr, bend left, "(\,,\,)_{x,\mathfrak{z}}"] & K_2(K_{x,\mathfrak{z}}) \arrow[r, "\partial_{x,\mathfrak{z}}"] & E^{\times}_{x,\mathfrak{z}} \arrow[r, "\widetilde{v}_{x,\mathfrak{z}}"] & \mathbb{Z}
\end{tikzcd} \tag{3.20}$$

where:

- By using the notation of appendix G.1, $\{\,,\,\}$ is the symbol map and $\partial_{x,\mathfrak{z}} := \partial_2$ is the boundary map. Remember that the composition $\partial_{x,\mathfrak{z}} \circ \{\,,\,\} = (\,,\,)_{K_{x,\mathfrak{z}}}$ is just the tame symbol associated to the complete discrete valuation field $K_{x,\mathfrak{z}}$.
- $\widetilde{v}_{x,\mathfrak{z}} := [k_{\mathfrak{z}}(x) : k] v^{(1)}_{x,\mathfrak{z}}$, where $v^{(1)}_{x,\mathfrak{z}}$ is the complete discrete valuation on $E_{x,\mathfrak{z}}$.
- We put $(\,,\,)_{x,\mathfrak{z}} := \widetilde{v}_{x,\mathfrak{z}} \circ \partial_{x,\mathfrak{z}} \circ \{\,,\,\} : K^{\times}_{x,\mathfrak{z}} \times K^{\times}_{x,\mathfrak{z}} \to \mathbb{Z}$.

By using finite "sums" over all local branches in $y(x)$, diagram (3.20) can be extended for $K_{x,y}$:

$$\begin{tikzcd}
K^{\times}_{x,y} \times K^{\times}_{x,y} \arrow[r, "(\{\,,\,\})_{\mathfrak{z}}"] \arrow[rrr, bend left, "(\,,\,)_{x,y} := \sum_{\mathfrak{z} \in y(x)} (\,,\,)_{x,\mathfrak{z}}"] & K_2(K_{x,y}) \arrow[r, "\partial_{x,y} := (\partial_{x,\mathfrak{z}})_{\mathfrak{z}}"] & E^{\times}_{x,y} \arrow[r, "\widetilde{v}_{x,y} := \sum_{\mathfrak{z}} \widetilde{v}_{x,\mathfrak{z}}"] & \mathbb{Z}
\end{tikzcd} \tag{3.21}$$

Now we study the main properties of the 2-dimensional symbol $(\,,\,)_{x,y}$. The first thing to notice is that it is closely related to residues and differentials:

Proposition 3.43. *Let $a, b \in K_{x,\mathfrak{z}}$, then*

$$\mathrm{res}_{x,\mathfrak{z}} \left(\frac{da}{a} \wedge \frac{db}{b} \right) = (a, b)_{x,\mathfrak{z}} \, .$$

In particular if x is a non singular point on a curve y and $a, b \in K_{x,y}$, then

$$\operatorname{res}_{x,y}\left(\frac{da}{a} \wedge \frac{db}{b}\right) = (a,b)_{x,y}\,.$$

Proof. Clearly it is enough to show a proof just of the first equation. Consider the identification $K_{x,\mathfrak{z}} = k_{\mathfrak{z}}(x)((u))((t))$ then for $m, n \in \mathbb{Z}$ we can write

$$a = t^m(a_0 + a_1 t + a_2 t^2 + \ldots) \quad \text{for } a_0 \in k_{\mathfrak{z}}(x)((u))^{\times};$$

$$b = t^n(b_0 + b_1 t + b_2 t^2 + \ldots) \quad \text{for } b_0 \in k_{\mathfrak{z}}(x)((u))^{\times}.$$

So we have:

$$\frac{da}{a} = \frac{mt^{m-1}dt + t^m(da_0 + d(a_1 t) + d(a_2 t^2) + \ldots)}{t^m(a_0 + a_1 t + a_2 t^2 + \ldots)} =$$

$$= m\frac{dt}{t} + \frac{da_0 + d(a_1 t) + d(a_2 t^2) + \ldots}{a_0 + a_1 t + a_2 t^2 + \ldots}$$

If we put

$$\alpha_0 + \alpha_1 t + \alpha_2 t^2 + \ldots := \frac{1}{a_0 + a_1 t + a_2 t^2 + \ldots}$$

where obviously $\alpha_0 = \frac{1}{a_0}$, then we obtain

$$\frac{da}{a} = m\frac{dt}{t} + \sum_{k \geq 0}\left(\sum_{j \geq 0} \alpha_j t^j\right) d\left(a_k t^k\right) = m\frac{dt}{t} + \frac{da_0}{a_0} + \text{“higher terms”}\,. \quad (3.22)$$

In the same way we can show that:

$$\frac{db}{b} = n\frac{dt}{t} + \frac{db_0}{b_0} + \text{“higher terms”}\,. \quad (3.23)$$

By using equations (3.22) and (3.23)we obtain the following expression:

$$\operatorname{res}_{x,\mathfrak{z}}\left(\frac{da}{a} \wedge \frac{db}{b}\right) = \operatorname{res}_{x,\mathfrak{z}}\left(\frac{dt}{t} \wedge \left(m\frac{db_0}{b_0} - n\frac{da_0}{a_0}\right)\right) = \operatorname{res}_{x,\mathfrak{z}}\left(\left(n\frac{da_0}{a_0} - m\frac{db_0}{b_0}\right) \wedge \frac{dt}{t}\right) \quad (3.24)$$

But again if

$$a_0 = u^{v^{(1)}_{x,\mathfrak{z}}(a_0)}(\alpha_0 + \alpha_1 u + \alpha_2 u^2 + \ldots) \quad \text{for } \alpha_0 \in k_{\mathfrak{z}}(x)^{\times};$$

$$b_0 = u^{v^{(1)}_{x,\mathfrak{z}}(b_0)}(\beta_0 + \beta_1 u + \beta_2 u^2 + \ldots) \quad \text{for } \beta_0 \in k_{\mathfrak{z}}(x)^{\times};$$

then we can write

$$\frac{da_0}{a_0} = v^{(1)}_{x,\mathfrak{z}}(a_0)\frac{du}{u} + \text{“higher terms”}$$

$$\frac{db_0}{b_0} = v^{(1)}_{x,\mathfrak{z}}(b_0)\frac{du}{u} + \text{“higher terms”}$$

At this point equation (3.24) assumes the following very simple form:

$$\mathrm{res}_{x,\mathfrak{z}}\left(\frac{da}{a}\wedge\frac{db}{b}\right) = \mathrm{Tr}_{k_\mathfrak{z}(x)|k}(nv^{(1)}_{x,\mathfrak{z}}(a_0) - mv^{(1)}_{x,\mathfrak{z}}(b_0)) = [k_\mathfrak{z}(x):k]\, v^{(1)}_{x,\mathfrak{z}}\left(\frac{a_0^n}{b_0^m}\right) =$$

$$= [k_\mathfrak{z}(x):k]\, v^{(1)}_{x,\mathfrak{z}}\left((-1)^{mn}\frac{a_0^n}{b_0^m}\right) .$$

Finally note that

$$\frac{a^n}{b^m} = \frac{a_0^n}{b_0^m} + \text{“higher terms”} \in k_\mathfrak{z}(x)[[u,t]]^\times$$

therefore we can conclude that

$$\mathrm{res}_{x,\mathfrak{z}}\left(\frac{da}{a}\wedge\frac{db}{b}\right) = [k_\mathfrak{z}(x):k]\, v^{(1)}_{x,\mathfrak{z}}\left((-1)^{mn}\overline{a^n b^{-m}}\right) = (a,b)_{x,\mathfrak{z}}.$$

□

Proposition 3.44. *The pairing* $(\ ,\ )_{x,y}$ *is a skew-symmetric bilinear form on* $K^\times_{x,y}$ *satisfying the following properties:*

1. *If* $r, s \in K^\times_x$*, then* $\sum_{y\ni x}(r,s)_{x,y} = 0$.
2. *If* $r, s \in K^\times_y$*, then* $\sum_{x\in y}(r,s)_{x,y} = 0$.
3. *If* $r, s \in \mathcal{O}^\times_{x,y}$*, then* $(r,s)_{x,y} = 0$.

Proof. (3) is obvious from definitions. For (1) and (2) see [50, Proposition 1]. □

Let's go back to the undetermined maps of diagram (3.17). Consider $a = (\alpha, \beta, \alpha^{-1}\beta^{-1}), b = (\gamma, \delta, \gamma^{-1}\delta^{-1}) \in \ker(d^1_\times) \subset A^\times_{01} \oplus A^\times_{02} \oplus A^\times_{12}$ and put:

$$\begin{aligned} \theta_1 : \ker(d^1_\times) \times \ker(d^1_\times) &\to K_2(\mathbf{A}_X) \\ (a,b) &\mapsto (\{\gamma_{x,y}, \beta_{x,y}\})_{\substack{x\in y,\\ y\subset X}} \end{aligned} \tag{3.25}$$

$$\begin{aligned} \theta_2 : K_2(\mathbf{A}_X) &\to \mathbb{Z} \\ c &\mapsto \sum_{\substack{x \in y, \\ y \subset X}} v_{x,y}(\partial_{x,y}(c_{x,y})) \end{aligned} \tag{3.26}$$

Definition 3.45. The composition

$$\langle\ ,\ \rangle_i := \theta_2 \circ \theta_1 : \ker(d^1_\times) \times \ker(d^1_\times) \to \mathbb{Z}$$

Can be written explicitly in the following way

$$\langle a, b \rangle_i = \sum_{\substack{x \in y, \\ y \subset X}} (\gamma_{x,y}, \beta_{x,y})_{x,y} \tag{3.27}$$

and it is called the *idelic intersection pairing.*

Note that the sums in equations (3.26) and (3.27) are finite because of idelic restricted product and proposition 3.44(3).

Theorem 3.46. *The pairing* $\langle\ ,\ \rangle_i$ *satisfies the following properties:*

(1) *It is bilinear and symmetric.*

(2) *Let* $a, b, a', b' \in \ker(d^1_\times)$ *such that* $p(a) = p(a')$ *and* $p(b) = p(b')$*, then* $\langle a, b \rangle_i = \langle a', b' \rangle_i$.

(3) *It descends naturally to a pairing* $H^1(\mathcal{A}_X^\times) \times H^1(\mathcal{A}_X^\times) \to \mathbb{Z}$.

Proof. Let's fix $a = (\alpha, \beta, \alpha^{-1}\beta^{-1})$, $b = (\gamma, \delta, \gamma^{-1}\delta^{-1}) \in \ker(d^1_\times)$.

(1) Bilinearity is clear by construction, so we have to prove just symmetry. For any flag $x \in y$ we have that: $\alpha_{x,y}^{-1}\beta_{x,y}^{-1} \in \mathcal{O}_{x,y}^\times$ and $\gamma_{x,y}^{-1}\delta_{x,y}^{-1} \in \mathcal{O}_{x,y}^\times$ so by proposition 3.44(3) we have that:

$$0 = \sum_{\substack{x \in y, \\ y \subset X}} (\alpha_{x,y}^{-1}\beta_{x,y}^{-1}, \gamma_{x,y}^{-1}\delta_{x,y}^{-1})_{x,y} =$$

$$= \sum_{\substack{x \in y, \\ y \subset X}} (\alpha_{x,y}, \gamma_{x,y})_{x,y} + \sum_{\substack{x \in y, \\ y \subset X}} (\alpha_{x,y}, \delta_{x,y})_{x,y} + \sum_{\substack{x \in y, \\ y \subset X}} (\beta_{x,y}, \gamma_{x,y})_{x,y} + \sum_{\substack{x \in y, \\ y \subset X}} (\beta_{x,y}, \delta_{x,y})_{x,y} =$$

$$= \underbrace{\sum_{y \subset X} \left(\sum_{x \in y} (\alpha_{x,y}, \gamma_{x,y})_{x,y} \right)}_{(i)} + \langle b, a \rangle_i - \langle a, b \rangle_i + \underbrace{\sum_{x \in X} \left(\sum_{y \ni x} (\beta_{x,y}, \delta_{x,y})_{x,y} \right)}_{(ii)} .$$

Note that in the above equalities we have used the the skew symmetry of the symbol $(\ ,\ )_{x,y}$ to write $(\beta_{x,y}, \gamma_{x,y})_{x,y} = -(\gamma_{x,y}, \beta_{x,y})_{x,y}$. Now, both the summations (i) and (ii) are equal to 0 thanks to 3.44(1) and 3.44(2), so the claim is proved.

(2) Let $a' = (\alpha', \beta', (\alpha')^{-1}(\beta')^{-1})$ and $b' = (\gamma', \delta', (\gamma')^{-1}(\delta')^{-1})$. Since $p(a) = p(a')$ and $p(b) = p(b')$, then $v_y(\alpha_{x,y}) = v_y(\alpha'_{x,y})$ and $v_y(\gamma_{x,y}) = v_y(\gamma'_{x,y})$. This means that $\gamma'_{x,y} = f_{x,y}\gamma_{x,y}$ and $\alpha'_{x,y} = g_{x,y}\alpha_{x,y}$ for $f_{x,y}, g_{x,y} \in \mathcal{O}_y^\times$ (for any $x \in y$). Then we have the following chain of equalities which depends on claim (1):

$$\langle a', b' \rangle_i = \sum_{\substack{x\in y,\\ y\subset X}} (f_{x,y}\gamma_{x,y}, \beta'_{x,y})_{x,y} = \sum_{\substack{x\in y,\\ y\subset X}} (f_{x,y}, \beta'_{x,y})_{x,y} + \sum_{\substack{x\in y,\\ y\subset X}} (\gamma_{x,y}, \beta'_{x,y})_{x,y} =$$

$$= \sum_{\substack{x\in y,\\ y\subset X}} (f_{x,y}, \beta'_{x,y})_{x,y} + \langle a', b \rangle_i = \sum_{\substack{x\in y,\\ y\subset X}} (f_{x,y}, \beta'_{x,y})_{x,y} + \langle b, a' \rangle_i =$$

$$= \sum_{\substack{x\in y,\\ y\subset X}} (f_{x,y}, \beta'_{x,y})_{x,y} + \sum_{\substack{x\in y,\\ y\subset X}} (g_{x,y}\alpha_{x,y}, \delta_{x,y})_{x,y} =$$

$$= \sum_{\substack{x\in y,\\ y\subset X}} (f_{x,y}, \beta'_{x,y})_{x,y} + \sum_{\substack{x\in y,\\ y\subset X}} (g_{x,y}, \delta_{x,y})_{x,y} + \sum_{\substack{x\in y,\\ y\subset X}} (\alpha_{x,y}, \delta_{x,y})_{x,y} =$$

$$= \underbrace{\sum_{\substack{x\in y,\\ y\subset X}} (f_{x,y}, \beta'_{x,y})_{x,y}}_{(i)} + \underbrace{\sum_{\substack{x\in y,\\ y\subset X}} (g_{x,y}, \delta_{x,y})_{x,y}}_{(ii)} + \langle a, b \rangle_i \ .$$

We have to show that the terms (i) and (ii) in the last line are 0. Since $(\alpha')^{-1}_{x,y}(\beta')^{-1}_{x,y} \in \mathcal{O}^\times_{x,y}$, we have:

$$0 = \sum_{\substack{x\in y,\\ y\subset X}} (f_{x,y}, (\alpha')^{-1}_{x,y}(\beta')^{-1}_{x,y})_{x,y} = \sum_{\substack{x\in y,\\ y\subset X}} (f_{x,y}, \alpha'_{x,y})_{x,y} + \sum_{\substack{x\in y,\\ y\subset X}} (f_{x,y}, \beta'_{x,y})_{x,y} =$$

$$\overset{\text{prop. } 3.44(2)}{=} \sum_{\substack{x\in y,\\ y\subset X}} (f_{x,y}, \beta'_{x,y})_{x,y} \ .$$

For (ii) the argument is the same.

(3) Let $a, b \in \mathrm{im}(d^0_\times)$. It means that $\alpha = rs^{-1}$, $\beta = tr^{-1}$, $\gamma = uv^{-1}$, $\delta = zu^{-1}$ for $r, u \in A_0^\times = K^\times$, $s, v \in A_1^\times$ and $t, z \in A_2^\times$. So:

$$\langle a, b \rangle_i = \sum_{\substack{x \in y, \\ y \subset X}} (uv_{x,y}^{-1}, t_{x,y} r^{-1})_{x,y} =$$

$$= \sum_{\substack{x \in y, \\ y \subset X}} (u, t_{x,y})_{x,y} + \sum_{\substack{x \in y, \\ y \subset X}} (u, r^{-1})_{x,y} + \sum_{\substack{x \in y, \\ y \subset X}} (v_{x,y}^{-1}, t_{x,y})_{x,y} + \sum_{\substack{x \in y, \\ y \subset X}} (v_{x,y}^{-1}, r^{-1})_{x,y} \,. \quad (3.28)$$

Now it is enough to use proposition 3.44, as we did previously, to show that each term of equation (3.28) is 0.

□

The following lemma allows to calculate the local intersection index $i_x(D, E)$ of two divisors in terms of valuations:

Lemma 3.47. *Assume that X is an Noetherian, integral, regular scheme of dimension 2. Let D, E two prime divisors on X and let $x \in D \cap E$ a nonsingular point for both D and E. Moreover let $d_x, e_x \in \mathscr{O}_{X,x}$ be the local equations at x of D and E respectively. Then we have the equality:*

$$v_{x,D}^{(1)}(\overline{e_x}) = i_x(D, E)$$

where $v_{x,D}^{(1)} : E_{x,D}^\times \to \mathbb{Z}$ is the one dimensional valuation and $\overline{e_x} \in E_{x,D}^\times$ is the natural projection through the map $\mathcal{O}_{x,D} \to E_{x,D}^\times$.

Proof. Put $y = D$ and $v = v_{x,D}^{(1)}$. First of all notice that $\mathscr{O}_{X,x} \subseteq \mathscr{O}_{X,y}$, therefore $\overline{e_x} \in \mathscr{O}_{y,x}$ and it is the image on the natural map $\mathscr{O}_{X,x} \to \mathscr{O}_{y,x} \subset k(y)$. We have to show that $v(\overline{e_x}) = \mathrm{length}_{\mathscr{O}_{X,x}} \frac{\mathscr{O}_{X,x}}{(d_x, e_x)}$, but we know that $\mathscr{O}_{y,x} = \frac{\mathscr{O}_{X,x}}{d_x \mathscr{O}_{X,x}}$, thus

$$\mathrm{length}_{\mathscr{O}_{X,x}} \frac{\mathscr{O}_{X,x}}{(d_x, e_x)} = \mathrm{length}_{\mathscr{O}_{y,x}} \frac{\mathscr{O}_{y,x}}{\overline{e_x} \mathscr{O}_{y,x}} = v(\overline{e_x}) \,.$$

□

Lemma 3.48. *Assume that X is an Noetherian, integral, regular scheme of dimension 2. Let D, E two prime divisors on X and let $x \in D \cap E$ a nonsingular point for both D and E. Moreover let $d_x, e_x \in \mathscr{O}_{X,x}$ be the local equations at x of D and E respectively. Then we have the equality:*

$$v_{x,D}^{(1)}(\overline{e_x}) = i_x(D, E)$$

where $v_{x,D}^{(1)} : E_{x,D}^\times \to \mathbb{Z}$ is the one dimensional valuation and $\overline{e_x} \in E_{x,D}^\times$ is the natural projection through the map $\mathcal{O}_{x,D} \to E_{x,D}^\times$.

Proof. Put $y = D$ and $v = v^{(1)}_{x,D}$. First of all notice that $\mathscr{O}_{X,x} \subseteq \mathscr{O}_{X,y}$, therefore $\overline{e_x} \in \mathscr{O}_{y,x}$ and it is the image on the natural map $\mathscr{O}_{X,x} \to \mathscr{O}_{y,x} \subset k(y)$. We have to show that $v(\overline{e_x}) = \operatorname{length}_{\mathscr{O}_{X,x}} \frac{\mathscr{O}_{X,x}}{(d_x,e_x)}$, but we know that $\mathscr{O}_{y,x} = \frac{\mathscr{O}_{X,x}}{d_x\mathscr{O}_{X,x}}$, thus

$$\operatorname{length}_{\mathscr{O}_{X,x}} \frac{\mathscr{O}_{X,x}}{(d_x, e_x)} = \operatorname{length}_{\mathscr{O}_{y,x}} \frac{\mathscr{O}_{y,x}}{\overline{e_x}\mathscr{O}_{y,x}} = v(\overline{e_x})\,.$$

□

There is another simpler expression for the pairing $\langle\, ,\, \rangle_i$:

$$\begin{aligned} -\sum_{\substack{x\in y,\\ y\subset X}} (\alpha_{x,y}, \gamma^{-1}_{x,y}\delta^{-1}_{x,y})_{x,y} &= \sum_{\substack{x\in y,\\ y\subset X}} (\alpha_{x,y}, \gamma_{x,y})_{x,y} + \sum_{\substack{x\in y,\\ y\subset X}} (\alpha_{x,y}, \delta_{x,y})_{x,y} = \\ &=^{\text{prop. } 3.44(2)} 0 + \sum_{\substack{x\in y,\\ y\subset X}} (\alpha_{x,y}, \delta_{x,y})_{x,y} = \langle b, a\rangle_i = \langle a, b\rangle_i \end{aligned} \tag{3.29}$$

More in detail, thanks to equation (3.29) we can write:

$$\begin{aligned} \langle a, b\rangle_i &= \sum_{\substack{x\in y,\\ y\subset X}} -\widetilde{v}_{x,y}\left((\alpha_{x,y}, \gamma^{-1}_{x,y}\delta^{-1}_{x,y})_{K_{x,y}}\right) = \\ &= \sum_{\substack{x\in y,\\ y\subset X}} v_{x,y}(\alpha_{x,y})\widetilde{v}_{x,y}\left(\overline{\gamma^{-1}_{x,y}\delta^{-1}_{x,y}}\right) = \sum_{\substack{x\in y,\\ y\subset X}} v_y(\alpha_{x,y})\widetilde{v}_{x,y}\left(\overline{\gamma^{-1}_{x,y}\delta^{-1}_{x,y}}\right) \end{aligned} \tag{3.30}$$

where in the last line we used the fact that $v_{x,y} = (v_{x,\mathfrak{z}})_{\mathfrak{z}\in y(x)}$ extends the valuation v_y on K_y.

Theorem 3.49. *If $a, b \in \ker(d^1_\times)$ such that $D = p(a)$ and $E = p(b)$ are two nonsingular prime divisors on X which meet transversally, then: $\langle a, b\rangle_i = D.E$*

Proof. Let $a = (\alpha, \beta, \alpha^{-1}\beta^{-1})$, $b = (\gamma, \delta, \gamma^{-1}\delta^{-1})$. By theorem 3.46(2) we can choose $\delta_{x,y} \in K_x$ in the following way: $\delta_{x,y} = 1$ if $x \notin D \cap E$ and $\delta_{x,y} = t^{-1}_x$, where $t_x \in \mathscr{O}_{X,x}$ is the local equation of E at x, if $x \in D \cap E$. Note that $v_y(\alpha_{x,y}) = 0$ if $y \neq D$ and $v_D(\alpha_{x,D}) = 1$ since $p(a) = D$, therefore by using equation (3.30) we obtain:

$$\langle a, b\rangle_i = \sum_{x\in D} \widetilde{v}_{x,D}\left(\overline{\gamma^{-1}_{x,D}\delta^{-1}_{x,D}}\right)\,.$$

But $\gamma^{-1}_{x,D} \in \mathcal{O}^\times_D$ and $\delta^{-1}_{x,D} \in \mathscr{O}_{X,x}$, therefore

$$\langle a, b\rangle_i = \sum_{x\in D} \widetilde{v}_{x,D}\left(\overline{\gamma^{-1}_{x,D}}\right) + \sum_{x\in D} \widetilde{v}_{x,D}\left(\overline{\delta^{-1}_{x,D}}\right)\,.$$

Note that $\sum_{x\in D}\widetilde{v}_{x,D}\left(\overline{\gamma_{x,D}^{-1}}\right)$ is the degree of a principal divisor on the curve D, so it is 0 and thanks to the choice of $\delta_{x,y}$ we can write:

$$\langle a,b\rangle_i=\sum_{x\in D\cap E}\widetilde{v}_{x,D}(\bar{t}_x)=\sum_{x\in D\cap E}[k(x):k]=D.E$$

where in the last equation we used lemma 3.48 and the hypothesis of transversal intersection between D and E. □

The following corollary says that the idelic representation of the intersection pairing is now complete:

Corollary 3.50. *Diagram (3.17) is commutative.*

Proof. For any two divisors $D,E\in\mathrm{Div}(X)$ define the pairing:

$$\Theta(D,E):=\langle a',b'\rangle_i$$

for a choice of $a',b'\in\ker(d^1_\times)$ such that $p(a')=D$ and $p(b')=E$. By theorem 3.46(2) Θ is well defined and moreover by 3.46(1), 3.46(3) and 3.49 we can conclude that $\Theta(D,E)=D.E$. Thus for any $a,b\in\ker(d^1_\times)$ we have that:

$$\langle a,b\rangle_i=\Theta(p(a),p(b))=p(a).p(b)\,.$$

□

3.2.8 Adelic intersection theory

We saw in section 2.2.4 that for algebraic curves it is possible to express the degree of a divisor D in terms of the characteristic of the one dimensional adelic complex:

$$\deg(D)=\chi(\mathcal{A}_X(D))-\chi(\mathcal{A}_X(0))$$

Similarly for the algebraic surface X we have the notion of adelic intersection number

Definition 3.51. Let $D,E\in\mathrm{Div}(X)$, the *adelic intersection number* is:

$$[D,E]_a:=\chi(\mathcal{A}_X(0))-\chi(\mathcal{A}_X(-D))-\chi(\mathcal{A}_X(-E))-\chi(\mathcal{A}_X(-D-E))\,.$$

Thanks to definition D.40 and theorem 3.34 it is evident to notice that

$$[D,E]_a=[D,E] \tag{3.31}$$

which means that the adelic intersection number actually calculates the ordinary intersection number.

In [19, 4], equality (3.31) is proved without using theorem 3.34 about cohomology. The proof uses only the moving lemma, 1-dimensional adelic theory and the following important relationship between the 2-dimensional complex and the 1-dimensional adelic complex:

Proposition 3.52. *Let $C \subset X$ be an integral curve and let D be a divisor not containing C in its support, then:*

$$\chi(\mathcal{A}_X(D)) = \chi(\mathcal{A}_X(D - C)) + \chi(\mathcal{A}_C(D|_C))$$

where $\mathcal{A}_C(D|_C)$ is the one dimensional complex on the curve C relative to the restricted divisor $D|_C \in \mathrm{Div}(C)$.

Proof. See [19, 4, lemma]. □

Let $\mathcal{K}$ be the canonical divisor of X then the *adelic Riemann-Roch* theorem is given by the following equality (true by equation (3.31)):

$$[-D, D-\mathcal{K}] = \chi(\mathcal{A}_X(0)) - \chi(\mathcal{A}_X(D)) - \chi(\mathcal{A}_X(\mathcal{K}-D)) - \chi(\mathcal{A}_X(\mathcal{K}))\,. \quad (3.32)$$

By applying theorem 3.34 and Serre's duality to equation 3.32 we get the usual Riemann-Roch formula for surfaces.

3.3 Arithmetic surfaces

In this section we fix a number field K and an arithmetic surface $\varphi : X \to B = \operatorname{Spec} O_K$. We use the notation of appendix D. The goal is to develop an adelic theory on X which extends Arakelov geometry. As usual, our philosophical guideline will be the 1-dimensional theory (see section 2.3) and we will see that the main differences with the adelic theory for algebraic surfaces will be of two types:

- *Local*: the 2-dimensional local field attached to a flag $x \in y \subset X$ will be of different nature with respect to geometric case. In fact here we have to distinguish between vertical and horizontal curves and in the former case we will deal with mixed characteristic higher local fields.
- *Global*: the central object involved in the study of Arakelov geometry is the completed surface $\widehat{X}$, so we have to improve the definition of the adelic ring and work with the ring of *completed adeles* denoted by $\mathbf{A}_{\widehat{X}}$, making sure that we include the fibres at infinity in the whole picture (note the analogy with the one dimensional case).

3.3.1 The “double nature” of $K_{x,\mathfrak{z}}$

Let’s fix a flag $x \in y \subset X$, and let $\mathfrak{z} \in y(x)$, then the two dimensional local field $K_{x,\mathfrak{z}}$ with its topology which can be formally constructed exactly as described in section 3.1, indeed as hypotheses we needed just a Noetherian and regular scheme of dimension 2. On the other hand, all the considerations about the structure of $K_{x,\mathfrak{z}}$ are completely different from subsection 3.2.1, since we are not in the category $\mathbf{TLF}^2(k)$ anymore. We distinguish two cases:

y is a vertical curve. If $\varphi(y) = b \in B$, then y is a projective curve over the finite field $k(b)$; we assume that $k(b)$ has characteristic p. $K_{x,\mathfrak{z}}$ has characteristic 0 since we have the embeddings $\mathbb{Q} \subset K \subset K(X) \subset K_{x,\mathfrak{z}}$ and the residue field $E_{x,\mathfrak{z}}$ has characteristic p since $k(b) \subset k(y) \subset E_{x,\mathfrak{z}}$. We conclude that $K_{x,\mathfrak{z}}$ is a two dimensional local field of type $(0, p, p)$ and by the classification theorem we have that $K_{x,\mathfrak{z}}$ is a finite extension of $K_p\{\{t\}\}$ where K_p is a finite extension of $\mathbb{Q}_p$.

y is a horizontal curve. In this case $K_{x,\mathfrak{z}}$ has still characteristic 0, but we have the embedding $K \subseteq k(y)$ given by the surjective map $y \to B$; therefore $E_{x,\mathfrak{z}}$ has characteristic 0. Moreover, if $\varphi(x) = b$, the local homomorphism $\varphi_x^\# : \mathscr{O}_{B,b} \to \mathscr{O}_{X,x}$ induces a field embedding $k(b) \subseteq k(x)$ and this implies that $k_\mathfrak{z}(x)$ has characteristic p. We conclude that $K_{x,\mathfrak{z}}$ is a two dimensional local field of type $(0, 0, p)$ and by the classification theorem we have that $K_{x,\mathfrak{z}} \cong K_p((t))$.

If $\varphi(x) = b$ we have an induced embedding $K_b \hookrightarrow K_{x,\mathfrak{z}}$ (see subsection 2.2.3) for details, so we can conclude that $K_{x,\mathfrak{z}}$ is an arithmetic 2-dimensional local field over K_b and we can apply the local theory developed in section 1.3. The topology on $K_{x,\mathfrak{z}}$ is the usual one obtained by the process (3.1).

Note that in the geometric case propositions 3.10, 3.13 and corollary 3.11 didn’t depend on any parametrization, therefore they remain true also for the field $K_{x,\mathfrak{z}}$ described here (in both cases). The construction of the ST-rings $K_{x,y}$ and $\mathcal{O}_{x,y}$ works in the usual way, so they are obtained by summing $K_{x,\mathfrak{z}}$ and $\mathcal{O}_{x,\mathfrak{z}}$ for $\mathfrak{z} \in y(x)$.

3.3.2 Adeles

One might think that a reasonable definition of the adelic ring $\mathbf{A}_{\widehat{X}}$ can be just $\mathbf{A}_X \oplus \prod_{\sigma \in B_\infty} \mathbf{A}_{X_\sigma}$, where: $\mathbf{A}_X$ is the adelic ring attached to the 2-dimensional scheme X (see remark 3.22) and $\mathbf{A}_{X_\sigma}$ is the one-dimensional adelic ring associated to the Riemann surface X_σ. But this construction would be wrong. When we add a fibre at infinity X_σ to the picture, we have to take in account all possible flags on the completed surface $\widehat{X}$: a point $p \in X_\sigma$ can be seen also as an "intersection point" between a completed horizontal curve and X_σ.

Let y be a curve on X, if y is vertical then $\overline{y} = y$, if y is horizontal, in this section, by $\overline{y}$ we mean:

$$\overline{y} = y \cup \bigcup_{\sigma \in B_\infty} y_\sigma$$

where $y_\sigma = \varphi_\sigma^*(y) \in \mathrm{Div}(X_\sigma)$; by simplicity we also put $y_\infty := \cup_{\sigma\in B_\infty} y_\sigma$, so we have the decomposition $\overline{y} = y \cup y_\infty$. Note that the definition of $\overline{y}$ is coherent with the notion of completed divisor given in appendix D. Any point $p \in X_\sigma$ lies on a completed horizontal curve $\overline{y}$ because we have the map $\varphi_\sigma : X_\sigma \to X_K \subset X$ and points of the generic fibre X_K are in bijective correspondence with horizontal curves. With the above wrong definition of $\mathbf{A}_{\widehat{X}}$, we totally forget about the flags of the type $p \in \overline{y} \subset \widehat{X}$ where y is horizontal and $p \in X_\infty$, but we only add the flags of the type $p \in X_\sigma \subset \widehat{X}$ to the usual geometric picture.

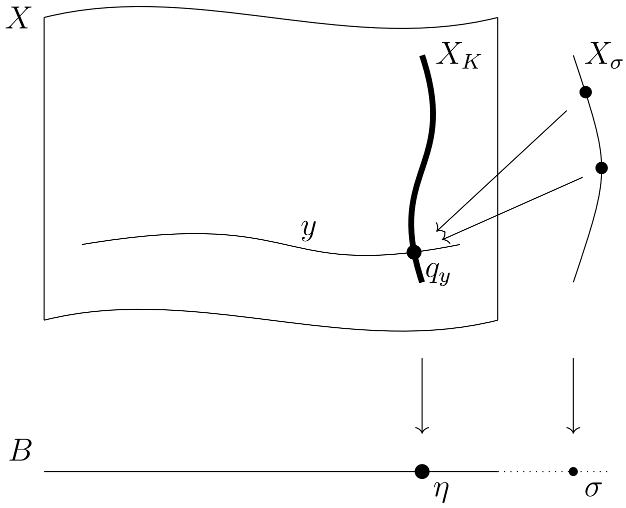

Figure 3.2: A visual example where y_σ is made of two points (marked on the curve X_σ).

For any point $p \in X_\sigma$ we put

$$K_{p,\sigma} := \mathrm{Frac}\left(\widehat{\mathscr{O}_{X_\sigma,p}}\right) .$$

In other words $K_{p,\sigma}$ is a complete discrete valuation field isomorphic to $\mathbb{C}((t))$ and we endow it with the complete discrete valuation topology. The valuation ring of $K_{p,\sigma}$ is $\mathcal{O}_{p,\sigma} \cong \mathbb{C}[[t]]$ and $E_{p,\sigma} \cong \mathbb{C}$ is the residue field.

Remark 3.53. Note that $K_{p,\sigma}$ is an archimedean 2-dimensional local field, but we insist on considering it as a 1-dimensional local field when topology is involved.

From now on, a curve on $\widehat{X}$ will be always a completed curve $\overline{y}$, and a point $x \in \overline{y}$ can be also a point lying on some "part at infinity" y_σ (when y is horizontal), if not explicitly said otherwise. If $p \in \overline{y}$ and $p \in y_\sigma$, we put

$$K_{p,\overline{y}} := K_{p,\sigma},\ \mathcal{O}_{p,\overline{y}} := \mathcal{O}_{p,\sigma},\ E_{p,\overline{y}} := E_{p,\sigma}\,;$$

for any other point $x \in y$ we have:

$$K_{x,\overline{y}} := K_{x,y},\ \mathcal{O}_{x,\overline{y}} := \mathcal{O}_{x,y},\ E_{x,\overline{y}} := E_{x,y},\ k_{\overline{y}}(x) = k_y(x)\,.$$

We are going to define a new adelic ring $\overline{\mathbf{A}}_X$ which will be a subspace of the big product $\prod_{\substack{x\in\overline{y},\\ \overline{y}\subset\widehat{X}}} K_{x,\overline{y}}$.

Remark 3.54. In the product $\prod_{\substack{x\in\overline{y},\\ \overline{y}\subset\widehat{X}}} K_{x,\overline{y}}$ we will find 3 different types of 2-dimensional local field: the ones described in subsection 3.3.1 and fields of the type $\mathbb{C}((t))$. Moreover note that *all* the 2-dimensional valuation fields described in table 1.4 (i.e. all possible 2-dimensional valuation fields) arise from algebraic surfaces or arithmetic surfaces. This means that the theory of higher local fields in dimension 2 is a "geometric theory".

Definition 3.55. for any completed curve $\overline{y}$ let's put:

$$\mathbb{A}_{\overline{y}} := \mathbb{A}_y \oplus \prod_{p\in y_\infty} K_{p,\overline{y}}$$

$$\mathbb{A}^{(0)}_{\overline{y}} := \mathbb{A}^{(0)}_y \oplus \prod_{p\in y_\infty} \mathcal{O}_{p,\overline{y}}$$

Again $\mathbb{A}_{\overline{y}}$ can be thought as a first restricted product performed on the completed curve $\overline{y}$ and we can use the notation:

$$\mathbb{A}_{\overline{y}} = \prod_{x\in\overline{y}}{}' K_{x,\overline{y}}$$

We have a topology on $\mathbb{A}_y$, therefore we can endow $\mathbb{A}_{\overline{y}}$ with the product topology. Like the geometric case we need a second restricted product running on all completed curves.

Definition 3.56. The modified version of $\mathbf{A}_X$ which takes in account the completed curve is:

$$\overline{\mathbf{A}}_X := \left\{ (\beta_{\overline{y}})_{\overline{y}\subset\widehat{X}} \in \prod_{\overline{y}\subset\widehat{X}} \mathbb{A}_{\overline{y}} \colon \beta_{\overline{y}} \in \mathbb{A}_{\overline{y}}^{(0)} \text{ for all but finitely many } \overline{y} \right\} \subset \prod_{\substack{x\in\overline{y},\\ \overline{y}\subset\widehat{X}}} K_{x,\overline{y}}$$

And we also introduce the formal notation

$$\overline{\mathbf{A}}_X = \prod_{\substack{x\in\overline{y}\\ \overline{y}\subset\widehat{X}}}'' K_{x,\overline{y}}$$

The topology on $\overline{A}_X$ is the restricted topology of the additive groups $\mathbb{A}_{\overline{y}}$ with respect to $\mathbb{A}_{\overline{y}}^{(0)}$.

Definition 3.57. The adelic ring attached to the completed surface $\widehat{X}$ is

$$\mathbf{A}_{\widehat{X}} := \overline{\mathbf{A}}_X \oplus \prod_{\sigma\in B_\infty} \mathbf{A}_{X_\sigma}$$

where clearly each $\mathbf{A}_{X_\sigma}$ is the adelic ring attached to the Riemann surface X_σ. The topology on $\mathbf{A}_{\widehat{X}}$ is the product topology.

The following proposition establishes a nice relationship between $\mathbf{A}_{\widehat{X}}$ and $\mathbf{A}_X$.

Proposition 3.58. *The following equality holds:*

$$\mathbf{A}_{\widehat{X}} = \mathbf{A}_X \oplus \prod_{\sigma\in B_\sigma} (\mathbf{A}_{X_\sigma} \oplus \mathbf{A}_{X_\sigma})$$

Proof. Let $\alpha \in \overline{\mathbf{A}}_X$, then it can be decomposed in the following way:

$$\alpha = (a_y)_{y\subset X} \times (a_{p,\sigma})_{\substack{p\in X_\sigma,\\ \sigma\in B_\infty}}$$

where:

- $a_y \in \mathbb{A}_y$ for all $y \subset X$ and $a_y \in \mathbb{A}_y^{(0)}$ for all but finitely many y.
- For any fixed σ we have $a_{p,\sigma} \in K_{p,\sigma}$ and $a_{p,\sigma} \in \mathcal{O}_{p,\sigma}$ for all but finitely many $p \in X_\sigma$.

This means that $\alpha \in \overline{\mathbf{A}}_X \subseteq \mathbf{A}_X \oplus \prod_{\sigma \in B\sigma} \mathbf{A}_{X_\sigma}$, so obviously

$$\mathbf{A}_{\widehat{X}} \subseteq \mathbf{A}_X \oplus \prod_{\sigma \in B_\sigma} (\mathbf{A}_{X_\sigma} \oplus \mathbf{A}_{X_\sigma}) \,.$$

Vice versa, let $\alpha \in \mathbf{A}_X \oplus \prod_{\sigma \in B\sigma} \mathbf{A}_{X_\sigma}$ then:

$$\alpha = (a_y)_{y \subset X} \times (a_{p,\sigma})_{\substack{p \in X_\sigma, \\ \sigma \in B_\infty}}$$

where a_y and $a_{p,\sigma}$ satisfy the conditions listed above. Since each $\varphi_\sigma : X_\sigma \to X_K$ is surjective and points of X_K correspond to horizontal curves on X, we can write easily:

$$\alpha = (a_y)_{y \subset X} \times (a_{p,\sigma})_{\substack{p \in X_\sigma, \\ \sigma \in B_\infty}} = (a_y)_{y \subset X} \times ((a_{p,\sigma})_{p \in y_\infty})_{y_\infty \subset X_\infty} = (a_{\overline{y}})_{\overline{y} \subset \widehat{X}} \in \overline{\mathbf{A}}_X \,.$$

□

3.3.3 Residues

We want to define spaces of differential forms and residues attached to the arithmetic surface $\widehat{X}$. We distinguish two cases:

(1) Fix a flag $x \in \overline{y} \subset \widehat{X}$:

  (1a) If $x \in X$ we assume $\mathfrak{z} \in \overline{y}(x)$ and $\varphi(x) = b \in B$ and we define the following spaces and residues maps:

$$\Omega^1_{x,\mathfrak{z}} := \Omega^{1,\mathrm{sep}}_{K_{x,\mathfrak{z}}|K_b} \,, \quad \Omega^1_{x,\overline{y}} := \bigoplus_{\mathfrak{z} \in \overline{y}(x)} \Omega^1_{x,\mathfrak{z}} \,;$$

$$\mathrm{res}_{x,\mathfrak{z}} := \psi^0_b \circ \mathrm{res}_{K_{x,\mathfrak{z}}|K_b} : \Omega^1_{x,\mathfrak{z}} \to \mathbb{C}^\times,$$

$$\mathrm{res}_{x,\overline{y}} := \prod_{\mathfrak{z} \in \overline{y}(x)} \mathrm{res}_{x,\mathfrak{z}} : \Omega^1_{x,\overline{y}} \to \mathbb{C}^\times \,.$$

  Where $\psi^0_b : K_b \to \mathbb{C}^\times$ is the standard character on K_b (see appendix G.2).

  (1b) If $x = p \in y_\sigma$, then:

$$\Omega^1_{x,\overline{y}} := \Omega^1_{p,\sigma} = \Omega^{1,\mathrm{sep}}_{K_{p,\sigma}|\mathbb{C}};$$

$$\mathrm{res}_{x,\overline{y}} := \psi^0_\sigma \circ \mathrm{res}_{p,\sigma} : \Omega^1_{p,\sigma} \to \mathbb{C}^\times \,.$$

  Where in the last line, $\mathrm{res}_{p,\sigma}$ is the one dimensional residue on the Riemann surface X_σ at the point p and $\psi^0_\sigma : \mathbb{C} \to \mathbb{C}^\times$ is the standard character of $\mathbb{C}$.

(2) Fix a flag $p \in X_\sigma$, then:

$$\operatorname{res}_{p,X_\sigma} := \psi^0_\sigma \circ (-\operatorname{res}_{p,\sigma}) : \Omega^1_{p,\sigma} \to \mathbb{C}^\times .$$

Remark 3.59. The choice of the minus sign in the definition of $\operatorname{res}_{p,X_\sigma}$ is coherent with the main theory if we consider X_σ as a legit vertical curve on $\widehat{X}$. See remark 1.35.

By the universal property of the module of differential forms we have a canonical map $\Omega^1_{K(X)|K} \to \Omega^1_{x,\overline{y}}$ therefore by abuse of notation, we can consider an element $\omega \in \Omega^1_{K(X)|K}$ as an element lying in $\Omega^1_{x,\overline{y}}$. Moreover, by base change we know that $\Omega^1_{\mathbb{C}(X_\sigma)|\mathbb{C}} \cong \Omega^1_{K(X)|K} \otimes_{K(X)} \mathbb{C}(X_\sigma)$, so again we have a canonical composition map:

$$\Omega^1_{K(X)|K} \to \Omega^1_{\mathbb{C}(X_\sigma)|\mathbb{C}} \to \Omega^1_{p,\sigma}$$

and when clear from the context we can consider $\omega \in \Omega^1_{K(X)|K}$ as an element lying in $\Omega^1_{p,\sigma}$. In other words, it always makes sense to take a residue of a "rational" differential form $\omega \in \Omega^1_{K(X)|K}$ for flags in X and in $\widehat{X}$.

Residues on arithmetic surfaces satisfy some global properties very similar to those described in theorem 3.23.

Theorem 3.60 (2D arithmetic reciprocity laws)**.** *Let $\omega \in \Omega^1_{K(X)|K}$ then:*

1. *Let $x \in X$, then $\prod_{\overline{y} \ni x} \operatorname{res}_{x,\overline{y}}(\omega) = 1$.*
2. *Let $p \in X_\sigma$, then $\operatorname{res}_{p,X_\sigma}(\omega) \cdot \prod_{\overline{y} \ni p} \operatorname{res}_{p,\overline{y}}(\omega) = 1$.*
3. *Let $\overline{y} \subset X$ be a vertical curve or $\overline{y} = X_\sigma$ for some $\sigma \in B_\infty$, then $\prod_{x \in \overline{y}} \operatorname{res}_{x,\overline{y}}(\omega) = 1$.*
4. *Let $\overline{y} \in \widehat{X}$ be a horizontal curve, then $\prod_{x \in \overline{y}} \operatorname{res}_{x,\overline{y}}(\omega) = 1$.*

Proof. See [46, 2.4], [46, 3] and [46, 5] for (1),(3) and (4) respectively. In the archimedean case of (3) it is enough to use the 1-dimensional reciprocity law. (2) For any fixed $p \in X_\sigma$ there is exactly one horizontal curve $\overline{y} \in \widehat{X}$ "passing by" p. Therefore

$$\operatorname{res}_{p,X_\sigma}(\omega) \cdot \prod_{\overline{y} \ni p} \operatorname{res}_{p,\overline{y}}(\omega) =$$

$$= \operatorname{res}_{p,X_\sigma}(\omega) \operatorname{res}_{p,\overline{y}}(\omega) = \psi^0(-\operatorname{res}_{p,\sigma}(\omega)) \cdot \psi^0(\operatorname{res}_{p,\sigma}(\omega)) = 1 .$$

□

Remark 3.61. Note that statements (1) and (2) of theorem 3.60 describe reciprocity laws around a point, whereas statements (3) and (4) describe reciprocity laws for a fixed curve. Archimedean data are taken in account without any special treatment: points on X_σ are considered as points of $\widehat{X}$ and achimedean fibres are considered as vertical curves on $\widehat{X}$.

