Lyapunov Approach on a Homogeneous Family of Controllers for Robotic Manipulator

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ABSTRACT

Second order sliding mode has been successfully implemented for solution of real problems for its inherent features such as finite-time convergence and robustness to disturbances. For the first order sliding modes, it is common to deal with the issues of stability, robustness, and convergence rate of the equilibrium by means of a Lyapunov approach. For higher order sliding modes, however, a similar treatment has not been developed until recently. The focus of this thesis is the construction of strong Lyapunov function, i.e. for which its time derivative can be upper bounded by negative-definite functions, for the design of control strategies for robotic manipulator, which is a nonlinear system, subject to combinations of parametric uncertainty, bounded disturbances, actuator saturation, and output feedback.

The first contribution of this work is the development of a strong Lyapunov function for a parameterized family of homogeneous sliding mode based controller comprising twisting algorithm, continuous finite time control, linear PD control law, and uniformly stable control law, all of which belongs to a general homogeneous family of control algorithms. The strict locally Lipschitz homogeneous Lyapunov function proposed permits the estimation of convergence time for the trajectories of the system to the origin, in finite-time, exponentially, or uniformly asymptotically, even in the case when it is affected by bounded non-vanishing or growth bounded vanishing external perturbations. Moreover, the relationship between the control gains and its convergence performance can be analyzed.

Leveraging on these results, a strong Lyapunov function is developed for a closely related second order sliding mode algorithm, the super-twisting algorithm based controller. In particular, the construction of these strong homogeneous Lyapunov function is able to show

the relationship between the twisting and super-twisting algorithms and allows linear combination of two homogeneous control of different degree.

Extending the results for MIMO robot manipulator, a type of Euler-Lagrange dynamic systems, a family of integral sliding mode-based controller is introduced for trajectory tracking. In particular, the homogeneous dynamics is employed as the desired error dynamics for the controller. Additionally, the conventional PID control is shown to be a special case and the present formulation presents the relationship between the gains of the controller and the desired performance, which provides a systematic method for gain selection for a robust PID control. In addition, for the special problem of regulation, employing the results of homogeneous control, finite-time regulation of the robot manipulator is achieved.

Since actuator saturation is a phenomenon that affects the performance of dynamic systems under closed-loop control, a saturated version of the controller is also developed that achieved global stability while maintaining the features of the unbounded version of the controller in terms of trajectory tracking and finite time regulation. Extending the results for system with position measurements only, a saturated output feedback version of the controller is introduced that can achieved global stability as well. Each of the proposed controllers provides advantages over the previous literature in their ability to design desired error dynamics and the time derivative of the disturbance is not required in the stability analyses.

Throughout the work, Lyapunov-based stability, in particular the nonsmooth Lyapunov analysis techniques, and numerical experiments are provided to highlight the performance of each controller design.

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TABLE OF CONTENTS

LIST OF FIGURES

Chapter 1: INTRODUCTION

1.1 Motivation

Sliding mode approach to control design has received great amount of attentions of late due to its inherent attractive features such as finite-time convergence and robustness to disturbances. It introduces a nonlinear discontinuous term where its gain must be designed so that the trajectories of the system are forced to remain on some user-defined sliding hyperplane in the error state space. The resulting motion is called sliding mode. It is this discontinuous term that provides the abilities to reject perturbations and some classes of uncertainties between the actual system and the nominal model used in the control design stages. In [1], a definition of the sliding mode order is given and consists of trajectories in the sense of Filippov [2]. The standard sliding mode is of the first order and is known as robust and very accurate with respect to various classes of internal and external perturbations, but it is restricted to the case in which the output relative degree is one. Besides, the high frequency switching that produces the sliding mode may cause chattering effect.

Higher order sliding modes (HOSM) appears sometimes in systems with traditional sliding mode control or they are deliberately introduced because it has been found that finite time convergent HOSMs preserve the features of the first order sliding modes and can improve them, if properly designed, by eliminating the chattering, for instance see [3] and [4]. While finite-time convergent arbitrary order sliding mode controllers are mostly still theoretically studied, 2-sliding controller or second order sliding mode(SOSM) with finite-time convergence have already been successfully implemented for solution of real problems.

However, there are only few SOSM that are widely used namely, the sub-optimal controller [3], [5], [6], the terminal sliding mode controllers [7], [8], [9], twisting controller, and the super-twisting controller; the last two being the most popular.

Particularly, the super-twisting control found its application on wind energy conversion system [10], velocity observer of mechanical systems [11], and uncertainty observers [12] [13]. On the other hand, application of twisting algorithm can be found in the adaptive tracking control of intelligent vehicle system [14] and trajectory tracking of crane [15] and [16]. Also of interest, are the development of family of controllers that are based upon the twisting and super-twisting algorithms [17] and [18]. For the first order sliding modes, it is common to deal with the issues of stability, robustness, and convergence rate of the equilibrium by means of a Lyapunov approach. For higher order sliding modes, however, a similar treatment has not been developed until recently. Instead it is usual to use majorant curves [11], homogeneity based methods [19], or a weak Lyapunov function together with geometric approach [20]. The focus of Chapter 2 is the development of strict Lyapunov functions for the super-twisting, twisting algorithms, and the corresponding family of controllers, which can be used as design and analysis tool whose time derivative can be bounded by negative definite functions.

The research on the control theory of serial mechanical systems has been a topic that is actively studied. The asymptotic stability of robot manipulators can be achieved by computed torque method or inverse-dynamics control [21]. While asymptotic stability implies that the system have convergence to the origin as time goes to infinity, finite-time stabilization can ensure convergence to the origin in finite time, as discussed above. On the other hand, robustness in control systems is an equally important property as stability and convergence. It is the property by which a system preserves a tolerable behaviour under the influence of uncertainty, perturbations, external disturbances, etc. In particular, it is known that finite-time stabilization of dynamical systems can provide high-precision performance and improved rejection of low-level persistent disturbances [22] [23]. This can be achieved by continuous non-Lipschitz feedback controllers such as twisting-based algorithm in [24]. However, the robustness issue is not specified clearly as the algorithms require exact knowledge of the dynamics of the manipulators. Leveraging the outcomes developed in Chapter 2, Chapter 3 presents full state feedback approach by integrating SOSM into the controller.

The previous discussion on robot manipulator control assumes that the joint velocity is available from measurement. If only position information is available, one has to employ output feedback control, which has received considerable interest in robotics literature due to its possibility to avoid the need of a tachometer hence simplifying the robot design [12]. The main problem in output feedback control is the need for the control law to not only compensate uncertainties of the system but also the lack of link joint velocity measurements. Employing the outcomes of Chapter 2 and 3, Chapter 4 presents an output feedback controller for robot manipulator.

While the robust control methods mentioned above for robot manipulators have been shown to be effective for the compensation of uncertainties and disturbance in their respective context, generally, the fact that the required input torques may command more actuation than is physically possible by the system for instances such as large perturbations, initial conditions that are far from the equilibriums, or fast desired trajectory. These may lead to unexpected or undesirable closed-loop behaviours, for instance, in the nonlinear PID control for the global regulation problem of robot manipulator of [25], it requires unbounded state dependent control gains, which may easily causes saturation of actuator if the initial conditions of the system is not restricted. Owing to these risks, control law that is bounded while ensuring performance when operating within actuator limits are motivated. By means of the results of Chapter 2 and 3, Chapter 5 presents a bounded full state feedback controller for robot manipulator which limits the control authority at or below an adjustable a priori limit. Bounded control designs are available in literature; however, the integration of SOSM into the bounded structure that has a strict Lyapunov function has remained an open problem.

Motivated by the same concerns presented in Chapter 4 on the lack of joint velocity measurements and that of actuator constraints in Chapter 5, Chapter 6 develops a control strategy for robot manipulators with a bounded control approach with output feedback. Previous techniques and outcomes obtained in Chapter 4 and 5 are utilized which allows for the bound on the control to be adjusted a priori provided through strict Lyapunov functions.

1.2 Literature Review

A literature review of Chapters 2-6 is presented below.

Chapter 2: Lyapunov approach on twisting and super-twisting based second order sliding mode: By means of strong Lyapunov functions, the study of stability on super-twisting algorithm and its finite time convergent characteristics was carried out for the first time by [26] and later in [27]. This approach allows a wider class of perturbations and uncertainties originally admitted by SOSM. Another advantage of the use of Lyapunov functions is that it is possible to obtain explicit relations for the design parameters. On the other hand, [17] developed a Lyapunov method for the analysis of a generic second order algorithm, which is a family of controllers of which the super-twisting algorithm is a special case. It extends the results of [26] by allowing the positive power of the control terms to range from less than one to more than one. By doing so, a range of stability results were obtained. Essentially, three types of stability can be achieved for the generic super-twisting system namely, finite-time, exponential, and uniform convergence. A remarkable fact from the approach is that the stability of the equilibrium of the system is completely determined by the stability of its associated linear counterpart. Additionally, non-homogeneous super-twisting algorithms have been studied using strict Lyapunov functions as well [28].

A closely related algorithm, which is equivalently important, is the twisting algorithm. The twisting algorithm [29] is given by

$$\dot{z}_1 = z_2,\ \dot{z}_2 = -k_1 sign(z_1) - k_2 sign(z_2)$$

where z_1 and $z_2 \in \mathbb{R}$ are scalar state variables, , $k_1 > 0$ and $k_2 > 0$ are control parameters.

Similar to that of super-twisting, it is common to find its stability analyses using the homogeneity approach [1], [20], [30], [31] or a weak Lyapunov function together with geometric approach [32]. The algorithm is globally uniformly finite time stable if the inequality $k_1 > k_2 > 0$ is satisfied [20]. Of recent, there is a growing interest in identifying strict Lyapunov function for this algorithm. A strict Lyapunov design and estimation for reaching time, based on Zubov method, is presented in [33]. It employs the idea of using the solution of a partial differential equation as the Lyapunov function. For design control purposes, this methodology becomes difficult. Indeed it requires some handicraft techniques, like fixing the discontinuities. In [34], a strict non smooth Lyapunov function is proposed for the twisting algorithm. The strictness of this function allows estimation of the convergence time of the closed loop system to the origin. While the in work of [35], an alternative proof is established with the same Lyapunov function as in [34]. It provides additional properties that are not shown in [34] as well as a simple rule of thumb pertaining to the relationship between the control gains and the finite settling time of the system.

Similar to the super-twisting algorithm, which has a generic second order algorithm developed based on it [17], for twisting algorithm, in [18], a parameterized family of homogeneous continuous controllers, inspired from the twisting algorithm, is proposed

$$\dot{z}_1 = z_2,\; \dot{z}_2 = -k_1 |z_1|^{\frac{\alpha}{2-\alpha}} sign(z_1) - k_2 |z_2|^{\alpha} sign(z_2) + \delta, \qquad (1 - 1)$$

where $\alpha \in [0,1)$, δ is an external disturbance, and $k_1, k_2 > 0$. Note that the twisting algorithm is a special case of this generalized controller. This continuous controller with unity gains was first proposed by [36] to develop a class of continuous second order finite time systems.

However, the approach employed there for finite time stability does not allow an upper bound of the settling time to be obtained.

Pertaining to the approaches mentioned above for the twisting algorithm, they cannot be applied directly for the parameterized family of homogeneous continuous controllers (1 - 1) that includes the twisting algorithm as its special case. In [18], for $\alpha \in (0, 1)$, the use of a weak Lyapunov function with invariance principle, only global asymptotic stability can be guaranteed. It requires application of Theorem 4.2 of [20], which depends on the weighted homogeneity properties to infer finite time convergence for the case of $\alpha = 0$.

In contrast, from the results of [37] and [38], a strict non-smooth Lyapunov function is proposed for the family of controllers (1 - 1), where the upper bound of the settling time can be obtained. On a related development in [24], a non-smooth proportional-derivative (PD) controller is proposed that is of the same form as the controllers (1 - 1) shown above. There, an explicit construction of Lyapunov function, based on the method of [39], is given. In spite of that, the Lyapunov function given cannot accept zero fractional power (i.e. $\alpha = 0$), the Lyapunov function does not work for the twisting algorithm. Within the work of [40] the conditions on the gains of the controllers (1-1) are developed for $\alpha \in (0, 1)$ that ensures finite time stability.

While in [41], an important link between asymptotic stability with finite-time stability is provided. It is shown that a homogeneous system is finite-time stable if and only if it is asymptotically stable and has a negative degree of homogeneity. On the other hand, a double integrator nonlinear system [42],

$$\dot{z}_1 = z_2, \dot{z}_2 = -f(z_1) - g(z_2)$$

where $f(0) = 0$ and $g(0) = 0$ are continuous functions verifying the sign conditions, $z_1 f(z_1) > 0$, for $z_1 \neq 0$, and $z_2 g(z_2) > 0$, for $z_2 \neq 0$, is shown to be globally asymptotically stable. Since, the parameterized controllers (1-1) for $\alpha \in (0, 1)$ with gains k_1 and $k_2 > 0$ satisfy the above sign conditions, the system is asymptotically stable.

In addition, it is noted that the parameterized family of system (1-1) is homogeneous of degree ($\alpha - 1$), which is negative for $\alpha \in (0,1)$, with respect to dilation ($(2-\alpha)$, 1) (see definitions of [13]). This implies that the family of controllers are finite time stable when k_1 and $k_2 > 0$ and $\alpha \in (0, 1)$. In [37] and [38] the gains are required to be greater than some positive values. Similarly the strict Lyapunov function of [24] requires k_2 to be sufficiently large, while k_1 has to belong to a certain positive interval in the results of [40]. Thus, it suggests that the gain restrictions given by these results are conservative.

Pertaining to the robustness of this family of controllers, two types of perturbation are analyzed in [18], namely $|\delta| \leq M$, and $|\delta| \leq \mu |z_2|^{\alpha}$ where δ is an external disturbance, M and μ are positive constants, and α has the same value as that of the controller (1-1). The first one is a uniform upper bounded type and the second one is a vanishing perturbation. Note that both conditions are identical when $\alpha = 0$. However, finite-time stability can only be proved for the uniform upper bounded type for $\alpha = 0$ (i.e. the discontinuous control). For the vanishing perturbation with $\alpha \neq 0$, the results of [18] can guarantee asymptotic stability only. Furthermore, the vanishing perturbation constraints are with respect to one state only. In [43], a recent survey on robust finite time stability is presented. It suggested that robustness of continuous finite time controllers to disturbances that are bounded by continuous bound with fractional exponent has not been proven. Furthermore, they noted that a recent result in this direction without proof appears in [37]. In [44], a similar form of perturbation is considered as well, however the range of α is limited and it requires additional condition on the gain of i.e. $k_1 > k_2 + \mu$. This system is also studied in [45] which provides asymptotic stability through explicit Lyapunov method, and concludes the finite-time stability through implicit Lyapunov method for the system (1-1) and that of the super-twisting observer. However, due to the structure of the Lyapunov function, the results are only reported for $\alpha \in [0, 1)$. While in [46], the same upper bound on the perturbation is considered but it allows discontinuity in the perturbation, which is not considered in prior contributions. However, the analyses are tedious since no strong Lyapunov function is available.

The vanishing perturbation that is with respect to both state variables,

$$|\delta| \le M_1 |z_1|^{\frac{\alpha}{2-\alpha}} + M_2 |z_2|^{\alpha} \tag{1 - 2}$$

where M_1 and M_2 are positive constants, are studied in [37] and [38]. Note that the vanishing perturbation considered in [18] is a special case of this when $M_1 = 0$. It should be noted that this perturbation has a non-Lipschitz continuous bounds that vanish at the origin. While [18] only manage to guarantee asymptotic stability under the vanishing perturbation, [38] manage to prove finite-time stability through its strict Lyapunov function. However, similar to its unperturbed analysis, the gains have to satisfy a more conservative constraints for the case of $\alpha = 0$ (i.e. twisting algorithm). In particular, when $\alpha = 0$, with $M_1 + M_2 = M$, the well known inequality [20] for the twisting algorithm under bounded perturbation, $k_1 - M > k_2 > M$ cannot be obtained from the results of [38], which suggest the stability properties of the system not being fully characterize by the Lyapunov function presented there.

Of recent, [47] put forward a stability analysis of second-order sliding mode for a discontinuous control by means of a Lyapunov function. The main benefits are to fully characterize the stability properties of the system so that finite-time convergence can be concluded without resorting to geometric methods, to provide a relationship between the gains and its estimation of convergence time. Pertaining to Lyapunov function candidate, [48] provides an important theorem that relates the finite-time stability of a system with the existence of a Lyapunov function. It states that there exists a strict Lyapunov function for a system if the origin of the system is a finite time equilibrium. In particular, the Lyapunov function satisfy the differential inequality $\dot{V} + kV^{\beta} \leq 0$ where $k > 0$ and $\alpha \in (0, 1)$. That contribution provides not only an existence of strict Lyapunov for the stability analysis, but also an estimate on the upper bound of the convergence time of the system based on the differential inequality.

Regarding the construction of Lyapunov function, [49] provides a method to construct strict Lyapunov function for a class of HOSM algorithms. In particular, the twisting algorithm is considered and a Lyapunov function is shown in the work. However, the construction is highly dependent on the knowledge of an expression for the solutions of the system, which is difficult to obtain for nonlinear systems in general. In [50] and [51], Lyapunov functions, obtained through solving partial differential equation, are applied to study the finite-time stability of some finite-time 2-sliding mode algorithms, including the twisting, and super-twisting algorithm, and terminal controllers [52]. It is found in a recent study of HOSM control schemes [53] that these Lyapunov functions are indeed homogeneous in nature. However, the form of the function makes it difficult to operate with for applications or developments [17].

Furthermore, recently a Lyapunov-based homogeneous controllers is presented by [54] for perturbed integrator chains, of which the results of universal SISO sliding mode of [55] are a special case of. In particular, greater simplicity in analyses can be achieved by taking advantage of the homogeneity properties of the system by using a homogeneous Lyapunov function. In [56], the existence of a homogeneous Lyapunov function for a homogeneous asymptotically stable system is shown. In particular, a homogeneous Lyapunov functions for a terminal sliding mode-like second order system is constructed in [57]. Due to the homogeneous nature of the Lyapunov function, several interesting properties can be obtained that simplifies the construction of a differential inequality that ultimately concludes finite-time stability.

Other than finite-time stability, there is another relevant development on homogeneous systems that provides a convergence time independent of the initial conditions. In [58], a lemma that relates the convergence rate of a homogeneous system with its degree of homogeneity is provided. In particular, the origin of a system is rationally stable if it is homogeneous with degrees greater than zero, exponentially stable if it is equal to zero, and finite-time stable if it is negative. For the rational stability, the states of the system will asymptotically converge to zero. It differs from the exponential convergence in the sense that it converge to a bounded region in finite time independent of the initial conditions of the system. The upper-bound of the settling time is dependent only on the said bounded region.

This condition is similar to the uniform convergence shown in [59].

This attractive feature of uniform convergence has attracted numerous attentions lately. For instance, by employing a control term having exponent greater than unity, fast and uniform convergence can be found in the work of double power reaching law for sliding mode control [60], uniform sliding dynamic of [59], generic second order algorithm of [17], and fixed-time stabilization of [61] and in [62] a fixed-time convergent super-twisting-like control inspired from [63] is presented, to name a few.

Since the parameterized family of controllers (1 - 1) is homogeneous, it is of interest to study its uniform convergence characteristic when its range of $\alpha \in [0, 1)$ is allowed to extend beyond unity, which to the best of the authors' knowledge, has not been reported in the published literature.

The properties of (1 - 1) and the kind of perturbations that each of the member of the algorithms can tolerate are different depending on the parameter α. The member with $\alpha \geq 1$ has correction terms that are stronger further away from the origin and not as strong near the origin as those members with $0 \leq \alpha < 1$. These differences are related to the kind of perturbations that each member of the algorithm is able to tolerate as suggested by the form of (1 - 2), although no conclusive result has been shown in the literature

Hence a natural question arises whether the linear combination of both members of the family can inherits the best properties of both. A result that is close to answering the question is found in [64], where a finite-time dynamics which has a fast transient process is introduced. Instead of having a linear sum of two homogeneous controls with different exponent, the fast transient process is achieved by modifying the finite-time homogeneous control law to have greater correctional values when the states are far away from the origin. On the other hand, when the states are smaller than a threshold, the control law is essentially a double integrator with control (1 - 1) of negative degree of homogeneity hence providing finite-time stability. Nevertheless, only weak Lyapunov function is given for analysis, as such no conclusion is obtained on its robustness. Another question is, if a strong homogeneous Lyapunov function can be developed for the family of algorithms, how can it be employed for this new combinational algorithm which is not homogeneous, that comprise of sum of two homogenous algorithm.

The above questions are answered positively however, not for the twisting based family of controllers (1 - 1), but for a closely-related SOSM based controllers, namely, a generalized super-twisting algorithm that comprises the super-twisting algorithm with additional linear correctional term found in [65]. In it a linear framework is put forward for the algorithm that allows the construction of a strong Lyapunov functions. While in [17] a generic second order sliding mode (super-twisting based), which extends the result of [65], where two Lyapunov functions, which are structurally different, are developed to show different aspects of convergence properties of the system. Both in [65] and [17], the control comprise of linear combination of two homogeneous control term of different homogeneity plus a additional term that has an exponent that depends on both the individual component. Hence, several questions arise here for the super-twisting based family of controllers, as whether the additional term in the linear combination is necessary in order to attain the best properties of both components, and whether the structurally different Lyapunov functions are required to obtain the same results. To that end, it is imperative to study the relationship between the super-twisting and the twisting algorithms and their respectively based family of controllers. In [32] and [66] the same mechanical energy are employed as the basis for constructing Lyapunov function to analyse the stability of both twisting and super-twisting algorithms. However, the resulting Lyapunov function is a weak one; i.e. its time derivative is only negative semi-definite.

While in [67] a super-twisting algorithm plus an adaptive term to compensate structured uncertainty is proposed. The additional adaptive term is based on the certainty equivalence principle, in which the controller is designed initially under the assumption of known parameters by means of a nominal Lyapunov function, in which case an adaptation law is derived from. There, it shows the importance of having a strong nominal Lyapunov function for the super-twisting algorithm, as a weak nominal Lyapunov function when applied to obtain adaptation law, stability of the system states cannot be concluded. At the same time, the strong Lyapunov function required here have to be at least Lipschitz continuous, since the strong Lyapunov function for the super-twisting algorithm developed in [26], which is non-Lipschitz, resulting singularity to appear in the adaptation law. Thus, it is of importance to not only develop a strong Lyapunov function but a locally Lipschitz one as well for the super-twisting algorithm in order for it to have a wider applications. In [67], a Lyapunov function that satisfies these criteria, developed by [68], is used. However, it is mentioned that the said Lyapunov function requires a more conservative condition on the controller gains than would be obtained employing the previously weak Lyapunov function or the non-Lipschitz one.

The preceding results and arguments point to the direction of finding strict and locally Lipschitz Lyapunov functions that are able to characterize the various stability properties of the family of controllers (1 - 1), which is a twisting-based algorithm, and correspondingly a super-twisting based algorithm, together with a derivative algorithm that are based on the sum of two homogeneous ones.

However, constructing Lyapunov function for a system is difficult, let alone for a family of system. In essence, some of the works above employ strict Lyapunov function for stability analyses without considering the homogeneity properties of the system. Interestingly, in [69] a constructive method is proposed for generating Lyapunov function for a class of homogeneous systems using Polya's theorem. However, there are some drawbacks mentioned in the method including the selection of monomials and the exponent in the Polya's theorem which are unknown in the initial selection process. On the other hand, others rely on a weak Lyapunov function for asymptotic stability, together with a homogeneity approach to prove finite-time stability. These methods inevitably provide restrictions on the range of usability of the Lyapunov function even when the controllers are from the same parameterized family.

At the same time, the system described above may have differential equations with discontinuous right-hand side (i.e. the twisting or the super-twisting algorithm or having perturbations that is discontinuous). According to Filippov's theory, a solution to a differential equation with discontinuous right-hand side is an absolutely continuous function that satisfies a suitable differential inclusion associated to the differential equation [2]. In particular, for some nonsmooth dynamic system, it is natural for the system to assume a nonsmooth Lyapunov function (Example 1, [70]). Due to the lack of differentiability of nonsmooth Lyapunov function, the usual Lyapunov's theorem [71] cannot be applied. Instead, we need some tools of generalized Lyapunov analysis for which the stability properties of nonsmooth dynamic systems can be determined such as in [70] [72] [73] [74].

Chapter 3: Robot manipulator control - full state feedback approach: An approach with sliding mode control has also been followed extensively on robot manipulator systems. However, such an approach will lead to discontinuous control [75]. In [76] the authors employ a smooth robust controller that comprises a proportional term and an integral term of a linear sliding mode variable for the trajectories control of robot manipulator. The stability analyses ensure the states to have asymptotic convergence. However, to ensure robustness, the robust gains of the control law have to dominate the first and second derivative of the uncertainties, which are difficult to obtain. While, in [77] the authors provides a chattering free sliding mode based control for trajectory tracking of robot manipulator. It can ensure global invariance by having the system to be on the sliding surface from the initial conditions. This is achieved through an integral sliding surface, hence producing a high order sliding manifold. However, due to the formulation, extended state variables, the acceleration, have to be available for computation o the control law. Higher order sliding mode algorithm for chattering reduction and finite-time stability for the control of robot manipulator is reported in [78]. However, the robust gains of the system have a singularity problem. In particular, when the sliding variable is zero while its derivative is non-zero, singularity will occur. In order to produce a smooth variable structure control on robot manipulator, the use of a low pass filter is presented [79]. Essentially, a virtual controller is designed based on sliding mode approach for a virtual plant that comprises the actual plant in cascade with a low pass filter. Hence, the switching action will be filtered before being applied to the actual plant.

Beside the discontinuous control, application of sliding mode control entails a certain reaching phase. The robustness property that sliding mode is well-known for can be achieved only after the occurrence of sliding mode. During the reaching phase, however, no guarantee on robustness is available [75]. In [80], a time-varying sliding manifold that comprise a conventional linear sliding manifold together with an exponentially decaying term is proposed to overcome this issue. By designing the initial condition of the decay term to be equal to the initial value of the linear sliding manifold, the time-varying manifold will be zero initially, hence avoiding reaching phase altogether. The concept of integral sliding mode to the trajectory tracking of robot manipulators is put forward in [75]. In particular, a low pass filter is added to reduce chattering effect of the discontinuous control term. By the means of adjusting the time constant of the low pass filter, the algorithm has a characteristic of a perturbation estimator to that of a pure integral sliding mode. Also, due to the use of an integral sliding surface, initial conditions of the algorithm can be chosen to match those of the robot manipulators so that the states of the systems are on the sliding manifold initially, thus having robustness throughout an entire response of the system starting from initial time instance [81]. Essentially, the integral sliding mode control leads to a sliding manifold that spans the whole state space [82]. Thus the tuning of the time constant of the filter, the trade-off between chattering reduction and robustness can be adjusted.

To apply second order sliding mode, in [83] a time varying nonsingular terminal sliding mode (NTSM) control for robot manipulator is presented. It is able to eliminate reaching phase by formulating a time-varying nonsingular terminal sliding surface. Essentially, the surface is augmented by an additional time varying function that provides the sliding function to be zero at the initial time instance and decay to zero in finite time. Together with a switching type robust control term, the system will be in sliding mode from the initial time instance. Also, in [84] the NTSM is use as the sliding manifold and a fast terminal sliding mode type reaching law for ensuring finite time convergence. The control law also comprise of nominal parts of the robot dynamics for compensation of the nonlinearities. Another example can be found in [85], where a type of fast nonsingular terminal sliding mode for the control of robot manipulators is presented. The sliding manifold consists of a nonsingular terminal sliding mode together with a proportional term.

Effectively, the main feature of sliding mode utilized in the previously mentioned controller is its inherent robustness properties to uncertainty and disturbances, in which case is also its main drawback because to have the said robustness properties, its gain for the discontinuous control term has to be sufficiently larger the upper bound of the disturbances. As such, a lot of effort has been done to research on the method of reducing the gain of the discontinuous term while at the same time having the same level of robustness in the system. Towards this direction, a sliding mode algorithm for the control of robot manipulator with an efficient on-line compensation for tracking of trajectories is found in [86]. The compensation is computed from the acceleration information and the torque applied to the robot manipulator. This method of uncertainties compensation is the essence of time-delay estimation, and the conventional sliding mode is applied not on the uncertainty of the system itself but on the error between the uncertainty and its compensated (estimated) form, which is assumed to be smaller.

A slightly different approach is found in [87], where a gradient estimator, instead of sliding mode approach, is applied on time-delay control (TDC) for improving the robustness of robot manipulator control under the presence of nonlinear friction. The control law consists of a time-delay estimation term to estimate the nonlinearities of the system, a desired error dynamics term, and a gradient estimator term as a compensator for the time-delay estimation error. The algorithm is shown to provide similar performance to the case of time-delay control with switching action which is sliding mode based. Similarly, an approach called time-delay control with ideal velocity feedback (TDCIVF) for controlling tracking problem of robot manipulators is given [88]. The control structure is simple; it has three distinct elements, namely the soft nonlinearity compensation term, hard nonlinearity cancelling term, and a desired error dynamics injection term. The soft nonlinearity compensation term is of the time-delay estimation form, while the hard nonlinearity is taken care of by the ideal velocity feedback. The so-called ideal velocity feedback term can be viewed as a proportional control term of an integral sliding surface variable, which its derivative is the desired error dynamics.

It is worth mentioning in [89], a simple decentralized linear time-invariant control for robot manipulator as a alternative to computation intensive computed torque method. The algorithm also uses a time-delayed control together with a specially designed constant diagonal gain matrix for the decoupling and linearization of the robot joint dynamics. The desired error dynamics are of a linear PD structure. Sufficient condition for ensuring stability for the design of the gain matrix is given, however, it requires knowledge of the inertial matrix of the robot manipulator. In [90], a similar robust control for the trajectory tracking of robot manipulators is developed that is based on a disturbance and uncertainty estimation (UDE), instead of time-delayed control. The control formulation consists of two parts, one to inject a desired linear error dynamics, and another term is based on the UDE [91] to compensate for the uncertainties and disturbances of the system. Essentially, the compensation includes the design of a constant diagonal matrix and a time constant for a first order low pass filter. The stability of the system is dependent on the effectiveness of this compensation. In particular, if exact compensation is achieved, exponential convergence can be achieved based on the injected desired linear error dynamics. On the other hand, if the derivatives of the uncertainties are non-zero but finite, uniform ultimate boundedness can be attained. However, the existence of the estimation term is not properly shown and it is mentioned that the compensation does not exist for systems that have discontinuous disturbances and uncertainties.

It is of interest to note that the time-delay estimation method of TDC has similar structure as the UDE algorithm above. Similar to the UDE, the time-delay-estimation method for the estimation of uncertainties includes the design of a constant diagonal matrix for decoupling of the nonlinear dynamics of the robot manipulator. In particular, it is shown by [92] that the time-delay estimation does indeed behave like a first-order digital low-pass filter, in the sense that the diagonal elements of the constant matrix is related to the cutoff frequency of the digital low-pass filter. Besides that, the time-delay estimation also has similar drawbacks as of the UDE. For instance in [93] the authors show that the time-delay estimation algorithm have difficulty in estimating hard nonlinearity or discontinuous uncertainty. Also, the time delay estimation has an inherent property of an integrator as shown by [94], which is similar to the integration action of the UDE. Hence, time-delay estimation can be seen as a discrete form of the UDE algorithm. In [95] UDE-based control was proposed as a replacement of the time-delay control [96]. Along the same line of estimating disturbance in robotic manipulators, a relationship between the discrete TDC and the discrete PID controller is established in [97]. In particular, the gains of the discrete PID controller can be selected such that it has same properties as the TDC. While in [98] and [99], based on the concept of modelling error compensation, the PID control is formulated as a composition of modelling error estimator and a certainty equivalent feedback function for regulation and tracking problem of robot manipulators. As most of the literature on the robot joint position control problem deal solely with the stability problem rather than the system performance in a transient situation, it has been early recognized that transient performance guarantees deserve further research as noted by [98], where a PID control scheme with acceptable transient performance guarantee is proposed. Several works on prescribed performance guarantees have been presented that utilised error transformation, see [100], [101]. A sliding mode controller with guaranteed transient performance is proposed for application on robot manipulators, [102], where by choosing proper initial value of the controller, reaching transient is eliminated

Note that TDE, UDE, and the modelling error observer discussed above, all have a similar structure, in which they can be reformulated to be PID control that inherently comprise of a linear desired error dynamics and a form of uncertainty compensation. The closed-loop error dynamics in all cases above involve the time derivative of the lumped disturbances and uncertainties, which its upper bound may not be easily evaluated in practical applications. Also, for both the TDE [97] and modelling error compensation [99] approaches it is necessary to choose an appropriate inertia matrix estimate term.

Alternatively, robustness of PID-controlled manipulators is studied differently in [103] as opposed to the formulation in [99]. It is shown that uniform semiglobal practical asymptotic stability can be achieved and a tuning procedure of the PID gains is given in order to obtain any given precision from any given bounded set of initial conditions. It is worth noting that the results are obtained through analysis of a strict Lyapunov function. Also, the perturbations that are considered there include the discontinuous functions of the state such as Coulomb friction and it does not require an appropriate inertia matrix estimate term. Furthermore, in [104], [105] a tuning procedure for the PID gains that ensures semiglobal asymptotic stability for the regulation problem of rigid robots. Although the tuning procedure there allows the selection of PID gains that ensures stability in a specified arbitrary domain, the transient performance of the closed-loop system is unclear from the gain selection procedures. As such, the performance in terms of desired error dynamics is not clear in these approaches.

Hence, a natural question arises here as to whether a particular formulation of stability analysis is available, such that it provides a strict Lyapunov function that ensures semiglobal practical stability of PID-controlled manipulators and at the same time, provides a linear desired error dynamics and an uncertainty compensation components that relates directly to the PID gains. This relationship between conventional PID control and the nonlinear formulation of TDE or modelling error estimator is useful, as it provides a systematic way of PID gain selection as opposed to heuristic gain tuning approach which has its own problems due to too many gains to tune simultaneously [97].

Next, it should be noted that in the previously mentioned integral sliding mode, TDE-based, UDE-based, and PID control of robot manipulators, even in the case of exact model compensation, only exponential convergence is attained due to the inherent linear desired error dynamics as opposed to that of NTSM, which can ensure finite time convergence [84]. The linear error dynamics is actually the response of linear second order system, and by designing the desired dynamics gains, the well-known responses such as stable node or stable focus in the phase portraits can be achieved (see Chapter 2 of [71]). However, due to the structure of NTSM, which comprise of two first order sliding mode, the type of desired error dynamics is limited. Essentially, under the conditions of no perturbations, it has two distinct phases, namely a reaching like phase, which will bring the trajectories towards the NTSM sliding surface if it is no already there, and once the NTSM surface, it will have a finite time sliding phase towards the origin, thus it is not possible for it to exhibit stable focus in its phase portraits. Hence, another question emerges here as to whether one can design a desired error dynamics that can retain finite time convergence property of NTSM while allowing the flexibility of selection of desired responses either stable node-like or stable focus-like in its dynamics. To answer it, one can consider the dynamics (1 - 1), which is a second order sliding mode based algorithm. In [106], for motion control of permanent-magnet linear motors, an integral sliding mode control with (1 - 1) as desired error dynamics is presented. It comprise of the usual discontinuous reaching law to enforce sliding mode, in which case, the system behaved like (1 - 1) in sliding mode. However, if the sliding surface is not reached exactly but reached within a bounded region (i.e. when a boundary layer method is employed to reduce chattering), the convergence of the states are not clearly shown due to the lack of strong Lyapunov function for the desired error dynamics of (1 - 1).

Chapter 4: Robot manipulator control - output feedback approach: Several outcomes in this direction have been developed. For instance, in [12], super-twisting algorithm is employed as observers for both velocity and uncertainties estimation. The same structure is employed in [24] as well, but the robot dynamics is required in its implementation. While in [90] a Luenberger-like plus UDE-based robust observer was proposed to solve the problem of requiring joint velocities for control. Additionally, a filter based on so-called "dirty-derivative" is used for finding the velocity from position measurements in [107], [108], [109], [110].

However, the Luenberger-like observer and the "dirty-derivative" can only provide asymptotical convergence. On using super-twisting algorithm as a velocity observer, the admissible upper bound of unknown disturbances and its finite-time convergence properties is studied in [111]. In addition, it is mentioned that finding the tradeoff between the gains of the algorithm to minimize chattering amplitude at the presence of unmodelled dynamics is an open problem. In [112] a modified super-twisting algorithm with double closed-loop feedback regulation is proposed.

$$\dot{\sigma} = -k_1|\sigma|\text{sign}(\sigma) - k_2\sigma + \omega,$$
$$\dot{\omega} = -k_3\text{sign}(\sigma) - k_4\omega + \phi(t),$$

Essentially, it added a linear correction term $-k_4\omega$ to the $\dot{\omega}$ dynamics as opposed to the linear correction term of the sliding variable proposed by [26]. It is reported in their results that the modified super-twisting algorithm can improve the convergence of the sliding variable by accelerating the approaching speed and at the same time limiting the overshoot. Note that in the observation error dynamics, both the Luenberger-like and the super-twisting algorithm do not have a velocity observation error term in the dynamics, which in part due to the lack of information on the velocity information, and hence the need of this observer in the first place. However, from the "dirty-derivative" formulation it is indeed possible to get a linear correction term into the observer error dynamics. Hence, another question arises as to whether the combination of super-twisting algorithm with "dirty-derivative" can complement each other, and if yes, can a strict Lyapunov function be developed that characterize its features.

Chapter 5: Robot manipulator control - bounded control approach with full-state feedback: Motivated by issues with actuator constraints for robot manipulator control, some efforts have been proposed in the literature. For instance, assuming exact value of robot manipulator parameters, a bounded static feedback for trajectory tracking for robot manipulator is proposed in [113]. In achieving semiglobal finite-time tracking, a saturated control law plus desired trajectories based dynamics term is found in [114]. It has the ability to ensure that actuator constraints are not violated by selecting control gains *a priori*. In [115] an asymptotic tracking control for robot manipulators with actuator saturation is presented. The control law comprises saturated hyperbolic tangent function and computed feed-forward of robot dynamics terms. In [116] a static nonlinear controller is added to an existing PD control plus exact gravity compensation to guarantee global asymptotic stability for the Euler-Lagrange system with input saturation. However, these methods require full dynamics of the system. This is not desirable because it requires *a priori* knowledge of parameter values of the system, including that of payload, which is particularly restrictive, because in typical tasks many different payloads are encountered and it is unrealistic to assume that the properties of all payloads are accurately known [117]. Also, the transient performance of these methods is not clearly linked to the control gains. Particularly, if the desired error dynamics is to be injected into the control, it should be expected for it to be modified due to the bounded nature of the actuator.

While in [118] global asymptotic stability for the tracking control of robot manipulators can be achieved in the presence of sufficiently large viscous friction by additionally including a feed-forward compensation term of the viscous friction. Additionally, in a frictionless setting, the control scheme of [119] is proven to only semi-globally stabilize the closed-loop system. Semiglobal stability for robot manipulator regulation problem is presented through a saturated linear PID control in [120]. In particular, their stability analysis showed that the semiglobal stability is due to the Coriolis term. Hence, it is concluded in [120] that the need for nonlinear integral function is justified to dominate the effects of Coriolis term at high velocities, for instance see [121], [122], [123], [124]. In the work of [125], theoretical justification is provided on the exponential stability for regulation problem of classical PID used in industrial robots in the presence of saturation effects, essentially the controller comprise nonlinear integral term as well.

Of interest is the new approach for integral action within a continuous sliding mode control design framework in [126]. The integrator presented is modified to provide integral action only inside the boundary layer. In particular, the anti-reset windup structure of conditional integrator is explored in [127]. In essence, the conditional integrator has an inherent anti-reset windup built-in. The conditional integrator bears resemblance to the above mentioned nonlinear integral in which it contains an integration of a saturation function.

Thus, by accounting the presence of viscous friction together with a nonlinear integral term, yet another question emerges as to whether a bounded control law for global tracking can be developed that answers all the previously mentioned questions and the modifications that is required, if necessary. Additionally, when no saturation occurs, it is expected for the system to behave similarly to its unbounded-control counterpart.

Chapter 6: Robot manipulator control - output feedback bounded control approach: Bounded controller without velocity measurements for robot manipulator have been studied in the literature. For example, for global regulation of robots using position measurements only are achieved in [128] [129]. However, both requires the desired gravity compensation term. While for global stabilization, in [130] a bounded output-feedback PID-type controller of robot manipulators is proposed. However, for the velocity observation, they share a structure similar to that of the "dirty-derivative".

For Luenberger-like observer, a saturated output feedback based PID control is proposed in [131]. The resulting controller is simple to implement. It is robust to parameter uncertainties, decentralized, and saturated. However, it only achieves semiglobal stability but for sufficiently high gains, the controller, locally, can achieve exponential stability for regulation problem. On the other hand, global regulation is achieved in [132] through adaptive control that is output feedback-based in a bounded control approach. As per the literature review of Chapter 4, question arises as to whether the combination of super-twisting algorithm of Chapter 2 with "dirty-derivative" can complement each other, and if yes, can a strict Lyapunov function be developed that characterize its features. Moreover, if the answer to the question positive, it is of interest to know as to whether it can be extended to a bounded control approach while maintaining its original features.

1.3 Problem Statement

With the above motivations, the problems investigated in this dissertation are now presented.

1. The twisting based dynamics (1 - 1) is to be explored not only for $\alpha \in [0, 1)$ as per the literature, but for all $\alpha \geq 0$. In particular, strict Lyapunov is to be constructed that can allows the full set of stabilizing constant gains.
2. Due to the different convergence properties of the above mentioned system when, $0 \leq \alpha < 1$ and that of $\alpha > 0$, a non homogeneous algorithms that comprise of linear sum of the two different system is to be studied, as to ascertain whether this new combination exhibits the properties of their individual components by means of strict Lyapunov function.
3. The super-twisting based homogeneous algorithm is revisited to study the possibility of having a single structure of locally-Lipschitz Lyapunov function that can fully characterize the various convergence properties. The locally-Lipschitz is important as to avoid singularity in its time derivative (as mentioned in Section 1.1).
4. Similar to point 2, the problem of a linear combination of two different degree of homogeneity of the super-twisting based algorithms is investigated. In particular, the necessity of an additional term found in the work of [26] is to be explored and the possibility of construction of a single strict Lyapunov function that fully characterize the system while at the same time avoiding singularity.
5. The similarities of the mechanical energy of the super-twisting and twisting based

algorithms are to be examined. In particular, the idea of combining both algorithms is to be explored through strict Lyapunov functions.

6. The types of disturbances that the systems can tolerate for each system (point 1-5) are to be considered. The possibility of robustness towards non-Lipschitz and discontinuous disturbances are looked into through the generalized Lyapunov framework and the construction of strict Lyapunov functions.
7. The control problem of the highly coupled nonlinear dynamics of robot manipulator, which is a second-order system, is to be dealt with by investigating the idea of injecting the second order sliding mode algorithms, mentioned previously, into the systems and how it affects the type of convergence attainable.
8. The application of the super-twisting and twisting based algorithms on the issue of lack of velocity measurements of the robot manipulator is to be investigated as well.
9. Another practical problem of the control of robot manipulator, the actuation limits of the control input to the system is looked into for the possibility of application of the said second order sliding mode based algorithms.
10. Finally, the idea of combining the constraints point 8 and 9 into the trajectory tracking control of robot manipulator is explored.

1.4 Contributions

The contributions of the main chapters of the dissertation are discussed as follows:

Chapter 2: Lyapunov Approach on Twisting and Super-twisting based Second Order Sliding Mode: Strict Lyapunov functions are developed for twisting and super-twisting based family of algorithms by using the generalized Lyapunov theorem for non-smooth systems and using the Filippov solutions. Due to the strictness of the proposed Lyapunov functions, whose time derivative can be bounded by negative definite functions, settling time for the finite time convergence member of the algorithms can be obtained. In addition, the strict Lyapunov functions are employed throughout the chapter to study the type of disturbances that the algorithms can tolerate, which include the non-Lipschitz type as well (Preliminary conference version of the some results here can be found in [133] and [134]) .

Chapter 3: Robot Manipulator Control: Full State Feedback Approach: The main contribution of Chapter 3 is the development of control law that generalised the well-known PID control, which comprises a desired error dynamics injection and uncertainty and disturbance compensation. Nonsmooth analysis methods introduced in Chapter 2 are used throughout the stability analysis. In particular the twisting based algorithms developed in Chapter 2 are to be employed as the desired error dynamics in the control law. The technical challenge presented by this aim, is the need to avoid differentiation of the non-Lipschitz twisting based algorithms, which will lead to singularity. To achieve this objective, an auxiliary desired error dynamics variable is introduced, which in effect avoid the singularity problem by having the non-Lipschitz desired error dynamics in to an integral term. Through this formulation, semiglobal practical trajectory tracking is achieved, where the region of attractions is directly dependent on a single gain parameter.

Chapter 4: Robot Manipulator Control: Output Feedback Approach: An observer based on the super-twisting based algorithms of Chapter 2 is utilised to tackle the problem of lack of velocity measurements. In particular, due to the non-Lipschitz nature, properties like finite time convergence of observation errors can be obtained rendering the controller development to be akin to that of full-state feedback in Chapter 3. However, the non-Lipschitz gains of the observer can be high even for large initial conditions of the closed-loop system. To overcome this issue, the proposed observer contains an additional linear damping term that can aids in reducing the said non-Lipschitz. Not only that, to increase the size of the region of attraction only the linear damping term of the observer gain has to be increased without requiring that of the non-Lipschitz gains.

Chapter 5: Robot Manipulator Control: Bounded Control Approach with Full-State Feedback: The controller in Chapter 3 is redesigned to accommodate the issue of saturation of actuators. The main problem that hinders global stability of robot manipulator is the quadratic nature of the Coriolis and centrifugal terms in the dynamics. To tackle this issue, the stability analysis takes into account of the inherent viscous friction of the robot manipulator, which has an additional damping effect on the system. In addition, the desired trajectory and the desired error dynamics that is to be injected to the system through the control law has to be modified accordingly to account for the bounded nature of the control. At the same time, it is desired for the control to have the same behaviours as their unbounded counterpart in Chapter 3 when the control is unsaturated. All of these are achieved through the special form of the integral term that injects saturated version of the twisting based desired error dynamics, and its inherent anti-windup nature that conditionally injects the desired error dynamics depending on the saturation condition of the control.

Chapter 6: Robot Manipulator Control: Output Feedback Bounded Control Approach: Based on the previous saturated controller utilized in Chapter 5 and the observer of Chapter 4, an output feedback version of saturated controller in Chapter 5 is developed. Due to the inherent boundedness of states of the closed-loop system (by taking into accounts of the issues mentioned in the previous paragraph), the observer applied here can achieve global stability results independent of initial conditions. Once the observation error is stabilized, it essentially acts as additional disturbances to the closed-loop system that affects the size of the ultimate bound of the states. Due to this nature, the proposed observer-controller structure is able to assure global practical stability for trajectory tracking, and if the actuator limits is sufficiently high, the desired error dynamics can be unsaturated, and similar behaviours observed in the unbounded output feedback control in Chapter 4 can acquired.

Chapter 2: LYAPUNOV APPROACH ON TWISTING AND SUPER-TWISTING BASED SECOND ORDER SLIDING MODE

In this chapter, several families of controllers based on twisting and super-twisting algorithms are presented. Strict Lyapunov functions, whose time derivative are negative definite, are developed for each family of controllers, with or without perturbations. The Lyapunov functions presented will fully characterize the type of stability and robustness properties of each algorithm. For the case of finite time convergence, the estimation of the finite settling time is provided by means of the Lyapunov function.

2.1 Preliminaries

In this section, several technical lemmas, some important definitions, and theorems for nonsmooth analysis [135], which will be employed throughout the thesis, are presented.

Lemma 2.1: (Young's inequality [17], and [136]) For every real numbers $a > 0$, $b > 0$, $c > 0$, $p > 1$, $q > 1$, with $\frac{1}{p}+\frac{1}{q}=1$ the following inequality is satisfied

$$ab = (ca)\left(\frac{b}{c}\right) \le c^{p}\,\frac{a^{p}}{p} + c^{-q}\,\frac{b^{q}}{q}$$

Lemma 2.2: If $a_i \ge 0$, for $i = 1, \ldots, n$, while $0 < c \le 1$, and $d \ge 1$, the following inequalities hold:

$$\sum_{i=1}^{n} a_i^{\,d} \le \left(\sum_{i=1}^{n} a_i\right)^d \le n^{d-1} \sum_{i=1}^{n} a_i^{\,d}, \qquad (2 - 1)$$

and

$$\frac{1}{n^{1-c}} \sum_{i=1}^{n} a_i^{\,c} \le \left(\sum_{i=1}^{n} a_i\right)^c \le \sum_{i=1}^{n} a_i^{\,c} \qquad (2 - 2)$$

Proof of lemma 2.2: For (2 - 1) note that when $d = 1$ or $\sum_{i=1}^{n} a_i = 0$, the results follow directly.

Now, consider the case when $d > 1$ and $\sum_{i=1}^{n} a_i \neq 0$. Let $A = \sum_{i=1}^{n} a_i$ and note that

$\frac{a_i}{A} \le 1, \quad \forall i = 1, \ldots, n,$ hence, $\sum_{i=1}^{n} \left(\frac{a_i}{A}\right)^d \le \sum_{i=1}^{n} \left(\frac{a_i}{A}\right) = 1, \quad$ since $d > 1,$

Consider the following,

$$\sum_{i=1}^{n} a_i^{\,d} = \sum_{i=1}^{n} \left(A \times \frac{a_i}{A}\right)^d = A^d \sum_{i=1}^{n} \left(\frac{a_i}{A}\right)^d \le A^d .1$$

$$\therefore \sum_{i=1}^{n} a_i^{\,d} \le \left(\sum_{i=1}^{n} a_i\right)^d$$

Next, let $B = \left(\sum_{i=1}^{n} a_i^{\,d}\right)^{\frac{1}{d}}$, and $C = n^{1-\frac{1}{d}}$, then,

$$\sum_{i=1}^{n} a_i = BC \cdot \sum_{i=1}^{n} \left(\frac{a_i}{B}\right)\left(\frac{1}{C}\right) \le BC \cdot \sum_{i=1}^{n} \left(\frac{1}{d}\left(\frac{a_i}{B}\right)^d + \frac{d-1}{d}\left(\frac{1}{C}\right)^{\frac{d}{d-1}} \right), \text{ from lemma 2.1, since } d > 1,$$

$$= BC \cdot \left(\frac{1}{d} \frac{\sum_{i=1}^{n} \left(a_i^{\ d}\right)}{B^d} + \frac{d-1}{d} \frac{\sum_{i=1}^{n} (1)}{C^{\frac{d}{d-1}}} \right)$$

$$= BC \cdot 1$$

$$\therefore \sum_{i=1}^{n} a_i \le n^{1-\frac{1}{d}} \cdot \left(\sum_{i=1}^{n} a_i^{\ d}\right)^{\frac{1}{d}} \Rightarrow \left(\sum_{i=1}^{n} a_i\right)^d \le n^{d-1} \cdot \sum_{i=1}^{n} a_i^{\ d}$$

Thus, we obtained (2 - 1).

Now, for (2 - 2), note that when $c = 1$, the results follow directly as well. So, consider the case when, $0 < c < 1$.

Now, consider,

$$\left(\sum_{i=1}^{n} a_i\right)^c = \left(\sum_{i=1}^{n} \left(a_i^{\ c}\right)^{\frac{1}{c}}\right)^c \le \left(\left(\sum_{i=1}^{n} a_i^{\ c}\right)^{\frac{1}{c}}\right)^c \quad \text{from the left hand side of (2 - 1), since } \frac{1}{c} > 1,$$

$$= \sum_{i=1}^{n} a_i^{\ c}$$

$$\therefore \left(\sum_{i=1}^{n} a_i\right)^c \le \sum_{i=1}^{n} a_i^{\ c}$$

Similarly,

$$\left(\sum_{i=1}^{n} a_i\right)^c = \left(\sum_{i=1}^{n}\left(a_i^{\,c}\right)^{\frac{1}{c}}\right)^c \geq \left(\frac{1}{n^{\frac{1}{c}-1}}\left(\sum_{i=1}^{n} a_i^{\,c}\right)^{\frac{1}{c}}\right)^c \quad \text{from the right hand side of (2 - 1), since } \frac{1}{c} > 1,$$

$$= \frac{1}{n^{1-c}}\sum_{i=1}^{n} a_i^{\,c},$$

$$\therefore \left(\sum_{i=1}^{n} a_i\right)^c \geq \frac{1}{n^{1-c}}\sum_{i=1}^{n} a_i^{\,c}$$

Thus, we obtained (2 - 2). ■

Lemma 2.3: Let $0 < p \leq r \leq q$ with $a \geq 0$, then the inequality hold, $|a|^p + |a|^q \geq |a|^r$.

Proof of lemma 2.3: For $0 \leq |a| \leq 1$, note that, $|a|^p \geq |a|^r$, since $p \leq r$, and $\max\{|a|^p, |a|^q\} = |a|^p$, since $p \leq q$, hence we have $\max\{|a|^p, |a|^q\} \geq |a|^r$. While for $|a| \geq 1$, note that, $|a|^q \geq |a|^r$, since $q \geq r$, and $\max\{|a|^p, |a|^q\} = |a|^q$, since $q \geq p$, hence, $\max\{|a|^p, |a|^q\} \geq |a|^r$. Thus, for all $|a| \geq 0$, we have $\max\{|a|^p, |a|^q\} \geq |a|^r$. Since $|a|^p + |a|^q \geq \max\{|a|^p, |a|^q\}$, hence for all $|a| \geq 0$, one obtains $|a|^p + |a|^q \geq |a|^r$. ■

Consider the vector differential equation,

$$\dot{x} = f(x,t), \ x(t_0) = x_0 \tag{2 - 3}$$

where $x \in \mathbb{R}^n$ is a state vector, and f: $\mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n$ is defined by almost all points on an open subset $Q \subset R^{n+1}$ and is measurable. For an arbitrary compact set $D \subset Q$, a Lebesgue integrable function $A(t)$ exists and satisfies the following:

$$\|f(x,t)\| \leq A(t), \text{ a.e. in } D.$$

Under the above conditions, the existence of a Filippov solution is guaranteed.

Remark 2.1. Throughout the subsequent discussion, for brevity of notation, let a.e. refer to almost everywhere, i.e., for almost all $t \in [0, \infty)$.

Definition 2.1: [70]When a vector function $x(t)$ meets the following conditions, the solution to (1) in the interval $[t_0, t_1]$ in Filippov's sense is called is called a Filippov solution.

a) $x(t)$ is a absolutely continuous on $[t_0, t_1]$.

b) For almost every $t \in [t_0, t_1]$, the following differential inclusion is satisfied:

$$\dot{x} \in K[f](x,t) \tag{2 - 4}$$

Here,

$$K[f](x,t) = \bigcap_{\delta > 0} \bigcap_{\mu N = 0} \overline{co}\, f(B(x,\delta) - N, t) \tag{2 - 5}$$

In addition, $\bigcap_{\mu N = 0}$ is the intersection over all sets N of Lebesgue measure zero , $\overline{\text{co}}$ is the convex hull, and $B(x,\delta) = \{y \in \mathbb{R}^n \mid \| y - x\| < \delta\}$ is an open sphere.

Definition 2.2: [72]A function f : $\mathbb{R}^n \to \mathbb{R}$**,** which is locally Lipschitz near $x \in \mathbb{R}^n$, is said to be regular at x if the following holds. For all directions $v \in \mathbb{R}^n$, there exists the usual one-sided directional derivative

$f'(x, v) = \lim_{\rho \downarrow 0} \dfrac{f(x + \rho v) - f(x)}{\rho}$ and we have $f\,'(x, v) = f^{\,0}(x, v)$, where

$$f^0(x, v) = \limsup_{\substack{y \to x \\ \rho \downarrow 0}} \frac{f(y + \rho v) - f(y)}{\rho}$$

is the generalized directional derivative of f at x in the direction v. The function is said to be regular in $\mathbb{R}^n$, if it is regular for any $x \in \mathbb{R}^n$.

Remark 2.2. A useful property is that a locally Lipschitz and convex function in $\mathbb{R}^n$ is also regular in $\mathbb{R}^n$ (see [137], Proposition 2.3.6). A feature of Filippov's solution is that it is defined by the condition (2 - 4) for $\dot{x}$. This approach leads to generalization of the Lyapunov stability theory so that the solution $x(t)$ of the differential equation is not needed explicitly.

Lemma 2.4*:* (Chain rule [70]) Let the vector function $x(t)$ be Filippov's solution to (2 - 3) on an interval containing t and V: $\mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be a Lipschitz-continuous regular function. In this case, $V(x(t), t)$ is absolutely continuous, $(d/dt)V(x(t), t)$ exists for almost everywhere (a.e.), i.e., for almost every t, and satisfies

$$\frac{d}{dt}V(x(t),t) \overset{a.e.}{\in} \dot{\tilde{V}}(x(t),t) = \bigcap_{\xi \in \partial V(x(t),t)} \xi^T \begin{bmatrix} K[f](x(t),t) \\ 1 \end{bmatrix},$$

Here $\dot{\tilde{V}}(x(t), t)$ is the generalized time derivative of $V(x(t), t)$, while $\partial V(x(t), t)$ is the Clarke's generalized gradient [138], defined as follows:

$$\partial V(x,t) = \overline{co}\left\{ \lim_{i \to \infty} \nabla V(x_i, t_i) \mid (x_i, t_i) \to (x,t), (x_i, t_i) \notin \Omega_V \right\}$$

where Ω_V is the set of measure zero where the gradient of V is not defined.

Lemma 2.5: (Lyapunov's Theorem Generalized [73], [70]). Suppose that V: $\mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is a Lipschitz-continuous regular function satisfying $V(0, t) = 0$, and

$$0 < V_1(\|x\|) \leq V(x,t) \leq V_2(\|x\|), \quad \text{for} \quad x \neq 0$$

for some V_1, $V_2 \in$ class $\mathcal{K}$. Then,

1) $\dot{\tilde{V}}$ $(x, t) \leq 0$ in implies $x(t) \equiv 0$ is a uniformly stable solution.

2) If in addition, there exists a class $\mathcal{K}$ function $\omega(.)$ that satisfies $\dot{\tilde{V}} \leq -\omega(x) < 0$,

   then the solution $x(t) \equiv 0$ is uniformly asymptotically stable.

3) Furthermore, if the function $\omega(x) = cV^a$, where $c > 0$ and $a > 0$, then we have

   a) Finite time convergence [72], for $0 < a < 1$, with the settling time estimate,

$$T(x_0) \le \frac{[V(x_0, t_0)]^{1-a}}{c(1-a)}$$, where x_0 is the initial states at $t = t_0$.

b) Exponential convergence [72], for $a = 1$.

c) Asymptotical convergence, for $a > 1$, with convergence time to bounded level set $V = \mu$, for any $\mu > 0$, uniformly upper bounded with respect to the initial condition [17],

$$T_{\max}(\mu) = \left(\frac{1}{c(a-1)}\right)\left(\frac{1}{\mu^{a-1}}\right)$$

Lemma 2.6: (Uniformly ultimate boundedness [139], Theorem 3.3 of [140]). Assume that there exists $r > 0$, $\sigma > 0$, and V: $\mathcal{D} \times R \to R$ is a globally Lipschitz continuous function such that for any initial condition, $\forall\ x(t_0) = x_0 \in \mathcal{D}$ be a domain that contains the origin and $\|x_0\| \le \sigma$, any Filippov solution of (2 - 3) $x(t) \in S(x_0)$ satisfies:

1) There exist two functions α_1 and α_2 of the class $\mathcal{K}_\infty$ such that

$$0 < \alpha_1(\|x\|) \le V(x,t) \le \alpha_2(\|x\|),$$

2) $\exists\ 0 < \mu < \alpha_2^{-1}(\alpha_1(r))$ while $\|x\| \ge \mu$, there exists a function α_3 of the class $\mathcal{K}$ such that $\dot{\tilde{V}} \le -\alpha_3(\|x\|)$

Then, the origin of the discontinuous system (2 - 3) is globally strongly uniformly ultimately bounded. In particular, there exist a finite $T(r, \sigma)$ such that $\forall\ t \ge t_0 + T(r, \sigma)$, all the Filippov solutions $x(t) \in S(x_0)$ of the system (2 - 3) with the initial condition x_0 satisfy $V(x(t), t) \leq \alpha_2(\|\mu\|)$, and by point (1) there holds $\|x(t)\| \leq \alpha_1^{-1}(\alpha_2(\mu)) < r$, $\forall t \geq t_0 + T(r, \sigma)$.

For the system described in (2 - 3) with a continuous right-hand side, existing Lyapunov theory can be used to examine the stability of the closed-loop system using continuous techniques such as those described in [71]. However, these theorems must be altered for the set-valued map $\dot{\tilde{V}}(x(t), t)$ for systems with right-hand sides which are not Lipschitz continuous [73], [70]. Lyapunov analysis for nonsmooth systems is analogous to the analysis used for continuous systems. The differences are that differential equations are replaced with inclusions, gradients are replaced with generalized gradients, and points are replaced with sets in several places.

In the following subsections, locally Lipschitz strict Lyapunov functions will be developed for second order sliding mode that are based on twisting and super-twisting algorithms.

2.2 Twisting Algorithm

In this section, a twisting based family of algorithms is developed. Particularly, the family of algorithms generalised the twisting and linear proportional-derivative (PD) algorithms. Different types of convergence (i.e. finite-time, exponential, and uniform) and the disturbances (i.e. which include non-Lipschitz type) that the algorithms can tolerate are presented by employing strict Lyapunov function throughout the stability analyses, in which its time derivative can be bounded by negative definite function. The inherent homogeneity properties of the algorithms and that of the proposed strict Lyapunov function are utilised, together with lemmas shown in the preliminaries (section 2.1), to obtained the results that characterize the algorithm.

2.2.1 System description

Consider the twisting based family of algorithms:

$$\begin{aligned} \dot{z}_1 &= z_2, \\ \dot{z}_2 &= -k_1|z_1|^b \operatorname{sign}(z_1) - k_2|z_2|^{\frac{2b}{1+b}} \operatorname{sign}(z_2) + d, \end{aligned} \tag{2 - 6}$$

where z_1, z_2 $\in$ $\mathbb{R}$, are the scalar state variables, k_1, k_2 are positive constants, $b \geq 0$ real number, and d is time-varying and/or nonlinear term of uncertainty bounded by

$$|d| \leq M_1|z_1|^b + M_2|z_2|^{\frac{2b}{1+b}} + M_3 \text{ with } |d| := \sup\{|\delta| : \delta \in \boldsymbol{K}[d]\},$$

where $M_1 \geq 0$, $M_2 \geq 0$, and $M_3 \geq 0$, with the same $b \geq 0$ as that in (2 - 6). Note that since no continuity assumption is made on d, it may contain discontinuities and hence we define its upper bound through Filippov set-valued map. Note that for the case of $b = 0$, the algorithm contains discontinuity and the uncertainty is upper bounded by nonvanishing constant.

2.2.2 Stability analysis

For system (2 - 6), the following function

$$V(z_1, z_2) = \left(\frac{1}{2}z_2{}^2 + \frac{k_1}{1+b}|z_1|^{1+b}\right)^2 + r|z_1|^{\frac{3+3b}{2}}|z_2| \operatorname{sign}(z_1 z_2)$$

where r is a positive constant scalar, will be shown as a strict Lyapunov function.

Remark 2.3. Note that the above function is locally Lipschitz and regular. It is differentiable everywhere for $b > 0$, and not differentiable on $z_1 = 0$ for $b = 0$. The proposed Lyapunov function comprises the energy function, which is known as a weak Lyapunov function, and an additional cross-term that consists of two states. It is worth mentioning that in studying the stability of system (2 - 6), only a single structure of Lyapunov function is employed in the following analysis for both unperturbed and perturbed cases. However, in [37] and [38], a separate Lyapunov function is required to study the perturbed system. In particular, their Lyapunov function comprise the upper bound of the disturbance, in which case making it unclear on extending the analysis to the cases of perturbations that might includes additive non-Lipschitz vanishing perturbations and non-vanishing perturbations that are considered here.

Note that the Lyapunov function V can be lower bounded by,

$$V(z_1, z_2) \geq \frac{1}{4}|z_2|^4 + \left(\frac{k_1}{1+b}\right)^2 |z_1|^{2+2b} + \left(\frac{k_1}{1+b}\right)|z_1|^{1+b}|z_2|^2 - r|z_1|^{\frac{3+3b}{2}}|z_2|$$

Using lemma 2.1

$$\frac{1}{8}|z_2|^4 + \frac{1}{2}\left(\frac{k_1}{1+b}\right)^2 |z_1|^{2+2b} \geq \left(\frac{4}{6}\left(\frac{k_1}{1+b}\right)^2\right)^{\frac{3}{4}} |z_1|^{\frac{3+3b}{2}} \left(\frac{4}{8}\right)^{\frac{1}{4}} |z_2|$$

Hence, for

$$\left(\frac{k_1}{1+b}\right)^{\frac{3}{2}}\left(\frac{2}{3}\right)^{\frac{3}{4}}\left(\frac{1}{2}\right)^{\frac{1}{4}} > r \qquad (2 - 7)$$

$$\therefore V(z_1, z_2) \geq \frac{1}{8}|z_2|^4 + \frac{1}{2}\left(\frac{k_1}{1+b}\right)^2 |z_1|^{2+2b} \geq \underline{\pi}_1\left(|z_2|^4 + |z_1|^{2+2b}\right)$$

where $\underline{\pi}_1 := \min\left\{\frac{1}{8}, \frac{1}{2}\left(\frac{k_1}{1+b}\right)^2\right\}$.

Similarly, it can be upper-bounded by,

$$V(z_1, z_2) \leq \frac{1}{4}|z_2|^4 + \left(\frac{k_1}{1+b}\right)^2 |z_1|^{2+2b} + \left(\frac{k_1}{1+b}\right)|z_1|^{1+b}|z_2|^2 + r|z_1|^{\frac{3+3b}{2}}|z_2|$$

Using lemma 2.1:

$$\left(\frac{k_1}{1+b}\right)|z_1|^{1+b}|z_2|^2 \leq \left(\frac{k_1}{2+2b}\right)|z_1|^{2+2b} + \left(\frac{k_1}{2+2b}\right)|z_2|^4$$

$$r|z_1|^{\frac{3+3b}{2}}|z_2| \leq \frac{3r}{4}|z_1|^{2+2b} + \frac{r}{4}|z_2|^4$$

Thus,

$$V(z_1, z_2) \leq \overline{\pi}_1\left(|z_1|^{2+2b} + |z_2|^4\right)$$

where $\overline{\pi}_1 := \max\left\{\frac{3r}{4} + \left(\frac{k_1}{1+b}\right)^2 + \left(\frac{k_1}{2+2b}\right), \quad \left(\frac{k_1}{2+2b}\right) + \frac{r}{4} + \frac{1}{4}\right\}$

Thus, V is positive definite and radially unbounded. Since (2 - 6) is a differential equation that has discontinuous right-hand side, i.e. when b = 0 and since no continuity assumption is made on d, its solutions are understood in the sense of Filippov (see definition 2.1). According to lemma 2.4, the time derivative of non-smooth Lyapunov function exists almost everywhere

$$\frac{d}{dt}V(\mathbf{z}(t),t) \overset{a.e.}{\in} \dot{\tilde{V}}(\mathbf{z}(t),t) = \bigcap_{\xi\in\partial V(\mathbf{z}(t),t)} \xi^T K\begin{bmatrix}\dot{z}_1\\ \dot{z}_2\end{bmatrix}(\mathbf{z},t),$$

where $\mathbf{z}=(z_1,z_2)^T$ and, $\xi=(\xi_1,\ \xi_2)^{\mathrm{T}} \in \partial V(\mathbf{z},t)$.

Since the controller (2 - 6) is discontinuous when $b = 0$, for ease of presentation, the analysis is separated for two different cases of $b \geq 0$, i.e. $b > 0$ and $b = 0$.

a) Case 1: For $b > 0$

Note that for $b > 0$, V is continuously differentiable, hence

$$\dot{\tilde{V}}(z_1,z_2) = \bigcap_{\xi\in\partial V(\mathbf{z}(t),t)} \xi^T K\begin{bmatrix}\dot{z}_1\\ \dot{z}_2\end{bmatrix} = \nabla V^T K\begin{bmatrix}\dot{z}_1\\ \dot{z}_2\end{bmatrix}$$

$$\begin{aligned}\subset & -\frac{2k_1k_2}{(1+b)}|z_1|^{1+b}|z_2|^{\frac{1+3b}{1+b}} - k_1 r|z_1|^{\frac{3+5b}{2}} - k_2 r|z_1|^{\frac{3+3b}{2}}|z_2|^{\frac{2b}{1+b}}\mathrm{sign}(z_1z_2) - k_2|z_2|^{\frac{3+5b}{1+b}}\\ & + \frac{3r(1+b)}{2}|z_1|^{\frac{1+3b}{2}}|z_2|^2\\ & + K[d]\left(|z_2|^3\mathrm{sign}(z_2) + r|z_1|^{\frac{3+3b}{2}}\mathrm{sign}(z_1) + \frac{2k_1}{(1+b)}|z_1|^{1+b}|z_2|\mathrm{sign}(z_2)\right)\end{aligned}$$

After rearrangement:

$$\begin{aligned}\dot{\tilde{V}}(z_1,z_2) \leq & -\frac{2k_1(k_2-M_2)}{(1+b)}|z_1|^{1+b}|z_2|^{\frac{1+3b}{1+b}} - \frac{r(k_1-M_1)}{2}|z_1|^{\frac{3+5b}{2}} - \frac{(k_2-M_2)}{2}|z_2|^{\frac{3+5b}{1+b}}\\ & + \dot{V}_1 + M_3\left(|z_2|^3 + r|z_1|^{\frac{3+3b}{2}} + \frac{2k_1}{(1+b)}|z_1|^{1+b}|z_2|\right)\end{aligned}$$

where

$$\dot{V}_1 = -\frac{r(k_1 - M_1)}{2}|z_1|^{\frac{3+5b}{2}} - \frac{(k_2 - M_2)}{2}|z_2|^{\frac{3+5b}{1+b}} + r(k_2 + M_2)|z_1|^{\frac{3+3b}{2}}|z_2|^{\frac{2b}{1+b}}$$
$$+ r\frac{3+3b}{2}|z_1|^{\frac{1+3b}{2}}|z_2|^2 + M_1|z_1|^b|z_2|^3 + \frac{2k_1}{(1+b)}M_1|z_1|^{1+2b}|z_2|$$

Applying lemma 2.1,

$$-|z_1|^{\frac{3+5b}{2}} - |z_2|^{\frac{3+5b}{1+b}} \le -\left(\frac{3+5b}{3+3b}\right)^{\frac{3+3b}{3+5b}}|z_1|^{\frac{3+3b}{2}}\left(\frac{3+5b}{2b}\right)^{\frac{2b}{3+5b}}|z_2|^{\frac{2b}{1+b}},$$

$$-|z_1|^{\frac{3+5b}{2}} - |z_2|^{\frac{3+5b}{1+b}} \le -\left(\frac{3+5b}{1+3b}\right)^{\frac{1+3b}{3+5b}}|z_1|^{\frac{1+3b}{2}}\left(\frac{3+5b}{2+2b}\right)^{\frac{2+2b}{3+5b}}|z_2|^2,$$

$$-|z_1|^{\frac{3+5b}{2}} - |z_2|^{\frac{3+5b}{1+b}} \le -\left(\frac{3+5b}{2b}\right)^{\frac{2b}{3+5b}}|z_1|^b\left(\frac{3+5b}{3+3b}\right)^{\frac{3+3b}{3+5b}}|z_2|^3,$$

$$-|z_1|^{\frac{3+5b}{2}} - |z_2|^{\frac{3+5b}{1+b}} \le -\left(\frac{3+5b}{2+4b}\right)^{\frac{2+4b}{3+5b}}|z_1|^{1+2b}\left(\frac{3+5b}{1+b}\right)^{\frac{1+b}{3+5b}}|z_2|,$$

Thus, if the following inequalities

$$\min\left\{\begin{array}{l}\lambda_1\dfrac{(k_1 - M_1)^{\frac{3+3b}{2b}}(k_2 - M_2)}{(k_2 + M_2)^{\frac{3+5b}{2b}}}, \\ \lambda_2(k_1 - M_1)^{\frac{1+3b}{2+2b}}(k_2 - M_2)\end{array}\right\} > r > \max\left\{\begin{array}{l}\lambda_3\dfrac{M_1^{\frac{3+5b}{2b}}}{(k_1 - M_1)(k_2 - M_2)^{\frac{3+3b}{2b}}}, \\ \lambda_4\dfrac{k_1^{\frac{3+5b}{2+4b}}M_1^{\frac{3+5b}{2+4b}}}{(k_1 - M_1)(k_2 - M_2)^{\frac{1+b}{2+4b}}}\end{array}\right\} \qquad (2\text{ - }8)$$

where

$$\lambda_1 = \left(\frac{3+5b}{24+24b}\right)^{\frac{3+3b}{2b}}\left(\frac{3+5b}{16b}\right), \quad \lambda_2 = \left(\frac{3+5b}{8+24b}\right)^{\frac{1+3b}{2+2b}}\left(\frac{3+5b}{16+16b}\right)\left(\frac{2}{3+3b}\right)^{\frac{3+5b}{2+2b}},$$

$$\lambda_3 = \left(\frac{16b}{3+5b}\right)\left(\frac{24+24b}{3+5b}\right)^{\frac{3+3b}{2b}}, \quad \lambda_4 = \left(\frac{16+32b}{3+5b}\right)\left(\frac{8+8b}{3+5b}\right)^{\frac{1+b}{2+4b}}\left(\frac{2}{1+b}\right)^{\frac{3+5b}{2+4b}}$$

hold then the function $\dot{V}_1$ is negative definite. Then,

$$\begin{aligned}\dot{\tilde{V}}(z_1,z_2) \le &-\frac{r(k_1-M_1)}{2}|z_1|^{\frac{3+5b}{2}} - \frac{(k_2-M_2)}{2}|z_2|^{\frac{3+5b}{1+b}} \\ &+ M_3\left(|z_2|^3 + r|z_1|^{\frac{3+3b}{2}} + \frac{2k_1}{(1+b)}|z_1|^{1+b}|z_2|\right)\end{aligned}$$

Applying lemma 2.1,

$$|z_1|^{1+b}|z_2| \le \frac{2}{3}|z_1|^{\frac{3+3b}{2}} + \frac{1}{3}|z_2|^3$$

together with lemma 2.2, and the bounds of the Lyapunov function, we have:

$$\begin{aligned}\dot{\tilde{V}}(z_1,z_2) &\le -\omega_1\left(|z_1|^{\frac{3+5b}{2}} + |z_2|^{\frac{3+5b}{1+b}}\right) + M_3\omega_2\left(|z_1|^{\frac{3+3b}{2}} + |z_2|^3\right) \\ &= -\omega_1\left(\left(|z_1|^{2+2b}\right)^{\frac{3+5b}{4+4b}} + \left(|z_2|^4\right)^{\frac{3+5b}{4+4b}}\right) + M_3\omega_2\left(\left(|z_1|^{2+2b}\right)^{\frac{3}{4}} + \left(|z_2|^4\right)^{\frac{3}{4}}\right) \\ &\le -\left(\frac{\omega_3}{\overline{\pi}_1^{\frac{3+5b}{4+4b}}}\right)V^{\frac{3+5b}{4+4b}} + M_3\left(\frac{2^{\frac{1}{4}}\omega_2}{\underline{\pi}_1^{\frac{3}{4}}}\right)V^{\frac{3}{4}}\end{aligned} \tag{2 - 9}$$

where

$$\omega_1 = \min\left\{\frac{r(k_1 - M_1)}{2}, \frac{(k_2 - M_2)}{2}\right\}, \quad \omega_2 = \max\left\{\left(r + \frac{4k_1}{3+3b}\right), \left(\frac{2k_1}{3+3b} + 1\right)\right\},$$

$$\omega_3 = \begin{cases} \omega_1 & \text{for } 0 < b \le 1, \\ \dfrac{\omega_1}{2^{\frac{b-1}{4+4b}}} & \text{for } b > 1 \end{cases}$$

Remark 2.4. Note that the nonlinear inequalities (2 - 7) and (2 - 8) are feasible for sufficiently large $k_1 > 0$, and $k_2 > 0$. Thus, an $r > 0$ can always exists. Consider some particular cases:

1. Note that (2 - 8) is feasible with respect to k_1, k_2 for any $M_1 \ge 0$ and $M_2 \ge 0$. This implies that for any given upper bound on the disturbances M_1 and M_2, sufficiently large k_1 and k_2 always exist to render V positive definite and $\dot{V}_1$ negative definite.
2. If $M_1 = 0$, then (2 - 8) is satisfied for any $k_2 > M_2$. This coincides with the conditions obtained through the weak Lyapunov function [18].

Remark 2.5. Hence, when $M_3 = 0$, i.e. without persistent perturbations, from lemma 2.5-3, the system (2 - 6) will have finite time convergence for $0 < b < 1$, with the settling time estimate,

$$T(z_{10}, z_{20}) \le \left(\frac{\bar{\pi}_1^{\frac{3+5b}{4+4b}}}{\omega_3}\right)\left(\frac{4+4b}{1-b}\right)[V(z_{10}, z_{20})]^{\frac{1-b}{4+4b}},$$

where (z_{10}, z_{20}) are the initial states of the system. Similarly, exponential and asymptotical convergence for $a = 1$ and $a > 1$ respectively, can be concluded from lemma 2.5 as well. The above results are possible due to the negative definiteness of the time derivative of the

Lyapunov function, i.e. strict Lyapunov function.

Remark 2.6. When persistent perturbations occur on the system, $M_3 \neq 0$, from (2 - 9)

$$\dot{\tilde{V}}(z_1, z_2) \leq -\frac{1}{2}\left(\frac{\omega_3}{\overline{\pi}_1^{\frac{3+5b}{4+4b}}}\right)V^{\frac{3+5b}{4+4b}} - V^{\frac{3}{4}}\left(\frac{1}{2}\left(\frac{\omega_3}{\overline{\pi}_1^{\frac{3+5b}{4+4b}}}\right)V^{\frac{b}{2+2b}} - M_3\left(\frac{2^{\frac{1}{4}}\omega_2}{\underline{\pi}_1^{\frac{3}{4}}}\right)\right)$$

$$\leq -\frac{1}{2}\left(\frac{\omega_3}{\overline{\pi}_1^{\frac{3+5b}{4+4b}}}\right)V^{\frac{3+5b}{4+4b}}, \text{ for } V \geq \left(2M_3\left(\frac{\overline{\pi}_1^{\frac{3+5b}{4+4b}}}{\omega_3}\right)\left(\frac{2^{\frac{1}{4}}\omega_2}{\underline{\pi}_1^{\frac{3}{4}}}\right)\right)^{\frac{2+2b}{b}}$$

hence, from lemma 2.6, the system (2 - 6) is uniformly ultimately bounded.

b) *Case 2: For b = 0*

For $b = 0$, V is not differentiable on $z_1 = 0$

$$\dot{\tilde{V}}(z_1, z_2) = \bigcap_{\xi \in \partial V(\mathbf{z}(t),t)} \xi^T K\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} \subset \dot{\tilde{V}}_1 + \dot{\tilde{V}}_2$$

where

$$\dot{\tilde{V}}_1 = \bigcap_{\xi \in \partial V(\mathbf{z}(t),t)} \xi^T K[f](z_1, z_2),\ \dot{\tilde{V}}_2 = \bigcap_{\xi \in \partial V(\mathbf{z}(t),t)} \xi^T K\begin{bmatrix} 0 \\ d \end{bmatrix},$$

$$K[f](z_1, z_2) = \begin{cases} \left\{ \begin{pmatrix} z_2, \\ -k_1 \mathrm{sign}(z_1) - k_2 \mathrm{sign}(z_2) \end{pmatrix} \right\} \forall z_1 \neq 0, z_2 \neq 0 \\ \left\{ \begin{pmatrix} z_2, \\ -k_1 [-1,\ 1] - k_2 \mathrm{sign}(z_2) \end{pmatrix} \right\} \forall z_1 = 0, z_2 \neq 0 \\ \left\{ \begin{pmatrix} 0, \\ -k_1 \mathrm{sign}(z_1) - k_2 [-1,\ 1] \end{pmatrix} \right\} \forall z_1 \neq 0, z_2 = 0 \\ \left\{ \begin{pmatrix} 0, \\ -k_1 [-1,\ 1] - k_2 [-1,\ 1] \end{pmatrix} \right\} \forall z_1 = 0, z_2 = 0 \end{cases}$$

$$\partial V = \boldsymbol{K}[\nabla V] = \boldsymbol{K} \begin{bmatrix} \frac{\partial V}{\partial z_1} \\ \frac{\partial V}{\partial z_2} \end{bmatrix}$$

$$\subset \begin{bmatrix} \boldsymbol{K}\left[\frac{\partial V}{\partial z_1}\right] \\ \boldsymbol{K}\left[\frac{\partial V}{\partial z_2}\right] \end{bmatrix} = \begin{cases} \left\{ \begin{pmatrix} 2k_1^{\ 2} z_1 + \frac{3}{2} r |z_1|^{\frac{1}{2}} z_2 + k_1 \mathrm{sign}(z_1) |z_2|^2 \\ |z_2|^3 \mathrm{sign}(z_2) + r |z_1|^{\frac{3}{2}} \mathrm{sign}(z_1) + 2k_1 |z_1| z_2 \end{pmatrix} \right\} \forall z_1 \neq 0, z_2 \neq 0, \\ \left\{ \begin{pmatrix} [-1,\ 1] k_1 z_2^{\ 2} \\ z_2^{\ 3} \end{pmatrix} \right\} \forall z_1 = 0,\ z_2 \neq 0, \\ \left\{ \begin{pmatrix} 2k_1^{\ 2} z_1 \\ r |z_1|^{\frac{3}{2}} \mathrm{sign}(z_1) \end{pmatrix} \right\} \forall z_1 \neq 0,\ z_2 = 0, \\ \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\} \forall z_1 = 0,\ z_2 = 0, \end{cases}$$

Let us define:

$\left|\frac{\partial V}{\partial z_1}\right| := \sup\left\{ |\xi_1| : \xi_1 \in \boldsymbol{K}\left[\frac{\partial V}{\partial z_1}\right] \right\}$, and $\left|\frac{\partial V}{\partial z_2}\right| := \sup\left\{ |\xi_2| : \xi_2 \in \boldsymbol{K}\left[\frac{\partial V}{\partial z_2}\right] \right\}$, with

$$\left|\frac{\partial V}{\partial z_2}\right| \leq |z_2|^3 + r|z_1|^{\frac{3}{2}} + 2k_1 |z_1| |z_2|$$

Thus, the term

$$\dot{\tilde{V}}_2 = \bigcap_{\xi_2 \in \boldsymbol{K}\left[\frac{\partial V}{\partial z_2}\right]} \xi_2 K[d] \le \left|\frac{\partial V}{\partial z_2}\right| |d| \le M\left(|z_2|^3 + r|z_1|^{\frac{3}{2}} + 2k_1|z_1||z_2|\right)$$

where M be defined as, $M := M_1 + M_2 + M_3$, since for $b = 0$, $|d| \le M_1 + M_2 + M_3$.

Computing $\dot{\tilde{V}}_1$ for each case, we have

For $z_1 \ne 0$ and $z_2 \ne 0$:

$$\dot{\tilde{V}}_1 = -2k_1k_2|z_1||z_2| - k_1r|z_1|^{\frac{3}{2}} - k_2r|z_1|^{\frac{3}{2}}\operatorname{sign}(z_1z_2) - k_2|z_2|^3 + \frac{3}{2}r|z_1|^{\frac{1}{2}}|z_2|^2$$

For $z_1 = 0$ and $z_2 \ne 0$: Let $(\xi_2 k_1 z_2^2, z_2^3)^T$ with $\xi_2 \in [-1, 1]$ be an arbitrary element of $\partial V(\mathbf{z}, t)$, then

$$\xi^T K[f](z_1, z_2) = [\xi_2 - 1,\ \xi_2 + 1]k_1 z_2^3 - k_2|z_2|^3$$

hence

$$\dot{\tilde{V}}_1 = \bigcap_{\xi_2 \in [-1,\ 1]} [\xi_2 - 1,\ \xi_2 + 1]k_1 z_2^3 - k_2|z_2|^3 = -k_2|z_2|^3$$

For $z_1 \ne 0$ and $z_2 = 0$:

$$\dot{\tilde{V}}_1 = -k_1r|z_1|^{\frac{3}{2}} - [-1,\ 1]k_2r|z_1|^{\frac{3}{2}}\operatorname{sign}(z_1)$$

For $z_1 = 0$ and $z_2 = 0$: $\dot{\tilde{V}}_1 = 0$.

Thus, for all $(z_1, z_2) \in \mathbb{R}^2$

$$\dot{\tilde{V}}_1 = -2k_1k_2|z_1||z_2| - k_1r|z_1|^{\frac{3}{2}} - k_2r|z_1|^{\frac{3}{2}}\operatorname{sign}(z_1)\operatorname{SGN}(z_2) - k_2|z_2|^3 + \frac{3r}{2}|z_1|^{\frac{1}{2}}|z_2|^2$$

Hence, the generalized time derivative of the Lyapunov function, after rearrangement:

$$\dot{\tilde{V}} = \dot{\tilde{V}}_1 + \dot{\tilde{V}}_2 \le -2k_1(k_2 - M)|z_1||z_2| - r(k_1 - k_2 - M)|z_1|^{\frac{3}{2}} - (k_2 - M)|z_2|^3 + \frac{3r}{2}|z_1|^{\frac{1}{2}}|z_2|^2$$

Applying lemma 2.1,

$$-|z_1|^{\frac{3}{2}} - |z_2|^3 \le -3^{\frac{1}{3}}|z_1|^{\frac{1}{2}}\left(\frac{3}{2}\right)^{\frac{2}{3}}|z_2|^2$$

Thus, if the following inequality holds

$$\frac{(k_1 - k_2 - M)^{\frac{1}{2}}(k_2 - M)}{2} > r \tag{2 - 10}$$

employing lemma 2.2 and the bounds on the Lyapunov function, then

$$\dot{\tilde{V}} \le -\omega_4\left(|z_1|^{\frac{3}{2}} + |z_2|^3\right) = -\omega_4\left(\left(|z_1|^2\right)^{\frac{3}{4}} + \left(|z_2|^4\right)^{\frac{3}{4}}\right) \le -\omega_4\left(|z_1|^2 + |z_2|^4\right)^{\frac{3}{4}} \le -\left(\frac{\omega_4}{\bar{\pi}_1^{\frac{3}{4}}}\right)V^{\frac{3}{4}}$$

where $\omega_4 = \min\left\{\frac{r(k_1 - k_2 - M)}{2}, \frac{(k_2 - M)}{2}\right\}$.

Remark 2.7. Note that the nonlinear inequalities (2 - 7) and (2 - 10) can be satisfied for any $k_1 > k_2 + M$, and $k_2 > M$, which guarantees the existence of an $r > 0$. The conditions on the gains obtained here for the special case of $b = 0$ coincides with [20]. Utilising lemma 2.5-3, the system (2 - 6) will have finite time convergence for $b = 0$, with the settling time estimate,

$$T(z_{10}, z_{20}) \le \left(\frac{4\bar{\pi}_1^{\frac{3}{4}}}{\omega_4} \right) [V(z_{10}, z_{20})]^{\frac{1}{4}}$$, where (z_{10}, z_{20}) are the initial states of the system.

2.3 Generic Twisting

Leveraging the results of section 2.2.1, a generic twisting based family of controllers is presented. Essentially, the algorithm comprise linear sum of two system that have different homogeneity (hence different convergence properties), thus yielding a family of non-homogeneous algorithms. By means of strict Lyapunov functions, the family of algorithms are shown to exhibit the properties of their individual components, i.e. finite-time and uniform convergence, thus, yielding uniform finite-time convergence that is independent of initial conditions of the system.

2.3.1 System description

Consider the following generic twisting based dynamics:

$$\begin{aligned} \dot{z}_1 &= z_2, \\ \dot{z}_2 &= -k_1 |z_1|^{2p-1} \operatorname{sign}(z_1) - k_{1n} |z_1|^{2q-1} \operatorname{sign}(z_1) - k_2 |z_2|^{2-\frac{1}{p}} \operatorname{sign}(z_2) - k_{2n} |z_2|^{2-\frac{1}{q}} \operatorname{sign}(z_2) + d \end{aligned} \tag{2 - 11}$$

where z_1, $z_2 \in \mathbb{R}$, are the scalar state variables, k_1, $k_{1\mathrm{n}}$, k_2, $k_{2\mathrm{n}}$ are positive constants, $0.5 \le p \le 1$, and $1 \le q$. Also, d is time-varying and/or nonlinear term of uncertainty bounded by:

$$|d| \le M_1|z_1|^{2p-1} + M_2|z_1|^{2q-1} + M_3|z_2|^{2-\frac{1}{p}} + M_4|z_2|^{2-\frac{1}{q}} + M_5 \text{ with } |d| := \sup\{|\delta| : \delta \in \boldsymbol{K}[d]\},$$

where $M_1 \ge 0$, $M_2 \ge 0$, $M_3 \ge 0$, $M_4 \ge 0$, and $M_5 \ge 0$ with the same p and q as that in (2 - 11). Note that for the case of $p = 0.5$, the algorithm contains discontinuity and the uncertainty is upper bounded by non-vanishing constant.

2.3.2 Stability analysis

The system above (2 - 11) is essentially a summation of two different degree-of-homogeneity of the twisting-based algorithm considered in section 2.2.1. Hence, a Lyapunov function, which consists of linear summation of two different degree-of-homogeneity Lyapunov functions based on section 2.2.1, is proposed for system (2 - 11).

$$V(z_1, z_2) = \left(\frac{1}{2}z_2^{\ 2} + \frac{k_1}{2p}|z_1|^{2p} + \frac{k_{1n}}{2q}|z_1|^{2q}\right)^2 + r|z_1|^{3p}|z_2|\text{sign}(z_1 z_2) + r|z_1|^{3q}|z_2|\text{sign}(z_1 z_2)$$

where r is a positive constant scalar. Note that the above function is locally Lipschitz and regular. It is differentiable everywhere for $p > 0.5$, and not differentiable on $z_1 = 0$ for $p = 0.5$.

Note that the Lyapunov function V can be lower bounded by,

$$\begin{aligned} V(z_1, z_2) \ge & \frac{1}{4}|z_2|^4 + \frac{k_1^{\ 2}}{4p^2}|z_1|^{4p} + \frac{k_{1n}^{\ 2}}{4q^2}|z_1|^{4q} + \frac{k_1}{2p}|z_1|^{2p}|z_2|^2 + \frac{k_{1n}}{2q}|z_1|^{2q}|z_2|^2 + \frac{k_1 k_{1n}}{2pq}|z_1|^{2p+2q} \\ & - r|z_1|^{3p}|z_2| - r|z_1|^{3q}|z_2| \end{aligned}$$

Using lemma 2.1,

$$\left|z_1\right|^{4p}+\left|z_2\right|^4\geq\left(\frac{4}{3}\right)^{\frac{3}{4}}\left|z_1\right|^{3p}(4)^{\frac{1}{4}}\left|z_2\right|$$

$$\left|z_1\right|^{4q}+\left|z_2\right|^4\geq\left(\frac{4}{3}\right)^{\frac{3}{4}}\left|z_1\right|^{3q}(4)^{\frac{1}{4}}\left|z_2\right|$$

Hence, for

$$\min\left\{\left(\frac{k_1{}^2}{6p^2}\right)^{\frac{3}{4}}\left(\frac{1}{3}\right)^{\frac{1}{4}},\ \left(\frac{k_{1n}{}^2}{6q^2}\right)^{\frac{3}{4}}\left(\frac{1}{3}\right)^{\frac{1}{4}}\right\}>r \qquad (2\text{ - }12)$$

$$V(z_1,z_2)\geq\frac{1}{12}\left|z_2\right|^4+\frac{k_1{}^2}{8p^2}\left|z_1\right|^{4p}+\frac{k_{1n}{}^2}{8q^2}\left|z_1\right|^{4q}\geq\underline{\pi}_1\left(\left|z_1\right|^{4p}+\left|z_1\right|^{4q}+\left|z_2\right|^4\right)$$

where $\underline{\pi}_1=\min\left\{\frac{1}{12},\frac{k_1{}^2}{8p^2},\frac{k_{1n}{}^2}{8q^2}\right\}$.

Also, it can be upper-bounded by,

$$\begin{aligned}V(z_1,z_2)\leq&\frac{1}{4}\left|z_2\right|^4+\frac{k_1{}^2}{4p^2}\left|z_1\right|^{4p}+\frac{k_{1n}{}^2}{4q^2}\left|z_1\right|^{4q}+\frac{k_1}{2p}\left|z_1\right|^{2p}\left|z_2\right|^2+\frac{k_{1n}}{2q}\left|z_1\right|^{2q}\left|z_2\right|^2+\frac{k_1k_{1n}}{2pq}\left|z_1\right|^{2p+2q}\\&+r\left|z_1\right|^{3p}\left|z_2\right|+r\left|z_1\right|^{3q}\left|z_2\right|\end{aligned}$$

Using lemma 2.1,

$$\frac{k_1}{2p}|z_1|^{2p}|z_2|^2 \le \frac{k_1}{4p}|z_1|^{4p} + \frac{k_1}{4p}|z_2|^4,$$
$$\frac{k_{1n}}{2q}|z_1|^{2q}|z_2|^2 \le \frac{k_{1n}}{4q}|z_1|^{4q} + \frac{k_{1n}}{4q}|z_2|^4,$$
$$\frac{k_1 k_{1n}}{2pq}|z_1|^{2p+2q} \le \frac{k_1 k_{1n}}{4pq}|z_1|^{4p} + \frac{k_1 k_{1n}}{4pq}|z_1|^{4q},$$
$$r|z_1|^{3p}|z_2| \le \frac{3r}{4}|z_1|^{4p} + \frac{r}{4}|z_2|^4,$$
$$r|z_1|^{3q}|z_2| \le \frac{3r}{4}|z_1|^{4q} + \frac{r}{4}|z_2|^4.$$

Then,

$$V(z_1, z_2) \le \bar{\pi}_1\left(|z_1|^{4p} + |z_1|^{4q} + |z_2|^4\right)$$

where

$$\bar{\pi}_1 = \max\left\{\left(\frac{k_1{}^2}{4p^2} + \frac{k_1}{4p} + \frac{k_1 k_{1n}}{4pq} + \frac{3r}{4}\right), \left(\frac{k_{1n}{}^2}{4q^2} + \frac{k_{1n}}{4q} + \frac{k_1 k_{1n}}{4pq} + \frac{3r}{4}\right), \left(\frac{1}{4} + \frac{k_1}{4p} + \frac{k_{1n}}{4q} + \frac{r}{4} + \frac{r}{4}\right)\right\}$$

Thus, V is positive definite and radially unbounded. Since (2 - 11) is a differential equation that has discontinuous right-hand side, i.e. when $p = 0.5$ and since no continuity assumption is made on d, its solution are understood in the sense of Filippov (see definition 2.1). According to lemma 2.4, the time derivative of non-smooth Lyapunov function exists almost everywhere

$$\frac{d}{dt}V(\mathbf{z}(t), t) \overset{a.e.}{\in} \dot{\tilde{V}}(\mathbf{z}(t), t) = \bigcap_{\xi \in \partial V(\mathbf{z}(t), t)} \xi^T K\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix}(\mathbf{z}, t).$$

Since the controller (2 - 11) is discontinuous when $p = 0.5$, for ease of presentation, the analysis is separated for the case of $p > 0.5$ and $p = 0.5$.

a) Case 1: For 0.5 < p ≤ 1, and 1 ≤ q

Note that for $p > 0.5$, V is continuously differentiable, hence

$$\dot{\tilde{V}}(z_1, z_2) = \bigcap_{\xi \in \partial V(\mathbf{z}(t),t)} \xi^T K\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \nabla V^T K\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix}$$

$$\begin{aligned}
&\subset -\frac{k_1 k_2}{p}|z_1|^{2p}|z_2|^{3-\frac{1}{p}} - \frac{k_{1n}k_2}{q}|z_1|^{2q}|z_2|^{3-\frac{1}{p}} - \frac{k_1 k_{2n}}{p}|z_1|^{2p}|z_2|^{3-\frac{1}{q}} \\
&- \frac{k_{1n}k_{2n}}{q}|z_1|^{2q}|z_2|^{3-\frac{1}{q}} - rk_{1n}|z_1|^{2q+3p-1} - rk_1|z_1|^{2p+3q-1} - k_2|z_2|^{5-\frac{1}{p}} \\
&- k_{2n}|z_2|^{5-\frac{1}{q}} - rk_1|z_1|^{5p-1} - rk_{1n}|z_1|^{5q-1} - rk_2|z_1|^{3p}|z_2|^{2-\frac{1}{p}}\operatorname{sign}(z_1 z_2) \\
&- rk_{2n}|z_1|^{3p}|z_2|^{2-\frac{1}{q}}\operatorname{sign}(z_1 z_2) + r3p|z_1|^{3p-1}|z_2|^2 \\
&- rk_2|z_1|^{3q}|z_2|^{2-\frac{1}{p}}\operatorname{sign}(z_1 z_2) - rk_{2n}|z_1|^{3q}|z_2|^{2-\frac{1}{q}}\operatorname{sign}(z_1 z_2) + r3q|z_1|^{3q-1}|z_2|^2 \\
&+ K[d]\begin{pmatrix} |z_2|^3\operatorname{sign}(z_2) + \dfrac{k_1}{p}|z_1|^{2p}z_2 + \dfrac{k_{1n}}{q}|z_1|^{2q}z_2 \\ + r|z_1|^{3p}\operatorname{sign}(z_1) + r|z_1|^{3q}\operatorname{sign}(z_1) \end{pmatrix}
\end{aligned}$$

After rearrangement,

$$\begin{aligned}
\dot{\tilde{V}}(z_1, z_2) \le &-\frac{k_1}{p}(k_2 - M_3)|z_1|^{2p}|z_2|^{3-\frac{1}{p}} - \frac{k_{1n}}{q}(k_2 - M_3)|z_1|^{2q}|z_2|^{3-\frac{1}{p}} \\
&- \frac{k_1}{p}(k_{2n} - M_4)|z_1|^{2p}|z_2|^{3-\frac{1}{q}} - \frac{k_{1n}}{q}(k_{2n} - M_4)|z_1|^{2q}|z_2|^{3-\frac{1}{q}} \\
&- r(k_{1n} - M_2)|z_1|^{3p+2q-1} - r(k_1 - M_1)|z_1|^{2p+3q-1} - \frac{(k_2 - M_3)}{2}|z_2|^{5-\frac{1}{p}} \\
&- \frac{(k_{2n} - M_4)}{2}|z_2|^{5-\frac{1}{q}} - \frac{r(k_1 - M_1)}{2}|z_1|^{5p-1} - \frac{r(k_{1n} - M_2)}{2}|z_1|^{5q-1} \\
&+ \dot{V}_1 + M_5\left(|z_2|^3 + \frac{k_1}{p}|z_1|^{2p}|z_2| + \frac{k_{1n}}{q}|z_1|^{2q}|z_2| + r|z_1|^{3p} + r|z_1|^{3q}\right)
\end{aligned}$$

where

$$
\begin{aligned}
\dot{V}_1 = & -\frac{(k_2 - M_3)}{2}|z_2|^{5-\frac{1}{p}} - \frac{(k_{2n} - M_4)}{2}|z_2|^{5-\frac{1}{q}} - \frac{r(k_1 - M_1)}{2}|z_1|^{5p-1} - \frac{r(k_{1n} - M_2)}{2}|z_1|^{5q-1} \\
& + r(k_2 + M_3)|z_1|^{3p}|z_2|^{2-\frac{1}{p}} + r(k_{2n} + M_4)|z_1|^{3p}|z_2|^{2-\frac{1}{q}} + r3p|z_1|^{3p-1}|z_2|^2 \\
& + r(k_2 + M_3)|z_1|^{3q}|z_2|^{2-\frac{1}{p}} + r(k_{2n} + M_4)|z_1|^{3q}|z_2|^{2-\frac{1}{q}} + r3q|z_1|^{3q-1}|z_2|^2 \\
& + M_1|z_1|^{2p-1}|z_2|^3 + \frac{k_1}{p}M_1|z_1|^{4p-1}|z_2| + \left(M_1\frac{k_{1n}}{q} + M_2\frac{k_1}{p}\right)|z_1|^{2p+2q-1}|z_2| \\
& + M_2|z_1|^{2q-1}|z_2|^3 + M_2\frac{k_{1n}}{q}|z_1|^{4q-1}|z_2|
\end{aligned}
$$

Applying lemma 2.1,

$$-|z_1|^{5p-1} - |z_2|^{5-\frac{1}{p}} \le -\left(\frac{5p-1}{3p}\right)^{\frac{3p}{5p-1}}|z_1|^{3p}\left(\frac{5p-1}{2p-1}\right)^{\frac{2p-1}{5p-1}}|z_2|^{2-\frac{1}{p}},$$

$$-|z_1|^{5p-1} - |z_2|^{5-\frac{1}{p}} \le -\left(\frac{5p-1}{3p-1}\right)^{\frac{3p-1}{5p-1}}|z_1|^{3p-1}\left(\frac{5p-1}{2p}\right)^{\frac{2p}{5p-1}}|z_2|^{2},$$

$$-|z_1|^{5p-1} - |z_2|^{5-\frac{1}{p}} \le -\left(\frac{5p-1}{2p-1}\right)^{\frac{2p-1}{5p-1}}|z_1|^{2p-1}\left(\frac{5p-1}{3p}\right)^{\frac{3p}{5p-1}}|z_2|^{3},$$

$$-|z_1|^{5p-1} - |z_2|^{5-\frac{1}{p}} \le -\left(\frac{5p-1}{4p-1}\right)^{\frac{4p-1}{5p-1}}|z_1|^{4p-1}\left(\frac{5p-1}{p}\right)^{\frac{p}{5p-1}}|z_2|,$$

$$-|z_1|^{5q-1} - |z_2|^{5-\frac{1}{q}} \le -\left(\frac{5q-1}{3q}\right)^{\frac{3q}{5q-1}}|z_1|^{3q}\left(\frac{5q-1}{2q-1}\right)^{\frac{2q-1}{5q-1}}|z_2|^{2-\frac{1}{q}},$$

$$-|z_1|^{5q-1} - |z_2|^{5-\frac{1}{q}} \le -\left(\frac{5q-1}{3q-1}\right)^{\frac{3q-1}{5q-1}}|z_1|^{3q-1}\left(\frac{5q-1}{2q}\right)^{\frac{2q}{5q-1}}|z_2|^{2},$$

$$-|z_1|^{5q-1} - |z_2|^{5-\frac{1}{q}} \le -\left(\frac{5q-1}{2q-1}\right)^{\frac{2q-1}{5q-1}} |z_1|^{2q-1} \left(\frac{5q-1}{3q}\right)^{\frac{3q}{5q-1}} |z_2|^3,$$

$$-|z_1|^{5q-1} - |z_2|^{5-\frac{1}{q}} \le -\left(\frac{5q-1}{4q-1}\right)^{\frac{4q-1}{5q-1}} |z_1|^{4q-1} \left(\frac{5q-1}{q}\right)^{\frac{q}{5q-1}} |z_2|,$$

$$-|z_1|^{5p-1} - |z_1|^{5q-1} - |z_2|^{5-\frac{1}{q}} \le -|z_1|^{5p-\frac{p}{q}} - |z_2|^{5-\frac{1}{q}} \le -\left(\frac{5q-1}{3q}\right)^{\frac{3q}{5q-1}} |z_1|^{3p} \left(\frac{5q-1}{2q-1}\right)^{\frac{2q-1}{5q-1}} |z_2|^{2-\frac{1}{q}},$$

$$-|z_1|^{5p-1} - |z_1|^{5q-1} - |z_2|^{5-\frac{1}{p}} \le |z_1|^{5q-\frac{q}{p}} - |z_2|^{5-\frac{1}{p}} \le -\left(\frac{5p-1}{3p}\right)^{\frac{3p}{5p-1}} |z_1|^{3q} \left(\frac{5p-1}{2p-1}\right)^{\frac{2p-1}{5p-1}} |z_2|^{2-\frac{1}{p}},$$

$$\begin{aligned}-|z_1|^{5p-1} - |z_1|^{5q-1} - |z_2|^{5-\frac{1}{p}} &\le |z_1|^{(2p+2q-1)\left(\frac{5p-1}{4p-1}\right)} - |z_2|^{5-\frac{1}{p}} \\ &\le -\left(\frac{5p-1}{4p-1}\right)^{\frac{4p-1}{5p-1}} |z_1|^{2p+2q-1} \left(\frac{5p-1}{p}\right)^{\frac{p}{5p-1}} |z_2|,\end{aligned}$$

where lemma 2.3 has been employed in the last three inequalities since

$5p-1 \le (2p+2q-1)\left(\dfrac{5p-1}{4p-1}\right) \le 5q-1,$ $5p-1 \le 5q - \dfrac{q}{p} \le 5q-1,$ and

$5p-1 \le 5p - \dfrac{p}{q} \le 5q-1,$

Thus, if the following inequalities

$$
\min\left\{
\begin{array}{l}
\lambda_1 \dfrac{(k_1 - M_1)^{\frac{3p}{2p-1}}(k_2 - M_3)}{(k_2 + M_3)^{\frac{5p-1}{2p-1}}}, \\
\lambda_2 (k_1 - M_1)^{\frac{3p-1}{2p}}(k_2 - M_3), \\
\lambda_3 \dfrac{(k_{1n} - M_2)^{\frac{3q}{2q-1}}(k_{2n} - M_4)}{(k_{2n} + M_4)^{\frac{5q-1}{2q-1}}}, \\
\lambda_4 (k_{1n} - M_2)^{\frac{3q-1}{2q}}(k_{2n} - M_4), \\
\lambda_5 \dfrac{\omega_1^{\frac{3q}{2q-1}}(k_{2n} - M_4)}{(k_{2n} + M_4)^{\frac{5q-1}{2q-1}}}, \\
\lambda_6 \dfrac{\omega_1^{\frac{3p}{2p-1}}(k_2 - M_3)}{(k_2 + M_3)^{\frac{5p-1}{2p-1}}}
\end{array}
\right\} > r > \max\left\{
\begin{array}{l}
\lambda_7 \dfrac{M_1^{\frac{5p-1}{2p-1}}}{(k_1 - M_1)(k_2 - M_3)^{\frac{3p}{2p-1}}}, \\
\lambda_8 \dfrac{k_1^{\frac{5p-1}{4p-1}} M_1^{\frac{5p-1}{4p-1}}}{(k_1 - M_1)(k_2 - M_3)^{\frac{p}{4p-1}}}, \\
\lambda_9 \dfrac{M_2^{\frac{5q-1}{2q-1}}}{(k_{1n} - M_2)(k_{2n} - M_4)^{\frac{3q}{2q-1}}}, \\
\lambda_{10} \dfrac{k_{1n}^{\frac{5q-1}{4q-1}} M_2^{\frac{5q-1}{4q-1}}}{(k_{1n} - M_2)(k_{2n} - M_4)^{\frac{q}{4q-1}}}, \\
\lambda_{11} \dfrac{\left(M_1 \dfrac{k_{1n}}{q} + M_2 \dfrac{k_1}{p}\right)^{\frac{5p-1}{4p-1}}}{\omega_1 (k_2 - M_3)^{\frac{p}{4p-1}}}
\end{array}
\right\} \tag{2 - 13}
$$

where

$$
\lambda_1 = \left(\frac{5p-1}{42p}\right)^{\frac{3p}{2p-1}}\left(\frac{5p-1}{24p-12}\right), \quad \lambda_2 = \left(\frac{5p-1}{42p-14}\right)^{\frac{3p-1}{2p}}\left(\frac{5p-1}{24p}\right)\left(\frac{1}{3p}\right)^{\frac{5p-1}{2p}},
$$

$$
\lambda_3 = \left(\frac{5q-1}{42q}\right)^{\frac{3q}{2q-1}}\left(\frac{5q-1}{20q-10}\right), \quad \lambda_4 = \left(\frac{5q-1}{42q-14}\right)^{\frac{3q-1}{2q}}\left(\frac{5q-1}{20q}\right)\left(\frac{1}{3q}\right)^{\frac{5q-1}{2q}},
$$

$$
\lambda_5 = \left(\frac{5q-1}{42q}\right)^{\frac{3q}{2q-1}}\left(\frac{5q-1}{20q-10}\right), \quad \lambda_6 = \left(\frac{5p-1}{42p}\right)^{\frac{3p}{2p-1}}\left(\frac{5p-1}{24p-12}\right),
$$

$$
\lambda_7 = \left(\frac{28p-14}{5p-1}\right)\left(\frac{36p}{5p-1}\right)^{\frac{3p}{2p-1}}, \quad \lambda_8 = \left(\frac{56p-14}{5p-1}\right)\left(\frac{12p}{5p-1}\right)^{\frac{p}{4p-1}}\left(\frac{1}{p}\right)^{\frac{5p-1}{4p-1}},
$$

$$
\lambda_9 = \left(\frac{28q-14}{5q-1}\right)\left(\frac{30q}{5q-1}\right)^{\frac{3q}{2q-1}}, \quad \lambda_{10} = \left(\frac{56q-14}{5q-1}\right)\left(\frac{10q}{5q-1}\right)^{\frac{q}{4q-1}}\left(\frac{1}{q}\right)^{\frac{5q-1}{4q-1}},
$$

$$
\lambda_{11} = \left(\frac{56p-14}{5p-1}\right)\left(\frac{12p}{5p-1}\right)^{\frac{p}{4p-1}}, \quad \omega_1 = \min\{(k_1 - M_1), (k_{1n} - M_2)\}
$$

hold then function $\dot{V}_1$ is negative definite. Then,

$$\dot{\tilde{V}}(z_1, z_2) \le -\frac{(k_2 - M_3)}{2}|z_2|^{5-\frac{1}{p}} - \frac{(k_{2n} - M_4)}{2}|z_2|^{5-\frac{1}{q}} - \frac{r(k_1 - M_1)}{2}|z_1|^{5p-1} - \frac{r(k_{1n} - M_2)}{2}|z_1|^{5q-1}$$
$$+ M_5\left(|z_2|^3 + \frac{k_1}{p}|z_1|^{2p}|z_2| + \frac{k_{1n}}{q}|z_1|^{2q}|z_2| + r|z_1|^{3p} + r|z_1|^{3q}\right)$$

Applying lemma 2.1,

$$|z_1|^{2p}|z_2| \le \frac{2}{3}|z_1|^{3p} + \frac{1}{3}|z_2|^3,$$
$$|z_1|^{2q}|z_2| \le \frac{2}{3}|z_1|^{3q} + \frac{1}{3}|z_2|^3,$$

and lemma 2.3:

$$-|z_1|^{5p-1} - |z_1|^{5q-1} \le -|z_1|^{5p-\frac{p}{q}},$$

$$-|z_1|^{5p-1} - |z_1|^{5q-1} \le -|z_1|^{5q-\frac{q}{p}},$$

together with lemma 2.2, and the bounds of the Lyapunov function, we have:

$$\begin{aligned}
\dot{\tilde{V}}(z_1, z_2) &\le -\omega_2\left(|z_1|^{5p-1} + |z_1|^{5q-\frac{q}{p}} + |z_2|^{5-\frac{1}{p}}\right) - \omega_3\left(|z_1|^{5q-1} + |z_1|^{5p-\frac{p}{q}} + |z_2|^{5-\frac{1}{q}}\right) \\
&\quad + M_5\omega_4\left(|z_2|^3 + |z_1|^{3p} + |z_1|^{3q}\right) \\
&= -\omega_2\left(\left(|z_1|^{4p}\right)^{\frac{5p-1}{4p}} + \left(|z_1|^{4q}\right)^{\frac{5p-1}{4p}} + \left(|z_2|^4\right)^{\frac{5p-1}{4p}}\right) \\
&\quad - \omega_3\left(\left(|z_1|^{4q}\right)^{\frac{5q-1}{4q}} + \left(|z_1|^{4p}\right)^{\frac{5q-1}{4q}} + \left(|z_2|^4\right)^{\frac{5q-1}{4q}}\right) \\
&\quad + M_5\omega_4\left(\left(|z_2|^4\right)^{\frac{3}{4}} + \left(|z_1|^{4p}\right)^{\frac{3}{4}} + \left(|z_1|^{4q}\right)^{\frac{3}{4}}\right)
\end{aligned}$$

$$\le -\left(\frac{\omega_2}{\bar{\pi}_1^{\frac{5p-1}{4p}}}\right)V^{\frac{5p-1}{4p}} - \left(\frac{\omega_3}{3^{\frac{q-1}{4q}}\,\bar{\pi}_1^{\frac{5q-1}{4q}}}\right)V^{\frac{5q-1}{4q}} + M_5\left(\frac{3^{\frac{1}{4}}\omega_4}{\underline{\pi}_1^{\frac{3}{4}}}\right)V^{\frac{3}{4}} \tag{2 - 14}$$

where

$$\omega_2 = \min\left\{\frac{r(k_1 - M_1)}{4}, \frac{r}{8}\omega_1, \frac{(k_2 - M_3)}{2}\right\} \omega_3 = \min\left\{\frac{r(k_{1n} - M_2)}{4}, \frac{r}{8}\omega_1, \frac{(k_{2n} - M_4)}{2}\right\},$$

$$\omega_4 = \max\left\{\left(r + \frac{2k_1}{3p}\right), \left(r + \frac{2k_{1n}}{3q}\right), \left(1 + \frac{k_1}{3p} + \frac{k_{1n}}{3q}\right)\right\}$$

Remark 2.8. Note that the nonlinear inequalities (2 - 12) and (2 - 13) are feasible for sufficiently large $k_1 > 0$, $k_{1n} > 0$, $k_2 > 0$, $k_{2n} > 0$. Thus, an $r > 0$ always exists. Consider some particular cases.

1. Note that (2 - 13) is feasible with respect to k_1, k_{1n}, k_2, k_{2n} for any $M_1 \geq 0$, $M_2 \geq 0$, $M_3 \geq 0$, and $M_4 \geq 0$.
2. If $M_1 = M_2 = 0$, then (2 - 13) is satisfied for any $k_2 > M_3$ and $k_{2n} > M_4$. This coincides with the conditions obtained through the weak Lyapunov function, i.e. the energy function.

Remark 2.9. Hence, when $M_5 = 0$, i.e. without persistent perturbations, from lemma 2.5-3,

1. For $0 < p < 1$, and $q > 1$the system exhibit uniform asymptotical convergence, where

$$T_{\max}(\mu)=\left(\frac{3^{\frac{q-1}{4q}}\,\overline{\pi}_1^{\frac{5q-1}{4q}}}{\omega_3}\right)\left(\frac{4q}{q-1}\right)\left(\frac{1}{\mu^{\frac{q-1}{4q}}}\right),$$

is the time at which the trajectories reach the surface level $V=\mu$. At the same time, the system (2 - 11) will have finite time convergence, in particular after reaching the surface level $V=\mu$, from lemma 2.5-3, the settling time estimate,

$$T(\mu)\leq\left(\frac{\overline{\pi}_1^{\frac{5p-1}{4p}}}{\omega_2}\right)\left(\frac{4p}{1-p}\right)[\mu]^{\frac{1-p}{4p}},$$

where the initial starting states is changed to $V=\mu$. Hence, the total time to reach the origin can be estimated as

$$T_{total}(\mu)=\left(\frac{3^{\frac{q-1}{4q}}\,\overline{\pi}_1^{\frac{5q-1}{4q}}}{\omega_3}\right)\left(\frac{4q}{q-1}\right)\left(\frac{1}{\mu^{\frac{q-1}{4q}}}\right)+\left(\frac{\overline{\pi}_1^{\frac{5p-1}{4p}}}{\omega_2}\right)\left(\frac{4p}{1-p}\right)[\mu]^{\frac{1-p}{4p}}.$$

The minimum of this function can be found at $\mu=\left(\frac{3^{\frac{q-1}{4q}}\,\overline{\pi}_1^{\frac{5q-1}{4q}}}{\omega_3}\right)^{\frac{4pq}{q-p}}\left(\frac{\omega_2}{\overline{\pi}_1^{\frac{5p-1}{4p}}}\right)^{\frac{4pq}{q-p}}$.

Substituting into the function, a finite settling time independent of initial conditions is obtained,

$$T_{total}=\left(\frac{3^{\frac{q-1}{4q}}\,\bar{\pi}_1^{\frac{5q-1}{4q}}}{\omega_3}\right)\left(\frac{4q}{q-1}\right)\left(\frac{\omega_3}{3^{\frac{q-1}{4q}}\,\bar{\pi}_1^{\frac{5q-1}{4q}}}\right)^{\frac{p(q-1)}{q-p}}\left(\frac{\bar{\pi}_1^{\frac{5p-1}{4p}}}{\omega_2}\right)^{\frac{p(q-1)}{q-p}}$$

$$+\left(\frac{\bar{\pi}_1^{\frac{5p-1}{4p}}}{\omega_2}\right)\left(\frac{4p}{1-p}\right)\left(\frac{3^{\frac{q-1}{4q}}\,\bar{\pi}_1^{\frac{5q-1}{4q}}}{\omega_3}\right)^{\frac{q(1-p)}{q-p}}\left(\frac{\omega_2}{\bar{\pi}_1^{\frac{5p-1}{4p}}}\right)^{\frac{q(1-p)}{q-p}}$$

2. For $0 < p < 1$, and $q = 1$, finite time convergence can be concluded.

3. For $p = 1$, and $q \geq 1$, exponential convergence can be concluded.

These results are possible due to the negative definiteness of the time derivative of the Lyapunov function, i.e. strict Lyapunov function.

