An investigation into how low achieving secondary students learn fractions through visual representations

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Abstract

The gap between low and high achievers is a worldwide concern in Education, especially when it comes to mathematics. One way of facing this issue is by investigating the learning processes of those disadvantaged students at a classroom level. Bearing this in mind, I started my research by observing lessons for low achieving students in an underperforming school in England. After getting acquainted with the context, I designed lesson plans to teach fraction addition and subtraction following three design principles: lessons should enable students to build their knowledge about fractions on visual representations, students should have opportunities to solve tasks without being told how to do it beforehand and lesson plans should maintain some coherence with participant teachers' current practices. The first principle is the most relevant for my findings, and its choice was based on the growing evidence pointing out the relevance of visual representations for mathematical learning and as a potential pathway to overcome some difficulties faced by low achieving students. Three teachers enacted the lesson plans with a different low achieving group each. Data was collected of the pupils' working out, as registered in the worksheets, and also in the form of audio recordings, taken during the lessons, of my interactions with students about their thinking while solving the tasks. The data analysis revealed aspects of students' learning through visual representations that were grouped into two major findings. Firstly, the lessons were successful in promoting reasoning anchored in visual representations, and enabled students to extend their knowledge beyond what was explicitly taught to them. Secondly, an apparent lack of visual skills and prior knowledge on multiplication restricted their engagement with some tasks. The final discussion focuses on the role of visual representations in the learning of mathematics in general, but mainly for low achieving students, and how this can be implemented in classrooms.

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1 INTRODUCTION

1.1 The researcher

My involvement with mathematics education started when I left computer science and started my degree in mathematics teaching in Brazil. During this degree course, I had the opportunity to get involved in research, and straight after that I started my Master in Mathematics Education degree in a Brazilian university.

Following that, I became involved in a huge educational resource development project that lasted for three years. During this period, I was teaching mathematics only as a secondary activity.

As this project faded away, I focused more on my career as a teacher. That is when I taught mathematics at Secondary and Higher Education and in courses related to mathematics, education and computer science. However, I never had teaching as my only profession, and during this period I worked on some initiatives related to professional development for teachers at city and state level in Brazil.

Considering the six years since the end of my Master's degree and the beginning of my PhD studies, I experienced a very diverse range of activities related to teaching mathematics, even though some of them were brief. At this point I felt that a PhD degree could not only add a lot to my skills and knowledge, but also enable me to reach higher positions when involved in such activities.

1.2 Initial motivation for my research interest

The motivation for this research proposal came from my experience as a teacher and teacher educator in Brazil, combined with the results obtained by Brazilian students in international and national assessment programs.

The last report from PISA (OECD, 2012b) showed that the situation in Brazilian education, especially in mathematics, is alarming to say the least. The report points out that around 67% of 15 year old students performed below level 2, which is the baseline for students at this age[1]. In other words, these students "at best [...] can

1 OECD countries' mean is 23% and general mean is 32%

extract relevant information from a single source and can use basic algorithms, formulae, procedures or conventions to solve problems involving whole numbers" (OECD, 2012b). This result is not isolated. In fact, the scenario is even worse if the numbers from SAEB (Basic Education Assessment System, implemented nationwide in Brazil) are considered. According to the most recent report, 83% of Year 9 students (around 15 years old) achieved lower than expected for their age (Todos pela Educação, 2012). I think it is important to stress that the severity of the Brazilian situation arises not from comparisons with other countries, as is commonly done by some European countries in relation to some East Asian countries, but from the perception that *most* Brazilian students are below baseline levels according to international and national parameters.

Additionally, PISA and SAEB highlight a positive correlation between socio-economic status and achievement, which means that students from families with low socio-economic status (SES) are in an even worse condition.

Since the 70s, when the movement towards the universalisation of Basic Education in Brazil started, Brazilian politicians have utilised mottos such as "school for everybody," or "the nation of Education," but in accordance with Penteado & Skovsmose (2009), I believe that "social inclusion is an attractive motto, but it could easily remain no more than a motto if it is not considered how inclusion can be worked on in practice" (p. 218).

From this point of view, classroom practices have to be central to any attempt to address the issue of low achievement in Brazil. Although the country is the home of renowned scholars such as Ubiratan D'Ambrósio and Paulo Freire, whose contributions to philosophical, cultural and social aspects of education are internationally recognized, my perception is that there is a lack of classroom-based research and interventions aimed at low-achieving students in Brazil.

This was the starting point to my research: **a broad interest in the nature of classroom practices that could be used with low-achieving students in mathematics in order to help them to succeed within the current educational system**.

1.3 The choice of context

On one hand, the interest described before strongly suggests that my research project should be carried out in Brazil, while on the other, my experience strongly recommends the opposite.

As highlighted by Valero and Vithal (1998), usually the reality of developing countries is that they are not so stable and predictable as European countries. This poses methodological challenges for researchers who intend to develop their research in developing countries, such as Brazil. While the authors point out the need to face these challenges, in order to develop new and more adequate ways of developing research in such contexts, due to limitations in terms of time and resources during PhD research, I decided to carry out my research in the UK.

This decision revealed itself to be very fruitful, due to a fact previously unknown to me: the adoption of ability setting for Mathematics in British secondary schools (Francis et al., 2016). The environment of a low set classroom in the UK is considerably close to the environment of a classroom in a Brazilian public school (meaning comprehensive) in several aspects: relatively low student engagement, low teacher expectations, a restricted curriculum, and a general feeling that the current approaches for teaching do not work with these students (Boaler & Wiliam, 2001).

Of course my findings cannot be directly transferred to the Brazilian reality, but I believe these similarities approximate the experience that I will face in the future back in Brazil.

1.4 Structure of the thesis

Instead of opening this document with a long literature review chapter about the main topics of my research, I decided to structure this document reflecting the temporal evolution of my research process. This decision was based on two factors. Firstly, I believe the story behind the thesis will be easier to understand if I tell it this way.

Secondly, my research is not based on any overarching theory of learning that could serve as a basis for everything that will be discussed in the final chapters (Analysis, Discussion and Conclusions); thus, I cannot think of a good reason to open this text with a big chapter about theory or a literature review. Instead, it seems more sensible to introduce literature when it is necessary to explain terminology, to justify the focus and relevance of my analysis, and to explain my decisions throughout the research process.

For this reason, the first theoretical section appears after I present my pre-field work stage (Chapter 2), based on which I was able to refine the research interest presented above. Then, after reporting the preliminary study (Chapter 3), some theoretical sections are necessary before stating the research question (Chapter 4).

Throughout Chapters 5 (Design of the lessons) and 6 (Data collection), literature is discussed, when necessary, to justify the methodological choices and clarify their characteristics. During Data Analysis (Chapter 7) some theoretical sections are necessary in order to clarify nomenclature and the focus of the analysis. Finally, during Chapter 8, literature is used to locate the contributions of the research within the wider mathematics education scientific community.

Finally, the lesson plans used during data collection are available at http://dx.doi.org/10.17639/nott.353 instead of as appendices of this document. This decision was made because, as the lesson plans are themselves text documents, they would not fit properly in here. Also, the navigation offered by the website referred before seems more adequate for such a long list of documents (more than 40).

1.5 Related publications

I would like to highlight that some sections of this document are very similar to texts that I have published during the course of my PhD studies.

"Possible parallels between visual representations and informal knowledge" (Barichello, 2015), published in the informal proceedings of the Day Conference of the British Society for Research into the Learning of Mathematics, was based on the data collected during the preliminary study and reported on in Chapter 3.

"Implications of Giaquinto's epistemology of visual thinking for teaching and learning of fractions" (Barichello, 2017), also published in the informal proceedings of the Day Conference of the British Society for Research into the Learning of Mathematics, was inspired by the data collected during the main study, and the result of my theoretical engagement with Giaquinto's (2007) ideas.

Finally, an early version of my findings, together with an overview of the lesson plans, were presented in the oral session "Learning fractions through visual representations: a PhD research with low-achieving secondary students" during the 9th British Congress of Mathematics Education in 2018.

1.6 Regarding "low achieving" and "ability grouping"

The fact that I am using the expression "low achieving" to refer to a certain group of students does not reflect any sort of agreement with the methods used to determine levels of achievement in the United Kingdom.

My decision to use this term is based on the fact that those methods exist, are known by the students, and adopted throughout the system, by policy-makers to teachers, and to students. Since primary school, students are evaluated, and in general they know that they are being, or will be evaluated, and the outcome of such evaluations.

The same can be said about the use of the term "ability grouping". By using it, I am just referring to the practice widely adopted in this country of grouping students according to their score in an exam, and not subscribing to any principle underlying its adoption, such as a fixed view of mathematical ability or the identification of ability with a score in an exam.

2 The pre-fieldwork stage

In the beginning of the 2013-2014 Academic Year, Dr. Peter Gates was looking for secondary schools to develop a long term research project about supporting low-achieving and disadvantaged learners of mathematics. The Head of the Mathematics Department at Purple Valley (pseudonym) showed interest in taking part in the project. Besides him, another two mathematics teachers at that school accepted the invitation, and Dr. Peter Gates started observing lessons taught by them. In October 2014, I and another PhD student, Rita Santos Guimarães, joined him in observing the lessons.

This initiative can be thought of as an overarching research project encompassing both mine and Guimarães' PhD research projects. Although focusing on different aspects, the three of us shared more than just context, but also interests, such as teaching and learning of disadvantaged students. Thus, we could constantly discuss impressions, plans for action and pieces of data.

This stage had two main aims. Firstly, it allowed me to get acquainted with the new context of a British school. Secondly, it enabled me to refine my research interest into "a set of questions to which an answer could be given" (Hammersley & Atkinson, 2007, p. 24). Not only did the experience itself contribute to these aims, but also the opportunity to interact with other members of the research team.

The next sections are important, not only as a report of the pre-fieldwork stage, but also as an introduction to the context of the actual data collection, since it took place at the same school.

2.1 The context

2.1.1 School

Purple Valley is located in the East Midlands, in an area described by the staff of the school as being a typical white working-class neighbourhood. According to official statistics, the social classification of the residents in this area encompasses managerial, technical and skilled occupations, both non-manual and manual.

When I started visiting the school, in the 2014-2015 academic year, it was "under special measures" by Ofsted (indeed most of the secondary schools in the city were similarly graded on the basis of the percentage of pupils gaining 5 or more A*-C grades at GCSE). In the following academic year the school was taken over by an Academy Trust.

The majority of the students are of white British origin and around one fifth are from minority ethnic groups. Less than 4% of the students speak English as a second language. The percentage of students in receipt of free school meals (around 45%) is significantly higher than the national average (28%). The Head of the Mathematics Department perceives the school to be ethnically homogeneous in comparison to other schools in which he had worked before. In terms of attainment, he stated that the school receives students slightly below the national average, and just before leaving the school, they also perform slightly below national average at GCSE.

The school is considered to be average in size, attending to about 1000 students, which means 5 or 6 groups in each Year. The Mathematics Department is composed of 6 teachers, with one being part-time. Each teacher has her/his own classroom, where she/he teaches all the lessons. This room also serves as a personal office, where teachers spend time preparing lessons, marking exams and having occasional meetings.

In terms of infrastructure, the school is very well served. The buildings are quite new and properly designed for their purpose. Every room in the building where the mathematics lessons took place is equipped with a computer for the teacher, an electronic white board, a regular board and a desk visualizer. Although not often utilized, the school also has sets of tablets and notebooks to be used during lessons. Teachers never complain about lack of equipment or basic materials.

2.1.2 Teachers

During this stage, I observed lessons from 3 different teachers: David, Julia and Oscar (pseudonyms).

David is the Head of the Mathematics Department. He is an experienced teacher and has taught in different schools before moving to Purple Valley four years ago. He is respected by the other teachers, and has no trouble with discipline during his lessons, even though he is not strict about behaviour. The students see him as relaxed and approachable. Based on what I have observed during the time I visited the school, he is the teacher that interacts the most with students around issues not related to mathematics and school.

Julia has taught in Purple Valley her whole career. She has 11 years of experience, and says openly that she identifies herself with Purple Valley's students and enjoys teaching there. She has never assumed any managerial position and shows no interest in doing so. Julia is more strict in terms of behaviour than David and her lessons are rarely disrupted. According to her, she is consciously more strict with Years 7 and 8, and tries to adopt a more relaxed attitude with the older students. She is considered successful by the other teachers and is recurrently asked to take over "critical" groups, such as challenging groups in terms of behaviour or middle achieving Year 11 groups that are perceived to be achieving below their potential.

Oscar is an experienced teacher who recently moved to Purple Valley. He has worked in several different schools during his career and seems to be not as integrated with the other teachers in Purple Valley. He is very eloquent during meetings and informal talks. Short disruptions for behaviour management are common during his lessons, and he seems more distant from the students in issues not related to school than Julia and David.

Most of the lessons observed during this stage were for Year 7, 8 and 9 following the recommendation of Dr. Peter Gates not to insist on observing lessons for Year 10 and 11, because as they are closer to the GCSE exams, both teachers and students are under more pressure and the school could be less inclined to accept interventions with these years.

2.1.3 Ability setting at Purple Valley

At Purple Valley students are placed in sets (each Year group is divided into 5 or 6 sets, depending on the number of students) according to prior attainment in Mathematics from Year 7 to Year 11, and in English in Years 10 and 11. As my research interest involves low achieving students, I progressively concentrated my observations into the bottom sets.

In general, lower sets are small in terms of numbers of students. For instance, David's Year 8 was composed only of 12 students, while the top sets usually reach the maximum that the classroom can hold: about 32 students.

Lower sets also follow a restricted version of the curriculum, in alignment with the curriculum expected by their final (external) evaluation by the end of Year 11.

It is also common for teachers during lessons for low set groups to have a teacher assistant available. In general, she/he will focus on students with specific needs, such as autism, behavioural problems or extremely low results in exams. This might happen just by sitting together with that particular student (or small group of students) and helping them with regular activities throughout the lesson, or by taking them out of the classroom to engage in other activities developed specifically for them.

Students are aware of the set in which they have been placed. From one academic year to the next, it is common to have students being moved up or down to other sets. Although I did not have access to official numbers, I would say that this happens to one or two students on each extreme of the achievement spectrum per year group.

Finally, based on the time I spent at the school, there was no preference in terms of teacher allocation to high or low sets. The only situation where this happened was for Year 11: the teacher perceived as most effective would be allocated to the middle sets, which were decisive for the school to reach a higher proportion of students achieving 5 or more A*-C grades on GCSE.

2.2 Lesson observations

During my visits (usually once a week), I observed one or two lessons by one of the three teachers. In total, about 25 lessons were observed from November 2014 to July 2015.

Also, I usually talked to the teachers between lessons, not only about issues observed during the lessons, but also about general topics regarding school practices, professional habits and their views on the English educational system. For all these observations I generated field notes during the lessons and complementary notes that were written after the visits.

At first, the complementary notes were "accounts of" the events observed, including descriptions of what was observed and my personal impressions. Progressively, these notes moved towards "accounts for", providing some tentative explanations for what was observed and reflections on the events, as suggested by Mason (2002). This movement was also propelled by my increasing familiarity with the practices in the school and with the British educational system in general, as well as by my reading during this period.

Along with this movement, the observations became more focused on specific aspects of the lessons, such as the nature of tasks posed by the teachers, the language used by teachers and students and the use of visual representations in general. These topics came from my personal interest, from issues that interested Dr. Peter Gates and from events observed in the school.

In terms of the balance between observer and participant, I take the view that there is no such thing as a fully neutral observer in this context (Wragg, 1999). Bearing this in mind, I was open to assume the role required by the teacher according to his/her judgement. For example, the teachers often asked for my assistance during specific moments in the lessons, to help students with doubts, check answers and distribute worksheets. Also, I regard this as a way to repay them for the fact that they were opening their lessons to my presence.

During this period, it seemed that the teachers felt comfortable about my presence in the classroom, to the point that they would ask me to not come to a particular lesson if they had any reason to believe that it could interfere with their plan. Students also felt more comfortable in asking me questions and helping to solve tasks. I can safely say that there were no incidents in this period to suggest that teachers or students were changing their behaviour due to my presence, or that there were any events that suggested I needed to take extra precautions in terms of ethical clearance.

By January, due to my observations and readings, two issues attracted my attention as potential points of focus for my research. They are the relationship between socio-economic status and achievement, and the role that visual representations could play in supporting low achieving students. Both issues will be discussed in the next sections.

2.3 Socio-economic status and achievement

As pointed out by West & Pennell (2003):

> Whilst in any population of pupils some will perform less well than others, there are links between achievement and a variety of different forms of disadvantage and other factors. (p. 25)

This statement opens space for several questions regarding the expected distribution of achievement over a population: should teachers expect that within a group, achievement is normally, equally or homogeneously distributed? How should teachers define what is normal, acceptable, or average achievement? Is it inevitable that there will always be some students below this level? How much variability is it reasonable to expect from one context to another?

Even though researchers are trying to tackle these questions from different perspectives, such as international comparative studies (OECD, 2016), or by investigating the genetic influence on achievement (Selzam et al., 2016), this is not the focus of the research. My interest is in what emerges when one considers not the expected distribution of achievement over a population, but the observed distribution, the variables that are correlated to that, and the possible causal mechanisms behind these correlations. There is much literature indicating the connection between achievement and some socio-economic variables, such as household income, ethnicity, gender, parental education, social class, and so on, at local, national and international levels (Valero et al., 2012; Valero & Meaney, 2014).

An interesting starting point from which to assess the influence of some of these variables on achievement are the results from PISA. The country notes for the United Kingdom based on the 2012 edition (the latest with a focus on mathematics), point out that:

> In the United Kingdom, equity in education outcomes is at the OECD average, with 13% of the variation in student performance in mathematics attributed to differences in students' socio-economic status. (OECD, 2012a, p. 4)

In order to make sense of the relevance of this percentage, it is necessary to compare it with similar figures related to other variables. OCDE (2012a) also considers gender and immigrant status, where both figures are weaker than for socio-economic status. The conclusion is the same if the data from the 2016 report is considered, even though it is focused on science instead of mathematics. Also, as highlighted by Valero & Meaney (2014), every large scale international comparative study reifies this same finding.

It is worth noting that recent studies investigating the relationship between genetics and educational achievement among the British population found that the former can explain 9% of the latter at age 16 (Selzam et al., 2016). Comparisons between this percentage and the one reported by OCDE (2012a) should be treated very cautiously, since the methods applied differed greatly in nature. However, when taken together, they suggested that the effect of socio-economic status in mathematics achievement is very relevant, if not the most relevant factor.

Dunne et al. (2011, 2007) analysed the relationship between set placement and social and economic variables and concluded that:

> data on pupils' allocation to groups confirms prior attainment as the main, albeit a relatively poor predictor of set placement [...] Social class is a significant predictor of set placement. Pupils from higher socio-economic status (SES) backgrounds are more likely to be assigned to higher sets and less likely to be assigned to lower sets. (Dunne et al., 2007, p. xii)

Although this relation seems to be well established in the educational literature from a statistical perspective, pointing it out is not enough (Valero & Meaney, 2014). I agree with Gates (2015) when he suggests that "we probably can't do much about improving their social and economic backgrounds; we might however be able to do something about enhancing some of the key skills, which they have not previously been required to focus on" (p.7). In order to do so, it is necessary to unveil the process (or processes) behind these associations (West & Pennell, 2003).

Jorgensen, Gates, & Roper (2014) and Noyes (2007) used Pierre Bourdieu's ideas to highlight why students from different social classes experience schooling differently. The core of their argument is that the dispositions, habits and preferences of certain social groups, namely the middle class, are more aligned with schools' and teachers' expectations than the dispositions, habits and preferences of working class children. As a result, students in these groups experience school in very different ways. Bourdieu's proposition is that these dispositions, habits and preferences, together with explicit forms of capital, such as availability of material goods (books, for instance), and financial capital (that could allow the family to hire private tuition, for instance), also play the role of capital that "can be exchanged for success in the classroom" (Jorgensen et al., 2014, p. 227).

However, as posited by Noyes (2007), "although Bourdieu's tools offer a convincing theorisation of the way things are [...], they are not so useful in generating emancipatory pathways" (p. 45). In order to do so, it is necessary to understand the mechanisms through which these forms of capital benefit the group that possesses it in school mathematics.

The authors offer some glimpses of such mechanisms. Jorgensen et al. (2014) point out how a simple habit, such as walking down a street towards the school (something that is more common among middle class families, since the nature of their jobs allows them to adjust their times with school times) pointing out the numbers of the houses, could promote "some sense of what bigger numbers mean and perhaps an intuitive sense of place value and a sense of odd and even numbers" (p.226). Noyes (2007) shows how Edward, a middle class student, "mathematized" his coin collection through ordering, categorizing and emphasizing the quantities and values when talking about it.

While both papers focused on a very fine grained analysis of single cases, that may sound too specific for a general comprehension of the mechanisms behind the phenomenon under discussion here. Other researchers have investigated broader factors.

Zevenbergen (2001) focuses on how the different uses of language by working-class and middle-class families influence pupils' experiences in schools within an Australian context. She relies strongly on the theories of the sociologist Basil Bernstein to understand how "the language used by some students position[s] them as marginal within the context of contemporary mathematics classrooms" (Zevenbergen, 2001, p. 40). She points out that the language can affect not only the understanding of mathematical statements, explanations and questions, but also the comprehension of the hidden rules of the classroom, concluding that in general:

> [W]orking-class children encounter forms of language in the home environment different from that which they encounter in the school. Hence, it is not valid to assume that problems in the levels of attainment of working-class students arise solely from any deficiency in their mathematical ability. Within this perspective, it becomes important to recognise the difference in home-school languages and to build bridges in order that students can access the mathematical language (p. 43).

An example is given in the classic paper by Labov (1969) when he compares the discourse by an American working class black person to the discourse of an upper middle class educated person. The former uses a non-standard version of the English language, and gives the impression of being impulsive in his opinions, while the latter sounds more tempered. However, when it comes to logic and semantics, the former is richer, and the latter just elongates less sophisticated thoughts that could be expressed very concisely. This characteristic, that the author calls verbosity, is typical of the middle class and is considered academically valuable, since according to Labov (1969), it highlights that the speaker is educated. What the author is proposing is that middle class language is considered better, not because it is richer in form, content and logic, but because it fits a standard that is expected in certain contexts, such as schools.

Lubienski (2002) brings an interesting contribution to this discussion when analysing adopting a reform-oriented approach during her lessons.

> The use of open, contextualized problems seems sensible at many levels. Rather than have students complete meaningless exercises and memorize what the teacher tells them, why not have them learn key mathematical ideas while solving interesting problems? (p. 172)

However, instead of assuming that this approach would benefit all students equally, she specifically observed possible differences regarding socio-economic status. After implementing the new teaching style, she noticed that although some previously uninterested students presented a more positive attitude towards mathematics, students in different SES groups responded differently to key features of the approach. In general, students from disadvantaged low SES backgrounds, especially the girls, felt lost with the lack of orientation in the open-ended questions and focused only on a particular problem without seeing the general ideas connecting various problems.

A similar conclusion was reached by Cooper (2001), when analysing answers to contextualized and "realistic" questions in a national exam in England, taking into account the socio-economic status of the students. The author is not suggesting that disadvantaged students should be given only "work out" type of questions, but that the solutions of contextualized questions might rely on skills that are more common among students from more affluent backgrounds.

But how could this be the case? In a classroom conversation, as in any other context, a lot is left unsaid, and a lot is assumed as "common ground". For instance, when asking "Could you do the dishes tonight?" a parent is not actually asking a question, but suggesting that the person responsible for doing so, starts the action. By giving the order in a question format, the parent sounds more tempered and polite even though the actual content is different from what is said, just like the upper middle class person discussed by Labov (1969). Returning to the issue of open and realistic problems, this type of task is (by definition) less structured and prescriptive, thus demanding a better understanding of what is unsaid, and demanding more "common ground" between teacher and students. Of course this can be achieved by the teacher during the lessons, but if not approached explicitly, could result in reinforcing social differences between different groups.

Gates (2015, 2018) builds on the argument presented by Zevenbergen (2001) regarding language, to suggest the use of visual representations, which are particularly effective in reducing the achievement gap between students from more, and less affluent backgrounds. His argument goes beyond the already desirable effect of reducing the reliance on verbal communication in the classroom based on the arguments presented above. His argument is based on three key points: first, the growing corpus of evidence showing that visual skills are central to the learning of mathematics; second, such skills are not part of the regular curriculum; and third, that practices that promote such skills, as in playing with building blocks, reading maps and playing with jigsaw puzzles, are more readily seen in middle class families.

The conclusion that visual representations could be especially beneficial for low achieving students sounded quite compelling and, at the same time, not satisfactorily explored at a classroom level. Therefore, it seemed to me at that stage of my research project that it could occupy a significant role in my investigation. For that reason, I will further develop the topic in the next sections.

2.4 Visual representations come into play

As emphasized by Bruce et al. (2017), the topic of visual representations has a long history, and is of interest to at least three big fields of study, Psychology, Education and Mathematics. Consequently, terminology varies hugely. The goal of the next section is to set the vocabulary, and results that will be used throughout my thesis regarding visual representations.

2.4.1 Visual representations

The term visualisation is associated with a wide range of expressions, such as visual skills, spatial ability, representations, diagrams, and even imagination and insight (Bruce et al., 2017; Macnab, Phillips, & Norris, 2012; Reed, 2013). Thus, it is necessary to clarify the meaning of visual representations in this paper.

First, it is important to distinguish between internal and external representations. The latter refers to representations (or signs) produced by human beings and available publicly to others, while the former refers to products of our cognition, therefore available only "inside people's heads". According to Larkin & Simon (1987):

> When they are solving problems, human beings use both internal representations, stored in their brains, and external representations, recorded on a paper, on a blackboard, or on some other medium. (p. 66)

The concreteness and availability of the external representations not only make their existence obvious, but make it possible to identify, classify, measure and discuss their features. On the other hand, internal representations are not directly tangible and can be discussed only through models, such as the dual coding model by Paivio (1986) or Baddeley's (2000) working memory model. Both models assume that our cognition needs some kind of representation in order to manage the cognitive processes available to human beings, and that those internal representations are somehow similar to external representations. Paivio (1986) assumes explicitly that:

> internal (mental) representations have their developmental origins in perceptual, motor, and affective experience and that they retain those experientially derived characteristics (p. 55).

It is beyond the scope of this paper to engage in the debate about similarities between internal and external representations, but I share this assumption. Consequently, there is no need to use the adjectives internal and external to qualify the representations.

The second distinction refers to visual and textual representations and is key to my work. Although it is difficult to establish a set of precise criteria to distinguish them, due to the fluidity between the extremes, it is possible to identify in the literature some characteristics that are commonly used to define the extremes, as shown in the table below.

Textual representations	Visual representations
Symbolic	Iconic
Based on natural language	Based on topological disposition of elements in a 2D or 3D space
Manifested through speaking and writing	Manifested through some kind of drawing, concrete objects and gestures.
Captured through viewing and hearing	Captured through seeing
Essentially sequential	The disposition of the elements is not sequential, but carries meanings in terms of the relationship between them.

Table 1: Differences between visual and textual representations

This duality may appear under different names in the literature, such as textual-visual, linguistic-graphic, sequential-spatial, and digital-analog (Shimojima, 2001), but the characteristics listed above are almost universally consensual, and if not, are the characteristics that I am adopting to define what, in this document, I call visual representations.

In the specific context of mathematics education, Skemp (1987) suggests the distinction between verbal-algebraic, and visual, and offers the table below to characterize them.

Visual	Verbal-algebraic
Abstracts spatial properties, such as shape, position	Abstracts properties that are independent of spatial configuration, such as number
Harder to communicate	Easier to communicate
May represent more individual thinking	May represent more socialized thinking
Integrative, showing structure	Analytic, showing detail
Simultaneous	Sequential
Intuitive	Logical

Table 2: Distinction offered by Skemp (1987).

Even though the characteristics given above are of a speculative nature (no empirical evidence is provided), Skemp's proposal resonates not only with Table 1, but also with other researchers who try to understand the apparently wide spread "gut feeling" among mathematicians that suggests that visual representations are fundamental to create and understand mathematics.

The distinction becomes relevant when you take into account experiments usually conducted by researchers in cognitive sciences (Baddeley, 2000; Paivio, 1986; Reed, 2013), showing that stimuli generated by visual or textual external representations are apparently treated differently by the brain.

> The most general assumption in dual coding theory is that there are two classes of phenomena handled cognitively by separate subsystems: one specialized for the representation and processing of information concerning nonverbal objects and events, the other specialized for dealing with language (Paivio, 1986, p. 53)

The same is perceived in the working memory model proposed by Baddeley (2000), in which there are two different components that deal separately with visual and textual representation: the visuospatial sketchpad, and the phonologic loop, respectively. Although presented as an assumption by Paivio (1986), this distinction is reinforced by extensive experimental results, as discussed by Reed (2013).

Larkin & Simon (1987) adopt an equivalent distinction between diagrammatic and sentential representations. According to the authors:

> The fundamental difference between [...] diagrammatic and sentential representations is that the diagrammatic representation preserves explicitly the information about the topological and geometric relations among the components of the problem, while the sentential representation does not. A sentential representation may, of course, preserve other kinds of relations, for example, temporal or logical sequence. (p. 66)

The reference to geometry and topology should not restrict the use of the term visual representation to refer to certain components of the mathematics curriculum. The definition adopted in this thesis includes geometric structures, as well as what are ordinarily called pictures, graphs, drawings, manipulatives, diagrams, and even particular ways to arrange the solution of a problem that take advantage of the space in the paper to suggest the order of the steps or relationships between them. Also, the category of textual representations includes, as suggested by Paivio (1986), representations composed by symbolic logic, computer languages and mathematical formal writing (such as equations).

For the sake of clarity, the expression "visual representation" will always be used to refer to an external representation that matches the characteristics of the second column of Table 1. However, as pointed out by Goldin & Kaput (1996), "representations do not occur in isolation [...] They usually belong to highly structured systems, either personal and idiosyncratic or cultural and conventional" (p. 398). These systems that govern representations (either textual or visual) encompass not only the signs, but also the properties that allow a user to decide if a particular sign belongs to that particular system, the rules that enable the translation of signs in a system to another, and the actions that can be carried out in its symbols. As Goldin (2014) explains:

> An essential feature of mathematical representations is that not only do they have signification, but they belong to or are situated within structured systems of representation within which other configurations have similar signifying relationships. This is analogous to the way words and sentences occur, not as discrete entities in isolation from each other, but within natural languages endowed with grammar, syntax, and networks of semantic relationships. (p. 410)

Once the meaning of visual representations is established, it is important to clarify the meaning of another expression very common in works related to visualization: visual (or spatial) ability.

In order to do so, let us consider Macnab et al.'s (2012) distinction between visualisation objects, introspective visualisation and interpretative visualisation. The first two concepts are similar to what was defined previously as visual external representations and visual internal representations respectively, and the third is defined as the "cognitive functions in visual perception, manipulation and transformation of visual representations by the mind" (Macnab et al., 2012, p. 114). This is the definition of visual ability adopted in this thesis.

Although using different names, such as competencies (Goldin, 1998) and visual thinking (Reed, 2013), authors would typically include in this set of cognitive functions a range of abilities (also called visual skills), such as rotating an image mentally, picturing an arrangement of objects described verbally, identifying patterns, decomposing a given object into simpler objects, transforming a geometric figure into a new equivalent figure, capturing the properties of a 3D object drawn on a 2D surface, identifying the relevant features of a diagram, etc.

These abilities are a common object of interest for cognitive researchers. For instance, Wai, Lubinski & Benbow (2009) used four components to create a measure they called spatial ability: folding of a two-dimensional image into a threedimensional object, performing mental rotational and reflection, visualizing the consequences of mechanical forces and movements, and identifying figure patterns.

There is one final distinction that is important for this thesis, and it refers to expressions, such as visual and spatial reasoning. Although some authors may use them as synonyms for visual ability, I will use visual reasoning to refer to any reasoning based on elements or properties of a visual representation, as used by Watanabe (2015).

After establishing the basic concepts regarding visualisation, the next subsection discusses how this issue is approached in the field of mathematics education.

2.4.2 Visualisation in mathematics teaching and learning

Any person involved with mathematics, from mathematicians to teachers and enthusiasts, would recognize the important role that visualisation plays within the field (Newcombe, 2010). In fact, it is easy to find emblematic quotations from famous characters, such as the physicist Albert Einstein and the mathematician Henry Poincare testifying to the claim that visualisation is more important than rigour and formalism in this field.

When it comes to educational research, researchers have repeatedly reinforced the importance of visualisation to cognition in general (Larkin & Simon, 1987; Reed, 2013) and in mathematics education in particular (Arcavi, 2003; Presmeg, 2006). Not surprisingly, when Bruce et al. (2017) traced the historical emergence of a concept related to this field, namely spatial reasoning, it was possible to identify research reaching back to the 16^{th} century in different scientific and philosophical fields, as well as the utilisation of a huge variety of nomenclature.

Similarly, it is difficult to conceive of a teacher who would not recognise the importance of elements related to visualisation, such as spatial abilities and diagrams. However, the use teachers make of visual representations may reveal a very different scenario. Dreher, Kuntze, & Lerman (2015) concluded from a sample of more than 300 German and English teachers that teachers were "mostly not able to recognise the learning potential of tasks focusing on conversions of representations, in comparison with tasks including rather unhelpful pictorial representations" (p. 8). Morgan (1991) showed that even though teachers say visual representations are important, when judging the quality of solutions with and without diagrams, the latter is usually better assessed. Stylianou (2010) reached a similar conclusion after interviewing 18 middle school American teachers about their views on visual representations. She concluded that:

> symbolic notation is not regarded as one of the ways in which a concept is represented, but is the concept itself. Other representations (often visual) are informal objects that assist in the actual work of doing mathematics. This idea of separating visual representations of a concept from the concept itself brings to the forefront teachers' tendency to attach value to representations. That is, certain representations (most notably symbolic or numeric) are considered more central to the learning of and doing mathematics, while graphic and visual representations are only secondary (p. 335)

Beyond the realization that teachers generally have a restricted view of the role of visual representations, Sinclair, Mamolo & Whiteley (2011) point out that teachers also "have substantial anxiety about working with these approaches" (p. 137). In fact, Verdine et al. (2014) report a study showing that among several career paths, those who ended up in education presented the lowest level of visual abilities 11 years before, when they were high school students.

When it comes to students, Arcavi (2003) lists three sources of challenge that have to be considered: a) cultural, which refers to beliefs and values about mathematics; b) cognitive, which refers to the high cognitive demand of visualising conceptually rich images; and c) sociological, which refers to the cultural diversity of students' backgrounds in a classroom.

Moyer (2001) highlights the cultural challenge when she shows teachers' views on working with manipulatives. After following 10 teachers for a year trying to use manipulatives, she reported that they saw it as "little more than a diversion" (p. 175).

The cognitive challenge can be illustrated by the research conducted by Steenpaß & Steinbring (2014). The authors analysed the answers given by a primary student to a question that asked her to associate number lines to sums that could be represented by that diagram. The question was asked before and after a classroom intervention focused on that type of visual representation. Their analysis showed that the student interpreted the elements of the representation as concrete single objects, isolated from the other elements. The authors highlight the cognitive challenge behind teaching and learning using visual representations by concluding that:

> the 'effect' of the intervention depends not only on the intervention tasks themselves. Rather, Sonja's learning process arises from an individual

> sense-making interpretation background (Steenpaß & Steinbring, 2014, p. 13)

Finally, the sociological challenge can be illustrated by exemplary research discussed by Reed (2013). After analysing 4 year old children engaged in a task that consisted of placing whole numbers in a 0 to 10 line, the authors observed that children from low-income families did worse than children from high-income families. In further studies, they found that this gap could be overcome by providing experiences, such as playing board games that require moving tokens in numbered squares. This study shows that the social background can affect how students deal with visual resources, such as a number line, even in early ages. This is not surprising after the discussion presented in Section 4.1 about the conflicts between schools and different forms of knowledge (Bourdieu, 1986; Gates & Noyes, 2014; Zevenbergen, 2001).

Besides those challenges listed by Arcavi (2003), there are challenges related to the lack of understanding about the actual mechanisms behind the benefits that visual representations can bring to cognition. As pointed out by Glenberg & Langston (1992):

> The literature is overflowing with work investigating the facilitative effects of pictures on text comprehension. And yet, no one has a clear idea of the cognitive processes underlying these effects (p. 129)

Although the statement refers to a more specific aspect of visualisation, it seems reasonable to generalise and claim that it holds for visualisation and mathematics education. An interesting example to illustrate this state of affairs is also given by Reed (2013) in an experiment where the subjects are expected to decide if a pair of words, such as mouth and nose, are related in meaning. Precise measures of response time showed that people need more time to make a decision if the pair mouth-nose are shown as in the right alignment.

Illustration 1: Options for the response time experiment

The hypothesis that the authors suggested as an explanation to this phenomenon is that, somehow, human perception is facilitated if the position of the words matches the usual visual arrangement of the body parts. The argument behind this example is that there is still much to know regarding visual representations, how they are processed by the brain, and how they affect human cognition.

All examples presented above were chosen to highlight the challenge of incorporating visual resources into mathematics classroom practices, even though most people recognise them as an important part of this subject. Despite all these challenges, Gates (2015) points to evidence suggesting that a more visual approach could not only be beneficial for mathematics students, but also could be more effective in closing the gap due to social background than verbal approaches. A similar claim is made by Mayer (1997) to explain the greater gain exhibited by low-prior knowledge students in relation to their peers when images are combined with text in multimedia educational material. This assertion lies at the core of this research project and it will be further explored in the next section.

2.4.3 Visual representations and achievement in mathematics

Before talking about the relationship between visual representations and achievement in mathematics, it is important to highlight the well-established relationship between visual ability and achievement in mathematics.

On one hand, as stated by Mix & Cheng (2012), "the relation between spatial ability and mathematics is so well established that it no longer makes sense to ask whether they are related" (p. 206). On the other, when it comes to causal mechanisms, the scenario is not so clear. However, constructs such as number sense (Dehaene, 2011), evidence generated by brain imaging techniques (Amalric & Dehaene, 2016; Dehaene, Spelke, Pinel, Stanescu, & Tsivkin, 1999), and longitudinal studies (Gunderson, Ramirez, Beilock, & Levine, 2012) are shedding some light on these mechanisms.

In summary, these studies suggest that our brain uses visual representations to represent some basic mathematical objects and visual abilities to manipulate them mentally. For instance, Gunderson et al. (2012) say that their results "are consistent with the hypothesis that spatial skill can improve children's development of numerical knowledge by helping them to acquire a linear spatial representation of numbers" (p. 1229).

From these results, it seems natural to conclude that the use of visual representations would be beneficial for the learning of mathematics. After all, the use of visual representations should promote the practice of visual skills. For instance, by representing a fraction in the rectangular area model, a student would need to mentally decompose the whole into pieces, practising this particular visual skill. I agree with this view: visual representations may naturally promote the use of visual skills, while textual representations (by definition based on abstract symbols instead of intuitive icons) do not necessarily promote it because these representations can be treated by means of abstract transformations.

But the benefits of using some visual representations may go beyond practising visual skills. As said in Section 2.4.1, evidence suggests that mental representations reflect external representations. Therefore, using external visual representations would enable the formation and development of mental visual representations. Dehaene (2011), for instance, proposes that the mental number line that supports his concept of number sense is the result of complex interactions between culturally constructed representations and innate faculties.

It is interesting to notice that recent research is proposing a distinction that goes beyond verbalizers (those who prefer textual representations) and visualizers (those who prefer visual representations). The new distinction differentiates between those who use more pictorial visual representations and those who use more schematic visual representations.

In a continuum, pictures would be at the pictorial extreme, with diagrams in the schematic extreme.

Van Garderen (2006) concluded that:

> Use of visual images was positively correlated with higher mathematical word problem–solving performance. Furthermore, the use of schematic imagery was significantly and positively correlated with higher performance on each spatial visualization measure; conversely, it was negatively correlated with the use of pictorial images. (p. 496)

Gray et al. (2000) reported that students demonstrated qualitatively different thinking when shown simple images, words or sounds, and were asked to explain what they had seen or listened. They described this difference in the following manner:

> The qualitatively different responses to the words, icons, and symbols suggest that the "low achievers" were reluctant to reject information. If

> there was little to describe, they created description by building stories around the items using images from their known physical world. Often they were participants in the image, elaborating the detail whenever it seemed that such embellishment was required. In some instances, they drew upon one image, which acted as a symbol, for example, "my football," "my mother's car" (p.408)

Note that this type of response is similar to a response to a picture, while high achievers "filtered out the superficial to concentrate on the more abstract qualities of the items" (Gray et al., 2000, p. 408), getting closer to a diagram.

These results suggest that the use of more schematic visual representations (diagrams from now on) is also related to achievement in mathematics. One possible causal explanation for this relationship could be the idea that this type of visual representation enables inferences by offering a holistic view of the relations between its elements (Larkin & Simon, 1987; Skemp, 1987).

Therefore, the use of schematic visual representations is beneficial for the learning of mathematics because: a) it promotes the practice of visual skills, b) it enables the acquisition of mental visual representations, and c) it facilitates key aspects of thinking, such as inferences.

The issues discussed in the last two sections regarding socio-economic status, visual representations and teaching and learning of mathematics informed my focus during the preliminary study that took place during the last term of the academic year, and will be discussed in the next chapter.

3 A PRELIMINARY STUDY

By February 2015, I had already observed about 30 lessons at Purple Valley and my research interest had been refined in order to include the educational potential of visual representation for low achieving students. Then the research team and teachers decided to move forward to what could be characterized as a preliminary study for my research project.

The initial idea was to start a series of meetings aimed at developing some lesson plans that would be enacted by one of the teachers in a style similar to the Japanese lesson studies. After three meetings, it was agreed that Julia would work on lesson plans covering fraction addition for her Year 7 Set 4 (out of 5). From this point on, all the meetings and informal talks between us focused on discussing her lesson plans, which were intended to make intense use of visual representations. She developed two lesson plans and I developed a third one. As these lessons greatly influenced the design of the lesson plans I used during my data collection, I will describe them in detail in the following sections.

It is important to keep in mind that at this stage, the activities were not being developed systematically by the research team, because we were still getting a feeling for what the teachers would be willing to do, and be interested in doing. Therefore, what may seem "too loose" for research, was actually intentionally conducted that way to create rapport and synergy between researchers and teachers.

3.1 First lesson

The first lesson was mostly designed based on one principle and around one task.

The principle, originally suggested by Dr. Peter Gates, was for the teacher to try to avoid talking as much as possible. This principle is aligned with the ideas regarding the over-reliance of teachers on verbal language and how this over-reliance may be differentially affecting students from different social backgrounds (Gates, 2015; Zevenbergen, 2001). At the same time, this principle tries to maximize the affordances of visual representations.

The task was proposed initially by Julia and derived from the animation, also conceived by her, available at https://youtu.be/adiA9_8-Mrc. Her idea was to show this animation and then ask students to create sums of fractions that add up to 1. This task became a pivotal element for this lesson plan, and all the other tasks were conceived as a preparation for, or an extension of it.

The lesson was taught in May. All members of the research team attended, but none of the other teachers could. During the lesson, students had a set of coloured cardboard pre-cut shapes representing the fractions 1/2, 1/4, 1/8 and 1/16 in the rectangular area model (the unit was a square with sides measuring 12 cm) that could be used throughout the lesson to solve the tasks.

It started with some warm-up questions on topics from previous lessons, as Julia usually does at the beginning of every lesson. Then she played an animation (https://youtu.be/U1O9uhCIIZ8) showing how to represent the fractions 1/2, 1/4, 1/8 and 1/16 in the rectangular area model, and some sums adding up to 1. Following the animation, she introduced the task below giving minimal instructions regarding what was seen and what students should do to solve it.

Fraction Sums Name:..

Write down a fraction sum which equals 1

Can you write another?

And another?

Fraction Sums: Extension Task 1

Can you write a fraction sum which equals 1 ½ ?

Can you write a fraction sum which equals to 2?

Illustration 2: First worksheet

After 14 minutes with students working by themselves on the task above, she played another animation (https://youtu.be/QKWuQ5JzQds) showing equivalent fractions in the rectangular area model. Following it, Julia posed a matching cards activity (see image below) involving diagrams and equivalent fractions.