Fix a rational differential form $\omega \in \Omega^1_{K(X)|K}$ let's define the map:

$$\begin{aligned} \xi^\omega : \mathbf{A}_{\widehat{X}} &\to \mathbb{C}^\times \\ (a_{x,\overline{y}})_{\substack{x\in\overline{y},\\ \overline{y}\subset\widehat{X}}} \times (a_{p,\sigma})_{\substack{p\in X_\sigma,\\ \sigma\in B_\infty}} &\mapsto \prod_{\substack{x\in\overline{y},\\ \overline{y}\subset\widehat{X}}} \operatorname{res}_{x,\overline{y}}(\omega a_{x,\overline{y}}) \prod_{\substack{p\in X_\sigma,\\ \sigma\in B_\infty}} \operatorname{res}_{p,\sigma}(\omega a_{p,\sigma})\,. \end{aligned} \tag{3.33}$$

Conjecture 3.62. Fix $\omega \in \Omega^1_{K(X)|K}$, then ξ^ω is a standard character of $\mathbf{A}_{\widehat{X}}$, so in particular $\mathbf{A}_{\widehat{X}}$ is self dual.

Conjecture 3.63. $K(X)$ is discrete in $\mathbf{A}_{\widehat{X}}$.

3.3.4 Idelic Deligne pairing

We saw in appendix D that there is no intersection theory for arithmetic surfaces, if we consider just scheme theory, but the intersection pairing is partially substituted by the Deligne pairing. The aim of this section is to give an idelic extension of the Deligne pairing for our arithmetic surface $\varphi : X \to B$. In this subsection we won't use the completion of X and B at all, so we will deal with the standard idelic theory for schemes. First of all let's see how the Deligne pairing can be lifted to a pairing at the level of divisors on X and with target in $\operatorname{Pic}(B)$:

Proposition 3.64. *There exists a unique pairing*

$$[[\,,\,]]\colon\ \operatorname{Div}(X) \times \operatorname{Div}(X) \to \operatorname{Pic}(B)$$

satisfying the following properties:

(1) *It is bilinear and symmetric.*

(2) *It descends to the Deligne pairing.*

$$\langle\,,\,\rangle :\ \operatorname{Pic}(X) \times \operatorname{Pic}(X) \to \operatorname{Pic}(B)$$

(3) *If $D, E \in \operatorname{Div}(X)$ are two divisors with no common components then $[[D,E]]$ is equal to the class of the divisor $\langle D, E\rangle$ in $\operatorname{Pic}(B)$ (here the bracket $\langle\,,\,\rangle$ is the one defined in D.55). In other words:*

$$[[D,E]] = \sum_{x\in D\cap E} [k(x) : k(\varphi(x))] i_x(D,E)\, [\varphi(x)] \in \operatorname{Pic}(B)\,.$$

Proof. For any $D, E \in \mathrm{Div}(X)$ it is enough to put:

$$[[D, E]] := \langle \mathscr{O}_X(D), \mathscr{O}_X(E) \rangle$$

where on the right hand side we have the Deligne pairing between invertible sheaves. Uniqueness follows from properties (1)-(3) and the moving lemma. □

At this point we will try to work in complete analogy to the geometric case and we will use the Kato symbol defined in section 1.3 to obtain the map, denoted below with a question mark, which makes the following diagram commutative:

$$\begin{tikzcd}
\ker(d^1_\times) \times \ker(d^1_\times) \arrow[d, two heads, "p\times p"] \arrow[drr, "?"] & & \mathbf{A}_B^\times \arrow[dd, two heads] \\
\mathrm{Div}(X) \times \mathrm{Div}(X) \arrow[d, two heads] \arrow[dr, "{[[\,,\,]]}"] & & \\
\mathrm{Pic}(X) \times \mathrm{Pic}(X) \arrow[r, "\langle\,,\,\rangle"] & \mathrm{Pic}(B) \cong \mathrm{CH}^1(B) &
\end{tikzcd} \tag{3.34}$$

As usual, fix a flag $x \in y$ with $\mathfrak{z} \in y(x)$ and assume that $\varphi(x) = b$, then we define

$$(\,,\,)_{x,\mathfrak{z}} := (\,,\,)_{K_{x,\mathfrak{z}}|K_b} : K_{x,\mathfrak{z}}^\times \times K_{x,\mathfrak{z}}^\times \to K_b^\times$$

where $(\,,\,)_{K_{x,\mathfrak{z}}|K_b}$ is the Kato symbol defined in section 1.3. Remember that depending on whether y is horizontal or vertical, we have a different expression for $(\,,\,)_{x,\mathfrak{z}}$. Then we put:

$$(\,,\,)_{x,y} := \prod_{\mathfrak{z}\in y(x)} (\,,\,)_{x,\mathfrak{z}} : K_{x,y}^\times \times K_{x,y}^\times \to K_b^\times$$

Proposition 3.65. *The pairing $(\,,\,)_{x,y}$ is a skew-symmetric bilinear form on $K_{x,y}^\times$ satisfying the following properties:*

(1) *Let $r, s \in K_x^\times$, then for all but finitely many curves y containing x we have that $(r, s)_{x,y} = 1$ and moreover $\prod_{y\ni x}(r, s)_{x,y} = 1$.*

(2) *Let y be a vertical curve and let $s, t \in K_y^\times$, then for all but finitely many $x \in y$ we have that $(r, s)_{x,y} = 1$ and moreover $\prod_{x\in y}(r, s)_{x,y} = 1$.*

Proof. Skew symmetry and bilinearity are clear. See [36, Theorem 4.3] for (1); note that in [36] the proof is made for $r, s \in K(X)$, but it is easy to see that it actually works also for $r, s \in K_x^\times$. See [36, Theorem 5.1] for (2). □

Definition 3.66. The *idelic Deligne pairing*

$$\langle\, ,\, \rangle_i : \ker(d^1_\times) \times \ker(d^1_\times) \to \mathrm{CH}^1(B)$$

is given by:

$$(r,s) \mapsto \langle r,s\rangle_i := \sum_{b\in B} n_b(r,s)[b] \in \mathrm{CH}^1(B) \tag{3.35}$$

such than:

$$n_b(r,s) := \sum_{\substack{x\in X_b,\\ y\ni x}} v_b\left((\gamma_{x,y},\beta_{x,y})_{x,y}\right) \tag{3.36}$$

for $r = (\alpha,\beta,\alpha^{-1}\beta^{-1})$, $s = (\gamma,\delta,\gamma^{-1}\delta^{-1}) \in \ker(d_1^\times)$ and where v_b is the complete discrete valuation on K_b. It is crucial to emphasize the fact the we consider $\sum_{b\in B} n_b(r,s)[b]$ in its linear equivalence class in $\mathrm{CH}^1(B)$ and not just as a divisor. By simplicity of notation we avoid to mention the canonical map $\mathrm{Div}(B) \to \mathrm{CH}^1(B)$.

We have to show that definition 3.66 makes sense: for any $b \in B$ we have to check that the sum in equation (3.36) is finite. Thanks to the adelic restricted product, for all but finitely many flags $x \in y$ we have that $\gamma_{x,y}, \beta_{x,y} \in \mathcal{O}^\times_{x,y}$. If y is horizontal we know that $(\, ,\,)_{x,y}$ is trivial on $\mathcal{O}^\times_{x,y} \times \mathcal{O}^\times_{x,y}$; if y is vertical, then $(\, ,\,)_{x,y}$ restricts to a map $\mathcal{O}^\times_{x,y} \times \mathcal{O}^\times_{x,y} \to \mathcal{O}^\times_b$ and $v(\mathcal{O}^\times_b) = 0$. Again thanks to the adelic restricted product it follows that $n_b(r,s) = 0$ for all but finitely many $b \in B$.

Remark 3.67. For any $b \in B$ we have the following decomposition for the big product (3.36):

$$\sum_{\substack{x\in X_b,\\ y\ni x}} v_b\left((\, ,\,)_{x,y}\right) = \sum_{\substack{y\subset X_b,\\ x\in y}} v_b\left((\, ,\,)_{x,y}\right) + \sum_{\substack{x\in X_b,\\ y\ni x,\\ y \text{ horiz.}}} v_b\left((\, ,\,)_{x,y}\right)$$

We put $\langle\, ,\, \rangle_i$ as the undetermined function in diagram (3.34) and we have the following fundamental result:

Theorem 3.68. *Consider the notation of diagram (3.34). The pairing* $\langle\, ,\, \rangle_i$ *satisfies the following properties:*

(1) *It is bilinear and symmetric.*

(2) *Let* $r,s,r',s' \in \ker(d^1_\times)$ *such that* $p(r) = p(r')$ *and* $p(s) = p(s')$*, then* $\langle r,s\rangle_i = \langle r',s'\rangle_i$.

(3) *It descends naturally to a pairing* $H^1(\mathcal{A}^\times_X) \times H^1(\mathcal{A}^\times_X) \to \mathrm{Pic}(B)$.

Proof. Let's fix $r = (\alpha, \beta, \alpha^{-1}\beta^{-1})$, $s = (\gamma, \delta, \gamma^{-1}\delta^{-1}) \in \ker(d^1_\times)$; moreover we can fix $b \in B$ and work componentwise.
(1) Bilinearity is clear. We will show that as elements of $\mathrm{Div}(B)$ we have $\langle r, s\rangle_i = \langle s, r\rangle_i + (f)$ with $f \in K^\times$.

For any flag $x \in y$: $\alpha^{-1}_{x,y}\beta^{-1}_{x,y} \in \mathcal{O}^\times_{x,y}$ and $\gamma^{-1}_{x,y}\delta^{-1}_{x,y} \in \mathcal{O}^\times_{x,y}$ so we have that:

$$\begin{aligned} 0 &= \sum_{\substack{x\in X_b,\\ y\ni x}} v_b\left((\alpha^{-1}_{x,y}\beta^{-1}_{x,y}, \gamma^{-1}_{x,y}\delta^{-1}_{x,y})_{x,y}\right) = \\ &= \underbrace{\sum_{\substack{x\in X_b,\\ y\ni x}} v_b\left((\alpha_{x,y}, \gamma_{x,y})_{x,y}\right)}_{(i)} + \underbrace{\sum_{\substack{x\in X_b,\\ y\ni x}} v_b((\alpha_{x,y}, \delta_{x,y})_{x,y})}_{(ii)} + \\ &\quad + \underbrace{\sum_{\substack{x\in X_b,\\ y\ni x}} v_b((\beta_{x,y}, \gamma_{x,y})_{x,y})}_{(iii)} + \underbrace{\sum_{\substack{x\in X_b,\\ y\ni x}} v_b((\beta_{x,y}, \delta_{x,y})_{x,y})}_{(iv)} \,. \end{aligned} \tag{3.37}$$

Now we analyze in detail the underbraced terms in equation (3.37): for (i) we have the following decomposition thanks to remark 3.67:

$$\begin{aligned} (i) &= \sum_{\substack{y\subset X_b,\\ x\in y}} v_b((\alpha_{x,y}, \gamma_{x,y})_{x,y}) + \sum_{\substack{x\in X_b,\\ y\ni x,\\ y \text{ horiz.}}} v_b((\alpha_{x,y}, \gamma_{x,y})_{x,y}) = \\ &{}^{\text{(prop.3.65(2))}} = 0 + \sum_{x\in X_b} \sum_{\substack{y\ni x,\\ y \text{ horiz.}}} v_b((\alpha_{x,y}, \gamma_{x,y})_{x,y}) \,. \end{aligned}$$

By definition we have that $(ii) = n_b(s, r)$ and $(iii) = -n_b(r, s)$. Finally:

$$(iv) = \sum_{x\in X_b} \sum_{y\ni x} (\beta_{x,y}, \delta_{x,y})_{x,y} =^{\text{(prop.3.65(1))}} 0$$

By substituting in equation (3.37) we conclude that

$$n_b(r, s) = n_b(s, r) + \sum_{\substack{x\in X_b,\\ y\ni x,\\ y \text{ horiz.}}} v_b((\alpha_{x,y}, \gamma_{x,y})_{x,y}) \,. \tag{3.38}$$

Let y be an horizontal curve and let $x \in y$ such that $\varphi(x) = b$, then the coefficient field of $K_{x,\mathfrak{z}}$ is $k(y)_x$. The two dimensional valuation $v_{x,\mathfrak{z}}$ extends the valuation v_y on $k(y)$ and moreover that the norm $N_{k(y)_x|K_b}$ extends $N_{k(y)|K}$. It follows that $(\ ,\ )_{x,\mathfrak{z}}$ extends the one dimensional tame symbol

$$(\ ,\ )_y := N_{k(y)|K} \circ (\ ,\ )_{k(y)} : K^\times_y \times K^\times_y \to k(y)^\times \to K^\times \,.$$

This means that for any two elements $u, v \in K_y$, where y is horizontal, we have that:

$$(u, v)_{x,y} = (u, v)_y \in K$$

for any $x \in y$. Therefore we can rewrite equation (3.38):

$$n_b(r, s) = n_b(s, r) + \sum_{y \text{ horiz.}} v_b((\alpha_{x,y}, \gamma_{x,y})_y)\,. \tag{3.39}$$

Let's put $f = \prod_{y \text{ horiz.}} (\alpha_{x,y}, \gamma_{x,y})_y \in K^\times$, then equation (3.39) implies the following equality:

$$\langle r, s\rangle_i = \langle s, r\rangle_i + \sum_{b \in B} v_b(f)[b] = \langle s, r\rangle_i + (f)\,.$$

(2) Let $r' = (\alpha', \beta', (\alpha')^{-1}(\beta')^{-1})$ and $s' = (\gamma', \delta', (\gamma')^{-1}(\delta')^{-1})$. Since $p(r) = p(r')$ and $p(s) = p(s')$, then $v_y(\alpha_{x,y}) = v_y(\alpha'_{x,y})$ and $v_y(\gamma_{x,y}) = v_y(\gamma'_{x,y})$. This means that $\gamma'_{x,y} = f_{x,y}\gamma_{x,y}$ and $\alpha'_{x,y} = g_{x,y}\alpha_{x,y}$ for $f_{x,y}, g_{x,y} \in \mathcal{O}_y^\times$ (for any $x \in y$). Then we have the following chain of equalities depending on what we showed in claim (1):

$$n_b(r', s') = \sum_{\substack{x \in X_b,\\ y \ni x}} v_b((f_{x,y}\gamma_{x,y}, \beta'_{x,y})_{x,y}) =$$

$$= \sum_{\substack{x \in X_b,\\ y \ni x}} ((f_{x,y}, \beta'_{x,y})_{x,y}) + \sum_{\substack{x \in X_b,\\ y \ni x}} v_b((\gamma_{x,y}, \beta'_{x,y})_{x,y}) =$$

$$= \sum_{\substack{x \in X_b,\\ y \ni x}} v_b((f_{x,y}, \beta'_{x,y})_{x,y}) + n_b(r', s) = \sum_{\substack{x \in X_b,\\ y \ni x}} v_b((f_{x,y}, \beta'_{x,y})_{x,y}) + n_b(s, r') + v_b(f) =$$

$$= \sum_{\substack{x \in X_b,\\ y \ni x}} v_b((f_{x,y}, \beta'_{x,y})_{x,y}) + \sum_{\substack{x \in X_b,\\ y \ni x}} v_b((g_{x,y}\alpha_{x,y}, \delta_{x,y})_{x,y}) + v_b(f) = (*)$$

Where $f \in K^\times$.

$$(*) = \sum_{\substack{x \in X_b,\\ y \ni x}} v_b((f_{x,y}, \beta'_{x,y})_{x,y}) + \sum_{\substack{y \subset X_b,\\ x \in y}} v_b((g_{x,y}, \delta_{x,y})_{x,y}) + \sum_{\substack{x \in X_b,\\ y \ni x}} v_b((\alpha_{x,y}, \delta_{x,y})_{x,y}) + v_b(f) =$$

$$= \underbrace{\sum_{\substack{x \in X_b,\\ y \ni x}} v_b((f_{x,y}, \beta'_{x,y})_{x,y})}_{(i)} + \underbrace{\sum_{\substack{x \in X_b,\\ y \ni x}} v_b((g_{x,y}, \delta_{x,y})_{x,y})}_{(ii)} + n_b(r, s) + v_b(fg)\,.$$

Note that in the last line we used the fact that $n_b(s,r) = n_b(r,s) + v_b(g)$ for $g \in K^\times$. We have to show that the terms (i) and (ii) are valuations at b of elements of $K^\times$. Since $(\alpha')^{-1}_{x,y}(\beta')^{-1}_{x,y} \in \mathcal{O}^\times_{x,y}$, we have:

$$0 = \sum_{\substack{x\in X_b,\\ y\ni x}} v_b((f_{x,y}, (\alpha')^{-1}_{x,y}(\beta')^{-1}_{x,y})_{x,y}) =$$

$$= \sum_{\substack{x\in X_b,\\ y\ni x}} v_b((f_{x,y}, \alpha'_{x,y})_{x,y}) + \sum_{\substack{x\in X_b,\\ y\ni x}} v_b((f_{x,y}, \beta'_{x,y})_{x,y}) =$$

$$= v_b(h) + \sum_{\substack{y\subset X_b,\\ x\in y}} v_b((f_{x,y}, \beta'_{x,y})_{x,y})\,.$$

with $h = \prod_{y \text{ horiz.}} (f_{x,y}, \alpha'_{x,y})_y \in K^\times$. For (ii) the argument is similar.

(3) Let $r, s \in \operatorname{im}(d^0_\times)$. It means that $\alpha = lm^{-1}$, $\beta = tl^{-1}$, $\gamma = uv^{-1}$, $\delta = zu^{-1}$ for $l, u \in A_0^\times = K(X)^\times$, $m, v \in A_1^\times$ and $t, z \in A_2^\times$. So:

$$\begin{aligned} n_b(r,s) &= \sum_{\substack{x\in X_b,\\ y\ni x}} v_b((uv^{-1}_{x,y}, t_{x,y}l^{-1})_{x,y}) = \\ &= \sum_{\substack{x\in X_b,\\ y\ni x}} v_b((u, t_{x,y})_{x,y}) + \sum_{\substack{x\in X_b,\\ y\ni x}} v_b((u, l^{-1})_{x,y}) + \\ &+ \sum_{\substack{x\in X_b,\\ y\ni x}} v_b(v^{-1}_{x,y}, t_{x,y})_{x,y}) + \sum_{\substack{x\in X_b,\\ y\ni x}} v_b((v^{-1}_{x,y}, l^{-1})_{x,y})\,. \end{aligned} \tag{3.40}$$

Now it is enough to appeal to one of the arguments previously used to conclude that each summand of equation (3.40) is either 0 or of the form $v_b(f)$ for $f \in K^\times$. It means that $\langle r, s\rangle_i = 0$ in $\mathrm{CH}^1(B)$. □

We want to give an alternative formula for the coefficient $n_b(r,s)$. Notice that:

$$- \sum_{\substack{x\in X_b,\\ y\ni x}} v_b((\alpha_{x,y}, \gamma^{-1}_{x,y}\delta^{-1}_{x,y})_{x,y}) =$$

$$= \sum_{\substack{x\in X_b,\\ y\ni x}} v_b((\alpha_{x,y}, \gamma_{x,y})_{x,y}) + \sum_{\substack{x\in X_b,\\ y\ni x}} v_b((\alpha_{x,y}, \delta_{x,y})_{x,y}) =$$

$$= v_b(f) + \sum_{\substack{x\in X_b,\\ y\ni x}} v_b((\alpha_{x,y}, \delta_{x,y})_{x,y}) = v_b(f) + n_b(s,r) = v_b(fg) + n_b(r,s) \tag{3.41}$$

for $f, g \in K^{\times}$. Therefore, we can also express:

$$n_b(r,s) = -\sum_{\substack{x\in X_b,\\ y\ni x}} v_b((\alpha_{x,y}, \gamma_{x,y}^{-1}\delta_{x,y}^{-1})_{x,y}) \,. \tag{3.42}$$

In particular if y is a horizontal curve:

$$-v_b((\alpha_{x,y}, \gamma_{x,y}^{-1}\delta_{x,y}^{-1})_{x,y}) = v_y(\alpha_{x,y}) v_b\left(N_{k(y)_x|K_b}\left(\overline{\gamma_{x,y}^{-1}\delta_{x,y}^{-1}}\right)\right) \,. \tag{3.43}$$

The following algebraic geometry lemma is fundamental in order to understand the relationship between $\langle r, s\rangle_i$ and Deligne pairing.

Lemma 3.69. *Let $X_b \subset X$ the fiber over $b \in B$ and assume that X_b has at least two irreducible components. If $D \subset X_b$ is an integral curve, then there exists a divisor $D' \sim D$ such that D' doesn't have components contained in X_b.*

Proof. Consider Γ running amongst all irreducible components of X_b, then put

$$S := \bigcup_{\substack{\Gamma\subset X_b\\ \Gamma\neq D}} (\Gamma \cap D) \,.$$

By the moving lemma we can find $D' \sim D$ not passing by S. It is clear by the definition of S that D' cannot have vertical components contained in X_b. □

Theorem 3.70. *If $r, s \in \ker(d^1_{\times})$ such that $D = p(r)$ and $E = p(s)$ are two nonsingular prime divisors on X with no common components, then $\langle r, s\rangle_i = [[D, E]]$.*

Proof. Fix $r = (\alpha, \beta, \alpha^{-1}\beta^{-1})$, $s = (\gamma, \delta, \gamma^{-1}\delta^{-1})$. We want to show that it is enough to restrict to the case when either D or E is horizontal. In any case, by theorem 3.68(2) we always choose $\delta_{x,y}$ in the following way: $\delta_{x,y} = 1$ if $x \notin D \cap E$ and $\delta_{x,y} = t_x^{-1}$, where $t_x \in \mathscr{O}_{X,x}$ is the local equation of E at x, if $x \in D \cap E$. For any $y \neq D$, $\alpha_{x,y} \in \mathcal{O}_y^{\times}$, since $p(a) = D$, therefore:

$$n_b(r,s) = -\sum_{x\in D\cap X_b} v_b((\alpha_{x,D}, \gamma_{x,D}^{-1}\delta_{x,D}^{-1})_{x,D}) \,. \tag{3.44}$$

If $D \subseteq X_b$ and $E \subseteq X_{b'}$ with $b \neq b'$, then by proposition 3.65(2) and the choice of $\delta_{x,y}$ we have:

$$n_b(r,s) = -\sum_{x\in D\cap X_b} v_b((\alpha_{x,D}, \gamma_{x,D}^{-1})_{x,D}) = 0 \,. \tag{3.45}$$

So in such a particular case $\langle r,s\rangle_i = [[D,E]] = 0$.
If $D,E \in X_b$ we can apply lemma 3.69 and find a divisor $D' = \sum_j n_j\Gamma_j \sim D$ such that $\Gamma_j \not\subset X_b$. Clearly

$$[[\Gamma_j, E]] = \sum_j n_j[[\Gamma_j, E]]$$

therefore from now on we can restrict our calculation to the case where either D or E is horizontal. By symmetry we can fix D to be horizontal and we denote with $K(D)$ its function field. In this case we have an explicit expression given by equation (3.43):

$$n_b(r,s) = \sum_{x\in D\cap X_b} v_b\left(N_{K(D)_x|K_b}\left(\overline{\gamma_{x,D}^{-1}t_x}\right)\right) = \sum_{x\in D\cap X_b} v_b\left(N_{K(D)|K}\left(\overline{\gamma_{x,D}^{-1}t_x}\right)\right) =$$

$$= \sum_{x\in D\cap X_b} v_b\left(N_{K(D)_x|K_b}\left(\overline{\gamma_{x,D}^{-1}}\right)\right) + \sum_{x\in D\cap E\cap X_b} v_b\left(N_{K(D)_x|K_b}\left(\overline{t_x}\right)\right)\,.$$

Now by the theory of extensions of valuation fields (see [21, II(2.5)]), we know that if $v_x := v_{x,D}^{(1)}$ is the valuation on $K(D)_x$, then:

$$v_x = \frac{1}{[k(x):k(b)]} v_b \circ N_{K(D)_x|K_b}\,.$$

Therefore we obtain:

$$n_b(r,s) = \sum_{x\in D\cap X_b} [k(x):k(b)] v_x\left(\overline{\gamma_{x,D}^{-1}}\right) + \sum_{x\in D\cap E\cap X_b} [k(x):k(b)] v_x\left(\overline{t_x}\right)\,. \tag{3.46}$$

Put by simplicity $f = \overline{\gamma_{x,D}^{-1}} \in K(D)^\times$, consider the restricted morphism of arithmetic curves $\varphi : D \to B$ and the principal divisor $(f) \in \mathrm{Princ}(D)$, then:

$$\varphi_*((f)) = \sum_{b\in B}\left(\sum_{x\in D\cap X_b} [k(x):k(b)] v_x(f)\right)[b]\,.$$

Moreover $v_x\left(\overline{t_x}\right) = i_x(D,E)$ by lemma 3.48. Equation 3.46 implies that in $\mathrm{Div}(B)$ we have the following equality:

$$\langle r,s\rangle_i = \varphi_*((f)) + [[D,E]]\,.$$

But by [37, 7 Remark 2.19] we know that $\varphi_*((f)) = \left(N_{K(D)|K}(f)\right) \in \mathrm{Princ}(B)$, so the proof is complete. □

We obtained the idelic representation of Deligne pairing:

Corollary 3.71. *Diagram (3.34) is commutative.*

Proof. For any two divisors $D, E \in \mathrm{Div}(X)$ define the pairing:

$$\Theta(D, E) := \langle r', s' \rangle_i$$

for a choice of $r', s' \in \ker(d^1_\times)$ such that $p(r') = D$ and $p(s') = E$. By theorem 3.68(2) Θ is well defined and moreover by 3.68(1), 3.68(3) and 3.70 we can conclude that $\Theta(D, E) = D.E$. Thus for any $a, b \in \ker(d^1_\times)$ we have that:

$$\langle r, s \rangle_i = \Theta(p(r), p(s)) = p(r).p(s)\,.$$

□

3.3.5 Idelic interpretation of Arakelov intersection theory

In this section we will deal with complex analytic theory for Riemann surfaces and we will use the notation introduced in appendix D.

The contribution at infinity to the Arakelov intersection pairing is given by the $*$-product between Green functions, so the next step in our theory is to find an idelic description of it. The infinity part of the full adelic ring $\mathbf{A}_X \oplus \prod_{\sigma \in B_\sigma} (\mathbf{A}_{X_\sigma} \oplus \mathbf{A}_{X_\sigma})$ is given by $\mathbf{A}_{X_\sigma} \oplus \mathbf{A}_{X_\sigma}$ (for each σ), so by working in complete analogy with the previous work, we want to find a surjective map:

$$(\mathbf{A}^\times_{X_\sigma} \oplus \mathbf{A}^\times_{X_\sigma}) \supseteq S \to \mathbb{Z}G(X_\sigma)$$

where S is an adequate subset of $\mathbf{A}^\times_{X_\sigma} \oplus \mathbf{A}^\times_{X_\sigma}$ still to be determined and $\mathbb{Z}G(X_\sigma)$ is the vector space of Green functions on X_σ with integer orders.

Remark 3.72. First of all let's introduce a notation. For any $a = (a_x) \in \mathbf{A}_{X_\sigma}$, with $a(x)$ we denote the projection of a_x onto the residue field $\mathbb{C}$ (when it is well defined).

Let $\mathcal{F}(X_\sigma, \mathbb{R})'$ be the set of functions $f : U \subseteq X_\sigma \to \mathbb{R}$ whose domain U is the whole X_σ minus a finite set of points, then we have the following map:

$$\begin{array}{rcl} \Theta : \mathbf{A}^\times_{X_\sigma} \times \mathbf{A}^\times_{X_\sigma} & \to & \mathcal{F}(X_\sigma, \mathbb{R})' \\ (a, b) & \mapsto & -\log(ba\overline{a}) := [x \mapsto -\log\Big(b(x)a(x)\overline{a(x)}\Big)] \end{array}$$

where $\overline{a(x)}$ denotes the complex conjugate. Note that $\mathbb{Z}G(X_\sigma) \subset \mathcal{F}(X_\sigma, \mathbb{R})'$, then put

$$G(\mathbf{A}^\times_{X_\sigma}) := \{(a, b) \in \Theta^{-1}(\mathbb{Z}G(X_\sigma))\colon v_x(a_x) = \mathrm{ord}^G_x(\Theta(a, b)),\ \forall x \in X_\sigma\}\,.$$

We get the map:

$$\pi_\sigma := \Theta|_{G(\mathbf{A}^\times_{X\sigma})} : G(\mathbf{A}^\times_{X\sigma}) \to \mathbb{Z}G(X_\sigma)\,.$$

Proposition 3.73. *The map π_σ is surjective.*

Proof. Let $g \in \mathbb{Z}G(X_\sigma)$, by proposition D.17, there exist a C^∞ hermitian invertible sheaf $(\mathscr{L}, h)$ on X and a meromorphic section $s = \{(s_j, U_j)\}$ of $\mathscr{L}$ such that we can write:

$$g = -\log(h(s,s))\,.$$

We can choose $a \in \mathbf{A}^\times_{X_\sigma}$ such that $a(x) = s(x)$ (when $s(x)$ is well defined) and $v_x(a_x) = \mathrm{ord}_x(s)$ for any $x \in X_\sigma$. Now we can write

$$g(x) = -\log(h_x(a(x), a(x)))\,.$$

Since $z \mapsto h_x(z\overline{z})$ is a complex absolute value, we have $h_x(z\overline{z}) = w_x z\overline{z}$ with $w_x \in \mathbb{C}$. Let's choose $b = (b_x) \in \mathbf{A}^\times_{X_\sigma}$ such that $b(x) = w_x$, then

$$g(x) = -\log\left(b(x)a(x)\overline{a(x)}\right)\,.$$

The fact that $v_x(a_x) = \mathrm{ord}^G_x(g)$ follows directly from the fact that for any hermitian metric h and meromorphic section s we have the equality:

$$\mathrm{div}^G(-\log(h(s,s))) = \mathrm{div}(s)\,.$$

(See proposition D.14) ☐

So far we have the idelic description of Green functions with integer orders thanks to the projection π_σ. Now let's fix a Kähler fundamental form Ω_σ on X_σ and consider $G^{\Omega_\sigma}_0(\mathbf{A}^\times_{X_\sigma}) := \pi_\sigma^{-1}(\mathbb{Z}G^{\Omega_\sigma}_0(X_\sigma))$, $G^{\Omega_\sigma}(\mathbf{A}^\times_{X_\sigma}) := \pi_\sigma^{-1}(\mathbb{Z}G^{\Omega_\sigma}(X_\sigma))$. For pairs $(\alpha,\beta) \in G^{\Omega_\sigma}(\mathbf{A}^\times_{X_\sigma}) \times G^{\Omega_\sigma}(\mathbf{A}^\times_{X_\sigma})$ such that $\mathrm{div}^G((\pi_\sigma(\alpha))$ and $\mathrm{div}^G(\pi_\sigma(\beta))$ have no common components we want to find a product $\alpha *_i \beta$ such that the following equality holds:

$$\begin{tikzcd}
(\alpha,\beta) \arrow[d, mapsto] \arrow[dr, mapsto] & \\
(\pi_\sigma(\alpha),\pi_\sigma(\beta)) \arrow[r, mapsto] & \alpha *_i \beta = \pi_\sigma(\alpha) * \pi_\sigma(\beta)
\end{tikzcd}$$

As a consequence of the symmetry of the $*$-product we will get also the symmetry of $*_i$. For any $\alpha = (a,b) \in G^{\Omega_\sigma}(\mathbf{A}^\times_{X_\sigma})$ let's put:

$$\xi(\alpha) := e^{\int_{X_\sigma} \log(ba\overline{a})\Omega_\sigma}$$

Definition 3.74. Let $\alpha = (a, b), \beta = (c, d) \in G^{\Omega_\sigma}(\mathbf{A}^{\times}_{X_\sigma})$, then the idelic $*$-product is defined as:

$$\alpha *_i \beta := -\sum_{x \in X_\sigma} v_x(c_x) \log\left(b(x)a(x)\overline{a(x)}\xi(\alpha)\right) + \log(\xi(\alpha))\,\mathrm{ideg}(c) + \log(\xi(\beta))\,\mathrm{ideg}(a)\,.$$

Proposition 3.75. $(\alpha, \beta) \in G^{\Omega_\sigma}(\mathbf{A}^{\times}_{X_\sigma}) \times G^{\Omega_\sigma}(\mathbf{A}^{\times}_{X_\sigma})$ *such that* $\mathrm{div}^G((\pi_\sigma(\alpha))$ *and* $\mathrm{div}^G(\pi_\sigma(\beta))$ *have no common component; then* $\alpha *_i \beta = \pi_\sigma(\alpha) * \pi_\sigma(\beta)$.

Proof. Put $g_1 = \pi_\sigma(\alpha)$ and $g_2 = \pi_\sigma(\beta)$, then by proposition D.22 we can write $g_1 = g_{1,0} + c_1$ and $g_2 = g_{2,0} + c_2$ for, $g_{1,0}, g_{2,0} \in G_0^{\Omega_\sigma}(X_\sigma)$, $c_1 = \log(\xi(\alpha))$ and $c_2 = \log(\xi(\beta))$. An easy calculation shows that:

$$g_1 * g_2 = \sum_{x \in X_\sigma} \mathrm{ord}_x^G(g_{2,0}) g_{1,0}(x) + c_1 \sum_{x \in X_\sigma} \mathrm{ord}_x^G(g_{2,0}) + c_2 \sum_{x \in X_\sigma} \mathrm{ord}_x^G(g_{1,0})\,.$$

Then it is enough to note the following equalities:

$$\mathrm{ord}_x^G(g_{1,0}) = \mathrm{ord}_x^G(g_1) = v_x(a_x)\,,$$

$$\mathrm{ord}_x^G(g_{2,0}) = \mathrm{ord}_x^G(g_2) = v_x(c_x)\,,$$

$$g_{1,0}(x) = g_1(x) - \log(\xi(\alpha)) = -\log(b(x)a(x)\overline{a(x)}) - \log(\xi(\alpha))\,.$$

□

Let's write an element $\alpha \in \mathbf{A}^{\times}_{\widehat{X}} = \mathbf{A}^{\times}_X \oplus \prod_{\sigma \in B_\infty}(\mathbf{A}^{\times}_{X_\sigma} \oplus \mathbf{A}^{\times}_{X_\sigma})$ in the following way:

$$\alpha = \alpha_X \times (\alpha_\sigma)_\sigma$$

with $\alpha_X \in \mathbf{A}^{\times}_X$ and $\alpha_\sigma \in \mathbf{A}^{\times}_{X_\sigma} \oplus \mathbf{A}^{\times}_{X_\sigma}$, then we have a surjective map:

$$\begin{aligned} \widehat{p} : \ker(d^1_\times) \oplus \prod_\sigma G(\mathbf{A}^{\times}_{X_\sigma}) &\rightarrow \mathrm{Div}(X) \oplus \bigoplus_\sigma G(X_\sigma) \\ \alpha = \alpha_X \times (\alpha_\sigma)_\sigma &\mapsto \left(p(\alpha_X), \sum_\sigma \pi_\sigma(\alpha_\sigma) X_\sigma\right) \end{aligned}$$

where $p : \ker(d^1_\times) \to \mathrm{Div}(X)$ is the usual projection on usual divisors and $\pi_\sigma : G(\mathbf{A}^{\times}_{X_\sigma}) \to G(X_\sigma)$ is the projection on Green functions.

Definition 3.76. Let's put

$$\mathrm{Div}\left(\mathbf{A}^{\times}_{\widehat{X}}\right) := \widehat{p}^{\,-1}\left(\mathrm{Div}_{\mathrm{Ar}}(X, \Omega)\right)\,,$$

and let $\alpha, \beta \in \operatorname{Div}\left(\mathbf{A}_{\widehat{X}}^{\times}\right)$ such that $(\widehat{p}(\alpha), \widehat{p}(\beta)) \in \Upsilon_{\mathrm{Ar}}$ then the *idelic Arakelov intersection pairing* is given by:

$$\alpha.\beta := \deg\left(\langle \alpha_X, \beta_X \rangle_i\right) + \frac{1}{2} \sum_{\sigma} \varepsilon_\sigma \, \alpha_\sigma *_i \beta_\sigma$$

where deg is the usual degree of line bundles, $\langle \, , \, \rangle_i$ is the idelic Deligne pairing and $\alpha_\sigma *_i \beta_\sigma$ is the idelic $*$-product.

We have to check that definition 3.76 gives the correct extension of the Arakelov pairing.

Theorem 3.77. *Let $\alpha, \beta \in \operatorname{Div}\left(\mathbf{A}_{\widehat{X}}^{\times}\right)$ such that $\widehat{p}(\alpha) = \widehat{D}$ and $\widehat{p}(\beta) = \widehat{E}$, with $(\widehat{D}, \widehat{E}) \in \Upsilon_{\mathrm{Ar}}$, then $\alpha.\beta = \widehat{D}.\widehat{E}$. In other words the idelic Arakelov intersection pairing extends to a pairing:*

$$\operatorname{Div}\left(\mathbf{A}_{\widehat{X}}^{\times}\right) \times \operatorname{Div}\left(\mathbf{A}_{\widehat{X}}^{\times}\right) \;\to\; \mathbb{R}$$

and the following diagram is commutative:

$$\begin{tikzcd}
\operatorname{Div}\left(\mathbf{A}_{\widehat{X}}^{\times}\right) \times \operatorname{Div}\left(\mathbf{A}_{\widehat{X}}^{\times}\right) \arrow[d, "\hat{p}\times\hat{p}"] \arrow[dr] & \\
\operatorname{Div}_{\mathrm{Ar}}(X) \times \operatorname{Div}_{\mathrm{Ar}}(X) \arrow[r] & \mathbb{R}
\end{tikzcd}$$

Proof. It follows easily from the definitions. □

Appendices

Appendix A

Complex differential geometry

This appendix is a short introduction to complex differential geometry. The reader is assumed to be familiar with real differential geometry and sheaf theory.

A.1 Complex manifolds and tangent spaces

There are two equivalent approaches to complex manifolds: "by charts" and "by sheaves". The former is very useful when one has to perform local explicit computation, whereas the latter gives general results is a very elegant and compact way. We will use indistinctly both approaches without any further specification since the setting will be clear from the context. Let's just recall the definition which employs sheaf theory:

Definition A.1 (complex manifold "by sheaves")**.** Let $(M, \mathscr{O}_M)$ be a locally $\mathbb{C}$-ringed space such that M is T2 and second countable. Then it is a n-*dimensional complex manifold* if there exists an open covering $M = \bigcup_i U_i$ such that for every index i there exist an open subspace $Y \subseteq \mathbb{C}^n$ and an isomorphism of locally $\mathbb{C}$-ringed spaces:

$$(U_i, \mathscr{O}_M|_{U_i}) \cong (Y, \mathscr{O}_Y^{\text{hol}})\,.$$

Here $\mathscr{O}_Y^{\text{hol}}$ is te sheaf of holomorphic functions on Y.

A n-dimensional complex manifold M is in the obvious way a real manifold of dimension $2n$, in particular at each point $p \in M$, the *real tangent space* T_pM is a $\mathbb{R}$-vector space of dimension $2n$. If $\left\{ \frac{\partial}{\partial x_1}\big|_p, \dots, \frac{\partial}{\partial x_n}\big|_p, \frac{\partial}{\partial y_1}\big|_p, \dots, \frac{\partial}{\partial y_n}\big|_p \right\}$ is a real basis for T_pM, then the maps:

$$\left.\frac{\partial}{\partial x_j}\right|_p \mapsto \left.\frac{\partial}{\partial y_j}\right|_p$$

$$\left.\frac{\partial}{\partial y_j}\right|_p \mapsto -\left.\frac{\partial}{\partial x_j}\right|_p$$

induce an *almost complex structure* I on T_pM, i.e. an endomorphism such that $I^2 = -\operatorname{id}$. Now consider the *complexified tangent space* $(T_pM)_{\mathbb{C}} := T_pM \otimes_{\mathbb{R}} \mathbb{C}$ which has complex dimension $2n$, and let's extend I by linearity on $(T_pM)_{\mathbb{C}}$. The eigenvectors of $I : (T_pM)_{\mathbb{C}} \to (T_pM)_{\mathbb{C}}$ are i and $-i$, and their eigenspaces are called respectively $T_pM^{1,0}$ and $T_pM^{0,1}$ (they are both complex vector spaces of dimension n); note that $\overline{T_pM^{1,0}} = T_pM^{0,1}$. Clearly we have the direct decomposition:

$$(T_pM)_{\mathbb{C}} = (T_pM)^{1,0} \oplus (T_pM)^{0,1}\,. \tag{A.1}$$

usually $(T_pM)^{1,0}$ is called the *holomorphic tangent space* and $(T_pM)^{0,1}$ the *antiholomorphic tangent space*. Moreover, if we put $\frac{\partial}{\partial z_j}\big|_p := \frac{\partial}{\partial x_j}\big|_p - i\frac{\partial}{\partial y_j}\big|_p$, and $\frac{\partial}{\partial \bar{z}_j}\big|_p := \frac{\partial}{\partial x_j}\big|_p + i\frac{\partial}{\partial y_j}\big|_p$, then

$$T_pM^{1,0} = \left\langle \left.\frac{\partial}{\partial z_1}\right|_p, \dots, \left.\frac{\partial}{\partial z_n}\right|_p \right\rangle$$

$$T_pM^{0,1} = \left\langle \left.\frac{\partial}{\partial \bar{z}_1}\right|_p, \dots, \left.\frac{\partial}{\partial \bar{z}_n}\right|_p \right\rangle$$

The *complexified cotangent space* is $(T_pM^*)_{\mathbb{C}} = (T_pM)^*_{\mathbb{C}}$ and we have the decomposition

$$(T_pM^*)_{\mathbb{C}} = (T_pM^*)^{1,0} \oplus (T_pM^*)^{0,1}\,. \tag{A.2}$$

where if $\{dx_1|_p, \dots dx_n|_p, dy_1|_p, \dots dy_n|_p\}$ is the dual basis for T_pM^* and we put $dz_j|_p := dx_j|_p + dy_j|_p$, $d\bar{z}_j|_p := dx_j|_p - dy_j|_p$, then:

$$(T_pM^*)^{1,0} = \langle dz_1|_p, \dots, dz_n|_p \rangle\ ,$$

$$(T_pM^*)^{0,1} = \langle d\bar{z}_1|_p, \dots, d\bar{z}_n|_p \rangle\ .$$

From now on we fix a n-dimensional complex manifold M.

A.2 Holomorphic vector bundles and locally free sheaves

Definition A.2. A *holomorphic vector bundle* of rank r over M is a complex manifold E equipped with a holomorphic surjective morphism $\pi : E \to M$ satisfying the following conditions:

- For any point $p \in M$, the fibre $E_x := \pi^{-1}(x)$ is a complex vector space of dimension r.
- For any point $x \in M$, there exsist an open neighborhood U and a biholomorphic map $\phi_U : \pi^{-1}(U) \to U \times \mathbb{C}^r$ such that the followind diagram is commutative:

$$\begin{tikzcd}
\pi^{-1}(U) \arrow[rr, "\phi_U"] \arrow[dr, "\pi"'] & & U \times \mathbb{C}^r \arrow[dl, "\pi_1"] \\
& U &
\end{tikzcd}$$

  where π_1 is the projection on the first component. The couple (U, ϕ_U) is called *trivialization chart.*
- The restriction $\phi_U : E_x \to \{x\} \times \mathbb{C}^r$ is an isomorphism of complex vector spaces.

A holomorphic [resp. C^∞] section of E is a holomorphic [resp. C^∞] map $s : M \to E$ such that $\pi \circ s = \text{id}$. Two holomorphic vector bundles $\pi : E \to M$ and $\pi' : E \to M$ are isomorphic if there is a biholomorphic map $\theta : E \to E'$ such that $\pi' \circ \theta = \pi$. A holomorphic vector bundle of rank 1 is called a *line bundle.*

Definition A.3. A C^∞ *complex vector bundle*, or by simplicity a *complex vector bundle*, is the same object described in definition A.2, but where we substitute the word “holomorphic” with “C^∞”.

Example A.4. The spaces $(TM)^{1,0} := \bigsqcup_{p\in M}(T_pM)^{1,0}$, $(TM)^{0,1} := \bigsqcup_{p\in M}(T_pM)^{0,1}$ and their duals have a unique structure of holomorphic vector bundles over M (of rank n). $(TM)^{1,0}$ is called the *holomorphic tangent bundle.*

If (U, ϕ_U) and (V, ϕ_V) are two trivialization charts of $\pi : E \to M$ such that $U \cap V \neq \emptyset$, there is a holomorphic map $t : U \cap V \to \mathrm{GL}(\mathbb{C}, r)$, satisfying the following property: the composition

$$\phi_V \circ \phi_U^{-1} : (U \cap V) \times \mathbb{C}^r \to (U \cap V) \times \mathbb{C}^r$$

arising by the commutative diagram

$$\begin{tikzcd}
(U \cap V) \times \mathbb{C}^r \arrow[dr, "\pi_1"'] & \pi^{-1}(U \cap V) \arrow[l, "\phi_U"'] \arrow[r, "\phi_V"] \arrow[d, "\pi"] & (U \cap V) \times \mathbb{C}^r \arrow[dl, "\pi_1"] \\
& U \cap V &
\end{tikzcd}$$

is given by

$$\phi_V \circ \phi_U^{-1}(x, v) = (x, t(x)v)\,.$$

The map $t : U \cap V \to \mathrm{GL}(\mathbb{C}, r)$ is called *transition map* relative to U and V. Suppose that $\{(U_\alpha, \phi_\alpha)\}$ is a set of trivialization charts such that $M = \bigcup_\alpha U_\alpha$, moreover we denote with $t_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(\mathbb{C}, r)$ the transition map relative to U_α and U_β. Then it is not difficult to show that for any three indices α, β, γ we have the so called *cocycle relation*:

$$t_{\alpha\beta} \circ t_{\beta\gamma} = t_{\alpha\gamma} \quad \text{on } U_\alpha \cap U_\beta \cap U_\gamma \tag{A.3}$$

Viceversa suppose on M we give the following glueing data:

- An open cover $M = \bigcup_\alpha U_\alpha$.
- Set of holomorphic maps $\{t_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(\mathbb{C}, r)\}_{\alpha,\beta}$ satisfying the cocycle relation (A.3).

Then there is a vector bundle or rank r $\pi : E \to M$ with trivialization charts $\{(U_\alpha, \phi_\alpha)\}$ such that the transition maps are the functions $t_{\alpha\beta}$. The total space E is given by:

$$E := \bigsqcup_\alpha U_\alpha \times \mathbb{C}^r \big/ \sim$$

where we say that $(x, v)_\alpha \sim (y, w)_\beta$ if $x = y$ and $w = t_{\alpha\beta}(x)v$. Two glueing data $\{t_{\alpha\beta}\}$ $\{t'_{\alpha\beta}\}$ on a fixed open cover $M = \bigcup_\alpha U_\alpha$ are called equivalent if there is a collection of holomorphic maps $\{\lambda_\alpha : U_\alpha \to \mathrm{GL}(\mathbb{C}, r)\}$ such that $t_{\alpha\beta} = \lambda_\alpha t'_{\alpha\beta} \lambda_\beta^{-1}$. Equivalent glueing data define isomorphic vector bundles. In other words a set of transition maps on a open covering (up to equivalence) satisfying the cocycle relation determines univocally (up to isomorphism) a holomorphic vector bundle.