Remark 2.10. While if persistent perturbations occur on the system, $M_5 \neq 0$, from (2 - 14),

$$\dot{\tilde{V}}(z_1,z_2)\leq -\frac{1}{2}\left(\frac{\omega_2}{\bar{\pi}_1^{\frac{5p-1}{4p}}}\right)V^{\frac{5p-1}{4p}}-\left(\frac{\omega_3}{3^{\frac{q-1}{4q}}\,\bar{\pi}_1^{\frac{5q-1}{4q}}}\right)V^{\frac{5q-1}{4q}}-V^{\frac{3}{4}}\left(\frac{1}{2}\left(\frac{\omega_2}{\bar{\pi}_1^{\frac{5p-1}{4p}}}\right)V^{\frac{2p-1}{4p}}-M_5\left(\frac{3^{\frac{1}{4}}\,\omega_4}{\underline{\pi}_1^{\frac{3}{4}}}\right)\right)$$

$$\leq -\frac{1}{2}\left(\frac{\omega_2}{\bar{\pi}_1^{\frac{5p-1}{4p}}}\right)V^{\frac{5p-1}{4p}}-\left(\frac{\omega_3}{3^{\frac{q-1}{4q}}\,\bar{\pi}_1^{\frac{5q-1}{4q}}}\right)V^{\frac{5q-1}{4q}},\ \text{for}\ \ V\geq\left(2M_5\left(\frac{\bar{\pi}_1^{\frac{5p-1}{4p}}}{\omega_2}\right)\left(\frac{3^{\frac{1}{4}}\,\omega_4}{\underline{\pi}_1^{\frac{3}{4}}}\right)\right)^{\frac{4p}{2p-1}}$$

or

$$\dot{\tilde{V}}(z_1,z_2)\le-\left(\frac{\omega_2}{\overline{\pi}_1^{\frac{5p-1}{4p}}}\right)V^{\frac{5p-1}{4p}}-\frac{1}{2}\left(\frac{\omega_3}{3^{\frac{q-1}{4q}}\,\overline{\pi}_1^{\frac{5q-1}{4q}}}\right)V^{\frac{5q-1}{4q}}$$

$$-V^{\frac{3}{4}}\left(\frac{1}{2}\left(\frac{\omega_3}{3^{\frac{q-1}{4q}}\,\overline{\pi}_1^{\frac{5q-1}{4q}}}\right)V^{\frac{2q-1}{4q}}-M_5\left(\frac{3^{\frac{1}{4}}\omega_4}{\underline{\pi}_1^{\frac{3}{4}}}\right)\right)$$

$$\le-\left(\frac{\omega_2}{\overline{\pi}_1^{\frac{5p-1}{4p}}}\right)V^{\frac{5p-1}{4p}}-\frac{1}{2}\left(\frac{\omega_3}{3^{\frac{q-1}{4q}}\,\overline{\pi}_1^{\frac{5q-1}{4q}}}\right)V^{\frac{5q-1}{4q}},$$

$$\text{for } V\ge\left(2M_5\left(\frac{3^{\frac{q-1}{4q}}\,\overline{\pi}_1^{\frac{5q-1}{4q}}}{\omega_3}\right)\left(\frac{3^{\frac{1}{4}}\omega_4}{\underline{\pi}_1^{\frac{3}{4}}}\right)\right)^{\frac{4q}{2q-1}}.$$

Thus,

$$\dot{\tilde{V}}(z_1,z_2)<0,\ \text{ for }\ V\ge\min\left\{\begin{array}{c}\left(2M_5\left(\frac{\overline{\pi}_1^{\frac{5p-1}{4p}}}{\omega_2}\right)\left(\frac{3^{\frac{1}{4}}\omega_4}{\underline{\pi}_1^{\frac{3}{4}}}\right)\right)^{\frac{4p}{2p-1}},\\ \left(2M_5\left(\frac{3^{\frac{q-1}{4q}}\,\overline{\pi}_1^{\frac{5q-1}{4q}}}{\omega_3}\right)\left(\frac{3^{\frac{1}{4}}\omega_4}{\underline{\pi}_1^{\frac{3}{4}}}\right)\right)^{\frac{4q}{2q-1}}\end{array}\right\},$$

in which case, from lemma 2.6, the system (2 - 11) is uniformly ultimately bounded.

b) Case 2: For p = 0.5, and 1 ≤ q

For $p = 0.5$, V is not differentiable on $z_1 = 0$:

$$\dot{\tilde{V}}(z_1, z_2) = \bigcap_{\xi \in \partial V(\mathbf{z}(t),t)} \xi^T K\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} \subset \dot{\tilde{V}}_1 + \dot{\tilde{V}}_2$$

where

$$\dot{\tilde{V}}_1 = \bigcap_{\xi \in \partial V(\mathbf{z}(t),t)} \xi^T K[f](z_1, z_2),\ \dot{\tilde{V}}_2 = \bigcap_{\xi \in \partial V(\mathbf{z}(t),t)} \xi^T K\begin{bmatrix} 0 \\ d \end{bmatrix},$$

$$K[f](z_1, z_2) \in \begin{bmatrix} z_2 \\ \left(-k_1 \mathrm{SGN}(z_1) - k_{1n}|z_1|^{2q-1}\mathrm{sign}(z_1) - k_2 \mathrm{SGN}(z_2) - k_{2n}|z_2|^{\frac{2q-1}{q}}\mathrm{sign}(z_2)\right) \end{bmatrix}$$

$$\begin{aligned}
\partial V &= \boldsymbol{K}[\nabla V] \\
&\subset \begin{bmatrix} \boldsymbol{K}\left[\dfrac{\partial V}{\partial z_1}\right] \\ \boldsymbol{K}\left[\dfrac{\partial V}{\partial z_2}\right] \end{bmatrix} \\
&= \begin{cases} \forall z_1 \neq 0, z_2 \in \mathbb{R}: \\ \begin{bmatrix} \left(|z_2|^2 + 2k_1|z_1| + \dfrac{k_{1n}}{q}|z_1|^{2q}\right)\left(k_1 + k_{1n}|z_1|^{2q-1}\right)\mathrm{sign}(z_1) + r\left(1.5|z_1|^{0.5} + 3q|z_1|^{3q-1}\right)z_2 \\ \left(|z_2|^2 + 2k_1|z_1| + \dfrac{k_{1n}}{q}|z_1|^{2q}\right)(z_2) + r|z_1|^{1.5}\mathrm{sign}(z_1) + r|z_1|^{3q}\mathrm{sign}(z_1) \end{bmatrix} \\ \forall z_1 = 0, z_2 \in \mathbb{R}: \\ \begin{bmatrix} |z_2|^2 k_1[-1,1] \\ {z_2}^3 \end{bmatrix} \end{cases}
\end{aligned}$$

Let us define:

$$\left|\frac{\partial V}{\partial z_1}\right| := \sup\left\{|\xi_1| : \xi_1 \in \boldsymbol{K}\left[\frac{\partial V}{\partial z_1}\right]\right\},\ \text{and}\ \left|\frac{\partial V}{\partial z_2}\right| := \sup\left\{|\xi_2| : \xi_2 \in \boldsymbol{K}\left[\frac{\partial V}{\partial z_2}\right]\right\},$$

with $\left|\dfrac{\partial V}{\partial z_2}\right| \leq |z_2|^3 + 2k_1|z_1||z_2| + \dfrac{k_{1n}}{q}|z_1|^{2q}|z_2| + r|z_1|^{1.5} + r|z_1|^{3q}$

Thus, the term

$$
\begin{aligned}
\dot{\tilde{V}}_2 &= \bigcap_{\xi_2 \in \boldsymbol{K}\left[\frac{\partial V}{\partial z_2}\right]} \xi_2 K[d] \\
&\le \left|\frac{\partial V}{\partial z_2}\right| |d| \\
&\le \left(M + M_2 |z_1|^{2q-1} + M_4 |z_2|^{2-\frac{1}{q}} + M_5 \right)\left(|z_2|^3 + 2k_1 |z_1||z_2| + \frac{k_{1n}}{q}|z_1|^{2q}|z_2| + r|z_1|^{1.5} + r|z_1|^{3q} \right)
\end{aligned}
$$

where M be defined as, $M := M_1 + M_3$, since for p = 0.5, $|d| \le M_1 + M_3 + M_2 |z_1|^{2q-1} + M_4 |z_2|^{2-\frac{1}{q}} + M_5$.

Computing $\dot{\tilde{V}}_1$ for each case, we have

For $z_1 \neq 0$, $z_2 \neq 0$:

$$
\begin{aligned}
\dot{\tilde{V}}_1 = {} & \left(z_2{}^2 + 2k_1|z_1| + \frac{k_{1n}}{q}|z_1|^{2q} \right) \times \left(-k_2|z_2| - k_{2n}|z_2|^{\frac{3q-1}{q}} \right) - rk_{1n}|z_1|^{2q+0.5} - rk_1|z_1|^{3q} - rk_1|z_1|^{1.5} \\
& - rk_{1n}|z_1|^{5q-1} - rk_2|z_1|^{1.5}\operatorname{sign}(z_1 z_2) - rk_{2n}|z_1|^{1.5}|z_2|^{2-\frac{1}{q}}\operatorname{sign}(z_1 z_2) + 1.5r|z_1|^{0.5}|z_2|^2 \\
& - rk_2|z_1|^{3q}\operatorname{sign}(z_1 z_2) - rk_{2n}|z_1|^{3q}|z_2|^{2-\frac{1}{q}}\operatorname{sign}(z_1 z_2) + 3rq|z_1|^{3q-1}|z_2|^2
\end{aligned}
$$

For $z_1 = 0$, $z_2 \neq 0$:

Let $(\xi_2 k_1 z_2{}^2, z_2{}^3)^T$ with $\xi_2 \in [-1, 1]$ be an arbitrary element of $\partial V(z_1, z_2)$, then

$$
\begin{aligned}
\xi^{\mathrm{T}} K[f](z_1, z_2) &= \begin{bmatrix} \xi_2 k_1 z_2{}^2 \\ z_2{}^3 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} z_2 \\ \left(-[-1,\, 1]k_1 - k_2 \operatorname{sign}(z_2) - k_{2n}|z_2|^{\frac{2q-1}{q}} \operatorname{sign}(z_2) \right) \end{bmatrix} \\
&= [\xi_2 - 1, \xi_2 + 1]k_1 z_2{}^3 - k_2|z_2|^3 - k_{2n}|z_2|^{\frac{5q-1}{q}}
\end{aligned}
$$

hence

$$\dot{\tilde{V}}_1 = \bigcap_{\xi_2 \in [-1,\, 1]} [\xi_2 - 1,\ \xi_2 + 1] k_1 z_2^{\,3} - k_2 |z_2|^3 - k_{2n} |z_2|^{\frac{5q-1}{q}} = -k_2 |z_2|^3 - k_{2n} |z_2|^{\frac{5q-1}{q}}$$

For $z_1 \neq 0$, $z_2 = 0$:

$$\dot{\tilde{V}}_1 = -rk_1 |z_1|^{1.5} - rk_1 |z_1|^{3q} - rk_{1n} |z_1|^{2q+0.5} - rk_{1n} |z_1|^{5q-1} - rk_2 [-1,1] |z_1|^{1.5} \operatorname{sign}(z_1) \\ - rk_2 [-1,1] |z_1|^{3q} \operatorname{sign}(z_1)$$

For $z_1 = 0$, $z_2 = 0$:

$$\dot{\tilde{V}}_1 = 0$$

Thus, for all $(z_1, z_2) \in \mathbb{R}^2$:

$$\dot{\tilde{V}}_1 = \left(z_2^{\,2} + 2k_1 |z_1| + \frac{k_{1n}}{q} |z_1|^{2q} \right) \times \left(-k_2 |z_2| - k_{2n} |z_2|^{\frac{3q-1}{q}} \right) - rk_{1n} |z_1|^{2q+0.5} - rk_1 |z_1|^{3q} - rk_1 |z_1|^{1.5} \\ - rk_{1n} |z_1|^{5q-1} - rk_2 |z_1|^{1.5} \operatorname{sign}(z_1) \mathrm{SGN}(z_2) - rk_{2n} |z_1|^{1.5} |z_2|^{2-\frac{1}{q}} \operatorname{sign}(z_1 z_2) + 1.5r |z_1|^{0.5} |z_2|^2 \\ - rk_2 |z_1|^{3q} \operatorname{sign}(z_1) \mathrm{SGN}(z_2) - rk_{2n} |z_1|^{3q} |z_2|^{2-\frac{1}{q}} \operatorname{sign}(z_1 z_2) + 3rq |z_1|^{3q-1} |z_2|^2$$

Hence, the generalized time derivative of the Lyapunov function can be obtained, after rearrangement:

$$
\begin{aligned}
\dot{\tilde{V}} &= \dot{\tilde{V}}_1 + \dot{\tilde{V}}_2 \\
&\le -2k_1(k_2 - M)|z_1||z_2| - \frac{k_{1n}}{q}(k_2 - M)|z_1|^{2q}|z_2| - r(k_1 - k_2 - M)|z_1|^{3q} \\
&\quad - 2k_1(k_{2n} - M_4)|z_1||z_2|^{3-\frac{1}{q}} - \frac{k_{1n}}{q}(k_{2n} - M_4)|z_1|^{2q}|z_2|^{3-\frac{1}{q}} - r(k_{1n} - M_2)|z_1|^{2q+0.5} \\
&\quad - \frac{(k_2 - M)}{2}|z_2|^3 - \frac{(k_{2n} - M_4)}{2}|z_2|^{5-\frac{1}{q}} - r\frac{(k_1 - k_2 - M)}{2}|z_1|^{1.5} - \frac{r(k_{1n} - M_2)}{2}|z_1|^{5q-1} \\
&\quad + \dot{V}_2 + M_5\left(|z_2|^3 + 2k_1|z_1||z_2| + \frac{k_{1n}}{q}|z_1|^{2q}|z_2| + r|z_1|^{1.5} + r|z_1|^{3q}\right)
\end{aligned}
$$

where

$$
\begin{aligned}
\dot{V}_2 &= -\frac{(k_2 - M)}{2}|z_2|^3 - \frac{(k_{2n} - M_4)}{2}|z_2|^{5-\frac{1}{q}} - r\frac{(k_1 - k_2 - M)}{2}|z_1|^{1.5} \\
&\quad - \frac{r(k_{1n} - M_2)}{2}|z_1|^{5q-1} + r(k_{2n} + M_4)|z_1|^{1.5}|z_2|^{2-\frac{1}{q}} + 1.5r|z_1|^{0.5}|z_2|^2 \\
&\quad + r(k_{2n} + M_4)|z_1|^{3q}|z_2|^{2-\frac{1}{q}} + 3rq|z_1|^{3q-1}|z_2|^2 \\
&\quad + M_2|z_1|^{2q-1}|z_2|^3 + M_2\frac{k_{1n}}{q}|z_1|^{4q-1}|z_2| + 2M_2k_1|z_1|^{2q}|z_2|
\end{aligned}
$$

Applying lemma 2.1,

$$
-|z_1|^{1.5} - |z_2|^3 \le -3^{\frac{1}{3}}|z_1|^{0.5}\left(\frac{3}{2}\right)^{\frac{2}{3}}|z_2|^2,
$$

$$
-|z_1|^{5q-1} - |z_2|^{5-\frac{1}{q}} \le -\left(\frac{5q-1}{3q}\right)^{\frac{3q}{5q-1}}|z_1|^{3q}\left(\frac{5q-1}{2q-1}\right)^{\frac{2q-1}{5q-1}}|z_2|^{2-\frac{1}{q}},
$$

$$
-|z_1|^{5q-1} - |z_2|^{5-\frac{1}{q}} \le -\left(\frac{5q-1}{3q-1}\right)^{\frac{3q-1}{5q-1}}|z_1|^{3q-1}\left(\frac{5q-1}{2q}\right)^{\frac{2q}{5q-1}}|z_2|^2,
$$

$$-\left|z_1\right|^{5q-1}-\left|z_2\right|^{5-\frac{1}{q}}\le-\left(\frac{5q-1}{2q-1}\right)^{\frac{2q-1}{5q-1}}\left|z_1\right|^{2q-1}\left(\frac{5q-1}{3q}\right)^{\frac{3q}{5q-1}}\left|z_2\right|^{3},$$

$$-\left|z_1\right|^{5q-1}-\left|z_2\right|^{5-\frac{1}{q}}\le-\left(\frac{5q-1}{4q-1}\right)^{\frac{4q-1}{5q-1}}\left|z_1\right|^{4q-1}\left(\frac{5q-1}{q}\right)^{\frac{q}{5q-1}}\left|z_2\right|,$$

$$\begin{aligned}-\left|z_1\right|^{1.5}-\left|z_1\right|^{5q-1}-\left|z_2\right|^{5-\frac{1}{q}}&\le-\left|z_1\right|^{\frac{5q-1}{2q}}-\left|z_2\right|^{5-\frac{1}{q}}\\&\le-\left(\frac{5q-1}{3q}\right)^{\frac{3q}{5q-1}}\left|z_1\right|^{1.5}\left(\frac{5q-1}{2q-1}\right)^{\frac{2q-1}{5q-1}}\left|z_2\right|^{2-\frac{1}{q}},\end{aligned}$$

$$\begin{aligned}-\left|z_1\right|^{1.5}-\left|z_1\right|^{5q-1}-\left|z_2\right|^{5-\frac{1}{q}}&\le-\left|z_1\right|^{2q\left(\frac{5q-1}{4q-1}\right)}-\left|z_2\right|^{5-\frac{1}{q}}\\&\le-\left(\frac{5q-1}{4q-1}\right)^{\frac{4q-1}{5q-1}}\left|z_1\right|^{2q}\left(\frac{5q-1}{q}\right)^{\frac{q}{5q-1}}\left|z_2\right|,\end{aligned}$$

where lemma 2.3 has been employed in the last three inequalities since

$$1.5\le\frac{5q-1}{2q}\le 5q-1,\ \text{ and } 1.5\le 2q\left(\frac{5q-1}{4q-1}\right)\le 5q-1.$$

Thus, if the following inequalities

$$\min\left\{\begin{array}{l}\left(\frac{1}{2}\right)\left(\frac{1}{3^{\frac{1}{2}}}\right)\left(k_1-k_2-M\right)^{\frac{1}{2}}\left(k_2-M\right),\\ \lambda_{12}\frac{\left(k_{1n}-M_2\right)^{\frac{3q}{2q-1}}\left(k_{2n}-M_4\right)}{\left(k_{2n}+M_4\right)^{\frac{5q-1}{2q-1}}},\\ \lambda_{13}\left(k_{1n}-M_2\right)^{\frac{3q-1}{2q}}\left(k_{2n}-M_4\right),\\ \lambda_{14}\frac{\omega_5^{\frac{3q}{2q-1}}\left(k_{2n}-M_4\right)}{\left(k_{2n}+M_4\right)^{\frac{5q-1}{2q-1}}}\end{array}\right\}>r>\max\left\{\begin{array}{l}\lambda_{15}\frac{M_2^{\frac{5q-1}{2q-1}}}{\left(k_{1n}-M_2\right)\left(k_{2n}-M_4\right)^{\frac{3q}{2q-1}}},\\ \lambda_{16}\frac{k_{1n}^{\frac{5q-1}{4q-1}}M_2^{\frac{5q-1}{4q-1}}}{\left(k_{1n}-M_2\right)\left(k_{2n}-M_4\right)^{\frac{q}{4q-1}}},\\ \lambda_{17}\frac{k_1^{\frac{5q-1}{4q-1}}M_2^{\frac{5q-1}{4q-1}}}{\omega_5\left(k_{2n}-M_4\right)^{\frac{q}{4q-1}}}\end{array}\right\}\qquad(2-15)$$

where

$$\lambda_{12}=\left(\frac{5q-1}{36q}\right)^{\frac{3q}{2q-1}}\left(\frac{5q-1}{24q-12}\right),\ \lambda_{13}=\left(\frac{5q-1}{36q-12}\right)^{\frac{3q-1}{2q}}\left(\frac{5q-1}{24q}\right)\left(\frac{1}{3q}\right)^{\frac{5q-1}{2q}},$$

$$\lambda_{14}=\left(\frac{5q-1}{3q}\right)^{\frac{3q}{2q-1}}\left(\frac{5q-1}{24q-12}\right),\ \lambda_{15}=\left(\frac{24q-12}{5q-1}\right)\left(\frac{36q}{5q-1}\right)^{\frac{3q}{2q-1}},$$

$$\lambda_{16}=\left(\frac{48q-12}{5q-1}\right)\left(\frac{12q}{5q-1}\right)^{\frac{q}{4q-1}}\left(\frac{1}{q}\right)^{\frac{5q-1}{4q-1}},\quad \lambda_{17}=2^{\frac{5q-1}{4q-1}}\left(\frac{4q-1}{5q-1}\right)\left(\frac{12q}{5q-1}\right)^{\frac{q}{4q-1}},$$

$$\omega_5=\min\left\{\frac{(k_1-k_2-M)}{6},\frac{(k_{1n}-M_2)}{12}\right\}$$

hold then function $\dot{V}_2$ is negative definite. Then,

$$\dot{\tilde{V}}\le -r(k_1-k_2-M)|z_1|^{3q}-\frac{(k_2-M)}{2}|z_2|^3-\frac{(k_{2n}-M_4)}{2}|z_2|^{5-\frac{1}{q}}-r\frac{(k_1-k_2-M)}{2}|z_1|^{1.5}$$
$$-\frac{r(k_{1n}-M_2)}{2}|z_1|^{5q-1}+M_5\left(|z_2|^3+2k_1|z_1||z_2|+\frac{k_{1n}}{q}|z_1|^{2q}|z_2|+r|z_1|^{1.5}+r|z_1|^{3q}\right)$$

Applying lemma 2.1,

$$|z_1||z_2|\le\frac{2}{3}|z_1|^{1.5}+\frac{1}{3}|z_2|^3,$$

$$|z_1|^{2q}|z_2|\le\frac{2}{3}|z_1|^{3q}+\frac{1}{3}|z_2|^3,$$

and lemma 2.3:

$$-|z_1|^{5q-1}-|z_1|^{1.5}\le-|z_1|^{\frac{5q-1}{2q}},$$

together with lemma 2.2, and the bounds of the Lyapunov function, we have:

$$\dot{\tilde{V}} \le -\omega_7\left(|z_1|^{1.5}+|z_1|^{3q}+|z_2|^3\right)-\omega_8\left(|z_1|^{5q-1}+|z_1|^{\frac{5q-1}{2q}}+|z_2|^{5-\frac{1}{q}}\right)+M_5\omega_6\left(|z_1|^{1.5}+|z_1|^{3q}+|z_2|^3\right)$$

$$=-\omega_7\left(\left(|z_1|^2\right)^{\frac{3}{4}}+\left(|z_1|^{4q}\right)^{\frac{3}{4}}+\left(|z_2|^4\right)^{\frac{3}{4}}\right)-\omega_8\left(\left(|z_1|^{4q}\right)^{\frac{5q-1}{4q}}+\left(|z_1|^2\right)^{\frac{5q-1}{4q}}+\left(|z_2|^4\right)^{\frac{5q-1}{4q}}\right)$$

$$+M_5\omega_6\left(\left(|z_1|^2\right)^{\frac{3}{4}}+\left(|z_1|^{4q}\right)^{\frac{3}{4}}+\left(|z_2|^4\right)^{\frac{3}{4}}\right)$$

$$\le -\left(\frac{\omega_7}{\overline{\pi}_1^{\frac{3}{4}}}\right)V^{\frac{3}{4}}-\left(\frac{\omega_8}{3^{\frac{q-1}{4q}}\,\overline{\pi}_1^{\frac{5q-1}{4q}}}\right)V^{\frac{5q-1}{4q}}+M_5\left(\frac{3^{\frac{1}{4}}\omega_6}{\underline{\pi}_1^{\frac{3}{4}}}\right)V^{\frac{3}{4}} \tag{2 - 16}$$

where

$$\omega_6=\max\left\{\left(\frac{4k_1}{3}+r\right),\left(\frac{2k_{1n}}{3q}+r\right),\left(1+\frac{2k_1}{3}+\frac{k_{1n}}{3q}\right)\right\},$$
$$\omega_7=\min\left\{\frac{r(k_1-k_2-M)}{4},\, r(k_1-k_2-M),\, \frac{(k_2-M)}{2}\right\},$$
$$\omega_8=\min\left\{\frac{2r(k_{1n}-M_2)}{5},\, \frac{(k_{2n}-M_4)}{2},\, r\omega_5\right\}$$

Remark 2.11. Note that the nonlinear inequalities (2 - 12) and (2 - 15) is feasible with respect to k_1, k_{1n}, k_2, k_{2n} for any $M \ge 0$, $M_2 \ge 0$, and $M_4 \ge 0$. As such an $r > 0$ always exists for sufficiently large k_1, k_{1n}, k_2, k_{2n}. In particular, consider the case where $M_2 = 0$, inequalities (2 - 15) can be easily satisfied for any $k_1 > k_2 + M > 0$, $k_2 > M > 0$, $k_{2n} > M_4$.

Remark 2.12. Hence, when $M_5 = 0$, it is not difficult to show that the system achieved finite time convergence for $q \ge 1$. In particular for the case of $q > 1$, following similar arguments in section 2.2.2(a), finite convergence time independent of initial conditions for $q > 1$,

$$
\begin{aligned}
T_{total} = {} & \left(\frac{3^{\frac{q-1}{4q}} \bar{\pi}_1^{\frac{5q-1}{4q}}}{\omega_8} \right) \left(\frac{4q}{q-1} \right) \left(\frac{\omega_8}{3^{\frac{q-1}{4q}} \bar{\pi}_1^{\frac{5q-1}{4q}}} \right)^{\frac{0.5(q-1)}{q-0.5}} \left(\frac{\bar{\pi}_1^{\frac{3}{4}}}{\omega_7} \right)^{\frac{0.5(q-1)}{q-0.5}} \\
& + \left(\frac{4\bar{\pi}_1^{\frac{3}{4}}}{\omega_7} \right) \left(\frac{3^{\frac{q-1}{4q}} \bar{\pi}_1^{\frac{5q-1}{4q}}}{\omega_8} \right)^{\frac{0.5q}{q-0.5}} \left(\frac{\omega_7}{\bar{\pi}_1^{\frac{3}{4}}} \right)^{\frac{0.5q}{q-0.5}}
\end{aligned}
$$

and when $M_5 \neq 0$, from (2 - 16),

$$
\begin{aligned}
\dot{\tilde{V}} & \leq -\left(\frac{\omega_7}{\bar{\pi}_1^{\frac{3}{4}}} \right) V^{\frac{3}{4}} - \frac{1}{2} \left(\frac{\omega_8}{3^{\frac{q-1}{4q}} \bar{\pi}_1^{\frac{5q-1}{4q}}} \right) V^{\frac{5q-1}{4q}} - V^{\frac{3}{4}} \left(\frac{1}{2} \left(\frac{\omega_8}{3^{\frac{q-1}{4q}} \bar{\pi}_1^{\frac{5q-1}{4q}}} \right) V^{\frac{2q-1}{4q}} - M_5 \left(\frac{3^{\frac{1}{4}} \omega_6}{\underline{\pi}_1^{\frac{3}{4}}} \right) \right) \\
& \leq -\left(\frac{\omega_7}{\bar{\pi}_1^{\frac{3}{4}}} \right) V^{\frac{3}{4}} - \frac{1}{2} \left(\frac{\omega_8}{3^{\frac{q-1}{4q}} \bar{\pi}_1^{\frac{5q-1}{4q}}} \right) V^{\frac{5q-1}{4q}}, \quad \text{for} \quad V \geq \left(2M_5 \left(\frac{3^{\frac{q-1}{4q}} \bar{\pi}_1^{\frac{5q-1}{4q}}}{\omega_8} \right) \left(\frac{3^{\frac{1}{4}} \omega_6}{\underline{\pi}_1^{\frac{3}{4}}} \right) \right)^{\frac{4q}{2q-1}}
\end{aligned}
$$

and uniform ultimate boundedness is implied from lemma 2.6.

Remark 2.13. Note that even in the case of $M_5 = 0$, the system is able to be exactly robust with respect to persistent perturbations with an upper bound of M. This interesting feature is possible due to the discontinuous nature of the algorithm when $p = 0.5$.

2.4 Super-twisting

In this section, the super-twisting based algorithm is revisited by presenting a locally-Lipschitz strict Lyapunov function. The various convergence properties of the algorithm can be fully described through a single Lyapunov function structure. The time derivative of the Lyapunov function is able to avoid singularity due to its locally Lipschitz property. Due to the strictness of the Lyapunov function, different types of disturbances that the algorithm can withstand are shown, including non-Lipschitz disturbances.

2.4.1 System description

Consider the following super-twisting based family of algorithms:

$$\begin{aligned}\dot{z}_1 &= -k_3|z_1|^p \operatorname{sign}(z_1) + z_2 + d_1, \\ \dot{z}_2 &= -k_1|z_1|^{2p-1} \operatorname{sign}(z_1) + d_2\end{aligned} \tag{2 - 17}$$

where z_1, $z_2 \in \mathbb{R}$, are the scalar state variables, k_1, k_3 are positive constants, $p \geq 0.5$, and d_1 and d_2 are time-varying and/or nonlinear term of uncertainties bounded by:

$$|d_1| \leq M_1|z_1|^p + M_2, \text{ and } |d_2| \leq M_3|z_1|^{2p-1} + M_4,$$

with $|d_1| := \sup\{|\delta| : \delta \in \boldsymbol{K}[d_1]\}$, and $|d_2| := \sup\{|\delta| : \delta \in \boldsymbol{K}[d_2]\}$,

where $M_1 \geq 0$, $M_2 \geq 0$, $M_3 \geq 0$, and $M_4 \geq 0$ with the same $p \geq 0$ as that in (2 - 17). Note that for the case of p = 0.5, the algorithm contains discontinuity and the uncertainty d_2 is upper bounded by nonvanishing constant.

2.4.2 Stability analysis

For system (2 - 17), the following function

$$V(z_1, z_2) = \left(\frac{1}{2} z_2{}^2 + \frac{k_1}{2p}|z_1|^{2p}\right)^2 - r|z_1||z_2|^{4-\frac{1}{p}} \operatorname{sign}(z_1 z_2)$$

where r is a positive constant scalar, will be shown as a strict Lyapunov function. Note that the above function is locally Lipschitz and regular. It is differentiable everywhere for $p > 0.5$, and not differentiable on $z_1 = 0$ for $p = 0.5$.

Remark 2.14. It should be noted that the above Lyapunov function is still locally Lipschitz even when the algorithm is discontinuous, i.e. $p = 0.5$ (the conventional super-twisting algorithm). On the other hand, the Lyapunov functions presented in the literature, (i.e. [26] and [27]), for the conventional super-twisting algorithm, are not locally-Lipschitz, which in turn will cause singularity in its time derivative. This singularity will cause issue as noted by [77], when the conventional super-twisting algorithm is use in tandem with adaptive controller. It is also mentioned in [77], that the problem may be resolved whenever a continuously differentiable Lyapunov function is available. Particularly, they proposed to use the continuously differentiable Lyapunov function of [68], albeit, it produces a more conservative condition on the controller gains than would be obtained from the non-Lipschitz one. Thus, it is important to develop a locally Lipschitz Lyapunov function for this system, while conforming to the gain conditions of the non-Lipschitz Lyapunov function.

Note that the Lyapunov function V can be lower bounded by,

$$V(z_1,z_2)\geq\frac{1}{4}|z_2|^4+\left(\frac{k_1}{2p}\right)^2|z_1|^{4p}+\left(\frac{k_1}{2p}\right)|z_1|^{2p}|z_2|^2-r|z_1||z_2|^{4-\frac{1}{p}}$$

Using lemma 2.1,

$$|z_1|^{4p}+|z_2|^4\geq(4p)^{\frac{1}{4p}}|z_1|\left(\frac{4p}{4p-1}\right)^{\frac{4p-1}{4p}}|z_2|^{4-\frac{1}{p}}$$

Hence, for

$$\left(\frac{p}{8p-2}\right)^{\frac{4p-1}{4p}}\left(\frac{1}{2p}\right)^{\frac{1}{4p}}k_1^{\frac{1}{2p}}>r \tag{2 - 18}$$

$$V(z_1,z_2)\geq\frac{1}{8}|z_2|^4+\frac{1}{2}\left(\frac{k_1}{2p}\right)^2|z_1|^{4p}\geq\underline{\pi}_1\left(|z_2|^4+|z_1|^{4p}\right)$$

where $\underline{\pi}_1=\min\left\{\frac{1}{8},\frac{1}{2}\left(\frac{k_1}{2p}\right)^2\right\}$.

Similarly, it can be upper-bounded by,

$$V(z_1,z_2)\leq\frac{1}{4}|z_2|^4+\left(\frac{k_1}{2p}\right)^2|z_1|^{4p}+\left(\frac{k_1}{2p}\right)|z_1|^{2p}|z_2|^2+r|z_1||z_2|^{4-\frac{1}{p}}$$

Using lemma 2.1,

$$|z_1|^{2p}|z_2|^2\leq\frac{1}{2}|z_1|^{4p}+\frac{1}{2}|z_2|^4$$
$$|z_1||z_2|^{4-\frac{1}{p}}\leq\frac{1}{4p}|z_1|^{4p}+\left(\frac{4p-1}{4p}\right)|z_2|^4$$

Thus,

$$V(z_1, z_2) \le \bar{\pi}_1 \left(|z_1|^{4p} + |z_2|^4 \right)$$

where $\bar{\pi}_1 = \max\left\{ \left(\frac{k_1}{2p}\right)^2 + \left(\frac{k_1}{4p}\right) + \frac{r}{4p},\ \frac{1}{4} + \left(\frac{k_1}{4p}\right) + r\left(\frac{4p-1}{4p}\right) \right\}$.

Thus, V is positive definite and radially unbounded. Since (2 - 17) is a differential equation that has discontinuous right-hand side, i.e. when $p = 0.5$ and since no continuity assumption is made on d_1, and d_2, its solution are understood in the sense of Filippov (see definition 2.1). According to lemma 2.4, the time derivative of non-smooth Lyapunov function exists almost everywhere

$$\frac{d}{dt} V(\mathbf{z}(t), t) \overset{a.e.}{\in} \dot{\tilde{V}}(\mathbf{z}(t), t) = \bigcap_{\xi \in \partial V(\mathbf{z}(t), t)} \xi^T K \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} (\mathbf{z}, t),$$

Since the controller (2 - 17) is discontinuous when $p = 0.5$, for ease of presentation, the analysis is separated for the case of $p > 0.5$ and $p = 0.5$.

a) Case 1: For $p > 0.5$

Note that for $p > 0$, V is continuously differentiable, hence

$$\dot{\tilde{V}}(z_1, z_2) = \bigcap_{\xi \in \partial V(\mathbf{z}(t), t)} \xi^T K \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \nabla V^T K \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix}$$

$$
\begin{aligned}
&\subset -\frac{k_1{}^2 k_3}{p}|z_1|^{5p-1} - k_1 k_3 |z_1|^{3p-1}|z_2|^2 - r|z_2|^{5-\frac{1}{p}} + r k_3 |z_1|^p |z_2|^{4-\frac{1}{p}}\operatorname{sign}(z_1 z_2) \\
&\quad + r\left(4-\frac{1}{p}\right) k_1 |z_1|^{2p}|z_2|^{3-\frac{1}{p}} \\
&\quad + \boldsymbol{K}[d_1]\left(k_1|z_1|^{2p-1}|z_2|^2\operatorname{sign}(z_1) + \frac{k_1{}^2}{p}|z_1|^{4p-1}\operatorname{sign}(z_1) - r|z_2|^{4-\frac{1}{p}}\operatorname{sign}(z_2)\right) \\
&\quad + \boldsymbol{K}[d_2]\left(z_2{}^3 + \frac{k_1}{p}|z_1|^{2p} z_2 - r\left(4-\frac{1}{p}\right)|z_1||z_2|^{3-\frac{1}{p}}\operatorname{sign}(z_1)\right)
\end{aligned}
$$

After rearrangement:

$$
\begin{aligned}
\dot{\tilde{V}}(z_1,z_2) &\le -k_1(k_3 - M_1)|z_1|^{3p-1}|z_2|^2 - \frac{k_1{}^2(k_3-M_1)}{2p}|z_1|^{5p-1} - \frac{r}{2}|z_2|^{5-\frac{1}{p}} \\
&\quad + \dot{V}_1 + M_2\left(k_1|z_1|^{2p-1}|z_2|^2 + \frac{k_1{}^2}{p}|z_1|^{4p-1} + r|z_2|^{4-\frac{1}{p}}\right) \\
&\quad + M_4\left(|z_2|^3 + \frac{k_1}{p}|z_1|^{2p}|z_2| + r\left(4-\frac{1}{p}\right)|z_1||z_2|^{3-\frac{1}{p}}\right)
\end{aligned}
$$

where

$$
\begin{aligned}
\dot{V}_1 &= -\frac{k_1{}^2(k_3-M_1)}{2p}|z_1|^{5p-1} - \frac{r}{2}|z_2|^{5-\frac{1}{p}} + r(k_3+M_1)|z_1|^p|z_2|^{4-\frac{1}{p}} \\
&\quad + r\left(4-\frac{1}{p}\right)(k_1+M_3)|z_1|^{2p}|z_2|^{3-\frac{1}{p}} + M_3|z_1|^{2p-1}|z_2|^3 + M_3\frac{k_1}{p}|z_1|^{4p-1}|z_2|
\end{aligned}
$$

Applying lemma 2.1,

$$
-|z_1|^{5p-1} - |z_2|^{5-\frac{1}{p}} \le -\left(\frac{5p-1}{p}\right)^{\frac{p}{5p-1}}|z_1|^p\left(\frac{5p-1}{4p-1}\right)^{\frac{4p-1}{5p-1}}|z_2|^{4-\frac{1}{p}},
$$

$$
-|z_1|^{5p-1} - |z_2|^{5-\frac{1}{p}} \le -\left(\frac{5p-1}{2p}\right)^{\frac{2p}{5p-1}}|z_1|^{2p}\left(\frac{5p-1}{3p-1}\right)^{\frac{3p-1}{5p-1}}|z_2|^{3-\frac{1}{p}},
$$

$$-|z_1|^{5p-1}-|z_2|^{5-\frac{1}{p}}\le -\left(\frac{5p-1}{2p-1}\right)^{\frac{2p-1}{5p-1}}|z_1|^{2p-1}\left(\frac{5p-1}{3p}\right)^{\frac{3p}{5p-1}}|z_2|^3,$$

$$-|z_1|^{5p-1}-|z_2|^{5-\frac{1}{p}}\le -\left(\frac{5p-1}{4p-1}\right)^{\frac{4p-1}{5p-1}}|z_1|^{4p-1}\left(\frac{5p-1}{p}\right)^{\frac{p}{5p-1}}|z_2|,$$

Thus, if the following inequalities

$$\min\left\{\lambda_1\frac{k_1^{\,2}(k_3-M_1)}{(k_3+M_1)^{\frac{5p-1}{p}}},\ \lambda_2\frac{k_1^{\,2}(k_3-M_1)}{(k_1+M_3)^{\frac{5p-1}{2p}}}\right\}>r>\max\left\{\begin{array}{l}\lambda_3\dfrac{M_3^{\frac{5p-1}{3p}}}{k_1^{\frac{4p-2}{3p}}(k_3-M_1)^{\frac{2p-1}{3p}}},\\ \lambda_4\dfrac{M_3^{\frac{5p-1}{p}}}{k_1^{\frac{3p-1}{p}}(k_3-M_1)^{\frac{4p-1}{p}}}\end{array}\right\} \tag{2 - 19}$$

where

$$\lambda_1=\left(\frac{5p-1}{8p^2}\right)\left(\frac{5p-1}{8p-8}\right)^{\frac{4p-1}{p}},\quad \lambda_2=\left(\frac{5p-1}{16p^2}\right)\left(\frac{5p-1}{24p-8}\right)^{\frac{3p-1}{2p}}\left(\frac{p}{4p-1}\right)^{\frac{5p-1}{2p}},$$

$$\lambda_3=\left(\frac{16p^2-8p}{5p-1}\right)^{\frac{2p-1}{3p}}\left(\frac{24p}{5p-1}\right),\quad \lambda_4=\left(\frac{32p^2-8p}{5p-1}\right)^{\frac{4p-1}{p}}\frac{8p}{(5p-1)p^{\frac{5p-1}{p}}}$$

hold then the function $\dot{V}_1$ is negative definite. Then,

$$\begin{aligned}\dot{\tilde{V}}(z_1,z_2)\le&-\frac{k_1^{\,2}(k_3-M_1)}{2p}|z_1|^{5p-1}-\frac{r}{2}|z_2|^{5-\frac{1}{p}}\\&+M_2\left(k_1|z_1|^{2p-1}|z_2|^2+\frac{k_1^{\,2}}{p}|z_1|^{4p-1}+r|z_2|^{4-\frac{1}{p}}\right)\\&+M_4\left(|z_2|^3+\frac{k_1}{p}|z_1|^{2p}|z_2|+r\left(4-\frac{1}{p}\right)|z_1||z_2|^{3-\frac{1}{p}}\right)\end{aligned}$$

Applying lemma 2.1,

$$|z_1|^{2p-1}|z_2|^2 \le \left(\frac{2p-1}{4p-1}\right)|z_1|^{4p-1} + \left(\frac{2p}{4p-1}\right)|z_2|^{4-\frac{1}{p}},$$

$$|z_1|^{2p}|z_2| \le \frac{2}{3}|z_1|^{3p} + \frac{1}{3}|z_2|^3,$$

$$|z_1||z_2|^{3-\frac{1}{p}} \le \frac{1}{3p}|z_1|^{3p} + \left(\frac{3p-1}{3p}\right)|z_2|^3,$$

together with lemma 2.2, and the bounds of the Lyapunov function, we have:

$$\begin{aligned}\dot{\tilde{V}}(z_1,z_2) &\le -\omega_1\left(|z_1|^{5p-1} + |z_2|^{5-\frac{1}{p}}\right) + M_2\omega_2\left(|z_1|^{4p-1} + |z_2|^{4-\frac{1}{p}}\right) + M_4\omega_3\left(|z_1|^{3p} + |z_2|^3\right) \\ &= -\omega_1\left(\left(|z_1|^{4p}\right)^{\frac{5p-1}{4p}} + \left(|z_2|^4\right)^{\frac{5p-1}{4p}}\right) + M_2\omega_2\left(\left(|z_1|^{4p}\right)^{\frac{4p-1}{4p}} + \left(|z_2|^4\right)^{\frac{4p-1}{4p}}\right) \\ &\quad + M_4\omega_3\left(\left(|z_1|^{4p}\right)^{\frac{3}{4}} + \left(|z_2|^4\right)^{\frac{3}{4}}\right) \\ &\le -\left(\frac{\omega_4}{\overline{\pi}_1^{\frac{5p-1}{4p}}}\right)V^{\frac{5p-1}{4p}} + M_2\left(\frac{2^{\frac{1}{4p}}\omega_2}{\underline{\pi}_1^{\frac{4p-1}{4p}}}\right)V^{\frac{4p-1}{4p}} + M_4\left(\frac{2^{\frac{1}{4}}\omega_3}{\underline{\pi}_1^{\frac{3}{4}}}\right)V^{\frac{3}{4}} \qquad (2 - 20)\end{aligned}$$

where

$$\omega_1 = \min\left\{\frac{k_1^2(k_3 - M_1)}{2p},\ \frac{r}{2}\right\}, \quad \omega_2 = \max\left\{k_1\left(\frac{2p-1}{4p-1}\right) + \frac{k_1^2}{p},\ \ k_1\left(\frac{2p}{4p-1}\right) + r\right\}$$

$$\omega_3 = \max\left\{\frac{2k_1}{3p} + \left(4 - \frac{1}{p}\right)\left(\frac{r}{3p}\right),\ \ 1 + \frac{k_1}{3p} + r\left(4 - \frac{1}{p}\right)\left(\frac{3p-1}{3p}\right)\right\},$$

$$\omega_4 = \begin{cases} \omega_1 & \text{for } 0.5 < p \le 1, \\ \dfrac{\omega_1}{2^{\frac{p-1}{4p}}} & \text{for } p > 1 \end{cases}$$

Remark 2.15. Note that the nonlinear inequalities (2 - 18) and (2 - 19) are feasible for sufficiently large $k_1 > 0$, and $k_3 > 0$. Thus, an $r > 0$ can always exist. Consider some particular cases:

1. Note that (2 - 19) is feasible with respect to k_1, k_3 for any $M_1 \geq 0$ and $M_3 \geq 0$.
2. If $M_3 = 0$, then (2 - 19) is satisfied for any $k_3 > M_1$. This coincides with the conditions obtained through the weak Lyapunov function, i.e. the energy function $E(z_1, z_2) = \dfrac{1}{2} z_2{}^2 + \dfrac{k_1}{2p} |z_1|^{2p}$.

Remark 2.16. Hence, when $M_2 = M_4 = 0$, i.e. without persistent perturbations, from lemma 2.5-3, the system (2 - 17) will have finite time convergence for $0.5 < p < 1$, with the settling time estimate,

$T(z_{10}, z_{20}) \leq \left(\dfrac{\bar{\pi}_1^{\frac{5p-1}{4p}}}{\omega_4} \right) \left(\dfrac{4p}{1-p} \right) [V(z_{10}, z_{20})]^{\frac{1-p}{4p}}$, where (z_{10}, z_{20}) are the initial states of the system.

Similarly, exponential and asymptotical convergence for $p = 1$ and $p > 1$ respectively, can be concluded from lemma 2.5 as well. The above results are possible due to the negative definiteness of the time derivative of the Lyapunov function, i.e. strict Lyapunov function.

Remark 2.17. When persistent perturbations occur on the system, $M_2 \neq 0$ and/or $M_4 \neq 0$, from (2 - 20)

$$\dot{\tilde{V}}(z_1,z_2)\leq -\frac{1}{3}\left(\frac{\omega_4}{\bar{\pi}_1^{\frac{5p-1}{4p}}}\right)V^{\frac{5p-1}{4p}}-V^{\frac{4p-1}{4p}}\left(\frac{1}{3}\left(\frac{\omega_4}{\bar{\pi}_1^{\frac{5p-1}{4p}}}\right)V^{\frac{1}{4}}-M_2\left(\frac{2^{\frac{1}{4p}}\omega_2}{\underline{\pi}_1^{\frac{4p-1}{4p}}}\right)\right)$$

$$-V^{\frac{3}{4}}\left(\frac{1}{3}\left(\frac{\omega_4}{\bar{\pi}_1^{\frac{5p-1}{4p}}}\right)V^{\frac{2p-1}{4p}}-M_4\left(\frac{2^{\frac{1}{4}}\omega_3}{\underline{\pi}_1^{\frac{3}{4}}}\right)\right)$$

$$\leq -\frac{1}{3}\left(\frac{\omega_4}{\bar{\pi}_1^{\frac{5p-1}{4p}}}\right)V^{\frac{5p-1}{4p}},\quad \text{for}\quad V\geq \max\left\{\begin{array}{c}\left(3M_2\left(\frac{\bar{\pi}_1^{\frac{5p-1}{4p}}}{\omega_4}\right)\left(\frac{2^{\frac{1}{4p}}\omega_2}{\underline{\pi}_1^{\frac{4p-1}{4p}}}\right)\right)^4,\\ \left(3M_4\left(\frac{\bar{\pi}_1^{\frac{5p-1}{4p}}}{\omega_4}\right)\left(\frac{2^{\frac{1}{4}}\omega_3}{\underline{\pi}_1^{\frac{3}{4}}}\right)\right)^{\frac{4p}{2p-1}}\end{array}\right\},$$

hence, from lemma 2.6, the system (2 - 17) is uniformly ultimately bounded.

b) Case 2: For $p = 0.5$

For $p = 0.5$, V is not differentiable on $z_1 = 0$:

$$\dot{\tilde{V}}(z_1,z_2)=\bigcap_{\xi\in\partial V(\mathbf{z}(t),t)}\xi^T K\begin{bmatrix}\dot{z}_1\\ \dot{z}_2\end{bmatrix}\subset \dot{\tilde{V}}_1+\dot{\tilde{V}}_2$$

where

$$\dot{\tilde{V}}_1=\bigcap_{\xi\in\partial V(\mathbf{z}(t),t)}\xi^T K[f](z_1,z_2),\ \dot{\tilde{V}}_2=\bigcap_{\xi\in\partial V(\mathbf{z}(t),t)}\xi^T K\begin{bmatrix}d_1\\ d_2\end{bmatrix},$$

$$K[f](z_1,z_2)\in\begin{bmatrix}-k_3|z_1|^{0.5}\operatorname{sign}(z_1)+z_2\\ -k_1\mathrm{SGN}(z_1)\end{bmatrix}$$

$$
\partial V = \boldsymbol{K}[\nabla V] = \boldsymbol{K}\begin{bmatrix} \dfrac{\partial V}{\partial z_1} \\ \dfrac{\partial V}{\partial z_2} \end{bmatrix}
\subset \begin{bmatrix} \boldsymbol{K}\left[\dfrac{\partial V}{\partial z_1}\right] \\ \boldsymbol{K}\left[\dfrac{\partial V}{\partial z_2}\right] \end{bmatrix}
= \begin{cases} \begin{bmatrix} \left(z_2{}^2 + 2k_1|z_1|\right)\left(k_1 \operatorname{sign}(z_1)\right) - r|z_2|^2 \operatorname{sign}(z_2) \\ \left(z_2{}^2 + 2k_1|z_1|\right)(z_2) - 2r\, z_1|z_2| \end{bmatrix}, & \forall z_1 \neq 0,\ z_2 \in \mathbb{R} \\ \begin{bmatrix} z_2{}^2 k_1[-1,1] - r|z_2|^2 \operatorname{sign}(z_2) \\ z_2{}^3 \end{bmatrix}, & \forall z_1 = 0,\ z_2 \in \mathbb{R} \end{cases}
$$

Let us define:

$$
\left|\frac{\partial V}{\partial z_1}\right| := \sup\left\{|\xi_1| : \xi_1 \in \boldsymbol{K}\left[\frac{\partial V}{\partial z_1}\right]\right\}, \text{ and } \left|\frac{\partial V}{\partial z_2}\right| := \sup\left\{|\xi_2| : \xi_2 \in \boldsymbol{K}\left[\frac{\partial V}{\partial z_2}\right]\right\},
$$

with $\left|\dfrac{\partial V}{\partial z_1}\right| \leq k_1|z_2|^2 + 2k_1{}^2|z_1| + r|z_2|^2$, and $\left|\dfrac{\partial V}{\partial z_2}\right| \leq |z_2|^3 + 2k_1|z_1||z_2| + 2r|z_1||z_2|$.

Thus, the term

$$
\begin{aligned}
\dot{\tilde{V}}_2 &= \bigcap_{(\xi_1,\xi_2)^T \in \partial V(\mathbf{z}(t),t)} \xi_1 K[d_1] + \xi_2 K[d_2] \\
&\leq \left|\frac{\partial V}{\partial z_1}\right||d_1| + \left|\frac{\partial V}{\partial z_2}\right||d_2| \\
&\leq \left(M_1|z_1|^{0.5} + M_2\right)\left(2k_1{}^2|z_1| + (k_1 + r)|z_2|^2\right) + M\left(|z_2|^3 + 2k_1|z_1||z_2| + 2r|z_1||z_2|\right)
\end{aligned}
$$

where M be defined as, $M := M_3 + M_4$, since for p = 0.5, $|d_1| \leq M_1|z_1|^{0.5} + M_2$, and $|d_2| \leq M_3 + M_4$.

Computing $\dot{\tilde{V}}_1$ for each case, we have

For $z_1 \neq 0$, $z_2 \in \mathbb{R}$:

$$\dot{\tilde{V}}_1 = -2k_1^2 k_3 |z_1|^{1.5} - k_1 k_3 |z_1|^{0.5} |z_2|^2 - r|z_2|^3 + rk_3 |z_1|^{0.5} |z_2|^2 \operatorname{sign}(z_1 z_2) + 2rk_1 |z_1||z_2|$$

For $z_1 = 0$, $z_2 \in \mathbb{R}$:

Let $(\xi_2 k_1 z_2^2 - r|z_2|^2 \operatorname{sign}(z_2), z_2^3)^T$ with $\xi_2 \in [-1, 1]$ be an arbitrary element of $\partial V(z_1, z_2)$, then

$$\begin{aligned}\xi^{\mathrm{T}} K[f](z_1, z_2) &= \begin{bmatrix} \xi_2 k_1 z_2^2 - r|z_2|^2 \operatorname{sign}(z_2) \\ z_2^3 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} z_2 \\ -[-1, 1]k_1 \end{bmatrix} \\ &= [\xi_2 - 1, \xi_2 + 1] k_1 z_2^3 - r|z_2|^3\end{aligned}$$

hence $\dot{\tilde{V}}_1 = \bigcap_{\xi_2 \in [-1, 1]} [\xi_2 - 1, \xi_2 + 1] k_1 z_2^3 - r|z_2|^3 = -r|z_2|^3$

Thus, for all $(z_1, z_2) \in \mathbb{R}^2$:

$$\dot{\tilde{V}}_1 = -2k_1^2 k_3 |z_1|^{1.5} - k_1 k_3 |z_1|^{0.5} |z_2|^2 - r|z_2|^3 + rk_3 |z_1|^{0.5} |z_2|^2 \operatorname{sign}(z_1 z_2) + 2rk_1 |z_1||z_2|$$

Hence, the generalized time derivative of the Lyapunov function, after rearrangement:

$$\dot{\tilde{V}} = \dot{\tilde{V}}_1 + \dot{\tilde{V}}_2 \leq -k_1^2 (k_3 - M_1)|z_1|^{1.5} - \frac{r}{2}|z_2|^3 + \dot{V}_2 + M_2 \left(2k_1^2 |z_1| + (k_1 + r)|z_2|^2\right)$$

where

$$\begin{aligned}\dot{V}_2 = &-k_1^2 (k_3 - M_1)|z_1|^{1.5} - \frac{r}{4}|z_2|^3 - \left(\frac{r}{4} - M\right)|z_2|^3 - \left(k_1 (k_3 - M_1) - r(k_3 + M_1)\right)|z_1|^{0.5} |z_2|^2 \\ &+ 2r(k_1 + M)|z_1||z_2| + 2Mk_1 |z_1||z_2|\end{aligned}$$

Applying lemma 2.1

$$-|z_1|^{1.5} - |z_2|^3 \le -(1.5)^{\frac{1}{1.5}}|z_1|(3)^{\frac{1}{3}}|z_2|$$

Thus, if the following inequalities

$$\min\left\{\begin{array}{l}\dfrac{k_1(k_3-M_1)}{(k_3+M_1)}, \\ \left(\dfrac{3}{4}\right)\left(\dfrac{3}{8}\right)^{\frac{1}{2}}\left(\dfrac{1}{2^{\frac{3}{2}}}\right)\dfrac{k_1{}^2(k_3-M_1)}{(k_1+M)^{\frac{3}{2}}}\end{array}\right\} > r > \max\left\{\begin{array}{l}4M, \\ \left(\dfrac{4}{3}\right)^2\left(\dfrac{8}{3}\right)2^3\,\dfrac{M^3}{k_1(k_3-M_1)^2}\end{array}\right\} \quad (2\text{ - }21)$$

hold then the function $\dot{V}_2$ is negative definite. Then,

$$\dot{\tilde{V}} \le -k_1{}^2(k_3-M_1)|z_1|^{1.5} - \frac{r}{2}|z_2|^3 + M_2\left(2k_1{}^2|z_1| + (k_1+r)|z_2|^2\right)$$

Employing lemma 2.2, and the bounds of the Lyapunov function, we have:

$$\begin{aligned}\dot{\tilde{V}} &\le -\omega_5\left(|z_1|^{1.5} + |z_2|^3\right) + M_2\omega_6\left(|z_1| + |z_2|^2\right) \\ &= -\omega_5\left(\left(|z_1|^2\right)^{\frac{3}{4}} + \left(|z_2|^4\right)^{\frac{3}{4}}\right) + M_2\omega_6\left(\left(|z_1|^2\right)^{\frac{1}{2}} + \left(|z_2|^4\right)^{\frac{1}{2}}\right) \\ &\le -\omega_5\left(|z_1|^2 + |z_2|^4\right)^{\frac{3}{4}} + M_2 2^{\frac{1}{2}}\omega_6\left(|z_1|^2 + |z_2|^4\right)^{\frac{1}{2}} \\ &\le -\left(\frac{\omega_5}{\overline{\pi}_1{}^{\frac{3}{4}}}\right)V^{\frac{3}{4}} + M_2\left(\frac{2^{\frac{1}{2}}\omega_6}{\underline{\pi}_1{}^{\frac{1}{2}}}\right)V^{\frac{1}{2}} \end{aligned} \quad (2\text{ - }22)$$

where

$$\omega_5 = \min\left\{k_1{}^2(k_3-M_1),\ \frac{r}{2}\right\}, \quad \omega_6 = \max\left\{2k_1{}^2,\ (k_1+r)\right\}$$

Remark 2.18. Note that the nonlinear inequalities (2 - 18) and (2 - 21) is feasible with respect to k_1 , k_3 for any $M \geq 0$ and $M_1 \geq 0$, which guarantees the existence of an $r > 0$. Also, when $M = 0$, (2 - 21) is easily satisfied for any $k_1 > 0$ and $k_3 > M_1$.

Remark 2.19. Thus, when $M_2 = 0$, utilising lemma 2.5-3, the system (2 - 17) will have finite time convergence for $p = 0.5$, with the settling time estimate,

$$T(z_{10}, z_{20}) \leq \left(\frac{4\overline{\pi}_1^{\frac{3}{4}}}{\omega_5} \right) [V(z_{10}, z_{20})]^{\frac{1}{4}},$$

where (z_{10}, z_{20}) are the initial states of the system, in which case the system is able to withstand persistent perturbations, bounded by M.

Remark 2.20. While for the case of $M_2 \neq 0$, from (2 - 22)

$$\begin{aligned} \dot{\hat{V}} &\leq -\frac{1}{2}\left(\frac{\omega_5}{\overline{\pi}_1^{\frac{3}{4}}}\right)V^{\frac{3}{4}} - V^{\frac{1}{2}}\left(\frac{1}{2}\left(\frac{\omega_5}{\overline{\pi}_1^{\frac{3}{4}}}\right)V^{\frac{1}{4}} - M_2\left(\frac{2^{\frac{1}{2}}\omega_6}{\underline{\pi}_1^{\frac{1}{2}}}\right)\right) \\ &\leq -\frac{1}{2}\left(\frac{\omega_5}{\overline{\pi}_1^{\frac{3}{4}}}\right)V^{\frac{3}{4}}, \quad \text{for} \quad V \geq \left(2M_2\left(\frac{\overline{\pi}_1^{\frac{3}{4}}}{\omega_5}\right)\left(\frac{2^{\frac{1}{2}}\omega_6}{\underline{\pi}_1^{\frac{1}{2}}}\right)\right)^4 \end{aligned}$$

where uniformly ultimately bounded is concluded from employing lemma 2.6.

2.5 Generic Super-twisting

Employing the results of section 2.2.3, a generic super-twisting based family of controllers is presented. In a similar spirit as per section 2.2.2 (generic twisting), the algorithm comprise linear sum of two different homogeneity super-twisting based algorithms. By means of strict and locally-Lipschitz Lyapunov functions, the family of algorithms are shown to exhibit the properties of their individual components while at the same time, singularity is avoided in its time derivative.

2.5.1 System description

Consider the following generic super-twisting based dynamics:

$$\begin{aligned}\dot{z}_1 &= -k_3|z_1|^p \operatorname{sign}(z_1) - k_{3n}|z_1|^q \operatorname{sign}(z_1) + z_2 + d_1, \\ \dot{z}_2 &= -k_1|z_1|^{2p-1} \operatorname{sign}(z_1) - k_{1n}|z_1|^{2q-1} \operatorname{sign}(z_1) + d_2\end{aligned} \qquad (2 - 23)$$

where z_1, $z_2 \in \mathbb{R}$, are the scalar state variables, k_1, $k_{1\mathrm{n}}$, k_3, $k_{3\mathrm{n}}$ are positive constants, $0.5 \le p \le 1$, and $1 \le q$. Also, d_1 and d_2 are time-varying and/or nonlinear term of uncertainties bounded by:

$|d_1| \le M_1|z_1|^p + M_2|z_1|^q + M_3$, and $|d_2| \le M_4|z_1|^{2p-1} + M_5|z_1|^{2q-1} + M_6$,

with $|d_1| := \sup\{|\delta| : \delta \in \boldsymbol{K}[d_1]\}$, and $|d_2| := \sup\{|\delta| : \delta \in \boldsymbol{K}[d_2]\}$,

where $M_1 \ge 0$, $M_2 \ge 0$, $M_3 \ge 0$, $M_4 \ge 0$, $M_5 \ge 0$, $M_6 \ge 0$ with the same p and q as that in (2 - 23). Note that for the case of $p = 0.5$, the algorithm contains discontinuity and the uncertainty d_2 is upper bounded by nonvanishing constant.

2.5.2 Stability analysis

The system above is essentially a summation of two different degree of homogeneity of the super-twisting-based algorithm considered in section 2.2.3. Hence, a summation of two different degree of homogeneity Lyapunov function based on section 2.2.3 is proposed for system (2 - 23), namely

$$V(z_1,z_2)=\left(\frac{1}{2}z_2^{\,2}+\frac{k_1}{2p}|z_1|^{2p}+\frac{k_{1n}}{2q}|z_1|^{2q}\right)^2-r|z_1||z_2|^{4-\frac{1}{p}}\operatorname{sign}(z_1z_2)$$
$$-r|z_1||z_2|^{4-\frac{1}{q}}\operatorname{sign}(z_1z_2)$$

where r is a positive constant scalar. Note that the above function is locally Lipschitz and regular. It is differentiable everywhere for $p > 0.5$, and not differentiable on $z_1 = 0$ for $p = 0.5$.

Note that the Lyapunov function, V can be lower bounded by,

$$V(z_1,z_2)\geq\frac{1}{4}|z_2|^4+\frac{k_1^{\,2}}{4p^2}|z_1|^{4p}+\frac{k_{1n}^{\,2}}{4q^2}|z_1|^{4q}+\frac{k_1}{2p}|z_1|^{2p}|z_2|^2+\frac{k_{1n}}{2q}|z_1|^{2q}|z_2|^2$$
$$+\frac{k_1k_{1n}}{2pq}|z_1|^{2p+2q}-r|z_1||z_2|^{4-\frac{1}{p}}-r|z_1||z_2|^{4-\frac{1}{q}}$$

Using lemma 2.1,

$$|z_1|^{4p}+|z_2|^4\geq(4p)^{\frac{1}{4p}}|z_1|\left(\frac{4p}{4p-1}\right)^{\frac{4p-1}{4p}}|z_2|^{4-\frac{1}{p}},$$

$$|z_1|^{4q}+|z_2|^4\geq(4q)^{\frac{1}{4q}}|z_1|\left(\frac{4q}{4q-1}\right)^{\frac{4q-1}{4q}}|z_2|^{4-\frac{1}{q}},$$

Hence, for

$$\min\left\{\left(\frac{k_1^{\ 2}}{2p}\right)^{\frac{1}{4p}}\left(\frac{4p}{4p-1}\frac{1}{12}\right)^{\frac{4p-1}{4p}},\ \left(\frac{k_{1n}^{\ 2}}{2q}\right)^{\frac{1}{4q}}\left(\frac{4q}{4q-1}\frac{1}{12}\right)^{\frac{4q-1}{4q}}\right\} > r, \tag{2 - 24}$$

$$V(z_1, z_2) \geq \underline{\pi}_1\left(|z_1|^{4p} + |z_1|^{4q} + |z_2|^4\right)$$

where $\underline{\pi}_1 = \min\left\{\frac{k_1^{\ 2}}{8p^2}, \frac{k_{1n}^{\ 2}}{8q^2}, \frac{1}{12}\right\}$.

Also, its upper bound can be obtained as,

$$V(z_1, z_2) \leq \frac{1}{4}|z_2|^4 + \frac{k_1^{\ 2}}{4p^2}|z_1|^{4p} + \frac{k_{1n}^{\ 2}}{4q^2}|z_1|^{4q} + \frac{k_1}{2p}|z_1|^{2p}|z_2|^2 + \frac{k_{1n}}{2q}|z_1|^{2q}|z_2|^2$$
$$+ \frac{k_1 k_{1n}}{2pq}|z_1|^{2p+2q} + r|z_1||z_2|^{4-\frac{1}{p}} + r|z_1||z_2|^{4-\frac{1}{q}}$$

Using lemma 2.1, we have,

$$\frac{k_1}{2p}|z_1|^{2p}|z_2|^2 \leq \frac{k_1}{4p}|z_1|^{4p} + \frac{k_1}{4p}|z_2|^4,$$

$$\frac{k_{1n}}{2q}|z_1|^{2q}|z_2|^2 \leq \frac{k_{1n}}{4q}|z_1|^{4q} + \frac{k_{1n}}{4q}|z_2|^4,$$

$$\frac{k_1 k_{1n}}{2pq}|z_1|^{2p+2q} \leq \frac{k_1 k_{1n}}{4pq}|z_1|^{4p} + \frac{k_1 k_{1n}}{4pq}|z_1|^{4q},$$

$$r|z_1||z_2|^{4-\frac{1}{p}} \leq \frac{r}{4p}|z_1|^{4p} + \frac{r(4p-1)}{4p}|z_2|^4,$$

$$r|z_1||z_2|^{4-\frac{1}{q}} \le \frac{r}{4q}|z_1|^{4q} + \frac{r(4q-1)}{4q}|z_2|^4$$

Then,

$$V(z_1, z_2) \le \bar{\pi}_1 \left(|z_1|^{4p} + |z_1|^{4q} + |z_2|^4\right)$$

where

$$\bar{\pi}_1 = \max\left\{ \begin{array}{l} \left(\frac{k_1{}^2}{4p^2} + \frac{k_1}{4p} + \frac{r}{4p} + \frac{k_1 k_{1n}}{4pq}\right), \left(\frac{k_{1n}{}^2}{4q^2} + \frac{k_{1n}}{4q} + \frac{r}{4q} + \frac{k_1 k_{1n}}{4pq}\right), \\ \left(\frac{1}{4} + \frac{k_1}{4p} + \frac{k_{1n}}{4q} + \frac{r(4p-1)}{4p} + \frac{r(4q-1)}{4q}\right) \end{array} \right\}.$$

Thus, V is positive definite and radially unbounded. Since (2 - 23) is a differential equation that has discontinuous right-hand side, i.e. when $p = 0.5$ and since no continuity assumption is made on d_1, and d_2, its solution are understood in the sense of Filippov (see definition 2.1). According to lemma 2.4, the time derivative of non-smooth Lyapunov function exists almost everywhere

$$\frac{d}{dt}V(\mathbf{z}(t), t) \overset{a.e.}{\in} \dot{\tilde{V}}(\mathbf{z}(t), t) = \bigcap_{\xi \in \partial V(\mathbf{z}(t), t)} \xi^T K\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix}(\mathbf{z}, t).$$

Since the controller (2 - 23) is discontinuous when $p = 0.5$, for ease of presentation, the analysis is separated for two different cases of $p > 0.5$ and $p = 0.5$.

a) Case 1: For $0.5 < p \le 1$, and $1 \le q$

Note that for $p > 0.5$, V is continuously differentiable, hence

$$\dot{\tilde{V}}(z_1,z_2)=\bigcap_{\xi\in\partial V(\mathbf{z}(t),t)}\xi^T K\begin{bmatrix}\dot{z}_1\\ \dot{z}_2\end{bmatrix}=\nabla V^T K\begin{bmatrix}\dot{z}_1\\ \dot{z}_2\end{bmatrix}$$

$$\begin{aligned}
&\subset -k_1k_3|z_1|^{3p-1}|z_2|^2-k_1k_{3n}|z_1|^{2p+q-1}|z_2|^2-k_{1n}k_3|z_1|^{p+2q-1}|z_2|^2\\
&-k_{1n}k_{3n}|z_1|^{3q-1}|z_2|^2-\frac{k_1}{p}k_1k_3|z_1|^{5p-1}-\frac{k_1}{p}k_1k_{3n}|z_1|^{4p+q-1}\\
&-\frac{k_1}{p}k_{1n}k_3|z_1|^{3p+2q-1}-\frac{k_1}{p}k_{1n}k_{3n}|z_1|^{2p+3q-1}-\frac{k_{1n}}{q}k_1k_3|z_1|^{2q+3p-1}\\
&-\frac{k_{1n}}{q}k_1k_{3n}|z_1|^{2p+3q-1}-\frac{k_{1n}}{q}k_{1n}k_3|z_1|^{p+4q-1}-\frac{k_{1n}}{q}k_{1n}k_{3n}|z_1|^{5q-1}\\
&-r|z_2|^{5-\frac{1}{p}}-r|z_2|^{5-\frac{1}{q}}+rk_3|z_1|^{p}|z_2|^{4-\frac{1}{p}}\operatorname{sign}(z_1z_2)\\
&+rk_{3n}|z_1|^{q}|z_2|^{4-\frac{1}{p}}\operatorname{sign}(z_1z_2)+r\left(4-\frac{1}{p}\right)k_1|z_1|^{2p}|z_2|^{3-\frac{1}{p}}\\
&+r\left(4-\frac{1}{p}\right)k_{1n}|z_1|^{2q}|z_2|^{3-\frac{1}{p}}+rk_3|z_1|^{p}|z_2|^{4-\frac{1}{q}}\operatorname{sign}(z_1z_2)\\
&+rk_{3n}|z_1|^{q}|z_2|^{4-\frac{1}{q}}\operatorname{sign}(z_1z_2)+r\left(4-\frac{1}{q}\right)k_1|z_1|^{2p}|z_2|^{3-\frac{1}{q}}\\
&+r\left(4-\frac{1}{q}\right)k_{1n}|z_1|^{2q}|z_2|^{3-\frac{1}{q}}\\
&+K[d_1]\left(\begin{aligned}&\left(k_1|z_1|^{2p-1}|z_2|^2+\frac{k_1^{\ 2}}{p}|z_1|^{4p-1}+\frac{k_1k_{1n}}{q}|z_1|^{2p+2q-1}\right)\operatorname{sign}(z_1)\\
&-r|z_2|^{4-\frac{1}{p}}\operatorname{sign}(z_2)-r|z_2|^{4-\frac{1}{q}}\operatorname{sign}(z_2)\\
&+\left(k_{1n}|z_1|^{2q-1}|z_2|^2+\frac{k_1k_{1n}}{p}|z_1|^{2p+2q-1}+\frac{k_{1n}^{\ 2}}{q}|z_1|^{4q-1}\right)\operatorname{sign}(z_1)\end{aligned}\right)\\
&+K[d_2]\left(\begin{aligned}&z_2^{\ 3}+\frac{k_1}{p}|z_1|^{2p}z_2+\frac{k_{1n}}{q}|z_1|^{2q}z_2-r\left(4-\frac{1}{p}\right)|z_1||z_2|^{3-\frac{1}{p}}\operatorname{sign}(z_1)\\
&-r\left(4-\frac{1}{q}\right)|z_1||z_2|^{3-\frac{1}{q}}\operatorname{sign}(z_1)\end{aligned}\right)
\end{aligned}$$

After rearrangement:

$$
\begin{aligned}
\dot{\tilde{V}} \le & -k_1\left(k_3 - M_1\right)\left|z_1\right|^{3p-1}\left|z_2\right|^2 - k_{1n}\left(k_{3n} - M_2\right)\left|z_1\right|^{3q-1}\left|z_2\right|^2 \\
& - k_1\left(k_{3n} - M_2\right)\left|z_1\right|^{q+2p-1}\left|z_2\right|^2 - k_{1n}\left(k_3 - M_1\right)\left|z_1\right|^{p+2q-1}\left|z_2\right|^2 \\
& - k_1k_{1n}\left(k_3 - M_1\right)\left(\frac{1}{p}+\frac{1}{q}\right)\left|z_1\right|^{3p+2q-1} - k_1k_{1n}\left(k_{3n} - M_2\right)\left(\frac{1}{q}+\frac{1}{p}\right)\left|z_1\right|^{2p+3q-1} \\
& - \frac{k_1{}^2}{p}\left(k_{3n} - M_2\right)\left|z_1\right|^{q+4p-1} - \frac{k_{1n}{}^2}{q}\left(k_3 - M_1\right)\left|z_1\right|^{p+4q-1} \\
& - \frac{k_1{}^2}{2p}\left(k_3 - M_1\right)\left|z_1\right|^{5p-1} - \frac{k_{1n}{}^2}{2q}\left(k_{3n} - M_2\right)\left|z_1\right|^{5q-1} - \frac{r}{2}\left|z_2\right|^{5-\frac{1}{p}} - \frac{r}{2}\left|z_2\right|^{5-\frac{1}{q}} + \dot{V}_1 \\
& + M_3\begin{pmatrix} \frac{k_1{}^2}{p}\left|z_1\right|^{4p-1} + \frac{k_{1n}{}^2}{q}\left|z_1\right|^{4q-1} + k_1\left|z_1\right|^{2p-1}\left|z_2\right|^2 + k_{1n}\left|z_1\right|^{2q-1}\left|z_2\right|^2 \\ + k_1k_{1n}\left(\frac{1}{p}+\frac{1}{q}\right)\left|z_1\right|^{2p+2q-1} + r\left|z_2\right|^{4-\frac{1}{p}} + r\left|z_2\right|^{4-\frac{1}{q}} \end{pmatrix} \\
& + M_6\begin{pmatrix} \left|z_2\right|^3 + \frac{k_1}{p}\left|z_1\right|^{2p}\left|z_2\right| + \frac{k_{1n}}{q}\left|z_1\right|^{2q}\left|z_2\right| \\ + r\left(4-\frac{1}{p}\right)\left|z_1\right|\left|z_2\right|^{3-\frac{1}{p}} + r\left(4-\frac{1}{q}\right)\left|z_1\right|\left|z_2\right|^{3-\frac{1}{q}} \end{pmatrix}
\end{aligned}
$$

where

$$
\begin{aligned}
\dot{V}_1 = & -\frac{k_1{}^2}{2p}\left(k_3 - M_1\right)\left|z_1\right|^{5p-1} - \frac{k_{1n}{}^2}{2q}\left(k_{3n} - M_2\right)\left|z_1\right|^{5q-1} - \frac{r}{2}\left|z_2\right|^{5-\frac{1}{p}} - \frac{r}{2}\left|z_2\right|^{5-\frac{1}{q}} \\
& + r\left(k_3 + M_1\right)\left|z_1\right|^{p}\left|z_2\right|^{4-\frac{1}{p}} + r\left(k_{3n} + M_2\right)\left|z_1\right|^{q}\left|z_2\right|^{4-\frac{1}{p}} \\
& + r\left(4-\frac{1}{p}\right)\left(k_1 + M_4\right)\left|z_1\right|^{2p}\left|z_2\right|^{3-\frac{1}{p}} + r\left(4-\frac{1}{p}\right)\left(k_{1n} + M_5\right)\left|z_1\right|^{2q}\left|z_2\right|^{3-\frac{1}{p}} \\
& + r\left(k_3 + M_1\right)\left|z_1\right|^{p}\left|z_2\right|^{4-\frac{1}{q}} + r\left(k_{3n} + M_2\right)\left|z_1\right|^{q}\left|z_2\right|^{4-\frac{1}{q}} \\
& + r\left(4-\frac{1}{q}\right)\left(k_1 + M_4\right)\left|z_1\right|^{2p}\left|z_2\right|^{3-\frac{1}{q}} + r\left(4-\frac{1}{q}\right)\left(k_{1n} + M_5\right)\left|z_1\right|^{2q}\left|z_2\right|^{3-\frac{1}{q}} \\
& + M_4\left|z_1\right|^{2p-1}\left|z_2\right|^3 + M_4\frac{k_1}{p}\left|z_1\right|^{4p-1}\left|z_2\right| + \left(M_4\frac{k_{1n}}{q} + M_5\frac{k_1}{p}\right)\left|z_1\right|^{2p+2q-1}\left|z_2\right| \\
& + M_5\left|z_1\right|^{2q-1}\left|z_2\right|^3 + M_5\frac{k_{1n}}{q}\left|z_1\right|^{4q-1}\left|z_2\right|
\end{aligned}
$$

Applying lemma 2.1,

$$-|z_1|^{5p-1}-|z_2|^{5-\frac{1}{p}}\le-\left(\frac{5p-1}{p}\right)^{\frac{p}{5p-1}}|z_1|^{p}\left(\frac{5p-1}{4p-1}\right)^{\frac{4p-1}{5p-1}}|z_2|^{4-\frac{1}{p}},$$

$$-|z_1|^{5p-1}-|z_2|^{5-\frac{1}{p}}\le-\left(\frac{5p-1}{2p}\right)^{\frac{2p}{5p-1}}|z_1|^{2p}\left(\frac{5p-1}{3p-1}\right)^{\frac{3p-1}{5p-1}}|z_2|^{3-\frac{1}{p}},$$

$$-|z_1|^{5p-1}-|z_2|^{5-\frac{1}{p}}\le-\left(\frac{5p-1}{2p-1}\right)^{\frac{2p-1}{5p-1}}|z_1|^{2p-1}\left(\frac{5p-1}{3p}\right)^{\frac{3p}{5p-1}}|z_2|^{3},$$

$$-|z_1|^{5p-1}-|z_2|^{5-\frac{1}{p}}\le-\left(\frac{5p-1}{4p-1}\right)^{\frac{4p-1}{5p-1}}|z_1|^{4p-1}\left(\frac{5p-1}{p}\right)^{\frac{p}{5p-1}}|z_2|,$$

$$-|z_1|^{5q-1}-|z_2|^{5-\frac{1}{q}}\le-\left(\frac{5q-1}{q}\right)^{\frac{q}{5q-1}}|z_1|^{q}\left(\frac{5q-1}{4q-1}\right)^{\frac{4q-1}{5q-1}}|z_2|^{4-\frac{1}{q}},$$

$$-|z_1|^{5q-1}-|z_2|^{5-\frac{1}{q}}\le-\left(\frac{5q-1}{2q}\right)^{\frac{2q}{5q-1}}|z_1|^{2q}\left(\frac{5q-1}{3q-1}\right)^{\frac{3q-1}{5q-1}}|z_2|^{3-\frac{1}{q}},$$

$$-|z_1|^{5q-1}-|z_2|^{5-\frac{1}{q}}\le-\left(\frac{5q-1}{2q-1}\right)^{\frac{2q-1}{5q-1}}|z_1|^{2q-1}\left(\frac{5q-1}{3q}\right)^{\frac{3q}{5q-1}}|z_2|^{3},$$

$$-|z_1|^{5q-1}-|z_2|^{5-\frac{1}{q}}\le-\left(\frac{5q-1}{4q-1}\right)^{\frac{4q-1}{5q-1}}|z_1|^{4q-1}\left(\frac{5q-1}{q}\right)^{\frac{q}{5q-1}}|z_2|,$$

$$-|z_1|^{5p-1}-|z_1|^{5q-1}-|z_2|^{5-\frac{1}{p}}\le-|z_1|^{5q-\frac{q}{p}}-|z_2|^{5-\frac{1}{p}}$$
$$\le-\left(\frac{5p-1}{p}\right)^{\frac{p}{5p-1}}|z_1|^{q}\left(\frac{5p-1}{4p-1}\right)^{\frac{4p-1}{5p-1}}|z_2|^{4-\frac{1}{p}},$$

$$-|z_1|^{5p-1}-|z_1|^{5q-1}-|z_2|^{5-\frac{1}{p}}\le -|z_1|^{5q-\frac{q}{p}}-|z_2|^{5-\frac{1}{p}}$$

$$\le -\left(\frac{5p-1}{2p}\right)^{\frac{2p}{5p-1}}|z_1|^{2q}\left(\frac{5p-1}{3p-1}\right)^{\frac{3p-1}{5p-1}}|z_2|^{3-\frac{1}{p}},$$

$$-|z_1|^{5p-1}-|z_1|^{5q-1}-|z_2|^{5-\frac{1}{p}}\le -|z_1|^{(2p+2q-1)\left(\frac{5p-1}{4p-1}\right)}-|z_2|^{5-\frac{1}{p}}$$

$$\le -\left(\frac{5p-1}{4p-1}\right)^{\frac{4p-1}{5p-1}}|z_1|^{(2p+2q-1)}\left(\frac{5p-1}{p}\right)^{\frac{p}{5p-1}}|z_2|,$$

$$-|z_1|^{5p-1}-|z_1|^{5q-1}-|z_2|^{5-\frac{1}{q}}\le -|z_1|^{5p-\frac{p}{q}}-|z_2|^{5-\frac{1}{q}}\le -\left(\frac{5q-1}{q}\right)^{\frac{q}{5q-1}}|z_1|^{p}\left(\frac{5q-1}{4q-1}\right)^{\frac{4q-1}{5q-1}}|z_2|^{4-\frac{1}{q}},$$

$$-|z_1|^{5p-1}-|z_1|^{5q-1}-|z_2|^{5-\frac{1}{q}}\le -|z_1|^{5p-\frac{p}{q}}-|z_2|^{5-\frac{1}{q}}\le -\left(\frac{5q-1}{2q}\right)^{\frac{2q}{5q-1}}|z_1|^{2p}\left(\frac{5q-1}{3q-1}\right)^{\frac{3q-1}{5q-1}}|z_2|^{3-\frac{1}{q}},$$

where lemma 2.3 has been employed in the last five inequalities since

$$5p-1\le(2p+2q-1)\left(\frac{5p-1}{4p-1}\right)\le 5q-1,\quad 5p-1\le 5q-\frac{q}{p}\le 5q-1,$$

$$5p-1\le 5p-\frac{p}{q}\le 5q-1\,.$$

Thus, if the following inequalities

$$\min\left\{\begin{array}{l}\lambda_1 \dfrac{k_1{}^2\left(k_3-M_1\right)}{\left(k_3+M_1\right)^{\frac{5p-1}{p}}}, \\ \lambda_2 \dfrac{k_1{}^2\left(k_3-M_1\right)}{\left(k_1+M_4\right)^{\frac{5p-1}{2p}}}, \\ \lambda_3 \dfrac{k_{1n}{}^2\left(k_{3n}-M_2\right)}{\left(k_{3n}+M_2\right)^{\frac{5q-1}{q}}}, \\ \lambda_4 \dfrac{k_{1n}{}^2\left(k_{3n}-M_2\right)}{\left(k_{1n}+M_5\right)^{\frac{5q-1}{2q}}}, \\ \lambda_5 \dfrac{\omega_1}{\left(k_{3n}+M_2\right)^{\frac{5p-1}{p}}}, \\ \lambda_6 \dfrac{\omega_1}{\left(k_{1n}+M_5\right)^{\frac{5p-1}{2p}}}, \\ \lambda_7 \dfrac{\omega_1}{\left(k_3+M_1\right)^{\frac{5q-1}{q}}}, \\ \lambda_8 \dfrac{\omega_1}{\left(k_1+M_4\right)^{\frac{5q-1}{2q}}} \end{array}\right\} > r > \max\left\{\begin{array}{l}\lambda_9 \dfrac{M_4{}^{\frac{5p-1}{3p}}}{k_1{}^{\frac{4p-2}{3p}}\left(k_3-M_1\right)^{\frac{2p-1}{3p}}}, \\ \lambda_{10} \dfrac{M_4{}^{\frac{5p-1}{p}}}{k_1{}^{\frac{3p-1}{p}}\left(k_3-M_1\right)^{\frac{4p-1}{p}}}, \\ \lambda_{11} \dfrac{M_5{}^{\frac{5q-1}{3q}}}{k_{1n}{}^{\frac{4q-2}{3q}}\left(k_{3n}-M_2\right)^{\frac{2q-1}{3q}}}, \\ \lambda_{12} \dfrac{M_5{}^{\frac{5q-1}{q}}}{k_{1n}{}^{\frac{3q-1}{q}}\left(k_{3n}-M_2\right)^{\frac{4q-1}{q}}}, \\ \lambda_{13} \dfrac{\left(M_4 \dfrac{k_{1n}}{q}+M_5 \dfrac{k_1}{p}\right)^{\frac{5p-1}{p}}}{\omega_1{}^{\frac{4p-1}{p}}} \end{array}\right\} \qquad (2 - 25)$$

where

$$\lambda_1=\left(\frac{5p-1}{18p^2}\right)\left(\frac{5p-1}{56p-14}\right)^{\frac{4p-1}{p}},\ \lambda_2=\left(\frac{5p-1}{36p^2}\right)\left(\frac{5p-1}{42p-14}\right)^{\frac{3p-1}{2p}}\left(\frac{p}{4p-1}\right)^{\frac{5p-1}{2p}},$$

$$\lambda_3=\left(\frac{5q-1}{18q^2}\right)\left(\frac{5q-1}{48q-12}\right)^{\frac{4q-1}{q}},\ \lambda_4=\left(\frac{5q-1}{36q^2}\right)\left(\frac{5q-1}{36q-12}\right)^{\frac{3q-1}{2q}}\left(\frac{q}{4q-1}\right)^{\frac{5q-1}{2q}},$$

$$\lambda_5=\left(\frac{5p-1}{p}\right)\left(\frac{5p-1}{56p-14}\right)^{\frac{4p-1}{p}},\ \lambda_6=\left(\frac{5p-1}{2p}\right)\left(\frac{5p-1}{42p-14}\right)^{\frac{3p-1}{2p}}\left(\frac{p}{4p-1}\right)^{\frac{5p-1}{2p}},$$

$$\lambda_7=\left(\frac{5q-1}{q}\right)\left(\frac{5q-1}{48q-12}\right)^{\frac{4q-1}{q}},\ \lambda_8=\left(\frac{5q-1}{2q}\right)\left(\frac{5q-1}{36q-12}\right)^{\frac{3q-1}{2q}}\left(\frac{q}{4q-1}\right)^{\frac{5q-1}{2q}},$$

$$\lambda_9=\left(\frac{36p^2-18p}{5p-1}\right)^{\frac{2p-1}{3p}}\left(\frac{42p}{5p-1}\right),\ \lambda_{10}=\left(\frac{72p^2-18p}{5p-1}\right)^{\frac{4p-1}{p}}\left(\frac{14p}{5p-1}\right)\left(\frac{1}{p}\right)^{\frac{5p-1}{p}},$$

$$\lambda_{11}=\left(\frac{36q^2-18q}{5q-1}\right)^{\frac{2q-1}{3q}}\left(\frac{36q}{5q-1}\right),\ \lambda_{12}=\left(\frac{72q^2-18q}{5q-1}\right)^{\frac{4q-1}{q}}\left(\frac{12q}{5q-1}\right)\left(\frac{1}{q}\right)^{\frac{5q-1}{q}},$$

$$\lambda_{13}=\left(\frac{4p-1}{5p-1}\right)^{\frac{4p-1}{p}}\left(\frac{14p}{5p-1}\right),\quad \omega_1=\min\left\{\frac{k_1^{\ 2}}{18p}(k_3-M_1),\ \frac{k_{1n}^{\ 2}}{18q}(k_{3n}-M_2)\right\}$$

hold then the function $\dot{V}_1$ is negative definite. Then,

$$\begin{aligned}\dot{\tilde{V}}\le&-\frac{k_1^{\ 2}}{2p}(k_3-M_1)|z_1|^{5p-1}-\frac{k_{1n}^{\ 2}}{2q}(k_{3n}-M_2)|z_1|^{5q-1}-\frac{r}{2}|z_2|^{5-\frac{1}{p}}-\frac{r}{2}|z_2|^{5-\frac{1}{q}}\\&+M_3\left(\begin{array}{l}\frac{k_1^{\ 2}}{p}|z_1|^{4p-1}+\frac{k_{1n}^{\ 2}}{q}|z_1|^{4q-1}+k_1|z_1|^{2p-1}|z_2|^2+k_{1n}|z_1|^{2q-1}|z_2|^2\\+k_1k_{1n}\left(\frac{1}{p}+\frac{1}{q}\right)|z_1|^{2p+2q-1}+r|z_2|^{4-\frac{1}{p}}+r|z_2|^{4-\frac{1}{q}}\end{array}\right)\\&+M_6\left(\begin{array}{l}|z_2|^3+\frac{k_1}{p}|z_1|^{2p}|z_2|+\frac{k_{1n}}{q}|z_1|^{2q}|z_2|+r\left(4-\frac{1}{p}\right)|z_1||z_2|^{3-\frac{1}{p}}\\+r\left(4-\frac{1}{q}\right)|z_1||z_2|^{3-\frac{1}{q}}\end{array}\right)\end{aligned}$$

Applying lemma 2.1,

$$|z_1|^{2p-1}|z_2|^2 \le \left(\frac{2p-1}{4p-1}\right)|z_1|^{4p-1} + \left(\frac{2p}{4p-1}\right)|z_2|^{\frac{4p-1}{p}},$$

$$|z_1|^{2q-1}|z_2|^2 \le \left(\frac{2q-1}{4q-1}\right)|z_1|^{4q-1} + \left(\frac{2q}{4q-1}\right)|z_2|^{\frac{4q-1}{q}},$$

$$|z_1|^{2p+2q-1} \le \frac{1}{2}|z_1|^{4p-1} + \frac{1}{2}|z_1|^{4q-1},$$

$$|z_1|^{2p}|z_2| \le \frac{2}{3}|z_1|^{3p} + \frac{1}{3}|z_2|^3,$$

$$|z_1|^{2q}|z_2| \le \frac{2}{3}|z_1|^{3q} + \frac{1}{3}|z_2|^3,$$

$$|z_1||z_2|^{3-\frac{1}{p}} \le \left(\frac{1}{3p}\right)|z_1|^{3p} + \left(\frac{3p-1}{3p}\right)|z_2|^3,$$

$$|z_1||z_2|^{3-\frac{1}{q}} \le \left(\frac{1}{3q}\right)|z_1|^{3q} + \left(\frac{3q-1}{3q}\right)|z_2|^3,$$

and lemma 2.3,

$$-|z_1|^{5p-1} - |z_1|^{5q-1} \le -|z_1|^{5p-\frac{p}{q}}, \text{ and } -|z_1|^{5p-1} - |z_1|^{5q-1} \le -|z_1|^{5q-\frac{q}{p}},$$

together with lemma 2.2, and the bounds of the Lyapunov function, we have:

$$
\begin{aligned}
\dot{\tilde{V}} \le & -\omega_2\left(\left|z_1\right|^{5p-1}+\left|z_1\right|^{5q-\frac{q}{p}}+\left|z_2\right|^{5-\frac{1}{p}}\right)-\omega_3\left(\left|z_1\right|^{5q-1}+\left|z_1\right|^{5p-\frac{p}{q}}+\left|z_2\right|^{5-\frac{1}{q}}\right) \\
& +M_3\omega_4\left(\left|z_1\right|^{4p-1}+\left|z_1\right|^{4q-1}+\left|z_2\right|^{\frac{4p-1}{p}}+\left|z_2\right|^{\frac{4q-1}{q}}\right)+M_6\omega_5\left(\left|z_1\right|^{3p}+\left|z_1\right|^{3q}+\left|z_2\right|^{3}\right) \\
= & -\omega_2\left(\left(\left|z_1\right|^{4p}\right)^{\frac{5p-1}{4p}}+\left(\left|z_1\right|^{4q}\right)^{\frac{5p-1}{4p}}+\left(\left|z_2\right|^{4}\right)^{\frac{5p-1}{4p}}\right) \\
& -\omega_3\left(\left(\left|z_1\right|^{4q}\right)^{\frac{5q-1}{4q}}+\left(\left|z_1\right|^{4p}\right)^{\frac{5q-1}{4q}}+\left(\left|z_2\right|^{4}\right)^{\frac{5q-1}{4q}}\right) \\
& +M_3\omega_4\left(\left(\left|z_1\right|^{4p}\right)^{\frac{4p-1}{4p}}+\left(\left|z_1\right|^{4q}\right)^{\frac{4q-1}{4q}}+\left(\left|z_2\right|^{4}\right)^{\frac{4p-1}{4p}}+\left(\left|z_2\right|^{4}\right)^{\frac{4q-1}{4q}}\right) \\
& +M_6\omega_5\left(\left(\left|z_2\right|^{4}\right)^{\frac{3}{4}}+\left(\left|z_1\right|^{4p}\right)^{\frac{3}{4}}+\left(\left|z_1\right|^{4q}\right)^{\frac{3}{4}}\right)
\end{aligned}
$$

$$
\begin{aligned}
\le & -\left(\frac{\omega_2}{\bar{\pi}_1^{\frac{5p-1}{4p}}}\right)V^{\frac{5p-1}{4p}}-\left(\frac{\omega_3}{3^{\frac{q-1}{4q}}\,\bar{\pi}_1^{\frac{5q-1}{4q}}}\right)V^{\frac{5q-1}{4q}}+M_3\left(\frac{2^{\frac{1}{4p}}\omega_4}{\underline{\pi}_1^{\frac{4p-1}{4p}}}\right)V^{\frac{4p-1}{4p}} \\
& +M_3\left(\frac{2^{\frac{1}{4q}}\omega_4}{\underline{\pi}_1^{\frac{4q-1}{4q}}}\right)V^{\frac{4q-1}{4q}}+M_6\left(\frac{3^{\frac{1}{4}}\omega_5}{\underline{\pi}_1^{\frac{3}{4}}}\right)V^{\frac{3}{4}}
\end{aligned}
\tag{2 - 26}
$$

where

$$\omega_2 = \min\left\{\frac{7k_1^{\ 2}}{18p}(k_3 - M_1),\ \omega_1,\ \frac{r}{2}\right\},\ \omega_3 = \min\left\{\frac{7k_{1n}^{\ 2}}{18q}(k_{3n} - M_2),\ \omega_1,\ \frac{r}{2}\right\}$$

$$\omega_4 = \max\left\{\begin{array}{l} \left(\frac{k_1^{\ 2}}{p} + k_1\left(\frac{2p-1}{4p-1}\right) + \frac{k_1 k_{1n}}{2}\left(\frac{1}{p}+\frac{1}{q}\right)\right), \\ \left(\frac{k_{1n}^{\ 2}}{q} + k_{1n}\left(\frac{2q-1}{4q-1}\right) + \frac{k_1 k_{1n}}{2}\left(\frac{1}{p}+\frac{1}{q}\right)\right), \\ \left(r + k_1\left(\frac{2p}{4p-1}\right)\right),\ \left(r + k_{1n}\left(\frac{2q}{4q-1}\right)\right) \end{array}\right\}$$

$$\omega_5 = \max\left\{\begin{array}{l} \left(\frac{2k_1}{3p} + r\left(4 - \frac{1}{p}\right)\left(\frac{1}{3p}\right)\right),\ \left(\frac{2k_{1n}}{3q} + r\left(4 - \frac{1}{q}\right)\left(\frac{1}{3q}\right)\right), \\ \left(1 + \frac{k_1}{3p} + \frac{k_{1n}}{3q} + r\left(4 - \frac{1}{p}\right)\left(\frac{3p-1}{3p}\right) + r\left(4 - \frac{1}{q}\right)\left(\frac{3q-1}{3q}\right)\right) \end{array}\right\}$$

Remark 2.21. Note that the nonlinear inequalities (2 - 24) and (2 - 25) is feasible for sufficiently large k_1, k_{1n}, k_3, k_{3n}. Thus, an $r > 0$ always exists. Consider some particular cases.

1. Note that (2 - 25) is feasible with respect to k_1, k_{1n}, k_3, k_{3n} for any $M_1 \geq 0$, $M_2 \geq 0$, $M_4 \geq 0$, and $M_5 \geq 0$.
2. If $M_4 = M_5 = 0$, then (2 - 25) is satisfied for any $k_3 > M_1$ and $k_{3n} > M_2$. This coincides with the conditions obtained through the weak Lyapunov function, i.e. the energy function $E(z_1, z_2) = \frac{1}{2}z_2^{\ 2} + \frac{k_1}{2p}|z_1|^{2p} + \frac{k_{1n}}{2q}|z_1|^{2q}$.

Remark 2.22. Hence, when $M_3 = M_6 = 0$, i.e. without persistent perturbations, from lemma 2.5-3,

1. For $0 < p < 1$, and $q > 1$the system exhibit uniform asymptotical convergence, where

$$T_{\max}(\mu)=\left(\frac{3^{\frac{q-1}{4q}}\bar{\pi}_1^{\frac{5q-1}{4q}}}{\omega_3}\right)\left(\frac{4q}{q-1}\right)\left(\frac{1}{\mu^{\frac{q-1}{4q}}}\right),$$

is the time at which the trajectories reach the surface level $V=\mu$. At the same time, the system (2 - 23) will have finite time convergence, in particular after reaching the surface level $V=\mu$, from lemma 2.5-3, the settling time estimate, $T(\mu)\le\left(\frac{\bar{\pi}_1^{\frac{5p-1}{4p}}}{\omega_2}\right)\left(\frac{4p}{1-p}\right)[\mu]^{\frac{1-p}{4p}}$,

where the initial starting states is changed to $V=\mu$. Hence, the total time to reach the origin can be estimated as

$$T_{total}(\mu)=\left(\frac{3^{\frac{q-1}{4q}}\bar{\pi}_1^{\frac{5q-1}{4q}}}{\omega_3}\right)\left(\frac{4q}{q-1}\right)\left(\frac{1}{\mu^{\frac{q-1}{4q}}}\right)+\left(\frac{\bar{\pi}_1^{\frac{5p-1}{4p}}}{\omega_2}\right)\left(\frac{4p}{1-p}\right)[\mu]^{\frac{1-p}{4p}}.$$

The minimum of this function can be found at $\mu=\left(\frac{3^{\frac{q-1}{4q}}\bar{\pi}_1^{\frac{5q-1}{4q}}}{\omega_3}\right)^{\frac{4pq}{q-p}}\left(\frac{\omega_2}{\bar{\pi}_1^{\frac{5p-1}{4p}}}\right)^{\frac{4pq}{q-p}}$.

Substituting into the function, a finite settling time independent of initial conditions is obtained,

$$T_{total} = \left(\frac{3^{\frac{q-1}{4q}} \bar{\pi}_1^{\frac{5q-1}{4q}}}{\omega_3} \right) \left(\frac{4q}{q-1} \right) \left(\frac{\omega_3}{3^{\frac{q-1}{4q}} \bar{\pi}_1^{\frac{5q-1}{4q}}} \right)^{\frac{p(q-1)}{q-p}} \left(\frac{\bar{\pi}_1^{\frac{5p-1}{4p}}}{\omega_2} \right)^{\frac{p(q-1)}{q-p}}$$

$$+ \left(\frac{\bar{\pi}_1^{\frac{5p-1}{4p}}}{\omega_2} \right) \left(\frac{4p}{1-p} \right) \left(\frac{3^{\frac{q-1}{4q}} \bar{\pi}_1^{\frac{5q-1}{4q}}}{\omega_3} \right)^{\frac{q(1-p)}{q-p}} \left(\frac{\omega_2}{\bar{\pi}_1^{\frac{5p-1}{4p}}} \right)^{\frac{q(1-p)}{q-p}}$$

2. For $0 < p < 1$, and $q = 1$, finite time convergence can be concluded.

3. For $p = 1$, and $q \geq 1$, exponential convergence can be concluded.

The above results are possible due to the negative definiteness of the time derivative of the Lyapunov function, i.e. strict Lyapunov function.

Remark 2.23. While if persistent perturbations occur on the system, $M_3 \neq 0$ and/or $M_6 \neq 0$, from (2 - 26),

$$
\begin{aligned}
\dot{\tilde{V}} \le & -\frac{1}{2}\left(\frac{\omega_2}{\overline{\pi}_1^{\frac{5p-1}{4p}}}\right)V^{\frac{5p-1}{4p}} - \frac{1}{3}\left(\frac{\omega_3}{3^{\frac{q-1}{4q}}\,\overline{\pi}_1^{\frac{5q-1}{4q}}}\right)V^{\frac{5q-1}{4q}} - V^{\frac{4p-1}{4p}}\left(\frac{1}{2}\left(\frac{\omega_2}{\overline{\pi}_1^{\frac{5p-1}{4p}}}\right)V^{\frac{1}{4}} - M_3\left(\frac{2^{\frac{1}{4p}}\,\omega_4}{\underline{\pi}_1^{\frac{4p-1}{4p}}}\right)\right) \\
& - V^{\frac{4q-1}{4q}}\left(\frac{1}{3}\left(\frac{\omega_3}{3^{\frac{q-1}{4q}}\,\overline{\pi}_1^{\frac{5q-1}{4q}}}\right)V^{\frac{1}{4}} - M_3\left(\frac{2^{\frac{1}{4q}}\,\omega_4}{\underline{\pi}_1^{\frac{4q-1}{4q}}}\right)\right) - V^{\frac{3}{4}}\left(\frac{1}{3}\left(\frac{\omega_3}{3^{\frac{q-1}{4q}}\,\overline{\pi}_1^{\frac{5q-1}{4q}}}\right)V^{\frac{2q-1}{4q}} - M_6\left(\frac{3^{\frac{1}{4}}\,\omega_5}{\underline{\pi}_1^{\frac{3}{4}}}\right)\right) \\
\le & -\frac{1}{2}\left(\frac{\omega_2}{\overline{\pi}_1^{\frac{5p-1}{4p}}}\right)V^{\frac{5p-1}{4p}} - \frac{1}{3}\left(\frac{\omega_3}{3^{\frac{q-1}{4q}}\,\overline{\pi}_1^{\frac{5q-1}{4q}}}\right)V^{\frac{5q-1}{4q}},
\end{aligned}
$$

$$
\text{for} \quad V \ge \max\left\{
\begin{array}{l}
\left(2M_3\left(\frac{\overline{\pi}_1^{\frac{5p-1}{4p}}}{\omega_2}\right)\left(\frac{2^{\frac{1}{4p}}\,\omega_4}{\underline{\pi}_1^{\frac{4p-1}{4p}}}\right)\right)^4, \left(3M_3\left(\frac{3^{\frac{q-1}{4q}}\,\overline{\pi}_1^{\frac{5q-1}{4q}}}{\omega_3}\right)\left(\frac{2^{\frac{1}{4q}}\,\omega_4}{\underline{\pi}_1^{\frac{4q-1}{4q}}}\right)\right)^4, \\
\left(3M_6\left(\frac{3^{\frac{q-1}{4q}}\,\overline{\pi}_1^{\frac{5q-1}{4q}}}{\omega_3}\right)\left(\frac{3^{\frac{1}{4}}\,\omega_5}{\underline{\pi}_1^{\frac{3}{4}}}\right)\right)^{\frac{4q}{2q-1}}
\end{array}
\right\}
$$

thus, uniform ultimate boundedness is achieved by applying lemma 2.6.

b) Case 2: For $p = 0.5$, and $1 \le q$

For $p = 0.5$, V is not differentiable on $z_1 = 0$, hence

$$
\dot{\tilde{V}}(z_1, z_2) = \bigcap_{\xi \in \partial V(\mathbf{z}(t),t)} \xi^T K \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} \subset \dot{\tilde{V}}_1 + \dot{\tilde{V}}_2
$$

where

$$
\dot{\tilde{V}}_1 = \bigcap_{\xi \in \partial V(\mathbf{z}(t),t)} \xi^T K[f](z_1, z_2), \quad \dot{\tilde{V}}_2 = \bigcap_{\xi \in \partial V(\mathbf{z}(t),t)} \xi^T K \begin{bmatrix} d_1 \\ d_2 \end{bmatrix},
$$

$$K[f](z_1,z_2)\in\begin{bmatrix}-k_3|z_1|^{0.5}\operatorname{sign}(z_1)-k_{3n}|z_1|^{q}\operatorname{sign}(z_1)+z_2\\ -k_1\mathrm{SGN}(z_1)-k_{1n}|z_1|^{2q-1}\operatorname{sign}(z_1)\end{bmatrix}$$

$$\partial V=\boldsymbol{K}[\nabla V]=\boldsymbol{K}\begin{bmatrix}\dfrac{\partial V}{\partial z_1}\\ \dfrac{\partial V}{\partial z_2}\end{bmatrix}$$

$$\subset\begin{bmatrix}\boldsymbol{K}\left[\dfrac{\partial V}{\partial z_1}\right]\\ \boldsymbol{K}\left[\dfrac{\partial V}{\partial z_2}\right]\end{bmatrix}$$

$$=\begin{cases}\forall z_1\neq 0, z_2\in\mathbb{R}:\\ \begin{bmatrix}\left(z_2^{\,2}+2k_1|z_1|+\dfrac{k_{1n}}{q}|z_1|^{2q}\right)\left(k_1+k_{1n}|z_1|^{2q-1}\right)\operatorname{sign}(z_1)-r|z_2|^2\operatorname{sign}(z_2)-r|z_2|^{4-\frac{1}{q}}\operatorname{sign}(z_2)\\ \left(z_2^{\,2}+2k_1|z_1|+\dfrac{k_{1n}}{q}|z_1|^{2q}\right)(z_2)-2rz_1|z_2|-r\left(4-\dfrac{1}{q}\right)z_1|z_2|^{3-\frac{1}{q}}\end{bmatrix},\\ \forall z_1=0, z_2\in\mathbb{R}:\\ \begin{bmatrix}z_2^{\,2}k_1[-1,1]-r|z_2|^2\operatorname{sign}(z_2)-r|z_2|^{4-\frac{1}{q}}\operatorname{sign}(z_2)\\ z_2^{\,3}\end{bmatrix}\end{cases}$$

Let us define:

$$\left|\frac{\partial V}{\partial z_1}\right|:=\sup\left\{|\xi_1|:\xi_1\in\boldsymbol{K}\left[\frac{\partial V}{\partial z_1}\right]\right\},\text{ and }\left|\frac{\partial V}{\partial z_2}\right|:=\sup\left\{|\xi_2|:\xi_2\in\boldsymbol{K}\left[\frac{\partial V}{\partial z_2}\right]\right\},$$

with

$$\left|\frac{\partial V}{\partial z_1}\right|\leq 2k_1^{\,2}|z_1|+k_1k_{1n}\left(\frac{1}{q}+2\right)|z_1|^{2q}+\frac{k_{1n}^{\,2}}{q}|z_1|^{4q-1}+k_{1n}|z_1|^{2q-1}|z_2|^2+k_1|z_2|^2+r|z_2|^2+r|z_2|^{4-\frac{1}{q}},$$

and $\left|\dfrac{\partial V}{\partial z_2}\right| \le |z_2|^3 + 2k_1|z_1||z_2| + \dfrac{k_{1n}}{q}|z_1|^{2q}|z_2| + 2r|z_1||z_2| + r\left(4-\dfrac{1}{q}\right)|z_1||z_2|^{3-\frac{1}{q}}$

Thus, the term

$$
\begin{aligned}
\dot{\tilde{V}}_2 &= \bigcap_{(\xi_1,\xi_2)^T \in \partial V(\mathbf{z}(t),t)} \xi_1 K[d_1] + \xi_2 K[d_2] \\
&\le \left|\frac{\partial V}{\partial z_1}\right||d_1| + \left|\frac{\partial V}{\partial z_2}\right||d_2| \\
&\le \left(M_1|z_1|^{0.5} + M_2|z_1|^q + M_3\right)\begin{pmatrix} 2k_1^{\,2}|z_1| + k_1k_{1n}\left(\dfrac{1}{q}+2\right)|z_1|^{2q} + \dfrac{k_{1n}^{\,2}}{q}|z_1|^{4q-1} + k_{1n}|z_1|^{2q-1}|z_2|^2 \\ + k_1|z_2|^2 + r|z_2|^2 + r|z_2|^{4-\frac{1}{q}} \end{pmatrix} \\
&\quad + \left(M_4 + M_5|z_1|^{2q-1} + M_6\right)\left(|z_2|^3 + 2k_1|z_1||z_2| + \frac{k_{1n}}{q}|z_1|^{2q}|z_2| + 2r|z_1||z_2| + r\left(4-\frac{1}{q}\right)|z_1||z_2|^{3-\frac{1}{q}}\right)
\end{aligned}
$$

Computing $\dot{\tilde{V}}_1$ for each case, we have

For $z_1 \ne 0$, $z_2 \in \mathbb{R}$:

$$
\begin{aligned}
\dot{\tilde{V}}_1 &= -k_1k_3|z_1|^{0.5}|z_2|^2 - k_1k_{3n}|z_1|^q|z_2|^2 - k_{1n}k_3|z_1|^{2q-0.5}|z_2|^2 - k_{1n}k_{3n}|z_1|^{3q-1}|z_2|^2 - 2k_1^{\,2}k_3|z_1|^{1.5} \\
&\quad - 2k_1^{\,2}k_{3n}|z_1|^{1+q} - k_1k_{1n}k_3\left(2+\frac{1}{q}\right)|z_1|^{0.5+2q} - 2k_1k_{1n}k_{3n}|z_1|^{3q} - \frac{k_{1n}k_1k_{3n}}{q}|z_1|^{3q} \\
&\quad - \frac{k_{1n}k_{1n}k_3}{q}|z_1|^{4q-0.5} - \frac{k_{1n}k_{1n}k_{3n}}{q}|z_1|^{5q-1} - r|z_2|^3 - r|z_2|^{5-\frac{1}{q}} + rk_3|z_1|^{0.5}|z_2|^2\operatorname{sign}(z_1z_2) \\
&\quad + rk_{3n}|z_1|^q|z_2|^2\operatorname{sign}(z_1z_2) + r2k_1|z_1||z_2| + r2k_{1n}|z_1|^{2q}|z_2| \\
&\quad + rk_3|z_1|^{0.5}|z_2|^{4-\frac{1}{q}}\operatorname{sign}(z_1z_2) + rk_{3n}|z_1|^q|z_2|^{4-\frac{1}{q}}\operatorname{sign}(z_1z_2) \\
&\quad + r\left(4-\frac{1}{q}\right)k_1|z_1||z_2|^{3-\frac{1}{q}} + r\left(4-\frac{1}{q}\right)k_{1n}|z_1|^{2q}|z_2|^{3-\frac{1}{q}}
\end{aligned}
$$

For $z_1 = 0$, $z_2 \in \mathbb{R}$:

Let $\begin{bmatrix} z_2{}^2 k_1 \xi_2 - r|z_2|^2 \operatorname{sign}(z_2) - r|z_2|^{4-\frac{1}{q}} \operatorname{sign}(z_2) \\ z_2{}^3 \end{bmatrix}$ with $\xi_2 \in [-1, 1]$ be an arbitrary element of $\partial V(z_1, z_2)$, then

$$\xi^{\mathrm{T}} K[f](z_1, z_2) = \begin{bmatrix} \left(z_2{}^2 k_1 \xi_2 - r|z_2|^2 \operatorname{sign}(z_2) - r|z_2|^{4-\frac{1}{q}} \operatorname{sign}(z_2) \right) \\ z_2{}^3 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} z_2 \\ -k_1[-1, 1] \end{bmatrix}$$
$$= [\xi_2 - 1, \xi_2 + 1] k_1 z_2{}^3 - r|z_2|^3 - r|z_2|^{5-\frac{1}{q}}$$

hence,

$$\dot{\tilde{V}}_1 = \bigcap_{\xi_2 \in [-1,\, 1]} [\xi_2 - 1, \xi_2 + 1] k_1 z_2{}^3 - r|z_2|^3 - r|z_2|^{5-\frac{1}{q}} = -r|z_2|^3 - r|z_2|^{5-\frac{1}{q}}$$

Thus, for all $(z_1, z_2) \in \mathbb{R}^2$:

$$\begin{aligned}
\dot{\tilde{V}}_1 = & -k_1 k_3 |z_1|^{0.5} |z_2|^2 - k_1 k_{3n} |z_1|^q |z_2|^2 - k_{1n} k_3 |z_1|^{2q-0.5} |z_2|^2 - k_{1n} k_{3n} |z_1|^{3q-1} |z_2|^2 - 2k_1{}^2 k_3 |z_1|^{1.5} \\
& - 2k_1{}^2 k_{3n} |z_1|^{1+q} - k_1 k_{1n} k_3 \left(2 + \frac{1}{q} \right) |z_1|^{0.5+2q} - 2k_1 k_{1n} k_{3n} |z_1|^{3q} - \frac{k_{1n} k_1 k_{3n}}{q} |z_1|^{3q} \\
& - \frac{k_{1n} k_{1n} k_3}{q} |z_1|^{4q-0.5} - \frac{k_{1n} k_{1n} k_{3n}}{q} |z_1|^{5q-1} - r|z_2|^3 - r|z_2|^{5-\frac{1}{q}} + rk_3 |z_1|^{0.5} |z_2|^2 \operatorname{sign}(z_1 z_2) \\
& + rk_{3n} |z_1|^q |z_2|^2 \operatorname{sign}(z_1 z_2) + r2k_1 |z_1| |z_2| + r2k_{1n} |z_1|^{2q} |z_2| + rk_3 |z_1|^{0.5} |z_2|^{4-\frac{1}{q}} \operatorname{sign}(z_1 z_2) \\
& + rk_{3n} |z_1|^q |z_2|^{4-\frac{1}{q}} \operatorname{sign}(z_1 z_2) + r\left(4 - \frac{1}{q} \right) k_1 |z_1| |z_2|^{3-\frac{1}{q}} + r\left(4 - \frac{1}{q} \right) k_{1n} |z_1|^{2q} |z_2|^{3-\frac{1}{q}}
\end{aligned}$$

Hence, the generalized time derivative of the Lyapunov function, after rearrangement:

$$
\begin{aligned}
\dot{\tilde{V}} &= \dot{\tilde{V}}_1 + \dot{\tilde{V}}_2 \\
&\le -k_{1n}(k_3 - M_1)|z_1|^{2q-0.5}|z_2|^2 - k_{1n}(k_{3n} - M_2)|z_1|^{3q-1}|z_2|^2 - k_1 k_{1n}\left(2 + \frac{1}{q}\right)(k_3 - M_1)|z_1|^{2q+0.5} \\
&\quad - 2k_1{}^2(k_{3n} - M_2)|z_1|^{q+1} - \frac{k_{1n}{}^2}{q}(k_3 - M_1)|z_1|^{4q-0.5} - \frac{k_1 k_{1n}}{2}\left(2 + \frac{1}{q}\right)(k_{3n} - M_2)|z_1|^{3q} \\
&\quad - k_1{}^2(k_3 - M_1)|z_1|^{1.5} - \frac{k_{1n}{}^2}{2q}(k_{3n} - M_2)|z_1|^{5q-1} - \frac{r}{2}|z_2|^3 - \frac{r}{2}|z_2|^{5-\frac{1}{q}} + \dot{V}_2 \\
&\quad + M_3\left(2k_1{}^2|z_1| + k_1 k_{1n}\left(\frac{1}{q} + 2\right)|z_1|^{2q} + \frac{k_{1n}{}^2}{q}|z_1|^{4q-1} + k_{1n}|z_1|^{2q-1}|z_2|^2 + (k_1 + r)|z_2|^2 + r|z_2|^{4-\frac{1}{q}}\right) \\
&\quad + M_6\left(|z_2|^3 + \frac{k_{1n}}{q}|z_1|^{2q}|z_2| + 2(k_1 + r)|z_1||z_2| + r\left(4 - \frac{1}{q}\right)|z_1||z_2|^{3-\frac{1}{q}}\right)
\end{aligned}
$$

where

$$
\begin{aligned}
\dot{V}_2 &= -\frac{k_1 k_{1n}}{2}\left(2 + \frac{1}{q}\right)(k_{3n} - M_2)|z_1|^{3q} - k_1{}^2(k_3 - M_1)|z_1|^{1.5} - \frac{k_{1n}{}^2}{2q}(k_{3n} - M_2)|z_1|^{5q-1} \\
&\quad - \frac{r}{4}|z_2|^3 - \frac{r}{2}|z_2|^{5-\frac{1}{q}} - \left(k_1(k_3 - M_1) - r(k_3 + M_1)\right)|z_1|^{0.5}|z_2|^2 \\
&\quad - \left(k_1(k_{3n} - M_2) - r(k_{3n} + M_2)\right)|z_1|^q|z_2|^2 - \left(\frac{r}{4} - M_4\right)|z_2|^3 \\
&\quad + 2r(k_1 + M_4)|z_1||z_2| + 2r(k_{1n} + M_5)|z_1|^{2q}|z_2| + r(k_3 + M_1)|z_1|^{0.5}|z_2|^{4-\frac{1}{q}} \\
&\quad + r(k_{3n} + M_2)|z_1|^q|z_2|^{4-\frac{1}{q}} + r\left(4 - \frac{1}{q}\right)(k_{1n} + M_5)|z_1|^{2q}|z_2|^{3-\frac{1}{q}} \\
&\quad + r\left(4 - \frac{1}{q}\right)(k_1 + M_4)|z_1||z_2|^{3-\frac{1}{q}} + 2M_4 k_1|z_1||z_2| \\
&\quad + \left(\frac{k_{1n}}{q}M_4 + 2k_1 M_5\right)|z_1|^{2q}|z_2| + M_5|z_1|^{2q-1}|z_2|^3 + M_5\frac{k_{1n}}{q}|z_1|^{4q-1}|z_2|
\end{aligned}
$$

Applying lemma 2.1,

$$
-|z_1|^{1.5} - |z_2|^3 \le -\left(\frac{3}{2}\right)^{\frac{2}{3}}|z_1|(3)^{\frac{1}{3}}|z_2|,
$$

$$-|z_1|^{5q-1}-|z_2|^{5-\frac{1}{q}}\le-\left(\frac{5q-1}{q}\right)^{\frac{q}{5q-1}}|z_1|^{q}\left(\frac{5q-1}{4q-1}\right)^{\frac{4q-1}{5q-1}}|z_2|^{4-\frac{1}{q}},$$

$$-|z_1|^{5q-1}-|z_2|^{5-\frac{1}{q}}\le-\left(\frac{5q-1}{2q}\right)^{\frac{2q}{5q-1}}|z_1|^{2q}\left(\frac{5q-1}{3q-1}\right)^{\frac{3q-1}{5q-1}}|z_2|^{3-\frac{1}{q}},$$

$$-|z_1|^{5q-1}-|z_2|^{5-\frac{1}{q}}\le-\left(\frac{5q-1}{2q-1}\right)^{\frac{2q-1}{5q-1}}|z_1|^{2q-1}\left(\frac{5q-1}{3q}\right)^{\frac{3q}{5q-1}}|z_2|^{3},$$

$$-|z_1|^{5q-1}-|z_2|^{5-\frac{1}{q}}\le-\left(\frac{5q-1}{4q-1}\right)^{\frac{4q-1}{5q-1}}|z_1|^{4q-1}\left(\frac{5q-1}{q}\right)^{\frac{q}{5q-1}}|z_2|,$$

$$-|z_1|^{3q}-|z_2|^{3}\le-\left(\frac{3}{2}\right)^{\frac{2}{3}}|z_1|^{2q}(3)^{\frac{1}{3}}|z_2|,$$

$$-|z_1|^{1.5}-|z_1|^{5q-1}-|z_2|^{5-\frac{1}{q}}\le-|z_1|^{\frac{5q-1}{2q}}-|z_2|^{5-\frac{1}{q}}\le-\left(\frac{5q-1}{q}\right)^{\frac{q}{5q-1}}|z_1|^{0.5}\left(\frac{5q-1}{4q-1}\right)^{\frac{4q-1}{5q-1}}|z_2|^{4-\frac{1}{q}},$$

$$-|z_1|^{1.5}-|z_1|^{5q-1}-|z_2|^{5-\frac{1}{q}}\le-|z_1|^{\frac{5q-1}{2q}}-|z_2|^{5-\frac{1}{q}}\le-\left(\frac{5q-1}{2q}\right)^{\frac{2q}{5q-1}}|z_1|\left(\frac{5q-1}{3q-1}\right)^{\frac{3q-1}{5q-1}}|z_2|^{3-\frac{1}{q}},$$

where lemma 2.3 has been employed in the last two inequalities since

$$1.5\le\frac{5q-1}{2q}\le 5q-1,$$

Thus, if the following inequalities

$$\min\left\{\begin{array}{l}\dfrac{k_1(k_3-M_1)}{(k_3+M_1)},\ \dfrac{k_1(k_{3n}-M_2)}{(k_{3n}+M_2)},\\ \lambda_{14}\dfrac{k_1^{\,2}(k_3-M_1)}{(k_1+M_4)^{\frac{3}{2}}},\ \lambda_{15}\dfrac{k_{1n}^{\,2}(k_{3n}-M_2)}{(k_{3n}+M_2)^{\frac{5q-1}{q}}},\\ \lambda_{16}\dfrac{k_{1n}^{\,2}(k_{3n}-M_2)}{(k_{1n}+M_5)^{\frac{5q-1}{2q}}},\ \lambda_{17}\dfrac{k_1k_{1n}(k_{3n}-M_2)}{(k_{1n}+M_5)^{\frac{3}{2}}},\\ \lambda_{18}\dfrac{\omega_6}{(k_3+M_1)^{\frac{5q-1}{q}}},\ \lambda_{19}\dfrac{\omega_6}{(k_1+M_4)^{\frac{5q-1}{2q}}}\end{array}\right\} > r > \max\left\{\begin{array}{l}4M_4,\ \lambda_{20}\dfrac{M_4^{\,3}}{k_1(k_3-M_1)^2},\\ \lambda_{21}\dfrac{M_5^{\frac{5q-1}{3q}}}{k_{1n}^{\frac{4q-2}{3q}}(k_{3n}-M_2)^{\frac{2q-1}{3q}}},\\ \lambda_{22}\dfrac{M_5^{\frac{5q-1}{q}}}{k_{1n}^{\frac{3q-1}{q}}(k_{3n}-M_2)^{\frac{4q-1}{q}}},\\ \lambda_{23}\dfrac{\left(\dfrac{k_{1n}}{q}M_4+2k_1M_5\right)^3}{k_1^{\,2}k_{1n}^{\,2}(k_{3n}-M_2)^2},\end{array}\right\} \quad (2\text{ - }27)$$

where

$$\lambda_{14}=\left(\frac{3}{8}\right)\left(\frac{3}{16}\right)^{\frac{1}{2}}\frac{1}{2^{\frac{3}{2}}},\quad \lambda_{15}=\left(\frac{5q-1}{12q^2}\right)\left(\frac{5q-1}{48q-12}\right)^{\frac{4q-1}{q}},$$

$$\lambda_{16}=\left(\frac{5q-1}{24q^2}\right)\left(\frac{5q-1}{36q-12}\right)^{\frac{3q-1}{2q}}\left(\frac{q}{4q-1}\right)^{\frac{5q-1}{2q}},\quad \lambda_{17}=\left(\frac{3}{8}\right)\left(\frac{2q+1}{q}\right)\left(\frac{3}{16}\right)^{\frac{1}{2}}\left(\frac{1}{2}\right)^{\frac{3}{2}},$$

$$\lambda_{18}=\left(\frac{5q-1}{q}\right)\left(\frac{5q-1}{48q-12}\right)^{\frac{4q-1}{q}},\ \lambda_{19}=\left(\frac{5q-1}{2q}\right)\left(\frac{5q-1}{36q-12}\right)^{\frac{3q-1}{2q}}\left(\frac{q}{4q-1}\right)^{\frac{5q-1}{2q}},$$

$$\lambda_{20}=\left(\frac{8}{3}\right)^2\left(\frac{16}{3}\right)2^3,\ \lambda_{21}=\left(\frac{24q^2-12q}{5q-1}\right)^{\frac{2q-1}{3q}}\left(\frac{36q}{5q-1}\right),$$

$$\lambda_{22}=\left(\frac{48q^2-12q}{5q-1}\right)^{\frac{4q-1}{q}}\left(\frac{12q}{5q-1}\right)\left(\frac{1}{q}\right)^{\frac{5q-1}{q}},\quad \lambda_{23}=\left(\frac{8}{3}\right)^2\left(\frac{q}{2q+1}\right)^2\left(\frac{16}{3}\right),$$

$$\omega_6=\min\left\{\frac{k_1^{\,2}(k_3-M_1)}{4},-\frac{k_{1n}^{\,2}(k_{3n}-M_2)}{12q}\right\}$$

hold then the function $\dot{V}_2$ is negative definite. Then,

$$
\begin{aligned}
\dot{\tilde{V}} \le & -\frac{k_1 k_{1n}}{2}\left(2+\frac{1}{q}\right)(k_{3n}-M_2)|z_1|^{3q} - k_1^{\,2}(k_3-M_1)|z_1|^{1.5} - \frac{k_{1n}^{\;\;2}}{2q}(k_{3n}-M_2)|z_1|^{5q-1} \\
& -\frac{r}{2}|z_2|^3 - \frac{r}{2}|z_2|^{5-\frac{1}{q}} \\
& + M_3\begin{pmatrix} 2k_1^{\,2}|z_1| + k_1 k_{1n}\left(\frac{1}{q}+2\right)|z_1|^{2q} + \frac{k_{1n}^{\;\;2}}{q}|z_1|^{4q-1} + k_{1n}|z_1|^{2q-1}|z_2|^2 \\ +(k_1+r)|z_2|^2 + r|z_2|^{4-\frac{1}{q}} \end{pmatrix} \\
& + M_6\left(|z_2|^3 + \frac{k_{1n}}{q}|z_1|^{2q}|z_2| + 2(k_1+r)|z_1||z_2| + r\left(4-\frac{1}{q}\right)|z_1||z_2|^{3-\frac{1}{q}}\right)
\end{aligned}
$$

Applying lemma 2.1,

$$
|z_1|^{2q-1}|z_2|^2 \le \left(\frac{2q-1}{4q-1}\right)|z_1|^{4q-1} + \left(\frac{2q}{4q-1}\right)|z_2|^{\frac{4q-1}{q}},
$$

$$
|z_1|^{2q} = |z_1|^{0.5}|z_1|^{2q-0.5} \le \frac{1}{2}|z_1| + \frac{1}{2}|z_1|^{4q-1},
$$

$$
|z_1|^{2q}|z_2| \le \frac{2}{3}|z_1|^{3q} + \frac{1}{3}|z_2|^3,
$$

$$
|z_1||z_2| \le \frac{2}{3}|z_1|^{\frac{3}{2}} + \frac{1}{3}|z_2|^3,
$$

$$
|z_1||z_2|^{3-\frac{1}{q}} \le \left(\frac{1}{3q}\right)|z_1|^{3q} + \left(\frac{3q-1}{3q}\right)|z_2|^3,
$$

and lemma 2.3

$$
-|z_1|^{1.5} - |z_1|^{5q-1} \le -|z_1|^{\frac{5q-1}{2q}}
$$

together with lemma 2.2, and the bounds of the Lyapunov function, we have:

$$
\begin{aligned}
\dot{\tilde{V}} \le& -\omega_7\left(|z_1|^{1.5} + |z_1|^{3q} + |z_2|^3\right) - \omega_8\left(|z_1|^{5q-1} + |z_1|^{\frac{5q-1}{2q}} + |z_2|^{5-\frac{1}{q}}\right) \\
& + M_3\omega_9\left(|z_1| + |z_2|^2 + |z_1|^{4q-1} + |z_2|^{4-\frac{1}{q}}\right) + M_6\omega_{10}\left(|z_1|^{\frac{3}{2}} + |z_1|^{3q} + |z_2|^3\right) \\
=& -\omega_7\left(\left(|z_1|^2\right)^{\frac{3}{4}} + \left(|z_1|^{4q}\right)^{\frac{3}{4}} + \left(|z_2|^4\right)^{\frac{3}{4}}\right) - \omega_8\left(\left(|z_1|^{4q}\right)^{\frac{5q-1}{4q}} + \left(|z_1|^2\right)^{\frac{5q-1}{4q}} + \left(|z_2|^4\right)^{\frac{5q-1}{4q}}\right) \\
& + M_3\omega_9\left(\left(|z_1|^2\right)^{\frac{1}{2}} + \left(|z_2|^4\right)^{\frac{1}{2}} + \left(|z_1|^{4q}\right)^{\frac{4q-1}{4q}} + \left(|z_2|^4\right)^{\frac{4q-1}{4q}}\right) \\
& + M_6\omega_{10}\left(\left(|z_1|^2\right)^{\frac{3}{4}} + \left(|z_1|^{4q}\right)^{\frac{3}{4}} + \left(|z_2|^4\right)^{\frac{3}{4}}\right)
\end{aligned}
$$

$$
\begin{aligned}
\le& -\left(\frac{\omega_7}{\bar{\pi}_1^{\frac{3}{4}}}\right)V^{\frac{3}{4}} - \left(\frac{\omega_8}{3^{\frac{q-1}{4q}}\,\bar{\pi}_1^{\frac{5q-1}{4q}}}\right)V^{\frac{5q-1}{4q}} + M_3\left(\frac{2^{\frac{1}{2}}\omega_9}{\underline{\pi}_1^{\frac{1}{2}}}\right)V^{\frac{1}{2}} \\
& + M_3\left(\frac{2^{\frac{1}{4q}}\omega_9}{\underline{\pi}_1^{\frac{4q-1}{4q}}}\right)V^{\frac{4q-1}{4q}} + M_6\left(\frac{3^{\frac{1}{4}}\omega_{10}}{\underline{\pi}_1^{\frac{3}{4}}}\right)V^{\frac{3}{4}}
\end{aligned} \qquad (2 - 28)
$$

where

$$
\omega_7 = \min\left\{\frac{k_1^{\,2}(k_3 - M_1)}{4},\ \frac{k_1k_{1n}(k_{3n} - M_2)}{2}\left(2 + \frac{1}{q}\right),\ \frac{r}{2}\right\},
$$

$$
\omega_8 = \min\left\{\frac{5k_{1n}^{\,2}(k_{3n} - M_2)}{12q},\ \omega_6,\ \frac{r}{2}\right\},
$$

$$
\omega_9 = \max\left\{\begin{array}{l}\left(2k_1^{\,2} + \dfrac{k_1k_{1n}}{2}\left(\dfrac{2q+1}{q}\right)\right),\ (k_1 + r), \\ \left(\dfrac{k_{1n}^{\,2}}{q} + k_{1n}\left(\dfrac{2q-1}{4q-1}\right) + \dfrac{k_1k_{1n}}{2}\left(\dfrac{2q+1}{q}\right)\right),\ \left(k_{1n}\left(\dfrac{2q}{4q-1}\right) + r\right)\end{array}\right\},
$$

$$
\omega_{10} = \max\left\{\frac{4(k_1 + r)}{3},\ \left(\frac{k_{1n}}{q}\frac{2}{3} + r\left(4 - \frac{1}{q}\right)\left(\frac{1}{3q}\right)\right),\ \left(1 + \frac{k_{1n}}{3q} + \frac{2(k_1 + r)}{3} + r\left(4 - \frac{1}{q}\right)\left(\frac{3q-1}{3q}\right)\right)\right\}
$$

Remark 2.24. Note that the nonlinear inequalities (2 - 24) and (2 - 27) is feasible with respect to k_1, k_{1n}, k_3, k_{3n} for any $M_1 \geq 0$, $M_2 \geq 0$, $M_4 \geq 0$, and $M_5 \geq 0$. As such an $r > 0$ always exists for sufficiently large k_1, k_{1n}, k_3, k_{3n}. In particular, consider the case where $M_4 = M_5 = 0$, inequalities (2 - 27) can be easily satisfied for any $k_3 > M_1$ and $k_{3n} > M_2$. This coincides with the conditions obtained through the weak Lyapunov function, i.e. the energy function $E(z_1, z_2) = \frac{1}{2} z_2^{\ 2} + \frac{k_1}{2p}|z_1| + \frac{k_{1n}}{2q}|z_1|^{2q}$ (similar to that observed when $p > 0$).