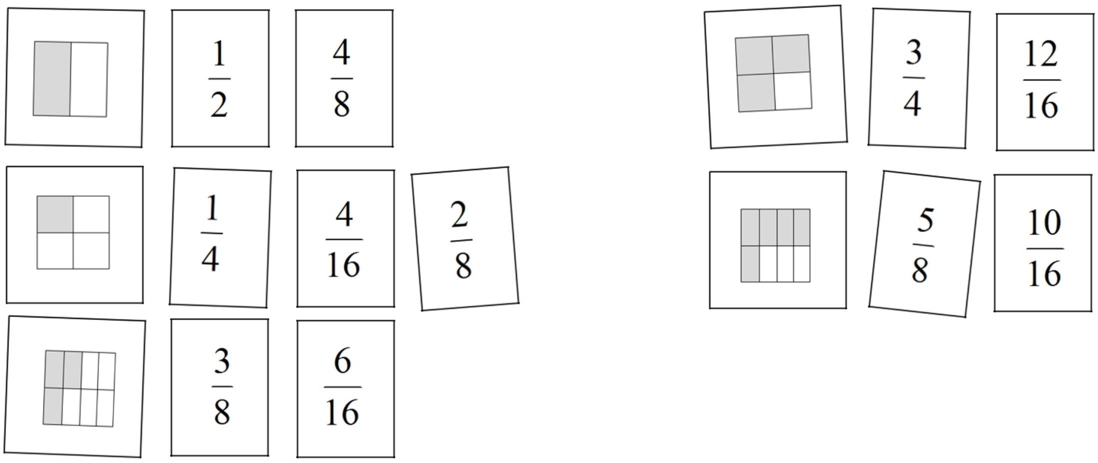
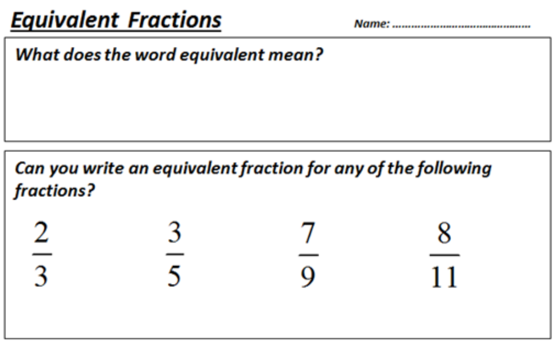


Illustration 3: Matching cards activity

Students worked on this task individually for 11 minutes and then she distributed the worksheet below and students worked on it for 8 more minutes.



Equivalent Fractions Name: ....

What does the word equivalent mean?

Can you write an equivalent fraction for any of the following fractions?

$\frac{2}{3}$ $\frac{3}{5}$ $\frac{7}{9}$ $\frac{8}{11}$

Illustration 4: Third worksheet

Julia then promoted a final discussion around answers given by the students to the question, "can you give me a sum that adds up to 1?" She got 4 examples, wrote them on the board and discussed each one briefly with the whole class. During this stage, they should not use the cut-outs.

The table below summarizes the main stage of each lesson. The width of each column is proportional to the time spend on the respective stage.

Warm-up questions	First worksheet	Matching cards activity	Third worksheet	Final discussion

Table 3: Scheme summarizing the first lesson of the preliminary study

After the lesson, we had a meeting with all the teachers and the research team. Julia was satisfied with students' engagement and apparent understanding. Based on their reactions and answers during the final discussion, she was convinced the students were actually thinking visually about fractions: "I could see some of them imagining the shapes in their heads".

3.2 Second lesson

The second lesson was designed to capitalize on the work done on the first lesson to introduce sums with the fractions involving 1/2, 1/4, 1/8 and 1/16 not necessarily adding up to 1. The research team and the teachers had time to meet twice after the previous lesson and before this one, to discuss the lesson plan. The lesson was taught in July.

It was decided that students would have the same set of cut-outs available, but there was less concern regarding the amount of talk by the teacher. As a result, instead of animations, the teacher used static slides to show diagrams.

The lesson started with the warm-up questions below followed by a whole class discussion about the answers.

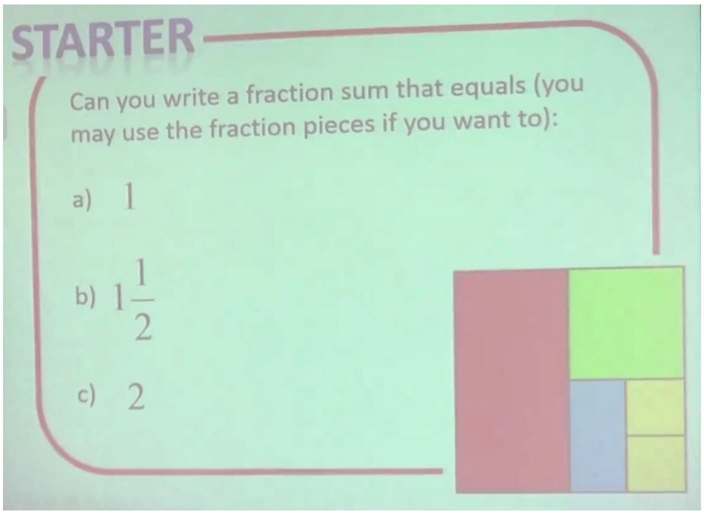


Illustration 5: Warm-up question

Then the teacher presented the three diagrams shown below, one at a time, and asked the students "what fraction of this shape is shaded?" Before explaining how the question could be solved, Julia asked the students to have a go and try to figure out how they could answer that question and present their solution to somebody else.

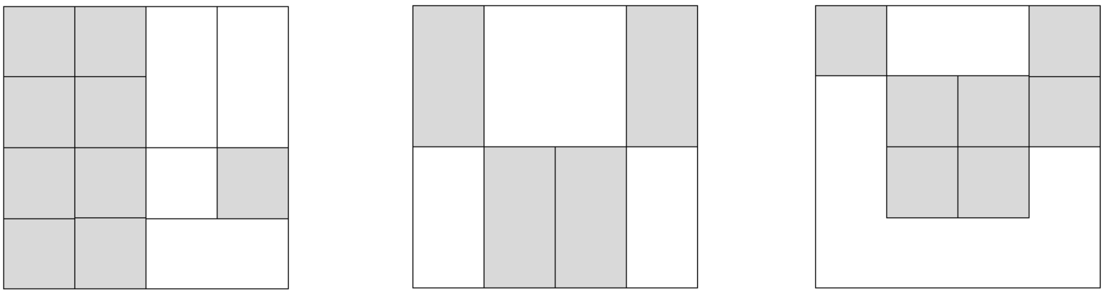


Illustration 6: The three sums discussed by the teacher

She spent 12 minutes posing the question, and waiting for the students to solve and discuss these three questions. Following that, the worksheet below was distributed to the students. They had 7 minutes to solve the questions and she spent 6 minutes discussing the solutions.

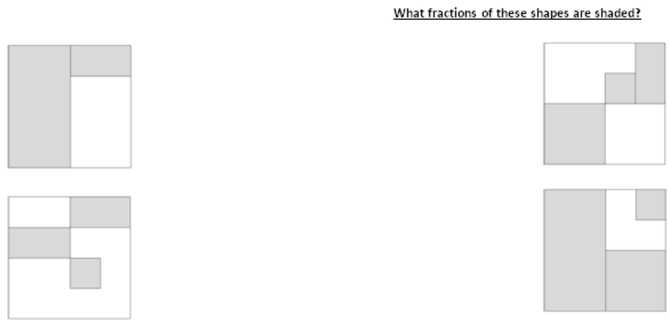


Illustration 7: First worksheet

Finally, she posed the final task with fraction sums given symbolically, but the students could use the cut-outs or diagrams to solve them if they wanted.

Can you add together the following fractions?

$$\frac{3}{16}+\frac{1}{4}$$

$$\frac{5}{8}+\frac{1}{16}$$

$$\frac{1}{2}+\frac{5}{8}$$

$$\frac{3}{4}+\frac{7}{16}$$

$$\frac{1}{2}+\frac{7}{8}$$

$$\frac{3}{8}+\frac{11}{16}$$

Illustration 8: Second worksheet

Students spent 12 minutes solving these sums and she concluded the lesson by discussing the answer for 4 minutes. The table below summarizes the main stages of the lesson.

Warm-up questions	First diagrams	Sums given diagrammatically	Sums given symbolically

Table 4: Scheme summarizing the second lesson of the preliminary study

Julia's evaluation of this lesson was very positive. She was impressed by how easily students were paraphrasing her, and able to transfer their knowledge from the previous lesson to this one.

3.3 Third lesson

After the second lesson, I offered to design a third lesson myself, extending the approach to other fractions, namely 1/3, 1/6 and 1/9. I was personally interested in seeing if Julia and the other teachers would be open to trying a lesson designed by somebody else and how this process would work.

At this moment, the academic year was very close to its end, and there was not much time for meetings, so I designed the full lesson plan and had time to discuss it with Julia only once before she enacted it. The lesson was video-recorded, and the research team attended with David and Alice, a new pre-service teacher at the school.

The lesson started with a warm-up task involving sums, such as those presented in the second lesson, $\frac{1}{4}+\frac{7}{16}$ and $\frac{1}{2}+\frac{3}{8}$.

Students were then asked to draw diagrams for 1/3, 1/6 and 1/9, and Julia discussed their answers in depth, asking for several answers for each diagram and commenting not only about the correctness, but also advantages and limitations of each. This stage alone took 16 minutes.

After that, the matching cards task shown below was posed.

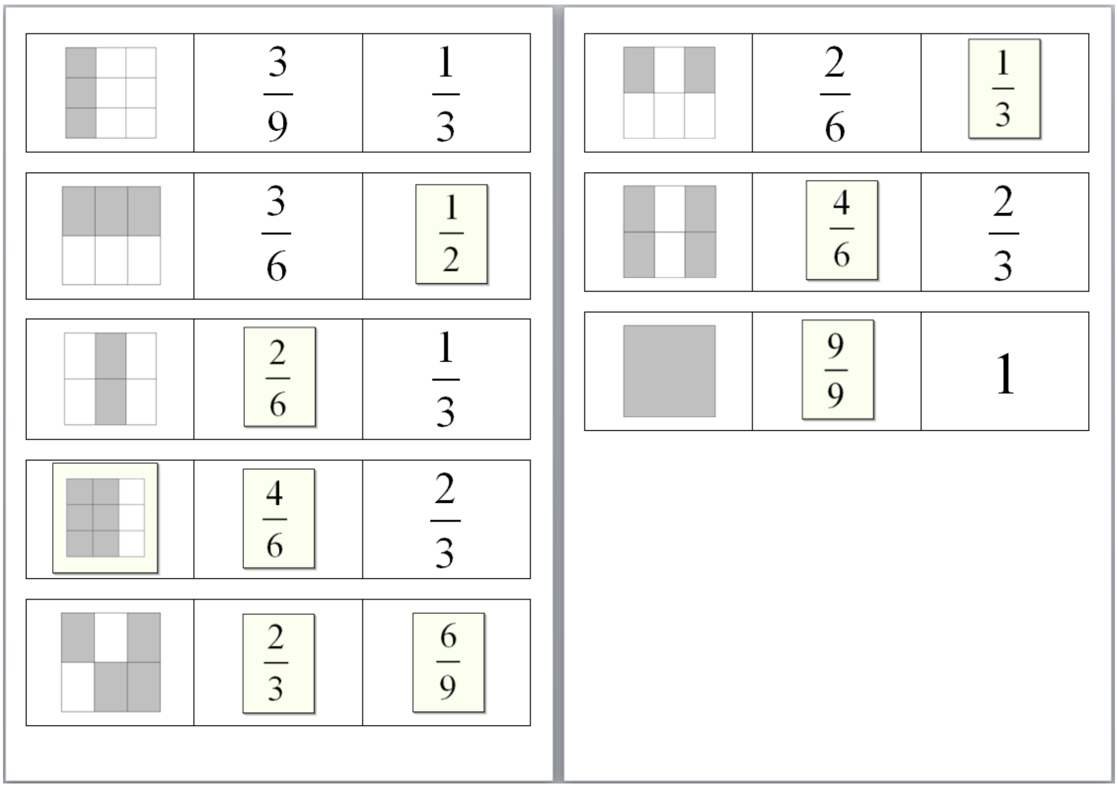


Illustration 9: The yellow rectangles represent the cards and the other diagrams and fractions were printed in the worksheet

Students had 8 minutes to work on this task and then Julia introduced the final worksheet.

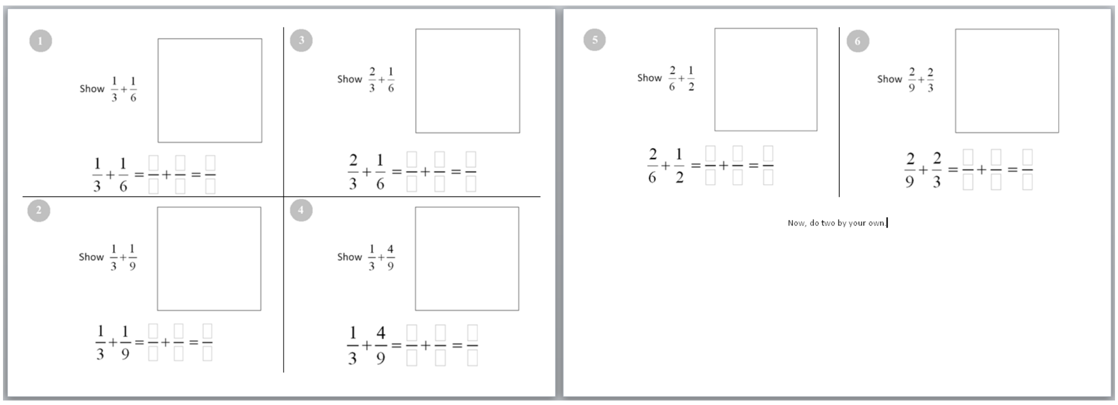


Illustration 10: Final worksheets

They worked on this worksheet for 13 minutes and the lesson was over. The table below summarizes the main stages of it.

Warm-up questions	Diagrams for 1/3, 1/6 and 1/9	Matching cards	Sums

Table 5: Scheme summarizing the third lesson of the preliminary study

After the lesson we had a meeting to discuss it, and Julia said she was less impressed with students' engagement and work this time. However, the teachers and research team engaged in a much deeper discussion about what worked and what did not, how the tasks could be improved, and what else would need to be discussed in other lessons in order to develop them into a full program to teach fractions.

By this lesson, the initial excitement by teachers due to the novelty had started to fade away and was replaced by a more reflexive view focused on students' learning. Although they were less excited with the third lesson specifically, it was clear that they were satisfied with the general proposal.

3.4 An episode of reasoning anchored on visual representations

Although these activities served as an exploratory experience for everyone involved, it was possible to identify episodes illustrating students making use of the visual representations to reason about mathematics. These episodes became even more salient because reasoning was not usually present in the lessons I observed, specially for low achieving students, at Purple Valley.

The episode below shows one of the Year 7 student's reasoning based on the diagrams (Barichello, 2015). It took place at the end of the third lesson and was triggered by the question: "Show $\frac{1}{9}+\frac{5}{6}$ ". This question was not included in the lesson plan and was posed to one student, L, because he had already finished all the tasks.

It is important to highlight that during the lessons the students only worked on sums involving like fractions. Therefore, this question was beyond what was explicitly discussed in the lessons, and the research team had no ideas about how the student would approach it. Image 11 shows L's solution.

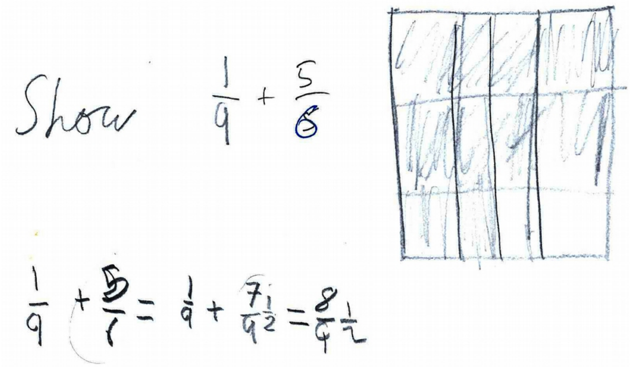


Illustration 11: L's solution to the sum 1/9+5/6

His final answer is not conventional and at no point during the lessons was this sort of approach suggested. However, it seems reasonable to interpret it as below and, therefore, he is right.

$$\frac{8\frac{1}{2}}{9}=\frac{8.5}{9}=\frac{17}{18}$$

The most impressive feature in his solution is how it is anchored on properties and elements of the diagram. The re-construction below was based on what was observed by one of the teachers attending the lesson, and by careful analysis of the solution itself. The writing at the bottom of his solution suggests that he used the diagram to find a fraction equivalent to 5/6 with a denominator equal to 9.

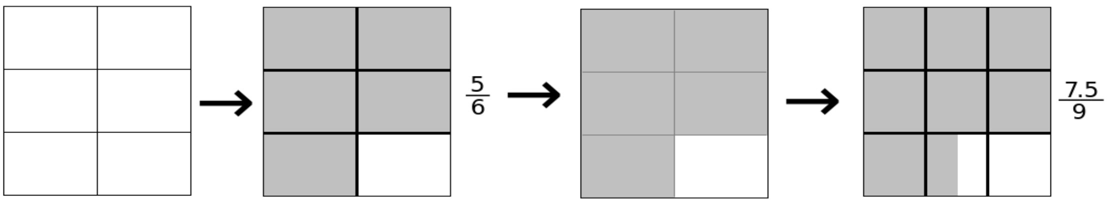


Illustration 12: Reconstruction of the diagram in Graham's solution

From this interpretation, it is safe to say that the student could not have solved the question without the support of the diagram, and at the same time, that visual properties of the diagram seem to have enabled him to move beyond what was explicitly taught.

This episode resonates with the idea of diagrammatic reasoning proposed by Rivera (2011) and with Rodd's (2000) view of diagrams acting as warrants for mathematical arguments. It illustrates the potential of visual representation to offer a gateway for low-achieving students to reason mathematically about relatively sophisticated topics, such as fractions.

This possibility gains even more importance if contrasted with the fact that reasoning was not commonly observed during lessons for low achieving students at Purple Valley. In general, its students experienced mathematics broken into lesson-sized portions and with very low levels of agency in terms of knowledge construction. Not only they were not used to be asked questions that demanded reasoning but they rarely showed reasoning when such questions were asked. These issues will be further discussed when the main study of my thesis is explained.

3.5 Other conclusions from the preliminary study

While the episode discussed above served as a starting point for re-elaboration of the research question, as will be discussed in Section 4.1, the preliminary study also served to inform some more practical aspects of the research design.

On the one hand, it became clear that the lesson study style would not work. Even though the Head of Department, David, was actively participating in all the meetings and encouraging the teachers to engage in the meetings and lessons, there was always 'something' happening in the school that impeded the teachers from observing each other's lessons. On the other hand, the teachers were very keen to open their classrooms for us to observe, to meet with the research team after lesson time with reasonable frequency (about once every half-term), to talk to us between lessons and to use lesson plans designed by us.

Finally, Julia and the other teachers were satisfied with the perceived results of the three lessons presented before, and this feeling paved the way for a longer and more systematic intervention in the next academic year, as will be described in the following chapters.

4 DEFINING THE RESEARCH QUESTION

As mentioned in the Introduction, I started the research process with a broad interest in low-achieving students' learning and would like to investigate at a classroom level through an interventionist approach.

The experience reported in the two previous chapters enabled me to refine this interest. During the preliminary study, the possibility of using visual representations as a basis on which to build knowledge about fractions became more tangible for me, as well as for the teachers. Additionally, the episode that took place during the preliminary study raised the issue of students being able to use visual representations to reason mathematically. As a result, visual representations were consolidated as a central issue of the research, and a new element was brought to it: reasoning.

Therefore, before presenting my research question, I will elaborate on these two issues. The content of the next three sections is a further development of the theoretical discussion presented in Barichello (2017).

4.1 The role of visual representations

In this section, I will deepen the discussion on how visual representations can have a fundamental role in the learning of mathematics further than that in Section 2.4. In order to do so, I will focus on the ideas of Marcus Giaquinto regarding the epistemological role of visualization in mathematical knowledge acquisition, and George Lakoff's and Rafael E. Núñez's concept of grounding metaphors.

4.1.1 The role of visualization in knowledge acquisition

Marcus Giaquinto is a British philosopher who concentrated his work for a long time in discussing the epistemological status of visualization in Mathematics. He is not commonly referred to in the mathematics education literature, as evidenced by the table below: only 8 papers are identified when searching his name on three major European journals in the field.

Journal	Number of results of a search by "Giaquinto" in their websites
Educational Studies in Mathematics	3
ZDM	3
Mathematics Thinking and Learning	2

Table 6: References to Giaquinto in mathematics education journals.

Most of the papers above focus on the issue of the use of figures and diagrams in proofs, which is not of interest for me. However, Giaquinto focused not only on this issue but also on discovery and knowledge acquisition in mathematics. For this reason, as highlighted by Gutiérrez, Llewellyn & Mendick (2009), his ideas are also of interest for mathematics educators.

His work around visualization was built since the 1990s and culminated with his book, "Visual thinking in mathematics: an epistemological study" published in 2007. In this book, in order to explain the epistemological role played by visualization in mathematics, he started with the concept of *category specification*. A category specification is less than a concept, as it is only a list of characteristics for something. The example he chooses is the square, and the category specification he proposes (one of many theoretically possible) is composed only of characteristics (parallelism, symmetry and reference system) that can be perceived by humans innately, or from a very early age. Here is the category specification:

> Plane surface region enclosed by straight edges;
> Edges parallel to H (one of the axes of the reference system established by our visual system), one above and other below;
> Edges parallel to V, one each side.
> Symmetrical about V;
> Symmetrical about H;
> Symmetrical about each axis bisecting angles of V and H. (Giaquinto, 2007, p. 23)

He calls this a visual category specification for squares, and states that:

> when, in seeing (perception) or visualizing (mental) something, the description set for the visual representation contains descriptions of all the features in this category specification, what is seen or visualized is seen or visualized as a square (Giaquinto, 2007, p. 24).

This means that a person can perceive the squareness of a figure without possessing a concept for a square or any other antecedent concept, since all the characteristics above can be perceived innately by our visual system. However,

> the capacity to reason about squares is distinct from the capacity to recognize perceptually something as a square. The capacity to reason about squares requires that one has a concept for squares. (Giaquinto, 2007, p. 24)

It is important to explain the author's definition of a concept. According to Giaquinto, a concept is a constituent of a thought and a thought is a mental state that has inferential relations between its constituents. In his view, "to possess a concept one must be disposed to find certain inferences cogent without supporting reasons" (Giaquinto, 2007, p. 25), i.e. one must be disposed to form a thought involving this concept. For example, consider the concept of uncle. One possesses this concept if and only if one is disposed to form a thought such as '*x* is an uncle if, for some person, *x* is a brother or brother-in-law of a parent of this person'.

The problem is that in order to possess a concept (uncle), one has to be able to form a thought with inferential relations between this concept and at least one other (brother and brother-in-law, in this example). But, if we always need a precedent concept to possess a new concept, how does the concept possession start?

Giaquinto's answer to this question is what I consider one of the main points of his theory: non-conceptual contents can be constituents of a thought.

Now consider the thought, 'an item *x* is a square if it is perceived in such a way that it matches the category specification described above'. Note that in this case, the formation of the thought does not depend on other concepts. Once the thought is formed in one's mind, the concept is acquired.

This mechanism allows us to acquire a concept, such as 'square', from the experience of perceiving something that fits the category specification above, and by being told that it is a square.

The same idea can be applied to new knowledge acquisition.

First, let's clarify the difference between concept and knowledge. According to Giaquinto, knowledge is a thought that a person believes to be true (before being true, the thought is called a belief by the author), it is reliable (the same stimulus would result in the same thought), and rational (the thought is not contradictory to any other already established knowledge).

The process of acquisition of new knowledge starts by possessing the concepts necessary to form the thought. This gives rise to what he calls a disposition, and dispositions are "bound" (Giaquinto, 2007, p. 45) to become thoughts due to certain stimuli. Here comes the second main point of Giaquinto's proposal: visual experience is capable of triggering the disposition to become a belief and if the person has no reason to question the experience this belief is considered true and, therefore, becomes a thought.

Note that the visual experience did not work as empirical evidence or ground for the thought, but as a trigger that activated a disposition to acquire that particular knowledge. This way, the author suggests that this mechanism can be an answer to the philosophical debate of how knowledge acquisition starts.

> Here, then, is one possible role for sense experience: together with certain innate mental propensities it results in our forming the geometrical concepts involved in the belief. [...] On this account, sense experience does enter into the causal pre-history of the belief, but not as evidence; rather it is the raw material from which the mind forms our geometrical concepts (Giaquinto, 1992, p. 389)

The next step is to show that the thought acquired through this mechanism is reliable and rational. Giaquinto (2007) discusses these issues in depth and shows that it is actually the case, but it is beyond the scope of my thesis to go through the details of it.

In summary, knowledge acquisition starts with possession of the necessary concepts, which results in a disposition, and this disposition can be triggered by visual experience to form a thought that has all the characteristics necessary to be considered knowledge. He concludes this discussion with the following summarizing statement:

> This manner of acquiring the belief is non-empirical, because the role of experience is not to provide evidence. At the same time, some visual experience is essential for activating the relevant belief-forming disposition; and it is clear that this way of reaching the belief does not involve unpacking definitions, conceptual analysis, or logical deduction. Hence it must count as non-analytical. Given that "non-analytical and non-empirical" translates as "synthetic a priori", we have arrived at a view that is a [...] synthetic a priori knowledge. (Giaquinto, 2007, p. 47)

The second role of sense-experience identified by the author is "to provide components on which the mind operates in producing a visualizing experience" (Giaquinto, 1992, p. 389). He reinforces that, once again, it is not a matter of providing empirical evidence, but basic components that can be manipulated mentally leading us to new knowledge.

It is also important to highlight that he explored these ideas in domains beyond geometry, such as arithmetic (Giaquinto, 1993b) and real analysis (Giaquinto, 1994) with special emphasis on the process of discovery, not proof.

Finally, there are two aspects that should be clarified regarding his proposal.

Firstly, he never stated that this is the only way to acquire concepts or new knowledge. The explanation below comes after a thorough discussion of the episode reported by Plato known as Meno's paradox (Giaquinto, 1993a). He argues that the episode illustrates a person acquiring new knowledge through visualization. He concludes the paper by stating that:

> It is possible that a reader of the text arrives at the theorem in the empirical manner, by making observational judgements about the areas of figures enclosed by the relevant ink-marks on the page and so on, thence inferring the theorem. The point here is merely that this is not the only way of following the text and using the diagrams to arrive at the theorem (Giaquinto, 1993a, p. 88)

Secondly, although sense (visual) experience and innate (visual) capacities are very important in his theory, it does not mean that knowledge acquisition through visualization should be seen as automatic or independent of instruction.

> My hypothesis is that the hidden process involves the activation of dispositions that come with possession of certain geometrical concepts (e.g. for square, diagonal, congruent). What triggers the activation of these dispositions is conscious, indeed attentive, visual experience; but the presence and operation of these dispositions is hidden from the subject. (Giaquinto, 2008, p. 33)

Although Giaquinto (2007) is much more concerned with philosophical issues than with teaching and learning, the implication of his ideas is that visualization, and visual representations, may occupy a more central status in the learning of mathematics than just being accessories to proofs and explanations, or ways of representing a given mathematical object. As Rodd (2000) puts it, "his visualizing can constitute a warrant for mathematical belief [...], a knowledge-generating process" (p.

238). In terms of my study, Giaquinto's discussions highlight the solid foundations for the possibility of building knowledge of fractions on visualization and visual representations, using them as means to discovery and knowledge acquisition and as warrants for mathematical reasoning.

4.1.2 Grounding metaphors

Although Giaquinto (2007) sometimes uses general terms such as sense-experience and mental propensities, his work focuses exclusively on visualization, leaving out of his proposals other senses and bodily experiences. Why did he not expand towards other senses?

The answer I provide to this question lies in the nature of the evidence that backs Giaquinto's arguments. At the beginning of his book, he highlights that his ideas were only possible due to recent advances in cognitive psychology showing the connections between seeing and visualizing. To the best of my knowledge, there is no empirical evidence and not even methods to capture such evidence regarding other senses and bodily experiences and, for that reason, he limited himself to visualization.

However, authors such as George Lakoff and Mark Johnson, coming from a linguistic background and relying on a different set of evidence, developed a theory proposing that bodily experiences in general are fundamental for human cognition. Instead of emphasizing innate or very basic mental abilities, they focus on the human capacity of transferring the functioning structure of concrete, physical, bodily experiences to other more abstract contexts via metaphors.

According to Lakoff (1993), his proposals regarding the role of metaphor for human cognition was originally inspired by the analysis presented by Michael J. Reddy in his classical paper "The conduit metaphor — a case of frame conflict in our language about language" (Reddy, 1979), where he shows that the way people use English language to talk about language suggests that we conceive language as a conduit for meaning and knowledge. This analysis gave rise to what Lakoff (1993) termed *contemporary theory of metaphor*, in which:

> metaphor is not just a matter of language, but of thought and reason. The language is secondary. The mapping is primary, in that it sanctions the use of source domain language and inference patterns for target domain concepts. (p. 208)

According to this view, metaphors allow us to transfer inference patterns and functioning structures from concepts in a domain (the source domain) to concepts in another domain (the target domain), not only in linguistic and poetic terms but most importantly in terms of reasoning and behaviour. What Reddy (1979) showed was that the way we talk about language is impregnated with metaphors that link this abstract domain (language) with a more concrete one (conduit). The movement from the local claim made by Reddy (1979) towards a whole view on human cognition, now called embodied cognition, was given by Lakoff (1993) when he claimed that "as soon as one gets away from concrete physical experience and starts talking about abstractions or emotions, metaphorical understanding is the norm" (p. 205) and that "our everyday behaviour reflects our metaphorical understanding of experience" (p. 204).

This claim means that we use metaphors to transfer what we know regarding concrete concepts, due to our bodily experience in the world, to transfer ways of thinking, inferences and relationships to more abstract domains that are less clearly delineated. Thus, metaphors here are not seen as poetic tools, but as fundamental cognitive tools that allow us to extend what we experience physically to other domains.

In 1980, George Lakoff and Mark Johnson published the book "Metaphors we live by" which is the first one to fully explore this contemporary theory of metaphor and the emergent ideas of embodied cognition. In this book, the authors discuss metaphors such as propositional logic as containers and the widespread use of the orientation metaphor (up and down) across several different domains. Their goal is to show how our understanding in different domains is grounded on physical experience and how this can be unveiled by linguistic analysis. Later in the book, they clarify that:

> We are not claiming that physical experience is in any way more basic than other kinds of experience, whether emotional, mental, cultural, or whatever. All of these experiences may be just as basic as physical experiences. Rather, what we are claiming about grounding is that we typically conceptualize the nonphysical in terms of the physical—that is, we conceptualize the less clearly delineated in terms of the more clearly delineated. (Lakoff & Johnson, 1980, p. 59)

Returning to the case explored by Reddy (1979), language is an abstract domain so, according to this view, we conceptualize it in terms of a more delineated domain such as the idea of conduit, which can be experienced in a very concrete level. Moreover, as mentioned before, the author shows that it is not only a linguistic issue, but the metaphor influences the way we reason and, ultimately, behave when it comes to language.

In order to understand the mechanisms through which metaphors work cognitively, Johnson (1987) proposes the concept of schema.

> A schema is a recurrent pattern, shape, and regularity in, or of, these ongoing ordering activities. These patterns emerge as meaningful structures for us chiefly at the level of our bodily movements through space, our manipulation of objects, and our perceptual interactions. (p. 29)

The author explores the container schema that emerges from our concrete, physical experience with containment, and shows through a linguistic analysis how propositional logic emerges from it. Other schemas, such as balance and path, are also discussed extensively in this book and in other related publications by the author.

At this point, even though the main ideas were already established, embodied cognition was mainly restricted to philosophers, due to their claims regarding the nature of knowledge, and to linguists, due to the original fields of interest of the main authors and the nature of their analysis. Although the main step towards mathematics was given in the book "Where mathematics comes from" published in 2000 by George Lakoff and Rafael E. Núñez, the first step was given by Núñez, Edwards & Filipe Matos (1999). The authors discuss how embodied cognition, treated as a new way of conceiving human cognition, can be understood in contrast to behaviourist and situated views. According to them, while situated perspectives:

> acknowledged that learning and teaching take place, and have always taken place, within embedding social contexts that do not just influence, but essentially determine the kinds of knowledge and practices that are constructed. [...] Research and theoretical frameworks based on a situated approach to cognition insist that linguistic, social, and interactional factors be included in any account of subject matter learning, including the learning of mathematics. (p. 45)

Although they recognize the advances promoted by the situated view in opposition to more traditional behaviourists views, they propose that:

> the nature of situated learning and cognition cannot be fully understood by attending only to contextual or social factors considered as inter-individual processes. Thinking and learning are also situated within biological and experiential contexts, contexts which have shaped, in a

> non-arbitrary way, our characteristic ways of making sense of the world. (Núñez et al., 1999, p. 46)

Next, they revisit the concept of schemata, reinforcing the importance of conceptual metaphors in mathematics.

> Conceptual metaphors are 'mappings' that preserve the inferential structure of a source domain as it is projected onto a target domain. Thus the target domain is understood, often unconsciously, in terms of the relations that hold in the source domain. For instance, within mathematics, Boolean logic is an extension of the container schema, realized through a conceptual metaphorical projection of the logic of containers. This metaphorical projection preserves the original inferential structure of IN, OUT, and transitivity, developed originally via physical experiences with actual containers, and later unconsciously mapped to a set of abstract mathematical concepts (Lakoff and Núñez, forthcoming). The 'projections' or 'mappings' involved in conceptual metaphors are not arbitrary, and can be studied empirically and stated precisely. They are not arbitrary, because they are motivated by our everyday experience — especially bodily experience, which is biologically constrained. (p. 52)

The paper ends with an analysis, that later was named "mathematical idea analysis" (Nuñez, 2009), of the concept of continuity showing how it is grounded on bodily experiences via metaphors.

This approach is extended and deepened by Lakoff and Núñez (2000). Although the book is quite controversial in terms of its philosophical claims regarding the nature of mathematics (Goldin, 2001), their proposal regarding the role of metaphors in mathematical knowledge has been used and explored by many authors in mathematics education with different emphasis (Sfard, 1994; N. Sinclair & Schiralli, 2003; Tall, 2013).

Lakoff & Núñez (2000) argue that all mathematical knowledge is built upon metaphors ultimately grounded in physical experiences that precede formal training, and this became one of the key assumptions of the subdomain of embodied cognition in mathematics education (Sriraman & Wu, 2014).

In order to develop their ideas, they differentiate between two types of metaphors: linking and grounding metaphors. The first is the type of metaphor that connects two different domains inside mathematics. As the authors explain, linking metaphors:

> occur whenever one branch of mathematics is used to model another, as happens frequently. Moreover, linking metaphors are central to the creation not only of new mathematical concepts but often of new branches of mathematics. As we shall see, such classical branches of mathematics as analytic geometry, trigonometry, and complex analysis owe their existence to linking metaphors. (p. 150)

An example discussed by the authors is the development of boolean algebra, establishing a connection between arithmetic and classes.

The other type of metaphor is the grounding metaphor. Lakoff & Núñez (2000) define them as the "metaphors that allow you to project from everyday experiences (like putting things into piles) onto abstract concepts (like addition)" (p. 52-53). Consider the example below given by the authors:

> When we conceptualize numbers as collections, we project the logic of collections onto numbers. In this way, experiences like grouping that correlate with simple numbers give further logical structure to an expanded notion of number (p. 54)

Note that it is not only an issue of representing numbers as collections but being able to transfer the structure of collections (arguably very intuitive due to our experiences in the physical world) to the context of numbers (more abstract). The importance of metaphors does not come only from the fact that 5+3 can be seen as 5 flowers being put together with another 3 flowers (change of representation), but from the fact that our experience with collections allows us to realize that putting 5 flowers together with 3 flowers has the same result as putting together 3 flowers and 5 flowers, and therefore, 5+3 must be equal to 3+5 (transference of inferential roles).

Lakoff & Núñez (2000) only explore grounding metaphors in the first chapters of the book, when discussing arithmetic and set theory. From that point on, other areas of mathematics, such as limits, continuity, calculus, trigonometry and complex numbers are explored via linking metaphors. However, these linking metaphors could all be traced back to arithmetic and set theory. Thus, from their perspective, all mathematics is ultimately grounded on a small number of grounding metaphors which, by definition, are anchored on bodily experiences.

Thus, even though linking metaphors are more salient in the work of mathematician and seem to be richer and leading to more complex results, grounding metaphors seem to be more relevant when it comes to basic mathematics. Moreover, linking metaphors do not sound promising for low achieving students, since they lack prior mathematical knowledge, while grounding metaphors, exactly for relying on concrete experiences, may present itself as a promising path for these students to build a way towards some topics. As emphasized by Núñez et al. (1999) if one admits the basic premises of embodied cognition there are entailments for the teaching of mathematics.

> Rather than looking for better ways to help students learn 'rigorous' definitions of pre-given mathematical ideas, we need to examine the kinds of understanding and sense-making we want students to develop. We should look at the everyday experiences that provide the initial grounding for the abstractions that constitute mathematics. (p. 61)

However, this claim should not be taken as a naive appraisal for teaching with concrete materials or emphasizing a daily use of mathematical ideas.

> At times, this grounding can be found in immediate physical experience, as in the case of work with early arithmetic, space, size, and motion. At other times, the grounding for a mathematical idea takes place indirectly, through a chain of conceptual mappings (Núñez et al., 1999, p. 62)

Johansen (2014) proposes a tool that can be used to help in the process of building metaphors. After analysing the cognitive function of symbols, figures and diagrams, he concludes that figures and diagrams can both serve as "material anchors for conceptual structures" (p. 89). His proposal is that the likeness of some visual representations with material objects is capable of enabling the conceptual mapping from one domain to another. Taken together with Giaquinto's (2007) ideas, this claim reinforces the relevance of visual representations in mathematical knowledge acquisition.

According to authors throughout this section, metaphors are a key element for human cognition. They serve as basis for us to extend the functioning structure from a more concrete, more clearly delineated domain to more abstract domains. In mathematics, even though metaphors linking different concepts are more frequent in the work of mathematicians and necessary for the advance of the field, basic concepts, are connected to concrete domains via grounding metaphors. Although these metaphors may not be obvious or directly accessible, Johansen (2014) claims that certain visual representations can elicit conceptual mappings from concrete to abstract domains. His claim, therefore, reinforces the possibility of grounding mathematics topics, such as fractions, in concrete experiences.

The great advantage of using such approach would be that it does not rely on prior knowledge about other mathematical topics, which can be an issue for low achieving students. Therefore, grounding metaphors may offer an alternative pathway to some mathematical concepts, such as fractions.

4.2 Reasoning

Since the processes of knowledge acquisition and learning are essentially internal and not directly accessible for analysis, my data analysis will focus on students' reasoning as a way to access their learning. Thus, I present the main ideas regarding reasoning below and then, in the next section, connect these ideas with what was discussed above.

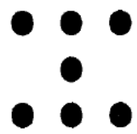
I define reasoning as an externalized sequence of arguments that supports the solution of a task, as the result of cognitive processes that enable the solution of the task, and is socially constructed as part of the classroom culture established between teacher and students. For me, reasoning is different from proof. In fact, proof is a specific form of reasoning, whose norms are dictated by the community of mathematicians. Reasoning, on the other hand, is any attempt at supporting an assertion rationally, from the perspective of the one presenting the arguments.

Stephen E. Toulmin, a philosopher concerned with reasoning in a broad range of formal and informal settings, proposed a structure (or layout) that could be used to break an argument into different components in order to enable analysis. His ideas are widely used in science education and were brought to mathematics education by Krummheuer (1995). Since then, his ideas have been adopted by researchers with varied interests within the field (Jeannotte & Kieran, 2017; Lithner, 2008; Yackel & Cobb, 1996).

Krummheuer's (1995) version of Toulmin's layout divides an argument into four components: conclusion, data, warrant and backing.

- Conclusion is the final claim, the answer that is considered correct by the reasoner;
- Data offers support for the conclusion. This component typically emerges when 'how' questions are asked. Further on, it is possible to question any particular information, action or claim in the data. In that case, the reasoner has to provide different or additional data to support that claim;

- Warrant is about the explanatory relevance of the data. It elicits the reasons why the data provided explains the conclusion. Typically, warrants emerge from 'why' questions;
- Backing is the justification of why the warrant should be accepted.



In order to understand the four components, Yackel (2004) illustrated them with the following example generated from the question "How many dots do you see?" followed by a quick display of the image on the right.

> Using the language of Toulmin's argumentation scheme, we would say that the conclusion is, [There are] seven [dots]. The data is, there are three and one and another three. This data is adequate for those children who know immediately that 3+1+3=7. However, those children who do not just know may require a warrant. A warrant would provide the explanatory relevance of the data, that is why three and one and three have anything to do with the conclusion, seven. For these children, counting the totality would be further (perhaps necessary) backing. (Yackel, 2004, p. 10)

It may be useful to consider the two examples below. They are fictional but inspired by lesson observations at Purple Valley. Both were triggered by the question 'Work out the sum 1/2+3/8'.

Reasoning sequence	**Components according to Toulmin's layout**

`Teacher: What is the answer?`
`Student: 14/16`
`Teacher: How do you know that?`
`Student: I did cross and smiles`[2]`.`
`Teacher: Why have you done that?`
`Student: Because that is how you add fractions.`
`Teacher: How do you know that?`
`Student: Because you told us so.`
	
	`Conclusion`
	
	`Data`
	
	`Warrant`
	
	`Backing`

`Teacher: What is the answer?`
`Student: 7/8`
`Teacher: How do you know that?`
`Student: Because I did 4/8+3/8`
`Teacher: Why have you done that?`
`Student: Because 1/2 is the same as 4/8.`
`Teacher: And how do you know that?`
`Student: Because four eights go into a half`
	
	`Conclusion`
	
	`Data`
	
	`Warrant`
	
	`Backing`

Table 7: Two examples of reasoning sequences

2 This is a mnemonic for the procedure to add two fractions.

Toulmin's original proposal actually has two more components, rebuttal and qualifier, that were not used by Krummheuer (1995), and are usually not even mentioned in research in elementary mathematics education, even though they seem to be relevant to understand mathematical reasoning in higher education (Inglis, Mejia-Ramos, & Simpson, 2007). The rebuttal refers to conditions added to the conclusion in order to clarify its scope, and the qualifier refers to the level of the reasoner's certainty regarding the conclusion.

Considering Toulmin's layout of an argument, one could ask what is a good mathematical argument.

Yackel & Cobb (1996) argue that a good argument should be based on mathematical rationale instead of classroom authority. I agree that classroom authority, although inevitable in certain situations, should not be the core of mathematics lessons (Hewitt, 1999). However, in accordance with Lithner (2008), I argue that being based on mathematical rationale is not sufficient. He illustrates this objection with an example from Schoenfeld (1985) showing that novices usually utilise naive empiricism to judge the correctness of geometrical constructions based on their appearance, but using mathematical elements to make the decisions (Does it look tangent? Does it look congruent?). He concludes that being based on mathematical elements is not enough: the content is also important.

Then, he introduces the concept of anchoring, "which refers not to the logical coherence of the warrant, but to its fastening [...] in relevant mathematical properties of the components one is reasoning about" (Lithner, 2008, p. 261). Returning to the examples given above, the one given by Yackel (2004) is anchored in counting strategies, while the first one on Table 7 is anchored in the authority of the teacher and the last one in the visual representation.

Note that the author defines anchoring related to warrants, but the same idea can be applied to backing (Toulmin himself recognizes that warrant and data are not always easy to distinguish and that this does not create any theoretical problem). Also, it seems reasonable to assume that the nature of data would be similar to the nature of its warrant and backing. Therefore, one can talk in more general terms about the anchoring of an argument.