Definition A.5. A C^∞ *hermitian vector bundle* (E, h) is a holomorphic vector bundle $\pi : E \to M$ equipped with a collection $h = \{h_x\}_{x \in M}$ of hermitian inner products $h_x : E_x \times E_x \to \mathbb{C}$ such that for any couple of local holomorphic sections $s, t : U \subseteq M \to E$, the map:

$$\begin{array}{rcl} h(s,t) : U & \to & \mathbb{C} \\ x & \mapsto & h_x(s(x), t(x)) \end{array}$$

is C^∞ on U. The collection $h = \{h_x\}_{x \in M}$ is an *hermitian metric* on E.

Remark A.6. A hermitian inner product $h_x : E_x \times E_x \to \mathbb{C}$ is a positive definite bilinear map $h_x : E_x \times \overline{E}_x \to \mathbb{C}$. But we have an isomorphism of vector spaces

$$\begin{array}{rcl} E_x^* \otimes \overline{E}_x^* & \to & B(E_x, \overline{E}_x; \mathbb{C}) \\ \eta_1 \otimes \eta_2 & \mapsto & \big[(v, w) \mapsto \eta_1(v)\eta_2(w)\big] . \end{array}$$

Therefore an hermitian metric h on E can be described as a C^∞ section:

$$\begin{array}{rcl} h : M & \to & E^* \otimes \overline{E}^* \\ x & \mapsto & h_x \end{array}$$

such that $h_x(\eta, \bar{\eta}) > 0$ for any $x \in M$ and any $\eta \neq 0$.

Remark A.7. Any holomorphic vector bundle admits a C^∞ hermitian metric (see [28, Proposition 4.1.4]).

If we consider a complex manifold $(M, \mathscr{O}_M)$ in the category of locally $\mathbb{C}$-ringed spaces then the equivalent object of an holomorphic vector bundle of rank r is a locally free $\mathscr{O}_M$-module of rank r. Now we briefly establish this correspondence: suppose that $\mathscr{E}$ is a locally free sheaf of rank r, then for any $x \in M$ we put

$$E_x := {}^{\mathscr{E}_x}\!\big/_{\mathfrak{m}_x \mathscr{E}_x} \cong \mathscr{E}_x \otimes_{\mathscr{O}_{M,x}} \mathbb{C} \cong \mathbb{C}^r$$

so $E = \bigsqcup_{x \in M} E_x$ and $\pi : E \to M$ is given by $\pi(s_x) = x$. Since $\mathscr{E}$ is locally free, there is an open covering $M = \bigcup_\alpha U_\alpha$ such that $\mathscr{E}|_{U\alpha} \cong \mathscr{O}_M^r$; moreover fix a local basis $\{b_1^\alpha, \ldots, b_r^\alpha\}$ of $\mathscr{E}$ over U_α then the structure of holomorphic vector bundle on E is induced by the following commutative diagram (for each α):

$$\begin{tikzcd} \pi^{-1}(U) = \bigsqcup_{x \in U_\alpha} E_x \arrow[rr, "\phi_U"] \arrow[dr, "\pi"'] & & U_\alpha \times \mathbb{C}^r \arrow[dl, "\pi_1"] \\ & U_\alpha & \end{tikzcd}$$

$$\begin{tikzcd} s_x = \lambda_1 b_{1,x}^\alpha + \ldots + \lambda_r b_{r,x}^\alpha \arrow[rr, mapsto, "\phi_U"] \arrow[dr, mapsto, "\pi"'] & & (x, (\lambda_1, \ldots, \lambda_r)) \arrow[dl, mapsto, "\pi_1"] \\ & x & \end{tikzcd}$$

Vice versa, given a holomorphic vector bundle $\pi : E \to M$ then the locally free sheaf is given by:

$$\mathscr{E}(U) := \{ s : U \to E \ : \ s \text{ is holomorphic and } \pi \circ s = \mathrm{id} \} .$$

Namely $\mathscr{E}$ is the sheaf of the holomorphic local sections of E.

Remark A.8. In this text we will deal mostly with the case $r = 1$, where we have the correspondence between line bundles and invertible sheaves.

Clearly there is the equivalent of the definition A.5 in the category of sheaves.

Definition A.9. A *C^∞ hermitian locally free $\mathscr{O}_M$-module* is a couple $(\mathscr{E}, h)$ where, $\mathscr{E}$ is a locally free $\mathscr{O}_M$-module and $h = \{h_x\}_{x\in M}$ is a collection of hermitian inner products $h_x : E_x \times E_x \to \mathbb{C}$ (remember that $E_x := {}^{\mathscr{E}_x}\!/\!_{\mathfrak{m}_x \mathscr{E}_x}$) such that for any open set $U \subseteq M$ and any two elements $s, t \in \mathscr{E}(U)$, the map

$$\begin{array}{rcl} h(s,t) : U & \to & \mathbb{C} \\ x & \mapsto & h_x(s(x), t(x)) \end{array}$$

is C^∞. Here by the notations $s(x)$ and $t(x)$ we mean the canonical images of s and t in E_x. The collection $h = \{h_x\}_{x\in M}$ is called *an hermitian metric on $\mathscr{E}$*.

Clearly any locally free $\mathscr{O}_M$-module admits an hermitian metric thanks to remark A.7.

A.3 Differential forms

A *C^∞ differential form of bidegree (p,q)*, or simply a *(p,q)-differential form* is a C^∞ section of the complex vector bundle:

$$\bigwedge^{p,q} M := \bigwedge^{p} (TM^*)^{1,0} \otimes \bigwedge^{q} (TM^*)^{0,1} .$$

On the other hand the C^∞ sections of $\bigwedge^k (TM^*)_{\mathbb{C}}$ are the *C^∞ (complex) k-differential forms*. Since equation (A.2) holds, then we have a canonical isomorphism of vector bundles:

$$\bigoplus_{p+q=k} \bigwedge^{p,q} M \cong \bigwedge^{k} (TM^*)_{\mathbb{C}} .$$

Therefore, on a chart, a (p, q)-form can be written as:

$$\omega = \sum_{\substack{i_1 \le \ldots \le i_p \\ j_1 \le \ldots \le j_q}} \omega_{i_1,\ldots,i_p,j_1,\ldots,j_q} dz_{i_1} \wedge \ldots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \ldots \wedge d\bar{z}_{j_q} \tag{A.4}$$

where the coefficients $\omega_{i_1,\ldots,i_p,j_1,\ldots,j_q}$ are C^∞ functions on the chart. Let's introduce the multi-index notations:

$$I := (i_1, \ldots, i_p \colon i_1 \leq \ldots \leq i_p)\,,$$

$$J := (j_1, \ldots, j_q \colon j_1 \leq \ldots \leq j_q)\,,$$

$$dz_I := dz_{i_1} \wedge \ldots \wedge dz_{i_p} \quad \text{for } I = (i_1, \ldots i_p)\,,$$

$$d\bar{z}_J := d\bar{z}_{j_1} \wedge \ldots \wedge d\bar{z}_{j_q} \quad \text{for } J = (j_1, \ldots j_q)\,,$$

Moreover let's indicate with $|I|$ and $|J|$ the length of the sequences I and J respectively. Then equation (A.4) can be written in the following compact form:

$$\omega = \sum_{|I|=p,|J|=q} \omega_{I,J} dz_I \wedge d\bar{z}_J\,.$$

A complex manifold is always orientable, and we fix an orientation on M, therefore one can define the integral of a (n, n)-form ω. By simplicity we assume that $\omega = f\, dz_1 \wedge \ldots \wedge dz_n \wedge d\bar{z}_1 \ldots \wedge \bar{d}z_n$ is a compactly supported on a chart with domain $U \subseteq M$, then the integral can be generalized to any ω by using the partition of unity. If $f = \alpha + i\beta$ with α and β C^∞ functions on M, then

$$\omega = (-2i)^n(\alpha + i\beta)\, dx_1 \wedge \ldots \wedge dx_n \wedge dy_1 \wedge \ldots \wedge dy_n\,.$$

Let's define the real $2n$-forms

$$\Re(\omega) := \alpha\, dx_1 \wedge \ldots \wedge dx_n \wedge dy_1 \wedge \ldots \wedge dy_n\,,$$

$$\Im(\omega) := \beta\, dx_1 \wedge \ldots \wedge dx_n \wedge dy_1 \wedge \ldots \wedge dy_n\,,$$

at this point $\int_M \omega$ can be defined by using the integration theory on real manifolds:

$$\int_M \omega = (-2i)^n \int_M (\Re(\omega) + i\Im(\omega)) = (-2i)^n \left(\int_M \Re(\omega) + i \int_M \Im(\omega) \right)\,.$$

The C^∞ local sections of a complex vector bundle define a sheaf, hence we have the following sheaves of $\mathbb{C}$-vector spaces:

$$\mathscr{A}^{p,q}(U) := \{C^\infty \text{ differential forms of bidegree } (p, q) \text{ on } U\}\,,$$

$$\mathscr{A}^k(U) := \{C^\infty\ k\text{-differential forms on } U\}\,.$$

Let's define the operators:

$$\partial : \mathscr{A}^{p,q}(U) \to \mathscr{A}^{p+1,q}(U)\,,$$

$$\bar{\partial} : \mathscr{A}^{p,q}(U) \to \mathscr{A}^{p,q+1}(U)\,,$$
$$d = \partial + \bar{\partial} : \mathscr{A}^{p,q}(U) \to \mathscr{A}^{p+1,q}(U) \oplus \mathscr{A}^{p,q+1}(U)\,,$$

such that, locally on a chart we have

$$\partial\omega = \partial\left(\sum_{|I|=p,|J|=q} \omega_{I,J} dz_I \wedge d\bar{z}_J\right) = \sum_{|I|=p,|J|=q} \sum_{r=1}^{n} \frac{\partial \omega_{I,J}}{\partial z_r} dz_r \wedge dz_I \wedge d\bar{z}_J\,,$$

$$\bar{\partial}\omega = \bar{\partial}\left(\sum_{|I|=p,|J|=q} \omega_{I,J} dz_I \wedge d\bar{z}_J\right) = \sum_{|I|=p,|J|=q} \sum_{r=1}^{n} \frac{\partial \omega_{I,J}}{\partial \bar{z}_r} d\bar{z}_r \wedge dz_I \wedge d\bar{z}_J\,,$$

Proposition A.10. *The following properties hold:* $\partial^2 = 0$, $\bar{\partial}^2 = 0$, $\partial\bar{\partial} + \bar{\partial}\partial = 0$. *In particular also* $d^2 = (\partial + \bar{\partial})^2 = 0$.

Proof. It is enough to do the calculation locally and use the properties of the wedge product. ☐

Let's fix any $p \in \{0, \dots, n\}$, then we have the following complex of $\mathbb{C}$-vector spaces

$$\cdots \longrightarrow \mathscr{A}^{p,q-1}(U) \xrightarrow{\bar{\partial}} \mathscr{A}^{p,q}(U) \xrightarrow{\bar{\partial}} \mathscr{A}^{p,q+1}(U) \longrightarrow \cdots$$

Define

$$Z^{p,q}(U) := \{\omega \in \mathscr{A}^{p,q}(U) \colon \bar{\partial}(\omega) = 0\}$$

then we have the *Dolbeaut cohomology groups*

$$H^{p,q}_{\bar{\partial}}(U) := Z^{p,q}(U) \big/ \bar{\partial}(\mathscr{A}^{p,q-1}(U))$$

Clearly $H^{p,q}_{\bar{\partial}}(U) = 0$ if $q > n$ and moreover by convention we put $\bar{\partial}(\mathscr{A}^{p,0}(U)) = 0$. The elements of $H^{p,0}_{\bar{\partial}}(U)$ are the *holomorphic p-forms* on U and locally they can be written as $\omega = \sum_{|I|=p} \omega_I dz_I$ with ω_I holomorphic function.

Other important operators on differential forms which are worth mentioning are the followings:

$$d^c := \frac{1}{4\pi i}(\partial - \bar{\partial}) : \mathscr{A}^{p,q}(U) \to \mathscr{A}^{p+1,q}(U) \oplus \mathscr{A}^{p,q+1}(U) \qquad \text{(A.5)}$$

$$dd^c = \frac{i}{2\pi}\partial\bar{\partial} : \mathscr{A}^{p,q}(U) \to \mathscr{A}^{p+1,q+1}(U) \qquad \text{(A.6)}$$

Remark A.11. The normalization $\frac{1}{4\pi i}$ in the definition of d^c is not standard and in the literature one may find many variants.

A.4 Hermitian and Kähler manifolds

Definition A.12. An *hermitian manifold* is a couple (M, h) where M is a complex manifold and h is an hermitian metric on the holomorphic tangent bundle $TM^{1,0}$. Sometimes we simply say that h is an hermitian metric on M.

By using remark A.6 we locally can write:

$$h = \sum_{j \leq n,\, k \leq n} h_{jk}\, dz_j \otimes d\bar{z}_k$$

where h_{jk} is a C^∞ function and for any $x \in M$ the matrix $(h_{jk}(x))_{jk}$ is hermitian and positive definite. The *fundamental form associated to* h is the $(1,1)$-form which is defined locally by

$$\Omega := \frac{i}{2} \sum_{j \leq n,\, k \leq n} h_{jk}\, dz_j \wedge d\bar{z}_k$$

note that thanks to the properties of h_{jk} it follows that $\Omega = \overline{\Omega}$ (i.e. Ω is real)

Remark A.13. An hermitian manifold is often indicated as the couple (M, Ω).

Definition A.14. An hermitian metric h on M with fundamental form Ω is called a *Kähler metric* if $d\Omega = 0$, in this case Ω is called a *Kähler form.* An hermitian manifold is a *Kähler manifold* if it admits a Kähler metric.

Remark A.15. On a Riemann surface an hermitian metric is always Kälher because the exterior derivative of a $(1,1)$-form is always zero.

The following theorem says that an hermitian metric is Kähler if and only if it is "euclidean at the first order".

Theorem A.16. *An hermitian metric h with fundamental form Ω is Kähler if and only if for each point $x \in M$ there is a holomorphic chart $(U, z = (z_1, \ldots, z_n))$ centred in x such that on U:*

$$\Omega = \frac{i}{2} \sum_{j \leq n,\, k \leq n} \omega_{jk}\, dz_j \wedge d\bar{z}_k \quad \textit{with} \;\; \omega_{jk} = \delta_{jk} + o(|z|^2)$$

where $\delta_{jk} = 1$ if $j = k$ and $\delta_{jk} = 0$ otherwise.

Proof. See cite[Theorem 4.8]Dema. □

Remember that if $(\omega, \eta) \in \bigwedge^{p,q} M \times \bigwedge^{r,s} M$, then $\omega \wedge \eta \in \bigwedge^{p+r,q+s} M$, in particular

$$\omega^k := \underbrace{\omega \wedge \ldots \wedge \omega}_{k\text{-times}} \in \bigwedge^{kp,kq} M\,.$$

Let (M, Ω) be an hermitian manifold; let's define the (n, n)-form

$$dV := \frac{1}{n!}\Omega^n$$

which is called the *hermitian volume element.*

Remark A.17. Here the notation dV is purely symbolic, we are not claiming that this is the exterior derivative of a form V.

For any integer $1 \leq r \leq n$ and any point $p \in M$ there is a hermitian product:

$$\langle\ ,\ \rangle_p : \bigwedge^r (T_pM^*)_{\mathbb{C}} \times \bigwedge^r (T_pM^*)_{\mathbb{C}} \to \mathbb{C}$$

such that

$$\langle \omega_1 \wedge \ldots \wedge \omega_r, \eta_1 \wedge \ldots \wedge \eta_r \rangle_p := \det((h_p(\omega_j|_p, \eta_k|_p))_{jk})\,.$$

This induces naturally a map

$$\langle\ ,\ \rangle : \bigwedge^r (TM^*)_{\mathbb{C}} \times \bigwedge^r (TM^*)_{\mathbb{C}} \to C^\infty(M)\,.$$

Given a (p, q)-form $\eta \in \bigwedge^{p,q} M$ there is a unique $(n-q, n-p)$-form, that we denote as $\star\eta$, such that

$$\omega \wedge \overline{\star\eta} = \langle \omega, \eta \rangle\, dV \quad \text{for any } \omega \in \bigwedge^{p,q} M\,.$$

This defines the so called *Hodge star operator*:

$$\begin{array}{rcl} \star : \bigwedge^{p,q} M & \to & \bigwedge^{n-q,n-p} M \\ \eta & \mapsto & \star\eta \end{array}$$

At this point we can define some additional operators for differential forms:

$$\partial^\star := -\star\bar{\partial}\star : \mathscr{A}^{p,q}(U) \to \mathscr{A}^{p-1,q}(U)\,,$$

$$\bar{\partial}^\star := -\star\partial\star : \mathscr{A}^{p,q}(U) \to \mathscr{A}^{p,q-1}(U)\,,$$

$$d^\star := -\star d\star : \mathscr{A}^{p,q}(U) \to \mathscr{A}^{p,q-1}(U) \oplus \mathscr{A}^{p-1,q}(U)\,,$$

$$\Delta_\partial := \partial\partial^\star + \partial^\star\partial : \mathscr{A}^{p,q}(U) \to \mathscr{A}^{p,q}(U)\,,$$

$$\Delta_{\bar{\partial}} := \bar{\partial}\bar{\partial}^\star + \bar{\partial}^\star\bar{\partial} : \mathscr{A}^{p,q}(U) \to \mathscr{A}^{p,q}(U)\,,$$

$$\Delta_d := dd^\star + d^\star d : \mathscr{A}^{p,q}(U) \to \mathscr{A}^{p+1,q-1}(U) \oplus \mathscr{A}^{p,q}(U) \oplus \mathscr{A}^{p-1,q+1}(U)\,.$$

The operators Δ_∂, $\Delta_{\bar\partial}$ and Δ_d are called respectively *∂-laplacian*, *$\bar\partial$-laplacian* and *d-laplacian.* Moreover for any open set $U \subseteq M$ we can define the spaces:;

$$\mathcal{H}^{p,q}_{\partial}(U) := \{\eta \in \mathscr{A}^{p,q}(U) \colon \Delta_\partial(\eta) = 0\}\,,$$

$$\mathcal{H}^{p,q}_{\bar\partial}(U) := \{\eta \in \mathscr{A}^{p,q}(U) \colon \Delta_{\bar\partial}(\eta) = 0\}\,,$$

$$\mathcal{H}^{p,q}_{d}(U) := \{\eta \in \mathscr{A}^{p,q}(U) \colon \Delta_d(\eta) = 0\}\,,$$

which are respectively the spaces of *∂-harmonic forms*, *$\bar\partial$-harmonic forms* and *d-harmonic forms.*
When M is compact the map:

$$\begin{array}{rcl} [\ ,\ ]^{p,q} : \mathscr{A}^{p,q}(M) & \to & \mathbb{C} \\ (\omega,\eta) & \mapsto & \displaystyle\int_M \omega \wedge \overline{\star\eta} \end{array}$$

defines an hermitian inner product.

Theorem A.18 (Hodge's theorem)**.** *If M is a compact hermitian manifold then the following statements hold for any p, q:*

(1) *$\mathcal{H}^{p,q}_{\bar\partial}$ is a finite dimensional complex vector space. In particular there exists the orthogonal projection $P^{p,q}_{\bar\partial} : \mathscr{A}^{p,q}(M) \to \mathcal{H}^{p,q}_{\bar\partial}(M)$*

(2) *There is a unique operator $G_{\bar\partial} : \mathscr{A}^{p,q}(M) \to \mathscr{A}^{p,q}(M)$ such that:*

  (2.1) *$G_{\bar\partial}(\mathcal{H}^{p,q}_{\bar\partial}) = 0$, $\bar\partial G_{\bar\partial} = G_{\bar\partial}\bar\partial$ and $\bar\partial^\star G_{\bar\partial} = G_{\bar\partial}\bar\partial^\star$.*

  (2.2) *$P^{p,q}_{\bar\partial} + \Delta_{\bar\partial} G_{\bar\partial} = \mathrm{id}_{\mathscr{A}^{p,q}(M)}$.*

Proof. See [25, 0.6, "Hodge Thoerem"]. □

Corollary A.19. *Let (M, Ω) be a compact hermitian manifold. If $\int_M f\Omega^n = 0$ for a function $f \in C^\infty(M)$, then there is $g \in C^\infty(M)$ such that $f = \Delta_{\bar\partial}(g)$.*

Proof. First of all one shows that the map:

$$f \mapsto \frac{\int_M f\Omega^n}{\int_M \Omega^n}$$

is the orthogonal projection $P^{0,0}_{\bar\partial} : C^\infty(M) \to \mathcal{H}^{0,0}_{\bar\partial}(M) = \mathbb{C}$. Then by property (2.2) of theorem A.18:

$$\frac{\int_M f\Omega^n}{\int_M \Omega^n} + \Delta_{\bar\partial} G_{\bar\partial}(f) = f\,.$$

By putting $g := G_{\bar\partial}(f)$ the corollary is proved. □

Finally, as a consequence of Hodge theorem we get the following useful proposition which relates the $\bar{\partial}$-Laplacian and the operator $\partial\bar{\partial}$ when M is compact and Kähler

Proposition A.20. *Let (M, Ω) be a compact Kähler manifold, then:*

$$\Delta_{\bar{\partial}}(f)\Omega^n = -ni\partial\bar{\partial}(f) \wedge \Omega^{n-1} \quad \text{for } f \in C^\infty(M)\,.$$

Proof. See [43, proposition 1.37]. ☐

A.5 Linear connections and curvature

In this subsection we consider M with its C^∞ structure and not the holomorphic one, in other words we work on the ringed space $(M, \mathscr{A})$, where $\mathscr{A}$ is the sheaf of complex valued C^∞ functions on M. Let $\mathscr{E}$ be a locally free $\mathscr{A}$-modules on M of rank r and let's define the *sheaf of C^∞ differential forms with values in $\mathscr{E}$*:

$$\mathscr{A}^{p,q}(\mathscr{E}) := \mathscr{A}^{p,q} \otimes_{\mathscr{A}} \mathscr{E}$$

Moreover let's put

$$\mathscr{A}^k(\mathscr{E}) := \bigoplus_{p+q=k} \mathscr{A}^{p,q}(\mathscr{E})\,,$$

$$E^\bullet(\mathscr{E}) := \bigoplus_k \mathscr{A}^k(\mathscr{E}).$$

Clearly $\mathscr{A}^0(\mathscr{E}) = \mathscr{E}$. One can show that for any open set U:

$$\mathscr{A}^k(\mathscr{E})(U) = \mathscr{A}^k(U) \otimes_{\mathscr{A}(U)} \mathscr{E}(U)\,.$$

Any element $\eta \in \mathscr{A}^k(\mathscr{E})(U)$ can be written as a finite sum $\eta = \sum_j \omega_j \otimes s_j$ for $\omega_j \in \mathscr{A}^k(U)$ and $s_j \in \mathscr{A}^0(\mathscr{E})(U)$. Moreover if $\xi \in \mathscr{A}^r(U)$ then

$$\xi \wedge \eta := \sum_j (\xi \wedge \omega_j) \otimes s_j \in \mathscr{A}^{k+r}(\mathscr{E})(U)\,.$$

Definition A.21. A *(linear) connection* on $\mathscr{E}$ is a $\mathbb{C}$-linear map of sheaves:

$$\nabla : E^\bullet(\mathscr{E}) \to E^\bullet(\mathscr{E})$$

which satisfies the following two conditions:

- If we still denote with ∇ the restriction $\nabla|_{\mathscr{A}^k(\mathscr{E})}$, then

$$\nabla : \mathscr{A}^k(\mathscr{E}) \to \mathscr{A}^{k+1}(\mathscr{E})\,.$$

- For any $\omega \in \mathscr{A}^k(U)$ and $s \in \mathscr{A}^0(\mathscr{E})(U)$, the following Leibniz rule holds:
$$\nabla(\omega \otimes s) = d\omega \otimes s + (-1)^k \omega \wedge \nabla(s)\,.$$

Suppose that $\{e_i\}_{i=1,\dots,r}$ is a local basis for $\mathscr{E}$ on U and let $s \in \mathscr{A}^0(E)(U)$ then by the Leibniz rule applied in the case $k=0$:

$$\nabla(s) = \nabla\left(\sum_{i=1}^{r} f_j e_j\right) = \sum_{i=1}^{r} df_i \otimes e_i + f_i \nabla(e_i) \tag{A.7}$$

for some functions $f_i \in \mathscr{A}(U)$. But $\nabla(e_i) = \sum_{j=1}^{r} \theta_{ij} \otimes e_j$ for $\theta_{ij} \in \mathscr{A}^1(U)$, hence by substituting in equation (A.7) and by changing the indexes we obtain the local explicit expression of ∇:

$$\nabla(s) = \sum_{j=1}^{r}\left(df_j + \sum_{i=1}^{r} f_i\theta_{ij}\right) \otimes e_j\,. \tag{A.8}$$

The matrix of 1-forms (θ_{ij}) is called *the connection matrix on* U, it obviously depends on the choice of the local basis $\{e_i\}$, but it is not difficult to study how it changes under a different choice. Here the important point is that a connection ∇ can be determined entirely from the data $\{e_i\}$ and (θ_{ij}).

Remark A.22. A connection ∇ on $\mathscr{E}$ can be used to calculate the "derivative along a fixed direction" of global sections $s \in \Gamma(\mathscr{A}^0(\mathscr{E}))$. Indeed let's fix a C^∞ vector field $Y : M \to TM_{\mathbb{C}}$, it will be the direction of derivation, then the *covariant derivative along* Y is the map:

$$\begin{array}{rcl} \nabla_Y : \Gamma(\mathscr{A}^0(\mathscr{E})) & \to & \Gamma(\mathscr{A}^0(\mathscr{E})) \\ s & \mapsto & \nabla(s)Y \end{array}$$

The element $\nabla(s)Y$ is defined by means contraction of tensor fields, that is if $\nabla(s) = \sum_j \omega_j \otimes s_j$ for $\omega_j \in \Gamma(\mathscr{A}^1)$, then

$$\nabla(s)Y = \sum_j (\omega_j Y) s_j$$

where $\omega_j Y \in C^\infty(M)$ is such that $(\omega_j Y)(x) = \omega_{j,x}(Y_x)$.

The local expression (A.8) of the connection operator implies that any C^∞ vector bundle admits a linear connection, but if $\mathscr{E}$ is hermitian and ∇ satisfies certain compatibility conditions there is a unique connection on $\mathscr{E}$. Let's define two operators:

$$\bar{\partial}_{\mathscr{E}} : \mathscr{A}^{p,q}(\mathscr{E}) \to \mathscr{A}^{p,q+1}(\mathscr{E})$$

given by $\bar{\partial}_{\mathscr{E}}(\omega \otimes s) := \bar{\partial}\omega \otimes s$ for $\omega \in \mathscr{A}^{p,q}(U)$ and $s \in \mathscr{E}(U)$. If $(\mathscr{E}, h)$ is hermitian consider

$$\{,\} : \mathscr{A}^k(\mathscr{E}) \otimes \mathscr{A}^j(\mathscr{E}) \to \mathscr{A}^{k+j}$$

defined by $\{\omega \otimes s, \eta \otimes t\} := h(s,t)\omega \wedge \bar{\eta}$.

Definition A.23. Let $\bar{\mathscr{E}} = (\mathscr{E}, h)$ be an hermitian locally free sheaf. A connection ∇ on $\bar{\mathscr{E}}$ is called *compatible* or an *h-connection* if the following conditions are satisfied:

- Consider the map $\nabla : \mathscr{A}^0(\mathscr{E}) \to \mathscr{A}^1(\mathscr{E}) = \mathscr{A}^{1,0}(\mathscr{E}) \oplus \mathscr{A}^{0,1}(\mathscr{E})$ which can be obviuosly decomposed in a sum $\nabla^{1,0} + \nabla^{0,1}$. We require that $\nabla^{0,1} = \bar{\partial}_{\mathscr{E}}$.
- For any $\eta \in \mathscr{A}^k(\mathscr{E})$ and $\xi \in A^j(\mathscr{E})$:
$$d\{\eta, \xi\} = \{\nabla(\eta), \xi\} + (-1)^k\{\eta, \nabla(\xi)\}$$

Theorem A.24. *If $\bar{\mathscr{E}} = (\mathscr{E}, h)$ is an hermitian locally free $\mathscr{A}$-module, then there exists a unique h-connection on $\bar{\mathscr{E}}$.*

Proof. See [25, 0.5, "Metrics..."]. ☐

Remark A.25. From now on if ∇ is a connection on an hermitian locally free sheaf, we will tacitly assume that it is the h-connection.

Definition A.26. Let ∇ be a connection on $\mathscr{E}$. The map

$$\nabla^2 : \mathscr{A}^0(\mathscr{E}) \xrightarrow{\nabla} \mathscr{A}^1(\mathscr{E}) \xrightarrow{\nabla} \mathscr{A}^2(\mathscr{E})$$

is called *the curvature* of ∇.

A connection is only $\mathbb{C}$-linear, but the next proposition show that its curvature is $\mathscr{A}$-linear.

Proposition A.27. *The curvature $\nabla^2 : \mathscr{A}^0(\mathscr{E}) \to \mathscr{A}^2(\mathscr{E})$ is a morphism of $\mathscr{A}$-modules.*

Proof. Let $f \in \mathscr{A}(U)$ and $s \in \mathscr{A}^0(\mathscr{E})(U)$:

$$\nabla^2(fs) = \nabla(df \otimes s + f\nabla(s)) = \nabla(df \otimes s) + \nabla(f\nabla(s)) =$$

$$d^2f \otimes s - df \otimes \nabla(s) + df \otimes \nabla(s) + f\nabla^2(s) = f\nabla^2(s)\,.$$

☐

If we restrict our attention on global sections we have that

$$\nabla^2 \in \mathrm{Hom}_{C^\infty(M)}\left(\Gamma(\mathscr{A}^0(\mathscr{E}), \Gamma(\mathscr{A}^2(\mathscr{E}))\right) \cong \Gamma(\mathscr{E})^\vee \otimes \Gamma(\mathscr{A}^2) \otimes \Gamma(\mathscr{E}) \cong$$

$$\cong \Gamma(\mathscr{A}^2) \otimes (\Gamma(\mathscr{E}) \otimes \Gamma(\mathscr{E})^\vee)\,.$$

We have defined ∇^2 as a map of sheaves but by abuse of notations we consider it also as an element of $\Gamma(\mathscr{A}^2) \otimes (\Gamma(\mathscr{E}) \otimes \Gamma(\mathscr{E})^\vee)$. When $\mathscr{E} = \mathscr{L}$ is a line bundle, then $\Gamma(\mathscr{E}) \otimes \Gamma(\mathscr{L})^\vee \cong \mathscr{A}$, hence $\nabla^2 \in \Gamma(\mathscr{A}^2)$ which means that the curvature can be thought as a global 2-form Θ attached to $\mathscr{L}$ and ∇; we will cat it *the curvature form*.

Definition A.28. If $(\mathscr{L}, h)$ is an hermitian line bundle on M, the curvature form of $\mathscr{L}$ with respect to the h-connection is denoted with the symbol $\Theta(\mathscr{L}, h)$.

A.6 First Chern class

Given two invertible sheaves on a complex manifold $(M, \mathscr{O})$, a fundamental issue is to establish whether they are isomorphic. This problem can be attacked by introducing the concept of first Chern class.

With the symbol $H^*(M, \mathscr{F})$ we denote the sheaf cohomology and if G is a group, then $\underline{G}$ is the locally constant sheaf on M induced by G. $H^*_s(M, G)$ is the singular cohomology on M with coefficients in G. We will use some well known results relating the various cohomologies on M:

- $H^*(M, \underline{\mathbb{Z}}) \cong H^*_s(M, \mathbb{Z})$.
- $H^*_{\mathrm{DR}}(M, \mathbb{R}) \cong H^*_s(M, \mathbb{R})$ (De Rham theorem).

From the fundamental exact sequence of sheaves

$$0 \to \underline{\mathbb{Z}} \to \mathscr{O} \to \mathscr{O}^\times \to 0$$

we get a homomorphism $\delta : H^1(M, \mathscr{O}^\times) \to H^2(M, \underline{\mathbb{Z}})$ which can be viewed as a morphism $\delta : \mathrm{Pic}_h(M) \to H^*_s(M, \mathbb{Z})$ where $\mathrm{Pic}_h(M)$ is the group of isomorphic invertible sheaves on M up to isomorphism. If we consider the natural morphism $H^*_s(M, \mathbb{Z}) \to H^*_s(M, \mathbb{R})$, then the map δ induces the following homomorphism of groups:

$$\begin{array}{rcl} c_1 : \mathrm{Pic}_h(M) & \to & H^2_{\mathrm{DR}}(M, \mathbb{R}) \\ \mathscr{L} & \mapsto & c_1(\mathscr{L})\,. \end{array}$$

Definition A.29. The element $c_1(\mathscr{L})$ is called the *first Chern class of* $\mathscr{L}$. Furthermore the *degree of* $\mathscr{L}$ is $\deg(\mathscr{L}) := \int_X c_1(\mathscr{L})$.

The following important result says that the first Chern class can be explicitly calculated from any connection on $\mathscr{L}$, in other words we have an effective way to check whether two invertible sheaves are isomorphic.

Proposition A.30. *Let $\mathscr{L}$ an invertible sheaf and let Θ be the curvature form of $\mathscr{L}$ with respect to any connection ∇. Then:*

$$c_1(\mathscr{L}) = \left[\frac{i}{2\pi}\Theta\right]_{\mathrm{DR}}$$

Proof. See [25, 1.1 “Chern classes...”]. □

Remark A.31 (First Chern class in algebraic geometry). If $(X, \mathscr{O}_X)$ is an integral scheme and $\mathscr{L}$ is an invertible sheaves on $\mathscr{L}$, the first Chern class of $\mathscr{L}$ denoted as $c_1(\mathscr{L}) \in \mathrm{CH}^1(X)$ is the image of $\mathscr{L}$ under the isomorphism $\mathrm{Pic}(X) \to \mathrm{CH}^1(X)$. Unfortunately the notation is the same, but the two different notions of first Chern classes are closely related. In subsection D.1.2 this relation is made explicit in the case of Riemann Surfaces.

Appendix B

Divisors on schemes

Basic scheme theory is assumed to be a prerequisite to read this thesis. In this appendix we just present some crucial topics about divisors. On a scheme X one can define two types of divisors: Cartier divisors and Weil divisors. The former have a local nature and they don't require any further assumption on X, the latter are more intuitive but X must fulfill some regularity conditions. In many cases Cartier divisors and Weil divisors coincide.

B.1 Cartier divisors

First of all we need generalize the concept of the function field when X is not an integral scheme.

Definition B.1. Let X be a scheme, then for every open set $U \subseteq X$ we put:

$$\mathcal{R}_X(U) := \{\, s \in \mathscr{O}_X(U) \colon s_x \text{ is not a zero divisor of } \mathscr{O}_{X,x},\ \forall x \in U \,\} \ .$$

Then we define the presheaf of $\mathscr{O}_X$-algebras $\mathscr{K}'_X$ such that for any open set $U \subseteq X$

$$\mathscr{K}'_X(U) := \mathcal{R}_X(U)^{-1}\mathscr{O}_X(U)\,.$$

The *sheaf of stalks of meromorphic functions* $\mathscr{K}_X$ is the sheafification of $\mathscr{K}'_X$. A *meromorphic function* on X is an element $f \in H^0(X, \mathscr{K}_X)$.

Remark B.2. One might notice that $\mathscr{K}_X$ could have been defined in a more natural way as $\mathrm{Frac}\,(\mathscr{O}_X(U))$ for every open set U. This construction would be wrong and the main obstruction is that the natural restriction map would be in general not well defined. Indeed the homomorphic image of a an element which is not a zero divisor can become a zero divisor. See [32] for more details about the construction of $\mathscr{K}_X$.

$\mathscr{K}'_X$ and $\mathscr{K}_X$ have the following properties (see [37, 7.1.1]):

- (i) If U is an affine open then $\mathscr{K}'_X(U) = \operatorname{Frac}(\mathscr{O}_X(U))$.
- (ii) For every open set U the natural homomorpshism $\mathscr{K}'_X(U) \to \prod_{x \in U} \mathscr{K}'_{X,x}$ is injective.

Note that property (ii) implies that the sheafification morphism $\mathscr{K}'_X \to \mathscr{K}_X$ is injective, therefore from now we can assume that $\mathscr{O}_X \subset \mathscr{K}_X$ as sheaves of $\mathscr{O}_X$-algebras.

Proposition B.3. *If X is an integral scheme and K is its function field, then $\mathscr{K}_X$ is isomorphic to the constant sheaf given by $U \mapsto K$ for every open set U.*

Proof. Omitted. □

Definition B.4. The (abelian) group of *Cartier divisor* on X is:

$$\operatorname{Div}(X) := H^0\left(X, \mathscr{K}_X^\times / \mathscr{O}_X^\times\right)$$

and an element $D \in \operatorname{Div}(X)$ is called a *Cartier divisor*.

Even if the operation in $\operatorname{Div}(X)$ is the multiplication we write it in an additive manner, therefore if $D_1, D_2 \in \operatorname{Div}(X)$, the notations $D_1 \pm D_2$ make perfect sense.

Definition B.5. The subgroup of *principal Cartier divisors* is the image of the natural homomorphism:

$$H^0\left(X, \mathscr{K}_X^\times\right) \to H^0\left(X, \mathscr{K}_X^\times / \mathscr{O}_X^\times\right) .$$

The quotient group of Cartier divisors modulo the subgroups of principal Cartier divisors is denoted as $\operatorname{CaCl}(X)$. In particular if $D_1, D_2 \in \operatorname{Div}(X)$ and $D_1 - D_2$ is principal we say that D_1 and D_2 are *linearly equivalent*; in symbols $D_1 \sim D_2$.

Definition B.6. The monoid of *effective Cartier divisors* is the image of the natural map

$$H^0\left(X, (\mathscr{K}_X^\times \cap \mathscr{O}_X)/\mathscr{O}_X^\times\right) \to H^0\left(X, \mathscr{K}_X^\times / \mathscr{O}_X^\times\right) .$$

and it is indicated as $\operatorname{Div}_+(X)$. When D is an effective Cartier divisor we write by simplicity $D \geq 0$.

The following remark about the sheafification of a presheaf is crucial in order to give an explicit presentation of Cartier divisors:

Remark B.7. Suppose that $\mathscr{F}$ is a separated sheaf on a topological space. The sheafification $\mathscr{F}^+$ is explicitly given by:

$$\mathscr{F}^+(U) := \left\{ \begin{array}{l} (\alpha_x)_{x\in U} \in \prod\limits_{x\in U} \mathscr{F}_x \colon \forall y \in U, \exists \text{ an open neighbourhood } U_y \subseteq U \\ \text{and a section } f \in \mathcal{F}(U_y) \text{ such that } f_z = \alpha_z \ \forall z \in U_y \end{array} \right\}$$

By taking all the couples (U_y, f) of the above definition and by changing the indexes for clarity of notations, it is evident that to each element $t \in \mathscr{F}^+(U)$ we can associate a collection of couples $\{(U_i, f_i)\}_i$, satisfying the following properties:

- $U = \bigcup_i U_i$ is an open cover.
- $f_i \in \mathscr{F}(U_i)$.
- $f_i|_{U_i\cap U_j} = f_j|_{U_i\cap U_j}$ for every i, j. It is important to point out that this condition is true because $\mathscr{F}$ is separated.

We say that two collections $\{(U_i, f_i)\}_i$ and $\{(V_j, g_j)\}_j$ satisfying the above two conditions are equivalent if $f_i|_{U_i\cap V_j} = g_j|_{U_i\cap V_j}$ for any i, j. So, to each collection $\{(U_i, f_i)\}$ we can associate an element $t := (f_{i,x})_{x\in U} \in \mathscr{F}^+(U)$, and this assignment is well defined up to the equivalence relation.
In other words any section $t \in \mathscr{F}^+(U)$ in the sheafification can be presented uniquely (up to equivalence) as a collection of "compatible local data" $\{(U_i, f_i)\}$.

The quotient presheaf of two sheaves of abelian groups is always separated, in particular the presheaf defined by

$$U \mapsto \mathscr{K}^\times(U)/\mathscr{O}^\times(U)$$

is separated. Therefore thanks to remark B.7 a Cartier divisor can be given as a collection $D = \{(U_i, f_i)\}_i$, satisfying the following properties:

(*cd*1) $X = \bigcup_i U_i$ is an open covering.

(*cd*2) $f_i \in \mathscr{K}_X^\times(U_i)$.

(*cd*3) $\frac{f_i|_{U_i\cap U_j}}{f_j|_{U_i\cap U_j}} \in \mathscr{O}_X^\times(U_i \cap U_j)$ for every i, j.

Moreover two collections $\{(U_i, f_i)\}_i$ and $\{(V_j, g_j)\}_j$ satisfying the above properties represent the same Cartier divisor if $\frac{f_i|_{U_i\cap V_j}}{g_j|_{U_i\cap V_j}} \in \mathscr{O}_X^\times(U_i \cap V_j)$ for every i, j. It is evident that $D \in \mathrm{Div}(X)$ is principal when $D = (X, f)$ for $f \in H^0\left(X, \mathscr{K}_X^\times\right)$; on the other hand D is effective when $D = \{(U_i, f_i)\}_i$ and condition (*cd*2) is replaced by: (*cd*2′) $f_i \in \mathscr{O}_X(U_i) \cap K_X^\times(U_i)$ for every i.

Definition B.8. Given a Cartier divisor $D = \{(U_i, f_i)\}$ we can define the following glueing data of sheaves on the open covering $X = \bigcup_i U_i$:

- $\mathscr{F}_i := f_i^{-1}\mathscr{O}_X|_{U_i}$ is sheaf of $\mathscr{O}_X|_{U_i}$-modules over U_i.
- $\varphi_{ij} : \mathscr{F}_i|_{U_i \cap U_j} \to \mathscr{F}_j|_{U_i \cap U_j}$ is an isomorphism of sheaves defined as:

$$\begin{array}{rcl} \varphi_{ij}(U) : \mathscr{F}_i|_{U_i \cap U_j}(U) & \to & \mathscr{F}_j|_{U_i \cap U_j}(U) \\ \dfrac{a}{f_i|_U} & \mapsto & \dfrac{a}{f_j|_U} \end{array}$$

  for every open set $U \subseteq U_i \cap U_j$.

The glueing conditions are easily verified, therefore the sheaves $\mathscr{F}_i$ glue to a unique invertible sheaf denoted as $\mathscr{O}_X(D)$.

Remark B.9. As an exercise one can verify that the above definition of $\mathscr{O}_X(D)$ is independent from the choice of the local data $\{(U_i, f_i)\}$ for D.

Let's keep the notation of the above definition and consider a point $x \in X$, then there exists a certain U_i such that $x \in U_i$, and we have the equality $\mathscr{O}_X(D)_x = \left(f_i^{-1}\right)_x \mathscr{O}_{X,x}$. Note that $D = \{(U_i, f_i)\}_i$ is an effective Cartier divisor if and only if $\mathscr{O}_X(-D)$ is a sheaf of ideals if $\mathscr{O}_X$ and in this case $\mathscr{O}_X(-D)_x = (f_i)_x \mathscr{O}_{X,x}$ where $(f_i)_x \in \mathscr{O}_{X,x}$ is called *a local equation of D at x*. If we don't write explicitly the local data of an effective Cartier divisor D, a local equation of D at a point x is simply a generator for the principal ideal $\mathscr{O}_X(-D)_x$ of $\mathscr{O}_{X,x}$.

Definition B.10. Let $\mathscr{F}$ be a sheaf of $\mathscr{O}_X$-modules on X, then we define the sheaf

$$\mathscr{K}_X(\mathscr{F}) := \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{K}_X$$

A *meromorphic section* of $\mathscr{F}$ is an element of $H^0(X, \mathscr{K}_X(\mathscr{F}))$.

Remark B.11. Note that $\mathscr{K}_X(\mathscr{F})$ is the sheafification of the presheaf given by $U \mapsto \mathcal{R}_X(U)^{-1}\mathscr{F}(U)$ for every open set U.

Proposition B.12. *if X is an integral scheme with generic point η and $\mathscr{F}$ is a quasi-coherent sheaf, then $\mathscr{K}_X(\mathscr{F})$ is isomorphic to the constant sheaf given by $U \mapsto \mathscr{F}_\eta$ for every open set U.*

Proof. Omitted. □

Let $\mathscr{L}$ be an invertible sheaf on an integral scheme X and consider a non-zero meromorphic section $s \in H^0(X, \mathscr{K}_X(\mathscr{L}))$; we want to show that s defines a Cartier divisor.

Let $X = \bigcup_i U_i$ an open covering such that $\mathscr{L}|_{U_i} = e_i \mathscr{O}_X|_{U_i}$, for $e_i \in \mathscr{L}(U_i)$, then

$$s|_{U_i} \in H^0(U_i, e_i \mathscr{O}_X|_{U_i} \otimes_{\mathscr{O}_X|_{U_i}} K) = e_i \mathscr{O}_X(U_i) \otimes_{\mathscr{O}_X(U_i)} K$$

and one can write $s|_{U_i} = e_i \otimes f_i$ with $f_i \in K$. Moreover $f_i \neq 0$ for every i, because otherwise we would have $s|_{U_i} = 0$ and since $\mathscr{K}_X(\mathscr{L})$ is isomorphic to a constant sheaf (see proposition B.12) this would imply $s = 0$ against the assumptions. Therefore it is evident that the data $\{(U_i, f_i)\}_i$ defines a Cartier divisor.

Definition B.13. Let $\mathscr{L}$ be an invertible sheaf on an integral scheme X and let $s \in H^0(X, \mathscr{K}_X(\mathscr{L}))$ be a nonzero meromorphic section. The Cartier divisor associated to s in the way showed above is denoted as $\operatorname{div}(s)$.

It is worth mentioning that $\mathscr{O}_X(\operatorname{div}(s)) \cong \mathscr{L}$, indeed for any open set $U \subset U_i$ we have the map

$$\begin{array}{rcl} f_i^{-1}|_U \mathscr{O}_X(U) & \to & \mathscr{L}(U) = e_i|_U \mathscr{O}_X(U) \\ \dfrac{a}{f_i|_U} & \mapsto & e_i|_U \cdot a \end{array}$$

An invertible sheaf may not have nonzero global sections but the existence of nonzero meromorphic section is guaranteed by the following proposition.

Proposition B.14. *Let $\mathscr{L}$ be an invertible sheaf on an integral scheme X, then there exists a nonzero meromorphic section of $\mathscr{L}$.*

Proof. Let $X = \bigcup_i U_i$ an open covering such that $\mathscr{L}|_{U_i} = e_i \mathscr{O}_X|_{U_i}$, for $e_i \in \mathscr{L}(U_i)$. We first show that $\mathscr{K}_X(\mathscr{L})|_{U_i}$ is a flasque sheaf for every i. Indeed let $U \subseteq V$ be two open set contained in U_i and consider an element $e_i|_U \otimes f \in \mathscr{K}_X(\mathscr{L})(U)$, then obviously $e_i|_V \otimes f \in \mathscr{K}_X(V)$ is such that:

$$(e_i|_V \otimes f)|_U = e_i|_U \otimes f \,.$$

Since $\mathscr{K}_X(\mathscr{L})$ is flasque, then we can find nonzero elements $s_i \in \mathscr{K}_X(\mathscr{L})(U_i)$ such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for every i, j. Finally, all the elements s_i glue to a nonzero global section $s \in H^0(X, \mathscr{K}_X(\mathscr{L}))$. ☐

B.2 Weil divisors

In this section we fix a Noetherian integral n-dimensional scheme X with function field denoted by K.