Remark 2.25. Hence, when $M_3 = M_6 = 0$, it is not difficult to show that the system achieved finite time convergence for $q \geq 1$. In particular for the case of $q > 1$, following similar arguments in section 2.2.4(a), finite convergence time independent of initial conditions for $q >$ 1,

$$T_{total} = \left(\frac{3^{\frac{q-1}{4q}} \bar{\pi}_1^{\frac{5q-1}{4q}}}{\omega_8} \right) \left(\frac{4q}{q-1} \right) \left(\frac{\omega_8}{3^{\frac{q-1}{4q}} \bar{\pi}_1^{\frac{5q-1}{4q}}} \right)^{\frac{0.5(q-1)}{q-0.5}} \left(\frac{\bar{\pi}_1^{\frac{3}{4}}}{\omega_7} \right)^{\frac{0.5(q-1)}{q-0.5}}$$

$$+ \left(\frac{4\bar{\pi}_1^{\frac{3}{4}}}{\omega_7} \right) \left(\frac{3^{\frac{q-1}{4q}} \bar{\pi}_1^{\frac{5q-1}{4q}}}{\omega_8} \right)^{\frac{0.5q}{q-0.5}} \left(\frac{\omega_7}{\bar{\pi}_1^{\frac{3}{4}}} \right)^{\frac{0.5q}{q-0.5}}$$

and when $M_3 \neq 0$ or $M_6 \neq 0$, from (2 - 28),

$$
\begin{aligned}
\dot{\tilde{V}} \le & -\frac{1}{2}\left(\frac{\omega_7}{\bar{\pi}_1^{\frac{3}{4}}}\right)V^{\frac{3}{4}} - \frac{1}{3}\left(\frac{\omega_8}{3^{\frac{q-1}{4q}}\,\bar{\pi}_1^{\frac{5q-1}{4q}}}\right)V^{\frac{5q-1}{4q}} - V^{\frac{1}{2}}\left(\frac{1}{2}\left(\frac{\omega_7}{\bar{\pi}_1^{\frac{3}{4}}}\right)V^{\frac{1}{4}} - M_3\left(\frac{2^{\frac{1}{2}}\omega_9}{\underline{\pi}_1^{\frac{1}{2}}}\right)\right) \\
& - V^{\frac{4q-1}{4q}}\left(\frac{1}{3}\left(\frac{\omega_8}{3^{\frac{q-1}{4q}}\,\bar{\pi}_1^{\frac{5q-1}{4q}}}\right)V^{\frac{1}{4}} - M_3\left(\frac{2^{\frac{1}{4q}}\omega_9}{\underline{\pi}_1^{\frac{4q-1}{4q}}}\right)\right) \\
& - V^{\frac{3}{4}}\left(\frac{1}{3}\left(\frac{\omega_8}{3^{\frac{q-1}{4q}}\,\bar{\pi}_1^{\frac{5q-1}{4q}}}\right)V^{\frac{2q-1}{4q}} - M_6\left(\frac{3^{\frac{1}{4}}\omega_{10}}{\underline{\pi}_1^{\frac{3}{4}}}\right)\right) \\
\le & -\frac{1}{2}\left(\frac{\omega_7}{\bar{\pi}_1^{\frac{3}{4}}}\right)V^{\frac{3}{4}} - \frac{1}{3}\left(\frac{\omega_8}{3^{\frac{q-1}{4q}}\,\bar{\pi}_1^{\frac{5q-1}{4q}}}\right)V^{\frac{5q-1}{4q}},
\end{aligned}
$$

$$
\text{for} \quad V \ge \max\left\{ \begin{array}{l} \left(2M_3\left(\dfrac{\bar{\pi}_1^{\frac{3}{4}}}{\omega_7}\right)\left(\dfrac{2^{\frac{1}{2}}\omega_9}{\underline{\pi}_1^{\frac{1}{2}}}\right)\right)^4, \\ \left(3M_3\left(\dfrac{3^{\frac{q-1}{4q}}\,\bar{\pi}_1^{\frac{5q-1}{4q}}}{\omega_8}\right)\left(\dfrac{2^{\frac{1}{4q}}\omega_9}{\underline{\pi}_1^{\frac{4q-1}{4q}}}\right)\right)^4, \quad \left(3M_6\left(\dfrac{3^{\frac{q-1}{4q}}\,\bar{\pi}_1^{\frac{5q-1}{4q}}}{\omega_8}\right)\left(\dfrac{3^{\frac{1}{4}}\omega_{10}}{\underline{\pi}_1^{\frac{3}{4}}}\right)\right)^{\frac{4q}{2q-1}} \end{array} \right\}
$$

thus, uniform ultimate boundedness is implied from lemma 2.6.

Remark 2.26. Note that even in the case of $M_3 = M_6 = 0$, the system is able to be exactly robust with respect to persistent perturbations in d_2 with an upper bound of M_4. This interesting feature is possible due to the discontinuous nature of the algorithm when $p = 0.5$.

Remark 2.27. It is worth mentioning that the system presented here is able to achieve the same type of convergence and robustness properties as the one in [17], in which a second order system also based on the super-twisting algorithm is studied. Of significance is in [17], an additional term in the $\dot{z}_2$ dynamics that has an exponent value $p + q + 1$, which satisfy $2p-1 \le p+q+1 \le 2q-1$, for $0.5 \le p \le 1$, and $q \ge 1$. The function of this additional term is not clearly stated there, besides that of a notational simplification in the stability analysis. Indeed if that is the case, then it is clearly not needed in our development, lemma 2.3 is available in our development. In addition, there, it requires two different Lyapunov function structure in order to extract different stability properties of the system. In particular, for ascertaining the finite time property, their strict Lyapunov function exhibit singularity in its time derivative, due to the non Lipschitzness of the said Lyapunov function. This is not desirable, as mentioned in [141], if such a system is to be applied for further applications. On the other hand, the results presented here do not have such a problem as the proposed strict Lyapunov function is locally Lipschitz.

2.6 Generic second order algorithm

Due to the similarities of their mechanical energy function in forming the respective strict Lyapunov function for twisting and super-twisting based algorithms, in this section, a generic second order algorithm, which consists of linear sum of the generic twisting algorithm (2 - 11) and generic super-twisting algorithm (2 - 23), is presented. Leveraging the results of previous sections, a strict and locally-Lipschitz is presented for the algorithm to study its stability and robustness properties.

2.6.1 System description

Consider the following generic second order algorithm dynamics:

$$\begin{aligned}\dot{z}_1 &= -k_3|z_1|^p \operatorname{sign}(z_1) - k_{3n}|z_1|^q \operatorname{sign}(z_1) + z_2 + d_1, \\ \dot{z}_2 &= -k_1|z_1|^{2p-1}\operatorname{sign}(z_1) - k_{1n}|z_1|^{2q-1}\operatorname{sign}(z_1) - k_2|z_2|^{\frac{2p-1}{p}}\operatorname{sign}(z_2) - k_{2n}|z_2|^{\frac{2q-1}{q}}\operatorname{sign}(z_2) + d_2\end{aligned} \tag{2 - 29}$$

where z_1, z_2 $\in \mathbb{R}$, are the scalar state variables, k_1, $k_{1\mathrm{n}}$, k_2, $k_{2\mathrm{n}}$, k_3, k_{3n} are positive constants, $0.5 \leq p \leq 1$, and $1 \leq q$. While d_1 and d_2 are time-varying and/or nonlinear terms of uncertainty bounded by:

$$|d_1| \leq M_1|z_1|^p + M_2|z_1|^q + M_3, \text{ and}$$

$$|d_2| \leq M_4|z_1|^{2p-1} + M_5|z_1|^{2q-1} + M_6|z_2|^{2-\frac{1}{p}} + M_7|z_2|^{2-\frac{1}{q}} + M_8,$$

with $|d_1| := \sup\{|\delta| : \delta \in \boldsymbol{K}[d_1]\}$, and $|d_2| := \sup\{|\delta| : \delta \in \boldsymbol{K}[d_2]\}$,

where $M_1 \geq 0$, $M_2 \geq 0$, $M_3 \geq 0$, $M_4 \geq 0$, $M_5 \geq 0$, $M_6 \geq 0$, $M_7 \geq 0$, and $M_8 \geq 0$ with the same p and q as that in (2 - 29). Note that for the case of p = 0.5, the algorithm contains discontinuity and the uncertainty is upper bounded by nonvanishing constant.

2.6.2 Stability analysis

The following Lyapunov function is proposed for the system (2 - 29),

$$V(z_1, z_2) = \left(\frac{1}{2}z_2^{\,2} + \frac{k_1}{2p}|z_1|^{2p} + \frac{k_{1n}}{2q}|z_1|^{2q}\right)^2$$

Remark 2.28. The system (2 - 29) can be viewed as the combination of the generic twisting algorithm and the generic super-twisting algorithm considered in section 2.2.2 and 2.2.4 respectively. Observed that the structure of Lyapunov functions proposed in the previous sections are motivated by the mechanical energy of the system [66], which by itself is a weak Lyapunov function, i.e. only negative semidefinite can be attained in its time derivative taken along the solutions of the system. To ensure negative definiteness, a cross term, that contains both state variables, is added accordingly. In particular, for the case of twisting based algorithm (Section 2.2.1 and 2.2.2), the cross term can only have a z_2 with unity exponent, in order to obtain a negative definite z_1 term in the time derivative of the Lyapunov function. While the exponent of the z_1 part of the cross term, it is selected as to maintain the homogeneity of the energy function. However, in using the original energy function, the resulting cross term is non-Lipschitz in order to maintain homogeneity of the function. Hence, to overcome it, the energy function is simply squared, consequently a locally Lipschitz and strict Lyapunov function is produced. The same arguments applied to super-twisting based algorithm as well. Of interest, is that both twisting and super-twisting based algorithm embodies the same mechanical energy term, moreover in this section, the algorithm (2 - 29) has the same energy function as well. As will be shown in the following development, the energy function is complete by itself without the need for cross term, to generate a negative definite time derivative.

Note that the Lyapunov function, V can be lower and upper bounded by

$$\underline{\pi}_1^{\,2}\left(|z_1|^{2p}+|z_1|^{2q}+|z_2|^2\right)^2 \le V(z_1,z_2) \le \overline{\pi}_1^{\,2}\left(|z_1|^{2p}+|z_1|^{2q}+|z_2|^2\right)^2$$

where $\overline{\pi}_1 = \max\left\{\frac{1}{2},\frac{k_1}{2p},\frac{k_{1n}}{2q}\right\}$, and $\underline{\pi}_1 = \min\left\{\frac{1}{2},\frac{k_1}{2p},\frac{k_{1n}}{2q}\right\}$

Thus, V is positive definite and radially unbounded. Since (2 - 29) is a differential equation that has discontinuous right-hand side, i.e. when $p = 0.5$ and since no continuity assumption is made on both d_1 and d_2, its solution are understood in the sense of Filippov (see definition 2.1). According to lemma 2.4, the time derivative of non-smooth Lyapunov function exists almost everywhere,

$$\frac{d}{dt}V(\mathbf{z}(t),t) \overset{a.e.}{\in} \dot{\tilde{V}}(\mathbf{z}(t),t) = \bigcap_{\xi\in\partial V(\mathbf{z}(t),t)} \xi^T K\begin{bmatrix}\dot{z}_1\\ \dot{z}_2\end{bmatrix}(\mathbf{z},t),$$

In accordance to the analysis performed in previous sections, the case of $p > 0.5$ and $p = 0.5$ are analyzed separately, due to discontinuity of the controller (2 - 29) when $p = 0.5$.

a) *Case 1: For 0.5 < p ≤ 1, and 1 ≤ q*

Note that for $p > 0.5$, V is continuously differentiable, hence

$$\dot{\tilde{V}}(z_1,z_2) = \bigcap_{\xi\in\partial V(\mathbf{z}(t),t)} \xi^T K\begin{bmatrix}\dot{z}_1\\ \dot{z}_2\end{bmatrix} = \nabla V^T K\begin{bmatrix}\dot{z}_1\\ \dot{z}_2\end{bmatrix}$$

$$
\begin{aligned}
&\subset -k_1 k_3 |z_1|^{3p-1} |z_2|^2 - k_{1n} k_{3n} |z_1|^{3q-1} |z_2|^2 - k_1 k_{3n} |z_1|^{2p+q-1} |z_2|^2 \\
&\quad - k_{1n} k_3 |z_1|^{2q+p-1} |z_2|^2 - \frac{k_1}{p} k_2 |z_1|^{2p} |z_2|^{\frac{3p-1}{p}} - \frac{k_1}{p} k_{2n} |z_1|^{2p} |z_2|^{\frac{3q-1}{q}} \\
&\quad - \frac{k_{1n}}{q} k_2 |z_1|^{2q} |z_2|^{\frac{3p-1}{p}} - \frac{k_{1n}}{q} k_{2n} |z_1|^{2q} |z_2|^{\frac{3q-1}{q}} \\
&\quad - k_1 k_{1n} k_3 \left(\frac{1}{p} + \frac{1}{q} \right) |z_1|^{3p+2q-1} - k_1 k_{1n} k_{3n} \left(\frac{1}{p} + \frac{1}{q} \right) |z_1|^{2p+3q-1} \\
&\quad - \frac{k_1}{p} k_1 k_{3n} |z_1|^{4p+q-1} - \frac{k_{1n}}{q} k_{1n} k_3 |z_1|^{4q+p-1} \\
&\quad - \frac{k_1}{p} k_1 k_3 |z_1|^{5p-1} - \frac{k_{1n}}{q} k_{1n} k_{3n} |z_1|^{5q-1} - k_2 |z_2|^{\frac{5p-1}{p}} - k_{2n} |z_2|^{\frac{5q-1}{q}} \\
&\quad + K[d_1] \left(\begin{aligned} & k_1 |z_1|^{2p-1} |z_2|^2 \operatorname{sign}(z_1) + \frac{k_1{}^2}{p} |z_1|^{4p-1} \operatorname{sign}(z_1) \\ & + \frac{k_1 k_{1n}}{q} |z_1|^{2p+2q-1} \operatorname{sign}(z_1) + k_{1n} |z_1|^{2q-1} |z_2|^2 \operatorname{sign}(z_1) \\ & + \frac{k_1 k_{1n}}{p} |z_1|^{2q+2p-1} \operatorname{sign}(z_1) + \frac{k_{1n}{}^2}{q} |z_1|^{4q-1} \operatorname{sign}(z_1) \end{aligned} \right) \\
&\quad + K[d_2] \left(z_2{}^3 + \frac{k_1}{p} |z_1|^{2p} z_2 + \frac{k_{1n}}{q} |z_1|^{2q} z_2 \right)
\end{aligned}
$$

After rearrangement:

$$
\begin{aligned}
\dot{\tilde{V}} \le & -k_1\left(k_3 - M_1\right)\left|z_1\right|^{3p-1}\left|z_2\right|^2 - k_{1n}\left(k_{3n} - M_2\right)\left|z_1\right|^{3q-1}\left|z_2\right|^2 \\
& -k_1\left(k_{3n} - M_2\right)\left|z_1\right|^{2p+q-1}\left|z_2\right|^2 - k_{1n}\left(k_3 - M\right)_1\left|z_1\right|^{p+2q-1}\left|z_2\right|^2 \\
& -\frac{k_1}{p}\left(k_2 - M_6\right)\left|z_1\right|^{2p}\left|z_2\right|^{3-\frac{1}{p}} - \frac{k_1}{p}\left(k_{2n} - M_7\right)\left|z_1\right|^{2p}\left|z_2\right|^{3-\frac{1}{q}} \\
& -\frac{k_{1n}}{q}\left(k_2 - M_6\right)\left|z_1\right|^{2q}\left|z_2\right|^{3-\frac{1}{p}} - \frac{k_{1n}}{q}\left(k_{2n} - M_7\right)\left|z_1\right|^{2q}\left|z_2\right|^{3-\frac{1}{q}} \\
& -k_1k_{1n}\left(\frac{1}{p}+\frac{1}{q}\right)\left(k_3 - M_1\right)\left|z_1\right|^{2q+3p-1} - k_1k_{1n}\left(\frac{1}{p}+\frac{1}{q}\right)\left(k_{3n} - M_2\right)\left|z_1\right|^{3q+2p-1} \\
& -\frac{k_1{}^2}{p}\left(k_{3n} - M_2\right)\left|z_1\right|^{4p+q-1} - \frac{k_{1n}{}^2}{q}\left(k_3 - M_1\right)\left|z_1\right|^{p+4q-1} - \frac{k_1{}^2}{2p}\left(k_3 - M_1\right)\left|z_1\right|^{5p-1} \\
& -\frac{k_{1n}{}^2}{2q}\left(k_{3n} - M_2\right)\left|z_1\right|^{5q-1} - \frac{\left(k_2 - M_6\right)}{2}\left|z_2\right|^{5-\frac{1}{p}} - \frac{\left(k_{2n} - M_7\right)}{2}\left|z_2\right|^{5-\frac{1}{q}} + \dot{V}_1 \\
& +M_3\begin{pmatrix} \frac{k_1{}^2}{p}\left|z_1\right|^{4p-1} + \frac{k_{1n}{}^2}{q}\left|z_1\right|^{4q-1} + k_1\left|z_1\right|^{2p-1}\left|z_2\right|^2 + k_{1n}\left|z_1\right|^{2q-1}\left|z_2\right|^2 \\ + k_1k_{1n}\left(\frac{1}{p}+\frac{1}{q}\right)\left|z_1\right|^{2p+2q-1} \end{pmatrix} \\
& +M_8\left(\left|z_2\right|^3 + \frac{k_1}{p}\left|z_1\right|^{2p}\left|z_2\right| + \frac{k_{1n}}{q}\left|z_1\right|^{2q}\left|z_2\right|\right)
\end{aligned}
$$

where

$$
\begin{aligned}
\dot{V}_1 = & -\frac{k_1{}^2}{2p}\left(k_3 - M_1\right)\left|z_1\right|^{5p-1} - \frac{k_{1n}{}^2}{2q}\left(k_{3n} - M_2\right)\left|z_1\right|^{5q-1} - \frac{\left(k_2 - M_6\right)}{2}\left|z_2\right|^{5-\frac{1}{p}} \\
& -\frac{\left(k_{2n} - M_7\right)}{2}\left|z_2\right|^{5-\frac{1}{q}} + M_4\left|z_1\right|^{2p-1}\left|z_2\right|^3 + M_4\frac{k_1}{p}\left|z_1\right|^{4p-1}\left|z_2\right| + M_5\left|z_1\right|^{2q-1}\left|z_2\right|^3 \\
& +\frac{k_{1n}}{q}M_5\left|z_1\right|^{4q-1}\left|z_2\right| + \left(\frac{k_{1n}}{q}M_4 + \frac{k_1}{p}M_5\right)\left|z_1\right|^{2p+2q-1}\left|z_2\right|
\end{aligned}
$$

Applying lemma 2.1,

$$
-\left|z_1\right|^{5p-1} - \left|z_2\right|^{5-\frac{1}{p}} \le -\left(\frac{5p-1}{2p-1}\right)^{\frac{2p-1}{5p-1}}\left|z_1\right|^{2p-1}\left(\frac{5p-1}{3p}\right)^{\frac{3p}{5p-1}}\left|z_2\right|^3,
$$

$$-|z_1|^{5p-1}-|z_2|^{5-\frac{1}{p}}\le-\left(\frac{5p-1}{4p-1}\right)^{\frac{4p-1}{5p-1}}|z_1|^{4p-1}\left(\frac{5p-1}{p}\right)^{\frac{p}{5p-1}}|z_2|,$$

$$-|z_1|^{5q-1}-|z_2|^{5-\frac{1}{q}}\le-\left(\frac{5q-1}{2q-1}\right)^{\frac{2q-1}{5q-1}}|z_1|^{2q-1}\left(\frac{5q-1}{3q}\right)^{\frac{3q}{5q-1}}|z_2|^{3},$$

$$-|z_1|^{5q-1}-|z_2|^{5-\frac{1}{q}}\le-\left(\frac{5q-1}{4q-1}\right)^{\frac{4q-1}{5q-1}}|z_1|^{4q-1}\left(\frac{5q-1}{q}\right)^{\frac{q}{5q-1}}|z_2|,$$

$$\begin{aligned}-|z_1|^{5p-1}-|z_1|^{5q-1}-|z_2|^{5-\frac{1}{p}}&\le-|z_1|^{(2p+2q-1)\left(\frac{5p-1}{4p-1}\right)}-|z_2|^{5-\frac{1}{p}}\\&\le-\left(\frac{5p-1}{4p-1}\right)^{\frac{4p-1}{5p-1}}|z_1|^{(2p+2q-1)}\left(\frac{5p-1}{p}\right)^{\frac{p}{5p-1}}|z_2|,\end{aligned}$$

where lemma 2.3 has been employed in the last inequality since

$$5p-1\le(2p+2q-1)\left(\frac{5p-1}{4p-1}\right)\le 5q-1.$$

Thus, if the following inequalities

$$\min\left\{\begin{array}{l}\lambda_1 k_1^{\frac{4p-2}{5p-1}}(k_3-M_1)^{\frac{2p-1}{5p-1}}(k_2-M_6)^{\frac{3p}{5p-1}},\\ \lambda_2 k_1^{\frac{3p-1}{5p-1}}(k_3-M_1)^{\frac{4p-1}{5p-1}}(k_2-M_6)^{\frac{p}{5p-1}},\\ \lambda_3 k_{1n}^{\frac{4q-2}{5q-1}}(k_{3n}-M_2)^{\frac{2q-1}{5q-1}}(k_{2n}-M_7)^{\frac{3q}{5q-1}},\\ \lambda_4 k_{1n}^{\frac{3q-1}{5q-1}}(k_{3n}-M_2)^{\frac{4q-1}{5q-1}}(k_{2n}-M_7)^{\frac{q}{5q-1}},\\ \lambda_5\left(\frac{pq}{pk_{1n}+qk_1}\right)\omega_1^{\frac{4p-1}{5p-1}}(k_2-M_6)^{\frac{p}{5p-1}}\end{array}\right\}>\max\{M_4,M_5\}\qquad(2\text{ - }30)$$

where

$$\lambda_1=\left(\frac{5p-1}{12p^2-6p}\right)^{\frac{2p-1}{5p-1}}\left(\frac{5p-1}{18p}\right)^{\frac{3p}{5p-1}},\ \lambda_2=p\left(\frac{5p-1}{24p^2-6p}\right)^{\frac{4p-1}{5p-1}}\left(\frac{5p-1}{6p}\right)^{\frac{p}{5p-1}},$$

$$\lambda_3=\left(\frac{5q-1}{12q^2-6q}\right)^{\frac{2q-1}{5q-1}}\left(\frac{5q-1}{12q}\right)^{\frac{3q}{5q-1}},\ \lambda_4=q\left(\frac{5q-1}{24q^2-6q}\right)^{\frac{4q-1}{5q-1}}\left(\frac{5q-1}{4q}\right)^{\frac{q}{5q-1}},$$

$$\lambda_5=\left(\frac{5p-1}{4p-1}\right)^{\frac{4p-1}{5p-1}}\left(\frac{5p-1}{6p}\right)^{\frac{p}{5p-1}},\ \omega_1=\min\left\{\frac{k_1^{\,2}(k_3-M_1)}{6p},\frac{k_{1n}^{\,2}(k_{3n}-M_2)}{6q}\right\}$$

hold then the function $\dot{V}_1$ is negative definite. Then,

$$\begin{aligned}\dot{\tilde{V}}\le&-\frac{k_1^{\,2}}{2p}(k_3-M_1)|z_1|^{5p-1}-\frac{k_{1n}^{\,2}}{2q}(k_{3n}-M_2)|z_1|^{5q-1}\\&-\frac{(k_2-M_6)}{2}|z_2|^{5-\frac{1}{p}}-\frac{(k_{2n}-M_7)}{2}|z_2|^{5-\frac{1}{q}}\\&+M_3\left(\begin{aligned}&\frac{k_1^{\,2}}{p}|z_1|^{4p-1}+\frac{k_{1n}^{\,2}}{q}|z_1|^{4q-1}+k_1|z_1|^{2p-1}|z_2|^2\\&+k_{1n}|z_1|^{2q-1}|z_2|^2+k_1k_{1n}\left(\frac{1}{p}+\frac{1}{q}\right)|z_1|^{2p+2q-1}\end{aligned}\right)\\&+M_8\left(|z_2|^3+\frac{k_1}{p}|z_1|^{2p}|z_2|+\frac{k_{1n}}{q}|z_1|^{2q}|z_2|\right)\end{aligned}$$

Applying lemma 2.1,

$$|z_1|^{2p-1}|z_2|^2\le\left(\frac{2p-1}{4p-1}\right)|z_1|^{4p-1}+\left(\frac{2p}{4p-1}\right)|z_2|^{\frac{4p-1}{p}},$$

$$|z_1|^{2q-1}|z_2|^2\le\left(\frac{2q-1}{4q-1}\right)|z_1|^{4q-1}+\left(\frac{2q}{4q-1}\right)|z_2|^{\frac{4q-1}{q}},$$

$$|z_1|^{2q+2p-1}\le\frac{1}{2}|z_1|^{4q-1}+\frac{1}{2}|z_1|^{4p-1},$$

$$|z_1|^{2p}|z_2|\le\left(\frac{2}{3}\right)|z_1|^{3p}+\left(\frac{1}{3}\right)|z_2|^3,$$

$$|z_1|^{2q}|z_2| \le \left(\frac{2}{3}\right)|z_1|^{3q} + \left(\frac{1}{3}\right)|z_2|^3,$$

and lemma 2.3,

$$-|z_1|^{5p-1} - |z_1|^{5q-1} \le -|z_1|^{5q-\frac{q}{p}},\ -|z_1|^{5p-1} - |z_1|^{5q-1} \le -|z_1|^{5p-\frac{p}{q}},$$

since $5p-1 \le 5q - \dfrac{q}{p} \le 5q-1,$ and $5p-1 \le 5p - \dfrac{p}{q} \le 5q-1,$

together with lemma 2.2, and the bounds of the Lyapunov function, we have:

$$\begin{aligned}
\dot{\tilde{V}} &\le -\omega_2\left(|z_1|^{5p-1} + |z_1|^{5q-\frac{q}{p}} + |z_2|^{5-\frac{1}{p}}\right) - \omega_3\left(|z_1|^{5q-1} + \omega_1|z_1|^{5p-\frac{p}{q}} + |z_2|^{5-\frac{1}{q}}\right) \\
&\quad + M_3\omega_4\left(|z_1|^{4p-1} + |z_2|^{\frac{4p-1}{p}} + |z_1|^{4q-1} + |z_2|^{\frac{4q-1}{q}}\right) + M_8\omega_5\left(|z_1|^{3p} + |z_1|^{3q} + |z_2|^3\right) \\
&= -\omega_2\left(\left(|z_1|^{2p}\right)^{\frac{5p-1}{2p}} + \left(|z_1|^{2q}\right)^{\frac{5p-1}{2p}} + \left(|z_2|^2\right)^{\frac{5p-1}{2p}}\right) \\
&\quad - \omega_3\left(\left(|z_1|^{2q}\right)^{\frac{5q-1}{2q}} + \left(|z_1|^{2p}\right)^{\frac{5q-1}{2q}} + \left(|z_2|^2\right)^{\frac{5q-1}{2q}}\right) \\
&\quad + M_3\omega_4\left(\left(|z_1|^{2p}\right)^{\frac{4p-1}{2p}} + \left(|z_2|^2\right)^{\frac{4p-1}{2p}} + \left(|z_1|^{2q}\right)^{\frac{4q-1}{2q}} + \left(|z_2|^2\right)^{\frac{4q-1}{2q}}\right) \\
&\quad + M_8\omega_5\left(\left(|z_1|^{2p}\right)^{\frac{3}{2}} + \left(|z_1|^{2q}\right)^{\frac{3}{2}} + \left(|z_2|^2\right)^{\frac{3}{2}}\right)
\end{aligned}$$

$$\begin{aligned}
&\le -\left(\frac{\omega_2}{3^{\frac{3p-1}{2p}}\,\overline{\pi}_1^{\frac{5p-1}{2p}}}\right)V^{\frac{5p-1}{4p}} - \left(\frac{\omega_3}{3^{\frac{3q-1}{2q}}\,\overline{\pi}_1^{\frac{5q-1}{2q}}}\right)V^{\frac{5q-1}{4q}} + M_3\left(\frac{\omega_4}{\underline{\pi}_1^{\frac{4p-1}{2p}}}\right)V^{\frac{4p-1}{4p}} \\
&\quad + M_3\left(\frac{\omega_4}{\underline{\pi}_1^{\frac{4q-1}{2q}}}\right)V^{\frac{4q-1}{4q}} + M_8\left(\frac{\omega_5}{\underline{\pi}_1^{\frac{3}{2}}}\right)V^{\frac{3}{4}}
\end{aligned} \qquad (2-31)$$

where

$$\omega_2 = \min\left\{\frac{k_1^{\,2}(k_3 - M_1)}{6p},\ \omega_1,\ \frac{(k_2 - M_6)}{2}\right\},\ \omega_3 = \min\left\{\frac{k_{1n}^{\,2}(k_{3n} - M_2)}{6q},\ \omega_1,\frac{(k_{2n} - M_7)}{2}\right\},$$

$$\omega_4 = \max\left\{\begin{array}{l}\left(\left(\frac{k_1^{\,2}}{p} + k_1\left(\frac{2p-1}{4p-1}\right) + \frac{k_1 k_{1n}}{2}\left(\frac{1}{p} + \frac{1}{q}\right)\right),\right. \\ \left(\frac{k_{1n}^{\,2}}{q} + k_{1n}\left(\frac{2q-1}{4q-1}\right) + \frac{k_1 k_{1n}}{2}\left(\frac{1}{p} + \frac{1}{q}\right)\right),\ k_1\left(\frac{2p}{4p-1}\right), k_{1n}\left(\frac{2q}{4q-1}\right)\end{array}\right\},$$

$$\omega_5 = \max\left\{\frac{2k_1}{3p},\ \frac{2k_{1n}}{3q},\ \left(1 + \frac{k_1}{3p} + \frac{k_{1n}}{3q}\right)\right\}$$

Remark 2.29. Note that the nonlinear inequalities (2 - 30) is feasible with respect to k_1, k_{1n} , k_2, k_{2n} , k_3, k_{3n} for any $M_1 \geq 0$, $M_2 \geq 0$, $M_4 \geq 0$, $M_5 \geq 0$, $M_6 \geq 0$, and $M_7 \geq 0$. If $M_4 = M_5 = 0$, (2 - 30) is satisfied for any $k_2 > M_6$, $k_{2n} > M_7$, $k_3 > M_1$, $k_{3n} > M_2$ and any $k_1 > 0$, $k_{1n} > 0$. This coincides with the conditions obtained through energy function for the generic twisting and generic super-twisting algorithm studied in the previous sections.

Remark 2.30. Hence, when $M_3 = M_8 = 0$, i.e. without persistent perturbations, from lemma 2.5-3,

1. For $0 < p < 1$, and $q > 1$the system exhibit uniform asymptotical convergence, where

$$T_{\max}(\mu)=\left(\frac{3^{\frac{3q-1}{2q}}\pi_1^{\frac{5q-1}{2q}}}{\omega_3}\right)\left(\frac{4q}{q-1}\right)\left(\frac{1}{\mu^{\frac{q-1}{4q}}}\right),$$

is the time at which the trajectories reach the surface level $V = \mu$. At the same time, the system (2 - 29) will have finite time convergence, in particular after reaching the surface level $V=\mu$, from lemma 2.5-3, the settling time estimate,

$$T(\mu)\leq\left(\frac{3^{\frac{3p-1}{2p}}\,\overline{\pi}_1^{\frac{5p-1}{2p}}}{\omega_2}\right)\left(\frac{4p}{1-p}\right)[\mu]^{\frac{1-p}{4p}},$$

where the initial starting states is changed to $V=\mu$. Hence, the total time to reach the origin can be estimated as

$$T_{total}(\mu)=\left(\frac{3^{\frac{3q-1}{2q}}\,\overline{\pi}_1^{\frac{5q-1}{2q}}}{\omega_3}\right)\left(\frac{4q}{q-1}\right)\left(\frac{1}{\mu^{\frac{q-1}{4q}}}\right)+\left(\frac{3^{\frac{3p-1}{2p}}\,\overline{\pi}_1^{\frac{5p-1}{2p}}}{\omega_2}\right)\left(\frac{4p}{1-p}\right)[\mu]^{\frac{1-p}{4p}}.$$

The minimum of this function can be found at

$$\mu=\left(\frac{3^{\frac{3q-1}{2q}}\,\overline{\pi}_1^{\frac{5q-1}{2q}}}{\omega_3}\right)^{\frac{4pq}{q-p}}\left(\frac{\omega_2}{3^{\frac{3p-1}{2p}}\,\overline{\pi}_1^{\frac{5p-1}{2p}}}\right)^{\frac{4pq}{q-p}}.$$

Substituting into the function, a finite settling time independent of initial conditions is obtained,

$$\begin{aligned}T_{total}=&\left(\frac{3^{\frac{3q-1}{2q}}\,\overline{\pi}_1^{\frac{5q-1}{2q}}}{\omega_3}\right)\left(\frac{4q}{q-1}\right)\left(\frac{\omega_3}{3^{\frac{3q-1}{2q}}\,\overline{\pi}_1^{\frac{5q-1}{2q}}}\right)^{\frac{p(q-1)}{q-p}}\left(\frac{3^{\frac{3p-1}{2p}}\,\overline{\pi}_1^{\frac{5p-1}{2p}}}{\omega_2}\right)^{\frac{p(q-1)}{q-p}}\\&+\left(\frac{3^{\frac{3p-1}{2p}}\,\overline{\pi}_1^{\frac{5p-1}{2p}}}{\omega_2}\right)\left(\frac{4p}{1-p}\right)\left(\frac{3^{\frac{3q-1}{2q}}\,\overline{\pi}_1^{\frac{5q-1}{2q}}}{\omega_3}\right)^{\frac{q(1-p)}{q-p}}\left(\frac{\omega_2}{3^{\frac{3p-1}{2p}}\,\overline{\pi}_1^{\frac{5p-1}{2p}}}\right)^{\frac{q(1-p)}{q-p}}\end{aligned}$$

2. For $0<p<1$, and $q=1$, finite time convergence can be concluded.

3. For $p = 1$, and $q \geq 1$, exponential convergence can be concluded.

The above results are possible due to the negative definiteness of the time derivative of the Lyapunov function, i.e. strict Lyapunov function.

Remark 2.31. While if persistent perturbations occur on the system, $M_3 \neq 0$ and/or $M_8 \neq 0$, from (2 - 31),

$$
\begin{aligned}
\dot{\tilde{V}} \leq & -\frac{1}{2}\left(\frac{\omega_2}{3^{\frac{3p-1}{2p}}\,\bar{\pi}_1^{\frac{5p-1}{2p}}}\right)V^{\frac{5p-1}{4p}} - \frac{1}{3}\left(\frac{\omega_3}{3^{\frac{3q-1}{2q}}\,\bar{\pi}_1^{\frac{5q-1}{2q}}}\right)V^{\frac{5q-1}{4q}} \\
& - V^{\frac{4p-1}{4p}}\left(\frac{1}{2}\left(\frac{\omega_2}{3^{\frac{3p-1}{2p}}\,\bar{\pi}_1^{\frac{5p-1}{2p}}}\right)V^{\frac{1}{4}} - M_3\left(\frac{\omega_4}{\underline{\pi}_1^{\frac{4p-1}{2p}}}\right)\right) \\
& - V^{\frac{4q-1}{4q}}\left(\frac{1}{3}\left(\frac{\omega_3}{3^{\frac{3q-1}{2q}}\,\bar{\pi}_1^{\frac{5q-1}{2q}}}\right)V^{\frac{1}{4}} - M_3\left(\frac{\omega_4}{\underline{\pi}_1^{\frac{4q-1}{2q}}}\right)\right) \\
& - V^{\frac{3}{4}}\left(\frac{1}{3}\left(\frac{\omega_3}{3^{\frac{3q-1}{2q}}\,\bar{\pi}_1^{\frac{5q-1}{2q}}}\right)V^{\frac{2q-1}{4q}} - M_8\left(\frac{\omega_5}{\underline{\pi}_1^{\frac{3}{2}}}\right)\right)
\end{aligned}
$$

$$\le -\frac{1}{2}\left(\frac{\omega_2}{3^{\frac{3p-1}{2p}}\,\overline{\pi}_1^{\frac{5p-1}{2p}}}\right)V^{\frac{5p-1}{4p}}-\frac{1}{3}\left(\frac{\omega_3}{3^{\frac{3q-1}{2q}}\,\overline{\pi}_1^{\frac{5q-1}{2q}}}\right)V^{\frac{5q-1}{4q}},$$

$$\text{for } V\ge \max\left\{\begin{array}{l}\left(2M_3\left(\frac{3^{\frac{3p-1}{2p}}\,\overline{\pi}_1^{\frac{5p-1}{2p}}}{\omega_2}\right)\left(\frac{\omega_4}{\underline{\pi}_1^{\frac{4p-1}{2p}}}\right)\right)^4,\left(3M_3\left(\frac{3^{\frac{3q-1}{2q}}\,\overline{\pi}_1^{\frac{5q-1}{2q}}}{\omega_3}\right)\left(\frac{\omega_4}{\underline{\pi}_1^{\frac{4q-1}{2q}}}\right)\right)^4,\\ \left(3M_8\left(\frac{3^{\frac{3q-1}{2q}}\,\overline{\pi}_1^{\frac{5q-1}{2q}}}{\omega_3}\right)\left(\frac{\omega_5}{\underline{\pi}_1^{\frac{3}{2}}}\right)\right)^{\frac{4q}{2q-1}}\end{array}\right\}$$

uniform ultimate boundedness is concluded by using lemma 2.6.

b) Case 2: For p = 0.5, and 1 ≤ q

For $p = 0.5$, V is not differentiable on $z_1 = 0$, hence

$$\dot{\tilde{V}}(z_1,z_2)=\bigcap_{\xi\in\partial V(\mathbf{z}(t),t)}\xi^T K\begin{bmatrix}\dot{z}_1\\ \dot{z}_2\end{bmatrix}\subset \dot{\tilde{V}}_1+\dot{\tilde{V}}_2$$

where

$$\dot{\tilde{V}}_1=\bigcap_{\xi\in\partial V(\mathbf{z}(t),t)}\xi^T K[f](z_1,z_2),\ \dot{\tilde{V}}_2=\bigcap_{\xi\in\partial V(\mathbf{z}(t),t)}\xi^T K\begin{bmatrix}d_1\\ d_2\end{bmatrix},$$

$$K[f](z_1,z_2)\in\begin{bmatrix}-k_3|z_1|^{0.5}\operatorname{sign}(z_1)-k_{3n}|z_1|^q\operatorname{sign}(z_1)+z_2\\ -k_1\mathrm{SGN}(z_1)-k_{1n}|z_1|^{2q-1}\operatorname{sign}(z_1)-k_2\mathrm{SGN}(z_2)-k_{2n}|z_2|^{\frac{2q-1}{q}}\operatorname{sign}(z_2)\end{bmatrix}$$

where

$$\operatorname{SGN}(x)=\begin{cases}-1 & x<0\\ [-1,\,1] & x=0\\ 1 & x>0\end{cases}$$

$$\partial V=\boldsymbol{K}[\nabla V]=\boldsymbol{K}\begin{bmatrix}\dfrac{\partial V}{\partial z_1}\\ \dfrac{\partial V}{\partial z_2}\end{bmatrix}\subset\begin{bmatrix}\boldsymbol{K}\left[\dfrac{\partial V}{\partial z_1}\right]\\ \boldsymbol{K}\left[\dfrac{\partial V}{\partial z_2}\right]\end{bmatrix}$$

$$=\begin{cases}\forall z_1\neq 0, z_2\in\mathbb{R}:\\ \begin{bmatrix}\left(z_2{}^2+2k_1|z_1|+\dfrac{k_{1n}}{q}|z_1|^{2q}\right)\left(k_1+k_{1n}|z_1|^{2q-1}\right)\operatorname{sign}(z_1)\\ \left(z_2{}^2+2k_1|z_1|+\dfrac{k_{1n}}{q}|z_1|^{2q}\right)(z_2)\end{bmatrix},\\ \forall z_1=0, z_2\in\mathbb{R}:\\ \begin{bmatrix}z_2{}^2k_1[-1,1]\\ z_2{}^3\end{bmatrix}\end{cases}$$

Let us define:

$$\left|\frac{\partial V}{\partial z_1}\right|:=\sup\left\{|\xi_1|:\xi_1\in\boldsymbol{K}\left[\frac{\partial V}{\partial z_1}\right]\right\},\text{and}\left|\frac{\partial V}{\partial z_2}\right|:=\sup\left\{|\xi_2|:\xi_2\in\boldsymbol{K}\left[\frac{\partial V}{\partial z_2}\right]\right\},$$

with $\left|\dfrac{\partial V}{\partial z_1}\right|\le k_1|z_2|^2+2k_1{}^2|z_1|+\dfrac{k_1k_{1n}}{q}|z_1|^{2q}+k_{1n}|z_1|^{2q-1}|z_2|^2+2k_1k_{1n}|z_1|^{2q}+\dfrac{k_{1n}{}^2}{q}|z_1|^{4q-1}$,

and $\left|\dfrac{\partial V}{\partial z_2}\right|\le|z_2|^3+2k_1|z_1||z_2|+\dfrac{k_{1n}}{q}|z_1|^{2q}|z_2|$.

Thus, the term

$$
\begin{aligned}
\dot{\tilde{V}}_2 &= \bigcap_{(\xi_1,\xi_2)^T \in \partial V(\mathbf{z}(t),t)} \xi_1 K[d_1] + \xi_2 K[d_2] \\
&\le \left|\frac{\partial V}{\partial z_1}\right| |d_1| + \left|\frac{\partial V}{\partial z_2}\right| |d_2| \\
&\le \left(M_1 |z_1|^{0.5} + M_2 |z_1|^q + M_3\right)\begin{pmatrix} k_1 |z_2|^2 + 2k_1^2 |z_1| + \frac{k_1 k_{1n}}{q} |z_1|^{2q} + k_{1n} |z_1|^{2q-1} |z_2|^2 \\ + 2k_1 k_{1n} |z_1|^{2q} + \frac{k_{1n}^2}{q} |z_1|^{4q-1} \end{pmatrix} \\
&\quad + \left(M + M_5 |z_1|^{2q-1} + M_7 |z_2|^{2-\frac{1}{q}} + M_8\right)\left(|z_2|^3 + 2k_1 |z_1| |z_2| + \frac{k_{1n}}{q} |z_1|^{2q} |z_2|\right)
\end{aligned}
$$

where M be defined as, $M := M_4 + M_6$, since for $p = 0.5$, $|d_1| \le M_1 |z_1|^{0.5} + M_2 |z_1|^q + M_3$, and

$|d_2| \le M_4 + M_5 |z_1|^{2q-1} + M_6 + M_7 |z_2|^{2-\frac{1}{q}} + M_8$.

Computing $\dot{\tilde{V}}_1$ for each case, we have

For $z_1 \ne 0, z_2 \ne 0$:

$$
\begin{aligned}
\dot{\tilde{V}}_1 &= -k_1 k_3 |z_1|^{0.5} |z_2|^2 - k_1 k_{3n} |z_1|^q |z_2|^2 - k_{1n} k_3 |z_1|^{2q-0.5} |z_2|^2 - k_{1n} k_{3n} |z_1|^{3q-1} |z_2|^2 \\
&\quad - 2k_1 k_2 |z_1| |z_2| - \frac{k_{1n}}{q} k_2 |z_1|^{2q} |z_2| - 2k_1 k_{2n} |z_1| |z_2|^{\frac{3q-1}{q}} - \frac{k_{1n}}{q} k_{2n} |z_1|^{2q} |z_2|^{\frac{3q-1}{q}} \\
&\quad - 2k_1^2 k_{3n} |z_1|^{1+q} - k_1 k_{1n} k_3 \left(2 + \frac{1}{q}\right) |z_1|^{2q+0.5} - k_1 k_{1n} k_{3n} \left(2 + \frac{1}{q}\right) |z_1|^{3q} - \frac{k_{1n}^2 k_3}{q} |z_1|^{4q-0.5} \\
&\quad - 2k_1^2 k_3 |z_1|^{1.5} - \frac{k_{1n}^2 k_{3n}}{q} |z_1|^{5q-1} - k_2 |z_2|^3 - k_{2n} |z_2|^{\frac{5q-1}{q}}
\end{aligned}
$$

For $z_1 = 0, z_2 \ne 0$:

Let $(\xi_2 k_1 z_2^2, z_2^3)^T$ with $\xi_2 \in [-1, 1]$ be an arbitrary element of $\partial V(z_1, z_2)$, then

$$\xi^{\mathrm{T}} K[f](z_1, z_2) = \begin{bmatrix} \xi_2 k_1 z_2{}^2 \\ z_2{}^3 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} z_2 \\ -[-1,\, 1]k_1 - k_2 \operatorname{sign}(z_2) - k_{2n}|z_2|^{\frac{2q-1}{q}} \operatorname{sign}(z_2) \end{bmatrix}$$
$$= [\xi_2 - 1, \xi_2 + 1]k_1 z_2{}^3 - k_2 |z_2|^3 - k_{2n}|z_2|^{\frac{5q-1}{q}}$$

hence

$$\dot{\tilde{V}}_1 = \bigcap_{\xi_2 \in [-1,\, 1]} [\xi_2 - 1,\ \xi_2 + 1]k_1 z_2{}^3 - k_2|z_2|^3 - k_{2n}|z_2|^{\frac{5q-1}{q}} = -k_2|z_2|^3 - k_{2n}|z_2|^{\frac{5q-1}{q}}$$

For $z_1 \neq 0$, $z_2 = 0$:

$$\dot{\tilde{V}}_1 = -2k_1{}^2 k_{3n}|z_1|^{1+q} - k_1 k_{1n} k_3 \left(2\frac{1}{q} \right) |z_1|^{2q+0.5} - k_1 k_{1n} k_{3n} \left(2 + \frac{1}{q} \right) |z_1|^{3q} - \frac{k_{1n}{}^2 k_3}{q} |z_1|^{4q-0.5}$$
$$- 2k_1{}^2 k_3 |z_1|^{1.5} - \frac{k_{1n}{}^2 k_{3n}}{q} |z_1|^{5q-1}$$

For $z_1 = 0$, $z_2 = 0$: $\dot{\tilde{V}}_1 = 0$.

Thus, for all $(z_1, z_2) \in \mathbb{R}^2$:

$$\dot{\tilde{V}}_1 = -k_1 k_3 |z_1|^{0.5} |z_2|^2 - k_1 k_{3n} |z_1|^q |z_2|^2 - k_{1n} k_3 |z_1|^{2q-0.5} |z_2|^2 - k_{1n} k_{3n} |z_1|^{3q-1} |z_2|^2$$
$$- 2k_1 k_2 |z_1||z_2| - \frac{k_{1n}}{q} k_2 |z_1|^{2q} |z_2| - 2k_1 k_{2n} |z_1||z_2|^{\frac{3q-1}{q}} - \frac{k_{1n}}{q} k_{2n} |z_1|^{2q} |z_2|^{\frac{3q-1}{q}}$$
$$- 2k_1{}^2 k_{3n} |z_1|^{1+q} - k_1 k_{1n} k_3 \left(2 + \frac{1}{q} \right) |z_1|^{2q+0.5} - k_1 k_{1n} k_{3n} \left(2 + \frac{1}{q} \right) |z_1|^{3q} - \frac{k_{1n}{}^2 k_3}{q} |z_1|^{4q-0.5}$$
$$- 2k_1{}^2 k_3 |z_1|^{1.5} - \frac{k_{1n}{}^2 k_{3n}}{q} |z_1|^{5q-1} - k_2 |z_2|^3 - k_{2n} |z_2|^{\frac{5q-1}{q}}$$

Hence, the generalized time derivative of the Lyapunov function, after rearrangement:

$$
\begin{aligned}
\dot{\tilde{V}} &= \dot{\tilde{V}}_1 + \dot{\tilde{V}}_2 \\
&\le -k_1\left(k_3 - M_1\right)\left|z_1\right|^{0.5}\left|z_2\right|^2 - k_{1n}\left(k_{3n} - M_2\right)\left|z_1\right|^{3q-1}\left|z_2\right|^2 - k_1\left(k_{3n} - M_2\right)\left|z_1\right|^q\left|z_2\right|^2 \\
&\quad - k_{1n}\left(k_3 - M_1\right)\left|z_1\right|^{2q-0.5}\left|z_2\right|^2 - 2k_1\left(k_2 - M\right)\left|z_1\right|\left|z_2\right| - 2k_1\left(k_{2n} - M_7\right)\left|z_1\right|\left|z_2\right|^{3-\frac{1}{q}} \\
&\quad - \frac{k_{1n}}{q}\left(k_2 - M\right)\left|z_1\right|^{2q}\left|z_2\right| - \frac{k_{1n}}{q}\left(k_{2n} - M_7\right)\left|z_1\right|^{2q}\left|z_2\right|^{3-\frac{1}{q}} - k_1k_{1n}\left(2 + \frac{1}{q}\right)\left(k_3 - M_1\right)\left|z_1\right|^{2q+0.5} \\
&\quad - \frac{k_1k_{1n}}{2}\left(2 + \frac{1}{q}\right)\left(k_{3n} - M_2\right)\left|z_1\right|^{3q} - 2k_1^{\,2}\left(k_{3n} - M_2\right)\left|z_1\right|^{q+1} \\
&\quad - \frac{k_{1n}^{\,2}}{q}\left(k_3 - M_1\right)\left|z_1\right|^{4q-0.5} - 2k_1^{\,2}\left(k_3 - M_1\right)\left|z_1\right|^{1.5} - \frac{k_{1n}^{\,2}}{2q}\left(k_{3n} - M_2\right)\left|z_1\right|^{5q-1} \\
&\quad - \frac{\left(k_2 - M\right)}{2}\left|z_2\right|^3 - \frac{\left(k_{2n} - M_7\right)}{2}\left|z_2\right|^{5-\frac{1}{q}} + \dot{V}_2 \\
&\quad + M_3\left(k_1\left|z_2\right|^2 + 2k_1^{\,2}\left|z_1\right| + k_1k_{1n}\left(\frac{1}{q} + 2\right)\left|z_1\right|^{2q} + k_{1n}\left|z_1\right|^{2q-1}\left|z_2\right|^2 + \frac{k_{1n}^{\,2}}{q}\left|z_1\right|^{4q-1}\right) \\
&\quad + M_8\left(\left|z_2\right|^3 + 2k_1\left|z_1\right|\left|z_2\right| + \frac{k_{1n}}{q}\left|z_1\right|^{2q}\left|z_2\right|\right)
\end{aligned}
$$

where

$$
\begin{aligned}
\dot{V}_2 = &-\frac{k_1k_{1n}}{2}\left(2 + \frac{1}{q}\right)\left(k_{3n} - M_2\right)\left|z_1\right|^{3q} - \frac{k_{1n}^{\,2}}{2q}\left(k_{3n} - M_2\right)\left|z_1\right|^{5q-1} - \frac{\left(k_{2n} - M_7\right)}{2}\left|z_2\right|^{5-\frac{1}{q}} \\
&- \frac{\left(k_2 - M\right)}{2}\left|z_2\right|^3 + M_5\left|z_1\right|^{2q-1}\left|z_2\right|^3 + 2k_1M_5\left|z_1\right|^{2q}\left|z_2\right| + \frac{k_{1n}}{q}M_5\left|z_1\right|^{4q-1}\left|z_2\right|
\end{aligned}
$$

Applying lemma 2.1,

$$
-\left|z_1\right|^{5q-1} - \left|z_2\right|^{5-\frac{1}{q}} \le -\left(\frac{5q-1}{2q-1}\right)^{\frac{2q-1}{5q-1}}\left|z_1\right|^{2q-1}\left(\frac{5q-1}{3q}\right)^{\frac{3q}{5q-1}}\left|z_2\right|^3,
$$

$$
-\left|z_1\right|^{5q-1} - \left|z_2\right|^{5-\frac{1}{q}} \le -\left(\frac{5q-1}{4q-1}\right)^{\frac{4q-1}{5q-1}}\left|z_1\right|^{4q-1}\left(\frac{5q-1}{q}\right)^{\frac{q}{5q-1}}\left|z_2\right|,
$$

$$-|z_1|^{3q} - |z_2|^3 \le -\left(\frac{3}{2}\right)^{\frac{2}{3}} |z_1|^{2q} (3)^{\frac{1}{3}} |z_2|,$$

Thus, if the following inequalities

$$\min \left\{ \begin{array}{l} \lambda_6 k_{1n}^{\frac{4q-2}{5q-1}} (k_{3n} - M_2)^{\frac{2q-1}{5q-1}} (k_{2n} - M_7)^{\frac{3q}{5q-1}}, \\ \lambda_7 k_{1n}^{\frac{3q-1}{5q-1}} (k_{3n} - M_2)^{\frac{4q-1}{5q-1}} (k_{2n} - M_7)^{\frac{q}{5q-1}}, \\ \lambda_8 \dfrac{k_{1n}^{\frac{2}{3}}}{k_1^{\frac{1}{3}}} (k_{3n} - M_2)^{\frac{2}{3}} (k_2 - M)^{\frac{1}{3}} \end{array} \right\} > M_5 \qquad (2 - 32)$$

where

$$\lambda_6 = \left(\frac{5q-1}{8q^2 - 4q}\right)^{\frac{2q-1}{5q-1}} \left(\frac{5q-1}{12q}\right)^{\frac{3q}{5q-1}}, \ \lambda_7 = q\left(\frac{5q-1}{16q^2 - 4q}\right)^{\frac{4q-1}{5q-1}} \left(\frac{5q-1}{4q}\right)^{\frac{q}{5q-1}},$$

$$\lambda_8 = \left(\frac{1}{2}\right)\left(\frac{3}{2}\right)^{\frac{1}{3}} \left(\frac{6q+3}{4q}\right)^{\frac{2}{3}},$$

hold then the function $\dot{V}_2$ is negative definite. Then,

$$\begin{aligned} \dot{\tilde{V}} \le & -2k_1^2 (k_3 - M_1)|z_1|^{1.5} - \frac{k_1 k_{1n}}{2}\left(2 + \frac{1}{q}\right)(k_{3n} - M_2)|z_1|^{3q} - \frac{(k_2 - M)}{2}|z_2|^3 \\ & - \frac{k_{1n}^2}{2q}(k_{3n} - M_2)|z_1|^{5q-1} - \frac{(k_{2n} - M_7)}{2}|z_2|^{5-\frac{1}{q}} \\ & + M_3\left(k_1|z_2|^2 + 2k_1^2|z_1| + k_1 k_{1n}\left(\frac{1}{q} + 2\right)|z_1|^{2q} + k_{1n}|z_1|^{2q-1}|z_2|^2 + \frac{k_{1n}^2}{q}|z_1|^{4q-1}\right) \\ & + M_8\left(|z_2|^3 + 2k_1|z_1||z_2| + \frac{k_{1n}}{q}|z_1|^{2q}|z_2|\right) \end{aligned}$$

Applying lemma 2.1,

$$|z_1|^{2q-1}|z_2|^2 \le \left(\frac{2q-1}{4q-1}\right)|z_1|^{4q-1} + \left(\frac{2q}{4q-1}\right)|z_2|^{\frac{4q-1}{q}},$$

$$|z_1|^{2q} = |z_1|^{0.5}|z_1|^{2q-0.5} \le \frac{1}{2}|z_1| + \frac{1}{2}|z_1|^{4q-1},$$

$$|z_1||z_2| \le \left(\frac{2}{3}\right)|z_1|^{\frac{3}{2}} + \left(\frac{1}{3}\right)|z_2|^3,$$

$$|z_1|^{2q}|z_2| \le \left(\frac{2}{3}\right)|z_1|^{3q} + \left(\frac{1}{3}\right)|z_2|^3,$$

and lemma 2.3,

$$-|z_1|^{1.5} - |z_1|^{5q-1} \le -|z_1|^{\frac{5q-1}{2q}} \text{ since } 1.5 \le \frac{5q-1}{2q} \le 5q-1,$$

together with lemma 2.2, and the bounds of the Lyapunov function, we have:

$$\begin{aligned}
\dot{\tilde{V}} &\le -\omega_6\left(|z_1|^{1.5} + |z_1|^{3q} + |z_2|^3\right) - \omega_7\left(|z_1|^{5q-1} + |z_1|^{\frac{5q-1}{2q}} + |z_2|^{5-\frac{1}{q}}\right) \\
&\quad + M_3\omega_8\left(|z_1| + |z_2|^2 + |z_1|^{4q-1} + |z_2|^{\frac{4q-1}{q}}\right) + M_8\omega_9\left(|z_1|^{\frac{3}{2}} + |z_1|^{3q} + |z_2|^3\right) \\
&= -\omega_6\left(|z_1|^{1.5} + \left(|z_1|^{2q}\right)^{1.5} + \left(|z_2|^2\right)^{1.5}\right) - \omega_7\left(\left(|z_1|^{2q}\right)^{\frac{5q-1}{2q}} + |z_1|^{\frac{5q-1}{2q}} + \left(|z_2|^2\right)^{\frac{5q-1}{2q}}\right) \\
&\quad + M_3\omega_8\left(|z_1| + |z_2|^2 + \left(|z_1|^{2q}\right)^{\frac{4q-1}{2q}} + \left(|z_2|^2\right)^{\frac{4q-1}{2q}}\right) + M_8\omega_9\left(|z_1|^{\frac{3}{2}} + \left(|z_1|^{2q}\right)^{\frac{3}{2}} + \left(|z_2|^2\right)^{\frac{3}{2}}\right)
\end{aligned}$$

$$\begin{aligned}
&\le -\left(\frac{\omega_6}{3^{0.5}\,\overline{\pi}_1^{\,1.5}}\right)V^{\frac{3}{4}} - \left(\frac{\omega_7}{3^{\frac{3q-1}{2q}}\,\overline{\pi}_1^{\,\frac{5q-1}{2q}}}\right)V^{\frac{5q-1}{4q}} + M_3\left(\frac{\omega_8}{\underline{\pi}_1}\right)V^{\frac{1}{2}} \\
&\quad + M_3\left(\frac{\omega_8}{\underline{\pi}_1^{\,\frac{4q-1}{2q}}}\right)V^{\frac{4q-1}{4q}} + M_8\left(\frac{\omega_9}{\underline{\pi}_1^{\,\frac{3}{2}}}\right)V^{\frac{3}{4}}
\end{aligned} \tag{2 - 33}$$

where

$$\omega_6 = \min\left\{k_1^{\,2}(k_3 - M_1),\ \frac{k_1 k_{1n}(k_{3n} - M_2)}{2}\left(2+\frac{1}{q}\right),\ \frac{(k_2 - M)}{2}\right\},$$

$$\omega_7 = \min\left\{\frac{k_{1n}^{\,2}}{4q}(k_{3n} - M_2),\ \min\left\{k_1^{\,2}(k_3 - M_1),\ \frac{k_{1n}^{\,2}}{4q}(k_{3n} - M_2)\right\},\ \frac{(k_{2n} - M_7)}{2}\right\},$$

$$\omega_8 = \max\left\{\begin{array}{l}\left(2k_1^{\,2} + \frac{k_1 k_{1n}}{2}\left(\frac{1}{q}+2\right)\right),\ k_1, \\ \left(\frac{k_1 k_{1n}}{2}\left(\frac{1}{q}+2\right) + k_{1n}\left(\frac{2q-1}{4q-1}\right) + \frac{k_{1n}^{\,2}}{q}\right),\ k_{1n}\left(\frac{2q}{4q-1}\right)\end{array}\right\},$$

$$\omega_9 = \max\left\{\left(\frac{4k_1}{3}\right),\ \frac{2k_{1n}}{3q},\ \left(1+\frac{2k_1}{3}+\frac{k_{1n}}{3q}\right)\right\}$$

Remark 2.32. Note that the nonlinear inequalities (2 - 32) is feasible with respect to k_1, k_{1n}, k_2, k_{2n}, and k_{3n} for any $M \geq 0$, $M_2 \geq 0$, $M_5 \geq 0$ and $M_7 \geq 0$. In particular, consider the case where $M_5 = 0$, inequalities (2 - 32) can be easily satisfied for any $k_2 > M$, $k_{2n} > M_7$, $k_{3n} > M_2$ and any $k_1 > 0$, $k_{1n} > 0$.

Remark 2.33. Hence, when $M_3 = M_8 = 0$, together with $k_3 > M_1$, it is not difficult to show that the system achieved finite time convergence for $q \geq 1$. In particular for the case of $q > 1$, following similar arguments in section 2.6.1, finite convergence time independent of initial conditions can be found to be,

$$
T_{total} = \left(\frac{3^{\frac{3q-1}{2q}} \overline{\pi}_1^{\frac{5q-1}{2q}}}{\omega_7} \right) \left(\frac{4q}{q-1} \right) \left(\frac{\omega_7}{3^{\frac{3q-1}{2q}} \overline{\pi}_1^{\frac{5q-1}{2q}}} \right)^{\frac{0.5(q-1)}{q-0.5}} \left(\frac{3^{0.5} \overline{\pi}_1^{1.5}}{\omega_6} \right)^{\frac{0.5(q-1)}{q-0.5}}
$$

$$
+ \left(\frac{3^{0.5} \overline{\pi}_1^{1.5}}{\omega_6} \right) (4) \left(\frac{3^{\frac{2q-1}{2q}} \overline{\pi}_1^{\frac{5q-1}{2q}}}{\omega_7} \right)^{\frac{0.5q}{q-0.5}} \left(\frac{\omega_6}{3^{0.5} \overline{\pi}_1^{1.5}} \right)^{\frac{0.5q}{q-0.5}}
$$

and when $M_3 \neq 0$ and/or $M_8 \neq 0$, from (2 - 33),

$$
\dot{\tilde{V}} \leq -\frac{1}{2} \left(\frac{\omega_6}{3^{0.5} \overline{\pi}_1^{1.5}} \right) V^{\frac{3}{4}} - \frac{1}{3} \left(\frac{\omega_7}{3^{\frac{3q-1}{2q}} \overline{\pi}_1^{\frac{5q-1}{2q}}} \right) V^{\frac{5q-1}{4q}} - V^{\frac{1}{2}} \left(\frac{1}{2} \left(\frac{\omega_6}{3^{0.5} \overline{\pi}_1^{1.5}} \right) V^{\frac{1}{4}} - M_3 \left(\frac{\omega_8}{\underline{\pi}_1} \right) \right)
$$

$$
-V^{\frac{4q-1}{4q}} \left(\frac{1}{3} \left(\frac{\omega_7}{3^{\frac{3q-1}{2q}} \overline{\pi}_1^{\frac{5q-1}{2q}}} \right) V^{\frac{1}{4}} - M_3 \left(\frac{\omega_8}{\underline{\pi}_1^{\frac{4q-1}{2q}}} \right) \right) - V^{\frac{3}{4}} \left(\frac{1}{3} \left(\frac{\omega_7}{3^{\frac{3q-1}{2q}} \overline{\pi}_1^{\frac{5q-1}{2q}}} \right) V^{\frac{2q-1}{4q}} - M_8 \left(\frac{\omega_9}{\underline{\pi}_1^{\frac{3}{2}}} \right) \right)
$$

$$
\leq -\frac{1}{2} \left(\frac{\omega_6}{3^{0.5} \overline{\pi}_1^{1.5}} \right) V^{\frac{3}{4}} - \frac{1}{3} \left(\frac{\omega_7}{3^{\frac{3q-1}{2q}} \overline{\pi}_1^{\frac{5q-1}{2q}}} \right) V^{\frac{5q-1}{4q}},
$$

$$
\text{for } \; V \geq \max \left\{ \begin{array}{l} \left(2M_3 \left(\frac{3^{0.5} \overline{\pi}_1^{1.5}}{\omega_6} \right) \left(\frac{\omega_8}{\underline{\pi}_1} \right) \right)^4, \; \left(3M_3 \left(\frac{3^{\frac{3q-1}{2q}} \overline{\pi}_1^{\frac{5q-1}{2q}}}{\omega_7} \right) \left(\frac{\omega_8}{\underline{\pi}_1^{\frac{4q-1}{2q}}} \right) \right)^4, \\ \left(3M_8 \left(\frac{3^{\frac{3q-1}{2q}} \overline{\pi}_1^{\frac{5q-1}{2q}}}{\omega_7} \right) \left(\frac{\omega_9}{\underline{\pi}_1^{\frac{3}{2}}} \right) \right)^{\frac{4q}{2q-1}} \end{array} \right\},
$$

uniform ultimate boundedness is implied from lemma 2.6.

Remark 2.34. Note that even in the case of $M_3 = M_8 = 0$, the system is able to be exactly robust with respect to persistent perturbations with an upper bound of M. This interesting feature is possible due to the discontinuous nature of the algorithm when $p = 0.5$.

2.7 Numerical Simulations

In this section, numerical simulations pertaining to the algorithms discussed in the previous sections are presented. The simulation setups for each algorithm are described. Discussion and analysis of the results are presented accordingly.

2.7.1 Simulation Setup

1) Twisting based algorithm: Recall from (2 - 6), the dynamics of the algorithm are,

$$\begin{aligned} \dot{z}_1 &= z_2, \\ \dot{z}_2 &= -k_1|z_1|^b \operatorname{sign}(z_1) - k_2|z_2|^{\frac{2b}{1+b}} \operatorname{sign}(z_2) + d, \end{aligned} \qquad \text{with } |d| \le M_1|z_1|^b + M_2|z_2|^{\frac{2b}{1+b}} + M_3 .$$

The parameter values of the dynamics are $k_1 = 1,\ k_2 = 2,$ and $d = |z_2|^{\frac{2b}{1+b}} \operatorname{sign}(z_1).\sin(t).$ The simulations are performed for three different values of $b = \{0.6,\ 0.7,\ 1.2\}$. The initial conditions are $z_1(0) = -2$ and $z_2(0) = -2.$ Note that the disturbance satisfy the upper bound of the system description of section 2.2 with $M_1 = M_3 = 0,$ and $M_2 = 1.$ The type of disturbance chosen here is typical in the literature (see [18], [37], [38], [46]). The specific nature of the disturbance is selected to show that past results on this algorithm are conservative. Similarly, the values of b are to demonstrate that the present results are less conservative.

Then, the another set of parameter values, $b = 0, k_1 = 4,\ k_2 = 2,$ and $d = 1$, is performed to show the exact robustness property of the algorithm under persistent disturbance.

2) Super-twisting based algorithm: Recall from (2 - 17), the dynamics of the algorithm are,

$$\begin{aligned}\dot{z}_1 &= -k_3|z_1|^p \operatorname{sign}(z_1) + z_2 + d_1, \\ \dot{z}_2 &= -k_1|z_1|^{2p-1}\operatorname{sign}(z_1) + d_2\end{aligned} \quad \text{with } |d_1| \le M_1|z_1|^p + M_2, \text{ and } |d_2| \le M_3|z_1|^{2p-1} + M_4.$$

The parameter values of the dynamics are $k_3 = 2,\ k_1 = 0.3,$ and $d_1 = |z_1|^p \operatorname{sign}(z_2).\sin(t)$. The simulations are performed for three different values of $p = \{0.5,\ 0.7,\ 1.2\}$. The initial conditions are $z_1(0) = -2$ and $z_2(0) = -2$. Note that the disturbances satisfy the upper bound of the system description of section 2.4 with $M_2 = M_3 = M_4 = 0,$ and $M_1 = 1.$ The type of disturbance chosen here is typical in the literature (see [26], [27], [68]). The disturbance and values of p are selected as such to show the conservativeness of previous results.

Another simulation is performed with the parameter values of $k_3 = 1,\ k_1 = 1,$ and $d = 0$ with $p = 0.5$ for the unperturbed case.

3) Generic super-twisting based algorithm: Recall from (2 - 23), the dynamics of the algorithm are,

$$\begin{aligned}\dot{z}_1 &= -k_3|z_1|^p \operatorname{sign}(z_1) - k_{3n}|z_1|^q \operatorname{sign}(z_1) + z_2 + d_1, \\ \dot{z}_2 &= -k_1|z_1|^{2p-1} \operatorname{sign}(z_1) - k_{1n}|z_1|^{2q-1} \operatorname{sign}(z_1) + d_2\end{aligned} \quad \text{with} \quad \begin{aligned}|d_1| &\le M_1|z_1|^p + M_2|z_1|^q + M_3 \\ |d_2| &\le M_4|z_1|^{2p-1} + M_5|z_1|^{2q-1} + M_6\end{aligned}.$$

The parameter values of the dynamics are $k_1 = k_{1n} = 4,\ k_3 = k_{3n} = 2,\ p = \frac{1}{1.4}$, and $q = \frac{1}{0.4}$ under perturbations $d_1 = \operatorname{sign}(z_2).\sin(t)\left(|z_1|^p + |z_1|^q\right)$ and $d_2 = \operatorname{sign}(z_2).\sin(t)\left(|z_1|^{2p-1} + |z_1|^{2q-1}\right)$. The simulations are performed with three different values of initial conditions $(z_1(0), z_2(0)) = \{(-0.1, -0.1), (-1, -1), (-10, -10)\}$. Note that the disturbances satisfy the upper bound of the system description of section 2.5 with $M_1 = M_2 = 1,\ M_3 = 0, M_4 = M_5 = 1$, and $M_6 = 0$. The type of disturbances chosen are typical in the literature (see [46], [17]). The three different initial conditions, each being an order of magnitude greater than the previous one, are selected as such to show the uniform finite-time capabilities of the algorithm.

4) Generic twisting based algorithm: Recall from (2 - 11), the dynamics of the algorithm are,

$$\begin{aligned}\dot{z}_1 &= z_2, \\ \dot{z}_2 &= -k_1|z_1|^{2p-1} \operatorname{sign}(z_1) - k_{1n}|z_1|^{2q-1} \operatorname{sign}(z_1) - k_2|z_2|^{2-\frac{1}{p}} \operatorname{sign}(z_2) - k_{2n}|z_2|^{2-\frac{1}{q}} \operatorname{sign}(z_2) + d\end{aligned}$$

with $|d| \le M_1|z_1|^{2p-1} + M_2|z_1|^{2q-1} + M_3|z_2|^{2-\frac{1}{p}} + M_4|z_2|^{2-\frac{1}{q}} + M_5$.

The parameter values of the dynamics are $k_1 = k_{1n} = 4, k_2 = k_{2n} = 2, p = \frac{1}{1.4}$, and $q = \frac{1}{0.4}$

under perturbations $d = \mathrm{sign}(z_2).\sin(t).\left(|z_1|^{2p-1} + |z_1|^{2q-1}\right) + \mathrm{sign}(z_1).\sin(t).\left(|z_2|^{2-\frac{1}{p}} + |z_2|^{2-\frac{1}{q}}\right)$.

The simulation is performed with the initial conditions $(z_1(0), z_2(0))$= (–3, –3). Note that the disturbances satisfy the upper bound of the system description of section 2.3 with $M_1 = M_2 = 1,\ M_3 = M_4 = 1,$ and $M_5 = 0$. The type of disturbances chosen is based on the sum of disturbances considered in twisting-based simulations that are of different exponent.

5) Generic second-order based algorithm: Recall from (2 - 29), the dynamics of the algorithm are,

$$\dot{z}_1 = -k_3|z_1|^p \mathrm{sign}(z_1) - k_{3n}|z_1|^q \mathrm{sign}(z_1) + z_2 + d_1,$$
$$\dot{z}_2 = -k_1|z_1|^{2p-1}\mathrm{sign}(z_1) - k_{1n}|z_1|^{2q-1}\mathrm{sign}(z_1) - k_2|z_2|^{\frac{2p-1}{p}}\mathrm{sign}(z_2) - k_{2n}|z_2|^{\frac{2q-1}{q}}\mathrm{sign}(z_2) + d_2$$

with

$|d_1| \le M_1|z_1|^p + M_2|z_1|^q + M_3$ and

$|d_2| \le M_4|z_1|^{2p-1} + M_5|z_1|^{2q-1} + M_6|z_2|^{2-\frac{1}{p}} + M_7|z_2|^{2-\frac{1}{q}} + M_8$.

The parameter values of the dynamics are

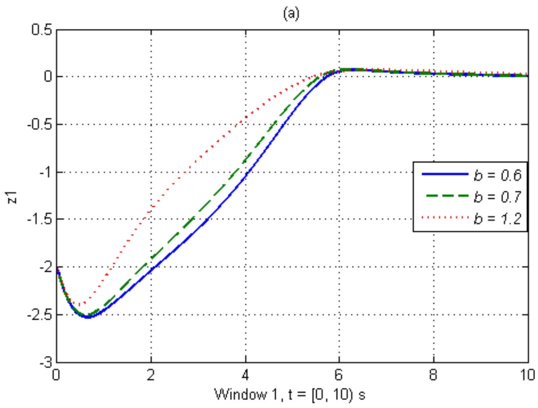
$k_1 = k_{1n} = k_2 = k_{2n} = 4, k_3 = k_{3n} = 2, p = \dfrac{1}{1.4}$, and $q = \dfrac{1}{0.4}$

under perturbations $d_1 = \mathrm{sign}(z_2).\sin(t).\left(|z_1|^p + |z_1|^q\right)$, and $d_2 = \mathrm{sign}(z_2).\sin(t).\left(|z_1|^{2p-1} + |z_1|^{2q-1}\right)$ $+ \mathrm{sign}(z_1).\sin(t).\left(|z_2|^{2-\frac{1}{p}} + |z_2|^{2-\frac{1}{q}}\right)$. The simulation is performed with the initial conditions of

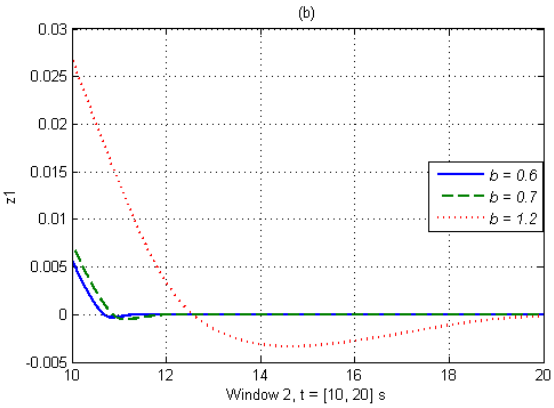
$(z_1(0), z_2(0))$ = (−3,−3). Note that the disturbances satisfy the upper bound of the system description of section 2.6 with $M_1 = M_2 = 1$, $M_3 = 0$, $M_4 = M_5 = M_6 = M_7 = 1$, and $M_8 = 0$. The type of disturbances chosen is based on the sum of disturbances considered in generic super-twisting and generic twisting-based simulations that are of different exponent.

2.7.2 Results and Discussions

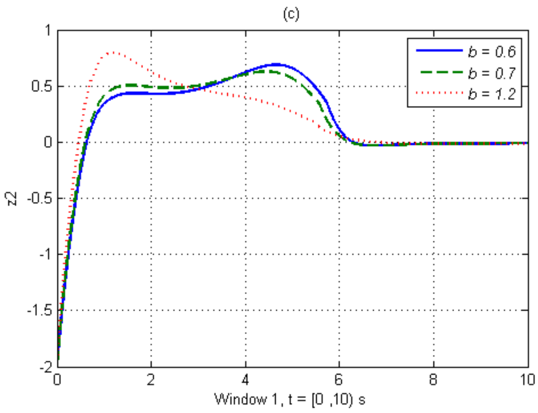
For better visualization of the plots, some figures are shown in two windows; each with different time intervals.



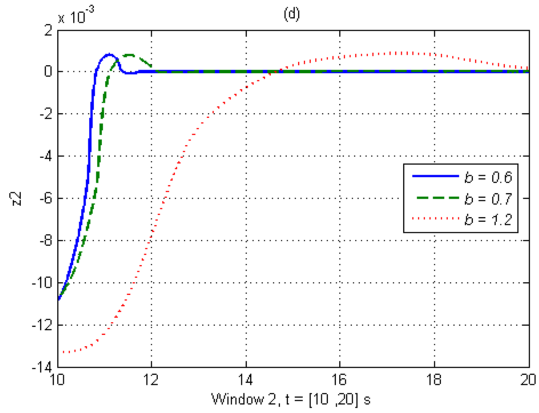
(a) State z_1 for time $t = [0, 10)$ s.



(b) State z_1 for time $t = [10, 20]$ s.

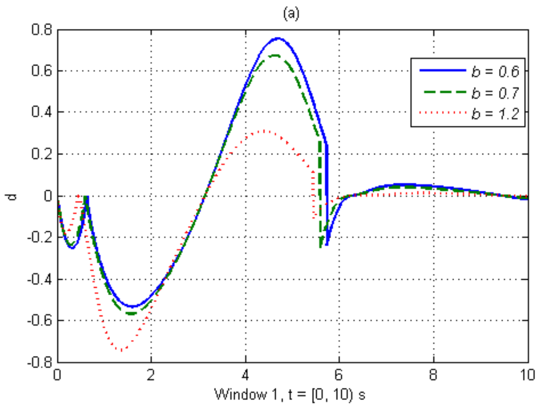


(c) State z_2 for time $t = [0, 10)$ s.

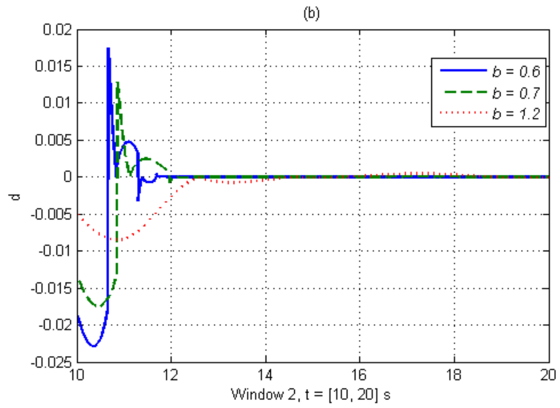


(d) State z_2 for time $t = [10, 20]$ s.

Figure 2.1 Twisting based algorithm. States of (2 - 6) with $k_1 = 1,\ k_2 = 2,$ and three different values of $b = \{0.6,\ 0.7,\ 1.2\}$.

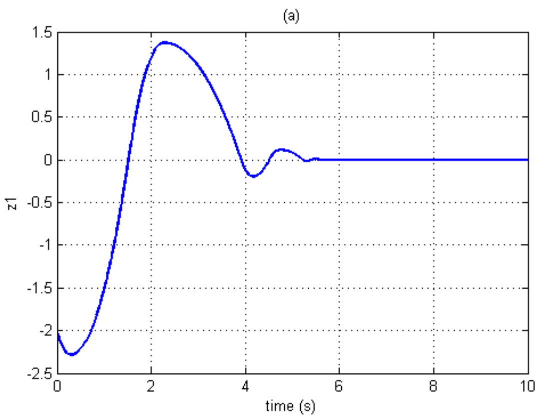


(a) Disturbance, d for time t = [0, 10) s.

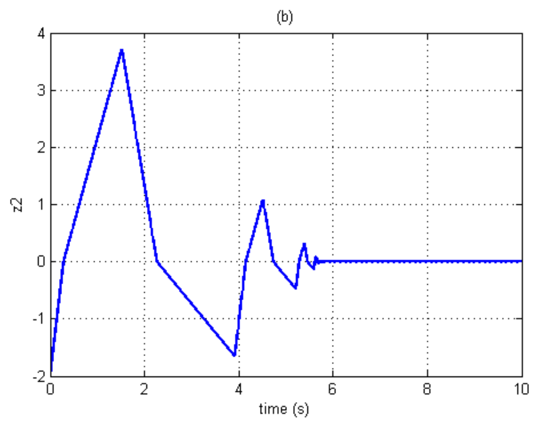


(b) Disturbance d for time t = [10, 20] s.

Figure 2.2 Twisting based algorithm. Disturbance $d = |z_2|^{\frac{2b}{1+b}} \operatorname{sign}(z_1).\sin(t)$ for $b = \{0.6,\ 0.7,\ 1.2\}$.

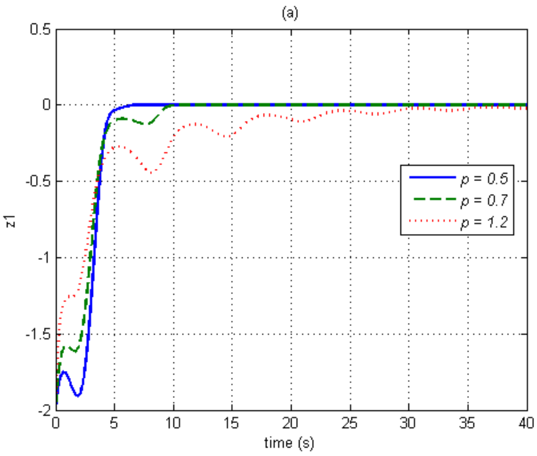


(a) State z_1 for time t = [0, 10] s.

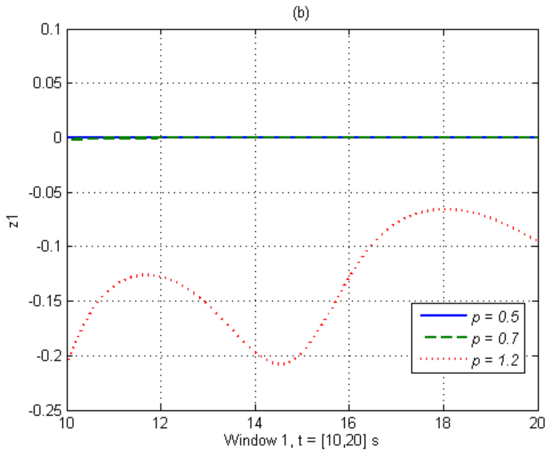


(b) State z_2 for time t = [0, 10] s.

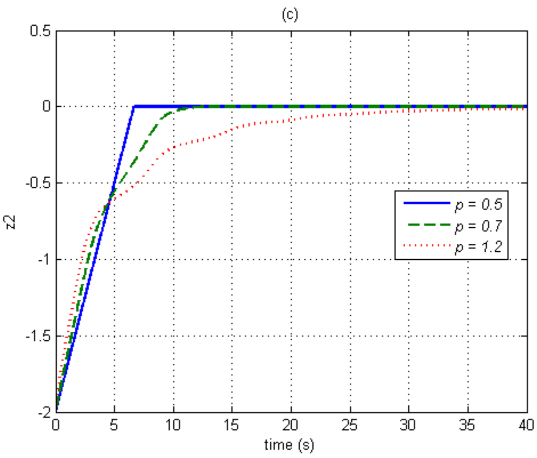
Figure 2.3 Twisting based algorithm. States of (2 - 6) for b = 0 with $k_1 = 4$, and $k_2 = 2$ under persistent disturbance d = 1.



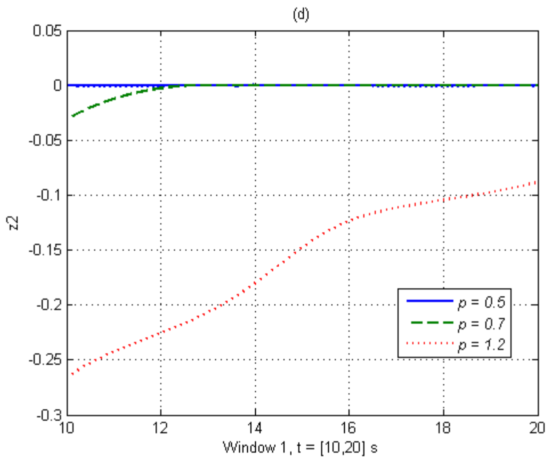
(a) State z_1 for time $t = [0, 40]$ s.



(b) State z_1 for time $t = [10, 20]$ s.

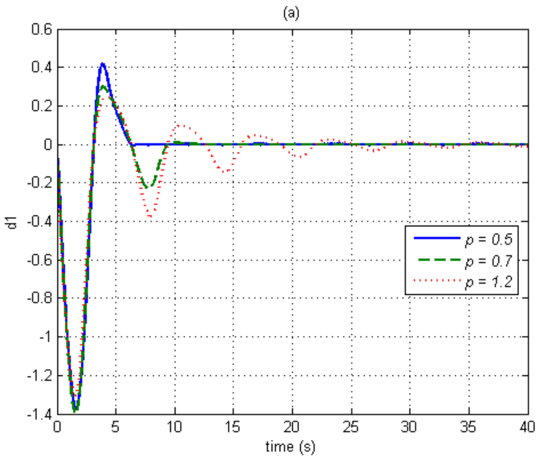


(c) State z_2 for time $t = [0, 40]$ s.

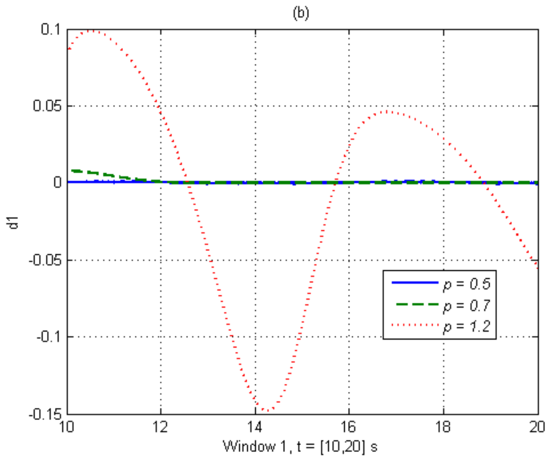


(d) State z_2 for time $t = [10, 20]$ s.

Figure 2.4 Super-twisting based algorithm. States of (2 - 17) with $k_3 = 2,\ k_1 = 0.3,$ and three different values of $p = \{0.5,\ 0.7,\ 1.2\}$.

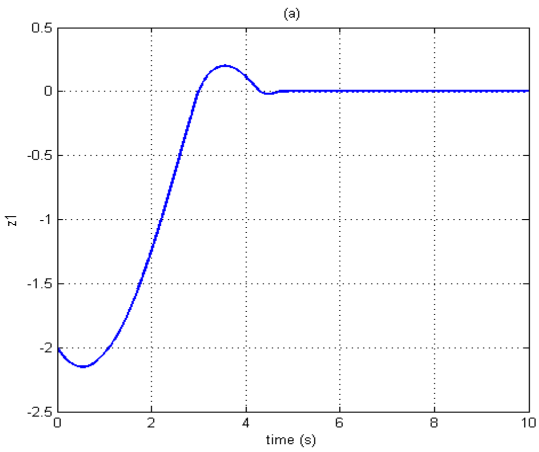


(a) Disturbance d_1 for time t = [0, 40] s.

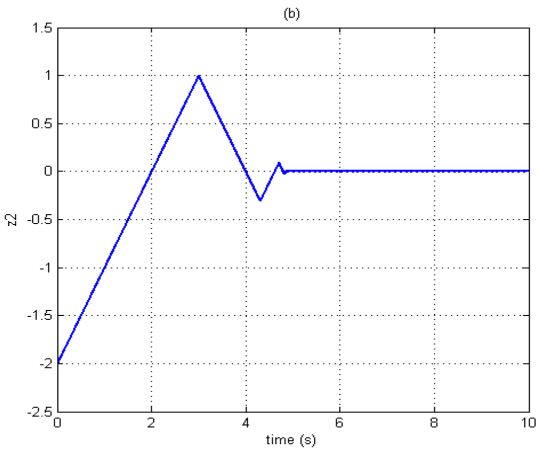


(b) Disturbance d_1 for time t = [10, 20] s.

Figure 2.5 Super-twisting based algorithm. Disturbance $d_1 = |z_1|^p \operatorname{sign}(z_2).\sin(t)$ for $p = \{0.5,\ 0.7,\ 1.2\}$.

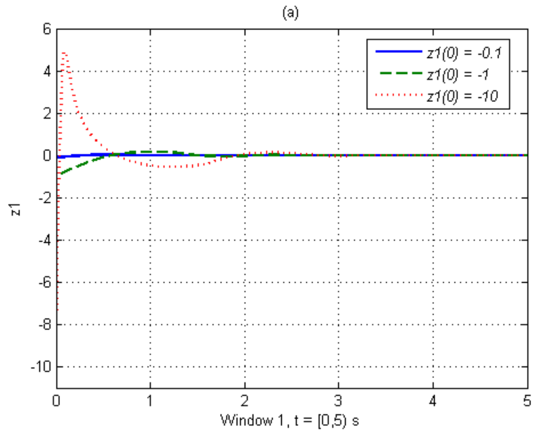


(a) State z_1 for time t = [0, 10] s.

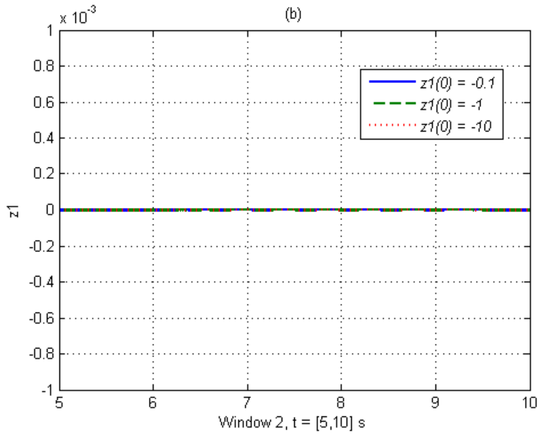


(b) State z_2 for time t = [0, 10] s.

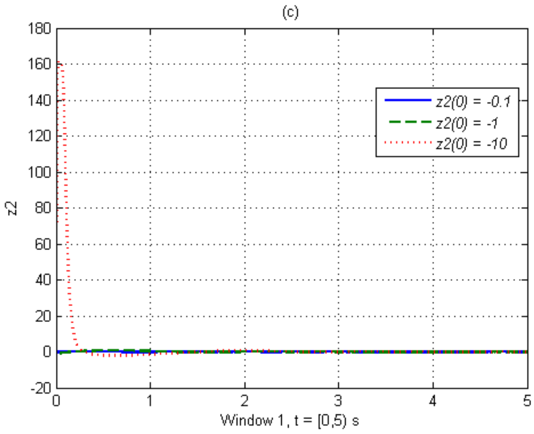
Figure 2.6 Super-twisting based algorithm. States of (2 - 17) for p = 0.5 with $k_3 = 1$ and $k_1 = 1$ without perturbation.



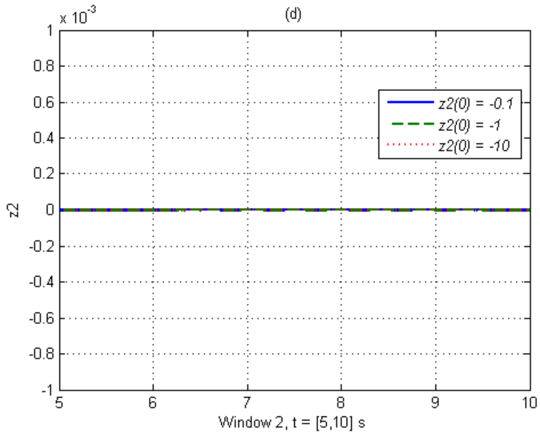
(a) State z_1 for time $t = [0, 5)$ s.



(b) State z_1 for time $t = [5, 10]$ s.

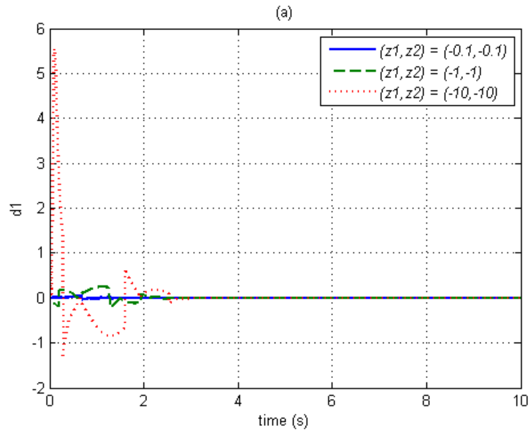


(c) State z_2 for time $t = [0, 5)$ s.

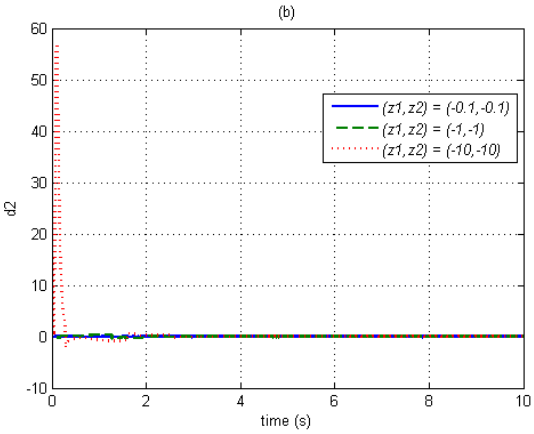


(d) State z_2 for time $t = [5, 10]$ s.

Figure 2.7 Generic super-twisting based algorithm. States of (2 - 23) with $k_1 = k_{1n} = 4$, and $k_3 = k_{3n} = 2$ under three values of initial conditions, $(z_1(0), z_2(0)) = \{(-0.1, -0.1), (-1, -1), (-10, -10)\}$.

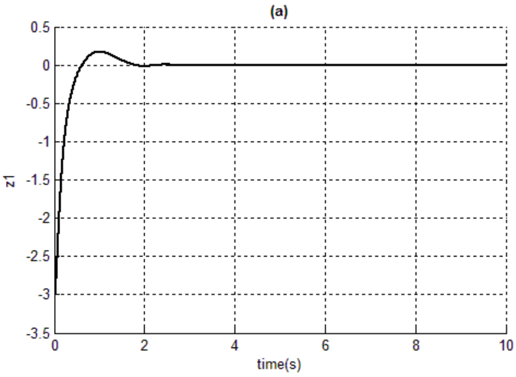


(a) Disturbance d_1 for time $t = [0, 10]$ s.

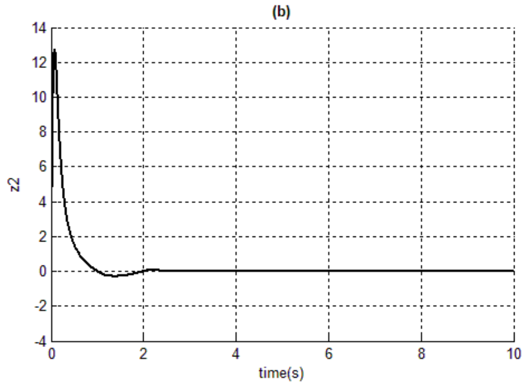


(b) Disturbance d_2 for time $t = [0, 10]$ s.

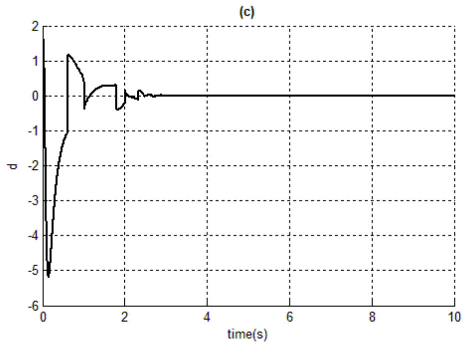
Figure 2.8 Generic super-twisting based algorithm. Disturbance $d_1 = \text{sign}(z_2).\sin(t)\left(|z_1|^p + |z_1|^q\right)$ and $d_2 = \text{sign}(z_2).\sin(t)\left(|z_1|^{2p-1} + |z_1|^{2q-1}\right)$ for three different values of initial conditions, $(z_1(0), z_2(0)) = \{(-0.1, -0.1), (-1, -1), (-10, -10)\}$.



(a) State z_1 for time $t = [0, 10]$ s.



(b) State z_2 for time $t = [0, 10]$ s.

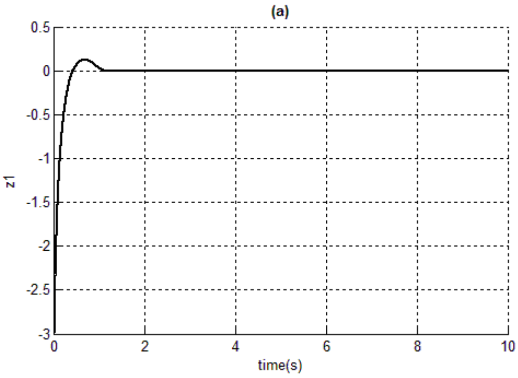


(c) Disturbance d for time $t = [0, 10]$ s.

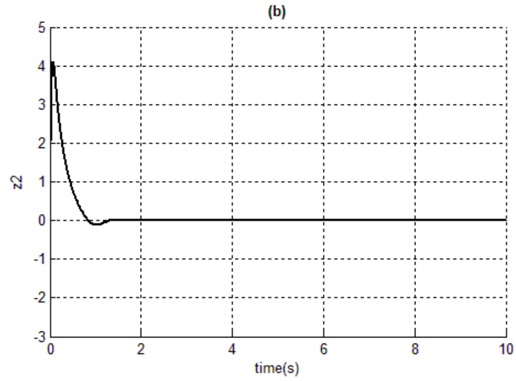
Figure 2.9 Generic twisting based algorithm. States of (2 - 11) with $p = \frac{1}{1.4}, q = \frac{1}{0.4}$,

$k_1 = k_{1n} = 4$, and $k_2 = k_{2n} = 2$ under the disturbance,

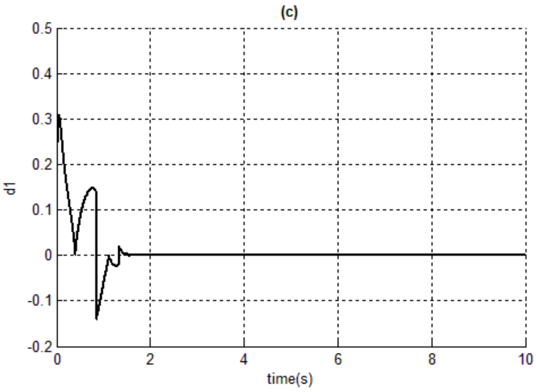
$$d = \text{sign}(z_2).\sin(t).\left(|z_1|^{2p-1} + |z_1|^{2q-1}\right) + \text{sign}(z_1).\sin(t).\left(|z_2|^{2-\frac{1}{p}} + |z_2|^{2-\frac{1}{q}}\right).$$



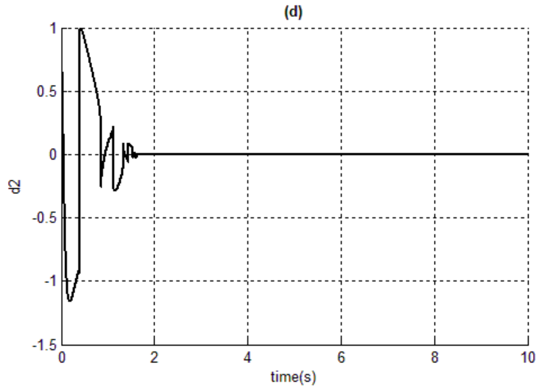
(a) State z_1 for time $t = [0, 10]$ s.



(b) State z_2 for time $t = [0, 10]$ s.



(c) Disturbance d_1 for time $t = [0, 10]$ s.



(d) Disturbance d_2 for time $t = [0, 10]$ s.

Figure 2.10 Generic second order based algorithm. States of (2 - 29) with $p = \frac{1}{1.4}, q = \frac{1}{0.4}$, $k_1 = k_{1n} = k_2 = k_{2n} = 4$, and $k_3 = k_{3n} = 2$ under the disturbances,

$$d_1 = \mathrm{sign}(z_2).\sin(t)\left(|z_1|^p + |z_1|^q\right) \text{ and}$$

$$d_2 = \mathrm{sign}(z_2).\sin(t)\left(|z_1|^{2p-1} + |z_1|^{2q-1}\right) + \mathrm{sign}(z_1).\sin(t).\left(|z_2|^{2-\frac{1}{p}} + |z_2|^{2-\frac{1}{q}}\right).$$

1) Twisting based algorithm:

Figure 2.1 shows the convergence of the states, z_1 and z_2 in finite time for b = 0.6 and 0.7. For b = 1.2 the states are converging to the origin asymptotically. Figure 2.2 shows the non-Lipschitz disturbances acting on the system. For the system (2 - 6), with parameters as stated in the simulation setup section, i.e. if $M_1 = M_3 = 0$, as per weak Lyapunov function [18], the requirements for finite time stability are $k_2 > M_2$ with any $k_1 > 0$. The gains of the simulation are selected to satisfy these conditions.

The following are the conditions on the gains, of the same dynamics under the same perturbations, obtained from the literature for comparison purposes.

1. In the results of [37] (see Theorem 2 of [37]), the conditions on the gains are

$$k_1(k_2 - M_2) > \frac{6}{(1+\alpha)(2-\alpha)}, \quad k_2 > \max\left\{M_2, M_2 + \frac{3}{2(1+\alpha)}\right\},$$
$$k_1 > 0, \quad \frac{k_1}{2} > \frac{(k_2 - M_2)}{1+\alpha} > 0$$

with $\alpha = \dfrac{2b}{1+b}$ for $b \in [0,1)$.

2. In Theorem 2 and 3 of [38], the conditions on gains are

$$k_1 > 0, \quad k_2 > \max\left\{M_2, M_2 + \frac{3}{2(1+\alpha)}\right\},$$
$$\frac{k_1}{2} > \frac{(k_2 - M_2)}{1+\alpha} > 0, \quad k_1(k_2 - M_2) > \frac{6}{(1+\alpha)(2-\alpha)}, \quad k_1^{\,3} > 1$$

and

$$k_2 > M_2 + \frac{3(1-\alpha)}{2}, \ k_1 > \frac{1}{1+\alpha}\max\{M_2, \alpha\}, \ k_1(k_2 - M_2) > \frac{3}{2}\left(\frac{1+\alpha}{(2-\alpha)}\right)$$

with $\alpha = \dfrac{2b}{1+b}$ for $b \in [0,1)$.

3. In [44] (see Theorem 1 of [44]) and [46] (see theorem 3.2 of [46]), the conditions are $k_1 - M_2 > k_2 > M_2 > 0$ with are need only applicable for $\frac{2b}{1+b} \in \left(\frac{2}{3}, 1\right)$.

4. In [45] only for $\frac{2b}{1+b} \in [0,1)$ it requires

$$k_1 > (k_2 + M_2)\frac{(2-\alpha)}{2} \quad \text{and} \quad k_2 > M_2$$

Clearly, the simulation choice of $k_1 = 1$, $k_2 = 2$ with $M_2 = 1$ does not satisfy all of the above conditions. Also, the simulation parameters of $b = \{0.6, 0.7, 1.2\}$ do not belong to the range considered there. However, as per remark 2.4, the strict Lyapunov function presented in section 2.2 allows the conditions imposed on the simulations, showing different convergence properties dependent on the parameter *b*. Thus, the results show that conservativeness of previous results mentioned above. In particular, the Lyapunov results of [18], [37] and [38] are not able to extend to exponent greater than 1. It is also worth mentioning that the types of disturbances are also extended, specifically, in [18], only global asymptotic stability is achieved for $0 < b < 1$ under disturbances upper bounded by one state variable only, which in part due to the weak Lyapunov function employed. In contrast, the strict Lyapunov function of section 2.2 allows disturbances upper bounded by the sum of both states.

Figure 2.3 shows the finite time convergence of system (2 - 6) under persistent disturbance $d = 1$ for $b = 0$ with $k_1 = 4$, and $k_2 = 2$. From remark 2.7, finite time convergence is achieved when for any $k_1 > k_2 + M$, and $k_2 > M$, which coincides with the well-known result [20]. On the other hand, the results of [37] and [38], while applicable to $b = 0.5$, it is unclear on how to obtain this conditions.

As a result, the strict Lyapunov function presented here is able to fully characterize the stability of the twisting based algorithm for any $b \geq 0$, which essentially fills the gaps in the literature, in the sense that those prior results are only applicable to certain range of b while at the same time requiring conservative conditions on gains with respect to disturbances.

2) Super-twisting based algorithm:

Figure 2.4 shows the convergence of the states, z_1 and z_2 in finite time for $p = 0.5$ and 0.7. For $p = 1.2$ the states are converging to the origin asymptotically. Figure 2.5 shows the non-Lipschitz disturbances acting on the system. For the system (2 - 17), with parameters as stated in the simulation setup section, i.e. if $M_2 = M_3 = M_4 = 0$, as per energy based Lyapunov function (see remark 2.15), the requirements for stability (finite time if $p = [0,1)$, exponential if $p = 1$, and asymptotical if $p > 1$) are $k_3 > M_1$ with any $k_1 > 0$. The gains of the simulation are selected to satisfy these conditions.

In [26], the Lyapunov function presented is for the special case of $p = 0.5$ of the super-twisting based algorithm studied here. Under the same disturbances as per the simulation, the conditions given for finite time stability are

$$k_1 > k_3 \frac{5M_1 k_3 + 4\left(\frac{M_1}{k_3}\right)^2}{2(k_3 - 2M_1)} \text{ and } k_3 > 2M_1$$

which is clearly not satisfied by the simulation parameter values for the gains, which shows the conservativeness of prior results.

Meanwhile, the Lyapunov function for the super-twisting algorithm presented by [26] and [27] (for p = 0.5) will cause singularity issue when used in further application such as with certainty-equivalence method in [67]. This issue is due to the non-Lipschitz nature of their proposed Lyapunov functions, in which case, is not a issue with our proposed Lyapunov function which is locally Lipschitz and strict (see section 2.4). Pertaining to the issue of solving singularity, a continuously differentiable Lyapunov function for the super-twisting algorithm (p = 0.5) is presented in [68], albeit, it comes with conservative gain conditions, namely, for the unperturbed system, $k_3^2 > 2k_1 > 0$, while in the present results, as per remark 2.18 and (2 -21), for the unperturbed system, finite time is achieved with any $k_1 > 0$ and $k_3 > 0$, which is shown in Figure 2.6 with $k_3 = 1$ and $k_1 = 1$. Also worth mentioning is that the mentioned prior results is only applicable to the case of p = 0.5, while the results shown here is for any $p \geq 0.5$ as per results shown in Figure 2.4.