The idea of anchoring sounds a more sensible way to analyse reasoning presented by lower secondary students while solving tasks about relatively foundational topics, such as fractions, than the full layout of an argument proposed by Toulmin (1969). My interest is in identifying whether the participant students will be able to reason about fractions, and if that reasoning is anchored in the visual representations.

4.3 Visual representations and reasoning

In this section, I will discuss what would be the possibilities for anchoring, in order to justify one of the reasons why I expect visual representations to be particularly helpful with low achieving students.

As mentioned in the previous section, two possibilities for anchoring are authority, and mathematical elements (Yackel & Cobb, 1996).

Although authority may be seen as the least valuable anchor for mathematical reasoning, it is important not to dismiss it totally, but to understand its role in mathematics. To do so, the concepts of arbitrary and necessary knowledge proposed by Hewitt (1999, 2001a, 2001b) seem appropriate. According to Hewitt, mathematical knowledge can be divided into these two groups. Some pieces of knowledge are *arbitrary*

> if someone could only come to know it to be true by being informed of it by some external means — whether by a teacher, a book, the internet, etc. [...] It is not only labels, symbols or names which are arbitrary. The mathematics curriculum is full of conventions, which are based on choices which have been made at some time in the past. (Hewitt, 1999, p. 3)

Complementarily, the author defines *necessary* knowledge as:

> things which students can work out for themselves and know to be correct. They are parts of the mathematics curriculum which are not social conventions but rather are properties which can be worked out from what someone already knows. (Hewitt, 1999, p. 4)

Hewitt (1999) suggests that "mathematics does not lie with the arbitrary, but is found in what is necessary" (p. 5). However, he recognizes the importance of arbitrary knowledge *per se,* and as a stepping stone for some necessary knowledge to be worked out.

The connection between arbitrary and necessary knowledge and anchoring in authority and mathematical elements is quite clear. On one hand, arbitrary knowledge is unavoidably anchored on some type of authority. It could be the authority of a person such as a teacher ("because the teacher told me so"), a cultural convention or historical reasons (the measure in degrees of a given angle) or linguistic reasons (the way polygons are named in Chinese). On the other hand, necessary knowledge (if presented by the teacher as such, and not converted into something arbitrary) is anchored on knowledge previously known by the person working it out. Therefore, new necessary knowledge is anchored in previous mathematical knowledge (arbitrary or necessary), in a dynamic very similar to the idea of proofs based on axioms or properties already proven.

This could lead to the same question posed in Section 4.1 regarding how this process starts, but the aspect I want to discuss here relates to what happens if we are dealing with low achieving students, who in general lack prior mathematical knowledge. If it is not desirable to transform necessary knowledge into arbitrary knowledge, and if the students lack the knowledge that could be the anchor of a new piece of knowledge, what can be done?

The answer that I will explore in this thesis is grounding metaphors connecting concrete experience to abstract concepts via visual representations, as argued by Johansen (2014):

> figures do not only provide a material anchor for the conceptual structure at hand; they provide an anchor that grounds our understanding of the conceptual structure in every-day sensory-motor experience of the physical world. (p. 94)

In his paper, Johansen (2014) shows how this process unfolds by carefully analysing one example: circles as a means to the grounding metaphor of `sets as containers`. He concludes that:

> The circles might not have a direct likeness with mathematical sets, but they do have a direct likeness with containers, and when we conceptualize sets as containers, the diagram gets an indirect or metaphorical likeness with mathematical sets as well. (p. 102)

According to the Johansen (2014), the circle diagram enables the conceptual mapping of one domain into the other. Analogously, I will explore the possibility of using a visual representation to enable the conceptual mapping from area of surfaces into fractions or, in Lakoff & Núñez's (2000) terms, to promote the `fractions as areas` metaphor.

In summary, the arguments presented by Giaquinto (2007) show that humans are capable of acquiring new knowledge based on visualization (visual experience plus fundamental visual skills), while the arguments presented by Lakoff & Núñez (2000) suggest that the mechanism of transferring the basic functioning structure from a more intuitive context to a more abstract context is key to human cognition. This combination offers a third option for anchoring: not authority (undesirable when possible), not previous knowledge (barrier for low achieving students), but grounding metaphors via carefully chosen visual representations. This approach resonates with Johansen's (2014) ideas regarding the epistemological role of diagrams, Rodd's (2000) view on visual warrants and Rivera's (2011) proposal of diagrammatic reasoning.

4.4 The research question

At this point, based on the experiences reported on in Chapters 2 and 3, and the theoretical considerations presented above, I defined my research question as the following: **what is the effect of a set of lessons based on a carefully chosen visual representation on low achieving students' reasoning about fraction?**

First of all, the question above presupposes an intervention at a classroom level. Second, it is intentionally exploratory in nature and, in such a scenario, as discussed by Hammersley & Atkinson (2007), it is common to start with a broader question or goal and refine it during the research process, according to the possibilities and limitations offered by the environment.

However, it is important to specify some other features of the research question in order to understand the choices that will be described later regarding methods and data collection, as well as the scope of the conclusions I may reach by the end of this thesis.

Firstly, since I am interested in classroom level research, my focus is more on learning as it takes place in an educational environment than on cognitive development *per se*. This has some implications:

- During the intervention, I will maintain the integrity of the teachers of the groups, instead of assuming the role of the teacher myself. This would increase the ecological validity of the findings and reduce my influence on the data collected;

- The data should be collected during lessons or in an environment analogous to a regular classroom. Details will be discussed on Section 4.5;
- The data collection should take place during an extended period of time, as teachers expect mathematical knowledge to be acquired or constructed by students over time. The natural timescale would be the academic year.

Secondly, the choice of focusing on low-achieving students implies collecting data from students placed in low sets. The specificities of such students will be discussed in Section 6.3.

Thirdly, the characteristics of the lessons that compose my intervention are of utmost importance to the conclusions. Therefore, I will discuss the whole process of designing the lesson plans in depth in Chapter 5.

4.4.1 On what this research is not about

Since the beginning of my research process, I have been asked by colleagues about how I would compare "my lessons" with some sort of control group. Although it was always very clear to me that the research would not follow any sort of experimental design, the recurrence of the question made me aware that I would need to justify why that is the case.

Firstly, my research goal is not to compare two or more approaches, but to understand how a particular approach, enacted through a set of lesson plans, would unfold within a specific context. Therefore, description, discussion and understanding of this process should inform choices of methods and analysis.

Secondly, I believe there is a huge methodological gap regarding assessment of low achieving students that is, to my knowledge, not discussed in the literature. These students, especially in the UK, due to the wide adoption of ability setting in mathematics, are exactly those who have constructed a negative attitude towards school as a result of their successive experience of failure (Boaler & Wiliam, 2001; Zevenbergen, 2003). For this reason, I argue that the scores these students get in a test are a worse indication of their learning than could be considered for high achieving students, since they may fail at a question because of their lack of interest or engagement with the test rather than because they do not know how to solve it.

Unfortunately, to my knowledge, this issue is neglected in the literature in mathematics education. In my opinion, this is due to:

> Researchers avoiding research in 'messy' environments [...] when they feel that this choice of context can lead to their research being considered as methodologically poor (Stylianides & Stylianides, 2013).

While the quotation above refers to "actual classrooms", following Skovsmose's (2011) suggestion that "90% of research in mathematics education concentrates on the 10% most affluent classroom environments in the world" (p. 18), my perception is that the problem is even bigger for low-achieving groups. Although I have no systematic data to support my position, there is an incident that can illustrate my position. It took place by the end of the first term, when one of the teachers, David, asked me if I could prepare a new version of the diagnostic test I used at the beginning of the academic year (see Appendix 10.1 for the original version of the test). His intention was to evaluate the progress of the group. Even though this was not part of my data collection, I prepared a new, but very similar version of the test, which he applied to the students and I marked. The table below shows the results for the 8 students that took both tests.

Questions	1A	1B	1C	1D	2B	2D	2E	2F	2G	3A	3C	4
Test 1	8	1	1	1	5	5	6	3	7	6	6	1
Test 2	8	6	3	6	3	6	8	6	7	8	6	1

Table 8: Number of correct answers for each question that had a clear equivalent question in the second version of the test.

David was disappointed by the results, since topics related to questions 1 to 4 were discussed during the lessons already taught to this group. In fact, students have successfully solved questions similar to these in previous lessons, and even though there was some improvement in the scores, David was expecting more.

At the end of the next term, David made the same request. I created a new version of the test, but this time he applied and marked it himself. Unfortunately, I do not have the scores because David was even more disappointed this time. However, I was present when the test was administered, and what became clear during the 15 minutes the students spent with the test, was that most of them were not even trying to solve the questions. They were visibly just waiting for the time to pass, and when David or I walked near them, they would look at the test and act as if they were engaging with it, but only while we were near them.

Even though some students presented a similar off-task behaviour sometimes during the lessons, this was much more salient during the test. I would say that these students have realized that the tests make little difference to them, because 1) the mobility from set to set was minimal in the school, 2) they were already in the bottom sets and 3) the reward they could get in the final examination at the end of Year 11 seemed very unlikely. Therefore, they do not engage with it. If I am right, their score is much more a reflection of their lack of engagement than of their learning.

This behaviour poses an extra methodological challenge to researchers willing to use experimental designs with low achieving students. However, it does not affect greatly the methods I will present in the next section, since they are focused on data collected during the lessons.

4.5 Methods

As explained in the previous chapter, my research goal was to investigate the changes provoked by a classroom-based intervention designed for low achieving students to learn addition and subtraction of fractions through an approach emphasizing visual representations.

Once this was defined, following the scheme proposed by Hammersley (1992), the next step would be to choose my selection strategy.

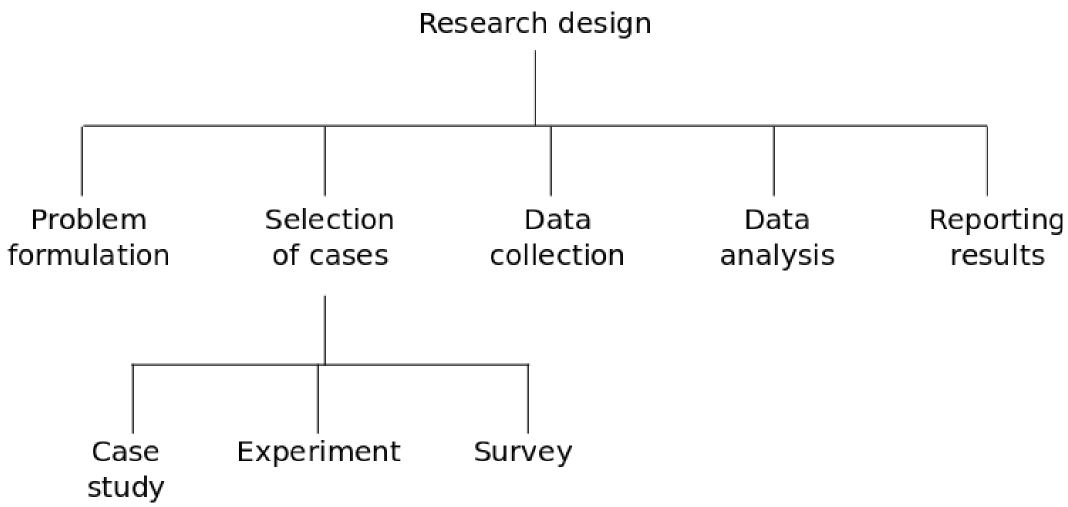


Illustration 13: Research design, according to Hammersley (1992, p. 184)

As can be seen in the scheme above, Hammersley adopts "a narrower definition of the term 'case study' than is conventional" (p. 185) by defining it as a selection strategy in opposition to experiment and survey. He points out that:

> There is no implication here that case studies always involve the use of participant observation, the collection and analysis of qualitative rather than quantitative data, that they focus on meaning rather than on behaviour, or that case study inquiry is inductive or idiographic rather than deductive or nomothetic (Hammersley, 1992, p. 185).

According to Hammersley (1992), the strategies for case selection have advantages and disadvantages that should be weighed bearing in mind the research goal. Experiments are based on cases created by the researcher in order to control theoretical and extraneous variables; consequently, experiments assist in identifying causal relationships, but lose ecological validity. Surveys are based on a relatively large number of naturally occurring cases, increasing the ease of generalisability but decreasing the details and the degree of likely accuracy. Finally, case studies are based on small numbers of naturally occurring cases, enabling a high degree of accuracy and detail, at the same time as maintaining the ecological validity of the findings, but with limitations in terms of generalisability due to the lack of variable control and to the specificity of the findings.

Considering the exploratory nature of my research goal, case study seemed to be the most appropriate selection strategy, the case being the group of low achieving students that participated in my data collection (see Section 6.3). It is important to reinforce that I am not focusing on any specific student, but on the whole group of students. For that reason, I decided not to use names for the students, but to identify each of them by a letter during the Data Analysis chapter.

In summary, my research can be characterized as an exploratory study (Hammersley & Atkinson, 2007) based on a classroom-based intervention (Stylianides & Stylianides, 2013) designed for a specific group of low achieving students. The methods of data collection will be presented and discussed in the next sections.

4.5.1 Within-class clinical interviews

Due to my focus on learning, methods derived from classical clinical interviews (Ginsburg, 1997), such as task based interviews (Carolyn, Sigley, & Davis, 2014), student observation and think aloud methods (Schoenfeld, 1985), and teaching experiments (Steffe & Thompson, 2000) seemed suitable.

However, before the beginning of my data collection two issues emerged.

The first was the Head of Department's hesitation when I suggested interviews with students outside of regular lesson times. According to him, their students were not used to such activities, and the school was not prepared for them. Also, interviews during lesson time, but outside the classroom would imply that staff personnel had been allocated for it, since I am not a teacher and would need to be accompanied during the interviews, and that would also have practical implications for the school. For those reasons, activities outside of the regular lesson context had to be discarded.

The second issue came from the perception that cultural differences between me, (a non-native speaker teacher used to the student-teacher interactions and relations as they are considered normal in Brazil) and the students at Purple Valley could create discomfort, especially if our interaction were to take place in small groups separated from the rest of the group.

Apart from these two pragmatic issues, there was another consideration taken into account when I was defining my methods. It refers to a bias that acts when researchers are choosing students to participate in methods similar to clinical interviews. Unfortunately, to my knowledge, this bias is not documented. However, it can be identified when researchers present their studies. An anecdotal example happened during a summer school for researchers in mathematics education in which I participated. Three researchers used the same methods to collect data for their research project: choose groups of two or three students to work together while instructed to "think aloud" when solving the tasks, and video record the group interaction during the whole lesson. All three researchers reported (in their papers) that the students were chosen according to their attainment levels. However, when I questioned them about the criteria during their presentations, all of them added that they explicitly selected talk-active students.

This is not a surprise, as their methods rely heavily on the students' capacity for communication (between them or to the teacher). However, this may be introducing a bias in terms of which students are being selected by researchers and, consequently, informing their conclusions. This potential bias is particularly important for this research if one takes into account the complex interactions between language, social class and underachievement in mathematics (Gates & Noyes, 2014; Zevenbergen, 2001).

Kvale (2008) gives an interesting example of this phenomenon when discussing interviews. He conducted interviews with young students about academic achievement; one participant suggested that students who talk more get higher grades. Out of curiosity, the author decided to analyse the correlation between the length of the interview by each student with his/her grades and found a statistically significant positive correlation of 0.65. This means that students with higher grades talked more during their interviews, even though there was nothing in the interview protocol that would promote this behaviour. The example illustrates perfectly the bias I was referring to above: researchers may be subconsciously choosing students with certain characteristics to take part in their research just because they are more talk-active and open to interactions, but this can be hiding a social class and achievement bias that is then propagated to their findings.

Pye (1988) also offers an interesting characterization of a particular group of students: the invisible children. According to her, these students occasionally engage in off-task behaviour, but not enough to attract attention, doing enough work to get by, but not enough to stand out. They seem to consciously and actively act to make themselves invisible to the teacher. Reciprocally, teachers do not pay much attention to them because they do not demand it, the other students are usually enough to keep him/her busy, and the reward is usually minimal. If the author is right, this last point could be even more relevant for researchers, who usually have practical constraints that force them to make decisions aiming to optimize their data collection. Once again, this could be promoting the same bias mentioned before, especially considering that this profile was very common among the students participating in my research (more details in Section 6.3).

The alternative I adopted to minimize the three issues mentioned above was to place my data collection during the lessons. As I have never planned to be the actual teacher, I would have enough time to approach students during the lesson and talk to them about the tasks they would be solving. The talk could be audio recorded by a smartpen[3] and complemented by students' worksheets and field notes.

3 Smartpen is a device that is shaped like a pen, records digitally everything that is written down with it and captures the ambient audio. The specific one I used for my data collection enabled me to watch my notes being written as a video synchronized with the audio.

This strategy is similar to what Steffe and Thompson (2000) describe as a teaching experiment, since this approach emphasizes the collection of data while teaching is taking place. Another similarity refers to the focus on students' progress during an extended period of time, instead of local portraits of their knowledge. However, there is an important difference. Steffe and Thompson (2000) emphasize the experimental component of a teaching experiment: the researcher should have a hypothesis and use the interaction with students to test this hypothesis locally. That is not the case for my research; although I have hypothesis and expectations that influenced my research design as a whole, my focus is more of exploratory in nature than hypothesis testing.

Due to all the specificities mentioned above, instead of subscribing to one of the methods discussed, I decided to term my approach "within-class clinical interview". It can be characterized as a middle ground between clinical interviews and teaching experiments, because it has a focus on learning (as both), it takes place during teaching (as the latter), but has an exploratory focus instead of a hypothesis testing focus (as the former).

Although my approach is similar to existing approaches, its novelty demanded extra caution and reflexivity during my data collection, especially in the beginning when its dynamic was unfolding for the first time. In Section 6.6.4, I will describe this process, and based on my experience, reflect on the approach as a whole.

4.5.2 Lesson observations

The second method for data collection I adopted was lesson observations.

As discussed in Section 2.2, this instrument was central during the pre-field work stage and preliminary study to help me familiarize myself with the English educational context, school ethos, teachers' practice and students. During the main data collection, this method had a similar purpose: to help me to build rapport with the groups of students taking part in my research and observe teachers' practices with them. With these purposes in mind, I planned to observe at least one lesson per teacher per week for each group that would take part in my research, as well as other occasional lessons for other low achieving groups of the same teachers.

When I could not identify in the literature any observation protocol that would fit my research goals, I decided to take unstructured field notes paying special attention to: a) use of visual representations by teachers and students, b) any mathematical reasoning expressed by students, c) the nature of the tasks posed to students, and d) how these tasks were used by the teachers. The field notes were complemented by further notes made as soon as possible after the lesson.

Additionally, I planned to audio record all the lessons about fractions, as my focus would be on the within class clinical interviews. This way, it would be possible to reconstruct the overall dynamic of the lessons in the future if necessary.

4.5.3 Complementary data

Apart from the two main methods described above, I also planned to collect data from another two different sources.

The first were audio records of the meetings with teachers. My plan was to promote meetings between the research team and teachers regularly during the academic year. Based on the availability they had in the previous year, it seemed feasible to have about three meetings per term. As it will be explained in the next section, they were important to give the opportunity to the teachers to get in touch with the lesson plans before the actual lessons, to understand my approach to visual representations and to collect their impressions of the lessons as a group.

The second complementary data source for my research was audio recording informal talks with the teachers before and after the observed lessons, and not only those about fractions. The importance of these talks became evident during the pre-field work stage, as moments when the teachers could externalize their impressions and opinions about what had just happened, or expectations and plans for what was going to happen. As stated by Hammersley and Atkinson (2007):

> these 'naturally occurring' oral accounts are a useful source both of direct information about the setting and of evidence about the perspectives, concerns, and discursive practices of the people who produce them. (p. 99)

Although these two sources of data had a secondary importance for the research project, they enabled me to record potentially interesting research data, such as conversations between the teachers and the students that were later described to me by the teachers.

5 Design of the lessons

In this section I will present the design principles behind the lesson plans that I developed for my research (targeted lesson plans from now on), discuss their implications, and present how the observations during the pre-field work and preliminary study stages were taken into account during this process.

Before discussing the design principles, I want to clarify two details. The first one refers to the design of my research. Even though the structure of this chapter may suggest that I adopted a design-based approach to my research project, this is not actually the case. According to Swan (2014b), the distinctive feature of design-based research is the use of "cycles of enactment, observation, analysis, and redesign, with systematic feedback from end users" (p. 148) aiming at refining a product or process. Although it is possible to identify a certain level of redesigning of lesson plans after enactment, observation and analysis during the data collection, my research project was not designed explicitly to enhance this process, and the refinement of the lesson plans was not part of my main research goals.

The second issue refers to the choice of the topic for the lessons. It was agreed by teachers and the research team that the main learning goal of the targeted lessons should be addition and subtraction of fractions, but they should also cover other topics, such as equivalent fractions, necessary to achieve that goal.

5.1 The design principles

There were three principles at the core of the design process of the targeted lesson plans. Two of them are based on empirical results and theoretical considerations related to mathematics teaching and learning, while the third is a personal principle (even though it resonates with some research in education) that I wanted to bring into my research.

The first principle is: **the lessons should enable students to build their knowledge about fractions on visual representations**. It derives from the discussion presented in Section 4.1 showing that it is epistemologically possible (Giaquinto, 2007), from the perception that it is didactically feasible (Rivera, 2011; Watanabe, 2015) to construct knowledge from visual representations, and that this approach has the potential to be particularly effective for low achieving students (Gates, 2015; Lowrie & Jorgensen, 2018).

The implication of this principle is that students should be taught not only how to read and write according to a particular representation, which is called 'treatment' by Duval (2006), but also how to operate with, and transform elements of a particular representation, which Duval calls 'conversion' (2006). Because this takes time, instead of presenting several different representations related to fractions, my lesson plans will explicitly focus on one visual representation, with all the intended concepts, properties and operations regarding fractions (such as equivalence, ordering, addition and subtraction) being taught through operations and transformations on elements of that visual representation in connection with the standard symbolic representation.

The use of multiple representations may lead students to what Rau and Matthews (2017) call "representational dilemma", where students are overwhelmed by the demands of learning the topic itself and the representations attached to it at the same time. The decision of focusing on a single representational model aims at promoting a deep understanding of the topic based on the elements, properties and actions afforded by the representation. Using the ideas of Lakoff & Núñez (2000), it aims at using a more concrete representation as a metaphor for learning a more abstract mathematical topic.

The second principle is: **students should have opportunities to solve the tasks without being told how to do it beforehand**. This principle should be understood in contrast to the common practice among the participant teachers of always showing the students how a particular type of question can be solved right before posing a series of questions of the same type, and will be discussed in the upcoming sections.

At first, this principle sounds aligned to principles related to student centred, investigative, or open-ended approaches to mathematics teaching. However, the principle does not imply that the lessons will be based on open-ended questions, collaborative work and discussions, as it is usually associated with these approaches (Watson, 2008). Actually, a quick look through the lesson plans will evidence that this is not the case. The majority of the tasks proposed in the lesson plans, when seen in isolation, would be considered closed and aligned to teacher centred, traditional approaches (Watson, 2008). The difference here is that the lesson plans were designed in such a way that the students have a chance to solve the questions before seeing a ready-to-use procedure, and after getting acquainted with the elements and operations allowed in the visual representation being used.

Being more specific, two factors are key to satisfying this design principle: the immediacy and specificity of the instruction received by the students before having a chance to solve a particular question. My intention is to avoid very specific instructions (in terms of how to solve the tasks that will be posed during a lesson) being presented right before a task. Instead, students should have opportunities to face a task to which they have never seen a solution specific enough to be simply mimicked, but to which they have had access to the necessary knowledge to solve. In very pragmatic terms, this principle makes explicit my intention of avoiding procedure-focused lessons.

Finally, the third principle is: **keep the lesson plans reasonably coherent with participant teachers' current practices**. This principle is mostly based on personal motivations. My perception is that if the lessons are too different from the practices with which the teachers feel comfortable, it may force the teachers too far from their comfort zones. Some authors may argue that this is necessary to induce change (Penteado, 2001). However, I believe that it is neither fair nor sustainable, because it may put the teacher in an uncomfortable position, and does not consider the knowledge and experience the teacher has accumulated, and the hidden reasons that have shaped her/his practice as it currently is.

Therefore, instead of suggesting a complete revolution in how a teacher teaches, I want to propose something that is familiar, but incorporates some new research based elements. This attitude resonates with Guimarães' (2015) ideas of trying to aim at an "innovation zone", which would be a middle ground between comfort and risk zones, where the teacher still has a certain feeling of control, but at the same time, pushes his/her boundaries towards new practices regarding some specific aspects.

Once the design principles are set, I will discuss literature related to teaching and learning fractions.

5.2 An overview about fractions

Fractions is a recurrent topic in mathematics education and it is easy to find reviews referring to studies from several decades ago showing that the topic is challenging for teachers and students. However, the study provided by Zhang et al. (2014) is particularly intriguing because they revisit research that is 200 years old. The authors show that fractions is considered a challenging topic at least since the end of the 18th century to the point of being consciously omitted from some arithmetic books and of researchers and teachers arguing against its inclusion in regular curriculum even in a very basic level.

It is not the focus of this thesis to engage in such discussions, but it is important to recognize the challenge faced by teachers and students when the topic of the lesson is fractions and for researchers trying to understand how this scenario can be improved.

One of the explanations proposed by researchers to explain the challenge of fractions is summarized by Charalambous and Pitta-Patanzi (2007) when they commented that:

> researchers and scholars agree that one of the predominant factors contributing to the complexities of teaching and learning fractions lies in the fact that fractions comprise a multifaceted construct (p. 293)

This view started with Kieren (1976) and was further explored by researchers that became members of the Rational Number Project[4], such as in Behr et al. (1983). The project extended for about three decades, generating several academic papers, many instructional materials and became hugely influential around the world.

Based on a semantic analysis of the topic (Olive & Lobato, 2008), Behr et al. (1983) proposed that fractions are composed of five different sub-constructs: part-whole, operator, ratio, quotient and measure. Behr et al. (1983) also linked the sub-constructs with different representations, such as number lines for the measurement sub-construct and area for the part-whole sub-construct, and with operations and properties, such as equivalence for ratio, addition and subtraction with measure, and multiplication with operator.

According to these authors, the different meanings and associations related to rational numbers would be responsible for the difficulty of teaching and learning fractions. A common reaction to this claim is to assume that a full understanding of fractions is only possible if students learn different representations. The reasoning is basically that only multiple representations can cover such variety of meanings encompassed by the concept. This became a basic assumption for many studies, such as Charalambous & Pitta-Pantazi (2007), Deliyianni & Gagatsis (2013) and Cramer & Wyberg (2009) and even influenced official documents on mathematics teaching and

4 www.cehd.umn.edu/ci/rationalnumberproject

learning in some countries, where knowledge of multiple representations is seen as a condition (or even as synonym) of conceptual knowledge.

However, this approach may lead to a new problem: the representational dilemma. It occurs when students have to learn domain knowledge from different representations while learning about the representations themselves. The dilemma may take place when working with a single representation, but it can be magnified if students have to deal with more than one representation because they usually rely on different sets of properties and operations: what is allowed in one representation may not be in another (Rau, 2016).

Another limitation regarding the Rational Number Project refers to the causes behind the conclusions. Although they developed instructional materials covering all topics and operations usually taught in basic education regarding fractions that were scrutinized by many studies using different approaches, it is difficult to isolate the causes behind the conclusions because their interventions and materials are usually multifaceted. Nonetheless, some results obtained within this initiative will be used to inform some of the design choices that will be presented in the next sections.

A second internationally renowned research initiative that approached the topic of fractions is the Realistic Mathematics Education (RME) developed in the Netherlands since the early 70s (Van den Heuvel-Panhuizen & Drijvers, 2014). RME is a domain-specific instruction theory for mathematics education that follows certain principles. The theory was originally proposed by Hanz Freudenthal and can be summarized, according to Van den Heuvel-Panhuizen & Drijvers (2014), as: 1) students should be active participants in their learning process, 2) mathematics should start from situations meaningful for students, 3) students develop through stages from informal approaches to more systematic tools and models play an important role in the transition, 4) different topics should not be studied in isolation, 5) learning should be seen as a social activity, and 6) learning opportunities should guide students towards the discovery of relevant mathematics.

These principles and their implications were applied to different topics, from primary to secondary mathematics, resulting in collections of instructional materials developed and refined based on evidence collected in classrooms, in a process that is widely recognized as exemplary for design-based research.

Within this context, Streefland (1991) reports the process of initial development of the unit on fractions. The starting point is the perception that the teaching of fractions:

> consisted of rigid, rule-oriented instruction, and neglected the students` own fragmentary and informal bank of knowledge. It rested on a superficial, brief and one-sided concrete introduction, considerably detached from reality, and isolated within the broader context of arithmetic/mathematics curriculum as a whole. (Streefland, 1991, p. 12)

Their proposal to face this challenge is, as expected, based on the wider principles behind RME. However, there is one aspect related to the third principle that deserves special attention considering the features of my study: their didactical use of models. First of all, it is important to highlight that, in this context, "the term 'model' is not taken in a very literal way. Materials, visual sketches, paradigmatic situations, schemes, diagrams and even symbols can serve as models" (Van Den Heuvel-Panhuizen, 2003, p. 12). Bearing that in mind, the author distinguishes between "models of" and "models for". The first refers to models that emerge from a situation being explored by students maintaining a close connection to it. The latter occurs when "models of" are generalized and "can be used to organize related and new problem situations and to reason mathematically" (Van Den Heuvel-Panhuizen, 2003, p. 14). This distinction is compatible with the idea defended in this thesis of using a model (or visual representation) as more than just a way to represent concepts, but as a means to build new mathematical knowledge.

However, in RME a great emphasis is given to students developing their models from carefully chosen situations, in resonance with the first principle mentioned above, while my approach is to pose a carefully chosen model to be explored by students enabling them to actively build new mathematical knowledge. Although this is a big difference, the details of the development process of their fractions unit reported by Streefland (1991) and the discussion about their use of models presented by Van den Heuvel-Panhuizen (2003) influenced the design of my lessons.

Since 2011, Robert Siegler and collaborators are developing another body of scientific work about fractions that is worth discussion. Siegler et al. (2011) initiated this by introducing the "integrated theory of whole number and fraction development", which posits the magnitude of a number and the number line as central aspects unifying the development of real numbers. Their proposal has its origin in the works of Dehaenne (2011) regarding the importance of number line as a mental representation for numbers and its cognitive role. Instead of emphasizing the disruption in learning from whole number to fractions, as advocates of the whole number bias do, Siegler et al. (2011) emphasize the possible continuity enabled by the concept of magnitude of a number (whole or fractional) connected to the number line representation.

> we propose an alternative theory of numerical development that emphasizes a key developmental continuity across all types of real numbers. This theory proposes that numerical development is at its core a process of progressively broadening the class of numbers that are understood to possess magnitudes and of learning the functions that connect that increasingly broad and varied set of numbers to their magnitudes. In other words, numerical development involves coming to understand that all real numbers have magnitudes that can be ordered and assigned specific locations on number lines. (Siegler et al., 2011, p. 2)

Siegler and contributors conducted several experiments aiming at testing the consequences of such theory. Siegler et al. (2013) summarize these results, restating the adequacy of their theory. A key argument in this paper is a conclusion reached by Siegler et al. (2012) according to which knowledge about fraction magnitude is the best predictor of future achievement in mathematics, even when controlled for knowledge on fraction arithmetic and general achievement in mathematics. Later, Torbyens et al. (2015) concluded the same based on a sample of students in three different countries, USA, Belgium and China.

However, the fact that fraction magnitude is a good predictor of later mathematics achievement does not imply directly that it is a key element for learning of fractions. In an attempt to strengthen their argument, Siegler et al. (2013) cite the study by Fuchs et al. (2013), where the researchers developed an instructional approach based on their theory of integrated development, with great emphasis on fraction magnitude and number line representation. By using an experimental design, the authors not only detected greater gains in the intervention group but also that "improvement in the accuracy of children's measurement interpretation of fractions mediated intervention effects" (Fuchs et al., 2013, p. 683). This result actually strengthens their theory, however, a closer look at the intervention shows that it was composed by several elements apart from the focus on magnitude and number line: "the intervention was designed to address the working memory, attentive behavior, processing speed, and listening comprehension deficits" (p. 687). Therefore, it seems premature to attribute the effects to their theory and more research in a classroom level could help to fill some gaps and strengthen the relevance of their proposal for educational contexts.

In summary, the three initiatives mentioned above show that there are still questions to be answered when it comes to the development, teaching and learning of fractions. Although they do not provide definitive answers, the potentially different constructs within fraction highlighted by the Rational Number Project, the different uses of models promoted by the Realistic Mathematics Education and the emphasis on magnitude given by Siegler and contributors were taken into account when designing the lessons, as it will be discussed in the next section, and when analysing the data in search for meaningful phenomena, as it will be discussed in Chapter 7.

5.3 The choice of the rectangular area model

Following the implications of the first design principle discussed above, I had to decide which model to adopt. The word model here is being used as a synonym to a visual representational system (see Section 2.4.1), which encompasses not only the symbols, but also the rules that allow a person to create new symbols and to transform and operate on them.

The five most widely known models to represent fractions are: rectangular area models, circular area models, number line models, bar models and discrete models. The image below shows the fraction 3/8 represented in each one of the models.

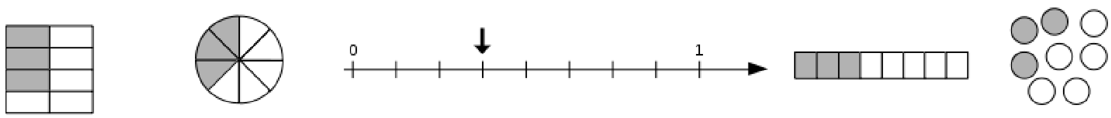


Illustration 14: 3/8 represented, from left to right, in the rectangular, circular, number line, bar and discrete models

During the preliminary study, when it was clear that visual representations were a central aspect of the overarching research project, the rectangular area model (RAM) emerged from the meetings and discussions between the research team and teachers as the preferred model to represent fractions. Later, when the design of the lesson plans actually started, I decided to search the literature for results regarding models representing fractions to confirm that the rectangular area model was actually a good choice and to see how it could actually be implemented.

Authors related to the Rational Number Project, recommend the use of the circular model (Cramer, Wyberg, & Leavitt, 2008). This recommendation comes from studies with experimental or quasi-experimental design aimed at comparing the effectiveness of different models. However, when comparing the use of different models, the way each model was realized was significantly different. For instance, Cramer et al. (2008) compared how students solved addition and subtraction questions using a set of acrylic shapes for the circular, ready-drawn diagrams for the discrete model, and dotted paper for the rectangular area model. Note that the authors varied not only the models, but how they were materialized. All materializations are very different in terms of the affordances and constraints they offer to the students, and therefore, the conclusions may refer not to the models themselves, but to the way they were materialized.

The same could be said about the conclusion reached by Zhang et al. (2014) stating that models based on area (rectangular and circular) should not be the only focus while teaching fractions. They used an experimental approach to compare the outcomes of a group receiving instruction based on a standard textbook that emphasized only area models, and an innovative approach that used "multiple embodiments" of fractions. The problem is that the two instructional approaches differed not only in this aspect. For instance, the latter incorporated elements of games during some lessons.

Martin and Schwartz (2005) also utilized a series of experiments to compare the circular and rectangular area models and concluded that students that used the rectangular area model were more able to transfer their knowledge to new situations. However, the authors recognize that the conclusion may come not from the model itself, but from one difference in how it was realized: while the circular model was composed of ready-to-use edges representing several different unit fractions, the rectangular area model was composed only of congruent squares, with which, the authors argue, students could be forced into figuring out how to represent a given fraction. According to Martin and Schwartz (2005), this extra cognitive exercise, imposed by the way the rectangular area model was realized, may explain the higher levels of transferability.

Other studies have shown that the discrete model is less effective for teaching fractions (Behr, Wachsmuth, & Post, 1988), or that students "operate with the number line representation with more difficulty when compared to circle and rectangle representations" (Tunç-Pekkan, 2015, p. 438).

This brief summary of results shows that it is not simple, and maybe not possible, to draw a clear conclusion regarding which model is more effective for teaching and learning fractions. As Gersten et al. (2009) concluded, the models are usually "part of a complex multicomponent intervention [...] So, it is difficult to judge the impact of the representation component alone" (p. 30). Therefore, I decided to adopt the rectangular area model based on my experience as a mathematics teacher, on the acceptance by the teachers, and on the perceived affordances of the model and of one of its materializations, as will be discussed in Section 5.3.3.

5.3.1 Two characteristics of the RAM

The rectangular area model has two characteristics that seem adequate to the learning goals of my lessons and to the design principles I described before. The first one is the fact that the model is truly bi-dimensional. This characteristic facilitates the combination of two fractions, as shown below.

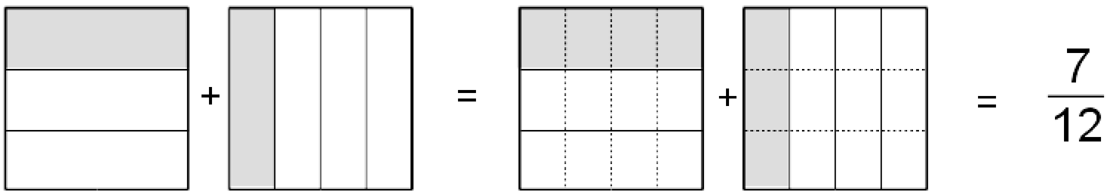


Illustration 15: Working out 1/3+1/4 using the rectangular area model

As addition and subtraction, as well as multiplication and division, are essentially binary operations (in the sense that two fractions are involved), the bi-dimensionality enables students to combine the fractions and then operate them.

One could argue that the circular area model is also bi-dimensional, since the value of a fraction is also represented by the area of a sector. However, because the lines utilised to represent a given fraction have to go through the centre of the circle, the model ends up behaving as a uni-dimensional model. The image below illustrates how 1/4+1/3 could be visualized in this model. Note that this representation is not useful to understand why twelfths can be used to obtain an answer to the same extent that the rectangular area model allows.

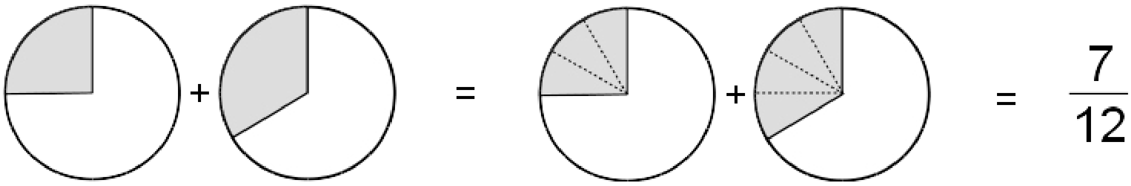


Illustration 16: Working out 1/4+1/3 using the circular area model

The image above could only be used as a way to represent a calculation that had already been made in a different representation. As discussed by Herman et al. (2004), the representation in this case can work as a *post-hoc* justification for a calculation, but fail to inform the process of adding two fractions. The same could be said about the number line and the bar models. Note that although the bar model also uses area to represent fractions, the second dimension (usually the height), is not actually used.

The second characteristic of the rectangular area model is the fact that its parts are similar to the whole in terms of shape, facilitating the visualization of relationships between different fractions, and not only between fractions and the unit. For instance, noticing that 1/8 is half of 1/4 is very similar to understanding 1/2 (of the unit).

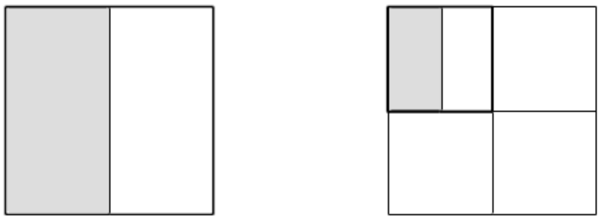


Illustration 17: Similarity between 1/2 and 1/8

The same property does not hold for the circular model where the comparison between 1/8 and 1/4, and between 1/2 and the unit is different in nature, since 1/4 is not similar to the unit. This way, the RAM seems to be able to convey the meaning of the relation between fractions better than the circular model.

These two characteristics reinforce RAM's flexibility and potential for generalization. These two characteristics are highlighted by Van Den Heuvel-Panhuizen (2003) as key for a given model to work as a more than a "model of" a certain situation, but also as a "model for" building new knowledge.

5.3.2 One or multiple representations?

A substantial part of the literature in mathematics education in recent decades seems to assume that using multiple representations is a key aspect of any instruction aimed at conceptual understanding, even though several authors recognize that the use of multiple representations may bring out challenges for teaching and learning. As Ainsworth (2006) puts it: "recently, attention has been focused on learning with more than one representation, seemingly predicated on the notion 'that two representations are better than one'" (p. 183).

Rau and Matthews (2017) is an example of this stance. On one hand, the authors defend the use of multiple representations because "no single visual representation perfectly depicts the complexity of mathematical concepts [and] different representations emphasize complementary conceptual aspects" (p. 531). On the other hand, they recognize that "multiple visual representations are not always more effective for promoting learning" (p. 531) and that its adoption may bring out the "representational dilemma", referring to the phenomena of having to learn the content from visual representations, while having to learn the representations themselves.

Bearing that in mind, I agree with Ainsworth's (2006) conclusion:

> It seems wise to use the minimum number of representations consistent with the pedagogical function of the system. In many cases it may not be appropriate to use [multiple external representations] at all, since one representation may be sufficient and will minimise the split attention affect. (p. 192)

In the same paper mentioned above, Rau and Matthews (2017) recognize that there is research pointing out the existence of privileged representations, that "convey meaning more intuitively than others because the human brain seems to be sensitive to perceiving their referents from their physical structures" (p. 540). From this perspective, other representations should come into play only if they are necessary. The necessity could come from curricular demands, because a particular representation is common in daily life or necessary for future topics, or to highlight specific aspects of certain multi-faceted topics, as some researchers would argue is the case with negative numbers (Ball, 1993).

In terms of my research, even though the students probably had already had lessons about fractions in the past few years, the lesson plans should treat the topic as if new. This decision was made because of teachers' perceptions that even if the students had some knowledge about fractions, they were still struggling with basic concepts, such as equivalence. Therefore, the option of using a single visual representation seemed sensible.

This is also coherent with the emphasis given to the development of powerful models by the researchers at the Realistic Mathematics Education (Van Den Heuvel-Panhuizen, 2003), since it allows students to get familiar with a certain model to the point of using it to approach new situations.

Moreover, the characteristics that will be discussed in the next two sections support my view that the representation chosen could act as a privileged representation (Rau & Matthews, 2017), capable of conveying meaning intuitively and covering most of the aspects necessary for the educational goals of the targeted lesson plans.

5.3.3 The materialization of the model

It is important to clarify the differences between the model itself from the way it is materialized.

When I stated that I have chosen the rectangular area model as the model for the targeted lessons, I meant that fractions would be represented as sections of rectangles in such a way that the ratio between the area of the section and the area of the rectangle defined as the unit is numerically equal to the intended fraction.

The students will not be asked to calculate areas and compare them numerically. Instead, the areas will be compared visually, by composing and decomposing sections of the unit, overlapping and juxtaposing them. For instance, a given section could be identified as representing the fraction $\frac{1}{3}$ by verifying that the unit can be fully covered by three instances of it.

Even though this description is enough to understand the model, it leaves open the question of how it will be materialized. By materialized I mean how symbols pertaining to this model will be physically constructed by students, or which objects and tools will be offered in order to enable them to construct and manipulate these symbols.

From the research mentioned in Section 5.3, it is possible to list some possibilities, such as pre-cut shapes in different sizes representing different fractions, pre-cut shapes all with the same size, previously drawn diagrams, and diagrams drawn by the students. The latter can be further unfolded into several possibilities if one takes into account details of how the diagrams would be produced by the students, for instance, by using squared, dotted or blank paper.

Each option has its own possibilities and limitations that would impact students' learning. My decision was informed by my design principles and by Giaquinto's (2007) and Lakoff & Núñez's (2000) ideas.