Definition B.15. A *k-cycle* on X is a finite formal sum:

$$\sum_{Y \in X^{(k)}} n_Y [Y]$$

where $n_i \in \mathbb{Z}$ and the Y's range over the set of all closed integral k-dimensional subschemes of X (such a set is denoted by $X^{(k)}$). A *prime k-cycle* is a k-cycle of the form $[Y]$ for $Y \in X^{(k)}$. Clearly k-cycles form an abelian group that we indicate by the symbol $Z_k X$. When $k = n-1$ then we say that $Z_{n-1}X$ (which is often denoted by $Z^1 X$) is the group of *Weil Divisors* on X. A Weil divisor it is said *positive* if all the coefficients in the formal sum are non-negative.

If Y is a prime Weil divisor then we have a well defined discrete valuation v_Y on K.

Definition B.16. A *principal Weil divisor* is a Weil divisor of the type:

$$(f) := \sum_{Y \in X^{(n-1)}} v_Y(f)[Y] \quad \text{for } f \in K^\times .$$

The quotient of $Z_{n-1}X$ with the subgroup of principal Weil divisors is denoted by $\mathrm{CH}^1(X)$. Two Weil divisors are said *linearly equivalent* if they have the same class in $\mathrm{CH}^1(X)$.

When X is a regular scheme we can identify Weil divisors and Cartier divisors, thanks to the following theorem:

Theorem B.17. *Let X be also a regular scheme, then* $\mathrm{Div}(X) \cong Z_{n-1}X$ *and* $\mathrm{CaCl}(X) \cong \mathrm{CH}^1(X)$.

Proof. See [37, proposition 7.2.16]. □

So, from now on we can simply use $\mathrm{Div}(X)$ to denote Weil divisors. If $D = \sum_Y n_Y[Y] \in \mathrm{Div}(X)$ and $Y = \overline{\{y\}}$ then $\mathrm{mult}_y(D) := n_Y$.

For general morphisms of schemes $f : X \to S$ Weil divisors and cycles don't have nice properties of pullback and push-forward, but if we add some hypotheses on f, then the situation is different.

Proper push-forward of cycles. If f is a proper map, then for any prime k-cycle $[Y] \in X^{(k)}$ we have that $f(Y)$ is an integral closed subscheme of S such that $\dim f(Y) \leq k$. Let K_Y be the function field of Y and let $K_{f(Y)}$ the function field of $f(Y)$, then we put:

$$\deg(Y/f(Y)) := \begin{cases} 0 & \text{if } \dim f(Y) < k, \\ [K_Y : K_{f(Y)}] & \text{if } \dim f(Y) = k\,. \end{cases}$$

Thus the push-forward of $[Y]$ is defined as:

$$f_*[Y] := \deg(Y/f(Y))[f(Y)]\,.$$

By $\mathbb{Z}$-linearity we can extend f_* to a group homomorphism $f_* : Z_k X \to Z_k S$.

Flat pullback of cycles. Assume that f is flat of relative dimension m, then for any prime k-cycle $[Y] \in S^{(k)}$ we put:

$$f^*[Y] := [f^{-1}(Y)] \in Z_{k+m}X$$

again by $\mathbb{Z}$-bilinearity this extends to a homomorphism $f^* : Z_k S \to Z_{k+m}X$. Roughly speaking the pullback of cycles "preserves the codimensions", thus we have a well defined homomorphism $f^* : \mathrm{Div}(S) \to \mathrm{Div}(X)$. We define the *ramification index* of f at x to be:

$$e_{x|f(x)} := \mathrm{length}_{\mathscr{O}_{X,x}} \frac{\mathscr{O}_{X,x}}{\mathfrak{m}_{f(x)}\mathscr{O}_{X,x}}\,,$$

Then it can be shown (see [37, Exercise 7.2.3(b)]), that the following explicit equation hold for $D \in \mathrm{Div}(S)$:

$$f^*D = \sum_{x \in T} e_{x|f(x)}\, \mathrm{mult}_{f(x)}(D)\overline{\{x\}}$$

where:

$$T := \Big\{ x \in X \colon\ \dim \overline{\{x\}} = n - 1 \text{ and } \dim \overline{\{f(x)\}} = n - m - 1 \Big\}\,.$$

Appendix C

Arakelov geometry in dimension one

Arakelov geometry in dimension 1 is a geometric viewpoint of basic algebraic number theory.

Let X be a regular projective curve over a finite field $\mathbb{F}_q$ and let K be its function field, then there is a bijection between the closed points of X and the places of K. This means that we can study the geometry of X by using arithmetic properties of K and vice versa. This perfect correspondence between arithmetic and geometry apparently seems to disappear for purely arithmetic objects. Fix our prototype of arithmetic curve, which is $B = \operatorname{Spec} O_K$ where O_K is the ring of integers of a number field K; notice that there are more places of K than closed points in B. The archimedean places of K cannot be recovered from B, so in order to solve this issue, we formally complete B with points corresponding to the missing archimedean places. We obtain a new space $\widehat{B}$ which is not a scheme anymore.

On an arithmetic surface we will find both arithmetic curves (the horizontal ones) and algebraic curves (the vertical ones, including the fibers at infinity). Such curves may admit singularities, for this reason we have to enlarge a bit the arithmetic theory to take in account the singularities. The prototype of (possibly) singular arithmetic curve is $B = \operatorname{Spec} A$ where $A \subseteq O_K$ is an order of K and the goal of this section is to develop the basic tools for a geometric theory on $\widehat{B}$.

We assume that the reader is familiar with basics notions of algebraic number theory at the level of [47, I,II].

Notation. We fix a number field K such that $[K : \mathbb{Q}] = d$. O_K is the ring of integers of K and we fix $A \subseteq O_K$ to be an order of K. A is in particular a one-dimensional, Notherian integral domain whose field of fractions is K.

When $A = O_K$, then it is clearly a Dedekind domain and it also called the maximal order of K. We put $B = \operatorname{Spec} A$, and the crucial point to keep in mind is that for $\mathfrak{p} \in B$, the local ring $\mathscr{O}_{B,\mathfrak{p}} = A_\mathfrak{p}$ may not be a DVR.

C.1 Arakelov divisors

Recall that for any nonzero ideal $\mathfrak{a} \subseteq A$, the quotient $A/\mathfrak{a}$ has finite cardinality, so we can define the norm function:

$$\mathfrak{N}(\mathfrak{a}) := \#(A/\mathfrak{a})$$

that turns out to be multiplicative (with respect to the product of two ideals). By definition we put $\mathfrak{N}(0) = 0$. For any element $\mathfrak{p} \in B$ the function:

$$\begin{aligned} \operatorname{ord}_\mathfrak{p} : K^\times &\to \mathbb{Z} \\ \frac{f}{g} &\mapsto \operatorname{length}_{A_\mathfrak{p}}(A_\mathfrak{p}/fA_\mathfrak{p}) - \operatorname{length}_{A_\mathfrak{p}}(A_\mathfrak{p}/gA_\mathfrak{p}) \end{aligned}$$

is a well defined group homomorphism (see [37, pag. 258-260]) and when $A_\mathfrak{p}$ is a DVR then $\operatorname{ord}_\mathfrak{p}$ is actually a discrete valuation that is denoted as $v_\mathfrak{p}$.

A *place of* K is an equivalence class of absolute values on K (two absolute values are equivalent if they generate the same topology). The set of field embeddings of K in $\mathbb{C}$, here denoted as $\mathbb{C}^K$, has cardinality $d = r_1 + 2r_2$ where r_1 is the number of real embeddings and $2r_2$ is the number of complex embeddings. Let's introduce the following equivalence relation $\sim$ on $\mathbb{C}^K$:

$$\sigma \sim \tau \quad \text{if and only if} \quad \sigma = \tau \ \text{ or } \ \tau = \bar{\sigma}\,.$$

The quotient set $B_\infty := \mathbb{C}^K/\sim$ has cardinality $r + s$. Define the *completion of* B to be the set $\widehat{B} := B \cup B_\infty$.

Remark C.1. When $A = O_K$ we have a bijective correspondence (see [47, II §8] for details):

$$\{\,\text{Places on } K\,\} \leftrightarrow \widehat{B} \setminus \{0\}\,.$$

For the non-archimedean place associated to $\mathfrak{p} \in B$ we choose the representative

$$|\cdot|_\mathfrak{p} := \mathfrak{N}(\mathfrak{p})^{-\operatorname{ord}_\mathfrak{p}(\cdot)}$$

Moreover:

- For any real embedding $\tau : K \to \mathbb{R}$ we consider the absolute value:

$$|\cdot|_\tau := |\tau(\cdot)|$$

where on the right hand side we mean the usual absolute value on $\mathbb{R}$. In this case we define the real valuation associated to τ as

$$v_\tau(\cdot) := -\log|\cdot|_\tau$$

- For any couple of conjugate embeddings $\sigma, \overline{\sigma} : K \to \mathbb{C}$ we choose:

  $$|\cdot|_\sigma := |\sigma(\cdot)|$$

  where on the right hand side we have the usual absolute value on $\mathbb{C}$[1]. Note that $|\cdot|_\sigma$ doesn't depend on the choice of σ, since its conjugate $\overline{\sigma}$ gives the same absolute value. The associated real valuation is

  $$v_\sigma(\cdot) := -\log|\cdot|_\sigma\,.$$

From now on we *always* fix a set of representatives in B_∞ (as explained above this choice doesn't affect the set of canonical absolute values). Therefore B_∞ is simply a set of $d = r_1 + r_2$ embeddings intended as points at infinity of B.

Furthermore, let's introduce a very important constant, associated to each $\sigma \in B_\infty$, which will be used throughout the whole text:

$$\epsilon_\sigma := \begin{cases} 1 & \text{if } \sigma \text{ is real} \\ 2 & \text{if } \sigma \text{ is complex}\,. \end{cases}$$

Now we are going to prove the product formula for orders and we need the following general lemma:

Lemma C.2. *Let M be a finitely generated free $\mathbb{Z}$-module and let $\phi : M \to M$ be an injective $\mathbb{Z}$-homomorphism. Then $\#(M/\phi(M)) = |\det(\phi)|$.*

Proof. See [43, Lemma 3.4]. □

Theorem C.3 (Product formula for orders)**.** *Let $f \in K^\times$, then $|f|_\mathfrak{p} = 1$ for all but finitely many $\mathfrak{p} \in B$ and*

$$\prod_{\mathfrak{p}\in B} |f|_\mathfrak{p} \cdot \prod_{\sigma\in B_\infty} |f|_\sigma^{\epsilon_\sigma} = 1\,. \tag{C.1}$$

Clearly equation (C.1) can be rewritten in the following way:

$$\prod_{\mathfrak{p}\in B} \mathfrak{N}(\mathfrak{p})^{-\operatorname{ord}_\mathfrak{p}(f)} \cdot \prod_{\sigma\in\mathbb{C}^K} |f|_\sigma = 1\,. \tag{C.2}$$

---

[1] Many authors in this case take the square of the complex absolute value to keep track of the fact that point at infinity induced by $|\cdot|_\sigma$ is "complex", so roughly speaking "of order two". We will fix this by using the coefficient 2 when necessary.

Proof. We can restrict the proof to the case $f \in A \setminus \{0\}$. Consider the injective $\mathbb{Z}$-homomorphism $\phi_f : A \to A$ given by $a \mapsto af$. By using Galois theory (See [41, Theorem II.8.12]) we know that $\det(\phi_f) = \prod_{\sigma \in \mathbb{C}^K} \sigma(f)$, so thanks to lemma C.2 we can write:

$$\mathfrak{N}(fA) = \#(A/fA) = |\det(\phi_f)| = \prod_{\sigma \in \mathbb{C}^K} |\sigma(f)| = \prod_{\sigma \in \mathbb{C}^K} |f|_\sigma \,.$$

Now in order to conclude the proof we have to show only that

$$\#(A/fA) = \prod_{\mathfrak{p} \in B} \#(A/\mathfrak{p})^{\mathrm{ord}_\mathfrak{p}(f)} \tag{C.3}$$

where the right hand side of equation (C.3) is a actually a finite product. Thanks to the Chinese remainder theorem ([47, Proposition I.12.3]) we know that

$$A/fA = \bigoplus_{\mathfrak{p} \in B} A_\mathfrak{p}/fA_\mathfrak{p}$$

where the direct sum on the right hand side is finite. Note that, for any $\mathfrak{p}$, $A_\mathfrak{p}/fA_\mathfrak{p}$ is an Artinian $A_\mathfrak{p}$-module, and suppose that it has a composition series:

$$0 = M_0 \subsetneq M_1 \subsetneq \ldots \subsetneq M_{\mathrm{ord}_\mathfrak{p}(f)} = A_\mathfrak{p}/fA_\mathfrak{p} \,.$$

For any $j = 0, \ldots, \mathrm{ord}_\mathfrak{p}(f)$, the composition factor M_{j+1}/M_j is a simple $A_\mathfrak{p}$-module, hence

$$M_{j+1}/M_j \cong A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p} \cong A/\mathfrak{p} \,.$$

We can conclude that $\#(A_\mathfrak{p}/fA_\mathfrak{p}) = \#(A/\mathfrak{p})^{\mathrm{ord}_\mathfrak{p}(f)}$, so equation C.3 is proved. □

Definition C.4. The group of *Arakelov divisors* on B is:

$$\mathrm{Div}_{\mathrm{Ar}}(B) := \mathrm{Div}(B) \oplus \mathbb{R}^{(B_\infty)}$$

where $\mathbb{R}^{(B_\infty)}$ is the free $\mathbb{R}$-module with basis B_∞. So, any $\widehat{D} \in \mathrm{Div}_{\mathrm{Ar}}(B)$ is a formal sum:

$$\widehat{D} = \sum_{\mathfrak{p} \in B} n_\mathfrak{p}[\mathfrak{p}] + \sum_{\sigma \in B_\infty} \alpha_\sigma[\sigma] \tag{C.4}$$

where $[\mathfrak{p}]$ and $[\sigma]$ are just symbols, $n_\mathfrak{p} \in \mathbb{Z}$ and $\alpha_\sigma \in \mathbb{R}$. Consider $f \in K^\times$, the Arakelov divisor:

$$\widehat{(f)} = \sum_{\mathfrak{p} \in B} \mathrm{ord}_\mathfrak{p}(f)[\mathfrak{p}] + \sum_{\sigma \in B_\infty} 2v_\sigma(f)[\sigma] \tag{C.5}$$

is called a *Principal Arakelov divisor*, moreover we put $\widehat{(0)} := 0$. The subgroup of principal Arakelov divisors is denoted as $\mathrm{Princ}_{\mathrm{Ar}}(B)$ and the quotient

$$\mathrm{CH}^1_{\mathrm{Ar}}(B) := \frac{\mathrm{Div}_{\mathrm{Ar}}(B)}{\mathrm{Princ}_{\mathrm{Ar}}(B)}$$

is called the *(first) Arakelov Chow group of B.*

We have the notion of degree for Arakelov divisors:

Definition C.5. If $\widehat{D}$ is given as in equation (C.4), then:

$$\deg_{\mathrm{Ar}}(\widehat{D}) := \sum_{\mathfrak{p}\in B} n_{\mathfrak{p}} \log \mathfrak{N}(\mathfrak{p}) + \frac{1}{2} \sum_{\sigma\in B_\infty} \epsilon_\sigma \alpha_\sigma \,.$$

If $A \neq O_K$, then it is not a Dedekind domain, in particular the set of fractional ideals of A is not a group and the unique factorization of fractional ideals fails. For this reason we restrict our attention to *invertible ideals* of A which are those fractional ideals $\mathfrak{a}$ admitting the inverse $\mathfrak{a}^{-1}$ (i.e. $\mathfrak{a}\mathfrak{a}^{-1} = A$). Invertible ideals have a nice characterization:

Proposition C.6. *A fractional ideal $\mathfrak{a}$ of A is invertible if and only if, for any $\mathfrak{p} \in B$, $\mathfrak{a}A_{\mathfrak{p}}$ is a fractional principal ideal of $A_{\mathfrak{p}}$.*

Proof. See [47, Proposition I.12.4]. □

Thus for any invertible ideal $\mathfrak{a}$ of A we have that $\mathfrak{a}A_{\mathfrak{p}} = a_{\mathfrak{p}}A_{\mathfrak{p}}$ for some $a_{\mathfrak{p}} \in K^\times$ and we can define the following Arakelov divisor without components at infinity:

$$\widehat{\mathfrak{a}} := \sum_{\mathfrak{p}\in B} -\mathrm{ord}_{\mathfrak{p}}(a_{\mathfrak{p}})[\mathfrak{p}] \,.$$

It is evident that $\widehat{\mathfrak{a}}$ is independent from the choices of the generators $a_{\mathfrak{p}}$.

Proposition C.7. *The degree of a principal Arakelov divisor is* 0*.*

Proof. It is a straightforward application of the product formula (theorem C.3), indeed if $\widehat{(f)}$ is given by equation (C.5), then:

$$\deg_{\mathrm{Ar}}(\widehat{(f)}) = \sum_{\mathfrak{p}\in B} \mathrm{ord}_{\mathfrak{p}}(f) \log \mathfrak{N}(\mathfrak{p}) + \sum_{\sigma\in B_\infty} -\epsilon_\sigma \log|\sigma(f)| =$$

$$= -\log\left(\prod_{\mathfrak{p}\in B} |f|_{\mathfrak{p}} \cdot \prod_{\sigma\in B_\infty} |f|_\sigma^{\epsilon_\sigma}\right) = 0 \,.$$

□

It follows that the Arakelov degree descends to a well defined homomorphism to the quotient:

$$\deg_{\mathrm{Ar}} : \mathrm{CH}^1_{\mathrm{Ar}}(B) \to \mathbb{R}\,.$$

C.2 Picard-Arakelov group

The usual (non-Arakelov) Chow group $\mathrm{CH}^1(B)$ can be identified with the Picard group $\mathrm{Pic}(B)$. In this section we want to define the group $\mathrm{Pic}_{\mathrm{Ar}}(B)$ which will be an enlargement of $\mathrm{Pic}(B)$ and moreover will be isomorphic to $\mathrm{CH}^1_{\mathrm{Ar}}(B)$. A useful simplification of the theory is due to the fact that B is an affine scheme, indeed the following theorem says that instead of working with invertible sheaves, it is enough to consider just the A-modules of global sections.

Theorem C.8. *Let R be a ring. The category of coherent sheaves on $Z = \operatorname{Spec} R$ is equivalent to the category of finitely generated R-modules.*

Proof. We just write down the definition of the functor on objects, all the details can be found in [53, §4].
Let $\mathscr{M}$ be a quasi coherent sheaf on Z, then we associate the R-module of global sections $M = H^0(Z, \mathscr{M})$. Vice versa let M be a finitely generated R-module and consider it as a constant sheaf on Z, then we associate the sheaf $\mathscr{M} = \widetilde{M}$, where $\widetilde{M}$ is the usual $\sim$-construction of a sheaf by starting from an R-module (see [37, 5.1.2]). $\square$

Let's recall some notions from commutative algebra:

Proposition C.9. *Let R be a ring and let M be a finitely generated R-module, then the following are equivalent:*

(1) *M is projective.*

(2) *$M_{\mathfrak{p}}$ is a free $R_{\mathfrak{p}}$-module for any $\mathfrak{p} \in \operatorname{Spec} R$ and the map*

$$\operatorname{Spec} R \ni \mathfrak{p} \mapsto \dim_{k(\mathfrak{p})}(M \otimes_R k(\mathfrak{p})) = \mathrm{rk}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}})$$

is locally constant.

Proof. See [57, Lemma 10,77.2]. $\square$

In our particular case, $R = A$ is an integral domain, so B is connected and the aforementioned local dimension map for a projective A-module is constant. Therefore the following definition makes sense:

Definition C.10. The *rank* of a finitely generated projective A-module is $\dim_{k(\mathfrak{p})} M \otimes_A k(\mathfrak{p})$ for any choice of $\mathfrak{p} \in B$ (also $\mathfrak{p} = 0$).

Definition C.11. An *invertible A-module* is a finitely generated projective A-module of rank 1.

Let R be any ring, then the support of an R-module M is defined as

$$\operatorname{supp}(M) := \{\mathfrak{p} \in \operatorname{Spec} R \colon M_{\mathfrak{p}} \neq 0\}$$

and it turns out to be a finite set for $A = R$; this will be an important property.

Lemma C.12. *Let R be any ring, let M be a finitely generated R-module and let $\mathfrak{p} \in \operatorname{Spec} R$. $M_{\mathfrak{p}} = 0$ if and only if $\operatorname{Ann}_R(M) \not\subseteq \mathfrak{p}$.*

Proof. Suppose that $b \in \operatorname{Ann}_R(M) \setminus \mathfrak{p}$, then $bm = 0$ $\forall m \in M$ which means that $M_{\mathfrak{p}} = 0$. Vice versa let $\{m_1, \dots, m_t\}$ a set of generators for M, $M_{\mathfrak{p}} = 0$ implies that we can find $b_i \in R \setminus \mathfrak{p}$ such that $b_i m_i = 0$ for all $i = 1, \dots, m$. Now consider $b = b_1 b_2 \dots b_t$, then $b \in \operatorname{Ann}_R(M) \setminus \mathfrak{p}$. ☐

Proposition C.13. *Let M be a finitely generated module over a Noetherian ring R and assume that $\operatorname{supp}(M)$ is contained in the set of maximal ideals of R, then $\operatorname{supp}(M)$ is a finite set. In particular if $A = R$, then $\operatorname{supp}(M)$ is a finite set.*

Proof. Since all ideals in $\operatorname{supp}(M)$ are maximal in R, then they are all minimal in $\operatorname{supp}(M)$. By lemma C.12 $\operatorname{supp}(M)$ is the set of minimal prime ideals containing $\operatorname{Ann}_R(M)$, so $\operatorname{supp}(M)$ corresponds to the minimal prime ideals of $R/\operatorname{Ann}_R(M)$. The claim follows from the fact that Noetherian rings admit only finitely many minimal prime ideals. ☐

From now on we identify $\operatorname{Pic}(B)$ with the group of invertible A-modules up to isomorphism. The identity element is the isomorphism class of A and the operation is the tensor product. Given any divisor $D = \sum_{\mathfrak{p}} n_{\mathfrak{p}}[\mathfrak{p}]$ on B, the associated invertible A-module can be obtained simply as

$$\mathscr{O}_B(D) = \{f \in K^{\times} \colon \operatorname{ord}_{\mathfrak{p}}(f) \geq -n_{\mathfrak{p}}, \ \forall \mathfrak{p} \in B\} \cup \{0\}$$

Let $L \in \operatorname{Pic}(B)$, the following argument shows how any nonzero element $s \in L \otimes_A K$ can be seen as "a rational section", indeed it generates a divisor $\operatorname{div}(s) \in \operatorname{Div}(B)$ such that $\mathscr{O}_B(\operatorname{div}(s)) \cong L$. For any $\mathfrak{p} \in \operatorname{supp}(L)$ we can choose a nonzero element $\omega_{\mathfrak{p}} \in L_{\mathfrak{p}}$ such that $L_{\mathfrak{p}} = \omega_{\mathfrak{p}} A_{\mathfrak{p}}$. Therefore we can write $s = \omega_{\mathfrak{p}} f_{\mathfrak{p}}$ for $f_{\mathfrak{p}} \in K$ (here we got rid of the tensor product by commodity). Let's define

$$\operatorname{ord}_{\mathfrak{p}}(s) := \begin{cases} \operatorname{ord}_{\mathfrak{p}}(f_{\mathfrak{p}}) & \text{if } \mathfrak{p} \in \operatorname{supp}(L) \\ 0 & \text{if } \mathfrak{p} \notin \operatorname{supp}(L)\,. \end{cases}$$

It is easy to see that $\operatorname{ord}_{\mathfrak{p}}(s)$ doesn't depend from the choice of $\omega_{\mathfrak{p}}$, indeed assume that $\omega'_{\mathfrak{p}}$ is another basis such that $\omega'_{\mathfrak{p}} = \alpha\omega_{\mathfrak{p}}$ and $\omega_{\mathfrak{p}} = \beta\omega'_{\mathfrak{p}}$; this implies immediately $\alpha\beta = 1$ and $\operatorname{ord}_{\mathfrak{p}}(\alpha) = 0$. By proposition C.13 the following divisor is well defined:

$$\operatorname{div}(s) := \sum_{\mathfrak{p} \in B} \operatorname{ord}_{\mathfrak{p}}(s)[\mathfrak{p}] \quad \in \operatorname{Div}(B)\,.$$

Proposition C.14. *Let L be an invertible A-module and let $s \in L \otimes_A K$ such that $s \neq 0$. Then $\mathscr{O}_B(\operatorname{div}(s)) \cong L$.*

Proof. Define the map:

$$\begin{aligned} \theta_s : \mathscr{O}_B(\operatorname{div}(s)) &\rightarrow L \otimes_A K \\ f &\mapsto fs \end{aligned}$$

First of all we show that the image is contained L. Choose a local $A_{\mathfrak{p}}$-basis $\omega_{\mathfrak{p}}$ for $L_{\mathfrak{p}}$ and write $s = \omega_{\mathfrak{p}} f_{\mathfrak{p}}$. At each prime we have $t := fs = \omega_{\mathfrak{p}} f f_{\mathfrak{p}}$ (when $\mathfrak{p} \notin \operatorname{supp}(L)$ everything is 0), but $\operatorname{ord}_{\mathfrak{p}}(ff_{\mathfrak{p}}) \geq 0$, so $t \in L_{\mathfrak{p}}$ for any $\mathfrak{p} \in B$. This implies that $t \in L$, indeed suppose by contradiction that $t \notin L$, then the ideal $\mathfrak{a} := \{r \in A \colon rt \in L\}$ is proper. Let $\mathfrak{q}$ be a prime containing $\mathfrak{a}$, then $t \in L_{\mathfrak{q}}$, thus there exists $a \notin \mathfrak{q}$ such that $at \in L$. We have found an element $a \in \mathfrak{a}$ such that $a \notin \mathfrak{q}$ and this is a contradiction with the fact that $\mathfrak{a} \subseteq \mathfrak{q}$. The map θ_s is evidently an injective homomorphism, so it remains to check that it is surjective on L. Let $t \in L$, since $L \otimes_A K$ has dimension 1 we can write $t = fs$ for $f \in K$, hence at each $\mathfrak{p}$ we have $t = \omega_{\mathfrak{p}} f_{\mathfrak{p}} f$ where $f_{\mathfrak{p}} f$ has to be an element of $A_{\mathfrak{p}}$. This is equivalent to say that $\operatorname{ord}_{\mathfrak{p}}(f) \geq -\operatorname{ord}_{\mathfrak{p}}(f_{\mathfrak{p}}) = -\operatorname{ord}_{\mathfrak{p}}(s)$. $\square$

Invertible A-modules keep track just of the ordinary divisors on B and they don't give any information about the archimedean data on B. We need an enriched notion of invertible A-modules:

Definition C.15. An *hermitian invertible A-module* is a couple $(L, \{h_\sigma\}_{\sigma \in B_\infty})$ where L is an invertible A-module and h_σ is an hermitian inner product on the complex one dimensional vector space $L_\sigma := L \otimes_A^\sigma \mathbb{C} \cong (L \otimes_A K) \otimes_K^\sigma \mathbb{C}$.

Remark C.16. If M is a K-module and we construct $M \otimes_K^\sigma \mathbb{C}$, then $\mathbb{C}$ is considered a K-vector space with the multiplication induced by the field embedding σ. In other words $a \cdot v = \sigma(a)v$ for any $a \in K$ and $v \in \mathbb{C}$. A pure tensor of $M \otimes_K^\sigma \mathbb{C}$ is denoted as $m \otimes^\sigma v$.

For any $\sigma \in B_\infty$ the tensor product induces a functor from the category of A-modules to the category of $\mathbb{C}$-vector spaces which acts on objects as $L \mapsto L_\sigma$. On the other hand if $\psi : L \to L'$ is a homomorphism of A-modules, we have a homomorphism of $\mathbb{C}$-vector spaces $\psi_\sigma : L_\sigma \to L'_\sigma$ given by $x \otimes^\sigma 1 \mapsto \psi(x) \otimes^\sigma 1$.

Definition C.17. Two hermitian invertible A-modules $(L, \{h_\sigma\}_\sigma)$ and $(L', \{h'_\sigma\}_\sigma)$ are *isometric* if there exists an isomorphism of A-modules $\psi : L \to L'$ such that $\psi_\sigma : L_\sigma \to L'_\sigma$ is an isometry of $\mathbb{C}$-vector spaces for any $\sigma \in B_\infty$.

If $\overline{L} = (L, \{h_\sigma\}_\sigma)$ is an hermitian invertible A-module, then any nonzero $s \in L \otimes_A K$ induces an Arakelov divisor:

$$\widehat{\operatorname{div}(s)} := \operatorname{div}(s) + \sum_\sigma -\log\left(h_\sigma(s \otimes^\sigma 1, s \otimes^\sigma 1)\right)[\sigma] \quad \in \operatorname{Div}_{\mathrm{Ar}}(B)\,. \tag{C.6}$$

Remark C.18. When $L = A$, and $f \in K$, then $\operatorname{div}(f) = (f)$ and $\widehat{\operatorname{div}(f)} = \widehat{(f)}$.

Definition C.19. The *Picard-Arakelov group* on $\widehat{B}$, denoted by $\operatorname{Pic}_{\mathrm{Ar}}(B)$ is the group of isometry classes of hermitian invertible A-modules.

Let L be an invertible A-module contained in K, then we define the *canonical metric on* L_σ as $h_{0_\sigma}(x, y) = \sigma(x)\overline{\sigma(y)}$. Then the identity element of $\operatorname{Pic}_{\mathrm{Ar}}(B)$ is given by the isometry class of $\overline{A} = (A, \{h_{0_\sigma}\})$. The operation between two hermitian invertible A-modules $\overline{L} = (L, \{h_\sigma\})$ and $\overline{L'} = (L', \{h'_\sigma\})$ is defined in the following way:

$$\overline{L} \otimes \overline{L'} := (L \otimes_A L', \{h_\sigma \otimes h'_\sigma\}_\sigma)$$

where $h_\sigma \otimes h'_\sigma : (L \otimes_A L')_\sigma \to \mathbb{R}$ is defined as:

$$h_\sigma \otimes h'_\sigma(l_1 \otimes l'_1, l_2 \otimes l'_2) = h_\sigma(l_1, l_2) h'_\sigma(l'_1, l'_2)\,.$$

Moreover from any Arakelov divisor $\widehat{D} = D + \sum_\sigma \alpha_\sigma[\sigma] = \sum_\mathfrak{p} n_\mathfrak{p}[\mathfrak{p}] + \sum_\sigma \alpha_\sigma[\sigma]$ we define an hermitian invertible A-module in the following way:

$$\widehat{D} \mapsto (\mathscr{O}_B(D), \{e^{-\alpha_\sigma} h_{0_\sigma}\}_\sigma)\,. \tag{C.7}$$

Now we can prove the fundamental result of this subsection:

Theorem C.20. $\mathrm{CH}^1_{\mathrm{Ar}}(B) \cong \operatorname{Pic}_{\mathrm{Ar}}(B)$.

Proof. Let's consider the map $\Psi : \mathrm{Div}_{\mathrm{Ar}}(B) \to \mathrm{Pic}_{\mathrm{Ar}}(B)$ induced by equation C.7. It is easy to verify that Ψ is a group homomorphism, so let's show that $\ker(\Psi) = \mathrm{Princ}_{\mathrm{Ar}}(B)$. Let $\widehat{(f)} \in \mathrm{Princ}_{\mathrm{Ar}}(B)$, for $f \in K$, then $\Psi(\widehat{(f)})$ is the isometry class of

$$\left(f^{-1}A, \{|f|_\sigma h_{0_\sigma}\}_\sigma\right) \tag{C.8}$$

and it is evident that there is an isometry between the above hermitian A-module and $(A, \{h_{0_\sigma}\})$ induced by the multiplication by f^{-1}. Vice versa if $\psi(\widehat{D})$ is isometric to $(O_K, \{h_{0_\sigma}\})$, the usual isomorphism $\mathrm{CH}^1(B) \cong \mathrm{Pic}(B)$ forces $\Psi(\widehat{D})$ to be as in equation (C.8) for some $f \in K$.
Finally let's show the surjectivity. Take an hermitian invertible A-module $(L, \{h_\sigma\})$, and consider a nonzero element $s \in L \otimes_A K$, then we show that

$$\Psi(\widehat{\mathrm{div}(s)}) = (\mathscr{O}_B(\mathrm{div}(s)), \{h_\sigma(s \otimes^\sigma 1, s \otimes^\sigma 1)h_{0_\sigma}\}_\sigma)$$

is isometric to $(L, \{h_\sigma\})$. By looking at the proof of proposition C.14 we know that the isomorphism $\mathscr{O}_B(\mathrm{div}(s)) \to L$ is given by $f \mapsto fs$. This map induces the required isometry at the archimedean points, indeed any element of $\mathscr{O}_B(\mathrm{div}(s))_\sigma$ of the type $f \otimes^\sigma 1$ (where f is inside K) is sent to $fs \otimes^\sigma 1 \in L_\sigma$ and moreover for any other element $f' \otimes^\sigma 1$:

$$h_\sigma(fs\otimes^\sigma 1, f's\otimes^\sigma 1) = h_\sigma(s\otimes^\sigma 1, s\otimes^\sigma 1)\sigma(f)\overline{\sigma(f')} = h_\sigma(s\otimes^\sigma 1, s\otimes^\sigma 1)h_{0_\sigma}(f, f')\,.$$

□

C.3 Arithmetic Riemann-Roch theorem

The Riemann-Roch theorem for projective curves is a powerful tools which allows to calculate the dimension of the vector space $H^0(\mathscr{L})$ of global sections of any invertible sheaf. In the arithmetic case, Riemann-Roch theorem involves the calculation of volumes related to an hermitian invertible A-module $\overline{L}$.

For any $\sigma \in B_\infty$ let's denote with K_σ, the completion of K with respect the absolute value $|\cdot|_\sigma$; hence $K_\sigma = \mathbb{R}$ or $K_\sigma = \mathbb{C}$ depending whether σ is a real or a complex embedding. The product $V_{\mathbb{R}} := \prod_{\sigma \in B_\infty} K_\sigma$ is a real vector space of dimension $d = [K : \mathbb{Q}]$ and given the map

$$\begin{array}{rcl} j : A & \to & V_{\mathbb{R}} \\ a & \mapsto & j(a) = (\sigma(a))_\sigma \end{array}$$

it is a well known fact that $j(A)$ is a lattice of $V_{\mathbb{R}}$. From now on we will omit any reference to the map j and by abuse of notation we will identify

A with $j(A)$ so we will consider $A \subset V_{\mathbb{R}}$ as a lattice of $V_{\mathbb{R}}$. Lets fix for the entire subsection an hermitian invertible A-module $\overline{L} = (L, \{h_\sigma\}_\sigma)$ and for an element $x \in L \setminus \{0\}$ lets define the injective A-homomorphism

$$\begin{array}{rcl} m_x : A & \to & L \\ a & \mapsto & ax\,, \end{array}$$

It is straightforward to notice that, as we did for A, L can be embedded in the d-dimensional real vector space $V_{\mathbb{R}}(L) = \prod_{\sigma \in B_\infty} L \otimes_A^\sigma K_\sigma$ thanks to the map $l \mapsto (l \otimes^\sigma 1)_\sigma$. Clearly A is considered an hermitian A-module with its canonical hermitian structure and $V_{\mathbb{R}}(A) = V_{\mathbb{R}}$. On the vector space $V_{\mathbb{R}}(L)$ we now fix a norm and a measure. Each hermitian product h_σ on L_σ induces in the obvious way a norm $||\cdot||_\sigma$ on the real vector space $L \otimes_A^\sigma K_\sigma$, so we can define the following norm on the whole $V_{\mathbb{R}}(L)$:

$$||\cdot||_{\sup} := \sup_{\sigma \in B_\infty} \{||\cdot||_\sigma\}\,.$$

The (closed) unit ball with respect to $||\cdot||_{\sup}$ is denoted by $\mathcal{B}^1_{V_{\mathbb{R}}(L)}$ and it is simply the product of the units balls in each $L \otimes_A^\sigma K_\sigma$. The following notion will be very useful later:

Definition C.21. The set of *small sections* of $\overline{L}$ is

$$H^0(\overline{L}) := \{x \in L\colon ||x||_{\sup} \leq 1\} = L \cap \mathcal{B}^1_{V_{\mathbb{R}}(L)}$$

Let's define the measure $\mu := \prod_\sigma \lambda_\sigma$ on $V_{\mathbb{R}}(L)$ where λ_σ is the Lebesgue measure on $L \otimes_A^\sigma K_\sigma$ (constructed by using the norm $||\cdot||_\sigma$).

Lemma C.22. *For any $x \in L \setminus \{0\}$, the quotient L/Ax has finite cardinality.*

Proof. The quotient L/Ax is a finitely generated torsion A-module, so it is a finitely generated module over a quotient of A, which is a finite ring. □

Proposition C.23. *L is a lattice in $V_{\mathbb{R}}(L)$.*

Proof. We need to show that L is discrete, so it is enough to show that for any compact set $C \subset V_{\mathbb{R}}(L)$, the intersection $L \cap C$ is finite. Let $v \in L \cap C$ and assume that $\#(L/Ax) = r$ (we are using lemma C.22), then $rv \in Ax \cap rC$. Now note that rC is compact and Ax is a lattice in $V_{\mathbb{R}}(L)$, thus $Ax \cap rC$ is finite. We can conclude that $L \cap C$ cannot be infinite, otherwise $r(L \cap C) \subset (Ax \cap rC)$ would be infinite too. □

Definition C.24. The volume of an invertible A-module L is defined as the volume of the lattice L. In other words:

$$\mathrm{vol}(L) := \mu_*(V_{\mathbb{R}}(L)/L)$$

where μ_* is the pushforward measure on the quotient $V_{\mathbb{R}}(L)/L$.

Remark C.25. If T is the fundamental region of the lattice L, recall that the volume of L can be also calculated as $\mu(T)$.

Now we are going to define the Arakelov characteristic of L and the degree of L; the arithmetic Riemann-roch theorem will relate these two notions.

Definition C.26. The *Arakelov characteristic* of $\overline{L}$ is:

$$\chi_{\mathrm{Ar}}(\overline{L}) := -\log(\mathrm{vol}(L))\,.$$

Definition C.27. Let $x \in L \setminus \{0\}$, then the *Arakelov degree* of $\overline{L}$ is:

$$\deg_{\mathrm{Ar}}(\overline{L}) := \log\left(\frac{\#(L/Ax)}{\prod_{\sigma\in B_\infty} ||x \otimes^\sigma 1||_\sigma^{\epsilon_\sigma}}\right)$$

Thanks to lemma C.22 we already know that $\#(L/Ax)$ is a finite number, but we still have to prove that $\deg(\overline{L})$ is independent on the choice of x. First of all let's see a way to rewrite the degree $\deg_{\mathrm{Ar}}(\overline{L})$; we will use the fact that for a finitely generated A-module we have the following isomorphism (See [43, Lemma 1.6]):

$$M \cong \bigoplus_{\mathfrak{p}\in B} M_{\mathfrak{p}}\,,$$

so in particular when M has finite cardinality

$$\#(M) = \prod_{\mathfrak{p}\in B} \#(M_{\mathfrak{p}}) = \prod_{\mathfrak{p}\in B} \mathfrak{N}(\mathfrak{p})^{\mathrm{length}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}})}\,.$$

At this point if we consider $x \in L \setminus \{0\}$ as an element of $L \otimes_A K$, we can write:

$$\deg_{\mathrm{Ar}}(\overline{L}) = \log(\#(L/Ax)) + \sum_{\sigma\in B_\infty} -\epsilon_\sigma \log ||x \otimes 1||_\sigma =$$

$$= \sum_{\mathfrak{p}\in B} \mathrm{length}_{A_{\mathfrak{p}}}((L/Ax)_{\mathfrak{p}}) \log \mathfrak{N}(\mathfrak{p}) + \frac{1}{2} \sum_{\sigma\in B_\infty} -\epsilon_\sigma \log(h_\sigma(x \otimes^\sigma 1, x \otimes^\sigma 1))_\sigma =$$

$$= \sum_{\mathfrak{p}\in B} \mathrm{length}_{A_{\mathfrak{p}}}(L_{\mathfrak{p}}/xA_{\mathfrak{p}}) \log \mathfrak{N}(\mathfrak{p}) + \frac{1}{2} \sum_{\sigma\in B_\infty} -\epsilon_\sigma \log(h_\sigma(x \otimes^\sigma 1, x \otimes^\sigma 1))_\sigma =$$

$$= \deg_{\mathrm{Ar}}\left(\widehat{\mathrm{div}(x)}\right) .$$

For any other $y \in L \setminus \{0\}$ we can write $y = fx$ for $f \in K^\times$, thus:

$$\log\left(\frac{\#(L/Ay)}{\prod_{\sigma\in B_\infty} ||y \otimes^\sigma 1||_\sigma^{\epsilon_\sigma}}\right) = \deg_{\mathrm{Ar}}\left(\widehat{\mathrm{div}(fx)}\right) = \deg_{\mathrm{Ar}}(\widehat{(f)})+\deg_{\mathrm{Ar}}\left(\widehat{\mathrm{div}(x)}\right) =$$

$$= \deg\left(\widehat{\mathrm{div}(x)}\right) = \deg_{\mathrm{Ar}}(\overline{L})$$

Theorem C.28 (Arithmetic Riemann-Roch theorem)**.** $\deg_{\mathrm{Ar}}(\overline{L}) = \log\left(\frac{\mathrm{vol}(A)}{\mathrm{vol}(L)}\right)$. *In particular the following equation holds:*

$$\chi_{\mathrm{Ar}}(\overline{L}) = \deg_{\mathrm{Ar}}(\overline{L}) + \chi_{\mathrm{Ar}}(\overline{A}) .$$

Proof. For any $x \in L \setminus \{0\}$ consider the short exact sequence:

$$0 \to Ax \to L \to L/Ax \to 0 ,$$

and using a bit of theory of lattices it is easy to see that

$$\mathrm{vol}(Ax) = \mathrm{vol}(L) \cdot \#(L/Ax) . \tag{C.9}$$

Now let's denote with λ'_σ and λ''_σ the Lebesgue measures respectively on $A \otimes_A K_\sigma$ and $Ax \otimes_A K_\sigma$. Moreover $(\lambda'_\sigma)_*$ is the pushforward measure with respect to the map $\widetilde{m}_x := m_x \otimes^\sigma 1 : A \otimes_A K_\sigma \to Ax \otimes_A K_\sigma$. Note that $\widetilde{m}_x$ sends the unit ball onto the ball of radius $||x \otimes^\sigma 1||_\sigma$, thus the following relationship between measures holds:

$$\lambda''_\sigma = (\lambda'_\sigma)_* \cdot ||x \otimes^\sigma 1||_\sigma^{\epsilon_\sigma} .$$

Hence, at this point we can conclude that equation (C.9) becomes:

$$\mathrm{vol}(A) \cdot \prod_{\sigma\in B_\infty} ||x \otimes^\sigma 1||_\sigma^{\epsilon_\sigma} = \mathrm{vol}(L) \cdot \#(L/Ax) .$$

□

We conclude the chapter explaining the relationship between the arithmetic Riemann-Roch theorem and Arakelov divisors. In particular we will see how it is possible to deduce some propositions similar to those that we have in algebraic geometry for divisors on projective curves.

Let's fix an Arakelov divisor $\widehat{D} = \sum_{\mathfrak{p}\in B} n_{\mathfrak{p}}[\mathfrak{p}] + \sum_{\sigma\in B_\infty} \alpha_\sigma[\sigma]$ and let $\overline{L} = (\mathscr{O}_B(D), \{e^{-\alpha_\sigma} h_{0_\sigma}\}_\sigma)$, where obviously $L = \mathscr{O}_B(D)$, the hermitian invertible

A-module associated to $\widehat{D}$. For any nonzero $s \in L \otimes_A K$ we have that $\deg_{\mathrm{Ar}}(\widehat{D}) = \deg_{\mathrm{Ar}}\left(\widehat{\mathrm{div}(s)}\right)$ and moreover there is $a \in A \setminus \{0\}$ such that $as \in L$, so:

$$\deg_{\mathrm{Ar}}(\widehat{D}) = \deg_{\mathrm{Ar}}\left(\widehat{\mathrm{div}(s)}\right) = \deg_{\mathrm{Ar}}\left(\widehat{\mathrm{div}(as)}\right) = \deg_{\mathrm{Ar}}(\overline{L}) \,.$$

In other words the definition of degree for elements $\mathrm{Pic}_{\mathrm{Ar}}(B)$ is the right one since it corresponds perfectly to the degree in $\mathrm{Div}_{\mathrm{Ar}}(B)$. In classical algebraic geometry, for a divisor $D = \sum n_x[x]$ on a projective curve X over k, we have the important notion of $H^0(D)$ which is the vector space of global sections of the invertible sheaf $\mathscr{O}_X(D)$; explicitly it can be written as:

$$H^0(D) = \{f \in K(X)^\times \colon \ \mathrm{ord}_x(f) \geq -n_x \ \ \forall x \in X'\} \cup \{0\} \,.$$

Now we give the equivalent arithmetic notion of $H^0(\widehat{D})$ for an Arakelov divisor just by emulating the geometric case:

$$H^0(\widehat{D}) := \{f \in K^\times \colon \ \mathrm{ord}_{\mathfrak{p}}(f) \geq -n_{\mathfrak{p}} \ \forall \mathfrak{p} \in B, \ 2v_\sigma(f) \geq -\alpha_\sigma \ \forall \sigma \in B_\infty\} \cup \{0\} \,.$$

One should immediately notice that the condition $2v_\sigma(f) \geq -\alpha_\sigma$ is equivalent to say $|f|_\sigma \leq e^{\frac{\alpha_\sigma}{2}}$ and archimedean balls are not closed under the addition; it means that $H^0(\widehat{D})$ is not an additive group but just a set. By looking at the explicit construction of $\overline{L}$ given by the equation (C.7) it is easy to see that $H^0(\widehat{D}) = H^0(\overline{L})$. In algebraic geometry the Riemann-Roch theorem is a powerful tool which allows to give an estimation of $h^0(D) := \dim_k(H^0(D))$, in particular we have the inequality:

$$h^0(D) \geq \deg(D) + \chi(\mathscr{O}_X) \,.$$

In our case the define $h^0(\widehat{D}) := \log \#(H^0(\hat{D}))$ and we now show how a result of Minkoswki theory of lattices and the arithmetic Riemann-Roch theorem give a lower bound for $h^0(\widehat{D})$. In particular we will use the following classical result:

Proposition C.29. *Let V be a real normed vector space of dimension d endowed with an Haar measure μ. Moreover let Λ be a lattice in V and $C \subset V$ a convex, bounded and symmetric subset. Then the following inequality holds:*

$$\#(C \cap \Lambda) \geq 2^{-d} \frac{\mu(C)}{\mathrm{vol}(\Lambda)} \,.$$

Furthermore, if C is closed, then the above inequality is strict.