3) Generic super-twisting based algorithm:

Figure 2.7 shows the uniform finite time convergence of the states, z_1 and z_2 under three different values of initial conditions that differ by an order of magnitude. In fact, for all three different initial conditions cases, all of them reach the origin in time, t less than 5 seconds even though the furthest initial conditions are of two order of magnitudes difference than the closest one. This strong convergence feature is due to the strong control terms that have exponent greater than 1, namely, $|z_1|^q \operatorname{sign}(z_1)$ and $|z_1|^{2q-1}\operatorname{sign}(z_1)$.

Figure 2.8 shows the non-Lipschitz disturbances acting on the system. Similar forms of disturbances are also studied in [46] and [17]. Particularly, the Lyapunov function proposed in [17] is able to show uniform finite time convergence as well under disturbances that are upper-bounded in a form similar to that in the simulation. However, these prior results require two distinct Lyapunov functions to ascertain different convergence properties of the system, i.e. uniform and finite time convergence, in which case for the finite time convergence, their Lyapunov function suffer the same singularity issue mentioned in the super-twisting based results above. In addition, due to the two different structure of Lyapunov functions employed there, the control requires an additional term in the $\dot{z}_2$ dynamics, i.e. $|z_1|^{p+q-1}\operatorname{sign}(z_1)$, to ensure strictness of Lyapunov functions. The proposed strict and locally-Lipschitz Lyapunov function (see section 2.5) is able to overcome these shortcomings.

4) Generic twisting based algorithm:

Figure 2.9 shows the uniform finite time convergence of the states, z_1 and z_2 under the influence of non-Lipschitz disturbances. The disturbances, which comprise both states of the system, considered in this simulation satisfy the upper bound of the system description in section 2.3. In fact, as per remark 2.9, since p = 1/1.4 < 1 and q = 1/0.4 > 1, uniform finite time convergence is guaranteed. The generic twisting algorithm is based on the results of twisting based algorithm by combining control term of different homogeneity with the intent of combining different stability properties within an algorithm

5) Generic second order based algorithm:

Figure 2.10 shows the uniform finite time convergence of the states, z_1 and z_2 under the influence of non-Lipschitz disturbances. The disturbances, which comprise both states of the system, considered in this simulation satisfy the upper bound of the system description in section 2.6. In fact, as per remark 2.30, since $p = 1/1.4 < 1$ and $q = 1/0.4 > 1$, uniform finite time convergence is guaranteed. The generic second order algorithm is based on the results of combining the generic twisting and generic super-twisting based algorithm. Through such combination, the inherent energy based function of the system (which is found to be a weak Lyapunov function when applied on either twisting or super-twisting based algorithm alone), is a strict and locally Lipschitz Lyapunov function as per remark 2.28.

Remark 2.35. The disturbances considered here (see Figure 2.2, 2.5, 2.8, 2.9(c), and 2.10(c)) comprise bounded discontinuity, which corroborates with the results of the stability analysis which employ the generalized Lyapunov theorem (see section 2.1).

2.8 Summary

Two twisting-based family of algorithms, two super-twisting based family of algorithms and a generic second order algorithm that combines the super-twisting and twisting algorithms are developed. In each case, strict Lyapunov functions have been introduced, that can fully characterize different stability properties of a parameterized family of algorithms. For the twisting based algorithm, it generalize the well-known twisting algorithm, continuous finite-time second order system, linear PD control, and uniform convergence algorithm. While for the super-twisting based algorithm, it generalize the super-twisting algorithm, homogeneous and linear PI control, and uniform convergence algorithm. Then we show that the linear combination of two homogeneous algorithms with different degree of homogeneity can indeed produce a system that has the characteristics of its individual component. Due to the availability of strict Lyapunov functions, settling time for finite time convergence can also be obtained. Finally, the robustness to different classes of perturbations can be easily considered as well.

Chapter 3: ROBOT MANIPULATOR CONTROL: FULL STATE FEEDBACK APPROACH

In this chapter, a family of controllers is developed for the trajectory tracking of robot manipulator. Based on the twisting based family of algorithms presented in the previous chapter, the proposed controller is able to generalised PID control to a homogeneous PID-like control. Semiglobal practical tracking stability is achieved despite uncertainty and additive disturbances in the robot dynamics. For the special case of regulation problem, the proposed controller is able to achieve finite-time or exponential convergence, depending on the chosen parameters. The stability analysis allows selection of control gains based on desired performance instead of gains tuning. Numerical simulations using two-link robot manipulator demonstrate the performance of the proposed controller.

3.1 System Description

A nonlinear mechanical system with n-degree of freedom in closed loop with a nonlinear controller that generalised the PID control is considered.

3.1.1 Manipulator Dynamics

A class of rigid, fully actuated, unconstrained mechanical systems which can be modelled by the Euler-Lagrange principle that results in a class of nonlinear systems modelled by a set of highly coupled nonlinear differential equations is considered. The dynamics of n-joint serial rigid robotic manipulators can be described by the following differential equation [117]

$$M(q)\ddot{q}+C(q,\dot{q})\dot{q}+F\dot{q}+G(q)+D(q,\dot{q},t)=\tau, \qquad (3 - 1)$$

where $q \in \mathbb{R}^n$ is the vector of generalized joint coordinates, $M(q) \in \mathbb{R}^{n\times n}$ is the inertia matrix, and $C(q, \dot{q})\ \dot{q}$, $F\dot{q}$, $G(q)$, $D(q, \dot{q}, t)$, $\tau \in \mathbb{R}^n$ are, respectively, the vectors of Coriolis and centrifugal, viscous friction, gravity, disturbances, and input generalized forces, with F being a constant, positive definite , diagonal (viscous friction coefficient) matrix and $D(q, \dot{q}, t)$ being a locally bounded disturbances. The terms of the robot manipulator dynamics satisfy some well-known properties (see for instance [117], [142]), in which several are recollected here.

Property 3.1: The inertia matrix $M(q)$ is a positive definite symmetric matrix satisfying $\underline{m} \le \|M(q)\| \le \overline{m},\ \forall q \in \mathbb{R}^n$, for some positive constants $\underline{m} \le \overline{m}$.

Property 3.2: The Coriolis matrix $C(q, \dot{q})$ satisfies:

3.2.1. $x^T\left[\frac{1}{2}\dot{M}(q,\dot{q})-C(q,\dot{q})\right]x=0, \forall x,q,\dot{q}\in\mathbb{R}^n$;

3.2.2. $\dot{M}(q,\dot{q})=C(q,\dot{q})+C^T(q,\dot{q}),\ \forall q,\dot{q}\in\mathbb{R}^n$;

3.2.3. $C(w,x+y)z=C(w,x)z+C(w,y)z,\quad \forall w,x,y,z\in\mathbb{R}^n$;

3.2.4. $C(x,y)z=C(x,z)y,\quad \forall x,y,z\in\mathbb{R}^n$;

3.2.5. $\|C(x,y)z\|\le C_m\|y\|\|z\|,\ \forall x,y,z\in\mathbb{R}^n$, for some constant $C_m \ge 0$.

Property 3.3: The gravitational torques vector for robots having only revolute joints satisfies (see [142] page 101):

3.1. $\|G(q)\| \le G_m$, $\forall q \in \mathbb{R}^n$, for some constant $G_m > 0$.

3.2. $\|G(x) - G(y)\| \le k_g \|x - y\|$, $\forall x, y \in \mathbb{R}^n$, for some constant $k_g > 0$.

Property 3.4: The viscous friction coefficient matrix satisfies $\underline{f} \le \|F\| \le \overline{f}$, where $0 < \underline{f} := \min_i \{f_i\} \le \max_i \{f_i\} := \overline{f}$.

In this chapter, it is assume that both joint positions and velocities are available from measurement, i.e. full state feedback is viable. The control objective here is to design a robust full state feedback controller that ensures the robot configuration vector q tracks a desired trajectory vector, q_d (t) with an ultimately bounded error that can be made as small as required, from any initial conditions that belong to an arbitrarily large compact set.

The desired trajectory vector, q_d (t) is assumed to be twice continuously differentiable vector-function such that $\|q_d(t)\|$, $\|\dot{q}_d(t)\|$, and $\|\ddot{q}_d(t)\|$ are bounded by *a priori* known constants. This is a standard assumption in the trajectory tracking control of robot manipulator (see for instance [103], [131], [113]).

Note that no continuity assumption is made on $D(q, \dot{q}, t)$, so it may have discontinuity, such as Coulomb friction. In particular, the form of $D(q, \dot{q}, t)$ considered here is assumed to be upper bounded by the function

$$\|D\| \le p_0 + p_1\|e_1\| + p_2\|\dot{q}\| + p_3\|e_1\|^2 + p_4\|\dot{q}\|^2, \text{ with } \|D\| := \sup\{\|\varsigma\| : \varsigma \in K[D]\}$$

where p_0, p_1, p_2, p_3, and p_4 are some nonnegative constants, while $e_1 := q - q_d \in \mathbb{R}^n$, and $e_2 = \dot{q} - \dot{q}_d \in \mathbb{R}^n$.

3.1.2 Control Development

The following notions, which were used in [36] and [7], are introduced for simplicity of notation and will be used in the analysis and design of the controller.

$$sig(x)^a = \left[|x_1|^a \operatorname{sign}(x_1), \ldots, |x_n|^a \operatorname{sign}(x_n)\right]^T, \forall x \in \mathbb{R}^n .$$

The controller proposed is given by

$$\tau = -K \operatorname{sig}(s)^a , \tag{3 - 2}$$

where K is a positive definite diagonal matrix, i.e. $K = \operatorname{diag}\{k_i\}_{i=1}^n$, with $k_i > 0, \forall i = 1, \ldots, n$, $a \geq 0$ constant, and $s \in \mathbb{R}^n$ is the desired error dynamics defined as $s = e_2 + \sigma$, with

$$\dot{\sigma} = K_2 \operatorname{sig}(e_2)^{\frac{2b}{1+b}} + K_1 \operatorname{sig}(e_1)^b , \tag{3 - 3}$$

where K_1 and K_2 are positive definite diagonal matrices, i.e. $K_1 = \operatorname{diag}\{k_{1i}\}_{i=1}^n$, with $k_{1i} > 0$, $K_2 = \operatorname{diag}\{k_{2i}\}_{i=1}^n$, with $k_{2i} > 0 \ \ \forall i = 1, \ldots, n$, $b \geq 0$ constant.

3.2 Stability Analysis

The closed-loop system of (3 - 1), (3 - 2), and (3 - 3) can be written as

$$\begin{aligned}
&\dot{\sigma} = K_2 \operatorname{sig}(e_2)^{\frac{2b}{1+b}} + K_1 \operatorname{sig}(e_1)^b , \\
&\dot{e}_1 = e_2 , \\
&\dot{e}_2 = -M^{-1}(q) K \operatorname{sig}(s)^a - M^{-1}(q)\left(C(q,\dot{q})\dot{q} + F\dot{q} + G(q) + D\right) - \ddot{q}_d ,
\end{aligned}$$

To rewrite the closed-loop system in a form more convenient for analysis, let us define the change of variable $z_1 = e_1$, and $z_2 = -\sigma$, and we obtain the following form of closed-loop system,

$$\begin{aligned}
\dot{z}_1 &= z_2 + s, \\
\dot{z}_2 &= -K_2 \mathrm{sig}(z_2 + s)^{\frac{2b}{1+b}} - K_1 \mathrm{sig}(z_1)^b, \\
\dot{s} &= -M^{-1}(z_1 + q_d) K \mathrm{sig}(s)^a + \Delta(\cdot),
\end{aligned} \tag{3 - 4}$$

where

$$\begin{aligned}
\Delta(\cdot) = &-M^{-1}(z_1 + q_d) C(z_1 + q_d, z_2 + s + \dot{q}_d)(z_2 + s + \dot{q}_d) - M^{-1}(z_1 + q_d) F(z_2 + s + \dot{q}_d) \\
&- M^{-1}(z_1 + q_d) G(z_1 + q_d) - M^{-1}(z_1 + q_d) D - \ddot{q}_d + K_2 \mathrm{sig}(z_2 + s)^{\frac{2b}{1+b}} + K_1 \mathrm{sig}(z_1)^b
\end{aligned}$$

3.2.1 Construction of Lyapunov Function

From the closed-loop dynamics (3 - 4), the structure (z_1, z_2) is essentially the desired error dynamics injected by the controller through (3 - 3). Essentially, for any $i \in 1,\ldots,n$, when $s_i = 0$, the dynamics of the subsystem (z_{1i}, z_{2i}) is identical to that of the twisting-based family of algorithm studied in section 2.2 of Chapter 2. In other words, s_i can be viewed as a perturbations on the (z_{1i}, z_{2i}) dynamics. While in the s-dynamics, its structure is akin to that of sliding mode control.

Since the differential equations (3 - 4) have discontinuous right-hand side, i.e. when $a = 0$ and/or $b = 0$, or D, and since no continuity assumption is made on D, its solutions are understood in the sense of Filippov (see definition 2.1).

The following Lyapunov functions will be used in the analysis:

for $i = 1,\ldots,n$,

$$V_{zi}(z_{1i}, z_{2i}) = \frac{k_{1i}^{\;2}}{(1+b)^2}|z_{1i}|^{2+2b} + \frac{1}{4}|z_{2i}|^4 + r_{zi}|z_{1i}|^{\frac{3+3b}{2}}\operatorname{sign}(z_{1i})z_{2i} + \frac{k_{1i}}{(1+b)}|z_{1i}|^{1+b}|z_{2i}|^2,$$

and

$$V_s(s,q) = \frac{1}{2}s^T M(q)s$$

Remark 3.1. Note that the Lyapunov function for the (z_{1i}, z_2) -subsystem is a strict Lyapunov function proposed for the twisting-based family of algorithms (section 2.2.2) of Chapter 2, where $\forall q \in \mathbb{R}^n$:

$$\underline{\pi}_{1i}\left(|z_{1i}|^{2+2b}|z_{2i}|^4\right) \le V_{zi}(z_{1i}, z_{2i}) \le \overline{\pi}_{1i}\left(|z_{1i}|^{2+2b} + |z_{2i}|^4\right)$$

with

$$\underline{\pi}_{1i} := \min\left\{\frac{1}{8}, \frac{1}{2}\left(\frac{k_{1i}}{1+b}\right)^2\right\},\ \overline{\pi}_{1i} := \max\left\{\frac{3r_{zi}}{4} + \left(\frac{k_{1i}}{1+b}\right)^2 + \left(\frac{k_{1i}}{2+2b}\right),\quad \left(\frac{k_{1i}}{2+2b}\right) + \frac{r_{zi}}{4} + \frac{1}{4}\right\},$$

while for V_s,

$$\frac{1}{2}\underline{m}\|s\|^2 \le V_s \le \frac{1}{2}\overline{m}\|s\|^2.$$

Let us define the following sets,

$$\Omega_z(R) = \left\{(z_1, z_2) \in \mathbb{R}^{2n} : V_{zi}(z_{1i}, z_{2i}) \le \rho_{1i}(R), \ldots, V_{zn}(z_{1n}, z_{2n}) \le \rho_{1i}(R), \text{ for } i = 1,\ldots,n\right\},$$

$$\Omega_s(R) = \left\{s \in \mathbb{R}^n : V_s \le R^2/2\right\},$$

$$\Omega(R)=\{(z_1,z_2,s)\in\mathbb{R}^{3n}:\Omega_z(R),\ \Omega_s(R)\},$$

where

for $i=1,\dots,n$

$$\rho_{1i}(R)=\begin{cases}\max\left\{\begin{array}{l}\left(\left(\dfrac{4\overline{\pi}_{1i}^{\frac{3+5b}{4+4b}}}{\omega_{4i}}\right)^4\left(\dfrac{2^{\frac{1}{2+2b}}\omega_{2i}}{\underline{\pi}_{1i}^{\frac{1+2b}{2+2b}}}\right)^4,\right.\\ \left(a_{1i}\left(\dfrac{4\overline{\pi}_{1i}^{\frac{3+5b}{4+4b}}}{\omega_{4i}}\right)\left(\dfrac{2^{\frac{1}{4}}\omega_{3i}}{\underline{\pi}_{1i}^{\frac{3}{4}}}\right)\right)^{\frac{2+2b}{b}}\end{array}\right\}\times\left(\dfrac{R}{\sqrt{\underline{m}}}\right)^4, & \text{for } b>0,\\ \max\limits_{(z_{1i},z_{2i})\in \mathrm{bd}\ \phi_{3i}\left(\frac{R}{\sqrt{\underline{m}}}\right)} V_{zi}(z_{1i},z_{2i}), & \text{for } b=0,\end{cases}$$

$\mathrm{bd}\ \phi_{3i}\left(\dfrac{R}{\sqrt{\underline{m}}}\right)$ is the boundary of the region

$$\phi_{3i}\left(\frac{R}{\sqrt{\underline{m}}}\right)=\left\{(z_{1i},z_{2i})\in\mathbb{R}^2:|z_{1i}|\le a_{3i}\left(\frac{R}{\sqrt{\underline{m}}}\right)^2,\ |z_{2i}|\le\frac{2k_{1i}}{k_{2i}}\left(\frac{R}{\sqrt{\underline{m}}}\right)\right\},$$

which is a compact set, with the constant $a_{3i}>0$ (see Appendix A.1).

Let also $\gamma_1(R)=\mathrm{diag}\{\gamma_{1i}(R)\}_{i=1}^n$ with $\gamma_{1i}(R)>0,\ \forall i=1,\dots,n$, such that $\forall(z_1,z_2,s)\in\Omega(R)$:

$$s^T M(\cdot)\boldsymbol{K}[\Delta(\cdot)]+\frac{1}{2}s^T\dot{M}(\cdot)s\le\sum_{i=1}^n\gamma_{1i}(R)|s_i|,$$

Remark 3.2. Note that such an upper bound always exists for any given compact set Ω(*R*) since *M*, *C*, *F*, *G*, *D*, and the desired trajectories are locally bounded, it implies that Δ(.) is locally bounded as well (i.e. it compose of summation of locally bounded function). From [143], the multi-valued function ***K***[Δ(.)] is locally bounded as well. Also, using the skew-symmetry property 3.2.2, the function, $\frac{1}{2}s^T\dot{M}(\cdot)s = s^T C(\cdot)s$ is locally bounded as well. Hence, within a compact set, an upper bound on the above function exists.

3.2.2 Stability Criterion Determination

The time derivatives of the Lyapunov functions, in accordance to lemma 2.4, $\forall(z_1, z_2, s) \in \Omega(R)$ of the closed-loop system satisfy the following inequalities:

Differential inequalities for the z-subsystem (see Appendix A.1):

$\forall i = 1, \ldots, n,$

for $b > 0$:

$$
\begin{aligned}
\dot{V}_{zi}(z_{1i}, z_{2i}) \overset{a.e.}{\le} & -\frac{1}{2}\left(\frac{\omega_{4i}}{\overline{\pi}_{1i}^{\frac{3+5b}{4+4b}}}\right)V_{zi}^{\frac{3+5b}{4+4b}} - \frac{1}{4}\left(\frac{\omega_{4i}}{\overline{\pi}_{1i}^{\frac{3+5b}{4+4b}}}\right)V_{zi}^{\frac{1+2b}{2+2b}}\left(V_{zi}^{\frac{1}{4}} - |s_i|\left(\frac{4\overline{\pi}_{1i}^{\frac{3+5b}{4+4b}}}{\omega_{4i}}\right)\left(\frac{2^{\frac{1}{2+2b}}\omega_{2i}}{\underline{\pi}_{1i}^{\frac{1+2b}{2+2b}}}\right)\right) \\
& -\frac{1}{4}\left(\frac{\omega_{4i}}{\overline{\pi}_{1i}^{\frac{3+5b}{4+4b}}}\right)V_{zi}^{\frac{3}{4}}\left(V_{zi}^{\frac{b}{2+2b}} - a_{1i}|s_i|^{\frac{2b}{1+b}}\left(\frac{4\overline{\pi}_{1i}^{\frac{3+5b}{4+4b}}}{\omega_{4i}}\right)\left(\frac{2^{\frac{1}{4}}\omega_{3i}}{\underline{\pi}_{1i}^{\frac{3}{4}}}\right)\right)
\end{aligned}
\quad (3 - 5)
$$

for $b = 0$:

$$\dot{V}_{zi}\left(z_{1i},z_{2i}\right)\overset{a.e.}{\in}\dot{\tilde{V}}_{zi}\leq 2k_{1i}{}^{2}\left|z_{1i}\right|\left|s_{i}\right|+\frac{3}{2}r_{i}\left|z_{1i}\right|^{\frac{1}{2}}\left|z_{2i}\right|\left|s_{i}\right|+k_{1i}\left|z_{2i}\right|^{2}\left|s_{i}\right|+\frac{3}{2}r_{i}\left|z_{1i}\right|^{\frac{1}{2}}\left|z_{2i}\right|^{2}$$
$$-2k_{1i}k_{2i}\left|z_{1i}\right|z_{2i}\mathrm{SGN}\left(z_{2i}+s_{i}\right)$$
$$-k_{2i}\left|z_{2i}\right|^{3}\mathrm{sign}\left(z_{2i}\right)\mathrm{SGN}\left(z_{2i}+s_{i}\right)-r_{i}\left(k_{1i}-k_{2i}\right)\left|z_{1i}\right|^{\frac{3}{2}}$$
$$<0\quad\text{for}\quad V_{zi}\geq\max_{\left(z_{1i},z_{2i}\right)\in\mathrm{bd}\ \phi_{3i}\left(\left|s_{i}\right|\right)}V_{zi}\left(z_{1i},z_{2i}\right)\tag{3 - 6}$$

Differential inequality for the s-subsystem:

$$\dot{V}_{s}\overset{a.e}{\in}\dot{\tilde{V}}_{s}=\bigcap_{\xi\in\partial V_{s}}\xi^{\mathrm{T}}\begin{pmatrix}\boldsymbol{K}[f](s)\\1\end{pmatrix}=\nabla V_{s}{}^{\mathrm{T}}\boldsymbol{K}[f](s)+\frac{1}{2}s^{\mathrm{T}}\dot{M}(\cdot)s$$
$$\subset s^{\mathrm{T}}M(\cdot)\boldsymbol{K}\left[-M^{-1}(\cdot)K\mathrm{sig}(s)^{a}+\Delta(\cdot)\right]+\frac{1}{2}s^{\mathrm{T}}\dot{M}(\cdot)s$$
$$=-s^{\mathrm{T}}\boldsymbol{K}\left[K\mathrm{sig}(s)^{a}\right]+s^{\mathrm{T}}M(\cdot)\boldsymbol{K}[\Delta(\cdot)]+\frac{1}{2}s^{\mathrm{T}}\dot{M}(\cdot)s$$
$$=-\sum_{i=1}^{n}k_{i}\left|s_{i}\right|^{1+a}+s^{\mathrm{T}}M(\cdot)\boldsymbol{K}[\Delta(\cdot)]+\frac{1}{2}s^{\mathrm{T}}\dot{M}(\cdot)s$$
$$\leq-\sum_{i=1}^{n}k_{i}\left|s_{i}\right|^{1+a}+\sum_{i=1}^{n}\gamma_{1i}(R)\left|s_{i}\right|$$

Remark 3.3. Note that the above analysis for the s-subsystem also apply to the case of a = 0. In particular, when a = 0, using the following notation,

$N^{+}(s)=\{i\in\{1,\ldots,n\}:\left|s_{i}\right|\neq 0\},\ \ N^{0}(s)=\{i\in\{1,\ldots,n\}:s_{i}=0\},$

if $N^{+}(s)\neq\emptyset$, then for at least one index $i\in\{i,\ldots,n\}$, $\left|s_{i}\right|\neq 0$, observe that

$$-s^{\mathrm{T}}K\boldsymbol{K}[\mathrm{sign}(s)]=-\sum_{i\in N^{+}(s)}k_{i}\left|s_{i}\right|-\sum_{i\in N^{0}(s)}k_{i}(0)\times[-1,+1]=-\sum_{i\in N^{+}(s)}k_{i}\left|s_{i}\right|-0=-\sum_{i=i}^{n}k_{i}\left|s_{i}\right|,$$

if $N^{+}(s)=\emptyset \Rightarrow s_i = 0, \forall i \in \{i,\ldots,n\}$, hence, $-s^{\mathrm{T}} K\boldsymbol{K}[\mathrm{sign}(s)]=0$.

Thus, when $a = 0$, $\forall\, s_i \in \mathbb{R}$, $\forall\, i \in \{i, \ldots, n\}$: $-s^{\mathrm{T}} K\boldsymbol{K}[\mathrm{sign}(s)] = -\sum_{i=i}^{n} k_i |s_i|$

Hence,

for $a = 0$:

$$\dot{\tilde{V}}_s \le -\sum_{i=1}^{n} |s_i| (k_i - \gamma_1(R)) \le -\min_i (k_i - \gamma_{1i}(R)) \sum_{i=1}^{n} \left(|s_i|^2\right)^{\frac{1}{2}}$$

$$\le -\min_i (k_i - \gamma_{1i}(R)) \left(\sum_{i=1}^{n} |s_i|^2\right)^{\frac{1}{2}}, \qquad \text{using lemma 2.2 of chapter 2,}$$

$$= -\min_i (k_i - \gamma_{1i}(R)) \|s\| \tag{3 - 7}$$

for $a > 0$:

$$\begin{aligned}
\dot{\tilde{V}}_s &\le -\lambda_{\min}(K) \sum_{i=1}^{n} |s_i|^{1+a} + \sum_{i=1}^{n} \gamma_{1i}(R) |s_i| \\
&\le -\frac{\lambda_{\min}(K)}{n^a} \left(\sum_{i=1}^{n} |s_i|\right)^{1+a} + \lambda_{\max}(\gamma_{1i}(R)) \sum_{i=1}^{n} |s_i|, \quad \text{using lemma 2.2 of chapter 2,} \\
&= -\frac{\lambda_{\min}(K)}{n^a} \left(\sum_{i=1}^{n} \left(|s_i|^2\right)^{\frac{1}{2}}\right)^{1+a} + \lambda_{\max}(\gamma_{1i}(R)) \sum_{i=1}^{n} \left(|s_i|^2\right)^{\frac{1}{2}} \\
&\le -\frac{\lambda_{\min}(K)}{n^a} \left(\left(\sum_{i=1}^{n} |s_i|^2\right)^{\frac{1}{2}}\right)^{1+a} + \lambda_{\max}(\gamma_{1i}(R)) \sqrt{n} \left(\sum_{i=1}^{n} |s_i|^2\right)^{\frac{1}{2}}, \quad \text{using lemma 2.2 of chapter 2,}
\end{aligned}$$

$$= -\frac{\lambda_{\min}(K)}{n^a} \|s\|^{1+a} + \lambda_{\max}(\gamma_{1i}(R)) \sqrt{n} \|s\| \tag{3 - 8}$$

Remark 3.4. For ease of presentation, note that the above differential inequalities are stated for two cases for each subsystems. This is due to the fact that the subsystems are discontinuous when $b = 0$ and $a = 0$ respectively.

Theorem 3.1: For any given K_1, $K_2 > 0$, suppose that the initial conditions for the closed-loop system (3 - 4) belong to a given compact set, there always exists a $c > 0$ such that initially, $(z_1, z_2, s) \in \Omega(c)$. Depending on the value of a:

1. For the case of $a > 0$, a $\mu > 0$ can be selected such that $c > \mu > 0$ and by selecting K such that

   $$\lambda_{\min}(K) > \frac{\lambda_{\max}(\gamma_1(c)) n^{a+0.5} \overline{m}^{\frac{a}{2}}}{\mu^a},$$

   all the trajectories will enter the compact set $\Omega(\mu)$, in finite time, and stay there for all future times.

2. For the case of $a = 0$, by selecting K such that

   $$k_i > \gamma_{1i}(c), \ \forall i = 1, \ldots, n, ,.$$

   all the trajectories will enter the compact set $\Omega_z(c) \times \Omega_s(0)$ in finite time, and stay there for all future times. Additionally, the states (z_1, z_2), have finite time convergence for $0 \le b < 1$, exponential convergence for $b = 1$, or for asymptotical convergence for $b > 1$.

Proof of Theorem 3.1: The stability analysis proceeds in two steps.

1. Obviously, for a given compact set of initial conditions, there exists $c > 0$ such that initially (z_1, z_2, s) belong to some compact set strictly inside $\Omega(c)$. A trajectory may leave the set $\Omega(c)$ only through one of the boundaries:

   $V_{zi}(z_{1i}, z_{2i}) = \rho_{1i}(c), \ \forall\, i = 1, \ldots, n$, or $V_s = c^2/2$.

   Let us show that it is impossible.

   (a) For the s-subsystem, $V_s = \frac{1}{2} s^{\mathrm{T}} M(q) s = \frac{1}{2} c^2$ implies $\|s\| \geq \frac{c}{\sqrt{m}}$.

   Hence, for $a = 0$, from (3 - 7), $\dot{\tilde{V}}_s \leq -\min_i \left(k_i - \gamma_{1i}(c) \right) \|s\|$

   $\therefore \dot{\tilde{V}}_s \leq -c_1$, where $c_1 = \min_i \left(k_i - \gamma_{1i}(c) \right) \left(\frac{c}{\sqrt{m}} \right)$,

   For $a > 0$, from (3 - 8)

$$\begin{aligned} \dot{V}_s &\overset{a.e.}{\leq} -\frac{\lambda_{\min}(K)}{n^a} \|s\|^{1+a} + \lambda_{\max}(\gamma_{1i}(c)) \sqrt{n} \|s\| \\ &= -\|s\| \left[\frac{\lambda_{\min}(K)}{n^a} \|s\|^a - \lambda_{\max}(\gamma_1(c)) \sqrt{n} \right] \end{aligned}$$

$$\therefore \dot{V}_s \overset{a.e.}{\leq} -c_2$$

   where $$c_2 = \frac{c}{\sqrt{m}} \left[\frac{\lambda_{\min}(K)}{n^a} \left| \frac{c}{\sqrt{m}} \right|^a - \lambda_{\max}(\gamma_1(c)) \sqrt{n} \right] \qquad (3 - 9)$$

   Hence, for k_i sufficiently big, c_1 is positive for $a = 0$, or $\lambda_{\min}(K)$ sufficiently big, c_2 is positive for $a > 0$, such that, $\dot{V}_s < 0$ almost everywhere. If the gain K satisfy the conditions of the Theorem 3.1, then, V_s is a decreasing function of t, so s stays in $\Omega_s(c)$ and

$\|s\| \le \frac{c}{\sqrt{\underline{m}}}$, together with the fact that $|s_i| \le \|s\|$, for $i = 1,\ldots,n$, we have

$|s_i| \le \frac{c}{\sqrt{\underline{m}}}$, for $i = 1,\ldots,n$.

(b) For the z-subsystem, for $b > 0$, at the boundary, it implies

$\forall\ i = 1,\ldots,n$:

$$V_{zi} = \rho_{1i}(c) = \max\left\{\left(\frac{4\overline{\pi}_{1i}^{\frac{3+5b}{4+4b}}}{\omega_{4i}}\right)^4\left(\frac{2^{\frac{1}{2+2b}}\omega_{2i}}{\underline{\pi}_{1i}^{\frac{1+2b}{2+2b}}}\right)^4,\ \left(a_{1i}\left(\frac{4\overline{\pi}_{1i}^{\frac{3+5b}{4+4b}}}{\omega_{4i}}\right)\left(\frac{2^{\frac{1}{4}}\omega_{3i}}{\underline{\pi}_{1i}^{\frac{3}{4}}}\right)\right)^{\frac{2+2b}{b}}\right\}\times\left(\frac{c}{\sqrt{\underline{m}}}\right)^4$$

$$\ge \max\left\{\left(\frac{4\overline{\pi}_{1i}^{\frac{3+5b}{4+4b}}}{\omega_{4i}}\right)^4\left(\frac{2^{\frac{1}{2+2b}}\omega_{2i}}{\underline{\pi}_{1i}^{\frac{1+2b}{2+2b}}}\right)^4,\ \left(a_{1i}\left(\frac{4\overline{\pi}_{1i}^{\frac{3+5b}{4+4b}}}{\omega_{4i}}\right)\left(\frac{2^{\frac{1}{4}}\omega_{3i}}{\underline{\pi}_{1i}^{\frac{3}{4}}}\right)\right)^{\frac{2+2b}{b}}\right\}\times|s_i|^4$$

Thus, we have from (3 - 5),

$$\dot{V}_{zi} \overset{a.e.}{\le} -\frac{1}{2}\left(\frac{\omega_{4i}}{\overline{\pi}_{1i}^{\frac{3+5b}{4+4b}}}\right)V_{zi}^{\frac{3+5b}{4+4b}} < 0$$

So, $\dot{V}_{zi}$ is a decreasing function of t for all $i = 1, \ldots, n$ and (z_1, z_2) stay in Ω_z(c).

While, for $b = 0$, at the boundary,

$V_{zi} = \rho_{1i}(c),\ \ \forall\ i = 1,\ldots,n,$ it implies

$$\left(\frac{c}{\sqrt{\underline{m}}}\right) \geq \|s\| \geq |s_i| \Rightarrow \phi_{3i}\left(\frac{c}{\sqrt{\underline{m}}}\right) \supset \phi_{3i}(|s_i|), \text{ (see the above definition of the region } \phi_{3i}(\cdot))$$

$$\Rightarrow V_{zi} = \rho_{1i}(c) \geq \rho_{1i}\left(\sqrt{\underline{m}}|s_i|\right), \quad \text{for } i = 1, \ldots, n$$

since $\max_{(z_{1i}, z_{2i}) \in \text{bd } \phi_{3i}(|s_i|)} V_{zi}(z_{1i}, z_{2i}) = \rho_{1i}\left(\sqrt{\underline{m}}|s_i|\right)$ (see the above definition of function $\rho_{1i}(.)$)

Hence, from (3 - 6), it implies $\dot{V}_{zi} \overset{a.e.}{<} 0, \ \forall i = 1, \ldots, n$.

So, $\dot{V}_{zi}$ is a decreasing function of t for all $i = 1, ..., n$ and (z_1, z_2) stay in $\Omega_z(c)$ for the case of $b = 0$.

As a result, the set $\Omega(c)$ is positively invariant, i.e. the trajectories of (3 - 4) stay in it once they have entered it.

2. Now we have shown boundedness, next is to show convergence to a smaller compact set $\Omega(\mu)$ with $c > \mu > 0$.

   (a) For the s-subsystem, for $a = 0$, from (3 - 7)

   $$\dot{\tilde{V}}_s \leq -\min_i\left(k_i - \gamma_{1i}(c)\right)\|s\|$$

   Hence, for the case of case of $a = 0$, $\forall\ i = 1, ..., n$, if $k_i > \gamma_{1i}(c)$, $\dot{\tilde{V}}_s < 0$ for $s \neq 0$, which implies that the s trajectory will converge to zero in finite time and stay there for all future times, i.e. all $(z_1, z_2, s) \in \Omega_z(c) \times \Omega_s(c)$ will enter the set $\Omega_z(c) \times \Omega_s(0)$ in finite time.

While, for the case of or $a > 0$, observed that $(z_1, z_2, s) \in \Omega(c) \setminus \Omega(\mu)$ implies that $\|s\| \geq \frac{\mu}{\sqrt{m}}$, hence from (3 - 8)

$$\dot{V}_s \overset{a.e.}{\leq} -\frac{\lambda_{\min}(K)}{n^a}\|s\|^{1+a} + \lambda_{\max}(\gamma_{1i}(c))\sqrt{n}\|s\|$$
$$= -\|s\|\left(\frac{\lambda_{\min}(K)}{n^a}\|s\|^a - \lambda_{\max}(\gamma_{1i}(c))\sqrt{n}\right)$$

$$\therefore \dot{V}_s \overset{a.e.}{\leq} -c_3 \qquad (3 - 10)$$

where $c_3 = \frac{\mu}{\sqrt{m}}\left[\frac{\lambda_{\min}(K)}{n^a}\left|\frac{\mu}{\sqrt{m}}\right|^a - \lambda_{\max}(\gamma_1(c))\sqrt{n}\right]$ (3 - 11)

Thus, for the case of $a > 0$, with $\lambda_{\min}(K)$ sufficiently big, c_3 is positive, such that, $\dot{V}_s < 0$ almost everywhere. Then, V_s is a decreasing function of t, so s enters the set $\Omega_s(\mu)$ in finite time and stays in it for all future times, in particular $\|s\| \leq \frac{\mu}{\sqrt{m}}$ together with the fact that $|s_i| \leq \|s\|$, for $i = 1, \ldots, n$, we have $|s_i| \leq \frac{\mu}{\sqrt{m}}$, for $i = 1, \ldots, n$.

(b) For the z-subsystem, for all $b \geq 0$, if $a = 0$, once the s reaches zero (inside the set $\Omega_z(c) \times \Omega_s(0)$), the z-subsystem becomes:

$$\dot{z}_1 = z_2,$$
$$\dot{z}_2 = -K_2\mathrm{sig}(z_2)^{\frac{2b}{1+b}} - K_1\mathrm{sig}(z_1)^b,$$

which is a multi-dimensional version of the planar homogeneous twisting-based control proposed in Chapter 2 (see section 2.2.1). Its convergence properties is based on the b parameters, where for $0 \leq b < 1$ finite time convergence can be achieved, for $b = 1$ will provide exponential convergence, while for $b > 1$ the system exhibits asymptotical convergence.

While, for $b \geq 0$, if $a > 0$, following the same arguments presented in point (1) - (b), it is not hard to see that, once $s(t)$ is inside the set $\Omega_s(\mu)$, (z_1, z_2) will enter the set $\Omega_z(\mu)$ in finite time and stay in it for all future times. Hence, the set $\Omega_s(\mu) \times \Omega_z(\mu)$ is positively invariant and attracting for all trajectories of the system (3 - 4) originating inside the set $\Omega_s(\mathrm{c}) \times \Omega_z(c)$.

■

Remark 3.5. Note that the results above show that the control law achieves semiglobal practical stability for the case when $a > 0$. Its region of attraction can be given as the set $\Omega_s(\mathrm{c}) \times \Omega_z(c)$, while its ultimate invariant set given by $\Omega_s(\mu) \times \Omega_z(\mu)$. The semiglobal nature of the control law can be seen where the estimate of region of attraction for each set of initial states can be increased by choosing a sufficiently big gain K. While the practical stability nature, pertaining to the set where the solutions converge is stable and may be reduced at will, can be achieved also by tuning the gain K (see the conditions on K in Theorem 3.1). On the other hand, when $a = 0$, the control law achieves semiglobal stability in which its convergence is dependent on the parameter b (i.e. finite time with $0 \leq b < 1$, exponential for $b = 1$, and asymptotical for $b > 1$). Similarly, its region of attraction is $\Omega_s(\mathrm{c}) \times \Omega_z(c)$.

Remark 3.6. Note that for the special case of $a = b = 1$, the proposed controller (3 - 2) becomes the well-known PID control, indeed when written in original coordinates

$$\tau = -K_d e_2 - K_p e_1 - K_i \int_{t_0}^{t} e_1(\varsigma) d\varsigma + K_i e_1(t_0),$$

where $K_d = K$, $K_p = KK_2$, and $K_i = KK_1$. Thus, based on the stability analysis above, one can select the gains of the conventional PID control based on the desired error dynamics (K_1, K_2) and the desired region of attraction with respect to the ultimate bound (K). This is a great simplification to the heuristic PID gain tuning method, where tuning a particular gain will affect the tuning of all the other gains of the system [97]. It is worth mentioning that in [99], a similar gain selection method is presented, but due to their specific formulation, the bounds on the inertia matrix are required to compute the PID gains. The need for the inertia matrix is echoed by TDE and UDE based PID approach as well [90] [88]. On the other hand, our proposed controller is model-free. Another issue in [99], is its stability formulation requires the time derivative of the robot dynamics, in which case is not directly applicable if discontinuity is to be considered in the type of disturbances affecting the system. Additionally, in the work of [103], semiglobal PID control results is achieved which is model-free without requiring time derivative of the robot dynamics, however, the gains selection only pertains to the desired region of attraction and ultimate bound, the transient performance or the desired error dynamics is not clearly specified with respect to the PID gains.

Theorem 3.2: In addition to the conditions in Theorem 3.1, consider the special case of regulation problem, where the desired trajectory of the robot dynamics (3 - 1) is a constant value, q_0, (i.e. $q_d = q_0$, $\dot{q}_d = \ddot{q}_d = 0$), the control (3 - 2) with the parameter b restricted to $0 < b \le 1$, and parameter a selected as $a = \dfrac{2b}{1+b}$, while the additive disturbance is upper bounded by $\|D\| \le p_1\|e_1\| + p_2\|\dot{q}\| + p_3\|e_1\|^2 + p_4\|\dot{q}\|^2$, i.e. vanishing perturbation. Then, for $b = 1$, semiglobal exponential regulation is guaranteed, provided that K is large enough with respect to initial error conditions. While, for $0 < b < 1$, semiglobal finite-time regulation is assured, provided that K is large enough with respect to initial error conditions, and the gravity vector at the constant desired position, $G(q_d)$ is zero.

Proof of Theorem 3.2: For this section, the exponent of the control law (3 - 2), a is selected as $a = \dfrac{2b}{1+b}$ for $0 < b \le 1$. Let us define the following variables,

$$\bar{z}_1 = 0,\ \ \bar{z}_2 = -\bar{s},\ \ \bar{s} = -\mathrm{sig}\left(K^{-1}G(q_0)\right)^{\frac{1+b}{2b}},$$
$$\tilde{z}_1 = z_1 - \bar{z}_1,\ \ \tilde{z}_2 = z_2 - \bar{z}_2,\ \ \tilde{s} = s - \bar{s},$$

where $\bar{s}_i = -\left|k_i^{-1} g_i(q_0)\right|^{\frac{1+b}{2b}} \mathrm{sign}\left(k_i^{-1} g_i(q_0)\right)$ and $g_i(q_0)$ is the i - th element of the vector $G(q_0)$.

Remark 3.7. Note that q_0 is a constant vector, and as a result the vector, $\bar{s}$ is a constant vector as well since the matrix K comprises constants as well. Also, $\bar{s}$ is a constant that is defined for stability analyses only, its actual value, which require knowledge of gravity vector, $G(q)$, is not required in the control law. Also note that D comprise vanishing perturbations only, i.e. $p_0 = 0$ (note that this is the general assumption on regulation problem , however, if constant perturbation do exists, the $\bar{s}$ can be redefined to accommodate this extra constant term).

Hence the closed-loop system (3 - 4) could be rewritten as

$$\begin{aligned}
\dot{\tilde{z}}_1 &= \tilde{z}_2 + \tilde{s}, \\
\dot{\tilde{z}}_2 &= -K_2 \mathrm{sig}(\tilde{z}_2 + \tilde{s})^{\frac{2b}{1+b}} - K_1 \mathrm{sig}(\tilde{z}_1)^b \\
\dot{\tilde{s}} &= -M^{-1}(\tilde{z}_1 + q_d) K \mathrm{sig}(\tilde{s})^{\frac{2b}{1+b}} + M^{-1}(\tilde{z}_1 + q_d) K \left(\mathrm{sig}(\tilde{s})^{\frac{2b}{1+b}} - \mathrm{sig}(s)^{\frac{2b}{1+b}} + \mathrm{sig}(\bar{s})^{\frac{2b}{1+b}} \right) + \tilde{\Delta}(\cdot)
\end{aligned}$$

where

$$\begin{aligned}
\tilde{\Delta}(\cdot) = & -M^{-1}(\tilde{z}_1 + q_d) C(\tilde{z}_1 + q_d, \tilde{z}_2 + \tilde{s} + \dot{q}_d)(\tilde{z}_2 + \tilde{s} + \dot{q}_d) - M^{-1}(\tilde{z}_1 + q_d) F(\tilde{z}_2 + \tilde{s} + \dot{q}_d) \\
& - M^{-1}(\tilde{z}_1 + q_d)(G(\tilde{z}_1 + q_d) - G(q_0)) - M^{-1}(\tilde{z}_1 + q_d) D - \ddot{q}_d + K_2 \mathrm{sig}(\tilde{z}_2 + \tilde{s})^{\frac{2b}{1+b}} + K_1 \mathrm{sig}(\tilde{z}_1)^b
\end{aligned}$$

Remark 3.8. Note that the fact of $\mathrm{sig}(\bar{s})^{\frac{2b}{1+b}} = -K^{-1} G(q_0)$ has been employed based on the above definition. Also, observe that the $\tilde{s}$ and $\tilde{z}$ subsystem have the same form as the s and z subsystem discussed in Theorem 3.1 (i.e. by replacing (s, z_1, z_2) with $(\tilde{s}, \tilde{z}_1, \tilde{z}_2)$).

Hence using a similar Lyapunov function structure, consider the following Lyapunov function

$$V(\tilde{s}, \tilde{z}_1, \tilde{z}_2) = [V_{\tilde{s}}(\tilde{s})]^2 + V_{\tilde{z}}(\tilde{z}_1, \tilde{z}_2),$$

with $V_{\tilde{s}}(\tilde{s}) = \frac{1}{2}\tilde{s}^{\mathrm{T}} M\tilde{s}$, and $V_{\tilde{z}}(\tilde{z}_1, \tilde{z}_2) = \sum_{i=1}^{n} V_{\tilde{z}i}(\tilde{z}_{1i}, \tilde{z}_{2i})$,

where $V_{\tilde{z}i}(\tilde{z}_{1i}, \tilde{z}_{2i}) = \frac{{k_{1i}}^2}{(1+b)^2}|\tilde{z}_{1i}|^{2+2b} + \frac{1}{4}|\tilde{z}_{2i}|^4 + r_{\tilde{z}i}|\tilde{z}_{1i}|^{\frac{3+3b}{2}}\operatorname{sign}(\tilde{z}_{1i})\tilde{z}_{2i} + \frac{k_{1i}}{(1+b)}|\tilde{z}_{1i}|^{1+b}|\tilde{z}_{2i}|^2$

Note the following properties of the Lyapunov functions:

$$\frac{1}{2}\underline{m}\|\tilde{s}\|^2 \le V_{\tilde{s}}(\tilde{s}) \le \frac{1}{2}\overline{m}\|\tilde{s}\|^2 \Leftrightarrow \frac{\sqrt{2}}{\sqrt{\overline{m}}}[V_{\tilde{s}}(\tilde{s})]^{\frac{1}{2}} \le \|\tilde{s}\| \le \frac{\sqrt{2}}{\sqrt{\underline{m}}}[V_{\tilde{s}}(\tilde{s})]^{\frac{1}{2}}$$

Also, from Appendix A.1,

$$\begin{aligned}
V_{\tilde{z}}(\tilde{z}_1, \tilde{z}_2) &\ge \sum_{i=1}^{n} \underline{\pi}_{1i}\left(|\tilde{z}_{1i}|^{2+2b} + |\tilde{z}_{2i}|^4\right) \\
&\ge \underline{\pi}_1\left(\sum_{i=1}^{n}|\tilde{z}_{1i}|^{2+2b} + \sum_{i=1}^{n}|\tilde{z}_{2i}|^4\right), \text{ where } \underline{\pi}_1 := \min_i\{\underline{\pi}_{1i}\}, \\
&\ge \underline{\pi}_1\left(\frac{1}{n^b}\left(\sum_{i=1}^{n}\left(|\tilde{z}_{1i}|^2\right)\right)^{\frac{2+2b}{2}} + \frac{1}{n}\left(\sum_{i=1}^{n}\left(|\tilde{z}_{2i}|^2\right)\right)^{\frac{4}{2}}\right), \quad \text{using lemma 2.2 of chapter 2,} \\
&= \underline{\pi}_1\left(\frac{1}{n^b}\|\tilde{z}_1\|^{2+2b} + \frac{1}{n}\|\tilde{z}_2\|^4\right) \\
&\ge \frac{\underline{\pi}_1}{n}\left(\|\tilde{z}_1\|^{2+2b} + \|\tilde{z}_2\|^4\right), \text{ since } \min\left\{\frac{1}{n^b}, \frac{1}{n}\right\} = \frac{1}{n} \text{ for } 0 < b \le 1 \text{ and } n \ge 1
\end{aligned}$$

and the term,

$$
\begin{aligned}
\|\tilde{z}_1\|^b + \|\tilde{z}_2\|^{\frac{2b}{1+b}} &= \left(\|\tilde{z}_1\|^{2+2b}\right)^{\frac{b}{2+2b}} + \left(\|\tilde{z}_2\|^4\right)^{\frac{b}{2+2b}} \\
&\le n^{\frac{2+b}{2+2b}}\left(\|\tilde{z}_1\|^{2+2b} + \|\tilde{z}_2\|^4\right)^{\frac{b}{2+2b}}, \quad \text{using lemma 2.2 of chapter 2,} \\
&\le n^{\frac{2+b}{2+2b}}\left(\frac{n}{\underline{\pi}_i} V_{\tilde{z}}(\tilde{z}_1, \tilde{z}_2)\right)^{\frac{b}{2+2b}} \\
&= \left(\frac{n}{\underline{\pi}_i^{\frac{b}{2+2b}}}\right)\left[V_{\tilde{z}}(\tilde{z}_1, \tilde{z}_2)\right]^{\frac{b}{2+2b}} .
\end{aligned}
$$

Consider the time derivative of the Lyapunov function $V_{\tilde{s}}$ along the solutions of the system for the $\tilde{s}$ subsystem:

$$
\begin{aligned}
&\dot{V}_{\tilde{s}} \overset{a.e}{\in} \dot{\tilde{V}}_{\tilde{s}} \\
&= \bigcap_{\xi \in \partial V_{\tilde{s}}} \xi^{\mathrm{T}} \begin{pmatrix} \boldsymbol{K}[f](\tilde{s}) \\ 1 \end{pmatrix} \\
&= \nabla V_{\tilde{s}}^{\mathrm{T}} \boldsymbol{K}[f](\tilde{s}) + \frac{1}{2}\tilde{s}^{\mathrm{T}}\dot{M}(q)\tilde{s} \\
&\subset \tilde{s}^{\mathrm{T}} M \begin{pmatrix} -M^{-1}(\tilde{z}_1 + q_d)K\mathrm{sig}(\tilde{s})^{\frac{2b}{1+b}} \\ +M^{-1}(\tilde{z}_1 + q_d)K\left(\mathrm{sig}(\tilde{s})^{\frac{2b}{1+b}} - \mathrm{sig}(s)^{\frac{2b}{1+b}} + \mathrm{sig}(\bar{\mathrm{s}})^{\frac{2b}{1+b}}\right) + \boldsymbol{K}\left[\tilde{\Delta}(\cdot)\right] \end{pmatrix} + \frac{1}{2}\tilde{s}^{\mathrm{T}}\dot{M}\tilde{s} \qquad (3 - 12)
\end{aligned}
$$

Remark 3.9. Note that the following three properties that will simplify (3 - 12):

1. From Theorem 3.1, the states (s, z_1, z_2) will reach and stay inside the compact set, $\Omega_s(\mu) \times \Omega_z(\mu)$ in finite time, and note that $\bar{s}, \bar{z}_1, \bar{z}_2$ are constants, we have, for $0 < b \le 1$, the following upper bounds (Note that upper bound of the perturbation, $\|D\|$ has

been expressed in terms of state variables of the closed-loop system):

$$
\begin{aligned}
\tilde{s}^{\mathrm{T}} M(\cdot)\boldsymbol{K}\left[\tilde{\Delta}(\cdot)\right]+\frac{1}{2}\tilde{s}^{\mathrm{T}}\dot{M}(\cdot)\tilde{s} \le & -\underline{f}\|\tilde{s}\|^{2}+\overline{f}\|\tilde{s}\|\|\tilde{z}_{2}\|+\overline{f}\|\tilde{s}\|\|\dot{q}_{d}\|+\|\tilde{s}\|C_{m}\|\tilde{z}_{2}\|^{2} \\
& +C_{m}\|\tilde{z}_{2}\|\|\tilde{s}\|^{2}+C_{m}\|\dot{q}_{d}\|\|\tilde{s}\|^{2}+\|\tilde{s}\|C_{m}\|\dot{q}_{d}\|^{2} \\
& +2\|\tilde{s}\|C_{m}\|\dot{q}_{d}\|\|\tilde{z}_{2}\|+\|\tilde{s}\|k_{g}\|\tilde{z}_{1}\|+\|\tilde{s}\|k_{g}\|q_{d}-q_{0}\| \\
& +\|\tilde{s}\|p_{1}\|\tilde{z}_{1}\|+\|\tilde{s}\|p_{2}\|\tilde{z}_{2}\|+p_{2}\|\tilde{s}\|^{2}+\|\tilde{s}\|p_{2}\|\dot{q}_{d}\| \\
& +\|\tilde{s}\|p_{3}\|\tilde{z}_{1}\|^{2}+\|\tilde{s}\|p_{4}\|\tilde{z}_{2}\|^{2}+p_{4}\|\tilde{s}\|^{3}+\|\tilde{s}\|p_{4}\|\dot{q}_{d}\|^{2} \\
& +2p_{4}\|\tilde{z}_{2}\|\|\tilde{s}\|^{2}+2\|\tilde{s}\|p_{4}\|\tilde{z}_{2}\|\|\dot{q}_{d}\|+2p_{4}\|\tilde{s}\|^{2}\|\dot{q}_{d}\| \\
& +\overline{m}\|\tilde{s}\|\|\ddot{q}_{d}\|+\overline{m}\bar{k}_{1}\|\tilde{s}\|n^{\frac{1-b}{2}}\|\tilde{z}_{1}\|^{b}+\overline{m}\bar{k}_{2}\|\tilde{s}\|n^{\frac{1-b}{2+2b}}\|\tilde{z}_{2}\|^{\frac{2b}{1+b}} \\
& +\overline{m}\bar{k}_{2}\|\tilde{s}\|n^{\frac{1-b}{2+2b}}\|\tilde{s}\|^{\frac{2b}{1+b}} \\
\le & -\underline{f}\|\tilde{s}\|^{2}+\gamma_{2}(\mu)\|\tilde{s}\|^{\frac{1+3b}{1+b}}+\gamma_{3}(\mu)\|\tilde{s}\|\left(\|\tilde{z}_{1}\|^{b}+\|\tilde{z}_{2}\|^{\frac{2b}{1+b}}\right) \\
& +\alpha(\|\eta(t)\|)\gamma_{4}(\mu)\|\tilde{s}\|
\end{aligned}
$$

where

$$
\bar{k}_{1}=\lambda_{\max}(K_{1}),\quad \bar{k}_{2}=\lambda_{\max}(K_{2}),
$$

$$
\left(p_{2}\|\tilde{s}\|^{2}+3p_{4}\|\tilde{s}\|^{3}+\overline{m}\bar{k}_{2}n^{\frac{1-b}{2+2b}}\|\tilde{s}\|\|\tilde{s}\|^{\frac{2b}{1+b}}\right)\le\gamma_{2}(\mu)\|\tilde{s}\|^{\frac{1+3b}{1+b}}
$$

$$
\left(\begin{aligned}
&\overline{f}\|\tilde{s}\|\|\tilde{z}_{2}\|+\overline{m}\bar{k}_{1}n^{\frac{1-b}{2}}\|\tilde{s}\|\|\tilde{z}_{1}\|^{b}+\overline{m}\bar{k}_{2}n^{\frac{1-b}{2+2b}}\|\tilde{s}\|\|\tilde{z}_{2}\|^{\frac{2b}{1+b}} \\
&+k_{g}\|\tilde{s}\|\|\tilde{z}_{1}\|+p_{1}\|\tilde{s}\|\|\tilde{z}_{1}\|+p_{2}\|\tilde{s}\|\|\tilde{z}_{2}\| \\
&+2C_{m}\|\tilde{s}\|\|\tilde{z}_{2}\|^{2}+3p_{4}\|\tilde{s}\|\|\tilde{z}_{2}\|^{2}+p_{3}\|\tilde{s}\|\|\tilde{z}_{1}\|^{2}+C_{m}\|\tilde{z}_{2}\|\|\tilde{s}\|^{2}
\end{aligned}\right)\le\gamma_{3}(\mu)\|\tilde{s}\|\left(\|\tilde{z}_{1}\|^{b}+\|\tilde{z}_{2}\|^{\frac{2b}{1+b}}\right)
$$

$$
\left(\begin{aligned}
&k_{g}\|\tilde{s}\|\|q_{d}-q_{0}\|+C_{m}\|\dot{q}_{d}\|\|\tilde{s}\|^{2}+3p_{4}\|\tilde{s}\|\|\dot{q}_{d}\|^{2}+p_{2}\|\tilde{s}\|\|\dot{q}_{d}\| \\
&+2C_{m}\|\tilde{s}\|\|\dot{q}_{d}\|^{2}+\overline{f}\|\tilde{s}\|\|\dot{q}_{d}\|+\overline{m}\|\tilde{s}\|\|\ddot{q}_{d}\|
\end{aligned}\right)\le\alpha(\|\eta(t)\|)\gamma_{4}(\mu)\|\tilde{s}\|
$$

with $\gamma_2(.)$, $\gamma_3(.)$, $\gamma_4(.)$ are positive functions and $\alpha(.)$ is a class $\mathcal{K}$ function, and the vector $\eta(t)$ is defined as:

$$\eta^{\mathrm{T}}(t)=\left[\left(q_d(t)-q_0\right)^{\mathrm{T}},\dot{q}_d^{\ \mathrm{T}}(t),\ddot{q}_d^{\ \mathrm{T}}(t)\right]\in\mathbb{R}^{3n}$$

2. Next, let us define the diagonal matrix,

$$\Lambda:=\operatorname{diag}\left(\operatorname{sig}\left(\frac{\tilde{s}_i}{\bar{s}_i}\right)^{\frac{2b}{1+b}}-\operatorname{sig}\left(\frac{\tilde{s}_i}{\bar{s}_i}+1\right)^{\frac{2b}{1+b}}+1\right)$$, and since, $\operatorname{sig}(\bar{s})^{\frac{2b}{1+b}}=-K^{-1}G(q_0)$,

then, $$\sum_{i=1}^{n}\tilde{s}_i k_i\left(\left|\tilde{s}_i\right|^{\frac{2b}{1+b}}\operatorname{sign}(\tilde{s}_i)-\left|s_i\right|^{\frac{2b}{1+b}}\operatorname{sign}(s_i)+\left|\bar{s}_i\right|^{\frac{2b}{1+b}}\operatorname{sign}(\bar{s}_i)\right)=-\tilde{s}^{\mathrm{T}}\Lambda G(q_0)\ .$$

3. Next, using lemma 2.2,

$$\sum_{i=1}^{n}\left|\tilde{s}_i\right|^{\frac{1+3b}{1+b}}\geq\frac{1}{n^{\frac{2b}{1+b}}}\left\|\tilde{s}\right\|^{\frac{1+3b}{1+b}},$$

Substituting the properties of remark 3.9 into (3 - 12), $\dot{\tilde{V}}_{\tilde{s}}$:

$$\dot{\tilde{V}}_{\tilde{s}}\leq-\frac{\underline{k}}{n^{\frac{2b}{1+b}}}\left\|\tilde{s}\right\|^{\frac{1+3b}{1+b}}+\left\|\tilde{s}\right\|\left\|\Lambda G(q_0)\right\|-\underline{f}\left\|\tilde{s}\right\|^2+\gamma_2(\mu)\left\|\tilde{s}\right\|^{\frac{1+3b}{1+b}}$$
$$+\gamma_3(\mu)\left\|\tilde{s}\right\|\left(\left\|\tilde{z}_1\right\|^{b}+\left\|\tilde{z}_2\right\|^{\frac{2b}{1+b}}\right)+\alpha\left(\left\|\eta(t)\right\|\right)\gamma_4(\mu)\left\|\tilde{s}\right\|$$

where $\underline{k}=\lambda_{\min}(K)$.

Employing the bounds on the Lyapunov functions and lemma 2.2, one obtains,

$$\dot{\tilde{V}}_{\tilde{s}} \le -\left(\frac{k}{n^{\frac{2b}{1+b}}}\right)\left(\frac{\sqrt{2}}{\sqrt{\overline{m}}}\right)^{\frac{1+3b}{1+b}} V_{\tilde{s}}^{\frac{1+3b}{2+2b}} - \underline{f}\left(\frac{2}{\overline{m}}\right)V_{\tilde{s}} + \gamma_2(\mu)\left(\frac{\sqrt{2}}{\sqrt{\underline{m}}}\right)^{\frac{1+3b}{1+b}} V_{\tilde{s}}^{\frac{1+3b}{2+2b}}$$

$$+\gamma_3(\mu)\left(\frac{\sqrt{2}n}{\sqrt{\underline{m}}\,\underline{\pi}_i^{\frac{b}{2+2b}}}\right)V_{\tilde{s}}^{\frac{1}{2}}V_{\tilde{z}}^{\frac{b}{2+2b}} + \left[\alpha(\|\eta(t)\|)\gamma_4(\mu) + \|\Lambda G(q_0)\|\right]\left(\frac{\sqrt{2}}{\sqrt{\underline{m}}}\right)V_{\tilde{s}}^{\frac{1}{2}} \tag{3 - 13}$$

Next, we compute the time derivative of the Lyapunov function for the $\tilde{z}$ subsystem, recall that:

$$V_{\tilde{z}}(\tilde{z}_1, \tilde{z}_2) = \sum_{i=1}^{n} V_{\tilde{z}i}(\tilde{z}_{1i}, \tilde{z}_{2i})$$

From Appendix A.1, we have

$$\dot{\tilde{V}}_{\tilde{z}}(\tilde{z}_1, \tilde{z}_2) \le -d_0 \sum_{i=1}^{n} V_{\tilde{z}i}^{\frac{3+5b}{4+4b}} + d_1 \sum_{i=1}^{n} |\tilde{s}_i| V_{\tilde{z}i}^{\frac{1+2b}{2+2b}} + d_2 \sum_{i=1}^{n} |\tilde{s}_i|^{\frac{2b}{1+b}} V_{\tilde{z}i}^{\frac{3}{4}}$$

where

$$d_0 = \min_i \left(\frac{\omega_{4i}}{\overline{\pi}_{1i}^{\frac{3+5b}{4+4b}}}\right), \qquad d_1 = \max_i \left(\frac{2^{\frac{1}{2+2b}}\omega_{2i}}{\underline{\pi}_{1i}^{\frac{1+2b}{2+2b}}}\right), \text{ and } \quad d_2 = \max_i a_{1i}\left(\frac{2^{\frac{1}{4}}\omega_{3i}}{\underline{\pi}_{1i}^{\frac{3}{4}}}\right).$$

Using lemma 2.2,

$$-d_0 \sum_{i=1}^{n} V_{\tilde{z}i}^{\frac{3+5b}{4+4b}} \le -d_0\left(\sum_{i=1}^{n} V_{\tilde{z}i}\right)^{\frac{3+5b}{4+4b}},$$

$$d_1 \sum_{i=1}^{n} V_{\tilde{z}i}^{\frac{1+2b}{2+2b}} \le n^{\frac{1}{2+2b}} d_1\left(\sum_{i=1}^{n} V_{\tilde{z}i}\right)^{\frac{1+2b}{2+2b}},$$

$$d_2 \sum_{i=1}^{n} V_{\tilde{z}i}^{\frac{3}{4}} \le n^{\frac{1}{4}} d_2 \left(\sum_{i=1}^{n} V_{\tilde{z}i} \right)^{\frac{3}{4}}$$

and the fact that $|x_i| \le \|x\|, \forall i = 1,\ldots,n,\ x \in \mathbb{R}^n$, it follows that

$$\dot{V}_{\tilde{z}}(\tilde{z}_1, \tilde{z}_2) \overset{a.e.}{\le} -d_0 V_{\tilde{z}}^{\frac{3+5b}{4+4b}} + n^{\frac{1}{2+2b}} d_1 \|\tilde{s}\| V_{\tilde{z}}^{\frac{1+2b}{2+2b}} + n^{\frac{1}{4}} d_2 \|\tilde{s}\|^{\frac{2b}{1+b}} V_{\tilde{z}}^{\frac{3}{4}}$$

Employing the bounds on the Lyapunov functions and lemma 2.2, one obtains,

$$\dot{V}_{\tilde{z}}(\tilde{z}_1, \tilde{z}_2) \overset{a.e.}{\le} -d_0 V_{\tilde{z}}^{\frac{3+5b}{4+4b}} + n^{\frac{1}{2+2b}} d_1 \left(\frac{\sqrt{2}}{\sqrt{\underline{m}}} \right) V_{\tilde{s}}^{\frac{1}{2}} V_{\tilde{z}}^{\frac{1+2b}{2+2b}} + n^{\frac{1}{4}} d_2 \left(\frac{\sqrt{2}}{\sqrt{\underline{m}}} \right)^{\frac{2b}{1+b}} V_{\tilde{s}}^{\frac{b}{1+b}} V_{\tilde{z}}^{\frac{3}{4}} \qquad (3 - 14)$$

With the above results, we are now in a position to find the time derivative of the Lyapunov function for the closed-loop system, recall that

$$V(\tilde{s}, \tilde{z}_1, \tilde{z}_2) = [V_{\tilde{s}}(\tilde{s})]^2 + V_{\tilde{z}}(\tilde{z}_1, \tilde{z}_2)$$

hence

$$\dot{V}(\tilde{s}, \tilde{z}_1, \tilde{z}_2) = 2 V_{\tilde{s}} \dot{V}_{\tilde{s}} + \dot{V}_{\tilde{z}},$$

Thus, substituting results from (3 - 13) and (3 - 14), after rearrangement,

$$\begin{aligned}\dot{V} \overset{a.e.}{\le} & -\left(\frac{3}{2}\right)\left(\frac{k}{n^{\frac{2b}{1+b}}}\right)\left(\frac{\sqrt{2}}{\sqrt{\overline{m}}}\right)^{\frac{1+3b}{1+b}} V_{\tilde{s}}^{\frac{3+5b}{2+2b}} - \underline{f}\left(\frac{4}{\overline{m}}\right) V_{\tilde{s}}^{2} - \left(\frac{3}{4}\right) d_0 V_{\tilde{z}}^{\frac{3+5b}{4+4b}} + \dot{V}_1 \\ & + 2\left[\alpha(\|\eta(t)\|)\gamma_4(\mu) + \|\Lambda G(q_0)\|\right]\left(\frac{\sqrt{2}}{\sqrt{\underline{m}}}\right) V_{\tilde{s}}^{\frac{3}{2}}\end{aligned}$$

where

$$\dot{V}_1 = -\left(\frac{1}{4}\right)\left(\frac{\underline{k}}{n^{\frac{2b}{1+b}}}\right)\left(\frac{\sqrt{2}}{\sqrt{\underline{m}}}\right)^{\frac{1+3b}{1+b}} V_{\tilde{s}}^{\frac{3+5b}{2+2b}} - \frac{d_0}{4} V_{\tilde{z}}^{\frac{3+5b}{4+4b}}$$

$$-\left(\left(\frac{1}{4}\right)\left(\frac{\underline{k}}{n^{\frac{2b}{1+b}}}\right)\left(\frac{\sqrt{2}}{\sqrt{\underline{m}}}\right)^{\frac{1+3b}{1+b}} - 2\gamma_2(\mu)\left(\frac{\sqrt{2}}{\sqrt{\underline{m}}}\right)^{\frac{1+3b}{1+b}}\right) V_{\tilde{s}}^{\frac{3+5b}{2+2b}}$$

$$+2\gamma_3(\mu)\left(\frac{\sqrt{2}n}{\sqrt{\underline{m}}\,\underline{\pi}_i^{\frac{b}{2+2b}}}\right) V_{\tilde{s}}^{\frac{3}{2}} V_{\tilde{z}}^{\frac{b}{2+2b}} + n^{\frac{1}{2+2b}} d_1\left(\frac{\sqrt{2}}{\sqrt{\underline{m}}}\right) V_{\tilde{s}}^{\frac{1}{2}} V_{\tilde{z}}^{\frac{1+2b}{2+2b}}$$

$$+ n^{\frac{1}{4}} d_2\left(\frac{\sqrt{2}}{\sqrt{\underline{m}}}\right)^{\frac{2b}{1+b}} V_{\tilde{s}}^{\frac{b}{1+b}} V_{\tilde{z}}^{\frac{3}{4}}$$

For $\dot{V}_1$, applying lemma 2.1,

$$-V_{\tilde{s}}^{\frac{3+5b}{2+2b}} - V_{\tilde{z}}^{\frac{3+5b}{4+4b}} \le -\left(\frac{3+5b}{3+3b}\right)^{\frac{3+3b}{3+5b}} V_{\tilde{s}}^{\frac{3}{2}} \left(\frac{3+5b}{2b}\right)^{\frac{2b}{3+5b}} V_{\tilde{z}}^{\frac{b}{2+2b}},$$

$$-V_{\tilde{s}}^{\frac{3+5b}{2+2b}} - V_{\tilde{z}}^{\frac{3+5b}{4+4b}} \le -\left(\frac{3+5b}{1+b}\right)^{\frac{1+b}{3+5b}} V_{\tilde{s}}^{\frac{1}{2}} \left(\frac{3+5b}{2+4b}\right)^{\frac{2+4b}{3+5b}} V_{\tilde{z}}^{\frac{1+2b}{2+2b}},$$

$$-V_{\tilde{s}}^{\frac{3+5b}{2+2b}} - V_{\tilde{z}}^{\frac{3+5b}{4+4b}} \le -\left(\frac{3+5b}{2b}\right)^{\frac{2b}{3+5b}} V_{\tilde{s}}^{\frac{b}{1+b}} \left(\frac{3+5b}{3+3b}\right)^{\frac{3+3b}{3+5b}} V_{\tilde{z}}^{\frac{3}{4}},$$

thus if the following inequalities

$$\underline{k} > \max\left\{\lambda_4\gamma_3(\mu),\quad \lambda_5[\gamma_4(\mu)]^{\frac{3+5b}{3+3b}},\quad \lambda_6,\quad \lambda_7\right\},$$

where

$$\lambda_4 = 8n^{\frac{2b}{1+b}}\left(\frac{\sqrt{\overline{m}}}{\sqrt{\underline{m}}}\right)^{\frac{1+3b}{1+b}},$$

$$\lambda_5 = 2^{\frac{3+5b}{3+3b}}\left(\frac{36+36b}{3+5b}\right)\left(\frac{24b}{d_0(3+5b)}\right)^{\frac{2b}{3+3b}} n^{\frac{2b}{1+b}}\left(\frac{\sqrt{2}n}{\sqrt{\underline{m}}\,\underline{\pi}_i^{\frac{b}{2+2b}}}\right)^{\frac{3+5b}{3+3b}}\left(\frac{\sqrt{\overline{m}}}{\sqrt{2}}\right)^{\frac{1+3b}{1+b}},$$

$$\lambda_6 = \left(\frac{12+12b}{3+5b}\right)\left(\frac{24+48b}{d_0(3+5b)}\right)^{\frac{2+4b}{1+b}} n^{\frac{2b}{1+b}}\left(n^{\frac{1}{2+2b}} d_1\left(\frac{\sqrt{2}}{\sqrt{\underline{m}}}\right)\right)^{\frac{3+5b}{1+b}}\left(\frac{\sqrt{\overline{m}}}{\sqrt{2}}\right)^{\frac{1+3b}{1+b}},$$

$$\lambda_7 = \left(\frac{24b}{3+5b}\right)\left(\frac{36+36b}{d_0(3+5b)}\right)^{\frac{3+3b}{2b}} n^{\frac{2b}{1+b}}\left(n^{\frac{1}{4}} d_2\left(\frac{\sqrt{2}}{\sqrt{\underline{m}}}\right)^{\frac{2b}{1+b}}\right)^{\frac{3+5b}{2b}}\left(\frac{\sqrt{\overline{m}}}{\sqrt{2}}\right)^{\frac{1+3b}{1+b}},$$

hold then the function $\dot{V}_1$ is negative definite. Then, we have,

$$\begin{aligned}
\dot{V} &\overset{a.e.}{\leq} -\left(\frac{3}{2}\right)\left(\frac{k}{n^{\frac{2b}{1+b}}}\right)\left(\frac{\sqrt{2}}{\sqrt{\overline{m}}}\right)^{\frac{1+3b}{1+b}} V_{\tilde{s}}^{\frac{3+5b}{2+2b}} - \underline{f}\left(\frac{4}{\overline{m}}\right)V_{\tilde{s}}^{2} - \left(\frac{3}{4}\right)d_0 V_{\tilde{z}}^{\frac{3+5b}{4+4b}} \\
&\quad + 2\left[\alpha(\|\eta(t)\|)\gamma_4(\mu) + \|\Lambda G(q_0)\|\right]\left(\frac{\sqrt{2}}{\sqrt{\underline{m}}}\right)V_{\tilde{s}}^{\frac{3}{2}} \\
&\leq -\pi_4\left(V_{\tilde{s}}^{\frac{3+5b}{2+2b}} + V_{\tilde{z}}^{\frac{3+5b}{4+4b}}\right) - \underline{f}\left(\frac{4}{\overline{m}}\right)V_{\tilde{s}}^{2} + 2\left[\alpha(\|\eta(t)\|)\gamma_4(\mu) + \|\Lambda G(q_0)\|\right]\left(\frac{\sqrt{2}}{\sqrt{\underline{m}}}\right)V_{\tilde{s}}^{\frac{3}{2}} \\
&= -\pi_4\left(\left(V_{\tilde{s}}^{2}\right)^{\frac{3+5b}{4+4b}} + V_{\tilde{z}}^{\frac{3+5b}{4+4b}}\right) - \underline{f}\left(\frac{4}{\overline{m}}\right)V_{\tilde{s}}^{2} + 2\left[\alpha(\|\eta(t)\|)\gamma_4(\mu) + \|\Lambda G(q_0)\|\right]\left(\frac{\sqrt{2}}{\sqrt{\underline{m}}}\right)\left(V_{\tilde{s}}^{2}\right)^{\frac{3}{4}} \\
&\leq -\pi_4\left(V_{\tilde{s}}^{2} + V_{\tilde{z}}\right)^{\frac{3+5b}{4+4b}} + 2\left[\alpha(\|\eta(t)\|)\gamma_4(\mu) + \|\Lambda G(q_0)\|\right]\left(\frac{\sqrt{2}}{\sqrt{\underline{m}}}\right)\left(V_{\tilde{s}}^{2}\right)^{\frac{3}{4}} \\
&= -\pi_4 V^{\frac{3+5b}{4+4b}} + 2\left[\alpha(\|\eta(t)\|)\gamma_4(\mu) + \|\Lambda G(q_0)\|\right]\left(\frac{\sqrt{2}}{\sqrt{\underline{m}}}\right)V^{\frac{3}{4}} \qquad (3 - 15)
\end{aligned}$$

where

$$\pi_4 := \min\left\{\left(\frac{3}{2}\right)\left(\frac{k}{n^{\frac{2b}{1+b}}}\right)\left(\frac{\sqrt{2}}{\sqrt{\underline{m}}}\right)^{\frac{1+3b}{1+b}}, \left(\frac{3}{4}\right)d_0, \right\},$$

Note that lemma 2.2 with $\frac{3+5b}{4+4b} \le 1$ and the bounds of the Lyapunov functions $V = [V_{\tilde{s}}(\tilde{s})]^2 + V_{\tilde{z}}(\tilde{z}_1, \tilde{z}_2) \ge [V_{\tilde{s}}(\tilde{s})]^2$ have been employed in the above inequality.

Consider the case of $b = 1$, recall that the term from remark 3.9,

$$\Lambda G(q_0) = \operatorname{diag}\left(\operatorname{sig}\left(\frac{\tilde{s}_i}{\bar{s}_i}\right)^{\frac{2b}{1+b}} - \operatorname{sig}\left(\frac{\tilde{s}_i}{\bar{s}_i} + 1\right)^{\frac{2b}{1+b}} + 1\right) G(q_0) = \operatorname{diag}(-1+1)G(q_0) = 0,$$

while for the case of $0 < b < 1$, consider the case when the final desired position corresponds to the rest position of the manipulator where $G(q_0) = \mathbf{0}$, or the gravitational torque of the manipulator dynamics is absent (i.e. in space where gravity is absent or in a planar horizontal configuration) where $G(q) = 0$ for $\forall\ q \in \mathbb{R}^n$, the term $\Lambda G(q_0)$ vanishes to zero. From (3 - 15),

$$\dot{V}(\tilde{s}, \tilde{z}_1, \tilde{z}_2) \overset{a.e.}{\le} -\pi_4 V^{\frac{3+5b}{4+4b}} + 2[\alpha(\|\eta(t)\|)\gamma_4(\mu)]\left(\frac{\sqrt{2}}{\sqrt{\underline{m}}}\right)V^{\frac{3}{4}}$$

Thus, for $\|\eta(t)\| \to 0$ as $t \to \infty$ (where the desired trajectory approaches a constant final desired position), then we have $V(\tilde{s}, \tilde{z}_1, \tilde{z}_2) \to 0$ as $t \to \infty$ as well. Obviously, if $\|\eta(t)\| \equiv 0$ (regulation problem), when $b = 1$, the system will converge to the equilibrium exponentially, while for $0 < b < 1$, the system achieves finite-time regulation, in which the states reach the origin in finite time with the estimate of reaching time as,

$$T(\tilde{s}_0, \tilde{z}_{10}, \tilde{z}_{20}) \leq \left(\frac{4+4b}{\pi_4(1-b)} \right) [V(\tilde{s}_0, \tilde{z}_{10}, \tilde{z}_{20})]^{\frac{1-b}{4+4b}}$$

where $(\tilde{s}_0, \tilde{z}_{10}, \tilde{z}_{20})$ are the states of the system when it first enters the region $\Omega_s(\mu) \times \Omega_z(\mu)$. ■

Remark 3.10. The stability analysis presented above is akin to the conventional sliding mode one, in which the sliding variable is forced to zero or to be made as small as possible followed by the desired error dynamics, i.e. sliding manifold, being perturbed by that value of sliding variable. There is a major difference, in which the conventional sliding mode has a first order sliding manifold, and hence there is a reaching phase. While in our proposed controller, due to its integral nature, it has the same properties of an integral sliding mode, i.e. the ability to eliminate or reducing the effect of the reaching phase. In particular, the initial condition of the integrator can always be selected such that initially $s(0) = 0$. From the above analysis, having $s(0) = 0$ implies initially the states are inside $\Omega_s(\mu) \times \Omega_z(c)$ and will remain in it for all future times. Thus, from initial time, the system will behave as per the desired error dynamics under the influence of a bounded perturbations of $\|s\| \leq \mu / \sqrt{m}$.

3.3 Numerical Simulations

In this section, numerical simulations on a two-link robot manipulator were carried out to illustrate the results discussed in this chapter. The setups for each simulation are described. Discussion and analysis of the results are presented accordingly.

3.3.1 Simulation Setup

1) Simulation 1:

A two-link rigid robot manipulator is adopted in the simulation. The dynamics of robot manipulator (3 - 1) with the following parameter values (the dynamic parameters are from [144]):

$$M(q)=\begin{bmatrix}3.511+0.191\cos(q_2) & 0.072+0.096\cos(q_2)\\ 0.072+0.096\cos(q_2) & 0.072\end{bmatrix},$$
$$C(q,\dot{q})=\begin{bmatrix}-0.096\dot{q}_2\sin(q_2) & -0.096(\dot{q}_1+\dot{q}_2)\sin(q_2)\\ 0.096\dot{q}_1\sin(q_2) & 0\end{bmatrix},$$
$$G(q)=\begin{bmatrix}40.888\sin(q_1)+2.079\sin(q_1+q_2)\\ 2.079\sin(q_1+q_2)\end{bmatrix},$$
$$F=\begin{bmatrix}0.764 & 0\\ 0 & 0.328\end{bmatrix},$$

Note that Properties 3.1, 3.2, 3.3 and 3.4 are satisfied. The desired trajectory vector and the additive disturbances Coulomb friction vector was defined as,

$$q_d(t)=\begin{bmatrix}\pi+0.5\sin(t)\\ 0.5\cos(t)\end{bmatrix},\quad D(\dot{q})=\begin{bmatrix}0.7\mathrm{sign}(\dot{q}_1)\\ 0.3\mathrm{sign}(\dot{q}_2)\end{bmatrix},$$

The initial conditions were,

$$q(0)=\begin{bmatrix}0\\ 0\end{bmatrix},\ \dot{q}(0)=\begin{bmatrix}0\\ 0\end{bmatrix},$$
$$q_d(0)=\begin{bmatrix}\pi\\ 0.5\end{bmatrix},\ \dot{q}_d(0)=\begin{bmatrix}0.5\\ 0\end{bmatrix},$$

The control (3 - 2) gains were selected as follows,

$$K=\begin{bmatrix}150 & 0\\ 0 & 15\end{bmatrix},\ K_1=\begin{bmatrix}18 & 0\\ 0 & 18\end{bmatrix},\ K_2=\begin{bmatrix}11 & 0\\ 0 & 11\end{bmatrix}, b=0.6$$

where the simulations were performed for each value of $a=\{0.7, 0.8, 0.9\}$ to examine its effect on the maximal position error. The initial conditions of the vector σ were selected as

$$\sigma(0)=-e_2(0)=\dot{q}_d(0)-\dot{q}(0)=\begin{bmatrix}0.5\\ 0\end{bmatrix}-\begin{bmatrix}0\\ 0\end{bmatrix}=\begin{bmatrix}0.5\\ 0\end{bmatrix}$$

The $\sigma(0)$ is selected as such so that the state $s(0)$ is zero initially, i.e. $s(0) = e_2(0) + \sigma(0)$. Hence, by theorem 3.1, the state s will stay inside the region $\Omega_s(\mu)$ as per remark 3.10.

2) Simulation 2:

The setup of Simulation 2 is exactly the same as that of Simulation 1. In this simulation, the value of the parameter, a was fixed at $a = 0.9$, while the simulations were performed for each control gains of K

$$K=\left\{\begin{bmatrix}150 & 0\\ 0 & 15\end{bmatrix},\begin{bmatrix}225 & 0\\ 0 & 22.5\end{bmatrix},\begin{bmatrix}300 & 0\\ 0 & 30\end{bmatrix}\right\},$$

for examining its effect on the maximal position error.

3.3.2 Results and Discussions

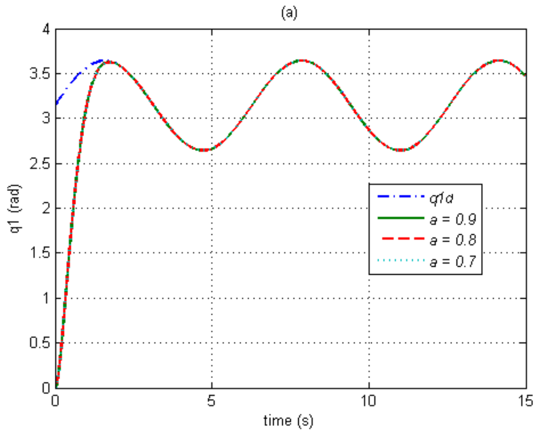
For better visualization of the plots, some figures are shown in two windows; each with different time intervals.

1) Simulation 1:

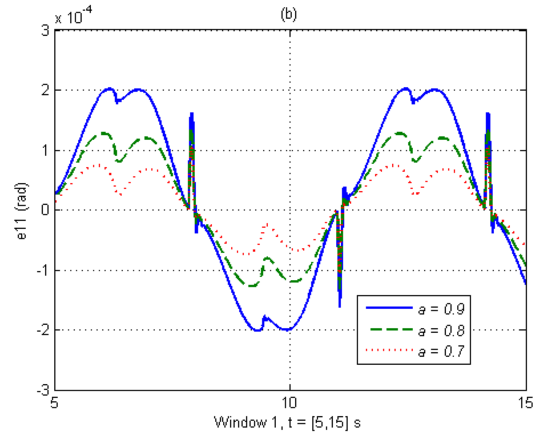
Figure 3.1 shows the tracking errors under control (3 - 2) with three different values of $a = \{0.7, 0.8, 0.9\}$. Particularly, the smaller the value of *a*, the smaller the ultimate bound on the position errors. Indeed, recall from the stability analysis (3 - 11), for c_3 to have a positive value,

$$\mu > \sqrt{m}\left(\frac{\lambda_{\max}(\gamma_1(c))n^{a+0.5}}{\lambda_{\min}(K)}\right)^{\frac{1}{a}}.$$

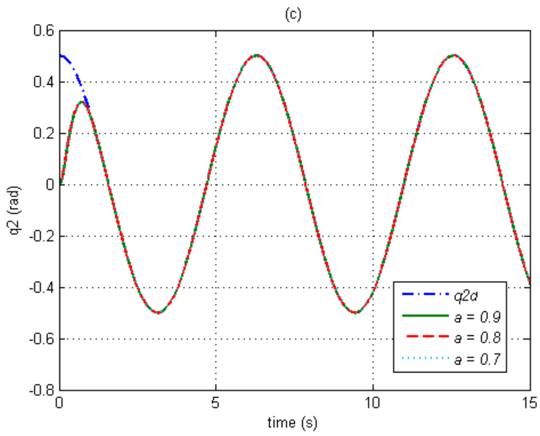
Thus, by satisfying the conditions of Theorem 3.1, the term within the bracket is less than unity, while for $0 < a < 1$, the exponent $1/a$ is greater than unity, which implies the ultimate bound on *s* can be reduced significantly by lowering *a* while maintaining the same control gain *K*. As can be seen in figure 3.2, reducing the value of *a* does not alter the control effort by much but the precision gained are more significant in comparison.



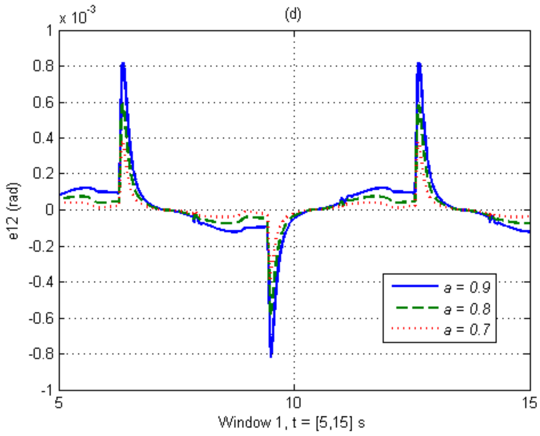
(a) Trajectory, q_1 for time t = [0, 15] s.



(b) Tracking error, e_{11} for time t = [5, 15] s.

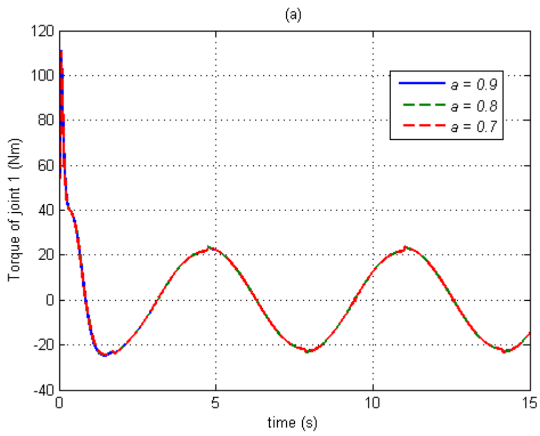


(c) Trajectory, q_2 for time t = [0,15] s.

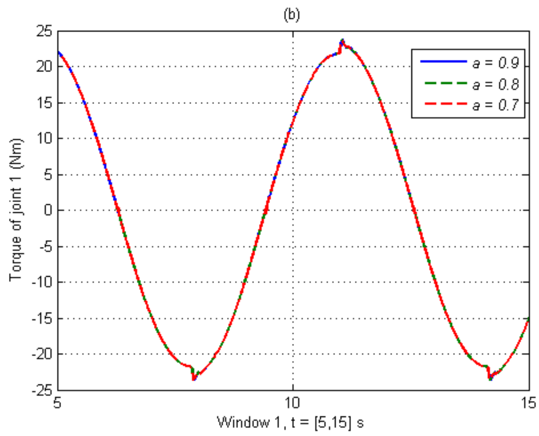


(d) Tracking error, e_{12} for time t = [5, 15] s.

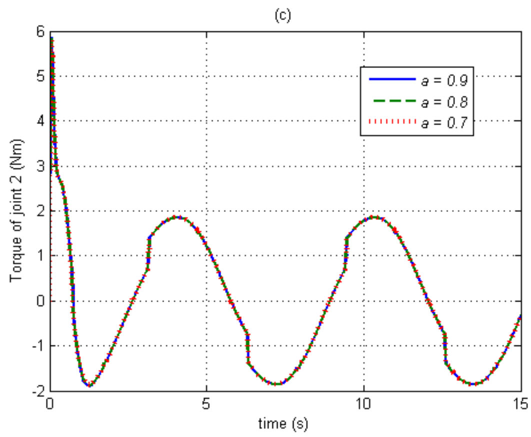
Figure 3.1 Simulation 1. Tracking errors using control (3 - 2) for three different values of $a = \{0.7, 0.8, 0.9\}$.



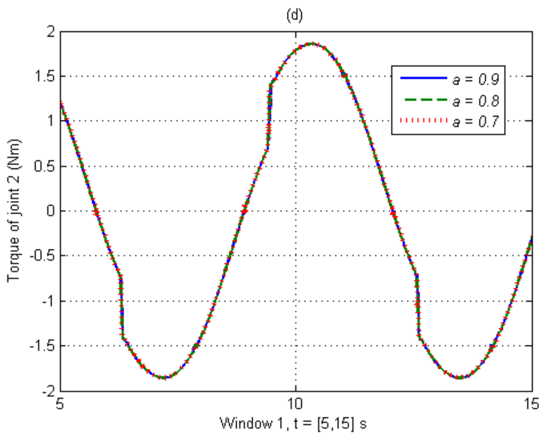
(a) Torque, τ_1 for t = [0, 15] s.



(b) Torque, τ_1 for t = [5, 15] s.

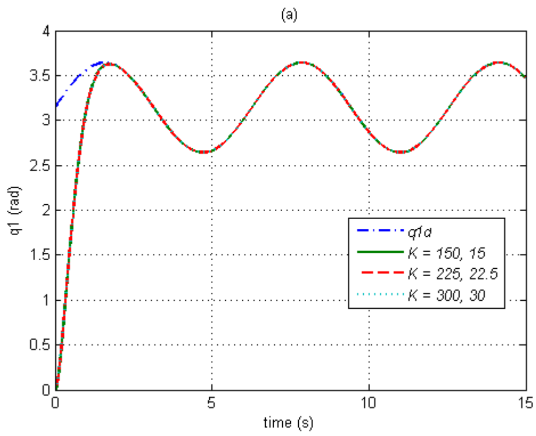


(c) Torque, τ_2 for t = [0, 15] s.

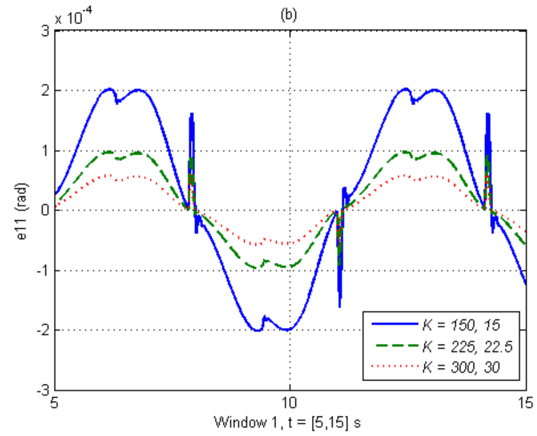


(d) Torque, τ_2 for t = [5, 15] s.

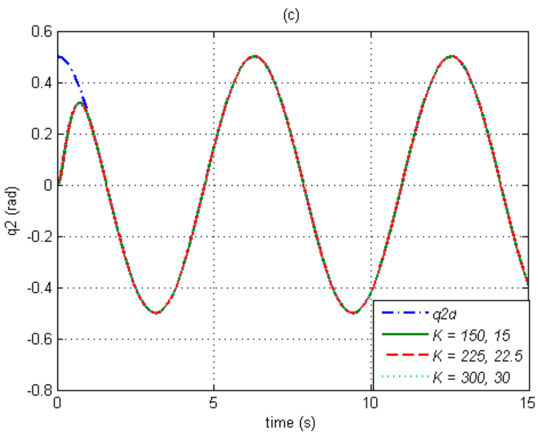
Figure 3.2 Simulation 1. Control input of joint 1 and joint 2.



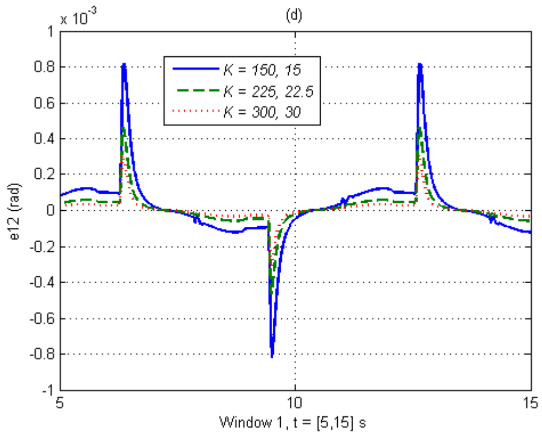
(a) Trajectory, q_1 for time t = [0, 15] s.



(b) Tracking error, e_{11} for time t = [5, 15] s.

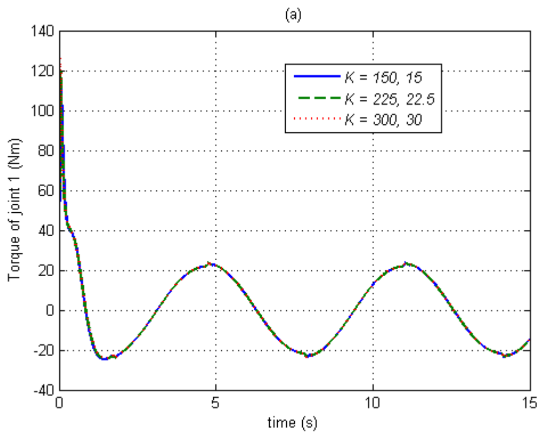


(c) Trajectory, q_2 for time t = [0,15] s.

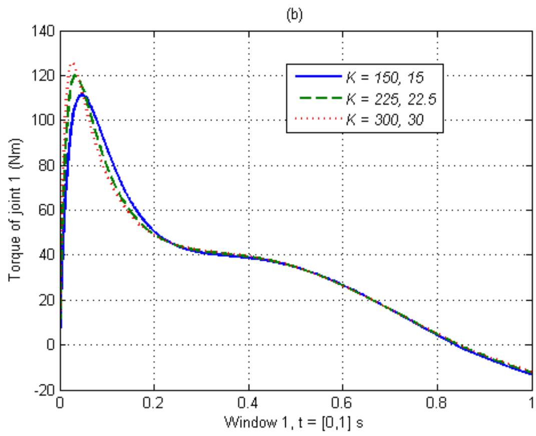


(d) Tracking error, e_{12} for time t = [5, 15] s.

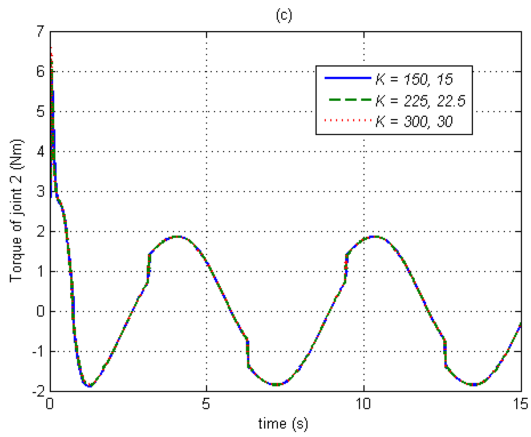
Figure 3.3 Simulation 2. Tracking errors using control (3 - 2) with a = 0.9 with three different values of control gains, K.



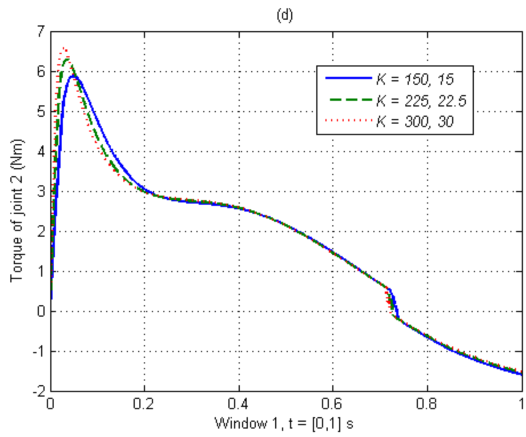
(a) Torque, τ_1 for $t = [0, 15]$ s.



(b) Torque, τ_1 for $t = [0, 1]$ s.



(c) Torque, τ_2 for $t = [0, 15]$ s.



(d) Torque, τ_2 for $t = [0, 1]$ s.

Figure 3.4 Simulation 2. Control input of joint 1 and joint 2.

2) Simulation 2:

Figure 3.3 shows the tracking errors under control (3 - 2) with a fixed at the value of 0.9 while changing the gain K. In essence, larger value of K results in smaller ultimate bound on the position error. Following (3 - 11), the ultimate bound μ indeed is smaller for larger K but not as dramatic as lowering the value *a* as shown in simulation 1. It is worth mentioning that in figure 3.4, the plots with larger control gain *K* exhibit larger slope initially. This high rate of change is not desirable as it may excite unmodelled dynamics in the system.

3) Comparative Study:

Comparative results are difficult to obtain because any comparative result can be dangerously biased. Besides, it is difficult to quantitatively compare controllers that are structurally different. Therefore, the comparative discussions presented here will be limited to PID controllers [90], [99], [103], [104], and [105] because the controller proposed here have a PID structure when *a* = b = 1 (see remark 3.6). Particularly, the discussions will focus on the analysis aspects of the controller, which governs the gains selection through the corresponding stability analysis.

1. In [90] the PID control is analysed to be a combination of feedback linearization term plus uncertainty disturbance estimation term. The structure allows injection of desired error dynamics (similar to the proposed controller) and an estimation term to cancel the effect of uncertainty. However, the existence of the estimation term is not shown and it is mentioned that the compensation does not exist for systems that have discontinuous disturbances and uncertainties.

2. Similar to [90], in [99] the PID control is analysed into a modelling error estimator and a desired error dynamics function. However, since the analysis requires taking the time derivative of the robot manipulator dynamics, the stability results is not applicable for discontinuous disturbances and uncertainties. In addition, the analysis

showed that it is necessary to have an accurate estimate, $\bar{M}$ of the inertia matrix of the robot manipulator (requires $\|I - M^{-1}\bar{M}\| < 1$) to ensure negative definiteness of the time derivative of Lyapunov function there. However, as remarked by the author, even when such condition is violated, stability is still observed in their simulations. Hence, it clearly shows the conservativeness of the results notwithstanding the aforementioned shortcomings.

3. In [103] a strict Lyapunov function is given that allows the selection of gains based on region of attraction and ultimate bound. Also, the analysis does not require taking the time derivative of the robot dynamics. However, it lacks the simple structure of desired error dynamics selection and modelling error compensation found in point 1 and 2. Particularly, although the author claimed the analysis is applicable for tracking control, the proof shown for regulation is not directly clear on how it can be extended for the tracking purposes. In fact, the analysis requires the time derivative of the gravity vector $G(q_d)$ in its auxiliary variable, which is zero for regulation problem(since q_d is a constant vector in regulation problem) but for tracking q_d is no longer a constant(it requires the time derivative of gravity vector) .

4. For regulation, in [104] and [105] , the procedure for PID tuning is extracted from the stability analysis which allows gain selection for stability to specified arbitrary domain. Nevertheless, its extension to tracking and the abilities to inject desired error dynamics are unclear.

Note that the stability analyses mentioned above are all for the same PID control structure for the control of robotic manipulators. The present results of this chapter are able to overcome the shortcomings mentioned. Particularly, recall from remark 3.6, for the special case of $a = b = 1$, the proposed controller (3 - 2) becomes the well-known PID control, indeed when written in original coordinates

$$\tau = -K_d e_2 - K_p e_1 - K_i \int_{t_0}^{t} e_1(\varsigma)d\varsigma + K_i e_1(t_0),$$

where $K_d = K,\ K_p = KK_2$, and $K_i = KK_1$. For the injection of desired error dynamics, it is done through the selection of gain K_1 and K_2. As per Theorem 3.1, after selecting K_1 and K_2, control gain K is selected to ensure stability and ultimate boundedness. Since the vector σ can be selected to have an initial value such that $s(0)$ is zero, from the stability analysis, desired error dynamics, under bounded s, will begin without reaching phase. Thus, the gain selections method is similar to that of point 1 and 2; albeit without requiring an estimate of inertia matrix and time derivative of robot dynamics (As such discontinuous disturbances are allowed in the analysis through generalized Lyapunov theorem). Additionally, the results pertains to semiglobal trajectory tracking(regulation is a special case of tracking), in which case the region of attraction and ultimate bound can be ascertained through the strict Lyapunov functions in section 3.2.1. namely the regions $\Omega(c)$ (which can be arbitrarily enlarged by increasing gain K)and $\Omega(\mu)$. Besides, the stability analysis presented here allows the extension of the results to allow non-Lipschitz desired error dynamics through strict Lyapunov functions of Chapter 2, which allows finite time regulation as per Theorem 3.2.

3.4 Summary

In this chapter, the trajectory tracking control of robot manipulator is developed. In particular, semiglobal practical stability is assured where the ultimate bound of the states can be made arbitrarily small and the region of attraction arbitrarily large by tuning a single parameter. Also, the stability analysis permits the disturbances to have discontinuity, i.e. hence controller is robust to disturbances such as Coulomb friction. Of interest is the ability of the proposed controller in generalizing the well-known PID control. From the stability analysis, the PID gains selection is transform into the selection of desired error dynamics and the selection of acceptable precision of error. For the special case of position regulation problem, sufficient conditions on the gains are obtained to ensure either finite-time or exponential convergence of the system towards the regulation point. In addition, due to integral nature of the controller, it is possible for the system to behave as per the desired error dynamics from the onset of control even in the presence of disturbances.

Chapter 4: ROBOT MANIPULATOR CONTROL: OUTPUT FEEDBACK APPROACH

This chapter considers the tracking control design of robot manipulator when joint velocity measurement is not available. Building on previous results in Chapters 2 and 3, an observer inspired from the super-twisting based family of algorithms is proposed to achieve semiglobal practical stability in the presence of unknown robotic model parameters and additive bounded disturbances. By adding a linear velocity observation error correction term into the proposed observer, the observer gains for the non-Lipschitz terms can be reduced without affecting the region of attractions. For the special case of regulation problem, the controller-observer structure is able to achieve finite-time or exponential convergence depending on parameter of the structure.

4.1 Observer Dynamics

Motivated by the results of section 2.4 and so-called "dirty-derivative" filter found in the literature, an observer dynamics is presented here. Essentially, it comprises a linear combination of the super-twisting based algorithm of Chapter 2 with a linear damping term. Its stability analysis that supports the main results of this chapter is described in this section.

4.1.1 System Description

Consider the super-twisting based dynamics,

$$\begin{aligned} \dot{\tilde{e}}_1 &= -L_1 \mathrm{sig}(\tilde{e}_1)^p + \tilde{e}_2 \quad , \\ \dot{\tilde{e}}_2 &= -L_2 \mathrm{sig}(\tilde{e}_1)^{2p-1} - L_3 \tilde{e}_2 + d \end{aligned}$$

where $\dot{\tilde{e}}_1, \dot{\tilde{e}}_2 \in \mathbb{R}^n$, are the vector state variables, L_1, L_2, $L_3 \in \mathbb{R}^{n \times n}$ are positive definite diagonal matrices, $0.5 \le p \le 1$, and d $\mathbb{R}^n$ is time-varying and/or nonlinear vector of bounded uncertainty and disturbances.

Consider element wise, $\forall\ i = 1,...,n$,

$$\begin{aligned} \dot{\tilde{e}}_{1i} &= -l_{1i}\mathrm{sig}(\tilde{e}_{1i})^p + \tilde{e}_{2i} \ , \\ \dot{\tilde{e}}_{2i} &= -l_{2i}\mathrm{sig}(\tilde{e}_{1i})^{2p-1} - l_{3i}\tilde{e}_{2i} + d_i \end{aligned} \tag{4 - 1}$$

with $|d_i| := \sup\{|\varepsilon_i| : \varepsilon_i \in \boldsymbol{K}[d_i]\}$.

4.1.2 Stability Analysis

Consider the following Lyapunov function:

$$V_{\tilde{e}i}(\tilde{e}_{1i}, \tilde{e}_{2i}) = \frac{1}{2}\tilde{e}_{2i}{}^2 + \frac{1}{2p}|\tilde{e}_{1i}|^{2p}$$

Note that the above function is locally Lipschitz and regular. It is differentiable everywhere for $p > 0.5$, and not differentiable on $\tilde{e}_{1i} = 0$ for $p = 0.5$.

It can be bounded by

$$\min\left\{\frac{1}{2}, \frac{1}{2p}\right\}\left(|\tilde{e}_{2i}|^2 + |\tilde{e}_{1i}|^{2p}\right) \le V_{\tilde{e}i} \le \max\left\{\frac{1}{2}, \frac{1}{2p}\right\}\left(|\tilde{e}_{2i}|^2 + |\tilde{e}_{1i}|^{2p}\right),$$

thus, $V_{\tilde{e}i}$ is positive definite and radially unbounded. Since (4 - 1) is a differential equation that has discontinuous right-hand side, i.e. when $p = 0.5$ and since no continuity assumption is made on d_i, its solutions are understood in the sense of Filippov (see definition 2.1).

According to lemma 2.4, the time derivative of non-smooth Lyapunov function exists almost everywhere

$$\frac{d}{dt}V_{\tilde{e}i}(\tilde{e}_{1i},\tilde{e}_{2i}) \overset{a.e.}{\in} \dot{\tilde{V}}_{\tilde{e}i} := \bigcap_{\xi\in\partial V_{\tilde{e}i}(\tilde{e}_{1i},\tilde{e}_{2i})} \xi^{\mathrm{T}} \boldsymbol{K}\begin{bmatrix}\dot{\tilde{e}}_{1i}\\ \dot{\tilde{e}}_{2i}\end{bmatrix}$$

Note that for $0.5 < p \le 1$, $V_{\tilde{e}i}$ is continuously differentiable:

$$\dot{\tilde{V}}_{\tilde{e}i} = \bigcap_{\xi\in\partial V_{\tilde{e}i}(\tilde{e}_{1i},\tilde{e}_{2i})} \xi^{\mathrm{T}} \boldsymbol{K}\begin{bmatrix}\dot{\tilde{e}}_{1i}\\ \dot{\tilde{e}}_{2i}\end{bmatrix} = \nabla V_{\tilde{e}i}{}^{T} K\begin{bmatrix}\dot{\tilde{e}}_{1i}\\ \dot{\tilde{e}}_{2i}\end{bmatrix}$$
$$\subset -l_{3i}|\tilde{e}_{2i}|^{2} - l_{1i}|\tilde{e}_{1i}|^{3p-1} + (1-l_{2i})|\tilde{e}_{1i}|^{2p-1}|\tilde{e}_{2i}|\mathrm{sign}(\tilde{e}_{1i}\tilde{e}_{2i}) + \boldsymbol{K}[d_i]\tilde{e}_{2i}$$

For $p = 0.5$, $V_{\tilde{e}i}$ is not differentiable on $\tilde{e}_{1i} = 0$:

$$\begin{bmatrix}\dot{\tilde{e}}_{1i}\\ \dot{\tilde{e}}_{2i}\end{bmatrix} \in \begin{bmatrix}-l_{1i}\mathrm{sig}(\tilde{e}_{1i})^{\frac{1}{2}} + \tilde{e}_{2i}\\ -l_{2i}\boldsymbol{K}[\mathrm{sign}(\tilde{e}_{1i})] - l_{3i}\tilde{e}_{2i} + \boldsymbol{K}[d_i]\end{bmatrix}, \quad \partial V_{\tilde{e}i}(\tilde{e}_{1i},\tilde{e}_{2i}) = \begin{bmatrix}\mathrm{SGN}(\tilde{e}_{1i})\\ \tilde{e}_{2i}\end{bmatrix},$$

for $\tilde{e}_{1i} \neq 0$, $\forall\, \tilde{e}_{2i} \in \mathbb{R}$:

$$\dot{\tilde{V}}_{\tilde{e}i} = -l_{3i}|\tilde{e}_{2i}|^{2} - l_{1i}|\tilde{e}_{1i}|^{3p-1} + (1-l_{2i})|\tilde{e}_{1i}|^{2p-1}|\tilde{e}_{2i}|\mathrm{sign}(\tilde{e}_{1i}\tilde{e}_{2i}) + \boldsymbol{K}[d_i]\tilde{e}$$

for $\tilde{e}_{1i} = 0$, $\forall\, \tilde{e}_{2i} \in \mathbb{R}$:

Let $[\xi_2, \tilde{e}_{2i}]^T$ with $\xi_2 \in [-1, 1]$ be an arbitrary element of $\partial V_{\tilde{e}i}$, then

$$\xi^{\mathrm{T}}\boldsymbol{K}\begin{bmatrix}\dot{\tilde{e}}_{1i}\\ \dot{\tilde{e}}_{2i}\end{bmatrix}=[\xi_2,\,\tilde{e}_{2i}]\begin{bmatrix}\tilde{e}_{2i}\\ -l_{2i}[-1,1]-l_{3i}\tilde{e}_{2i}+\boldsymbol{K}[d_i]\end{bmatrix}=([\xi_2-l_{2i},\xi_2+l_{2i}])\tilde{e}_{2i}-l_{3i}|\tilde{e}_{2i}|^2+\boldsymbol{K}[d_i]\tilde{e}_{2i}$$

implies

$$\dot{\tilde{V}}_{\tilde{e}i}=\bigcap_{\xi_2\in[-1,1]}([\xi_2-l_{2i},\xi_2+l_{2i}])\tilde{e}_{2i}-l_{3i}|\tilde{e}_{2i}|^2+\boldsymbol{K}[d_i]\tilde{e}_{2i}\leq|1-l_{2i}||\tilde{e}_{2i}|-l_{3i}|\tilde{e}_{2i}|^2+|d_i||\tilde{e}_{2i}|$$

Note that $\bigcap_{\xi_2\in[-1,1]}([\xi_2-l_{2i},\xi_2+l_{2i}])=\begin{cases}0, & \text{for } l_{2i}=1,\\ [1-l_{2i},l_{2i}-1], & \text{for } l_{2i}>1,\\ \emptyset, & \text{for } 0<l_{2i}<1\end{cases}$

where the convention for the empty set of $\max\dot{\tilde{V}}=-\infty$, if $\dot{\tilde{V}}=\emptyset$ is employed.(see [74])

Thus, for all $(\tilde{e}_{1i},\tilde{e}_{2i})\in\mathbb{R}^n$ and $\forall\ 0.5\leq p\leq 1$, we have

$$\dot{V}_{\tilde{e}i}(\tilde{e}_{1i},\tilde{e}_{2i})\overset{a.e.}{\leq}-l_{3i}|\tilde{e}_{2i}|^2-l_{1i}|\tilde{e}_{1i}|^{3p-1}+|1-l_{2i}||\tilde{e}_{1i}|^{2p-1}|\tilde{e}_{2i}|+|d_i||\tilde{e}_{2i}| \qquad (4\text{ - }2)$$

For the ease of analysis, the state space is divided into the following three regions:

$$\psi_1(|\zeta_i|,l_{3i})=\{(\tilde{e}_{1i},\tilde{e}_{2i})\in\mathbb{R}^2:\tilde{e}_{1i}\in\mathbb{R},|\tilde{e}_{2i}|\geq\beta_2(|\zeta_i|,l_{3i})\},$$

$$\psi_2(|\zeta_i|,l_{3i})=\{(\tilde{e}_{1i},\tilde{e}_{2i})\in\mathbb{R}^2:|\tilde{e}_{1i}|\geq\beta_1(|\zeta_i|,l_{3i}),|\tilde{e}_{2i}|\leq\beta_2(|\zeta_i|,l_{3i})\},$$

$$\psi_3(|\zeta_i|,l_{3i})=\{(\tilde{e}_{1i},\tilde{e}_{2i})\in\mathbb{R}^2:|\tilde{e}_{1i}|\leq\beta_1(|\zeta_i|,l_{3i}),|\tilde{e}_{2i}|\leq\beta_2(|\zeta_i|,l_{3i})\}$$

where $|\zeta_i|:=\max\{|1-l_{2i}|,|d_i|\}$ and the functions β_1 and β_2 are of class-$\mathcal{KL}$ (see [145]) defined by:

$$\beta_1\left(|\zeta_i|, l_{3i}\right) := \max\left\{\begin{array}{c} \left(\dfrac{3\beta_2\left(|\zeta_i|, l_{3i}\right)|\zeta_i|}{l_{1i}}\right)^{\frac{1}{p}}, \\ \left(\dfrac{3\beta_2\left(|\zeta_i|, l_{3i}\right)|\zeta_i|}{l_{1i}}\right)^{\frac{1}{3p-1}} \end{array}\right\},$$

$$\beta_2\left(|\zeta_i|, l_{3i}\right) := \begin{cases} \dfrac{3|\zeta_i|}{l_{3i}}, & \text{for } p = 1, \\ \max\left\{\dfrac{3|\zeta_i|}{l_{3i}},\ \lambda_1 \dfrac{|\zeta_i|^{\frac{3p-1}{1-p}}}{l_{1i}^{\frac{2p-1}{1-p}} l_{3i}^{\frac{p}{1-p}}}\right\}, & \text{for } 0.5 < p < 1, \\ \dfrac{4\left(|\zeta_i|\right)}{l_{3i}}, & \text{for } p = 0, \end{cases}$$

with $l_{1i} l_{3i} > \left(\dfrac{4p-2}{3p-1}\right)\left(\dfrac{3p}{3p-1}\right)\left|1 - l_{2i}\right|^2$, for $p = 1$, and $\lambda_1 = \left(\dfrac{4p-2}{3p-1}\right)^{\frac{2p-1}{1-p}} \left(\dfrac{3p}{3p-1}\right)^{\frac{p}{1-p}}$

In the region: $\psi_1\left(|\zeta_i|, l_{3i}\right) = \left\{\left(\tilde{e}_{1i}, \tilde{e}_{2i}\right) \in \mathbb{R}^2 : \tilde{e}_{1i} \in \mathbb{R}, |\tilde{e}_{2i}| \geq \beta_2\left(|\zeta_i|, l_{3i}\right)\right\}$

Applying lemma 2.1 (for $0.5 < p \leq 1$):

$$-\frac{l_{1i}}{2}|\tilde{e}_{1i}|^{3p-1} - \frac{l_{3i}}{3}|\tilde{e}_{2i}|^2 \leq -\left(\left(\frac{3p-1}{2p-1}\right)\frac{l_{1i}}{2}\right)^{\frac{2p-1}{3p-1}} |\tilde{e}_{1i}|^{2p-1} \left(\left(\frac{3p-1}{p}\right)\frac{l_{3i}}{3}\right)^{\frac{p}{3p-1}} |\tilde{e}_{2i}|^{\frac{2p}{3p-1}}$$

From (4 - 2), we have

$$\dot{V}_{\tilde{e}i}\left(\tilde{e}_{1i},\tilde{e}_{2i}\right)\overset{a.e.}{\leq}-\frac{l_{1i}}{2}\left|\tilde{e}_{1i}\right|^{3p-1}-\frac{l_{3i}}{3}\left|\tilde{e}_{2i}\right|^{2}-\left|\tilde{e}_{2i}\right|\left(\frac{l_{3i}}{3}\left|\tilde{e}_{2i}\right|-\left|d_{i}\right|\right)$$
$$-\left|\tilde{e}_{1i}\right|^{2p-1}\left|\tilde{e}_{2i}\right|\left(\left(\left(\frac{3p-1}{2p-1}\right)\frac{l_{1i}}{2}\right)^{\frac{2p-1}{3p-1}}\left(\left(\frac{3p-1}{p}\right)\frac{l_{3i}}{3}\right)^{\frac{p}{3p-1}}\left|\tilde{e}_{2i}\right|^{\frac{1-p}{3p-1}}-\left|1-l_{2i}\right|\right)$$

While for $p = 0.5$, from (4 - 2)

$$\dot{V}_{\tilde{e}i}\left(\tilde{e}_{1i},\tilde{e}_{2i}\right)\overset{a.e.}{\leq}-\frac{l_{3i}}{2}\left|\tilde{e}_{2i}\right|^{2}-l_{1i}\left|\tilde{e}_{1i}\right|^{0.5}-\left|\tilde{e}_{2i}\right|\left(\frac{l_{3i}}{2}\left|\tilde{e}_{2i}\right|-\left(\left|1-l_{2i}\right|+\left|d_{i}\right|\right)\right)$$

If the following inequalities hold, for different cases of p:

for $p = 1$:

$$\left|\tilde{e}_{2i}\right|\geq\frac{3\left|d_{i}\right|}{l_{3i}},\ \text{with}\ \ l_{1i}l_{3i}\geq\left(\frac{4p-2}{3p-1}\right)\left(\frac{3p}{3p-1}\right)\left|1-l_{2i}\right|^{2},\quad\text{for}\quad p=1,$$

or

for $0.5 < p < 1$:

$$\left|\tilde{e}_{2i}\right|\geq\max\left\{\frac{3\left|d_{i}\right|}{l_{3i}},\ \ \lambda_{1}\frac{\left|1-l_{2i}\right|^{\frac{3p-1}{1-p}}}{l_{1i}^{\frac{2p-1}{1-p}}\,l_{3i}^{\frac{p}{1-p}}}\right\}\ \text{where}\ \lambda_{1}=\left(\frac{4p-2}{3p-1}\right)^{\frac{2p-1}{1-p}}\left(\frac{3p}{3p-1}\right)^{\frac{p}{1-p}}$$

or for $p = 0.5$:

$$\left|\tilde{e}_{2i}\right|\geq\frac{2\left(\left|1-l_{2i}\right|+\left|d_{i}\right|\right)}{l_{3i}}$$

then $\dot{V}_{\tilde{e}i}\left(\tilde{e}_{1i},\tilde{e}_{2i}\right)\overset{a.e.}{<}0$, which is sufficiently satisfied by the states in this region.