Firstly, as I emphasized the importance of students learning not only how to read a visual representation, but also the transformations that could be carried out on it, it was clear to me that students should produce their own representations. Sometimes they would do it entirely by themselves, sometimes they would do it over partially drawn diagrams intentionally designed to reinforce some property or transformation that I wanted to emphasize, or with some other sort of support.

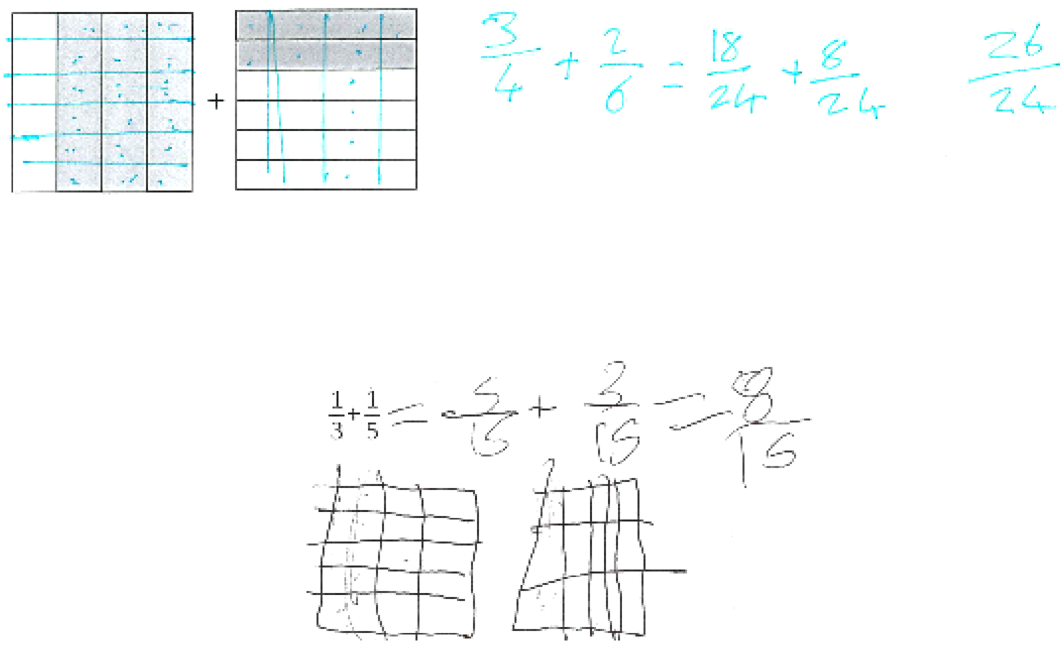


Illustration 18: Example of a solution built over a partially drawn diagram (top) and another fully drawn by the student (bottom)

I did not use squared or dotted paper, because I believe they reinforce the misconception reported by Cramer et al. (2008), where students represent all the unit fractions by the same size (the size of the basic square suggested by the grid or dots) and consequently end up with units of different sizes (as shown below).

In fact, I observed this effect during a regular lesson, taught between the first and second packs, when I decided to observe after the teacher told me she would include a starter with a question about fractions. As usual, students solved the questions in their notebooks, whose pages have a squared grid as a watermark, instead of solving them on a worksheet prepared by me. When I was checking their answers, I noticed one of the most successful students, when solving fraction questions with his own diagrams, drawing the diagrams shown below in order to compare the two given fractions.

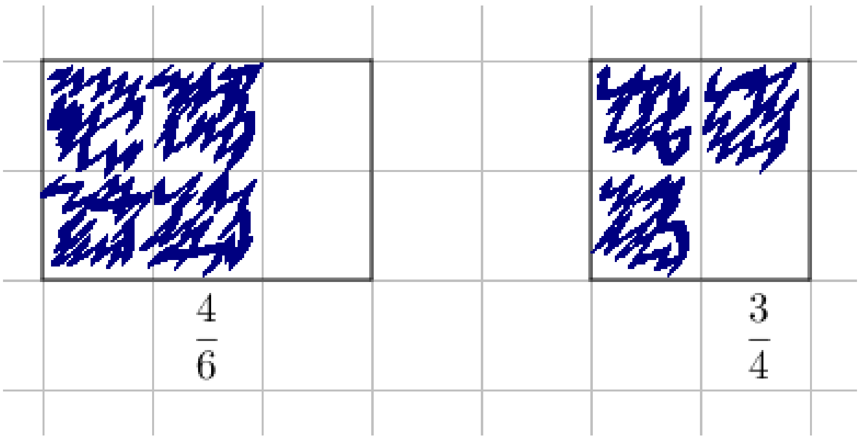


Illustration 19: Typical misconception when drawing using graph paper

Secondly, since I wanted to capitalize on the advantages of visual representations as discussed in Section 2.4, I needed to find a way to materialize the model that embedded the properties and transformations necessary for my lessons in the most intuitive way possible, otherwise I would be bound to explain verbally these properties and transformations, and would lose the very advantages I wanted to capitalize on.

The three inter-related properties and transformations I needed in the model were:

1. Composition and decomposition of shapes: students should be able to decompose a given shape into two or more shapes preserving the areas involved, and reversely, put together shapes (equal or different both in terms of shape and area) in order to obtain another bigger shape;
2. Comparison of areas: students should be able to compare the area of different shapes by composing and decomposing them;

3. Rigid movements: students should be able to move shapes around in order to obtain different configurations.

Although diagrams enable the first affordance (by adding and removing lines), they are not so suitable for the second and third ones. Pre-cut shapes, on the other hand, enable all the items by allowing students to juxtapose shapes (item 1), overlap shapes (item 2), and actually move the shapes around (item 3). If this analysis is correct, the materialization would enable the model to act as a grounding metaphor (Lakoff & Núñez, 2000), as will be discussed on the next section.

For that reason, I decided to use pre-cut shapes (in cardboard, called cut-outs from now on) as the introductory material throughout the lessons. This happened in the three first lessons of pack 1, when the model was introduced using fractions from the same family, and in the first lesson of pack 3, when students explored fractions with any denominators.

However, I recognise that the cut-outs are restrictive, since the students would depend on the existence of sections representing the fractions they need to operate with. For that reason, the lessons gradually progressed towards diagrams.

At this point, one could expect a final step towards purely symbolic approaches. If that was the case, the trajectory from cut-outs to diagrams, and then to symbols, would resonate with a widely accepted movement from enactive to iconic, to symbolic, which is extensively described and discussed by Jerome Bruner and other researchers (Hoong, Kin, & Pien, 2015). However, that was not the case for my lessons. Since I subscribe to Giaquinto's (2007) view that knowledge acquired by visual means should be accepted as mathematically valid, I see no need for moving towards purely symbolic representations. From my perspective, there is no epistemological difference between the two solutions to $\frac{1}{2}+\frac{1}{4}$ shown below.

$$\frac{1}{2}+\frac{1}{4}=\frac{2}{4}+\frac{1}{4}=\frac{2+1}{4}=\frac{3}{4}$$

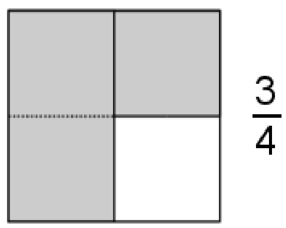


Illustration 20: A purely symbolic solution at the left and a diagrammatic solution on the right

Considering that the teachers were also satisfied with the solution presented on the right side, I can think of no justifiable reason to move towards purely symbolic approaches. It could be argued that it is important for the students to be familiar with the symbolic representation of fractions, since it occurs in some everyday situations and (mostly) in assessments, but the familiarity with symbolic and verbal forms was guaranteed by regular use in the worksheets and explanations from the teacher. What I want to emphasize here is that verbal and symbolic representations were not avoided in any way during the lessons, on the contrary, they were present in the worksheets, animations and explanations. However, there were no urge into using approaches solely dependent on them.

5.3.4 Arbitrary, necessary and grounding metaphors

Considering the reasons and characteristics presented above, it is important to emphasize a consequence of the choices regarding how the rectangular area model is utilized. This consequence is related to the three concepts mentioned in the title of this section.

The concepts of arbitrary and necessary knowledge, as proposed by Hewitt (1999), have previously been discussed in Section 4.3. In summary, arbitrary knowledge is the kind of knowledge that one can only come to know by being told, while necessary knowledge is the kind that can be worked out by a person.

From these definitions, it is clear that necessary knowledge is always the result of some previous knowledge. So, does all necessary knowledge come from a set of arbitrary knowledge that was, by definition, told to the students at some point? If the answer is affirmative, it could be argued that students need a lot of arbitrary knowledge before moving on to build necessary knowledge.

My answer, however, is that apart from these two categories, human cognition can capitalize on some built-in capacities, such as those referred by Giaquinto (2007) and Dehaene (2011) when discussing visual thinking and number sense, respectively. Acting upon that, our capacity to use metaphors to link objects in different domains would allow us to project the inferential structure from the more intuitive domain of areas onto the more abstract domain of fractions, enabling students to work out the intended necessary knowledge.

Bearing that in mind, my intention when designing the lessons was to minimize the amount of arbitrary knowledge, which is inevitably anchored in authority, and capitalize as much as possible on commonplace activities that I expect to act as grounding metaphors.

The one piece of knowledge I assumed as arbitrary was the definition of a fraction in the model. This definition can be summarized by the answer one gives to the question: how should the diagram below be read?

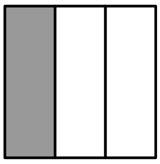


Illustration 21: Example of a diagram representing 1/3

Equivalence is the other key concept that could be presented as arbitrary, depending on how the teachers does it. However, the way I employed the rectangular area model allows the replacement of this arbitrary definition by comparison of area, which can be done in an arguably very intuitive way by overlapping cut-outs or adding lines to diagrams, depending on how the model was materialized. Thus, the concept of equivalent fraction becomes necessary, derived from the behaviour of areas.

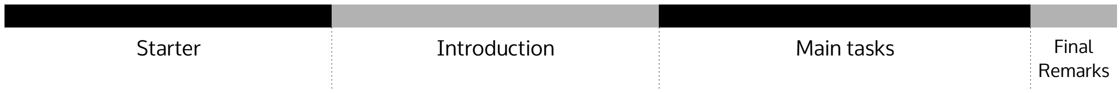
Based on these two starting points, I designed the targeted lessons in a way that all the other concepts and operations could be obtained by the students as necessary knowledge.

5.4 The targeted lessons

Before explaining the structure of the targeted lessons and lesson plans, because of the third design principle (keep the lesson plans coherent with participant teachers' current practices), it is important to understand the structure of the regular lessons for low achieving groups at Purple Valley.

5.4.1 The general structure of a regular lesson

The lessons observed during the pre-field work stage and preliminary study (about 30 in total) allowed me to identify a common structure for the mathematics lessons for low achieving students at Purple Valley. This structure can be summarized by the following scheme, where the relative length of the bars represents the relative duration of each stage in a typical lesson (1 hour).



The starter would be composed of three or four questions about topics discussed in previous lessons, or about arithmetic procedures. After some time, the teacher would discuss these questions with the whole class.

After that, the teacher would introduce the main topic of the lesson. This could vary a lot between teachers and between lessons, but the teacher would usually interact with the students similarly to Watson & De Geest's (2012) characterization of a typical British lesson: teacher asking "short closed questions, wait[ing] a very short time for replies and deal[ing] very briefly with student responses, whether right or wrong, conceptual or procedural" (p. 227). During this stage, the teacher would clarify new nomenclature and recall mathematical properties and procedures related to the topic. However, most of the time would be spent on solving and commenting on example questions.

Once that was done, the teacher would pose a longer task, usually very similar to the questions solved during the introduction and composed of several similar items, or a sequence of two or three closely related tasks.

Finally, the teacher would make some final remarks, checking the answers for some or all of the tasks, followed by some managerial issues, and the lesson would be over.

Although the scheme holds in general for the three participant teachers, below I will present one observed lesson for each teacher and comment about some differences between them.

Julia was more systematic in terms of checking the answers for all tasks during a lesson, which she usually did by asking for input from the students. Sometimes, this process could evolve into a discussion (as opposed to just a very direct exchange of questions and answers), but in general she would ask the students the final answer and sometimes the steps necessary for the solution. Also, during the 'main task' stage she would rarely use a single very long task. Instead, she would pose two or three tasks (each with several items) and use the transitions to check answers and make some remarks. Due to that, sometimes her introduction stage would be shorter than those of the other teachers. Table 9 shows one lesson by Julia to Year 8 set 4 about index laws that is representative of her regular lessons, according to my observations.

The general structure of David's lessons was very similar, except that he never made final remarks and would rarely make comments for the whole class after the introduction stage. His main task, usually one single task long enough to last the rest of the lesson, was always very similar to examples commented on in the introduction. He rarely checked answers with the whole class after the starter, although he interacted individually with some students during the main task stage and checked the answers when a student claimed to have completed the task. Table 10 represents a lesson about rounding decimal numbers that is representative of his regular practice, according to my observations.

Finally, like Julia, Alice usually broke the main task stage into several relatively smaller tasks and used the transitions between tasks for short "lecture-like" moments commenting on the solution of a few items and introducing the next task. Also, her introduction stage was relatively short compared to those of the other teachers. Through this practice she was able to vary the tasks more than David during the same lesson, but at the same time, she always solved an example before posing any task. Table 11 shows a lesson about multiplication using the box method that is representative of her regular lessons.

In the three schemes below, the height of each section is proportional to the time dedicated to that stage within that particular lesson.

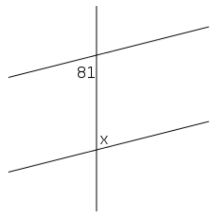
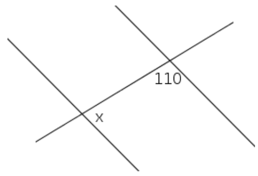
Starter (21 minutes)	The following questions were available to the students when they arrived at the classroom: 1 – Simplify the following by collecting like terms: a) 13a+5b-3a+2b b) 15c+8d-6c-5d c) 9e-6f-3e+10f 2 - Work out the following sum $1060 \div 8$ 3 – Write down the value of the following angles. Give a reason.   After 9 minutes, she started to discuss the questions, which took another 12 minutes.
Introduction (14 minutes)	She presented all the laws using slides and asked the students to copy them into their notebooks. Then, she posed three examples followed by some time for the students to solve them. When discussing each example, she asked for input from the students, but there was no attempt to understand their mistakes and strategies. She would ask until she got a correct answer adding some very short comments about the previous answer, almost as clues to lead them to the right answer. The examples were: $c^3 \times c^4$, $x^7 \times x$ and $a^5 b^2 \times a b^3$
Main tasks (21 minutes)	She posed two other questions very similar to the previous examples and just checked their answers after a brief interval. This could also be considered part of the introduction, but she was talking less and focusing more on getting the questions solved. At this point, I noticed that the lesson was focusing on only one of the index laws although she had presented three. She distributed a worksheet with 12 questions very similar to the examples before and waited some time for them to be solved. During this time, she walked around the classroom checking answers and making short specific comments to some students.
Final remarks (3 min)	A quick correction of the 12 questions with minor comments.

Table 9: Scheme representing a regular lesson by Julia

Starter (11 minutes)	1 – Answer the following: a) -16+7 b) -19-8 c) 11-17 2 – Answer the following: a) 0.37x10 b) 89÷10 c) 5.3x100 3 – Simplify: a) x+2y+x+8y b) 3a+9b+2a-7a After 7 minutes, he started to discuss the questions and this took another 4 minutes.
Introduction (5 minutes)	David introduced the concept of "significant figures" contrasting with "decimal places", which had been discussed in previous lessons. He solved three examples below. When doing so, he used coloured pens to indicate the digit that should be the last one in the answer and the digit that should be taken into account to decide if the number should be rounded up or down.
Main task (33 minutes)	The main task was a single question with 15 items similar to the examples shown above. During this period, the teacher walked around the classroom checking answers and giving brief suggestions to some students. At some point, he went to the board and tried to describe a list of steps that could be followed to solve the questions. However, the procedure was not valid for all the cases. He apparently noticed that and decided not to discuss it further. By the end of the lesson, the teacher had marked almost all answers of all students as right or wrong. A few minutes before the end, he asked them to stick the homework to their planners.

Table 10: Scheme representing a regular lesson by David

Starter (14 minutes)	The following sums were shown on the white board when the students arrived at the classroom: -4-4 4+-2 10-6 -7+8 -5x-2 72÷-3 She discussed only the first three items and only gave the final answer for the last three. The students struggled a lot with the positive and negative signs. She tried to help them with a mnemonic gesture with her arms to memorize that "two minus would join to make a plus".
Introduction (6 minutes)	She introduced the box method through examples: 4x32 and 24x46. When discussing it, she asked students to come to the board and show how to do it. None of the students that went to the board did the multiplication correctly. At some point, she showed how to do it by emphasizing the number of zeros in each cell of the multiplication box.
Main task 1 (25 minutes)	She posed six multiplications with an increasing number of digits to be solved by the students. While they were working out the sums, she walked around the classroom giving extra individual explanations to some students. After 17 minutes, she started to discuss the answers. She did it thoroughly for the first two sums and for the last one.
Main task 2 (9 min)	She posed the following word problem, gave some time to the students to solve it and then solved it by asking for input from the students. Most of them were not able to solve the question individually. I earn £34 a week doing a paper round. There are 52 weeks in a year. If I work every week, how much will I earn in a whole year? She posed another similar problem, but it was too close to the end of the lesson and there was no time to check answers. I could only identify one student that had solved this final question.

Table 11: Scheme representing a regular lesson by Alice

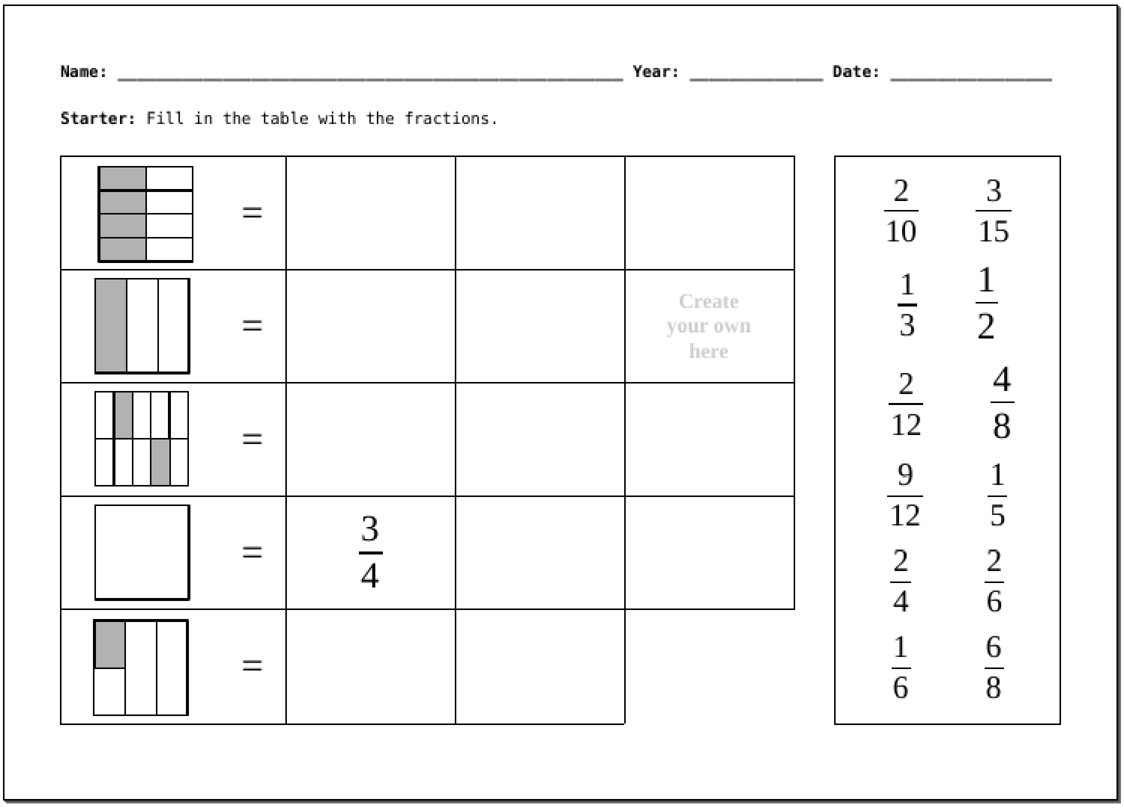
Based on this data, I decided to adopt the sequence starter-introduction-main tasks as the general structure of the targeted lessons. Considering that both Alice and David did not usually save time at the end of the lesson for final remarks, I decided not to include these moments in my lesson plans.

The most relevant difference of my lesson plans compared to their regular lessons refers to the nature of the introduction. As discussed in Section 5.1, one of my three design principles is to give opportunities for the students to engage with the visual representations and build what is necessary to solve the task by themselves. Therefore, instead of showing examples during the introduction and then posing tasks that could be solved following exactly what was shown before, I wanted the teachers to use the introduction to present as minimally as possible. This stage should focus on arbitrary knowledge, and avoid presenting necessary knowledge.

5.4.2 An example of a targeted lesson

The lesson chosen to illustrate the general structure of the targeted lessons was lesson 2.2 (the second lesson of the second pack). The topic of the lesson was addition of fractions with denominators equal to 2, 4, 8 and 16. At this point the students would be familiar with equivalent fractions (main topic of the first pack) and would have just had a lesson about how to decompose the unit into sums of fractions. Lesson 2.2 is the first one where students would have to actually express the addition of two fractions as a third fraction.

The image below shows the starter, the aim of which was to reinforce equivalent fractions.


Name: ______ Year: ______ Date: ______

Starter: Fill in the table with the fractions.

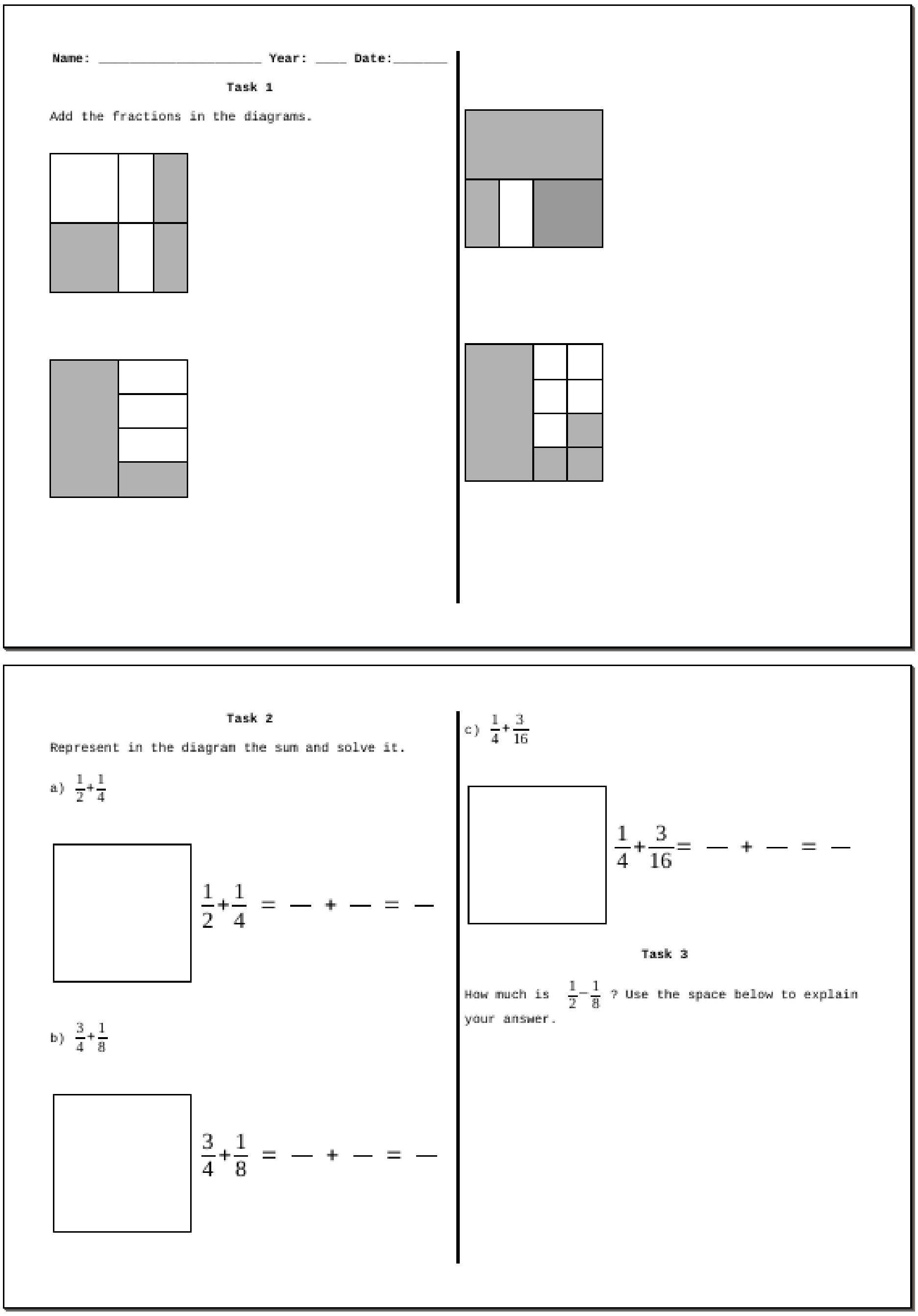
=				
=			Create your own here	
=				
=	$\frac{3}{4}$			
=				

$\frac{2}{10}$ $\frac{3}{15}$ $\frac{1}{3}$ $\frac{1}{2}$ $\frac{2}{12}$ $\frac{4}{8}$ $\frac{9}{12}$ $\frac{1}{5}$ $\frac{2}{4}$ $\frac{2}{6}$ $\frac{1}{6}$ $\frac{6}{8}$

Illustration 22: The starter designed for lesson 2.2

Following the starter, the teacher would check the answers with the whole class and discuss eventual mistakes. Special attention should be paid to the "create your own" item.

Then the teacher would show an animation (https://youtu.be/7FMhj3E1WzI) showing the strategy to add two fractions. If necessary, the teacher should add some comments, but I suggested that the teachers should try to give as minimal extra input as possible, before letting the students try to solve the main tasks (shown below) by themselves.



Name: ____________________ Year: ____ Date:_______

Task 1

Add the fractions in the diagrams.

Task 2

Represent in the diagram the sum and solve it.

a) $\frac{1}{2}+\frac{1}{4}$

$\frac{1}{2}+\frac{1}{4} = \_ + \_ = \_$

b) $\frac{3}{4}+\frac{1}{8}$

$\frac{3}{4}+\frac{1}{8} = \_ + \_ = \_$

c) $\frac{1}{4}+\frac{3}{16}$

$\frac{1}{4}+\frac{3}{16} = \_ + \_ = \_$

Task 3

How much is $\frac{1}{2}-\frac{1}{8}$? Use the space below to explain your answer.

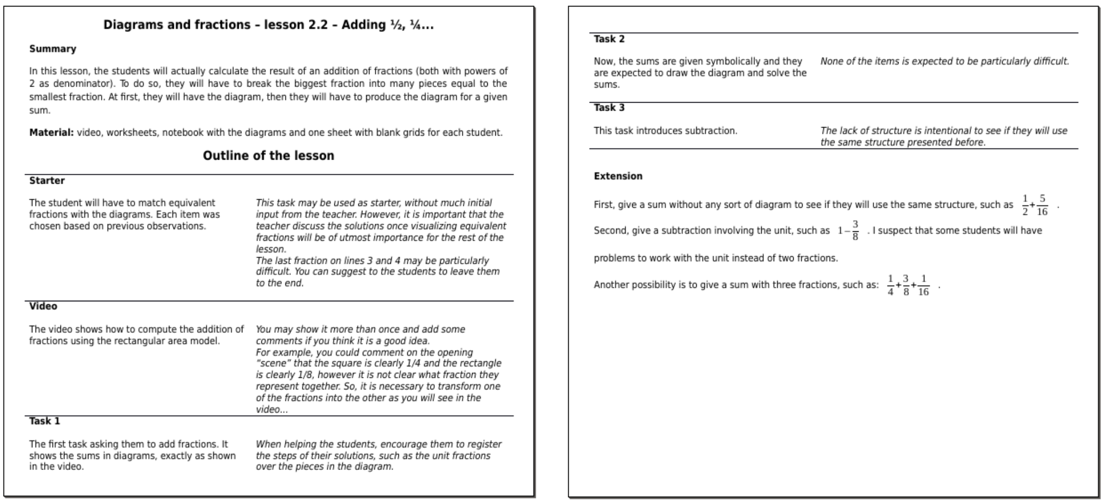
Illustration 23: The three main tasks of lesson 2.2

The main tasks asked the students to add fractions. At first, the fractions were given only in a diagram, and later symbolically but accompanied by a square to support the drawing of a diagram. The third task was a subtraction given symbolically. In addition, some extra sums were suggested in the "Comments for the teacher" (see next section) in case students needed.

Note that some questions posed in the main tasks were closely related to what was seen in the video, whereas others went a bit beyond that by not giving the visual representation or by introducing subtraction.

5.4.3 The lesson plans for the targeted lessons

Apart from the worksheet and eventual manipulatives and videos, all the lesson plans included a document called "Comments for the teachers", whose structure was inspired by the Teaching Materials designed by the ICCAMS project (http://iccams-maths.org/).



Diagrams and fractions – lesson 2.2 – Adding ½, ¼...

Summary

In this lesson, the students will actually calculate the result of an addition of fractions (both with powers of 2 as denominator). To do so, they will have to break the biggest fraction into many pieces equal to the smallest fraction. At first, they will have the diagram, then they will have to produce the diagram for a given sum.

Material: video, worksheets, notebook with the diagrams and one sheet with blank grids for each student.

Outline of the lesson

Starter

The student will have to match equivalent fractions with the diagrams. Each item was chosen based on previous observations.

This task may be used as starter, without much initial input from the teacher. However, it is important that the teacher discuss the solutions once visualizing equivalent fractions will be of utmost importance for the rest of the lesson.
The last fraction on lines 3 and 4 may be particularly difficult. You can suggest to the students to leave them to the end.

Video

The video shows how to compute the addition of fractions using the rectangular area model.

You may show it more than once and add some comments if you think it is a good idea.
For example, you could comment on the opening "scene" that the square is clearly 1/4 and the rectangle is clearly 1/8, however it is not clear what fraction they represent together. So, it is necessary to transform one of the fractions into the other as you will see in the video...

Task 1

The first task asking them to add fractions. It shows the sums in diagrams, exactly as shown in the video.

When helping the students, encourage them to register the steps of their solutions, such as the unit fractions over the pieces in the diagram.

Task 2

Now, the sums are given symbolically and they are expected to draw the diagram and solve the sums.

None of the items is expected to be particularly difficult.

Task 3

This task introduces subtraction.

The lack of structure is intentional to see if they will use the same structure presented before.

Extension

First, give a sum without any sort of diagram to see if they will use the same structure, such as $\frac{1}{2}+\frac{5}{16}$.

Second, give a subtraction involving the unit, such as $1-\frac{3}{8}$. I suspect that some students will have problems to work with the unit instead of two fractions.

Another possibility is to give a sum with three fractions, such as: $\frac{1}{4}+\frac{3}{8}+\frac{1}{16}$.

Illustration 24: Example of a "Comments for the teacher". Although it had been broken into two pages here to save space, the contents fit one A4 sheet

They were usually only one-page long and were composed of: a description of the learning objectives, a list with the material, a commented sequence of expected stages for the whole lesson and some extra questions to be used, if necessary, at the end of the lesson. In the comments I tried to anticipate critical moments and make general recommendations about how the teacher could discuss them with the students and what questions could be used to deepen their understanding.

This format was chosen based on remarks made by the teachers during meetings in the preliminary study, when several options were presented to them for discussion. One MAP lesson was provided as an example of a long and detailed lesson plan[5], one

5 http://map.mathshell.org/lessons.php?unit=6120&collection=8

activity adapted from NCTEM as an example of a very loose lesson plan[6], and one from ICCAMS as an example of a less detailed plan than the MAP lesson, but still with some remarks and suggestions aimed at the teacher[7]. They clearly appreciated the idea of a summary of the whole lesson with notes on critical moments, but disliked the length of more detailed descriptions and recommendations. Therefore, I decide to adopt a style similar to the third option listed above.

5.4.4 The overall learning path

In general, the mathematics scheme of work used by the teachers at Purple Valley groups lessons about the same topic in blocks of three or four lessons each, and revisits that topic every term, deepening some aspect of it, or just reinforcing what was taught before. Based on that, I decided to develop 3 packs of lesson plans to be used: one for each term of the academic year, with each pack being about 3 or 4 lessons long.

As said at the beginning of this chapter, the main learning goal of the lessons was addition and subtraction of fractions. This was agreed at the beginning of the overarching research project because of the common perception that fractions is a particularly challenging topic for low achieving students, and at the same time a topic suitable for a visual approach.

It was clear to me from my first design principle that I had to dedicate a considerable amount of time to clarifying the meaning and the transformations allowed in the rectangular area model before moving to addition. Also, authors such as Besuk and Cramer (1989) and Streefland (1991), from the Rational Number Project and from the Realistic Mathematics Education group respectively, recommend that an understanding of equivalence should precede the operations with fractions.

Based on that, I decided to dedicate all the lessons in the first pack to the introduction of the rectangular area model (lesson 1), the connection between the model and fractions (lessons 2 and 3), and equivalence and comparison of fractions (lessons 2 to 5).

The next step was to decide how to approach addition and subtraction. To my disappointment, I found that there are not enough details reported in the literature, such as Mack (1995), Cramer et al. (2008) and Izsák et al. (2008), in terms of design

6 http://barichello.coffee/public/uploads/extra/fair_shares.pdf
7 http://iccams-maths.org/multiplicative-reasoning

when it comes to these operations. However, reading the tasks used in these papers it is possible to notice a movement starting with pairs of fractions in which one denominator is a multiple of the other (I am calling these 'fractions from the same family'[8] from now on), and then moving on to fractions in which the denominators are not multiples of each other ('fractions from different families' from now on).

In terms of the rectangular area model, this sequence also makes sense, since it is possible to add two fractions from the same family just by altering the diagram of one of the fractions, while for fractions not from the same family both diagrams have to be altered.

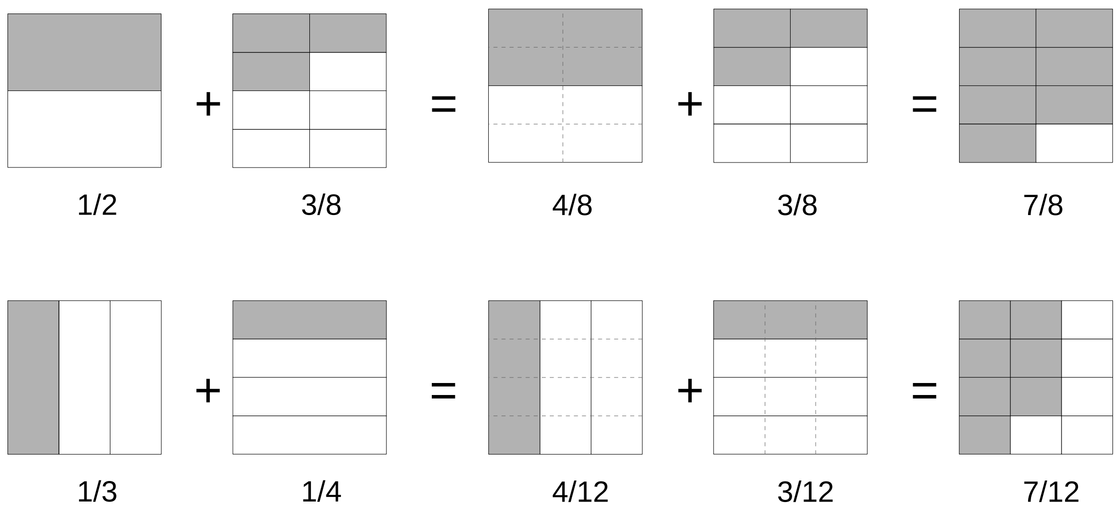


Illustration 25: How to solve 1/2+3/8 on the top and 1/3+1/4 on the bottom using the rectangular area model

Although based on a limited sample, Mack (2004) adopts the same sequencing and reinforces the importance of spending a considerable amount of time on fractions from the same family first. This way, according to her, the students may grasp the importance of operating on "like-size units [and] extend this big idea in small steps" (p. 227).

8 This concept is the same as the concept of "like fractions" in the United States of America, but as the latter seems to be unusual in England, I decide to adopt "fractions from the same family" in this thesis. Note that a fraction can belong to more than one family. For instance, 1/2 belong to the family composed of 1/2, 1/4, 1/8 and so on, but also to the family composed of 1/2, 1/6, 1/12 and so on and this does not imply that 1/4 and 1/6 belong to the same family. A family of fractions could be defined as a set of fractions in which for any pair of fractions chosen one of the denominators is a multiple of the other.

Based on these considerations, in the second pack of lessons I decided to focus on addition and subtraction of fractions from the same family, continuing the work done on the first pack. After that, I would progress to addition and subtraction of any two fractions in the third pack.

Another element that influenced the design of the lesson plans was the teachers' expectation that they should enable students to solve not only 'work out' questions, but also what they called 'word problems'. This concern apparently originates from their perception that both types of questions are being asked in mandatory external exams.

'Word problem':

Sheila ate $\frac{1}{3}$ of a giant cookie in the morning and $\frac{2}{9}$ in the evening. In total, had she eaten more than half of the cookie? Why?

'Work out' question:

Solve the sums below.

a) $\frac{2}{3}-\frac{1}{9}$

Illustration 26: Example of a 'word problem' (top) and of a 'work out' question (bottom)

From my perspective, the 'word problems' would be an opportunity to analyse whether, and how students transferred the knowledge they originally built on 'work out' questions to a different setting. Therefore, I decided to dedicate one lesson at the end of the second and third packs to such problems.

Finally, a special emphasis on comparison of fractions was given based on the proposal of Siegler (2011) regarding the unifying role that the idea of magnitude can play when it comes to whole numbers and fractions. This was done by including questions asking students to compare two given fractions throughout the three packs.

After several rounds of sketches based on a vast collection of research papers, teacher-oriented materials and lesson plans available online, I decided that it would be necessary to have 5 lessons on the first pack, 4 on the second, and 3 on the third in order to achieve the goal of each pack. The tables below summarize the main topics and material used on all the targeted lesson plans.

Lesson 1.1	**Lesson 1.2**	**Lesson 1.3**	**Lesson 1.4**	**Lesson 1.5**
- Introduce the rectangular area model;	- Introduce fractions from the same family as 1/2 through the model; - Equivalent fractions;	- Introduce fractions from the same family as 1/3 through the model; - Equivalent fractions;	- Fractions from other families; - Equivalent fractions; - Comparison of fractions;	- Equivalent fractions; - Create a booklet with diagrams;
- Cut-outs	- Cut-outs	- Cut-outs	- Cut-outs	- Diagrams

Table 12: Lessons from pack 1

Lesson 2.1	**Lesson 2.2**	**Lesson 2.3**	**Lesson 2.4**
- Decompose the unit as a sum of fractions; - Comparison of fractions;	- Add and subtract fractions from the same family as 1/2;	- Add and subtract fractions from the same family as 1/3 and 1/5;	Word problems
- Diagrams	- Diagrams	- Diagrams	- Diagrams

Table 13: Lessons from pack 2

Lesson 3.1	**Lesson 3.2**	**Lesson 3.3**
- Introduce addition and subtraction of fractions from different families	- Add and subtract fractions from different families;	- Word problems
- Cut-outs	- Diagrams	- Diagrams

Table 14: Lesson from pack 3

All the initial versions of each lesson can be accessed at http://dx.doi.org/10.17639/nott.353. Although most of them were adapted before enactment, due to observations made in the previous lessons and to impressions from the teachers and research team, the changes were generally small and preserved the lesson objectives and the general design.

Once the design of the targeted lessons is discussed and the learning path outlined, I will describe how the data collection unfolded in the next chapter.

6 DATA COLLECTION

6.1 Updates on the context

The school where the data collection took place was the same as for the pre-field work and preliminary study, Purple Valley. After the end of the 2014-2015 academic year, and as a result of being "under special measures", the school was taken over by an Academy[9]. This event had no direct impact on my data collection and throughout the academic year I could barely notice any change that could be a result of the incorporation, apart from changes in uniform.

There were two changes in terms of the participant teachers. One was that a new teacher, Alice, who observed the last lesson of the preliminary study, joined the group for the next academic year. At that moment, she was getting her certification through the "Teach First" programme and was responsible for 5 groups at Purple Valley. She had a brief experience as P.E. teacher assistant, but had never taught mathematics before. She voluntarily agreed to take part in my research with her Year 8 (set 5 out of 5) group.

The other change was that Oscar's participation faded out towards the end of the 2014-2015 academic year, and he did not take part in the main data collection. No reason was given for that.

The other teachers, already described in Section 2.1.2, were David and Julia with their Year 9 class (set 5 out of 5) and Year 8 class (set 4 out of 5) respectively. The group chosen by Julia was the same group that participated in the preliminary study (see Chapter 3).

9 In short, being "under special measures" means that the monitoring agency would be paying special attention to that school during the upcoming year. In the case of the school not improving as expected, one possible outcome is to be "taken over by an Academy", which essentially means that some decisions would be transferred to another supposedly high quality school (called an Academy) experienced in helping schools "under special measures".

6.2 Ethics

My activities at Purple Valley started as part of the overarching research project that was being developed, and whose ethics procedures were initially taken care of, by Dr. Peter Gates.

When the 2015-2016 academic year started and my data collection was near, I started to consider ethical implications related to my research. My main concern was with students, since my focus is on low-achieving students and this group often overlaps with other groups that may be considered to be in a vulnerable situation. However, since my methods resemble regular teaching activities, I had already built a good rapport with the teachers and was familiar both with the school and for all the students potentially participating in my research, I anticipated no major ethical issues. Since the end of the previous academic year, it was clear to me that the good relationship that had been already established with the school would be the foundation for the ethical standards of my research. For these reasons, I was planning to adopt an opt-out approach for students' consent, which was approved by the Ethical Committee at the School of Education (University of Nottingham).

However, David suggested a different approach to avoid raising excessive concerns from parents. It is important to remember that David was the Head of the Mathematics Department and an experienced teacher. His position was motivated by the fact that the teachers knew about my proposal and agreed on using the lesson plans, my lessons would not be substantially different from what is commonly done in the school, and that every activity I would carry out in the school would be accompanied by a teacher. Therefore, he could extend the blanket permission students had already given to the school for audio and video recordings, as well as use of their notebooks and worksheets for educational purposes, to my activities. Once all the students of the groups that were about to take part in my research had accepted the school's terms, I could carry out my data collection without further consent.

After discussing this issue with my supervisor and considering the rapport already established with David and teachers, I decided to go through with his suggestion but not without being sure that the teachers were aware of the arrangement and students were informed about my activities. In order to do so, three measures were taken. First, on top of the consent form and information sheet that he agreed upon and signed as a teacher, David agreed upon and signed a second one covering the details of this arrangement. Second, the other teachers were informed of this decision and agreed with it. They were informed of the possibility of withdrawing themselves from the research at any time as well as the possibility of students asking to withdraw at any time. Third, although all students already knew me, the teachers re-introduced me to them before the beginning of my data collection, explaining the activities I would carry out during some lessons, their right to not participate in my data collection activities and making sure students knew they could reach out for the teachers in case of any discomfort.

The issues of confidentiality, anonymity, non-traceability of students' identities would be properly dealt with following the guidelines adopted at the School of Education regarding all data collected from students.

Additionally, I was attentive during all my data collection to any student showing signs of discomfort or distress during the lessons. Since I was visiting the school at least one day per week, I had plenty opportunities to talk to the teachers about any issues that could arise, related or not to my research. Any event that could suggest that students were in a sensitive situation or that was abnormal in any sense was reported to and discussed with the teachers.

Although David had provided what he saw as consent for me to carry out my research with students at Purple Valley, my approach was to treat their consent as a process, being open to revisit it if necessary. Fortunately, no student showed signs of discomfort or distress and the data collection ran smoothly during the whole academy year.