Proof. See [43, Corollary 2.2]. □

The immediate consequence is that:

$$h^0(\widehat{D}) = \log \#(H^0(\overline{L})) = \log \# \left(L \cap \mathcal{B}^1_{V_{\mathbb{R}}(L)}\right) > \log 2^{-d} + \log \mu\left(\mathcal{B}^1_{V_{\mathbb{R}}(L)}\right) - \log(\mathrm{vol}(L)) =$$

$$=^{(\text{thm. C.28})} \log 2^{-d} + \log \mu\left(\mathcal{B}^1_{V_{\mathbb{R}}(L)}\right) + \deg_{\mathrm{Ar}}(\widehat{D}) + \chi_{\mathrm{Ar}}(A)\,.$$

Note that

$$\mu\left(\mathcal{B}^1_{V_{\mathbb{R}}(L)}\right) = \prod_{\sigma \text{ compl.}} \pi e^{\alpha_\sigma} \cdot \prod_{\sigma \text{ real.}} e^{\frac{\alpha_\sigma}{2}}\,,$$

thus if r_2 is the number of complex embeddings of K up to conjugation we get the following inequality:

$$h^0(\widehat{D}) > \deg_{\mathrm{Ar}}(\widehat{D}) + \chi_{\mathrm{Ar}}(A) + \frac{1}{2} \sum_{\sigma \in B_\infty} \epsilon_\sigma \alpha_\sigma + \log\left(\frac{\pi^{r_2}}{2^d}\right).$$

Appendix D

Arakelov geometry in dimension two

This appendix is a first introduction to Arakelov geometry for arithmetic surfaces. It can be seen as a rework and extension of the original Arakelov papers [2] and [1]. A particular emphasis is placed on giving an intuitive explanation of the Arakelov intersection pairing. We don't include any discussion about Faltings-Riemann-Roch theorem because it is not needed in the main part of this thesis and its complete presentation (and proof) would be quite lengthy; however good references are [43], [34] and [12].

D.1 Preliminaries on Riemann surfaces

In the first two subsections we give a quick review about some topics regarding Riemann surfaces, the main references are [25] and [40]. Subsection D.1.3 covers in detail the theory of Greeen functions that is of crucial importance in Arakelov geometry.

D.1.1 Divisors and holomorphic line bundles

Let's explain the link between holomorphic line bundles, invertible sheaves and divisors on a compact Riemann surface. We will work in the framework of 1-dimensional complex manifolds, but remember that this theory is related to the scheme theory of non-singular projective curves thanks to Serre's GAGA.

Let's fix a compact Riemann surface $(X, \mathscr{O})$ where $\mathscr{O}$ is the sheaf of holomorphic functions on X. The sheaf of regular functions on X is $\mathscr{O}_X$ and clearly we have $\mathscr{O}_X \subseteq \mathscr{O}$. Let $\mathscr{M}$ be the sheaf of meromorphic functions on

X; for any $f \in \mathscr{M}(X)^\times$ we can associate a divisor

$$(f) := \sum_{x \in X} \operatorname{ord}_x(f)[x]$$

where $\operatorname{ord}_x(f) = \min\{j \colon a_j \neq 0\}$ if $\sum_{j \in \mathbb{Z}} a_j z^j$ is the Laurent expansion around x of f in some holomorphic chart centred in x[1]. Let's sketch how we can identify $\operatorname{Div}(X)$ with $H^0(X, \mathscr{M}^\times / \mathscr{O}^\times)$. An element of $H^0(X, \mathscr{M}^\times / \mathscr{O}^\times)$ is a collection $\{(f_\alpha, U_\alpha)\}_\alpha$ where $f_\alpha \in \mathscr{M}(U_\alpha)$, $\{U_\alpha\}_\alpha$ is an open cover of X and $\frac{f_\alpha}{f_\beta} \in \mathscr{O}^\times(U_\alpha \cap U_\beta)$[2]. Hence given $\{(f_\alpha, U_\alpha)\}_\alpha$, we can define the divisor $D = \sum_{x \in X} n_x[x]$ such that $n_x = \operatorname{ord}_x(f_\alpha)$ if $x \in U_\alpha$. Note that the fact that $\frac{f_\alpha}{f_\beta} \in \mathscr{O}^\times(U_\alpha \cap U_\beta)$ implies that D is well defined. Vice versa, given a divisor $D = n_1[x_1] + \ldots + n_r[x_r]$ with $n_i \neq 0$, we can find an atlas $\{(U_i, z_i)\}_i$ of X such that $x_i \in U_i$ for any $i = 1, \ldots, r$ and $x_j \notin U_i$ if $i \neq j$. For any index i consider the function $f_i : U_i \to \mathbb{C}$ defined as

$$f_i := \prod_{j=1}^{r} (z_i - z_j(x_j))^{n_j} ;$$

it is clear that the collection $\{(f_i, U_i)\}_i$ is an element of $H^0(X, \mathscr{M}^\times / \mathscr{O}^\times)$.

The group of invertible sheaves on $(X, \mathscr{O})$ modulo the isomorphism relation is denoted by $\operatorname{Pic}(X)_h$ and it is obviously identified with the group of holomorphic line bundles up to isomorphism. As in the case of invertible sheaves on schemes, we give the definition of a meromorphic section of an invertible sheaf $\mathscr{L}$ on X:

Definition D.1. A *meromorphic section* of $\mathscr{L}$ is an element of $H^0(X, \mathscr{L} \otimes_{\mathscr{O}} \mathscr{M})$.

Let $X = \bigcup_\alpha U_\alpha$ be an open covering such that $\mathscr{L}|_{U\alpha} = b_\alpha \mathscr{O}|_{U_\alpha}$ for $b_\alpha \in \mathscr{L}(U_\alpha)$. Then, any nonzero meromorphic section s of $\mathscr{L}$ can be given as a collection of compatible elements $\{g_\alpha b_\alpha\}_\alpha$ where $g_\alpha \in \mathscr{M}(U_\alpha)$. Note that for any couple of indexes α, β we have that $\frac{g_\alpha}{g_\beta} \in \mathscr{O}(U_\alpha \cap U_\beta)^\times$. Let's put

$$\operatorname{ord}_x(s) := \operatorname{ord}_x(g_\alpha) \quad \text{if } x \in U_\alpha \, .$$

It is easy to show that this is a well defined equation, namely $\operatorname{ord}_x(s)$ is independent from the open set U_α containing x and from the basis b_α. The finite subset of X such that $\operatorname{ord}_x(s) \neq 0$ is called the support of s and it is denoted by $\operatorname{supp}(s)$. So, we have the divisor associated to s:

$$\operatorname{div}(s) := \sum_{x \in X} \operatorname{ord}_x(s)[x] \, .$$

---

[1] Remember that $\operatorname{ord}_x(f)$ is independent from the chosen chart for the Laurent expansion.

[2] See appendix B to see why an element of $H^0(X, \mathscr{M}^\times / \mathscr{O}^\times)$ has this form.

Remark D.2. For all but finitely many points $x \in X$ we have that $s_x \in \mathscr{L}_x$, so at these points is well defined the map $s \mapsto s(x) \in \mathscr{L}_x/\mathfrak{m}_x\mathscr{L}_x \cong \mathbb{C}$.

Remark D.3. Let s be a meromorphic section of $\mathscr{L}$ and let (L, π) the holomorphic line bundle associated to $\mathscr{L}$. Then s can be interpreted as a map $s : X \to L$ such that:

- $\pi \circ s = \mathrm{id}$.
- For any trivialization chart (U, ϕ_U), the composition $\pi_2 \circ \phi_U \circ s|_U : U \to \mathbb{C}$ is meromorphic (remember that $\pi_2 : U \times \mathbb{C} \to \mathbb{C}$ is the projection on the second component).

To each divisor $D = \sum_{x \in X} n_x[x] \in \mathrm{Div}(X)$ we can associate an invertible sheaf $\mathscr{O}(D)$ defined as follows for any open set $U \subseteq X$:

$$\mathscr{O}(D)(U) := \{f \in \mathscr{M}(U)^\times :\ \mathrm{ord}_x(f) + n_x \geq 0 \quad \forall x \in U\} \cup \{0\}\,.$$

If $D = \{(f_\alpha, U_\alpha)\}_\alpha$, the sheaf $\mathscr{O}(D)$ is the glueing of the sheaves $f_\alpha^{-1}\mathscr{O}|_{U_\alpha}$. Thus we have a map $\mathrm{Div}(X) \to \mathrm{Pic}(X)_h$ whose image is the set of invertible sheaves admitting a meromorphic section (up to isomorphism). This is clear by noticing that the meromorphic functions $f_\alpha \in \mathscr{M}(U_\alpha)$ give a meromorphic section which will be denoted as 1_D. The fundamental result is that invertible sheaves on X always admits nonzero meromorphic sections (see the appendix B for the equivalent result in the algebraic setting).

Theorem D.4. *Any invertible sheaf $\mathscr{L}$ on $(X, \mathscr{O})$ admits a nonzero meromorphic section.*

Proof. Let $p \in X$ be a fixed point and consider the divisor $m[p]$ for some integer $m > 0$. Moreover let $\mathbb{C}^m_{[p]}$ the skyscraper sheaf concentrated in p with value $\mathbb{C}^m$. Now we describe how to obtain a surjective map of sheaves

$$\varphi : \mathscr{L} \otimes_{\mathscr{O}} \mathscr{O}(m[p]) \to \mathbb{C}^m_{[p]}\,.$$

It is enough to give a "good" collection of homomorphisms $\{\varphi_x\}_{x \in X}$ on the stalks, then the reader can verify that this collection originates a morphism of sheaves. First of all fix a holomorphic chart (V, z) centred in p and a local basis b for $\mathscr{L}$ around p, then construct φ_x as follows: $\varphi_x = 0$ if $x \neq p$, otherwise $\varphi_p(b_p \otimes f_p) = (a_{-m}, \ldots, a_{-1})$ where $\sum_{i \geq -m} a_i z^i$ is the Laurent expression of f_p around p. The following sequence of sheaves on X turns out to be exact:

$$0 \to \mathscr{L} \to \mathscr{L} \otimes_{\mathscr{O}} \mathscr{O}(m[p]) \xrightarrow{\varphi} \mathbb{C}^m_{[p]} \to 0\,.$$

By using the long exact cohomology sequence we get another exact sequence:

$$H^0(X, \mathscr{L} \otimes_{\mathscr{O}} \mathscr{O}(m[p])) \to \mathbb{C}^m \to H^1(X, \mathcal{L}) \,.$$

If we choose $m > h^1(X, \mathcal{L})$, then $\mathbb{C}^m \to H^1(X, \mathcal{L})$ has nontrivial kernel, therefore $H^0(X, \mathscr{L} \otimes_{\mathscr{O}} \mathscr{O}(m[p]))$ contains a nonzero element. Finally, since

$$H^0(X, \mathscr{L} \otimes_{\mathscr{O}} \mathscr{O}(m[p])) \subseteq H^0(X, \mathscr{L} \otimes_{\mathscr{O}} \mathscr{M}) \,,$$

the theorem is proved. ☐

Corollary D.5. *The following sequence of abelian groups is exact:*

$$0 \to \mathbb{C}^\times \to \mathscr{M}(X)^\times \to \mathrm{Div}(X) \to \mathrm{Pic}(X)_h \to 0 \,.$$

Proof. The only nontrivial thing to show is the surjectivity of $\mathrm{Div}(X) \to \mathrm{Pic}(X)_h$, but this follows from theorem D.4. ☐

In particular $\mathrm{Pic}(X)_h \cong \mathrm{Pic}(X)$, which means that any holomorphic line bundle is algebraic.

D.1.2 Currents

In this subsection we allow the Riemann surface X to be noncompact. Here we show how differential forms and divisors define currents on X. For our purposes we don't need the full generality of the theory of distributions and currents on complex manifolds, by the way keep in mind that this theory is essential in order to study Arakelov geometry on arithmetic varieties of any dimension ([55] and [43]).
Let's fix $p, q \in \{0, 1\}$ and denote with $\mathscr{D}^{p,q}$ the sheaf of compactly supported C^∞ (p, q)-differential forms on X (when X is compact, then $\mathscr{D}^{p,q} = \mathscr{A}^{p,q}$).

Definition D.6. The $\mathbb{C}$-vector space of currents of bidegree (p, q) is the algebraic dual of $\mathscr{D}^{1-p,1-q}(X)$, which we denote as $(\mathscr{D}^{1-p,1-q}(X))^*$. In other words a *current of bidegree* (p, q) (or simply a (p, q)-current) on X is a $\mathbb{C}$-linear map:

$$\Psi : \mathscr{D}^{1-p,1-q}(X) \to \mathbb{C}$$

We can also define the "derivatives" of a (p, q)-current Ψ in the following way.

- $\partial\Psi$ is a $(1, q)$-current defined as $\partial\Psi(\eta) := (-1)^q \Psi(\partial\eta)$.
- $\bar{\partial}\Psi$ is a $(p, 1)$-current defined as $\bar{\partial}\Psi(\eta) := (-1)^p \Psi(\bar{\partial}\eta)$.

- $\partial\bar{\partial}\Psi$ is the composition of the above two derivatives and it is necessarily a $(1,1)$-current which can be written as $\partial\bar{\partial}\Psi(g) = (-1)^{(p+1)}\Psi(\bar{\partial}\partial g)$ for any $g \in C^\infty(X)$.

Now we want to describe some very important currents on X arising from global differential forms and divisors. If ω is a global (p,q)-differential form on X which locally can be expressed by means of locally integrable functions (for example $\omega \in \mathscr{A}^{p,q}(X)$), then *the (p,q)-current induced by* ω is:

$$\begin{array}{rrcl} [\omega]: & \mathscr{D}^{1-p,1-q}(X) & \to & \mathbb{C} \\ & \eta & \mapsto & \displaystyle\int_X \omega \wedge \eta \,. \end{array}$$

For any $f \in C^\infty(X)$, the operator $\partial\bar{\partial}$ applied on $[f]$ has the following explicit form:

$$\partial\bar{\partial}[f](g) = -[f](\bar{\partial}\partial g) = -\int_X f\bar{\partial}\partial g = \int_X f\partial\bar{\partial}g \,.$$

Proposition D.7. *Let f be a C^∞ function on X, then $\partial\bar{\partial}[f] = [\partial\bar{\partial}f]$.*

Proof. We have to show that for any compactly supported C^∞ function g on X we have the equality

$$\int_X f\partial\bar{\partial}g = \int_X g\partial\bar{\partial}f \,.$$

First of all notice that:

$$d(g\bar{\partial}f) = dg \wedge \bar{\partial}f + gd\bar{\partial}f = dg \wedge \bar{\partial}f + g(\partial + \bar{\partial})\bar{\partial}f = dg \wedge \bar{\partial}f + g\partial\bar{\partial}f \,.$$

$$d(f\partial g) = df \wedge \partial g + fd\partial g = df \wedge \partial g + f(\partial + \bar{\partial})\partial g = df \wedge \partial g - f\partial\bar{\partial}g \,,$$

At this point consider the submanifold $Y = \mathrm{supp}(g) \subseteq X$; g is 0 on the boundary ∂Y. By Stoke's theorem we get:

$$0 = \int_{\partial Y} g\bar{\partial}f = \int_Y d(g\bar{\partial}f) = \int_X d(g\bar{\partial}f)$$

and similarly $0 = d(f\partial g)$. Therefore we can conclude

$$\int_X g\partial\bar{\partial}f = -\int_X dg \wedge \bar{\partial}f = -\int_X (\partial g \wedge \bar{\partial}f + \bar{\partial}g \wedge \bar{\partial}f) =$$

$$= -\int_X \partial g \wedge \bar{\partial}f = \int_X \bar{\partial}f \wedge \partial g = \int_X df \wedge \partial g = \int_X f\partial\bar{\partial}g \,.$$

□

Consider a real divisor of X

$$D = \sum_{x \in X} \lambda_x [x] \in \mathrm{Div}(X)_{\mathbb{R}} = \mathrm{Div}(X) \otimes_{\mathbb{Z}} \mathbb{R}\,,$$

then *the* $(1,1)$*-current induced by* D is:

$$\begin{array}{rrcl} [D]: & \mathscr{D}^{0,0}(X) & \to & \mathbb{C} \\ & g & \mapsto & \widetilde{g}(D) := \displaystyle\sum_{x \in X} \lambda_x \, g(x)\,. \end{array}$$

If $D = [p]$ for a point $p \in X$, then the current induced by D is denoted by δ_p and it is the Dirac distribution concentrated in p.

The Poincare-Lelong formula shows a nice relation between the Laplacian of a current "originated" by a meromorphic function $f \in \mathscr{M}(X)^{\times}$ and the current induced by the divisor (f). Remember that $dd^c := \frac{i}{2\pi}\partial\bar{\partial}$.

Theorem D.8 (Poincare-Lelong formula)**.** *Let f be a nonzero meromorphic function on X then:*

$$dd^c[\log|f|^2] = [(f)]\,.$$

Equivalently, for any $g \in \mathscr{D}^{0,0}(X)$ we have:

$$\int_X \log|f|^2 \, dd^c(g) = \widetilde{g}((f))\,.$$

Proof. See [14, III.2.15]. □

The following theorem generalizes the Poincare-Lelong formula for meromorphic sections of hermitian invertible sheaves (recall the definition of the curvature form $\Theta(\mathscr{L}, h)$ given in appendix A.5):

Theorem D.9 (Poincare-Lelong formula for meromorpic sections)**.** *Let $(\mathscr{L}, h)$ be a C^∞ hermitian invertible sheaf on X and let s be a nonzero meromorphic section of $\mathscr{L}$, then*[3]*:*

$$dd^c[\log(h(s,s))] = [\mathrm{div}(s)] - \frac{i}{2\pi}[\Theta(\mathscr{L}, h)]\,. \tag{D.1}$$

Equivalently for any $g \in \mathscr{D}^{0,0}(X)$ we have:

$$\int_X \log(h(s,s)) \, dd^c(g) = \widetilde{g}(\mathrm{div}(s)) - \frac{i}{2\pi}\int_X g\,\Theta(\mathscr{L}, h) \tag{D.2}$$

---

[3] Recall that the map $h(s,s)$ is defined as $x \mapsto h_x(s(x), s(x))$.

Proof. See [14, pag. 271]. $\square$

Corollary D.10. *Let s be a meromorphic section of $(\mathscr{L}, h)$, then on $X \setminus \operatorname{supp}(s)$ we have the equality $\Theta(\mathscr{L}, h) = -\partial\bar{\partial}\log(h(s,s))$.*

Proof. Apply theorem D.9 on the Riemann surface $U = X \setminus \operatorname{supp}(s)$ to get the equation of currents $\partial\bar{\partial}[\log(h(s,s))] = -[\Theta(\mathscr{L}, h)]$. Proposition D.7 implies that $[\partial\bar{\partial}\log(h(s,s))] = -[\Theta(\mathscr{L}, h)]$ which is equivalent to

$$[\partial\bar{\partial}\log(h(s,s)) + \Theta(\mathscr{L}, h)] = 0\,.$$

It remains to show that if ω is a $(1,1)$-form on U such that $[\omega] = 0$, then $\omega = 0$. Suppose that $\omega \neq 0$; without loosing generality we can assume the existence of an open set V inside the support of ω such that $\omega = f(z,\overline{z})dz \wedge d\bar{z}$ on V with $\Re(f) > 0$ and $\Im(f) > 0$. Now let C be any non empty closed set of V and let g be the smooth bump function for C supported in V, then evidently $\int_U g\omega \neq 0$. $\square$

Since $\Theta(\mathscr{L}, h)$ is defined on the whole X, we can extend $-\partial\bar{\partial}\log(h(s,s))$ on X as well. Thus, from now on we consider the equality of differential forms $\Theta(\mathscr{L}, h) = -\partial\bar{\partial}\log(h(s,s))$ on X, for any meromorphic section s.

Theorem D.9 relates the two different notions of first Chern class on X, since the class of $\operatorname{div}(s)$ in $\mathrm{CH}^1(X)$ is the "geometric" first Chern class of $\mathscr{L}$ (see remark A.31). In particular for $g = 1$ equation (D.2) becomes

$$\deg(\operatorname{div}(s)) = \frac{i}{2\pi}\int_X \Theta(\mathscr{L}, h) = \deg(\mathscr{L})$$

so, the degree maps on $\mathrm{CH}^1(X)$ and on $\mathrm{Pic}_h(X)$ coincide as expected.

D.1.3 Green functions

Definition D.11. A *Green function* on X is a map $g : U \subseteq X \to \mathbb{R}$ satisfying the following properties:

1. $U = X \setminus \{x_1, \ldots, x_r\}$ for $r \in \mathbb{N}$.
2. g is a C^∞ function on U.
3. For any point $x \in X$ there exist a real number $a \in \mathbb{R}$ and a C^∞ function u on an open neighborhood of x such that the equality:

   $$g = a\log|z|^2 + u$$

   holds in an open (punctured) neighborhhod of x contained in a holomorphic chart (V, z) centred in x.

Proposition D.12. *Let g be a Green function on X and fix a point $x \in X$. Then the number $a \in \mathbb{R}$ arising in condition (3) of definition D.11 depends only on the point x and it is uniquely defined.*

Proof. First we check the independence on the chosen holomorphic chart. By hypothesis we know that inside (V, z) we have $g = a \log |z|^2 + u$; we want to show that inside any other holomorphic chart (V', w) centred in x we can write

$$g = a \log |w|^2 + u'$$

where u' is a C^∞ map around x. The function zw^{-1} is bihomolorphic in a neighborhood of 0 and such that $zw^{-1}(0) = 0$. It follows that on an open set $W \subseteq V \cap V'$ we have the equality $z = wf(w)$, where f is a holomorphic function on a neighborhood of 0 such that $f(0) \neq 0$. Then

$$g = a \log |z|^2 + u = a \log |wf(w)|^2 + u = a \log |w|^2 + \big(a \log |f(w)|^2 + u\big)$$

where $a \log |f(w)|^2 + u$ is C^∞ around x.
In order to prove that a is uniquely defined by x, it is enough to show that if A and A' are two open neighborhood of x contained in a chart (V, z) such that:

$$g = a \log |z|^2 + u \quad \text{on } A$$

$$g = b \log |z|^2 + u' \quad \text{on } A'$$

then $a = b$. But on $A \cap A'$ we have the equality $(a - b) \log |z|^2 = u' - u$ which is true only when $a - b = 0$, otherwise we would have that $u' - u$ is a C^∞ continuation of $(a - b) \log |z|^2$ at x. This is impossible since $\lim\limits_{t \to x}(a - b) \log |z(t)|^2 = -\infty$. $\square$

Proposition D.12 ensures that the following definition makes sense:

Definition D.13. Let g be a Green function on X such that around a point $x \in X$ it can be written as $g = a \log |z|^2 + u$. Then we put $\operatorname{ord}_x^G(g) := -a$ and we call it *the Green order of g at x*.

Clearly $\operatorname{ord}_x^G(g) \neq 0$ if and only if x is a point out from the domain of g, i.e. only at a finite number of points. The Green functions on X form a real vector space $G(X)$, and for any $g, g' \in G(X)$

$$\operatorname{ord}_x^G(\lambda g) = \lambda \operatorname{ord}_x^G(g) \quad \text{for any } \lambda \in \mathbb{R}\,,$$

$$\operatorname{ord}_x^G(g + g') = \operatorname{ord}_x^G(g) + \operatorname{ord}_x^G(g')\,.$$

Let's denote with $\mathrm{Div}(X)_{\mathbb{R}} := \mathrm{Div}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ the vector space of $\mathbb{R}$ divisors on X, then we have a $\mathbb{R}$-linear map:

$$\begin{array}{rcl} \mathrm{div}^G : G(X) & \to & \mathrm{Div}(X)_{\mathbb{R}} \\ g & \mapsto & \sum_{x \in X} \mathrm{ord}_x^G(g)[x] \end{array}$$

Proposition D.14. *Let* $(\mathscr{L}, h)$ *be a* C^∞ *hermitian invertible sheaf on* X*, and let* s *be a nonzero meromorphic section of* $\mathscr{L}$*, then the map* $-\log(h(s,s))$ *is a Green function on* X *such that* $\mathrm{div}^G(-\log(h(s,s))) = \mathrm{div}(s)$.

Proof. The meromorphic section s is given by a collection $\{g_\alpha b_\alpha\}_\alpha$ where: $X = \bigcup U_\alpha$ is a trivialization covering for $\mathscr{L}$, $g_\alpha \in \mathscr{M}_X(U_\alpha)$ and $b_\alpha \in \mathscr{L}(U_\alpha)$ is a local basis on U_α. The map $-\log(h(s,s))$ given by $x \mapsto -\log(h_x(s(x), s(x)))$ is well defined and C^∞ on a set of the type $U = X \setminus \{x_1, \dots, x_r\}$. If $x \in U_\alpha$ then we can find a holomorphic chart (V, z) chart centred in x such that $V \subseteq U_\alpha$ and $g_\alpha|_V = z^a f(z)$ where $a \in \mathbb{Z}$ and f is a holomorphic function around 0 such that $f(0) \neq 0$. It follows that on V we have

$$\begin{array}{rcl} h(s,s) & = & h(g_\alpha b_\alpha, g_\alpha b_\alpha) = h(z^a f(z) b_\alpha, z^a f(z) b_\alpha) = \\ & = & |z|^{2a} h(f(z) b_\alpha, f(z) b_\alpha) = |z|^{2a} u \end{array}$$

where $u := h(f(z)b_\alpha, f(z)b_\alpha)$ is C^∞ around x and $u(x) \neq 0$. Hence locally

$$-\log(h(s,s)) = -a \log |z|^2 - \log u$$

and this shows that $-\log(h(s,s))$ is a Green function.
Moreover it is clear from the local analysis that $a = \mathrm{ord}_x(s)$. Since $\mathrm{div}(s) = \sum_{x \in X} \mathrm{ord}_x(s)[x]$, then we have $\mathrm{div}^G(-\log(h(s,s))) = \mathrm{div}(s)$. □

The following result is an immediate consequence of proposition D.14:

Proposition D.15. *The map* $\mathrm{div}^G : G(X) \to \mathrm{Div}(X)_{\mathbb{R}}$ *is surjective.*

Proof. Let $D = \sum_{j=1}^r \lambda_j [x_j] \in \mathrm{Div}(X)_{\mathbb{R}}$, then for each j there exists a meromorphic section s_j of $\mathscr{O}_X([x_j])$ such that $\mathrm{div}(s_j) = [x_j]$. By proposition *D*.14 it follows that

$$\mathrm{div}^G \left(-\sum_{j=1}^r \lambda_j \log(h(s_j, s_j)) \right) = D\,.$$

□

Let's define a very important subspace of $\mathbb{Z}G(X)$:

Definition D.16. The vector space of Green functions with integer orders on X is:

$$\mathbb{Z}G(X) := \big\{g \in G(X) \colon \operatorname{ord}_x^G(g) \in \mathbb{Z} \ \forall x \in X\big\}$$

The next result shows that any Green function which induces a divisor on X is actually of the form $-\log(h(s,s))$ for some meromorphic section s of a C^∞ hermitian invertible sheaf $(\mathscr{L}, h)$.

Proposition D.17. *Let $g \in \mathbb{Z}G(X)$, then there exist a C^∞ hermitian invertible sheaf $(\mathscr{L}, h)$ on X and a meromorphic section s of $\mathscr{L}$ such that $g = -\log(h(s,s))$.*

Proof. Let $\operatorname{div}^G(g) = D \in \operatorname{Div}(X)$ and let's choose an hermitian metric h' on $\mathscr{O}_X(D)$. If s is a meromorphic section of $\mathscr{O}_X(D)$, then by proposition D.14

$$\operatorname{div}^G(-\log(h'(s,s))) = \operatorname{div}(s) = D\,. \tag{D.3}$$

Consider the Green function $f = g + \log(h'(s,s))$, then $\operatorname{div}^G(f) = D - D = 0$ by equation (D.3), which means that f is a C^∞ function on X. Hence if we put

$$h_x := e^{-f(x)} h'_x\,,$$

then $h = \{h_x\}_{x\in X}$ is a C^∞ hermitian metric on $\mathscr{O}_X(D)$. Finally, for any $x \in X$ it holds that:

$$\begin{aligned} -\log(h_x(s(x), s(x))) &= -\log\left(e^{-f(x)} h'_x(s(x), s(x))\right) = \\ &= f(x) - \log(h'_x(s(x), s(x))) = \\ &= g(x)\,. \end{aligned}$$

□

If $f : X \to \mathbb{C}$ is any function, then we define:

$$\begin{aligned} \widetilde{f} : \operatorname{Div}(X)_{\mathbb{R}} &\to \mathbb{C} \\ \sum_{x\in X} \lambda_x x &\mapsto \sum_{x \in X} \lambda_x f(x)\,. \end{aligned}$$

In particular $\widetilde{1}(D) = \deg(D)$ for any $D \in \operatorname{Div}(X)_{\mathbb{R}}$. If $g \in G(X)$, then $dd^c(g)$ is locally written as

$$dd^c(g) = \frac{i}{2\pi}\partial\bar{\partial}(a \log|z|^2 + u)$$

but $\partial\bar{\partial}\log(|z|^2) = \partial\bar{\partial}\log(z\bar{z}) = 0$, namely $dd^c(g) \in \mathscr{A}^{1,1}(X)$. This means that we can always integrate the $(1,1)$-form $dd^c(g)$ on the whole X.

Remark D.18. Note that if $g \in G(X)$, then $\int_X dd^c(g)$ is a real number.

Theorem D.19 (Poincare-Lelong formula for Green functions)**.** *Let g be a Green function on X, then the following equality of currents holds:*

$$dd^c[g] + [\mathrm{div}^G(g)] = [dd^c(g)]\,.$$

Equivalently for any C^∞ function f we have:

$$\int_X g\, dd^c(f) + \widetilde{f}(\mathrm{div}^G(g)) = \int_X f\, dd^c(g)$$

Proof. X can be covered by a finite number of charts $\{(U_j, z_j)\}$ for $j = 1, \ldots, m$ such that on U_j we have $g = a_j \log|z_j|^2 + u_j$ where u_j is a C^∞ function. Furthermore we can choose the charts in a way that if $p \in X$ is a point such that $\mathrm{ord}_p^G(g) \neq 0$ then p is a center of some chart (U_j, z_j). Let $\{\psi_j\}$ be a C^∞ partition of the unity relative to the covering $\{U_j\}$, then for any $f \in C^\infty(X)$ by the "chain rule" applied on dd^c we have:

$$dd^c[\psi_j g](f) = dd^c[g](\psi_j f) - dd^c[fg](\psi_j)\,. \tag{D.4}$$

But on the other hand

$$dd^c[g](\psi_j f) = dd^c[a_j \log|z|^2 + u_j](\psi_j f) = dd^c[a_j \log|z|^2](\psi_j f) + dd^c[u_j](\psi_j f)\,. \tag{D.5}$$

By the Poicare-Lelong formula applied to the Riemann surface U_j

$$dd^c[a \log|z|^2](\psi_j f) = \widetilde{\psi_j f}((z_j^a)) = \widetilde{\psi_j f}(-\mathrm{div}^G(g))\,.$$

Moreover by proposition D.7 and the fact that $dd^c(\log|z|^2) = 0$ we deduce the following expression for $dd^c[u_j](\psi_j f)$:

$$dd^c[u_j](\psi_j f) = [dd^c(u_j)](\psi_j f) = [dd^c(a_j \log|z|^2 + u_j)](\psi_j f) =$$

$$= [dd^c(g)](\psi_j f) = [dd^c(\psi_j g)](f) - [dd^c(\psi_j)](fg)\,.$$

Now the new values of $dd^c[a \log|z|^2](\psi_j f)$ and $dd^c[u_j](\psi_j f)$ can be substituted in equation (D.5) to get:

$$dd^c[g](\psi_j f) = \widetilde{\psi_j f}(-\mathrm{div}^G(g)) + [dd^c(\psi_j g)](f) - [dd^c(\psi_j)](fg)\,. \tag{D.6}$$

Then we substitute equation (D.6) in equation (D.4), in order to obtain:

$$dd^c[\psi_j g](f) = \widetilde{\psi_j f}(-\mathrm{div}^G(g)) + [dd^c(\psi_j g)](f) - [dd^c(\psi_j)](fg) - dd^c[fg](\psi_j)\,. \tag{D.7}$$

Finally:

$$dd^c[g](f) = dd^c\left[\sum_j \psi_j g\right](f) = \sum_j dd^c[\psi_j g](f) =$$

$$= \underbrace{\sum_j \widetilde{\psi_j f}(-\operatorname{div}^G(g))}_{(i)} + \underbrace{\sum_j [dd^c(\psi_j g)](f)}_{(ii)} - \underbrace{\sum_j [dd^c(\psi_j)](fg)}_{(iii)} - \underbrace{\sum_j dd^c[fg](\psi_j)}_{(iv)} . \tag{D.8}$$

Let's analyze all the sums in equation (D.8):

(i)

$$\sum_j \widetilde{\psi_j f}(-\operatorname{div}^G(g)) = -\sum_j \sum_{x\in X} \operatorname{ord}_x^G(g)\psi_j(x)f(x) =$$

$$= -\sum_{x\in X} \operatorname{ord}_x^G(g)f(x)\sum_j \psi_j(x) = -\sum_{x\in X} \operatorname{ord}_x^G(g)f(x) = -\widetilde{f}(\operatorname{div}^G(g)) .$$

(ii)

$$\sum_j [dd^c(\psi_j g)](f) = \left[dd^c\left(\sum_j \psi_j g\right)\right](f) = [dd^c(g)](f) .$$

(iii)

$$\sum_j [dd^c(\psi_j)](fg) = \sum_j \int_X fg\, dd^c(\psi_j) = \int_X fg\, dd^c\left(\sum_j \psi_j\right) =$$

$$= \int_X fg\, dd^c(1) = 0 .$$

(iv)

$$\sum_j dd^c[fg](\psi_j) = \sum_j \int_X fg\, dd^c(\psi_j) = 0 .$$

Therefore equation (D.8) become

$$dd^c[g](f) = -\widetilde{f}(\operatorname{div}^G(g)) + [dd^c(g)](f)$$

and the proof is complete. □

Corollary D.20. *For any $g \in G(X)$:*

$$\deg\left(\operatorname{div}^G(g)\right) = \int_X dd^c(g) .$$

Proof. Use the Poincare-Lelong formula for Green functions with $f = 1$. □

From now on, in this subsection we fix a Kähler fundamental form Ω on X such that $\int_X \Omega = 1$. By proposition A.20 for any Green function g on X we have

$$dd^c(g) = -\frac{1}{2\pi}\Delta_{\bar{\partial}}(g)\Omega\,, \tag{D.9}$$

so in particular $\Delta_{\bar{\partial}}(g)$ is a C^∞ function on X. In practice we have a map

$$\Delta_{\bar{\partial}} : G(X) \to C^\infty(X)\,.$$

Let's define the two linear subspaces of $G(X)$:

$$G^\Omega(X) := \{g \in G(X) \colon \Delta_{\bar{\partial}}(g) \text{ is constant}\}$$

$$G_0^\Omega(X) := \{g \in G^\Omega(X) \colon \int_X g\Omega = 0\}$$

$$\mathbb{Z}G^\Omega(X) := \mathbb{Z}G(X) \cap G^\Omega(X)$$

$$\mathbb{Z}G_0^\Omega(X) := \mathbb{Z}G(X) \cap G_0^\Omega(X)$$

Theorem D.21. *The map* $\mathrm{div}^G|_{G_0^\Omega(X)} : G_0^\Omega(X) \to \mathrm{Div}(X)_{\mathbb{R}}$ *is an isomorphism.*

Proof. Assume that $g \in G_0^\Omega(X)$ and $\mathrm{div}^G(g) = 0$. This means that $g \in C^\infty(X)$. By equation (D.9) there is a constant $\alpha \in \mathbb{C}$ such that $dd^c(g) = \alpha\Omega$ and corollary D.20 says that

$$0 = \deg(\mathrm{div}^G(g)) = \int_X dd^c(g) = \int_X \alpha\Omega = \alpha\,.$$

The equality $dd^c(g) = 0$ means that g is harmonic, so by using the maximum principle for harmonic functions on Riemann surfaces, we deduce that $g \in \mathbb{R}$. On the other hand $\int_X g\Omega = 0$, so we conclude that $g = 0$. This shows that the map $\mathrm{div}^G|_{G_0^\Omega(X)}$ is injective.

Let $D \in \mathrm{Div}(X) \in \mathbb{R}$, then by proposition D.15 there exists a Green function $g_1 \in G(X)$ such that $\mathrm{div}^G(g_1) = D$. Let's put

$$f = \Delta_{\bar{\partial}}(g_1) - \int_X \Delta_{\bar{\partial}}(g_1)\Omega\,.$$

Then

$$\int_X f\Omega = \int_X \Delta_{\bar{\partial}}(g_1)\Omega - \int_X \Delta_{\bar{\partial}}(g_1)\Omega = 0\,.$$

By corollary A.19 there exists $u \in C^\infty(X)$ such that $\Delta_{\bar\partial}(u) = f$. Now put $g_2 := g_1 - u \in G(X)$, then

$$\Delta_{\bar\partial}(g_2) = \Delta_{\bar\partial}(g_1) - \Delta_{\bar\partial}(u) = \int_X \Delta_{\bar\partial}(g_1)\Omega\,,$$

namely $g_2 \in G^\Omega(X)$. Finally if

$$g := g_2 - \int_X g_2\Omega \in G(X)$$

it follows that $\Delta_{\bar\partial}(g)$ is a constant and $\int_X g\Omega = 0$, so $g \in G_0^\Omega(X)$. Furthermore

$$\mathrm{div}^G(g) = \mathrm{div}^G\left(g_1 - u - \int_X g_2\Omega\right) = \mathrm{div}^G(g_1) = D\,,$$

and this hows that $\mathrm{div}^G|_{G_0^\Omega(X)}$ is surjective. ☐

Proposition D.22. *For any $g \in G^\Omega(X)$ there exists a unique decomposition $g = g_0 + c$ for $g_0 \in G_0^\Omega(X)$ and $c \in \mathbb{R}$.*

Proof. Such a decomposition exists, indeed put $g_0 := g - \int_X g\Omega$ and $c := \int_X g\Omega$. Consider two different decompositions

$$g = g_0 + c = g_0' + c'\,.$$

By integrating $g\Omega$ we conclude that $c = c'$ and $g_0 = g_0'$. ☐

Definition D.23. The inverse map of $\mathrm{div}^G|_{G_0^\Omega(X)}$ is denoted as:

$$\begin{array}{rcl} \mathcal{G}^\Omega : \mathrm{Div}(X)_{\mathbb{R}} & \to & G_0^\Omega(X) \\ D & \mapsto & \mathcal{G}^\Omega(D) \end{array}$$

and we can define the following function:

$$\begin{array}{rcl} g^\Omega : (X \times X) \setminus \Delta_{X\times X} & \to & \mathbb{R} \\ (p,q) & \mapsto & g^\Omega(p,q) := \mathcal{G}^\Omega([p])(q) \end{array}$$

where $\Delta_{X\times X}$ denotes the diagonal subset of $X \times X$.

By construction g^Ω is C^∞ in the variable q, but, as we will see soon (corollary D.28), g^Ω turns out to be symmetric, therefore it is C^∞. Since $g^\Omega(p,\cdot) \in G_0^\Omega(X) \subset G^\Omega(X)$, then $dd^c(g^\Omega(p,\cdot)) = \alpha\Omega$ for a constant $\alpha \in \mathbb{C}$, but

$$1 = \deg^G(g^\Omega(p,\cdot)) = \int_X dd^c(g^\Omega(p,\cdot)) = \int_X \alpha\Omega = \alpha\,.$$

Hence $\alpha = 1$ and

$$dd^c(g^\Omega(p,\cdot)) = \Omega. \tag{D.10}$$

Thus, amongst all Green functions, those of the form $g^\Omega(p,\cdot)$ satisfy the Poisson differential equation (D.10). this feature will be very useful for intersection theory.

Another important property is that for any fixed $p \in X$:

$$\int_X g^\Omega(p,\cdot)\Omega = \int_X \mathcal{G}^\Omega([p])\Omega = 0 \tag{D.11}$$

because $\mathcal{G}^\Omega([p]) \in G_0^\Omega(X)$.

Remark D.24. g^Ω can be defined as the *unique* function on $(X \times X) \setminus \Delta_{X\times X}$ with values in $\mathbb{R}$ satisfying the following properties:

(1) Around any point $p \in X$ we can write $g^\Omega(p,\cdot) = -\log|z|^2 + u$, where z is a chart centred in p and u is C^∞.

(2) $dd^c(g^\Omega(p,\cdot)) = \Omega$.

(3) $\int_X g^\Omega(p,\cdot)\Omega = 0$.

This is how Arakelov defined g^Ω in [2] and [1]. In the literature g^Ω is usually called *the Green function of X (with respect to* Ω*)*[4]. Here we used a different approach (and notations), indeed g^Ω was constructed directly by using the isomorphism $\mathrm{Div}(X)_\mathbb{R} \cong G_0^\Omega(X)$.

Definition D.25. Let $g_1, g_2 \in G(X)$ such that $\mathrm{div}^G(g_1)$ and $\mathrm{div}^G(g_2)$ have no common components then the $*$*-product* between g_1 and g_2 is the real number:

$$g_1 * g_2 := \widetilde{g_1}(\mathrm{div}^G(g_2)) + \int_X dd^c(g_1)g_2\,.$$

Remark D.26. It is necessary to assume that $\mathrm{div}^G(g_1)$ and $\mathrm{div}^G(g_2)$ have no common components otherwise $\widetilde{g_1}(\mathrm{div}(g_2))$ wouldn't be well defined.

The following theorem is crucial for the whole Arakelov theory; it says that the $*$-product is commutative.

Theorem D.27. *Let $g_1, g_2 \in G(X)$ such that $\mathrm{div}^G(g_1)$ and $\mathrm{div}^G(g_2)$ have no common components, then $g_1 * g_2 = g_2 * g_1$.*

---

[4] Actually the conditions which uniquely define g^Ω in [2] and [1] are slightly different from the ones listed here, and moreover they may vary in other references. For instance it is common to find different constants for the differential Poisson equation, or the Green function might be defined as $G = \exp(g^\Omega)$. Of course these discrepancies are fixed when the Green function is applied for intersection theory.

Proof. The strategy consists in performing the calculation of $g_1 * g_2$ and $g_2 * g_1$ by using smooth modified versions of g_1 and g_2, then the thesis will follow by a limit process.

Let $\{x_1, \dots, x_r\}$ and $\{y_1, \dots, y_s\}$ the set of singular points respectively of g_1 and g_2. Around each x_i consider an open neighborhood $D(x_i, \delta)$ which is the homeomorphic preimage, through a local chart z, of a small open disk in $\mathbb{C}$ of center $z(x_i)$ and radius δ. $D(y_i, \delta)$ is defined in the same way. Consider the following data:

- Define $A_1 := X \setminus \bigcup_{i=1}^{r} D(x_i, \delta)$ and $U_1 := X \setminus \bigcup_{i=1}^{r} \overline{D(x_i, \delta/2)}$. Let ρ_1^δ be a smooth bump function for A_1 supported in U_1; moreover $g_1^\delta := \rho_1^\delta g_1$ and $\omega_1^\delta := dd^c(g_1 - g_1^\delta)$.
- Define $A_2 := X \setminus \bigcup_{i=1}^{s} D(y_i, \delta)$ and $U_2 := X \setminus \bigcup_{i=1}^{s} \overline{D(y_i, \delta/2)}$. Let ρ_2^δ be a smooth bump function for A_2 supported in U_2; moreover $g_2^\delta := \rho_2^\delta g_2$ and $\omega_2^\delta := dd^c(g_2 - g_2^\delta)$.

Note that g_j^δ is a smooth function and ω_j^δ is a smooth C^∞ form, for $j = 1, 2$.

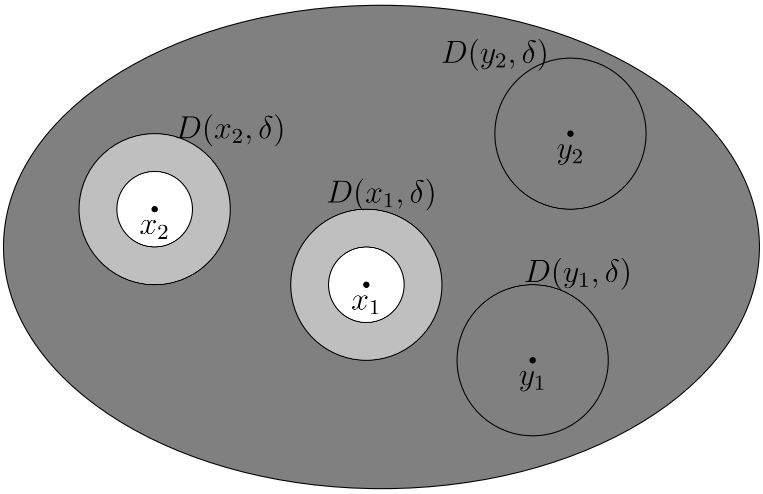

Figure D.1: The diagram represents the value of g_1^δ on its domain: on the dark grey region we have the equality $g_1^\delta = g_1$; on the light grey annuli $0 \le g_1^\delta \le g_1$ and on the white discs $g_1^\delta = 0$.

Let's calculate the following integral for any smooth function f on X:

$$\int_X f\omega_2^\delta = \int_X f dd^c(g_2) - \int_X f dd^c(g_2^\delta) =^{(\text{prop.} D.7)} \int_X f dd^c(g_2) - \int_X g_2^\delta dd^c(f)\,. \tag{D.12}$$

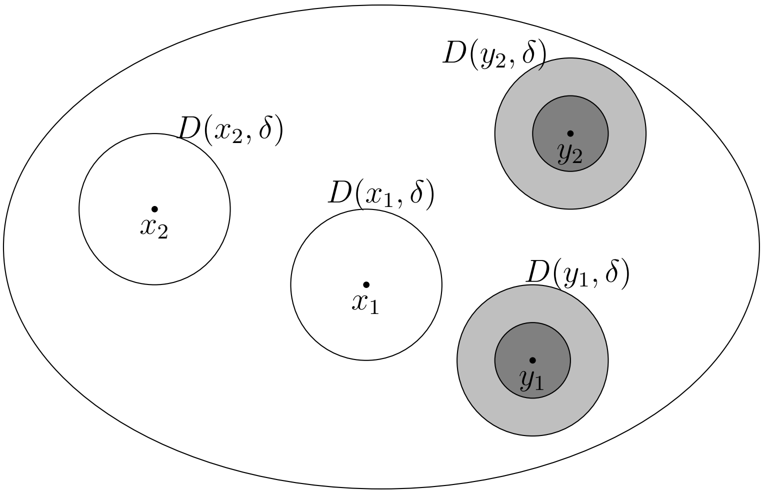

Figure D.2: The diagram represents the value of ω_2^δ on its domain: on the white region we have the equality $\omega_2^\delta = 0$; on the light grey annuli $0 \leq \omega_2^\delta \leq 1$ and on the white discs $\omega_2^\delta = 1$.