Next, consider the region: $\psi_2\left(|\zeta_i|, l_{3i}\right) = \left\{\left(\tilde{e}_{1i}, \tilde{e}_{2i}\right) \in \mathbb{R}^2 : |\tilde{e}_{1i}| \geq \beta_1\left(|\zeta_i|, l_{3i}\right), |\tilde{e}_{2i}| \leq \beta_2\left(|\zeta_i|, l_{3i}\right)\right\}$

Thus, from (4 - 2),

$$
\begin{aligned}
\dot{V}_{\tilde{e}i}\left(\tilde{e}_{1i}, \tilde{e}_{2i}\right) &\overset{a.e.}{\leq} -l_{3i}|\tilde{e}_{2i}|^2 - l_{1i}|\tilde{e}_{1i}|^{3p-1} + |1 - l_{2i}||\tilde{e}_{1i}|^{2p-1}\beta_2\left(|\zeta_i|, l_{3i}\right) + |d_i|\beta_2\left(|\zeta_i|, l_{3i}\right) \\
&= -l_{3i}|\tilde{e}_{2i}|^2 - \frac{l_{1i}}{3}|\tilde{e}_{1i}|^{3p-1} - |\tilde{e}_{1i}|^{2p-1}\left(\frac{l_{1i}}{3}|\tilde{e}_{1i}|^p - |1 - l_{2i}|\beta_2\left(|\zeta_i|, l_{3i}\right)\right) \\
&\quad - \left(\frac{l_{1i}}{3}|\tilde{e}_{1i}|^{3p-1} - |d_i|\beta_2\left(|\zeta_i|, l_{3i}\right)\right) \\
&< 0
\end{aligned}
$$

if $|\tilde{e}_{1i}| \geq \max\left\{\left(\frac{3\beta_2\left(|\zeta_i|, l_{3i}\right)|1 - l_{2i}|}{l_{1i}}\right)^{\frac{1}{p}}, \left(\frac{3\beta_2\left(|\zeta_i|, l_{3i}\right)|d_i|}{l_{1i}}\right)^{\frac{1}{3p-1}}\right\}$

which is sufficiently satisfied for the states in this region.

Next consider the compact set:

$$\psi_3\left(|\zeta_i|, l_{3i}\right) = \left\{\left(\tilde{e}_{1i}, \tilde{e}_{2i}\right) \in \mathbb{R}^2 : |\tilde{e}_{1i}| \leq \beta_1\left(|\zeta_i|, l_{3i}\right), |\tilde{e}_{2i}| \leq \beta_2\left(|\zeta_i|, l_{3i}\right)\right\}$$

Note that, $\dot{V}_{\tilde{e}i}\left(\tilde{e}_{1i}, \tilde{e}_{2i}\right) \overset{a.e.}{<} 0$ for $\left(\tilde{e}_{1i}, \tilde{e}_{2i}\right) \notin \psi_3\left(|\zeta_i|, l_{3i}\right)$.

Next, define a Lyapunov level set $\Sigma_{\tilde{e}i}\left(|\zeta_i|, l_{3i}\right) = \left\{\left(\tilde{e}_{1i}, \tilde{e}_{2i}\right) \in \mathbb{R}^2 : V_{\tilde{e}i} \leq \rho_{3i}\left(|\zeta_i|, l_{3i}\right)\right\}$, where the class $\mathcal{KL}$ function ρ_{3i} is defined as,

$$\rho_{3i}(|\zeta_i|, l_{3i}) = \max_{(\tilde{e}_{1i}, \tilde{e}_{2i}) \in \mathrm{bd}\ \psi_3(|\zeta_i|, l_{3i})} V_{\tilde{e}i}(\tilde{e}_{1i}, \tilde{e}_{2i})$$

$$= \max_{(\tilde{e}_{1i}, \tilde{e}_{2i}) \in \mathrm{bd}\ \psi_3(|\zeta_i|, l_{3i})} \left(\frac{1}{2} \tilde{e}_{2i}^{\ 2} + \frac{1}{2p} |\tilde{e}_{1i}|^{2p} \right)$$

$$= \left(\frac{1}{2} \left[\beta_2(|\zeta_i|, l_{3i}) \right]^2 + \frac{1}{2p} \left[\beta_1(|\zeta_i|, l_{3i}) \right]^{2p} \right)$$

which exists since the boundary of the set is compact and $V_{\tilde{e}i}$ is continuous. Then we observe that $\psi_3(|\zeta_i|, l_{3i}) \subset \Sigma_{\tilde{e}i}(|\zeta_i|, l_{3i})$. As a result, we have

$$\frac{d}{dt} V_{\tilde{e}i}(\tilde{e}_{1i}, \tilde{e}_{2i}) \overset{a.e.}{\in} \dot{\tilde{V}}_{\tilde{e}i}(\tilde{e}_{1i}, \tilde{e}_{2i}) < 0 \ \ for \ V_{\tilde{e}i} \geq \rho_{3i}(|\zeta_i|, l_{3i})$$

which implies that each of the trajectories for the i-th planar system will enter their respective compact level set $\Sigma_{\tilde{e}i}(|\zeta_i|, l_{3i})$ in finite time and stay in it once entered.

Remark 4.1. Note that for any given l_{1i} , $l_{2i} > 0$, and a bounded $|\zeta_i|$, $\rho_{3i}(|\zeta_i|, l_{3i})$ can be made arbitrarily small by increasing $l_{3i} > 0$. Hence, it can be observed that for $0.5 \leq p \leq 1$, the increase of observer gain L_3 will result in a smaller upper bound on the observation errors.

Remark 4.2. The finite time property of the observer is not shown here yet. The homogeneity of $0.5 \leq p < 1$ will be utilised for showing the finite time capability of the observer in the following sections of this chapter, as the controller is developed.

4.2 Manipulator Dynamics

Recall the dynamics of an n-joint serial rigid robotic manipulators as in (3 - 1) of Chapter 3

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + F\dot{q} + G(q) + D(q,\dot{q},t) = \tau \tag{4 - 3}$$

where $q \in \mathbb{R}^n$ is the vector of generalized joint coordinates, $M(q) \in \mathbb{R}^{n\times n}$ is the inertia matrix, and $C(q, \dot{q})\ \dot{q}$, $F\dot{q}$, $G(q)$, $D(q, \dot{q}, t)$, $\tau \in \mathbb{R}^n$ are, respectively, the vectors of Coriolis and centrifugal, viscous friction, gravity, disturbances, and input generalized forces, with F being a constant, positive definite , diagonal (viscous friction coefficient) matrix and $D(q, \dot{q}, t)$ being a locally bounded disturbances. The robot manipulator satisfies the same properties as that in Chapter 3 and will not be repeated here. Also, similar assumptions on the bound of $D(q, \dot{q}, t)$ and twice differentiability of the desired trajectory $q_d(t)$ apply here. The only exception is that only joint positions measurement is available. No velocity information from the system is assumed in this chapter. As such, the following development will follow an output feedback approach.

4.2.1 Control Development

The proposed controller has the following form,

$$\tau = -K\,\mathrm{sig}(\hat{s})^a, \tag{4 - 4}$$

where K is a positive definite diagonal matrix, i.e. $K = \mathrm{diag}\{k_i\}_{i=1}^n$, with $k_i > 0, \forall i = 1,\ldots,n$, $a \geq 0$ constant, and $\hat{s} \in \mathbb{R}^n$ is the velocity-estimate-based desired error dynamics defined as $\hat{s} = \hat{e}_2 + \sigma$, with

$$\dot{\sigma} = K_2 \mathrm{sig}(\hat{e}_2)^{\frac{2b}{1+b}} + K_1 \mathrm{sig}(e_1)^b \tag{4 - 5}$$

where K_1 and K_2 are positive definite diagonal matrices, i.e. $K_1 = \mathrm{diag}\{k_{1i}\}_{i=1}^{n}$, with $k_{1i} > 0$, $K_2 = \mathrm{diag}\{k_{2i}\}_{i=1}^{n}$, with $k_{2i} > 0 \ \ \forall i = 1,\ldots,n$, $b \geq 0$ constant, and $e_1 := q - q_d \in \mathbb{R}^n$, $\hat{e}_2 \in \mathbb{R}^n$ is the output of the observer defined as,

$$\begin{aligned} \dot{\hat{e}}_1 &= -L_1 \mathrm{sig}(\tilde{e}_1)^p + \hat{e}_2 \ , \\ \dot{w} &= -L_2 \mathrm{sig}(\tilde{e}_1)^{2p-1} - L_3 \hat{e}_2 \\ \hat{e}_2 &= w + L_3 e_1 \end{aligned} \tag{4 - 6}$$

where $\hat{e}_1, \hat{e}_2 \in \mathbb{R}^n$, L_1, L_2, and L_3 are positive definite diagonal matrices, i.e. $L_1 = \mathrm{diag}\{l_{1i}\}_{i=1}^{n}$, with $l_{1i} > 0$, $L_2 = \mathrm{diag}\{l_{2i}\}_{i=1}^{n}$, with $l_{2i} > 0$, and $L_3 = \mathrm{diag}\{l_{3i}\}_{i=1}^{n}$, with $l_{3i} > 0$, $\forall i = 1,\ldots,n$, and $0.5 \leq p \leq 1$. Let us define $e_2 = \dot{q} - \dot{q}_d \in \mathbb{R}^n$, $\tilde{e}_1 = \hat{e}_1 - e_1 \in \mathbb{R}^n$, and $\tilde{e}_2 = \hat{e}_2 - e_2 \in \mathbb{R}^n$, then the closed-loop system of (4 - 3), (4 - 4), (4 - 5) and (4 - 6) can be written as

$$\begin{aligned} \dot{\sigma} &= K_2 \mathrm{sig}(\hat{e}_2)^{\frac{2b}{1+b}} + K_1 \mathrm{sig}(e_1)^b \\ \dot{e}_1 &= e_2 \\ \dot{e}_2 &= M^{-1}(q)\tau - M^{-1}(q)\left(C(q,\dot{q})\dot{q} + F\dot{q} + G(q) + D\right) - \ddot{q}_d \\ \dot{\tilde{e}}_1 &= -L_1 \mathrm{sig}(\tilde{e}_1)^p + \tilde{e}_2 \ , \\ \dot{\tilde{e}}_2 &= -L_2 \mathrm{sig}(\tilde{e}_1)^{2p-1} - L_3 \tilde{e}_2 - \left(M^{-1}(q)\tau - M^{-1}(q)\left(C(q,\dot{q})\dot{q} + F\dot{q} + G(q) + D\right) - \ddot{q}_d\right) \end{aligned}$$

To rewrite the closed-loop system in a form convenient for analysis, let us define $z_1 = e_1$, $z_2 = -\sigma$, with $s = e_2 + \sigma = \hat{s} - (\hat{e}_2 - e_2) = \hat{s} - \tilde{e}_2$, and we obtain the following form of closed-loop system,

$$
\begin{aligned}
\dot{z}_1 &= z_2 + s, \\
\dot{z}_2 &= -K_2 \mathrm{sig}(z_2)^{\frac{2b}{1+b}} - K_1 \mathrm{sig}(z_1)^b - K_2\left(\mathrm{sig}(z_2 + s + \tilde{e}_2)^{\frac{2b}{1+b}} - \mathrm{sig}(z_2)^{\frac{2b}{1+b}} \right) \\
\dot{s} &= -M^{-1}(z_1 + q_d) K \mathrm{sig}(s)^a + \Delta_1(\cdot) - M^{-1}(z_1 + q_d) K \left(\mathrm{sig}(s + \tilde{e}_2)^a - \mathrm{sig}(s)^a \right) \\
&\quad + K_2 \left(\mathrm{sig}(z_2 + s + \tilde{e}_2)^{\frac{2b}{1+b}} - \mathrm{sig}(z_2 + s)^{\frac{2b}{1+b}} \right) \\
\dot{\tilde{e}}_1 &= -L_1 \mathrm{sig}(\tilde{e}_1)^p + \tilde{e}_2 \ , \\
\dot{\tilde{e}}_2 &= -L_2 \mathrm{sig}(\tilde{e}_1)^{2p-1} - L_3 \tilde{e}_2 + \Delta_2(\cdot)
\end{aligned}
\tag{4 - 7}
$$

where

$$
\begin{aligned}
\Delta_1(\cdot) = &-M^{-1}(z_1 + q_d) C(z_1 + q_d, z_2 + s + \dot{q}_d)(z_2 + s + \dot{q}_d) - M^{-1}(z_1 + q_d) F(z_2 + s + \dot{q}_d) \\
&- M^{-1}(z_1 + q_d) G(z_1 + q_d) - M^{-1}(z_1 + q_d) D - \ddot{q}_d + K_2 \mathrm{sig}(z_2 + s)^{\frac{2b}{1+b}} + K_1 \mathrm{sig}(z_1)^b,
\end{aligned}
$$

$$
\begin{aligned}
\Delta_2(\cdot) = &\, M^{-1}(z_1 + q_d) K \left(\mathrm{sig}(s + \tilde{e}_2)^a \right) + M^{-1}(z_1 + q_d) C(z_1 + q_d, z_2 + s + \dot{q}_d)(z_2 + s + \dot{q}_d) \\
&+ M^{-1}(z_1 + q_d) F(z_2 + s + \dot{q}_d) + M^{-1}(z_1 + q_d) G(z_1 + q_d) + M^{-1}(z_1 + q_d) D + \ddot{q}_d,
\end{aligned}
$$

4.2.2 Stability Analysis

From the closed-loop dynamics (4 - 7), the structure (z_1, z_2) is essentially the desired error dynamics injected by the controller through (4 - 5). Essentially, for any $i \in 1, ..., n$, when $s_i = \tilde{e}_{2i} = 0$, the dynamics of the subsystem (z_{1i}, z_{2i}) is identical to that of the twisting-based family of algorithm studied in section 2.2 of Chapter 2. In other words, s_i and $\tilde{e}_{2i}$ can be viewed as a perturbations on the (z_{1i}, z_{2i}) dynamics. While in the s-dynamics, its structure is akin to that of sliding mode control with $\tilde{e}_{2i}$ as perturbations.

Since the differential equations (4 - 7) have discontinuous right-hand side, i.e. when $a = 0$ or $b = 0$ or $p = 0.5$, and since no continuity assumption is made on D, its solutions are understood in the sense of Filippov (see definition 2.1).

The following Lyapunov functions will be used in the analysis:

$$V_{zi}(z_{1i}, z_{2i}) = \frac{k_{1i}{}^2}{(1+b)^2}|z_{1i}|^{2+2b} + \frac{1}{4}|z_{2i}|^4 + r_{zi}|z_{1i}|^{\frac{3+3b}{2}}\operatorname{sign}(z_{1i})z_{2i}$$
$$+ \frac{k_{1i}}{(1+b)}|z_{1i}|^{1+b}|z_{2i}|^2, \qquad \text{for } i = 1,\ldots,n,$$

$$V_s = \frac{1}{2}s^T M(q)s,$$

$$V_{\tilde{e}i}(\tilde{e}_{1i}, \tilde{e}_{2i}) = \frac{1}{2}\tilde{e}_{2i}{}^2 + \frac{1}{2p}|\tilde{e}_{1i}|^{2p}, \qquad \text{for } i = 1,\ldots,n.$$

Note that the Lyapunov function for the (z_{1i}, z_2)-subsystem is a strict Lyapunov function proposed for the twisting-based family of algorithms in Chapter 2, where $\forall q \in \mathbb{R}^n$:

$$\underline{\pi}_{1i}\left(|z_{1i}|^{2+2b}|z_{2i}|^4\right) \le V_{zi}(z_{1i}, z_{2i}) \le \overline{\pi}_{1i}\left(|z_{1i}|^{2+2b} + |z_{2i}|^4\right),$$

where

$$\underline{\pi}_{1i} := \min\left\{\frac{1}{8}, \frac{1}{2}\left(\frac{k_{1i}}{1+b}\right)^2\right\}, \ \overline{\pi}_{1i} := \max\left\{\frac{3r_{zi}}{4} + \left(\frac{k_{1i}}{1+b}\right)^2 + \left(\frac{k_{1i}}{2+2b}\right), \ \left(\frac{k_{1i}}{2+2b}\right) + \frac{r_{zi}}{4} + \frac{1}{4}\right\},$$

while for V_s,

$$\frac{1}{2}\underline{m}\|s\|^2 \le V_s \le \frac{1}{2}\overline{m}\|s\|^2,$$

and for $V_{\tilde{e}i}$

$$\min\left\{\frac{1}{2},\frac{1}{2p}\right\}\left(\left|\tilde{e}_{2i}\right|^2+\left|\tilde{e}_{1i}\right|^{2p}\right)\le V_{\tilde{e}i}\le\max\left\{\frac{1}{2},\frac{1}{2p}\right\}\left(\left|\tilde{e}_{2i}\right|^2+\left|\tilde{e}_{1i}\right|^{2p}\right)$$

Let us define the following sets:

$$\Omega_z(R)=\left\{(z_1,z_2)\in\mathbb{R}^{2n}:V_{zi}(z_{1i},z_{2i})\le\rho_{1i}(R),\ldots,V_{zn}(z_{1n},z_{2n})\le\rho_{1i}(R),\text{ for } i=1,\ldots,n\right\},$$

$$\Omega_s(R)=\left\{s\in\mathbb{R}^n:V_s\le\rho_2\frac{R^2}{2}\right\},$$

$$\Omega_{\tilde{e}}(R)=\left\{(\tilde{e}_1,\tilde{e}_2)\in\mathbb{R}^{2n}:V_{\tilde{e}i}\le\frac{R^2}{2},\ldots,V_{\tilde{e}n}\le\frac{R^2}{2},\quad\text{for } i=1,\ldots,n\right\},$$

$$\Omega(R)=\left\{(z_1,z_2,s,\tilde{e}_1,\tilde{e}_2)\in\mathbb{R}^{5n}:\Omega_z(R),\ \ \Omega_s(R),\ \ \Omega_{\tilde{e}}(R)\right\}$$

where

for $i=1,\ldots,n$

$$\rho_{1i}(R)=\begin{cases}\max\left\{\begin{array}{l}(\rho_2)^2\left(\dfrac{4\bar{\pi}_{1i}^{\frac{3+5b}{4+4b}}}{\omega_{4i}}\right)^4\left(\dfrac{2^{\frac{1}{2+2b}}\omega_{2i}}{\underline{\pi}_{1i}^{\frac{1+2b}{2+2b}}}\right)^4,\\[2ex]\left(\sqrt{\rho_2}+\sqrt{n\,\underline{m}}\right)^4\left(a_{1i}\left(\dfrac{4\bar{\pi}_{1i}^{\frac{3+5b}{4+4b}}}{\omega_{4i}}\right)\left(\dfrac{2^{\frac{1}{4}}\omega_{3i}}{\underline{\pi}_{1i}^{\frac{3}{4}}}\right)\right)^{\frac{2+2b}{b}}\end{array}\right\}\times\left(\dfrac{R}{\sqrt{\underline{m}}}\right)^4, & \text{for } b>0,\\[2ex]\max\limits_{(z_{1i},z_{2i})\in\operatorname{bd}\phi_{3i}\left(\left(\frac{\sqrt{\rho_2}}{\sqrt{\underline{m}}}+\sqrt{n}\right)R\right)}V_{zi}(z_{1i},z_{2i}), & \text{for } b=0,\end{cases}$$

$$\rho_2 = \begin{cases} \overline{m}\left(\dfrac{2a_4}{(1-a_5)}\dfrac{\lambda_{\max}(K)}{\lambda_{\min}(K)} n^{\frac{3a+1}{2}}\right)^{\frac{2}{a}}, & \text{for } a > 0, \\ \overline{m}\left(4n\dfrac{\lambda_{\max}(K)}{\lambda_{\min}(K)}\right)^2, & \text{for } a = 0, \end{cases}$$

bd $\phi_{3i}\left(\left(\dfrac{\sqrt{\rho_2}}{\sqrt{\underline{m}}}+\sqrt{n}\right)R\right)$ is the boundary of the region

$$\phi_{3i}\left(\left(\frac{\sqrt{\rho_2}}{\sqrt{\underline{m}}}+\sqrt{n}\right)R\right) = \left\{(z_{1i}, z_{2i}) \in \mathbb{R}^2 : |z_{1i}| \le a_{3i}\left(\left(\frac{\sqrt{\rho_2}}{\sqrt{\underline{m}}}+\sqrt{n}\right)R\right)^2, |z_{2i}| \le \frac{2k_{1i}}{k_{2i}}\left(\left(\frac{\sqrt{\rho_2}}{\sqrt{\underline{m}}}+\sqrt{n}\right)R\right)\right\},$$

which is a compact set, with the constant, $a_{3i} > 0$ (see Appendix A.1), while the positive constants a_4 and a_5 defined as

$$a_4 := \frac{a_5 + 2}{\left((a_5+1)^{\frac{1}{a}} - 1\right)^a} > 0, \text{ for any } 0 < a_5 < 1$$

Remark 4.3. By selecting $\left(\dfrac{\lambda_{\max}(K)}{\lambda_{\min}(K)}\right)$ to be a constant ratio, ρ_2 is a constant.

Define the positive diagonal matrix, $\gamma_1(R) = \mathrm{diag}\{\gamma_{1i}(R)\}_{i=1}^n$ with, $\gamma_{1i}(R) > 0, \ \forall i = 1,\ldots,n$, and the following positive vector

$\gamma_2(R)=(\gamma_{2i}(R),\ldots,\gamma_{2n}(R))^{\mathrm{T}}$ with $\gamma_{2i}(R)>0,\ \forall i=1,\ldots,n$, such that $\forall(z_1,z_2,s,\tilde{e}_1,\tilde{e}_2)\in\Omega(R)$:

$$s^{\mathrm{T}}M(\cdot)\boldsymbol{K}[\Delta_1(\cdot)]+\frac{1}{2}s^{\mathrm{T}}\dot{M}(\cdot)s+s^TM(\cdot)K_2\left(\operatorname{sig}(z_2+s+\tilde{e}_2)^{\frac{2b}{1+b}}-\operatorname{sig}(z_2+s)^{\frac{2b}{1+b}}\right)\leq\sum_{i=1}^{n}\gamma_{1i}(R)|s_i|,$$

$$\boldsymbol{K}[\Delta_2(\cdot)]\leq\gamma_2(R),$$

where such an upper bound always exist since *M*, *C*, *F*, *G*, *D* and desired trajectories are locally bounded, it implies that $\Delta_1(.)$, and $\Delta_2(.)$ are locally bounded as well (i.e. it compose of summation of locally bounded functions). From [143], the multi-valued function $\boldsymbol{K}[\Delta_1(.)]$ and $\boldsymbol{K}[\Delta_2(.)]$ are locally bounded as well. Also, using the skew-symmetry property 3.2.2, we have

$$\frac{1}{2}s^T\dot{M}(\cdot)s=s^TC(\cdot)s$$

which is locally bounded as well. Hence, within a compact set, an upper bound on the above function exists.

The time derivatives of the Lyapunov functions, in accordance to lemma 2.4, $\forall(z_1,z_2,s,\tilde{e}_1,\tilde{e}_2)\in\Omega(R)$ of the closed-loop system satisfy the following inequalities:

Differential inequalities for the *z*-subsystem: (see Appendix A.1)

$\forall i=1,\ldots,n$:

for $b > 0$:

$$\dot{V}_{zi}(z_{1i},z_{2i}) \overset{a.e.}{\leq} -\frac{1}{2}\left(\frac{\omega_{4i}}{\overline{\pi}_{1i}^{\frac{3+5b}{4+4b}}}\right)V_{zi}^{\frac{3+5b}{4+4b}} - \frac{1}{4}\left(\frac{\omega_{4i}}{\overline{\pi}_{1i}^{\frac{3+5b}{4+4b}}}\right)V_{zi}^{\frac{1+2b}{2+2b}}\left(V_{zi}^{\frac{1}{4}} - |s_i|\left(\frac{4\overline{\pi}_{1i}^{\frac{3+5b}{4+4b}}}{\omega_{4i}}\right)\left(\frac{2^{\frac{1}{2+2b}}\omega_{2i}}{\underline{\pi}_{1i}^{\frac{1+2b}{2+2b}}}\right)\right)$$

$$-\frac{1}{4}\left(\frac{\omega_{4i}}{\overline{\pi}_{1i}^{\frac{3+5b}{4+4b}}}\right)V_{zi}^{\frac{3}{4}}\left(V_{zi}^{\frac{b}{2+2b}} - a_{1i}|s_i + \tilde{e}_{2i}|^{\frac{2b}{1+b}}\left(\frac{4\overline{\pi}_{1i}^{\frac{3+5b}{4+4b}}}{\omega_{4i}}\right)\left(\frac{2^{\frac{1}{4}}\omega_{3i}}{\underline{\pi}_{1i}^{\frac{3}{4}}}\right)\right)$$

for $b = 0$:

$$\begin{aligned}\dot{V}_{zi}(z_{1i},z_{2i}) &\overset{a.e.}{\leq} 2k_{1i}^{2}|z_{1i}||s_i| + \frac{3}{2}r_i|z_{1i}|^{\frac{1}{2}}|z_{2i}||s_i| + k_{1i}|z_{2i}|^{2}|s_i| + \frac{3}{2}r_i|z_{1i}|^{\frac{1}{2}}|z_{2i}|^{2} \\ &- 2k_{1i}k_{2i}|z_{1i}|z_{2i}\mathrm{SGN}(z_{2i}+s_i+\tilde{e}_{2i}) \\ &- k_{2i}|z_{2i}|^{3}\operatorname{sign}(z_{2i})\mathrm{SGN}(z_{2i}+s_i+\tilde{e}_{2i}) - r_i(k_{1i}-k_{2i})|z_{1i}|^{\frac{3}{2}} \\ &< 0,\ \text{for}\ \ V_{zi} \geq \max_{(z_{1i},z_{2i})\in \mathrm{bd}\ \phi_{3i}(\max\{|s_i|,|s_i+\tilde{e}_{2i}|\})} V_{zi}(z_{1i},z_{2i})\end{aligned}$$

Differential inequalities for the s-subsystem:

$$\dot{V}_s \overset{a.e}{\in} \dot{\tilde{V}}_s = \bigcap_{\xi\in\partial V_s} \xi^{\mathrm{T}}\begin{pmatrix}\boldsymbol{K}[f](s)\\ 1\end{pmatrix} = \nabla V_s^{\mathrm{T}}\boldsymbol{K}[f](s) + \frac{1}{2}s^{\mathrm{T}}\dot{M}(\cdot)s$$

For $a = 0$:

Using the following notation:

$N^{+}(s) = \{i \in \{1,\ldots,n\} : s_i \neq 0\}$, $N^{0}(s) = \{i \in \{1,\ldots,n\} : s_i = 0\}$, observe that

$$-s^{\mathrm{T}}K\boldsymbol{K}[\operatorname{sign}(s)] = -\sum_{i\in N^{+}(s)} k_i|s_i| - \sum_{i\in N^{0}(s)} k_i(0)\times[-1,+1] = -\sum_{i\in N^{+}(s)} k_i|s_i| - 0 = -\sum_{i=i}^{n} k_i|s_i|,$$

Also, note that, $\boldsymbol{K}[(\operatorname{sign}(s_i+\tilde{e}_{2i}) - \operatorname{sign}(s_i))] \leq 2$, and observe that for $|s_i| > |\tilde{e}_{2i}| > 0$,

$$\boldsymbol{K}[(\operatorname{sign}(s_i+\tilde{e}_{2i})-\operatorname{sign}(s_i))]=(\operatorname{sign}(s_i+\tilde{e}_{2i})-\operatorname{sign}(s_i))=\operatorname{sign}(s_i)-\operatorname{sign}(s_i)=0$$
$$\therefore |s_i k_i \boldsymbol{K}[(\operatorname{sign}(s_i+\tilde{e}_{2i})-\operatorname{sign}(s_i))]|\le 2k_i|\tilde{e}_{2i}|$$

Hence,

$$\begin{aligned}
\dot{V}_s &\overset{a.e.}{\subset} s^{\mathrm{T}} M(\cdot)\boldsymbol{K}\begin{bmatrix} -M^{-1}(\cdot)K\operatorname{sign}(s)+\Delta_1(\cdot)-M^{-1}(\cdot)K(\operatorname{sign}(s+\tilde{e}_2)-\operatorname{sign}(s)) \\ +K_2\left(\operatorname{sig}(z_2+s+\tilde{e}_2)^{\frac{2b}{1+b}}-\operatorname{sig}(z_2+s)^{\frac{2b}{1+b}}\right)\end{bmatrix}+\frac{s^{\mathrm{T}}\dot{M}(\cdot)s}{2} \\
&\le -\sum_{i=1}^{n}k_i|s_i|+s^{\mathrm{T}}M(\cdot)\boldsymbol{K}[\Delta_1](\cdot)+\frac{1}{2}s^{\mathrm{T}}\dot{M}(\cdot)s-s^{T}K\,\boldsymbol{K}[(\operatorname{sign}(s+\tilde{e}_2)-\operatorname{sign}(s))] \\
&\quad +s^{T}M(\cdot)K_2\left(\operatorname{sig}(z_2+s+\tilde{e}_2)^{\frac{2b}{1+b}}-\operatorname{sig}(z_2+s)^{\frac{2b}{1+b}}\right) \\
&\le -\sum_{i=1}^{n}k_i|s_i|+\sum_{i=1}^{n}\gamma_{1i}(R)|s_i|-\sum_{i=1}^{n}2k_i|\tilde{e}_{2i}| \\
&= -\frac{1}{2}\sum_{i=1}^{n}k_i|s_i|+\sum_{i=1}^{n}2k_i|\tilde{e}_{2i}|-\sum_{i=1}^{n}|s_i|\left(\frac{k_i}{2}-\gamma_{1i}(R)\right) \\
&\le -\frac{\lambda_{\min}(K)}{2}\sum_{i=1}^{n}|s_i|+2\lambda_{\max}(K)\sum_{i=1}^{n}|\tilde{e}_{2i}|-\min_i\left(\frac{k_i}{2}-\gamma_{1i}(R)\right)\sum_{i=1}^{n}|s_i| \\
&= -\frac{\lambda_{\min}(K)}{2}\sum_{i=1}^{n}\left(|s_i|^2\right)^{\frac{1}{2}}+2\lambda_{\max}(K)\sum_{i=1}^{n}\left(|\tilde{e}_{2i}|^2\right)^{\frac{1}{2}}-\min_i\left(\frac{k_i}{2}-\gamma_{1i}(R)\right)\sum_{i=1}^{n}\left(|s_i|^2\right)^{\frac{1}{2}} \\
&\le -\frac{\lambda_{\min}(K)}{2}\|s\|+2\sqrt{n}\,\lambda_{\max}(K)\|\tilde{e}_2\|-\min_i\left(\frac{k_i}{2}-\gamma_{1i}(R)\right)\|s\|,\quad \text{(using lemma 2 of chapter 2)},
\end{aligned}$$

where lemma 2 of chapter 2 has been employed in the last inequality.

For $a > 0$:

From Appendix A.2-proposition 1, one obtains

$$
\begin{aligned}
-s^T K\left(\operatorname{sig}(s+\tilde{e}_2)^a - \operatorname{sig}(s)^a\right) = s^T K\left(\operatorname{sig}(s)^a - \operatorname{sig}(s+\tilde{e}_2)^a\right)& \\
= \sum_{i=1}^{n} s_i k_i \left(|s_i|^a \operatorname{sign}(s_i) - |s_i + \tilde{e}_{2i}|^a \operatorname{sign}(s_i + \tilde{e}_{2i})\right)& \\
\le \sum_{i=1}^{n} |s_i| k_i \left| |s_i|^a \operatorname{sign}(s_i) - |s_i + \tilde{e}_{2i}|^a \operatorname{sign}(s_i + \tilde{e}_{2i}) \right|& \\
\le \sum_{i=1}^{n} |s_i| k_i \left(a_4 |\tilde{e}_{2i}|^a + a_5 |s_i|^a\right)& \\
= a_4 \sum_{i=1}^{n} |s_i| k_i |\tilde{e}_{2i}|^a + a_5 \sum_{i=1}^{n} k_i |s_i|^{1+a}&
\end{aligned}
$$

where positive constants $a_4, a_5 \in \mathbb{R}^+$ are as defined earlier.

Then,

$$
\dot{V}_s \overset{a.e.}{\subset} s^{\mathrm{T}} M(\cdot) \boldsymbol{K} \begin{bmatrix} -M^{-1}(\cdot)K\mathrm{sig}(s)^a + \Delta_1(\cdot) - M^{-1}(\cdot)K\left(\mathrm{sig}(s+\widetilde{e}_2)^a - \mathrm{sig}(s)^a\right) \\ + K_2\left(\mathrm{sig}(z_2+s+\widetilde{e}_2)^{\frac{2b}{1+b}} - \mathrm{sig}(z_2+s)^{\frac{2b}{1+b}}\right) \end{bmatrix} + \frac{s^{\mathrm{T}}\dot{M}(\cdot)s}{2}
$$

$$
\le -\sum_{i=1}^{n} k_i |s_i|^{1+a} + s^{\mathrm{T}} M(\cdot)\boldsymbol{K}[\Delta_1](\cdot) + \frac{1}{2}s^{\mathrm{T}}\dot{M}(\cdot)s - s^T K\left(\mathrm{sig}(s+\widetilde{e}_2)^a - \mathrm{sig}(s)^a\right)
$$

$$
+ s^T M(\cdot)K_2\left(\mathrm{sig}(z_2+s+\widetilde{e}_2)^{\frac{2b}{1+b}} - \mathrm{sig}(z_2+s)^{\frac{2b}{1+b}}\right)
$$

$$
\le -\sum_{i=1}^{n} k_i |s_i|^{1+a} + \sum_{i=1}^{n}\gamma_{1i}(R)|s_i| + a_4\sum_{i=1}^{n}|s_i| k_i |\widetilde{e}_{2i}|^a + a_5\sum_{i=1}^{n} k_i |s_i|^{1+a} \quad \text{with } a_5 < 1,
$$

$$
= -(1-a_5)\sum_{i=1}^{n} k_i |s_i|^{1+a} + a_4\sum_{i=1}^{n}|s_i| k_i |\widetilde{e}_{2i}|^a + \sum_{i=1}^{n}\gamma_{1i}(R)|s_i|
$$

$$
\le -(1-a_5)\lambda_{\min}(K)\sum_{i=1}^{n}|s_i|^{1+a} + a_4\lambda_{\max}(K)\|\widetilde{e}_2\|^a\sum_{i=1}^{n}|s_i| + \lambda_{\max}(\gamma_{1i}(R))\sum_{i=1}^{n}|s_i|
$$

$$
\le -\frac{(1-a_5)\lambda_{\min}(K)}{n^a}\left(\sum_{i=1}^{n}|s_i|\right)^{1+a} + a_4\lambda_{\max}(K)\|\widetilde{e}_2\|^a\sum_{i=1}^{n}|s_i| + \lambda_{\max}(\gamma_{1i}(R))\sum_{i=1}^{n}|s_i|
$$

$$
= -\frac{(1-a_5)\lambda_{\min}(K)}{n^a}\left(\sum_{i=1}^{n}\left(|s_i|^2\right)^{\frac{1}{2}}\right)^{1+a} + a_4\lambda_{\max}(K)\|\widetilde{e}_2\|^a\sum_{i=1}^{n}\left(|s_i|^2\right)^{\frac{1}{2}} + \lambda_{\max}(\gamma_{1i}(R))\sum_{i=1}^{n}\left(|s_i|^2\right)^{\frac{1}{2}}
$$

$$
\le -\frac{(1-a_5)\lambda_{\min}(K)}{n^a}\left(\left(\sum_{i=1}^{n}|s_i|^2\right)^{\frac{1}{2}}\right)^{1+a}
$$

$$
+ a_4\lambda_{\max}(K)\sqrt{n}\|\widetilde{e}_2\|^a\left(\sum_{i=1}^{n}|s_i|^2\right)^{\frac{1}{2}} + \lambda_{\max}(\gamma_{1i}(R))\sqrt{n}\left(\sum_{i=1}^{n}|s_i|^2\right)^{\frac{1}{2}}
$$

$$
= -\|s\|\left(\frac{(1-a_5)\lambda_{\min}(K)}{2n^a}\|s\|^a - a_4\lambda_{\max}(K)\sqrt{n}\|\widetilde{e}_2\|^a\right)
$$

$$
-\|s\|\left(\frac{(1-a_5)\lambda_{\min}(K)}{2n^a}\|s\|^a - \lambda_{\max}(\gamma_{1i}(R))\sqrt{n}\right)
$$

Note that the property of $|\widetilde{e}_{2i}| \le \|\widetilde{e}_2\|, \forall i$ and lemma 2 of chapter 2 have been employed.

Differential inequalities for the $\tilde{e}$-subsystem: (see Section 4.1)

$\forall i = 1, \ldots, n$:

$$\dot{V}_{\tilde{e}i}\left(\tilde{e}_{1i}, \tilde{e}_{2i}\right) \overset{a.e.}{\leq} -l_{3i}\left|\tilde{e}_{2i}\right|^{2} - l_{1i}\left|\tilde{e}_{1i}\right|^{3p-1} + \left|1 - l_{2i}\right|\left|\tilde{e}_{1i}\right|^{2p-1}\left|\tilde{e}_{2i}\right| + \gamma_{2i}(R)\left|\tilde{e}_{2i}\right|$$
$$< 0, \quad \text{for } V_{\tilde{e}i} \geq \rho_{3i}\left(\left|\zeta_i\right|, l_{3i}\right),$$

where $\left|\zeta_i\right| := \max\left\{\left|1 - l_{2i}\right|, \gamma_{2i}(R)\right\}$, the class $\mathcal{KL}$ function,

$$\rho_{3i}\left(\left|\zeta_i\right|, l_{3i}\right) := \max_{\left(\tilde{e}_{1i}, \tilde{e}_{2i}\right) \in \text{bd } \psi_{3i}\left(\left|\zeta_i\right|, l_{3i}\right)} V_{\tilde{e}i}\left(\tilde{e}_{1i}, \tilde{e}_{2i}\right),$$

bd $\psi_{3i}\left(\left|\zeta_i\right|, l_{3i}\right)$ is the boundary of the region

$$\psi_3\left(\left|\zeta_i\right|, l_{3i}\right) = \left\{\left(\tilde{e}_{1i}, \tilde{e}_{2i}\right) \in \mathbb{R}^2 : \left|\tilde{e}_{1i}\right| \leq \beta_1\left(\left|\zeta_i\right|, l_{3i}\right), \left|\tilde{e}_{2i}\right| \leq \beta_2\left(\left|\zeta_i\right|, l_{3i}\right)\right\}$$

which is a compact set, with functions $\beta_1(.)$ and $\beta_2(.)$ are of class-$\mathcal{KL}$.

Theorem 4.1: By redefining the control gain K, as $K = \hat{k} \times \text{diag}\left\{\tilde{k}_i\right\}_{i=1}^{n} \in \mathbb{R}^{n \times n}$, where $\hat{k} > 0$ is the control gain, $\tilde{k}_i$ is positive constant selected *a priori* for all $i = 1, \ldots, n$. Then, the ratio,

$$\left(\frac{\lambda_{\max}(K)}{\lambda_{\min}(K)}\right) = \frac{\max_i k_i}{\min_i k_i} = \frac{\max_i \tilde{k}_i}{\min_i \tilde{k}_i} = \text{constant}$$

can be selected *a priori* independent of the control gain, $\hat{k} > 0$. Thus, for any given K_1, K_2, L_1, L_2, and $\tilde{k}_i$, suppose that the initial conditions for the closed-loop system (4 - 7) belong to a given compact set, there always exists a $c > 0$ such that initially, $\left(z_1, z_2, s, \tilde{e}_1, \tilde{e}_2\right) \in \Omega(c)$. Also, a $\mu > 0$ can be selected such that $c > \mu > 0$. Then, by selecting $\hat{k}$ such that

$$\hat{k} > \begin{cases} \dfrac{2\gamma_{1i}(c)}{\widetilde{k}_i}, & \text{for a} = 0, \\ \left(\dfrac{2\lambda_{\max}(\gamma_{1i}(c)) n^{a+0.5} \overline{m}^{\frac{a}{2}}}{\left(\min\limits_i \widetilde{k}_i\right)(1-a_5)\rho_2^{\frac{a}{2}}} \right) \times \dfrac{1}{\mu^a}, & \text{for a} > 0, \end{cases}$$

and L_3 such that

$$\frac{\mu^2}{2} \geq \rho_{3i}(|\zeta_i|, l_{3i}), \text{ with } |\zeta_i| := \max\{|1-l_{2i}|, \gamma_{2i}(c)\}$$

all the trajectories will enter the compact set $\Omega(\mu)$, in finite time, and stay there for all future times.

Proof of Theorem 4.1: The stability analysis proceeds in two steps.

1. Obviously, for a given compact set of initial conditions, there exists $c > 0$ such that initially $(z_1, z_2, s, \tilde{e}_1, \tilde{e}_2)$ belong to some compact set strictly inside $\Omega(c)$. A trajectory may leave the set $\Omega(c)$ only through one of the boundaries:

$$V_{zi}(z_{1i}, z_{2i}) = \rho_{1i}(c), \quad \forall\ i = 1, \ldots, n, \qquad V_s = \rho_2 \frac{c^2}{2}, \quad \text{or } V_{\tilde{e}i} = \frac{c^2}{2}, \qquad \forall\ i = 1, \ldots, n$$

Let us show that it is impossible:

(a) For the $\tilde{e}$-subsystem, on the boundary Ω(c): $\forall i = 1, \ldots, n$, $V_{\tilde{e}i} = \dfrac{c^2}{2}$,

now, in order for $\dot{V}_{\tilde{e}i}(\widetilde{e}_{1i}, \widetilde{e}_{2i}) \overset{a.e.}{<} 0$ on the boundary, we need:

$$\frac{c^2}{2} \geq \rho_{3i}\left(\max\left\{\left|1-l_{2i}\right|,\ \gamma_{2i}(c)\right\}, l_{3i}\right) \tag{4 - 8}$$

To satisfy the inequality, recall that (see Section 4.1) ρ_{3i} is a class $\mathcal{KL}$ function, thus, it can be observed that for any given L_1, L_2, and c, by increasing the observer gain, L_3, the value $\rho_{3i}\left(\max\left\{\left|1-l_{2i}\right|,\ \gamma_{2i}(c)\right\}, l_{3i}\right)$ can be arbitrarily reduced such that the inequality (4 - 8) is satisfied. Thus, selecting L_3 as per Theorem 4.1, (4 - 8) is satisfied.

Hence, at the boundary, $V_{\tilde{e}i} = \dfrac{c^2}{2}$, $\dot{V}_{\tilde{e}i}\left(\tilde{e}_{1i}, \tilde{e}_{2i}\right) \overset{a.e.}{<} 0$, so $V_{\tilde{e}i}$ is a decreasing function of time, and $\left(\tilde{e}_{1i}, \tilde{e}_{2i}\right)$ stays in $\Omega_{\tilde{e}}(c)$, and

$$\frac{c^2}{2} = V_{\tilde{e}i} \geq \frac{1}{2p}\left|\tilde{e}_{1i}\right|^{2p} \Rightarrow (p)^{\frac{1}{2p}} c^{\frac{1}{p}} \geq \left|\tilde{e}_{1i}\right| \Rightarrow \sqrt{n}(p)^{\frac{1}{2p}} c^{\frac{1}{p}} \geq \left\|\tilde{e}_1\right\|,$$

and $\quad \dfrac{c^2}{2} = V_{\tilde{e}i} \geq \dfrac{1}{2}\left|\tilde{e}_{2i}\right|^2 \Rightarrow c \geq \left|\tilde{e}_{2i}\right| \Rightarrow \sqrt{n}\, c \geq \left\|\tilde{e}_2\right\|$

(b) For the s-subsystem, on the boundary $\Omega(c)$: $V_s(\cdot) = \dfrac{1}{2} s^{\mathrm{T}} M(\cdot) s = \rho_2 \dfrac{c^2}{2}$ implies

$\left\|s\right\| \geq \dfrac{\sqrt{\rho_2}}{\sqrt{m}} c \qquad$ and $\qquad \Rightarrow \left|\tilde{e}_{2i}\right| \leq \left\|\tilde{e}_2\right\| \leq \sqrt{n}\, c$

For $a = 0$:

$$\dot{V}_s \overset{a.e.}{\leq} -\frac{\lambda_{\min}(K)}{2}\|s\| + 2\sqrt{n}\,\lambda_{\max}(K)\|\tilde{e}_2\| - \min_i\left(\frac{k_i}{2} - \gamma_{1i}(c)\right)\|s\|$$
$$= -\left(\frac{\lambda_{\min}(K)}{2}\|s\| - 2\sqrt{n}\,\lambda_{\max}(K)\|\tilde{e}_2\|\right) - \min_i\left(\frac{k_i}{2} - \gamma_{1i}(c)\right)\|s\|$$

Note that on the boundary of the set $\Omega(c)$, the first term of the above inequality is non-positive:

$$-\left(\frac{\lambda_{\min}(K)}{2}\|s\| - 2\sqrt{n}\,\lambda_{\max}(K)\|\tilde{e}_2\|\right) \leq -\left(\frac{\lambda_{\min}(K)}{2}\frac{\sqrt{\rho_2}}{\sqrt{\overline{m}}}c - 2n\lambda_{\max}(K)c\right) = 0$$

Hence,

$$\therefore \dot{V}_s \overset{a.e.}{\leq} -\min_i\left(\frac{k_i}{2} - \gamma_{1i}(c)\right)\|s\| \leq -c_1 \text{, where } c_1 = \min_i\left(\frac{k_i}{2} - \gamma_{1i}(c)\right)\left(\frac{\sqrt{\rho_2}}{\sqrt{\overline{m}}}c\right),$$

Hence, for k_i sufficiently big, c_1 is positive and, correspondingly, $\dot{V}_s < 0$ almost everywhere. If the gain K satisfy the conditions of the Theorem 4.1, then, V_s is a decreasing function of t, so s stays in $\Omega_s(c)$ and

$$\|s\| \leq \frac{\sqrt{\rho_2}}{\sqrt{\underline{m}}}c\,.$$

For $a > 0$:

$$\dot{V}_s \overset{a.e.}{\leq} -\|s\|\left(\frac{(1-a_5)\lambda_{\min}(K)}{2n^a}\|s\|^a - a_4\,\lambda_{\max}(K)\sqrt{n}\,\|\tilde{e}_2\|^a\right)$$
$$-\|s\|\left(\frac{(1-a_5)\lambda_{\min}(K)}{2n^a}\|s\|^a - \lambda_{\max}(\gamma_{1i}(c))\sqrt{n}\right)$$

Note that on the boundary of the set $\Omega(c)$, the first term of the above inequality is non-positive:

$$-\left(\frac{(1-a_5)\lambda_{\min}(K)}{2n^a}\|s\|^a - a_4\,\lambda_{\max}(K)\sqrt{n}\,\|\tilde{e}_2\|^a\right)$$

$$\leq -\left(\frac{(1-a_5)\lambda_{\min}(K)}{2n^a}\left(\frac{\sqrt{\rho_2}}{\sqrt{\underline{m}}}c\right)^a - a_4\,\lambda_{\max}(K)\sqrt{n}\left(\sqrt{n}\,c\right)^a\right) = 0$$

Hence,

$$\therefore \dot{V}_s \overset{a.e.}{\leq} -\|s\|\left(\frac{(1-a_5)\lambda_{\min}(K)}{2n^a}\|s\|^a - \lambda_{\max}(\gamma_{1i}(c))\sqrt{n}\right) \leq -c_2$$

where

$$c_2 = \left(\frac{\sqrt{\rho_2}}{\sqrt{\underline{m}}}c\right)\left(\frac{(1-a_5)\lambda_{\min}(K)}{2n^a}\left(\frac{\sqrt{\rho_2}}{\sqrt{\underline{m}}}c\right)^a - \lambda_{\max}(\gamma_{1i}(c))\sqrt{n}\right) \tag{4 - 9}$$

Hence, for $\lambda_{\min}(K)$ sufficiently big, c_2 is positive and, correspondingly, $\dot{V}_s < 0$ almost everywhere. Then, V_s is a decreasing function of t, so s stays in $\Omega_s(c)$ and

$$\|s\| \leq \frac{\sqrt{\rho_2}}{\sqrt{\underline{m}}}c\,.$$

(c) For the z-subsystem, on the boundary of the set $\Omega(c)$:

For $b > 0$: At the boundary, it implies

$\forall\, i = 1,\ldots,n:$

$$V_{zi} = \rho_{1i}(c) = \max\left\{\begin{array}{l}(\rho_2)^2\left(\dfrac{4\overline{\pi}_{1i}^{\frac{3+5b}{4+4b}}}{\omega_{4i}}\right)^4\left(\dfrac{2^{\frac{1}{2+2b}}\omega_{2i}}{\underline{\pi}_{1i}^{\frac{1+2b}{2+2b}}}\right)^4, \\ \left(\sqrt{\rho_2}+\sqrt{n\underline{m}}\right)^4\left(a_{1i}\left(\dfrac{4\overline{\pi}_{1i}^{\frac{3+5b}{4+4b}}}{\omega_{4i}}\right)\left(\dfrac{2^{\frac{1}{4}}\omega_{3i}}{\underline{\pi}_{1i}^{\frac{3}{4}}}\right)\right)^{\frac{2+2b}{b}}\end{array}\right\}\times\left(\frac{c}{\sqrt{\underline{m}}}\right)^4$$

$$\geq \max\left\{\begin{array}{l}\left(\dfrac{4\overline{\pi}_{1i}^{\frac{3+5b}{4+4b}}}{\omega_{4i}}\right)^4\left(\dfrac{2^{\frac{1}{2+2b}}\omega_{2i}}{\underline{\pi}_{1i}^{\frac{1+2b}{2+2b}}}\right)^4 |s_i|^4, \\ \left(a_{1i}\left(\dfrac{4\overline{\pi}_{1i}^{\frac{3+5b}{4+4b}}}{\omega_{4i}}\right)\left(\dfrac{2^{\frac{1}{4}}\omega_{3i}}{\underline{\pi}_{1i}^{\frac{3}{4}}}\right)\right)^{\frac{2+2b}{b}} \left(|s_i|+|\widetilde{e}_{2i}|\right)^4\end{array}\right\}$$

Noting that, $|s_i|+|\widetilde{e}_{2i}| \geq |s_i+\widetilde{e}_{2i}|$, hence, we have

$$\dot{V}_{zi}(z_{1i}, z_{2i}) \overset{a.e.}{\leq} -\frac{1}{2}\left(\frac{\omega_{4i}}{\overline{\pi}_{1i}^{\frac{3+5b}{4+4b}}}\right)V_{zi}^{\frac{3+5b}{4+4b}} < 0,\ \forall i = 1,\ldots,n,$$

So, $\dot{V}_{zi}$ is a decreasing function of t for all $i = 1, \ldots, n$ and (z_1, z_2) stay in $\Omega_z(c)$.

For $b = 0$: At the boundary,

$$V_{zi}(z_{1i}, z_{2i}) = \rho_{1i}(c), \quad \forall\ i = 1,\ldots,n,$$

it implies

$$\frac{\sqrt{\rho_2}}{\sqrt{\underline{m}}}c \geq \|s\| \geq |s_i| \text{ and } \left(\frac{\sqrt{\rho_2}}{\sqrt{\underline{m}}}+\sqrt{n}\right)c \geq |s_i|+|\widetilde{e}_{2i}| \geq |s_i+\widetilde{e}_{2i}|,$$

Hence,

$$\Rightarrow \phi_{3i}\left(\left(\frac{\sqrt{\rho_2}}{\sqrt{m}}+\sqrt{n}\right)c\right)\supset \phi_{3i}\left(\left|s_i+\tilde{e}_{2i}\right|\right), \text{ and } \phi_{3i}\left(\left(\frac{\sqrt{\rho_2}}{\sqrt{m}}+\sqrt{n}\right)c\right)\supset \phi_{3i}\left(\left|s_i\right|\right),$$

$$\Rightarrow V_{zi}=\rho_{1i}(c)\geq \rho_{1i}\left(\frac{\left|s_i+\tilde{e}_{2i}\right|}{\left(\frac{\sqrt{\rho_2}}{\sqrt{m}}+\sqrt{n}\right)}\right), \text{ and } V_{zi}=\rho_{1i}(c)\geq \rho_{1i}\left(\frac{\left|s_i\right|}{\left(\frac{\sqrt{\rho_2}}{\sqrt{m}}+\sqrt{n}\right)}\right)$$

Thus, $\dot{V}_{zi}(z_{1i}, z_{2i}) \overset{a.e.}{<} 0$, for $\forall i = 1,\ldots,n$. So, $\dot{V}_{zi}$ is a decreasing function of t for all i = 1, ..., n and (z_1, z_2) stay in Ω_z(c) for the case of b = 0.

In the (b) section for the s-subsystem, the trajectories will stay inside the boundary

$$V_s(\cdot)=\rho_2\frac{c^2}{2},$$

provided that $\lambda_{\min}(K)$ or k_i is sufficiently high while maintaining the ratio

$$\left(\frac{\lambda_{\max}(K)}{\lambda_{\min}(K)}\right)=\text{constant},$$

such that ρ_2 remains as a constant. The increment of the gain K will cause the upper bound of the function $\gamma_2(c)$ to increase as well, since vector Δ_2(.) consists of term proportional to the gain K and $\boldsymbol{K}[\Delta_2(\cdot)]\leq \gamma_2(c)$

This in turn, implies that the observer gains, elements of the vector L_3, have to be increased accordingly for the same compact region c, such that $\dot{V}_{\tilde{e}i}(\tilde{e}_{1i}, \tilde{e}_{2i}) \overset{a.e.}{<} 0$.

As a result, the set $\Omega(c)$ is positively invariant, i.e. the trajectories of (4 - 7) stay in it once they have entered it.

2. Now we have shown boundedness, next is to show convergence to a smaller compact set $\Omega(\mu)$ with $c > \mu > 0$.

   (a) For the $\tilde{e}$-subsystem:

   Recall from the above that by increasing the observer gains, one can satisfy the following inequality for a given c:

   $$\frac{c^2}{2} \geq \rho_{3i}\left(\max\left\{\left|1-l_{2i}\right|, \gamma_{2i}(c)\right\}, l_{3i}\right),$$

   now if the gain L_{3i} of the observer is much higher such that

   $$\frac{c^2}{2} > \frac{\mu^2}{2} \geq \rho_{3i}\left(\max\left\{\left|1-l_{2i}\right|, \gamma_{2i}(c)\right\}, l_{3i}\right),$$

   all the trajectories starting in $\Omega_{\tilde{e}}(c)$ will enter $\Omega_{\tilde{e}}(\mu)$ within finite time and stay in it for all future times. Then, the upper bound on $(\tilde{e}_1, \tilde{e}_2)$ can be found to be

   $$\frac{\mu^2}{2} = V_{\tilde{e}i} \geq \frac{1}{2p}\left|\tilde{e}_{1i}\right|^{2p} \Rightarrow (p)^{\frac{1}{2p}} \mu^{\frac{1}{p}} \geq \left|\tilde{e}_{1i}\right| \Rightarrow \sqrt{n}(p)^{\frac{1}{2p}} \mu^{\frac{1}{p}} \geq \left\|\tilde{e}_1\right\|,$$

   and $\quad \dfrac{\mu^2}{2} = V_{\tilde{e}i} \geq \dfrac{1}{2}\left|\tilde{e}_{2i}\right|^2 \Rightarrow \mu \geq \left|\tilde{e}_{2i}\right| \Rightarrow \sqrt{n}\,\mu \geq \left\|\tilde{e}_2\right\|.$

   (b) For the s-subsystem, consider the states $(z_1, z_2, s) \in \Omega_z(c) \times \Omega_s(c) \setminus \Omega_z(\mu) \times \Omega_s(\mu)$ and

$\left(\widetilde{e}_1, \widetilde{e}_2\right) \in \Omega_{\widetilde{e}}(\mu)$, it implies that, $\|s\| \geq \dfrac{\sqrt{\rho_2}}{\sqrt{m}}\mu$ and $\|\widetilde{e}_2\| \leq \sqrt{n}\,\mu$.

Hence, for $a = 0$:

$$\dot{V}_s \overset{a.e.}{\leq} -\left(\frac{\lambda_{\min}(K)}{2}\|s\| - 2\sqrt{n}\,\lambda_{\max}(K)\|\widetilde{e}_2\|\right) - \min_i\left(\frac{k_i}{2} - \gamma_{1i}(c)\right)\|s\|$$

Note that the first term of the above inequality is non-positive,

$$\therefore \dot{V}_s \overset{a.e.}{\leq} -\min_i\left(\frac{k_i}{2} - \gamma_{1i}(c)\right)\|s\| \leq -c_3, \quad \text{for } \|s\| \geq \frac{\sqrt{\rho_2}}{\sqrt{m}}\mu$$

where $c_3 = \min_i\left(\dfrac{k_i}{2} - \gamma_{1i}(c)\right)\left(\dfrac{\sqrt{\rho_2}}{\sqrt{m}}\mu\right)$ which is positive for $k_i, > 2\gamma_{1i}(c),\ \forall i = 1, \ldots, n,$

which implies that the s trajectory will enter the set $\Omega_s(\mu)$ in finite time and stay there for all future times.

For $a > 0$:

$$\begin{aligned}\dot{V}_s \overset{a.e.}{\leq} &-\|s\|\left(\frac{(1-a_5)\lambda_{\min}(K)}{2n^a}\|s\|^a - a_4\,\lambda_{\max}(K)\sqrt{n}\,\|\widetilde{e}_2\|^a\right)\\ &-\|s\|\left(\frac{(1-a_5)\lambda_{\min}(K)}{2n^a}\|s\|^a - \lambda_{\max}(\gamma_{1i}(c))\sqrt{n}\right)\end{aligned}$$

Note that the first term of the above inequality is non-positive

$$\therefore \dot{V}_s \overset{a.e.}{\leq} -\|s\|\left(\frac{(1-a_5)\lambda_{\min}(K)}{2n^a}\|s\|^a - \lambda_{\max}(\gamma_{1i}(c))\sqrt{n}\right) \leq -c_4, \quad \forall \|s\| \geq \frac{\sqrt{\rho_2}}{\sqrt{m}}\mu$$

where $c_4 = \left(\frac{\sqrt{\rho_2}}{\sqrt{m}}\mu\right)\left(\frac{(1-a_5)\lambda_{\min}(K)}{2n^a}\left(\frac{\sqrt{\rho_2}}{\sqrt{m}}\mu\right)^a - \lambda_{\max}(\gamma_{1i}(c))\sqrt{n}\right)$

Note that by choosing sufficiently big K, as per the conditions of the Theorem 4.1 (while maintaining the ratio $\left(\frac{\lambda_{\max}(K)}{\lambda_{\min}(K)}\right) = \text{constant}$) , c_4 is positive.

Thus, for all $a \geq 0$, the states $(z_1, z_2, s) \in \Omega_z(c) \times \Omega_s(c) \setminus \Omega_z(\mu) \times \Omega_s(\mu)$ and $(\tilde{e}_1, \tilde{e}_2) \in \Omega_{\tilde{e}}(\mu)$

will enter the set $(z_1, z_2, s, \tilde{e}_1, \tilde{e}_2) \in \Omega_z(c) \times \Omega_s(\mu) \times \Omega_{\tilde{e}}(\mu)$ in finite time and in it for all future times, in particular

$\|s\| \leq \mu \frac{\sqrt{\rho_2}}{\sqrt{m}}$ together with the fact that $|s_i| \leq \|s\|, \text{ for } i = 1, \ldots, n,$ we have

$|s_i| \leq \mu \frac{\sqrt{\rho_2}}{\sqrt{m}}, \text{ for } i = 1, \ldots, n$

(c) For the z-subsystem, consider the states $(z_1, z_2) \in \Omega_z(c) \setminus \Omega_z(\mu)$ and $(s, \tilde{e}_1, \tilde{e}_2) \in \Omega_s(\mu) \times \Omega_{\tilde{e}}(\mu)$. Following the same arguments presented in point (1)-(c), it is not hard to see that, once ($\tilde{e}_1(t)$, $\tilde{e}_2(t)$) enter the set $\Omega_{\tilde{e}}(\mu)$, and $s(t)$ is inside the set $\Omega_s(\mu)$, (z_1, z_2) will enter the set $\Omega_z(\mu)$ in finite time and stay in it for all future times.

Hence, the set $\Omega(\mu) = \Omega_{\tilde{e}}(\mu) \times \Omega_s(\mu) \times \Omega_z(\mu)$ is positively invariant and attracting for all trajectories of the system (4 - 7) originating inside the set $\Omega(c) = \Omega_{\tilde{e}}(c) \times \Omega_s(c) \times \Omega_z(c)$ with $\mu < c$. ■

Remark 4.4. Note that the results above show that the control law achieves semiglobal practical stability. Its region of attraction can be given as the set $\Omega_{\tilde{e}}(c) \times \Omega_s(c) \times \Omega_z(c)$, while its ultimate invariant set given by $\Omega_{\tilde{e}}(\mu) \times \Omega_s(\mu) \times \Omega_z(\mu)$. The semiglobal nature of the control law can be seen where the estimate of region of attraction for each set of initial states can be increased by choosing a sufficiently big gains $K,$ and L_3, for any K_1, K_2, and L_1, L_2. While the practical stability nature, pertaining to the set where the solutions converge is stable and may be reduced at will, can be achieved as well by tuning the gains K and L_3, (see the conditions on K and L_3 in Theorem 4.1).

Remark 4.5. The observation error dynamics comprise of the homogeneous super-twisting based observer plus a linear damping term. Note that the linear term depends on the velocity observation error, which is not available. However, it is not necessary in implementation by formulating as per (4 - 6).

Next, consider the special case of Theorem 4.1 with the observer parameter $p = 0.5$. In particular, when $p = 0.5$, the non-Lipschitz terms of the observer (4 - 7) will contain discontinuity and becomes a super-twisting observer with a linear damping term. From the results of Chapter 2 (see section 2.4.2), it has been shown that when $p = 0.5$, the super-twisting algorithm is able to be exactly robust with respect to persistent, non-vanishing, additive disturbances. Thus, these properties will be examined in the following.

Theorem 4.2: Consider the special case of the observer with $p = 0.5$. Using Theorem 4.1 with the following additional inequality,

$$\min\left\{l_{2i},\left(\frac{3}{4}\right)\left(\frac{3}{8}\right)^{\frac{1}{2}}\left(\frac{1}{2^{\frac{3}{2}}}\right)\frac{l_{1i}l_{2i}^{\ 2}}{\left(\gamma_{2i}(\mu)+l_{2i}\right)^{\frac{3}{2}}}\right\} > \max\left\{4\gamma_{2i}(\mu),\ \left(\frac{4}{3}\right)^{2}\left(\frac{8}{3}\right)2^{3}\frac{\gamma_{2i}(\mu)^{3}}{l_{1i}^{\ 2}l_{2i}}\right\}$$

which can be satisfied for sufficiently large L_1 and L_2, the observation error, $(\tilde{e}_1,\tilde{e}_2)$ will converge to zero in finite time and stay there for all future times.

Proof of Theorem 4.2: The inequality of Theorem 4.2 can be satisfied by sufficiently large L_1 and L_2. Then, using similar arguments as in the proofs of Theorem 4.1, for any compact set of initial conditions, there always exists a $c > 0$ such that initially, $(z_1,z_2,s,\tilde{e}_1,\tilde{e}_2)\in\Omega(c)$. Also, a $\mu > 0$ can be selected such that $c > \mu > 0$. Then, by selecting $\bar{k}$ and L_3 as per Theorem 4.1, all the trajectories starting in $\Omega(c)$ will enter the compact set $\Omega(\mu)$ in finite time and stay in it for all future times.

Next, recall that the observer error dynamics for $p = 0.5$,

$$\begin{aligned}\dot{\tilde{e}}_1 &= -L_1\mathrm{sig}(\tilde{e}_1)^{0.5}+\tilde{e}_2\ ,\\ \dot{\tilde{e}}_2 &= -L_2\mathrm{sig}(\tilde{e}_1)^{0}-L_3\tilde{e}_2+\Delta_2(\cdot)\end{aligned}$$

with element wise, $\forall\ i = 1,\ldots,n$,

$$\begin{aligned}\dot{\tilde{e}}_{1i} &= -l_{1i}\mathrm{sig}(\tilde{e}_{1i})^{0.5}+\tilde{e}_{2i}\ ,\\ \dot{\tilde{e}}_{2i} &= -l_{2i}\mathrm{sign}(\tilde{e}_{1i})-l_{3i}\tilde{e}_{2i}+\Delta_{2i}(\cdot)\end{aligned} \qquad (4 - 10)$$

Note that inside $\Omega(\mu)$, we have the upper bound

$$\boldsymbol{K}[\Delta_2(\cdot)] \leq \gamma_2(\mu),$$

Consider the Lyapunov function,

$$W_{\tilde{e}i}(\tilde{e}_{1i},\tilde{e}_{2i}) = \left(\frac{1}{2}\tilde{e}_{2i}^{\ 2} + l_{2i}|\tilde{e}_{1i}|\right)^2 - r_{\tilde{e}}|\tilde{e}_{1i}||\tilde{e}_{2i}|^2 \operatorname{sign}(\tilde{e}_{1i}\tilde{e}_{2i})$$

(Note that this Lyapunov function has appeared as a strict Lyapunov function for the super-twisting based algorithm in Section 2.4.2)

where $\underline{\pi}_2\left(|\tilde{e}_{2i}|^4 + |\tilde{e}_{1i}|^2\right) \leq W_{\tilde{e}i} \leq \overline{\pi}_2\left(|\tilde{e}_{2i}|^4 + |\tilde{e}_{1i}|^2\right)$

$$\underline{\pi}_2 := \min\left\{\frac{1}{8}, \frac{l_{2i}^{\ 2}}{2}\right\},\ \overline{\pi}_2 := \max\left\{l_{2i}^{\ 2} + \frac{l_{2i}}{2} + \frac{r_{\tilde{e}}}{2},\ \frac{1}{4} + \frac{l_{2i}}{2} + \frac{r_{\tilde{e}}}{2}\right\}$$

with (see section 2.4.2) $\forall\ i = 1,\ldots,n$,

$$\left(\frac{1}{4}\right)^{\frac{1}{2}} l_{2i} > r_{\tilde{e}}$$

Thus, $W_{\tilde{e}i}$ is positive definite and radially unbounded (see section 2.4.2). Since (4 - 10) is a differential equation that has discontinuous right-hand side, its solutions are understood in the sense of Filippov (see definition 2.1). According to lemma 2.4, the time derivative of non-smooth Lyapunov function exists almost everywhere

$$\frac{d}{dt} W_{\tilde{e}i}(\tilde{e}_{1i},\tilde{e}_{2i}) \overset{a.e.}{\in} \dot{\tilde{W}}_{\tilde{e}i} = \bigcap_{\xi \in \partial W_{\tilde{e}i}(\tilde{e}_{1i},\tilde{e}_{2i})} \xi^{\mathrm{T}} \boldsymbol{K}\begin{bmatrix} \dot{\tilde{e}}_{1i} \\ \dot{\tilde{e}}_{2i} \end{bmatrix}$$

Note that for $p = 0.5$, $W_{\tilde{e}i}$ is not differentiable on $\tilde{e}_{1i} = 0$:

$$\begin{bmatrix} \dot{\tilde{e}}_{1i} \\ \dot{\tilde{e}}_{2i} \end{bmatrix} \in \begin{bmatrix} -l_{1i}\,\mathrm{sig}(\tilde{e}_{1i})^{\frac{1}{2}} + \tilde{e}_{2i} \\ -l_{2i}\boldsymbol{K}[\mathrm{sign}(\tilde{e}_{1i})] - l_{3i}\tilde{e}_{2i} + \boldsymbol{K}[\Delta_{2i}(\cdot)] \end{bmatrix},$$

and $$\partial W_{\tilde{e}i}(\tilde{e}_{1i},\tilde{e}_{2i}) = \begin{bmatrix} \left(\tilde{e}_{2i}^{\ 2} + 2l_{2i}|\tilde{e}_{1i}|\right)\left(l_{2i}\mathrm{SGN}(\tilde{e}_{1i}) - r_{\tilde{e}}|\tilde{e}_{2i}|^2\,\mathrm{sign}(\tilde{e}_{2i})\right) \\ \left(\tilde{e}_{2i}^{\ 2} + 2l_{2i}|\tilde{e}_{1i}|\right)(\tilde{e}_{2i}) - 2r_{\tilde{e}}\tilde{e}_{1i}|\tilde{e}_{2i}| \end{bmatrix}$$

For $\tilde{e}_{1i} \neq 0$, $\forall\ \tilde{e}_{2i} \in \mathbb{R}$:

$$\begin{aligned}\dot{\tilde{W}}_{\tilde{e}i} &= -2l_{1i}l_{2i}^{\ 2}|\tilde{e}_{1i}|^{\frac{3}{2}} - l_{3i}|\tilde{e}_{2i}|^4 - l_{1i}l_{2i}|\tilde{e}_{1i}|^{\frac{1}{2}}|\tilde{e}_{2i}|^2 - 2l_{2i}l_{3i}|\tilde{e}_{1i}||\tilde{e}_{2i}|^2 \\ &\quad + r_{\tilde{e}}l_{1i}|\tilde{e}_{1i}|^{\frac{1}{2}}|\tilde{e}_{2i}|^2\,\mathrm{sign}(\tilde{e}_{1i}\tilde{e}_{2i}) - r_{\tilde{e}}|\tilde{e}_{2i}|^3 + 2r_{\tilde{e}}l_{2i}|\tilde{e}_{1i}||\tilde{e}_{2i}| \\ &\quad + 2r_{\tilde{e}}l_{3i}\tilde{e}_{1i}|\tilde{e}_{2i}|^2\,\mathrm{sign}(\tilde{e}_{2i}) + \boldsymbol{K}[\Delta_{2i}(\cdot)]\left(2l_{2i}|\tilde{e}_{1i}|\tilde{e}_{2i} + \tilde{e}_{2i}^{\ 3} - 2r_{\tilde{e}}\tilde{e}_{1i}|\tilde{e}_{2i}|\right)\end{aligned}$$

For $\tilde{e}_{1i} = 0$, $\forall\ \tilde{e}_{2i} \in \mathbb{R}$:

Let

$\left[\left(l_{2i}\tilde{e}_{2i}^{\ 2}\right)\xi_2 - r_{\tilde{e}}|\tilde{e}_{2i}|^2\,\mathrm{sign}(\tilde{e}_{2i}),\ \tilde{e}_{2i}^{\ 3}\right]^T$ with $\xi_2 \in [-1, 1]$ be an arbitrary element of $\partial W_{\tilde{e}i}$, then

$$\begin{aligned}\xi^{\mathrm{T}}\boldsymbol{K}\begin{bmatrix} \dot{\tilde{e}}_{1i} \\ \dot{\tilde{e}}_{2i} \end{bmatrix} &= \left[\left(l_{2i}\tilde{e}_{2i}^{\ 2}\right)\xi_2 - r_{\tilde{e}}|\tilde{e}_{2i}|^2\,\mathrm{sign}(\tilde{e}_{2i}),\ \tilde{e}_{2i}^{\ 3}\right]\begin{bmatrix} \tilde{e}_{2i} \\ -l_{2i}[-1,1] - l_{3i}\tilde{e}_{2i} + \boldsymbol{K}[\Delta_{2i}(\cdot)] \end{bmatrix} \\ &= \left([\xi_2 - 1, \xi_2 + 1]\right)l_{2i}\tilde{e}_{2i}^{\ 3} - r_{\tilde{e}}|\tilde{e}_{2i}|^3 - l_{3i}|\tilde{e}_{2i}|^4 + \boldsymbol{K}[\Delta_{2i}(\cdot)]\tilde{e}_{2i}^{\ 3}\end{aligned}$$

implies

$$\begin{aligned}\dot{\tilde{W}}_{\tilde{e}i} &= \bigcap_{\xi_2 \in [-1,1]}\left([\xi_2 - 1, \xi_2 + 1]\right)l_{2i}\tilde{e}_{2i}^{\ 3} - r_{\tilde{e}}|\tilde{e}_{2i}|^3 - l_{3i}|\tilde{e}_{2i}|^4 + \boldsymbol{K}[\Delta_{2i}(\cdot)]\tilde{e}_{2i}^{\ 3} \\ &= -r_{\tilde{e}}|\tilde{e}_{2i}|^3 - l_{3i}|\tilde{e}_{2i}|^4 + \boldsymbol{K}[\Delta_{2i}(\cdot)]\tilde{e}_{2i}^{\ 3}\end{aligned}$$

Thus, for all $(\tilde{e}_{1i}, \tilde{e}_{2i}) \in \mathbb{R}^n$, after rearrangement:

$$\dot{W}_{\tilde{e}i}(\tilde{e}_{1i},\tilde{e}_{2i}) \overset{a.e.}{\leq} -l_{1i}l_{2i}^{\ 2}|\tilde{e}_{1i}|^{1.5} - \frac{r_{\tilde{e}}}{2}|\tilde{e}_{2i}|^3 - l_{3i}|\tilde{e}_{2i}|^4 + \dot{V}_1$$

where

$$\dot{V}_1 = -l_{1i}l_{2i}^{\,2}\left|\tilde{e}_{1i}\right|^{1.5} - \frac{r_{\tilde{e}}}{4}\left|\tilde{e}_{2i}\right|^{3} - \left(\frac{r_{\tilde{e}}}{4} - \gamma_{2i}(\mu)\right)\left|\tilde{e}_{2i}\right|^{3} - l_{1i}\left(l_{2i} - r_{\tilde{e}}\right)\left|\tilde{e}_{1i}\right|^{0.5}\left|\tilde{e}_{2i}\right|^{2}$$
$$+2l_{2i}\gamma_{2i}(\mu)\left|\tilde{e}_{1i}\right|\left|\tilde{e}_{2i}\right| + 2r_{\tilde{e}}\left(\gamma_{2i}(\mu) + l_{2i}\right)\left|\tilde{e}_{1i}\right|\left|\tilde{e}_{2i}\right| - 2l_{3i}\left(l_{2i} - r_{\tilde{e}}\right)\left|\tilde{e}_{1i}\right|\left|\tilde{e}_{2i}\right|^{2}$$

Applying lemma 2.1:

$$-\left|\tilde{e}_{1i}\right|^{1.5} - \left|\tilde{e}_{2i}\right|^{3} \le -(1.5)^{\frac{1}{1.5}}\left|\tilde{e}_{1i}\right|(3)^{\frac{1}{3}}\left|\tilde{e}_{2i}\right|$$

Thus, if the following inequalities

$$\min\left\{l_{2i},\ \left(\frac{3}{4}\right)\left(\frac{3}{8}\right)^{\frac{1}{2}}\left(\frac{1}{2^{\frac{3}{2}}}\right)\frac{l_{1i}l_{2i}^{\,2}}{\left(\gamma_{2i}(\mu) + l_{2i}\right)^{\frac{3}{2}}}\right\} > r_{\tilde{e}} > \max\left\{4\gamma_{2i}(\mu),\ \left(\frac{4}{3}\right)^{2}\left(\frac{8}{3}\right)2^{3}\,\frac{\gamma_{2i}(\mu)^{3}}{l_{1i}^{\,2}l_{2i}}\right\}$$

hold then the function $\dot{V}_1$ is negative definite. Then,

$$\dot{W}_{\tilde{e}i}\left(\tilde{e}_{1i}, \tilde{e}_{2i}\right) \overset{a.e.}{\le} -l_{1i}l_{2i}^{\,2}\left|\tilde{e}_{1i}\right|^{1.5} - \frac{r_{\tilde{e}}}{2}\left|\tilde{e}_{2i}\right|^{3}$$

Employing lemma 2.2, and the bounds of the Lyapunov function, we have:

$$\begin{aligned}\dot{W}_{\tilde{e}i}\left(\tilde{e}_{1i}, \tilde{e}_{2i}\right) &\overset{a.e.}{\le} -\omega_5\left(\left|\tilde{e}_{1i}\right|^{1.5} + \left|\tilde{e}_{2i}\right|^{3}\right) = -\omega_5\left(\left(\left|\tilde{e}_{1i}\right|^{2}\right)^{\frac{3}{4}} + \left(\left|\tilde{e}_{2i}\right|^{4}\right)^{\frac{3}{4}}\right) \\ &\le -\omega_5\left(\left|\tilde{e}_{1i}\right|^{2} + \left|\tilde{e}_{2i}\right|^{4}\right)^{\frac{3}{4}} \\ &\le -\left(\frac{\omega_5}{\bar{\pi}_2^{\frac{3}{4}}}\right)W_{\tilde{e}i}^{\frac{3}{4}}\end{aligned}$$

where $\omega_5 = \min\left\{l_{1i}l_{2i}^{\,2},\ \frac{r_{\tilde{e}}}{2}\right\}$

Hence, the states $(\tilde{e}_{1i}, \tilde{e}_{2i})$ will converge to the origin in finite time. ■

Remark 4.6. The addition of a linear damping term to the observer, L_3 has the benefit of reducing the gain of the super-twisting part of the observer (i.e. when p = 0.5) in achieving exact robustness property with respect to persistent, non-vanishing disturbances. Essentially, from Theorem 4.1, the L_3 term is instrumental for all the trajectories starting in $\Omega(c)$ to enter a smaller compact set $\Omega(\mu)$. In particular, note that the term $\gamma_2(\mu) < \gamma_2(c)$ (see definition in (4 - 7) and section 4.3), thus, from the condition of Theorem 4.2, a smaller gain pair of L_1, L_2 is required to dominate $\gamma_2(\mu)$ as opposed to that of $\gamma_2(c)$.

Remark 4.7. Once the observation error converge to zero in finite time as per Theorem 4.2, the control law (4 - 4) will be identical to that of (3 - 2) presented in Chapter 3 for full-state feedback.

Theorem 4.3: In addition to the conditions in Theorem 4.1, consider the special case of regulation problem, where the desired trajectory is a constant value, q_0, of the robot dynamics (4 - 3) and control (4 - 4) and observer (4 - 6) with the parameter b restricted to $0 < b \leq 1$, and a and p selected as

$$a = \frac{2b}{1+b}, \ p = \frac{1+b}{2}$$

and the disturbance is upper bounded by $\|D\| \le p_1\|e_1\| + p_2\|\dot{q}\| + p_3\|e_1\|^2 + p_4\|\dot{q}\|^2$, i.e. vanishing perturbation. Then, for $b = 1$, semiglobal exponential regulation is guaranteed, provided that K and L_3 are large enough with respect to initial error conditions. While, for $0 < b < 1$, semiglobal finite-time regulation is assured, provided that K and L_3 are large enough with respect to initial error conditions, and the gravity vector at the constant desired position, $G(q_d)$ is zero.

Proof of Theorem 4.3: For this section, power of the control law a is selected as

$$a = \frac{2b}{1+b},\; p = \frac{1+b}{2} \text{ for } 0 < b \le 1.$$

Let,

$$\bar{z}_1 = 0,\quad \bar{z}_2 = -\bar{s},\quad \bar{s} = -\mathrm{sig}\left(K^{-1}G(q_0)\right)^{\frac{1+b}{2b}},$$

$$\tilde{z}_1 = z_1 - \bar{z}_1,\; \tilde{z}_2 = z_2 - \bar{z}_2,\; \tilde{s} = s - \bar{s},$$

where

$$\bar{s}_i = -\left|k_i^{-1} g_i(q_0)\right|^{\frac{1+b}{2b}} \mathrm{sign}\left(k_i^{-1} g_i(q_0)\right) \text{ and } g_i(q_0) \text{ is the } i\text{-th element of the vector } G(q_0).$$

Note that q_0 is a constant vector, and as a result the $\bar{s}$ vector is a constant vector since the matrix K comprises constants as well. Also, $\bar{s}$ is a constant that is defined for stability analyses only, its actual value, which require knowledge of gravity vector, $G(q)$ is not required in the control law. Also note that D comprise of vanishing perturbations only, i.e. $p_0 = 0$ (note that this is the general assumption on regulation problem, however, if constant perturbation do exists, the constant vector, $\bar{s}$ can be redefined to accommodate this extra constant term).

Hence the closed loop system (4 - 7) could be rewritten as

$$
\begin{aligned}
\dot{\tilde{z}}_1 &= \tilde{z}_2 + \tilde{s}, \\
\dot{\tilde{z}}_2 &= -K_2 \operatorname{sig}(\tilde{z}_2 + \tilde{s} + \tilde{e}_2)^{\frac{2b}{1+b}} - K_1 \operatorname{sig}(\tilde{z}_1)^b \\
\dot{\tilde{s}} &= -M^{-1}(\tilde{z}_1 + q_d) K \operatorname{sig}(\tilde{s})^{\frac{2b}{1+b}} + M^{-1}(\tilde{z}_1 + q_d) K \left(\operatorname{sig}(\tilde{s})^{\frac{2b}{1+b}} - \operatorname{sig}(s)^{\frac{2b}{1+b}} + \operatorname{sig}(\bar{s})^{\frac{2b}{1+b}} \right) \\
&\quad - M^{-1}(\tilde{z}_1 + q_d) K \left(\operatorname{sig}(s + \tilde{e}_2)^{\frac{2b}{1+b}} - \operatorname{sig}(s)^{\frac{2b}{1+b}} \right) + \tilde{\Delta}_1(\cdot) \\
\dot{\tilde{e}}_1 &= -L_1 \operatorname{sig}(\tilde{e}_1)^{\frac{1+b}{2}} + \tilde{e}_2 \ , \\
\dot{\tilde{e}}_2 &= -L_2 \operatorname{sig}(\tilde{e}_1)^b - L_3 \tilde{e}_2 + \tilde{\Delta}_2(\cdot)
\end{aligned}
\tag{4 - 11}
$$

where

$$
\begin{aligned}
\tilde{\Delta}_1(\cdot) = &-M^{-1}(\tilde{z}_1 + q_d) C(\tilde{z}_1 + q_d, \tilde{z}_2 + \tilde{s} + \dot{q}_d)(\tilde{z}_2 + \tilde{s} + \dot{q}_d) - M^{-1}(\tilde{z}_1 + q_d) F(\tilde{z}_2 + \tilde{s} + \dot{q}_d) \\
&- M^{-1}(\tilde{z}_1 + q_d)(G(\tilde{z}_1 + q_d) - G(q_0)) - M^{-1}(\tilde{z}_1 + q_d) D - \ddot{q}_d \\
&+ K_2 \operatorname{sig}(\tilde{z}_2 + \tilde{s} + \tilde{e}_2)^{\frac{2b}{1+b}} + K_1 \operatorname{sig}(\tilde{z}_1)^b
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\Delta}_2(\cdot) = &-M^{-1}(\tilde{z}_1 + q_d) K \operatorname{sig}(\bar{s})^{\frac{2b}{1+b}} + M^{-1}(\tilde{z}_1 + q_d) K \operatorname{sig}(s + \tilde{e}_2)^{\frac{2b}{1+b}} \\
&+ M^{-1}(\tilde{z}_1 + q_d) C(\tilde{z}_1 + q_d, \tilde{z}_2 + \tilde{s} + \dot{q}_d)(\tilde{z}_2 + \tilde{s} + \dot{q}_d) \\
&+ M^{-1}(\tilde{z}_1 + q_d) F(\tilde{z}_2 + \tilde{s} + \dot{q}_d) + M^{-1}(\tilde{z}_1 + q_d)(G(\tilde{z}_1 + q_d) - G(q_0)) \\
&+ M^{-1}(\tilde{z}_1 + q_d) D + \ddot{q}_d
\end{aligned}
$$

Note that the fact of $\operatorname{sig}(\bar{s})^{\frac{2b}{1+b}} = -K^{-1} G(q_0)$ has been employed based on the above definition.

Consider the Lyapunov function

$$
V(\tilde{s}, \tilde{z}_1, \tilde{z}_2, \tilde{e}_1, \tilde{e}_2) = [V_{\tilde{s}}(\tilde{s})]^2 + V_{\tilde{z}}(\tilde{z}_1, \tilde{z}_2) + W_{\tilde{e}}(\tilde{e}_1, \tilde{e}_2)
$$

with $V_{\tilde{s}}(\tilde{s}) = \frac{1}{2}\tilde{s}^{\mathrm{T}} M \tilde{s}$, $V_{\tilde{z}}(\tilde{z}_1, \tilde{z}_2) = \sum_{i=1}^{n} V_{\tilde{z}i}(\tilde{z}_{1i}, \tilde{z}_{2i})$, $W_{\tilde{e}}(\tilde{e}_1, \tilde{e}_2) = \sum_{i=1}^{n} W_{\tilde{e}i}(\tilde{e}_{1i}, \tilde{e}_{2i})$.

Note that the $\tilde{s}$ and $\tilde{z}$ subsystem have the same form as the s and z subsystem discussed above in Theorem 4.1 (i.e. by replacing (s, z_1, z_2) with ($\tilde{s}$, $\tilde{z}_1$, $\tilde{z}_2$). Hence, a similar Lyapunov function structure is employed here, where

$$V_{\tilde{z}i}(\tilde{z}_{1i}, \tilde{z}_{2i}) = \frac{k_{1i}^{\;2}}{(1+b)^2}|\tilde{z}_{1i}|^{2+2b} + \frac{1}{4}|\tilde{z}_{2i}|^4 + r_{\tilde{z}i}|\tilde{z}_{1i}|^{\frac{3+3b}{2}}\operatorname{sign}(\tilde{z}_{1i})\tilde{z}_{2i} + \frac{k_{1i}}{(1+b)}|\tilde{z}_{1i}|^{1+b}|\tilde{z}_{2i}|^2$$

and $W_{\tilde{e}i}(\tilde{e}_{1i}, \tilde{e}_{2i}) = \left(\frac{1}{2}\tilde{e}_{2i}^{\;2} + \frac{l_{2i}}{1+b}|\tilde{e}_{1i}|^{1+b}\right)^2 - r_{\tilde{e}}|\tilde{e}_{1i}||\tilde{e}_{2i}|^{\frac{2+4b}{1+b}}\operatorname{sign}(\tilde{e}_{1i}\tilde{e}_{2i})$

Note that $r_{\tilde{z}i} > 0$ always exists for any $k_{1i} > 0$ and $k_{2i} > 0$ such that $V_{\tilde{z}i} > 0$. While

$$\therefore W_{r\tilde{e}}(\tilde{e}_1, \tilde{e}_2) \geq 0 \quad \text{for } \forall i = 1, \ldots, n, \quad \left(\frac{1+b}{8b+4}\right)^{\frac{1+2b}{2+2b}}\left(\frac{1}{1+b}\right)^{\frac{1}{2+2b}} l_{2i}^{\;\frac{1}{1+b}} > r_{\tilde{e}}$$

where $r_{\tilde{e}} > 0$ is a constant . This Lyapunov function is positive definite (see Section 2.4.2 in Chapter 2).

Remark 4.8. Note that an additional term is included for the observer Lyapunov function (see Section 4.1) to reflect its finite time nature, which will be shown in the following development.

Note the following properties of the Lyapunov functions:

$$\frac{1}{2}\underline{m}\|\tilde{s}\|^2 \le V_{\tilde{s}}(\tilde{s}) \le \frac{1}{2}\overline{m}\|\tilde{s}\|^2 \Leftrightarrow \frac{\sqrt{2}}{\sqrt{\overline{m}}}\left[V_{\tilde{s}}(\tilde{s})\right]^{\frac{1}{2}} \le \|\tilde{s}\| \le \frac{\sqrt{2}}{\sqrt{\underline{m}}}\left[V_{\tilde{s}}(\tilde{s})\right]^{\frac{1}{2}}$$

Also, (see Appendix A.1)

$$\begin{aligned}
V_{\tilde{z}}(\tilde{z}_1,\tilde{z}_2) &\ge \sum_{i=1}^{n}\underline{\pi}_{1i}\left(|\tilde{z}_{1i}|^{2+2b} + |\tilde{z}_{2i}|^4\right) \\
&\ge \underline{\pi}_1\left(\sum_{i=1}^{n}|\tilde{z}_{1i}|^{2+2b} + \sum_{i=1}^{n}|\tilde{z}_{2i}|^4\right), \text{ where } \underline{\pi}_1 := \min_i\{\underline{\pi}_{1i}\}, \\
&\ge \underline{\pi}_1\left(\frac{1}{n^b}\left(\sum_{i=1}^{n}\left(|\tilde{z}_{1i}|^2\right)\right)^{\frac{2+2b}{2}} + \frac{1}{n}\left(\sum_{i=1}^{n}\left(|\tilde{z}_{2i}|^2\right)\right)^{\frac{4}{2}}\right), \quad \text{using lemma 2 of chapter 2,} \\
&= \underline{\pi}_1\left(\frac{1}{n^b}\|\tilde{z}_1\|^{2+2b} + \frac{1}{n}\|\tilde{z}_2\|^4\right) \\
&\ge \frac{\underline{\pi}_1}{n}\left(\|\tilde{z}_1\|^{2+2b} + \|\tilde{z}_2\|^4\right), \text{ since } \min\left\{\frac{1}{n^b},\frac{1}{n}\right\} = \frac{1}{n} \text{ for } 0 < b \le 1 \text{ and } n \ge 1,
\end{aligned}$$

and the term,

$$\begin{aligned}
\|\tilde{z}_1\|^b + \|\tilde{z}_2\|^{\frac{2b}{1+b}} &= \left(\|\tilde{z}_1\|^{2+2b}\right)^{\frac{b}{2+2b}} + \left(\|\tilde{z}_2\|^4\right)^{\frac{b}{2+2b}} \\
&\le n^{\frac{2+b}{2+2b}}\left(\|\tilde{z}_1\|^{2+2b} + \|\tilde{z}_2\|^4\right)^{\frac{b}{2+2b}}, \quad \text{using lemma 2 of chapter 2,} \\
&\le n^{\frac{2+b}{2+2b}}\left(\frac{n}{\underline{\pi}_i}V_{\tilde{z}}(\tilde{z}_1,\tilde{z}_2)\right)^{\frac{b}{2+2b}} \\
&= \left(\frac{n}{\underline{\pi}_i^{\frac{b}{2+2b}}}\right)\left[V_{\tilde{z}}(\tilde{z}_1,\tilde{z}_2)\right]^{\frac{b}{2+2b}}
\end{aligned}$$

Consider the time derivative of the Lyapunov function $V_{\tilde{s}}$ along the solutions of the system for the $\tilde{s}$-subsystem:

$$\dot{V}_{\tilde{s}} \overset{a.e}{\in} \dot{\tilde{V}}_{\tilde{s}} = \nabla V_{\tilde{s}}^{\mathrm{T}} \boldsymbol{K}[f](\tilde{s}) + \frac{1}{2} s^{\mathrm{T}} \dot{M}(\cdot) s$$

$$\subset \tilde{s}^{\mathrm{T}} M(\cdot) \begin{pmatrix} -M^{-1}(\tilde{z}_1 + q_d) K \operatorname{sig}(\tilde{s})^{\frac{2b}{1+b}} \\ + M^{-1}(\tilde{z}_1 + q_d) K\left(\operatorname{sig}(\tilde{s})^{\frac{2b}{1+b}} - \operatorname{sig}(s)^{\frac{2b}{1+b}} + \operatorname{sig}(\bar{s})^{\frac{2b}{1+b}} \right) \\ - M^{-1}(\tilde{z}_1 + q_d) K\left(\operatorname{sig}(s + \tilde{e}_2)^{\frac{2b}{1+b}} - \operatorname{sig}(s)^{\frac{2b}{1+b}} \right) + \boldsymbol{K}[\tilde{\Delta}_1(\cdot)] \end{pmatrix} + \frac{1}{2} \tilde{s}^{\mathrm{T}} \dot{M}(\cdot) \tilde{s}$$

Note the following four properties that will simplify the above expression:

1. From Theorem 4.1, the states (s, z_1, z_2) will reach and stay inside the compact set, $\Omega_{\hat{e}}(\mu) \times \Omega_s(\mu) \times \Omega_z(\mu)$, and $\bar{s}$, $\bar{z}_1$, $\bar{z}_2$ are constants, we have, for $0 < b \leq 1$, the following upper bounds (Note that the upper bound of the disturbance, $\|D\|$ has been expressed in terms of the state variables of the closed-loop system):

$$\begin{aligned}
\tilde{s}^{\mathrm{T}} M(\cdot)\boldsymbol{K}[\tilde{\Delta}_1(\cdot)] + \frac{1}{2}\tilde{s}^{\mathrm{T}}\dot{M}(\cdot)\tilde{s} \leq & -\underline{f}\|\tilde{s}\|^2 + \bar{f}\|\tilde{s}\|\|\tilde{z}_2\| + \bar{f}\|\tilde{s}\|\|\dot{q}_d\| + \|\tilde{s}\| C_m \|\tilde{z}_2\|^2 + C_m\|\tilde{z}_2\|\|\tilde{s}\|^2 \\
& + C_m\|\dot{q}_d\|\|\tilde{s}\|^2 + \|\tilde{s}\| C_m \|\dot{q}_d\|^2 + 2\|\tilde{s}\| C_m \|\dot{q}_d\|\|\tilde{z}_2\| \\
& + \|\tilde{s}\| k_g \|\tilde{z}_1\| + \|\tilde{s}\| k_g \|q_d - q_0\| + \|\tilde{s}\| p_1 \|\tilde{z}_1\| + \|\tilde{s}\| p_2 \|\tilde{z}_2\| \\
& + p_2\|\tilde{s}\|^2 + \|\tilde{s}\| p_2 \|\dot{q}_d\| + \|\tilde{s}\| p_3 \|\tilde{z}_1\|^2 + \|\tilde{s}\| p_4 \|\tilde{z}_2\|^2 \\
& + p_4\|\tilde{s}\|^3 + \|\tilde{s}\| p_4 \|\dot{q}_d\|^2 + 2p_4\|\tilde{z}_2\|\|\tilde{s}\|^2 + 2\|\tilde{s}\| p_4 \|\tilde{z}_2\|\|\dot{q}_d\| \\
& + 2p_4\|\tilde{s}\|^2\|\dot{q}_d\| + \bar{m}\|\tilde{s}\|\|\ddot{q}_d\| + \bar{m}\bar{k}_1\|\tilde{s}\| n^{\frac{1-b}{2}} \|\tilde{z}_1\|^b \\
& + \bar{m}\bar{k}_2\|\tilde{s}\| n^{\frac{1-b}{2+2b}} \|\tilde{z}_2\|^{\frac{2b}{1+b}} + \bar{m}\bar{k}_2\|\tilde{s}\| n^{\frac{1-b}{2+2b}} \|\tilde{s}\|^{\frac{2b}{1+b}} \\
& + \bar{m}\bar{k}_2\|\tilde{s}\| n^{\frac{1-b}{2+2b}} \|\tilde{e}_2\|^{\frac{2b}{1+b}} \\
\leq & -\underline{f}\|\tilde{s}\|^2 + \gamma_3(\mu)\|\tilde{s}\|^{\frac{1+3b}{1+b}} + \gamma_4(\mu)\|\tilde{s}\|\left(\|\tilde{z}_1\|^b + \|\tilde{z}_2\|^{\frac{2b}{1+b}} \right) \\
& + \bar{m}\bar{k}_2 n^{\frac{1-b}{2+2b}} \|\tilde{s}\|\|\tilde{e}_2\|^{\frac{2b}{1+b}} + \alpha(\|\eta(t)\|)\gamma_5(\mu)\|\tilde{s}\|
\end{aligned}$$

where

$$\bar{k}_1 = \lambda_{\max}(K_1), \quad \bar{k}_2 = \lambda_{\max}(K_2),$$

$$\left(p_2 \|\tilde{s}\|^2 + 3p_4 \|\tilde{s}\|^3 + \bar{m}\bar{k}_2 n^{\frac{1-b}{2+2b}} \|\tilde{s}\| \|\tilde{s}\|^{\frac{2b}{1+b}} \right) \leq \gamma_3(\mu) \|\tilde{s}\|^{\frac{1+3b}{1+b}}$$

$$\begin{pmatrix} \bar{f}\|\tilde{s}\|\|\tilde{z}_2\| + \bar{m}\bar{k}_1 n^{\frac{1-b}{2}} \|\tilde{s}\|\|\tilde{z}_1\|^b + \bar{m}\bar{k}_2 n^{\frac{1-b}{2+2b}} \|\tilde{s}\|\|\tilde{z}_2\|^{\frac{2b}{1+b}} \\ + k_g \|\tilde{s}\|\|\tilde{z}_1\| + p_1 \|\tilde{s}\|\|\tilde{z}_1\| + p_2 \|\tilde{s}\|\|\tilde{z}_2\| \\ + 2C_m \|\tilde{s}\|\|\tilde{z}_2\|^2 + 3p_4 \|\tilde{s}\|\|\tilde{z}_2\|^2 + p_3 \|\tilde{s}\|\|\tilde{z}_1\|^2 + C_m \|\tilde{z}_2\|\|\tilde{s}\|^2 \end{pmatrix} \leq \gamma_4(\mu)\|\tilde{s}\| \left(\|\tilde{z}_1\|^b + \|\tilde{z}_2\|^{\frac{2b}{1+b}} \right)$$

$$\begin{pmatrix} k_g \|\tilde{s}\|\|q_d - q_0\| + C_m \|\dot{q}_d\|\|\tilde{s}\|^2 + 3p_4 \|\tilde{s}\|\|\dot{q}_d\|^2 + p_2 \|\tilde{s}\|\|\dot{q}_d\| \\ + 2C_m \|\tilde{s}\|\|\dot{q}_d\|^2 + \bar{f}\|\tilde{s}\|\|\dot{q}_d\| + \bar{m}\|\tilde{s}\|\|\ddot{q}_d\| \end{pmatrix} \leq \alpha(\|\eta(t)\|)\gamma_5(\mu)\|\tilde{s}\|$$

with $\gamma_3(.)$, $\gamma_4(.)$, $\gamma_5(.)$ are positive functions and $\alpha(.)$ is a class $\mathcal{K}$ function, and the vector $\eta(t)$ is defined as:

$$\eta^{\mathrm{T}}(t) = \left[(q_d(t) - q_0)^{\mathrm{T}}, \dot{q}_d^{\mathrm{T}}(t), \ddot{q}_d^{\mathrm{T}}(t) \right] \in \mathbb{R}^{3n}.$$

2. Let, us define the diagonal matrix $\Lambda := \mathrm{diag}\left(-\left| \mathrm{sig}\left(\frac{\tilde{s}_i}{\bar{s}_i} \right)^{\frac{2b}{1+b}} - \mathrm{sig}\left(\frac{\tilde{s}_i}{\bar{s}_i} + 1 \right)^{\frac{2b}{1+b}} \right| + 1 \right)$, and since, $\mathrm{sig}(\bar{s})^{\frac{2b}{1+b}} = -K^{-1}G(q_0)$,

then, $\sum_{i=1}^{n} \tilde{s}_i k_i \left(|\tilde{s}_i|^{\frac{2b}{1+b}} \mathrm{sign}(\tilde{s}_i) - |s_i|^{\frac{2b}{1+b}} \mathrm{sign}(s_i) + |\bar{s}_i|^{\frac{2b}{1+b}} \mathrm{sign}(\bar{s}_i) \right) = -\tilde{s}^{\mathrm{T}} \Lambda G(q_0)$.

3. From lemma 2.2:

$\sum_{i=1}^{n} |\tilde{s}_i|^{\frac{1+3b}{1+b}} \geq \frac{1}{n^{\frac{2b}{1+b}}} \|\tilde{s}\|^{\frac{1+3b}{1+b}}$, and $\sum_{i=1}^{n} |\tilde{s}_i| \leq \sqrt{n}\|\tilde{s}\|$,

4. From Appendix A.3 - Proposition 2,

$$\left| |s_i + \tilde{e}_{2i}|^{\frac{2b}{1+b}} \operatorname{sign}(s_i + \tilde{e}_{2i}) - |s_i|^{\frac{2b}{1+b}} \operatorname{sign}(s_i) \right| \le 2^{\frac{1-b}{1+b}} |\tilde{e}_{2i}|^{\frac{2b}{1+b}},$$

and the fact that $|\tilde{e}_{2i}| \le \|\tilde{e}_2\|,\ \forall i \in 1,\ldots,n$.

Substituting these properties, $\dot{\tilde{V}}_{\tilde{s}}$ becomes,

$$\begin{aligned}
\dot{\tilde{V}}_{\tilde{s}} \le & -\sum_{i=1}^{n} k_i |\tilde{s}_i|^{\frac{1+3b}{1+b}} + \sum_{i=1}^{n} \tilde{s}_i k_i \left(|\tilde{s}_i|^{\frac{2b}{1+b}} \operatorname{sign}(\tilde{s}_i) - |s_i|^{\frac{2b}{1+b}} \operatorname{sign}(s_i) + |\bar{s}_i|^{\frac{2b}{1+b}} \operatorname{sign}(\bar{s}_i) \right) \\
& - \sum_{i=1}^{n} \tilde{s}_i \left(k_i |s_i + \tilde{e}_{2i}|^{\frac{2b}{1+b}} \operatorname{sign}(s_i + \tilde{e}_{2i}) - k_i |s_i|^{\frac{2b}{1+b}} \operatorname{sign}(s_i) \right) \\
& - \underline{f} \|\tilde{s}\|^2 + \gamma_3(\mu) \|\tilde{s}\|^{\frac{1+3b}{1+b}} + \gamma_4(\mu) \|\tilde{s}\| \left(\|\tilde{z}_1\|^b + \|\tilde{z}_2\|^{\frac{2b}{1+b}} \right) \\
& + \overline{m} \bar{k}_2 n^{\frac{1-b}{2+2b}} \|\tilde{s}\| \|\tilde{e}_2\|^{\frac{2b}{1+b}} + \alpha(\|\eta(t)\|) \gamma_5(\mu) \|\tilde{s}\|
\end{aligned}$$

$$\begin{aligned}
\le & -\frac{\underline{k}}{n^{\frac{2b}{1+b}}} \|s\|^{\frac{1+3b}{1+b}} - \underline{f} \|\tilde{s}\|^2 + \gamma_3(\mu) \|\tilde{s}\|^{\frac{1+3b}{1+b}} + \gamma_4(\mu) \|\tilde{s}\| \left(\|\tilde{z}_1\|^b + \|\tilde{z}_2\|^{\frac{2b}{1+b}} \right) \\
& + \alpha(\|\eta(t)\|) \gamma_5(\mu) \|\tilde{s}\| + \|\tilde{s}\| \|\Lambda G(q_0)\| + \left(\overline{m} \bar{k}_2 n^{\frac{1-b}{2+2b}} + 2^{\frac{1-b}{1+b}} \bar{k} \sqrt{n} \right) \|\tilde{s}\| \|\tilde{e}_2\|^{\frac{2b}{1+b}}
\end{aligned}$$

where

$$\bar{k} = \lambda_{\max}(K), \quad \underline{k} = \lambda_{\min}(K).$$

Employing the bounds on the Lyapunov functions and lemma 2.2, one obtains,

$$\dot{V}_{\tilde{s}} \leq -\left(\frac{k}{n^{\frac{2b}{1+b}}}\right)\left(\frac{\sqrt{2}}{\sqrt{\overline{m}}}\right)^{\frac{1+3b}{1+b}} V_{\tilde{s}}^{\frac{1+3b}{2+2b}} - \underline{f}\left(\frac{2}{\overline{m}}\right)V_{\tilde{s}} + \gamma_3(\mu)\left(\frac{\sqrt{2}}{\sqrt{\underline{m}}}\right)^{\frac{1+3b}{1+b}} V_{\tilde{s}}^{\frac{1+3b}{2+2b}}$$

$$+\gamma_4(\mu)\left(\frac{\sqrt{2}n}{\sqrt{\underline{m}}\,\underline{\pi}_i^{\frac{b}{2+2b}}}\right)V_{\tilde{s}}^{\frac{1}{2}}V_{\tilde{z}}^{\frac{b}{2+2b}} + \left[\alpha(\|\eta(t)\|)\gamma_5(\mu) + \|\Lambda G(q_0)\|\right]\left(\frac{\sqrt{2}}{\sqrt{\underline{m}}}\right)V_{\tilde{s}}^{\frac{1}{2}} \tag{4 - 12}$$

$$+\left(\overline{m}\bar{k}_2 n^{\frac{1-b}{2+2b}} + 2^{\frac{1-b}{1+b}}\bar{k}\sqrt{n}\right)\left(\frac{\sqrt{2}}{\sqrt{\underline{m}}}\right)V_{\tilde{s}}^{\frac{1}{2}}\|\tilde{e}_2\|^{\frac{2b}{1+b}}$$

Similarly, we compute the time derivative of the Lyapunov function for the $\tilde{z}$-subsystem, (see the appendix A.1):

$$V_{\tilde{z}}(\tilde{z}_1, \tilde{z}_2) = \sum_{i=1}^{n} V_{\tilde{z}i}(\tilde{z}_{1i}, \tilde{z}_{2i})$$

Hence,

$$\dot{V}_{\tilde{z}}(\tilde{z}_1, \tilde{z}_2) \leq -d_0\sum_{i=1}^{n} V_{\tilde{z}i}^{\frac{3+5b}{4+4b}} + d_1\sum_{i=1}^{n}|\tilde{s}_i|\,V_{\tilde{z}i}^{\frac{1+2b}{2+2b}} + d_2\sum_{i=1}^{n}|\tilde{s}_i + \tilde{e}_{2i}|^{\frac{2b}{1+b}}V_{\tilde{z}i}^{\frac{3}{4}}$$

where

$$d_0 = \min_i\left(\frac{\omega_{4i}}{\overline{\pi}_{1i}^{\frac{3+5b}{4+4b}}}\right), \qquad d_1 = \max_i\left(\frac{2^{\frac{1}{2+2b}}\omega_{2i}}{\underline{\pi}_{1i}^{\frac{1+2b}{2+2b}}}\right), \qquad d_2 = \max_i a_{1i}\left(\frac{2^{\frac{1}{4}}\omega_{3i}}{\underline{\pi}_{1i}^{\frac{3}{4}}}\right),$$

Using lemma 2.2:

$$-d_0\sum_{i=1}^{n} V_{\tilde{z}i}^{\frac{3+5b}{4+4b}} \leq -d_0\left(\sum_{i=1}^{n} V_{\tilde{z}i}\right)^{\frac{3+5b}{4+4b}},$$

$$d_1 \sum_{i=1}^{n} V_{\tilde{z}i}{}^{\frac{1+2b}{2+2b}} \le n^{\frac{1}{2+2b}} d_1 \left(\sum_{i=1}^{n} V_{\tilde{z}i} \right)^{\frac{1+2b}{2+2b}},$$

$$d_2 \sum_{i=1}^{n} V_{\tilde{z}i}{}^{\frac{3}{4}} \le n^{\frac{1}{4}} d_2 \left(\sum_{i=1}^{n} V_{\tilde{z}i} \right)^{\frac{3}{4}}$$

and the fact that $|x_i| \le \|x\|, \forall i = 1, \ldots, n,\ x \in \mathbb{R}^n$, thus it follows that

$$\begin{aligned} \dot{V}_{\tilde{z}}(\tilde{z}_1, \tilde{z}_2) &\overset{a.e.}{\le} -d_0 V_{\tilde{z}}{}^{\frac{3+5b}{4+4b}} + \|\tilde{s}\| n^{\frac{1}{2+2b}} d_1 V_{\tilde{z}}{}^{\frac{1+2b}{2+2b}} + \|\tilde{s} + \tilde{e}_2\|^{\frac{2b}{1+b}} n^{\frac{1}{4}} d_2 V_{\tilde{z}}{}^{\frac{3}{4}} \\ &\le -d_0 V_{\tilde{z}}{}^{\frac{3+5b}{4+4b}} + n^{\frac{1}{2+2b}} d_1 \|\tilde{s}\| V_{\tilde{z}}{}^{\frac{1+2b}{2+2b}} + n^{\frac{1}{4}} d_2 \|\tilde{s}\|^{\frac{2b}{1+b}} V_{\tilde{z}}{}^{\frac{3}{4}} + n^{\frac{1}{4}} d_2 \|\tilde{e}_2\|^{\frac{2b}{1+b}} V_{\tilde{z}}{}^{\frac{3}{4}} \end{aligned}$$

where in the last inequality, we have employed the following inequality:

$$\begin{aligned} \|\tilde{s} + \tilde{e}_2\|^{\frac{2b}{1+b}} &\le \left(\|\tilde{s}\| + \|\tilde{e}_2\| \right)^{\frac{2b}{1+b}}, \ \text{using triangle inequality} \\ &\le \|\tilde{s}\|^{\frac{2b}{1+b}} + \|\tilde{e}_2\|^{\frac{2b}{1+b}}, \ \text{for } 0 < b \le 1, \text{ using lemma 2 of chapter 2} \end{aligned}$$

Employing the bounds on the Lyapunov functions and lemma 2.2, one obtains,

$$\begin{aligned} \dot{V}_{\tilde{z}}(\tilde{z}_1, \tilde{z}_2) &\overset{a.e.}{\le} -d_0 V_{\tilde{z}}{}^{\frac{3+5b}{4+4b}} + n^{\frac{1}{2+2b}} d_1 \left(\frac{\sqrt{2}}{\sqrt{\underline{m}}} \right) V_{\tilde{s}}{}^{\frac{1}{2}} V_{\tilde{z}}{}^{\frac{1+2b}{2+2b}} \\ &\quad + n^{\frac{1}{4}} d_2 \left(\frac{\sqrt{2}}{\sqrt{\underline{m}}} \right)^{\frac{2b}{1+b}} V_{\tilde{s}}{}^{\frac{b}{1+b}} V_{\tilde{z}}{}^{\frac{3}{4}} + n^{\frac{1}{4}} d_2 \|\tilde{e}_2\|^{\frac{2b}{1+b}} V_{\tilde{z}}{}^{\frac{3}{4}} \end{aligned} \qquad (4 - 13)$$

Next for the $\tilde{e}$-subsystem, the derivative of the Lyapunov function is:

$$\dot{W}_{\tilde{e}} \overset{a.e}{\in} \dot{\tilde{W}}_{\tilde{e}} = \bigcap_{\xi \in \partial W_{\tilde{e}}} \xi^{\mathrm{T}} \boldsymbol{K} \begin{bmatrix} \dot{\tilde{e}}_1 \\ \dot{\tilde{e}}_2 \end{bmatrix} (\tilde{e}_1, \tilde{e}_2)$$

$$\dot{\tilde{W}}_{\tilde{e}} = \nabla W_{\tilde{e}}^{\mathrm{T}} \boldsymbol{K}\begin{bmatrix} \dot{\tilde{e}}_1 \\ \dot{\tilde{e}}_2 \end{bmatrix}(\tilde{e}_1, \tilde{e}_2)$$

$$\subset \sum_{i=1}^{n}\left(\begin{array}{l} -\dfrac{2l_{1i}l_{2i}^{\ 2}}{1+b}|\tilde{e}_{1i}|^{\frac{3+5b}{2}} - l_{3i}|\tilde{e}_{2i}|^4 - l_{1i}l_{2i}|\tilde{e}_{1i}|^{\frac{1+3b}{2}}|\tilde{e}_{2i}|^2 - \dfrac{2l_{2i}l_{3i}}{1+b}|\tilde{e}_{1i}|^{1+b}|\tilde{e}_{2i}|^2 - r_{\tilde{e}}|\tilde{e}_{2i}|^{\frac{3+5b}{1+b}} \\ + r_{\tilde{e}}l_{1i}|\tilde{e}_{1i}|^{\frac{1+b}{2}}|\tilde{e}_{2i}|^{\frac{2+4b}{1+b}}\operatorname{sign}(\tilde{e}_{1i}\tilde{e}_{2i}) + r_{\tilde{e}}\left(\dfrac{2+4b}{1+b}\right)l_{2i}|\tilde{e}_{1i}|^{1+b}|\tilde{e}_{2i}|^{\frac{1+3b}{1+b}} \\ + r_{\tilde{e}}\left(\dfrac{2+4b}{1+b}\right)l_{3i}|\tilde{e}_{1i}||\tilde{e}_{2i}|^{\frac{2+4b}{1+b}}\operatorname{sign}(\tilde{e}_{1i}\tilde{e}_{2i}) \end{array}\right)$$

$$+\sum_{i=1}^{n}\boldsymbol{K}\left[\tilde{\Delta}_{2i}(\cdot)\right]\left(\frac{2l_{2i}}{1+b}|\tilde{e}_{1i}|^{1+b}|\tilde{e}_{2i}| + |\tilde{e}_{2i}|^3 - r_{\tilde{e}}\left(\frac{2+4b}{1+b}\right)\tilde{e}_{1i}|\tilde{e}_{2i}|^{\frac{1+3b}{1+b}}\right)$$

Define $\|\tilde{\Delta}_2(.)\| := \sup\{\|\varepsilon\|: \varepsilon \in K[\tilde{\Delta}_2(.)]\ \}$, then, we obtain:

$$\dot{\tilde{W}}_{\tilde{e}} \leq -\sum_{i=1}^{n}\left(\frac{l_{1i}l_{2i}^{\ 2}}{1+b}|\tilde{e}_{1i}|^{\frac{3+5b}{2}}\right) - \frac{r_{\tilde{e}}}{2}\sum_{i=1}^{n}|\tilde{e}_{2i}|^{\frac{3+5b}{1+b}} + \dot{W}_1$$

$$+\left\|\tilde{\Delta}_2(\cdot)\right\|\sum_{i=1}^{n}\left(\frac{2l_{2i}}{1+b}|\tilde{e}_{1i}|^{1+b}|\tilde{e}_{2i}| + |\tilde{e}_{2i}|^3 + r_{\tilde{e}}\left(\frac{2+4b}{1+b}\right)|\tilde{e}_{1i}||\tilde{e}_{2i}|^{\frac{1+3b}{1+b}}\right)$$

where

$$\dot{W}_1 = \sum_{i=1}^{n}\left(\begin{array}{l} -\dfrac{l_{1i}l_{2i}^{\ 2}}{1+b}|\tilde{e}_{1i}|^{\frac{3+5b}{2}} - l_{3i}|\tilde{e}_{2i}|^4 - l_{1i}l_{2i}|\tilde{e}_{1i}|^{\frac{1+3b}{2}}|\tilde{e}_{2i}|^2 - \dfrac{2l_{2i}l_{3i}}{1+b}|\tilde{e}_{1i}|^{1+b}|\tilde{e}_{2i}|^2 - \dfrac{r_{\tilde{e}}}{2}|\tilde{e}_{2i}|^{\frac{3+5b}{1+b}} \\ + r_{\tilde{e}}l_{1i}|\tilde{e}_{1i}|^{\frac{1+b}{2}}|\tilde{e}_{2i}|^{\frac{2+4b}{1+b}} + r_{\tilde{e}}\left(\dfrac{2+4b}{1+b}\right)l_{2i}|\tilde{e}_{1i}|^{1+b}|\tilde{e}_{2i}|^{\frac{1+3b}{1+b}} + r_{\tilde{e}}\left(\dfrac{2+4b}{1+b}\right)l_{3i}|\tilde{e}_{1i}||\tilde{e}_{2i}|^{\frac{2+4b}{1+b}} \end{array}\right)$$

Applying lemma 2.1 we have $\forall\ i = 1, \ldots, n$:

$$-|\tilde{e}_{1i}|^{\frac{3+5b}{2}} - |\tilde{e}_{2i}|^{\frac{3+5b}{1+b}} \leq -\left(\frac{3+5b}{1+b}\right)^{\frac{1+b}{3+5b}}|\tilde{e}_{1i}|^{\frac{1+b}{2}}\left(\frac{3+5b}{2+4b}\right)^{\frac{2+4b}{3+5b}}|\tilde{e}_{2i}|^{\frac{2+4b}{1+b}},$$

$$-|\tilde{e}_{1i}|^{\frac{3+5b}{2}} - |\tilde{e}_{2i}|^{\frac{3+5b}{1+b}} \leq -\left(\frac{3+5b}{2+2b}\right)^{\frac{2+2b}{3+5b}}|\tilde{e}_{1i}|^{1+b}\left(\frac{3+5b}{1+3b}\right)^{\frac{1+3b}{3+5b}}|\tilde{e}_{2i}|^{\frac{1+3b}{1+b}},$$

$$-\left|\tilde{e}_{2i}\right|^4-\left|\tilde{e}_{1i}\right|^{1+b}\left|\tilde{e}_{2i}\right|^2 \le -\left(\frac{1+b}{b}\right)^{\frac{b}{1+b}}\left|\tilde{e}_{2i}\right|^{\frac{4b}{1+b}}(1+b)^{\frac{1}{1+b}}\left|\tilde{e}_{1i}\right|\left|\tilde{e}_{2i}\right|^{\frac{2}{1+b}}$$

Thus, if the following inequalities

$$\min\left\{\lambda_1\left(\frac{l_{2i}{}^2}{l_{1i}{}^{\frac{2+4b}{1+b}}}\right),\ \lambda_2\left(l_{1i}l_{2i}{}^{\frac{1-b}{2+2b}}\right),\ \lambda_3 l_{2i}{}^{\frac{1}{1+b}}\right\} \ge r_{\tilde{e}}$$

where

$$\lambda_1=\left(\frac{1}{6}\right)^{\frac{3+5b}{1+b}}\left(\frac{6+10b}{(1+b)^2}\right)\left(\frac{3+5b}{2+4b}\right)^{\frac{2+4b}{1+b}},$$

$$\lambda_2=\left(\frac{(1+b)}{12+24b}\right)^{\frac{3+5b}{2+2b}}\left(\frac{3+5b}{(1+b)^2}\right)\left(\frac{3+5b}{1+3b}\right)^{\frac{1+3b}{2+2b}},\ \ \lambda_3=\left(\frac{1+b}{b}\right)^{\frac{b}{1+b}}\left(\frac{1+b}{2+4b}\right)2^{\frac{1}{1+b}},$$

hold then the function $\dot{W}_1$ is negative definite. Note that such an $r_{\tilde{e}\,i}>0$ always exists for any $l_{1i}>0$, $l_{2i}>0$.