I also provided an information sheet to the teachers, explaining my research objectives, methods and activities that would be developed throughout the academic year as well as issues related to confidentiality, anonymity, non-traceability of their identities and their right to withdraw from the project at any time without risk or prejudice. Later, I obtained a signed consent form from all the teachers involved and I could carry out the research as I was planning to do.

6.3 The students

All the groups that took part in my research were low sets, which means that most of the students had usually scored low marks in external and internal exams. In fact, except for a few students in Year 8 Set 4, all the other students got marks below 30% on all mathematics exams that took place during the academic year.

The groups were relatively small when compared to other groups: 13 students in Year 8 set 5, 10 students in Year 9 Set 5, and 20 in Year 8 set 4[10]. Also, they experienced a limited curriculum focused only on topics expected for the low tier certificate (GSCE Foundation).

On one hand, Year 8 Set 4 and Year 9 Set 5 would not present any major problem for the teachers in terms of behaviour other than occasional off-task talk and small disturbances. On the other hand, Year 8 Set 5 was more challenging; Alice had to interrupt lessons several times for behavioural management and some students were excluded from the classroom during some lessons. Although these situations were usually started by the same three students and did not happen in most of the lessons, it was enough to disrupt the whole lesson and affect the behaviour of the other students.

Apart from these few disrupting students and a similarly small number of students that were more engaged and keen to participate in the lessons, most of the students in these groups would fit into what Pye (1988) calls "invisible children", as discussed in Section 4.5.1.

An important characteristic of the environment in which these students are immersed is the lack of agency when it comes to "doing mathematics". This characteristic can be seen in the structure of the lessons: presentation of the topic of the lessons, resolution of a paradigmatic example followed by a list of items that can be solved using the strategy shown in the example. This lack of agency, that probably has been experienced since primary school by low set students (Boaler & Wiliam, 2001; Marks, 2011), is ultimately incorporated by the students into a form of apparent apathy and lack of initiative when they are asked to solve new questions.

In terms of prior knowledge, as widely acknowledged by teachers and researchers (Gersten, Jordan, & Flojo, 2005), when it comes to low set students, they generally struggle with basic arithmetical knowledge. Before the lessons started, I asked the teachers to apply a diagnostic assessment about fractions to the targeted groups (see appendix 10.1). The questions were adapted from Hart et al. (1984), but it was a conscious decision to make the assessment shorter in such a way that it could be fitted into the beginning of a lesson as a starter, or at the end as a final activity. Just to illustrate students' previous knowledge regarding addition of fractions, the table below shows the percentage of correct answers for the three final questions of the diagnostic assessment.

10 These numbers changed throughout the academic year.

	5A) Work out $\frac{3}{8}+\frac{2}{8}$.	5B) Work out $\frac{1}{10}+\frac{3}{5}$.	5C) Work out $\frac{1}{3}+\frac{1}{4}$.	Number of students
Year 8 set 4	88%[11]	29%	0%	17
Year 8 set 5	8%	0%	0%	13
Year 9 set 5	0%	0%	0%	8
Across all groups	42%	13%	0%	38

Table 15: Percentage of right answers on the diagnostic assessment

The scores are considerably lower than the national average reported by Hart et al. (1981) for the same questions. Although the result is more than 30 years old, more recent studies showed that the levels of achievement in England remain almost the same (Hodgen, Küchemann, Brown, & Coe, 2008, 2010).

Finally, it is worth pointing out that according to the English curriculum, these students should have had lessons about all the topics that were to be covered during the targeted lesson in their primary schools. However, it was widely accepted by the participant teachers that the students probably have had only a limited exposition to concepts related to fractions, since they were probably labelled as low achieving students in their primary schools, therefore, having access to a restricted version of the curriculum, which according to them, probably excluded more advanced topics, such as comparison, addition and subtraction of fractions.

6.4 The original plan

My initial sketches of the lesson plans suggested that 12 lessons would be necessary to cover the topics necessary to get to addition and subtraction of fractions. Considering how the teachers used to organise the working plan, the lessons were grouped into 3 packs that should be spread throughout the academic year, each made of 3 to 5 lessons that should be enacted consecutively in a row. Remember that the initial version of the lesson plans can be accessed at https://dx.doi.org/10.17639/nott.353. As I was planning to incorporate changes in future lessons based on what would be observed in previous lessons, it felt sensible to plan some gaps between the packs.

11 This comparatively high score can be explained by the fact they this group participated in the preliminary study in the previous academic year, which covered essentially sums as presented on question 5A and 5B.

Based on that, my original plan was for the teachers to enact one pack of lessons per term, and more precisely, in the second half of the term, so the research team and the teachers would have time to meet and discuss the lesson plans before they were used.

It was also agreed that none of the groups would have had lessons about fractions during the 2015-2016 academic year apart from the targeted lessons. However, they may have had lessons covering topics marginally related, such as percentages and ratios.

6.5 The data collection

As mentioned before, my data came essentially from four different sources: observation of the targeted lessons, observation of other lessons taught by the three teachers, informal talks with the teachers, and meetings between the research team and the teachers.

The data collection started with the first meeting in September. From that moment on, I started to come to the school two or three times a week to observe lessons and talk to the teachers. This stage was important to establish rapport with the students and teachers, get familiar with the lessons for the targeted groups and discuss, even informally, aspects of the targeted lessons that I was designing in the meantime.

The timeline below shows an overview of the targeted lessons and group meetings. The lesson observations and informal talks took place over the whole academic year, making a total of 95 lessons observed apart from the targeted lessons.

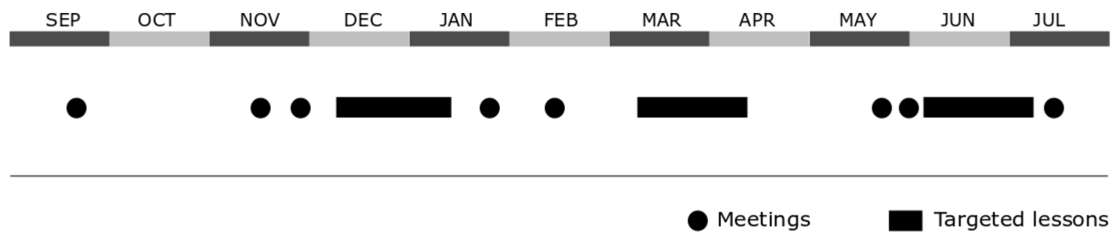


Illustration 27: Rough timeline

In the following sections I will give an overview of how the data collection went with each teacher during each term, paying special attention to the targeted lessons.

It is important to keep in mind that my research is not focused on the teachers, but on the students. For that reason some events reported in the next three sections will be left without further discussion. They are mentioned only because I believe they are useful to understand how my data collection actually happened.

6.5.1 First term

The first meeting was set to re-establish contact with the teachers, explain our research proposals, clarify our goals and needs for the next academic year, discuss a rough timetable and any possible constraints on their parts, and get their formal consent to take part in the research.

The next two meetings were focused on the upcoming targeted lessons. The research team brought excerpts of the lesson plans and the underlying core ideas to be discussed. My goal was to capture the teachers' impressions and adjust the lesson plans accordingly.

The first pack of lessons planned for this term was composed of five lessons and started to be enacted in December.

Alice's lessons

Due to the training that Alice was undergoing that academic year, she was asked to develop a sequence of lessons using some sort of innovation. For that reason, she decided to take the general description of the lessons in the first pack and develop the lesson plans by herself. Bearing that in mind, I asked her to start enacting the lessons before the other teachers, so I could use her lessons as a trial for my data collection strategies, especially regarding the within-class clinical interviews.

The lessons started on 1st December, and after the third lesson she decided to interrupt the targeted lessons. According to her, the decision of not going through all the five lessons was due to a perceived lack of impact on the students' learning. This view was based on her on the fly impressions during the lessons, since there was no formal mechanism to evaluate learning incorporated into the lesson plans. During talks after the lessons and in the next meeting, she said that the work with the cut-outs was very demanding for the students and they were too tired when the lessons finally got to fractions.

At that point, I agreed with her in terms of the demands of working with the cut-outs to the extent that it became one of the central issues discussed in my data analysis (Section 7.2). However, I did not agree with her perception of lack of impact on students' learning. This could be the result of a conscious design choice of starting by introducing the rectangular area model with not much emphasis on fractions that was not clear to Alice.

From the perspective of my research, these lessons were very useful. Firstly, being the first lessons of the main data collection to be enacted, they helped me to make final adjustments to the lesson plans. Secondly, through my interactions with the students, I had the opportunity to refine how I would collect the data.

Regarding the second point, it is important to clarify how this refinement happened. During the first lesson, I decided to turn on the audio recording function of the smartpen only if an interaction with a student was revealed to not be superficial. This strategy proved to be ineffective, since the decision to start recording could happen after a meaningful interaction had been started. During the second lesson, I decided to start recording just before approaching any student. This strategy also proved ineffective because the action of starting a recording disrupted my integration into the classroom environment, and my interactions with students. Finally, during the third lesson, I adopted the strategy that was used during the rest of my data collection: the audio recording function was left on during the whole lesson and I tried to take notes immediately after any meaningful interaction with a student. These notes always included the name of the student, her/his position in the classroom and could include details of the interaction, such as specific words, expressions and diagrams used by the student, short descriptions of the interaction, or later on in the data collection process, brief references to what sort of phenomenon I felt that the interaction was an example of.

David's lessons

Starting in December David taught five consecutive lessons about fractions, but we decided to divide the original lesson 1.2 into two lessons based on the progress made by the students. The final lesson of this pack was enacted right after the Christmas break.

He always came for the lessons with a printed version of the lesson plans with several notes. We usually had time for a quick talk before the lessons. During these talks he would usually explain the general objectives and structure of the lesson and highlight critical issues that could arise. I observed that he was very effective in paying attention to these issues during the lesson.

Julia's lessons

Since the beginning, Julia reinforced that she was willing to use the lessons exactly as I had planned, and in fact, she rarely made any changes to the lesson plans. This decision was motivated by her perception that this would "improve the quality" of my research. Nevertheless, she was still conscious of her role as the teacher and over the academic year felt comfortable enough to sometimes interfere in a lesson, or in the plans for the next lesson.

Her lessons also started in December, but due to external reasons, she could only teach three lessons before the Christmas break. The two final lessons of this pack were taught at the beginning of January, and the lesson 1.4 was divided into two lessons. This was done because in the first lesson she ended up spending a long time on the starter and the students had not enough time to conclude all the tasks.

6.5.2 Second term

This term started with the final lessons by Julia. During the first half-term, I kept observing lessons for the targeted groups, even though I reduced the number of visits to the school in order to dedicate more time to finishing the design of the second pack of lessons. These would focus on adding fractions from the same family. The pack was composed of four lessons, with the last one being different from the others because it was focused on word problems.

There was a meeting with the teachers at the end of January to discuss the lessons enacted during the previous term, and another at the beginning of February to discuss the upcoming lessons, which started on 29th February.

Julia's lessons

Julia taught the four lessons in a row and everything went as planned even though this group have had similar lessons during the preliminary study.

She evaluated that the students' levels of engagement started low, but increased significantly as the lessons progressed. Nevertheless, she was concerned about how the final lesson would turn out, considering that it was the first that focused on word problems. According to my observations, this type of lesson was not common in general for low sets at Purple Valley. However, considering the amount of work registered on the worksheets, it can be said that the lesson was successful, since most students solved all the questions proposed in the original lesson plan.

After the final lesson, Julia said to us that she wanted to do another lesson on word problems, because she felt that the students could go even further. As a result, she prepared the worksheet below for the next lesson and I went to the school to observe it.

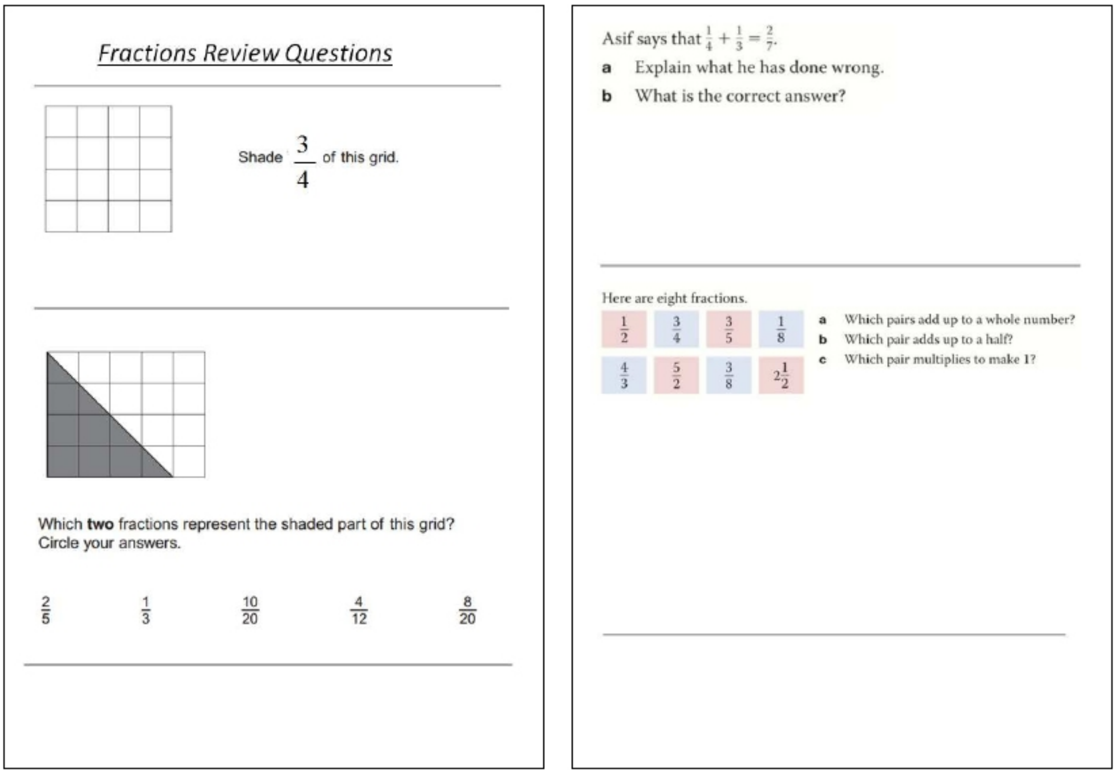


Illustration 28: Lesson plan for an extra lesson on fractions

It is remarkable how this lesson plan is different from Julia's regular lesson plans. Firstly, the number of questions to be answered is considerably smaller, so the students had more time to focus on each one of them. Secondly, the questions are considerably different from each other in terms of what is expected from the students. Finally, during the lesson, the questions were not preceded by examples showing how to solve them, as she usually did (see Section 5.4). It could be said that Julia was copying my lesson plans, however, even when compared to mine, she went further in terms of the variety of the questions. Considering that Julia is always very careful in the choice of tasks for her lessons, this lesson plan suggests that Julia considered that students' knowledge about fractions was somehow different from their knowledge about other topics.

David's lessons

The four lessons became six for David's groups. Firstly, lesson 2.1 was broken into two lessons, because the students were not able to conclude all the tasks. Secondly, lesson 2.2 was repeated after one week, when David was not able to teach this group for personal reasons. Both decisions were agreed between the teacher and the research team.

In relation to the final lesson with word problems, it was clear that the students asked for help more frequently than Julia's students. However, their progression during the lesson was satisfactory enough for us to consider that an additional lesson was not necessary.

Alice's lessons

The term started with Alice teaching the final lesson of the last pack. I could not observe this lesson due to conflict with Julia's lesson, but she was satisfied with it, because according to her, the lesson went more smoothly and she was able to notice progress. This time, she used my lesson plan. It is interesting to notice that this lesson was different to the lesson plans she enacted last term, and more focused on fractions than on the model.

Alice, as did David, needed two lessons to complete what was planned for lesson 2.1. After the second lesson, we talked about her impressions, and she thought the students needed more similar questions on each task, so they could get some fluency. As a result, I decided to change the lesson plan for lesson 2.2 more than I used to do from one lesson to the other, but still keeping the same design principles and objectives.

From my perspective, as the designer, the difference is that there were more items in the opening question (work out sums given diagrammatically), which allowed a slower transition from simpler to more complex cases. The consequence of this change is that there was less time to spend on the final questions of the original lesson plans (work out sums given symbolically and compare sums).

After this lesson, Alice was very satisfied with the result. According to her, students progressed more and were more engaged during the lesson. Consequently, I decided to adapt the style of lessons 2.3 and 2.4 to a similar format. By the end of this sequence of lessons (seven considering the one from the first pack), Alice was much more enthusiastic about the approach of the targeted lessons than she had been at the end of the previous pack.

6.5.3 Third term

We had two meetings before the third pack of lessons, one to discuss the previous and another for the third pack of lessons, which started in the middle of January.

Julia's lessons

The first and second lessons went as planned. The students made the transition from using the pieces (lesson 3.1) to using diagrams or purely numerical approaches (lesson 3.2). However, at the beginning of the third lesson, when the lesson plan suggested a discussion about how to add fractions without drawing the full diagrams, Julia asked me to explain it for the whole class. Later, she told me that she had no time to prepare for this lesson and could not remember what aspects should be emphasized during the discussion. As I was not expecting to lead the lesson, my explanation did not cover what was suggested in the lesson plan, and this apparently affected students' progress and engagement during the lesson.

Consequently, we decided to include a fourth lesson, very similar to the third one. Afterwards, Julia considered the fourth lesson very successful and we concluded the lessons on fractions with this group.

David's lessons

David enacted all the lessons in this pack consecutively and there was no need to add extra lessons. As had happened with Julia, he also perceived them to have been successful and so the targeted lessons for this group were concluded.

Alice's lessons

Alice was the last one to enact the lessons in the third pack, but she was able to do all of them consecutively and with no extra lesson needed.

Interestingly, during the first lesson, I noticed that the students were less cooperative than usual. As I have mentioned before, this was the most challenging group in terms of behaviour, as well as being the group with the highest number of students that were not open to talking to me. During the first lesson I noticed that some of them were avoiding talking to me and apparently joking about it when I went away. Apparently, some students were actively avoiding interactions with me. When I mentioned my impression to Alice, she said that she did not believe there was any special reason, just something normal for this group.

Consequently, I was not able to collect as much data as I could in previous lessons.

6.5.4 Extra lessons

After the end of the third pack, the research team had a meeting with all the teachers to discuss the lessons as a whole. By the end of the meeting, all the teachers asked us if it would be possible to use the lessons with other groups. Specifically, David wanted to use a condensed version of the lessons with his Year 9 Set 3 (out of 5), Alice wanted to use some lessons, especially from the first pack, with her Year 7 Set 3 (out of 3), and Julia wanted to teach multiplication of fractions using a similar approach to her Year 8 Set 4 (out of 5 — the same group as the targeted lesson).

For David and Alice, I selected some lessons and adapted them in such a way that they could be used as a coherent set considering the number of lessons each one of them had available (5 for David and 3 for Alice). I observed all these lessons and collected data during them as I had for the targeted lessons. However, due to the fact that I was not prepared for that, and the time for planning was very restricted, I decided not to include this data as part of my research.

With Julia, the approach was different. The research team met with her twice to discuss her ideas to teach multiplication of fractions based on the approach I developed throughout the targeted lessons. I also observed the two resulting lessons, but again decided not to include this data as part of my research.

This interest in extending the experience can be considered a sign of satisfaction with the results achieved with the targeted lessons.

Before the end of the academic year, the teachers asked me if I could organize a booklet including all the lessons to be used during the next academic year by them, and eventually by other teachers in the department. In August, I organised the lesson plans into a booklet, incorporated some changes based on my informal analysis of the experience, and passed it to them with the cut-outs. This new version of the lesson plans is available at http://barichello.coffee/about-my-research/lessons-fractions-visual-representations.

6.6 Reflection on the data collection process

6.6.1 The relationship with the school

I could not actually expect a better relationship with the school than the one I developed during the period of my pre-field work, preliminary study and data collection, as I had no problems in carrying out my research. Other members of the staff were aware of my activities in the school.

The other mathematics teachers also received me very well. I spent a significant amount of time in the staff room between lessons and during breaks and always felt very welcome, engaging in several conversations about many topics related, or not, to my research.

6.6.2 The relationship with the teachers

The same can be said about the relationship with the participant teachers. They were not only very open about me observing their lessons, but were also available to talk about any topic related to my research. They felt comfortable in asking me, for example, not to come to observe particular lessons (all of them did that at some point during the academic year), or to openly say that they had no time to prepare for a particular lesson (again, all of them did that at some point during the academic year). At the same time, I felt that I could ask them for anything related to my research and they would honestly ponder and say yes or no according to their judgement. Fortunately, the data collection went smoothly and I did not have to make any significant changes during it.

6.6.3 The relationship with the students

Regarding the students, it was clear to me that my rapport with them evolved throughout the year. This became even clearer during the extra lessons for David's Year 9 Set 3, as I had never observed this group before, and by contrast it was more difficult to engage in meaningful interactions with these students.

On the other hand, apart from the change in Alice's group's behaviour during the beginning of the third pack, I felt very comfortable when talking to all students, even though some of them would sometimes not be open to conversation. In that case, I would just walk away and try to approach the student again after some time. In fact, it became quite common for students to ask for help directly from me instead of just raising their hands and waiting for somebody to come.

Another sign that I had reached a good rapport with the participant students was the "small talks" in which the students had started to engage with me at the end of some lessons. As a former teacher in Brazil, I noticed that the relationships between teachers and students in this school were restricted to academic issues, with almost no interaction outside the classroom; this is very different in Brazil. However, after a while the students felt comfortable enough around me to initiate conversations about non-academic topics, or even better, to show me how much they had done by the end of a lesson, or brag about having solved a particular question.

6.6.4 Considerations about the within-class clinical interviews

It is important to clarify a few more aspects of the within-class clinical interviews than what was described in Section 4.5.1.

The first aspect refers to how I chose the students with whom I would interact. In general, during a lesson, I would walk around the classroom making myself available for the students to ask for assistance. When no student asked for my assistance, I would keep walking around checking discretely what tasks they were solving and what answers they were obtaining. Based on that, if I noticed anything that could be interesting in terms of my research goals, I would approach the student asking `why` and `how` questions (Ginsburg, 1997).

During this process, I did not follow any pattern in terms of which student would be approached next. My choice was based on what I could notice on their worksheets. Although I was intentionally trying to avoid focusing solely on talk-active students, I would not insist on engaging with a student who was not open to conversation.

In terms of the depth of the interactions, most of the time they were superficial (asking me to check an answer, for instance), but inspecting my notes retrospectively I can say that I collected at least 4 meaningful interactions per lesson, and that quantity seems enough to shed light onto my research goals.

Finally, it is worth discussing the balance of power between me (as interviewer) and the students (as interviewees) during my data collection. This is a common criticism of the work of Piaget (diSessa, 2007): are children's reactions and answers during a clinical interview a portrait of their cognitive resources, or are they the result of an uneven power relation between the children and the researcher?

Although I admit this question is also pertinent to within-class clinical interviews, my answer is that the power relation in this case is closer to the power relation existent between teacher and students, which is arguably an inevitable component of any educational endeavour in the current context (maybe apart from very innovative or specific settings). Therefore, the data I could access with this method is closer to the input a teacher could have during a lesson, rather than the data arising from traditional clinical interviews, meaning that it gains a higher ecological validity.

6.6.5 What could have been done better

During my data collection, I sometimes felt that it was too unstructured and could result in a set of data lacking rigour. However, as argued by Hammersley and Atkinson (2007), it is common in exploratory research to have a wide range of methods for data collection that may evolve or be refined during the research process. In fact, after a while I started to realise a certain coherence emerging from my data, and could then focus more on the within-class clinical interview, and less on students' answers on the worksheets.

Nevertheless, reflecting retrospectively, I can identify three aspects that could be improved in my data collection.

Firstly, some lessons, such as 1.1 and 3.1, generated little material apart from the audio recordings. This happened especially with lessons based on cut-outs, due to the fact that I did not use video recording, and the students used the same set of cut-outs to answer several questions, which limited drastically the possibility of leaving some sort of permanent register of their solutions. Although I do not think it was a problem for most of the lessons, some video recordings would complement the data and could be useful during analysis, even considering the disruption that the video may cause.

Secondly, some clinical interviews outside the classroom could have been useful to elucidate specific issues that arose during my data collection. I do not mean that this would provide deeper data, but that it would allow me to pursue some phenomena that I could not pursue during the lessons. However, I am not sure it would have been possible to identify the issues and prepare clinical interviews within the time constraints. Therefore, these clinical interviews can be seen as a choice of complementary method for a subsequent research project aiming at further investigate issues already identified in this one.

Thirdly, even though it was possible to avoid overlapping of lessons on the same day and at the same time, it was common to have overlaps in the periods during which the teachers decided to enact the lesson plans. For instance, the three lessons of the third pack were enacted by Julia and David in the same week. As a result, the time constraints to incorporate changes from one lesson to the next one were huge and there was no time to refine a lesson for the next teacher once it had been enacted by the first. Ideally, it would be desirable to have big gaps between the delivery of the packs, as I had, but also small gaps between when the different teachers enacted the lessons. However, I recognize that when it comes to research into real classrooms this may be not feasible.

7 Data analysis

7.1 First steps

7.1.1 Initial impressions from the data collection

As it is expected during any research with an ethnographic component (Hammersley & Atkinson, 2007), the data analysis inevitably starts during data collection. Every day after my visit to the school, I would produce some additional notes summarizing my impressions of what happened during that day and events that caught my attention for some reason. Moreover, the research team talked after every visit to the school. Even though these talks were essentially informal and mostly focused on practical issues regarding ongoing and future actions, they sometimes evolved into more reflexive discussions.

The interactions with the participant teachers were also important in this process. Their impressions about the lessons and opinions about what should be done next not only influenced the design of the lesson plans, but also directed my attention towards specific events and phenomena.

As a result of all of this, when I concluded my data collection, I had already tentatively identified three themes that seemed relevant to the research question (how students learn fraction addition and subtraction through an approach emphasizing visual representations), namely:

1. Students using vocabulary related to visual representations when reasoning about fractions;
2. Students being able to extend their knowledge to solve questions slightly beyond what was taught to them;
3. Specific difficulties that seemed to be related to the use of the chosen visual model.

With these issues in mind, but with no plans to restrict myself to them, I started the systematic data analysis.

7.1.2 The beginning of the systematic data analysis

After the end of the data collection, I started to analyse it systematically, organizing worksheets, audio recordings and notes. Considering the amount of data collected (about 2000 pages of worksheets used by students, more than 100 pages of notes accompanied by audio, and several hours of audio recordings of talks and meetings with the teachers), it took me some considerable time merely to organize it in a way that allowed me to identify and access quickly any piece of data[12].

After this stage, I started to describe further the within-class interviews that I had marked during my data collection as potentially interesting. This means that among all the interactions I recorded with students during the targeted lessons, these are the ones that caught my attention directly after their occurrence, and were, therefore, marked on my field notes as potentially interesting. This process generated a list with about 60 episodes. To each episode I attached any worksheet related to it, a brief description, a short comment on why I thought it could be interesting to the research question, and a code. The latter allowed me to create an initial grouping for the episodes, taking into account recurrent phenomena, the themes that emerged during the data collection and relevant aspects according to the literature.

Once I had concluded the organization of the episodes already identified during data collection, I moved on to a systematic inspection of all my field notes, searching for other meaningful within-class interviews in terms of the research questions. It is worth remembering that during all the lessons I did not adopt a stance of a neutral observer, so sometimes I would talk to students explicitly aiming at collecting data and at other times I would assume a role similar to that of an assistant teacher, encouraging, checking answers, offering apposite support, and keeping students on task, etc. Because of that, my recordings also contained lots of interactions that were not related to my research questions. By the end of the inspection, I identified 105 episodes altogether, with all of them receiving a description, reflective comment and a code.

Then I started to revisit all the episodes focusing on the codes I created. My intention was to refine them into a more coherent and shorter list. Similar codes were unified and others merged due to further consideration of the episodes listed under them, resulting in 8 different codes. This whole process of organising, revisiting, listing and coding was very immersive for me, and allowed me to consider the data available and decide what issues it would be possible to discuss.

12 This was done mainly using the free software Pipoca (https://github.com/barichello/openQDA).

I started to write the following sections analysing the data for each one of the eight codes. As it progressed, three of them ended up being satisfactorily covered by the other five. Consequently, my data analysis is divided into 5 sections, each referring to a different issue that was observed during data collection and for which I believe to have collected enough data for a rich understanding. The issues are:

1. **The interference of visual abilities** (Section 7.2). This refers to how an apparent lack of certain visual abilities seem to have interfered negatively in how students engaged with the tasks;

2. **Reasoning and visual representations** (Section 7.3). This section discusses the emergence of reasoning during the lessons and highlights how this reasoning was anchored in the visual representations being used;

3. **Generative reasoning** (Section 7.4). This refers to the capacity shown by students to extend their knowledge of solving questions that went beyond what was explicitly taught to them during the targeted lessons;

4. **The multiplicative aspect of fractions** (Section 7.5). In this section I discuss two issues related to prior knowledge on multiplication. First, I will show how the targeted lessons presented a low threshold in terms of previous knowledge of multiplication, and secondly, how this knowledge influenced the way students engaged with the tasks;

5. **Whole number bias and other rote procedures** (Section 7.6). The aim of this section is to discuss the occurrence of the whole number bias and other rote procedures during the targeted lessons.

Each one of these issues will be presented and discussed in the next sections in the light of the data collected.

7.2 The interference of visual abilities

As discussed in Section 2.4.3, the correlation between visual abilities and achievement in mathematics is well documented in the literature (Mix & Cheng, 2012), even though the causal mechanisms are still not completely clear. Since my research deals with visual representations and low achieving students, I was expecting that visual abilities would be a critical aspect during the lessons and that they would be salient during my observations.

The first episodes, when that interference was observed, took place during the very first few lessons, when the rectangular area model was introduced for 1/2, 1/4, 1/8 and 1/16 and for 1/3, 1/6, 1/9 and 1/12 (lessons 1.1, 1.2 and 1.3, see http://dx.doi.org/10.17639/nott.353). During these lessons, students were asked to compare the relative size (in terms of area) of cut-outs as shown in image 29. The lessons started with questions such as 'how many of this do you need to cover that?', and most of the combinations involved the white square on the right side of image 29 (later, this shape would be defined as the unit).

In general, students had no problem solving these questions when both shapes were rectangles. However, I observed some students struggling to fit the cut-outs when there were triangles involved.

Illustration 29: Some cut-outs used during lessons 1.1 and 1.2

On some occasions, I observed students staring at the cut-outs almost as if frozen for several seconds. That was the case for a pair of students working together during Alice's first lesson when trying to answer the question "how many yellows would you need to cover the white square?" with a configuration of cut-outs similar to the image below in front of them (the yellow shape is an isosceles triangle with an area equal to 1/16 of the square's area and the longest side equal to 1/2 of the side of the square).

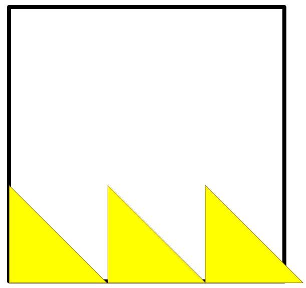


Illustration 30: Recreation of students' work

Before intervening, I observed them manipulating the cut-outs for a while. Their actions were limited to adding new yellow shapes to the current configuration or translating one of the shapes already being used. At some point, one of the students acted as if he had just had an insight, but he just removed the yellow shapes from over the white square and started over to position them in a similar configuration, but instead of starting by covering one of the sides of the square, he built yellow squares by joining the hypotenuses. After doing that to some triangles, he noticed that his approach would not work.

They seemed to be unable to use rotations and reflections to solve the task. After a while, I decided to intervene by verbally[13] suggesting 'why don't you try this?' and rotating one of the triangles, such that its hypotenuse would coincide with the base of the square. When I finished the movement, one of them reacted as if having an insight, discarded the other triangles and carried on the pattern my triangle suggested, thereby finding an answer to the question.

Similar situations, where students seemed not to be able to use reflections and rotations to solve such questions, were observed throughout the first three lessons with all the teachers. On other occasions, instead of showing the rotation with the shape, I tried to suggest the movement verbally using words such as 'rotate' or 'turn'.

13 The issue of using language as mean for my interventions is discussed in Section 8.4.1

While it worked for a few students, it did not for most students that were struggling with triangles.

My conjecture is that the prevalence of this struggle with triangles was because for rectangles the rotations (90, 180 and 270 degrees) do not add much new information, since the referent (the square unit) is symmetrical under such transformations. So in general, rotation (realized mentally or manually) was not really necessary to solve questions when only rectangles were involved. However, for some questions involving triangles, rotation was fundamental. In the task discussed above, the yellow shapes represented 1/16, as its longest side was half the length of the unit and the other two sides were equal to $\frac{\sqrt{2}}{4}$ times the length of the unit. Therefore, it would only fit the unit square if the longest side was aligned with the side of the unit.

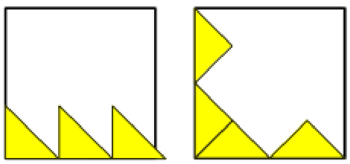


Illustration 31: On the left, an arrangement unsuccessfully used by students and, on the right, the position necessary to solve the task

It is important to emphasise that this was not generalised across all the students, but it happened with students in all the groups, and for those struggling it was a real impediment, to such an extent that I decided to remove the triangles from lesson 1.3 when David taught it as he was the last teacher to do so and I had a gap between lessons that allowed me to prepare the new cut-outs.

After lesson 1.3, students stopped using triangles to represent fractions because of the convenience of using rectangles. This was an explicit goal for the first pack of lessons. Therefore, it is natural to expect that the occurrence of this sort of episode would reduce, and it actually did.

However, towards the end of the first pack of lessons, I was still able to observe episodes that I consider related to this issue. This one happened with the student M, from David's group, during lesson 1.5, the first lesson with no cut-outs. Students were asked to produce a series of diagrams representing several fractions to keep in their notebooks for future reference (after this lesson a break was planned before the beginning of the second pack). After creating the diagrams, the students were asked to complete some equalities involving equivalent fractions, such as $\frac{1}{3} = \frac{\quad}{9}$. M had already created all the diagrams and was successful in answering the first few questions on equivalent fractions, when I noticed that she was stuck on the question $\frac{1}{5} = \frac{\quad}{10}$. From my perspective, this question was nothing special in comparison with the ones she had already answered, so I intervened by just reinforcing the question: "so, you have to compare one fifth to tenths. How many tenths would fit into one fifth?". At this point, she was looking at her list of diagrams, which contained an example for $\frac{1}{5}$ and one for $\frac{1}{10}$, as shown below.

Illustration 32: Diagrams created by M

To my surprise she was not able to answer, even though she had apparently identified both diagrams. I had no clue as to what could be behind her difficulty to solve this particular question, when I noticed that the diagram for $\frac{1}{10}$ was the only one that she had glued into her notebook in a different orientation. At this moment, I pointed that out to her and commented that this could happen and that she could imagine the diagram rotated. She did not show any sign of an insight from my comment, but found the correct answer and moved on successfully to the next questions.

Note that in this episode triangles were not involved, but it seems that even a student who was able to create the diagrams and answer some questions about equivalent fractions could get stuck on a question that demanded mental rotation of a rectangle by 90 degrees.

These two episodes illustrate how visual abilities, such as mental rotation, may interfere with students' learning. Visual representations are common in mathematics classrooms, maybe due to curricular demands (data representations), teachers' preferences (fraction models), or to the nature of some topics (Euclidean geometry), and when using them teachers or task designers may implicitly expect students to be able to freely use such seemingly simple visual abilities, such as mental rotation. The problem is that such abilities may not be readily available to all students.

For instance, consider the work of Cheng and Mix (2012) showing that training in mental rotation improved the achievement of young students in missing number questions ($4+?=7$). While at first it could be said that this question can be solved by purely arithmetical means, the study suggests that there could be a connection between basic arithmetical knowledge and visual skill mediated, arguably, by the use of some internal visual representation. The authors suggest that the effect could be explained by a diagrammatic view of the equations: students see the expression as a diagram that can be visually manipulated by, for instance, rotating the 4 around the =, to the other side of the expression obtaining the new version $?=7-4$. Uribe et al. (2017) uses a semiotic reference to search for other causal mechanisms to explain what was observed by Cheng and Mix (2012) and proposes three possibilities: an improvement in working memory due to a more efficient use of visual skills, better reading and understanding of the arithmetic-symbols due to an improved capacity of dealing with spatial information, and improved capacity to manipulate images could conceptually influence the capacity of association and distribution of numbers. Although more research is necessary to better understand the causal mechanisms, all possibilities point to ways in which visual skills could interfere with "doing mathematics", even when there is no obvious visual representation involved.

While the two episodes discussed in this section show a negative interference of a student lacking visual abilities, there were also episodes showing a positive influence of having fluency with them.

W, who was Julia's student, was well-behaved and seemed to engage with the tasks during all the lessons. However, according to Julia, he had severe literacy problems, to the extent of being allowed to ask for a reader during exams. Regarding my lessons, since my preliminary study, he always progressed very well, being one of the first students to complete the basic lesson plan and engage with extension questions. A quick check of his worksheets shows that W completed the vast majority of the lessons and got the right answer for most questions. This episode took place during the starters used by the teachers before the beginning of the second pack of lessons. The question asked the students to show 1/3 of three given diagrams. When I was walking around, I saw W's solution to this question.

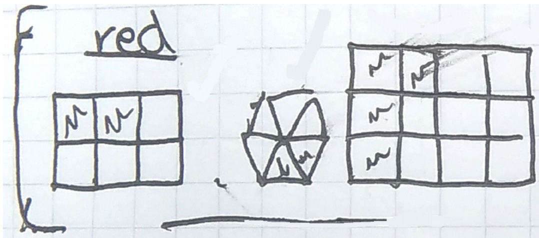


Illustration 33: Photo of student's answer

It caught my attention because I was expecting answers with one column shaded in the first diagram (1 out of the 3 columns available) and one line shaded in the third (1 out of the 3 lines available). When I asked him "why this is 1/3? [pointing to the third diagram]", he answered: "that one is for that column, that one is for that column, that one is for that column, that one is for that column [pointing to each one of the shaded squares successively]".

The spontaneity of his reasoning was striking for me. He sounded absolutely under control of where the shaded squares would be, since he could move them around mentally if necessary. Although this is just one episode, I would argue that it is not a coincidence that one of the students who showed great progress during my lessons was also able to exhibit a great ability to mentally transform a visual representation. Even though he was placed in a low set in mathematics (set 4 out of 5), W was promoted to set 3 by the end of the academic year.

Interestingly, with the second pack of lessons, this type of episode ceased and this could be due to two reasons. The first comes from the fact that the lesson plans moved away from triangular and towards rectangular sections to represent fractions in diagrams, and since rectangles reduced the need to perform mental rotation and more sophisticated transformations, there were fewer opportunities for such episodes to happen. The second relies on the expectation that increasing familiarity and exposure to the rectangular area model would enhance students' visual ability, or at least some components of it, which is coherent with findings indicating that this kind of skill is trainable at any age (Uttal et al., 2013).

7.2.1 Discussion

Originally, I decided to use triangles during the lessons as a way to transmit the message that it is not the shape of the pieces that is important, but their area in relation to the unit. For that reason, after the introduction of the model (the three first lessons of the first pack), the triangles faded away and the students would work only with rectangular sections, which were expected to be simpler to draw and manipulate.

The lessons started with Alice's group, and even though she did not use my lesson plans she used the same set of pieces that I suggested originally. The challenge represented by these pieces was translated by Alice in a talk we had just after the second lesson.

The lesson started with the same pieces used in the first lesson (representing 1, 1/2, 1/4, 1/8 and 1/16 — including three triangular shapes) and the tasks consisted of identifying how many of one type would be necessary to cover another set of pieces. By the second half of the lesson, fractions would be finally introduced by associating each one of the pieces with a fraction based on how many it would be necessary to cover the white square (unit). This was one of the lessons when I observed several students struggling to rotate the pieces.

After the lesson Alice said that after the first half of the lesson students were so mentally exhausted due to the working with the pieces, that they were not able to learn anything about fractions. Even though her judgement was based on her perception during the lesson, most of the students actually did not conclude the tasks posed and planned by her.

I can conceive of two reasons for that struggle: that students did not realize that those transformations were allowed, or that they were not able to see how the transformation could be useful.

The first hypothesis would mean that the model being used did not embed the transformations very well, thus demanding more explanations from the teacher in order for it to be successfully used by students. However, I do not think that is the case, because on no occasion did the students seem to be surprised when I showed them a transformation taking place (flipping over or turning around a cut-out), except when I approached the actual solution to the task, for instance, by positioning one cut-out in a way that started the pattern that would lead to the solution.

This perception suggests that the second hypothesis is actually correct: students were not able to project how the transformation could be useful to reach a solution to the tasks posed. This conclusion is also compatible with results showing that visual skills, such as mental rotation, should not be taken for granted, as I had done in a retrospective analysis when deciding to include triangles in the lesson plans.

Note that this remark does not contradict one of the main premises of this work, that the rectangular area model can work as a grounding metaphor for learning fractions. I believe that the model very successfully embedded the properties and transformations necessary to build the intended knowledge regarding fractions, due to the characteristics discussed in Section 5.3, that the lack of some visual abilities affect the fluency of some students with some transformations.

The next question that has to be answered is should teachers just avoid situations that demand visual abilities with low achieving students or should they intentionally use such situations to improve students' visual abilities?

First of all, it is important to remember that such abilities are trainable (Uttal et al., 2013) and correlate with achievement in mathematics (Wai et al., 2009). Therefore, the natural conclusion seems to be that those situations should not be avoided, but promoted. Additionally, as Gates (2015, 2018) argues, the gap in visual abilities can be even more significant for students from certain social backgrounds, since they may have experienced fewer opportunities to develop these abilities through playing with construction blocks, jig-saws, board games and other mathematics related activities (Levine, Ratliff, Huttenlocher, & Cannon, 2012; Tudge & Doucet, 2004).

However, what my data shows is that this could be challenging to an extent that may hinder the mathematical aims of a lesson, as highlighted by Alice's observation above. Finally, I cannot identify any situation in my observations where the previous contact with triangular shapes was clearly beneficial for students.

Thus, my recommendation is that if a teacher wants to focus on the mathematical topic of fractions, as in the case of my research, he/she should avoid situations with a high demand of visual abilities because they can be a major distractor and become the focus of the activity.

This is not to say that activities designed to promote visual abilities should be avoided at all. Actually, Newcombe (2016) highlights that the evidence showing the connection between visual skills and achievement in mathematics and science is so extensive that it is beyond time to "spatialize the curriculum". She points out that this can be achieved by explicitly including such goals in the curriculum, or by incorporating activities with such goals into regular disciplines.

My argument is that this should be done consciously and based on the merits of visual skills, rather than being left to be marginally covered during eventual lessons, and at risk of becoming another gatekeeper instead of an opportunity to learn.

7.3 Reasoning and visual representations

In Section 3.4, I presented and discussed an episode of a student working out the sum $\frac{1}{9}+\frac{5}{6}$ using diagrams. This episode was striking for two reasons. Firstly, his arguments were strongly anchored in the visual representation. Secondly, the visual representation seemed to have enabled him to extend his knowledge and solve a question that had not been covered during the lesson (so far, students had only added fractions from the same family). As discussed before, this observation influenced the focus of my research, such that reasoning became one of the main issues that I had in mind during the data collection.

As a result, in Chapter 4 I developed ideas related to reasoning towards what I call reasoning anchored in visual representations, based on concepts proposed by Toulmin (1969), Rivera (2011) and Lithner (2008). In summary, this term refers to chains of arguments whose warrants or backings are explicitly tied to visual representations, its elements, properties and transformations.

In this section, I will present some episodes with students explaining how they solved different tasks and why they had chosen the way they did so, i.e. reasoning. Beyond showing their reasoning, my goal is to evidence how this reasoning is anchored in the visual representations, and even further, is enabled by the visual representations.

7.3.1 Anchoring reasoning in visual representations

The first episode took place during lesson 1.4, after students used cut-outs to identify several rectangles as representing the fractions 1/6, 1/8, 1/10, 1/12 and 1/15. The task below was than posed to them. For the first item of this task two fractions were given, but for the other items only one fraction was given.