Notice that $\lim\limits_{\delta\to 0} g_j^\delta(x) = g_j(x)$ for any x nonsingular, so by taking the limit for $\delta \to 0$ in equation (D.12) we obtain:

$$\lim_{\delta\to 0}\int_X f\omega_2^\delta = \int_X f dd^c(g_2) - \int_X g_2 dd^c(f) =^{(\text{theo.}D.19)} \widetilde{f}(\mathrm{div}^G(g_2))\,. \tag{D.13}$$

Similarly we get

$$\lim_{\delta\to 0}\int_X f\omega_1^\delta = \widetilde{f}(\mathrm{div}^G(g_1))\,. \tag{D.14}$$

Put by simplicity $\omega_1 := dd^c(g_1)$, $\omega_2 := dd^c(g_2)$ and consider the function:

$$a(\delta) = \int_X g_1^\delta dd^c(g_2^\delta) = \int_X g_2^\delta dd^c(g_1^\delta)\,.$$

By the first equality we have

$$a(\delta) = \int_X g_1^\delta dd^c(g_2^\delta) = \int_X g_1^\delta(\omega_2 - \omega_2^\delta) = \int_X g_1^\delta \omega_2 - \int_X g_1^\delta\omega_2^\delta\,,$$

but remember that $\mathrm{supp}(\omega_2^\delta) \subseteq \bigcup_{i=1}^s D(y_i,\delta)$ and $\mathrm{supp}(g_1^\delta) \subseteq X\backslash\bigcup_{i=1}^r \overline{D(x_i,\delta/2)}$, so if we choose $\gamma < \delta$ we have the equality $\int_X g_1^\delta\omega_2^\delta = \int_X g_1^\gamma\omega_2^\delta$. By using equation (D.13) with $f = g_1^\gamma$ we have:

$$\lim_{\delta\to 0} a(\delta) = \int_X g_1^\delta\omega_2 - \int_X g_1^\gamma\omega_2^\delta = \int_X g_1\omega_2 - \widetilde{g_1^\gamma}(\mathrm{div}^G(g_2)) =$$

$$= \int_X g_1\omega_2 - \widetilde{g_1}(\mathrm{div}^G(g_2))\,.$$

In the same way we can show that

$$\lim_{\delta\to 0} a(\delta) = \int_X g_2^\delta dd^c(g_1^\delta) = \int_X g_2\omega_1 - \widetilde{g_2}(\mathrm{div}^G(g_1))\,,$$

therefore we get the equality

$$\int_X g_1\omega_2 + \widetilde{g_2}(\mathrm{div}^G(g_1)) = \int_X g_2\omega_1 + \widetilde{g_1}(\mathrm{div}^G(g_2))$$

which is exactly $g_2 * g_1 = g_1 * g_2$. ☐

Corollary D.28. *$g^\Omega(p,q) = g^\Omega(q,p)$ for any $p \neq q$.*

Proof. By using the properties of the elements in G_0^Ω, it is easy to verify that

$$\mathcal{G}^\Omega([p]) * \mathcal{G}^\Omega([q]) = g^\Omega(p,q); \quad \mathcal{G}^\Omega([q]) * \mathcal{G}^\Omega([p]) = g^\Omega(q,p)\,.$$

Hence the conclusion follows immediately from theorem D.27. ☐

Note that for any three different points $p, q, t \in X$ and coefficients $a, b \in \mathbb{R}$ we have that:

$$\mathcal{G}^\Omega(a[p] + b[q]) * \mathcal{G}^\Omega([t]) = a\mathcal{G}^\Omega([p]) * \mathcal{G}^\Omega([t]) + b\mathcal{G}^\Omega([q]) * \mathcal{G}^\Omega([t])\,.$$

Therefore if $D = \sum_{p\in X} a_p[p]$ and $E = \sum_{q\in X} b_q[q]$ are two real divisors of X with no common components, then it is customary to define:

$$g^\Omega(D,E) := \sum_{p\neq q} a_p b_q g^\Omega(p,q)\,. \tag{D.15}$$

Remark D.29. The important point to emphasize here is that for Green functions $g_1, g_2 \in G_0^\Omega$, i.e. coming from some real divisors on X, the integral appearing in $g_1 * g_2$ vanishes. This means that for such kind of Green functions, the nature of the $*$-product is “less analytic”, indeed it depends only on the value of g_1 or g_2 at a finite set of points.

We conclude the presentation of the general theory of Green functions by explaining how g^Ω can be used to compute the “pseudo-inverse” of the $\bar\partial$-Laplacian. First of all consider the function $\Delta_{\bar\partial} : C^\infty(X) \to C^\infty(X)$, thanks to proposition A.20 its kernel is exactly the set of harmonic functions on X, that is $\mathbb{C}$. On the other hand if we define

$$C^\infty_\Omega(X) = \{f \in C^\infty(X) \colon \int_X f\Omega = 0\}\,.$$

we have a direct decomposition $C^\infty(X) = \mathbb{C} \oplus C^\infty_\Omega(X)$, thus the restriction

$$\Delta_{\bar\partial}|_{C^\infty_\Omega(X)} : C^\infty_\Omega(X) \to C^\infty(X) \tag{D.16}$$

is injective. Fix a function $f \in C^\infty_\Omega(X)$, we want to show that the map:

$$p \mapsto \frac{1}{2\pi}[g^\Omega(p,\cdot)](\Delta_{\bar\partial}(f)\Omega) \quad \forall p \in X$$

where $[g^\Omega(p,\cdot)]$ denotes a current, is again f. this will imply that we have an effective way to compute f when $\Delta_{\bar\partial}(f)$ is given. In other words we invert the restricted $\bar\partial$-Laplacian described in equation (D.16). Here the computation:

$$\frac{1}{2\pi}[g^\Omega(p,\cdot)](\Delta_{\bar\partial}(f)\Omega) = \frac{1}{2\pi}\int_X \mathcal{G}^\Omega([p])\Delta_{\bar\partial}(f)\Omega =^{(\text{prop. A.20})}$$

$$= -\frac{i}{2\pi}\int_X \mathcal{G}^\Omega([p])\partial\bar\partial(f) = -\frac{i}{2\pi}\partial\bar\partial[\mathcal{G}^\Omega([p])](f) = -dd^c[\mathcal{G}^\Omega([p])](f) =^{(\text{th. D.19})}$$

$$= -[dd^c(\mathcal{G}^\Omega([p]))](f) + [\mathrm{div}^G(\mathcal{G}^\Omega([p]))](f) = -\int_X f dd^c(\mathcal{G}^\Omega([p])) + f(p) =$$

$$= -c\int_X f\Omega + f(p) = f(p)\,.$$

D.1.4 Admissible hermitian invertible sheaves

In this subsection we explain how hermitian invertible sheaves on the Riemann surface $(X, \mathscr{O})$ are related to Green functions. Fix on X a Khäler form Ω such that $\int_X \Omega = 1$ and moreover recall that by invertible sheaf on X we mean a holomorphic invertible sheaf.

Definition D.30. Let $\mathscr{L}$ be an invertible sheaf on X, a C^∞ hermitian metric h on $\mathscr{L}$ is said *admissible* (with respect to Ω) if there exists $a \in \mathbb{R}$ such that $\frac{1}{2\pi}\Theta(\mathscr{L}, h) = a\Omega$. A C^∞ hermitian invertible sheaf $(\mathscr{L}, h)$ on X is said admissible if h is admissible.

Clearly any positive real multiple of an admissible metric is again admissible, but the viceversa is also true thanks to the second part of the following theorem.

Theorem D.31. *Any invertible sheaf $\mathscr{L}$ on X has an admissible metric with respect to Ω. Moreover if h and h' are two admissible metric, then there is a constant $b \in \mathbb{R}_{>0}$ such that $h = bh'$ (obviously $bh' = \{bh'_x\}_{x\in X}$).*

Proof. Let h_1 be any C^∞ hermitian metric on $\mathscr{L}$ and let s be a nonzero meromorphic section of $\mathscr{L}$. By proposition D.14 $-\log(h_1(s,s))$ is a Green function such that $\operatorname{ord}^G(-\log(h_1(s,s)) = \operatorname{div}(s)$. By theorem D.21 there exists $g \in G_0^\Omega(X)$ such that $\operatorname{div}^G(g) = \operatorname{div}(s)$. Consider the Green function $f = g + \log(h_1(s,s))$, then $\operatorname{div}^G(f) = 0$. If we put

$$h := e^{-f} h_1$$

then

$$\frac{1}{2\pi}\Theta(\mathscr{L}, h) = dd^c(-\log(h(s,s))) = dd^c(f - \log(h_1(s,s))) = dd^c(g) = -\frac{1}{2\pi}\Delta_{\bar\partial}(g)\Omega\,.$$

But $\Delta_{\bar\partial}(g)$ is constant, therefore h is admissible with respect to Ω.
Let h and h' be two admissible C^∞ hermitian metrics and consider the C^∞ function

$$f' = \log(h(s,s)) - \log(h'(s,s))\,. \tag{D.17}$$

Since the two hermitian metrics are both admissible then

$$dd^c(f') = -\frac{1}{2\pi}\Theta(\mathscr{L}, h) + \frac{1}{2\pi}\Theta(\mathscr{L}, h') = r\Omega$$

for a real constant r. By using theorem D.9:

$$\int_X \frac{1}{2\pi}\Theta(\mathscr{L}, h) = \deg(\operatorname{div}(s)) = \int_X \frac{1}{2\pi}\Theta(\mathscr{L}, h')\,,$$

hence $r = \int_X r\Omega = \int_X dd^c(f') = 0$. In other words f' is harmonic, so $f' \in \mathbb{R}$. Finally equation (D.17) implies that $h = (e^{f'})h'$ an we put $b = e^{f'}$. $\square$

Corollary D.32. *Let h and h' be two admissible metrics on an invertible sheaf $\mathscr{L}$. Then $(\mathscr{L}, h)$ and $(\mathscr{L}, h')$ are isometric.*

Proof. By theorem D.31 $h' = bh$ with $b \in \mathbb{R}_+$, so for any open set $U \subset X$ is is enough to consider the map $\frac{1}{\sqrt{b}}\operatorname{id}_U : \mathscr{L}(U) \to \mathscr{L}(U)$.

Now we show that given a divisor $D \in \operatorname{Div}(X)$ there is a canonical choice of an admissible C^∞ metric on $\mathscr{O}(D)$.

Proposition D.33. *Let's fix a nonzero meromorphic section s_0 of $\mathscr{O}(D)$ such that $\operatorname{div}(s_0) = D$. Then there exists a unique admissible C^∞ metric h_D^Ω on $\mathscr{O}(D)$ satisfying the following property:*

$$-\log(h_D^\Omega(s_0, s_0)) = \mathscr{G}^\Omega(D)\,. \tag{D.18}$$

Proof. Let h_1 be any C^∞ hermitian metric on $\mathscr{O}(D)$. Define

$$f = \mathcal{G}^\Omega(D) + \log(h_1(s_0, s_0)) ,$$

then exactly as in the proof of theorem D.31 one shows that f is C^∞ and that $h_D^\Omega := e^{-f} h_1$ is an admissible metric. Furthermore we have the equality $-\log(h_D^\Omega(s_0, s_0)) = \mathcal{G}^\Omega(D)$. By the second part of theorem D.31 any other admissible metric for $\mathscr{O}(D)$ has the form $h' = b h_D^\Omega$ for $b \in \mathbb{R}_{>0}$, hence $-\log(h'(s_0, s_0)) \neq \mathcal{G}^\Omega(D)$. □

The metric h_D^Ω depends on both D and the meromorphic section s_0, so we have to fix a canonical choice of s_0.

Definition D.34. Let $D \in \mathrm{Div}(X)$ and fix $s_0 = 1_D$. The metric h_D^Ω arising uniquely from proposition D.33 is called *the canonical metric* of $\mathscr{O}(D)$ with respect to Ω.

The canonical metric $h_0 := h_0^\Omega$ on the trivial sheaf $\mathscr{O}$ is evidently given by:

$$h_0(s(x), t(x)) = s(x)\overline{t(x)}$$

where $\overline{t(x)}$ is the usual conjugation in $\mathbb{C}$. For any nonzero meromorphic function $f \in \mathscr{M}^\times$ it is easy to obtain the explicit expression of the canonical metric on $\mathscr{O}((f)) = f^{-1}\mathscr{O}$.

Proposition D.35. *Let (f) be a nonzero principal divisor on X then the canonical metric $h_f := h_{(f)}^\Omega$ is given by:*

$$h_f(f^{-1}s, f^{-1}t) = h_0(s, t)$$

where $h_0 := h_0^\Omega$ is the canonical metric associated to the zero divisor.

Proof. Consider the Green function $-\log|f|^2$, since $\partial\bar{\partial}(-\log|f|^2) = 0$ we have that $-\log|f|^2 \in G^\Omega(X)$. Moreover by theorem D.8 we know that:

$$\int_X -\log|f|^2\Omega = \deg((f)) = 0,$$

thus by proposition D.22 we know that $\mathcal{G}^\Omega((f)) = -\log|f|^2$. We have to show that h_f satisfies the condition expressed by equation (D.18):

$$-\log(h_f(1,1)) = -\log(h_f(f^{-1}f, f^{-1}f)) = -\log\left(|f|^2\right) = \mathcal{G}^\Omega((f)) .$$

□

D.2 Arakelov intersection theory

D.2.1 Prologue

Roughly speaking, the aim of the intersection theory on a two dimensional flat S-scheme $X \to S$ is to count the number of intersection points, considered with their multiplicities, between two Weil divisors on X (up to linear equivalence). Obviously we need to impose some conditions on X, but the crucial fact is that the nature of the base scheme S affects dramatically the approach to the whole intersection theory. This leads to an important distinction between algebraic geometry and arithmetic geometry; indeed, if $S = \operatorname{Spec} k$ then we have the classical intersection theory on algebraic surfaces, on the other hand if $S = \operatorname{Spec} O_K$ for a number field K, we have to introduce Arakelov intersection theory. In this prologue we first review some basic facts about intersection theory on algebraic surfaces, then we highlight the motivations that necessitate a novel approach in the arithmetic setting.

Let's start by recalling the notion of local intersection number which is well defined in a very general setting. So, just for now let X be a regular, integral scheme of dimension 2, then consider the set

$$\Upsilon := \{ (D, E) \in \operatorname{Div}(X) \times \operatorname{Div}(X) \colon D \text{ and } E \text{ have no common components} \},$$

and note that if $(D_j, E_j) \in \Upsilon$ with $j = 1, 2$, then $(D_1 + D_2, E_1 + E_2) \in \Upsilon$.

Definition D.36. Let $(D, E) \in \Upsilon$ such that D and E are both effective, then for any closed point $x \in X$ we put:

$$i_x(D, E) := \operatorname{length}_{\mathscr{O}_{X,x}} \mathscr{O}_{X,x} / \left(\mathscr{O}_X(-D)_x + \mathscr{O}_X(-E)_x \right) .$$

This is called the *local intersection number* of D and E at x.

Remark D.37. *$i_x(D, E)$ is a well defined non-negative integer as consequence of an easy result of commutative algebra* [37, Proposition 7.1.25].

The local intersection number assigns the multiplicity of the intersection at each point of X, and the following basic result summarizes its naive properties.

Proposition D.38. *Let $(E, D) \in \Upsilon$ and $(E_j, D_j) \in \Upsilon$ with $j = 1, 2$ such that all the divisors are effective, then*

(1) $i_x(D, E) = i_x(E, D)$.

(2) $i_x(D_1 + D_2, E_1 + E_2) = \displaystyle\sum_{j,k=1}^{2} i_x(D_j, E_k)$.

(3) *$i_x(D,E) \neq 0$ if and only if $x \in \mathrm{Supp}(D) \cap \mathrm{Supp}(E)$.*

(4) *If $x \in E$, $i_x(D,E) = \mathrm{mult}_x(D|_E)$.*

Proof. (1) and (3) are obvious. For (2) and (4) see [37, lemma 9.1.4]. □

Any divisor $D \in \mathrm{Div}(X)$ can be written in a unique way as $D = D_+ - D_-$ where both D_+ and D_- are effective and if $(D,E) \in \Upsilon$, then $(D_\pm, E_\pm) \in \Upsilon$. We can use definition D.36 in order to have the local intersection at x of D and E when (D,E) is any element of Υ (so not necessarily effective):

$$i_x(D,E) := i_x(D_+,E_+) - i_x(D_+,E_-) - i_x(D_-,E_+) + i_x(D_-,E_-)\,.$$

Now that the local intersection is defined, we would like to globalize this concept by putting all together this local data to get the intersection number. Our goal is to describe also the intersection between two divisors with common components and more importantly we want the intersection number to be invariant up to linear equivalence of divisors. In the framework of algebraic geometry this is a relatively simple task.

Assume that X is a regular, integral, projective surface over a field k and let D and E two effective divisors on on X such that $(D,E) \in \Upsilon$, then for any $x \in X$ consider the k-vector space

$$\mathscr{O}_{D\cap E,x} := \mathscr{O}_{X,x} / \left(\mathscr{O}_X(-D)_x + \mathscr{O}_X(-E)_x\right)\,,$$

then, by [37, Exercise 7.1.6 (d)] we have that $\dim_k \mathscr{O}_{D\cap E,x} = i_x(D,E)[k(x):k]$. Let's construct the sheaf of k-vector spaces $\mathscr{O}_{D\cap E}$ by the following rule:

$$U \mapsto \mathscr{O}_{D\cap E}(U) := \prod_{x\in U} \mathscr{O}_{D\cap E,x}\,.$$

The intersection number between D and E can be very naturally defined as

$$D.E := \dim_k H^0(X, \mathscr{O}_{D\cap E})\,. \tag{D.19}$$

It is precisely the sum of all local intersection numbers scaled by a factor depending on the field extensions $k \subseteq k(x)$; obviously when k is algebraically closed $D.E$ coincides with the naive notion of intersection number, indeed it is simply the sum of all local data. Notice that

$$\dim_k H^0(X, \mathscr{O}_{D\cap E}) = \chi_k(\mathscr{O}_{D\cap E})$$

since skyscraper sheaves have nonzero cohomology only in degree 0. Furthermore the following result is of key importance.

Proposition D.39. *Let X be a regular, integral, projective surface over a field k and let $(D, E) \in \Upsilon$. Then there is an exact sequence of sheaves*

$$0 \to \mathscr{O}_X(-D-E) \to \mathscr{O}_X(-D) \oplus \mathscr{O}_X(-E) \to \mathscr{O}_X \to \mathscr{O}_{D\cap E} \to 0 \qquad \text{(D.20)}$$

Proof. See the classical text [3, Lemma I.5] where the proof is made when X a complex surface. However it can be easily generalized to our case. ☐

Consider the exact sequence in equation D.20 and put:

$$f : \mathscr{O}_X(-D) \oplus \mathscr{O}_X(-E) \to \mathscr{O}_X\,,$$

$$g : \mathscr{O}_X \to \mathscr{O}_{D\cap E}\,,$$

and $\mathscr{G} = \operatorname{im}(f) = \ker(g)$. Then we get the following two short exact sequences:

$$0 \to \mathscr{O}_X(-D-E) \to \mathscr{O}_X(-D) \oplus \mathscr{O}_X(-E) \xrightarrow{f} \mathscr{G} \to 0\,; \qquad \text{(D.21)}$$

$$0 \to \mathscr{G} \to \mathscr{O}_X \xrightarrow{g} \mathscr{O}_{D\cap E} \to 0\,. \qquad \text{(D.22)}$$

By the properties of the Euler-Poincare characteristic for short exact sequences we get:

$$\chi_k(\mathscr{O}_X(-D) \oplus \mathscr{O}_X(-E)) = \chi(\mathscr{O}_X(-D-E)) + \chi_k(\mathscr{G})\,,$$

$$\chi_k(\mathscr{O}_X) = \chi_k(\mathscr{G}) + \chi_k(\mathscr{O}_{D\cap E})\,,$$

so, in other words

$$\chi_k(\mathscr{O}_{D\cap E}) = \chi_k(\mathscr{O}_X) - \chi_k(\mathscr{O}_X(-D)) - \chi_k(\mathscr{O}_X(-E)) + \chi_k(\mathscr{O}_X(-D-E))\,. \qquad \text{(D.23)}$$

At this point, in order to obtain our goal let's use the correspondence between divisors and invertible sheaves. For any $\mathscr{L}, \mathscr{M} \in \operatorname{Pic}(X)$, guided by equation (D.23) define the pairing

$$\mathscr{L}.\mathscr{M} := \chi_k(\mathscr{O}_X) - \chi_k(\mathscr{L}^{-1}) - \chi_k(\mathscr{M}^{-1}) + \chi_k(\mathscr{L}^{-1} \otimes \mathscr{M}^{-1})\,.$$

It is not difficult to show that the map

$$\begin{array}{rcl} \operatorname{Pic}(X) \times \operatorname{Pic}(X) & \to & \mathbb{Z} \\ (\mathscr{L}, \mathscr{M}) & \mapsto & \mathscr{L}.\mathscr{M} \end{array}$$

is well defined (i.e. $\mathscr{L}.\mathscr{M}$ invariant up to isomorphism of invertible sheaves), symmetric and $\mathbb{Z}$-bilinear (for the last claim see [3, Lemma I.6 and Fact I.7]).

Definition D.40. For any two divisor $D, E \in \mathrm{Div}(X)$, the intersection pairing is defined as:

$$D.E := \mathscr{O}_X(D).\,\mathscr{O}_X(E)$$

Note that it coincides with equation (D.19) when $(D, E) \in \Upsilon$ and they are both effective. Furthermore it is important to underline the obvious fact that the intersection pairing descends trivially to a pairing $\mathrm{CH}^1(X) \times \mathrm{CH}^1(X) \to \mathbb{Z}$ since any two linearly equivalent divisors induce isomorphic invertible sheaves.

Remark D.41. The above procedure is not the common one for introducing intersection theory on surfaces. Usually, the intersection number is first defined for distinct integral curves and then extended to all divisors by using the moving lemma.

The description of the intersection pairing for algebraic surfaces is now complete, and it is evident that relies heavily on the structure of the base scheme $S = \operatorname{Spec} k$. A moment of reflection will show that it is not possible to have *any* decent intersection theory when $S = \operatorname{Spec} O_K$, and the reason is due to the existence of horizontal divisors. Let be X be a regular, integral, projective surface flat over S and suppose that we have well defined bilinear and symmetric intersection pairing on $\mathrm{Div}(X)$. Consider a closed point $s = \mathfrak{p} \in S$, if $\varpi \in K^\times$ is the generator of $\mathfrak{p}$ then we have an equality of divisors $[s] = (\varpi)$. In other words $[s]$ is principal, so also the fibre X_s is a principal vertical divisor on X. Now Let D an irreducible horizontal divisor on X, then certainly D meets X_s, which means $D.X_s > 0$ because our phantomatic intersection pairing should count the number of intersection points with multiplicity. Thus, for any other divisor $E \in \mathrm{Div}(X)$ we obtain

$$D.E < D.(E + X_s)\,, \tag{D.24}$$

but $E \sim (E + X_s)$ since X_s is principal by construction. The issue can be solved by claiming that what we see on the surface X is not the entire picture, in particular there exists a notion of "intersection at infinity" that we don't see in the framework of scheme theory. If we consider this intersection at infinity, the inequality (D.24) should become an equality.

This kind of problems was has been faced several times in geometry, and they have always been solved with the same approach that should be clear from the following two examples:

Example D.42. The fist example is one dimensional, and it was analysed in detail in chapter 1. We needed to *add formally* the archimedean points B_∞ to B in order to preserve the correspondence between points and places. In the new space $\widehat{B}$ we were able to define a notion of divisor and a notion of degree stable under linear equivalence.

Example D.43. The second example is more classical and involves the formulation of the plane projective geometry. In this case the original problem was to find the "intersection at infinity" of two parallel lines in $\mathbb{R}^2$, and it was inspired by the notion of vanishing point in the graphic art. The solution consists in *adding formally* the line at infinity to the ordinary real plane to get the projective plane $\mathbb{P}^2_{\mathbb{R}}$. In this new space any two lines meet and in particular the lines that were parallel before, now will meet on the line at infinity.

The general idea behind Arakelov intersection theory on X is the same of the above examples, we want to complete our scheme X with some new data which will be interpreted as "the data at infinity". So, we formally add some curves at infinity to the scheme X and we get a new completed space (which is not a scheme) $\widehat{X}$. Here we define an extended notion of divisors, the Arakelov divisors, and we construct a reasonable intersection theory. In particular the new curves at infinity, artificially introduced, will be a particular case of Arakelov divisors.

Below we fix some notations that will be used in the remaining part of the chapter:

Notations. $B = \operatorname{Spec} O_K$ for a number field K, and regarding B the notations imposed in chapter 1 are valid. $\varphi : X \to B$ is a B-scheme satisfying the following properties:

- X is two dimensional, integral, and regular. The generic point of X is η and the function field of X is denoted by $K(X)$.
- φ is proper and flat.
- The generic fibre, denoted by X_K, is a geometrically integral, smooth, projective curve over K.

We say that X is an *arithmetic surface over* B.

Remark D.44. In the literature there is no accordance on a universal definition of arithmetic surface, so the reader can be find several slightly different definitions.

Thanks to [37, Theorem 8.3.16] it follows that φ is a projective morphism, so in particular also X is projective (see [37, Definition 3.1.12] for a simplified notion of projective morphism which is enough in our setting). Let's recall a useful result which characterize all points of dimension 1 in X:

Proposition D.45. *If x is a closed point of the curve X_K, then $\overline{\{x\}}$ is a horizontal (prime) divisor in X. Vice versa if D is a prime divisor on X,* *then either $D \subseteq X_b$ for a closed point $b \in B$ or $D = \overline{\{x\}}$ where x is a closed point of X_K.*

Proof. See [37, Proposition 8.3.4]. □

D.2.2 Arakelov divisors

For any $\sigma \in B_\infty$ consider the base change diagram:

$$\begin{tikzcd}
X_\sigma := X \times_B \operatorname{Spec}\mathbb{C} \arrow[r] \arrow[d, "\varphi_\sigma"] & \operatorname{Spec}\mathbb{C} \arrow[d, "\operatorname{Spec}\sigma"] \\
X \arrow[r, "\varphi"] & B\,.
\end{tikzcd} \tag{D.25}$$

By the properties of the fibred product it turns out that $X_\sigma \to \operatorname{Spec}\mathbb{C}$ is a complex integral[5] regular projective curve. By simplicity we identify X_σ with the set of closed points $X_\sigma(\mathbb{C})$, therefore by abuse of notation we will say that X_σ is a compact Riemann surface "lying" on the archimedean point σ. It will be clear from the context when we want to consider X_σ as scheme or as Riemann surface.

Remark D.46. Diagram (D.25) arises from the following rather obvious commutative diagram:

$$\begin{tikzcd}
X_\sigma \arrow[r] \arrow[d, "\beta"] \arrow[dd, bend right, "\varphi_\sigma"'] & \operatorname{Spec}\mathbb{C} \arrow[d, "\operatorname{Spec}\sigma"] \\
X_K \arrow[r] \arrow[d] & \operatorname{Spec}K \arrow[d, "\operatorname{Spec}\iota"] \\
X \arrow[r] & B
\end{tikzcd}$$

where $\iota : O_K \hookrightarrow K$ is the natural embedding and the map β is surjective. In other words φ_σ maps surjectively X_σ on the curve X_K. Since the morphisms ι and σ are both flat and flatness is preserved after base change we can conclude that φ_σ is flat. In particular it is possible to pullback Cartier divisors from X to X_σ through φ_σ.

With the notation $\widehat{X}$, we indicate the "completed surface"

$$\widehat{X} := X \cup \bigcup_{\sigma \in B_\sigma} X_\sigma\,.$$

Notice that $\widehat{X}$ is simply formal object which does not live in the category of schemes, therefore we have to use different theoretical tools respectively on X

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[5] Integrality is a consequence of the geometrical integrality of X_K.

and on the Riemann surfaces X_σ. Furthermore, on each Riemann surface X_σ we fix a Kähler form Ω_σ such that $\int_{X_\sigma} \Omega_\sigma = 1$, and we put $\Omega := \{\Omega_\sigma\}_{\sigma \in B_\infty}$. For any divisor $D \in \mathrm{Div}(X)$, $D_\sigma := \varphi_\sigma^* D \in \mathrm{Div}(X_\sigma)$ denotes its pullback through φ_σ.

Consider the additive group $\mathbf{G}(X) := \oplus_{\sigma \in B_\infty} G(X_\sigma)$ and its subgroup, depending on Ω, $\mathbf{G}(X, \Omega) := \bigoplus_{\sigma \in B_\infty} G^{\Omega_\sigma}(X_\sigma)$. By commodity we write any element of $\mathbf{G}(X)$ (or of $\mathbf{G}(X, \Omega)$) as a finite formal linear combination $\sum_\sigma g_\sigma X_\sigma$ for $g_\sigma \in \mathbf{G}(X)$ (or $g_\sigma \in \mathbf{G}(X, \Omega)$).

Definition D.47. The group of *Arakelov divisors* on $\widehat{X}$ is:

$$\mathrm{Div}_{\mathrm{Ar}}(X, \Omega) := \left\{ \left(D, \sum_\sigma g_\sigma X_\sigma \right) \in \mathrm{Div}(X) \times \mathbf{G}(X, \Omega) \colon \ \mathrm{div}^G(g_\sigma) = D_\sigma \right\}.$$

We often denote the element $(0, X_\sigma) \in \mathrm{Div}_{\mathrm{Ar}}(X, \Omega)$ simply with the symbol X_σ.

It is important to understand the geometry lying behind the above apparently mysterious definition. Fix an Arakelov divisor $(D, \sum_\sigma g_\sigma X_\sigma)$, by theorem D.21 and proposition D.22 we can write

$$g_\sigma = \mathcal{G}^{\Omega_\sigma}(D_\sigma) + \alpha_\sigma \tag{D.26}$$

where $\alpha_\sigma \in \mathbb{R}$ is uniquely determined. Figure D.3 highlights the fact that D_σ, which is a finite set of points on X_σ, can been interpreted as the "prolongation" of D on the curve X_σ; thus, it makes sense to define the Arakelov divisor

$$\overline{D} := \left(D, \sum_\sigma \mathcal{G}^{\Omega_\sigma}(D_\sigma) X_\sigma \right) \in \mathrm{Div}_{\mathrm{Ar}}(X, \Omega)$$

which will be called *completion* of D in $\widehat{X}$. By equation (D.26) we have the following unique decomposition of $(D, \sum_\sigma g_\sigma X_\sigma)$ in $\mathrm{Div}_{\mathrm{Ar}}(X)$:

$$\left(D, \sum_\sigma g_\sigma X_\sigma \right) = \overline{D} + \sum_\sigma \alpha_\sigma X_\sigma \tag{D.27}$$

where the linear combination $\sum_\sigma \alpha_\sigma X_\sigma$ can be evidently read as a "real divisor" on $\widehat{X}$ with support made of curves at infinity. In perfect analogy with the usual notion of divisor, equation (D.27) tells us that an Arakelov divisor can be interpreted as a formal linear combination of "curves" in $\widehat{X}$, such that the coefficients of the curves at infinity are in $\mathbb{R}$. The presence of this real coefficients underlines once again the fact that the curves at infinity have an analytic nature.

Remark D.48. As one may expect, if E is a vertical divisor on X, then $\overline{E} = E$.

From the above discussion we recover the original definition of the group of Arakelov divisors given in [2] and [1]:

Proposition D.49. *There is an isomorphism of groups:*

$$\mathrm{Div}_{\mathrm{Ar}}(X, \Omega) \cong \mathrm{Div}(X) \oplus \mathbb{R}^{(B_\infty)}$$

Proof. Thanks to equation (D.27) we can define the isomorphism:

$$\left(D, \sum_{\sigma} g_\sigma X_\sigma\right) \mapsto D + \sum_{\sigma} \alpha_\sigma [\sigma]\,.$$

□

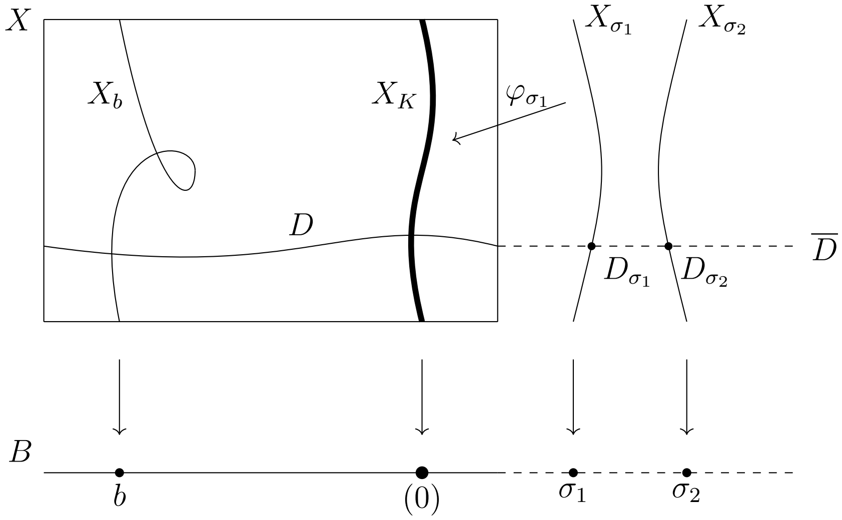

Figure D.3: A schematization of an arithmetic surfaces $\varphi : X \to B$ such that $B_\infty = \{\sigma_1, \sigma_2\}$ (for instance $B = \mathrm{Spec}\,\mathbb{Z}[i]$). X_b is a vertical divisor over the closed point b, D is a horizontal divisor such that D_{σ_1} and D_{σ_2} are prime divisors respectively on X_{σ_1} and X_{σ_2}.

Now we want to introduce the concept of principal Arakelov divisor, in other words we want to define an Arakelov divisor associated to an element of $K(X)$. Recall that $K(X)$ is also the function field of X_K, so the morphism $\varphi_\sigma : X_\sigma \to X_K$ induces a field embedding

$$\varphi_\sigma^{\#} : K(X) \hookrightarrow \mathbb{C}(X_\sigma)\,.$$

For any rational function $f \in K(X)$ we put by simplicity $f_\sigma := \varphi_\sigma^{\#}(f)$. Moreover let $\mathscr{O}_\sigma$ be the sheaf of regular functions on X_σ, then as usual f_σ

can be identified with a holomorphic map $X_\sigma \to \mathbb{C}$ at all but finitely many points:

$$p \mapsto f_{\sigma,p} \mapsto \overline{f}_{\sigma,p} \in k(p) \cong \mathbb{C}$$

Then it is easy to see that $-\log|f_\sigma|^2$ is a Green function on X_σ such that $\partial\bar{\partial}(-\log|f_\sigma|^2) = 0$, therefore $-\log|f_\sigma|^2 \in G^{\Omega_\sigma}(X_\sigma)$.

Proposition D.50. *Let* $f \in K(X)^\times$*, then* $\operatorname{div}^G(-\log|f_\sigma|^2) = (f)_\sigma$*, where* $(f)_\sigma$ *is the pullback of the principal divisor* (f)*.*

Proof. Fix a point $p \in X_\sigma$, let $x = \varphi_\sigma(p)$ and consider f as a rational function on X_K. If ϖ_σ is a local parameter in $\mathscr{O}_{\sigma,p}$ and ϖ is a local parameter in $\mathscr{O}_{X_K,x}$, then

$$f_\sigma = \varpi_\sigma^{v_p(\varphi_\sigma^\#(\varpi))v_x(f)} u \quad \text{for } u \in \mathscr{O}_{\sigma,p}\,.$$

This implies that $\operatorname{ord}_p^G(-\log|f_\sigma|^2) = v_p(\varphi_\sigma^\#(\varpi))v_x(f)$, but $v_p(\varphi_\sigma^\#(\varpi))$ is precisely the ramification index $e_{\varphi_\sigma,p}$, hence $\operatorname{ord}_p^G(-\log|f_\sigma|^2) = e_{\varphi_\sigma,p}v_x(f)$. So, we finally have:

$$\operatorname{div}^G(-\log|f_\sigma|^2) = \sum_{p\in X_\sigma} e_{\varphi_\sigma,p} v_{\varphi_\sigma(p)}(f)[p] = (f)_\sigma\,.$$

□

Now the following definition makes sense:

Definition D.51. Let $f \in K(X)^\times$ be a rational function. It induces an Arakelov divisor in the following way:

$$\widehat{(f)} := \left((f), \sum_\sigma -\log|f_\sigma|^2 X_\sigma\right) \in \operatorname{Div}_{\mathrm{Ar}}(X,\Omega)\,.$$

The group

$$\operatorname{Princ}_{\mathrm{Ar}}(X,\Omega) := \left\{\widehat{(f)} \,:\, f \in K(X)\right\}$$

is called the group of *principal Arakelov divisor* and $\mathrm{CH}^1_{\mathrm{Ar}}(X,\Omega) := \frac{\operatorname{Div}_{\mathrm{Ar}}(X,\Omega)}{\operatorname{Princ}_{\mathrm{Ar}}(X,\Omega)}$ is the *Arakelov Chow group.* Two Arakelov divisor are said *linearly equivalent* if they are contained in the same class in $\mathrm{CH}^1_{\mathrm{Ar}}(X,\Omega)$.

Moreover for any principal Arakelov divisor $\widehat{(f)}$ we get the following decomposition:

$$\widehat{(f)} = \overline{(f)} + \sum_\sigma \left(\int_{X_\sigma} -\log|f_\sigma|^2\Omega_\sigma\right) X_\sigma\,.$$

D.2.3 Finite intersection and Deligne pairing

It is clear from the definitions that an intersection pairing

$$\left(D, \sum_\sigma g_\sigma X_\sigma\right) \cdot \left(E, \sum_\sigma l_\sigma X_\sigma\right)$$

between two Arakelov divisors should be composed by two parts: the first one is an intersection number between D and E, and this will be called the *finite intersection*; the second one deals with the terms at infinity given by the Green functions. In this subsection we describe the finite intersection.

Definition D.52. Let (D, E) be an element of Υ, then we define the 0-cycle on X given by:

$$i(D, E) := \sum_{x \in X^{(0)}} i_x(D, E)[x],$$

where here $[x]$ is a shorthand of $[\overline{\{x\}}]$.

Remark D.53. The sum in definition D.52 is finite because if D and E are effective without common components, then $i_x(D, E) = \mathrm{mult}_x(D|_E)$ (proposition D.38(4)) and there is only a finite number of points on E at which the divisor $D|_E$ has nonzero multiplicity.

Proposition D.54. *If* $(D, E), (D_j, E_j) \in \Upsilon$ *with* $j = 1, 2$, *then the following properties hold for* $i(D, E)$*:*

- $i(D, E) = i(E, D)$ *(simmetry).*
- $i(D_1 + D_2, E_1 + E_2) = \sum_{j,k=1}^{2} i(D_j, E_k)$ *(bilinearity).*

Proof. It follows immediately from proposition D.38. □

Definition D.55. We have the symmetric and bilinear pairing on Υ:

$$\begin{array}{rcl} \Upsilon & \to & \mathrm{Div}(B) \\ (D, E) & \mapsto & \langle D, E \rangle \end{array}$$

where

$$\langle D, E \rangle := \varphi_* i(D, E) = \sum_{x \in X} [k(x) : k(\varphi(x))]\, i_x(D, E)\, [\varphi(x)]\,.$$

[6]Then the *finite intersection number* of D and E is:

$$D.E := \deg_{Ar}\left(\langle D, E\rangle\right) .$$

Clearly also the map $(D, E) \mapsto D.E$ is symmetric and bilinear.

Remark D.56. The pairing $(D, E) \mapsto \langle D, E\rangle$ is still well defined if in place of the base scheme B we use any irreducible Dedekind scheme S.

Let Γ be a prime divisor of X with generic point γ and consider a nonzero rational function $f \in K(X)^{\times}$ such that (f) and Γ have no common components, then define $N_{\Gamma}(f) \in K^{\times}$ in the following way:

$$N_{\Gamma}(f) := \begin{cases} N_{K(\Gamma)|K}(f|_{\Gamma}) & \text{if } \Gamma \text{ is horizontal} \\ 1 & \text{if } \Gamma \text{ is vertical} \end{cases}$$

where $N_{K(\Gamma)|K}$ is the usual field norm and $f|_{\Gamma}$ is defined as follows: since (f) and Γ have no common components it follows that $v_{\gamma}(f) = 0$, that is $f \in \mathscr{O}_{X,\gamma}^{\times}$. So $f|_{\Gamma}$, is the natural image of f in $k(\gamma) = K(\Gamma)$. At this point for any $D = \sum_i n_i \Gamma_i \in \mathrm{Div}(X)$ such that D and (f) have no common components we have:

$$N_D(f) := \prod_i N_{\Gamma_i}(f)^{n_i} \quad \in K^{\times}$$

Since $K(X)$ is the function field of any open subscheme $U \subseteq X$ and of X_K we can restrict the operator $N_*(\cdot)$ to U and to X_K.

Proposition D.57. *Let $f \in K(X)^{\times}$ and let $D \in Div(X)$ such that (f) and D have no common components, then the following claims hold:*

1. *Let $U \subseteq X$ be an open subscheme, then $N_{D|_U}(f) = N_D(f)$.*
2. *$N_{D|_{X_K}}(f) = N_D(f)$, where the left hand side is the one-dimensional operator defined in equation (G.2).*

Proof. In both items we can restrict to the case when $D = \Gamma$ is an irreducible horizontal divisor.
(1) The function fields and the generic points of Γ and $\Gamma|_U$ coincide, so the claim follows trivially.
(2) Let $\gamma \in X_K$ be the generic point of Γ, it is a closed point of X_K such that $k(\gamma) = K(\Gamma)$. By the bare definitions we can check the required equality. ☐

Proposition D.58. *Let $f \in K(X)^{\times}$ and let $D \in \mathrm{Div}(X)$ a divisor such that D and (f) have no common components, then*

$$\langle D, (f)\rangle = (N_D(f)) \ \in \mathrm{Princ}(B) .$$

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[6] In this case we see $\langle D, E\rangle$ as an Arakelov divisor on B without components at infinity.

Proof. See [43, Proposition 4.3]. □

Remark D.59. The Arakelov degree of an element of Princ(B) is not 0, so it is very important to point out that $D.(f) \neq 0$. On the other hand one cannot expect that intersection number to be 0, since in the prologue we said clearly that there is no way to define an intersection number on X invariant up to linear equivalence.

Now we interpret the pairing $\langle D, E\rangle$ (remember that we have to assume that D and E have no common components) in terms of invertible sheaves. We decide to work in a bit more general context, so, instead of B, as base scheme we consider any irreducible Dedekind scheme S. The morphism $\varphi : X \to S$ still have all the properties listed at the end of section D.2.1. For any pair of invertible sheaves $\mathscr{L}$ and $\mathscr{M}$ on X we construct an invertible sheaf $\langle \mathscr{L}, \mathscr{M}\rangle$ on S in a way that the map $(\mathscr{L}, \mathscr{M}) \to \langle \mathscr{L}, \mathscr{M}\rangle$, called the *Deligne pairing*, induces a well defined bilinear pairing

$$\langle \cdot, \cdot \rangle : \mathrm{Pic}(X) \times \mathrm{Pic}(X) \to \mathrm{Pic}(S)\,.$$

Remark D.60. The pairing was introduced by Deligne in [13] where he describes very shortly and without details how to obtain it. Here, by following the beautiful book [43] we construct the Deligne pairing in all details. By the way, be aware that [43], in this point, contains some mistakes.

The construction is a bit involved and we divide it in two steps:
Step 1. Definition of the K-vector space $\langle \mathscr{L}, \mathscr{M}\rangle_K$.

Consider the sets:

$$\Upsilon_K := \left\{ (D,E) \in \mathrm{Div}(X) \times \mathrm{Div}(X) : \begin{array}{c} D|_{X_K} \text{ and } E|_{X_K} \text{ have no} \\ \text{common components (as} \\ \text{divisors on } X_K) \end{array} \right\},$$

$$\Sigma_K := \left\{ (l,m) : \begin{array}{c} l \text{ and } m \text{ are nonzero meromorphic sections of } \mathscr{L} \\ \text{and } \mathscr{M} \text{ such that } (\mathrm{div}(l), \mathrm{div}(m)) \in \Upsilon_K \end{array} \right\}.$$

Note that Υ_K is just the set of couple of divisors with no common horizontal components. Now we define some vector spaces over K.

$$V := K^{(\Sigma_K)}\,,$$

namely V is the free K-vector space over Σ_K.

$$W' := \left\{ (fl, m) - N_{\mathrm{div}(m)|_{X_K}}(f) \cdot (l,m) \colon f \in K(X)^\times, (l,m), (fl,m) \in \Sigma_K \right\}, \tag{D.28}$$

$$T' := \left\{ (l, gm) - N_{\operatorname{div}(l)|_{X_K}}(g) \cdot (l, m) \colon g \in K(X)^\times, (l, m), (l, gm) \in \Sigma_K \right\} . \tag{D.29}$$

Note that the above "$N_*(\cdot)$" is the one-dimensional operator of definition G.2 considered on the curve X_K.

Remark D.61. $N_{\operatorname{div}(m)|_{X_K}}(f)$ and $N_{\operatorname{div}(l)|_{X_K}}(g)$ are well defined since (l, m), (fl, m), $(l, gm) \in \Sigma_K$, so $\operatorname{div}(m)|_{X_K}$ and (f) have no common components. The same holds for $\operatorname{div}(l)|_{X_K}$ and (g).

Define the free vector spaces $W := K^{(W')}$ and $T := K^{(T')}$; moreover put

$$\langle \mathscr{L}, \mathscr{M} \rangle_K := V/(W + T) ,$$

which is considered as a constant sheaf (of K-vector spaces) over X. The natural image of any element $(l, m) \in \Sigma_K \subset V$ in $\langle \mathscr{L}, \mathscr{M} \rangle_K$ is denoted as $\langle l, m \rangle_K$.