Then,

$$\begin{aligned}\dot{W}_{\tilde{e}} \overset{a.e.}{\le} &-\sum_{i=1}^{n}\left(\frac{l_{1i}l_{2i}{}^2}{1+b}\left|\tilde{e}_{1i}\right|^{\frac{3+5b}{2}}\right)-\frac{r_{\tilde{e}}}{2}\sum_{i=1}^{n}\left|\tilde{e}_{2i}\right|^{\frac{3+5b}{1+b}} \\ &+\left\|\tilde{\Delta}_2(\cdot)\right\|\sum_{i=1}^{n}\left(\frac{2l_{2i}}{1+b}\left|\tilde{e}_{1i}\right|^{1+b}\left|\tilde{e}_{2i}\right|+\left|\tilde{e}_{2i}\right|^3+r_{\tilde{e}}\left(\frac{2+4b}{1+b}\right)\left|\tilde{e}_{1i}\right|\left|\tilde{e}_{2i}\right|^{\frac{1+3b}{1+b}}\right)\end{aligned} \qquad (4\text{ - }14)$$

Note the following properties that will aid in simplifying the above expression:

1. Note the above definition of the diagonal matrix, Λ, we have

$$\left(K\mathrm{sig}(\tilde{s})^{\frac{2b}{1+b}}-K\mathrm{sig}(s)^{\frac{2b}{1+b}}+K\mathrm{sig}(\bar{s})^{\frac{2b}{1+b}}\right)=-\Lambda G(q_0),$$

2. While from Appendix A.3-Proposition 2,

$$\left| K\mathrm{sig}(s+\tilde{e}_2)^{\frac{2b}{1+b}} - K\mathrm{sig}(s)^{\frac{2b}{1+b}} \right| \le 2^{\frac{1-b}{1+b}} K \left|\tilde{e}_2\right|^{\frac{2b}{1+b}}, \quad \text{where} \quad \left|\tilde{e}_2\right| = \left(\left|\tilde{e}_{21}\right|, \ldots, \left|\tilde{e}_{2n}\right|\right)^{\mathrm{T}},$$

3. From the property 1 of the manipulator dynamics, $0 < \underline{m_I} \le \left\|M^{-1}\right\| \le \overline{m}_I$,

4. From lemma 2 of chapter 2,

$$\left\| \mathrm{sig}(\tilde{s})^{\frac{2b}{1+b}} \right\| = \left(\sum_{i=1}^{n} \left(\left|\tilde{s}_i\right|^{\frac{2b}{1+b}} \right)^2 \right)^{\frac{1}{2}} = \left(\sum_{i=1}^{n} \left(\left|\tilde{s}_i\right|^2 \right)^{\frac{2b}{1+b}} \right)^{\frac{1}{2}}$$

$$\le \left(n^{\frac{1-b}{1+b}} \left(\sum_{i=1}^{n} \left|\tilde{s}_i\right|^2 \right)^{\frac{2b}{1+b}} \right)^{\frac{1}{2}}, \quad \text{using lemma 2 of chapter 2,}$$

$$= n^{\frac{1-b}{2+2b}} \left(\left(\sum_{i=1}^{n} \left|\tilde{s}_i\right|^2 \right)^{\frac{1}{2}} \right)^{\frac{2b}{1+b}} = n^{\frac{1-b}{2+2b}} \left\|\tilde{s}\right\|^{\frac{2b}{1+b}}$$

and similar arguments we obtained $\left\| \left|\tilde{e}_2\right|^{\frac{2b}{1+b}} \right\| \le n^{\frac{1-b}{2+2b}} \left\|\tilde{e}_2\right\|^{\frac{2b}{1+b}}$,

5. From lemma 2.1 of chapter 2,

$$2\|\tilde{z}_2\|\|\tilde{s}\| + 2\|\tilde{z}_2\|\|\dot{q}_d\| + 2\|\tilde{s}\|\|\dot{q}_d\| \le 2\|\tilde{z}_2\|^2 + 2\|\tilde{s}\|^2 + 2\|\dot{q}_d\|^2,$$

6. From Theorem 4.1, the states (s, z_1, z_2) will reach and stay inside the compact set, $\Omega_{\hat{e}}(\mu) \times \Omega_s(\mu) \times \Omega_z(\mu)$, and since $\overline{s}$, $\overline{z}_1$, and $\overline{z}_2$ are constants, for $0 < b \le 1$, the following functions can be upper bounded by ,

$$\left(\bar{m}_I\,\bar{k}n^{\frac{1-b}{2+2b}}\|\tilde{s}\|^{\frac{2b}{1+b}}+3\bar{m}_I C_m\|\tilde{s}\|^2+\bar{m}_I\bar{f}\|\tilde{s}\|+\bar{m}_I p_2\|\tilde{s}\|+3\bar{m}_I p_4\|\tilde{s}\|^2\right)\le\gamma_6(\mu)\|\tilde{s}\|^{\frac{2b}{1+b}}$$

$$\begin{pmatrix}\bar{m}_I k_g\|\tilde{z}_1\|+\bar{m}_I p_1\|\tilde{z}_1\|+\bar{m}_I p_3\|\tilde{z}_1\|^2+\bar{m}_I p_2\|\tilde{z}_2\|\\+\bar{m}_I\bar{f}\|\tilde{z}_2\|+3\bar{m}_I p_4\|\tilde{z}_2\|^2+3\bar{m}_I C_m\|\tilde{z}_2\|^2\end{pmatrix}\le\gamma_7(\mu)\left(\|\tilde{z}_1\|^b+\|\tilde{z}_2\|^{\frac{2b}{1+b}}\right)$$

$$\begin{pmatrix}\bar{m}_I(p_2+\bar{f})\|\dot{q}_d\|+3\bar{m}_I(p_4+C_m)\|\dot{q}_d\|^2\\+\bar{m}_I k_g\|q_d-q_0\|+\|\ddot{q}_d\|+\bar{m}_I\|\Lambda G(q_0)\|\end{pmatrix}\le\alpha_2(\|\eta(t)\|)+\bar{m}_I\|\Lambda G(q_0)\|$$

with $\gamma_6(.)$, and $\gamma_7(.)$, are positive functions and $\alpha_2(.)$ is a class $\mathcal{K}$ function, and the vector $\eta(t)$ is defined as: $\eta^{\mathrm{T}}(t)=\left[(q_d(t)-q_0)^{\mathrm{T}},\dot{q}_d^{\mathrm{T}}(t),\ddot{q}_d^{\mathrm{T}}(t)\right]\in\mathbb{R}^{3n}$

7. Employing the above properties, after algebraic rearrangement,

$$\|\tilde{\Delta}_2(\cdot)\|\le\gamma_6(\mu)\|\tilde{s}\|^{\frac{2b}{1+b}}+\left(\bar{m}_I\,\bar{k}\,2^{\frac{1-b}{1+b}}n^{\frac{1-b}{2+2b}}\|\tilde{e}_2\|^{\frac{2b}{1+b}}\right)+\gamma_7(\mu)\left(\|\tilde{z}_1\|^b+\|\tilde{z}_2\|^{\frac{2b}{1+b}}\right)$$
$$+\alpha_2(\|\eta(t)\|)+\bar{m}_I\|\Lambda G(q_0)\|$$

Note that the upper bound of the disturbance, $\|D\|$ has been expressed in terms of the state variables of the closed-loop system in obtaining the above inequality.

8. From lemma 2.2 of chapter 2, the following can be obtained,

$$-\sum_{i=1}^{n}|\tilde{e}_{1i}|^{\frac{3+5b}{2}}\le-\frac{1}{n^{\frac{1+5b}{2}}}\|\tilde{e}_1\|^{\frac{3+5b}{2}},\quad -\sum_{i=1}^{n}|\tilde{e}_{2i}|^{\frac{3+5b}{1+b}}\le-\frac{1}{n^{\frac{2+4b}{1+b}}}\|\tilde{e}_2\|^{\frac{3+5b}{1+b}},\quad \sum_{i=1}^{n}|\tilde{e}_{2i}|^3\le n^{\frac{3}{2}}\|\tilde{e}_2\|^3,$$

$$\begin{aligned}\sum_{i=1}^{n}|\tilde{e}_{1i}|^{1+b}|\tilde{e}_{2i}|&\le\|\tilde{e}_1\|^{1+b}\sum_{i=1}^{n}|\tilde{e}_{2i}|,\quad\text{since}\,\|\tilde{e}_1\|\ge|\tilde{e}_{1i}|,\\&\le\sqrt{n}\|\tilde{e}_1\|^{1+b}\|\tilde{e}_2\|\end{aligned}$$

$$\sum_{i=1}^{n}|\tilde{e}_{1i}||\tilde{e}_{2i}|^{\frac{1+3b}{1+b}} \le \|\tilde{e}_2\|^{\frac{1+3b}{1+b}}\sum_{i=1}^{n}|\tilde{e}_{1i}|, \quad \text{since } \|\tilde{e}_2\| \ge |\tilde{e}_{2i}|,$$

$$\le \sqrt{n}\,\|\tilde{e}_1\|\|\tilde{e}_2\|^{\frac{1+3b}{1+b}}$$

Substituting the above properties into (4 - 14), one obtains,

$$\dot{W}_{\tilde{e}} \overset{a.e.}{\le} -\left(\frac{\underline{l_1}\,\underline{l_2}^2}{1+b}\right)\left(\frac{1}{n^{\frac{1+5b}{2}}}\right)\|\tilde{e}_1\|^{\frac{3+5b}{2}} - \left(\frac{r_{\tilde{e}}}{2}\right)\left(\frac{1}{n^{\frac{2+4b}{1+b}}}\right)\|\tilde{e}_2\|^{\frac{3+5b}{1+b}}$$

$$+\begin{pmatrix} \gamma_6(\mu)\|\tilde{s}\|^{\frac{2b}{1+b}} + \overline{m}_I\,\bar{k}\,2^{\frac{1-b}{1+b}}\,n^{\frac{1-b}{2+2b}}\|\tilde{e}_2\|^{\frac{2b}{1+b}} \\ +\gamma_7(\mu)\left(\|\tilde{z}_1\|^b + \|\tilde{z}_2\|^{\frac{2b}{1+b}}\right) + \alpha_2(\|\eta(t)\|) + \overline{m}_I\|\Lambda G(q_0)\| \end{pmatrix} \times \begin{pmatrix} \dfrac{2\overline{l_2}}{1+b}\sqrt{n}\|\tilde{e}_1\|^{1+b}\|\tilde{e}_2\| + n^{\frac{3}{2}}\|\tilde{e}_2\|^3 \\ + r_{\tilde{e}}\left(\dfrac{2+4b}{1+b}\right)\sqrt{n}\,\|\tilde{e}_1\|\|\tilde{e}_2\|^{\frac{1+3b}{1+b}} \end{pmatrix}$$

where

$\lambda_{\min}(L_1) = \underline{l_1}, \quad \lambda_{\min}(L_2) = \underline{l_2}, \quad \lambda_{\max}(L_2) = \overline{l_2},$

Employing the bounds on the Lyapunov functions and lemma 2.2, one obtains,

$$
\begin{aligned}
\dot{W}_{\tilde{e}} \overset{a.e.}{\leq} & -\left(\frac{\underline{l_1}\,\underline{l_2}^2}{1+b}\right)\left(\frac{1}{n^{\frac{1+5b}{2}}}\right)\|\tilde{e}_1\|^{\frac{3+5b}{2}} - \left(\frac{r_{\tilde{e}}}{2}\right)\left(\frac{1}{n^{\frac{2+4b}{1+b}}}\right)\|\tilde{e}_2\|^{\frac{3+5b}{1+b}} \\
& + \overline{l}_2\gamma_6(\mu)\left(\frac{\sqrt{2}}{\sqrt{\underline{m}}}\right)^{\frac{2b}{1+b}}\left(\frac{2\sqrt{n}}{1+b}\right)V_{\tilde{s}}^{\frac{b}{1+b}}\|\tilde{e}_1\|^{1+b}\|\tilde{e}_2\| + \gamma_6(\mu)n^{\frac{3}{2}}\left(\frac{\sqrt{2}}{\sqrt{\underline{m}}}\right)^{\frac{2b}{1+b}}V_{\tilde{s}}^{\frac{b}{1+b}}\|\tilde{e}_2\|^3 \\
& + r_{\tilde{e}}\gamma_6(\mu)\sqrt{n}\left(\frac{\sqrt{2}}{\sqrt{\underline{m}}}\right)^{\frac{2b}{1+b}}\left(\frac{2+4b}{1+b}\right)V_{\tilde{s}}^{\frac{b}{1+b}}\|\tilde{e}_1\|\|\tilde{e}_2\|^{\frac{1+3b}{1+b}} \\
& + \overline{l}_2\left(\frac{2\sqrt{n}}{1+b}\right)2^{\frac{1-b}{1+b}}\overline{m}_I\,\overline{k}\,n^{\frac{1-b}{2+2b}}\|\tilde{e}_1\|^{1+b}\|\tilde{e}_2\|^{\frac{1+3b}{1+b}} + 2^{\frac{1-b}{1+b}}\overline{m}_I\,\overline{k}\,n^{\frac{4+2b}{2+2b}}\|\tilde{e}_2\|^{\frac{3+5b}{1+b}} \\
& + r_{\tilde{e}}\left(\frac{2+4b}{1+b}\right)2^{\frac{1-b}{1+b}}\overline{m}_I\,\overline{k}\,n^{\frac{1}{1+b}}\|\tilde{e}_1\|\|\tilde{e}_2\|^{\frac{1+5b}{1+b}} \\
& + \overline{l}_2\gamma_7(\mu)\left(\frac{2\sqrt{n}}{1+b}\right)\left(\frac{n}{\underline{\pi}_i^{\frac{b}{2+2b}}}\right)V_{\tilde{z}}^{\frac{b}{2+2b}}\|\tilde{e}_1\|^{1+b}\|\tilde{e}_2\| \\
& + \gamma_7(\mu)\left(\frac{n^{\frac{5}{2}}}{\underline{\pi}_i^{\frac{b}{2+2b}}}\right)V_{\tilde{z}}^{\frac{b}{2+2b}}\|\tilde{e}_2\|^3 + r_{\tilde{e}}\gamma_7(\mu)\left(\frac{2+4b}{1+b}\right)\left(\frac{n^{\frac{3}{2}}}{\underline{\pi}_i^{\frac{b}{2+2b}}}\right)V_{\tilde{z}}^{\frac{b}{2+2b}}\|\tilde{e}_1\|\|\tilde{e}_2\|^{\frac{1+3b}{1+b}} \\
& + \left[\alpha_2(\|\eta(t)\|) + \overline{m}_I\|\Lambda G(q_0)\|\right]\times\left(\begin{array}{l}\dfrac{2\overline{l}_2}{1+b}\sqrt{n}\|\tilde{e}_1\|^{1+b}\|\tilde{e}_2\| + n^{\frac{3}{2}}\|\tilde{e}_2\|^3 \\ + r_{\tilde{e}}\left(\dfrac{2+4b}{1+b}\right)\sqrt{n}\|\tilde{e}_1\|\|\tilde{e}_2\|^{\frac{1+3b}{1+b}}\end{array}\right)
\end{aligned}
\tag{4 - 15}
$$

With the above results, we are now in a position to find the time derivative of the Lyapunov function for the closed-loop system, namely:

$V(\tilde{s},\tilde{z}_1,\tilde{z}_2,\tilde{e}_1,\tilde{e}_2) = [V_{\tilde{s}}(\tilde{s})]^2 + V_{\tilde{z}}(\tilde{z}_1,\tilde{z}_2) + W_{\tilde{e}}(\tilde{e}_1,\tilde{e}_2)$ and hence $\dot{V} = 2V_{\tilde{s}}\dot{V}_{\tilde{s}} + \dot{V}_{\tilde{z}} + \dot{W}_{\tilde{e}}$.

Thus, substituting results (4 - 12), (4 - 13) and (4 - 15), after rearrangement,

$$\dot{V} \overset{a.e.}{\leq} -\left(\frac{\underline{k}}{n^{\frac{2b}{1+b}}}\right)\left(\frac{\sqrt{2}}{\sqrt{\overline{m}}}\right)^{\frac{1+3b}{1+b}} V_{\tilde{s}}^{\frac{3+5b}{2+2b}} - \underline{f}\left(\frac{4}{\overline{m}}\right)V_{\tilde{s}}^{2} - \frac{d_0}{2}V_{\tilde{z}}^{\frac{3+5b}{4+4b}} - \left(\frac{1}{2}\right)\left(\frac{\underline{l_1}\ \underline{l_2}^{2}}{1+b}\right)\left(\frac{1}{n^{\frac{1+5b}{2}}}\right)\|\tilde{e}_1\|^{\frac{3+5b}{2}}$$

$$-\left(\frac{r_{\tilde{e}}}{4}\right)\left(\frac{1}{n^{\frac{2+4b}{1+b}}}\right)\|\tilde{e}_2\|^{\frac{3+5b}{1+b}} + \dot{V}_1 + \dot{V}_2 + 2\left[\alpha(\|\eta(t)\|)\gamma_5(\mu) + \|\Lambda G(q_0)\|\right]\left(\frac{\sqrt{2}}{\sqrt{\underline{m}}}\right)V_{\tilde{s}}^{\frac{3}{2}}$$

$$+\left[\alpha_2(\|\eta(t)\|) + \overline{m}_I\|\Lambda G(q_0)\|\right]\times\left(\begin{array}{l}\dfrac{2\overline{l}_2}{1+b}\sqrt{n}\|\tilde{e}_1\|^{1+b}\|\tilde{e}_2\| + n^{\frac{3}{2}}\|\tilde{e}_2\|^{3} \\ + r_{\tilde{e}}\left(\dfrac{2+4b}{1+b}\right)\sqrt{n}\,\|\tilde{e}_1\|\|\tilde{e}_2\|^{\frac{1+3b}{1+b}}\end{array}\right)$$

where

$$\dot{V}_1 = -\left(\frac{1}{4}\right)\left(\frac{\underline{k}}{n^{\frac{2b}{1+b}}}\right)\left(\frac{\sqrt{2}}{\sqrt{\overline{m}}}\right)^{\frac{1+3b}{1+b}} V_{\tilde{s}}^{\frac{3+5b}{2+2b}} - \frac{d_0}{4}V_{\tilde{z}}^{\frac{3+5b}{4+4b}}$$

$$-\left(\left(\frac{1}{4}\right)\left(\frac{\underline{k}}{n^{\frac{2b}{1+b}}}\right)\left(\frac{\sqrt{2}}{\sqrt{\overline{m}}}\right)^{\frac{1+3b}{1+b}} - 2\gamma_3(\mu)\left(\frac{\sqrt{2}}{\sqrt{\underline{m}}}\right)^{\frac{1+3b}{1+b}}\right)V_{\tilde{s}}^{\frac{3+5b}{2+2b}}$$

$$+2\gamma_4(\mu)\left(\frac{\sqrt{2}n}{\sqrt{\underline{m}}\,\underline{\pi}_i^{\frac{b}{2+2b}}}\right)V_{\tilde{s}}^{\frac{3}{2}}V_{\tilde{z}}^{\frac{b}{2+2b}} + n^{\frac{1}{2+2b}}d_1\left(\frac{\sqrt{2}}{\sqrt{\underline{m}}}\right)V_{\tilde{s}}^{\frac{1}{2}}V_{\tilde{z}}^{\frac{1+2b}{2+2b}}$$

$$+n^{\frac{1}{4}}d_2\left(\frac{\sqrt{2}}{\sqrt{\underline{m}}}\right)^{\frac{2b}{1+b}}V_{\tilde{s}}^{\frac{b}{1+b}}V_{\tilde{z}}^{\frac{3}{4}}$$

$$
\dot{V}_2 = -\left(\frac{1}{2}\right)\left(\frac{k}{n^{\frac{2b}{1+b}}}\right)\left(\frac{\sqrt{2}}{\sqrt{\underline{m}}}\right)^{\frac{1+3b}{1+b}} V_{\tilde{s}}^{\frac{3+5b}{2+2b}} - \frac{d_0}{4} V_{\tilde{z}}^{\frac{3+5b}{4+4b}} - \left(\frac{1}{2}\right)\left(\frac{\underline{l_1}\ \underline{l_2}^{2}}{1+b}\right)\left(\frac{1}{n^{\frac{1+5b}{2}}}\right)\left\|\tilde{e}_1\right\|^{\frac{3+5b}{2}}
$$

$$
-\left(\frac{r_{\tilde{e}}}{8}\right)\left(\frac{1}{n^{\frac{2+4b}{1+b}}}\right)\left\|\tilde{e}_2\right\|^{\frac{3+5b}{1+b}} - \left(\left(\frac{r_{\tilde{e}}}{8}\right)\left(\frac{1}{n^{\frac{2+4b}{1+b}}}\right) - 2^{\frac{1-b}{1+b}}\,\overline{m}_I\,\bar{k}\,n^{\frac{4+2b}{2+2b}}\right)\left\|\tilde{e}_2\right\|^{\frac{3+5b}{1+b}}
$$

$$
+ n^{\frac{1}{4}} d_2 \left\|\tilde{e}_2\right\|^{\frac{2b}{1+b}} V_{\tilde{z}}^{\frac{3}{4}} + 2\left(\overline{m}\bar{k}_2 n^{\frac{1-b}{2+2b}} + 2^{\frac{1-b}{1+b}}\,\bar{k}\sqrt{n}\right)\left(\frac{\sqrt{2}}{\sqrt{\underline{m}}}\right) V_{\tilde{s}}^{\frac{3}{2}} \left\|\tilde{e}_2\right\|^{\frac{2b}{1+b}}
$$

$$
+ \overline{l}_2 \gamma_6(\mu)\left(\frac{\sqrt{2}}{\sqrt{\underline{m}}}\right)^{\frac{2b}{1+b}}\left(\frac{2\sqrt{n}}{1+b}\right) V_{\tilde{s}}^{\frac{b}{1+b}} \left\|\tilde{e}_1\right\|^{1+b}\left\|\tilde{e}_2\right\| + \gamma_6(\mu) n^{\frac{3}{2}}\left(\frac{\sqrt{2}}{\sqrt{\underline{m}}}\right)^{\frac{2b}{1+b}} V_{\tilde{s}}^{\frac{b}{1+b}}\left\|\tilde{e}_2\right\|^{3}
$$

$$
+ r_{\tilde{e}}\gamma_6(\mu)\sqrt{n}\left(\frac{\sqrt{2}}{\sqrt{\underline{m}}}\right)^{\frac{2b}{1+b}}\left(\frac{2+4b}{1+b}\right) V_{\tilde{s}}^{\frac{b}{1+b}}\left\|\tilde{e}_1\right\|\left\|\tilde{e}_2\right\|^{\frac{1+3b}{1+b}}
$$

$$
+ \overline{l}_2\left(\frac{2\sqrt{n}}{1+b}\right) 2^{\frac{1-b}{1+b}}\,\overline{m}_I\,\bar{k}\,n^{\frac{1-b}{2+2b}}\left\|\tilde{e}_1\right\|^{1+b}\left\|\tilde{e}_2\right\|^{\frac{1+3b}{1+b}} + r_{\tilde{e}}\left(\frac{2+4b}{1+b}\right) 2^{\frac{1-b}{1+b}}\,\overline{m}_I\,\bar{k}\,n^{\frac{1}{1+b}}\left\|\tilde{e}_1\right\|\left\|\tilde{e}_2\right\|^{\frac{1+5b}{1+b}}
$$

$$
+ \overline{l}_2 \gamma_7(\mu)\left(\frac{2\sqrt{n}}{1+b}\right)\left(\frac{n}{\underline{\pi}_i^{\frac{b}{2+2b}}}\right) V_{\tilde{z}}^{\frac{b}{2+2b}}\left\|\tilde{e}_1\right\|^{1+b}\left\|\tilde{e}_2\right\|
$$

$$
+ \gamma_7(\mu)\left(\frac{n^{\frac{5}{2}}}{\underline{\pi}_i^{\frac{b}{2+2b}}}\right) V_{\tilde{z}}^{\frac{b}{2+2b}}\left\|\tilde{e}_2\right\|^{3} + r_{\tilde{e}}\gamma_7(\mu)\left(\frac{2+4b}{1+b}\right)\left(\frac{n^{\frac{3}{2}}}{\underline{\pi}_i^{\frac{b}{2+2b}}}\right) V_{\tilde{z}}^{\frac{b}{2+2b}}\left\|\tilde{e}_1\right\|\left\|\tilde{e}_2\right\|^{\frac{1+3b}{1+b}}
$$

For $\dot{V}_1$, applying lemma 2.1

$$
-V_{\tilde{s}}^{\frac{3+5b}{2+2b}} - V_{\tilde{z}}^{\frac{3+5b}{4+4b}} \le -\left(\frac{3+5b}{3+3b}\right)^{\frac{3+3b}{3+5b}} V_{\tilde{s}}^{\frac{3}{2}}\left(\frac{3+5b}{2b}\right)^{\frac{2b}{3+5b}} V_{\tilde{z}}^{\frac{b}{2+2b}},
$$

$$
-V_{\tilde{s}}^{\frac{3+5b}{2+2b}} - V_{\tilde{z}}^{\frac{3+5b}{4+4b}} \le -\left(\frac{3+5b}{1+b}\right)^{\frac{1+b}{3+5b}} V_{\tilde{s}}^{\frac{1}{2}}\left(\frac{3+5b}{2+4b}\right)^{\frac{2+4b}{3+5b}} V_{\tilde{z}}^{\frac{1+2b}{2+2b}},
$$

$$
-V_{\tilde{s}}^{\frac{3+5b}{2+2b}} - V_{\tilde{z}}^{\frac{3+5b}{4+4b}} \le -\left(\frac{3+5b}{2b}\right)^{\frac{2b}{3+5b}} V_{\tilde{s}}^{\frac{b}{1+b}}\left(\frac{3+5b}{3+3b}\right)^{\frac{3+3b}{3+5b}} V_{\tilde{z}}^{\frac{3}{4}},
$$

thus if the following inequalities

$$\underline{k} > \max\left\{\lambda_4\gamma_3(\mu),\quad \lambda_5[\gamma_4(\mu)]^{\frac{3+5b}{3+3b}},\quad \lambda_6,\quad \lambda_7\right\},$$

where

$$\lambda_4 = 8n^{\frac{2b}{1+b}}\left(\frac{\sqrt{\overline{m}}}{\sqrt{\underline{m}}}\right)^{\frac{1+3b}{1+b}},$$

$$\lambda_5 = 2^{\frac{3+5b}{3+3b}}\left(\frac{36+36b}{3+5b}\right)\left(\frac{24b}{d_0(3+5b)}\right)^{\frac{2b}{3+3b}} n^{\frac{2b}{1+b}}\left(\frac{\sqrt{2}n}{\sqrt{\underline{m}}\,\underline{\pi}_i^{\frac{b}{2+2b}}}\right)^{\frac{3+5b}{3+3b}}\left(\frac{\sqrt{\overline{m}}}{\sqrt{2}}\right)^{\frac{1+3b}{1+b}},$$

$$\lambda_6 = \left(\frac{12+12b}{3+5b}\right)\left(\frac{24+48b}{d_0(3+5b)}\right)^{\frac{2+4b}{1+b}} n^{\frac{2b}{1+b}}\left(n^{\frac{1}{2+2b}}d_1\left(\frac{\sqrt{2}}{\sqrt{\underline{m}}}\right)\right)^{\frac{3+5b}{1+b}}\left(\frac{\sqrt{\overline{m}}}{\sqrt{2}}\right)^{\frac{1+3b}{1+b}},$$

$$\lambda_7 = \left(\frac{24b}{3+5b}\right)\left(\frac{36+36b}{d_0(3+5b)}\right)^{\frac{3+3b}{2b}} n^{\frac{2b}{1+b}}\left(n^{\frac{1}{4}}d_2\left(\frac{\sqrt{2}}{\sqrt{\underline{m}}}\right)^{\frac{2b}{1+b}}\right)^{\frac{3+5b}{2b}}\left(\frac{\sqrt{\overline{m}}}{\sqrt{2}}\right)^{\frac{1+3b}{1+b}},$$

hold then the function $\dot{V}_1$ is negative definite. The above inequalities can be satisfied by a sufficiently large gain *K*.

For $\dot{V}_2$, applying lemma 2.1,

$$-\|\tilde{e}_2\|^{\frac{3+5b}{1+b}} - V_{\tilde{z}}^{\frac{3+5b}{4+4b}} \le -\left(\frac{3+5b}{2b}\right)^{\frac{2b}{3+5b}}\|\tilde{e}_2\|^{\frac{2b}{1+b}}\left(\frac{3+5b}{3+3b}\right)^{\frac{3+3b}{3+5b}} V_{\tilde{z}}^{\frac{3}{4}},$$

$$-\|\tilde{e}_2\|^{\frac{3+5b}{1+b}} - V_{\tilde{s}}^{\frac{3+5b}{2+2b}} \le -\left(\frac{3+5b}{2b}\right)^{\frac{2b}{3+5b}}\|\tilde{e}_2\|^{\frac{2b}{1+b}}\left(\frac{3+5b}{3+3b}\right)^{\frac{3+3b}{3+5b}} V_{\tilde{s}}^{\frac{3}{2}},$$

$$-\left\|\widetilde{e}_2\right\|^{\frac{3+5b}{1+b}}-V_{\widetilde{s}}^{\frac{3+5b}{2+2b}}\leq-\left(\frac{3+5b}{3+3b}\right)^{\frac{3+3b}{3+5b}}\left\|\widetilde{e}_2\right\|^{3}\left(\frac{3+5b}{2b}\right)^{\frac{2b}{3+5b}}V_{\widetilde{s}}^{\frac{b}{1+b}},$$

$$-\left\|\widetilde{e}_1\right\|^{\frac{3+5b}{2}}-\left\|\widetilde{e}_2\right\|^{\frac{3+5b}{1+b}}\leq-\left(\frac{3+5b}{2+2b}\right)^{\frac{2+2b}{3+5b}}\left\|\widetilde{e}_1\right\|^{1+b}\left(\frac{3+5b}{1+3b}\right)^{\frac{1+3b}{3+5b}}\left\|\widetilde{e}_2\right\|^{\frac{1+3b}{1+b}},$$

$$-\left\|\widetilde{e}_1\right\|^{\frac{3+5b}{2}}-\left\|\widetilde{e}_2\right\|^{\frac{3+5b}{1+b}}\leq-\left(\frac{3+5b}{2}\right)^{\frac{2}{3+5b}}\left\|\widetilde{e}_1\right\|\left(\frac{3+5b}{1+5b}\right)^{\frac{1+5b}{3+5b}}\left\|\widetilde{e}_2\right\|^{\frac{1+5b}{1+b}},$$

$$-\left\|\widetilde{e}_2\right\|^{\frac{3+5b}{1+b}}-V_{\widetilde{z}}^{\frac{3+5b}{4+4b}}\leq-\left(\frac{3+5b}{3+3b}\right)^{\frac{3+3b}{3+5b}}\left\|\widetilde{e}_2\right\|^{3}\left(\frac{3+5b}{2b}\right)^{\frac{2b}{3+5b}}V_{\widetilde{z}}^{\frac{b}{2+2b}},$$

$$\begin{aligned}-\left\|\widetilde{e}_1\right\|^{\frac{3+5b}{2}}-\left\|\widetilde{e}_2\right\|^{\frac{3+5b}{1+b}}-V_{\widetilde{s}}^{\frac{3+5b}{2+2b}}&\leq-\left(\frac{3}{2}\right)^{\frac{2}{3}}\left\|\widetilde{e}_1\right\|^{\frac{3+5b}{3}}(3)^{\frac{1}{3}}\left\|\widetilde{e}_2\right\|^{\frac{3+5b}{3+3b}}-V_{\widetilde{s}}^{\frac{3+5b}{2+2b}}\\&\leq-\left(\left(\frac{3+5b}{3+3b}\right)\left(\frac{3}{2}\right)^{\frac{2}{3}}(3)^{\frac{1}{3}}\right)^{\frac{3+3b}{3+5b}}\left\|\widetilde{e}_1\right\|^{1+b}\left\|\widetilde{e}_2\right\|\left(\frac{3+5b}{2b}\right)^{\frac{2b}{3+5b}}V_{\widetilde{s}}^{\frac{b}{1+b}},\end{aligned}$$

$$\begin{aligned}-\left\|\widetilde{e}_1\right\|^{\frac{3+5b}{2}}-\left\|\widetilde{e}_2\right\|^{\frac{3+5b}{1+b}}-V_{\widetilde{s}}^{\frac{3+5b}{2+2b}}&\leq-\left(\frac{3+3b}{2}\right)^{\frac{2}{3+3b}}\left\|\widetilde{e}_1\right\|^{\frac{3+5b}{3+3b}}\left(\frac{3+3b}{1+3b}\right)^{\frac{1+3b}{3+3b}}\left\|\widetilde{e}_2\right\|^{\left(\frac{(3+5b)(1+3b)}{(1+b)(3+3b)}\right)}-V_{\widetilde{s}}^{\frac{3+5b}{2+2b}}\\&\leq-\left(\left(\frac{3+5b}{3+3b}\right)\left(\frac{3+3b}{2}\right)^{\frac{2}{3+3b}}\left(\frac{3+3b}{1+3b}\right)^{\frac{1+3b}{3+3b}}\right)^{\frac{3+3b}{3+5b}}\left\|\widetilde{e}_1\right\|\left\|\widetilde{e}_2\right\|^{\frac{1+3b}{1+b}}\left(\frac{3+5b}{2b}\right)^{\frac{2b}{3+5b}}V_{\widetilde{s}}^{\frac{b}{1+b}},\end{aligned}$$

$$\begin{aligned}-\left\|\widetilde{e}_1\right\|^{\frac{3+5b}{2}}-\left\|\widetilde{e}_2\right\|^{\frac{3+5b}{1+b}}-V_{\widetilde{z}}^{\frac{3+5b}{4+4b}}&\leq-\left(\frac{3}{2}\right)^{\frac{2}{3}}\left\|\widetilde{e}_1\right\|^{\frac{3+5b}{3}}(3)^{\frac{1}{3}}\left\|\widetilde{e}_2\right\|^{\frac{3+5b}{3+3b}}-V_{\widetilde{z}}^{\frac{3+5b}{4+4b}}\\&\leq-\left(\left(\frac{3+5b}{3+3b}\right)\left(\frac{3}{2}\right)^{\frac{2}{3}}(3)^{\frac{1}{3}}\right)^{\frac{3+3b}{3+5b}}\left\|\widetilde{e}_1\right\|^{1+b}\left\|\widetilde{e}_2\right\|\left(\frac{3+5b}{2b}\right)^{\frac{2b}{3+5b}}V_{\widetilde{z}}^{\frac{b}{2+2b}}\end{aligned}$$

$$-\|\tilde{e}_1\|^{\frac{3+5b}{2}} - \|\tilde{e}_2\|^{\frac{3+5b}{1+b}} - V_{\tilde{z}}^{\frac{3+5b}{4+4b}} \le -\left(\frac{3+3b}{2}\right)^{\frac{2}{3+3b}}\|\tilde{e}_1\|^{\frac{3+5b}{3+3b}}\left(\frac{3+3b}{1+3b}\right)^{\frac{1+3b}{3+3b}}\|\tilde{e}_2\|^{\left(\frac{(3+5b)(1+3b)}{(1+b)(3+3b)}\right)} - V_{\tilde{z}}^{\frac{3+5b}{4+4b}}$$

$$\le -\left(\left(\frac{3+5b}{3+3b}\right)\left(\frac{3+3b}{2}\right)^{\frac{2}{3+3b}}\left(\frac{3+3b}{1+3b}\right)^{\frac{1+3b}{3+3b}}\right)^{\frac{3+3b}{3+5b}}\|\tilde{e}_1\|\|\tilde{e}_2\|^{\frac{1+3b}{1+b}}\left(\frac{3+5b}{2b}\right)^{\frac{2b}{3+5b}} V_{\tilde{z}}^{\frac{b}{2+2b}}$$

thus if the following inequalities

$$\min\left\{\begin{array}{l} \lambda_8 \underline{l}_1 \underline{l}_2^{\,2}, \\ \lambda_9 \dfrac{\underline{l}_1^{\frac{1}{1+b}} \underline{l}_2^{\frac{2}{1+b}}}{[\gamma_6(\mu)]^{\frac{3+5b}{2+2b}}}, \\ \lambda_{10} \dfrac{\underline{l}_1^{\frac{1}{1+b}} \underline{l}_2^{\frac{2}{1+b}}}{[\gamma_7(\mu)]^{\frac{3+5b}{2+2b}}} \end{array}\right\} > r_{\tilde{e}} > \max\left\{\begin{array}{l} \lambda_{11},\quad \lambda_{12},\quad \lambda_{13},\quad \lambda_{14}[\gamma_6(\mu)]^{\frac{3+5b}{3+3b}}, \\ \lambda_{15}\left(\dfrac{1}{\underline{l}_1^{\frac{2+2b}{1+3b}} \underline{l}_2^{\frac{1-b}{1+3b}}}\right)\left(\dfrac{\overline{l}_2}{\underline{l}_2}\right)^{\frac{3+5b}{1+3b}}, \lambda_{16}[\gamma_7(\mu)]^{\frac{3+5b}{3+3b}}, \\ \lambda_{17}\dfrac{[\gamma_6(\mu)]^{\frac{3+5b}{1+b}}}{\underline{l}_1^{\,2} \underline{l}_2^{\frac{1-b}{1+b}}}\left(\dfrac{\overline{l}_2}{\underline{l}_2}\right)^{\frac{3+5b}{1+b}}, \quad \lambda_{18}\dfrac{[\gamma_7(\mu)]^{\frac{3+5b}{1+b}}}{\underline{l}_1^{\frac{2+2b}{1+b}} \underline{l}_2^{\frac{1-b}{1+b}}}\left(\dfrac{\overline{l}_2}{\underline{l}_2}\right)^{\frac{3+5b}{1+b}} \end{array}\right\}, \qquad (4-16)$$

where

$$\lambda_8 = \left(\frac{3+5b}{24+24b}\right)\left(\frac{1}{n^{\frac{1+5b}{2}}}\right)\left(\frac{3+5b}{80+400b}\left(\frac{1}{n^{\frac{2+4b}{1+b}}}\right)\right)^{\frac{1+5b}{2}}\left(\frac{1+b}{2^{\frac{1-b}{1+b}}(2+4b)\overline{m}_I\,\bar{k}\,n^{\frac{1}{1+b}}}\right)^{\frac{3+5b}{2}},$$

$$\lambda_9 = \left(\frac{3+5b}{3+3b}\right)^{\frac{3+3b}{2+2b}}\left(\frac{1}{8n^{\frac{1+5b}{2}}}\right)^{\frac{1}{1+b}}\left(\frac{3+3b}{(80+240b)n^{\frac{2+4b}{1+b}}}\right)^{\frac{1+3b}{2+2b}}$$

$$\times\left(\left(\left(\frac{(3+5b)\underline{k}}{16bn^{\frac{2b}{1+b}}}\right)\left(\frac{\sqrt{2}}{\sqrt{\overline{m}}}\right)^{\frac{1+3b}{1+b}}\right)^{\frac{b}{1+b}}\left(\left(\frac{\sqrt{\underline{m}}}{\sqrt{2}}\right)^{\frac{2b}{1+b}}\left(\frac{1+b}{\sqrt{n}(2+4b)}\right)\right)\right)^{\frac{3+5b}{2+2b}},$$

$$\lambda_{10} = \left(\frac{3+5b}{3+3b}\right)^{\frac{3+3b}{2+2b}} \left(\frac{1}{8n^{\frac{1+5b}{2}}}\right)^{\frac{2}{2+2b}} \left(\left(\frac{3+3b}{80+240b}\right)\left(\frac{1}{n^{\frac{2+4b}{1+b}}}\right)\right)^{\frac{1+3b}{2+2b}}$$

$$\times \left(\frac{3+5b}{2b}\frac{d_0}{16}\right)^{\frac{2b}{2+2b}} \left(\frac{1+b}{2+4b}\right)^{\frac{3+5b}{2+2b}} \left(\frac{\underline{\pi}_i^{\frac{b}{2+2b}}}{n^{\frac{3}{2}}}\right)^{\frac{3+5b}{2+2b}},$$

$$\lambda_{11} = 2^{\frac{4+2b}{1+b}}\,\overline{m}_I\,\bar{k}\,n^{\frac{4+2b}{2+2b}} n^{\frac{2+4b}{1+b}}, \qquad \lambda_{12} = 80\left(\frac{2b}{3+5b}\right)\left(\frac{48+48b}{(3+5b)d_0}\right)^{\frac{3+3b}{2b}} n^{\frac{2+4b}{1+b}}\, n^{\frac{3+5b}{8b}}\, d_2^{\frac{3+5b}{2b}},$$

$$\lambda_{13} = 80\left(2^{\frac{3+5b}{2b}}\, n^{\frac{2+4b}{1+b}}\right)\left(\frac{2b}{3+5b}\right)\left(\left(\frac{24+24b}{3+5b}\right)\left(\frac{n^{\frac{2b}{1+b}}}{\underline{k}}\right)\left(\frac{\sqrt{\underline{m}}}{\sqrt{2}}\right)^{\frac{1+3b}{1+b}}\right)^{\frac{3+3b}{2b}}$$

$$\times \left(\overline{m}\bar{k}_2 n^{\frac{1-b}{2+2b}} + 2^{\frac{1-b}{1+b}}\,\bar{k}\sqrt{n}\right)^{\frac{3+5b}{2b}} \left(\frac{\sqrt{2}}{\sqrt{\underline{m}}}\right)^{\frac{3+5b}{2b}},$$

$$\lambda_{14} = 80\left(\frac{3+3b}{3+5b}\right)\left(\left(\frac{16b}{3+5b}\right)\left(\frac{n^{\frac{2b}{1+b}}}{\underline{k}}\right)\left(\frac{\sqrt{\underline{m}}}{\sqrt{2}}\right)^{\frac{1+3b}{1+b}}\right)^{\frac{2b}{3+3b}} n^{\frac{2+4b}{1+b}} \left(n^{\frac{3}{2}}\left(\frac{\sqrt{2}}{\sqrt{\underline{m}}}\right)^{\frac{2b}{1+b}}\right)^{\frac{3+5b}{3+3b}},$$

$$\lambda_{15} = 80\left(\frac{1+3b}{3+5b}\right) n^{\frac{2+4b}{1+b}} \left(\left(\frac{24+24b}{3+5b}\right)(1+b)\left(n^{\frac{1+5b}{2}}\right)\right)^{\frac{2+2b}{1+3b}} \left(\left(\frac{2\sqrt{n}}{1+b}\right)2^{\frac{1-b}{1+b}}\,\overline{m}_I\,\bar{k}\,n^{\frac{1-b}{2+2b}}\right)^{\frac{3+5b}{1+3b}},$$

$$\lambda_{16} = 80n^{\frac{2+4b}{1+b}}\left(\frac{3+3b}{3+5b}\right)\left(\frac{32b}{d_0(3+5b)}\right)^{\frac{2b}{3+3b}}\left(\frac{n^{\frac{5}{2}}}{\underline{\pi}_i^{\frac{b}{2+2b}}}\right)^{\frac{3+5b}{3+3b}},$$

$$\lambda_{17} = \left(\frac{24+24b}{3}\right)^2 \left(\frac{80}{3}\right) n^{\frac{2+4b}{1+b}} n^{1+5b} \left(\frac{3+3b}{3+5b}\right)^3$$
$$\times \left(\left(\frac{16b}{3+5b}\right) \left(\frac{n^{\frac{2b}{1+b}}}{\underline{k}}\right) \left(\frac{\sqrt{m}}{\sqrt{2}}\right)^{\frac{1+3b}{1+b}} \right)^{\frac{2b}{1+b}} \left(\left(\frac{\sqrt{2}}{\sqrt{\underline{m}}}\right)^{\frac{2b}{1+b}} \left(\frac{2\sqrt{n}}{1+b}\right) \right)^{\frac{3+5b}{1+b}},$$

$$\lambda_{18} = \left(\frac{80}{3}\right) \left(\frac{32b}{(3+5b)d_0}\right)^{\frac{2b}{1+b}} \left(\frac{3+3b}{3+5b}\right)^{\frac{3+3b}{1+b}} n^{\frac{2+4b}{1+b}} \left((8+8b) n^{\frac{1+5b}{2}}\right)^{\frac{2+2b}{1+b}} \left(\frac{2\sqrt{n}}{1+b}\right)^{\frac{3+5b}{1+b}} \left(\frac{n}{\underline{\pi}_i^{\frac{b}{2+2b}}}\right)^{\frac{3+5b}{1+b}},$$

hold then the function $\dot{V}_2$ is negative definite. Note that the inequalities (4 - 16) is feasible with respect to $\underline{l}_1, \underline{l}_2$. Thus, for a given K_1, K_2, and K, there exists a sufficiently large L_1, L_2, with the ratio,

$$\left(\frac{\bar{l}_2}{\underline{l}_2}\right) = \text{constant}$$

remaining constant, such that a positive $r_{\tilde{e}} > 0$ exists that satisfy all the above conditions.

Then, we have,

$$\dot{V} \overset{a.e.}{\leq} -\left(\frac{k}{n^{\frac{2b}{1+b}}}\right) \left(\frac{\sqrt{2}}{\sqrt{m}}\right)^{\frac{1+3b}{1+b}} V_{\tilde{s}}^{\frac{3+5b}{2+2b}} - \underline{f}\left(\frac{4}{m}\right) V_{\tilde{s}}^{2} - \frac{d_0}{2} V_{\tilde{z}}^{\frac{3+5b}{4+4b}} - \left(\frac{1}{2}\right) \left(\frac{\underline{l}_1\, \underline{l}_2^{\,2}}{1+b}\right) \left(\frac{1}{n^{\frac{1+5b}{2}}}\right) \|\tilde{e}_1\|^{\frac{3+5b}{2}}$$
$$-\left(\frac{r_{\tilde{e}}}{4}\right) \left(\frac{1}{n^{\frac{2+4b}{1+b}}}\right) \|\tilde{e}_2\|^{\frac{3+5b}{1+b}} + 2\left[\alpha(\|\eta(t)\|)\gamma_5(\mu) + \|\Lambda G(q_0)\|\right] \left(\frac{\sqrt{2}}{\sqrt{\underline{m}}}\right) V_{\tilde{s}}^{\frac{3}{2}}$$
$$+ \left[\alpha_2(\|\eta(t)\|) + \bar{m}_l \|\Lambda G(q_0)\|\right] \times \left(\frac{2\bar{l}_2}{1+b} \sqrt{n} \|\tilde{e}_1\|^{1+b} \|\tilde{e}_2\| + n^{\frac{3}{2}} \|\tilde{e}_2\|^3 + r_{\tilde{e}} \left(\frac{2+4b}{1+b}\right) \sqrt{n} \|\tilde{e}_1\| \|\tilde{e}_2\|^{\frac{1+3b}{1+b}} \right)$$

Note that from lemma 2.1, we have

$$\left(\frac{2\sqrt{n}}{1+b}\right)\overline{l_2}\|\widetilde{e}_1\|^{1+b}\|\widetilde{e}_2\|,+n^{\frac{3}{2}}\|\widetilde{e}_2\|^3+r_{\widetilde{e}}\left(\frac{2+4b}{1+b}\right)\sqrt{n}\,\|\widetilde{e}_1\|\|\widetilde{e}_2\|^{\frac{1+3b}{1+b}}\le\pi_3\left(\|\widetilde{e}_1\|^{\frac{3+3b}{2}}+\|\widetilde{e}_2\|^3\right)$$

where

$$\pi_3=\max\left\{\begin{array}{l}\left(\left(\frac{2\sqrt{n}\,\overline{l_2}}{1+b}\right)\left(\frac{2}{3}\right)+r_{\widetilde{e}}\sqrt{n}\left(\frac{2+4b}{1+b}\right)\left(\frac{2}{3+3b}\right)\right),\\ \left(\left(\frac{2\sqrt{n}\,\overline{l_2}}{1+b}\right)\left(\frac{1}{3}\right)+n^{\frac{3}{2}}+r_{\widetilde{e}}\sqrt{n}\left(\frac{2+4b}{1+b}\right)\left(\frac{1+3b}{3+3b}\right)\right)\end{array}\right\}$$

Next, note that (see Section 2.4 of chapter 2) we have the following bounds on the Lyapunov function, $W_{\tilde{e}}$,

$$\underline{\pi}_2\sum_{i=1}^{n}\left(|\widetilde{e}_{1i}|^{2+2b}+|\widetilde{e}_{2i}|^4\right)\le W_{\widetilde{e}}\left(\widetilde{e}_1,\widetilde{e}_2\right)\le\overline{\pi}_2\sum_{i=1}^{n}\left(|\widetilde{e}_{1i}|^{2+2b}+|\widetilde{e}_{2i}|^4\right)$$

where

$$\underline{\pi}_2=\min\left\{\frac{1}{8},\ \frac{1}{2}\left(\frac{\underline{l_2}}{1+b}\right)^2\right\},$$

$$\overline{\pi}_2=\max\left\{\left(\frac{\overline{l_2}}{1+b}\right)^2+\left(\frac{l_{2i}}{2+2b}\right)+\frac{r_{\widetilde{e}}}{2+2b},\ \frac{1}{4}+\left(\frac{\overline{l_2}}{2+2b}\right)+r_{\widetilde{e}}\left(\frac{1+2b}{2+2b}\right)\right\},$$

Using lemma 2.2,

$$\begin{aligned}\sum_{i=1}^{n}|\widetilde{e}_{1i}|^{2+2b}+\sum_{i=1}^{n}|\widetilde{e}_{2i}|^4&=\sum_{i=1}^{n}\left(|\widetilde{e}_{1i}|^2\right)^{\frac{2+2b}{2}}+\sum_{i=1}^{n}\left(|\widetilde{e}_{2i}|^2\right)^{\frac{4}{2}}\\&\ge\frac{1}{n^b}\left(\sum_{i=1}^{n}\left(|\widetilde{e}_{1i}|^2\right)\right)^{\frac{2+2b}{2}}+\frac{1}{n}\left(\sum_{i=1}^{n}\left(|\widetilde{e}_{2i}|^2\right)\right)^{\frac{4}{2}},\quad\text{using lemma 2.2},\\&=\frac{1}{n^b}\|\widetilde{e}_1\|^{2+2b}+\frac{1}{n}\|\widetilde{e}_2\|^4\\&\ge\frac{1}{n}\left(\|\widetilde{e}_1\|^{2+2b}+\|\widetilde{e}_2\|^4\right),\ \text{since}\min\left\{\frac{1}{n^b},\frac{1}{n}\right\}=\frac{1}{n}\ \text{ for }0<b\le1\ \text{ and }\ n\ge1,\end{aligned}$$

and

$$\sum_{i=1}^{n}\left|\tilde{e}_{1i}\right|^{2+2b}+\sum_{i=1}^{n}\left|\tilde{e}_{2i}\right|^{4}=\sum_{i=1}^{n}\left(\left|\tilde{e}_{1i}\right|^{2}\right)^{\frac{2+2b}{2}}+\sum_{i=1}^{n}\left(\left|\tilde{e}_{2i}\right|^{2}\right)^{\frac{4}{2}}$$

$$\leq\left(\sum_{i=1}^{n}\left|\tilde{e}_{1i}\right|^{2}\right)^{\frac{2+2b}{2}}+\left(\sum_{i=1}^{n}\left|\tilde{e}_{2i}\right|^{2}\right)^{\frac{4}{2}},\ \text{ using lemma 2.2,}$$

$$=\left\|\tilde{e}_{1}\right\|^{2+2b}+\left\|\tilde{e}_{2}\right\|^{4}$$

Hence,

$$\frac{\underline{\pi}_{2}}{n}\left(\left\|\tilde{e}_{1}\right\|^{2+2b}+\left\|\tilde{e}_{2}\right\|^{4}\right)\leq W_{\tilde{e}}\left(\tilde{e}_{1},\tilde{e}_{2}\right)\leq\overline{\pi}_{2}\left(\left\|\tilde{e}_{1}\right\|^{2+2b}+\left\|\tilde{e}_{2}\right\|^{4}\right)$$

Next, note the following inequalities (using lemma 2.2):

$$\left\|\tilde{e}_{1}\right\|^{\frac{3+3b}{2}}+\left\|\tilde{e}_{2}\right\|^{3}=\left(\left\|\tilde{e}_{1}\right\|^{2+2b}\right)^{\frac{3}{4}}+\left(\left\|\tilde{e}_{2}\right\|^{4}\right)^{\frac{3}{4}}$$

$$\leq 2^{\frac{1}{4}}\left(\left\|\tilde{e}_{1}\right\|^{2+2b}+\left\|\tilde{e}_{2}\right\|^{4}\right)^{\frac{3}{4}}$$

$$\leq 2^{\frac{1}{4}}\left(\frac{n}{\underline{\pi}_{2}}\right)^{\frac{3}{4}}W_{\tilde{e}}^{\frac{3}{4}}$$

$$\left\|\tilde{e}_{1}\right\|^{\frac{3+5b}{2}}+\left\|\tilde{e}_{2}\right\|^{\frac{3+5b}{1+b}}=\left(\left\|\tilde{e}_{1}\right\|^{2+2b}\right)^{\frac{3+5b}{4+4b}}+\left(\left\|\tilde{e}_{2}\right\|^{4}\right)^{\frac{3+5b}{4+4b}}$$

$$\geq\left(\left\|\tilde{e}_{1}\right\|^{2+2b}+\left\|\tilde{e}_{2}\right\|^{4}\right)^{\frac{3+5b}{4+4b}},\ \text{using lemma 2.2, since}\ \frac{3+5b}{4+4b}\leq 1,\ \text{for}\ 0<b\leq 1,$$

$$\geq\left(\frac{1}{\overline{\pi}_{2}}\right)^{\frac{3+5b}{4+4b}}W_{\tilde{e}}^{\frac{3+5b}{4+4b}}$$

Thus, the time derivative of the Lyapunov function becomes,

$$
\begin{aligned}
\dot{V} &\overset{a.e.}{\leq} -\pi_4\left(V_{\tilde{s}}^{\frac{3+5b}{2+2b}}+V_{\tilde{z}}^{\frac{3+5b}{4+4b}}+W_{\tilde{e}}^{\frac{3+5b}{4+4b}}\right)-\underline{f}\left(\frac{4}{\overline{m}}\right)V_{\tilde{s}}^{2}+\alpha_3(\|\eta(t)\|)\left(V_{\tilde{s}}^{\frac{3}{2}}+W_{\tilde{e}}^{\frac{3}{4}}\right)\\
&\quad+\pi_5\|\Lambda G(q_0)\|\left(V_{\tilde{s}}^{\frac{3}{2}}+W_{\tilde{e}}^{\frac{3}{4}}\right)\\
&=-\pi_4\left(\left(V_{\tilde{s}}^{2}\right)^{\frac{3+5b}{4+4b}}+V_{\tilde{z}}^{\frac{3+5b}{4+4b}}+W_{\tilde{e}}^{\frac{3+5b}{4+4b}}\right)-\underline{f}\left(\frac{4}{\overline{m}}\right)V_{\tilde{s}}^{2}+\alpha_3(\|\eta(t)\|)\left(\left(V_{\tilde{s}}^{2}\right)^{\frac{3}{4}}+W_{\tilde{e}}^{\frac{3}{4}}\right)\\
&\quad+\pi_5\|\Lambda G(q_0)\|\left(\left(V_{\tilde{s}}^{2}\right)^{\frac{3}{4}}+W_{\tilde{e}}^{\frac{3}{4}}\right)\\
&\leq-\pi_4\left(V_{\tilde{s}}^{2}+V_{\tilde{z}}+W_{\tilde{e}}\right)^{\frac{3+5b}{4+4b}}+2^{\frac{1}{4}}\alpha_3(\|\eta(t)\|)\left(V_{\tilde{s}}^{2}+W_{\tilde{e}}^{\frac{3}{4}}\right)^{\frac{3}{4}}\\
&\quad+2^{\frac{1}{4}}\pi_5\|\Lambda G(q_0)\|\left(V_{\tilde{s}}^{2}+W_{\tilde{e}}^{\frac{3}{4}}\right)^{\frac{3}{4}}\\
&=-\pi_4V^{\frac{3+5b}{4+4b}}+2^{\frac{1}{4}}\alpha_3(\|\eta(t)\|)V^{\frac{3}{4}}+2^{\frac{1}{4}}\pi_5\|\Lambda G(q_0)\|V^{\frac{3}{4}}
\end{aligned}
$$

where

$$
\pi_4:=\min\left\{\begin{array}{l}\left(\dfrac{\underline{k}}{n^{\frac{2b}{1+b}}}\right)\left(\dfrac{\sqrt{2}}{\sqrt{\overline{m}}}\right)^{\frac{1+3b}{1+b}},\ \dfrac{d_0}{2},\\ \left(\dfrac{1}{\overline{\pi}_2}\right)^{\frac{3+5b}{4+4b}}\left(\dfrac{1}{2}\right)\left(\dfrac{\underline{l_1}\,\underline{l_2}^{2}}{1+b}\right)\left(\dfrac{1}{n^{\frac{1+5b}{2}}}\right),\ \left(\dfrac{1}{\overline{\pi}_2}\right)^{\frac{3+5b}{4+4b}}\left(\dfrac{r_{\tilde{e}}}{4}\right)\left(\dfrac{1}{n^{\frac{2+4b}{1+b}}}\right)\end{array}\right\},
$$

$$
\pi_5:=\max\left\{\left(\frac{2^{\frac{3}{2}}}{\sqrt{\underline{m}}}\right),\ \overline{m}_I\left(\frac{2^{\frac{1}{4}}\pi_3 n^{\frac{3}{4}}}{\underline{\pi}_2^{\frac{3}{4}}}\right)\right\},
$$

$$
\alpha_3(\|\eta(t)\|):=\max\left\{\alpha(\|\eta(t)\|)\gamma_5(\mu)\left(\frac{2^{\frac{3}{2}}}{\sqrt{\underline{m}}}\right),\ \alpha_2(\|\eta(t)\|)\left(\frac{2^{\frac{1}{4}}\pi_3 n^{\frac{3}{4}}}{\underline{\pi}_2^{\frac{3}{4}}}\right)\right\},
$$

Note that lemma 2.2 with $\dfrac{3+5b}{4+4b}\leq 1$ and the bounds of the Lyapunov functions,

$$V=\left[V_{\tilde{s}}(\tilde{s})\right]^2+V_{\tilde{z}}(\tilde{z}_1,\tilde{z}_2)+W_{\tilde{e}}(\tilde{e}_1,\tilde{e}_2)\geq\left[V_{\tilde{s}}(\tilde{s})\right]^2+W_{\tilde{e}}\ ,$$

have been employed in the above inequality.

Note that when $b = 1$, recall that the term

$$\Lambda G(q_0)=\mathrm{diag}\left(-\left|\mathrm{sig}\left(\frac{\tilde{s}_i}{\bar{s}_i}\right)^{\frac{2b}{1+b}}-\mathrm{sig}\left(\frac{\tilde{s}_i}{\bar{s}_i}+1\right)^{\frac{2b}{1+b}}\right|+1\right)G(q_0)=\mathrm{diag}\left(-\left|-1\right|+1\right)G(q_0)=0\ .$$

Also for the case of $0 < b < 1$, consider the case when the final desired position vector q_0 corresponds to the rest position of the manipulator where $G(q_0) = (0, ..., 0)^{\mathrm{T}}$, or the gravitational torque of the manipulator dynamics is absent (i.e. in space where gravity is absent or in a planar horizontal configuration) where $G(q) = 0$ for $\forall\ q \in \mathbb{R}^n$, the term $\Lambda G(q_0)$ vanishes to zero. We obtain,

$$\dot{V}\overset{a.e.}{\leq}-\pi_4 V^{\frac{3+5b}{4+4b}}+2^{\frac{1}{4}}\alpha_3\left(\left\|\eta(t)\right\|\right)V^{\frac{3}{4}}$$

Thus, for $\|\eta(t)\| \to 0$ as $t \to \infty$ (where the desired trajectory approaches a constant final desired position), then we have $V(\tilde{s}, \tilde{z}_1, \tilde{z}_2, \tilde{e}_1, \tilde{e}_2) \to 0$ as $t \to \infty$ as well. Obviously, if $\|\eta(t)\| \equiv 0$ (regulation problem), when $b = 1$, the system will converge to the equilibrium exponentially, while for $0 < b < 1$, the system achieves finite-time regulation, in which the states reach the origin in finite time with the estimate of reaching time as,

$$T(\tilde{s}_0,\tilde{z}_{10},\tilde{z}_{20},\tilde{e}_{10},\tilde{e}_{20})\leq\left(\frac{4+4b}{\pi_4(1-b)}\right)\left[V(\tilde{s}_0,\tilde{z}_{10},\tilde{z}_{20},\tilde{e}_{10},\tilde{e}_{20})\right]^{\frac{1-b}{4+4b}}$$

where $(\tilde{s}_0, \tilde{z}_{10}, \tilde{z}_{20}, \tilde{e}_{10}, \tilde{e}_{20})$ are the states of the system when it first enters the region $\Omega_{\tilde{e}}(\mu) \times \Omega_s(\mu) \times \Omega_z(\mu)$. ■

Remark 4.9. Note that a smaller value of μ can help lower the required observer gains L_1, and L_2 (see (4 - 16)). This is useful for the case of $0 < b < 1$, since both gains L_1, and L_2 corresponds to the non-Lipschitz component of the observer, while μ can be reduced by increasing the linear observer gain L_3 (see Section 4.1).

Remark 4.10. As per remark 4.5 and 4.8, the "dirty-derivative" inspired linear damping term of the observer, $L_3\tilde{e}_2$ has a synergistic effect on the performance of the observer by allowing the reduction on the level of gains of the non-Lipschitz part of the observer while maintaining special properties of the super-twisting algorithm such as finite-time convergence and exact robustness.

4.3 Numerical Simulations

In this section, numerical simulations on a two-link robot manipulator were carried out to illustrate the results discussed in this chapter. The setups for each simulation are described. Discussion and analysis of the results are presented accordingly.

4.3.1 Simulation Setup

1) Simulation 1:

The same two-link rigid robot manipulator considered in section 3.3 is adopted in simulation. The dynamics of robot manipulator (4 - 3) have the same parameter values as that in section 3.3. The desired trajectory vector and the additive disturbances Coulomb friction vector were defined similarly as well.

The control (4 - 4) parameter values were selected as follows,

$$a = 0.9,\ K = \begin{bmatrix} 150 & 0 \\ 0 & 15 \end{bmatrix},\ K_1 = \begin{bmatrix} 18 & 0 \\ 0 & 18 \end{bmatrix},\ K_2 = \begin{bmatrix} 11 & 0 \\ 0 & 11 \end{bmatrix}, b = 0.6$$

The initial conditions of the vector σ were selected as

$$\sigma(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

since no velocity measurement is available.

The observer (4 - 6) parameter values were selected as follows,

$$L_1 = L_2 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, L_3 = \begin{bmatrix} 150 & 0 \\ 0 & 150 \end{bmatrix}, \ p = 0.8$$

The initial conditions of the observer were selected as

$$\hat{e}_1(0) = e_1(0),$$
$$w(0) = -L_3 e_1(0)$$

since position measurement is available, so that initially, the observation error

$$\tilde{e}_1(0) = 0,$$
$$\hat{e}_2(0) = 0,$$
$$\tilde{e}_2(0) = -e_2(0)$$

Note that the initial conditions for the velocity observation error were selected to be non zero to show the convergence of the observer through simulations, by having the initial conditions for the manipulator,

$$q(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \ \dot{q}(0) = \begin{bmatrix} 1.2 \\ 0.5 \end{bmatrix},$$

$$q_d(0) = \begin{bmatrix} \pi \\ 0.5 \end{bmatrix}, \ \dot{q}_d(0) = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix},$$

2) Simulation 2:

The setup of Simulation 2 is exactly the same as that of Simulation 1 except, in this simulation, only the parameter values of the observer (4 - 6) were changed to examine the effect of removing the linear damping term from the observer as follows, First case:

$$L_1 = L_2 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, L_3 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},\ p = 0.8,$$

Second case:

$$L_1 = \begin{bmatrix} 125 & 0 \\ 0 & 125 \end{bmatrix},\ L_2 = \begin{bmatrix} 800 & 0 \\ 0 & 800 \end{bmatrix},\ L_3 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},\ p = 0.8,$$

3) Simulation 3:

The setup of Simulation 3 is exactly the same as that of Simulation 1 except the observer parameter p was selected as $p = 0.5$ to demonstrate the observation errors converge to origin in finite time under uncertainties with

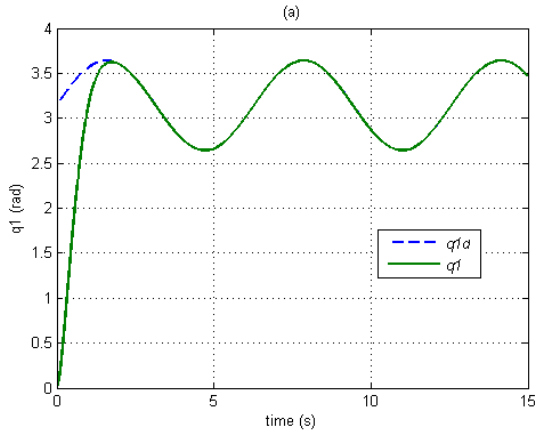
$$L_1 = L_2 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, L_3 = \begin{bmatrix} 150 & 0 \\ 0 & 150 \end{bmatrix},\ p = 0.5.$$

For comparison purposes, under the same setup, a simulation with the full-state feedback control (3 - 2) of Chapter 3 were performed with

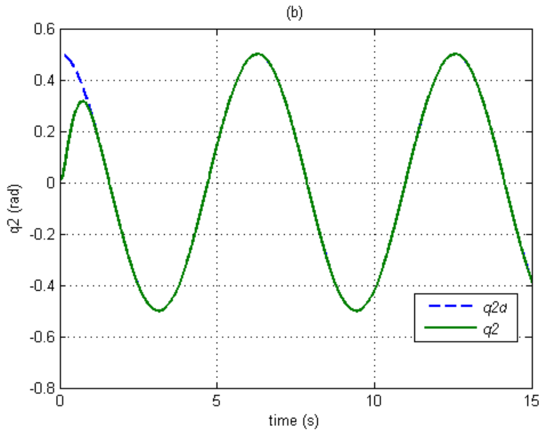
$\sigma(0) = -e_2(0)$, so that initially $s(0) = 0$, i.e. since full-state is available

4.3.2 Results and Discussions

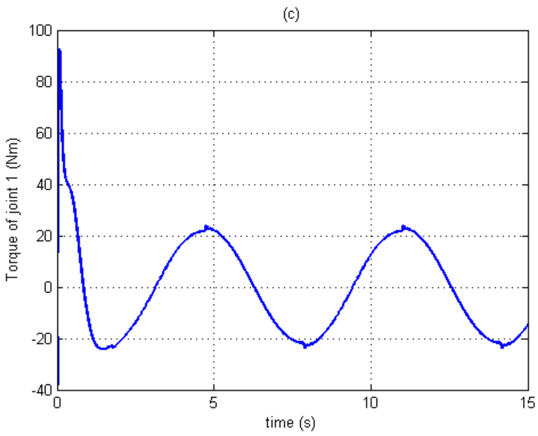
For better visualization of the plots, some figures are shown in two windows; each with different time intervals.



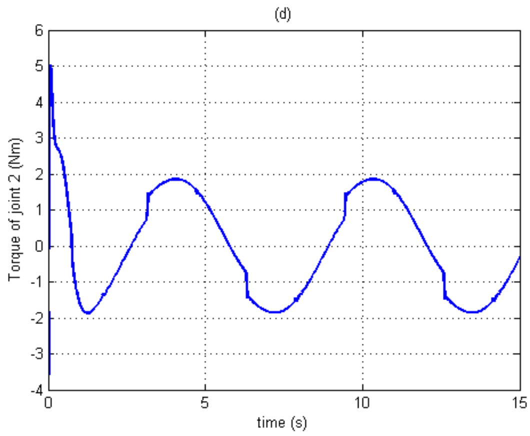
(a) Trajectory, q_1 for time t = [0, 15] s.



(b) Trajectory, q_2 for time t = [0, 15] s.

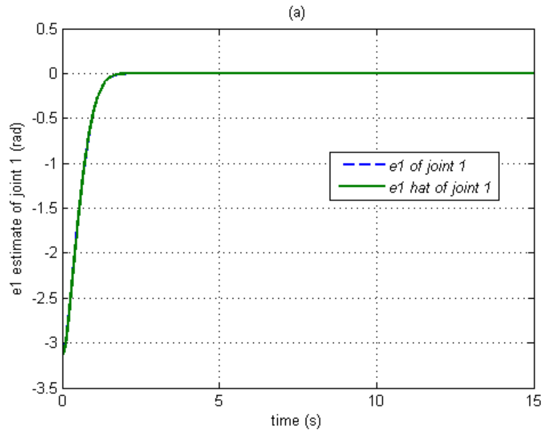


(c) Torque, τ_1 for time t = [0, 15] s.

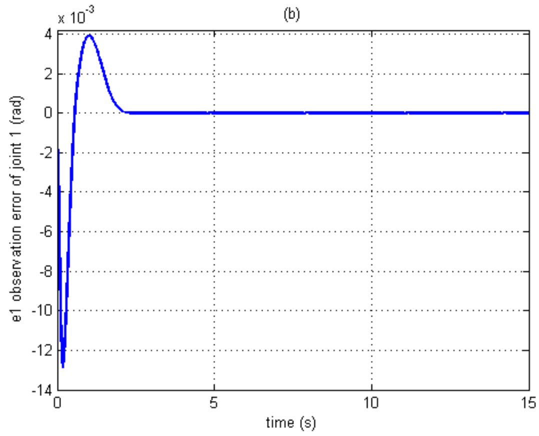


(d) Torque, τ_2 for time t = [0, 15] s.

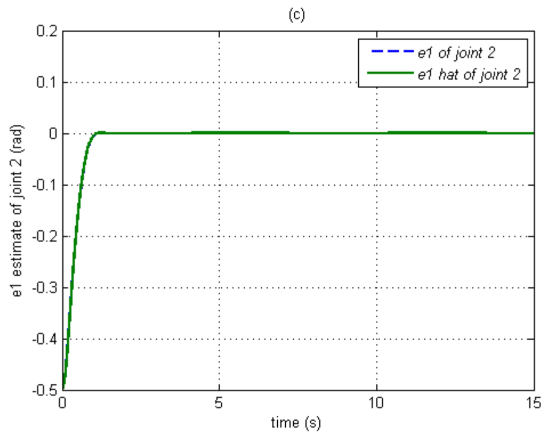
Figure 4.1 Simulation 1. Tracking errors and control input of joint 1 and joint 2.



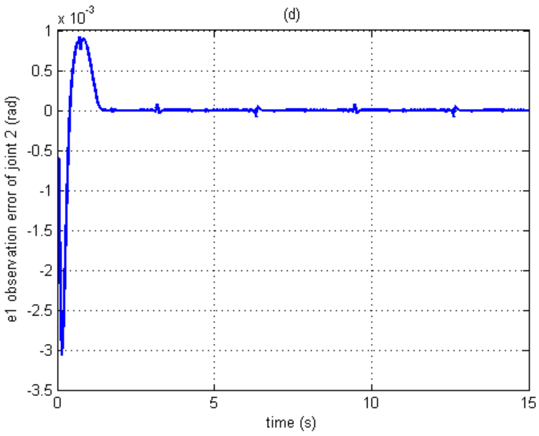
(a) Position estimate, $\hat{e}_{11}$ for time t = [0, 15] s.



(b) Position observation error, $\tilde{e}_{11}$ for time t = [0, 15] s.

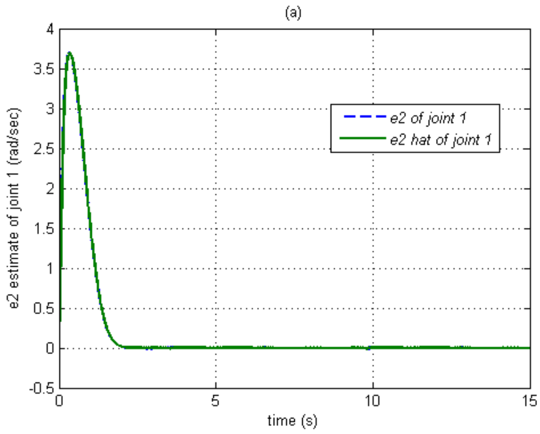


(c) Position estimate, $\hat{e}_{12}$ for time t = [0, 15] s.

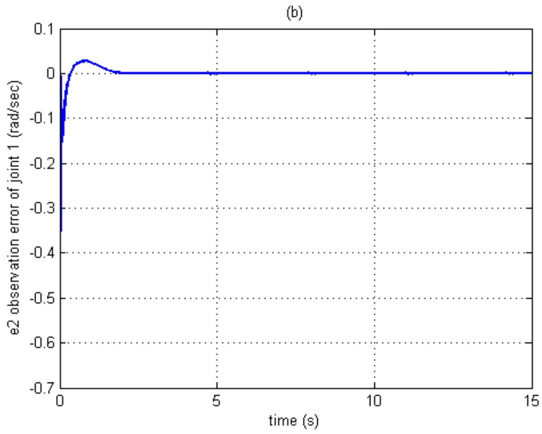


(d) Position observation error, $\tilde{e}_{12}$ for time t = [0, 15] s.

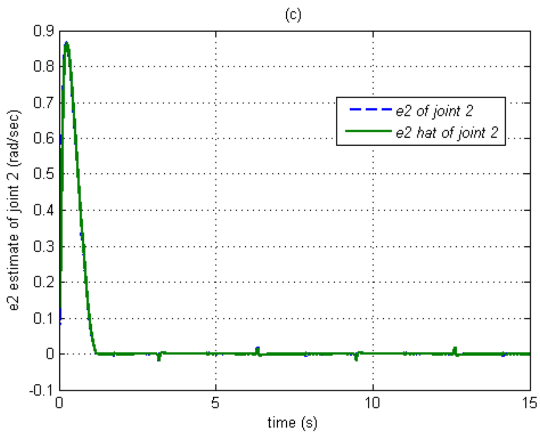
Figure 4.2 Simulation 1. Position observation errors using observer (4 - 6) of joint 1 and joint 2.



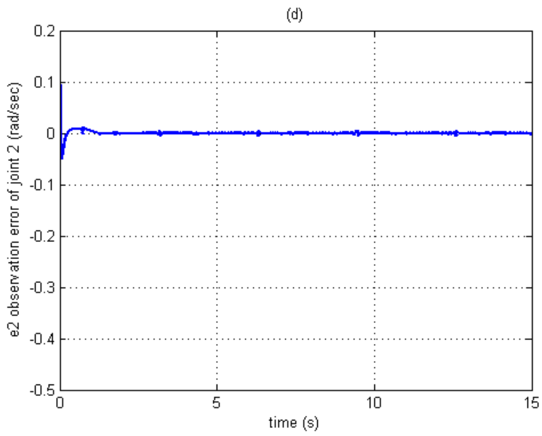
(a) Velocity estimate, $\hat{e}_{21}$ for time t = [0, 15] s.



(b) Velocity observation error, $\tilde{e}_{21}$ for time t = [0, 15] s.

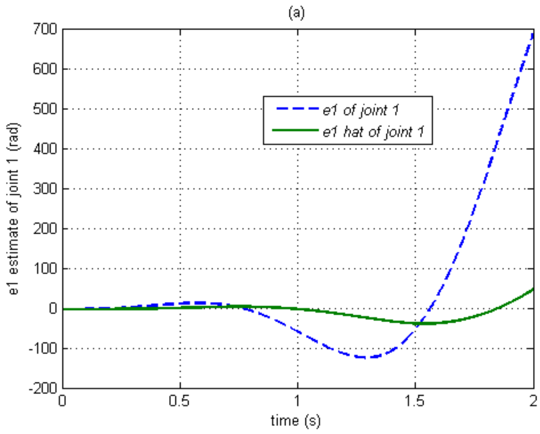


(c) Velocity estimate, $\hat{e}_{22}$ for time t = [0, 15] s.

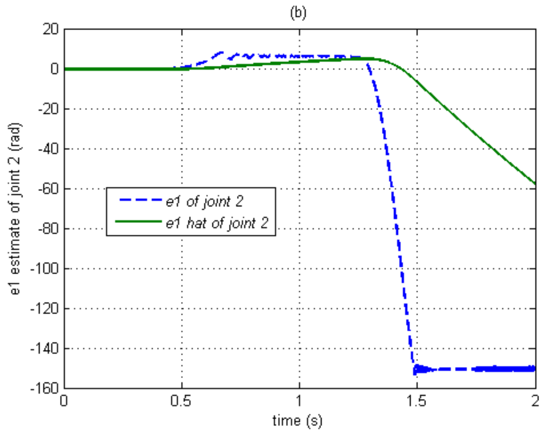


(d) Velocity observation error, $\tilde{e}_{22}$ for time t = [0, 15] s.

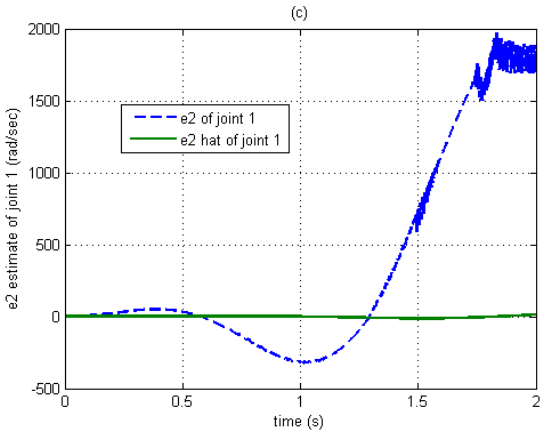
Figure 4.3 Simulation 1. Velocity observation errors using observer (4 - 6) of joint 1 and joint 2.



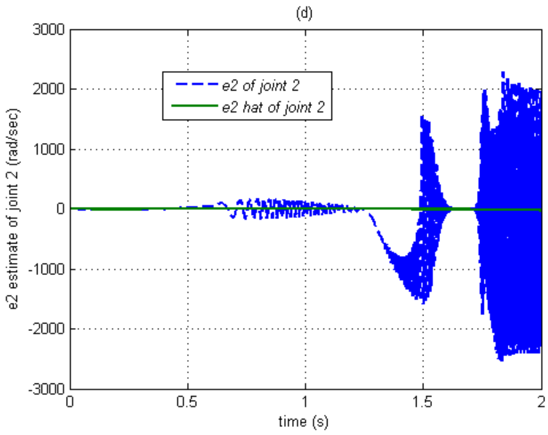
(a) Position estimate, $\hat{e}_{11}$ for time $t = [0, 2]$ s.



(b) Position estimate, $\hat{e}_{12}$ for time $t = [0, 2]$ s.

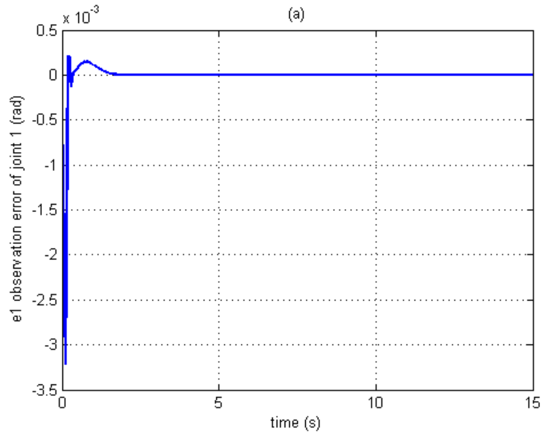


(c) Velocity estimate, $\hat{e}_{21}$ for time $t = [0, 2]$ s.

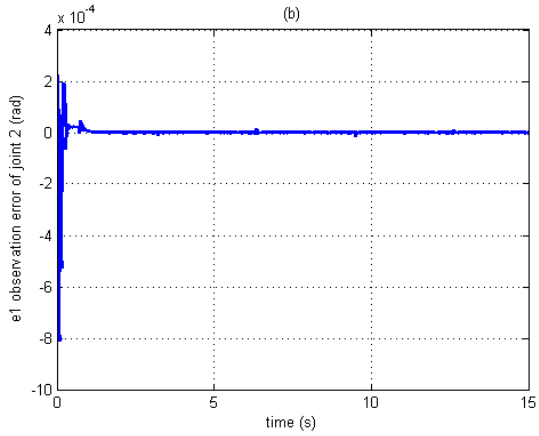


(d) Velocity estimate, $\hat{e}_{22}$ for time $t = [0, 2]$ s.

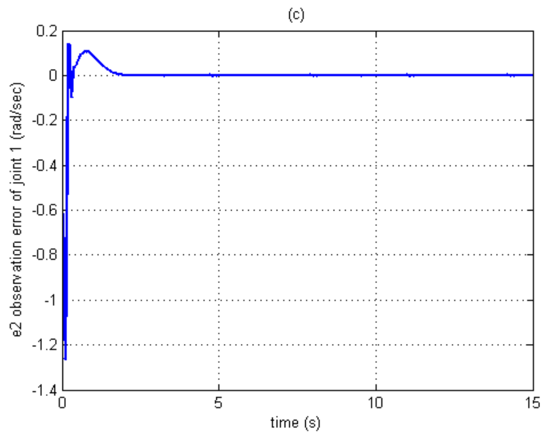
Figure 4.4 Simulation 2. Observation errors without L_3, small L_1 and L_2 .



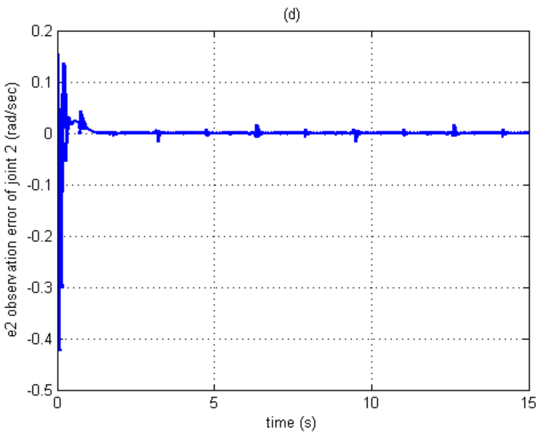
(a) Position observation error, $\tilde{e}_{11}$ for time t = [0, 15] s.



(b) Position observation error, $\tilde{e}_{12}$ for time t = [0, 15] s.

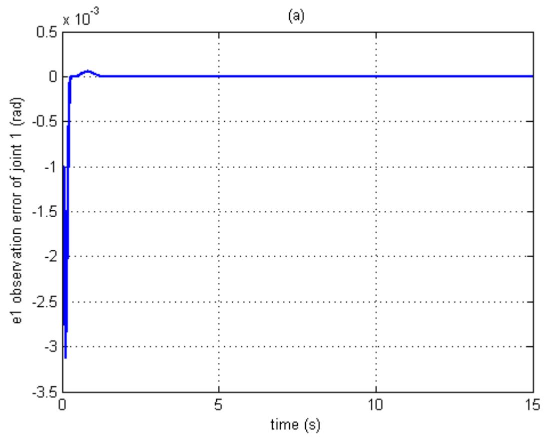


(c) Velocity observation error, $\tilde{e}_{21}$ for time t = [0, 15] s.

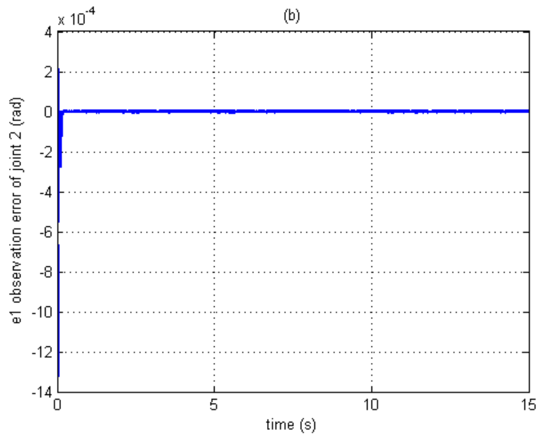


(d) Velocity observation error, $\tilde{e}_{22}$ for time t = [0, 15] s.

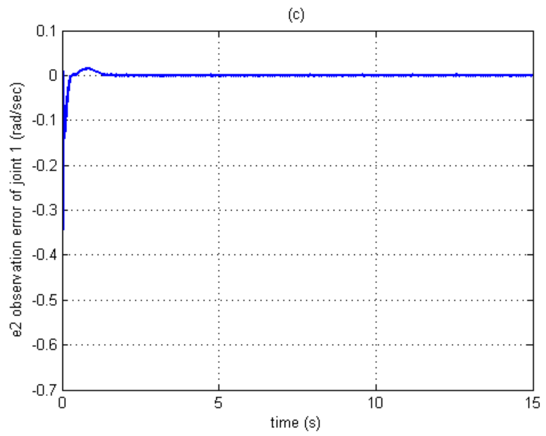
Figure 4.5 Simulation 2. Observation errors without L_3, large L_1 and L_2.



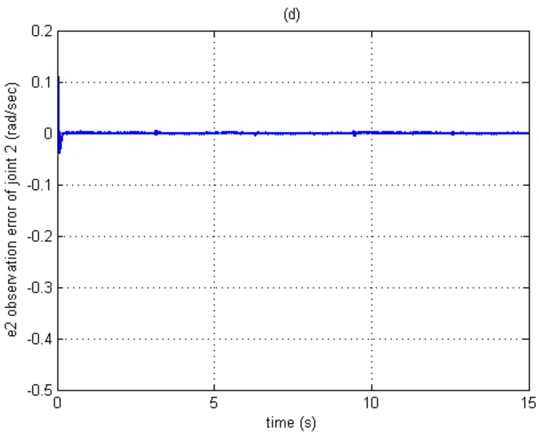
(a) Position observation error, $\tilde{e}_{11}$ for time t = [0, 15] s.



(b) Position observation error, $\tilde{e}_{12}$ for time t = [0, 15] s.

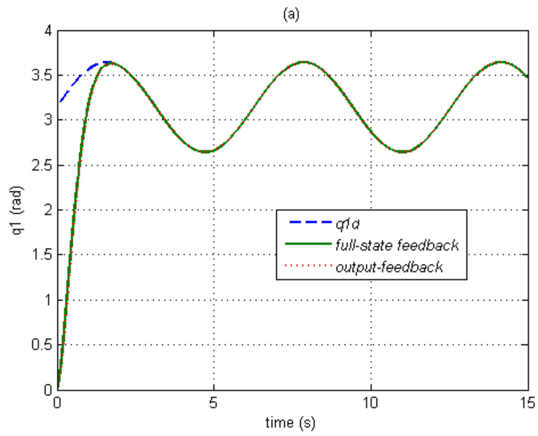


(c) Velocity observation error, $\tilde{e}_{21}$ for time t = [0, 15] s.

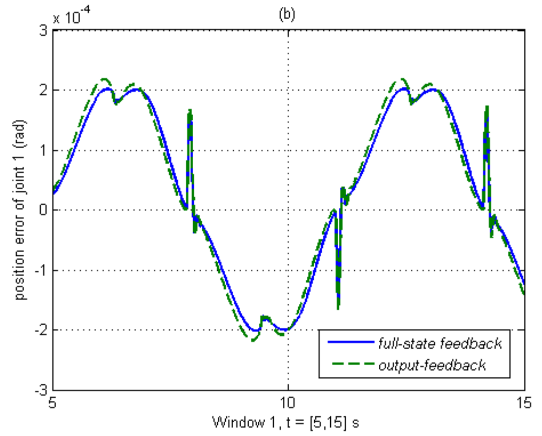


(d) Velocity observation error, $\tilde{e}_{22}$ for time t = [0, 15] s.

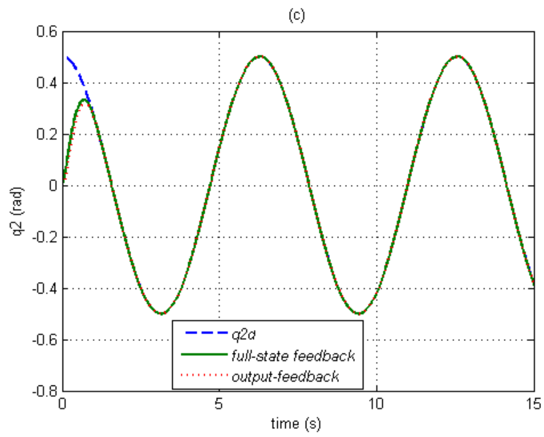
Figure 4.6 Simulation 3. Observation errors with p = 0.5.



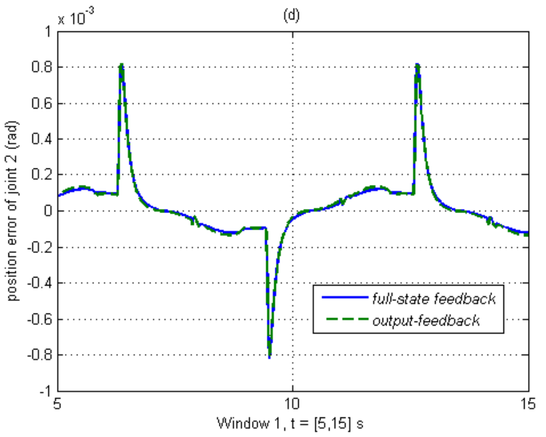
(a) Trajectory, q_1 for time $t = [0, 15]$ s.



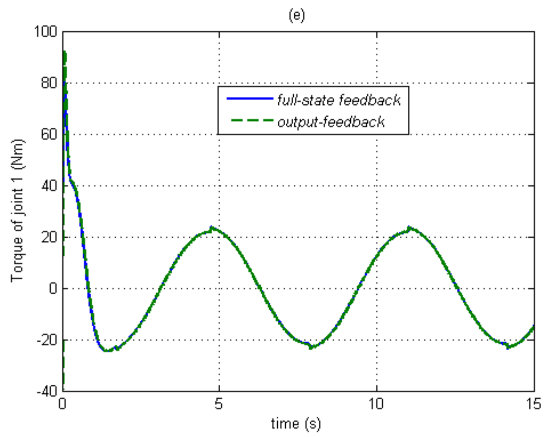
(b) Tracking error, e_{11} for time $t = [5, 15]$ s.



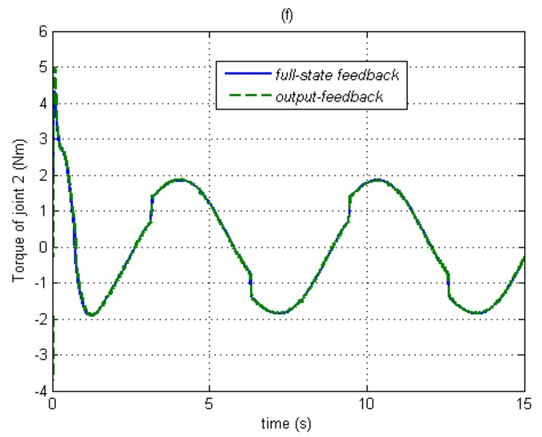
(c) Trajectory, q_2 for time $t = [0, 15]$ s.



(d) Tracking error, e_{12} for time $t = [5, 15]$ s.



(e) Torque, τ_1 for time $t = [0,15]$ s.



(f) Torque, τ_2 for time $t = [0,15]$ s.

Figure 4.7 Simulation 3. Tracking errors using control (3 - 2), control (4 - 4) with $p = 0.5$, and control input of joint 1 and 2.

1) Simulation 1:

Figure 4.1 shows the tracking errors and control inputs under the output-feedback control (4 - 4) and the observer (4 - 6). As per Theorem 4.1, for initial conditions inside the region $\Omega(c)$ will enter the region $\Omega(\mu)$ infinite time. In fact, in Figure 4.2 and Figure 4.3, the observation errors converge to a bounded region in about 2.5 seconds. Once the observation errors are sufficiently bounded, boundedness of the other states will follow provided gain K satisfy the sufficient conditions of Theorem 4.1. Particularly, the stability is highly dependent on the boundedness of the observation errors. In step 2 of the proof of Theorem 4.1 shows that the boundedness of observation error $\tilde{e}_2$ affects the ultimate bound of the tracking errors. In order to achieve better observer convergence, a linear damping term $L_3\,\tilde{e}_2$ is added. This term is instrumental to achieve semiglobal stability. As a matter of fact, from remark 4.1, increase in L_3 will reduce the ultimate bound on the observation error while at the same time the region $\Omega_{\tilde{e}}(c)$ remains unchanged due to its Lyapunov region $V_{\tilde{e}i}$ that is independent of L_3.

2) Simulation 2:

To study the impact of the linear damping term $L_3\,\tilde{e}_2$, Simulation 2 were performed under the same setup as Simulation 1 by removing the damping term from the observer (4 - 6). Figure 4.4 shows the observer estimates with respect to its actual values with the same small gains of L_1 and L_2 as per Simulation 1. The simulation was stopped at 2 seconds due to instability of the systems. The observation errors were too large for such a small gains to compensate as per the inequality in Theorem 4.2. In fact, to ensure stability the L_1 and L_2

gains have to be increased to a very large value to compensate for the nonlinearities as shown in Figure 4.5. This is in part due to the non-Lipschitz terms of the observer $\mathrm{sig}(\tilde{e}_{1_i})^p$ and $\mathrm{sig}(\tilde{e}_{1_i})^{2p-1}$ which are slow to grow when the observation error is from the origin due to $0.5 \leq p < 1$. However, there are some interesting features of the observer, particularly when $p = 0.5$ that enable the observer to be exactly robust to persistent disturbances and ensures convergence in finite time (see section 2.4). This feature will be discussed in the next simulation.

3) Simulation 3:

After showing the importance of $L_3\tilde{e}_2$ in the previous discussions, the exact robustness of the observer when $p = 0.5$ is shown in Figure 4.6. Note that the observation errors were able to converge in finite time with such small non-Lipschitz gains of 2, is due to the synergistic effect of the linear damping term. As per remark 4.6, after the convergence of the states of the system into the region $\Omega(\mu)$, the nonlinearities and state-dependent disturbances will be smaller. Thus, uncertainties that were initially unable to cope with by the non-Lipschitz gains (see Figure 4.4) are now much smaller when the states are inside $\Omega(\mu)$ thanks to the term, $L_3 \tilde{e}_2$ as per Theorem 4.1. In fact, when the observation errors converge to zero and stay in it in all future times, the control (4 - 4) becomes that of the full-state feedback of (3 - 2). In fact, from Figure 4.7, it can be observed that the ultimate bound on the tracking errors were the same for both controllers.

4.4 Summary

Trajectory tracking control of robot manipulator without velocity measurement in a semiglobal practical manner is achieved with the results of this chapter. Effectively, the proposed controller is an output feedback version of that develops in Chapter 3. The velocity measurement is substituted with the output of an observer that combines the super-twisting based algorithm develop in Chapter 2 with a damping term that is termed in the literature as "dirty-derivative". Through the non-Lipschitzness of the super-twisting based part of the observer, several finite time properties is produced, such as finite time exact robustness to achieve finite time convergence of observation error and finite time regulation. While the added linear damping term aids in reducing the level of gains of the non-Lipschitz part of the observer while maintaining performance.

Chapter 5: ROBOT MANIPULATOR CONTROL: BOUNDED CONTROL APPROACH WITH FULL-STATE FEEDBACK

In this chapter, a bounded controller is developed for the tracking control of robot manipulator. The proposed controller is based on the non-saturated results in chapter 3. By having an integral of nonlinear function in the controller, a conditional integrator-like behaviour is attained, and is able to achieve global practical stability results for trajectory tracking despite bounded uncertainties and disturbances. Similar performance as the non-saturated controller is obtained when the bounded controller is not saturated.

5.1 Nonlinear Robot Dynamics

The dynamic model of a rigid n-link serial robot manipulator, described in Section 3.1,

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + F\dot{q} + G(q) + D(q,\dot{q},t) = \tau \qquad (5 - 1)$$

where q, $\dot{q}$, $\ddot{q} \in \mathbb{R}^n$ are, respectively, the position (generalized coordinates), velocity and acceleration vectors, $M(q) \in \mathbb{R}^{n\times n}$ is the inertia matrix, and $C(q, \dot{q})\dot{q}$, $F\dot{q}$, $G(q)$, $\tau \in \mathbb{R}^n$ are, respectively, the vectors of Coriolis and centrifugal, viscous friction, gravity, and torque input, with F being a constant, positive definite, diagonal (viscous friction coefficient) matrix and $D(q, \dot{q}, t)$ being a bounded disturbances. The properties that the dynamics satisfied can be found in Section 3.1.

In this chapter, let us suppose that the absolute value of each joint input τ_i is constrained to be smaller than a given saturation bound $T_{i,\max} > 0$, i.e., $|\tau_i| \le T_{i,\max}$, $\forall i = 1,\dots, n$. The control objective here is to design a robust full state feedback saturated control that ensures the robot configuration vector q tracks a desired trajectory vector, $q_d\,(t)$ with an ultimately bounded error that can be made as small as required globally even under the presence of bounded disturbances.

The desired trajectory vector, $q_d\,(t)$ is assumed to be twice continuously differentiable vector-function such that $\|q_d(t)\|$, $\|\dot{q}_d(t)\|$, and $\|\ddot{q}_d(t)\|$ are bounded by *a priori* known constants. While the type of disturbances considered here has to be upper bounded by a constant, due to the limited actuation, (Note that no continuity assumption is made so that discontinuous models of friction may be used in D), i.e.

$$\|D\| \le \operatorname{sat}_{p_5}\left[p_0 + p_1\|e_1\| + p_2\|\dot{q}\| + p_3\|e_1\|^2 + p_4\|\dot{q}\|^2 \right]$$

where $\|D\| := \sup\{\|\varsigma\| : \varsigma \in K[D]\}$, p_0, p_1, p_2, p_3, p_4 , p_5 are some nonnegative constants, $e_1 = q - q_d \in \mathbb{R}^n$, and $e_2 = \dot{q} - \dot{q}_d \in \mathbb{R}^n$.

5.1.1 Control Development

Let us define the following scalar saturation function as

$$\operatorname{sat}_{\varepsilon}[x] = \begin{cases} x, & \text{if } |x| \le \varepsilon, \\ \varepsilon \operatorname{sign}[x], & \text{if } |x| > \varepsilon, \end{cases} \quad \forall x \in \mathbb{R},$$

if ε_i = 1, it will become the standard saturation function (see p.19 of [71]) and the subscript will be omitted, while the vector saturation function, given a set of positive number $\varepsilon \in \{\varepsilon_1$, …, ε_n }, define

$$\mathrm{sat}_{\varepsilon}[x] = \left[\mathrm{sat}_{\varepsilon_1}[x_1], \ldots, \mathrm{sat}_{\varepsilon_n}[x_n]\right], \forall x \in \mathbb{R}^n$$

and recall the following notation (see Section 3.1.2)

$$\mathrm{sig}(x)^a = \left[|x_1|^a \mathrm{sign}(x_1), \ldots, |x_n|^a \mathrm{sign}(x_n)\right]^T, \forall x \in \mathbb{R}^n.$$

Under the above assumptions, the proposed control law is of the form

$$\tau = -K\,\mathrm{sat}\left[\mathrm{sig}\left(\mu^{-1}s\right)^a\right], \; s \in \mathbb{R}^n, \tag{5 - 2}$$

where K and μ are positive definite diagonal matrix, i.e. $K = \mathrm{diag}\{k_i\}_{i=1}^n$, with $k_i > 0$, $\mu = \mathrm{diag}\{\mu_i\}_{i=1}^n$, with $\mu_i > 0$, $\forall i = 1, \ldots, n$ $a \geq 0$ constant, and $s \in \mathbb{R}^n$

i.e. $\tau_i = -k_i\,\mathrm{sat}\left[\dfrac{|s_i|^a}{\mu_i^a}\mathrm{sign}(s_i)\right]$ for $i = 1 \ldots n$.

When $a = 0$, the control becomes a discontinuous control law,

$$\tau = -K\,\mathrm{sign}(s), \; s \in \mathbb{R}^n.$$

Note that the control is bounded, i.e. such that $|\tau_i(t)| \leq k_i$, for $i = 1, \ldots, n$, $\forall\, t \geq 0$.

The s is the desired error dynamics defined as $s = e_2 + \sigma$, with

$$\dot{\sigma} = -K_a\sigma + K_a\mu\,\mathrm{sat}\left[\mu^{-1}s\right] - K_a\,\mathrm{sat}_{\left(\frac{\varepsilon_2}{k_2}\right)^{\frac{1+b}{2b}}}[e_2] + \mathrm{sat}_{\varepsilon_2}\left[K_2\mathrm{sig}(e_2)^{\frac{2b}{1+b}}\right] + \mathrm{sat}_{\varepsilon_1}\left[K_1\mathrm{sig}(e_1)^b\right] \tag{5 - 3}$$

where K_a, K_1, and K_2 are positive definite diagonal matrices, i.e. $K_a = \mathrm{diag}\{k_{ai}\}_{i=1}^n$, with

$k_{ai} > 0$, $K_1 = \mathrm{diag}\{k_{1i}\}_{i=1}^{n}$, with $k_{1i} > 0$, $K_2 = \mathrm{diag}\{k_{2i}\}_{i=1}^{n}$, with $k_{2i} > 0$, $k_2 \in \{k_{21}, \ldots, k_{2n}\}$, $\varepsilon_1 \in \{\varepsilon_{11}, \ldots, \varepsilon_{1n}\}$, with $\varepsilon_{1i} > 0$, $\varepsilon_2 \in \{\varepsilon_{21}, \ldots, \varepsilon_{2n}\}$, with $\varepsilon_{2i} > 0$, $\forall i = 1, \ldots, n$, and $b \geq 0$ constant, with $\varepsilon_{1i} > k_{ai}\left(\dfrac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+b}{2b}} + \varepsilon_{2i}$,

i.e.

$$\begin{aligned}\dot{\sigma}_i &= -k_{ai}\sigma_i + k_{ai}\mu_i \,\mathrm{sat}\left[\frac{s_i}{\mu_i}\right] - k_{ai}\,\mathrm{sat}_{\left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+b}{2b}}}\left[e_{2i}\right] \\ &\quad + \mathrm{sat}_{\varepsilon_{2i}}\left[k_{2i}\left|e_{2i}\right|^{\frac{2b}{1+b}}\mathrm{sign}(e_{2i})\right] + \mathrm{sat}_{\varepsilon_1}\left[k_{1i}\left|e_{1i}\right|^{b}\mathrm{sign}(e_{1i})\right]\end{aligned},$$

for $i = 1 \ldots n$.

Note that when $b = 0$, let the element of ε_1 equal to that of K_1, and ε_2 to that of K_2, i.e. $\varepsilon_{1n} = k_{1n}, \forall i \in N$ and $\varepsilon_{2n} = k_{2n}, \forall i \in N$, i.e.

$$\dot{\sigma}_i = -k_{ai}\sigma_i + k_{ai}\mu_i \,\mathrm{sat}\left[\frac{s_i}{\mu_i}\right] - k_{ai}\,\mathrm{sat}\left[e_{2i}\right] + k_{2i}\mathrm{sign}(e_{2i}) + k_{1i}\mathrm{sign}(e_{1i})$$

Then, the closed-loop system of (5 - 1), (5 - 2), and (5 - 3) can be written as

$$\begin{aligned}\dot{\sigma} &= -K_a\sigma + K_a\mu\,\mathrm{sat}\left[\mu^{-1}s\right] - K_a\,\mathrm{sat}_{\left(\frac{\varepsilon_2}{k_2}\right)^{\frac{1+b}{2b}}}\left[e_2\right] + \mathrm{sat}_{\varepsilon_2}\left[K_2\mathrm{sig}(e_2)^{\frac{2b}{1+b}}\right] + \mathrm{sat}_{\varepsilon_1}\left[K_1\mathrm{sig}(e_1)^{b}\right], \\ \dot{e}_1 &= e_2, \\ \dot{e}_2 &= -M^{-1}(q)K\,\mathrm{sat}\left[\mathrm{sig}(\mu^{-1}s)^{a}\right] - M^{-1}(q)\left(C(q,\dot{q})\dot{q} + F\dot{q} + G(q) + D\right) - \ddot{q}_d,\end{aligned}$$

To rewrite the closed-loop system in a form more convenient for analysis, let us define the change of variable $v_1 = e_1$, and $v_2 = -\sigma$, and we obtain the following form of closed-loop system,

$$
\begin{aligned}
\dot{v}_1 &= v_2 + s, \\
\dot{v}_2 &= -K_a v_2 - K_a \mu \operatorname{sat}\left[\mu^{-1} s\right] + K_a \operatorname{sat}_{\left(\frac{\varepsilon_2}{k_2}\right)^{\frac{1+b}{2b}}}\left[v_2 + s\right] \\
&\quad - \operatorname{sat}_{\varepsilon_2}\left[K_2 \operatorname{sig}\left(v_2 + s\right)^{\frac{2b}{1+b}}\right] - \operatorname{sat}_{\varepsilon_1}\left[K_1 \operatorname{sig}\left(v_1\right)^{b}\right], \qquad (5 - 4)\\
\dot{s} &= -M^{-1}\left(z_1 + q_d\right) K \operatorname{sat}\left[\operatorname{sig}\left(\mu^{-1} s\right)^{a}\right] - M^{-1}(q)\left(C(q,\dot{q})\dot{q} + F\dot{q} + G(q) + D\right) - \ddot{q}_d + \dot{\sigma},
\end{aligned}
$$

5.1.2 Stability Analysis

Theorem 5.1: Consider the robot dynamics (5 - 1), with the bounded controller given by (5 - 2), global practical trajectory tracking of the desired trajectory q_d can be assured, provided that the gain, K is sufficiently large up to the saturation bound $T_{i,\max}$, $\forall i = 1,\ldots,n$, and the desired trajectory sufficiently slow.

Proof of Theorem 5.1: The stability analysis proceeds in three steps.

1. First we will show the boundedness of σ, by the following Lyapunov function

   $$V_\sigma = \frac{1}{2}\sigma_i^{\,2}, \quad \forall i = 1,\ldots,n,$$

   the time derivative of the Lyapunov function, along its solution of (5 - 3), gives rise to:

$$\dot{V}_\sigma = -k_{ai}\sigma_i^{\,2} + \sigma_i k_{ai}\mu_i \operatorname{sat}\left(\frac{s_i}{\mu_i}\right) - \sigma_i k_{ai} \operatorname{sat}_{\left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+b}{2b}}}\left(e_{2i}\right)$$

$$+\sigma_i \operatorname{sat}_{\varepsilon_{2i}}\left(k_{2i}\left|e_{2i}\right|^{\frac{2b}{1+b}} \operatorname{sign}\left(e_{2i}\right)\right) + \sigma_i \operatorname{sat}_{\varepsilon_{1i}}\left(k_{1i}\left|e_{1i}\right|^{b} \operatorname{sign}\left(e_{1i}\right)\right)$$

$$\leq -\frac{k_{ai}}{2}\sigma_i^{\,2} - \frac{k_{ai}}{2}\sigma_i^{\,2} + \left|\sigma_i\right|\left(k_{ai}\mu_i + k_{ai}\left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+b}{2b}} + \varepsilon_{2i} + \varepsilon_{1i}\right)$$

$$\therefore \left|\sigma_i\right| \leq 2\left(\mu_i + \left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+b}{2b}} + \frac{\varepsilon_{2i}}{k_{ai}} + \frac{\varepsilon_{1i}}{k_{ai}}\right),$$

Since μ_i, ε_{1i}, $\varepsilon_{2i} > 0$ constants, $|\sigma_i|$ is bounded. Also, from (5 - 3), we can see that $|\dot{\sigma}_i|$ is bounded as well.

2. Now, we proceed to show the boundedness of s by the following Lyapunov function

$$V_s = \frac{1}{2} s^{\mathrm{T}} M(.) s \,.$$

Note that $\dot{q} = s + \left(\dot{q}_d - \sigma\right)$,

The s dynamics is described by,

$$\dot{s} = -M^{-1}(q) K \operatorname{sat}\left[\operatorname{sig}\left(\mu^{-1}s\right)^a\right] - M^{-1}(q)\left(C(q,\dot{q})\dot{q} + F\dot{q} + G(q) + D\right) - \ddot{q}_d + \dot{\sigma} \qquad (5 - 5)$$

Since the differential equation (5 - 5) has discontinuous right-hand side, i.e. when $a = 0$ or since no continuity assumption is made on D, its solutions are understood in the sense of Filippov (see definition 2.1), and in accordance to lemma 2.4, the time derivative of the Lyapunov function, V_s along the dynamics (5 - 5) for all $a \geq 0$:

$$\dot{V}_s \overset{a.e}{\in} \dot{\tilde{V}}_s$$

$$\begin{aligned}\dot{\tilde{V}}_s &= \bigcap_{\xi\in\partial V_s} \xi^{\mathrm{T}} \begin{pmatrix} \boldsymbol{K}[f](s) \\ 1 \end{pmatrix} = \nabla V_s{}^{\mathrm{T}} \boldsymbol{K}[f](s) + \frac{1}{2} s^{\mathrm{T}} \dot{M}(\cdot) s \\
&\subset \left[s^{\mathrm{T}} M(.)\right]\boldsymbol{K}\left[M^{-1}(q)\tau - M^{-1}(q)\left(C(q,\dot{q})\dot{q} + F\dot{q} + G(q) + D\right) - \ddot{q}_d + \dot{\sigma}\right] + \frac{1}{2} s^{\mathrm{T}} \dot{M}(\cdot) s \\
&= s^{\mathrm{T}} \boldsymbol{K}[\tau] + s^{\mathrm{T}}\left(-G(q) - M(q)\ddot{q}_d + M(q)\dot{\sigma}\right) - s^{\mathrm{T}}\boldsymbol{K}[D] - s^{\mathrm{T}} F\dot{q} - s^{\mathrm{T}} C(q,\dot{q})\dot{q} \\
&\quad + s^{\mathrm{T}} C(q,\dot{q}) s \\
&= s^{\mathrm{T}} \boldsymbol{K}[\tau] + s^{\mathrm{T}}\left(-G(q) - M(q)\ddot{q}_d + M(q)\dot{\sigma}\right) - s^{\mathrm{T}}\boldsymbol{K}[D] - s^{\mathrm{T}} F(\dot{q}_d - \sigma) \\
&\quad - s^{\mathrm{T}}\left[C(q,(\dot{q}_d - \sigma))\right](\dot{q}_d - \sigma) - s^{\mathrm{T}} F s - s^{\mathrm{T}}\left[C(q,(\dot{q}_d - \sigma))\right] s \\
&\leq s^{\mathrm{T}} \boldsymbol{K}[\tau] + s^{\mathrm{T}}\Delta - \|s\|^2\left(\underline{f} - C_m\|(\dot{q}_d - \sigma)\|\right),\end{aligned}$$

with the vector

$$\Delta = -G(q) - M(q)\ddot{q}_d + M(q)\dot{\sigma} - \boldsymbol{K}[D] - F(\dot{q}_d - \sigma) - C(q,(\dot{q}_d - \sigma))(\dot{q}_d - \sigma),$$

Remark 5.1. Observe that for $a > 0$, $s^{\mathrm{T}}\boldsymbol{K}[\tau] = -s^{\mathrm{T}} K \mathrm{sat}\left[\mu^{-1}\mathrm{sig}(s)^a\right]$, while for $a = 0$, using the following notation:

$$N^{+}(s) = \{i \in \{1,\ldots,n\} : s_i \neq 0\},\ N^{0}(s) = \{i \in \{1,\ldots,n\} : s_i = 0\},$$

$$\begin{aligned}s^{\mathrm{T}}\boldsymbol{K}[\tau] &= -s^{\mathrm{T}} K\boldsymbol{K}[\mathrm{sign}(s)] \\
&= -\sum_{i\in N^{+}(s)} k_i|s_i| - \sum_{i\in N^{0}(s)} k_i(0)\times[-1,+1] \\
&= -\sum_{i\in N^{+}(s)} k_i|s_i| - 0 \\
&= -\sum_{i=i}^{n} k_i|s_i|\end{aligned}$$

Also, let us define the set of real number, $N \in \{1,...,n\}$, and $|\Delta_i| := \sup\{|\varepsilon_i| : \varepsilon_i \in \boldsymbol{K}[\Delta_i]\}$, $\forall i \in N$, with Δ_i as elements of Δ, where

$$\boldsymbol{K}[\Delta] \subset \begin{bmatrix} \boldsymbol{K}[\Delta_1] \\ \vdots \\ \boldsymbol{K}[\Delta_n] \end{bmatrix}.$$

Since $\dot{q}_d$, $\ddot{q}_d$, are bounded from the assumptions and σ, $\dot{\sigma}$ are bounded (see step (1)), the vector Δ is bounded as well. Hence from [143], the multi-valued function $\boldsymbol{K}[\Delta]$ is bounded as well. Thus, if

$$\underline{f} > C_m \|(\dot{q}_d - \sigma)\|,$$

which is possible by selecting appropriate desired trajectory and the desired error dynamics, we have:

For $a = 0$:

$$\begin{aligned} \dot{V}_s &\overset{a.e.}{\leq} -\sum_{i=i}^{n} k_i |s_i| + s^{\mathrm{T}} \Delta \\ &\leq -\sum_{i=i}^{n} k_i |s_i| + \sum_{i=1}^{n} |s_i| |\Delta_i| \\ &= -\sum_{i=i}^{n} |s_i| (k_i - |\Delta_i|) \\ \therefore \dot{V}_s &\overset{a.e.}{<} 0 \quad \text{for} \ \forall |s_i| \neq 0 \end{aligned}$$

if the control gain is selected as $k_i > |\Delta_i|$. Thus, the state s will reach zero in finite time and stay there for all future times.

While for the case of $a > 0$:

$$\dot{V}_s \overset{a.e.}{\leq} -s^{\mathrm{T}} K \mathrm{sat}\left[\mathrm{sig}\left(\mu^{-1} s\right)^a\right] + s^{\mathrm{T}} \Delta$$

and it can be rewritten as

$$\begin{aligned}\dot{V}_s &\overset{a.e.}{\leq} \sum_{i \in N} -k_i |s_i| \mathrm{sat}\left[\frac{|s_i|^a}{\mu_i^a}\right] + |\Delta_i||s_i| \\ &= -k_j |s_j| \mathrm{sat}\left[\frac{|s_j|^a}{\mu_j^a}\right] + |\Delta_j||s_j| + \left(\sum_{i \in N \setminus j} |s_i| \left(-k_i \mathrm{sat}\left[\frac{|s_i|^a}{\mu_i^a}\right] + |\Delta_i|\right)\right)\end{aligned}$$

Remark 5.2. Note that the last equality has single out one of its element, $j \in N$ from the summation for ease of analysis.

Due to the saturated nature of the control, it is necessary for stability that $k_i > |\Delta_i|$, $\forall i \in N$. Then the maximum of the last term in the above inequality occurs when $|s_i| < \mu_i$. In particular, by taking the derivative of the term w.r.t. $|s_i|$ and equating it to zero, the maximum is found to occur at

$$|s_i| = \left|\frac{|\Delta_i|}{(1+a)k_i}\right|^{\frac{1}{a}} \mu_i, \ \ \forall i \in N \setminus j$$

and the corresponding maximum,

$$\left(\frac{a}{(1+a)^{\frac{1+a}{a}}}\right)\left(\frac{|\Delta_i|}{k_i}\right)^{\frac{1}{a}}\mu_i|\Delta_i|,\ \forall i\in N\setminus j$$

Hence, one obtains,

$$\dot{V}_s \overset{a.e.}{\leq} -k_j|s_j|\text{sat}\left[\frac{|s_j|^a}{\mu_j^{\ a}}\right]+|\Delta_j||s_j|+\left(\frac{a}{(1+a)^{\frac{1+a}{a}}}\right)\sum_{i\in N\setminus j}\left(\frac{|\Delta_i|}{k_i}\right)^{\frac{1}{a}}\mu_i|\Delta_i|$$

$$=-(1-\theta)k_j|s_j|\text{sat}\left[\frac{|s_j|^a}{\mu_j^{\ a}}\right]-\frac{\theta}{2}k_j|s_j|\text{sat}\left[\frac{|s_j|^a}{\mu_j^{\ a}}\right]+|\Delta_j||s_j|-\frac{\theta}{2}k_j|s_j|\text{sat}\left[\frac{|s_j|^a}{\mu_j^{\ a}}\right]$$
$$+\left(\frac{a}{(1+a)^{\frac{1+a}{a}}}\right)\sum_{i\in N\setminus j}\left(\frac{|\Delta_i|}{k_i}\right)^{\frac{1}{a}}\mu_i|\Delta_i|$$

with any constant $0<\theta<1$,

$$\dot{V}_s \overset{a.e.}{\leq} -(1-\theta)k_j|s_j|\text{sat}\left[\frac{|s_j|^a}{\mu_j^{\ a}}\right],\ \text{for}\ |s_j|\geq\pi_j,\ \text{and}\ |s_i|\in\mathbb{R},\forall i\in N\setminus j$$

with

$$\pi_j=\max\left\{\left(\frac{2}{\theta}\frac{|\Delta_j|}{k_j}\right)^{\frac{1}{a}}\mu_j,\ \left(\frac{2\mu_j^{\ a}}{\theta k_j}\right)^{\frac{1}{1+a}}\left(\frac{a}{(1+a)^{\frac{1+a}{a}}}\right)^{\frac{1}{1+a}}\left(\sum_{i\in N\setminus j}\left(\frac{|\Delta_i|}{k_i}\right)^{\frac{1}{a}}\mu_i|\Delta_i|\right)^{\frac{1}{1+a}}\right\},$$

$$\text{and } k_j>\max\left\{\frac{2|\Delta_j|}{\theta},\ \frac{2}{\theta\mu_j}\left(\frac{a}{(1+a)^{\frac{1+a}{a}}}\right)\sum_{i\in N\setminus j}\left(\frac{|\Delta_i|}{k_i}\right)^{\frac{1}{a}}\mu_i|\Delta_i|\right\} \tag{5 - 6}$$

Note that with the above conditions, we have $\pi_j<\mu_j$. Thus, selecting each k_j , $\forall\, j \in$

N that satisfy (5 - 6), we have

$$\dot{V}_s \overset{a.e.}{<} -c_1, \ \forall s \notin \Omega_\pi$$

where the compact set is defined as: $\Omega_\pi = \left\{ s \in \mathbb{R}^n : |s_i| \le \pi_i, \forall i \in N \right\}$

with

$$c_1 = \min_i \left\{ (1-\theta) k_i \pi_i \left(\frac{\pi_i}{\mu_j} \right)^a \right\}, \ \forall i \in N,$$

Note that this compact set can be made arbitrarily small by increasing k_i up to the maximal allowable control bound, $T_{i,\max}$. In particular note that by having large k_i implies a smaller π_i since

$$\pi_j = \max\left\{ \left(\frac{2}{\theta} \frac{|\Delta_j|}{k_j} \right)^{\frac{1}{a}} \mu_j, \left(\frac{2\mu_j^{\ a}}{\theta k_j} \right)^{\frac{1}{1+a}} \left(\frac{a}{(1+a)^{\frac{1+a}{a}}} \right)^{\frac{1}{1+a}} \left(\sum_{i \in N \setminus j} \left(\frac{|\Delta_i|}{k_i} \right)^{\frac{1}{a}} \mu_i |\Delta_i| \right)^{\frac{1}{1+a}} \right\}, \ \forall j \in N$$

which further implies that a smaller compact set Ω_π.