Task 2

Use the pieces and the information in the table to find new fractions that are the same as the fractions below.

$$\frac{1}{2} = \frac{3}{6} = \frac{4}{8} = \frac{5}{10} = \frac{6}{12}$$

Illustration 34: Task 2 from lesson 1.4 answered by M

Furthermore, the items of this task were chosen such that a unitary fraction would be asked before a non-unitary fraction with same denominator. For instance, before asking for fractions equivalent to 3/4, the task asked for fractions equivalent to 1/4. The intention was to create an opportunity for students to notice the relation between a unitary fraction (and its equivalent fractions) and non-unitary fractions with the same denominator (and its equivalent fractions).

When I was walking around the room, I noticed that M was solving Task 2 for the fractions 1/3. After talking to some other students, I noticed she had not only solved this item correctly, but also the next one regarding 2/3, as shown below.

$$\frac{1}{3} = \frac{2}{6} = \frac{4}{12} = \frac{5}{15}$$

$$\frac{2}{3} = \frac{10}{15} = \frac{8}{12} = \frac{4}{6}$$

Illustration 35: M's answers (the circle, underlining and tick were inserted later by the teacher)

Because she got the right answers very quickly, I decided to ask her how she got the answers to 2/3, to which she answered:

> M: You can fit five of these [pointing to 5/15] into that [pointing to 1/3]. So, you can fit 10 into that [pointing to 2/3].
> Me: What about the others?

M: The same. 4 and 8 [pointing to 4/12 and 8/12], 2 and 4 [pointing to 2/6 and 4/6]

First of all, note the meaning she attributed to the equal sign: "to fit into". This meaning is strongly connected to how equality is treated in the rectangular area model. Two fractions are considered equal if they cover the same area and this is operationalized by overlapping (fitting) the shapes one onto the other. This is very close to viewing non-unitary fractions as a collection of unitary fractions. Collection is seen by Lakoff and Núñez (2000) as one of the most fundamental grounding metaphors for mathematical cognition. However, note that there is more than the numerical view of a collection involved in her reasoning. The use of "fit" as the verb suggests that she is aware of the relation between different types of fractions in terms of areas, and that it is not only an issue of counting elements. Therefore, in terms of Lithner's (2008) proposal, her reasoning was anchored in the visual representations.

Another episode that shows that students' conceptualization of fractions was not restricted to counting, but considered the relation between different fractions, can be seen in the answer given by Z when justifying for the whole class why he thought 3/4 was bigger than 5/8. He said: "Though five eighths has more bits, they are smaller". Once again, the use of a word such as "bits" reinforces the anchoring in visual representations.

The second episode took place at the end of the second pack of lessons. L wrote the following as an answer to the question, "What is bigger, one half or six tenths?".

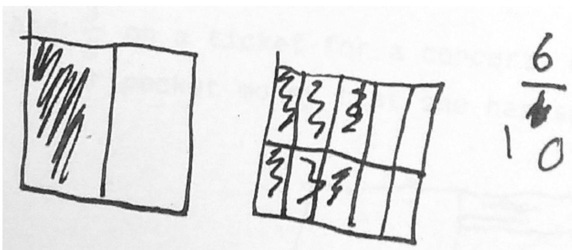


Illustration 36: L's solution

When asked how he concluded that 6/10 is bigger, he said, "because if this was a half [pointing to the second diagram], the line would be in the middle of that one [moving his finger along the red line]".

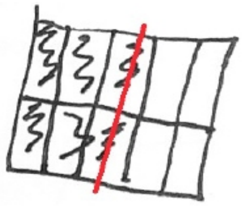


Illustration 37: L's gesturing

Note that his argument relies on a visual perception of what half should look like, so it is clearly an example of reasoning anchored in visual representations. I recognise that, maybe, this student would not be able to employ a similar strategy if the question did not involve half, although the episode shows that visual representations may also capitalize on the familiarity students show with certain fractions, especially half (Clarke & Roche, 2009). Because "halving" and "doubling" are emphasized in the English curriculum, a teacher would naturally expect students to easily calculate that 5 is half of 10 and, therefore, 6 tenths would be more than 5 tenths. However, this may not be case for low achieving students, who are usually not fluent with what are considered basic arithmetical facts. Nevertheless, this episode shows an alternative pathway, anchored in a visual representation that also relies on common knowledge regarding "half of something", but one that could be accessible to a low achieving student.

Similarly, when asked about the question "Is 5/8 bigger than 1/4+2/8?" during the first pack of questions, C answered: "This [pointing to 1/4+2/8] would fill up half, and this [pointing to 5/8] has one more". Even though he had no diagram drawn on his worksheet, his explanation is very visual, as indicated by the verb "fill up". Also, although not referred to explicitly in the question, half was again used as a benchmark, with the strategy employed by the student showing no sign of a numerical approach.

Third episode

The next episode involved student D (Year 9 Set 5), who usually engaged well during mathematics lessons. In lesson 2.3 I noticed he had correctly worked out the sum $\frac{2}{3}+\frac{5}{12}$, even though his diagram (below) only showed the unit divided into twelfths, with all of them shaded (as the answer is 13/12, I was expecting something else to represent the thirteenth twelfth).

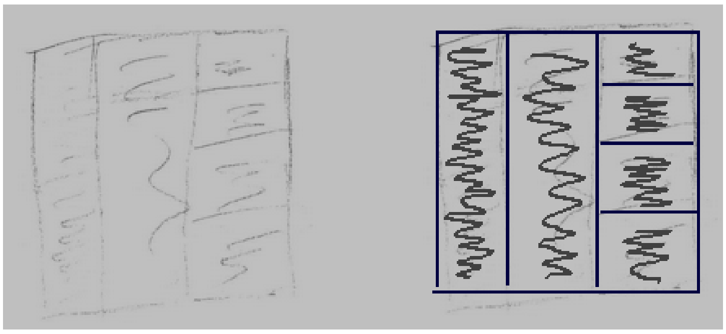


Illustration 38: D's diagram for the sum 2/3+5/12 and a recreation of it

I decided to approach him to ask how he had solved it, to which he answered:

> "Here is two thirds [running his finger along the two first columns in the diagram] and it is the same as eight twelfths. And then one, two, three, four [pointing to each one of the rectangles in the third column] and five [pointing again to the last rectangle]. So eight plus five is thirteen."

I was impressed by his explanation, but decided to add that he could have drawn another unit to show the last twelfth and he said promptly, "Yeah, I know," as he had consciously decided not to do it. Actually, when I checked his answer for a question involving the sum $\frac{3}{4}+\frac{3}{8}$ from the previous lesson, I noticed that he did draw and use a second square.

My argument is that his strategy of "double counting" a twelfth in the diagram above is anchored in the possibility of overlapping shapes that was extensively explored in early lessons, and that this action was naturally embedded in the chosen materialization of the rectangular area model.

7.3.2 Discussion

The three episodes presented above are just a selection that could have been used to illustrate students' reasoning based on visual representations. My intention when presenting them is to highlight the central role played by the visual representations manifested through the vocabulary used by the students, their gestures, and the content of their arguments. They all resonate with Rodd's (2000) reading of Giaquinto's ideas that visualization can work as warrant for mathematical reasoning.

However, this small sample would not be enough to draw conclusions. Although they are focused on other phenomena, the following sections will reinforce the close connection between reasoning and visual representation during the targeted lessons. Specifically, the next section will extend the argument presented here by exploring what I called the generativity of students' reasoning.

7.4 Generative reasoning

The idea of generativity came into play when I noticed the contrast between teachers' expectations regarding some specific questions included in my lesson plans, and how students actually responded to these questions. The moment this contrast was most evident was when I showed the teachers my ideas for lesson 2.5, with the first one being focused on word problems. Their first reaction was that students would struggle to solve the word problems due to their difficulty in applying mathematics, or to literacy limitations.

I did not feel they were discouraging me or would prefer not to use the lesson plan. The reactions sounded like honest diagnoses based on their experience with the students involved in my research. That means low achieving students, with low levels of confidence, from working class backgrounds and accustomed to being placed in low sets, at least since the beginning of secondary education.

Nevertheless, all the teachers were positively impressed with the student engagement and results after lesson 2.5. Most of the students solved most of the questions posed and obtained correct answers. They did not struggle more with these questions than they had done with the un-contextualized questions in previous lessons.

This phenomenon, where students were able to extend smoothly what they were taught in order to solve questions that were somehow new to them, appeared on other occasions throughout my data collection. This step beyond what was explicitly taught is what I am calling "generative thinking" and will be discussed in the next sections.

7.4.1 Generativity

The use of the term generativity occurred to me after reading Tsang et al. (2015). The authors used an experimental design to compare three approaches to teaching operations with negative numbers. However, instead of only using pre- and post-test measurements, they added a post-test composed of what they called generative questions. Their idea was to use the regular pre- and post-test to measure whether the three groups received minimally satisfactory instruction during intervention, i.e., the three groups should achieve similarly in the post-test (considering their achievement in the pre-test), otherwise it would mean that the quality of instruction was different, and any change detected in the generative test could be due to this factor and not to the approach employed.

The regular pre- and post-tests were composed of questions that were directly discussed during the intervention. Therefore, they established a baseline against which the researchers could disregard the quality of teaching as an influence in the results.

In contrast, the generative test was composed of questions that were not directly discussed during the intervention. All the questions had some new element embedded. For instance, while the regular pre- and post-tests posed questions, such as "work out 7-4" and "place -2 on a given number line", the generative test posed questions, such as "complete the number sentence $-5+-3=-4+[\quad]$ " and "place -3/7 on a given number line". Note that the latter questions are similar to the former, but include elements that even though not discussed during intervention, were familiar to students in other contexts (they have previously had lessons about missing number problems and fractions), but new in the context of negative numbers.

Personally, I would characterize such questions as "suitable topics for the next lesson", since neither do they demand new arbitrary knowledge to be solved, nor novel chains of reasoning. The novelty here is to apply the same chain of reasoning that was used to solve recent questions, to a broader set of objects, to extend what was seen in previous lessons to other known situations.

There are some concepts in the mathematics education literature that resemble the concept of generativity. One of them is transferability. This concept usually appears in research related to the use of different representations, such as Behr et al. (1988) who analysed how well students that received instruction on fractions using a continuous manipulative, were able to transfer their knowledge to solve tasks based on a discrete manipulative.

Another context in which this concept is common is problem solving. Here, transferability refers to students' capacity to mobilize concepts that they have already learned, to solve a problem which by definition, should be new to the student. In this context some researchers use the idea of transfer distance to recognize that novelties can be more or less challenging for students. For example, Fuchs et al. (2004) employed this concept to group problems used to evaluate an intervention aiming to help third graders solve real-life mathematical problems.

Note that neither of these contexts are compatible with Tsang et al's. (2015) use of generative questions. The same could be said about the use I intend to make of it in this thesis. Since I am not referring to transferences between different representations, and not specifically to transferences towards what are typically called problems, I believe the term generative is more adequate for my purposes. Therefore, I will clarify in the next sections what I mean by it.

7.4.2 Zone of proximal development

Another concept that seems to be related to generativity is the zone of proximal development (ZPD), originally proposed by Vygotsky in his seminal works (Roth, 2014). Vygotsky uses the concept to differentiate between the space of tasks that a learner can solve independently, from the space of tasks that a learner can solve with the help of a more knowledgeable person. He defined ZPD as being:

> the distance between the actual developmental level (independent problem solving) and the level of potential development (problem solving under adult guidance or in collaboration with more capable peers). (Vygotsky, 1978, p. 86)

The concept is key for his view of learning and development as essentially social acts: students are learning when they explore problems in their zone of proximal development, enabled by some sort of support, and they develop by progressively incorporating these problems into the space of problems they can solve independently. From his perspective, the role of teachers was to build opportunities for learners to act on tasks in their ZPD.

Vygotsky also considered that all human action is mediated by tools and signs. Even though he distinguished both by defining the former as externally oriented and the latter as internally oriented, these two concepts were collapsed into the concept of artefact (Vygotsky, 1978). Note that at this point, even though some emphasis had been placed on the artefacts, the creation of ZPD depended on interaction with another person: a more knowledgeable one. Later on, the role of artefacts in this process was reconsidered by researchers.

Abtahi, Graven & Lerman (2017) analysed an interaction between a child and her mother around the task of counting in threes. However, instead of considering only the child and the more knowledgeable mother, they took into account the role played by the artefact being used during the task: a remote control (displaying numbers from 1 to 9 in a 3×3 grid). As did other researchers, they concluded that the features of the artefact were determinant in the learning opportunities created during the interaction. They understood that:

> While we are not considering a tool (such as a ruler) to be an active agential participant in interactions, nevertheless we believe that a tool carries the knowledge of its designer (who draws on knowledge of standardized mathematical knowledge), and also the knowing and perception of the people who have used and modified it over time. It is in this respect that a tool can function as a knowledgeable other if the actions of the participants bestow on it that function. (Abtahi et al., 2017, p. 276)

Abtahi (2017) arrives at a similar conclusion after analysing two students working out a fraction sum using manipulatives:

> I suggest that the possibilities for learning in the ZPD exist not only between individuals, but also between individuals and tools, with tools at times being the more knowledgeable other. (p. 12)

In terms of my research, the visual representations (cut-outs and diagrams) can be seen as artefacts, and following the arguments presented above, they could enable students to act in their ZPD. This means that the visual representations could support students in solving questions beyond what were explicitly taught to them, and this is what I am going to define as generative. However, before that I will revisit one episode that I consider illustrative of the concept.

7.4.3 One example of generative thinking

Once again, I want to refer to the episode reported in Section 3.4. On that occasion, L worked out the sum $\frac{5}{6}+\frac{1}{9}$ and obtained the correct (although unusual) answer $\frac{8+½}{9}$, even though he had never explored sums with unlike fractions. Can this question, considering the context, elicit generative thinking?

First, it is important to clarify how students in this group were adding fractions at that time. They were only dealing with sums of fractions from the same family, such as $\frac{1}{6}+\frac{5}{12}$. In this case, students would typically represent $\frac{1}{6}$ in a diagram and then try to figure out how twelfths could be represented in the same diagram. The image below shows a hypothetical path towards the final answer for this sum.

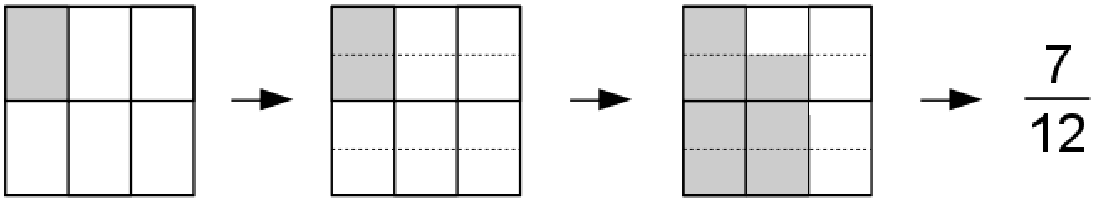


Illustration 39: Hypothetical path to work out the sum 1/6+5/12

Note that this strategy depends on the possibility of converting sixths into twelfths. However, this is not so simple in the case of sixths and ninths because nine is not a multiple of six. For this reason, I consider that this was actually a generative question when it was posed to the student. Note the situational nature of this characterization: the same question could be considered generative or not depending on what was taught in previous lessons, depending on students' prior knowledge.

Moreover, the generativity of the question can be confirmed by the generativity of L's solution. Since the student had no ready-known strategy to apply, it is expected to see some element of novelty in his solution too. In this case, L's solution arose from the possibility of using fractional multiples: $\frac{5}{6}=\frac{7½}{9}$. Although unconventional, his solution is logically consistent. It is worth mentioning that this approach was never shown to him during the lessons. Therefore, it could also be said that he showed generative thinking.

The argument that was made in Section 3.4, and that will be reinforced in the next section, is that this generative thinking was enabled by the visual representation in use.

7.4.4 Defining generative reasoning

Another framework that has to be considered is one proposed by Lithner (2008). He distinguishes between imitative reasoning, which can be further discriminated as memorised and algorithmic, and creative reasoning. He defines the latter (actually called creative mathematically founded reasoning) based on three criteria:

> 1) Novelty. A new (to the reasoner) reasoning sequence is created, or a forgotten one is re-created; 2) Plausibility. There are arguments supporting the strategy choice and/or strategy implementation motivating why the conclusions are true or plausible; and 3) Mathematical foundation.

> The arguments are anchored in intrinsic mathematical properties of the components involved in the reasoning. (p. 266)

The example presented in the previous section certainly satisfies the first criterion. I consider that it does satisfy the third, because from my perspective the visual representation counts as "mathematical foundation". However, the second is difficult to evaluate, since support for strategy choice and strategy implementation are unlikely to be captured in written solutions, and considering the characteristics of the students in my study, are also unlikely to be captured through conversations.

A similar result regarding the second and third criteria would be achieved if other episodes from my data were analysed against Lithner's framework. For that reason I believe his framework is not adequate.

I would define generative reasoning as a reasoning sequence, inferred from a written solution produced by a student and possibly complemented by conversations with him or her, that are:

1) new for the reasoner. The novelty does not have to be in terms of all its components, but mainly in terms of how known arguments are connected. This criterion is very similar to Lithner's first criterion;
2) not anchored in authority. This is a more flexible version of Lithner's third criterion. The main reason for this change is to make explicit the acceptance of arguments anchored in visual thinking, as proposed by Giaquinto (2007). This should be enough for my study, since I am intentionally focusing on visual representations, but arguments of a different nature could be included here, such as those discussed by Lakoff & Núñez (2000).

These two criteria seem to encapsulate the essence of what was presented in the example without relying on assumptions that are not reasonable, considering the data available.

Generative reasoning is likely to be elicited by questions that are, to some extent, new to that particular student, but not too distant from what he/she can already solve, similarly to Vygostky's ZPD. I would call these questions generative. Its main characteristic is to force students into a situation where they do not have the full prior knowledge to solve the problem, but can "generate" an approach given what they actually know.

However, generative reasoning could also emerge from familiar questions when approached from a different perspective.

7.4.5 Episodes showing generative thinking

In this section I will analyse some episodes grouped according to the conceptual structure of the questions that originated them, illustrating what I would qualify as generative thinking supported by visual representations.

Subtraction

To understand how this question could be generative, it is important to consider how addition and subtraction of fractions were taught to low achieving groups at Purple Valley. I had the opportunity to observe Julia teaching this lesson to a Year 9 Set 4 (out of 5) during my data collection, and essentially the lesson was limited to presenting the algorithm through the memory aid called cross-and-smile (shown below).

$$\frac{a}{b} + \frac{c}{d} = \frac{a \cdot d \ + \ b \cdot c}{b \cdot d}$$

Illustration 40: Cross-and-smile

What surprised me the most was not the dependency on the memorization of an algorithm or mnemonic device, but the fact that after showing how to use it for addition, Julia explicitly explained how to use it for subtraction. She did not explain the algorithm step by step, but she also did not establish any connections between the two cases. In fact, I would say that there was no meaning attached to any action or object throughout the algorithm, only a hidden message saying something like "if you see the symbol + between two fractions, you should follow this procedure; and if the symbol is —, you should use this other (very similar) procedure".

This lack of meaning, especially regarding the symbols + and -, was reinforced by two observations I made regarding the same student. First, I noticed the following answer to the questions, "Work out the sums shown in the diagram".

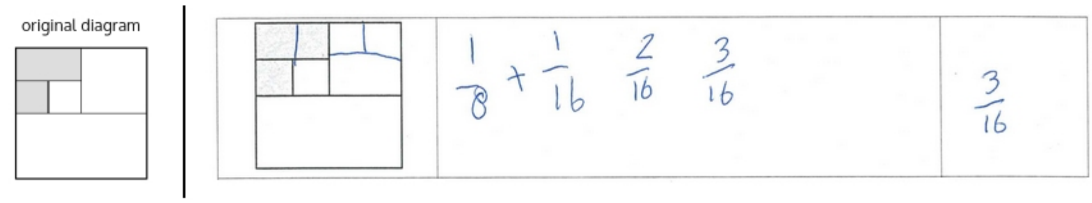


Illustration 41: Student's solution

Note how he registers the third and fourth fractions without connecting them to the sum written right before. He was answering this way even though the teacher had solved one example on the board for the whole class. It could be argued that he did not bother writing down everything and was just focusing on the numbers as a way to register partial results necessary to get to the final answer. However, in the next lesson I noticed the following answer on his sheet during the starter.

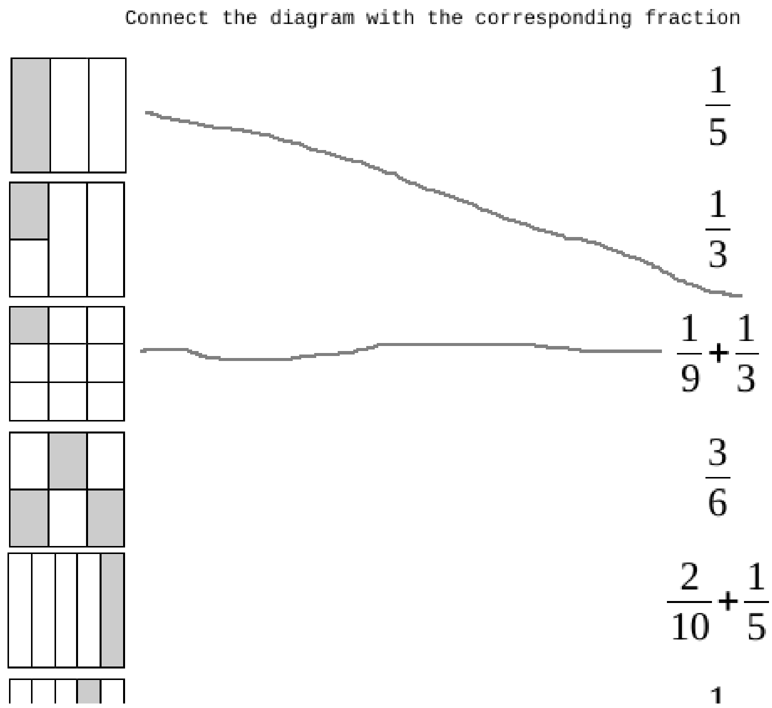


Illustration 42: Partial recreation of student's answer

Note that he was treating the fractions of the sum as individual objects. Finally, I observed the same student solving basic arithmetic questions successfully with whole numbers in several regular lessons. Taken together, these observations suggest a very fragmented understanding of the symbol +. It could mean an indication of which procedure to follow, something dispensable, or addition of whole numbers.

The same could be said about the symbol -. Actually, when subtractions were posed for the first time during my lessons, three questions were asked by several students: "What do I do here?", "Is this take away?" and "How do I show this in the diagram?" I was expecting the latter, since the way addition was represented did not suggest how subtraction could be represented, but the former two questions reinforced my perception of fragmented understandings of the symbol -.

In general, my reaction to these questions was to confirm that it meant take away. In comparison to the effort involved in Julia's re-explanation of the cross-and-smile approach, I consider my intervention to be quite short. Nonetheless, it proved enough for most of the students. Although the subtractions were represented in different ways (as can be seen in the image below), most of the students were able to work out these sums, after my sole clarification regarding the meaning of the symbol -.

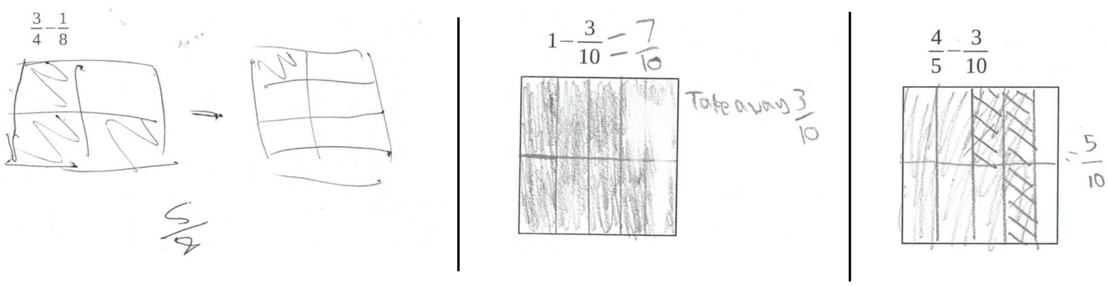


Illustration 43: Three different representations for subtraction: two diagrams, erasing and crossing

My argument is that this relative smoothness and success in transitioning from addition to subtraction should be seen as result of generative thinking and this is the result of the objects and actions involved in the questions being anchored on the visual representations, providing a meaningful reference for the symbols.

Bigger than the unit

Generativity can be manifested not only when solving questions that demand something beyond what students were taught, but also when they deal with didactical obstacles (Schneider, 2014), and that was the case when sums bigger than one were introduced. Due to the way sums of fractions were presented (the video shown on lesson 2.3 summarizes the approach: https://youtu.be/7FMhj3E1WzI), I expected that sums bigger than the unit would be challenging for students, but since one of my assumptions was that visual representations are a solid enough basis to build knowledge of fraction addition and subtraction, I decided to keep it, expecting it would lead students to productive struggle (Kapur, 2010).

For instance, consider what happened with C, one of Alice's students, when he first faced a sum whose result would be bigger than 1. He was raising his hand and looking at me asking for some assistance. When I approached him, he had the diagram below drawn.

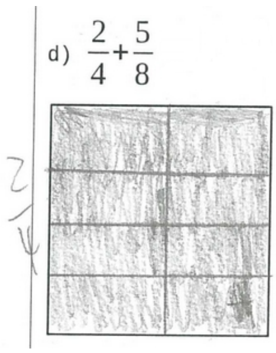


Illustration 44: C's diagram when he asked me for help

And here is the transcription of our conversation initiated by him:

> C: So, two quarters is a half. And five eights… I can only put four [pointing to the second column of his diagram].
> Me: Perfect. Now, you can add another square here [outlining a new unit square besides his diagram] and have an extra eighth right here.
> C: So, would it be… one… one… add one eighth?
> Me: Yes, you are right!

First of all, I consider it important to remember that the students in low sets at Purple Valley would rarely ask questions spontaneously and lacked confidence. Even though C was one of the students with the highest scores in external exams (but still below 25%), he was not an exception to this behaviour. However, he felt something was not right with his answer, and instead of making a mistake as will be shown below, he asked for assistance and was able to express his doubt. The fact that he needed assistance should not be seen as lack of generativity by him, especially if you take into account that he was able to conclude the solution after my modest assistance.

Another student from David's group, J, when solving a sum bigger than 1 for the first time, was approached by David when I happened to be passing by, so I captured their conversation based on the solution shown below.

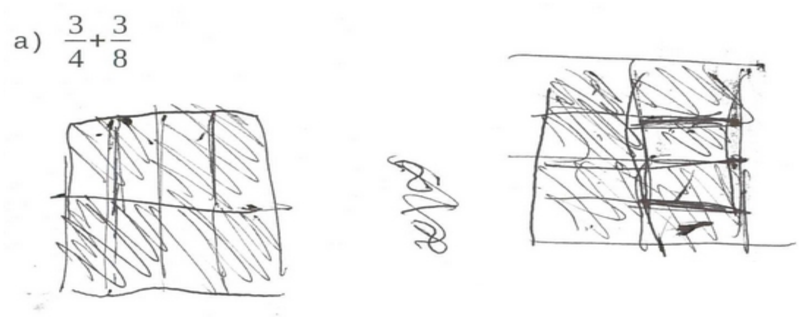


Illustration 45: J's diagrams done before the conversation with David (the fraction 8/8 was not crossed)

> David: Show me your thinking on this one.
> J: So you get your quarters and you do three of them. Because you already have three done, you put a line through the other one there [suggesting dividing the bottom-right quarter on the second diagram in half horizontally] and shade in one. And then to get the answer I have just put a line through all of them [referring to the lines dividing the other quarters into eighths] and add them up.
> David: Why is that right?
> J: Because it says three out of eight. [long pause] That is what is confusing me because there is not three [inaudible]...

At this point, David noticed my interest in the conversation and asked me to lead the conversation.

> Me: This is a bit weird... Can you show me the three quarters?
> J: Here, here and here [pointing to the diagram]
> Me: Okay, so how many eighths are in three quarters?
> J: Six. And then there is one left in the eights, so it is seven.

It seems reasonable that the last sentence in the dialogue above is an attempt by him to deal with the eight that did not fit in the unitary square. Since it is only possible to fit another two eighths, he may be imagining the third one overlapped on one of the eighths contained on the three quarters.

> Me: Wait... You have six eights here and you have to add another three...
> J: Oh! Nine... But then it is not right in there [referring to the fourth quarter in the diagram].

Instead of incurring the mistake made by some students of deforming the diagram in order to fit any fraction needed, he showed awareness that something was not right. Note that he had already drawn two diagrams before the conversation, suggesting that he was trying to find a better solution for the sum. It could be argued that the possibility of working with more than one unitary square should have been introduced before, but his struggle turned out to be productive, and he was able to expand his knowledge towards a situation with new elements and actively participate in the construction of a solution, even though my intervention was necessary.

In the next lesson, the sum $\frac{2}{3}+\frac{5}{12}$ was posed and when I was passing by, J called me and asked, "Is that one right, 'cause you can't go over the one, can you?" pointing to the diagrams below.

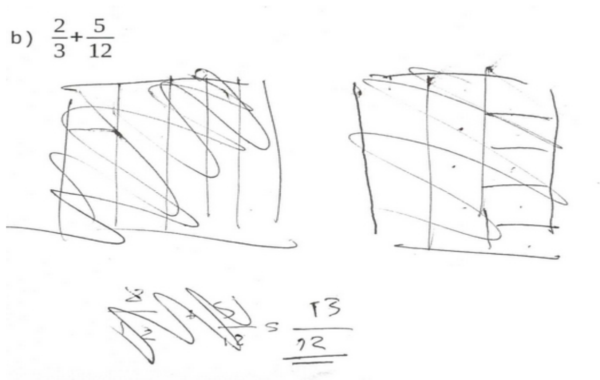


Illustration 46: J's solution before I talked to him (the sum in the bottom were crossed later)

Note that he was still struggling with sums bigger than one. The second diagram shows that he tried to fit 5/12 into 1/3 but the dots suggest that he counted the parts to check if it was right. Finally, the crossing all over the diagrams (done before he called for me) and the question he asked me, suggest that he was aware that something was wrong and was trying to find a better diagrammatic representation, even though he had got a numeric answer.

This type of behaviour was very common in all three groups: students noticing that something different was happening with the representations and being tentative in terms of how to approach this. My point is that even though these students needed some sort of assistance, in general, a suggestion from me or from the teacher was enough for them. This shows that generative thinking to expand the possibilities of the representation had been utilized.

Mixed numbers

By the end of every term, all students at Purple Valley undertook an exam developed by the teacher responsible for each specific Key Stage; for each year group there were two exams, one for the two bottom sets and another for the other sets, such that the exam would fit better to the expected curriculum for each set. In the case of the groups participating in my research, the teacher responsible for creating the test was not Julia, Alice, nor David.

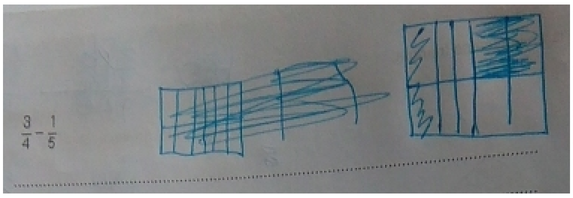
The exam applied at the end of the second term took place on a day that I was not in the school, but Julia told me afterwards that there were some questions about fractions.

1) Work out $\frac{1}{4}+\frac{2}{3}$.	4) Work out $\frac{3}{4}-\frac{1}{5}$.	6) Convert $2\frac{3}{4}$ to an improper fraction.
2) Work out $\frac{2}{5}\times\frac{1}{4}$.	5) Work out $1\frac{4}{5}+3\frac{3}{10}$.	7) Convert $\frac{15}{4}$ to a mixed numbers.
3) Work out $3\frac{1}{2}\times2\frac{1}{4}$.		

Unfortunately, none of the questions were comparable to what students were learning during the targeted lessons. Just to remember, at this point that students were adding and subtracting fractions from the same family. Note that questions 2 and 3 were about multiplication; questions 6 and 7 depended heavily on knowing the meaning of improper fractions and mixed numbers, which was not the focus of the targeted lessons; although questions 1 and 4 are about addition and subtraction, the denominators are not multiples. Finally, question 5 was a good fit, apart from the mixed numbers, that were never used during the targeted lessons and that I have never observed teachers using with low achieving groups at Purple Valley.

After marking the assessment, Julia invited me to have a look at students' answers for those questions. None of the other teachers offered me the possibility of looking at the exams and I did not feel comfortable asking them to allow me to do so. There is nothing to comment on questions 2 and 3, because they are not related to the targeted lessons, but the answers for the other questions are interesting in terms of my research.

Firstly, regarding questions 1 and 4, I think it is worth mentioning that no student in Julia's group fell back to 'add-tops-and-bottoms' even though they were not able to solve these questions. Actually, the diagrams made by some students, considering what they had learned by that day, show coherent attempts to find a solution. The images below show diagrams created for the sum $\frac{3}{4}-\frac{1}{5}$.



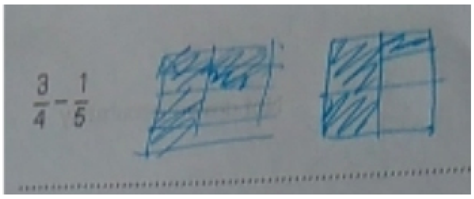


Illustration 47: Two students' answers

The picture to the left of Illustration 47 shows that the student shaded 2 over 10 (first column) and then a portion that resembles a quarter (apparently trying to make sense of how to represent $\frac{3}{4}$ on this diagram). On the right, the first diagram seems to be an attempt to represent $\frac{3}{4}$ in a diagram divided into tenths, and the second shows the same amount shaded minus one tenth (it seems that the student meant one fifth, as asked in the question); note that the final answer given by the student was $\frac{2\frac{1}{2}}{4}$, which is coherent with the second diagram.

Secondly, some students got right answers for question 5, even though it involved mixed numbers. The image below is an example.

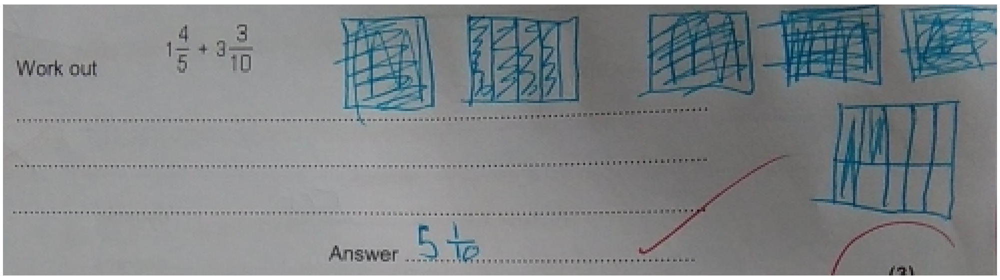


Illustration 48: Student's answer

Note that the student represented correctly the fractional part of each mixed number and left out the integer parts. However, the final answer not only considered all the parts but it is also presented as a mixed number instead of $4\frac{11}{10}$. I consider this to be a great example of generative thinking, because this student was able to mobilize not only the specific topics covered during the targeted lesson, but was also successful in interpreting the addition of mixed numbers and converting an improper fraction to a mixed number relying only on diagrams.

The following answer is also very revealing of how much students relied on diagrams. It seems that this particular student felt comfortable enough to expand the use of the model to whole numbers.

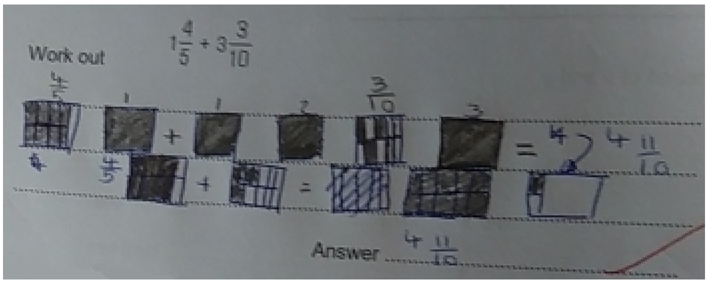


Illustration 49: Student's answer

A new comparison method

Comparison of fractions is usually done by equalling the denominators of the given fractions and then comparing the numerators. This approach also makes sense within the rectangular area model: when the fractions are from the same family, the student has to transform one of the diagrams into the other, but even when the fractions are not from the same family, the approach can be implemented by combining the grids, as the lessons in pack 3 suggest for addition and subtraction of fractions. However, this is not the only approach. For instance, you can equal the numerator and compare the denominators. The difficulty here is that the conclusion is the reverse of the direct comparison between denominators: the fraction with the biggest denominator is the smallest fraction. This is the reason why this approach is unusual, unless the given fractions already have the same numerator.

From pack 1 to 3, I included questions regarding comparison of fractions for three reasons. Firstly, it is part of the curriculum. Secondly, there is research showing that the knowledge of the magnitude is an important component for understanding rational numbers in general (Bezuk & Cramer, 1989; Torbeyns, Schneider, Xin, & Siegler, 2015). Thirdly, since comparison was never the main goal of a lesson, I felt I could use the topic to elicit reasoning not influenced by previous explanations.

The event I want to analyse here took place during lesson 1.4 in David's group. The third task posed four comparisons between two fractions right after a question in which students should list fractions equivalent to a given one. The comparison task was designed in such a way that all the fractions used had been also used in the previous task. The intention behind it was to allow students to use the list of equivalent fractions to identify equivalent fractions that enabled the comparisons. My expectation was to highlight the strategy of using fractions with the same denominator. For instance, the first pair of fractions was $\frac{1}{3}$ and $\frac{2}{5}$ and my expectation was that students would find, by checking the list of fractions equivalent to $\frac{1}{3}$ and $\frac{2}{5}$, that $\frac{1}{3}=\frac{5}{15}$ and that $\frac{2}{5}=\frac{6}{15}$. However, after the end of the lesson, David told us that a pair of students working together had used a different strategy: they noticed that $\frac{1}{3}$ is also equal to $\frac{2}{6}$ and that this fraction is smaller than $\frac{2}{5}$ because "sixths are smaller than fifths" (David's words describing students' explanation). Although unusual, their approach is actually more economic, since it can be executed (for this pair of fractions) by inspecting only one list of equivalent fractions.

From my perspective, even though these students had faced comparison questions before this lesson, their reasoning could be considered to be generative thinking. Is it reasonable to say that it emerged from the visual representations?

To answer that question, I would like to discuss David's description of the explanation given by the students: "sixths are smaller than fifths", even though I do not have a recording of their explanation, and David may not have been precise in his description. In a subsequent lesson I posed to some students three comparisons intentionally designed to tackle different approaches, using $\frac{5}{8}$ and $\frac{6}{8}$ for comparison of numerators, $\frac{1}{4}$ and $\frac{1}{3}$ for comparison of denominators, and $\frac{2}{3}$ and $\frac{5}{9}$ for equivalent fractions. I got data from some students using sentences just like "sixths are smaller than fifths" to justify their choices. For instance, when I asked about the three comparisons, A answered the following quite quickly, without much pondering:

> This has one more than the other [pointing to 5/8].
> These [pointing to 1/4] are smaller than these [pointing to 1/3].
> This [pointing to 2/3] is the same as… [pause for thinking] six ninths so it has got one more than this [pointing to 5/9].

After getting his verbal answers, I asked him to write them down and the result is shown below.

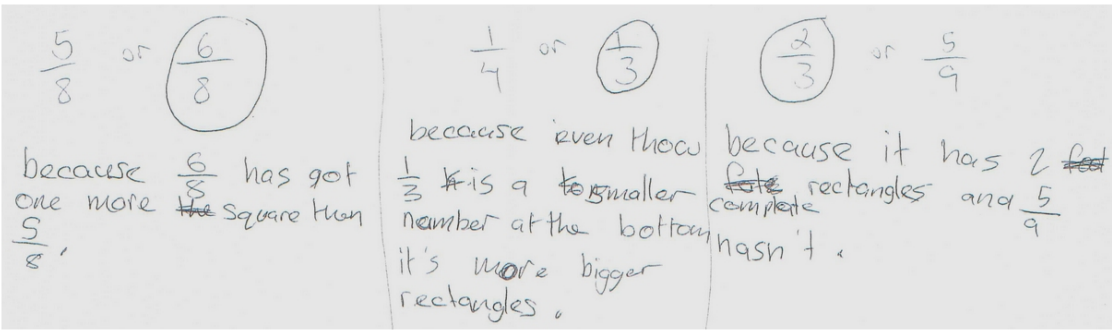


Illustration 50: Student's answer to my request to write down his explanations

The first two answers are quite clear, but the third needs some interpretation. I believe that "2 rectangles" refers to the shaded columns that he would obtain if he had drawn a diagram for 2/3 using vertical sections. Note that if you add horizontal sections to this diagram in order to divide it into ninths, 5/9 would not fill two full columns.

Note the similarity between the verbal reasoning and the written answers above, and David's description of the student's explanation before. The way they both referred to sixths and fifths as objects, and the use of "has got" reflects the way the rectangular area model was materialised: unitary fractions as basic elements (cut-outs) that can be directly compared via overlapping and non-unitary fractions as collections of such basic elements. Based on this analysis, I argue that the generativity shown by the pair of students who compared fractions by equalling the numerator of the fractions was anchored in the visual representations.

7.4.6 Discussion

When reading this section, it is important to keep in mind the students who participated in my research: low achieving students in a school that was "under special measures" due to the low achievement levels of its students in national exams.

At first, from a mathematical perspective, it may seem not much to extrapolate properties from the addition of fractions to subtraction, or to deal with sums bigger than one, with just a little extra input from the teacher. In an ideal scenario, even I would expect my students to do it seamlessly. However, the students in my research are students with low confidence and undeniable deficiencies in their prior knowledge when compared to their peers. Also, they are part of a school culture that emphasizes bite-sized instruction and presents mathematics as a collection of specific, and mostly unconnected procedures in which students have little agency and no space for creativity.

It is also important to take into account the impressions of the three teachers. As I mentioned at the beginning of this section, they were hesitant when I presented the lesson plans with word problems. However, after teaching these lessons, they were all very impressed by how much the students were able to accomplish. They may have not noticed consciously all the episodes that I discussed above, but they certainly felt the lessons were successful, since all three teachers asked me to use the lessons with other groups when I finished my official data collection.

In conclusion, I would extend my argument to say that what was shown in the episodes above represents generative thinking, and also to suggest that this generativity was actually enabled by the visual representations.

This conclusion is coherent with theoretical arguments posed by researchers, such as Richard Skemp, that visual representations are somehow more intuitive and give access to the structures underlying the representation itself (Skemp, 1987).

From the perspective of embodied cognition, as proposed by Lakoff & Núñez (2000), the model utilized can be seen as a metaphor that enabled students to transfer the inference structure from the source domain of flat surfaces (area, movement, composition and decomposition) to the target domain of fractions addition and subtraction. Considering the arguments presented by Giaquinto (2007) regarding some visual abilities, it can be argued that the model worked as a grounded metaphor, "allow[ing] us to ground our understanding of arithmetic in our prior understanding of extremely commonplace physical activity" (Lakoff & Núñez, 2000, p. 53). This approximation between the target domain from a meaningful source domain though visual representations, as proposed by Johansen (2014), not only enabled students to reason, but provided the flexibility necessary to allow them to do so generatively.

Of course other factors, such as the design of the lesson plans, played an important role in this process. Therefore, my conclusion should be understood under the constraints of the context in question and the characteristics of the implementation, from the choice of the model and its materialization, to the lesson plans and the way teachers enacted them.

7.5 The multiplicative aspect of fractions

Some researchers argue that fractions are part of the so called multiplicative conceptual field (Harel & Confrey, 1994), and therefore, cannot be separated from the broader idea of multiplicative reasoning (M. Brown, Küchemann, & Hodgen, 2010). In my approach, the multiplicative aspect of fractions arises from the importance given to equivalent fractions throughout the lessons.