Proposition D.62. *$\langle \mathscr{L}, \mathscr{M} \rangle_K$ is a one-dimensional vector space over K.*

Proof. Fix $(l_0, m_0) \in \Sigma_K$, then for any $(l, m) \in \Sigma_K$ there are two elements $f_0, g_0 \in K(X)^\times$ such that $l = f_0 l_0$, $m = g_0 m_0$ and moreover:

$$((f_0), (g_0)),\ ((f_0),\ \operatorname{div}(m_0)),\ ((g_0), \operatorname{div}(l_0)) \in \Upsilon_K .$$

By equations (D.28) and (D.29), in $\langle \mathscr{L}, \mathscr{M} \rangle_K$ we can write:

$$\langle l, m \rangle_K = \langle f_0 l_0, g_0 m_0 \rangle_K = [f_0, g_0]\, N_{\operatorname{div}(m_0)|_{X_K}}(f_0)\, N_{\operatorname{div}(l_0)|_{X_K}}(g_0)\, \langle l_0, m_0 \rangle_K . \tag{D.30}$$

where, in order to simplify the notations, we put $[f_0, g_0] := N_{(f_0)}(g_0)$ intended as operation on the curve X_K. This shows that $\langle \mathscr{L}, \mathscr{M} \rangle_K$ has dimension at most 1 over K. Define the homomorphism of K-vector spaces:

$$\theta : V \to K$$

such that

$$\theta(l, m) := [f_0, g_0] N_{\operatorname{div}(m_0)|_{X_K}}(f_0) N_{\operatorname{div}(l_0)|_{X_K}}(g_0) .$$

Note that θ is nontrivial, so surjective, since $\theta(l_0, m_0) = 1$. Now by using the Weil reciprocity law (corollary G.9) we prove that θ descends to a nontrivial morphism $\overline{\theta} : \langle \mathscr{L}, \mathscr{M} \rangle_K \to K$, indeed for $f, g \in K(X)^\times$:

$$\begin{aligned} \theta(fl, m) &= [f f_0, g_0]\, N_{\operatorname{div}(m_0)|_{X_K}}(f f_0)\, N_{\operatorname{div}(l_0)|_{X_K}}(g_0) = \\ &= [f, g_0]\, [f_0, g_0]\, N_{\operatorname{div}(m_0)|_{X_K}}(f)\, N_{\operatorname{div}(m_0)|_{X_K}}(f_0)\, N_{\operatorname{div}(l_0)|_{X_K}}(g_0) = \\ &= [g_0, f]\, N_{\operatorname{div}(m_0)|_{X_K}}(f)\, \theta(l, m) = \\ &= N_{\operatorname{div}(m)|_{X_K}}(f)\, \theta(l, m). \end{aligned}$$

Similarly it holds that

$$\theta(l, gm) = N_{\mathrm{div}(l)|_{X_K}}(g)\,\theta(l, m).$$

In other words equation D.30 can we written as:

$$\langle l, m\rangle_K = \overline{\theta}(\langle l, m\rangle_K)\,\langle l_0, m_0\rangle_K$$

hence, by the nontriviality of $\overline{\theta}$ we conclude that $\langle \mathscr{L}, \mathscr{M}\rangle_K$ has dimension 1. $\square$

Step 2. Definition of $\langle \mathscr{L}, \mathscr{M}\rangle$.
Let $U \subseteq S$ be a non-empty open subset and denote with X_U the schematic inverse image of U with respect to φ. We clearly have a flat map $X_U \to U$, so we define:

$$\Upsilon_U := \left\{ (D, E) \in \mathrm{Div}(X) \times \mathrm{Div}(X) \ : \begin{array}{c} D|_{X_U} \text{ and } E|_{X_U} \text{ have no} \\ \text{common components (as} \\ \text{divisors on } X_U) \end{array} \right\},$$

$$\Sigma_U := \left\{ (l, m) \ : \begin{array}{c} l \text{ and } m \text{ are nonzero meromorphic sections} \\ \text{of } \mathscr{L} \text{ and } \mathscr{M} \text{ such that} \\ (\mathrm{div}(l), \mathrm{div}(m)) \in \Upsilon_U \text{ and} \\ \langle \mathrm{div}(l)|_{X_U}, \mathrm{div}(m)|_{X_U}\rangle \text{ is effective on } U \end{array} \right\}.$$

Moreover notice that that $\Sigma_U \subset \Sigma_K$. We define a sheaf of $\mathscr{O}_S$-modules $\mathscr{A}$ on X given by:

$$\mathscr{A}|_U := \mathscr{O}_S|_U{}^{(\Sigma_U)}.$$

Finally consider the morphism of sheaves: $\Phi : \mathscr{A} \to \langle \mathscr{L}, \mathscr{M}\rangle_K$ which sends $(l, m) \in \Sigma_U$ to $\langle l, m\rangle_K$ and define

$$\langle \mathscr{L}, \mathscr{M}\rangle := {}^{\mathscr{A}}\!\big/_{\ker(\Phi)}.$$

The canonical image of $(l, m) \in \Sigma_U$ in $\mathscr{A}(U)$ is denoted as $\langle l, m\rangle_U$.

Proposition D.63. *Let* $(l, m) \in \Sigma_U$ *such that* $\langle \mathrm{div}(l)|_{X_U}, \mathrm{div}(m)|_{X_U}\rangle = 0 \in \mathrm{Div}(U)$. *Then for any* $(l', m') \in \Sigma_U$ *there exists an element* $a \in \mathscr{O}_S(U)$ *such that* $\langle l', m'\rangle_U = a\,\langle l, m\rangle_U$.

Proof. There are two elements $f, g \in K(X)^\times$ such that $l' = fl$, $m' = gm$ and moreover:

$$((f), (g)),\ ((f),\ \mathrm{div}(m)),\ ((g), \mathrm{div}(l)) \in \Upsilon_K\,.$$

Hence by using proposition D.58:

$$\langle \operatorname{div}(l')|_{X_U}, \operatorname{div}(m')|_{X_U} \rangle = \langle (f)|_{X_U} \operatorname{div}(l)|_{X_U}, (g)|_{X_U} \operatorname{div}(m)|_{X_U} \rangle =$$

$$= \langle (f)|_{X_U}, (g)|_{X_U} \rangle + \langle (f)|_{X_U}, \operatorname{div}(m)|_{X_U} \rangle + \left\langle \operatorname{div}(l)|_{X_U}, (g)|_{X_{U_i}} \right\rangle + 0 =$$

$$= \left(N_{(f)|_{X_U}}(g) \right) + \left(N_{\operatorname{div}(m)|_{X_U}}(f) \right) + \left(N_{\operatorname{div}(l)|_{X_U}}(g) \right) =$$

$$= \left(N_{(f)|_{X_U}}(g)\, N_{\operatorname{div}(m)|_{X_U}}(f)\, N_{\operatorname{div}(l)|_{X_U}}(g) \right).$$

Since $\langle \operatorname{div}(l')|_{X_U}, \operatorname{div}(m')|_{X_U} \rangle$ is effective, then

$$a := N_{(f)|_{X_U}}(g)\, N_{\operatorname{div}(m)|_{X_U}}(f)\, N_{\operatorname{div}(l)|_{X_U}}(g) \in \mathscr{O}_S(U).$$

On the other hand

$$\langle l', m' \rangle_K = [f, g] N_{\operatorname{div}(m)|_{X_K}}(f) N_{\operatorname{div}(l)|_{X_K}}(g) \, \langle l, m \rangle_K$$

therefore by proposition D.57 we can conclude that:

$$\langle l', m' \rangle_U = N_{(f)|_{X_U}}(g) N_{\operatorname{div}(m)|_{X_U}}(f) N_{\operatorname{div}(l)|_{X_U}}(g) \, \langle l, m \rangle_U = a \, \langle l, m \rangle_U \,.$$

□

We are ready to show that $\langle \mathscr{L}, \mathscr{M} \rangle$ is an invertible sheaf on S. By proposition D.62 $\langle \mathscr{L}, \mathscr{M} \rangle$ is nonzero; now assume $\mathscr{L} = \mathscr{O}_X(D)$, $\mathscr{M} = \mathscr{O}_X(E)$ and fix a point $s_0 \in S$. By the moving lemma we can find a divisor D' such that $D' \sim D$ and D' doesn't have components in X_{s_0}. Suppose that $x_1, \dots x_m$ are the intersection points of D' and X_{s_0}, by applying again the moving lemma we can find a divisor E' such that: $E' \sim E$, E' and $D' + X_{s_0}$ have no common components, and E doesn't pass by $x_1, \dots, x_m$. Consider the finite subset of S

$$C := \{ s \in S' \colon D' \cap E' \cap X_s \neq \emptyset \}$$

and note that its complement $U := S \setminus C$ has the following properties: $s_0 \in U$ and $\langle D'|_U, E'|_U \rangle = 0$. At this point any two meromorphic sections of $\mathscr{L}$ and $\mathscr{M}$ corresponding respectively to the divisors D' and E' will satisfy the hypothesis of proposition D.63 on U. This implies that $\langle \mathscr{L}, \mathscr{M} \rangle$ is an invertible sheaf.

Theorem D.64. *The Deligne pairing $(\mathscr{L}, \mathscr{M}) \to \langle \mathscr{L}, \mathscr{M} \rangle$ satisfies the properties listed below. We assume that $\mathscr{L}$ and $\mathscr{M}$ are two invertible sheaves on X.*

(1) *If* $\mathscr{M}'$ *and* $\mathscr{L}'$ *are two invertible sheaves such that* $\mathscr{L} \cong \mathscr{L}'$ *and* $\mathscr{M} \cong \mathscr{M}'$*, then* $\langle \mathscr{L}, \mathscr{M} \rangle \cong \langle \mathscr{L}', \mathscr{M}' \rangle$.

(2) *The induced map* $\operatorname{Pic}(X) \times \operatorname{Pic}(X) \to \operatorname{Pic}(S)$ *is bilinear and symmetric. By abuse of notation, this map will be also called the Deligne pairing.*

(3) *Let* l *and* m *be two nonzero meromorphic sections of* $\mathscr{L}$ *and* $\mathscr{M}$*, respectively, such that* $\operatorname{div}(l)$ *and* $\operatorname{div}(m)$ *have no common components. Then, there exists a nonzero meromorphic section* $\langle l, m \rangle$ *with the following properties:*

  (*i*) *If* $f, g \in K(X)^\times$ *such that* $(\operatorname{div}(fl), \operatorname{div}(m)), (\operatorname{div}(l), \operatorname{div}(gm)) \in \Upsilon$*, then:*

$$\langle fl, m \rangle = N_{\operatorname{div}(m)}(f) \, \langle l, m \rangle$$
$$\langle l, gm \rangle = N_{\operatorname{div}(l)}(g) \, \langle l, m \rangle$$

  (*ii*) *There is an isomorphism of invertible sheaves*

$$\langle \mathscr{L}, \mathscr{M} \rangle \cong \mathscr{O}_S(\langle \operatorname{div}(l), \operatorname{div}(m) \rangle)$$

  where the brackets $\langle \cdot, \cdot \rangle$ *on the right hand side indicate the pairing between divisors. Moreover, under the above isomorphism* $\langle l, m \rangle$ *corresponds to* $1_{\langle \operatorname{div}(l), \operatorname{div}(m) \rangle}$. *In particular:*

$$\operatorname{div}(\langle l, m \rangle) = \langle \operatorname{div}(l), \operatorname{div}(m) \rangle \ .$$

(4) *Let* $\psi : S' \to S$ *be a flat morphism between irreducible Dedekind schemes and consider the following canonical base change diagram:*

$$\begin{tikzcd} X' := X \times_S S' \arrow[r, "g"] \arrow[d, "f"] & S' \arrow[d, "\psi"] \\ X \arrow[r, "\varphi"] & S \ . \end{tikzcd}$$

Then:

$$\psi^*(\langle \mathscr{L}, \mathscr{M} \rangle) \cong \langle f^*(\mathscr{L}), f^*(\mathscr{M}) \rangle$$

as invertible sheaves on S'.

Proof. See [43, Theorem 4.7]. □

Remark D.65. Note that when $S = \operatorname{Spec} k$ for any field k (in other words X is an algebraic curve), then $\langle \mathscr{L}, \mathscr{M} \rangle$ is just a one dimensional k-vector space.

D.2.4 Arakelov intersection number

All the necessary tools for Arakelov intersection pairing. The plan is to give the abstract definition and properties first; then we will recover a geometric interpretation in terms of the decomposition given by equation (D.27).

Before going straight to the formal theory, it is good to see a motivational example:

Example D.66. In section D.2.1 we explained that the crucial point of Arakelov theory is to define the intersection at infinity of two horizontal divisors, in this example we will try to guess how we can achieve this goal. We will see later in this subsection how the formal definition matches perfectly our guess. Fix $\sigma \in B_\infty$ ad consider the situation depicted in figure D.4: suppose that $D, E \in \mathrm{Div}(X)$ are two different prime horizontal divisors such that $D_\sigma = [p]$ and $E_\sigma = [q]$ as divisors on X_σ. We want to find a number, that we denote (just here) as $(D.E)_\sigma$, which makes sense as "intersection number" on the curve at infinity X_σ. By proposition D.45, $D = \overline{\{x_D\}}$ and $E = \overline{\{x_E\}}$, where $x_D, x_E \in X_K$ are two different points (since D and E are different divisors). This implies that $[p]$ and $[q]$ are distinct because $\varphi_\sigma(p) = x_D$ and $\varphi_\sigma(p) = x_E$. In other words D and E will never "physically" meet on X_σ. But we could at least "measure" how much $[p]$ and $[q]$ are close to each other and then interpret this measure as the intersection number at infinity. For the point $p \in X$ we have the Green function $\mathcal{G}^\Omega([p]) = g^\Omega(p, \cdot)$ which satisfies the Poisson equation $dd^c(g^\Omega(p, \cdot)) = \Omega$, so in analogy with physics we can imagine $g^\Omega(p, \cdot)$

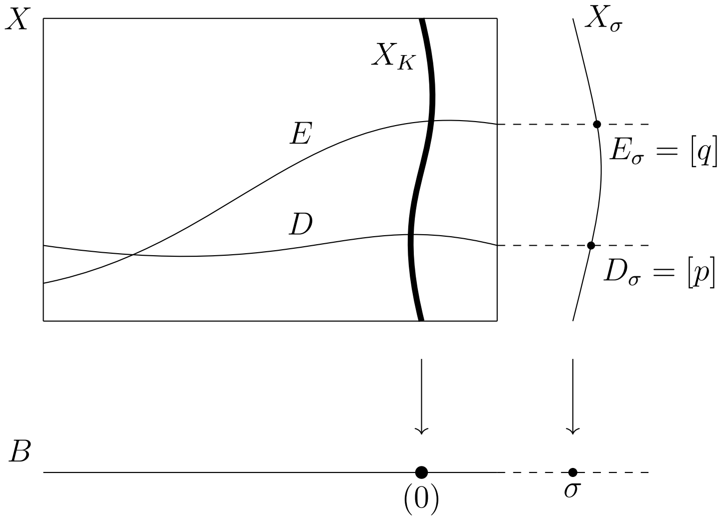

Figure D.4: D and E are two horizontal divisors. We already know how to deal with their finite intersection (in this case a point), but need an intersection number on X_σ.

as a gravitational (or electric) potential on X arising by a punctiform source situated in p. Since around p we can write $g^{\Omega}(p,q) = -\log|z(q)|^2 + u(q)$, note that $g^{\Omega}(p,q)$ is a very big positive real number when q is close to p. This means that $g^{\Omega_\sigma}(p,q)$ is a reasonable choice for measuring the closeness of $[p]$ and $[q]$, and we can put

$$(D.E)_\sigma = \frac{1}{2}\epsilon_\sigma g^{\Omega_\sigma}(p,q)\,.$$

where factor $\frac{1}{2}\epsilon_\sigma$ is just an arbitrary normalization which will make some formulae nicer. By extending this reasoning to all divisors of Υ, we expect the intersection pairing between $\overline{D}$ and $\overline{E}$ to be:

$$\overline{D}.\overline{E} = D.E + \frac{1}{2}\sum_\sigma \epsilon_\sigma g^{\Omega_\sigma}(D_\sigma, E_\sigma)\,. \tag{D.31}$$

Proposition D.67. *Let D, E be two finite divisors on X with no common components, then for any $\sigma \in B_\infty$ the divisors D_σ and E_σ on X_σ have no common components.*

Proof. Example D.66 contains the proof in the case when D and E are prime and horizontal. The proof for the general case follows trivially. ☐

Let's denote as $\Upsilon_{\mathrm{Ar}} \subset \mathrm{Div}_{\mathrm{Ar}}(X,\Omega) \times \mathrm{Div}_{\mathrm{Ar}}(X,\Omega)$ the set of couples of Arakelov divisors with no common components on X, then we can define the Arakelov intersection pairing on Υ_{Ar}:

Definition D.68. Let $\widehat{D} := (D, \sum_\sigma g_\sigma X_\sigma)\,, \widehat{E} := (E, \sum_\sigma l_\sigma X_\sigma)$ be two Arakelov divisors such that $(\widehat{D}, \widehat{E}) \in \Upsilon_{\mathrm{Ar}}$ (in other words $D, E \in \Upsilon$). Thanks to proposition D.67 we can define an Arakelov divisor on B:[7]

$$\left\langle \widehat{D}, \widehat{E} \right\rangle_{\mathrm{Ar}} := \langle D, E\rangle + \sum_\sigma g_\sigma * l_\sigma\,[\sigma] \quad \in \mathrm{Div}_{\mathrm{Ar}}(B)$$

where $\langle D, E\rangle$ is the finite intersection defined in the previous subsection and $*$ is the product between Green functions defined in subsection D.1.3. The *Arakelov intersection number* of $\widehat{D}$ and $\widehat{E}$ is:

$$\widehat{D}.\widehat{E} := \deg_{\mathrm{Ar}}\left(\left\langle \widehat{D}, \widehat{E} \right\rangle_{\mathrm{Ar}}\right) = D.E + \frac{1}{2}\sum_\sigma \epsilon_\sigma\, g_\sigma * l_\sigma \quad \in \mathbb{R}\,.$$

The following proposition summarizes some properties:

---

[7] Note that we assume D and E to have no common components in order to ensure that the $*$-product between green functions is well defined for any $\sigma \in B_\infty$.

Proposition D.69. *Let* $(\widehat{D}, \widehat{E}), (\widehat{D_j}, \widehat{E_j}) \in \Upsilon_{\mathrm{Ar}}$ *with* $j = 1, 2$*, then*

(1) $\widehat{D}.\widehat{E} = \widehat{E}.\widehat{D}$ *(symmetry).*

(2) $(\widehat{D_1} + \widehat{D_2}).(\widehat{E_1} + \widehat{E_2}) = \sum_{j,k=1}^{2} \widehat{D_j}.\widehat{E_k}$ *($\mathbb{Z}$-bilinearity).*

(3) *If* $\widehat{D} = (D, \sum_\sigma g_\sigma X_\sigma)$ *and* $f \in K(X)^\times$ *such that* $(D, (f)) \in \Upsilon$*, then*

$$\left\langle \widehat{D}, \widehat{(f)} \right\rangle_{\mathrm{Ar}} = (\widehat{N_D(f)}) \in \mathrm{Princ}_{\mathrm{Ar}}(B) .$$

In particular $\widehat{D}.\widehat{(f)} = 0$*.*

Proof. (1) and (2) follow immediately from the properties of the finite intersection and the $*$-product.
(3) By proposition D.58 we already know that $\langle D, (f) \rangle = (N_D(f))$ so let's concentrate our attention on the divisors at infinity. For $\sigma \in B_\infty$ write $D_\sigma = \sum n_p[p]$, then:

$$\begin{aligned} -\log|f_\sigma|^2 * g_\sigma &= -\widetilde{\log|f_\sigma}|^2(\mathrm{div}^G(g_\sigma)) + \int_X dd^c(-\log|f_\sigma|^2) g_\sigma = \\ &= -\widetilde{\log|f_\sigma}|^2(D_\sigma) = \sum_{p \in X_\sigma} -n_p \log|f_\sigma(p)|^2 = \\ &= -\log\left|\prod_{p \in X_\sigma} (f_\sigma(p))^{n_p}\right|^2 . \end{aligned}$$

Now note that $f_\sigma(p) = N_p(f_\sigma)$ (see defintion G.2), therefore we found the equality:

$$-\log|f_\sigma|^2 * g_\sigma = -\log|N_{D_\sigma}(f_\sigma)|^2 .$$

Let $x \in X_K$ be a point where f is defined and suppose that $p \in X_\sigma$ is such that $x = \varphi^\sigma(p)$. Remember that $f_\sigma(p) = f(x) \otimes^\sigma 1 \in k(x) \otimes_K^\sigma \mathbb{C}$, then by a simple calculation with a basis of the finite extension $k(x)|K$ we have that:

$$N_p(f_\sigma) = \sigma(N_x(f))$$

which in turn implies the equality

$$N_{D_\sigma}(f_\sigma) = \sigma(N_D(f)) .$$

By summarizing, we have shown that $-\log|f_\sigma|^2 * g_\sigma = -2\log|\sigma(N_D(f))|$ and we finally obtain:

$$\left\langle \widehat{D}, \widehat{(f)} \right\rangle_{\mathrm{Ar}} = (N_D(f)) + \sum_\sigma 2v_\sigma(N_D(f))[\sigma] = (\widehat{N_D(f)}) .$$

□

We finally can show the main feature of the Arakelov intersection number: it can be extended to an intersection pairing on the whole $\mathrm{Div}_{\mathrm{Ar}}(X,\Omega)$ and induces a natural intersection pairing on $\mathrm{CH}^1_{\mathrm{Ar}}(X,\Omega)$.

Proposition D.70. *The Arakelov intersection number extends to any two Arakelov divisors in* $\mathrm{Div}_{\mathrm{Ar}}(X,\Omega)\times\mathrm{Div}_{\mathrm{Ar}}(X,\Omega)$ *and moreover descends naturally to pairing on* $\mathrm{CH}^1_{\mathrm{Ar}}(X,\Omega)\times\mathrm{CH}^1_{\mathrm{Ar}}(X,\Omega)$.

Proof. Let $\widehat{D}:=(D,\sum_\sigma g_\sigma X_\sigma)$ and $\widehat{E}:=(E,\sum_\sigma l_\sigma X_\sigma)$ be any two Arakelov divisors. By the moving lemma we can find a rational function $f\in K(X)^\times$ such that $D+(f)$ and E have no common components, so $(\widehat{D}+\widehat{(f)},\widehat{E})\in\Upsilon_{\mathrm{Ar}}$. We define:

$$\widehat{D}.\widehat{E}:=(\widehat{D}+\widehat{(f)}).\widehat{E}$$

and we have to show that the intersection number doesn't depend on the choice of f. Let $f'\in K(X)^\times$ another rational function such that $D+(f')$ and E have no common components then by proposition D.69:

$$(\widehat{D}+\widehat{(f)}).\widehat{E}-(\widehat{D}+\widehat{(f')}).\widehat{E}=\widehat{(f/f')}.\widehat{E}=0\,.$$

Finally, the intersection number descends to $\mathrm{CH}^1_{\mathrm{Ar}}(X,\Omega)\times\mathrm{CH}^1_{\mathrm{Ar}}(X,\Omega)$ immediately by the definition. □

Now we interpret the Arakelov intersection pairing in a more geometric way by using the decomposition given in equation (D.27). Fix two Arakelov divisors $\widehat{D},\widehat{E}\in\Upsilon_{\mathrm{Ar}}$, then we can write

$$\widehat{D}=\overline{D}+\sum_\sigma\alpha_\sigma X_\sigma=\left(D,\sum_\sigma\mathcal{G}^{\Omega_\sigma}(D_\sigma)X_\sigma\right)+\left(0,\sum_\sigma\alpha_\sigma X_\sigma\right)\,,$$

$$\widehat{E}=\overline{E}+\sum_\sigma\beta_\sigma X_\sigma=\left(E,\sum_\sigma\mathcal{G}^{\Omega_\sigma}(E_\sigma)X_\sigma\right)+\left(0,\sum_\sigma\beta_\sigma X_\sigma\right)\,.$$

In order to find explicitly $\widehat{D}.\widehat{E}$, by bilinearity and symmetry of the intersection pairing it is enough to understand how calculate the following three terms:

(*i*) $\overline{D}.\overline{E}$; namely the intersection of two completed divisors (see example D.66).

(*ii*) $\overline{D}.(0,\sum_\sigma\beta_\sigma X_\sigma)$; namely the intersection between a completed divisor and a divisor at infinity. Clearly $(0,\sum_\sigma\alpha_\sigma X_\sigma).\overline{E}$ is obtained in the same way.

(iii) $(0,\sum_\sigma \alpha_\sigma X_\sigma).(0,\sum_\sigma \beta_\sigma X_\sigma)$; that is the intersection of divisors composed only by curves at infinity.

For (i) let's evaluate $\mathcal{G}^{\Omega_\sigma}(D_\sigma) * \mathcal{G}^{\Omega_\sigma}(E_\sigma)$. By the bare definition of the $*$-product and g^{Ω_σ}:

$$\mathcal{G}^{\Omega_\sigma}(D_\sigma) * \mathcal{G}^{\Omega_\sigma}(E_\sigma) = g^{\Omega_\sigma}(D_\sigma, E_\sigma) + \int_{X_\sigma} dd^c \left(\mathcal{G}^{\Omega_\sigma}(D_\sigma)\right) \mathcal{G}^{\Omega_\sigma}(E_\sigma)\,,$$

but since $\mathcal{G}^{\Omega_\sigma}(D_\sigma), \mathcal{G}^{\Omega_\sigma}(E_\sigma) \in G_0^{\Omega_\sigma}(X_\sigma)$, it is straightforward to verify that the integral on the right hand side is 0. Therefore we get:

$$\overline{D}.\overline{E} = D.E + \frac{1}{2}\sum_\sigma \epsilon_\sigma g^{\Omega_\sigma}(D_\sigma, E_\sigma)\,. \tag{D.32}$$

Note how this result matches perfectly equation (D.31). To calculate (ii) we need $\mathcal{G}^{\Omega_\sigma}(D_\sigma) * \beta_\sigma$:

$$\mathcal{G}^{\Omega_\sigma}(D_\sigma)*\beta_\sigma = \beta_\sigma*\mathcal{G}^{\Omega_\sigma}(D_\sigma) = \beta_\sigma \deg(D_\sigma)+\int_{X_\sigma} dd^c(\beta_\sigma)\mathcal{G}^{\Omega_\sigma}(D_\sigma) = \beta_\sigma \deg(D_\sigma)\,,$$

thus we obtain

$$\overline{D}.(0,\sum_\sigma \beta_\sigma X_\sigma) = \frac{1}{2}\sum_\sigma \epsilon_\sigma\beta_\sigma \deg(D_\sigma)\,. \tag{D.33}$$

Finally (iii) is trivial since $\alpha_\sigma * \beta_\sigma = 0$ and we have:

$$(0,\sum_\sigma \alpha_\sigma X_\sigma).(0,\sum_\sigma \beta_\sigma X_\sigma) = 0\,. \tag{D.34}$$

Remark D.71. Arakelov originally introduced the intersection pairing by distinguishing the three different cases (i), (ii), (iii) and by giving the equations (D.32), (D.33) and (D.34) as definitions. Here we recovered Arakelov's approach from a more abstract definition.

D.2.5 Picard-Arakelov group and Deligne-Arakelov pairing

So far we have enlarged the group $\mathrm{Div}(X)$ to the group $\mathrm{Div}_{\mathrm{Ar}}(X,\Omega)$, now we want to enlarge the Picard group $\mathrm{Pic}(X)$ to get the desired isomorphism with $\mathrm{CH}^1_{\mathrm{Ar}}(X,\Omega)$.

The first step consists in considering the right notion of hermitian invertible sheaf on $\widehat{X}$, and as we did in the previous sections we will use the pullback properties of the maps φ_σ. If $\mathscr{L}$ is an invertible sheaf on X then its pullback $\mathscr{L}_\sigma := \varphi_\sigma^*\mathscr{L}$ is an invertible sheaf over X_σ. Let $s \in \mathscr{L}_\eta$ be a meromorphic section of $\mathscr{L}$, then the natural embedding $\mathscr{L}_\eta \hookrightarrow \mathscr{L}_\eta \otimes^\sigma_{K(X)} \mathbb{C}(X_\sigma)$ induces the meromorphic section $s_\sigma := s \otimes^\sigma 1$ of $\mathscr{L}_\sigma$.

Definition D.72. An *hermitian invertible sheaf on* $\widehat{X}$ is a couple

$$\widehat{\mathscr{L}} = (\mathscr{L}, \{h_\sigma\}_{\sigma \in B_\infty})$$

such that $\mathscr{L}$ is an invertible sheaf on X and h_σ is a C^∞ hermitian metric on the pullback sheaf $\mathscr{L}_\sigma$. Moreover $\widehat{\mathscr{L}}$ is called admissible if each h_σ is an admissible metric.

For inverible sheaves on X, the natural notion of equivalence relation is the isomorphism. For hermitian invertible sheaves on $\widehat{X}$ some metrics at infinity are involved so we need an equivalence relation which includes isometries at infinity.

Definition D.73. Let $\widehat{\mathscr{L}} = (\mathscr{L}, \{h_\sigma\}_\sigma)$ and $\widehat{\mathscr{L}'} = (\mathscr{L}', \{h'_\sigma\}_\sigma)$ be two invertible herimitan sheaves on $\widehat{X}$. They are said *isometric* if there is an isomorphism of sheaves $\psi : \mathscr{L} \to \mathscr{L}'$ such that for any $\sigma \in B_\infty$, the map given by the pullback $\psi_\sigma : (\mathscr{L}_\sigma, h_\sigma) \to (\mathscr{L}'_\sigma, h'_\sigma)$ is an isometry of hermitian sheaves on X_σ.

Note that hermitian invertible sheaves on $\widehat{X}$ still form a group with respect to the following operation:

$$(\mathscr{L}, \{h_\sigma\}_\sigma) \otimes (\mathscr{L}', \{h'_\sigma\}_\sigma) := (\mathscr{L} \otimes \mathscr{L}', \{h_\sigma \otimes h'_\sigma\}_\sigma) \ .$$

The identity element is $\widehat{\mathscr{O}_X} := (\mathscr{O}_X, \{h_{0_\sigma}\}_\sigma)$ where h_{0_σ} is the canonical metric on the trivial divisor $\mathscr{O}_\sigma$.

Definition D.74. The *Picard-Arakelov group* on $\widehat{X}$, denoted by $\mathrm{Pic}_{\mathrm{Ar}}(X, \Omega)$, is the group of isometry classes of hermitian invertible sheaves on $\widehat{X}$.

Let $\widehat{\mathscr{L}} = (\mathscr{L}, \{h_\sigma\})$ be an invertible hermitian sheaf and fix a meromorphic section s of $\mathscr{L}$; then $-\log(h_\sigma(s_\sigma, s_\sigma)) \in G^{\Omega_\sigma}(X_\sigma)$ and moreover $\mathrm{div}^G(-\log(h_\sigma(s_\sigma, s_\sigma))) = \mathrm{div}(s)_\sigma$ for any $\sigma \in B_\infty$. These facts were proved in subsection D.2.2 in the case $\widehat{\mathscr{L}} = \widehat{\mathscr{O}_X}$ and $s = f \in K(X)$, so rather obvious modifications of those arguments work for the general situation. Thus we can construct the Arakelov divisor associated to s:

$$\begin{aligned} \widehat{\mathrm{div}(s)} &:= \left(\mathrm{div}(s), \sum_\sigma (-\log(h_\sigma(s_\sigma, s_\sigma))X_\sigma\right) = \\ &= \overline{\mathrm{div}(s)} + \sum_\sigma \left(\int_{X_\sigma} -\log(h_\sigma(s_\sigma, s_\sigma))\Omega_\sigma\right) X_\sigma \,. \end{aligned}$$

Theorem D.75. $\mathrm{CH}^1_{\mathrm{Ar}}(X,\Omega) \cong \mathrm{Pic}_{\mathrm{Ar}}(X,\Omega)$.

Proof. First of all we have to define a map $\mathrm{Div}_{\mathrm{Ar}}(X,\Omega) \to \mathrm{Pic}_{\mathrm{Ar}}(X,\Omega)$. For any divisor $D \in \mathrm{Div}(X)$ consider the pullback $D_\sigma \in \mathrm{Div}(X_\sigma)$ and endow the invertible sheaf $\mathscr{O}_\sigma(D_\sigma)$ with its canonical admissible metric $h_{D_\sigma} := h^{\Omega_\sigma}_{D_\sigma}$. So we can define the map Θ:

$$\mathrm{Div}_{\mathrm{Ar}}(X,\Omega) \ni \overline{D} + \sum_\sigma \alpha_\sigma X_\sigma \mapsto \left(\mathscr{O}_X(D), \left\{e^{-\alpha_\sigma} h_{D_\sigma}\right\}_\sigma\right) \in \mathrm{Pic}_{\mathrm{Ar}}(X,\Omega)$$

where of course on the right hand side we take the equivalence class of isometry. It is easy to check that it is a morphism of groups, so let's show that its kernel is $\mathrm{Princ}_{\mathrm{Ar}}(X,\Omega)$. If $\widehat{(f)} \in \mathrm{Princ}_{\mathrm{Ar}}(X,\Omega)$, then:

$$\widehat{(f)} = \overline{(f)} + \sum_\sigma \deg((f)_\sigma) X_\sigma \,,$$

so Θ sends $\widehat{(f)}$ to the isometric class of $\left(\mathscr{O}_X((f)), \{e^{-\alpha_\sigma} h_{f_\sigma}\}_\sigma\right)$ which the same class of the identity element thanks to proposition D.35. Viceversa if $(\mathscr{L}, \{h_\sigma\}_\sigma)$ is isometric to $(\mathscr{O}_X, \{h_{0_\sigma}\}_\sigma)$, then by the usual theory of finite divisors we have that $\mathscr{L} = \mathscr{O}_X((f))$ for $f \in K(X)$. The metrics h_σ and $e^{-\alpha_\sigma} h_{f_\sigma}$ are both admissible on $\mathscr{L}_\sigma$, thus by corollary D.32 we have an isometry between $(\mathscr{L}, \{h_\sigma\}_\sigma)$ and $\left(\mathscr{L}, \{e^{-\alpha_\sigma} h_{f_\sigma}\}_\sigma\right)$. This shows that $\ker \Theta \subseteq \mathrm{Princ}_{\mathrm{Ar}}(X,\Omega)$.

Finally, we have to show the surjectivity of Θ. Let $(\mathscr{L}, \{h_\sigma\}_\sigma)$ an admissible hermitian sheaf, then consider a meromorphic section s of $\mathscr{L}$ and let $D = \mathrm{div}(s)$. For any σ we have the pullback meromorphic section s_σ of $\mathscr{L}_\sigma$ such that $\mathscr{L}_\sigma = \mathscr{O}_\sigma(\mathrm{div}(s_\sigma))$. Consider the Arakelov divisor:

$$\widehat{D} = \left(\mathrm{div}(s), \sum_\sigma -\log(h_\sigma(s_\sigma, s_\sigma)) X_\sigma\right) \in \mathrm{Div}_{\mathrm{Ar}}(X,\Omega) \,,$$

its image through Θ is:

$$\left(\mathscr{L}, \left\{e^{-\alpha_\sigma} h_{D_\sigma}\right\}_\sigma\right) \in \mathrm{Pic}_{\mathrm{Ar}}(X,\Omega).$$

But for any σ, $e^{-\alpha_\sigma} h_{D_\sigma}$ and h_σ are both admissible metrics on the same invertible sheaf $\mathscr{L}_\sigma$; this implies that $(\mathscr{L}_\sigma, h_\sigma)$ and $(\mathscr{L}_\sigma, e^{-\alpha_\sigma} h_{D_\sigma})$ are isometric thanks to corollary D.32. We can conclude that $\Theta(\widehat{D})$ and $(\mathscr{L}, \{h_\sigma\}_\sigma)$ are isometric. □

In subsection D.2.1 we have introduced intersection theory on an algebraic surface by defining a pairing between invertible sheaves, now we want to show that the same approach is possible in the case of Arakelov intersection theory. In other words we want to define a pairing:

$$\langle\!\langle \, , \, \rangle\!\rangle : \mathrm{Pic}_{\mathrm{Ar}}(X, \Omega) \times \mathrm{Pic}_{\mathrm{Ar}}(X, \Omega) \to \mathrm{Pic}_{\mathrm{Ar}}(B)$$

such that if $\hat{D}, \hat{E} \in \mathrm{Div}_{\mathrm{Ar}}(X)$ and $\overline{\mathscr{L}}$ and $\overline{\mathscr{M}}$ are the associated hermitian invertible sheaves, then:

$$\deg_{\mathrm{Ar}} \left(\langle\!\langle \overline{\mathscr{L}}, \overline{\mathscr{M}} \rangle\!\rangle \right) = \widehat{D}.\widehat{E}$$

For obvious reason we will call pairing $\langle\!\langle \, , \, \rangle\!\rangle$ the *Deligne-Arakelov pairing.*[8]

Let C be a compact Riemann surface and fix two hermitian invertible sheaves $(\mathscr{L}, h_{\mathscr{L}})$, $(\mathscr{M}, h_{\mathscr{M}})$ on it, then we are going to define an hermitian inner product on the one dimensional $\mathbb{C}$-vector space $\langle \mathscr{L}, \mathscr{M} \rangle$ (here we mean the usual Deligne pairing in the one dimensional case), involving the metrics $h_{\mathscr{L}}$ and $h_{\mathscr{M}}$. Let l and m be two meromorphic sections of $\mathscr{L}$ and $\mathscr{M}$ respectively such that $\mathrm{div}(l)$ and $\mathrm{div}(m)$ have no common components. Consider the element $\langle l, m \rangle \in \langle \mathscr{L}, \mathscr{M} \rangle$ constructed in theorem D.64 (here remark D.65 is crucial), then let's define:

$$h_{\langle \mathscr{L}, \mathscr{M} \rangle}(\langle l, m \rangle, \langle l, m \rangle) := e^{(-\log h_{\mathscr{L}}(l,l)) * (\log h_{\mathscr{L}}(m,m))} \tag{D.35}$$

where on the right hand side we have the usual $*$-product between Green functions. Clearly we can extend equation (D.35) to an hermitian inner product $h_{\langle \mathscr{L}, \mathscr{M} \rangle}$ on the whole $\langle \mathscr{L}, \mathscr{M} \rangle$, but still we have to show that equation (D.35) doesn't depend on the choice of the meromorphic sections l and m. Let's start with the following lemma which gives an explicit description of the $*$-product in a special case:

Lemma D.76. *Let $\phi \in \mathbb{C}(C)^{\times}$ and $g \in G(C)$, then:*

$$\log |\phi| * g = \log |N_{\mathrm{div}^G(g)}(\phi)|$$

Proof. Put $D = \mathrm{div}^G(g) = \sum_j a_j [x_j]$ with $a_j \neq 0$ for any j, then by definition:

$$\log |\phi| * g = \widetilde{\log |\phi|}(D) + \int_C dd^c(\log |\phi|) g = \widetilde{\log |\phi|}(D) \, .$$

The last equality follows from directly by the fact that $dd^c(\log |\phi|) = 0$. But now

$$\widetilde{\log |\phi|}(D) = \sum_j a_j \log |\phi(x_j)| = \log \left| \prod_j \phi(x_j)^{a_j} \right| = \log |N_D(\phi)| \, .$$

□

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[8] Be aware that this terminology is not used in the literature.

Let l' and m' any other two meromorphic sections of $\mathscr{L}$ and $\mathscr{M}$ such that $\operatorname{div}(l')$ and $\operatorname{div}(m')$ have no common components, then there are $f, t \in \mathbb{C}(C)^\times$ such that $l' = fl$ and $m' = tm$. At this point we obtain:

$$(-\log h_{\mathscr{L}}(l',l')) * (\log h_{\mathscr{L}}(m',m')) = (-\log h_{\mathscr{L}}(l,l) - \log|f|^2) * (\log h_{\mathscr{M}}(m,m) + \log|t|^2) =$$

$$= (-\log h_{\mathscr{L}}(l,l)) * (\log h_{\mathscr{L}}(m,m)) - (\log h_{\mathscr{L}}(l,l)) * \log|t|^2 - (\log h_{\mathscr{L}}(m,m)) * \log|f|^2 +$$
$$- \log|f|^2 * \log|t|^2 =^{\text{lem. D.76}}$$
$$= (-\log h_{\mathscr{L}}(l,l)) * (\log h_{\mathscr{L}}(m,m)) + \log\left(\left|[f,g]N_{\operatorname{div}(m)}(l)N_{\operatorname{div}(l)}(m)\right|^2\right)$$

However, let's recall that by construction:

$$\langle l', m' \rangle = [f,g]N_{\operatorname{div}(m)}(l)N_{\operatorname{div}(l)}(m) \langle l, m \rangle \ ,$$

thus

$$h_{\langle \mathscr{L}, \mathscr{M} \rangle}(\langle l', m' \rangle, \langle l,' m' \rangle) = \left|[f,g]N_{\operatorname{div}(m)}(l)N_{\operatorname{div}(l)}(m)\right|^2 h_{\langle \mathscr{L}, \mathscr{M} \rangle}(\langle l, m \rangle, \langle l, m \rangle)$$

and the claim is proved. We now have a way to put an hermitian product on the result of the 1-dimensional Deligne pairing for a Riemann surface.

Let's go back to our completed arithmetic surface $\widehat{X}$, for any $\overline{\mathscr{L}}, \overline{\mathscr{M}} \in \operatorname{Pic}_{\text{Ar}}(X)$ let's define the element $\langle\langle \overline{\mathscr{L}}, \overline{\mathscr{M}} \rangle\rangle \in \operatorname{Pic}_{\text{Ar}}(B)$:

$$\langle\langle \overline{\mathscr{L}}, \overline{\mathscr{M}} \rangle\rangle := \left(\langle \mathscr{L}, \mathscr{M} \rangle, h_{\langle \mathscr{L}, \mathscr{M} \rangle}\right)$$

where:

- $\langle \mathscr{L}, \mathscr{M} \rangle \in \operatorname{Pic}(B)$ is the 2-dimensional Deligne pairing.
- $h_{\langle \mathscr{L}, \mathscr{M} \rangle}$ is the collection of metrics on $\{\langle \mathscr{L}, \mathscr{M} \rangle_\sigma\}_{\sigma \in B_\infty}$ defined in the following way: by the properties of Deligne pairing with respect to base change we know that $\langle \mathscr{L}, \mathscr{M} \rangle_\sigma = \langle \mathscr{L}_\sigma, \mathscr{M}_\sigma \rangle$; on the latter vector space we impose the hermitian inner product $h_{\langle \mathscr{L}_\sigma, \mathscr{M}_\sigma \rangle}$ defined in equation (D.35).

It remains to show the relation between Deligne-Arakelov pairing and Arakelov intersection number:

Proposition D.77. *Let $\widehat{D}$ and $\widehat{E}$ be two Arakelov divisors and let $\overline{\mathscr{L}}$ and $\overline{\mathscr{M}}$ the two associated elements in* $\operatorname{Pic}_{Ar}(X)$*, respectively. Then*

$$\deg_{\text{Ar}}\left(\langle\langle \overline{\mathscr{L}}, \overline{\mathscr{M}} \rangle\rangle\right) = \widehat{D}.\widehat{E}$$

Proof. Let l and m be two meromorphic sections of $\mathscr{L}$ and $\mathscr{M}$ such that $\operatorname{div}(l)$ and $\operatorname{div}(m)$ have no common components. Then

$$\begin{gathered}\widehat{D}.\widehat{E}=\widehat{\operatorname{div}(l)}.\widehat{\operatorname{div}(m)}=\operatorname{div}(l).\operatorname{div}(m)+\\ +\frac{1}{2}\sum_{\sigma}\epsilon_{\sigma}(-\log h_{\mathscr{L}_{\sigma}}(l_{\sigma},l_{\sigma}))*(-\log h_{\mathscr{M}_{\sigma}}(m_{\sigma},m_{\sigma}))\,.\end{gathered}$$

But:

$$\operatorname{div}(l).\operatorname{div}(m)=\deg_{\mathrm{Ar}}\left(\langle\mathscr{L},\mathscr{M}\rangle\right)=\deg_{\mathrm{Ar}}\left(\operatorname{div}\langle l,m\rangle\right)$$

and we can conclude that

$$\begin{gathered}\widehat{D}.\widehat{E}=\deg_{\mathrm{Ar}}\left(\operatorname{div}\langle l,m\rangle\right)=\frac{1}{2}\sum_{\sigma}-\epsilon_{\sigma}\log h_{\langle\mathscr{L}_{\sigma},\mathscr{M}_{\sigma}\rangle}\left(\langle l,m\rangle\,,\langle l,m\rangle\right)=\\ =\deg_{\mathrm{Ar}}\left(\widehat{\operatorname{div}\langle l,m\rangle}\right)=\deg_{\mathrm{Ar}}\left(\langle\langle\overline{\mathscr{L}},\overline{\mathscr{M}}\rangle\rangle\right)\ .\end{gathered}$$

□

Appendix E

Semi-topological algebraic structures

E.1 Basic notions

Definition E.1. A topological abelian group (G, τ) is *linearly topologized* (or has a *linear topology*) if there is a local basis at 0 made of subgroups. A morphism between linearly topologized groups is a continuous homomorphism. The category of linearly topologized group is denoted by **LTAb**.

Proposition E.2. *Let G be an abelian group and fix a non-empty collection of subgroups $\mathcal{F} = \{U_i\}_{i \in I}$. If G is endowed with the topology τ generated by $\{x + U_i\}_{i \in I, x \in G}$, then it becomes a linearly topologized group.*

Proof. First we show that G is a topological group: we want the inversion $\iota : G \to G$ and the sum $\sigma : G \times G \to G$ to be continuous. We check this for the subbase $\{x + U_i\}_{i \in I, x \in G}$. Obviously $\iota^{-1}(U_i + x) = U_i - x \in \tau$. Then we prove that the following equality holds:

$$\sigma^{-1}(U_i + x) = \bigcup_{y \in G} (U_i + y) \times (U_i + x - y) \, .$$

The inclusion $\supseteq$ is evident, so let $(z, z') \in \sigma^{-1}(U_i + x)$, then $z = u + (x - z')$ for $u \in U_i$. If we write $z' = 0 + x - (x - z')$ and we put $y = x - z'$ we finally get $(z, z') = (u + y, 0 + x - y) \in (U_i + y) \times (U_i + x - y)$.
For the last statement consider the family

$$\mathcal{B} := \{U \in \tau : \text{ U is finite intersection of elements of } \mathcal{F}\} \, .$$

Then $\mathcal{B}$ is a local basis at 0 made of subgroups. $\square$

Definition E.3. The linear topology on an abelian group G obtained from a family of subgroups $\{U_i\}_{i \in I}$, as it is described in proposition E.2, is called *the linear topology generated by* $\{U_i\}_{i \in I}$.

In this setting, concepts like initial and final topologies are well defined. Let G be an abelian group and consider some homomorphisms of groups $\{\varphi_\alpha : G \to H_\alpha\}_\alpha$ and $\{\psi_\beta : H_\beta \to G\}_\beta$, where the H_α and H_β are all linearly topologized. The *initial linear topology* on G with respect to $\{\varphi_\alpha\}_\alpha$ is the linear topology generated by

$$\left\{\varphi_\alpha^{-1}(V_\alpha)\colon V_\alpha \subseteq H_\alpha \text{ is an open subgroup}\right\}_\alpha .$$

This is the coarsest linear topology which makes all the φ_α continuous. The *final linear topology* on G with respect to $\{\psi_\beta\}_\beta$ is the linear topology generated by

$$\left\{U \subseteq G\colon U \text{ is a subgroup and } \psi_\beta^{-1}(U) \text{ is open for any } \beta\right\} .$$

This is the finest linear topology which makes all the ψ_β continuous.

Proposition E.4. LTAb *is an additive category and moreover it admits inverse and direct limits.*

Proof. The nontrivial statements are those involving the categorical limits. In particular $\varprojlim_i G_i$ and $\varinjlim_j G_j$ are the usual limits in the category of groups, endowed respectively with the inital and final linear topology.

Remark E.5. By commodity, in the category of linearly topologized groups, we call the limits $\varprojlim_i G_i$ and $\varinjlim_j G_j$ respectively *linear inverse limit* and *linear direct limit.*

Definition E.6. A *ST ring* (ST stands for semi-topological) is a ring A endowed with a topology satisfying the following two properties:

- $(A, +)$ is a linearly topologized abelian group.
- For any $a \in A$ the map $\lambda_a : A \to A$, such that $\lambda_a(x) = ax$, is continuous.