Now with each given μ_i, let c_1 be a positive constant $0 < c_1 < 1$, consider the following compact set

$$\Omega_\mu = \left\{ s \in \mathbb{R}^n : |s_i| \le c_1 \mu_i, \forall i \in N \right\},$$

$$\Sigma_\mu = \left\{ s \in \mathbb{R}^n : V_s \le k_\mu \right\},$$

where $k_\mu > 0$ is defined as $k_\mu = \min\limits_{s \in \mathrm{bd}\ \Omega_\mu} V_s$, which exists since the boundary, bd Ω_μ is a compact set. Note that $\Sigma_\mu \subset \Omega_\mu$. Since Σ_μ is a Lyapunov level set, if the states, s can be confined within this set, all s_i's are not saturated. Hence, if the states s stay inside the Lyapunov level set Σ_μ, all s_i's are not saturated. To achieve this, we simply need the set $\Omega_\pi \subset \Sigma_\mu$, which can be attained when k_i is large enough, such that $\dot{V}_s$ being negative the outside of the set Σ_μ,

$$\dot{V}_s \overset{a.e.}{<} 0, \text{ for } V_s \geq k_\mu$$

which implies that the trajectories of s will enter the set Σ_μ in finite time and stay in it once entered. Hence, we have

$$|s_i| \leq c_1\mu_i < \mu_i \Leftrightarrow |s_i| < \mu_i, \ \forall i \in N$$

In particular, a sufficient condition on k_i such that each s_i is unsaturated can be shown as following:

Note that:

$$k_\mu = \min_{s \in \text{bd } \Omega_\mu} V_s \geq \min_{s \in \text{bd } \Omega_\mu} \frac{1}{2}\underline{m}\|s\|^2 = \frac{1}{2}\underline{m}\left(\min_i c_1\mu_i\right)^2$$

Also,

$$\max_{s\in \text{bd }\Omega_\pi} V_s \le \max_{s\in \text{bd }\Omega_\pi} \frac{1}{2}\overline{m}\|s\|^2$$

$$= \frac{1}{2}\overline{m}\left(\max_{s\in \text{bd }\Omega_\pi}\|s\|\right)^2 = \frac{1}{2}\overline{m}\left(\left(\sum_i^n |\pi_i|^2\right)^{\frac{1}{2}}\right)^2 = \frac{1}{2}\overline{m}\left(\sum_i^n |\pi_i|^2\right)$$

$$\le \frac{1}{2}\overline{m}\left(\sum_i^n |\pi_i|\right)^2, \quad \text{from lemma 2,}$$

$$= \frac{1}{2}\overline{m}\left(\sum_j^n \max\left\{\begin{array}{l}\left(\dfrac{2}{\theta}\dfrac{|\Delta_j|}{k_j}\right)^{\frac{1}{a}}\mu_j, \\ \left(\dfrac{2\mu_j^a}{\theta k_j}\right)^{\frac{1}{1+a}}\left(\dfrac{a}{(1+a)^{\frac{1+a}{a}}}\right)^{\frac{1}{1+a}}\left(\displaystyle\sum_{i\in N\backslash j}\left(\dfrac{|\Delta_i|}{k_i}\right)^{\frac{1}{a}}\mu_i|\Delta_i|\right)^{\frac{1}{1+a}}\end{array}\right\}\right)^2$$

$$\le \frac{1}{2}\overline{m}\left(\frac{1}{\underline{k}}\right)^{\frac{2}{a}}\left(\sum_j^n \max\left\{\begin{array}{l}\left(\dfrac{2|\Delta_j|}{\theta}\right)^{\frac{1}{a}}\mu_j, \\ \left(\dfrac{2\mu_j^a}{\theta}\right)^{\frac{1}{1+a}}\left(\dfrac{a}{(1+a)^{\frac{1+a}{a}}}\right)^{\frac{1}{1+a}}\left(\displaystyle\sum_{i\in N\backslash j}|\Delta_i|^{\frac{1}{a}}\mu_i|\Delta_i|\right)^{\frac{1}{1+a}}\end{array}\right\}\right)^2$$

where $\underline{k} = \min_i k_i$

Hence, to satisfy the condition $\Omega_\pi \subset \Sigma_\mu$, it is sufficient for:

$$k_\mu \ge \max_{s\in \text{bd }\Omega_\pi} V_s$$

$$\Rightarrow \frac{1}{2}\underline{m}\left(\min_i c_1\mu_i\right)^2$$

$$\ge \frac{1}{2}\overline{m}\left(\frac{1}{\underline{k}}\right)^{\frac{2}{a}}\left(\sum_j^n \max\left\{\begin{array}{l}\left(\dfrac{2|\Delta_j|}{\theta}\right)^{\frac{1}{a}}\mu_j, \\ \left(\dfrac{2\mu_j^a}{\theta}\right)^{\frac{1}{1+a}}\left(\dfrac{a}{(1+a)^{\frac{1+a}{a}}}\right)^{\frac{1}{1+a}}\left(\displaystyle\sum_{i\in N\backslash j}|\Delta_i|^{\frac{1}{a}}\mu_i|\Delta_i|\right)^{\frac{1}{1+a}}\end{array}\right\}\right)^2,$$

$$\Leftrightarrow \underline{k} \geq \frac{\left(\sum_{j}^{n} \max \left\{ \begin{aligned} &\left(\frac{2\left|\Delta_j\right|}{\theta}\right)^{\frac{1}{a}} \mu_j, \\ &\left(\frac{2\mu_j{}^{a}}{\theta}\right)^{\frac{1}{1+a}} \left(\frac{a}{(1+a)^{\frac{1+a}{a}}}\right)^{\frac{1}{1+a}} \left(\sum_{i \in N \backslash j} \left|\Delta_i\right|^{\frac{1}{a}} \mu_i \left|\Delta_i\right|\right)^{\frac{1}{1+a}} \end{aligned} \right\}\right)^{a}}{\left(\frac{\underline{m}}{\overline{m}}\right)^{\frac{a}{2}} \left(\min_i c_1 \mu_i\right)^{a}} \tag{5 - 7}$$

It is not difficult to show that selecting K that satisfies (5 - 7) implies that (5 - 6) is sufficiently satisfied as well. Thus, for the case of $a > 0$ the control will be unsaturated in finite time and remain so thereafter.

3. Having shown the boundedness of σ in step 1 and s being unsaturated and bounded or zero through step 2, depending on the parameter a, we are going to show the stability analysis of the desired error dynamics (5 - 3).

   For the case of $b > 0$:

   For $a = 0$, we have, since s will reach zero in finite time, from (5 - 4) the (v_1, v_2) dynamics

$$\dot{v}_1 = v_2,$$
$$\dot{v}_2 = -K_a v_2 + K_a \operatorname{sat}_{\left(\frac{\varepsilon_2}{k_2}\right)^{\frac{1+b}{2b}}}\left[v_2\right] - \operatorname{sat}_{\varepsilon_2}\left[K_2 \operatorname{sig}(v_2)^{\frac{2b}{1+b}}\right] - \operatorname{sat}_{\varepsilon_1}\left[K_1 \operatorname{sig}(v_1)^{b}\right]$$

   where element-wise, we have

$$
\begin{aligned}
\dot{v}_{1i} &= v_{2i} \\
\dot{v}_{2i} &= -k_{ai}\left((v_{2i}) - \operatorname{sat}_{\left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+b}{2b}}}\left[v_{2i}\right] \right) - \operatorname{sat}_{\varepsilon_{2i}}\left[k_{2i}\left|v_{2i}\right|^{\frac{2b}{1+b}} \operatorname{sign}(v_{2i}) \right] \\
&\quad - \operatorname{sat}_{\varepsilon_{1i}}\left[k_{1i}\left|v_{1i}\right|^{b} \operatorname{sign}(v_{1i}) \right]
\end{aligned}
$$

While for $a > 0$:

Thus, we have, since the control will be unsaturated in finite time, $\operatorname{sat}\left[\mu^{-1}s\right] = \mu^{-1}s$, from (5 - 4) the ($v_1$, v_2) dynamics

$$
\begin{aligned}
\dot{v}_1 &= v_2 + s, \\
\dot{v}_2 &= -K_a v_2 - K_a s + K_a \operatorname{sat}_{\left(\frac{\varepsilon_2}{k_2}\right)^{\frac{1+b}{2b}}}\left[v_2 + s\right] - \operatorname{sat}_{\varepsilon_2}\left[K_2 \operatorname{sig}(v_2 + s)^{\frac{2b}{1+b}} \right] \\
&\quad - \operatorname{sat}_{\varepsilon_1}\left[K_1 \operatorname{sig}(v_1)^{b} \right]
\end{aligned}
$$

where element-wise,

$$
\begin{aligned}
\dot{v}_{1i} &= v_{2i} + s_i \\
\dot{v}_{2i} &= -k_{ai}\left((v_{2i} + s_i) - \operatorname{sat}_{\left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+b}{2b}}}\left[v_{2i} + s_i\right] \right) - \operatorname{sat}_{\varepsilon_{2i}}\left[k_{2i}\left|v_{2i} + s_i\right|^{\frac{2b}{1+b}} \operatorname{sign}(v_{2i} + s_i) \right] \\
&\quad - \operatorname{sat}_{\varepsilon_{1i}}\left[k_{1i}\left|v_{1i}\right|^{b} \operatorname{sign}(v_{1i}) \right]
\end{aligned}
\tag{5 - 8}
$$

Remark 5.3. Note that the desired error dynamics, after the convergence of s, for the case of $a = 0$ is identical to that of the case of $a > 0$ when $s = 0$.

From Appendix B.1, with the Lyapunov function

$$V = \left(\frac{1}{2} v_{2i}{}^2 + \int_0^{v_{1i}} \mathrm{sat}_{\varepsilon_{1i}} \left[k_{1i} |r|^b \,\mathrm{sign}(r) \right] dr \right)^2 + r_1 \left(\frac{3+3b}{2} \right) \left(\int_0^{v_{1i}} |r|^{\frac{1}{2}} \,\mathrm{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3}{2}}} [|r|^{\frac{3b}{2}}] \; dr \right) v_{2i}$$

$$+ r_1 (1+b) k_{ai} \left(\frac{\varepsilon_{1i}}{k_{1i}} \right)^{\frac{3}{2}} \left(\int_0^{v_{1i}} |r|^{\frac{3}{2}} \mathrm{sign}(r) - \mathrm{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3}{2b}}} \left[|r|^{\frac{3}{2}} \mathrm{sign}(r) \right] dr \right)$$

where $r_1 > 0$ is a constant, it is shown that

$$\dot{\tilde{V}} < 0, \quad \text{for} \quad V \geq \rho_{1i}(|s_i|),$$

where the class $\mathcal{K}$ function, ρ_{1i} is defined as, $\rho_{1i}(|s_i|) = \max_{(v_{1i}, v_{2i}) \in \mathrm{bd}\ \Psi_{3i}(|s_i|)} V$,

with bd $\Psi_{3i}(|s_i|)$ as the boundary of the compact set

$\Psi_{3i}(|s_i|) = \{(v_{1i}, v_{2i}) \in \mathbb{R}^2 : |v_{1i}| \leq \alpha_1(|s_i|), |v_{2i}| \leq \alpha_2(|s_i|)\}$, and

α_1, α_2 are class $\mathcal{K}$ functions defined in Appendix B.1. Hence, invoking lemma 2.6

together with the boundedness of s (see step 2), the states (v_1, v_2) are uniformly ultimately bounded.

For the special case of $b = 0$:

Let the elements of ε_1 equal to that of K_1, and ε_2 to that of K_2, i.e. $\varepsilon_{1n} = k_{1n}, \forall i \in N$ and $\varepsilon_{2n} = k_{2n}, \forall i \in N,$ hence the desired dynamics becomes

$$\begin{aligned} \dot{v}_{1i} &= v_{2i} + s_i \\ \dot{v}_{2i} &= -k_{ai}\left((v_{2i} + s_i) - \mathrm{sat}[v_{2i} + s_i]\right) - k_{2i}\mathrm{sign}(v_{2i} + s_i) - k_{1i}\mathrm{sign}(v_{1i}) \end{aligned} \qquad (5 - 9)$$

From Appendix B.1, with the Lyapunov function

$$V = \left(\frac{1}{2} v_{2i}{}^2 + k_{1i} \int_0^{v_{1i}} \mathrm{sign}(r)\, dr \right)^2 + r_1 |v_{1i}|^{\frac{3}{2}} \mathrm{sign}(v_{1i})\, v_{2i} + r_1 \left(\frac{2}{5} \right) k_{ai} |v_{1i}|^{\frac{5}{2}}$$
$$- r_1 k_{ai} \int_0^{v_{1i}} \left[\mathrm{sat}_1 [|r|^{\frac{3}{2}}] \right] \mathrm{sign}(r)\, dr$$

where $r_1 > 0$ is a constant, it is shown that

$$\dot{\tilde{V}} < 0, \text{ for } V \geq \rho_{2i}(|s_i|),$$

where the class $\mathcal{K}$ function, ρ_{2i} is defined as $\rho_{2i}(|s_i|) = \max_{(v_{1i}, v_{2i}) \in \mathrm{bd}\ \phi_{3i}(|s_i|)} V$,

with bd $\phi_{3i}(|s_i|)$ as the boundary of the compact set

$\phi_{3i}(|s_i|) = \{(v_{1i}, v_{2i}) \in \mathbb{R}^2 : |v_{1i}| \leq \alpha_{11}(|s_i|), |v_{2i}| \leq \alpha_{12}(|s_i|)\}$ and

α_{11}, α_{12} are class $\mathcal{K}$ functions defined in Appendix B.1. Hence, invoking lemma 2.6

together with the boundedness of s (see step 2 0), the states (v_1, v_2) are uniformly ultimately bounded.

■

Remark 5.4. Hence, for K satisfying (5 - 7), the trajectories of the closed-loop system (5 - 4), are globally stable, with the ultimate bound being reduced as desired up to the saturation limit of the actuator, $T_{i,\max} > 0$, $\forall i = 1,...,n$. Another way to view the stability results is, for a given bounds on the actuator limit, and hence K, if the bounds on the uncertainties,

$$\Delta = -G(q) - M(q)\ddot{q}_d + M(q)\dot{\sigma} - \boldsymbol{K}[D] - F(\dot{q}_d - \sigma) - C(q,(\dot{q}_d - \sigma))(\dot{q}_d - \sigma)$$

are sufficiently small to satisfy (5 - 7), the global practical stability results are still assured. Essentially, besides the bounds on the parameters of the robot manipulator dynamics and disturbances, the term Δ is also dependent on the desired acceleration, velocity and the desired error dynamics through the term σ and $\dot{\sigma}$. In which case, a slower desired trajectory or slower desired error dynamics can, in effect, produce a smaller upper bound of $|\Delta|$.

Remark 5.5. It is desired for the controller (5 - 2) to behave as per its non-bounded counterpart (Chapter 3) when the controller is not saturated so that it exhibits the same properties such as the ability to inject desired error dynamics, and desired performance. Indeed from stability proof above (step 2), the control (5 - 2) will be unsaturated in finite time and stay so in all future times. Particularly, if the upper bound of $|s_i|$ is sufficiently small (by selecting gain K satisfying (5 - 7) up to the allowable control bound, $T_{i,\max}$), from step 3 above, the states (v_1, v_2) of the dynamics (5 - 8) and (5 - 9) will become unsaturated as well (see Appendix B.1),

for $b > 0$,

$$\begin{aligned}
\dot{v}_{1i} &= v_{2i} + s_i \\
\dot{v}_{2i} &= -k_{1i}|v_{1i}|^b \operatorname{sign}(v_{1i}) - k_{2i}|v_{2i} + s_i|^{\frac{2b}{1+b}} \operatorname{sign}(v_{2i} + s_i)
\end{aligned}$$

for $b = 0$,

$$\begin{aligned}
\dot{v}_{1i} &= v_{2i} + s_i \\
\dot{v}_{2i} &= -k_{2i}\operatorname{sign}(v_{2i} + s_i) - k_{1i}\operatorname{sign}(v_{1i})
\end{aligned}$$

which is similar to the z-subsystem (3 - 4) considered in Chapter 3, and hence will exhibit the same properties, since the states will remain unsaturated for all future times, provided $|s_i|$ is sufficiently small.

Remark 5.6. Note that the desired error dynamics injected by the controller through (5 - 3) is a modified version of (3 - 3) in Chapter 3 to account for the bounded control structure considered in this chapter. Specifically, it is an integral of saturation functions that provides boundedness of σ as per step 1 of the proof of Theorem 5.1. This in turn, helps to dominate the effects of Coriolis term as shown in step 2 above, which is essential for global practical stability by accounting the inherent viscous friction of the robot manipulator.

Remark 5.7. Observe that in step 2 above, it is necessary for the viscous friction term to satisfy the inequality $\underline{f} > C_m \|(\dot{q}_d - \sigma)\|$. This condition that is commonly found in the literature of global stability of robot manipulator with bounded control (see, for instance, [118] and [113]). This condition can be met by selecting the desired trajectories (through the desired velocity vector) and desired error dynamics (through (5 - 3)) appropriately. This condition restricts the desired velocity and acceleration (i.e. through the desired error dynamics) vectors but not the location of the desired task (i.e. q_d). Essentially, the desired trajectory may be defined anywhere within the workspace of the robotic manipulator provided it has sufficiently slow motions.

Remark 5.8. It is worth mentioning that the desired error dynamics injection term have an inherent anti-windup structure, rewriting (5 - 3) by adding and subtracting $K_a s$,

$$\dot{\sigma} = -K_a\sigma + K_a s - \underbrace{K_a\left(s - \mu\,\mathrm{sat}\left[\mu^{-1}s\right]\right)}_{anti-windup} - K_a\,\mathrm{sat}_{\left(\frac{\varepsilon_2}{k_2}\right)^{\frac{1+b}{2b}}}\left[e_2\right]$$
$$+\,\mathrm{sat}_{\varepsilon_2}\left[K_2\,\mathrm{sig}(e_2)^{\frac{2b}{1+b}}\right] + \mathrm{sat}_{\varepsilon_1}\left[K_1\,\mathrm{sig}(e_1)^b\right],$$

hence when the control is saturated, the above feedback signal tries to drive the "control input error" $\left(s - \mu\,\mathrm{sat}\left[\mu^{-1}s\right]\right)$ to zero. Anti-windup prevents control loop from severe stability and performance degradation induced by integral action winding [117].

5.2 Numerical Simulations

In this section, numerical simulations on a two-link robot manipulator were carried out to illustrate the results discussed in this chapter. The setups for each simulation are described. Discussion and analysis of the results are presented accordingly.

5.2.1 Simulation Setup

1) Simulation 1:

The same two-link rigid robot manipulator considered in section 3.3 is adopted in simulation. The dynamics of robot manipulator (5 - 1) have the same parameter values as that in section 3.3. The desired trajectory vector and the additive disturbances Coulomb friction vector were defined similarly as well. The initial conditions of the robot manipulator were selected as,

$$q(0)=\begin{bmatrix}0\\0\end{bmatrix},\ \dot{q}(0)=\begin{bmatrix}10\\16\end{bmatrix},\ q_d(0)=\begin{bmatrix}\pi\\0.5\end{bmatrix},\ \dot{q}_d(0)=\begin{bmatrix}0.5\\0\end{bmatrix}$$

The control (5- 2) parameter values were selected as follows,

$$a=0.9,\ K=\begin{bmatrix}150 & 0\\0 & 15\end{bmatrix},\ K_1=\begin{bmatrix}18 & 0\\0 & 18\end{bmatrix},\ K_2=\begin{bmatrix}11 & 0\\0 & 11\end{bmatrix}, b=0.6$$

$$\mu=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix},\ \varepsilon_{11}=\varepsilon_{12}=18,\ \varepsilon_{21}=\varepsilon_{22}=11,\ K_a=\begin{bmatrix}10 & 0\\0 & 10\end{bmatrix},$$

The initial conditions of the vector σ were selected as

$$\sigma(0)=-e_2(0)=\begin{bmatrix}0.5\\0\end{bmatrix}-\begin{bmatrix}10\\16\end{bmatrix}=\begin{bmatrix}-9.5\\-16\end{bmatrix}$$

Note that the initial velocity is purposefully made to be far from the origin in order to bring the controller to saturation.

2) Simulation 2:

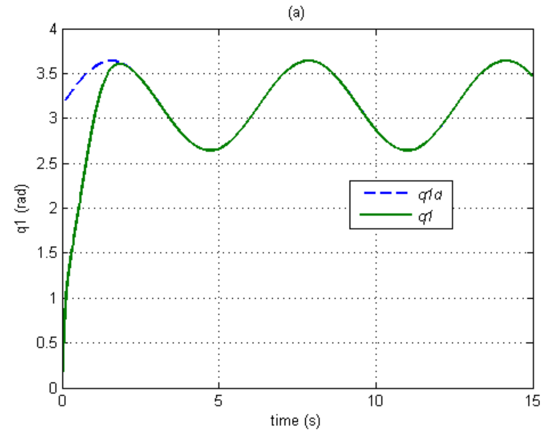
The setup of Simulation 2 is exactly the same as that of Simulation 1. The simulation is repeated using the unbounded full-state feedback control (3 - 2), to examine its maximal tracking errors compared to that of control (5 - 2), with the parameter values selected as

$$a=0.9,\ K=\begin{bmatrix}150 & 0\\0 & 15\end{bmatrix},\ K_1=\begin{bmatrix}18 & 0\\0 & 18\end{bmatrix},\ K_2=\begin{bmatrix}11 & 0\\0 & 11\end{bmatrix}, b=0.6$$

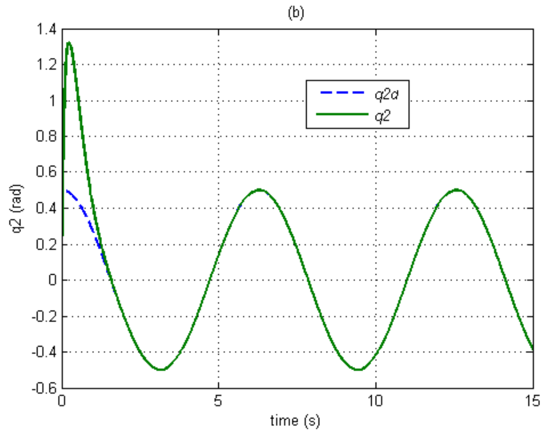
Note that the same parameter values are attained when the control (5 - 2) vectors of Simulation 1 are unsaturated.

5.2.2 Results and Discussions

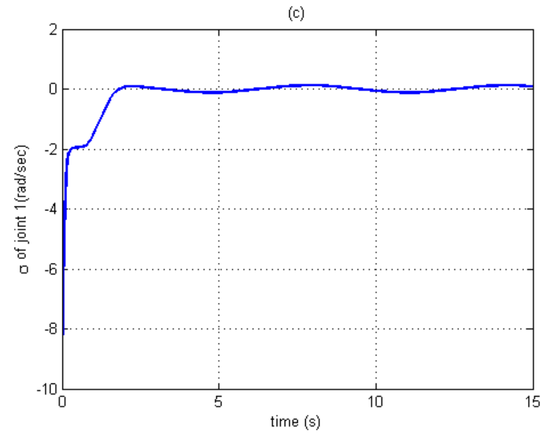
For better visualization of the plots, some figures are shown in two windows; each with different time intervals.



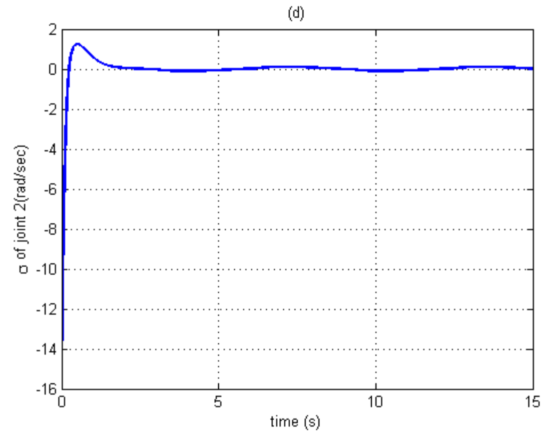
(a) Trajectory, q_1 for time t = [0,15] s.



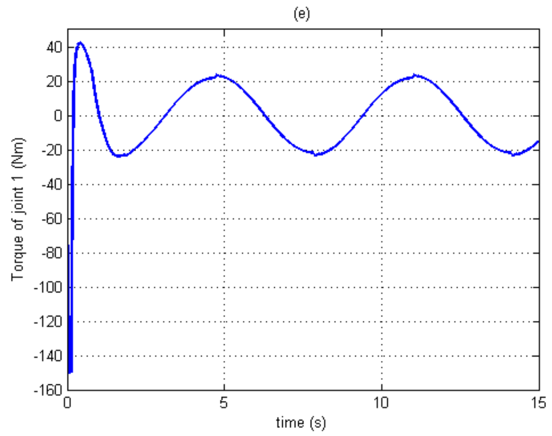
(b) Trajectory, q_2 for time t = [0, 15] s.



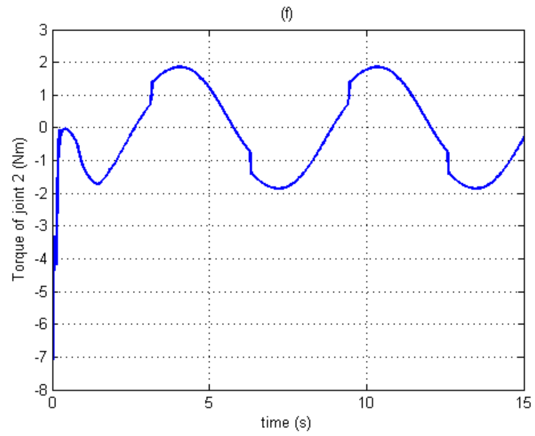
(c) Sigma, σ_1 for time t = [0, 15] s.



(d) Sigma, σ_2 for time t = [0, 15] s.

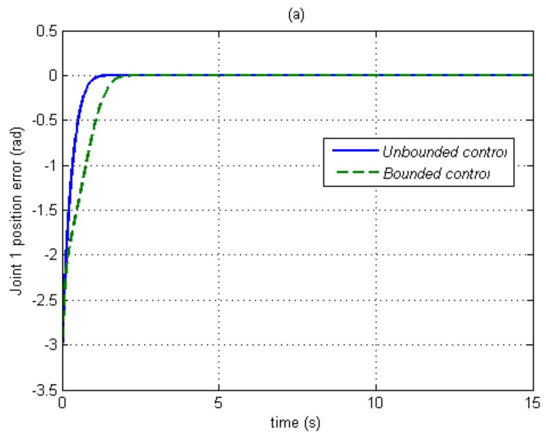


(e) Torque, τ_1 for time t = [0, 15] s.

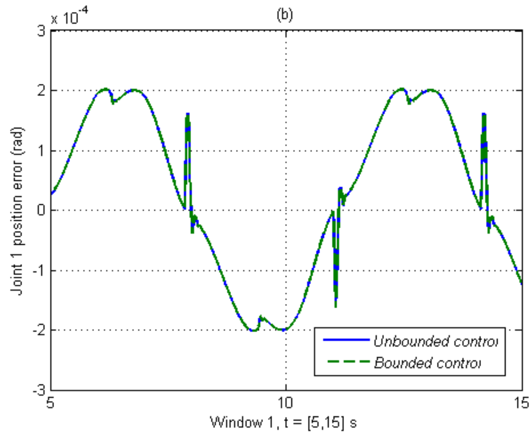


(f) Torque, τ_2 for time t = [0, 15] s.

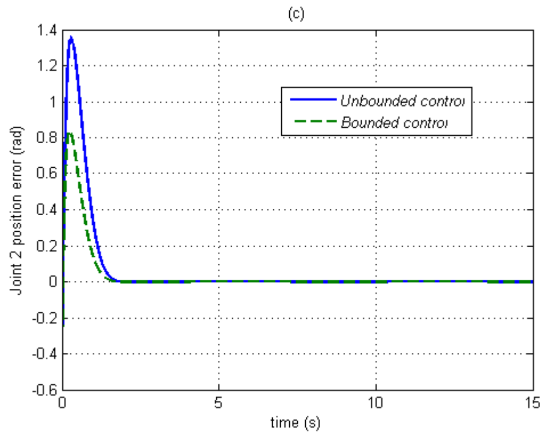
Figure 5.1 Simulation 1. Tracking errors, sigma, and control input of joint 1 and joint 2.



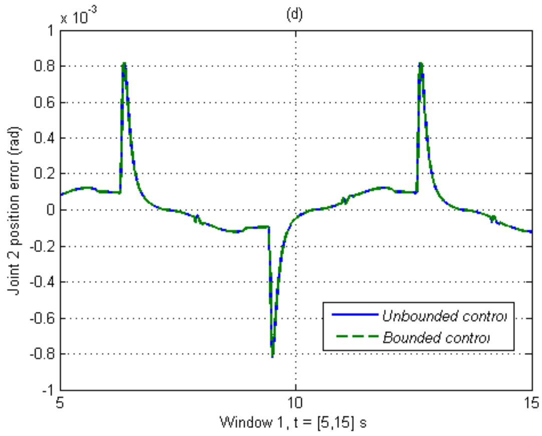
(a) Tracking error, e_{11} for time t = [0, 15] s.



(b) Tracking error, e_{11} for time t = [5, 15] s.



(c) Tracking error, e_{12} for time t = [0, 15] s.



(d) Tracking error, e_{12} for time t = [5, 15] s.

Figure 5.2 Simulation 2. Tracking errors using control (3 - 2), control (5 - 2).

1) Simulation 1:

As can be observed Figure 5.1(e), the large initial conditions caused the control (5 - 2) to saturate. From Theorem 5.1, the state σ will converge to a bounded region in finite time, due to the saturated nature of the integral (see (5 - 3)), which is clearly shown in Figure 5.1(c)-(d). According to the stability proof, once the state σ converge to a bounded region and stay in it for all future times, with control gain K satisfying (5 - 6), the control will be unsaturated in finite time and stay in it for all future times. In fact, from Figure 5.1(e)-(f) the control inputs for both joints of the robot manipulator remained unsaturated after the initial saturation.

2) Simulation 2:

Figure 5.2 shows that both controllers (3 - 2) and (5 - 2) have similar ultimate bound on the tracking errors. This is expected as when the controller (5 - 2) becomes unsaturated, it has an identical structure of the unbounded control (3 - 2). However, a more important issue is how the control (5 - 2) can remain unsaturated once it becomes unsaturated. The controller (5 - 2) solve this issue in two-fold. Firstly, through its saturated integral structure (5 - 3), it ensures the boundedness of the states of σ. This in turns, provide boundedness of the vector Δ(including the Coriolis term), together with the inherent viscous friction of the dynamics, the global boundedness of both σ and s are achieved. Secondly, through sufficiently high saturation levels of the control, i.e. such that for $a = 0$, $k_i > |\Delta_i|$ while for $a > 0$, need (5 - 6), by means of strict Lyapunov functions, the state s will become unsaturated and remain so for all future times.

Hence, due to the specific design (5 - 3) of control (5 -2), a saturated controller that inherits interesting properties (such as gain selection based on desired error dynamics injection and performance bound) of its unbounded counterpart is shown.

5.3 Summary

A saturated controller is developed for the trajectory tracking or robot manipulators under the influence of bounded disturbances. Global stability is assured by taking into account of the viscous friction and the proposed nonlinear integrator. Effectively, the controller allows the injection of desired error dynamics similar to the unbounded control of Chapter 3, hence allowing simple gain selection as well. When the saturation level is sufficiently high for the user-defined speed of desired trajectory, global finite-time and exponential regulation can be achieved.

Chapter 6: ROBOT MANIPULATOR CONTROL: OUTPUT FEEDBACK BOUNDED CONTROL APPROACH

Leveraging the results of Chapter 4 and 5, this chapter explore an output feedback bounded tracking control of robot manipulators with bounded disturbances. Lyapunov based stability analysis is provided to show global practical tracking results, while global finite time or exponential convergence can be obtained as well for regulation problem. Simulation results are provided to display the control performance.

6.1 Observer Dynamics

Inspired by the results of Chapter 4, the observer that exhibits properties of super-twisting algorithms with the addition of a linear damping term will be employed. Essentially, the same observer dynamics of section 4.1 will be considered in this chapter for the robot manipulator under bounded control approach.

6.1.1 System Description

Recall from section 4.1, the observer dynamics,

$$\begin{aligned}\dot{\tilde{e}}_1 &= -L_1 \operatorname{sig}(\tilde{e}_1)^p + \tilde{e}_2 \ , \\ \dot{\tilde{e}}_2 &= -L_2 \operatorname{sig}(\tilde{e}_1)^{2p-1} - L_3 \tilde{e}_2 + d\end{aligned}$$

where $\dot{\tilde{e}}_1, \dot{\tilde{e}}_2 \in \mathbb{R}^n$, are the vector state variables, L_1, L_2, L_3 $\in$ $\mathbb{R}^{n \times n}$ are positive definite diagonal matrices, $0.5 \leq p \leq 1$, and d comprise elements of bounded perturbations.

Consider element wise, $\forall\ i = 1,\ldots, n$,

$$\begin{aligned} \dot{\tilde{e}}_{1i} &= -l_{1i}\operatorname{sig}(\tilde{e}_{1i})^{p} + \tilde{e}_{2i} \quad , \\ \dot{\tilde{e}}_{2i} &= -l_{2i}\operatorname{sig}(\tilde{e}_{1i})^{2p-1} - l_{3i}\tilde{e}_{2i} + d_i \end{aligned} \qquad (6 - 1)$$

with $|d_i| := \sup\{|\varepsilon_i| : \varepsilon_i \in \boldsymbol{K}[d_i]\}$.

The stability analysis that supports the main results of this chapter is described in the following.

6.1.2 Stability Analysis

Consider the following Lyapunov function:

$$V_{\tilde{e}i}(\tilde{e}_{1i}, \tilde{e}_{2i}) = \left(\frac{1}{2}\tilde{e}_{2i}{}^{2} + \frac{l_{2i}}{2p}|\tilde{e}_{1i}|^{2p}\right)^{2} - r_{\tilde{e}}|\tilde{e}_{1i}||\tilde{e}_{2i}|^{\frac{4p-1}{p}}\operatorname{sign}(\tilde{e}_{1i}\tilde{e}_{2i})$$

(Note that this Lyapunov function is presented for the super-twisting based algorithms in section 2.4.2)

where $\underline{\pi}_{2i}\left(|\tilde{e}_{2i}|^{4} + |\tilde{e}_{1i}|^{4p}\right) \le V_{\tilde{e}i} \le \overline{\pi}_{2i}\left(|\tilde{e}_{2i}|^{4} + |\tilde{e}_{1i}|^{4p}\right)$

$$\underline{\pi}_{2i} := \min\left\{\frac{1}{8}, \frac{1}{2}\left(\frac{l_{2i}}{2p}\right)^{2}\right\}, \quad \overline{\pi}_{2i} := \max\left\{\left(\frac{l_{2i}}{2p}\right)^{2} + \frac{l_{2i}}{4p} + \frac{r_{\tilde{e}}}{4p}, \ \frac{1}{4} + \frac{l_{2i}}{4p} + \frac{r_{\tilde{e}}(4p-1)}{4p}\right\},$$

thus, $V_{\tilde{e}i}$ is positive definite and radially unbounded. Since (6 - 1) is a differential equation that has discontinuous right-hand side, i.e. when $p = 0.5$ and since no continuity assumption is made on d_i, its solutions are understood in the sense of Filippov (see definition 2.1).

Remark 6.1. The stability analysis of (6 - 1) will be considered with a different Lyapunov function as in section 4.1. This is due to the fact that in Chapter 4, the results pertain to semiglobal stability, while on a bounded control approach of Chapter 5, global stability is achieved. Since in this chapter, the observer (6 - 1) will be utilised under a bounded control approach in the subsequent sections, to show global stability, the observer dynamics analysis is repeated with a different Lyapunov function.

According to lemma 2.4, the time derivative of non-smooth Lyapunov function exists almost everywhere

$$\frac{d}{dt} V_{\tilde{e}i}\left(\tilde{e}_{1i}, \tilde{e}_{2i}\right) \overset{a.e.}{\in} \dot{\tilde{V}}_{\tilde{e}i} := \bigcap_{\xi \in \partial V_{\tilde{e}i}(\tilde{e}_{1i}, \tilde{e}_{2i})} \xi^{\mathrm{T}} \boldsymbol{K} \begin{bmatrix} \dot{\tilde{e}}_{1i} \\ \dot{\tilde{e}}_{2i} \end{bmatrix}$$

Since (6 - 1) is discontinuous when p = 0.5, for ease of presentation, the analysis is separated for the case of $0.5 < p \leq 1$ and $p = 0.5$.

a) Case 1: For $0.5 < p \leq 1$*:*

Note that for $0.5 < p \leq 1$, $V_{\tilde{e}i}$ is continuously differentiable:

$$\begin{aligned} \dot{\tilde{V}}_{\tilde{e}i} &= \bigcap_{\xi \in \partial V_{\tilde{e}i}(\tilde{e}_{1i}, \tilde{e}_{2i})} \xi^{\mathrm{T}} \boldsymbol{K} \begin{bmatrix} \dot{\tilde{e}}_{1i} \\ \dot{\tilde{e}}_{2i} \end{bmatrix} \\ &= \nabla V_{\tilde{e}i}{}^{T} K \begin{bmatrix} \dot{\tilde{e}}_{1i} \\ \dot{\tilde{e}}_{2i} \end{bmatrix} \end{aligned}$$

$$\subset -\frac{l_{1i}l_{2i}^{\ 2}}{p}\left|\tilde{e}_{1i}\right|^{5p-1} - l_{3i}\left|\tilde{e}_{2i}\right|^{4} - l_{1i}l_{2i}\left|\tilde{e}_{1i}\right|^{3p-1}\left|\tilde{e}_{2i}\right|^{2} - \frac{l_{2i}l_{3i}}{p}\left|\tilde{e}_{1i}\right|^{2p}\left|\tilde{e}_{2i}\right|^{2} - r_{\tilde{e}}\left|\tilde{e}_{2i}\right|^{\frac{5p-1}{p}}$$
$$+ r_{\tilde{e}}l_{1i}\left|\tilde{e}_{1i}\right|^{p}\left|\tilde{e}_{2i}\right|^{\frac{4p-1}{p}}\operatorname{sign}(\tilde{e}_{1i}\tilde{e}_{2i}) + r_{\tilde{e}}l_{2i}\left(\frac{4p-1}{p}\right)\left|\tilde{e}_{1i}\right|^{2p}\left|\tilde{e}_{2i}\right|^{\frac{3p-1}{p}}$$
$$+ r_{\tilde{e}}l_{3i}\left(\frac{4p-1}{p}\right)\left|\tilde{e}_{1i}\right|\left|\tilde{e}_{2i}\right|^{\frac{4p-1}{p}}\operatorname{sign}(\tilde{e}_{1i}\tilde{e}_{2i})$$
$$+ \boldsymbol{K}[d_i]\left(\frac{l_{2i}}{p}\left|\tilde{e}_{1i}\right|^{2p}\tilde{e}_{2i} + \tilde{e}_{2i}^{\ 3} - r_{\tilde{e}}\left(\frac{4p-1}{p}\right)\tilde{e}_{1i}\left|\tilde{e}_{2i}\right|^{\frac{3p-1}{p}}\right)$$

After rearrangement,

$$\dot{V}_{\tilde{e}i}(\tilde{e}_{1i},\tilde{e}_{2i}) \overset{a.e.}{\leq} -\frac{l_{1i}l_{2i}^{\ 2}}{3p}\left|\tilde{e}_{1i}\right|^{5p-1} - \frac{r_{\tilde{e}}}{2}\left|\tilde{e}_{2i}\right|^{\frac{5p-1}{p}} + \dot{V}_1 + \dot{V}_2$$

where

$$\dot{V}_1 = -\frac{l_{1i}l_{2i}^{\ 2}}{3p}\left|\tilde{e}_{1i}\right|^{5p-1} - \frac{l_{3i}}{2}\left|\tilde{e}_{2i}\right|^{4} - \frac{l_{2i}l_{3i}}{2p}\left|\tilde{e}_{1i}\right|^{2p}\left|\tilde{e}_{2i}\right|^{2} - l_{1i}l_{2i}\left|\tilde{e}_{1i}\right|^{3p-1}\left|\tilde{e}_{2i}\right|^{2}$$
$$-\frac{r_{\tilde{e}}}{2}\left|\tilde{e}_{2i}\right|^{\frac{5p-1}{p}} + r_{\tilde{e}}l_{1i}\left|\tilde{e}_{1i}\right|^{p}\left|\tilde{e}_{2i}\right|^{\frac{4p-1}{p}}$$
$$+ r_{\tilde{e}}l_{2i}\left(\frac{4p-1}{p}\right)\left|\tilde{e}_{1i}\right|^{2p}\left|\tilde{e}_{2i}\right|^{\frac{3p-1}{p}} + r_{\tilde{e}}l_{3i}\left(\frac{4p-1}{p}\right)\left|\tilde{e}_{1i}\right|\left|\tilde{e}_{2i}\right|^{\frac{4p-1}{p}}$$

$$\dot{V}_2 = -\frac{l_{1i}l_{2i}^{\ 2}}{3p}\left|\tilde{e}_{1i}\right|^{5p-1} - \frac{l_{3i}}{2}\left|\tilde{e}_{2i}\right|^{4} - \frac{l_{2i}l_{3i}}{2p}\left|\tilde{e}_{1i}\right|^{2p}\left|\tilde{e}_{2i}\right|^{2}$$
$$+\left|d_i\right|\left(\frac{l_{2i}}{p}\left|\tilde{e}_{1i}\right|^{2p}\left|\tilde{e}_{2i}\right| + \left|\tilde{e}_{2i}\right|^{3} + r_{\tilde{e}}\left(\frac{4p-1}{p}\right)\left|\tilde{e}_{1i}\right|\left|\tilde{e}_{2i}\right|^{\frac{3p-1}{p}}\right)$$

For $\dot{V}_1$:

Applying lemma 2.1,

$$-\left|\tilde{e}_{1i}\right|^{5p-1}-\left|\tilde{e}_{2i}\right|^{\frac{5p-1}{p}} \le -\left(\frac{5p-1}{p}\right)^{\frac{p}{5p-1}}\left|\tilde{e}_{1i}\right|^{p}\left(\frac{5p-1}{4p-1}\right)^{\frac{4p-1}{5p-1}}\left|\tilde{e}_{2i}\right|^{\frac{4p-1}{p}},$$

$$-\left|\tilde{e}_{1i}\right|^{5p-1}-\left|\tilde{e}_{2i}\right|^{\frac{5p-1}{p}} \le -\left(\frac{5p-1}{2p}\right)^{\frac{2p}{5p-1}}\left|\tilde{e}_{1i}\right|^{2p}\left(\frac{5p-1}{3p-1}\right)^{\frac{3p-1}{5p-1}}\left|\tilde{e}_{2i}\right|^{\frac{3p-1}{p}},$$

$$-\left|\tilde{e}_{2i}\right|^{4}-\left|\tilde{e}_{1i}\right|^{2p}\left|\tilde{e}_{2i}\right|^{2} \le -(2p)^{\frac{1}{2p}}\left|\tilde{e}_{1i}\right|\left|\tilde{e}_{2i}\right|^{\frac{1}{p}}\left(\frac{2p}{2p-1}\right)^{\frac{2p-1}{2p}}\left|\tilde{e}_{2i}\right|^{\frac{4p-2}{p}},$$

Thus, if the following inequalities

$$\min\left\{\lambda_1 \frac{l_{2i}{}^{2}}{l_{1i}{}^{\frac{4p-1}{p}}},\ \lambda_2 l_{1i} l_{2i}{}^{\frac{1-p}{2p}},\ \lambda_3 l_{2i}{}^{\frac{1}{2p}}\right\} > r_{\tilde{e}} > 0$$

where

$$\lambda_1 = \left(\frac{5p-1}{6p^2}\right)\left(\frac{5p-1}{16p-4}\right)^{\frac{4p-1}{p}},\ \lambda_2 = \left(\frac{5p-1}{12p^2}\right)\left(\frac{5p-1}{12p-4}\right)^{\frac{3p-1}{2p}}\left(\frac{p}{4p-1}\right)^{\frac{5p-1}{2p}},$$
$$\lambda_3 = \left(\frac{p}{2p-1}\right)^{\frac{2p-1}{2p}}\left(\frac{p}{4p-1}\right),$$

hold then the function $\dot{V}_1$ is negative definite. Note that such an $r_{\tilde{e}} > 0$ always exists for any $l_{1i} > 0$, $l_{2i} > 0$.

For $\dot{V}_2$:

Applying lemma 2.1,

$$\left|\tilde{e}_{1i}\right|\left|\tilde{e}_{2i}\right|^{\frac{3p-1}{p}} = \left|\tilde{e}_{1i}\right|\left|\tilde{e}_{2i}\right|^{\frac{1}{2p}}\left|\tilde{e}_{2i}\right|^{\frac{6p-3}{2p}} \le \left(\frac{1}{2p}\right)\left|\tilde{e}_{1i}\right|^{2p}\left|\tilde{e}_{2i}\right| + \left(\frac{2p-1}{2p}\right)\left|\tilde{e}_{2i}\right|^{3}$$

Then,

$$\dot{V}_2 \le -\frac{l_{1i}l_{2i}^{\;2}}{3p}\left|\widetilde{e}_{1i}\right|^{5p-1} - \frac{l_{3i}}{2}\left|\widetilde{e}_{2i}\right|^{4} - \frac{l_{2i}l_{3i}}{2p}\left|\widetilde{e}_{1i}\right|^{2p}\left|\widetilde{e}_{2i}\right|^{2}$$
$$+\left|d_i\right|\left(\left(\frac{l_{2i}}{p}+r_{\widetilde{e}}\left(\frac{4p-1}{2p^2}\right)\right)\left|\widetilde{e}_{1i}\right|^{2p}\left|\widetilde{e}_{2i}\right|+\left(1+r_{\widetilde{e}}\left(\frac{4p-1}{p}\right)\left(\frac{2p-1}{2p}\right)\right)\left|\widetilde{e}_{2i}\right|^{3}\right)$$

The state space is divided into the following three regions for ease of analysis:

$$\phi_{1i}\left(\left|d_i\right|,l_{3i}\right)=\left\{\left(\widetilde{e}_{1i},\widetilde{e}_{2i}\right)\in\mathbb{R}^2:\widetilde{e}_{1i}\in\mathbb{R},\left|\widetilde{e}_{2i}\right|\ge\beta_2\left(\left|d_i\right|,l_{3i}\right)\right\},$$

$$\phi_{2i}\left(\left|d_i\right|,l_{3i}\right)=\left\{\left(\widetilde{e}_{1i},\widetilde{e}_{2i}\right)\in\mathbb{R}^2:\left|\widetilde{e}_{1i}\right|\ge\beta_1\left(\left|d_i\right|,l_{3i}\right),\left|\widetilde{e}_{2i}\right|\le\beta_2\left(\left|d_i\right|,l_{3i}\right)\right\},$$

$$\phi_{3i}\left(\left|d_i\right|,l_{3i}\right)=\left\{\left(\widetilde{e}_{1i},\widetilde{e}_{2i}\right)\in\mathbb{R}^2:\left|\widetilde{e}_{1i}\right|\le\beta_1\left(\left|d_i\right|,l_{3i}\right),\left|\widetilde{e}_{2i}\right|\le\beta_2\left(\left|d_i\right|,l_{3i}\right)\right\},$$

where the functions β_1 and β_2 are of class-$\mathcal{KL}$ defined by:

$$\beta_1\left(\left|d_i\right|,l_{3i}\right):=\max\left\{\begin{array}{c}\left(\dfrac{\beta_2\left(\left|d_i\right|,l_{3i}\right)\left|d_i\right|}{l_{1i}l_{2i}^{\;2}}\left(6l_{2i}+r_{\widetilde{e}}\left(\dfrac{12p-3}{p}\right)\right)\right)^{\frac{1}{3p-1}},\\ \left(\dfrac{\left|d_i\right|\left[\beta_2\left(\left|d_i\right|,l_{3i}\right)\right]^3}{l_{1i}l_{2i}^{\;2}}\left(6p+r_{\widetilde{e}}\left(\dfrac{4p-1}{p}\right)\left(6p-3\right)\right)\right)^{\frac{1}{5p-1}}\end{array}\right\},$$

$$\beta_2\left(\left|d_i\right|,l_{3i}\right):=\frac{\left|d_i\right|}{l_{3i}}\times\max\left\{\left(2+r_{\widetilde{e}}\left(\frac{4p-1}{p}\right)\left(\frac{2p-1}{p}\right)\right),\ \left(2+\frac{r_{\widetilde{e}}}{l_{2i}}\left(\frac{4p-1}{p}\right)\right)\right\}$$

Consider the region: $\phi_{1i}\left(\left|d_i\right|,l_{3i}\right)=\left\{\left(\widetilde{e}_{1i},\widetilde{e}_{2i}\right)\in\mathbb{R}^2:\widetilde{e}_{1i}\in\mathbb{R},\left|\widetilde{e}_{2i}\right|\ge\beta_2\left(\left|d_i\right|,l_{3i}\right)\right\}$

Thus,

$$\dot{V}_2 = -\frac{l_{1i}{l_{2i}}^2}{3p}\left|\tilde{e}_{1i}\right|^{5p-1} - \left|\tilde{e}_{2i}\right|^3\left(\frac{l_{3i}}{2}\left|\tilde{e}_{2i}\right| - \left|d_i\right|\left(1 + r_{\tilde{e}}\left(\frac{4p-1}{p}\right)\left(\frac{2p-1}{2p}\right)\right)\right)$$

$$-\left|\tilde{e}_{1i}\right|^{2p}\left|\tilde{e}_{2i}\right|\left(\frac{l_{2i}l_{3i}}{2p}\left|\tilde{e}_{2i}\right| - \left|d_i\right|\left(\frac{l_{2i}}{p} + r_{\tilde{e}}\left(\frac{4p-1}{2p^2}\right)\right)\right)$$

$$\le 0$$

for the states in this region.

Next, consider the region: $\phi_{2i}\left(\left|d_i\right|, l_{3i}\right) = \left\{\left(\tilde{e}_{1i}, \tilde{e}_{2i}\right) \in \mathbb{R}^2 : \left|\tilde{e}_{1i}\right| \ge \beta_1\left(\left|d_i\right|, l_{3i}\right), \left|\tilde{e}_{2i}\right| \le \beta_2\left(\left|d_i\right|, l_{3i}\right)\right\}$

Thus,

$$\dot{V}_2 \le -\frac{l_{1i}{l_{2i}}^2}{3p}\left|\tilde{e}_{1i}\right|^{5p-1} - \frac{l_{3i}}{2}\left|\tilde{e}_{2i}\right|^4 - \frac{l_{2i}l_{3i}}{2p}\left|\tilde{e}_{1i}\right|^{2p}\left|\tilde{e}_{2i}\right|^2$$

$$+\left|d_i\right|\left(\left(\frac{l_{2i}}{p} + r_{\tilde{e}}\left(\frac{4p-1}{2p^2}\right)\right)\left|\tilde{e}_{1i}\right|^{2p}\beta_2\left(\left|d_i\right|, l_{3i}\right) + \left(1 + r_{\tilde{e}}\left(\frac{4p-1}{p}\right)\left(\frac{2p-1}{2p}\right)\right)\left[\beta_2\left(\left|d_i\right|, l_{3i}\right)\right]^3\right)$$

$$= -\left|\tilde{e}_{1i}\right|^{2p}\left(\frac{l_{1i}{l_{2i}}^2}{6p}\left|\tilde{e}_{1i}\right|^{3p-1} - \left|d_i\right|\left(\frac{l_{2i}}{p} + r_{\tilde{e}}\left(\frac{4p-1}{2p^2}\right)\right)\beta_2\left(\left|d_i\right|, l_{3i}\right)\right)$$

$$-\left(\frac{l_{1i}{l_{2i}}^2}{6p}\left|\tilde{e}_{1i}\right|^{5p-1} - \left|d_i\right|\left(1 + r_{\tilde{e}}\left(\frac{4p-1}{p}\right)\left(\frac{2p-1}{2p}\right)\right)\left[\beta_2\left(\left|d_i\right|, l_{3i}\right)\right]^3\right)$$

where $\dot{V}_2 \le 0$ for

$$\left|\tilde{e}_{1i}\right| \ge \max\left\{\begin{array}{l}\left(\frac{\beta_2\left(\left|d_i\right|, l_{3i}\right)\left|d_i\right|}{l_{1i}{l_{2i}}^2}\left(6l_{2i} + r_{\tilde{e}}\left(\frac{12p-3}{p}\right)\right)\right)^{\frac{1}{3p-1}}, \\ \left(\frac{\left|d_i\right|\left[\beta_2\left(\left|d_i\right|, l_{3i}\right)\right]^3}{l_{1i}{l_{2i}}^2}\left(6p + r_{\tilde{e}}\left(\frac{4p-1}{p}\right)\left(6p-3\right)\right)\right)^{\frac{1}{5p-1}}\end{array}\right\}$$

which is sufficiently satisfied for the states in this region.

Next consider the compact set:

$$\phi_{3i}\left(|d_i|, l_{3i}\right) = \left\{\left(\tilde{e}_{1i}, \tilde{e}_{2i}\right) \in \mathbb{R}^2 : |\tilde{e}_{1i}| \le \beta_1\left(|d_i|, l_{3i}\right), |\tilde{e}_{2i}| \le \beta_2\left(|d_i|, l_{3i}\right)\right\}$$

Note that, $\dot{V}_2 \le 0$ for $\left(\tilde{e}_{1i}, \tilde{e}_{2i}\right) \notin \phi_{3i}\left(|d_i|, l_{3i}\right)$.

Thus,

$$\dot{V}_{\tilde{e}i}\left(\tilde{e}_{1i}, \tilde{e}_{2i}\right) \overset{a.e.}{\le} -\frac{l_{1i} l_{2i}^{\;2}}{3p} |\tilde{e}_{1i}|^{5p-1} - \frac{r_{\tilde{e}}}{2} |\tilde{e}_{2i}|^{\frac{5p-1}{p}} < 0, \text{ for } \left(\tilde{e}_{1i}, \tilde{e}_{2i}\right) \notin \phi_{3i}\left(|d_i|, l_{3i}\right)$$

Next, define a Lyapunov level set:

$$\Sigma_{\tilde{e}i}\left(|d_i|, l_{3i}\right) = \left\{\left(\tilde{e}_{1i}, \tilde{e}_{2i}\right) \in \mathbb{R}^2 : V_{\tilde{e}i} \le \rho_{1i}\left(|d_i|, l_{3i}\right)\right\}$$

where the class $\mathcal{KL}$ function ρ_{1i} ($|d_\mathrm{i}|$, l_{3i}) is defined as follows:

$$\rho_{1i}\left(|d_i|, l_{3i}\right) = \max_{\left(\tilde{e}_{1i}, \tilde{e}_{2i}\right) \in \mathrm{bd}\ \phi_3\left(|d_i|, l_{3i}\right)} V_{\tilde{e}i}\left(\tilde{e}_{1i}, \tilde{e}_{2i}\right)$$

which exists since the boundary of the set is compact and $V_{\tilde{e}i}$ is continuous. Then we observe that ϕ_{3i} ($|d_\mathrm{i}|$, l_{3i}) $\subset \Sigma_{\tilde{e}i}$ ($|d_\mathrm{i}|$, l_{3i}). As a result, we have

$$\frac{d}{dt} V_{\tilde{e}i}\left(\tilde{e}_{1i}, \tilde{e}_{2i}\right) \overset{a.e.}{\in} \dot{\tilde{V}}_{\tilde{e}i}\left(\tilde{e}_{1i}, \tilde{e}_{2i}\right) \le -\frac{l_{1i} l_{2i}^{\;2}}{3p} |\tilde{e}_{1i}|^{5p-1} - \frac{r_{\tilde{e}}}{2} |\tilde{e}_{2i}|^{\frac{5p-1}{p}}, \text{ for } V_{\tilde{e}i} \ge \rho_{1i}\left(|d_i|, l_{3i}\right)$$

which implies that each of the trajectories for the *i*-th planar system will enter their respective compact level set $\Sigma_{\tilde{e}i}$ ($|d_\mathrm{i}|$,l_{3i}) in finite time and stay in it once entered.

b) Case 2: For $p = 0.5$

Note that or $p = 0.5$, $V_{\tilde{e}i}$ is not differentiable on $\tilde{e}_{1i} = 0$:

$$\begin{bmatrix} \dot{\tilde{e}}_{1i} \\ \dot{\tilde{e}}_{2i} \end{bmatrix} \in \begin{bmatrix} -l_{1i}\operatorname{sig}(\tilde{e}_{1i})^{\frac{1}{2}} + \tilde{e}_{2i} \\ -l_{2i}\boldsymbol{K}[\operatorname{sign}(\tilde{e}_{1i})] - l_{3i}\tilde{e}_{2i} + \boldsymbol{K}[d_i] \end{bmatrix}$$

$$\partial V_{\tilde{e}i}(\tilde{e}_{1i}, \tilde{e}_{2i}) = \begin{bmatrix} (\tilde{e}_{2i}^{\ 2} + 2l_{2i}|\tilde{e}_{1i}|)(l_{2i}\operatorname{SGN}(\tilde{e}_{1i}) - r_{\tilde{e}}|\tilde{e}_{2i}|^2 \operatorname{sign}(\tilde{e}_{2i})) \\ (\tilde{e}_{2i}^{\ 2} + 2l_{2i}|\tilde{e}_{1i}|)(\tilde{e}_{2i}) - 2r_{\tilde{e}}\tilde{e}_{1i}|\tilde{e}_{2i}| \end{bmatrix}$$

for $\tilde{e}_{1i} \neq 0$, $\forall\ \tilde{e}_{2i} \in \mathbb{R}$:

$$\begin{aligned} \dot{\tilde{V}}_{\tilde{e}i} = &-2l_{1i}l_{2i}^{\ 2}|\tilde{e}_{1i}|^{\frac{3}{2}} - l_{3i}|\tilde{e}_{2i}|^4 - l_{1i}l_{2i}|\tilde{e}_{1i}|^{\frac{1}{2}}|\tilde{e}_{2i}|^2 - 2l_{2i}l_{3i}|\tilde{e}_{1i}||\tilde{e}_{2i}|^2 + r_{\tilde{e}}l_{1i}|\tilde{e}_{1i}|^{\frac{1}{2}}|\tilde{e}_{2i}|^2 \operatorname{sign}(\tilde{e}_{1i}\tilde{e}_{2i}) \\ &- r_{\tilde{e}}|\tilde{e}_{2i}|^3 + 2r_{\tilde{e}}l_{2i}|\tilde{e}_{1i}||\tilde{e}_{2i}| + 2r_{\tilde{e}}l_{3i}\tilde{e}_{1i}|\tilde{e}_{2i}|^2 \operatorname{sign}(\tilde{e}_{2i}) + \boldsymbol{K}[d_i](2l_{2i}|\tilde{e}_{1i}|\tilde{e}_{2i} + \tilde{e}_{2i}^{\ 3} - 2r_{\tilde{e}}\tilde{e}_{1i}|\tilde{e}_{2i}|) \end{aligned}$$

for $\tilde{e}_{1i} = 0$, $\forall\ \tilde{e}_{2i} \in \mathbb{R}$:

Let

$\left[(l_{2i}\tilde{e}_{2i}^{\ 2})\xi_2 - r_{\tilde{e}}|\tilde{e}_{2i}|^2 \operatorname{sign}(\tilde{e}_{2i}),\ \tilde{e}_{2i}^{\ 3}\right]^T$ with $\xi_2 \in [-1, 1]$ be an arbitrary element of $\partial V_{\tilde{e}i}$, then

$$\begin{aligned} \xi^{\mathrm{T}}\boldsymbol{K}\begin{bmatrix} \dot{\tilde{e}}_{1i} \\ \dot{\tilde{e}}_{2i} \end{bmatrix} &= \left[(l_{2i}\tilde{e}_{2i}^{\ 2})\xi_2 - r_{\tilde{e}}|\tilde{e}_{2i}|^2 \operatorname{sign}(\tilde{e}_{2i}),\ \tilde{e}_{2i}^{\ 3}\right]\begin{bmatrix} \tilde{e}_{2i} \\ -l_{2i}[-1,1] - l_{3i}\tilde{e}_{2i} + \boldsymbol{K}[d_i] \end{bmatrix} \\ &= ([\xi_2 - 1, \xi_2 + 1])l_{2i}\tilde{e}_{2i}^{\ 3} - r_{\tilde{e}}|\tilde{e}_{2i}|^3 - l_{3i}|\tilde{e}_{2i}|^4 + \boldsymbol{K}[d_i]\tilde{e}_{2i}^{\ 3} \end{aligned}$$

implies

$$\dot{\tilde{V}}_{\tilde{e}i} = \bigcap_{\xi_2 \in [-1,1]} ([\xi_2 - 1, \xi_2 + 1])l_{2i}\tilde{e}_{2i}^{\ 3} - r_{\tilde{e}}|\tilde{e}_{2i}|^3 - l_{3i}|\tilde{e}_{2i}|^4 + \boldsymbol{K}[d_i]\tilde{e}_{2i}^{\ 3} = -r_{\tilde{e}}|\tilde{e}_{2i}|^3 - l_{3i}|\tilde{e}_{2i}|^4 + \boldsymbol{K}[d_i]\tilde{e}_{2i}^{\ 3}$$

Thus, for all $(\tilde{e}_{1i}, \tilde{e}_{2i}) \in \mathbb{R}^n$, after rearrangement:

$$\dot{V}_{\tilde{e}i}(\tilde{e}_{1i}, \tilde{e}_{2i}) \overset{a.e.}{\leq} -\frac{2l_{1i}l_{2i}^{\ 2}}{3}|\tilde{e}_{1i}|^{1.5} - \frac{r_{\tilde{e}}}{2}|\tilde{e}_{2i}|^3 + \dot{V}_3 + \dot{V}_4$$

where

$$\dot{V}_3 = -\frac{2l_{1i}l_{2i}^{\;2}}{3}\left|\tilde{e}_{1i}\right|^{1.5} - \frac{r_{\tilde{e}}}{2}\left|\tilde{e}_{2i}\right|^{3} - l_{3i}\left(l_{2i} - 2r_{\tilde{e}}\right)\left|\tilde{e}_{1i}\right|\left|\tilde{e}_{2i}\right|^{2} - l_{1i}\left(l_{2i} - r_{\tilde{e}}\right)\left|\tilde{e}_{1i}\right|^{0.5}\left|\tilde{e}_{2i}\right|^{2} + 2r_{\tilde{e}}l_{2i}\left|\tilde{e}_{1i}\right|\left|\tilde{e}_{2i}\right|,$$

$$\dot{V}_4 = -\frac{2l_{1i}l_{2i}^{\;2}}{3}\left|\tilde{e}_{1i}\right|^{1.5} - l_{3i}\left|\tilde{e}_{2i}\right|^{4} - l_{2i}l_{3i}\left|\tilde{e}_{1i}\right|\left|\tilde{e}_{2i}\right|^{2} + \left|d_i\right|\left(2\left(l_{2i} + r_{\tilde{e}}\right)\left|\tilde{e}_{1i}\right|\left|\tilde{e}_{2i}\right| + \left|\tilde{e}_{2i}\right|^{3}\right)$$

For $\dot{V}_3$:

Applying lemma 2.1,

$$-\left|\tilde{e}_{1i}\right|^{1.5} - \left|\tilde{e}_{2i}\right|^{3} \le -\left(\frac{3}{2}\right)^{\frac{2}{3}}\left|\tilde{e}_{1i}\right|(3)^{\frac{1}{3}}\left|\tilde{e}_{2i}\right|$$

Thus, if the following inequalities

$$\min\left\{l_{2i},\ \frac{l_{2i}}{2},\ \left(\frac{3^{\frac{1}{2}}}{2^{2}}\right)l_{1i}l_{2i}^{\frac{1}{2}}\right\} > r_{\tilde{e}}$$

hold then the function $\dot{V}_3$ is negative definite. Note that such an $r_{\tilde{e}} > 0$ always exists for any $l_{1i} > 0$, $l_{2i} > 0$.

For $\dot{V}_4$:

The state space is divided into the following three regions for ease of analysis:

$$\phi_{1i}\left(\left|d_i\right|, l_{3i}\right) = \left\{\left(\tilde{e}_{1i}, \tilde{e}_{2i}\right) \in \mathbb{R}^2 : \tilde{e}_{1i} \in \mathbb{R}, \left|\tilde{e}_{2i}\right| \ge \beta_4\left(\left|d_i\right|, l_{3i}\right)\right\},$$

$$\phi_{2i}\left(\left|d_i\right|, l_{3i}\right) = \left\{\left(\tilde{e}_{1i}, \tilde{e}_{2i}\right) \in \mathbb{R}^2 : \left|\tilde{e}_{1i}\right| \ge \beta_3\left(\left|d_i\right|, l_{3i}\right), \left|\tilde{e}_{2i}\right| \le \beta_4\left(\left|d_i\right|, l_{3i}\right)\right\},$$

$$\phi_{3i}\left(\left|d_i\right|, l_{3i}\right) = \left\{\left(\tilde{e}_{1i}, \tilde{e}_{2i}\right) \in \mathbb{R}^2 : \left|\tilde{e}_{1i}\right| \le \beta_3\left(\left|d_i\right|, l_{3i}\right), \left|\tilde{e}_{2i}\right| \le \beta_4\left(\left|d_i\right|, l_{3i}\right)\right\},$$

where the functions $\beta_1(.)$ and $\beta_2(.)$ are class κ functions defined by:

$$\beta_3(|d_i|, l_{3i}) := \max\left\{ \left(\frac{\beta_4(|d_i|, l_{3i})|d_i|}{l_{1i} l_{2i}^{\;2}} 3(l_{2i} + r_{\tilde{e}}) \right)^2, \left(\frac{3|d_i|[\beta_4(|d_i|, l_{3i})]^3}{l_{1i} l_{2i}^{\;2}} \right)^{\frac{2}{3}} \right\},$$

$$\beta_4(|d_i|, l_{3i}) := \frac{|d_i|}{l_{3i}} \times \max\left\{ 1,\ 2 + \frac{2r_{\tilde{e}}}{l_{2i}} \right\}$$

Consider the region: $\phi_{1i}(|d_i|, l_{3i}) = \{(\tilde{e}_{1i}, \tilde{e}_{2i}) \in \mathbb{R}^2 : \tilde{e}_{1i} \in \mathbb{R}, |\tilde{e}_{2i}| \geq \beta_4(|d_i|, l_{3i})\}$

Thus,

$$\begin{aligned} \dot{V}_4 &= -\frac{2 l_{1i} l_{2i}^{\;2}}{3} |\tilde{e}_{1i}|^{1.5} - |\tilde{e}_{2i}|^3 (l_{3i}|\tilde{e}_{2i}| - |d_i|) - |\tilde{e}_{1i}||\tilde{e}_{2i}|(l_{2i} l_{3i} |\tilde{e}_{2i}| - 2|d_i|(l_{2i} + r_{\tilde{e}})) \\ &\leq 0 \end{aligned}$$

for the states in this region.

Next, consider the region: $\phi_{2i}(|d_i|, l_{3i}) = \{(\tilde{e}_{1i}, \tilde{e}_{2i}) \in \mathbb{R}^2 : |\tilde{e}_{1i}| \geq \beta_3(|d_i|, l_{3i}), |\tilde{e}_{2i}| \leq \beta_4(|d_i|, l_{3i})\}$

Thus,

$$\begin{aligned} \dot{V}_4 &\leq -\frac{2 l_{1i} l_{2i}^{\;2}}{3} |\tilde{e}_{1i}|^{1.5} - l_{3i}|\tilde{e}_{2i}|^4 - l_{2i} l_{3i} |\tilde{e}_{1i}||\tilde{e}_{2i}|^2 + |d_i|\left(2(l_{2i} + r_{\tilde{e}})|\tilde{e}_{1i}| \beta_4(|d_i|, l_{3i}) + [\beta_4(|d_i|, l_{3i})]^3 \right) \\ &\leq -|\tilde{e}_{1i}|\left(\frac{l_{1i} l_{2i}^{\;2}}{3} |\tilde{e}_{1i}|^{0.5} - |d_i|(l_{2i} + r_{\tilde{e}})\beta_4(|d_i|, l_{3i}) \right) - \left(\frac{l_{1i} l_{2i}^{\;2}}{3} |\tilde{e}_{1i}|^{1.5} - |d_i|[\beta_4(|d_i|, l_{3i})]^3 \right) \end{aligned}$$

where $\dot{V}_4 \leq 0$ for

$$|\tilde{e}_{1i}| \geq \max\left\{ \left(\frac{\beta_4(|d_i|, l_{3i})|d_i|}{l_{1i} l_{2i}^{\;2}} 3(l_{2i} + r_{\tilde{e}}) \right)^2, \left(\frac{3|d_i|[\beta_4(|d_i|, l_{3i})]^3}{l_{1i} l_{2i}^{\;2}} \right)^{\frac{2}{3}} \right\}$$

which is sufficiently satisfied for the states in this region.

Next consider the compact set:

$$\phi_{3i}\left(\left|d_i\right|, l_{3i}\right)=\left\{\left(\tilde{e}_{1i}, \tilde{e}_{2i}\right)\in\mathbb{R}^2 : \left|\tilde{e}_{1i}\right|\le\beta_3\left(\left|d_i\right|, l_{3i}\right), \left|\tilde{e}_{2i}\right|\le\beta_4\left(\left|d_i\right|, l_{3i}\right)\right\}$$

Note that, $\dot{V}_4 \le 0 \;\; for \;\; \left(\tilde{e}_{1i}, \tilde{e}_{2i}\right)\notin\phi_3\left(\left|d_i\right|, l_{3i}\right)$.

Thus,

$$\dot{V}_{\tilde{e}i}\left(\tilde{e}_{1i}, \tilde{e}_{2i}\right)\overset{a.e.}{\le} -\frac{2l_{1i}l_{2i}^{\;2}}{3}\left|\tilde{e}_{1i}\right|^{1.5}-\frac{r_{\tilde{e}}}{2}\left|\tilde{e}_{2i}\right|^{3}<0 \; for \;\; \left(\tilde{e}_{1i}, \tilde{e}_{2i}\right)\notin\phi_{3i}\left(\left|d_i\right|, l_{3i}\right)$$

Next, define a Lyapunov level set:

$$\Sigma_{\tilde{e}i}\left(\left|d_i\right|, l_{3i}\right)=\left\{\left(\tilde{e}_{1i}, \tilde{e}_{2i}\right)\in\mathbb{R}^2 : V_{\tilde{e}i}\le\rho_{2i}\left(\left|d_i\right|, l_{3i}\right)\right\}$$

where the positive definite function ρ_{2i} ($|d_i|$, l_{3i}) is defined as follows:

$$\rho_{2i}\left(\left|d_i\right|, l_{3i}\right)=\max_{\left(\tilde{e}_{1i}, \tilde{e}_{2i}\right)\in \mathrm{bd}\;\phi_3\left(\left|d_i\right|, l_{3i}\right)} V_{\tilde{e}i}\left(\tilde{e}_{1i}, \tilde{e}_{2i}\right)$$

which exists since the boundary of the set is compact and $V_{\tilde{e}i}$ is continuous. Then we observe that ϕ_3 ($|d_i|$, l_{3i}) $\subset \Sigma_{\tilde{e}i}$ ($|d_i|$, l_{3i}) As a result, we have

$$\frac{d}{dt}V_{\tilde{e}i}\left(\tilde{e}_{1i}, \tilde{e}_{2i}\right)\overset{a.e.}{\in}\dot{\tilde{V}}_{\tilde{e}i}\left(\tilde{e}_{1i}, \tilde{e}_{2i}\right)\le -\frac{2l_{1i}l_{2i}^{\;2}}{3}\left|\tilde{e}_{1i}\right|^{1.5}-\frac{r_{\tilde{e}}}{2}\left|\tilde{e}_{2i}\right|^{3}, \text{ for } V_{\tilde{e}i}\ge\rho_{2i}\left(\left|d_i\right|, l_{3i}\right)$$

which implies that each of the trajectories for the i-th planar system will enter their respective compact level set $\Sigma_{\tilde{e}i}$ ($|d_i|$, l_{3i}) in finite time and stay in it once entered.

6.2 Manipulator Dynamics

The dynamic model of a rigid n-link serial nonredundant robot manipulator, with all actuated revolute joints described in joint coordinates, is given as follows:

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + F\dot{q} + G(q) + D(q,\dot{q},t) = \tau \tag{6 - 2}$$

where q, $\dot{q}$, $\ddot{q} \in \mathbb{R}^n$ are, respectively, the position (generalized coordinates), velocity and acceleration vectors, $M(q) \in \mathbb{R}^{n\times n}$ is the inertia matrix, and $C(q, \dot{q})\dot{q}$, $F\dot{q}$, $G(q)$, $\tau \in \mathbb{R}^n$ are, respectively, the vectors of Coriolis and centrifugal, viscous friction, gravity, and torque input, with F being a constant, positive definite, diagonal (viscous friction coefficient) matrix. $D(q, \dot{q}, t)$ is a an additional bounded uncertainty or perturbation term. (Note that no continuity assumption is made so that discontinuous models of friction may be used in D). The robot manipulator satisfies the same properties as that in Chapter 3 and will not be repeated here. In this chapter, only joint positions measurement is available. No velocity information from the system is assumed in this chapter. As such, the following development will follow an output feedback approach. In addition, let us suppose that the absolute value of each joint input τ_i is constrained to be smaller than a given saturation bound $T_{i,\max} > 0$, i.e., $|\tau_i| \le T_{i,\max}$, $\forall i = 1, ..., n$.

The control objective here is to design a robust output feedback saturated control that ensures the robot configuration vector q tracks a desired trajectory vector, $q_d\,(t)$ with an ultimately bounded error that can be made as small as required globally even under the presence of bounded disturbances.

The desired trajectory vector, $q_d\,(t)$ is assumed to be twice continuously differentiable vector-function such that $\|q_d(t)\|$, $\|\dot{q}_d(t)\|$, and $\|\ddot{q}_d(t)\|$ are bounded by *a priori* known constants. While the type of disturbances considered here has to be upper bounded by a constant, due to the limited actuation (Note that no continuity assumption is made so that discontinuous models of friction may be used in D), i.e.

$$\|D\| \le \mathrm{sat}_{p_5}\left[p_0 + p_1\|e_1\| + p_2\|\dot{q}\| + p_3\|e_1\|^2 + p_4\|\dot{q}\|^2 \right]$$

where $\|D\| := \sup\{\|\varsigma\| : \varsigma \in K[D]\}$, p_0, p_1, p_2, p_3, p_4 , p_5 are some nonnegative constants, $e_1 = q - q_d \in \mathbb{R}^n$, and $e_2 = \dot{q} - \dot{q}_d \in \mathbb{R}^n$.

6.2.1 Control Development

Under these constraints, the following controller is proposed,

$$\tau = -K\,\mathrm{sat}\left[\mathrm{sig}\left(\mu^{-1}\hat{s}\right)^a\right], \tag{6 - 3}$$

where K and μ are positive definite diagonal matrices, i.e. $K = \mathrm{diag}\{k_i\}_{i=1}^n$, with $k_i > 0$, $\mu = \mathrm{diag}\{\mu_i\}_{i=1}^n$, with $\mu_i > 0$, $a \ge 0$ constant, and $\hat{s} \in \mathbb{R}^n$

i.e. $\tau_i = -k_i\,\mathrm{sat}\left(\dfrac{|\hat{s}_i|^a}{\mu_i^{\,a}}\mathrm{sign}(\hat{s}_i)\right)$ for $i = 1 \ldots n$.

When $a = 0$, the control becomes a discontinuous control law,

$$\tau = -K\,\mathrm{sign}(\hat{s}),\ \ \hat{s} \in \mathbb{R}^n$$

Note that the control is bounded, i.e. such that $|\tau_i(t)| \leq k_i$, for $i = 1, \ldots, n$, $\forall\, t \geq 0$. The $\hat{s}$ is the velocity-estimate-based desired error dynamics defined as $\hat{s} = \hat{e}_2 + \sigma$, with

$$\dot{\sigma} = -K_a\sigma + K_a\mu\,\mathrm{sat}\left[\mu^{-1}\hat{s}\right] - K_a\,\mathrm{sat}_{\left(\frac{\varepsilon_2}{k_2}\right)^{\frac{1+b}{2b}}}\left[\hat{e}_2\right] + \mathrm{sat}_{\varepsilon_2}\left[K_2\mathrm{sig}(\hat{e}_2)^{\frac{2b}{1+b}}\right] + \mathrm{sat}_{\varepsilon_1}\left[K_1\mathrm{sig}(e_1)^b\right] \quad (6\text{ - }4)$$

where K_a, K_1, and K_2 are positive definite diagonal matrices, i.e. $K_a = \mathrm{diag}\{k_{ai}\}_{i=1}^n$, with $k_{ai} > 0$, $K_1 = \mathrm{diag}\{k_{1i}\}_{i=1}^n$, with $k_{1i} > 0$, $K_2 = \mathrm{diag}\{k_{2i}\}_{i=1}^n$, with $k_{2i} > 0$, $k_2 \in \{k_{21}, \ldots, k_{2n}\}$, $\varepsilon_1 \in \{\varepsilon_{11}, \ldots, \varepsilon_{1n}\}$, with $\varepsilon_{1i} > 0$, $\varepsilon_2 \in \{\varepsilon_{21}, \ldots, \varepsilon_{2n}\}$, with $\varepsilon_{2i} > 0$, $\forall i = 1, \ldots, n$, and $b \geq 0$ constant, with $\varepsilon_{1i} > k_{ai}\mu_i + k_{ai}\left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+b}{2b}} + \varepsilon_{2i}$, and $\hat{e}_2 \in \mathbb{R}^n$,

i.e.

$$\dot{\sigma}_i = -k_{ai}\sigma_i + k_{ai}\mu_i\,\mathrm{sat}\left[\frac{\hat{s}_i}{\mu_i}\right] - k_{ai}\,\mathrm{sat}_{\left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+b}{2b}}}\left[\hat{e}_{2i}\right]$$
$$+\,\mathrm{sat}_{\varepsilon_{2i}}\left[k_{2i}|\hat{e}_{2i}|^{\frac{2b}{1+b}}\mathrm{sign}(\hat{e}_{2i})\right] + \mathrm{sat}_{\varepsilon_1}\left[k_{1i}|e_{1i}|^b\,\mathrm{sign}(e_{1i})\right]$$

for $i = 1 \ldots n$.

Note that when $b = 0$, , let the element of ε_1 equal to that of K_1, and ε_2 to that of K_2, i.e. $\varepsilon_{1n} = k_{1n}, \forall i \in N$ and $\varepsilon_{2n} = k_{2n}, \forall i \in N$, i.e.

$$\dot{\sigma}_i = -k_{ai}\sigma_i + k_{ai}\mu_i\,\mathrm{sat}\left[\frac{\hat{s}_i}{\mu_i}\right] - k_{ai}\,\mathrm{sat}\,\left[\hat{e}_{2i}\right] + k_{2i}\mathrm{sign}(\hat{e}_{2i}) + k_{1i}\mathrm{sign}(e_{1i})$$

for $i = 1 \ldots n$.

Note that $\hat{e}_2$ is the output of the observer defined as:

$$\begin{aligned}
\dot{\hat{e}}_1 &= -L_1 \operatorname{sig}(\tilde{e}_1)^p + \hat{e}_2 \ , \\
\dot{w} &= -L_2 \operatorname{sig}(\tilde{e}_1)^{2p-1} - L_3 \hat{e}_2 \\
\hat{e}_2 &= w + L_3 e_1
\end{aligned} \tag{6 - 5}$$

where $\hat{e}_1, \hat{e}_2 \in \mathbb{R}^n$, L_1, L_2, and L_3 are positive definite diagonal matrices, i.e. $L_1 = \operatorname{diag}\{l_{1i}\}_{i=1}^n$, with $l_{1i} > 0$, $L_2 = \operatorname{diag}\{l_{2i}\}_{i=1}^n$, with $l_{2i} > 0$, and $L_3 = \operatorname{diag}\{l_{3i}\}_{i=1}^n$, with $l_{3i} > 0$, $\forall i = 1, \ldots, n$, and $0.5 \le p \le 1$. Let us define $\tilde{e}_1 = \hat{e}_1 - e_1 \in \mathbb{R}^n$, and $\tilde{e}_2 = \hat{e}_2 - e_2 \in \mathbb{R}^n$, then, the closed-loop system (6 - 2), (6 - 3), (6 - 4) and (6 - 5) can be written as

$$\begin{aligned}
\dot{\sigma} &= -K_a \sigma + K_a \mu \operatorname{sat}\left[\mu^{-1}\hat{s}\right] - K_a \operatorname{sat}_{\left(\frac{\varepsilon_2}{k_2}\right)^{\frac{1+b}{2b}}}\left[\hat{e}_2\right] \\
&\quad + \operatorname{sat}_{\varepsilon_2}\left[K_2 \operatorname{sig}(\hat{e}_2)^{\frac{2b}{1+b}}\right] + \operatorname{sat}_{\varepsilon_1}\left[K_1 \operatorname{sig}(e_1)^b\right], \\
\dot{e}_1 &= e_2, \\
\dot{e}_2 &= -M^{-1}(q) K \operatorname{sat}\left[\operatorname{sig}\left(\mu^{-1}\hat{s}\right)^a\right] - M^{-1}(q)\left(C(q,\dot{q})\dot{q} + F\dot{q} + G(q) + D\right) - \ddot{q}_d, \\
\dot{\tilde{e}}_1 &= -L_1 \operatorname{sig}(\tilde{e}_1)^p + \tilde{e}_2 \ , \\
\dot{\tilde{e}}_2 &= -L_2 \operatorname{sig}(\tilde{e}_1)^{2p-1} - L_3 \tilde{e}_2 \\
&\quad - \left(-M^{-1}(q) K \operatorname{sat}\left[\operatorname{sig}\left(\mu^{-1}\hat{s}\right)^a\right] - M^{-1}(q)\left(C(q,\dot{q})\dot{q} + F\dot{q} + G(q) + D\right) - \ddot{q}_d\right)
\end{aligned}$$

To rewrite the closed-loop system in a form convenient for analysis, let us define $v_1 = e_1$, $v_2 = -\sigma$, with $s = e_2 + \sigma = \hat{s} - (\hat{e}_2 - e_2) = \hat{s} - \tilde{e}_2$, and we obtain the following form of closed-loop system,

$$
\begin{aligned}
\dot{v}_1 &= v_2 + s, \\
\dot{v}_2 &= -K_a v_2 - K_a \mu \operatorname{sat}\left[\mu^{-1}\left(s + \tilde{e}_2\right)\right] + K_a \operatorname{sat}_{\left(\frac{\varepsilon_2}{k_2}\right)^{\frac{1+b}{2b}}}\left[v_2 + s + \tilde{e}_2\right] \\
&\quad - \operatorname{sat}_{\varepsilon_2}\left[K_2 \operatorname{sig}\left(v_2 + s + \tilde{e}_2\right)^{\frac{2b}{1+b}}\right] - \operatorname{sat}_{\varepsilon_1}\left[K_1 \operatorname{sig}\left(v_1\right)^b\right], \\
\dot{s} &= -M^{-1}\left(z_1 + q_d\right) K \operatorname{sat}\left[\operatorname{sig}\left(\mu^{-1}\hat{s}\right)^a\right] - M^{-1}(q)\left(C(q,\dot{q})\dot{q} + F\dot{q} + G(q) + D\right) - \ddot{q}_d + \dot{\sigma}, \qquad (6\text{ - }6) \\
\dot{\tilde{e}}_1 &= -L_1 \operatorname{sig}\left(\tilde{e}_1\right)^p + \tilde{e}_2 \ , \\
\dot{\tilde{e}}_2 &= -L_2 \operatorname{sig}\left(\tilde{e}_1\right)^{2p-1} - L_3 \tilde{e}_2 \\
&\quad - \left(-M^{-1}(q) K \operatorname{sat}\left[\operatorname{sig}\left(\mu^{-1}\hat{s}\right)^a\right] - M^{-1}(q)\left(C(q,\dot{q})\dot{q} + F\dot{q} + G(q) + D\right) - \ddot{q}_d\right)
\end{aligned}
$$

6.2.2 Stability Analysis

Theorem 6.1: Consider the robot dynamics (6 - 2) with the bounded controller given by (6 - 3), and the observer (6 - 5), global practical trajectory tracking of the desired trajectory, q_d can be assured, even under bounded disturbances and without velocity measurement, provided that the gain K is sufficiently large up to the saturation bound $T_{i,\max}$, $\forall i = 1,\ldots,n$, and the observer gain L_3 sufficiently large and the desired trajectory sufficiently slow.

Proof of Theorem 6.1: The stability analysis will proceeds in three steps.

1. First we will show that σ and s are bounded globally, and then the observer is perturbed by bounded disturbances only.

    a. First we will show the boundedness of σ, by the following Lyapunov function

$$V_\sigma = \frac{1}{2}\sigma_i^{\,2}, \quad \forall i = 1,\ldots,n.$$

Its time derivative along its solution (6 - 4), gives rise to

$$\dot{V}_\sigma = -k_{ai}\sigma_i^{\,2} + \sigma_i k_{ai}\mu_i \,\mathrm{sat}\left(\frac{\hat{s}_i}{\mu_i}\right) - \sigma_i k_{ai}\,\mathrm{sat}_{\left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+b}{2b}}}\left(\hat{e}_{2i}\right) + \sigma_i \mathrm{sat}_{\varepsilon_{2i}}\left(k_{2i}\left|\hat{e}_{2i}\right|^{\frac{2b}{1+b}}\mathrm{sign}\left(\hat{e}_{2i}\right)\right)$$

$$+ \sigma_i \mathrm{sat}_{\varepsilon_{1i}}\left(k_{1i}\left|e_{1i}\right|^{b}\mathrm{sign}\left(e_{1i}\right)\right)$$

$$\le -\frac{k_{ai}}{2}\sigma_i^{\,2} - \frac{k_{ai}}{2}\sigma_i^{\,2} + \left|\sigma_i\right|\left(k_{ai}\mu_i + k_{ai}\left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+b}{2b}} + \varepsilon_{2i} + \varepsilon_{1i}\right)$$

$$\therefore \left|\sigma_i\right| \le 2\left(\mu_i + \left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+b}{2b}} + \frac{\varepsilon_{2i}}{k_{ai}} + \frac{\varepsilon_{1i}}{k_{ai}}\right),$$

Since μ_i, ε_{1i}, $\varepsilon_{2i} > 0$ are constants, $|\sigma_i|$ is bounded. Also, from (6 - 4), we can see that $|\dot{\sigma}_i|$ is bounded as well.

b. Next, we proceed to show the boundedness of s by the following Lyapunov function

$$V_s = \frac{1}{2}s^{\mathrm{T}}M(.)s, \text{ with } \frac{1}{2}\underline{m}\|s\|^2 \le V_s \le \frac{1}{2}\overline{m}\|s\|^2,$$

Note that $\dot{q} = s + \left(\dot{q}_d - \sigma\right)$,

The s dynamics is described by,

$$\dot{s} = -M^{-1}(q)K\mathrm{sat}\left[\mathrm{sig}\left(\mu^{-1}\hat{s}\right)^a\right] - M^{-1}(q)\left(C(q,\dot{q})\dot{q} + F\dot{q} + G(q) + D\right) - \ddot{q}_d + \dot{\sigma} \qquad (6\text{ - }7)$$

Since the differential equation (6 - 7) has discontinuous right-hand side, i.e. when a = 0 or since no continuity assumption is made on D, its solutions are understood in the sense of Filippov (see definition 2.1), and in accordance to lemma 2.4, the time derivative of the Lyapunov function V_s along the dynamics (6 - 7) for all $a \ge 0$:

$$\dot{V}_s \overset{a.e}{\in} \dot{\tilde{V}}_s$$

$$\begin{aligned}
\dot{\tilde{V}}_s &= \bigcap_{\xi \in \partial V_s} \xi^{\mathrm{T}} \begin{pmatrix} \boldsymbol{K}[f](s) \\ 1 \end{pmatrix} = \nabla V_s^{\mathrm{T}} \boldsymbol{K}[f](s) + \frac{1}{2} s^{\mathrm{T}} \dot{M}(\cdot) s \\
&\subset \left[s^{\mathrm{T}} M(.)\right] \boldsymbol{K} \begin{bmatrix} -M^{-1}(q) K \mathrm{sat}\left[\mathrm{sig}\left(\mu^{-1}\hat{s}\right)^a\right] \\ \quad -M^{-1}(q)\left(C(q,\dot{q})\dot{q} + F\dot{q} + G(q) + D\right) - \ddot{q}_d + \dot{\sigma} \end{bmatrix} + \frac{1}{2} s^{\mathrm{T}} \dot{M}(\cdot) s \\
&= -s^{\mathrm{T}} K\, \boldsymbol{K}\left[\mathrm{sat}\left[\mathrm{sig}\left(\mu^{-1}\hat{s}\right)^a\right]\right] + s^{\mathrm{T}}\left(-G(q) - M(q)\ddot{q}_d + M(q)\dot{\sigma}\right) - s^{\mathrm{T}} \boldsymbol{K}[D] - s^{\mathrm{T}} F\dot{q} \\
&\quad - s^{\mathrm{T}} C(q,\dot{q})\dot{q} + s^{\mathrm{T}} C(q,\dot{q}) s \\
&= -s^{\mathrm{T}} K\, \boldsymbol{K}\left[\mathrm{sat}\left[\mathrm{sig}\left(\mu^{-1}\hat{s}\right)^a\right]\right] + s^{\mathrm{T}}\left(-G(q) - M(q)\ddot{q}_d + M(q)\dot{\sigma}\right) - s^{\mathrm{T}} \boldsymbol{K}[D] \\
&\quad - s^{\mathrm{T}} F(\dot{q}_d - \sigma) - s^{\mathrm{T}}\left[C(q,(\dot{q}_d - \sigma))\right](\dot{q}_d - \sigma) - s^{\mathrm{T}} F s - s^{\mathrm{T}}\left[C(q,(\dot{q}_d - \sigma))\right] s \\
&\le -s^{\mathrm{T}} K\, \boldsymbol{K}\left[\mathrm{sat}\left[\mathrm{sig}\left(\mu^{-1}\hat{s}\right)^a\right]\right] + s^{\mathrm{T}} \Delta_1(\cdot) - \|s\|^2\left(\underline{f} - C_m\|(\dot{q}_d - \sigma)\|\right) \\
&\le \|s\|\left(\|K\| + \|\Delta_1(\cdot)\|\right) - \|s\|^2\left(\underline{f} - C_m\|(\dot{q}_d - \sigma)\|\right) \qquad (6 - 8)
\end{aligned}$$

with $\|\Delta_1\| \coloneqq \sup\{\|\varepsilon\| : \varepsilon \in \boldsymbol{K}[\Delta_1(.)]\}$, and the vector

$$\Delta_1(\cdot) = -G(q) - M(q)\ddot{q}_d + M(q)\dot{\sigma} - \boldsymbol{K}[D] - F(\dot{q}_d - \sigma) - C(q,(\dot{q}_d - \sigma))(\dot{q}_d - \sigma),$$

and $\left\| \boldsymbol{K}\left[\mathrm{sat}\left[\mathrm{sig}\left(\mu^{-1}\hat{s}\right)^a\right] \right] \right\| \le 1$.

Note that the upper bound of the vector $\Delta_1(.)$ is determined by the physical properties of the robot dynamics, desired trajectory, and the desired error dynamics. Since $\dot{q}_d$, $\ddot{q}_d$, are bounded from the assumptions and σ, $\dot{\sigma}$ are bounded (see step (1-a)), the vector Δ_1 is bounded as well. Hence from [143], the multi-valued function $\boldsymbol{K}[\Delta_1]$ is bounded as well. Thus, if

$$\underline{f} > C_m \left\| (\dot{q}_d - \sigma) \right\|$$

Define $k_c := \underline{f} - C_m \left\| (\dot{q}_d - \sigma) \right\| > 0$, which is possible by selecting appropriate desired trajectory and the desired error dynamics. Then we have,

$$\dot{V}_s \overset{a.e.}{\le} -\left(\frac{k_c}{2}\right)\|s\|^2 - \left(\frac{k_c}{2}\right)\|s\|\left(\|s\| - \frac{2(\|K\| + \|\Delta_1\|)}{k_c}\right)$$

$$\dot{V}_s \overset{a.e.}{<} 0, \text{ for } \|s\| \ge \left(\frac{2(\|K\| + \|\Delta_1\|)}{k_c}\right) \Rightarrow \text{for } V_s \ge \frac{\overline{m}}{2}\left(\frac{2(\|K\| + \|\Delta_1\|)}{k_c}\right)^2$$

Hence, $||s||$ is bounded by

$$\Rightarrow \|s\| \le \frac{\sqrt{\overline{m}}}{\sqrt{\underline{m}}}\left(\frac{2(\|K\| + \|\Delta_1\|)}{k_c}\right)$$

Note that from above:

$$\dot{q} = s + (\dot{q}_d - \sigma),$$
$$\therefore \|\dot{q}\| = \|s + (\dot{q}_d - \sigma)\| \le \|s\| + \|\dot{q}_d\| + \|\sigma\|$$

where the velocity is bounded globally since the right hand side of the inequality is bounded, i.e. s is upper bounded by the $\Delta_1(.)$ and the control gain K, while desired trajectory and desired error dynamics are upper bounded by design.

c. The observer dynamics, from (6 - 6),

For ∀ i=1, ..., n:

$$\dot{\tilde{e}}_{1i} = -l_{1i}\mathrm{sig}(\tilde{e}_{1i})^{p} + \tilde{e}_{2i} \quad ,$$
$$\dot{\tilde{e}}_{2i} = l_{2i}\mathrm{sig}(\tilde{e}_{1i})^{2p-1} - l_{3i}\tilde{e}_{2i} + \Delta_{2i}(\cdot)$$

where

$\Delta_2(\cdot) = (\Delta_{2i}(\cdot),\ldots,\Delta_{2n}(\cdot))^{\mathrm{T}}$, $\forall i = 1,\ldots n$, and

$$\begin{aligned}\Delta_2(\cdot) = {} & M^{-1}(q)K\mathrm{sat}\left[\mu^{-1}\mathrm{sig}(\hat{s})^{a}\right] + M^{-1}(q)C(q,\dot{q})\dot{q} + M^{-1}(q)F\dot{q} \\ & + M^{-1}(q)G(q) + M^{-1}(q)D + \ddot{q}_d,\end{aligned} \tag{6 - 9}$$

Let us define $|\Delta_{2i}| := \sup\{|\varepsilon_i| : \varepsilon_i \in \boldsymbol{K}[\Delta_{2i}(\cdot)]\}$.

Remark 6.2. Note that $\Delta_2(.)$ is upper bounded by the physical properties of the robot dynamics, desired trajectory, control gain K, and velocity of the robot manipulator. Since $\dot{q}_d$, $\ddot{q}_d$, are bounded from the assumptions, σ, $\dot{q}$ and s are bounded (see step (1-a to 1-b)), the vector Δ_2 is bounded as well. Hence from [143], the multi-valued function $\boldsymbol{K}[\Delta_{2i}]$ is bounded as well.

To show the boundedness of ($\tilde{e}_1$, $\tilde{e}_2$), consider the following Lyapunov function (see section 6.1), $V_{\tilde{e}i}(\tilde{e}_{1i},\tilde{e}_{2i}) = \left(\frac{1}{2}\tilde{e}_{2i}^{\,2} + \frac{l_{2i}}{2p}|\tilde{e}_{1i}|^{2p}\right)^2 - r_{\tilde{e}}|\tilde{e}_{1i}||\tilde{e}_{2i}|^{\frac{4p-1}{p}}\mathrm{sign}(\tilde{e}_{1i}\tilde{e}_{2i})$

The differential inequalities for the ($\tilde{e}_1$, $\tilde{e}_2$)-subsystem satisfy (see section 6.1):

$$\dot{V}_{\tilde{e}i}(\tilde{e}_{1i},\tilde{e}_{2i}) \overset{a.e.}{\leq} \begin{cases} -\dfrac{l_{1i}l_{2i}^{\ 2}}{3p}|\tilde{e}_{1i}|^{5p-1} - \dfrac{r_{\tilde{e}}}{2}|\tilde{e}_{2i}|^{\frac{5p-1}{p}} & \text{for } V_{\tilde{e}i} \geq \rho_{1i}(|\Delta_{2i}|, l_{3i}) \text{ and } 0.5 < p \leq 1, \\ -\dfrac{2l_{1i}l_{2i}^{\ 2}}{3}|\tilde{e}_{1i}|^{1.5} - \dfrac{r_{\tilde{e}}}{2}|\tilde{e}_{2i}|^{3}, & \text{for } V_{\tilde{e}i} \geq \rho_{2i}(|\Delta_{2i}|, l_{3i}) \text{ and } p = 0.5, \end{cases}$$

where the functions ρ_{1i} and ρ_{2i} are of class-$\mathcal{KL}$,

$$\rho_{1i}(|\Delta_{2i}|, l_{3i}) := \max_{(\tilde{e}_{1i},\tilde{e}_{2i}) \in \text{bd } \phi_3(|\Delta_{2i}|, l_{3i})} V_{\tilde{e}i}(\tilde{e}_{1i},\tilde{e}_{2i}),$$

$$\rho_{2i}(|\Delta_{2i}|, l_{3i}) = \max_{(\tilde{e}_{1i},\tilde{e}_{2i}) \in \text{bd } \phi_3(|\Delta_{2i}|, l_{3i})} V_{\tilde{e}i}(\tilde{e}_{1i},\tilde{e}_{2i}),$$

bd $\phi_{3i}(|\Delta_{2i}|, l_{3i})$ is the boundary of the region

$$\phi_{3i}(|\Delta_{2i}|, l_{3i}) = \begin{cases} \{(\tilde{e}_{1i},\tilde{e}_{2i}) \in \mathbb{R}^2 : |\tilde{e}_{1i}| \leq \beta_1(|\Delta_{2i}|, l_{3i}), |\tilde{e}_{2i}| \leq \beta_2(|\Delta_{2i}|, l_{3i})\}, & \text{for } 0.5 < p \leq 1, \\ \{(\tilde{e}_{1i},\tilde{e}_{2i}) \in \mathbb{R}^2 : |\tilde{e}_{1i}| \leq \beta_3(|\Delta_{2i}|, l_{3i}), |\tilde{e}_{2i}| \leq \beta_4(|\Delta_{2i}|, l_{3i})\}, & \text{for } p = 0.5, \end{cases}$$

which is a compact set, with functions β_1, β_2, β_3, and β_4 are of class-$\mathcal{KL}$. (See section 6.1).

Define the following Lyapunov level set,

$$\Sigma_{\tilde{e}i}(|\Delta_{2i}|, l_{3i}) = \begin{cases} \{(\tilde{e}_{1i},\tilde{e}_{2i}) \in \mathbb{R}^2 : V_{\tilde{e}i} \leq \rho_{1i}(|\Delta_{2i}|, l_{3i})\}, & \text{for } 0.5 < p \leq 1, \\ \{(\tilde{e}_{1i},\tilde{e}_{2i}) \in \mathbb{R}^2 : V_{\tilde{e}i} \leq \rho_{2i}(|\Delta_{2i}|, l_{3i})\}, & \text{for } p = 0.5, \end{cases}$$

Thus, it implies that each of the trajectories for the i-th planar system will enter their respective compact level set $\Sigma_{\tilde{e}i}(|d_i|, l_{3i})$ in finite time and stay in it once entered.

In particular, once inside the compact set $\Sigma_{\tilde{e}i}(|d_i|, l_{3i})$ and since , $\underline{\pi}_{2i}(|\tilde{e}_{2i}|^4 + |\tilde{e}_{1i}|^{4p}) \leq V_{\tilde{e}i}$, it implies

$\forall i = 1,\ldots,n:$

$$|\tilde{e}_{2i}| \le \begin{cases} \left(\dfrac{\rho_{1i}\left(|\Delta_{2i}|, l_{3i}\right)}{\underline{\pi}_{2i}}\right)^{\frac{1}{4}}, & \text{for } 0.5 < p \le 1, \\ \left(\dfrac{\rho_{2i}\left(|\Delta_{2i}|, l_{3i}\right)}{\underline{\pi}_{2i}}\right)^{\frac{1}{4}}, & \text{for } p = 0.5, \end{cases}$$

and
$$|\tilde{e}_{1i}| \le \begin{cases} \left(\dfrac{\rho_{1i}\left(|\Delta_{2i}|, l_{3i}\right)}{\underline{\pi}_{2i}}\right)^{\frac{1}{4p}}, & \text{for } 0.5 < p \le 1, \\ \left(\dfrac{\rho_{2i}\left(|\Delta_{2i}|, l_{3i}\right)}{\underline{\pi}_{2i}}\right)^{\frac{1}{4p}}, & \text{for } p = 0.5, \end{cases} \tag{6 - 10}$$

Hence, it can be observed that for $0.5 \le p \le 1$, the increase of observer gain L_3 will result in a smaller upper bound on the observation errors, since the functions ρ_{1i} and ρ_{2i} are of class-$\mathcal{KL}$.

Remark 6.3. It is worth mentioning that increasing the control gain K, the upper bound of the vector $\Delta_2(.)$ will increase as well (see (6 - 9)), requiring a larger observer gains L_3 to obtain the same upper bounds on the observation errors.

2. Now recall the analysis for s-dynamics, with this new-found bound on observation error, we are going to show that the bound on s can be made arbitrarily small; in particular the

control will be unsaturated.

Recall that from (6 - 8), since $k_c := F_m - C_m \|(\dot{q}_d - \sigma)\| > 0$

$$\dot{V}_s \overset{a.e.}{\leq} -s^{\mathrm{T}} K\, \boldsymbol{K}\left[\mathrm{sat}\left[\mathrm{sig}\left(\mu^{-1}\hat{s}\right)^{a}\right]\right] + s^{\mathrm{T}}\Delta_1(\cdot)$$

and $s = \hat{s} - \tilde{e}_2$,

a. For $a = 0$:

Then we can rewrite the above inequality as:

$$\begin{aligned}\dot{V}_s &\overset{a.e.}{\leq} -s^{\mathrm{T}} K\, \boldsymbol{K}[\mathrm{sign}(s+\tilde{e}_2)] + s^{\mathrm{T}}\Delta_1(\cdot) \\ &= -s^{\mathrm{T}} K\, \boldsymbol{K}[\mathrm{sign}(s)] + s^{\mathrm{T}}\Delta_1(\cdot) - s^{\mathrm{T}} K\, \boldsymbol{K}[\mathrm{sign}(s+\tilde{e}_2)] + s^{\mathrm{T}} K\, \boldsymbol{K}[\mathrm{sign}(s)]\end{aligned}$$

Using the following notation:

$$N^{+}(s) = \{i \in \{1,\ldots,n\} : s_i \neq 0\}, \quad N^{0}(s) = \{i \in \{1,\ldots,n\} : s_i = 0\},$$

$$\begin{aligned}-s^{\mathrm{T}} K\boldsymbol{K}[\mathrm{sign}(s)] &= -\sum_{i\in N^{+}(s)} k_i|s_i| - \sum_{i\in N^{0}(s)} k_i(0)\times[-1,+1] \\ &= -\sum_{i\in N^{+}(s)} k_i|s_i| - 0 \\ &= -\sum_{i=i}^{n} k_i|s_i|\end{aligned}$$

Thus,

$$\begin{aligned}\dot{V}_s &\overset{a.e.}{\leq} -\sum_{i=i}^{n} k_i|s_i| + s^{\mathrm{T}}\Delta_1(\cdot) - s^{\mathrm{T}} K\left(\boldsymbol{K}[\mathrm{sign}(s+\tilde{e}_2)] - \boldsymbol{K}[\mathrm{sign}(s)]\right) \\ &\leq -\sum_{i=i}^{n} k_i|s_i| + s^{\mathrm{T}}\Delta_1(\cdot) - \sum_{i=1}^{n} s_i k_i\left(\boldsymbol{K}[\mathrm{sign}(s_i+\tilde{e}_{2i})] - \boldsymbol{K}[\mathrm{sign}(s_i)]\right)\end{aligned}$$

Note that:

$$\boldsymbol{K}[\mathrm{sign}(s_i + \tilde{e}_{2i})] - \boldsymbol{K}[\mathrm{sign}(s_i)] \le 2,$$

$$\begin{aligned}
&\text{Also, for } |s_i| > |\tilde{e}_{2i}| > 0: \\
&\boldsymbol{K}[\mathrm{sign}(s_i + \tilde{e}_{2i})] - \boldsymbol{K}[\mathrm{sign}(s_i)] = (\mathrm{sign}(s_i + \tilde{e}_{2i}) - \mathrm{sign}(s_i)) \\
&\qquad\qquad\qquad\qquad\qquad\qquad = \mathrm{sign}(s_i) - \mathrm{sign}(s_i) \\
&\qquad\qquad\qquad\qquad\qquad\qquad = 0 \\
&\therefore |s_i k_i \boldsymbol{K}[(\mathrm{sign}(s_i + \tilde{e}_{2i}) - \mathrm{sign}(s_i))]| \le 2k_i |\tilde{e}_{2i}|
\end{aligned}$$

Hence we have:

$$\begin{aligned}
\dot{V}_s &\overset{a.e.}{\le} -\sum_{i=i}^{n} k_i |s_i| + \sum_{i=1}^{n} |s_i||\Delta_{1i}| + \sum_{i=1}^{n} 2k_i |\tilde{e}_{2i}| \\
&\le -(1-\theta)\sum_{i=i}^{n} k_i |s_i| + \sum_{i=1}^{n} 2k_i |\tilde{e}_{2i}| - \sum_{i=i}^{n} \theta k_i |s_i| + \sum_{i=1}^{n} |s_i||\Delta_{1i}| \\
&\le -(1-\theta)\lambda_{\min}(K)\|s\| + 2\sqrt{n}\,\lambda_{\max}(K)\|\tilde{e}_2\| - \sum_{i=i}^{n} |s_i|(\theta k_i - |\Delta_{1i}|) \\
&\le -(1-\theta)\lambda_{\min}(K)\frac{\sqrt{2}}{\sqrt{\overline{m}}} V_s^{\frac{1}{2}} + 2\sqrt{n}\,\lambda_{\max}(K)\|\tilde{e}_2\| - \sum_{i=i}^{n} |s_i|(\theta k_i - |\Delta_{1i}|)
\end{aligned}$$

where the following property has been employed, $\|x\|_2 \le \|x\|_1 \le \sqrt{n}\|x\|_2, \forall x \in \mathbb{R}^n$

$$\therefore \dot{V}_s \overset{a.e.}{\le} -\sum_{i=i}^{n} |s_i|(\theta k_i - |\Delta_{1i}|) < 0, \quad \text{for} \quad V_s \ge \left(\frac{\overline{m}}{2}\right)\left(\left(\frac{2\sqrt{n}}{(1-\theta)}\right)\left(\frac{\lambda_{\max}(K)}{\lambda_{\min}(K)}\right)\|\tilde{e}_2\|\right)^2, \text{ with}$$

$$k_i > \frac{|\Delta_{1i}|}{\theta}, \ \forall i \in N.$$

Remark 6.4. Note that for a given desired trajectory and given desired error dynamics (hence an upper bound on the vector $\Delta_1(.)$), there exist k_i such that the above condition on control gains K is satisfied. Accordingly, for a given control gain K, desired trajectory, and desired error dynamics (hence an upper bound on the vector $\Delta_2(.)$), there exist observer gains L_3 such that the observation velocity error $\tilde{e}_2$ is bounded.

Since $\tilde{e}_2$ is bounded, the state s will reach the above Lyapunov level set in finite time and stay in it for all future times. In particular, the ultimate bound on s can be obtained as:

$$\|s\| \leq \left(\frac{\overline{m}}{\underline{m}}\right)^{\frac{1}{2}} \left(\frac{2\sqrt{n}}{(1-\theta)}\right)\left(\frac{\lambda_{\max}(K)}{\lambda_{\min}(K)}\right)\|\tilde{e}_2\|. \tag{6 - 11}$$

Note that the desired dynamics (6 - 4) comprise a term of the form:

$$k_{ai}\mu_i \ \mathrm{sat}\left(\frac{\hat{s}_i}{\mu_i}\right) = k_{ai}\mu_i \ \mathrm{sat}\left(\frac{s_i + \tilde{e}_{2i}}{\mu_i}\right),$$

For this term to be unsaturated, it is sufficient for

$$|s_i| + |\tilde{e}_{2i}| \leq \mu_i,$$

and since

$$|s_i + \tilde{e}_{2i}| \leq |s_i| + |\tilde{e}_{2i}|,$$

while the ultimate bound on s is dependent on that of the observation error for a given K, recall from (6 - 10) and (6 - 11),

$$|\tilde{e}_{2i}| \le \left(\frac{\rho_i(|\Delta_{2i}|, l_{3i})}{\underline{\pi}_{2i}} \right)^{\frac{1}{4}}, \text{ where } \rho_i(|\Delta_{2i}|, l_{3i}) := \begin{cases} \rho_{1i}(|\Delta_{2i}|, l_{3i}), & \text{for } 0.5 < p \le 1, \\ \rho_{2i}(|\Delta_{2i}|, l_{3i}), & \text{for } p = 0.5, \end{cases}$$

$$|s_i| \le \|s\| \le \left(\frac{\overline{m}}{\underline{m}} \right)^{\frac{1}{2}} \left(\frac{2\sqrt{n}}{(1-\theta)} \right) \left(\frac{\lambda_{\max}(K)}{\lambda_{\min}(K)} \right) \|\tilde{e}_2\|,$$

$$\therefore |s_i| + |\tilde{e}_{2i}| \le \left(\frac{\rho_i(|\Delta_{2i}|, l_{3i})}{\underline{\pi}_{2i}} \right)^{\frac{1}{4}} + \left(\frac{\overline{m}}{\underline{m}} \right)^{\frac{1}{2}} \left(\frac{2\sqrt{n}}{(1-\theta)} \right) \left(\frac{\lambda_{\max}(K)}{\lambda_{\min}(K)} \right) \sqrt{n} \max_i \left\{ \left(\frac{\rho_i(|\Delta_{2i}|, l_{3i})}{\underline{\pi}_{2i}} \right)^{\frac{1}{4}} \right\}$$

$$\le \max_i \left\{ \left(\frac{\rho_i(|\Delta_{2i}|, l_{3i})}{\underline{\pi}_{2i}} \right)^{\frac{1}{4}} \right\} \times \left(1 + \left(\frac{\overline{m}}{\underline{m}} \right)^{\frac{1}{2}} \left(\frac{2n}{(1-\theta)} \right) \left(\frac{\lambda_{\max}(K)}{\lambda_{\min}(K)} \right) \right)$$

where the following property has been employed: $\|x\|_\infty \le \|x\|_2 \le \sqrt{n}\|x\|_\infty$, $\forall x \in \mathbb{R}^n$, then the sufficient condition such that no saturation occur, is

$$\max_i \left\{ \left(\frac{\rho_i(|\Delta_{2i}|, l_{3i})}{\underline{\pi}_{2i}} \right)^{\frac{1}{4}} \right\} \times \left(1 + \left(\frac{\overline{m}}{\underline{m}} \right)^{\frac{1}{2}} \left(\frac{2n}{(1-\theta)} \right) \left(\frac{\lambda_{\max}(K)}{\lambda_{\min}(K)} \right) \right) \le \mu_i \, , \tag{6 - 12}$$

Thus, it is possible for the term $\text{sat}(s_i + \tilde{e}_{2i})$ to be unsaturated in finite time, provided a sufficiently big observer gain L_3.

b. For $a > 0$:

Then we can rewrite the above inequality as:

$$\dot{V}_s \overset{a.e.}{\leq} -s^{\mathrm{T}} K \mathrm{sat}\left[\mathrm{sig}\left(\mu^{-1} s\right)^a\right] + s^{\mathrm{T}} \Delta_1(\cdot) - s^{\mathrm{T}} K\left(\mathrm{sat}\left[\mathrm{sig}\left(\mu^{-1} \hat{s}\right)^a\right] - \mathrm{sat}\left[\mathrm{sig}\left(\mu^{-1} s\right)^a\right]\right)$$

$$\leq -s^{\mathrm{T}} K \mathrm{sat}\left[\mathrm{sig}\left(\mu^{-1} s\right)^a\right] + s^{\mathrm{T}} \Delta_1(\cdot) - s^{\mathrm{T}} K\left(\mathrm{sat}\left[\mathrm{sig}\left(\mu^{-1} (s + \tilde{e}_2)\right)^a\right] - \mathrm{sat}\left[\mathrm{sig}\left(\mu^{-1} s\right)^a\right]\right)$$

$$\leq \sum_{i=1}^{n} - k_i |s_i| \mathrm{sat}\left[\frac{|s_i|^a}{\mu_i^{\,a}}\right] + |s_i||\Delta_{1i}|$$

$$+ |s_i| k_i \left| \mathrm{sat}\left[\frac{|s_i + \tilde{e}_{2i}|^a \operatorname{sign}(s_i + \tilde{e}_{2i})}{\mu_i^{\,a}}\right] - \mathrm{sat}\left[\frac{|s_i|^a \operatorname{sign}(s_i)}{\mu_i^{\,a}}\right] \right|$$

where

$$\Delta_1(\cdot) = \left(\Delta_{1i}(\cdot), \ldots, \Delta_{1n}(\cdot)\right)^{\mathrm{T}}, \ \ \forall i = 1, \ldots n,$$

Note that from Appendix B.2-proposition 1, the following inequality is satisfied

$$\left| \mathrm{sat}\left[\frac{|s_i + \tilde{e}_{2i}|^a \operatorname{sign}(s_i + \tilde{e}_{2i})}{\mu_i^{\,a}}\right] - \mathrm{sat}\left[\frac{|s_i|^a \operatorname{sign}(s_i)}{\mu_i^{\,a}}\right] \right| \leq \mathrm{sat}_2\left[\beta\left(|\tilde{e}_{2i}|\right)\right]$$

where β is a class $\mathcal{K}$ function.

Now, let us define the set of real number $N \in \{1, \ldots, n\}$ and $|\Delta_{1i}| := \sup\{|\varepsilon_i| : \varepsilon_i \in \boldsymbol{K}[\Delta_{1i}]\}$, $\forall i \in N$, then $\dot{V}_s$ becomes

$$\dot{V}_s \overset{a.e.}{\le} \sum_{i \in N} -k_i |s_i| \operatorname{sat}\left[\frac{|s_i|^a}{\mu_i^a}\right] + |s_i||\Delta_{1i}| + |s_i| k_i \operatorname{sat}_2\left[\beta(|\tilde{e}_{2i}|)\right]$$

$$= \underbrace{\left(-\frac{k_j}{2}|s_j| \operatorname{sat}\left[\frac{|s_j|^a}{\mu_j^a}\right] + |s_j||\Delta_{1j}| + \left(\sum_{i \in N \setminus j} |s_i| \left(-\frac{k_i}{2} \operatorname{sat}\left[\frac{|s_i|^a}{\mu_i^a}\right] + |\Delta_{1i}| \right)\right)\right)}_{\omega_{s1}}$$

$$+ \underbrace{\left(\begin{array}{l} -\dfrac{k_j}{2}|s_j| \operatorname{sat}\left[\dfrac{|s_j|^a}{\mu_j^a}\right] + |s_j| k_j \operatorname{sat}_2\left[\beta(|\tilde{e}_{2j}|)\right] \\ + \left(\displaystyle\sum_{i \in N \setminus j} k_i |s_i| \left(-\frac{1}{2} \operatorname{sat}\left[\frac{|s_i|^a}{\mu_i^a}\right] + \operatorname{sat}_2\left[\beta(|\tilde{e}_{2i}|)\right] \right)\right) \end{array} \right)}_{\omega_{s2}} \qquad (6 - 13)$$

Note that the last equality has single out one of its element, $j \in N$ from the summation and rearranged to separate terms due to observation errors and uncertainties. To show sign definiteness, the two terms are considered separately:

For ω_{s1}:

$$\omega_{s1} = -\frac{k_j}{2}|s_j| \operatorname{sat}\left[\frac{|s_j|^a}{\mu_j^a}\right] + |s_j||\Delta_{1j}| + \left(\sum_{i \in N \setminus j} |s_i| \left(-\frac{k_i}{2} \operatorname{sat}\left[\frac{|s_i|^a}{\mu_i^a}\right] + |\Delta_{1i}| \right)\right)$$

Note that it is necessary that $0.5k_i > |\Delta_i|$, $\forall i \in N$, then the maximum of the last term in the above equation occurs when $|s_i| < \mu_i$. In particular, by taking the derivative of the term w.r.t. $|s_i|$ and equating it to zero, the maximum is found to occur at

$$|s_i| = \left| \frac{2|\Delta_{1i}|}{(1+a)k_i} \right|^{\frac{1}{a}} \mu_i, \ \forall i \in N \setminus j$$

and the corresponding maximum,

$$\left(\frac{a}{(1+a)^{\frac{1+a}{a}}}\right)\left(\frac{2|\Delta_{1i}|}{k_i}\right)^{\frac{1}{a}}\mu_i|\Delta_{1i}|,\ \ \forall i\in N\setminus j$$

Hence, one obtains,

$$\omega_{s1}\le-\frac{k_j}{2}|s_j|\mathrm{sat}\left[\frac{|s_j|^a}{\mu_j{}^a}\right]+|s_j||\Delta_{1j}|+\left(\frac{a}{(1+a)^{\frac{1+a}{a}}}\right)\sum_{i\in N\setminus j}\left(\frac{2|\Delta_{1i}|}{k_i}\right)^{\frac{1}{a}}\mu_i|\Delta_{1i}|$$

$$=-(1-\theta)\frac{k_j}{2}|s_j|\mathrm{sat}\left[\frac{|s_j|^a}{\mu_j{}^a}\right]-\frac{\theta}{4}k_j|s_j|\mathrm{sat}\left[\frac{|s_j|^a}{\mu_j{}^a}\right]+|s_j||\Delta_{1j}|-\frac{\theta}{4}k_j|s_j|\mathrm{sat}\left[\frac{|s_j|^a}{\mu_j{}^a}\right]$$
$$+\left(\frac{a}{(1+a)^{\frac{1+a}{a}}}\right)\sum_{i\in N\setminus j}\left(\frac{2|\Delta_{1i}|}{k_i}\right)^{\frac{1}{a}}\mu_i|\Delta_{1i}|,$$

$$\le-(1-\theta)\frac{k_j}{2}|s_j|\mathrm{sat}\left[\frac{|s_j|^a}{\mu_j{}^a}\right],\ \text{for}\,|s_j|\ge\pi_{1j},\ \text{and}\ |s_i|\in\mathbb{R},\forall i\in N\setminus j$$

with

$$\pi_{1j}=\max\left\{\left(\frac{4}{\theta}\frac{|\Delta_{1j}|}{k_j}\right)^{\frac{1}{a}}\mu_j,\ \left(\frac{4\mu_j{}^a a}{\theta k_j(1+a)^{\frac{1+a}{a}}}\right)^{\frac{1}{1+a}}\left(\sum_{i\in N\setminus j}\left(\frac{2|\Delta_{1i}|}{k_i}\right)^{\frac{1}{a}}\mu_i|\Delta_{1i}|\right)^{\frac{1}{1+a}}\right\},$$

$$\text{and}\ k_j>\max\left\{\frac{4|\Delta_{1j}|}{\theta},\ \frac{4}{\theta\mu_j}\left(\frac{a}{(1+a)^{\frac{1+a}{a}}}\right)\sum_{i\in N\setminus j}\left(\frac{2|\Delta_{1i}|}{k_i}\right)^{\frac{1}{a}}\mu_i|\Delta_{1i}|\right\},\tag{6 - 14}$$

where $0 < \theta < 1$ constant. Note that with the above conditions, we have $\pi_{1j} < \mu_j$.