From a numerical perspective, the relation between equivalent fractions and multiplication is quite obvious: two fractions are equivalent if, and only if, one could be obtained by multiplying the numerator and the denominator of the other by the same scalar number. At one extreme, if a teacher adopts a so called conceptual approach to the topic, emphasizing the meaning behind the procedures, students will deal with equivalent fractions all the time, to add, subtract, compare, multiply and divide them. At the other extreme, if a teacher adopts an approach emphasizing procedures without much concern about meaning, multiplications and division will be used all the time, since equivalence is necessary for most of the procedures. Either way, knowledge of fractions seems to depend on knowledge of multiplication.

Considering the approach utilized in the targeted lessons, which is less numerical and more visual, I argue that the relation between fraction and multiplication becomes more complex. On one hand, multiplication seems to be less of a pre-requisite, since it is possible to obtain equivalent fractions via transformations that can be realized without any numerical use of multiplication (adding lines to, removing lines from and rearranging a given diagram). For instance, the transformation below shows that 1/3 is equivalent to 2/6 without any numerical multiplication being necessary.

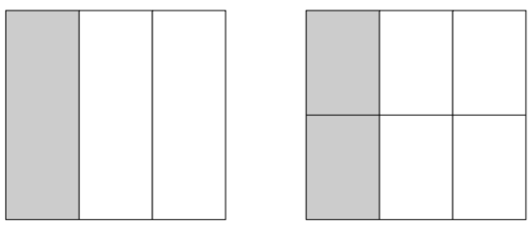


Illustration 51: Diagram showing equivalent fractions

However, fluency with multiplication could enable students to use the rectangular area model more efficiently. For instance, when asked to work out $\frac{1}{3}+\frac{1}{12}$ using diagrams, a student may notice that twelfths can be represented in a diagram representing thirds by adding some extra lines.

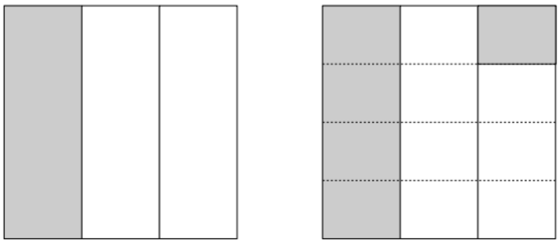


Illustration 52: Diagram to work out 1/3+1/12

The relationship that has to be noticed in order to use the diagrams as shown above is that 12 is a multiple of 3, and even further, that $12=3\times4$, so the diagram has to be further divided into 4 new cross-sections. This example suggests that knowledge of multiplication is actually a pre-requisite for a visual approach to teaching fractions. However, the rectangular area model is closely related to the array model for multiplication (Küchemann, Hodgen, & Konstantine, 2016). Therefore, it was expected that by using the rectangular area model for fractions, students could improve their knowledge of multiplication. This way, the perception that 'learning fractions depends on knowing multiplication', could be reversed to 'learning fractions may promote learning of multiplication'.

As discussed in Section 6.3, the participant students, as usually expected from low achieving students, lacked some prior knowledge regarding arithmetical facts, including multiplication. I observed students from the three groups struggling to recall the result of simple multiplications such as 3×3 and 6×4 and struggling to execute the procedure for grid multiplication. Therefore, when designing the lesson plans, I aimed at reducing the dependence on numerical skills, and at the same time, helped to promote them through the use of the rectangular area model. The first aim can be seen in the use of a limited range of small denominators and in the careful introduction of new denominators. For example, I rarely posed questions with denominators larger than 20 in the first and second packs of lessons, and new denominators were only introduced during the first and second packs together with other fractions from the same family. The second aim can be seen in the questions related to equivalent fractions intensively used in the first pack, when the rectangular area model was being established.

In the next two sections, I will analyse some episodes to try to shed some light on the relationship between fractions and multiplication throughout the approach utilized in the targeted lessons.

7.5.1 The emergence of multiplication

From the first lessons until the last, I was able to observe several students solving the questions, from the "work out" to the "word problems" posed in the targeted lessons, without using multiplication explicitly. This was possible by obtaining a relevant diagram and then counting the pieces that it had been divided into. An example that took place in the third pack of lessons is shown in the image below.

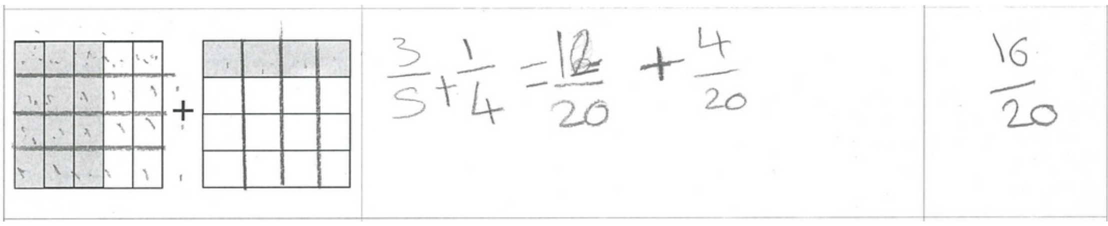


Illustration 53: Diagram with signs of counting

This question asked the students to work out the sum shown in the diagrams below.

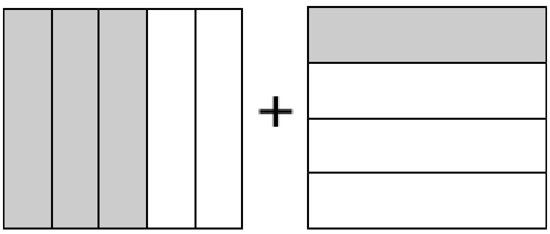


Illustration 54: original diagram given in the question

Although the student made a mistake when converting the second diagram to the same denominator as the first, the marks on the unshaded pieces suggest that she counted not only to determine the numerators of the fractions, but also the denominator, even though the total to be counted was not so small (20) and the answer could be obtained by calculating 4×5 . The answer below, from the same student in the same lesson, confirms her reliance on counting.

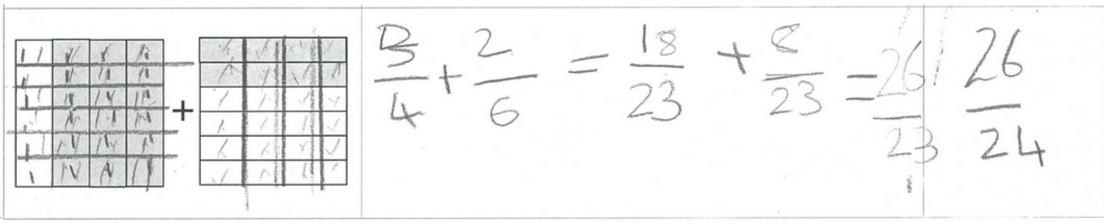


Illustration 55: Diagram with evidence of a counting error

It seems likely that her mistake in the first answer (26/23) is due to miscounting, which confirms that she was counting when solving the questions above, instead of multiplying. This behaviour is noticeable in all her worksheets for this lesson and some worksheets in other lessons, whenever she was working with ready made diagrams or diagrams drawn by her. The same was observed for several other students.

Similar behaviour could be observed for several students throughout the targeted lessons. I interpret that as being a sign of students dealing with fractions based on the visual representations and not on numerical properties, which allowed them to get into this field regardless of their difficulties with topics such as multiplication. This could only be achieved because the visual representations were not being used only as a form of representing mathematical objects, but also as conveyors of meaning, as grounding metaphors (Lakoff & Núñez, 2000). Following the arguments proposed by Giaquinto (2007), what is happening here is that students are acquiring new knowledge through visual thinking, instead of working with visual representations just as an accessory.

It could be argued that this reliance on counting is problematic, since it is very basic and may be ineffective or even inappropriate for more advanced topics. Therefore, it would be desirable to observe students progressing to more advanced strategies throughout the targeted lessons. It actually was possible to observe steps towards such progress, as will be shown in the next episodes.

The first step was the use of repeated addition. Later in the same lesson, where I observed the episodes referred to above, when she was working out a sum whose result had 24 as denominator, I asked her how she could find out the denominator without counting each bit in the diagrams. Immediately she started to count in blocks, using three fingers to point to each column and saying in a low voice: "3, 6, 9 [short pause] 12, 15 [short pause], 18 [short pause], 21 [short pause], 24". The fact that she used her fingers pointing to the diagram suggests that the visual arrangement of the elements in the paper was relevant to her realization that repeated addition could be used to answer my question. Therefore, I would argue that the visual representation enabled her, a low achieving student with weak prior knowledge, who was apparently relying on counting to solve the questions so far, to use repeated addition, which is a first step towards multiplicative reasoning.

A similar example happened during the starter of a lesson in the second pack, when students were asked to draw a diagram representing the unitary fractions that would be used in the lesson.

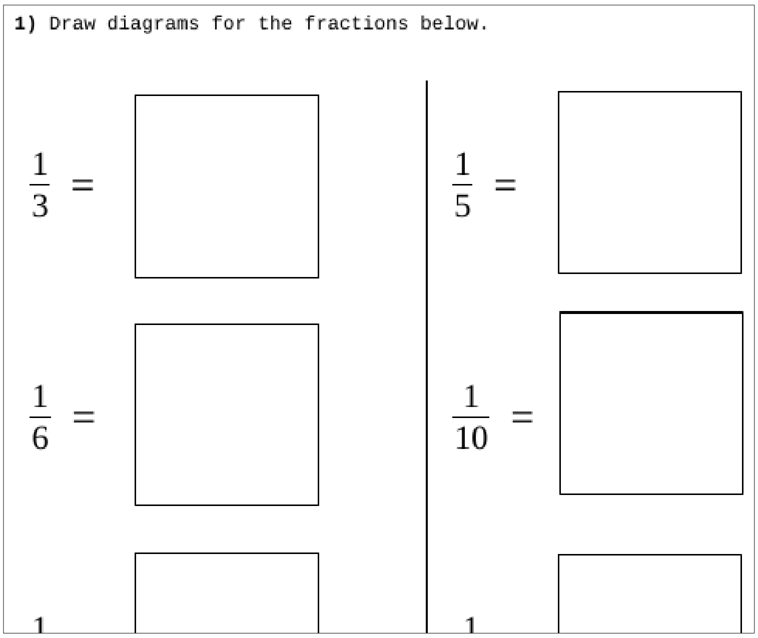


Illustration 56: Part of the task on unitary fractions

I was watching when M (Year 9 set 6) finished her diagram for 1/5, using five vertical strips, as expected, and was about to move to the next question (1/10). I asked her how she would draw it. After a brief moment thinking, the following conversation took place:

M: I would put five there and put a line across
Me: How do you know that?

> M: Because you already got five, then you put a line across the middle, 5 there and 5 there... 10

Another example took place even earlier in the year, during the first pack. When T (Year 9 set 6) was about to complete the equality $\frac{5}{4}=\frac{}{16}$, I intervened with the question "How could you know the answer if you know that there are four sixteenths in a quarter?" (the same conclusion had just been reached in the previous question), to which she answered: "Because, it is... 4... [silence while she was moving her hand rhythmically, like knocking in the table, a few times]... 20. It is twenty sixteenths". The pause and the rhythmic movement of her arm suggested that she was counting, however the length of the pause and the fact that she only moved her hand a few times suggested that she counted not one by one, but in blocks: 4, 8, 12, 16, 20.

The three episodes above show students using repeated addition when dealing with questions that did not ask or suggest explicitly the use of such a strategy. It is also possible to notice references to visual elements in their explanations, especially in the first one, when the student used her fingers to point to the columns in her diagram while counting, and in the second one, when M used words such as, "put a line across", "here", "there". These evidences support my argument that this is the result of the rectangular area model working also as a model for multiplication.

Beyond repeated addition, there were students who apparently noticed the multiplicative nature of equivalent fractions. Student J, for instance, was using a combination of counting and multiplying. I noticed that his answer to the sum 3/4+1/6 was wrong (10/12 instead of 11/12), so I asked him to explain how he had solved it, which he did as follows:

> I shaded three here and one here. I can use twelfths, because 12 is in both times tables, 4 and 6. So, I did two lines across [referring to the first diagram] and one down [referring to the second diagram]. And I've got ten twelfths.
> Me: I think you may have miscounted...
> J: Oh... It is [counting bits on the first diagram] eleven [pause] eleven twelfths.

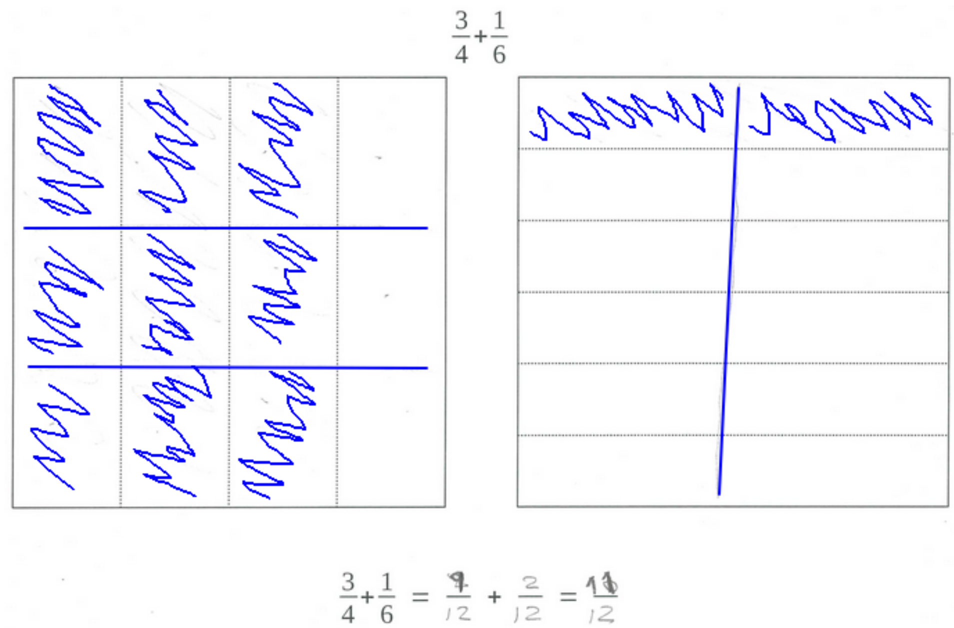


Illustration 57: Recreation of student's answer

Note that he used multiplication (times tables) to figure out the denominator of his answer. It seems reasonable to say that it was not a *post factum* explanation, because if he had used the diagrams, as he did in the previous lesson, it would have led him to 24 as denominator, so he probably relied on his knowledge of multiplication to choose 12 as a denominator for this sum.

Similarly, during the final lesson of the first pack, when students were asked to draw their own diagrams for the first time, based on some previously made grids (showing halves, thirds, quarters and fifths), part of the challenge was to decide which grid should be used for each fraction. For instance, to obtain a diagram representing 1/10, students could use the grid showing 1/5 and divide it further in half, as shown below.

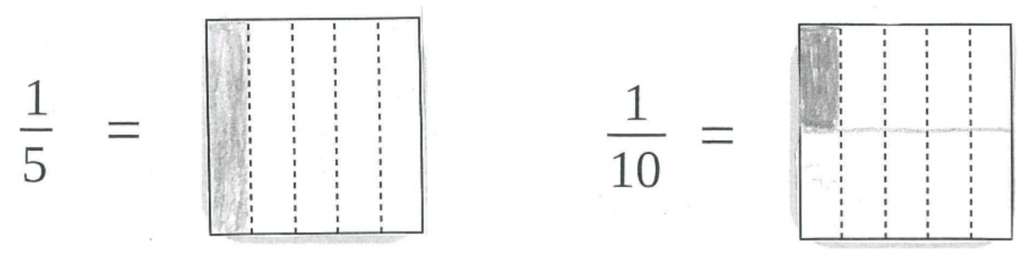


Illustration 58: Diagrams showing 1/10 obtained from 1/5

Although there was no explicit reference to multiplication in the task, it was actually designed to enable them to notice the relationship between how to obtain a certain diagram and the idea of multiples and factors. When I approached W (Year 8 set 4), he had completed the task. I asked him why he decided to use the grids divided into fifths to draw the diagrams for 1/10 and 1/15, to which he answered: "because 5 goes into 10 and 15."

Together, these episodes show that the approach utilized enabled at least some students to notice the multiplicative aspect of fractions, even though this was never explicitly covered during the lessons. In fact, an increasing number of students gave me numerical explanations to their answers as the lessons progressed. It is also easy to identify them in their worksheet solutions without a diagram, suggesting the use of a purely numerical approach.

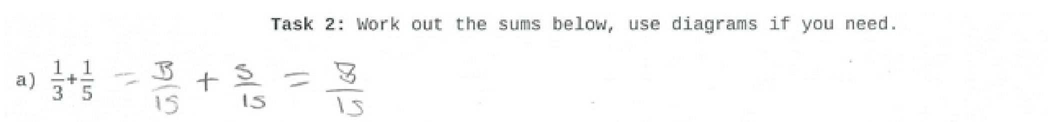


Illustration 59: Example of purely numerical answer

The episodes above suggest that the lesson plans were successful in enabling students to get into the field of fractions without much prior knowledge of multiplication, and at the same time create a fertile terrain to promote knowledge of this topic through visual representations.

However, it was also possible to identify episodes in which students were apparently limited by their lack of knowledge about multiplication.

7.5.2 The influence of prior knowledge on multiplication

During the four first lesson in the first pack, students had cut-outs available to solve all the questions posed. Actually, the questions started depending totally on the cut-outs (using their colours as identification, for instance) and progressively used more references to symbolic fractions. The students were expected to figure out which cut-out they should use, if any. From the fifth lesson onwards, students started to use diagrams.

During teaching of the second pack, there were no cut-outs available and students alternated between reading diagrams, completing partially drawn diagrams and drawing their own diagrams. At this stage, a new challenge arose: how to obtain a diagram to represent a fraction with a given denominator?

The episodes below illustrate the magnitude of this challenge for the students. It is important to point out that all three students involved in the episodes were engaging really well during the targeted lessons. They systematically finished the tasks included in the lesson plans and engaged with some extra questions during the final minutes of the lessons. When I noticed that this was happening frequently, I started to plan some extension questions to include in the "Comments for the teacher", so there was no need to improvise, and I could use them to generate more data.

The first episode happened with student G (Year 9 Set 6) during the second lesson of the second pack. This lesson was about adding halves, quarters, eighths and sixteenths. By the end of the lesson, she had completed all the tasks and I posed the sum $\frac{1}{3}+\frac{1}{6}$. It was the first time she was asked to draw a diagram involving thirds. Even though she was able to solve several questions involving halves, quarters, eighths and sixteenths, she struggled to create a diagram for 1/3. After thinking for a while, she managed to get such a diagram by dividing the whole by two vertical lines. For the next step, she needed to realize that sixths could be obtained by drawing a line across the diagram, but she was not able to realize that by herself, and was completely stuck until I asked her, "What would happen if you add a line across the middle?", while referring to her diagram showing thirds.

The second and third episodes took place in the final minutes of the third lesson of the second pack, after some work with fractions from the same family as 1/3 and 1/5. Firstly, I asked J how he would represent 1/24. My intention was to check if he had developed any strategy that was not only guessing or counting, since 24 was a relatively large number in comparison to the denominators used so far and was novel. After some minutes, I returned to him and he had the answer below.

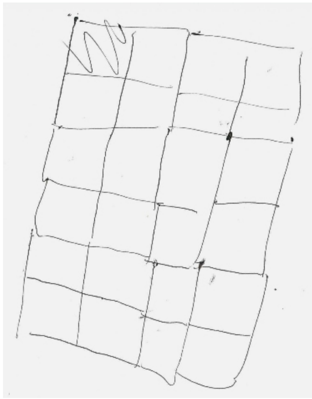


Illustration 60: Student's diagram for 1/24

When I asked him to explain how he had solved this he said, "I know 4 goes into 24, so I did like this (pointing consecutively to 2×2 groups as shown below) until I get 24". Note that the drawing suggests that he had actually drawn 2×2 groups where indicated, but the last two groups do not look like they were drawn in the same manner. If he had calculated $24\div 4=6$ he would have known that he needed exactly six groups, so I asked him how he knew that "4 goes into 24", to which he answered: "because it ends in 4", in an apparent reference to a wrong generalization of the rule to identify if a number is divisible by 2.

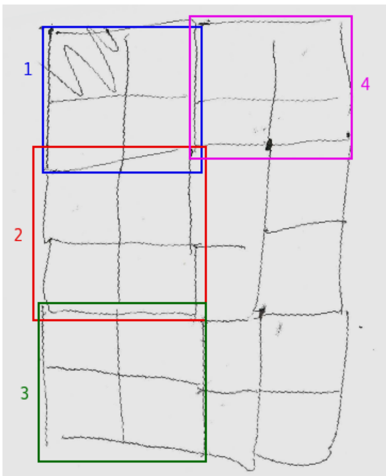


Illustration 61: 2x2 groups indicated by the student

The third episode took place when I asked C (Year 8 Set 6) to work out the sum $\frac{1}{7}+\frac{1}{14}$. I returned to him after some minutes and found him completely stuck on the question. When I approached him, he promptly said, "I don't know how to get sevenths," and his worksheet had several incomplete drawings of diagrams representing sixths, eights and ninths. When I helped him by saying "Why don't you go all like this?", suggesting several vertical strips, he was able to complete the rest of solution by himself.

Remember that these three episodes took place with students who were successful in the targeted lessons so far. In the first episode, even though $6 \div 3 = 2$ is supposedly a simple calculation, the student was not able to obtain sixths from thirds. In the second episode, the student was clearly aware of the multiplicative aspect of fractions, but was not able to use it fully to his advantage. In the third episode, the several incomplete diagrams on his worksheet suggest that he was looking for a multiplication that was equal to 7 and was unable to realize that the only option was 1×7 ; it could be argued that something else was involved here, such as a tendency to avoid diagrams with too many strips, but the realization that 7 is a prime number could have helped him to abandon this search and decide for the correct representation. For these reasons, my interpretation is that their difficulty in figuring out how to obtain the diagrams is a result of a poor knowledge of multiplication and division.

Based on these cases, it seems reasonable to conjecture that a more profound knowledge of multiplication would be helpful to students, and that the targeted lessons were not enough to develop such understanding, even though there was some improvement.

There were also cases in which the students seemed not to have noticed the multiplicative aspect of fractions at all. For instance, two students (Year 8 Set 4) were working on how to represent twelfths using some already drawn grids (half, thirds and fifths). When I approached them, they were struggling with the grid divided into fifths, so I said, "Why don't you try this one?" and moved a grid with thirds closer to them. After a few seconds contemplating the grid, one of them had an "A-ha moment" and promptly drew the exact number of extra lines to obtain twelfths. In this episode, it seems that she was able to calculate $12 \div 3 = 4$, but had not yet realized how this can be used to predict which grid was appropriate.

7.5.3 Discussion

At first, the two sets of episodes presented above could be interpreted as contradictory. However, each is highlighting different aspects of the targeted lessons in terms of students' prior knowledge on multiplication.

The first set of episodes shows that the approach utilized enabled students to get into the field of fractions even though some of them lacked prior knowledge of a topic that is usually considered a pre-requisite to learn fractions: multiplication. This seems to be a result of anchoring all the content taught during the targeted lessons in visual representations instead of in numerical knowledge. Furthermore, the lessons seemed to have created some situations that promoted multiplicative knowledge for some students.

The second set shows that despite the low initial threshold, some students were not able to appreciate the multiplicative aspect of fractions, or were not able to make use of such an aspect when solving particular questions. As presented before, this issue often emerged when students faced fractions with new denominators without a lesson introducing them through the use of other fractions from the same family. Considering that the ability to work with any fraction is desired, I would say that the targeted lessons did not fill this gap.

It was said at the beginning of this section that the similarity between the rectangular area model (for fractions) and the array model (for multiplication) suggests that by working with the former, students could acquire knowledge regarding multiplication. Actually, as illustrated by episodes showing students counting in blocks or performing other rudimentary multiplicative reasoning strategies, this effect happened, but not to the extent that was necessary for all students to make full use of the affordances provided by the visual representations. Bearing that in mind, my understanding is that students would benefit greatly from lessons specifically designed to explore the connection between the two models, i.e., between fractions and multiplication.

This design implication will be further discussed in Section 8.2 together with other issues that point towards design improvements for the lesson plans.

7.6 Whole number bias and other rote procedures

Whole number bias is the tendency of students to extend rules and properties of whole numbers to fractions. Two examples of whole number bias are to: compare fractions by judging the magnitude of numerators and denominators in isolation and add or subtract numerators and denominators when asked to add or subtract two fractions (Ni & Zhou, 2005).

These behaviours are widely reported in children, students, adults and even pre-service teachers (Siegler & Lortie-Forgues, 2015). One of the hypotheses to explain this behaviour is that:

> Much of the mathematics that is taught and learned in schools is focused on written mathematical symbols. However, a substantial body of literature has suggested that many students perform operations on symbolic representations with little understanding of the meaning underlying the representation. (Mack, 1995, p. 422)

The argument would be that since fractions are represented basically using whole numbers (in the form $\frac{a}{b}$) and teaching usually focuses on symbolic manipulation, students would be biased towards using the properties of whole numbers they already know and have been using successfully in mathematics lessons. This interpretation is reinforced by more recent studies showing that the difficulties with fractions seems to arise when symbolic notations are introduced (Siegler et al., 2013). Another study commented on by Vamvakoussi, (2015) investigated adults trained to associate random symbols to fractions, instead of their standard symbolic representation. The authors concluded that when comparing fractions represented by those random symbols, adults would automatically access their magnitude, but not when the fractions are presented with usual symbolic representation.

Although Vamvakoussi (2015) recognizes that more research is necessary to understand how and why this bias happens, he proposes that:

> Because human reasoning is often intuitive in nature, the bias will occasionally manifest itself, sometimes in an error, sometimes in a time-consuming attempt to inhibit the initial intuitive response, and sometimes

> as illusion of understanding, depending on the task at hand, the individual's familiarity with the task, and their level of engagement. (p. 52)

Intuitive or not, there is still the challenge of overcoming the whole number bias in the classroom. Mack (1995) highlights that a "number of researchers concur that students can develop a deep understanding of mathematical symbols by relating symbolic representations to referents that are meaningful to them" (p. 422), and pursues this recommendation by using informal knowledge about equal sharing as a meaningful referent candidate for third and fourth-grade students.

The similarities between her work (Mack, 1990, 1995, 2001) and mine were discussed in Barichello (2015). In summary, both of us are looking for some knowledge that can be used as basis to build understanding of fraction addition and subtraction and is available for students in general. Her choice is based on the perception that even though informal knowledge is ill-structured and usually very circumstantial, it is widespread and meaningful. My choice is based on the ideas discussed in Sections 2.4.3 and 4.1 that suggest that mental visual representations play a fundamental part in how humans apprehend mathematics, and that some visual skills are widely shared by humans. For Mack (2001), multiplication of fractions can be seen as equal sharing of fractional totals. For me, addition of fractions can be seen as juxtaposing sections of a whole represented by a rectangle. Drawing on Lakoff and Núñez (2000), we are both looking for some knowledge that can work as a metaphor for fractions addition and subtraction, so by using the metaphor we would be borrowing properties, transformations and inferences from the source domain (informal knowledge or visual representations) and using them in the target domain (fractions). For that reason, some of her conclusions could be relevant to my work. The dialogue below illustrates the dual conclusion Mack (1990) obtained.

I: When you add fractions, how do you add fractions?

Aaron: Well, you go across. You add the top numbers together and the bottom numbers together.

I: Now I want you to solve this problem (shows Aaron a piece of paper with 4 – 7/8 printed on it).

Aaron: (Writes 4 – 7/8 on his paper) Well, you change this (the 4) to 4/4.

I: Why 4/4?

Aaron: 'Cause you need a whole, so you have to have a fraction and that's that fraction, and then you have to reduce, or whatever that's called, that (the 4) times two, so you'll have 8/8. Eight eighths minus seven, so it's 1/8.

I: Now suppose I told you you have four cookies and you eat 7/8 of one cookie, how many cookies do you have left?

Aaron: You don't have any cookies left. You have an eighth of a cookie left.

I: If you have four cookies...

Aaron: (interrupting) Oh! Four cookies!

I: ...and you eat 7/8 of one cookie, how many cookies do you have left?

Aaron: Seven-eighths of one cookie? Three and one eighth.

I: Now how come you got 3 1/8 here (referring to what Aaron had just said) and you got 1/8 there (referring to paper)?

Aaron: (Pauses, looking over problem) I don't know. (Contemplates problem; repeats problem.) Well, because on this you're talking about four cookies, and on this you're talking about one.

Illustration 62: Dialogue from Mack (1990, p. 23)

Note the use of expressions, such as, "You have to have a fraction," and "You have to reduce," as a sign that he is trying to recall and follow some sort of procedure, even though he did not need that to solve the contextualized (and supposedly meaningful) question presented right after that. After discussing several interactions like this one, Mack (1990) concludes that informal knowledge about equal sharing can be a solid basis for learning fraction arithmetic, however, what she calls "rote procedures" keep getting in the way. The use of the expression "rote procedure" suggests procedures that were learned in the past without being connected to a meaningful referent. In fact, she finishes her paper presenting "arguments in favour of teaching concepts prior to procedures" (p. 30).

My argument is that fraction instruction suffers from more than the interference of procedures learned previously, but also from phenomena, such as the whole number bias, which seems not to be created by previous teaching, but results from how fractions are symbolically represented and how our brains work.

Regarding my research, my expectation was that the approach aiming at building all the concepts, properties and operations with fractions in visual representations, would allow students to overcome whole number bias or interference of rote procedures. This is because the standard symbolic representation was only used as a way to represent the fraction, while the meaning was anchored in the visual representations, which as discussed before, were expected to act as grounding metaphors.

Before the targeted lessons started, I applied a diagnostic test to assess how much students knew about fractions (see Appendix 10.1). The results were discussed in Section 6.3 (see Table 8 on page 71), but I would like to discuss further the results from the questions involving fraction addition.

Julia's group (Year 8 Set 4 out of 5) had participated in the preliminary study (see Chapter 3), during which they had three lessons on addition of fractions that were very similar to the three first lessons of the second pack. Even though they performed better than the other two groups on questions 5A and 5B, only 28% got a right answer for question 5B (Work out $\frac{1}{10}+\frac{3}{5}$) and 0% for question 5C (Work out $\frac{1}{3}+\frac{1}{4}$).

However, when you compare the solutions from this group with Alice's group, (Year 8 Set 5 out of 5) something else becomes salient. While all of Alice's students who did not leave the questions blank (10 out of 13) on 5C, used some sort of variation of the whole number bias for addition of fractions (added numerators and denominators obtaining a new fraction as answer or added numerators and denominators altogether obtaining a whole number as answer), only 4 (out of 17) of Julia's students used a strategy that could reasonably be associated with the whole number bias. Even though the data available does not allow me to reach a firm conclusion regarding causes, it seems reasonable to conjecture that this difference was a result of the lessons during the preliminary study.

In order to confirm my impression, I checked my field notes for annotations regarding any answer that resembled the whole number bias, as I was aware of the issue during the data collection, because it is strongly documented in the literature. However, I could only identify two occurrences for all three groups through the whole academic year. One was with a student from another group who was moved to David's group temporarily, and another with a student who missed the first three lessons of the second pack and made that mistake in the fourth lesson. This reinforces my argument that the targeted lessons enabled students to overcome whole number bias for addition and subtraction of fractions.

Nonetheless, another "rote procedure" interfered with my lessons: doubling. This became evident in lesson 1.4 during the second task, shown below.

Task 2

Use the pieces and the information in the table to find new fractions that are the same as the fractions below.

$$\frac{1}{2} = \frac{3}{6} = \frac{\quad}{\quad} =$$

Illustration 63: First item of Task 2, lesson 1.4

The table referred to in the task contained information about the colours of the cut-outs representing a half, thirds, quarters, fifths, sixths, eights, tenths, twelfths and fifteenths. Beyond the item above, the task was composed of another seven items, such as $\frac{1}{3}=$ and $\frac{2}{3}=$.

What I observed with several students when solving the item above is that after identifying $\frac{2}{4}$ as another equivalent fraction, they carried on doubling the numerator and denominator, in order to generate more equivalent fractions, even though they did not have cut-outs for sixteenths this lesson. This behaviour may have been caused by an insistence from me and from the teacher that they should find all fractions equivalent to the given one, since this would be important for the next task. Note that the strategy is correct as a way to generate equivalent fractions, but because it is not compatible with what the task asked them to do, I insisted that they should use the cut-outs and the information on the table to find the answers. However, in general, once students used the strategy they were very reluctant to abandon it, probably because it resonates with procedures previously learned by rote.

Apart from this particular situation, it was possible to identify several other examples of students "doubling" the numerator and denominator of a fraction throughout the targeted lessons. Although this is only based on my experience as a teacher in Brazil and as a researcher in England, I would argue that the persistence of "doubling" is due to a comparatively strong emphasis on "doubling", "halving" and "count in twos" in the English national curriculum for mathematics in primary school.

Another topic covered during my lessons where whole number bias could be expected, is comparison. Regarding Alice's group, the one in which most students presented whole number bias in the diagnostic test, comparison questions appeared in two lessons during the second pack, which means they had already had five of the targeted lessons when such questions appeared for the first time. On this occasion, 11 students showed some working out in their worksheets. After checking all the worksheets, I could identify only 2 students who made a mistake that seems related to whole number bias, such as in the answer below.

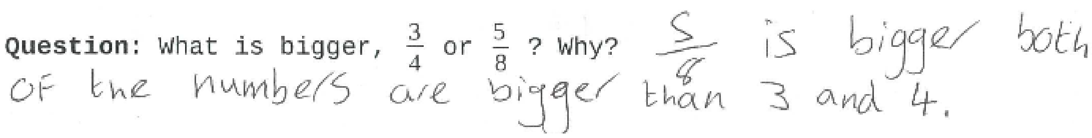


Illustration 64: An answer with explicit reference to whole number bias

Among the right answers, the example below is representative of the most common strategy.

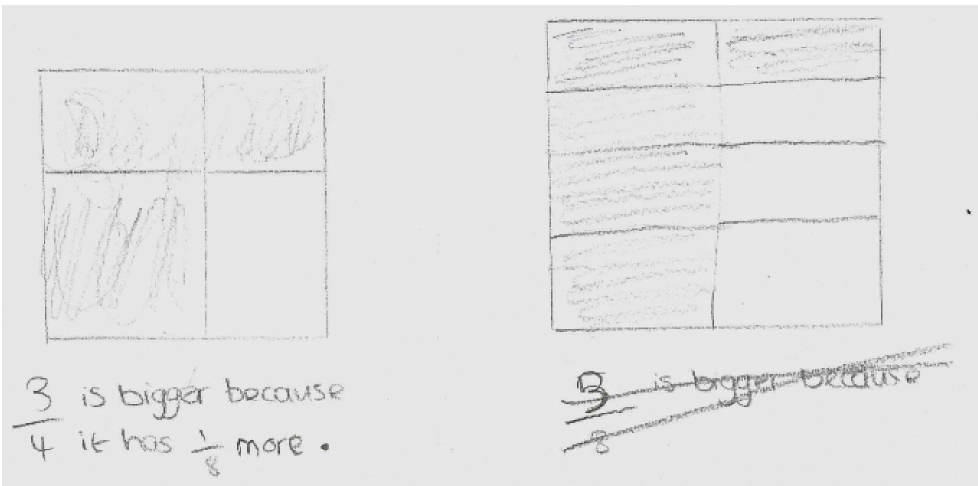


Illustration 65: An example of right answer for a comparison question

On the next occasion where comparison questions were posed (3 lessons later), only one student made a mistake that resulted in answers compatible with the whole number bias, but as she only wrote a fraction as an answer, it is not reasonable to infer about what may have influenced her.

My argument is that after 5 lessons on fractions using visual representations, students from a low achieving group that presented, almost unanimously, whole number bias when adding fractions, were able to overcome this bias on a related topic (comparison) without being explicitly taught about it. This, together with the other evidence presented in this section, suggest that the approach utilized in the targeted lessons was very effective in overcoming the whole number bias for addition, subtraction and comparison of fractions. However, other procedures, such as "doubling", arguably interfered as an easy fall-back option.

7.6.1 Discussion

Mack (1995) suggests that this negative interference of "rote procedure" is the result of an over emphasis on procedures before conceptual understanding, and my data seems to support her conclusion. However, my argument is that the issue is more complex than the dichotomy of procedural versus conceptual knowledge. Although I do not have data to support this view directly, I interpret this behaviour of "falling back" to simple procedures as the result of a long culture of learning to solve tasks by simply applying procedures without meaningful referents for the choice of procedure, and for the procedure itself.

An illustrative example was given by student J (Year 9 Set 6) when solving the word question below. The question was presented in the final (sixth) lesson of the second pack after a sequence of lessons during which this particular student presented an exemplar level of engagement and success.

Task 3

Instead of giving Marc's full allowance at the beginning of the month, Marc's father said that he will receive two fifths in the first week, three tenths in the second week and the rest in the third week. What fraction of his allowance will Marc receive in the third week?

Illustration 66: Task 3 from the final lesson of the second pack

When I observed that he had answered $\frac{4}{15}$, a totally unexpected answer from my perspective, I asked him why and he justified it by pointing out a regularity in the numerators and denominators given in the task: "add one to the numerator and five to the denominator". This was a huge surprise for me due to his success in the last lessons and the fact that he had solved correctly the previous questions.

My argument is that for this student, the association of this strategy (looking for additive patterns, which is very common when working with number sequences), to some type of question is meaningless, and at the moment he spotted an easy pattern in the numbers printed in the task, the association was triggered. Also, he sees no problem in acting like that, since most of the associations he does during mathematics lessons may be of the same meaningless nature.

The conclusion I would draw from the data discussed in the previous section, is that even though some episodes similar to the one just described actually happened, the approach utilized was effective in reducing the influence of known "rote procedures", such as the whole number bias.

8 CONCLUSIONS

In the previous chapter I presented and discussed five themes that emerged from my data and are related to my research question: **what is the effect of a set of lessons based on a carefully chosen visual representation on low achieving students' reasoning about fraction?**

In this chapter I will discuss those themes presented in the previous chapter by grouping them into two main conclusions. The first, discussed in Section 8.1, refers to the qualities of the learning based on visual representations during the targeted lessons. This conclusion responds more clearly to the research question posed above. The second, discussed in Section 8.2, refers to aspects that seemed to have limited students' engagement during the targeted lessons and that could be interpreted as opportunities to improve the design of the lesson plans.

However, before these sections, I want to summarize the trajectory of this study in order to make sense of what was done and why.

At the beginning of this thesis I introduced my general interest in investigating the learning processes of low achieving students. At that point it was not clear to me in what ways these processes would differ from those of other students, but there was a feeling that low achieving students are not achieving less than their peers simply because they are lazy, are having problems at home, are going through challenging times in their personal lives, or are intellectually limited. Even though I admit those issues can interfere in the achievement of some students, the literature shows that achievement is not homogeneously distributed in a population and is correlated to variables, such as family income and parents' education (West & Pennell, 2003), and these variables could be a way to better understand why they are low achieving.

In the case of the United Kingdom, where students are grouped in sets supposedly according to their prior achievement in mathematics, there is a strong correlation between social class and the set in which students are placed (Dunne et al., 2007; Gates & Noyes, 2014).

Inspired by the literature in mathematics education (Mayer, 1997; Gates, 2015, 2018; Lowrie & Jorgensen, 2018), visualization emerged as a possible tool to overcome the disadvantages that may be forcing this failure onto students from underprivileged social classes. An intense use of visual representations could: a) reduce the reliance on language, which is a factor strongly connected to social class (Zevenbergen, 2001), b) develop visual skills, which are correlated to achievement in mathematics and seem to develop via early experiences, such as playing with blocks and puzzles, which is much more common among children from privileged social classes (Verdine et al., 2014), and finally, c) reduce the dependence on prior mathematical knowledge (Barichello, 2017).

The experience during the preliminary study was important to confirm such expectations (Barichello, 2015), and establish a plan to implement a sequence of lessons on fraction addition and subtraction through visual representations. Also, this preliminary stage of my research was key to becoming acquainted with the British educational system at different levels: school, teachers, classroom and students. With these experiences in mind, I designed an intervention composed of 12 lesson plans, based on three main design principles: 1) keep coherence with current practices in the school, 2) use visual representations as the basis on which to build all the knowledge intended and 3) create opportunities for students to engage with tasks without being told how to solve them beforehand.

As the main data collection method, I used what I termed within-class clinical interviews, which are essentially task-based clinical interviews (Carolyn et al., 2014) taking place during a regular lesson, instead of in a special environment designed for research purposes. Worksheets solved by the students and field notes were also very important.

Then, during the next academic year, three teachers in the same school enacted the lessons, each with one low-achieving group from Years 8 or 9. In total, about 40 students were involved in my data collection.

All twelve lesson plans were used, some of them for more than one lesson and a few extra lessons were planned on demand. In total, 45 lessons were taught based on the lesson plans. Based on teachers' perceptions and researcher's impressions from the previous lessons, each lesson plan was tweaked from one lesson to the next, without compromising the main learning goal and the design principles. The twelve lesson plans were grouped in three packs in a way that the lessons in each pack were enacted consecutively and each pack was enacted in a different term.

Finally, the analysis of the episodes of within-class clinical interviews, together with my field notes and students' worksheets revealed five themes. Considered together, these themes showed that, even though the effect of the lesson plans were limited for some design choices, mathematical reasoning emerged during the targeted lessons. The data analysis showed that this reasoning was not only anchored on but seemed to have been enabled by the visual representations. Moreover, students were able to extend their knowledge beyond what was explicitly taught to them, solving questions and reasoning about situations that demanded generative thinking from them. Once again, the data analysis suggested that this behaviour was also enabled by the visual representations.

This observation was especially striking considering the participants of my research: students from three low-set groups in an under-performing school. These students presented serious limitations in their prior mathematical knowledge and were not used to reasoning in their lessons, as I could observe throughout the research process. However, the use of visual representations as basis to build their knowledge constituted a pathway for them to have access to fractions.

These issues will be further detailed and discussed in the next sections.

8.1 Reasoning anchored in visual representations

Section 7.3 of the data analysis chapter shows that students' reasoning was fundamentally anchored in the elements, properties and actions of the visual representations, when they were talking about the tasks posed during the targeted lessons. Moreover, Section 7.4 showed that their reasoning was generative, meaning that students were able to extend their knowledge to solve questions that were beyond what was explicitly taught to them.

The third characteristic of the learning elicited by the targeted lessons that I will take into account in this section, is the low threshold, in terms of previous mathematical knowledge created by the visual representations, and discussed in Section 7.5.1.

My data analysis suggests that these three characteristics of learning fractions through the lesson plans I designed are the result of two components of my research related to the design principles. Each one of them will be discussed separately in the next sections.

8.1.1 The rectangular area model as a grounding metaphor for fractions

The first component is the model chosen to serve as a basis on which to build the intended knowledge, both because of its general properties for being a visual model, and because of its specific properties as a rectangular area model. The analysis presented on chapter 7 suggests that together these properties enable the model to function as a grounding metaphor in the sense proposed by Lakoff & Núñez (2000). or, as suggested by Johansen (2014), as a material anchor for fractions into area of surfaces. Lakoff & Núñez (2000) proposed that metaphors are important because they allow us to transfer not only the direct meaning implied by a metaphor (in this case, the direct meaning would be how to represent a fraction given symbolically using the model and vice-versa), but also all the inferential structure from the source domain into the target domain (in this case, this would be the perception that by putting together two one quarters the result is equal to one half). Through this transfer mechanism, humans are capable of reasoning in new contexts based on arguments that we know from other more familiar contexts. Still, according to Lakoff & Núñez (2000), a grounding metaphor is a metaphor that allows us to "ground our understanding [...] in our prior understanding of extremely commonplace physical activities" (p. 53).

As discussed on Section 5.3, my argument is that the properties of the model that were used during my lessons (composition and decomposition of shapes by juxtaposing and cutting, and comparison of areas by overlapping) are commonplace physical activities, and when combined with some definitions, they were sufficient to explore the intended topics in the target domain. Therefore, and my data analysis supports this claim, the rectangular area model enabled this grounding metaphor for fractions.

This explains why mathematical reasoning emerged during the targeted lessons to an extent not observed during regular lessons: students were able to transfer the inferential structure from the cut-outs (and later the diagrams) to the domain of fractions. This observation is coherent with Johansen's (2014) view of certain visual representations as capable of promoting conceptual mappings.