A morphism of ST rings is a continuous homomorphisms of rings. The category of ST rings is denoted as **STRing**. Moreover B is a ST A-algebra if there is a morphism of ST rings $\varphi : A \to B$. The category of ST A-algebras is A-**STAlg**.

Proposition E.7. STRing *and* A-**STAlg** *admit inverse and direct limits.*

Proof. We show it only for rings. Let $A = \varprojlim_i A_i$ be the usual inverse limit in the category of rings and topologize its additive structure by taking the linear inverse limit topology. Thus we have the coarsest linear topology on $(A, +)$ such that the projections $\pi_j : A \to A_j$ are continuous. Assume that $\Lambda_{(a_i)}$ is the multiplication by $(\ldots, a_i, a_{i+1}, \ldots)$ in A and consider the composition: $A \xrightarrow{\Lambda_{(a_i)}} A \xrightarrow{\pi_j} A_j$, given by

$$x = (\ldots x_i, x_{i+1}, \ldots) \mapsto (\ldots a_i x_i, a_{i+1} x_{i+1}, \ldots) \mapsto a_j x_j \,.$$

Since $\pi_j \circ \Lambda_{(a_i)}(x) = \lambda_{a_j} \circ \pi_j(x)$, we can conclude that $\pi_j \circ \Lambda_{(a_j)}$ is continuous. Finally if $\pi_j^{-1}(V_j) \subset A$ is an element in the subbase of A, then $\Lambda_{(a_i)}^{-1}(\pi_j^{-1}(V_j))$ is open in A.

Let $A = \varinjlim_i A_i$ be the usual direct limit in the category of rings and topologize its additive structure by taking the linear direct limit topology. Thus we have the finest linear topology on $(A, +)$ such that the maps $\phi_i : A_i \to A$ are continuous. Let's denote with $\mu_{ij} : A_i \to A_j$ the continuous homomorpshims in the directed set $\{A_i\}_i$; moreover $\Lambda_{[(j,a)]}$ is the multiplication in $A = (\sqcup_i A_i) /_\sim$ for the fixed element $[(j, a)]$ where $a \in A_j$. Note that the composition: $A_i \xrightarrow{\phi_i} A \xrightarrow{\Lambda_{[j,a]}} A$, given by

$$x \mapsto [(i, x)] \mapsto [k, \mu_{jk}(a)\mu_{ik}(x)] \,.$$

is continuous. Thus if $U \subset A$ is open, then $\phi_1^{-1}\left(\Lambda_{[j,a]}^{-1}(U)\right)$ is open and by definition of final linear topology we can conclude that $\Lambda_{[j,a]}^{-1}(U)$ is open in A. □

In the next definition we describe how we can carry the structure of ST ring from A to the ring of Laurent power series $A((t))$.

Definition E.8. Let A be a ST ring, clearly there is an isomorphism:

$$\frac{t^r A[[t]]}{t^{r+m} A[[t]]} \cong A^{\oplus m}$$

which induces a (finite) product topology on $\frac{t^r A[[t]]}{t^{r+m} A[[t]]}$, then in **STRing** we consider

$$A((t)) = \varinjlim_r t^r A[[t]] = \varinjlim_r \varprojlim_m \frac{t^r A[[t]]}{t^{r+m} A[[t]]} \,.$$

We call this topology the *ind/pro-topology* on $A((t))$.

Note that for any $r \in \mathbb{Z}$, $t^r A[[t]] = \varprojlim_m \frac{t^r A[[t]]}{t^{r+m} A[[t]]}$ is endowed with the (infinite) product topology of A, hence the following result is quite straightforward:

Proposition E.9. *Let A be a ST ring which is a T1 space. The subring $t^m A[[t]]$ is closed in $A((t))$ for any $m \in \mathbb{Z}$.*

Proof. It is enough to show that for any $r < m$, $t^m A[[t]]$ is closed in $t^r A[[t]]$. Since $t^r A[[t]]$ is endowed with the product topology given by $\prod_{i \in \mathbb{N}} A$, then $t^m A[[t]]$ as subset of $t^r A[[t]]$ can identified as:

$$\underbrace{\{0\} \times \{0\} \times \ldots \times \{0\}}_{m-r} \times \prod_{i > m-r} A$$

which is evidently closed. $\square$

There is also an explicit way to define the topology on $A((t))$ by giving directly a local basis at 0:

Definition E.10. Let A be a ST ring and let $\{U_i\}_{i \in \mathbb{Z}}$ be a family of open subgroups of A such that there exists $r \in \mathbb{Z}$ for which $U_i = A$ when $i \geq r$; then:

$$\sum_{i \in \mathbb{Z}} U_i t^i := \left\{ \sum a_i t^i \in A((t)) \colon a_i \in U_i \right\}$$

is a local basis at 0 and the induced topology on $A((t))$ is called the *P-topology.*

Proposition E.11. *The ind/pro-topology and the P-topology are equal.*

Proof. Let $\sum_{i \in \mathbb{Z}} U_i t^i$ an element of the basis at 0 of the P-topology. Then for any $r \in \mathbb{Z}$, the intersection $t^r A[[t]] \cap \sum_{i \in \mathbb{Z}} U_i t^i$ is open in $t^r A[[t]]$ because of the definition of open sets in the product topology. Hence the P-topology is coarser then the ind/pro-topology.
Vice versa let U be an open subgroup of $A((t))$ for the ind/pro-topology, then $t^r A[[t]] \cap U$ is open in $t^r A[[t]]$ and we can write

$$t^r A[[t]] \cap U = \sum_{i \geq r} U_i t^i \quad \text{where } U_i = A \text{ when } i >> r\,.$$

By using the fact that $U = \bigcup_{r \in \mathbb{Z}} (U \cap t^r A[[t]])$ we can conclude that U is an element of the local basis of the P-topology. $\square$

If A has the discrete topology, then we recover on $A((t))$ the topology which has $\{t^r A[[t]]\}_{r \in \mathbb{Z}}$ as local basis at 0; hence in this particular case $A((t))$ is a topological ring. Let's give a quick proof: each $t^r A[[t]]$ is clearly open because singletons are open in A. Assume by contradiction that for an open set U there is no $t^r A[[t]]$ containing it; it means that for any $r \geq 0$ we can find $x_r \in t^r A[[t]] \setminus U$. Since $\lim_{r\to\infty} x_r = 0$, we conclude that $A((t)) \setminus U$ is not closed against the assumption on U. We want to emphasize the fact that when $A = F$ is a field endowed the discrete topology, then $F((T))$ has the usual topology of complete valuation field.

Remark E.12. From now on we always assume that $A((t))$ is topologized with the ind/pro or P-topology if A is a ST ring.

Definition E.13. Let A be a ST ring. A *ST A-module* is an A-module satisfying the following properties:

- M is a linearly topologized abelian group.
- For any $a \in A$ and any $m \in M$ the maps $\lambda_a^M : M \to M$ and $\rho_m : A \to M$ such that $\lambda_a(x) = ax$ and $\rho_m(x) = xm$ are continuous.

A morphism of ST modules is a continuous homomorphism of A-modules. If A is a ST field then M is called a *ST vector space.*

Given a ST A-module M, the subset $\overline{\{0\}}$ is a submodule because of the continuity of λ_a, therefore we define

$$M^{\text{sep}} := M/\overline{\{0\}}$$

which is again a ST A-module if endowed with the quotient topology.

Proposition E.14. *Let A be a ST ring, and M an A-module. If M is endowed with the final linear topology with respect to the group homomorphisms $\rho_m : A \to M$, then M is a ST A-module*

Proof. See [63, pag. 17]. ☐

Definition E.15. The topology on M described in proposition E.14 is called the *fine A-module topology.*

Now we present the crucial part of this very general theory. Given a ST ring A, we describe two procedures called (C) and (L) that give canonical topologies of ST rings respectively on $\varprojlim_r A/\mathfrak{p}^r$ and $A_{\mathfrak{p}}$ for any prime ideal $\mathfrak{p} \subset A$. We need the following lemma:

Lemma E.16. *Let $\varphi : A \to B$ be a ring homomorphism where A is a ST ring. Consider B as an A-module endowed with the fine A-module topology, then B is a ST ring.*

Proof. [63, Proposition 1.2.9.(b)]. ☐

(C) For any $r > 0$ we put on $A/\mathfrak{p}^r$ the fine A-module topology, so by lemma E.16 $A/\mathfrak{p}^r$ is a ST ring. By proposition E.7 we can endow $\varprojlim_r A/\mathfrak{p}^r$ with a structure of ST ring.

(L) $A_\mathfrak{p}$ is naturally an A-module, so we endow it with the fine A-module topology. Again by lemma E.16 we conclude that $A_\mathfrak{p}$ is a ST ring.

Example E.17. Let $(A, \mathfrak{m})$ be a DVR endowed with the $\mathfrak{m}$-adic topology. We show that the topology in $\mathrm{Frac}(\widehat{A})$ obtained as it follows is the usual complete discrete valuation topology.

$$A \overset{(C)}{\rightsquigarrow} \widehat{A} \overset{(L)}{\rightsquigarrow} \mathrm{Frac}(\widehat{A})$$

First of all note that the fine A-module topology on $A/\mathfrak{m}^r$ is the discrete topology, therefore with (C) we obtain the usual $\widehat{\mathfrak{m}}$-adic topology on $\widehat{A} = \varprojlim_r A/\mathfrak{m}^r$, and we denote the valuation on $\widehat{A}$ with the letter v.
The topology on $\mathrm{Frac}(\widehat{A})$ is the finest linear topology such that the maps

$$\begin{array}{rcl} \rho_b : \widehat{A} & \to & \mathrm{Frac}(\widehat{A}) \\ x & \mapsto & bx \end{array}$$

are continuous for any $b \in \mathrm{Frac}(\widehat{A})$. Assume that $\mathfrak{m} = tA$, then we first show that for any $r \in \mathbb{Z}$ the set $t^r\widehat{A} \subset \mathrm{Frac}(\widehat{A})$ is open. We need to check that $\rho_b^{-1}(t^r\widehat{A})$ is open in $\widehat{A}$ for any b. For $b = 0$ it is trivial, so let's assume $b \neq 0$:

$$\rho_b^{-1}(t^r\widehat{A}) = \left(t^{r-v(b)}\widehat{A}\right) \cap \widehat{A} = \begin{cases} \widehat{\mathfrak{m}}^{r-v(b)} & \text{if } r \geq v(b) \\ \widehat{A} & \text{if } r < v(b) \end{cases}$$

Thus we get a sequence of open sets in $\mathrm{Frac}(\widehat{A})$:

$$\ldots \supset t^{r-1}\widehat{A} \supset t^r\widehat{A} \supset t^{r+1}\widehat{A} \supset \ldots$$

which is a local basis at 0. In particular it is the same local basis of the topology induced by the valuation v.

E.2 Differential forms

We want to conclude the chapter by giving the essential notions regarding a modified version of differential forms on a ST K-algebra A (a comprehensive presentation can be found in [63, 1.5]). In this subsection we allow K to be any field endowed with a ST ring topology. The usual A-module of Kähler differentials $\Omega^1_{A|K}$ is not very useful because it might be "too big". The following example clarifies this point:

Example E.18. Let $A = K[t]$ be the ring of polynomials over K, then $\Omega^1_{A|K} = Adt$, but if we consider $B = K[[t]]$ then $\Omega^1_{B|K}$ is not a finitely generated B-module.

Therefore, we want to find a reasonable quotient of $\Omega^1_{A|K}$ with finite rank over S when $A = L((t_1))\dots((t_n))$, for a finite extension $L|K$. The topology of this new vector space that we are going to construct will be inherited from the topology of $\Omega^1_{A|K}$. So let's first explain how to topologize $\Omega^1_{A|K}$.

Definition E.19. Let M and N be two ST A-modules, then the topology on $M \otimes_A N$ is the finest linear topology such that the following maps are continuous for any $m \in M$ and $n \in N$:

$$\begin{aligned} \lambda_m : N &\to M \otimes_A N \\ x &\mapsto m \otimes x \end{aligned}$$

$$\begin{aligned} \rho_n : M &\to M \otimes_A N \\ x &\mapsto x \otimes n \end{aligned}$$

Let $T^*_K(A) := \bigoplus_{m\geq 0} T^m_K(A)$ where $T^m_K(A) := A \otimes_K A \otimes_K \dots \otimes_K A$ is the m-fold tensor product, and moreover as usual consider $\Omega^m_{A|K} = \bigwedge^m \Omega^1_{A|K}$. Let's consider the A-Algebra $\Omega^*_{A|K} = \bigoplus_{m\geq 0} \Omega^m_{A|K}$ and define the following homomorphism of A-algebras:

$$\begin{aligned} \Phi : A \otimes_A T^*_K(A) &\to \Omega^*_{A|K} \\ x_0 \otimes x_1 \otimes \dots \otimes x_m &\mapsto x_0 dx_1 \wedge \dots \wedge dx_m \end{aligned}$$

It is surjective, therefore if we endow $A \otimes_A T^*_K(A)$ with the (iterated) topology constructed by definition E.19, then we can put on $\Omega^*_{A|K}$ the topology induced by the quotient topology on $\frac{A\otimes_A T^*_K(A)}{\ker \Phi}$. Each $\Omega^m_{A|K}$ has the subspace topology.

Proposition E.20. *$\Omega^*_{A|K}$ is a ST A-algebra and each $\Omega^m_{A|K}$ is a ST A-module. Moreover, the topology on $\Omega^1_{A|K}$ is the finest topology such that the following homomorphisms are continuous for any $a \in A$:*

$$\begin{aligned} \lambda_a \circ d : A &\to \Omega^1_{A|K} \\ x &\mapsto adx\,, \end{aligned}$$

$$\begin{aligned} \rho_{da} : A &\to \Omega^1_{A|K} \\ x &\mapsto fdx\,. \end{aligned}$$

Proof. See [63, Lemma 1.5.2]. □

At this point we are ready for the "correct" definition of differential forms over A:

Definition E.21. The *module of separated differential forms of A over K* is a $T2$ semi-topological A-module $\Omega^{1,\mathrm{sep}}_{A|K}$ endowed with a continuous K-derivation $d : A \to \Omega^{1,\mathrm{sep}}_{A|K}$ satisfying the following universal property: for any $T2$ semi-topological A-module M and continuous K-derivation $d' : A \to M$ the map obtained by composition with d

$$\mathrm{Hom}^{\mathrm{cont}}_{A}(\Omega^{1,\mathrm{sep}}_{A|K}, M) \to \mathrm{Der}^{\mathrm{cont}}_{K}(A, M)$$

is bijective.

We have an explicit expression for the module of differential forms:

Proposition E.22. *The A-module $(\Omega^1_{A|K})^{\mathrm{sep}}$ endowed with the natural K-derivation $d : A \to \Omega^1_{A|K} \to (\Omega^1_{A|K})^{\mathrm{sep}}$ satisfies the universal property described in definition E.21.*

Proof. Omitted. □

So, from now on we can use the identification:

$$\Omega^{1,\mathrm{sep}}_{A|K} = (\Omega^1_{A|K})^{\mathrm{sep}}$$

and clearly we have the obvious notations:

$$\Omega^{m,\mathrm{sep}}_{A|K} := \bigwedge^{m} \Omega^{1,\mathrm{sep}}_{A|K} ,$$

$$\Omega^{*,\mathrm{sep}}_{A|K} := \bigoplus_{m\geq 0} \Omega^{m,\mathrm{sep}}_{A|K} .$$

Appendix F

Locally linearly compact vector spaces

In this section we fix a base field k which is endowed with the discrete topology. A topological k-vector space V is an k-vector space endowed with a T2 topology such that the vector spaces operations are continuous as functions of two variables. A topological vector space is obviously a topological group with respect to the addition.

Definition F.1. A topological vector space (V, τ) is *linearly topologized* (or has a *linear topology*) if there is a local basis at 0 made of linear subspaces.

Clearly a linearly topologized vector space is also a linearly topologized abelian group with respect to the addition. All open subspaces of a linearly topologized vector space V are also closed and moreover if V is finite dimensional, then it must be discrete.

Definition F.2. Let V be a linearly topologized k-vector space; then it is said *linearly compact* if one of the following equivalent conditions holds:

(1) For any filter of the form:

$$\{H_i = W + v \colon v \in V \text{ and } W \subseteq V \text{ is a closed vector subspace}\}_i$$

we have $\bigcap_i H_i \neq \emptyset$.

(2) V is complete and all its open vector subspaces have finite codimension.

(3) $V = \varprojlim_i V_i$ where $\{V_i\}_i$ is an inverse system of finite dimensional k-vector spaces all endowed with the discrete topology.

Remark F.3. A proof of the equivalence of the conditions in definition F.2 can be found in [5, Proposition 24.4].

Linear compactness is a weaker condition that usual compactness, but some important properties induced by compactness have the corresponding version for linear compactness:

Proposition F.4. *The following statements hold:*

1. *A closed linear subspace of a linearly compact vector space is again linearly compact.*
2. *If H is a closed linear subspace of a linearly compact vector space V then V/H is linearly compact.*
3. *A discrete linearly compact vector space is finite dimensional.*
4. *A finite dimensional vector space is linearly compact.*
5. *The arbitrary product of linearly compact vector spaces is again linearly compact.*
6. *The class of linearly compact vector spaces is closed under taking inverse limits.*

Proof. See [33, 10.9]. □

We also have the "local" version of linear compactness:

Definition F.5. A linearly topologized k-vector space V is called *locally linearly compact* if it contains an open linearly compact subspace.

Example F.6. Let $k \subset L$ be a finite extension and let $r \in \mathbb{Z}$, then the k-vector space

$$t^r L[[t]] = \varprojlim_m \frac{t^r L[[t]]}{t^{r+m} L[[t]]}$$

is linearly compact thanks to proposition F.2.(4). In other words if we topologize $L((t))$ with the ind/pro-topology we can conclude that it is locally linearly compact.

Definition F.7. Let V be a linearly topologized k-vector space, then we define its dual as:

$$V^\vee := \mathrm{Hom}_k^{\mathrm{cont}}(V, k)\,.$$

an element of $V^\vee$ is called *k-character*. We endow $V^\vee$ with the linearly compact-open topology, which is a linear topology.

For any subset $S \subset V$, let's put:

$$S^{\triangle} := \{\xi \in V^{\vee} \colon \xi(S) = 0\} ,$$

$$S^{\triangle\triangle} := \{v \in V \colon \xi(v) = 0,\ \forall \xi \in S^{\triangle}\} .$$

Clearly $S \subseteq S^{\triangle\triangle}$ and the moreover family

$$\{U^{\triangle} \colon U \subseteq V \text{ is open and linearly compact subspace}\}$$

forms a fundamental system of neighborhoods at 0 for the topology on $V^{\vee}$.

Definition F.8. Let V be a linearly topologized k-vector space. If V is algebraically and topologically isomorphic to $V^{\vee}$, then we say that V *is self dual.* If V is a ST k-algebra and $\xi \in V^{\vee}$ is a nontrivial k-character, then for any $a \in V$ the map

$$\begin{array}{rcl} \xi_a : V & \to & k \\ x & \mapsto & \xi(ax) \end{array}$$

is a k-character too because the product for a fixed element is continuous in V. If the map

$$\begin{array}{rcl} \theta_{\xi} \colon V & \to & V^{\vee} \\ a & \mapsto & \xi_a \end{array}$$

induces a self duality, we say that ξ is a *standard k-character.*

Proposition F.9. *If V is discrete then $V^{\vee}$ is linearly compact.*

Proof. See [61, Theorem 25.14]. □

The Pontryagin duality for locally compact groups is replaced by the Lefschetz duality in the theory of locally linearly compact vector spaces:

Theorem F.10 (Lefschetz duality)**.** *Let V be a locally linearly compact k-vector space, then the following statements hold:*

(1) *If V is finite dimensional then V is self dual.*

(2) *$V^{\vee}$ is locally linearly compact.*

(3) *There is an algebraic and topological isomorphism $V \cong V^{\vee\vee}$.*

(4) *$S^{\triangle}$ is closed in $V^{\vee}$ for any subset $S \subseteq V$.*

(5) *If $W \subset V$ is a linear subspace, then $W^{\triangle\triangle} = \overline{W}$.*

(6) *If* W *is a closed linear subspace of* V*, then:*

  (6*a*) *There is an algebraic and topological isomorphism* $(V/W)^\vee \cong W^\vartriangle$*.*

  (6*b*) *There is an algebraic and topological isomorphism* $V^\vee/W^\vartriangle \cong W^\vee$*.*

Proof. For 1,2,3,6 See [61, 25].
(4) Consider the continuous pairing:

$$\begin{array}{rcl} \langle\,,\,\rangle : V^\vee \times V & \to & k \\ (\xi, s) & \mapsto & \langle \xi, s\rangle := \xi(s)\,, \end{array}$$

then

$$S^\vartriangle = \bigcap_{s\in S}\{s\}^\vartriangle = \bigcap_{s\in S}\ker\langle\cdot, s\rangle\ ,$$

so $S^\vartriangle$ is closed.
(5) $W^{\vartriangle\vartriangle}$ is a closed subspace containing W. Let $C = \overline{W}$, then by using proposition F.4.(2) we know that V/C is locally linearly compact so by (3) we have

$$V/C \cong (V/C)^{\vee\vee}\,. \tag{F.1}$$

Let $a \in V \setminus C$, then by equation (F.1) there exists an element $\xi \in (V/C)^\vee$ such that $\xi(C + a) \neq 0$. If we lift ξ to a character $\widetilde{\xi} \in V^\vee$, we can conclude that $\widetilde{\xi}$ has the following properties:

- $\widetilde{\xi} \in C^\vartriangle \subset W^\vartriangle$.
- $\widetilde{\xi}(a) \neq 0$.

In other words $a \in V \setminus W^{\vartriangle\vartriangle}$. □

Now we assume that V is a locally linearly compact k-vector space and a ST k-algebra admitting a standard k-character ψ^0. The we have the continuous and symmetric pairing given by:

$$\begin{array}{rcl} d\colon V \times V & \to & k \\ (v, w) & \mapsto & \psi^0(vw) = \psi^0_v(w) = \psi^0_w(v)\,. \end{array}$$

For any subset $S \subseteq V$ we put:

$$S^\perp := \{v \in V \colon d(v, S) = 0\}\ ,$$

so by the self duality of V induced by ψ_0, for any subspace $W \subset V$ we have the isomorphisms $W^\perp \cong W^\vartriangle$ and $W^{\perp\perp} \cong W^{\vartriangle\vartriangle}$.

Proposition F.11. *Let V be a locally linearly compact k-vector space and a ST k-algebra admitting a standard k-character; moreover let $W, W_1, W_2 \subseteq V$ be closed subspaces. The following statements hold:*

(1) $W^{\perp\perp} = W$. *More in general, if W is any linear subspace,* $W^{\perp\perp} = \overline{W}$.

(2) $(V/W)^{\vee} \cong W^{\perp}$.

(3) $V/W^{\perp} \cong W^{\vee}$.

(4) $W_1^{\perp} \cap W_2^{\perp} = (W_1 + W_2)^{\perp}$.

(5) $\overline{(W_1^{\perp} + W_2^{\perp})} = (W_1 \cap W_2)^{\perp}$.

Proof. (1),(2) and (3) follow immediately from theorem F.10.
(4) If $v \in W_1^{\perp} \cap W_2^{\perp}$, then $d(v, W_1 + W_2) = d(v, W_1) + d(v, W_2) = 0$, so one inclusion is proved. Vice versa assume that $v \in (W_1 + W_2)^{\perp}$, then $d(v, W_i) = 0$ for $i = 1, 2$, so we have also the other inclusion.
(5)

$$\overline{(W_1^{\perp} + W_2^{\perp})} =^{(1)} (W_1^{\perp} + W_2^{\perp})^{\perp\perp} =^{(5)} (W_1 \cap W_2)^{\perp}.$$

□

In the category of locally compact group we have a well defined notion of restricted product. We introduce now the restricted product in the category of locally linearly compact vector spaces.

Definition F.12. Let I be a set of indexes and let $I_\infty \subset I$ a finite subset (possibly empty). Fix a family of locally linearly compact k-vector spaces $\{V_i\}_{i \in I}$ and for any $i \in I \setminus I_\infty$ let $H_i \subseteq V_i$ be an open linearly compact subspace. The restricted product of the V_i with respect to the H_i is defined as:

$$\prod_{i \in I}{}' V_i := \left\{ (v_i)_i \in \prod_{i \in I} V_i \colon v_i \in H_i \text{ for all but finitely many } i \in I \setminus I_\infty \right\}.$$

The topology on $\prod'_{i \in I} V_i$ is the usual restricted product topology and by F.4 we can conclude that it is again a linearly locally compact k-vector space.

Let V be a locally linearly compact vector space and let H be a compact open subspace; the quotient V/H is discrete, so $(V/H)^{\vee} \cong H^{\vartriangle}$ is linearly compact in V by proposition F.9. Hence If $\{V_i\}_{i \in I}$ and $\{H_i\}_{i \in I \setminus I_\infty}$ is the given data for a restricted product, it makes sense to consider the restricted product:

$$\prod_{i\in I}{}' V_i^\vee$$

with respect to the subspaces $H_i^\vartriangle$ (remember that $V^\vee$ is again locally linearly compact thanks to F.10.(2)). Like in the case of locally compact groups we have a very naive description for the dual of a restricted product:

Theorem F.13. *Let* $\prod'_{i\in I} V_i$ *be a restricted product of linearly locally compact* k*-vector spaces with respect to a family of open compact subspaces* $\{H_i\}_i$*. Consider the restricted product* $\prod'_{i\in I} V_i^\vee$ *with respect to* $\{H_i^\vartriangle\}$*, then the map:*

$$\begin{aligned} \prod_{i\in I}{}' V_i^\vee &\to \left(\prod_{i\in I}{}' V_i\right)^\vee \\ (\psi_i) &\mapsto \sum_i \psi_i \end{aligned}$$

is an isomorphism of topological vector spaces.

Proof. Just modify the proof of [52, Theorem 5-4] which is made in the category of locally compact abelian groups. □

Appendix G

Selected topics

G.1 “Ad hoc” K-theory

Algebraic K-theory is a very wide subject with a long history. It can be approached in many different ways and several links can be build between all approaches (see for example [56]). This appendix is not a short introduction to algebraic K-theory, but just a mere collection of definition and notations needed in this text.

Definition G.1. Let G be an abelian group, and fix an integer ≥ 1. A *r-Steinberg map* is an homomorphism of $\mathbb{Z}$-modules $f : (F^\times)^{\oplus_r} \to G$ such that $f(a_1, \ldots, a_r) = 0$ whenever there exist two indexes i, j such that $i \neq j$ and $a_i + a_j = 1$.

Let's denote with $\mathbf{St}(r)$ the category whose objects are the r-Steinberg maps $f : (F^\times)^{\oplus_r} \to G$ and the morphisms are the commutative diagrams:

$$\begin{tikzcd}
(F^\times)^{\oplus_r} \arrow[r, "f"] \arrow[d, "g"'] & G \\
H \arrow[ur, "\phi"'] &
\end{tikzcd}$$

where ϕ is a group homomorphism.

Proposition G.2. *The category* $\mathbf{St}(r)$ *has the initial object.*

Proof. We construct the initial objects by hands. Let's define

$$K_r(F) := \underbrace{F^\times \otimes_{\mathbb{Z}} \ldots \otimes_{\mathbb{Z}} F^\times}_{r \text{ times}} \,/\, S$$

where S is the (multiplicative) subgroup generated by the following set:

$$\{a_1 \otimes \ldots \otimes a_r \colon a_i + a_j = 1 \text{ for some } i \neq j\}\,.$$

The natural image of a pure tensor $a_1 \otimes \ldots \otimes a_r$ in $K_r(F)$ is denoted by $\{a_1, \ldots, a_r\}$. Clearly we have an induced map:

$$\begin{array}{rcl} \{\ \} : \ (F^\times)^{\oplus r} & \to & K_r(F) \\ (a_1, \ldots, a_r) & \mapsto & \{a_1, \ldots, a_r\} \end{array}$$

At this point it is straightforward to see that $\{\ \} : (F^\times)^{\oplus r} \to K_r(F)$ is the initial object for $\mathbf{St}(r)$. □

Definition G.3. For $r = 0$ we put $K_0(F) := \mathbb{Z}$ and in general we call the group $K_r(F)$ constructed in proposition G.2 *the r-th K-group of F*. Note that $K_1(F) = F^\times$. The map $\{\ \} : (F^\times)^{\oplus r} \to K_r(F)$ is called the r-th symbol map and in the cases $r = 0, 1$ it is just the identity.

Remark G.4. The groups introduced in definitons G.3 are usually called Milnor K-groups and the standard notation is K_r^M. However in this text we can simplify the notation.

The construction $K_r(\ )$ is functorial, in fact let $f : F^\times \to L^\times$ be a group homomorphism, then the composition:

$$(F^\times)^{\oplus r} \xrightarrow{f^{\oplus r}} (L^\times)^{\oplus r} \xrightarrow{\{,\}} K_r(L)$$

is evidently a Steinberg map. By the universal property it induces a morphism $K_r(f) : K_r(F) \to K_r(L)$.

When F is a complete discrete valuation field there exists a nice relationship between K-groups of F and K-groups of the residue fields:

Theorem G.5. *Let F be a discrete valuation field (not necessarily complete) then there is a unique group homomorphism:*

$$\partial_r : K_r(F) \to K_{r-1}(\overline{F})$$

satisfying the following property:

$$\partial_r(\{x_1, \ldots, x_{r-1}, \varpi\}) = \{\overline{x_1}, \ldots, \overline{x_{r-1}}\}$$

for any local parameter ϖ of F and any $x_1, \ldots, x_{r-1} \in \mathcal{O}_F^\times$.

Proof. See [39]. □

Definition G.6. The map ∂_r described in theorem G.5 is called *the (Milnor) r-th boundary map.*

Consider the *tame symbol* for a complete discrete valuation field (F, v):

$$\begin{aligned} (\ ,\ )_F : F^\times \times F^\times &\to \overline{F}^\times \\ (a, b) \mapsto (a, b)_F &= (-1)^{v(a)v(b)}\, \overline{a^{v(b)} b^{-v(a)}}\,. \end{aligned} \tag{G.1}$$

Proposition G.7. *The tame symbol* $(\ ,\ )_F$ *is a continuous* 2*-Steinberg map.*

Proof. It is very easy to verify that $(\ ,\ )_F$ is $\mathbb{Z}$-bilinear, so we just check that $(a, 1-a)_F = 1$. There are 3 cases:

(i) $v(a) > 0$. It means that $a \in \mathfrak{p}_F$, so $1-a \in \mathcal{O}_F^\times$ by standard commutative algebra. It follows that $(a, 1-a)_F = \overline{(1-a)^{-v(a)}}$, but:

$$(1-a)^{v(a)} = \sum_{j=0}^{v(a)} \binom{v(a)}{j} (-a)^j = 1 + \sum_{j=1}^{v(a)} \binom{v(a)}{j} (-a)^j \in 1 + \mathfrak{p}_F$$

and we can conclude that $(a, 1-a)_F = 1$.

(ii) $v(a) < 0$. Then in this case $v(1-a) = \min\{0, v(a)\} = v(a)$ and we write:

$$(a, 1-a)_F = (-1)^{v(a)^2}\, \overline{a^{v(a)}(1-a)^{-v(a)}} = \overline{(a^{-1}+1)^{-v(a)}}\,.$$

But $a^{-1} \in \mathfrak{p}_F$, so $a^{-1}+1 \in \mathfrak{p}_F$ and the claim is proved.

(iii) $v(a) = 0$. We have that $1-a \in \mathcal{O}_F$, but if $v(1-a) = 0$ the claim is obvious. When $v(1-a) > 0$, we have $(a, 1-a)_F = \overline{a^{v(1-a)}}$ and we can write $a = 1-(1-a)$. At this point just repeat the proof of case (i). □

The tame symbol has very important geometric properties. When X is nonisngular projective curve over a perfect field k, for any $x \in X$ we put:

$$(\ ,\ )_x := N_{k(x)|k} \circ (\ ,\ )_{K_x} : K_x^\times \times K_x^\times \to k^\times\,.$$

Theorem G.8 (1-dimensional reciprocity law)**.** *Let $f, g \in k(X)^\times$ two nonzero rational functions, then*

$$\prod_{x \in X} (f, g)_x = 1\,.$$

Proof. Se [54, III, §1.4] . □

Corollary G.9 (Weil reciprocity law)**.** *Let $f, g \in k(X)^\times$ two nonzero rational functions such that the principal divisors (f) and (g) have no common components, then:*

$$\prod_{x\in X} N_{k(x)|k}\left(f^{v_x(g)}(x)\right) = \prod_{x\in X} N_{k(x)|k}\left(g^{f_x(g)}(x)\right).$$

(Note that by hypothesis about (f) and (g) if $v_x(g) \neq 0$, then $v_x(f) = 0$, therefore $v_x\left(f^{v_x(g)}\right) = 0$ and $f^{v_x(g)}(x)$ is actually a well defined value of $k(x)^\times$. The same fact of course holds by exchanging the roles of f and g)

Proof. We have the two finite sums $(f) = \sum_i v_{x_i}(f)[x_i]$ with $v_{x_i}(f) \neq 0$ and $(g) = \sum_i v_{y_i}(g)[y_i]$ with $v_{x_i}(g) \neq 0$. Since (f) and (g) have no common components, then $x_i \neq y_j$ for any i, j. Moreover by theorem G.8:

$$1 = \prod_{x\in X}(f,g)_x =$$

$$= \left(\prod_{y_i} N_{k(y_i)|k}\left(f^{v_{y_i}(g)}(y_i)\right)\right)\left(\prod_{x_i} N_{k(x_i)|k}\left(g^{-v_{x_i}(f)}(x_i)\right)\right).$$

This concludes the proof. □

Weil reciprocity law can be expressed in a more compact and elegant way if we introduce some additional notations. Let's put

$$N_x(f) := N_{k(x)|k}\left(f(x)\right). \tag{G.2}$$

So If $D = \sum_{x\in X} v_x(D)[x] \in \mathrm{Div}(X)$ and $f \in k(X)^\times$ is a nonzero rational function such that (f) and D have no common components, then it is well defined the following element:

$$N_D(f) := \prod_{x\in X} N_x(f)^{v_x(D)} \quad \in k^\times$$

At this point Weil reciprocity law can be rewritten simply as the equation:

$$N_{(g)}(f) = N_{(f)}(g).$$

Coming back to general K-theory, we have a nice description of the boundary map ∂_2 in relation to the tame symbol. By the universal property of $K_2(F)$, the tame symbol $(\ ,\ )_F$ induces a unique map $\Psi : K_2(F) \to \overline{F}^\times = K_1(F)$ such that $\Psi(\{\ ,\ \}) = (\ ,\ )_F$. Let $a \in \mathcal{O}_F^\times$ and let ϖ be a local parameter for F, then $\Psi(\{a, \varpi\}) = (a, \varpi)_F = \overline{a}$; this actually means that $\partial_2 = \Psi$. In other words the 2-nd boundary map for a complete discrete valuation field is exactly the map induced naturally by the tame symbol.

For a discrete valuation field F (not necessarily complete) we have the multiplicative group $U_F^{(i)} := 1 + \mathfrak{p}_F^i$ for $i \geq 1$ and we have also the K-theoretic version of it:

$$U^i K_r(F) := \{\{a_1 \dots a_r\} \in K_r(F) \colon a_j \in U_F^{(i)} \ \forall j = 1, \dots, r\}$$

and we put:

$$\widehat{K}_r(F) := \varprojlim_i K_r(F)/U^i K_r(F) \,. \tag{G.3}$$

Clearly we have a natural homomorphism $K_r(F) \to \widehat{K}_r(F)$ and moreover if $\widehat{F}$ is the completion of F there is an isomorphism $\widehat{K}_r(F) \cong \widehat{K}_r(\widehat{F})$. Now put $L = \operatorname{Frac}(\mathcal{O}_F[[t]])$, for any prime ideal $\mathfrak{p}$ of height 1 in $\mathcal{O}_F[[t]]$ we have that $\mathcal{O}_F[[t]]_{\mathfrak{p}}$ is a discrete valuation ring and in particular $F\{\{t\}\}$ is the completion of L at $\mathfrak{p} = \mathfrak{p}_F \mathcal{O}_F[[t]]$. Consider the set:

$$\mathfrak{S} := \{\mathfrak{p} \in \operatorname{Spec}(\mathcal{O}_F[[t]]) \colon \operatorname{ht}\mathfrak{p} = 1, \mathfrak{p} \neq \mathfrak{p}_F \mathcal{O}_F[[t]]\} \,,$$

and for any $\mathfrak{p} \in \mathfrak{S}$ let's denote with $\partial_r^{(\mathfrak{p})} : K_r(L) \to K_{r-1}(k(\mathfrak{p}))$ the r-th boundary map relative to the evaluation defined by $\mathfrak{p}$.

Definition G.10. For $r \geq 1$, the r-th (Kato) residue map on L is given by the following composition:

$$\operatorname{res}_L^{(r)} : K_r(L) \xrightarrow{(\partial_r^{(\mathfrak{p})})_{\mathfrak{p} \in \mathfrak{S}}} \bigoplus_{\mathfrak{p} \in \mathfrak{S}} K_{r-1}(k(\mathfrak{p})) \xrightarrow{\sum_{\mathfrak{p} \in \mathfrak{S}} K_r(N_{k(\mathfrak{p})|F})} K_{r-1}(F)$$

Theorem G.11. *The r-th residue map satisfies:*

$$\operatorname{res}_L^{(r)} \left(U^{(i)} K_r(L)\right) \subseteq U^{(i)} K_{r-1}(F) \quad \forall i \geq 1$$

therefore it induces a homomorphism:

$$\operatorname{res}_{F\{\{t\}\}}^{(r)} : \widehat{K}_r(F\{\{t\}\}) \cong \widehat{K}_r(L) \to \widehat{K}_{r-1}(F) \,.$$

Proof. See [30]. □

In this text we will also need the notions of $K_0(A)$, $K_1(A)$ and $K_2(A)$ for a local ring A. The definition of these groups will resemble exactly the ones given for fields:

Definition G.12. If A is a local ring, the K-groups $K_0(A)$, $K_1(A)$ and $K_2(A)$ are defined in the same fashion of definition G.3, where we just substitute A in place of F. Note that also in this case K_0, K_1 and K_2 are functorial.

Remark G.13. The constructions of all K-groups for general rings is more complicated than the one described here, but we don't need the full theory.

In some particular cases we have a more explicit description of $K_2(A)$:

Proposition G.14. *Let F be a complete discrete valuation field, then $K_2(\mathcal{O}_F) = \ker(\partial_2)$, where $\partial_2 : K_2(F) \to K_1(\overline{F}) = \overline{F}^\times$ is the boundary map.*

Proof. See [48, Proposition 11]. □

Definition G.15. Let A be a local ring and let $\mathfrak{a} \subset A$ be an ideal. The canonical projection $p : A \to A/\mathfrak{a}$ induces a morphism $K_2(p) : K_2(A) \to K_2(A/\mathfrak{a})$. Then we put:

$$K_2(A, \mathfrak{a}) := \ker(K_2(p)) \,.$$

G.2 Tate's Fourier analysis

A very important result of real Fourier analysis is the Poisson summation formula which says that for any Schwartz function $f : \mathbb{R} \to \mathbb{C}$:

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n) \tag{G.4}$$

where $\hat{f}$ is the Fourier transform of f. J. Tate in [60] showed the equivalent of equation (G.4) respectively for the ring of adeles and ideles of a global field K. In particular, Tate's formulation of the Poisson summation formula for the ring of ideles is equivalent to the Riemann-Roch theorem when K is a function field. This gives beautiful bridge between the geometry of curves and arithmetic of global fields. In this appendix we briefly recall the general Fourier analysis on locally compact local fields and on the adelic ring.

Let K be a global field, so it is a number field or a finite extension of $\mathbb{F}_q(t)$. For any place v on K, the completion K_v is a locally compact local field and its dual $\widehat{K_v} = \mathrm{Hom}^{\mathrm{cont}}(K_v, S^1)$ is endowed with the usual compact-open topology. We can choose an additive character $\psi_v^0 \in \widehat{K_v}$ such that the map

$$\begin{array}{rcl} K_v & \to & \widehat{K_v} \\ a & \mapsto & \psi^0_{v,a} := (x \mapsto \psi_v^0(ax)) \end{array}$$

is an algebraic and topological isomorphism. The character ψ_v^0 is canonically chosen in the following way: by the classification theorem of locally compact local fields K_v is a finite extension of K_0, where $K_0 = \mathbb{Q}_p$, $K_0 = \mathbb{F}_p((t))$, or $K_0 = \mathbb{R}$.

- If $K_0 = \mathbb{R}$, put $\xi^0(x) := e^{-2\pi i x}$.
- If $K_0 = \mathbb{Q}_p$ and $x = \sum_{j \geq m} a_j p^j \in \mathbb{Q}_p$, then put
$$\xi^0(x) := e^{2\pi i \sum_{j=m}^{-1} a_j p^j} .$$
- If $K_0 = \mathbb{F}_p((t))$ and $x = \sum_{j \geq m} a_j t^j \in \mathbb{F}_p((t))$, then put
$$\xi^0(x) := e^{2\pi i \sum_{j=m}^{-1} a_j p^j} .$$

At this point it is enough to define $\psi_v^0 := \xi^0 \circ \mathrm{Tr}_{K_v|K_0}$. A Schwartz-Bruhat function on K_v is a function $f : K_v \to \mathbb{C}$ such that:

- If K_v is archimedean, then f is a usual Schwartz function.
- If K_v is nonarchimedean, then f is a locally constant function with compact support.

The $\mathbb{C}$-vector space of Schwartz-Bruhat functions on K_v is denoted by $\mathscr{S}(K_v)$. If μ_v is a Haar measure on K_v, for any $f \in \mathscr{S}(K_v)$ the Fourier transform is given by
$$\hat{f}(y) := \int_{K_v} f(x)\psi_v^0(xy)d\mu_v(x) \in \mathscr{S}(K_v) .$$
and $\hat{f} \in \mathscr{S}(K_v)$. The Fourier inversion formula and the isomorphism $K_v \cong \widehat{K_v}$ imply that
$$\hat{\hat{f}}(x) = rf(-x) \quad \text{for some } r \in \mathbb{R}_{>0} . \tag{G.5}$$
There exists a unique Haar measure on K_v such that $r = 1$ in equation (G.5), and this is called the self-dual Haar measure. It can be shown that the self-dual Haar measure μ_v is obtained as it follows:

- If $K_v = \mathbb{R}$, μ_v is the Lebesgue measure.
- If $K_v = \mathbb{C}$, μ_v is twice the Lebesgue measure.
- If K_v is nonarchimedean, μ_v is the Haar measure such that $\mu(\mathcal{O}_v) = \#(\mathcal{O}_v/\mathfrak{D})^{-\frac{1}{2}}$ where $\mathfrak{D}$ is the different of $K_v|K_0$.

From now on with μ_v we indicate the self-dual Haar measure on K_v.

Now it is time to globalize all this concepts. The adelic ring $\mathbf{A}_K$ is locally compact and we have a well defined character $\psi^0 := \prod_v \psi_v^0 \in \widehat{\mathbf{A}_K}$ which gives an algebraic and topological isomorphism like in the local case:

$$\begin{array}{rcl} \mathbf{A}_K & \to & \widehat{\mathbf{A}_K} \\ \alpha & \mapsto & \psi_\alpha^0 := (x \mapsto \psi^0(\alpha x)) \end{array}$$

The self-dual measures μ_v originate a well defined self-dual Haar measure $\mu := \prod_v \mu_v$ on $\mathbf{A}_K$, and moreover the Schwartz-Bruhat vector space $\mathscr{S}(\mathbf{A}_K)$ is defined as the $\mathbb{C}$-vector spaces generated by the functions of the type:

$$\prod_v f_v : \mathbf{A}_K \to \mathbb{C}$$

such that $f_v \in \mathscr{S}(K_v)$ and $f_v = 1_{\mathcal{O}_v}$ (i.e. the characteristic function of $\mathcal{O}_v$) for all but finitely many places v. Thus, for any $f \in \mathscr{S}(\mathbf{A}_K)$, the *adelic Fourier* transform is defined by:

$$\hat{f}(\alpha) := \int_{\mathbf{A}_K} f(x)\psi^0(\alpha x) d\mu(x)$$

and $\hat{f} \in \mathscr{S}(\mathbf{A}_K)$. For any $f \in \mathscr{S}(\mathbf{A}_K)$, consider

$$\widetilde{f}(x) := \sum_{k \in K} f(x+k) \quad \forall x \in \mathbf{A}_K \,,$$

clearly we have that $\widetilde{f}(x+K) = \widetilde{f}(x)$.

Lemma G.16. *For any $f \in \mathscr{S}(\mathbf{A}_K)$, the functions $\widetilde{f}$ and $\widetilde{\hat{f}}$ are both normally convergent in $\mathbf{A}_K$.*

Proof. See [52, lemma 7-6]. □

Theorem G.17. *(Adelic Poisson summation formula) Let $f \in \mathscr{S}(\mathbf{A}_K)$, then $\widetilde{f} = \widetilde{\hat{f}}$; that is:*

$$\sum_{k \in K} f(x+k) = \sum_{k \in K} \hat{f}(x+k)$$

Proof. See [52, lemma 7-7]. □

The idelic equivalent of the Poisson summation formula is an easy consequence of theorem G.17, and it is also referred as Tate's Riemann-Roch theorem. Remember that for any $x = (x_v)_v \in \mathbf{A}_K$ it is well defined the adelic "absolute value" $|x| = \prod_v |x_v|_v \in \mathbb{R}$.

Theorem G.18 (Tate's Riemann-Roch theorem). *Let $f \in \mathscr{S}(\mathbf{A}_K)$, then for any $x \in \mathbf{A}_K^\times$:*

$$\sum_{k\in K} f(kx) = \frac{1}{|x|}\sum_{k\in K} \hat{f}(kx^{-1})$$

Proof. Define the function $h(y) = f(yx) \in \mathscr{S}(\mathbf{A}_K)$, then

$$\hat{h}(k) = \int_{\mathbf{A}_K} h(y)\psi^0(yk)d\mu(y) = \int_{\mathbf{A}_K} f(yx)\psi^0(yk)d\mu(y)\,. \tag{G.6}$$

Now consider the change of variables in $\mathbf{A}_K$ given by $\varphi : y \mapsto x^{-1}y$, then the pushforward measure is given by $\mu_*(E) := \mu(xE)$ for any measurable set $E \subseteq \mathbf{A}_K$, but it is not difficult to show that $\mu(xE) = |x|\mu(E)$.Thus equation (G.6) becomes:

$$\hat{h}(k) = \frac{1}{|x|}\int_{\mathbf{A}_K} f(y)\psi^0(kx^{-1}y)d\mu(y) = \frac{1}{|x|}\hat{f}(kx^{-1})\,.$$

Finally, theorem G.17 says that:

$$\sum_{k\in K} h(k) = \sum_{k\in K} \hat{h}(k)\,,$$

so the proof is complete. □

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