Moreover, as argued on Section 7.4, it also explains the generativity presented by the students. As Rodd (2000) speculated when discussing possible connection between reasoning and visualization, since the warrants and backings of the arguments constructed during the targeted lessons were anchored in a grounding metaphor via visual representations, or ultimately in a commonplace physical activity, students were able to actually interact with the building blocks of the arguments being used and eventually re-use them to solve relatively new questions.

This behaviour was not common during regular lessons, because since the students lacked prior mathematical knowledge, the only option left to which the teachers could anchor their arguments, was authority (manifested through the mere presentation of information or through algorithms).

Based on that, I propose that the rectangular area model, manifested as it was in the targeted lessons, worked as a privileged representation for fractions. According to Rau and Matthews (2017), there is evidence that for some topics there could be representations that "convey meaning more intuitively" (p. 540) and could, therefore, be used as a way to solve the representation dilemma often highlighted by advocates of the use of multiple representations.

8.1.2 Opportunities for thinking

The discussion presented in the previous section explains the emergence of reasoning and its generativity based fundamentally on the first design principle, discussed in Section 5.1, that says that the **lessons should enable students to build their knowledge about fractions on visual representations**.

At this point, it is relevant to remember the other two principles. The second says that **students should have opportunities to solve the tasks without being told how to do it beforehand**, and the third was to **keep the lesson plans coherent with current practices in the school**. The latter was not expected to have impact on students' learning, but to facilitate the use of lesson plans by the teachers, increasing the fidelity of implementation, and to avoid drastic changes for the students.

However, the second principle could also be seen as responsible for the emergence of reasoning, as discussed in the previous section. It could be argued that all that was observed during the lessons was not the result of the use of visual representations, but of the fact that the lesson plans created situations in which the students had opportunities to solve tasks without having been told how to do so beforehand.

I recognize that the design of my research did not allow me to separate the effect of both principles. Actually, I expected them to act together.

The first principle without the second could result in an approach where the data of students' arguments would refer to visual representations, but any warrant or backing would be anchored in authority. In that scenario, I would not say that the model worked as a grounded metaphor, since the inferential structure (warrants and backings and how they can be combined) would not have come from the visual representations. Consequently, I would not expect much generativity in students' reasoning.

The second principle without the first would arguably result in difficulties regarding whatever prior knowledge would be used as the anchor for students' reasoning. The consequence, I believe, would be the well-known feeling by most teachers that her/his students do not know the pre-requisites for the topic of the lesson. This feeling is arguably the reason behind the reduced curriculum commonly adopted with low sets in the UK (Boaler & Wiliam, 2001).

In conclusion, the first and second design principles acting together were fundamental for the emergence of reasoning, its generativity and the low threshold in terms of prior mathematical knowledge observed during the targeted lessons. This conclusion resonates with some of the principles defended by the Realistic Mathematics Education movement and their way of using models (Van Den Heuvel-Panhuizen, 2003). For them, the active engagement of students with models is key to promote the movement from "models of", which are models tied to specific situations, to "models for", which are more abstract models that can be used to approach new problems and situations. My analysis showed that the students participating on my study used the rectangular area model as a "model for" fractions in a very broad sense, even expanding towards new questions and situations.

8.1.3 One final episode

Before moving on to the second major conclusion of my thesis, I will discuss a final episode collected during the targeted lessons. This episode took place during the second lesson of the third pack, which was also the second to last lesson. After the lesson, when checking the worksheets, I identified the three answers below given by the same student.

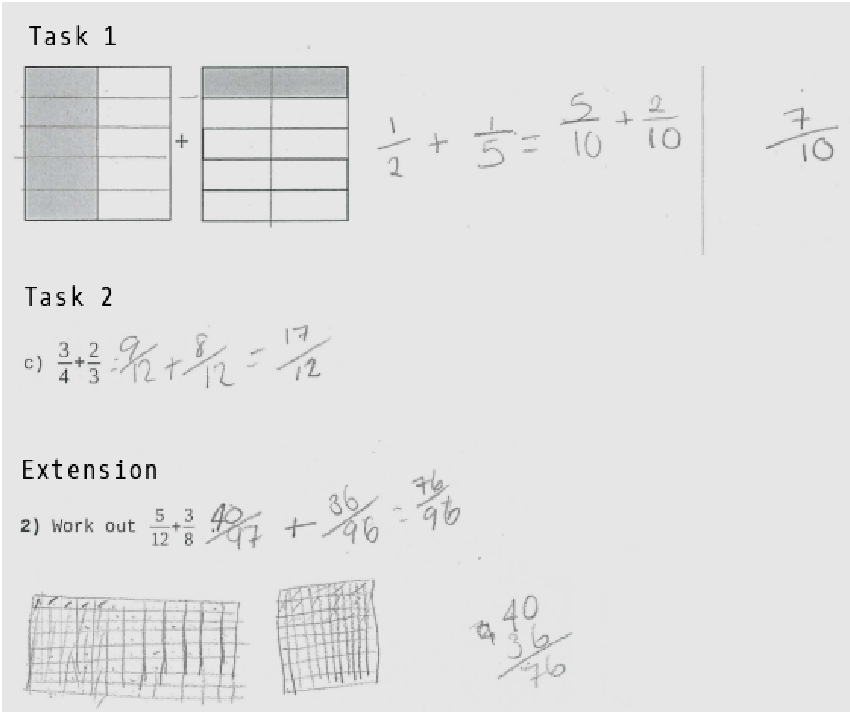


Illustration 67: Three answers by the same student

In the first question, the sums were given as diagrams. The students were expected to use them to obtain the final answer. My expectation was that they would add extra lines to each (mimicking the other) in order to get the same denominator for both fractions and then obtain the final answer; apparently, that is how the student solved the question (by adding the horizontal lines onto the first diagram and the vertical onto the second). In the second question, the sums were given symbolically and students had enough space to draw diagrams if they wanted to. In this case, there was no sign that this student had drawn diagrams to solve the question. Finally, the third question also posed a sum symbolically, but the student voluntarily drew a diagram to solve it; the counting marks in both diagrams and the mistake in the denominator of the first fraction (97 instead of 96) suggest that she actually relied on the diagrams to obtain the final answer.

I consider this episode quite revealing in terms of my research question because, even though I am presenting only one case, it is possible to concoct other similar episodes with other students and questions from other lessons. For this reason, I consider this a paradigmatic episode. It is important in the sense that it shows a student who was able to use the visual representation (first question), and was also able to use a purely symbolic approach when convenient (second question), even though such an approach was never emphasized in any lesson. Moreover, in the third question, which was intentionally designed with big denominators, a reasonable interpretation is that the student was not able to perform the operations using the symbolic representation, probably due to limitations in her knowledge on multiplication, as she had in the second. However, she felt comfortable enough with the visual representation to use it, even though it is arguable that the method is not so efficient due to the number and size of the subdivisions in the diagram.

This analysis is coherent with observations that a model should provide "students with opportunities for progress, without blocking the way back to the sources in which the understanding is grounded" (Van Den Heuvel-Panhuizen, 2003, p. 30) since the student seems to be able to move beyond, towards a numerical approach, and back to the model according to her needs.

To fully understand the significance of this episode as a synthesis of the learning promoted by the targeted lessons, it is important to contrast it with the episode reported by Ainsworth (2016) shown below. The student was explaining to the teacher what she did regarding the sum $\frac{1}{2}+\frac{2}{5}$.

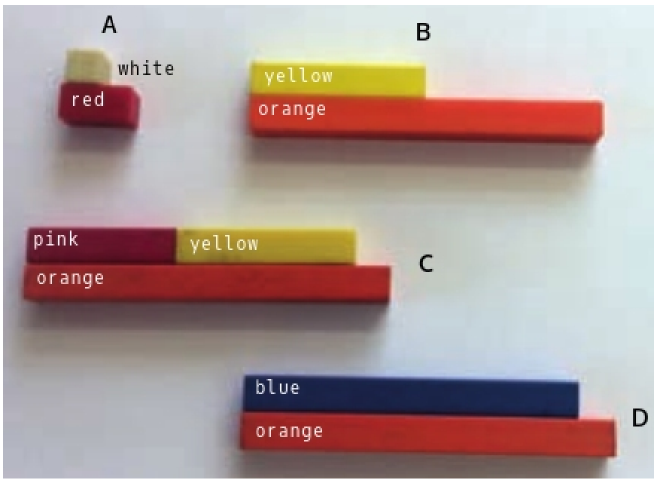


Illustration 68: Image adapted from C. Ainsworth (2016), p. 15

"This is half [picking up white and red - **A**] and so is this [yellow and orange - **B**]. This is one-fifth, [she taps red along the orange – **A then B** - with no attempt at accuracy as we both know its true], so this must be two-fifths [pink]." She places the pink and yellow end to end [**C**]. "So it makes nine-tenths altogether." (C. Ainsworth, 2016, p. 15- letters added)

Note how the student demonstrates fluency not only when explaining the process, but also by presenting more representations and relations that would be necessary to solve the question (the relation between white and red in A and the arrangement in D). However, from the information available, it seems that the choice of tenths for the final answer, which is a critical step when adding fractions, does not seem to follow from the diagrams. Quite the opposite, it seems reasonable to argue that she knew the final answer and represented (very competently) all the elements involved in it (9/10 and 1/10) as a *post-hoc* illustration to her solution. A similar behaviour was reported by Herman et al. (2004) in a study based on a large sample of secondary students that showed that they tend to use diagrams as static objects in *post-hoc* justifications of solutions obtained via symbolic methods.

As I discussed before, the scenario for the third solution on Illustration 67 is sharply different. When needed, the student used the diagrams to obtain the answer and relied on its properties and transformations to carry out the process.

In conclusion, I would say that this was the main characteristic of the learning that resulted from the approach adopted during the targeted lessons: students' reasoning was based on elements, properties and transformations of the visual representations, and this was the result of the first and second design principles. Therefore, my work contributes to the challenge of finding ways for "spatializing the curriculum", in the words of Newcombe (2016).

8.2 Limitations of the lesson plans

Sections 7.2 and 7.5 discussed issues that can be seen as limitations of the lesson plans in terms of students' learning. In the former, I showed how challenging it was for some students to mobilize certain visual skills to solve some of the very first tasks with cut-outs. In the latter, I showed how the lack of knowledge on multiplication apparently limited how students engaged with some tasks, especially towards the end of the third pack.

In this section I will discuss how I think these two issues are not limitations of the approach itself, but should be seen as opportunities to improve the lesson plans.

8.2.1 The challenge of rotating shapes

This issue emerged in the first lessons of the first pack, when students were exploring cut-outs, and refers specifically to rotation. Students struggled to figure out how to obtain certain arrangements when it was necessary to rotate the cut-outs. The struggle was more evident when triangles were involved, since they sometimes demanded rotation not by multiples of 90 degrees. The struggle was so salient that I decided to remove the triangles before David's third lesson (after observing all the lessons by Alice and by Julia). The change was perceived to be successful by me and the teachers, as students advanced more quickly through the worksheets without the triangles.

Later on, when the focus of the lesson plans was not so much on the rectangular area model as it was on fractions, and when diagrams had replaced the cut-outs, I observed students using visual skills to reason about tasks related to fraction equivalence and comparison.

I consider this example very relevant because it illustrates an arguably simpler connection between visual skills and mathematical achievement when compared, for instance, to the result reported by Cheng and Mix (2012). The authors reported an improvement in solving missing number problems by young students who participated in mental rotation training. Because the missing number questions were presented in a purely symbolic way ($4+[\ ]=11$) and do not suggest explicitly any relation with rotation, it is not simple to understand the causal mechanisms behind the result. Cheng and Mix (2012) propose two explanations: a) the mental rotation training improved students' general working memory and this facilitated the strategy employed by them to solve the task, and b) the training helped students to mentally move the number to the other side of the equal sign, obtaining $[\ ]=11-4$, which is one step closer to the final solution. Nevertheless, the authors recognize that their proposals are tentative.

Uribe et al. (2017) revisit Cheng and Mix's discussion and propose three explanations for their results. The first is a detailed discussion of their first proposals regarding a general improvement in working memory. The second relies on the idea that even though the missing number questions were essentially symbolic, the symbols are still spatially represented on the paper, therefore, an improvement in visual skills could lead to a better understanding of the symbols involved. The third is related to the second proposal by the original authors, according to which students could be manipulating pieces of the numeric expression as it is done in parts of a visual representation by decomposing, moving them around and composing something new. Note that the three explanations proposed are not specific for the missing number questions and could be extended to other mathematical topics. However, the underlying mechanisms supporting them are not fully understood yet.

This discussion illustrates the complexity of the causal mechanisms that act behind the relationship between visual skills and mathematics learning, which is a discussion away from a clear conclusion (Mix & Cheng, 2012). However, my research provides an example where this seems to be more direct. By the very nature of some curricular topics, mathematics lessons are full of visual representations, even when teachers do not intend to emphasize them: polygons, circles, grids, tables, number lines, dots, planes, axis, angles, etc. In addition, as it happened in the first lessons of the first pack, sometimes these objects have to be imagined in a different position or orientation in order to enable a particular approach, and teachers may inadvertently use transformations that rely on visual skills that are not readily available to the learners. Even in my research, which clearly revolved around visual representations, the episodes reported in Section 7.2 showed that to some extent, I took for granted that students would be able to deal with rotation of the cut-outs.

This discussion highlights a mechanism that could explain the well-established correlation between visual skill and achievement in mathematics: when visual representations are at play, due to their nature, it is inevitable that some transformations will be necessary in order to solve tasks, even those that may not be explicitly focusing on visual elements, and since the curriculum is permeated with topics that intrinsically rely on visual representations of some form, it is difficult, if not impossible, to avoid such transformations. However, they may be challenging for some students, especially those who are already considered to be low achieving (Gates, 2015).

The consequence of this discussion is that visual skills should be explicitly incorporated into lessons, especially if a teacher wants to explore topics and approaches relying on visual representations. However, at a pragmatical level, there may be no opportunities to do so while exploring a regular topic in the curriculum, such as fractions. In this scenario, since my results suggest that visual skills may interfere negatively, it may be necessary to choose paths that minimize this interference, as I did when removing the triangles from the third lesson for David's group. This situation opens up an opportunity for improvement in the lesson plans in case the teacher does not want, or simply cannot, focus on visual skills, as will be presented after the following section.

8.2.2 Multiplication and the rectangular area model

As discussed in Section 7.5, this aspect emerged as a two-fold issue. On one hand, I argued that the visual representations were able to lower the threshold in terms of previous mathematical knowledge during the targeted lessons. On the other hand, not all the students were able to contemplate the multiplicative aspect of fractions and this limited their engagement with some tasks, especially towards the final lessons.

It could be argued that this actually shows a limitation of the approach: the model and the lessons were not able to build the knowledge on multiplication necessary to fully explore the concepts and properties of fractions intended during the targeted lessons. However, it has to be taken into account that the lesson plans did not have multiplication as an explicit goal. Since some students were able to contemplate the multiplicative aspect of fractions quite successfully, even though it was not central to any lesson, it seems that there is potential to promote such knowledge within a sequence of lessons.

The fact that the lessons were able to lower the threshold regarding multiplication also highlights the potential for further explorations on this topic during the lessons. However, this should be made explicit and not only treated as a side effect as was the case in my original lessons plans.

This observation also has implications for the design of the lessons and they will be discussed in the next section.

8.2.3 Improvements in the lesson plans

The two sections previously pointed out aspects that could be improved in the lessons plans.

Firstly, the challenge resulting from the use of triangles in the first lessons suggested that it became a distractor from the main goal of those lessons. This effect was so evident during Alice's and Julia's lessons that I decided to remove the triangles before David started his lessons on 1/3, 1/6, 1/9 and 1/12 and this was perceived as a good decision, as students were able to move faster throughout the tasks. Later, after the end of the targeted lessons, when the teachers asked me to use the lesson plans with other groups[14], the triangles were also not included.

Based on that and on the considerations presented before, my recommendation would be not to use triangles in the lessons if the teacher wants to focus on fractions and does not want to spend more time on developing mental rotation. This could be achieved by simply replacing the triangular cut-outs in the first lessons by rectangles. As a side effect, the important message that the shape is not relevant (only the area)

14 These lessons were not considered as part of my data collection because there was not enough time to plan it properly.

could get weaker. In order to overcome this, I would recommend the inclusion of triangles in starters, to be discussed in the beginning of some lessons as examples of representations that can be done but may not be convenient.

It is important to highlight that I am not advocating against approaches that rely on mental rotation or other visual skills. On the contrary, throughout this thesis I have presented several arguments in favour of a greater emphasis on visual aspects in the teaching of mathematics. However, my data suggest that such a thing should be taken seriously, not as a secondary goal, but as a central topic in itself. As shown in literature, these skills are not only trainable, but also fairly transferable (Wright, Thompson, Ganis, Newcombe, & Kosslyn, 2008). If a teacher wants to keep the triangles in the lessons as a way of enhancing students' visual skills, I would recommend the inclusion of extra lessons explicitly focused on these skills before introducing fractions.

Secondly, the fact that the multiplicative aspect of fractions did not emerge for some students suggests that the connection between the rectangular area model and multiplication of whole numbers should be explicitly explored. This could be achieved by changing the focus of the final lesson of the first pack, when diagrams are introduced and students have already expanded the range of denominators beyond the families of 1/2 and 1/3.

The original lesson emphasized the creation of diagrams by the students based on the cut-outs and grids used in the previous lesson. The outcome of this lesson was a leaflet with the diagrams of several unitary fractions that could be used as reference at the beginning of the next pack. However, students seldom referred back to it. Therefore, I believe its substitution for a lesson that also introduced diagrams but did not generate the leaflet would not have greater impact in the overall trajectory.

The lesson could be centred around questions such as: "In how many ways can you represent 1/12?" This question would allow students to explore how to draw diagrams, as well as the original lesson plan, and could create more explicit situations for them to contemplate and discuss the multiplicative aspect.

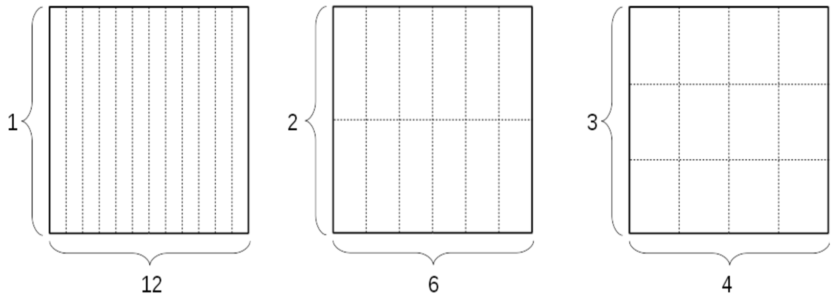


Illustration 69: Three possibilities to represent twelfths in the RAM

An example of a lesson plan built around this question can be seen in Appendix 10.2[15]. Note that this question opens up opportunities for discussions about multiplication (factors and multiples) and, leaning towards visual skills and the vocabulary associated with it, about rotation (reflections and symmetries).

Based on the suggestions presented above, the outline of the whole sequence of lessons, keeping the division into 3 packs, would look like the diagram below.

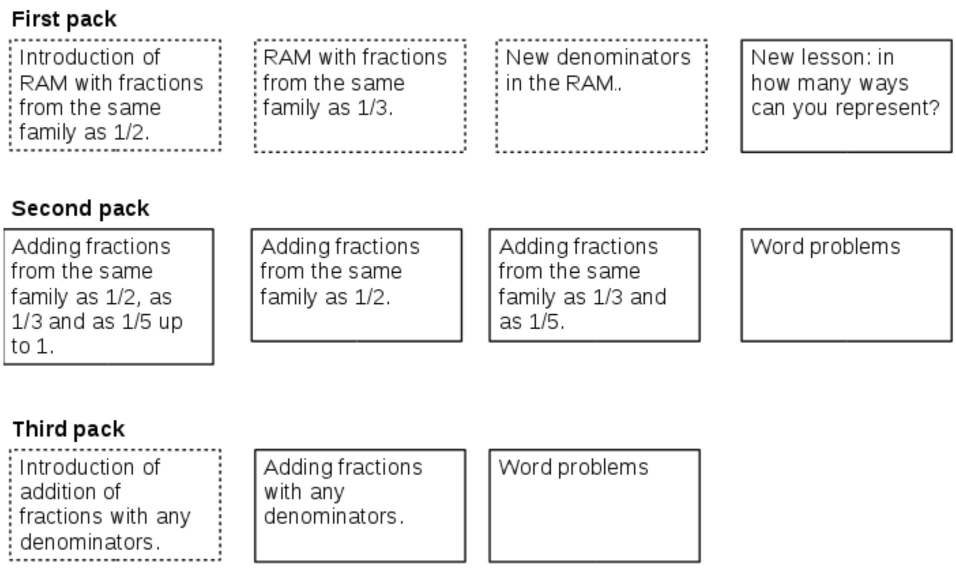


Illustration 70: New outline of the lessons (dotted boxes means lesson with cut-outs and solid boxes mean lesson with diagrams)

15 This lesson plan was designed due to teachers' request, after the end of my data collection, for a booklet with all the lessons to be used in the next academic year.

As it was said in Section 4.4.1, my research was not conceived as a design-based research study, therefore, it was beyond my objectives to perfect the lesson plans. This could be explored as a future research project, as will be discussed on Section 8.4.4.

However, I believe the discussion presented here can be informative for design purposes, as well as being a contribution towards a better understanding of the role of visual representations in teaching and learning mathematics and how this can be effectively achieved.

8.3 For a more visual pedagogy

In this section I will focus on a general issue that is broader than the conclusions that have been presented so far. I will make an argument for a more visual pedagogy in mathematics.

From the perspective of visual representations, there is a long standing understanding among educationalists that visual representations are fundamental for mathematics learning, see for instance the arguments presented in Skemp (1987). For him, visual representations are responsible for bringing meaning to the more abstract and concise symbolic representations, enabling students to approach mathematical objects in a more intuitive and holistic way. Even though his ideas are more speculative, they resonate with arguments presented by many other researchers in the field of mathematics education. For instance, this feeling that visual representations are more intuitive and holistic could be understood under the lens of Lakoff & Núñez's (2000) ideas regarding the role of metaphors in learning: visual representations are intuitive because they can work as metaphors connecting the abstract concepts with more concrete realms. Giaquinto's (2007) ideas can also help us to understand Skemp's argument: humans are equipped with some visual skills and visual perception can be the trigger to the acquisition of new knowledge.

Considering these arguments, I suggest that the first major conclusion I presented before, regarding the emergence of generative reasoning anchored in visual representations, is the result of students successfully transferring their (intuitive) knowledge on 2-D shapes into fractions, and consequently being able to extend it towards novel questions.

Therefore, a more visual pedagogy from the perspective of visual representations could make the learning of abstract topics more meaningful by offering an alternative to anchor it in intuitive prior knowledge.

From the perspective of visual skills, as was discussed in Section 2.4, there is extensive evidence that visual skills and achievement in mathematics (or STEM areas in general) are correlated both predictively and concurrently. Moreover, recent studies are starting to unveil the causal mechanisms behind this correlation. For instance, proposals, such as Dehaene's (2011) number sense, that places the number line, a visual way of representing numbers, as a central element in all mathematical cognition. This thesis provides a more direct example where the lack of visual skills interfered negatively in how students engaged with mathematical tasks. As discussed in Section 7.2, by not being able to rotate some shapes mentally, students were not able to solve some tasks involving equivalent fractions via comparison of areas. This difficulty was not anticipated by me when designing the lesson plans and resulted in students being intellectually distracted from the actual goals of the lesson. On the positive side, there is research showing that apparently simple activities involving visual skills, such as playing with linear numbered board games (Ramani & Siegler, 2008) and video games like tetris (Newcombe, 2010), or training in mental rotation (Cheng & Mix, 2012), and activities involving paper folding (Wright et al., 2008), have a positive impact for the learning of standard mathematical topics ranging from number estimation to missing number problems.

Although there is still a lot to understand regarding the complex relationship between visual skills and mathematics cognition, I agree with Newcombe (2016) when she states that there is enough evidence for researchers to investigate how to use it in teaching, and I believe my results contribute to fill this gap by suggesting two main features for such a pedagogy.

Firstly, many benefits may arise from the use of visual representations as grounding metaphors for a mathematical topic, instead of only using them as ways of representing mathematical objects. In this scenario, it is necessary to learn much more than only the elements of a representational system, but also the transformations allowed within it. This takes time, yet, it seems to be key to the success of an approach emphasizing visual representations. Consider, for instance, the case in Japan and Singapore. Watanabe (2015) shows how the Japanese textbooks capitalize on visual representations as tools for reasoning, and points out that this is possible by consistently supporting a cohesive use of visual representations since the early stages of education. A similar conclusion is reached through the description given by Ng & Lee (2009) of how Singapore uses the bar model for different topics and problems in different years and on how models are used in the Realistic Mathematics Education approach (Van Den Heuvel-Panhuizen, 2003).

Secondly, the development of visual skills is important. They can enable students to use the properties and transformations of the representational system more efficiently and even understand its element more clearly. As pointed out by Newcombe (2016), this development can be reached via lessons fully focused on visual skills, or via incorporating such activities within regular lessons. For both approaches, there are plenty of activities in the literature that can be helpful (L. Brown, Coles, & Hewitt, 2016; Newcombe, 2010).

In conclusion, a more visual curriculum does not mean just providing a multitude of different visual representations for each concept in the curriculum, but:

1. a change in the role played by visual representations in the teaching and learning of mathematical topics from accessories to source of meanings, and;
2. a recognition of the relevance of visual and spatial skills in learning mathematics.

Another aspect that has to be considered refers to the participants of my research: low achieving students. It could be said that my conclusions are bounded by my choice of focusing on this particular group of students (a detailed discussion of their characteristics beyond the `low achieving` label is presented on section 6.3). My answer is based on two arguments.

First, although my focus on visual representations was particularly inspired by some features of these students, the benefits of approaches emphasizing visual representations and skills are not limited to them. The ideas proposed by Giaquinto (2007) and by Lakoff and Núñez (2000), as well as proposals connecting these two body of knowledge such as Johansen (2014), refer to the nature of mathematical knowledge and the results emphasized by Newcombe (2016) refer to science and mathematics learning in general. It is also worth reinforcing that high achieving countries, such as Japan and Singapore, adopt strategies that emphasize visual elements as a general approach to teach mathematics.

Second, as a general ideological principle, I believe that not only research should focus more on unprivileged individuals but also education. Curriculum and didactics should be taught aiming at the struggling students and then expect that the others will also benefit from it.

One last aspect that has to be considered refers to choice of fractions as a topic. While it is true that my conclusions are limited to it, the main ideas on which my arguments are built are not limited to any specific topic. For that reason, I would expect that my conclusions could be extended to other topics taught at a similar school level. However, as emphasized in my research question, the choice of the visual representation and the design of the lessons were both very carefully, demanding a lot of considerations. Therefore, although possible, I do not think that the extension of my results should be taken for granted. It depends on work to be done in finding and developing a model suitable for other topics and designing lesson plans accordingly.

8.4 Final considerations

In this final section, I will present a brief discussion of some issues that emerged during the data collection and analysis, but were not central to my research question. For that reason, the amount of data available is relatively scarce, and the discussion proposed has a more tentative aspect than the conclusions presented in the previous chapter.

Also, I will reflect on some issues that I have experienced during my research project and finish by presenting some possibilities for further research related to what was explored throughout this thesis.

8.4.1 My own use of language

In Section 2.4, I define visual representations in opposition to textual representations by presenting a series of characteristics that would identify verbal communication with the latter. Moreover, the reduction of the use of verbal language is highlighted as a potential benefit of using visual representations to teach low achieving students.

Bearing that in mind, how can I justify the fact that most of the interactions discussed in my data analysis were essentially verbal?

Reflecting retrospectively, I believe the answer relies on my habits and unconscious preferences for verbal interactions. Although I have explicitly questioned the role of such a medium when designing the lesson plans, and have created, together with the teachers, an environment where arguments based on visual representations were accepted and encouraged, this questioning could have been extended further to cover the way I interacted with students when helping them to solve the tasks.

By the end of a paper on the importance of visualization for STEM education (Gates, 2018), Gates makes an interesting observation: "We all have a lot to learn — even me, who has written a chapter praising and encouraging visualisation — with only four diagrams" (p. 188). The fact that even I, a researcher explicitly concerned with the role of visual representations in the teaching and learning of mathematics, could have become more detached from verbal communication, but did not, only shows how pervasive this medium is inside classrooms.

A related anecdote happened during the preliminary study. After the first lesson (during which she was challenged by the researchers not to use much language), Julia told us that she felt tempted to use talk every time she noticed something that was not going according to the plan. Once again, it seems that when the context demanded unplanned action, the immediate response involved verbal communication.

Nevertheless, several episodes discussed in my data analysis also show that students felt this change and while reasoning, actually used visual elements, such as, vocabulary strongly charged with visual words, explicit references to the diagrams and gesturing. The observation that this kind of behaviour emerged even though the use of visual representation could have been more intense, encourages the exploration of more extreme approaches in terms of using less verbal and more visual communication.

8.4.2 My influence on the data

As mentioned in Section 2.2, I do not believe that it is possible to adopt a stance of pure observer, when it comes to research inside classrooms (Wragg, 1999). During the pre-field work stage, my interactions with the students were dictated by what the teacher asked or expected from me (going from managerial assistance, to checking answers and helping students with the tasks). As there was already a good rapport built between me and the teachers, during the preliminary study I adopted a more active stance, initiating conversation with students when I felt they would be open to talk or needed assistance. Finally, during the main data collection, I tried to initiate as many interactions as I could, in order to generate as much data as possible without causing discomfort for the students, or interfering in how the teacher was conducting the lesson.

The students got used to the kind of interaction I used to initiate during the targeted lessons. One of Julia's students said humorously, "Here comes Leo asking why." It could be argued that any conclusion I have drawn from my data is the result of my presence in the classroom and the way I interacted with the students instead of due to any specific characteristic of the lessons. While I cannot discard my influence as relevant, I can say that my attitude during the lessons was similar to the teacher's: 1) checking answers and giving punctual feedback, or 2) offering some help to a student when struggling with a question, by pointing out critical features, or offering some guidance towards the solution, or 3) asking "why" and "how" questions when I identified some solution that could be interesting to my research, or when I felt the student would be willing to elaborate on it.

My argument is that my attitude could be replaced by that of the teacher, or by a teaching assistant. In fact, David commented during one of the meetings that the approach of the lessons about fractions enabled him to interact with the students in much richer ways than during his regular lessons. This remark weakens the argument that my findings are the result of my presence instead of the characteristics of the approach being investigated.

Also, as discussed in Section 4.5.1, the within-class clinical interviews are closer to regular classroom interaction between teacher and student than a typical clinical interview, and since my research question refers to learning as it occurs in an educational environment, and not to cognitive development itself, this characteristic increases the ecological validity of my findings.

8.4.3 What I would do differently

This issue has already been discussed in Section 6.6.5 based on my first impressions right after the end of the data collection. The reflections that follow are somehow deeper and take into account the results of my research.

Reflecting retrospectively on all stages of the research, I can identify two changes that I believe would have been beneficial to my aims.

The first of these, and the most important in my opinion, refers to the timing of the lessons. Initially, I believed that the gaps between the packs of lessons (remember that each pack was enacted in a different term) would be enough for me to make sense of the data in time to inform the design of the lessons in the next pack. Although I think I was right in this respect, now I believe that my data collection would benefit from some interval between the lessons (within the same pack) from one teacher to the other. This gap would have to be short, otherwise it would not be possible to accommodate the three teachers and meetings with the research team in a term, but what happened during my data collection was that teachers decided to enact the lessons usually during the same weeks. Consequently, sometimes it was not possible to reflect properly on the lessons in time to incorporate changes for the next teacher who would use that lesson plan.

In general, every lesson plan was tweaked from one teacher to the next, but the tweaks were sometimes limited by the time available, for instance, to prepare more cut-outs or envisage new tasks. Also, these tweaks were often based on local evidence, from one lesson, instead of on all lessons from a pack, which limited the scope of what could be noticed and incorporated.

Considering that every term is about 12 weeks long and that the school did not follow a strict scheme of work, I believe it would have been possible, with some planning, to avoid overlaps and still accommodate meetings, lessons observations and the targeted lessons of each pack within a term.

The second change refers to instruments for data collection. At first, I discarded the use of video as a tool for data collection, based on my perception that it could intimidate students, especially considering their lack of confidence with mathematics, as discussed in Section 6.3. However, I believe this study would benefit from pictures of their solutions complementing my notes, and short videos recording their working out during the lessons. Moreover, the other members of the research team made occasional use of such tools and did not notice any disturbance when doing so. Considering that it all could have been done using a smartphone and that students are very familiar with such devices nowadays, I believe that they would not have felt intimidated and my data would have gained in richness.

Unfortunately, I did not notice the potential advantages of these two changes in time to incorporate them into my data collection and I believe the reason for that is the intensity of the activities during this period.

8.4.4 Possibilities for further research

Considering the topics discussed the context, data collection methods and findings of this study, I can identify three possibilities to continue the investigation presented in this thesis.

The first possibility would be to investigate the lesson plans being used in a similar context (ideally in the same school) after incorporating the changes discussed in Section 8.2. Even though I do not subscribe to the common view among design-based researchers of aiming for some sort of optimal, finely crafted lesson plan (Swan, 2014a), I believe that the issues related to visual abilities and to the multiplicative aspect of fractions deserve further investigation at a classroom level. In this scenario, the research question would move a bit away from fractions, and focus more on how low achieving students mobilize visual abilities in lessons involving cut-outs and diagrams, and how they develop knowledge on multiplication through visual representations. Both questions are similar in nature to my research question in this thesis, therefore, both could be investigated with methods very similar to those employed here.

The second possibility would be to explore different contexts, especially in terms of teachers' styles and preferences. This possibility is particularly enticing for me since my own teaching style is quite different from what I observed in Purple Valley's teachers, and for that reason, I caught myself throughout my data collection imagining how exactly I would enact the lessons I had designed. How would the approach look if teachers and students were more comfortable with whole class or peer discussions, or with more investigative questions? As I have discussed before, one of my design principles was to maintain a certain level of coherence with the current practices at the school. This principle had a massive impact on how I designed the lesson plans and some variation in this respect could provide new insights into my research question without compromising the advantages of keeping coherence with school practices.

This possibility is especially relevant considering that my career is probably going to continue in Brazil instead of in the United Kingdom, which implies changes in terms of preferred teaching styles, characteristics of students' behaviour and engagement, and schools' support to professional development. A combination of these would result in different starting points, which would in turn, imply different characteristics for the lesson plans within the boundaries set by the design principles adopted here.

Finally, the third possibility I envisage is to explore a different topic from the curriculum with a similar approach. I believe, based on my views of what mathematics education research should be, that this possibility should aim at understanding how to use visual representations in a more integrative way throughout the curriculum, instead of only aiming at expanding what was done here to new topics, in order to create a collection of lesson plans. Research in other countries, such as Japan (Watanabe, Takahashi, & Yoshida, 2010), Singapore (Clark, 2007) and Netherlands (Van Den Heuvel-Panhuizen, 2003), has already shown that it is possible to build such a curriculum. However, I would make a case for the importance of developing such a strategy rooted in current practices, instead of trying to transplant it from another context, as well as by focusing on the needs of low achieving students, instead of relying on the "misguided belief that by supporting the affluent, all will benefit through the trickle down principle" (Gates, 2015, p. 1).

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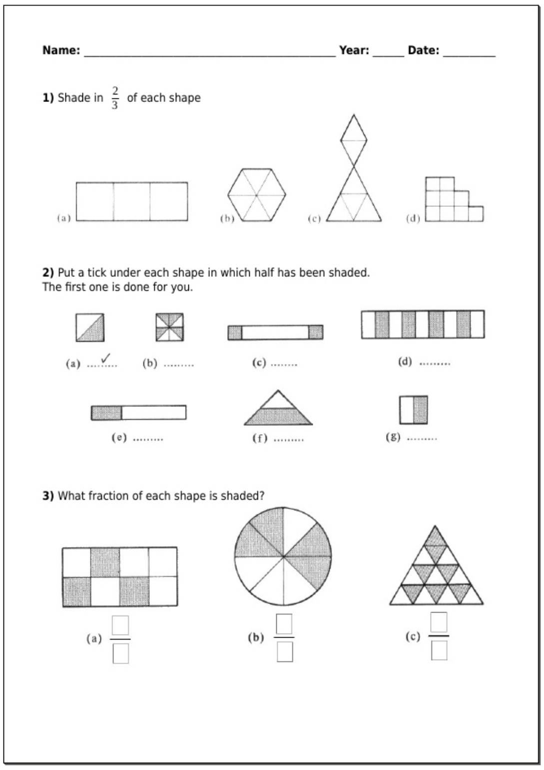
10 APPENDICES

10.1 Diagnostic assessment



Name: ____________________ **Year:** _____ **Date:** ________

1) Shade in $\frac{2}{3}$ of each shape

(a) (b) (c) (d)

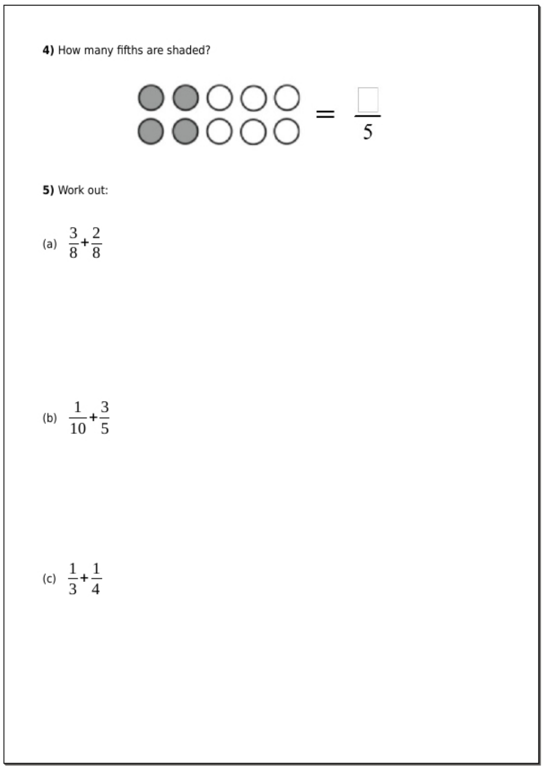
2) Put a tick under each shape in which half has been shaded.
The first one is done for you.

(a) ....✓.... (b) .... (c) .... (d) ....

(e) .... (f) .... (g) ....

3) What fraction of each shape is shaded?

(a) $\frac{\square}{\square}$ (b) $\frac{\square}{\square}$ (c) $\frac{\square}{\square}$



4) How many fifths are shaded?

$= \frac{\square}{5}$

5) Work out:

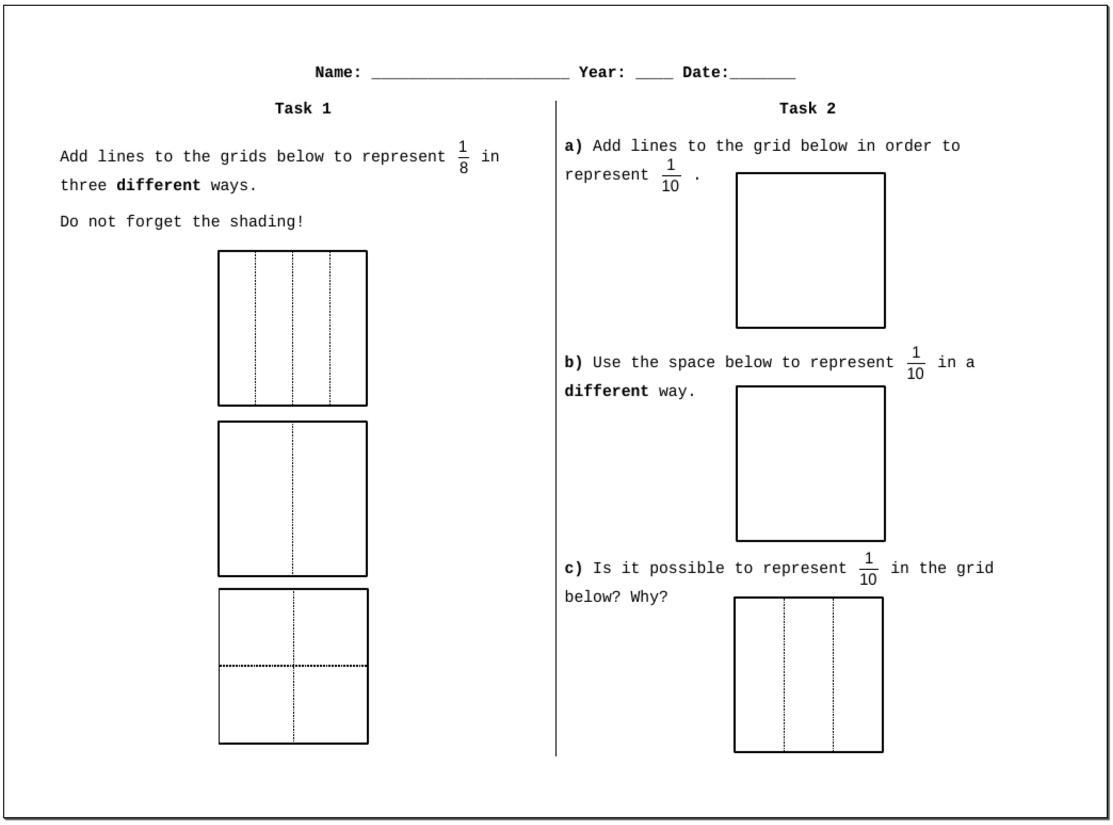
(a) $\frac{3}{8}+\frac{2}{8}$

(b) $\frac{1}{10}+\frac{3}{5}$

(c) $\frac{1}{3}+\frac{1}{4}$

10.2 Lesson plan on multiplication

This lesson plan is composed of four pages with seven tasks in total. As explained in this document, it was developed based on my initial analysis of the lessons but not used in my data collection.



Name: ____________________ **Year:** ____ **Date:**_______

Task 1

Add lines to the grids below to represent $\frac{1}{8}$ in three **different** ways.

Do not forget the shading!

Task 2

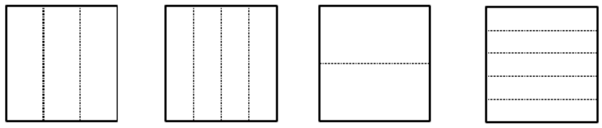
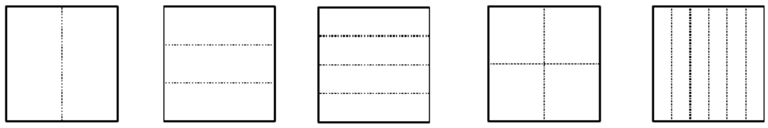
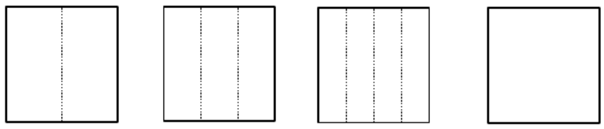
a) Add lines to the grid below in order to represent $\frac{1}{10}$.

b) Use the space below to represent $\frac{1}{10}$ in a **different** way.

c) Is it possible to represent $\frac{1}{10}$ in the grid below? Why?

Task 3

For each fraction below, decide which grid can be used to represent it and draw the representations.

$\frac{1}{6}$	
$\frac{1}{9}$	
$\frac{1}{5}$	
$\frac{1}{20}$	

Name: ____________________ **Year:** ____ **Date:**_______

Task 4

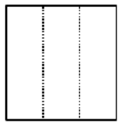
Use the space below to represent $\frac{1}{12}$ in **three different** ways.

Task 5

Use the space below to represent $\frac{1}{18}$ in as many **different** ways as you can.

Task 6

a) What fractions can you represent using the diagram on the right? Show them.



b) Is it possible to represent $\frac{1}{8}$ using the same diagram? Why?

Task 7

a) Use the space below to represent $\frac{1}{7}$.

b) Is it possible to represent $\frac{1}{7}$ in a different way? Why?

c) Can you think of a new fraction that can only be represented in one way?