For ω_{s2}:

$$\omega_{s2} = -\frac{k_j}{2}|s_j|\text{sat}\left[\frac{|s_j|^a}{\mu_j^a}\right] + |s_j|k_j\text{sat}_2\left[\beta\left(\left|\tilde{e}_{2j}\right|\right)\right]$$
$$+\left(\sum_{i\in N\setminus j} k_i|s_i|\left(-\frac{1}{2}\text{sat}\left[\frac{|s_i|^a}{\mu_i^a}\right] + \text{sat}_2\left[\beta\left(\left|\tilde{e}_{2j}\right|\right)\right]\right)\right),$$

Note that it is necessary that

$$\text{sat}_2\left[\beta\left(\left|\tilde{e}_{2j}\right|\right)\right] \leq 0.5, \quad \forall i = 1,\ldots,n$$

such that the maximum of the last term occurs when $|s_i| < \mu_i$.

This can be achieved by having sufficiently high observer gain, L_3 for a given control gain K (see step (1-c)). In particular, by taking the derivative of the term w.r.t. $|s_i|$ and equating it to zero, the maximum is found to occur at

$$|s_i| = \left(\frac{2\text{sat}_2\left[\beta\left(\left|\tilde{e}_{2j}\right|\right)\right]}{(1+a)}\right)^{\frac{1}{a}}\mu_i, \quad \forall i \in N\setminus j$$

and the corresponding maximum,

$$\left(\frac{2^{\frac{1}{a}}a}{(1+a)^{\frac{1+a}{a}}}\right)k_i\mu_i\left(\text{sat}_2\left[\beta\left(\left|\tilde{e}_{2j}\right|\right)\right]\right)^{\frac{1+a}{a}}, \quad \forall i \in N\setminus j$$

Hence, one obtains,

$$
\begin{aligned}
\omega_{s2} \le & -\frac{k_j}{2}|s_j|\operatorname{sat}\left[\frac{|s_j|^a}{\mu_j^{\,a}}\right] + |s_j| k_j \operatorname{sat}_2\left[\beta\left(|\widetilde{e}_{2j}|\right)\right] + \left(\frac{2^{\frac{1}{a}} a}{(1+a)^{\frac{1+a}{a}}}\right)\sum_{i\in N\setminus j} k_i \mu_i \left(\operatorname{sat}_2\left[\beta\left(|\widetilde{e}_{2j}|\right)\right]\right)^{\frac{1+a}{a}} \\
& = -(1-\theta)\frac{k_j}{2}|s_j|\operatorname{sat}\left[\frac{|s_j|^a}{\mu_j^{\,a}}\right] - \frac{\theta}{4}k_j|s_j|\operatorname{sat}\left[\frac{|s_j|^a}{\mu_j^{\,a}}\right] + |s_j| k_j \operatorname{sat}_2\left[\beta\left(|\widetilde{e}_{2j}|\right)\right] \\
& \quad -\frac{\theta}{4}k_j|s_j|\operatorname{sat}\left[\frac{|s_j|^a}{\mu_j^{\,a}}\right] + \left(\frac{2^{\frac{1}{a}} a}{(1+a)^{\frac{1+a}{a}}}\right)\sum_{i\in N\setminus j} k_i \mu_i \left(\operatorname{sat}_2\left[\beta\left(|\widetilde{e}_{2j}|\right)\right]\right)^{\frac{1+a}{a}} \\
& \le -(1-\theta)\frac{k_j}{2}|s_j|\operatorname{sat}\left[\frac{|s_j|^a}{\mu_j^{\,a}}\right], \quad \text{for } |s_j| \ge \pi_{2j}, \text{ and } |s_i| \in \mathbb{R}, \forall i \in N\setminus j,
\end{aligned}
$$

with

$$
\pi_{2j} = \max\left\{ \begin{aligned} & \left(\frac{4}{\theta}\operatorname{sat}_2\left[\beta\left(|\widetilde{e}_{2j}|\right)\right]\right)^{\frac{1}{a}} \mu_j, \\ & \left(\frac{4\mu_j^{\,a}}{\theta k_j}\left(\frac{2^{\frac{1}{a}} a}{(1+a)^{\frac{1+a}{a}}}\right)\sum_{i\in N\setminus j} k_i \mu_i \left(\operatorname{sat}_2\left[\beta\left(|\widetilde{e}_{2j}|\right)\right]\right)^{\frac{1+a}{a}}\right)^{\frac{1}{1+a}} \end{aligned} \right\},
$$

$$
\begin{aligned}
& \operatorname{sat}_2\left[\beta\left(|\widetilde{e}_{2j}|\right)\right] \le \min\left\{\frac{\theta}{4}, 0.5\right\}, \quad \forall i = 1,\ldots,n, \\
& \left\|\operatorname{sat}_2\left[\beta\left(|\widetilde{e}_2|\right)\right]\right\| \le \left(\frac{\theta}{4(n-1)}\left(\frac{k_j}{\lambda_{\max}(K)}\right)\left(\frac{\mu_j}{\lambda_{\max}(\mu)}\right)\left(\frac{(1+a)^{\frac{1+a}{a}}}{2^{\frac{1}{a}} a}\right)\right)^{\frac{a}{1+a}},
\end{aligned}
\qquad (6-15)
$$

where $\operatorname{sat}_2\left[\beta\left(|\widetilde{e}_2|\right)\right] = \left(\operatorname{sat}_2\left[\beta\left(|\widetilde{e}_{21}|\right)\right], \ldots, \operatorname{sat}_2\left[\beta\left(|\widetilde{e}_{2n}|\right)\right]\right)^{\mathrm{T}}$, and $0 < \theta < 1$.

Note that the above conditions can be satisfied by having sufficiently high observer gain L_3 for a given control gain K (see step (1-c)). With the above conditions, we have $\pi_{2j} < \mu_j$.

Now, from (6 - 13) we have:

$$\dot{V}_s \overset{a.e.}{\leq} \omega_{s1} + \omega_{s2},$$

and by selecting each k_j , $\forall j \in N$ that satisfy (6 - 14), and having the observer gain, L_3 sufficiently high that satisfy (6 - 15), we have

$$\omega_{s1} \leq -c_{s1}, \ \forall s \notin \Omega_{\pi 1}$$
$$\omega_{s2} \leq -c_{s2}, \ \forall s \notin \Omega_{\pi 2}$$

where the compact sets are defined as: $\Omega_{\pi 1} = \{s \in \mathbb{R}^n : |s_i| \leq \pi_{1i}, \forall i \in N\}$, and

$\Omega_{\pi 2} = \{s \in \mathbb{R}^n : |s_i| \leq \pi_{2i}, \forall i \in N\}$,

with positive constants,

$$c_{s1} = \min_i \left\{ (1-\theta)\frac{k_i}{2}\pi_{1i}\left(\frac{\pi_{1i}}{\mu_i}\right)^a \right\}, \text{ and } \quad c_{s2} = \min_i \left\{ (1-\theta)\frac{k_i}{2}\pi_{2i}\left(\frac{\pi_{2i}}{\mu_i}\right)^a \right\}, \forall i \in N,$$

Define $\max_j \{\pi_{1j}, \pi_{2j}\} = \pi_M, \ \forall j \in N$, then, we have:

$$\dot{V}_s \overset{a.e.}{<} 0, \ \forall s \notin \Omega_{\pi M}$$

where $\Omega_{\pi M} = \{s \in \mathbb{R}^n : |s_i| \leq \pi_M, \forall i \in N\}$, since $\Omega_{\pi M} \supset (\Omega_{\pi 1} \cup \Omega_{\pi 2})$.

Note that this compact set can be made arbitrarily small by increasing k_i up to the maximal allowable control bound $T_{i,\max}$, to reduce π_{1i} and the increase of the observer gain, L_3 (for a given control gain K) to reduce π_{2i}. Now with each given μ_i, let c_1 be a positive constant $0 < c_1 < 1$, consider the following compact set

$\Omega_\mu = \{s \in \mathbb{R}^n : |s_i| \le c_1\mu_i, \forall i \in N\}$ and $\Sigma_\mu = \{s \in \mathbb{R}^n : V_s \le k_\mu\}$,

where $k_\mu > 0$ is defined as

$$k_\mu = \min_{s \in \text{bd } \Omega_\mu} V_s$$

which exists since the boundary, bd Ω_μ is a compact set. Note that $\Sigma_\mu \subset \Omega_\mu$. Note that if the states, s can be confined within Σ_μ, which is a Lyapunov level set, $|s_i| \le c_1\mu_i$. To achieve this, we simply need the set $\Omega_{\pi M} \subset \Sigma_\mu$, which can be attained when K, and L_3 are large enough, such that $\dot{V}_s$ being negative the outside of the set Σ_μ,

$$\dot{V}_s \overset{a.e.}{<} 0, \text{ for } V_s \ge k_\mu$$

which implies that the trajectories of s will enter the set Σ_μ in finite time and stay in it once entered. Hence, we have

$$|s_i| \le c_1\mu_i < \mu_i \Leftrightarrow |s_i| < \mu_i, \ \forall i \in N$$

In particular, a sufficient condition on the control gain, k_i and observer gain, l_i such that the control is unsaturated can be obtained as follows,

Firstly note that:

$$k_\mu = \min_{s \in \text{bd } \Omega_\mu} V_s \ge \min_{s \in \text{bd } \Omega_\mu} \frac{1}{2}\underline{m}\|s\|^2 = \frac{1}{2}\underline{m}\left(\min_i c_1\mu_i\right)^2$$

Next, observed that for a given K, the observer gain, L_3 can be chosen to be sufficiently high such that

$\pi_{1j} > \pi_{2j}, \ \forall j \in N,$ to that effect, recall that

$$\pi_{2j} \le \left\| \left(\mathrm{sat}_2 \left[\beta\left(\left| \tilde{e}_2 \right| \right) \right] \right) \right\|^{\frac{1}{a}} \times \max \left\{ \left(\frac{4}{\theta} \right)^{\frac{1}{a}} \mu_j ,\ \left(\frac{4 \mu_j{}^a}{\theta k_j} \left(\frac{2^{\frac{1}{a}} a}{(1+a)^{\frac{1+a}{a}}} \right) \sum_{i \in N \setminus j} k_i \mu_i \right)^{\frac{1}{1+a}} \right\}$$

Thus, to satisfy $\pi_{1j} > \pi_{2j}, \ \forall j \in N,$ we need

$$\max \left\{ \begin{array}{l} \left(\frac{4}{\theta} \frac{\left| \Delta_{1j} \right|}{k_j} \right)^{\frac{1}{a}} \mu_j , \\ \left(\frac{4 \mu_j{}^a a}{\theta k_j (1+a)^{\frac{1+a}{a}}} \right)^{\frac{1}{1+a}} \left(\sum_{i \in N \setminus j} \left(\frac{2 \left| \Delta_{1i} \right|}{k_i} \right)^{\frac{1}{a}} \mu_i \left| \Delta_{1i} \right| \right)^{\frac{1}{1+a}} \end{array} \right\}$$

$$> \left\| \left(\mathrm{sat}_2 \left[\beta\left(\left| \tilde{e}_2 \right| \right) \right] \right) \right\|^{\frac{1}{a}} \times \max \left\{ \begin{array}{l} \left(\frac{4}{\theta} \right)^{\frac{1}{a}} \mu_j , \\ \left(\frac{4 \mu_j{}^a}{\theta k_j} \left(\frac{2^{\frac{1}{a}} a}{(1+a)^{\frac{1+a}{a}}} \right) \sum_{i \in N \setminus j} k_i \mu_i \right)^{\frac{1}{1+a}} \end{array} \right\}$$

$$\Leftrightarrow \left\| \left(\mathrm{sat}_2 \left[\beta\left(\left| \tilde{e}_2 \right| \right) \right] \right) \right\|^{\frac{1}{a}} < \frac{ \max \left\{ \left(\frac{4}{\theta} \frac{\left| \Delta_{1j} \right|}{k_j} \right)^{\frac{1}{a}} \mu_j ,\ \left(\frac{4 \mu_j{}^a a}{\theta k_j (1+a)^{\frac{1+a}{a}}} \right)^{\frac{1}{1+a}} \left(\sum_{i \in N \setminus j} \left(\frac{2 \left| \Delta_{1i} \right|}{k_i} \right)^{\frac{1}{a}} \mu_i \left| \Delta_{1i} \right| \right)^{\frac{1}{1+a}} \right\} }{ \max \left\{ \left(\frac{4}{\theta} \right)^{\frac{1}{a}} \mu_j ,\ \left(\frac{4 \mu_j{}^a}{\theta k_j} \left(\frac{2^{\frac{1}{a}} a}{(1+a)^{\frac{1+a}{a}}} \right) \sum_{i \in N \setminus j} k_i \mu_i \right)^{\frac{1}{1+a}} \right\} } \qquad (6 - 16)$$

Then, we have

$$\pi_M = \max_j \{\pi_{1j}, \pi_{2j}\} = \max_j \{\pi_{1j}\}, \ \forall j \in N$$

Hence,

$$\max_{s \in \mathrm{bd}\ \Omega_{\pi M}} V_s \le \max_{s \in \mathrm{bd}\ \Omega_{\pi M}} \frac{1}{2} \overline{m} \|s\|^2$$

$$= \frac{1}{2}\overline{m}\left(\max_{s \in \mathrm{bd}\ \Omega_{\pi M}} \|s\|\right)^2 = \frac{1}{2}\overline{m}\left(\left(\sum_i^n |\pi_M|^2\right)^{\frac{1}{2}}\right)^2 = \frac{n}{2}\overline{m}|\pi_M|^2$$

$$= \frac{n}{2}\overline{m}\left(\max_j \left\{\begin{array}{l} \left(\dfrac{4}{\theta}\dfrac{|\Delta_{1j}|}{k_j}\right)^{\frac{1}{a}} \mu_j, \\ \left(\dfrac{4\mu_j{}^a a}{\theta k_j (1+a)^{\frac{1+a}{a}}}\right)^{\frac{1}{1+a}} \left(\displaystyle\sum_{i \in N \backslash j} \left(\dfrac{2|\Delta_{1i}|}{k_i}\right)^{\frac{1}{a}} \mu_i |\Delta_{1i}|\right)^{\frac{1}{1+a}} \end{array}\right\}\right)^2, \text{ for } j \in N,$$

$$\le \frac{n}{2}\overline{m}\left(\frac{1}{\underline{k}}\right)^{\frac{2}{a}}\left(\max_j \left\{\begin{array}{l} \left(\dfrac{4|\Delta_{1j}|}{\theta}\right)^{\frac{1}{a}} \mu_j, \\ \left(\dfrac{4\mu_j{}^a a}{\theta (1+a)^{\frac{1+a}{a}}}\right)^{\frac{1}{1+a}} \left(\displaystyle\sum_{i \in N \backslash j} (2|\Delta_{1i}|)^{\frac{1}{a}} \mu_i |\Delta_{1i}|\right)^{\frac{1}{1+a}} \end{array}\right\}\right)^2$$

where $\underline{k} = \min_i k_i$

Hence, to satisfy the condition $\Omega_{\pi M} \subset \Sigma_\mu$, it is sufficient for:

$$k_\mu \geq \max_{s \in \text{bd}\,\Omega_{\pi M}} V_s$$

$$\Rightarrow \frac{1}{2}\underline{m}\left(\min_i c_1\mu_i\right)^2 \geq \frac{n}{2}\overline{m}\left(\frac{1}{\underline{k}}\right)^{\frac{2}{a}}\left(\max_j\left\{\begin{array}{l}\left(\dfrac{4|\Delta_{1j}|}{\theta}\right)^{\frac{1}{a}}\mu_j, \\ \left(\dfrac{4\mu_j{}^a a}{\theta(1+a)^{\frac{1+a}{a}}}\right)^{\frac{1}{1+a}}\left(\sum_{i\in N\setminus j}\left(2|\Delta_{1i}|\right)^{\frac{1}{a}}\mu_i|\Delta_{1i}|\right)^{\frac{1}{1+a}}\end{array}\right\}\right)^2$$

$$\Leftrightarrow \underline{k} \geq \frac{\left(\max_j\left\{\left(\dfrac{4|\Delta_{1j}|}{\theta}\right)^{\frac{1}{a}}\mu_j, \left(\dfrac{4\mu_j{}^a a}{\theta(1+a)^{\frac{1+a}{a}}}\right)^{\frac{1}{1+a}}\left(\sum_{i\in N\setminus j}\left(2|\Delta_{1i}|\right)^{\frac{1}{a}}\mu_i|\Delta_{1i}|\right)^{\frac{1}{1+a}}\right\}\right)^a}{\left(\dfrac{\underline{m}}{n\overline{m}}\right)^{\frac{a}{2}}\left(\min_i c_1\mu_i\right)^a} \qquad (6 - 17)$$

It is not difficult to show that selecting K that satisfies (6 - 17) implies that (6 - 14) is sufficiently satisfied as well. Thus, for the case of $a > 0$ the control will be unsaturated in finite time and remain so thereafter.

Note that for any $0 < c_1 < 1$, there always exist a positive constant c_2 such that $0 < c_2 < (1 - c_1)$.

Thus, with $|s_i| \leq c_1\mu_i,\ \forall i \in N$, a sufficient condition for the control to be unsaturated is, for, $|\tilde{e}_{2i}| \leq c_2\mu_i$ such that,

$$\begin{aligned}|\hat{s}_i| = |s_i + \tilde{e}_{2i}| &\leq |s_i| + |\tilde{e}_{2i}| \\ &\leq c_1\,\mu_i + c_2\mu_i \\ &< \mu_i,\ \text{ since } c_1 + c_2 < 1,\end{aligned}$$

Together with the above conditions, it is sufficient for the observer gains to satisfy the following (for a given control gain K) by combining (6 - 15), (6 - 16) and $\left|\tilde{e}_{2i}\right| \le c_2\mu_i$:

$$
\max\left\{\left|\tilde{e}_{2i}\right|, \left\|\mathrm{sat}_2\left[\beta\left(\left|\tilde{e}_2\right|\right)\right]\right\|\right\}
< \min\left\{
\begin{array}{l}
\left(c_2\mu_i,\ \dfrac{\theta}{4},\ 0.5,\ \left(\dfrac{\theta}{4(n-1)}\left(\dfrac{k_j}{\lambda_{\max}(K)}\right)\left(\dfrac{\mu_j}{\lambda_{\max}(\mu)}\right)\left(\dfrac{(1+a)^{\frac{1+a}{a}}}{2^{\frac{1}{a}}a}\right)\right)^{\frac{a}{1+a}}\right)^{a}, \\
\left(\dfrac{\max\left\{\begin{array}{l}\left(\dfrac{4}{\theta}\dfrac{\left|\Delta_{1j}\right|}{k_j}\right)^{\frac{1}{a}}\mu_j, \\ \left(\dfrac{4\mu_j{}^{a}a}{\theta k_j(1+a)^{\frac{1+a}{a}}}\right)^{\frac{1}{1+a}}\left(\displaystyle\sum_{i\in N\backslash j}\left(\dfrac{2\left|\Delta_{1i}\right|}{k_i}\right)^{\frac{1}{a}}\mu_i\left|\Delta_{1i}\right|\right)^{\frac{1}{1+a}}\end{array}\right\}}{\max\left\{\left(\dfrac{4}{\theta}\right)^{\frac{1}{a}}\mu_j,\ \left(\dfrac{4\mu_j{}^{a}}{\theta k_j}\left(\dfrac{2^{\frac{1}{a}}a}{(1+a)^{\frac{1+a}{a}}}\right)\displaystyle\sum_{i\in N\backslash j}k_i\mu_i\right)^{\frac{1}{1+a}}\right\}}\right)
\end{array}
\right\}, \quad (6 - 18)
$$

and the control will be unsaturated in finite time and remain so thereafter.

3. Having shown the boundedness of $(\tilde{e}_{1i}, \tilde{e}_{2i})$, and $\hat{s}$ being unsaturated, we are going to show the boundedness property of the desired error dynamics (6 - 4).

From step 2, we have $\left|s_i\right| < c_1\mu_i$, $\left|\tilde{e}_{2i}\right| \le c_2\mu_i \quad \forall i \in N$, for $a > 0$, and

$\left|s_i\right| + \left|\tilde{e}_{2i}\right| \le \mu_i$, $\forall i \in N$, for $a = 0$, then, in this region, the following is equivalent:

$$k_{ai}\mu_i \mathrm{sat}\left[\frac{(s_i+\tilde{e}_{2i})}{\mu_i}\right] = k_{ai}(s_i+\tilde{e}_{2i}) = k_{ai}s_i + k_{ai}\mathrm{sat}_{\mu_i}[\tilde{e}_{2i}], \text{ and}$$

$$v_{2i}+s_i+\tilde{e}_{2i} = v_{2i}+s_i+\mathrm{sat}_{\mu_i}[\tilde{e}_{2i}] \text{ , since } |\tilde{e}_{2i}|<\mu_i,$$

For the case of $b > 0$:

Then, the desired error dynamics can be written element-wise as (6 - 6), $\forall\, i = 1,...,n$,

$$\begin{aligned}
\dot{v}_{1i} &= v_{2i}+s_i \\
\dot{v}_{2i} &= -k_{ai}\left(v_{2i}+s_i+\mathrm{sat}_{\mu_i}[\tilde{e}_{2i}]\right) + k_{ai}\mathrm{sat}_{\left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+b}{2b}}}\left[v_{2i}+s_i+\mathrm{sat}_{\mu_i}[\tilde{e}_{2i}]\right] \\
&\quad -\mathrm{sat}_{\varepsilon_{2i}}\left[k_{2i}\left|v_{2i}+s_i+\mathrm{sat}_{\mu_i}[\tilde{e}_{2i}]\right|^{\frac{2b}{1+b}}\mathrm{sign}\left(v_{2i}+s_i+\mathrm{sat}_{\mu_i}[\tilde{e}_{2i}]\right)\right] \\
&\quad -\mathrm{sat}_{\varepsilon_{1i}}\left[k_{1i}|v_{1i}|^b\,\mathrm{sign}(v_{1i})\right]
\end{aligned} \tag{6 - 19}$$

From Appendix B.1, with the Lyapunov function,

$$\begin{aligned}
V &= \left(\frac{1}{2}{v_{2i}}^2 + \int_0^{v_{1i}}\mathrm{sat}_{\varepsilon_{1i}}\left[k_{1i}|r|^b\,\mathrm{sign}(r)\right]dr\right)^2 + r_1\left(\frac{3+3b}{2}\right)\left(\int_0^{v_{1i}}|r|^{\frac{1}{2}}\mathrm{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3}{2}}}[|r|^{\frac{3b}{2}}]\;dr\right)v_{2i} \\
&\quad + r_1(1+b)k_{ai}\left(\frac{\varepsilon_{1\mathrm{i}}}{k_{1i}}\right)^{\frac{3}{2}}\left(\int_0^{v_{1i}}|r|^{\frac{3}{2}}\mathrm{sign}(r) - \mathrm{sat}_{\left(\frac{\varepsilon_{1\mathrm{i}}}{k_{1i}}\right)^{\frac{3}{2b}}}\left[|r|^{\frac{3}{2}}\mathrm{sign}(r)\right]dr\right)
\end{aligned}$$

where $r_1 > 0$ is a constant, it is shown that

$$\dot{\tilde{V}} < 0, \quad \text{for} \quad V \geq \rho_{1i}(|\zeta_i|),$$

where $|\zeta_i|$ is defined as $|\zeta_i| = \max\{|s_i|, \mathrm{sat}_{\mu_i}[|\tilde{e}_{2i}|]\}$, and the class $\mathcal{K}$ function, ρ_{1i} is defined as, $\rho_{1i}(|\zeta_i|) = \max\limits_{(v_{1i},v_{2i})\in \mathrm{bd}\;\Psi_3(|\zeta_i|)} V$

with bd $\Psi_{3i}(|\zeta_i|)$ as the boundary of the compact set

$\Psi_{3i}(|\zeta_i|) = \{(v_{1i}, v_{2i}) \in \mathbb{R}^2 : |v_{1i}| \le \alpha_1(|\zeta_i|),\ |v_{2i}| \le \alpha_2(|\zeta_i|)\}$, and

α_1, α_2 are class $\mathcal{K}$ functions defined in Appendix B.1. Hence, invoking lemma 2.6, the states (v_1, v_2) are uniformly ultimately bounded.

For the special case of $b = 0$:

Let the elements of ε_1 equal to that of K_1, and ε_2 to that of K_2, i.e. $\varepsilon_{1n} = k_{1n}, \forall i \in N$ and $\varepsilon_{2n} = k_{2n}, \forall i \in N,$ hence the desired dynamics becomes

$$\begin{aligned}
\dot{v}_{1i} &= v_{2i} + s_i \\
\dot{v}_{2i} &= -k_{ai}\left(v_{2i} + s_i + \mathrm{sat}_{\mu_i}[\tilde{e}_{2i}]\right) + k_{ai}\mathrm{sat}\left[v_{2i} + s_i + \mathrm{sat}_{\mu_i}[\tilde{e}_{2i}]\right] \\
&\quad - k_{2i}\mathrm{sign}\left(v_{2i} + s_i + \mathrm{sat}_{\mu_i}[\tilde{e}_{2i}]\right) - k_{1i}\mathrm{sign}(v_{1i})
\end{aligned} \tag{6 - 20}$$

From Appendix B.1, with the Lyapunov function

$$\begin{aligned}
V &= \left(\frac{1}{2}v_{2i}{}^2 + k_{1i}\int_0^{v_{1i}} \mathrm{sign}(r)\, dr\right)^2 + r_1|v_{1i}|^{\frac{3}{2}}\mathrm{sign}(v_{1i})\, v_{2i} + r_1\left(\frac{2}{5}\right)k_{ai}|v_{1i}|^{\frac{5}{2}} \\
&\quad - r_1 k_{ai}\int_0^{v_{1i}}\left[\mathrm{sat}_1[|r|^{\frac{3}{2}}]\right]\mathrm{sign}(r)\, dr
\end{aligned}$$

where $r_1 > 0$ is a constant, it is shown that

$$\dot{\tilde{V}} < 0, \ \text{for } V \ge \rho_{2i}(|\zeta_i|),$$

where $|\zeta_i|$ is defined as $|\zeta_i| = \max\{|s_i|, \mathrm{sat}_{\mu_i}[|\tilde{e}_{2i}|]\}$, and the class $\mathcal{K}$ function, ρ_{2i} is defined as $\rho_{2i}(|\zeta_i|) = \max_{(v_{1i}, v_{2i}) \in \text{bd } \phi_{3i}(|\zeta_i|)} V$, with bd $\phi_{3i}(|\zeta_i|)$ as the boundary of the compact set $\phi_{3i}(|s_i|) = \{(v_{1i}, v_{2i}) \in \mathbb{R}^2 : |v_{1i}| \le \alpha_{11}(|s_i|), |v_{2i}| \le \alpha_{12}(|s_i|)\}$

and α_{11}, α_{12} are class $\mathcal{K}$ functions defined in Appendix B.1. Hence, invoking lemma 2.6, the states (v_1, v_2) are uniformly ultimately bounded. Hence, invoking lemma 2.6, the states (v_1, v_2) are uniformly ultimately bounded.

■

Remark 6.5. Hence, for K satisfying (6 - 17), and observer gain L_3 sufficiently high such that (6 - 18) is satisfied through (6 - 10), the trajectories of the closed-loop system (6 - 6), are globally stable, with the ultimate bound being reduced as desired up to the saturation limit of the actuator, $T_{i,\max} > 0$, $\forall i = 1,...,n$, and correspondingly that of the observer L_3. Another way to view the stability results is for a given bounds on the actuator limit, and hence K, if the bounds on the uncertainties,

$$\Delta_1(\cdot) = -G(q) - M(q)\ddot{q}_d + M(q)\dot{\sigma} - \boldsymbol{K}[D] - F(\dot{q}_d - \sigma) - C(q, (\dot{q}_d - \sigma))(\dot{q}_d - \sigma)$$

are sufficiently small to satisfy (6 - 17), and observer gain L_3 sufficiently large for (6 - 18) the global practical stability results are still assured. Essentially, besides the bounds on the parameters of the robot manipulator dynamics and disturbances, the term Δ_1 is also dependent on the desired acceleration, velocity and the desired error dynamics through the term σ and $\dot{\sigma}$. In which case, a slower desired trajectory or slower desired error dynamics can in effect produce a smaller upper bound of $|\Delta_1|$.

Remark 6.6. It is desired for the controller (6 - 3) to behave as per its non-bounded counterpart (Chapter 4) when the controller is not saturated so that it exhibits the same properties such as the ability to inject desired error dynamics, and desired performance. Indeed from stability proof above (step 2), the control (6 - 3) will be unsaturated in finite time and stay so in all future times. Particularly, if the upper bound of $|\zeta_i|$ is sufficiently small (by selecting gain K satisfying (6 - 17) up to the allowable control bound, $T_{i,\max}$ and observer gain L_3 satisfying (6 - 18) through (6 - 10)) , from step 3 above, the states (v_1, v_2) of the dynamics (6 - 19) and (6 - 20), will become unsaturated as well (see Appendix B.1),

for $b > 0$,

$$
\begin{aligned}
\dot{v}_{1i} &= v_{2i} + s_i \\
\dot{v}_{2i} &= -k_{1i}|v_{1i}|^{b}\operatorname{sign}(v_{1i}) - k_{2i}\left|v_{2i} + s_i + \operatorname{sat}_{\mu_i}[\tilde{e}_{2i}]\right|^{\frac{2b}{1+b}}\operatorname{sign}\left(v_{2i} + s_i + \operatorname{sat}_{\mu_i}[\tilde{e}_{2i}]\right)
\end{aligned}
$$

for $b = 0$,

$$
\begin{aligned}
\dot{v}_{1i} &= v_{2i} + s_i \\
\dot{v}_{2i} &= -k_{2i}\operatorname{sign}\left(v_{2i} + s_i + \operatorname{sat}_{\mu_i}[\tilde{e}_{2i}]\right) - k_{1i}\operatorname{sign}(v_{1i})
\end{aligned}
$$

which is similar to the z-subsystem (4 - 7) considered in Chapter 4, and hence will exhibit the same properties, since the states will remain unsaturated for all future times, provided $|\zeta_i|$ is sufficiently small.

Remark 6.7. Since the same observer structure of that in Chapter 4 is used here, for the special case of p = 0.5, if the gains of the observer, L_1 and L_2 are sufficiently large to satisfy conditions of Theorem 4.2, the observation error $(\tilde{e}_{1i}, \tilde{e}_{2i})$ will converge to zero in finite time and stay there for all future times (refer to step 1-c of proof of Theorem 6.1). In that case, the closed-loop system (6 - 6) will be identical to the bounded full-state feedback control of (5 - 4) considered in the previous chapter, Chapter 5.

Remark 6.8. Since the control (6 - 3) follows a similar bounded approach as Chapter 5 , it has some similar inherent properties as that of (5 - 2) such as the use of integral of saturation functions that give rise to the boundedness of σ (see step 1-a of the proof of Theorem 6.1), the need to satisfy inequality $\underline{f} > C_m \|(\dot{q}_d - \sigma)\|$ (see step 1-b of the proof of Theorem 6.1) by having sufficiently slow desired motions, and the anti-windup structure (recall from (6 - 4), by adding and subtracting $K_a\,\hat{s}$),

$$\dot{\sigma} = -K_a\sigma + K_a\hat{s} - \underbrace{K_a\left(\hat{s} - \mu\,\mathrm{sat}\left[\mu^{-1}\hat{s}\right]\right)}_{anti-windup} - K_a\,\mathrm{sat}_{\left(\frac{\varepsilon_2}{k_2}\right)^{\frac{1+b}{2b}}}\left[\hat{e}_2\right] + \mathrm{sat}_{\varepsilon_2}\left[K_2\mathrm{sig}(\hat{e}_2)^{\frac{2b}{1+b}}\right] + \mathrm{sat}_{\varepsilon_1}\left[K_1\mathrm{sig}(e_1)^b\right].$$

6.3 Numerical Simulations

In this section, numerical simulations on a two-link robot manipulator were carried out to illustrate the results discussed in this chapter. The setups for each simulation are described. Discussion and analysis of the results are presented accordingly.

6.3.1 Simulation Setups

1) Simulation 1:

The same two-link rigid robot manipulator considered in section 3.3 is adopted in simulation. The dynamics of robot manipulator (6 - 2) have the same parameter values as that in section 3.3. The desired trajectory vector and the additive disturbances Coulomb friction vector were defined similarly as well. The initial conditions of the robot manipulator were selected as,

$$q(0)=\begin{bmatrix}0\\0\end{bmatrix},\ \dot{q}(0)=\begin{bmatrix}10\\16\end{bmatrix},\ q_d(0)=\begin{bmatrix}\pi\\0.5\end{bmatrix},\ \dot{q}_d(0)=\begin{bmatrix}0.5\\0\end{bmatrix}$$

The control (6 - 3) parameter values were selected as follows,

$$a=0.9,\ K=\begin{bmatrix}150 & 0\\0 & 15\end{bmatrix},\ K_1=\begin{bmatrix}18 & 0\\0 & 18\end{bmatrix},\ K_2=\begin{bmatrix}11 & 0\\0 & 11\end{bmatrix}, b=0.6$$
$$\mu=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix},\ \varepsilon_{11}=\varepsilon_{12}=18,\ \varepsilon_{21}=\varepsilon_{22}=11,\ K_a=\begin{bmatrix}10 & 0\\0 & 10\end{bmatrix},$$

The initial conditions for the vector σ were selected as

$$\sigma(0)=\begin{bmatrix}0\\0\end{bmatrix}$$

The observer (6 - 5) parameter values were selected as,

$$L_1=L_2=\begin{bmatrix}2 & 0\\0 & 2\end{bmatrix}, L_3=\begin{bmatrix}150 & 0\\0 & 150\end{bmatrix},\ p=0.8,$$

with the following initial conditions,

$$\hat{e}_1(0) = e_1(0), w(0) = -L_3 e_1(0)$$

since position measurement is available.

2) Simulation 2:

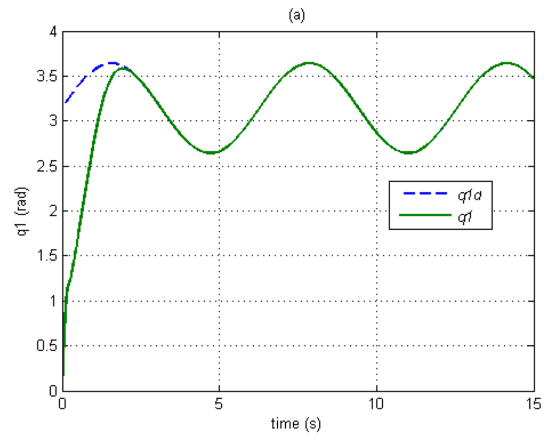
The setup of Simulation 2 is exactly the same as that of Simulation 1. The simulation is repeated using the unbounded output-state feedback control (4 - 4), to examine its maximal tracking errors compared to that of control (6 - 3), with the parameter values selected as

$$a = 0.9,\ K = \begin{bmatrix} 150 & 0 \\ 0 & 15 \end{bmatrix},\ K_1 = \begin{bmatrix} 18 & 0 \\ 0 & 18 \end{bmatrix},\ K_2 = \begin{bmatrix} 11 & 0 \\ 0 & 11 \end{bmatrix}, b = 0.6$$

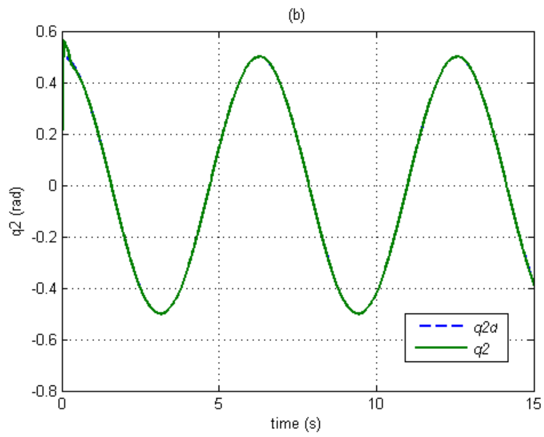
Note that when the control (6 - 3) vectors of Simulation 1 are unsaturated, it has the same parameter values as those employed in Simulation 2.

6.3.2 Results and Discussions

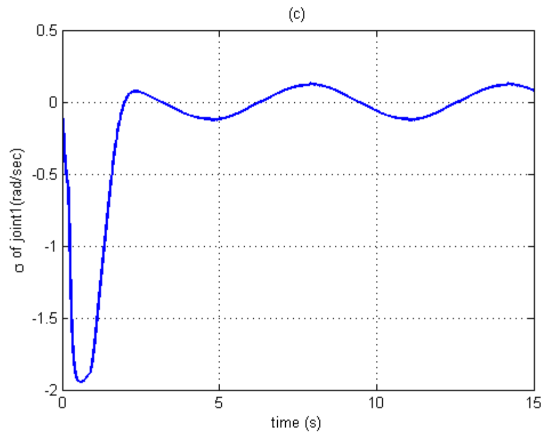
For better visualization of the plots, some figures are shown in two windows; each with different time intervals.



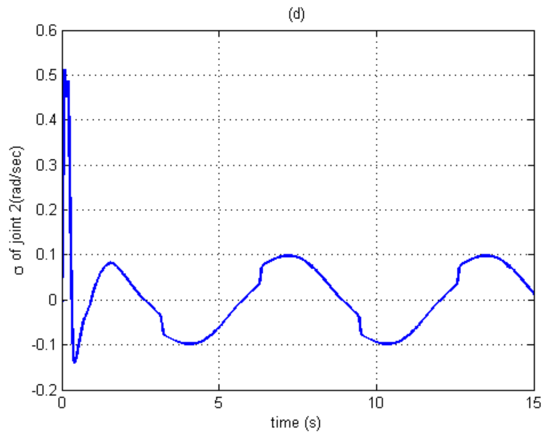
(a) Trajectory, q_1 for time t = [0, 15] s.



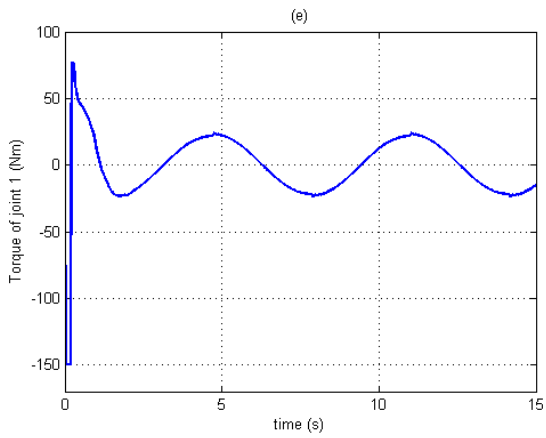
(b) Trajectory, q_2 for time t = [0, 15] s.



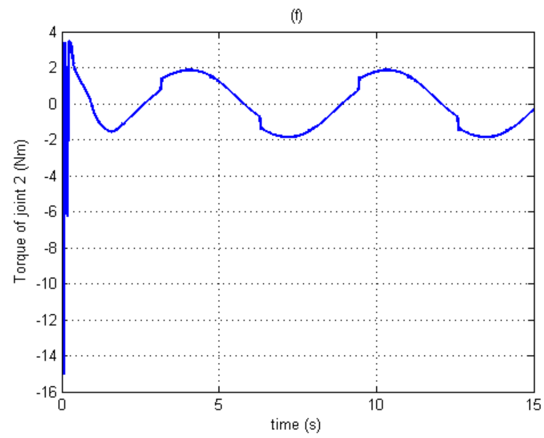
(c) Sigma, σ_1 for time t = [0, 15] s.



(c) Sigma, σ_2 for time t = [0, 15] s.

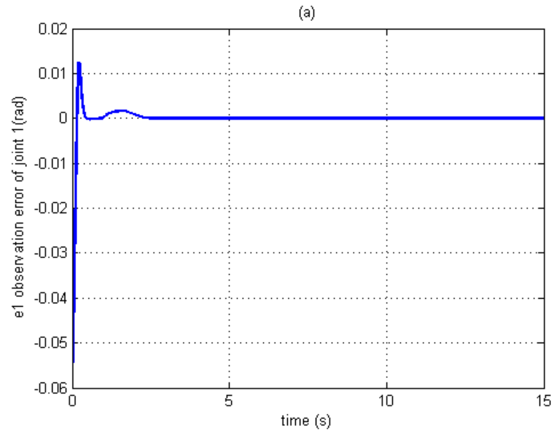


(e) Torque, τ_1 for time t = [0, 15] s.

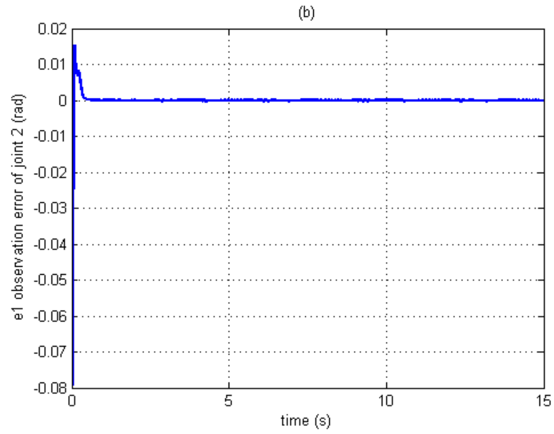


(f) Torque, τ_2 for time t = [0, 15] s.

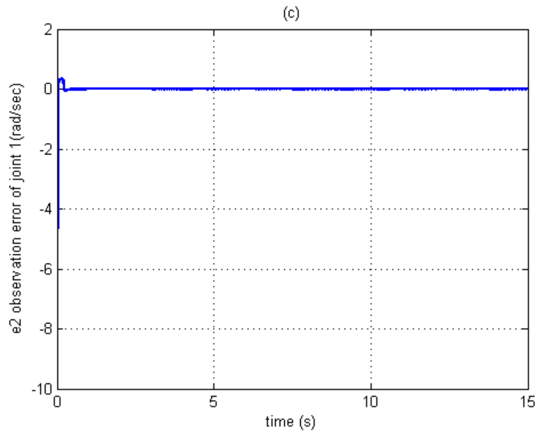
Figure 6.1 Simulation 1. Tracking errors, sigma and control input of joint 1 and joint 2.



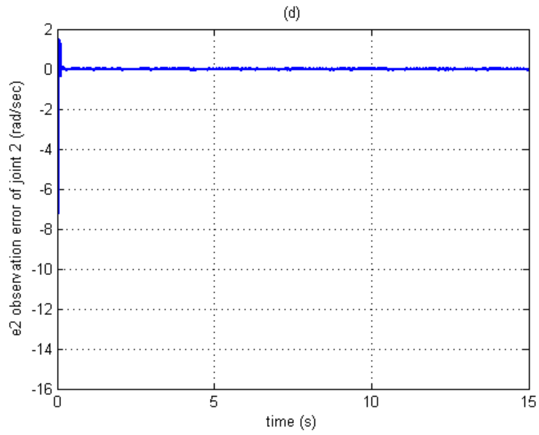
(a) Position observation error, $\tilde{e}_{11}$ for time t = [0, 15] s.



(b) Position observation error, $\tilde{e}_{12}$ for time t = [0, 15] s.

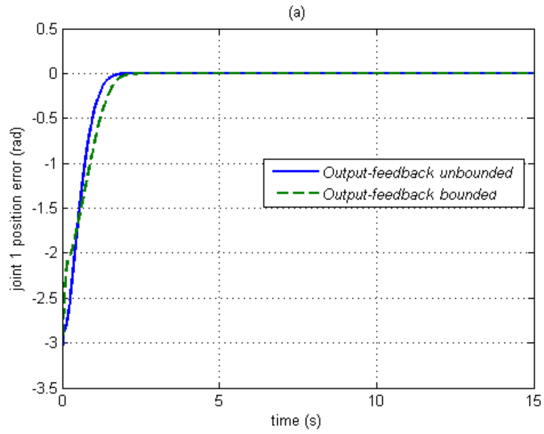


(c) Velocity observation error, $\tilde{e}_{21}$ for time t = [0, 15] s.

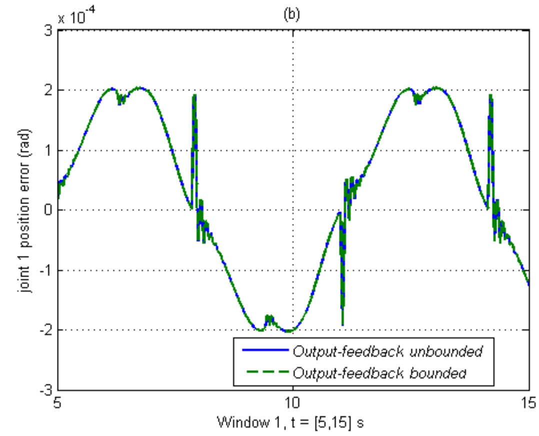


(d) Velocity observation error, $\tilde{e}_{22}$ for time t = [0, 15] s.

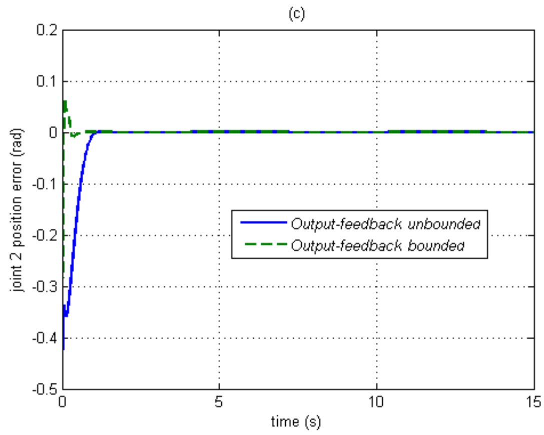
Figure 6.2 Simulation 1. Observation errors for joint 1 and joint 2.



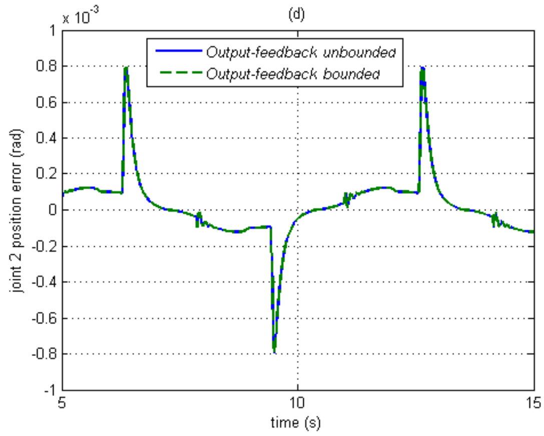
(a) Tracking error, e_{11} for time t = [0, 15] s.



(b) Tracking error, e_{11} for time t = [5, 15] s.



(c) Tracking error, e_{12} for time t = [0, 15] s.



(d) Tracking error, e_{12} for time t = [5, 15] s.

Figure 6.3 Simulation 2. Tracking errors using control (4 - 4), control (6 - 3).

1) Simulation 1:

As can be observed Figure 6.1(e)-(f), the large initial conditions caused the control (6 - 3) to saturate. From Theorem 6.1, the state σ will converge to a bounded region in finite time, due to the saturated nature of the integral (see (6 - 4)), which is clearly shown in Figure 6.1(c)-(d). According to the stability proof, once the state σ converge to a bounded region and stay in it for all future times, with control gain K satisfying (6 - 14), the control will be unsaturated in finite time and stay in it for all future times, provided the observation errors $\tilde{e}_2$ are sufficiently small, which in turn are governed by the observer linear damping gain L_3. In fact, from Figure 6.1(e)-(f) the control inputs for both joints of the robot manipulator remained unsaturated after the initial saturation. The performance of the observer, as shown in Figure 6.2, is very good in terms of estimation speed and residual error. This is due to the synergistic combination of the super-twisting based algorithm and a linear damping term.

2) Simulation 2:

Figure 6.3 shows that both controllers (4 - 4) and (6 - 3) have similar ultimate bound on the tracking errors, which is expected since the unsaturated (6 - 3) is identical to (4 - 4). Similar to the argument of the bounded control of Chapter 5, the vital point of the control (6 - 3) is for it to become unsaturated and remain so for all future times. The controller (6 - 3) is structurally identical to that of Chapter 5 (5 - 2), hence the design of the saturated integral (6 - 4) has the same properties as that in Chapter 5. The only difference is that the velocity measurement is considered not available in this chapter, hence the use of the observer introduced in Chapter 4. From the stability proof of Theorem 6.1, the stability of this control (6 - 3) boils down to the boundedness of the velocity observation errors, $\tilde{e}_{2;}$ i.e. if $\tilde{e}_2$ is sufficiently small, the bounded output-feedback control will be unsaturated and remain so for all future times.

Hence, by combining the observer of Chapter 4 together with the integral of saturated functions designed in Chapter 5, a bounded output-feedback control (6 - 3) that inherits the properties of it unbounded and full-state feedback controller (3 - 2) is shown.

6.4 Summary

In this chapter, globally stabilizing saturated controller for the trajectory tracking of robot manipulators without velocity measurements with additive bounded disturbances were proposed. The velocity is being observed through a super-twisting based plus linear damping observer. Strict Lyapunov functions developed in the previous chapter 2 are being modified to accommodate the bounded nature of the desired error dynamics. Essentially, after the controller forces the states to have unsaturated control, similar properties of their unbounded counterpart can be obtained.

Chapter 7: CONCLUSION AND FUTURE WORK

7.1 Dissertation Summary

The focus of this research is to develop strict Lyapunov functions for algorithms that are based on twisting and super-twisting algorithms and applying them to the control of robot manipulators with real world problem considerations including output feedback without velocity measurements and actuator saturation. Due to the non-Lipschitz nature of the algorithms and the type of disturbances are allowed to exhibit discontinuity, nonsmooth Lyapunov theorem is employed throughout the work to obtain important results from the proposed strict Lyapunov functions such as settling time estimate for the finite-time convergence and robustness to non-Lipschitz disturbances. Real world systems are always affected by nonlinear behaviours that are often not considered; i.e. such as saturation in control channels, lack of velocity measurements, Coulomb friction, the work in this dissertation aims to compensate for these phenomena with practical control designs that can be implemented easily.

Family of algorithms based on second order sliding mode algorithm, namely that of twisting and super-twisting, are introduced in Chapter 2. Due to the non-Lipschitzness of the system, using the generalized solutions in the sense of Filippov, strict Lyapunov functions proposed are analysed using the generalized Lyapunov theorem. In particular, the strict Lyapunov function can fully characterize different stability properties of a parameterized family of controllers. Effectively, the family of controllers generalized the proportional-derivative (PD) control and twisting algorithm, and that of proportional-integral (PI) control and super-twisting algorithm. At the same time the strict Lyapunov functions proposed can similarly generalized the type of stability of these systems, from finite-time convergence to exponential convergence to uniform asymptotic convergence. Leveraging on this results, algorithms that combines the family of controllers with different degree of homogeneity are develop, and the corresponding strict Lyapunov functions are similarly develop by combining the one from the individual family, and the stability properties of each algorithm remains in the new combination. In essence, it is possible to achieve finite-time with uniform convergence, which in effect produces finite time convergence with fixed- settling time that is independent of initial conditions of the system.

Chapter 3 focuses on the trajectory tracking control of robot manipulator. Semiglobal practical stability is assured where the ultimate bound of the states can be made arbitrarily small and the region of attraction arbitrarily large by tuning a single parameter. Due to the generalized Lyapunov theorem and the proposed Lyapunov function, the stability analysis permits the disturbances to have discontinuity, such as Coulomb friction. Of interest is the ability of the proposed controller in generalizing the well-known PID control. In particular, for this special case, the PID gains selection is transform into the selection of desired error dynamics and the selection of acceptable precision of error. This gain tuning simplification to gains selection is important, as it allows the nonlinear robust control of desired error dynamics injection and disturbance compensation into an existing PID control, which is of great benefit since PID control is widely used in industrial robot manipulator. In addition, for the special case of position regulation problem, sufficient conditions on the gains are obtained to ensure either finite-time or exponential convergence of the system towards the regulation point. In addition, due to integral nature of the controller, it is possible for the system to behave as per the desired error dynamics from the onset of control even in the presence of disturbances.

Trajectory tracking control of robot manipulator without velocity measurement is tackle in Chapter 4. The velocity measurement is substituted with the output of an observer that combines the super-twisting based algorithm develop in Chapter 2 with a damping term that is termed in the literature as "dirty-derivative". The addition of the linear damping term has the benefit of reducing the gain required of the non-Lipschitz part of the observer that is responsible for finite-time convergence. At the same time it allows the definition of the region of attraction to grow with the linear damping term, in which case is not possible if the term is not added and peaking phenomenon will occur if the initial observation error is not small enough. With the addition of this observer structure, the controller can maintain its useful feature from Chapter 3, where effectively, the controller here is an output feedback version of that proposed in Chapter 3.

The problem of saturated control is developed in Chapter 5 for the trajectory tracking or robot manipulators under the influence of bounded disturbances. By taking into account of the viscous friction and the proposed nonlinear integrator that injects a bounded desired error dynamics, global practical stability is achieved. Also, when the saturation level is sufficiently high for the user-defined speed of desired trajectory, the finite-time and exponential regulation of the unbounded control in Chapter 3 is recovered but in a global manner instead of semiglobal.

In the final chapter, the control robot manipulator is assumed to have both constraints; no velocity measurements available and the control is bounded. For the velocity observation, the observer proposed in Chapter 4 is employed while the framework of bounded control in Chapter 5 is applied. As a result, a globally stabilizing saturated controller for the trajectory tracking of robot manipulators without velocity measurements under the influence of additive bounded disturbances is developed. Essentially, after the controller forces the states to have unsaturated control, similar properties of its unbounded counterpart in Chapter 4, can be obtained.

7.2 Limitations and Future Work

The work in this dissertation complements the SOSM algorithms and control designs for the trajectory tracking of robot manipulator. At the same time, it reveals new information on existing nonlinear systems. Hence in this section, open problems related to the research in this dissertation are presented.

In Chapter 2, the strict Lyapunov proposed are for the algorithms with constant gains. In particular, the mechanical energy of the system is utilized as part of the strict Lyapunov function. However, in doing so, if the gains of the system is allowed to be time varying, the construction of Lyapunov function is not as straightforward. An example of a strict Lyapunov function for super-twisting algorithm can be found in [146] where the structure of the Lyapunov functions dictate the type and form of the variable gains, which is restrictive. Hence, it is of interest if strict Lyapunov function can be developed that allows a full range of variable gains. As the task of finding strict Lyapunov function is not straightforward especially for higher order sliding mode algorithms, it is hoped that the work in this dissertations help sheds light on through employing the inherent structure of the algorithms into the Lyapunov function.

For chapter 3, the stability result achieved is semiglobal which implies that the gains of the control are selected based on the initial conditions of the closed-loop system. If the gains are allowed to vary in accordance to the initial conditions by means of an adaptive algorithm, for instance, global stability may be attained. Particularly, given the strictness of the Lyapunov function proposed, utilizing certainty equivalence techniques similar to [147], an adaptive version of the controller may be possibly developed.

An output feedback version of the controller in Chapter 3 is presented in Chapter 4, by utilizing a super-twisting based algorithm plus a linear damping term observer. The stability analysis here is also dependent on satisfying sufficient conditions pertaining to the initial conditions of the closed-loop system. Hence, similar arguments as in Chapter 3 applies here, in which case, an adaptive version of the output feedback controller seems plausible through the strict Lyapunov functions of both the observer and controller produced here.

From Chapter 5, global results are achieved for the bounded controller by taking into account of the damping effect of the viscous friction and a bounded desired error dynamics in counteracting the effect of Coriolis and centrifugal terms of the dynamics. However, as a result, the type of desired trajectory, in particular the speed and acceleration of the desired trajectory is affected in order to satisfy the sufficient conditions for stability. Although, it is understood that, given a limit on the actuation, there is a bound on the speed at which the system can operate, it is useful if the limit on the desired trajectory be variable instead of a constant bound, depending on the configuration of the robot manipulator, so that the bandwidth of the control can be utilised optimally.

Chapter 6 combines the results of the observer based control in Chapter 4 and the bounded control framework of Chapter 5 to produce a global practical stability result on trajectory tracking with the real world considerations of actuator limits and the lack of velocity measurements from the system. All controllers proposed from Chapter 3 to Chapter 6 for robot manipulator can achieved finite time convergence for the regulation problems by satisfying sufficient conditions, including the need for the gravity vector, $G(q)$ to be zero at the point of regulation. However, such a need is not necessary for the exponential convergence for the regulation problem. This is mainly due to the non-Lipschitzness of the control, for finite-time convergence, occur at the origin, when a change of variable is applied to the desired gravity point, the new equilibrium point is locally Lipschitz, hence the control cannot render the convergence in finite time unless the desired gravity exactly compensated or zero.

Appendix A

A.1 PROOF OF DESIRED DYNAMICS (CH3 & 4)

Consider the following dynamics:

$$\begin{aligned}\dot{z}_1 &= z_2 + d_1, \\ \dot{z}_2 &= -k_1 \operatorname{sig}(z_1)^b - k_2 \operatorname{sig}(z_2)^{\frac{2b}{1+b}} + \delta\end{aligned} \qquad \text{(A1 - 1)}$$

where z_1, $z_2 \in \mathbb{R}$, are the scalar state variables, k_1, k_2 are positive constants, $b \geq 0$ real number with $\delta = k_2 |z_2|^{\frac{2b}{1+b}} \operatorname{sign}(z_2) - k_2 |z_2 + d_2|^{\frac{2b}{1+b}} \operatorname{sign}(z_2 + d_2)$ and d_1, $d_2 \in \mathbb{R}$ are bounded disturbances:

$|d_1| \coloneqq \sup\{|\varepsilon_1| : \varepsilon_1 \in K[\, d_1 \,]\}$, $|d_2| \coloneqq \sup\{|\varepsilon_2| : \varepsilon_2 \in K[\, d_2 \,]\}$ and $|\delta| \coloneqq \sup\{|\varepsilon_3| : \varepsilon_3 \in K[\, \delta]\}$(Note that no continuity assumption is made so that discontinuous disturbances may be used in d) .

The results in this section are applicable to the desired dynamics section of both chapter 3 & 4. In particular, by applying the following change of variable:

For chapter 3:

$z_1 = z_{1i}$, $z_2 = z_{2i}, d_1 = d_2 = s_i$,

For chapter 4:

$z_1 = z_{1i}$, $z_2 = z_{2i}, d_1 = d_2 = s_i$,

the same differential equations are obtained.

Next, we take the Lyapunov function candidate of the form

$$V(z_1,z_2)=\frac{k_1^{\,2}}{(1+b)^2}|z_1|^{2+2b}+\frac{1}{4}|z_2|^4+r|z_1|^{\frac{3+3b}{2}}\operatorname{sign}(z_1)z_2+\frac{k_1}{(1+b)}|z_1|^{1+b}|z_2|^2$$

where $r > 0$ is a constant. Note that V is positive definite (with $r_i > 0$ exist, see Chapter 1).

$$\underline{\pi}_1\left(|z_1|^{2+2b}|z_2|^4\right)\le V(z_1,z_2)\le \overline{\pi}_1\left(|z_1|^{2+2b}+|z_2|^4\right)$$

where $\underline{\pi}_1 := \min\left\{\frac{1}{8},\ \frac{1}{2}\left(\frac{k_1}{1+b}\right)^2\right\}$, $\overline{\pi}_1 := \max\left\{\frac{3r}{4}+\left(\frac{k_1}{1+b}\right)^2+\left(\frac{k_1}{2+2b}\right),\quad \left(\frac{k_1}{2+2b}\right)+\frac{r}{4}+\frac{1}{4}\right\}$,

In accordance to lemma 2.4, taking the time derivative of the Lyapunov function along the solutions of the system (A1 - 1) exists almost everywhere:

$$\frac{d}{dt}V(\mathbf{z}(t),t)\overset{a.e.}{\in}\dot{\tilde{V}}(\mathbf{z}(t),t)=\bigcap_{\xi\in\partial V(\mathbf{z}(t),t)}\xi^T K\begin{bmatrix}\dot{z}_1\\ \dot{z}_2\end{bmatrix}(\mathbf{z},t)$$

For ease of presentation, the analysis is separated for the case of $b > 0$ and $b = 0$.

<u>For the case of $b > 0$:</u>

Note that for $b > 0$, V is continuously differentiable:

$$\dot{\tilde{V}}(z_1,z_2)=\bigcap_{\xi\in\partial V(\mathbf{z}(t),t)}\xi^T K\begin{bmatrix}\dot{z}_1\\ \dot{z}_2\end{bmatrix}=\nabla V^T K\begin{bmatrix}\dot{z}_1\\ \dot{z}_2\end{bmatrix}$$

$$\subset\begin{pmatrix} -\frac{2k_1k_2}{(1+b)}|z_1|^{1+b}|z_2|^{\frac{1+3b}{1+b}}-k_2|z_2|^{\frac{3+5b}{1+b}}-rk_1|z_1|^{\frac{3+5b}{2}}+r\left(\frac{3+3b}{2}\right)|z_1|^{\frac{1+3b}{2}}|z_2|^2 \\ -rk_2|z_1|^{\frac{3+3b}{2}}|z_2|^{\frac{2b}{1+b}}\operatorname{sign}(z_1)\operatorname{sign}(z_2) \end{pmatrix}$$

$$+K[d_1]\left[\frac{2k_1^{\,2}}{(1+b)}|z_1|^{1+2b}\operatorname{sign}(z_1)+r\left(\frac{3+3b}{2}\right)|z_1|^{\frac{1+3b}{2}}z_2+k_1|z_1|^b|z_2|^2\operatorname{sign}(z_1)\right]$$

$$+K[\delta]\left[|z_2|^3 sign(z_2)+r|z_1|^{\frac{3+3b}{2}}\operatorname{sign}(z_1)+\frac{2k_1}{(1+b)}|z_1|^{1+b}|z_2|\operatorname{sign}(z_2)\right]$$

Also, from the results of the planar system (Appendix A.2-proposition 1):

$$|\delta|\le a_1|d_2|^{\frac{2b}{1+b}}+a_2|z_2|^{\frac{2b}{1+b}} \quad \text{with} \quad 0<a_2<k_2 \text{ and}$$

$$a_1=k_2\left(\frac{a_2+2k_2}{\left((a_2+k_2)^{\frac{1+b}{2b}}-k_2^{\frac{1+b}{2b}}\right)^{\frac{2b}{1+b}}}\right),$$

Then, after rearrangement:

$$\dot{\tilde{V}}(z_1,z_2)\le-\frac{2k_1}{(1+b)}(k_2-a_2)|z_1|^{1+b}|z_2|^{\frac{1+3b}{1+b}}-\frac{rk_1}{2}|z_1|^{\frac{3+5b}{2}}-\frac{(k_2-a_2)}{2}|z_2|^{\frac{3+5b}{1+b}}+\dot{V}_1$$
$$+|d_1|\left(\frac{2k_1^{\,2}}{(1+b)}|z_1|^{1+2b}+r\left(\frac{3+3b}{2}\right)|z_1|^{\frac{1+3b}{2}}|z_2|+k_1|z_1|^b|z_2|^2\right)$$
$$+a_1|d_2|^{\frac{2b}{1+b}}\left(|z_2|^3+r|z_1|^{\frac{3+3b}{2}}+\frac{2k_1}{(1+b)}|z_1|^{1+b}|z_2|\right)$$

where

$$\dot{V}_1=-\frac{rk_1}{2}|z_1|^{\frac{3+5b}{2}}-\frac{(k_2-a_2)}{2}|z_2|^{\frac{3+5b}{1+b}}+r\left(\frac{3+3b}{2}\right)|z_1|^{\frac{1+3b}{2}}|z_2|^2+r(k_2+a_2)|z_1|^{\frac{3+3b}{2}}|z_2|^{\frac{2b}{1+b}}$$

Note that from Section 2.2 (twisting algorithm): if the following inequalities

$$\min\left\{\lambda_1 \frac{k_1^{\frac{3+3b}{2b}}(k_2-a_2)}{(k_2+a_2)^{\frac{3+5b}{2b}}},\ \lambda_2 k_1^{\frac{1+3b}{2+2b}}(k_2-a_2)\right\} > r > 0$$

with $$\lambda_1 = \left(\frac{3+5b}{24+24b}\right)^{\frac{3+3b}{2b}}\left(\frac{3+5b}{16b}\right), \quad \lambda_2 = \left(\frac{3+5b}{8+24b}\right)^{\frac{1+3b}{2+2b}}\left(\frac{3+5b}{16+16b}\right)\left(\frac{2}{3+3b}\right)^{\frac{3+5b}{2+2b}},$$

hold, then the function $\dot{V}_1$ is negative definite. Such an $r > 0$ always exists for any $k_1 > 0$ and $k_2 > a_2 > 0$. Then,

$$\begin{aligned}\dot{\tilde{V}}(z_1,z_2) \le & -\frac{rk_1}{2}|z_1|^{\frac{3+5b}{2}} - \frac{(k_2-a_2)}{2}|z_2|^{\frac{3+5b}{1+b}} \\ & + |d_1|\left(\frac{2k_1^{\,2}}{(1+b)}|z_1|^{1+2b} + r\left(\frac{3+3b}{2}\right)|z_1|^{\frac{1+3b}{2}}|z_2| + k_1|z_1|^b|z_2|^2\right) \\ & + a_1|d_2|^{\frac{2b}{1+b}}\left(|z_2|^3 + r|z_1|^{\frac{3+3b}{2}} + \frac{2k_1}{(1+b)}|z_1|^{1+b}|z_2|\right)\end{aligned}$$

Applying lemma 2.1,

$$|z_1|^{1+b}|z_2| \le \frac{2}{3}|z_1|^{\frac{3+3b}{2}} + \frac{1}{3}|z_2|^3,$$

$$|z_1|^{\frac{1+3b}{2}}|z_2| \le \left(\frac{1+3b}{2+4b}\right)|z_1|^{1+2b} + \left(\frac{1+b}{2+4b}\right)|z_2|^{\frac{2+4b}{1+b}},$$

$$|z_1|^b|z_2|^2 \le \left(\frac{b}{1+2b}\right)|z_1|^{1+2b} + \left(\frac{1+b}{1+2b}\right)|z_2|^{\frac{2+4b}{1+b}},$$

together with lemma 2.2, and the bounds of the Lyapunov function, we have:

$$
\begin{aligned}
\dot{\tilde{V}}(z_1,z_2) &\le -\omega_1\left(|z_1|^{\frac{3+5b}{2}}+|z_2|^{\frac{3+5b}{1+b}}\right)+|d_1|\omega_2\left(|z_1|^{1+2b}+|z_2|^{\frac{2+4b}{1+b}}\right)+a_1|d_2|^{\frac{2b}{1+b}}\omega_3\left(|z_1|^{\frac{3+3b}{2}}+|z_2|^{3}\right)\\
&= -\omega_1\left(\left(|z_1|^{2+2b}\right)^{\frac{3+5b}{4+4b}}+\left(|z_2|^{4}\right)^{\frac{3+5b}{4+4b}}\right)+|d_1|\omega_2\left(\left(|z_1|^{2+2b}\right)^{\frac{1+2b}{2+2b}}+\left(|z_2|^{4}\right)^{\frac{1+2b}{2+2b}}\right)\\
&\quad + a_1|d_2|^{\frac{2b}{1+b}}\omega_3\left(\left(|z_1|^{2+2b}\right)^{\frac{3}{4}}+\left(|z_2|^{4}\right)^{\frac{3}{4}}\right)\\
&\le -\left(\frac{\omega_4}{\overline{\pi}_1^{\frac{3+5b}{4+4b}}}\right)V^{\frac{3+5b}{4+4b}}+|d_1|\left(\frac{2^{\frac{1}{2+2b}}\omega_2}{\underline{\pi}_1^{\frac{1+2b}{2+2b}}}\right)V^{\frac{1+2b}{2+2b}}+a_1|d_2|^{\frac{2b}{1+b}}\left(\frac{2^{\frac{1}{4}}\omega_3}{\underline{\pi}_1^{\frac{3}{4}}}\right)V^{\frac{3}{4}}
\end{aligned}
$$

where

$$
\omega_1=\min\left\{\frac{rk_1}{2},\frac{(k_2-l)}{2}\right\},\ \omega_2=\max\left\{\begin{array}{l}\left(\left(\frac{2k_1^{\ 2}}{(1+b)}+r\left(\frac{3+3b}{2}\right)\left(\frac{1+3b}{2+4b}\right)+k_1\left(\frac{b}{1+2b}\right)\right),\\ \left(r\left(\frac{3+3b}{2}\right)\left(\frac{1+b}{2+4b}\right)+k_1\left(\frac{1+b}{1+2b}\right)\right)\end{array}\right\},
$$

$$
\omega_3=\max\left\{\left(r+\frac{4k_1}{(3+3b)}\right),\left(1+\frac{2k_1}{(3+3b)}\right)\right\},\ \omega_4=\begin{cases}\omega_1 & \text{for } 0<b\le 1,\\ \dfrac{\omega_1}{2^{\frac{b-1}{4+4b}}} & \text{for } b>1\end{cases}
$$

It can be further arranged into the following form to dominate the positive term,

$$\dot{V}(z_1,z_2)\overset{a.e.}{\le}-\frac{1}{2}\left(\frac{\omega_4}{\bar{\pi}_1^{\frac{3+5b}{4+4b}}}\right)V^{\frac{3+5b}{4+4b}}-\frac{1}{4}\left(\frac{\omega_4}{\bar{\pi}_1^{\frac{3+5b}{4+4b}}}\right)V^{\frac{3+5b}{4+4b}}+|d_1|\left(\frac{2^{\frac{1}{2+2b}}\omega_2}{\underline{\pi}_1^{\frac{1+2b}{2+2b}}}\right)V^{\frac{1+2b}{2+2b}}-\frac{1}{4}\left(\frac{\omega_4}{\bar{\pi}_1^{\frac{3+5b}{4+4b}}}\right)V^{\frac{3+5b}{4+4b}}$$

$$+a_1|d_2|^{\frac{2b}{1+b}}\left(\frac{2^{\frac{1}{4}}\omega_3}{\underline{\pi}_1^{\frac{3}{4}}}\right)V^{\frac{3}{4}}$$

$$=-\frac{1}{2}\left(\frac{\omega_4}{\bar{\pi}_1^{\frac{3+5b}{4+4b}}}\right)V^{\frac{3+5b}{4+4b}}-\frac{1}{4}\left(\frac{\omega_4}{\bar{\pi}_1^{\frac{3+5b}{4+4b}}}\right)V^{\frac{1+2b}{2+2b}}\left(V^{\frac{1}{4}}-|d_1|\left(\frac{4\bar{\pi}_1^{\frac{3+5b}{4+4b}}}{\omega_4}\right)\left(\frac{2^{\frac{1}{2+2b}}\omega_2}{\underline{\pi}_1^{\frac{1+2b}{2+2b}}}\right)\right)$$

$$-\frac{1}{4}\left(\frac{\omega_4}{\bar{\pi}_1^{\frac{3+5b}{4+4b}}}\right)V^{\frac{3}{4}}\left(V^{\frac{b}{2+2b}}-a_1|d_2|^{\frac{2b}{1+b}}\left(\frac{4\bar{\pi}_1^{\frac{3+5b}{4+4b}}}{\omega_4}\right)\left(\frac{2^{\frac{1}{4}}\omega_3}{\underline{\pi}_1^{\frac{3}{4}}}\right)\right)$$

$$\le-\frac{1}{2}\left(\frac{\omega_4}{\bar{\pi}_1^{\frac{3+5b}{4+4b}}}\right)V^{\frac{3+5b}{4+4b}},\ \forall V(z_1,z_2)\ge\Delta:=\max\left\{\begin{array}{l}|d_1|^4\left(\frac{4\bar{\pi}_1^{\frac{3+5b}{4+4b}}}{\omega_4}\right)^4\left(\frac{2^{\frac{1}{2+2b}}\omega_2}{\underline{\pi}_1^{\frac{1+2b}{2+2b}}}\right)^4,\\ \left(a_1|d_2|^{\frac{2b}{1+b}}\left(\frac{4\bar{\pi}_1^{\frac{3+5b}{4+4b}}}{\omega_4}\right)\left(\frac{2^{\frac{1}{4}}\omega_3}{\underline{\pi}_1^{\frac{3}{4}}}\right)\right)^{\frac{2+2b}{b}}\end{array}\right\}$$

Hence, invoking lemma 2.6, the states will reach the compact set

$\Omega_\Delta=\{(z_1,z_2)\in\mathbb{R}^2:V(z_1,z_2)\le\Delta\}$ in finite time. Note that when $|d_1| = |d_2| = 0$, everywhere

$$\dot{V}(z)\le-\left(\frac{\omega_4}{\bar{\pi}_1^{\frac{3+5b}{4+4b}}}\right)V^{\frac{3+5b}{4+4b}},\ \forall(z_1,z_2)\in\mathbb{R}^2 \qquad \blacksquare$$

<u>For the case of $b = 0$:</u>

First we consider the planar case:

Now let us consider the planar dynamics with $b = 0$ and $k_1 > k_2 > 0$.

$$\dot{z}_1 = z_2 + d_1,$$
$$\dot{z}_2 = -k_1 \operatorname{sign}(z_1) - k_2 \operatorname{sign}(z_2 + d_2)$$

$$V(z_1, z_2) = k_1^2 |z_1|^2 + \frac{1}{4}|z_2|^4 + r|z_1|^{\frac{3}{2}} \operatorname{sign}(z_1) z_2 + k_1 |z_1||z_2|^2$$

Now, for $b = 0$, the Lyapunov function is not differentiable at $z_1 = 0$ but Lipschitz continuous and the system is described by the differential inclusion:

$$\dot{\tilde{V}}(\mathbf{z}(t), t) = \bigcap_{\xi \in \partial V(\mathbf{z}(t), t)} \xi^T K \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} (\mathbf{z}, t) \subset \dot{\tilde{V}}_1 + \dot{\tilde{V}},$$

where

$$\dot{\tilde{V}}_1 = \bigcap_{\xi \in \partial V(\mathbf{z}(t), t)} \xi^T K[f](z_1, z_2), \ \dot{\tilde{V}}_2 = \bigcap_{\xi \in \partial V(\mathbf{z}(t), t)} \xi^T K \begin{bmatrix} d_1 \\ 0 \end{bmatrix},$$

$$K[f](z_1, z_2) = \begin{cases} \left\{ \begin{pmatrix} z_2, \\ -k_1 \operatorname{sign}(z_1) - k_2 \operatorname{sign}(z_2) \end{pmatrix} \right\} \forall z_1 \neq 0, |z_2| > |d_2| \\ \left\{ \begin{pmatrix} z_2, \\ -k_1 [-1,\ 1] - k_2 \operatorname{sign}(z_2) \end{pmatrix} \right\} \forall z_1 = 0, |z_2| > |d_2| \\ \left\{ \begin{pmatrix} z_2, \\ -k_1 \operatorname{sign}(z_1) - k_2 \operatorname{SGN}(z_2 + d_2) \end{pmatrix} \right\} \forall z_1 \neq 0, |z_2| \leq |d_2| \\ \left\{ \begin{pmatrix} z_2, \\ -k_1 [-1,\ 1] - k_2 \operatorname{SGN}(z_2 + d_2) \end{pmatrix} \right\} \forall z_1 = 0, |z_2| \leq |d_2| \end{cases}$$

and

$$\operatorname{SGN}(x) = \begin{cases} -1 & x < 0 \\ [-1,\ 1] & x = 0 \\ 1 & x > 0 \end{cases}$$

$$\partial V = \boldsymbol{K}[\nabla V] = \boldsymbol{K}\begin{bmatrix} \dfrac{\partial V}{\partial z_1} \\ \dfrac{\partial V}{\partial z_2} \end{bmatrix}$$

$$\subset \begin{bmatrix} \boldsymbol{K}\left[\dfrac{\partial V}{\partial z_1}\right] \\ \boldsymbol{K}\left[\dfrac{\partial V}{\partial z_2}\right] \end{bmatrix} \left\{ \begin{array}{l} \left\{ \begin{pmatrix} 2k_1{}^2 z_1 + \dfrac{3}{2} r|z_1|^{\frac{1}{2}} z_2 + k_1 \operatorname{sign}(z_1)|z_2|^2 \\ z_2{}^3 + r|z_1|^{\frac{3}{2}} \operatorname{sign}(z_1) + 2k_1|z_1| z_2 \end{pmatrix} \right\} \forall z_1 \neq 0, z_2 \in \mathbb{R} \\ \left\{ \begin{pmatrix} [-1,1] k_1 z_2{}^2 \\ z_2{}^3 \end{pmatrix} \right\} \forall z_1 = 0, z_2 \in \mathbb{R} \end{array} \right.$$

Let us define:

$$\left|\frac{\partial V}{\partial z_1}\right| := \sup\left\{|\xi_1| : \xi_1 \in \boldsymbol{K}\left[\frac{\partial V}{\partial z_1}\right]\right\}, \text{ and } \left|\frac{\partial V}{\partial z_2}\right| := \sup\left\{|\xi_2| : \xi_2 \in \boldsymbol{K}\left[\frac{\partial V}{\partial z_2}\right]\right\},$$

with $\left|\dfrac{\partial V}{\partial z_1}\right| \leq 2k_1{}^2|z_1| + \dfrac{3}{2} r|z_1|^{\frac{1}{2}}|z_2| + k_1|z_2|^2$,

Thus, the term

$$\dot{\tilde{V}}_2 = \bigcap_{\xi_1 \in \boldsymbol{K}\left[\frac{\partial V}{\partial z_1}\right]} \xi_1 K[d_1] \leq \left|\frac{\partial V}{\partial z_1}\right| |d_1| \leq |d_1| \left(2k_1{}^2|z_1| + \frac{3}{2} r|z_1|^{\frac{1}{2}}|z_2| + k_1|z_2|^2 \right)$$

Computing $\dot{\tilde{V}}_1$ for each case, we have

For $z_1 \neq 0$, $|z_2| > |d_2|$:

$$\dot{\tilde{V}}_1 = \frac{3}{2} r|z_1|^{\frac{1}{2}} z_2{}^2 - 2k_1 k_2 |z_1||z_2| - k_2|z_2|^3 - rk_1|z_1|^{\frac{3}{2}} - rk_2|z_1|^{\frac{3}{2}} \operatorname{sign}(z_1 z_2)$$

For $z_1 = 0$, $|z_2| > |d_2|$:

Let $(\xi_2 k_1 z_2{}^2, z_2{}^3)^T$ with $\xi_2 \in [-1, 1]$ be an arbitrary element of $\partial V(\mathbf{z}, t)$, then

$$\dot{\tilde{V}}_1 = \bigcap_{\xi_2 \in [-1,\ 1]} [\xi_2 - 1,\ \xi_2 + 1] k_1 z_2^{\ 3} - k_2 |z_2|^3 = -k_2 |z_2|^3$$

For $z_1 \neq 0$, $|z_2| \leq |d_2|$:

$$\dot{\tilde{V}}_1 = \frac{3}{2} r |z_1|^{\frac{1}{2}} |z_2|^2 - 2k_1 k_2 |z_1| z_2 \mathrm{SGN}(z_2 + d_2) - k_2 |z_2|^3 \operatorname{sign}(z_2) \mathrm{SGN}(z_2 + d_2)$$
$$- r k_1 |z_1|^{\frac{3}{2}} - r k_2 |z_1|^{\frac{3}{2}} \operatorname{sign}(z_1) \mathrm{SGN}(z_2 + d_2)$$

For $z_1 = 0$, $|z_2| \leq |d_2|$:

$$\dot{\tilde{V}}_1 = \bigcap_{\xi_2 \in [-1,\ 1]} [\xi_2 - 1,\ \xi_2 + 1] k_1 |z_2|^3 \operatorname{sign}(z_2) - k_2 |z_2|^3 \operatorname{sign}(z_2) \mathrm{SGN}(z_2 + d_2)$$
$$= -k_2 |z_2|^3 \operatorname{sign}(z_2) \mathrm{SGN}(z_2 + d_2)$$

Thus, for $\forall\ (z_1, z_2) \in \mathbb{R}^2$

$$\dot{\tilde{V}}_1 \leq \frac{3}{2} r |z_1|^{\frac{1}{2}} |z_2|^2 - 2k_1 k_2 |z_1| z_2 \mathrm{SGN}(z_2 + d_2)$$
$$- k_2 |z_2|^3 \operatorname{sign}(z_2) \mathrm{SGN}(z_2 + d_2) - r k_1 |z_1|^{\frac{3}{2}} + r k_2 |z_1|^{\frac{3}{2}}$$

Hence, the generalized time derivative of the Lyapunov function, after rearrangement:

$$\dot{\tilde{V}} = \dot{\tilde{V}}_1 + \dot{\tilde{V}}_2 \leq 2k_1^{\ 2} |z_1| |d_1| + \frac{3}{2} r |z_1|^{\frac{1}{2}} |z_2| |d_1| + k_1 |z_2|^2 |d_1| + \frac{3}{2} r |z_1|^{\frac{1}{2}} |z_2|^2$$
$$- 2k_1 k_2 |z_1| z_2 \mathrm{SGN}(z_2 + d_2)$$
$$- k_2 |z_2|^3 \operatorname{sign}(z_2) \mathrm{SGN}(z_2 + d_2) - r(k_1 - k_2) |z_1|^{\frac{3}{2}}$$

For analysis, the state space is divided into three regions:

$$\phi_1(|\zeta|)=\left\{(z_1,z_2)\in\mathbb{R}^2 : z_1\in\mathbb{R}, |z_2|\ge\frac{2k_1}{k_2}|\zeta|\right\},$$

$$\phi_2(|\zeta|)=\left\{(z_1,z_2)\in\mathbb{R}^2 : |z_1|\ge a_3|\zeta|^2, |z_2|\le\frac{2k_1}{k_2}|\zeta|\right\},$$

$$\phi_3(|\zeta|)=\left\{(z_1,z_2)\in\mathbb{R}^2 : |z_1|\le a_3|\zeta|^2,\ |z_2|\le\frac{2k_1}{k_2}|\zeta|\right\},$$

where $|\zeta| =: \max\{|d_1|, |d_2|\}$ and

$$a_3 := \max\left\{\left(\frac{24k_1^{\ 2}}{r(k_1-k_2)}\right)^2, \frac{\frac{12k_1}{k_2}\left(1+\left(\frac{2k_1}{k_2}\right)\right)}{(k_1-k_2)}, \left(\frac{48\left(\frac{k_1^{\ 3}}{k_2^{\ 2}}\right)}{r(k_1-k_2)}\right)^{\frac{2}{3}}\right\}>0$$

since $k_1 > k_2$

Consider the region:

$$\phi_1(|\zeta|)=\left\{(z_1,z_2)\in\mathbb{R}^2 : z_1\in\mathbb{R}, |z_2|\ge\frac{2k_1}{k_2}|\zeta|\right\}$$

Note that in this region, the following properties apply:

$$z_2\mathrm{SGN}(z_2+d_2)=|z_2|\mathrm{sign}(z_2)\mathrm{SGN}(z_2+d_2)=|z_2|$$

since $\frac{2k_1}{k_2}|\zeta_i|\ge|d_2|$, where the equality only occur when $|d_1|=|d_2|=0$;

also when $|d_2|=0$, the above is trivially satisfied,

when $|d_2|\neq 0$, we have $\mathrm{SGN}(z_2+d_2)=\mathrm{sign}(z_2+d_2)$, (since $(z_2+d_2)\neq 0$ in this region)

$= \mathrm{sign}(z_2)$ since $|z_2|>|d_2|$ in this region

Then, the time derivative of the Lyapunov function becomes:

$$\dot{\tilde{V}} \le -k_1|z_1|\left(k_2|z_2| - 2k_1|d_1|\right) - |z_2|^2\left(\frac{k_2}{2}|z_2| - k_1|d_1|\right) - k_2|z_2|\left(\left(k_1|z_1|\right)^{\frac{1}{2}} - \left(\frac{1}{2}|z_2|^2\right)^{\frac{1}{2}}\right)^2$$
$$-\frac{1}{2}|z_1|^{\frac{1}{2}}|z_2|^2\left(2^{\frac{1}{2}}k_1^{\frac{1}{2}}k_2 - 3r\right) - \frac{1}{2}|z_1|^{\frac{1}{2}}|z_2|\left(2^{\frac{1}{2}}k_1^{\frac{1}{2}}k_2|z_2| - 3r|d_1|\right) - r\left(k_1 - k_2\right)|z_1|^{\frac{3}{2}}$$

and $\dot{\tilde{V}} < 0$ for any positive

$$0 < r < \frac{2^{\frac{1}{2}}}{3}k_1^{\frac{1}{2}}k_2 \text{ , with } k_1 > k_2 > 0$$

for all $\forall\, z_1 \in \mathbb{R}$, $|z_2| \ge \max\left\{\frac{2k_1|d_1|}{k_2}, \frac{3r}{2^{\frac{1}{2}}k_1^{\frac{1}{2}}k_2}|d_1|\right\} = \frac{2k_1|d_1|}{k_2} > |d_1|$ and $|z_2| > |d_2|$,

which is sufficiently satisfied for the states in this region.

Next, consider the region:

$$\phi_2\left(|\zeta|\right) = \left\{\left(z_1, z_2\right) \in \mathbb{R}^2 : |z_1| \ge a_3|\zeta|^2, |z_2| \le \frac{2k_1}{k_2}|\zeta|\right\}$$

Then, the time derivative of the Lyapunov function becomes:

$$\dot{\tilde{V}} \le 2k_1^2|z_1||d_1| + \frac{3}{2}r|z_1|^{\frac{1}{2}}|z_2||d_1| + k_1|z_2|^2|d_1| + \frac{3}{2}r|z_1|^{\frac{1}{2}}|z_2|^2$$
$$+ 2k_1k_2|z_1||z_2| + k_2|z_2|^3 - r\left(k_1 - k_2\right)|z_1|^{\frac{3}{2}}$$

After rearrangement,

$$\dot{\tilde{V}} \le |z_1|\left(2k_1^{\,2}|d_1| + 2k_1k_2|z_2|\right) + \frac{3}{2}r|z_1|^{\frac{1}{2}}|z_2|\left(|d_1| + |z_2|\right) + k_1|z_2|^2|d_1| + k_2|z_2|^3 - r(k_1 - k_2)|z_1|^{\frac{3}{2}}$$

Note that in this region, we have:

$$|z_2| \le \frac{2k_1}{k_2}|\zeta|, \quad \text{also,} |\zeta| \ge |d_1|$$

Hence,

$$\dot{\tilde{V}} \le -|z_1|\left(\frac{r}{4}(k_1 - k_2)|z_1|^{\frac{1}{2}} - 6k_1^{\,2}|\zeta_i|\right) - r|z_1|^{\frac{1}{2}}\left(\frac{1}{4}(k_1 - k_2)|z_1| - \frac{3k_1}{k_2}\left(1 + \frac{2k_1}{k_2}\right)|\zeta_i|^2\right)$$
$$-\left(\frac{r}{4}(k_1 - k_2)|z_1|^{\frac{3}{2}} - 12\left(\frac{k_1^{\,3}}{k_2^{\,2}}\right)|\zeta_i|^3\right) - \frac{r}{4}(k_1 - k_2)|z_1|^{\frac{3}{2}}$$

where $\dot{\tilde{V}} < 0$ for any positive $r > 0$, with $k_1 > k_2 > 0$

for all

$$|z_1| \ge \max\left\{\left(\frac{24k_1^{\,2}}{r(k_1 - k_2)}\right)^2, \frac{\frac{12k_1}{k_2}\left(1 + \left(\frac{2k_1}{k_2}\right)\right)}{(k_1 - k_2)}, \left(\frac{48\left(\frac{k_1^{\,3}}{k_2^{\,2}}\right)}{r(k_1 - k_2)}\right)^{\frac{2}{3}}\right\} \times |\zeta_i|^2, \text{ and } |z_2| \le \frac{2k_1}{k_2}|\zeta_i|$$

which is sufficiently satisfied for the states in this region.

Finally, consider the region $\phi_3(|\zeta|) = \left\{(z_1, z_2) \in \mathbb{R}^2 : |z_1| \le a_3|\zeta|^2, |z_2| \le \frac{2k_1}{k_2}|\zeta|\right\}$ which is a compact set. Hence, with the above results, we have

$$\dot{\tilde{V}} < 0, \text{ for } (z_1, z_2) \notin \phi_3(|\zeta|)$$

Now, define a Lyapunov level set:

$$\Omega_\mu = \{(z_1, z_2) \in \mathbb{R}^2 : V \leq \rho_1(|\zeta|)\}$$

where the positive definite function $\rho_1(|\zeta|)$ is defined as follows:

$$\rho_1(|\zeta|) = \max_{(z_1, z_2) \in \text{bd } \phi_3(|\zeta|)} V$$

which exists since the boundary of the set is compact and V is continuous. Then we observe that $\phi_3(|\zeta|) \subset \Omega_\mu$. As a result, we have

$$\frac{d}{dt} V(z_1, z_2) \overset{a.e.}{\in} \dot{\tilde{V}}(z_1, z_2) < 0 \quad \textit{for} \;\; V \geq \rho_1(|\zeta|)$$

which implies that the trajectories will enter the compact level set Ω_μ in finite time and stay in it once entered. ■

A.2 PROPOSITION 1

For every real numbers k_2, d_2, $z_2 \in \mathbb{R}$, $b > 0$, and $|d_2| := \sup\{|\varepsilon_2| : \varepsilon_2 \in K[\, d_2\,]\}$ the following inequalities is satisfied

$$\left| k_2 |z_2|^{\frac{2b}{1+b}} \operatorname{sign}(z_2) - k_2 |z_2 + d_2|^{\frac{2b}{1+b}} \operatorname{sign}(z_2 + d_2) \right|$$

$$\leq \max\left\{ \begin{array}{l} k_2 \big| |z_2| + |d_2| \big|^{\frac{2b}{1+b}} - k_2 |z_2|^{\frac{2b}{1+b}}, \\ k_2 |z_2|^{\frac{2b}{1+b}} - k_2 \big| |z_2| - |d_2| \big|^{\frac{2b}{1+b}} \operatorname{sign}(|z_2| - |d_2|) \end{array} \right\},$$

$$\left| k_2 |z_2|^{\frac{2b}{1+b}} \operatorname{sign}(z_2) - k_2 |z_2 + d_2|^{\frac{2b}{1+b}} \operatorname{sign}(z_2 + d_2) \right| \leq a_1 |d_2|^{\frac{2b}{1+b}} + a_2 |z_2|^{\frac{2b}{1+b}},$$

with some positive constants a_1, a_2 define by $0 < a_2 < k_2$ and

$$a_1 := k_2 \left(\frac{a_2 + 2k_2}{\left((a_2 + k_2)^{\frac{1+b}{2b}} - k_2^{\frac{1+b}{2b}} \right)^{\frac{2b}{1+b}}} \right) > 0$$

Proof of Proposition 1: Let $\delta = k_2 |z_2|^{\frac{2b}{1+b}} \operatorname{sign}(z_2) - k_2 |z_2 + d_2|^{\frac{2b}{1+b}} \operatorname{sign}(z_2 + d_2)$ and $|\delta| :=$ $\sup\{|\varepsilon_3| : \varepsilon_3 \in K[\, \delta\,]\}$.

First we are going to show that $\forall\, z_2 \in \mathbb{R}$, and any $\varepsilon_2 \in K[\, d_2\,]$,

$$|\varepsilon_3| \leq \max\left\{ k_2 \big| |z_2| + |\varepsilon_2| \big|^{\frac{2b}{1+b}} - k_2 |z_2|^{\frac{2b}{1+b}},\ k_2 |z_2|^{\frac{2b}{1+b}} - k_2 \big| |z_2| - |\varepsilon_2| \big|^{\frac{2b}{1+b}} \operatorname{sign}(|z_2| - |\varepsilon_2|) \right\}$$

Let us note that when $|\varepsilon_2| = 0$, the above is trivially satisfied. Hence, for $|\varepsilon_2| > 0$, we consider the two cases of $\text{sign}(\varepsilon_2) < 0$ or $\text{sign}(\varepsilon_2) > 0$.

For $\text{sign}(\varepsilon_2) > 0$:

$$|\varepsilon_3| = \left| k_2 |z_2|^{\frac{2b}{1+b}} \text{sign}(z_2) - k_2 \left| z_2 + |\varepsilon_2| \right|^{\frac{2b}{1+b}} \text{sign}(z_2 + |\varepsilon_2|) \right|$$

In the region of $z_2 \geq 0$, it becomes:

$$|\varepsilon_3| = \left| k_2 |z_2|^{\frac{2b}{1+b}} - k_2 \left| |z_2| + |\varepsilon_2| \right|^{\frac{2b}{1+b}} \right| = \left| k_2 \left| |z_2| + |\varepsilon_2| \right|^{\frac{2b}{1+b}} - k_2 |z_2|^{\frac{2b}{1+b}} \right|$$

In the region of $z_2 \leq 0$, it becomes:

$$\begin{aligned} |\varepsilon_3| &= \left| -k_2 |z_2|^{\frac{2b}{1+b}} - k_2 \left| -|z_2| + |\varepsilon_2| \right|^{\frac{2b}{1+b}} \text{sign}(-|z_2| + |\varepsilon_2|) \right| \\ &= \left| -k_2 |z_2|^{\frac{2b}{1+b}} + k_2 \left| |z_2| - |\varepsilon_2| \right|^{\frac{2b}{1+b}} \text{sign}(|z_2| - |\varepsilon_2|) \right| \\ &= \left| k_2 |z_2|^{\frac{2b}{1+b}} - k_2 \left| |z_2| - |\varepsilon_2| \right|^{\frac{2b}{1+b}} \text{sign}(|z_2| - |\varepsilon_2|) \right| \end{aligned}$$

For $\text{sign}(\varepsilon_2) < 0$:

$$|\varepsilon_3| = \left| k_2 |z_2|^{\frac{2b}{1+b}} \text{sign}(z_2) - k_2 \left| z_2 - |\varepsilon_2| \right|^{\frac{2b}{1+b}} \text{sign}(z_2 - |\varepsilon_2|) \right|$$

In the region of $z_2 \geq 0$, it becomes:

$$|\varepsilon_3| = \left| k_2 |z_2|^{\frac{2b}{1+b}} - k_2 \left| |z_2| - |\varepsilon_2| \right|^{\frac{2b}{1+b}} \text{sign}(|z_2| - |\varepsilon_2|) \right|$$

which is identical to the case of sign(ε_2) > 0 in the region $z_2 \le 0$.

In the region of $z_2 \le 0$, it becomes:

$$|\varepsilon_3| = \left| -k_2|z_2|^{\frac{2b}{1+b}} - k_2\left|-|z_2| - |\varepsilon_2|\right|^{\frac{2b}{1+b}} \operatorname{sign}\left(-|z_2| - |\varepsilon_2|\right)\right| = \left| -k_2|z_2|^{\frac{2b}{1+b}} + k_2\left||z_2| + |\varepsilon_2|\right|^{\frac{2b}{1+b}}\right|$$
$$= \left| k_2\left||z_2| + |\varepsilon_2|\right|^{\frac{2b}{1+b}} - k_2|z_2|^{\frac{2b}{1+b}}\right|$$

which is identical to the case of sign(ε_2) > 0 in the region $z_2 \ge 0$.

Hence, for all $|\varepsilon_2| \ge 0$ and $|z_2| \ge 0$:

$$|\varepsilon_3| \le \max\left\{\left| k_2\left||z_2| + |\varepsilon_2|\right|^{\frac{2b}{1+b}} - k_2|z_2|^{\frac{2b}{1+b}}\right|,\ \left| k_2|z_2|^{\frac{2b}{1+b}} - k_2\left||z_2| - |\varepsilon_2|\right|^{\frac{2b}{1+b}} \operatorname{sign}\left(|z_2| - |\varepsilon_2|\right)\right|\right\} \qquad \text{(A2 - 1)}$$

Next, note that the function $f(|\cdot|) = k_2|\cdot|^{\frac{2b}{1+b}}$ is strictly increasing for $|z_2| \ge 0$ and $\left||z_2| + |\varepsilon_2|\right| \ge |z_2|$, we have

$$k_2\left||z_2| + |\varepsilon_2|\right|^{\frac{2b}{1+b}} \ge k_2|z_2|^{\frac{2b}{1+b}} \Rightarrow k_2\left||z_2| + |\varepsilon_2|\right|^{\frac{2b}{1+b}} - k_2|z_2|^{\frac{2b}{1+b}} \ge 0$$

Also, note that the following is an increasing function, $k_2|z_2|^{\frac{2b}{1+b}} \operatorname{sign}(z_2)$ and since $|z_2| \ge |z_2| - |\varepsilon_2|$ we have

$$k_2|z_2|^{\frac{2b}{1+b}} \operatorname{sign}(|z_2|) \ge k_2\left||z_2| - |\varepsilon_2|\right|^{\frac{2b}{1+b}} \operatorname{sign}\left(|z_2| - |\varepsilon_2|\right)$$
$$\Rightarrow k_2|z_2|^{\frac{2b}{1+b}} - k_2\left||z_2| - |\varepsilon_2|\right|^{\frac{2b}{1+b}} \operatorname{sign}\left(|z_2| - |\varepsilon_2|\right) \ge 0$$

Hence we can write (A2 - 1) as:

$$|\varepsilon_3| \le \max\left\{k_2\left\|z_2\right|+\left|\varepsilon_2\right\|^{\frac{2b}{1+b}}-k_2\left|z_2\right|^{\frac{2b}{1+b}},\ \ k_2\left|z_2\right|^{\frac{2b}{1+b}}-k_2\left\|z_2\right|-\left|\varepsilon_2\right\|^{\frac{2b}{1+b}}\operatorname{sign}\left(\left|z_2\right|-\left|\varepsilon_2\right|\right)\right\}$$

Now, since $|d_2| \coloneqq \sup\{|\varepsilon_2| : \varepsilon_2 \in K[\, d_2\,]\}$, we have

$$\begin{aligned}
&\left|z_2\right|+\left|d_2\right| \ge \left|z_2\right|+\left|\varepsilon_2\right| \Rightarrow k_2\left\|z_2\right|+\left|d_2\right\|^{\frac{2b}{1+b}} \ge k_2\left\|z_2\right|+\left|\varepsilon_2\right\|^{\frac{2b}{1+b}} \\
&\qquad\qquad\qquad \Rightarrow k_2\left\|z_2\right|+\left|d_2\right\|^{\frac{2b}{1+b}}-k_2\left|z_2\right|^{\frac{2b}{1+b}} \ge k_2\left\|z_2\right|+\left|\varepsilon_2\right\|^{\frac{2b}{1+b}}-k_2\left|z_2\right|^{\frac{2b}{1+b}}
\end{aligned}$$

also, we have

$$\begin{aligned}
&\left|z_2\right|-\left|\varepsilon_2\right| \ge \left|z_2\right|-\left|d_2\right| \\
&\Rightarrow k_2\left\|z_2\right|-\left|\varepsilon_2\right\|^{\frac{2b}{1+b}}\operatorname{sign}\left(\left|z_2\right|-\left|\varepsilon_2\right|\right) \ge k_2\left\|z_2\right|-\left|d_2\right\|^{\frac{2b}{1+b}}\operatorname{sign}\left(\left|z_2\right|-\left|d_2\right|\right) \\
&\Rightarrow k_2\left|z_2\right|^{\frac{2b}{1+b}}-k_2\left\|z_2\right|-\left|d_2\right\|^{\frac{2b}{1+b}}\operatorname{sign}\left(\left|z_2\right|-\left|d_2\right|\right) \ge k_2\left|z_2\right|^{\frac{2b}{1+b}}-k_2\left\|z_2\right|-\left|\varepsilon_2\right\|^{\frac{2b}{1+b}}\operatorname{sign}\left(\left|z_2\right|-\left|\varepsilon_2\right|\right)
\end{aligned}$$

Hence we have $\forall\ \varepsilon_2 \in K[\, d_2\,]$, $\forall\ |z_2| \ge 0$:

$$|\varepsilon_3| \le |\delta| \le \max\left\{k_2\left\|z_2\right|+\left|d_2\right\|^{\frac{2b}{1+b}}-k_2\left|z_2\right|^{\frac{2b}{1+b}},\ \ k_2\left|z_2\right|^{\frac{2b}{1+b}}-k_2\left\|z_2\right|-\left|d_2\right\|^{\frac{2b}{1+b}}\operatorname{sign}\left(\left|z_2\right|-\left|d_2\right|\right)\right\}$$

Thus, to determine the upper bound of $|\delta|$, it is sufficient to consider two cases only for $|z_2| \ge 0$ where both cases are greater than or equal to zero for $|z_2| \ge 0$ and $\left|d_2\right| \ne 0$

Case 1:

Consider the following function:

$$\left(a_2+k_2\right)\left|z_2\right|^{\frac{2b}{1+b}},\ \text{ with }\ \ 0<a_2<k_2$$

and for the following inequality to be satisfied

$$k_2\left|\left|z_2\right|+\left|d_2\right|\right|^{\frac{2b}{1+b}} \le \left(a_2+k_2\right)\left|z_2\right|^{\frac{2b}{1+b}}$$

we need

$$\left|z_2\right| \ge C\left|d_2\right|, \quad \text{where } \mathrm{C} =: \left(\frac{k_2^{\frac{1+b}{2b}}}{\left(a_2+k_2\right)^{\frac{1+b}{2b}}-k_2^{\frac{1+b}{2b}}}\right)$$

where $C > 0$ for any $a_2 > 0$, which implies for $|z_2| \ge C\,\bar{d}$

$$k_2\left|\left|z_2\right|+\left|d_2\right|\right|^{\frac{2b}{1+b}}-k_2\left|z_2\right|^{\frac{2b}{1+b}} \le a_2\left|z_2\right|^{\frac{2b}{1+b}}$$

and, for $|z_2| \le C\,|d_2|$, we have

$$k_2\left|\left|z_2\right|+\left|d_2\right|\right|^{\frac{2b}{1+b}} \le k_2\left|C\left|d_2\right|+\left|d_2\right|\right|^{\frac{2b}{1+b}} = k_2\left(C+1\right)^{\frac{2b}{1+b}}\left|d_2\right|^{\frac{2b}{1+b}}$$

which also implies

$$k_2\left|\left|z_2\right|+\left|d_2\right|\right|^{\frac{2b}{1+b}}-k_2\left|z_2\right|^{\frac{2b}{1+b}} \le k_2\left(C+1\right)^{\frac{2b}{1+b}}\left|d_2\right|^{\frac{2b}{1+b}}-k_2\left|z_2\right|^{\frac{2b}{1+b}},$$

since $k_2\left(C+1\right)^{\frac{2b}{1+b}}\left|d_2\right|^{\frac{2b}{1+b}}-k_2\left|z_2\right|^{\frac{2b}{1+b}} \le k_2\left(C+1\right)^{\frac{2b}{1+b}}\left|d_2\right|^{\frac{2b}{1+b}}$

$$\therefore k_2\left|\left|z_2\right|+\left|d_2\right|\right|^{\frac{2b}{1+b}}-k_2\left|z_2\right|^{\frac{2b}{1+b}} \le k_2\left(C+1\right)^{\frac{2b}{1+b}}\left|d_2\right|^{\frac{2b}{1+b}}$$

where the equality holds only when both $|z_2| = |d_2| = 0$.

Hence, for all $|z_2| \ge 0$, we have, $k_2\left|\left|z_2\right|+\left|d_2\right|\right|^{\frac{2b}{1+b}}-k_2\left|z_2\right|^{\frac{2b}{1+b}} \le k_2\left(C+1\right)^{\frac{2b}{1+b}}\left|d_2\right|^{\frac{2b}{1+b}}+a_2\left|z_2\right|^{\frac{2b}{1+b}}$

Now consider case 2:

Consider the following function:

$$(a_2 + k_2)\left||z_2| - |d_2|\right|^{\frac{2b}{1+b}} \operatorname{sign}(|z_2| - |d_2|)$$

and for the following inequality to be satisfied:

$$k_2|z_2|^{\frac{2b}{1+b}} \le (a_2 + k_2)\left||z_2| - |d_2|\right|^{\frac{2b}{1+b}} \operatorname{sign}(|z_2| - |d_2|)$$

we need:

$$|z_2| \ge D|d_2|, \quad \text{where } D =: \left(\frac{(a_2 + k_2)^{\frac{1+b}{2b}}}{(a_2 + k_2)^{\frac{1+b}{2b}} - k_2^{\frac{1+b}{2b}}} \right)$$

where $D > 1$ for any $a_2 > 0$, which implies for $|z_2| \ge D\,|\overline{d}_2|$

$$k_2|z_2|^{\frac{2b}{1+b}} - k_2\left||z_2| - |d_2|\right|^{\frac{2b}{1+b}} \operatorname{sign}(|z_2| - |d_2|) \le a_2\left||z_2| - |d_2|\right|^{\frac{2b}{1+b}} \operatorname{sign}(|z_2| - |d_2|)$$

Note that $a_2\,|z_2|^{\frac{2b}{1+b}} \ge a_2\,\left||z_2| - |d_2|\right|^{\frac{2b}{1+b}} \operatorname{sign}(|z_2| - |d_2|)$

$$\Rightarrow k_2|z_2|^{\frac{2b}{1+b}} - k_2\left||z_2| - |d_2|\right|^{\frac{2b}{1+b}} \operatorname{sign}(|z_2| - |d_2|) \le a_2\,|z_2|^{\frac{2b}{1+b}}$$

Also, for $|z_2| \le D\,|d_2|$, we have

$$\begin{aligned} k_2|z_2|^{\frac{2b}{1+b}} - k_2\left||z_2| - |d_2|\right|^{\frac{2b}{1+b}} \operatorname{sign}(|z_2| - |d_2|) &\le k_2|z_2|^{\frac{2b}{1+b}} + k_2\left||z_2| - |d_2|\right|^{\frac{2b}{1+b}} \\ &\le k_2 D^{\frac{2b}{1+b}}|d_2|^{\frac{2b}{1+b}} + k_2(D-1)^{\frac{2b}{1+b}}|d_2|^{\frac{2b}{1+b}} \\ &= k_2\left(D^{\frac{2b}{1+b}} + (D-1)^{\frac{2b}{1+b}} \right)|d_2|^{\frac{2b}{1+b}} \end{aligned}$$

Note that $D > 1$ from above.

Hence, for all $|z_2| \geq 0$, we have

$$k_2 |z_2|^{\frac{2b}{1+b}} - k_2 \big||z_2| - |d_2|\big|^{\frac{2b}{1+b}} \operatorname{sign}\big(|z_2| - |d_2|\big) \leq k_2 \left(D^{\frac{2b}{1+b}} + (D-1)^{\frac{2b}{1+b}} \right) |d_2|^{\frac{2b}{1+b}} + a_2 |z_2|^{\frac{2b}{1+b}}$$

Thus,

$$|\delta| \leq \max \left\{ k_2 \big||z_2| + |d_2|\big|^{\frac{2b}{1+b}} - k_2 |z_2|^{\frac{2b}{1+b}},\ k_2 |z_2|^{\frac{2b}{1+b}} - k_2 \big||z_2| - |d_2|\big|^{\frac{2b}{1+b}} \operatorname{sign}\big(|z_2| - |d_2|\big) \right\}$$

$$\leq \max \left\{ k_2 (C+1)^{\frac{2b}{1+b}} |d_2|^{\frac{2b}{1+b}} + a_2 |z_2|^{\frac{2b}{1+b}},\ k_2 \left(D^{\frac{2b}{1+b}} + (D-1)^{\frac{2b}{1+b}} \right) |d_2|^{\frac{2b}{1+b}} + a_2 |z_2|^{\frac{2b}{1+b}} \right\}$$

$$\Rightarrow |\delta| \leq a_1 |d_2|^{\frac{2b}{1+b}} + a_2 |z_2|^{\frac{2b}{1+b}}$$

where

$$a_1 = k_2 \times \max \left\{ (C+1)^{\frac{2b}{1+b}},\ \left(D^{\frac{2b}{1+b}} + (D-1)^{\frac{2b}{1+b}} \right) \right\}$$

$$= k_2 \times \max \left\{ \frac{a_2 + k_2}{\left((a_2 + k_2)^{\frac{1+b}{2b}} - k_2^{\frac{1+b}{2b}} \right)^{\frac{2b}{1+b}}},\ \frac{a_2 + 2k_2}{\left((a_2 + k_2)^{\frac{1+b}{2b}} - k_2^{\frac{1+b}{2b}} \right)^{\frac{2b}{1+b}}} \right\}$$

$$= k_2 \left(\frac{a_2 + 2k_2}{\left((a_2 + k_2)^{\frac{1+b}{2b}} - k_2^{\frac{1+b}{2b}} \right)^{\frac{2b}{1+b}}} \right)$$

$$\Rightarrow |\delta| \leq a_1 |d_2|^{\frac{2b}{1+b}} + a_2 |z_2|^{\frac{2b}{1+b}}$$

where a_2 is chosen as $0 < a_2 < k_2$. ■

Remark: The reason for having $a_2 < k_2$ will be apparent in the subsequent stability analysis.

A.3 PROPOSITION 2

For $x \in \mathbb{R}, y \in \mathbb{R}, p \geq 1$, the following inequality is satisfied

$$|x+y| \leq 2^{1-\frac{1}{p}} \left|\operatorname{sig}(x)^p + \operatorname{sig}(y)^p\right|^{\frac{1}{p}}.$$

Proof of proposition 2: (We extend the proof from Lemma 2.3 of [148] where it is originally for integer p only.):

It is straightforward to prove that the function

$$f(\lambda) := \left(\operatorname{sig}(\lambda)^p + \operatorname{sig}(1-\lambda)^p\right)2^{p-1} - 1$$

arrives its minimal value at $\lambda = 1/2$. Thus, $f(\lambda) \geq f(1/2) = 0$. Consequently,

$$\left(\operatorname{sig}(\lambda)^p + \operatorname{sig}(1-\lambda)^p\right)2^{p-1} \geq 1, \quad \forall \lambda \in \mathbb{R}.$$

In the case when $x + y \neq 0$, set $\lambda = x/(x + y)$. Then, the proposition follows immediately. When $x + y = 0$, the proposition is trivial. Hence, the proposition is true $\forall x \in \mathbb{R}, y \in \mathbb{R}$. ■

Appendix B

B.1 PROOF OF DESIRED ERROR DYNAMICS (CH 5 &6)

For the case of $b > 0$:

Consider the following dynamics:

$$\dot{v}_{1i} = v_{2i} + s_i$$
$$\dot{v}_{2i} = -k_{ai}\left(v_{2i} + s_i + \mathrm{sat}_{\mu_i}\left[\tilde{e}_{2i}\right]\right) + k_{ai}\mathrm{sat}_{\left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+b}{2b}}}\left[v_{2i} + s_i + \mathrm{sat}_{\mu_i}\left[\tilde{e}_{2i}\right]\right]$$
$$-\mathrm{sat}_{\varepsilon_{2i}}\left[k_{2i}\left|v_{2i} + s_i + \mathrm{sat}_{\mu_i}\left[\tilde{e}_{2i}\right]\right|^{\frac{2b}{1+b}}\mathrm{sign}\left(v_{2i} + s_i + \mathrm{sat}_{\mu_i}\left[\tilde{e}_{2i}\right]\right)\right] - \mathrm{sat}_{\varepsilon_{1i}}\left[k_{1i}\left|v_{1i}\right|^{b}\mathrm{sign}\left(v_{1i}\right)\right]$$

where v_{1i}, v_{2i}, s_i, $\tilde{e}_{2i} \in \mathbb{R}$, are the scalar state variables, k_{1i}, k_{2i}, k_{ai}, ε_{1i}, ε_{2i}, μ_i are positive constants, $b \geq 0$ real number

with

$$\varepsilon_{3i} := \begin{cases} \varepsilon_{1i} - k_{ai}\mu_i - k_{ai}\left(\dfrac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+b}{2b}} - \varepsilon_{2i}, & \text{for } \left|\tilde{e}_{2i}\right| \neq 0, \\ \varepsilon_{1i} - k_{ai}\left(\dfrac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+b}{2b}} - \varepsilon_{2i}, & \text{for } \tilde{e}_{2i} = 0 \quad \text{(for the case of full state feedback)} \end{cases}$$

The results in this section are applicable to the desired dynamics section of both chapter 5 & 6. In particular, for Chapter 5, let $\tilde{e}_{2i} = 0$, while for chapter 6, no changes required, and the same differential equations are obtained.

Lyapunov function:

$$V = \left(\frac{1}{2} {v_{2i}}^2 + \int_0^{v_{1i}} \mathrm{sat}_{\varepsilon_{1i}} \left[k_{1i} |r|^b \,\mathrm{sign}(r) \right] dr \right)^2 + r_1 \left(\frac{3+3b}{2} \right) \left(\int_0^{v_{1i}} |r|^{\frac{1}{2}} \,\mathrm{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3}{2}}} [|r|^{\frac{3b}{2}}] \; dr \right) v_{2i}$$

$$+ r_1 (1+b) k_{ai} \left(\frac{\varepsilon_{1\mathrm{i}}}{k_{1i}} \right)^{\frac{3}{2}} \left(\int_0^{v_{1i}} |r|^{\frac{3}{2}} \,\mathrm{sign}(r) - \mathrm{sat}_{\left(\frac{\varepsilon_{1\mathrm{i}}}{k_{1i}}\right)^{\frac{3}{2b}}} \left[|r|^{\frac{3}{2}} \,\mathrm{sign}(r) \right] dr \right)$$

where r_1 is a positive constant scalar, will be shown as a strict Lyapunov function. Note that the above function is locally Lipschitz and regular. It is differentiable everywhere for $b > 0$, and not differentiable on $v_{1i} = 0$ for $b = 0$. In particular, when none of the terms of the dynamics are saturated, the Lyapunov function is indeed identical to that proposed in Section 2.2 for the twisting-based family of algorithms.

Sign definiteness of V

Note the following properties:

$$\int_0^{v_{1i}} \mathrm{sat}_{\varepsilon_{1i}} \left[k_{1i} |r|^b \,\mathrm{sign}(r) \right] dr = \frac{k_{1i}}{1+b} \mathrm{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{1+b}{b}}} \left[|v_{1i}|^{1+b} \right] + \varepsilon_{1i} \left(|v_{1i}| - \mathrm{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{1}{b}}} \left[|v_{1i}| \right] \right)$$

$$r_1\left(\frac{3+3b}{2}\right)\int_0^{v_{1i}}|r|^{\frac{1}{2}}\operatorname{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3}{2}}}[|r|^{\frac{3b}{2}}]\ dr$$

$$=\left(r_1\operatorname{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3+3b}{2b}}}\left[|v_{1i}|^{\frac{3+3b}{2}}\right]+r_1(1+b)\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3}{2}}\left(|v_{1i}|^{\frac{3}{2}}-\operatorname{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3}{2b}}}\left[|v_{1i}|^{\frac{3}{2}}\right]\right)\right)\operatorname{sign}(v_{1i})$$

$$=\left(r_1\operatorname{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3+3b}{2b}}}\left[|v_{1i}|^{\frac{3+3b}{2}}\right]+r_1(1+b)\operatorname{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3}{2}}}\left[|v_{1i}|^{\frac{3b}{2}}\right]\left(|v_{1i}|^{\frac{3}{2}}-\operatorname{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3}{2b}}}\left[|v_{1i}|^{\frac{3}{2}}\right]\right)\right)\operatorname{sign}(v_{1i})$$

$$=\left(r_1(1+b)\operatorname{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3}{2}}}\left[|v_{1i}|^{\frac{3b}{2}}\right]|v_{1i}|^{\frac{3}{2}}-r_1 b\operatorname{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3+3b}{2b}}}\left[|v_{1i}|^{\frac{3+3b}{2}}\right]\right)\operatorname{sign}(v_{1i})$$

Using lemma 2.2,

$$|v_{1i}|^{\frac{3}{2}}=\left(|v_{1i}|-\operatorname{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{1}{b}}}[\![v_{1i}|]\!]+\operatorname{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{1}{b}}}[\![v_{1i}|]\!]\right)^{\frac{3}{2}}\le 2^{\frac{1}{2}}\left(|v_{1i}|-\operatorname{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{1}{b}}}[\![v_{1i}|]\!]\right)^{\frac{3}{2}}+2^{\frac{1}{2}}\left(\operatorname{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{1}{b}}}[\![v_{1i}|]\!]\right)^{\frac{3}{2}}$$

$$=2^{\frac{1}{2}}\left(|v_{1i}|-\operatorname{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{1}{b}}}[\![v_{1i}|]\!]\right)^{\frac{3}{2}}+2^{\frac{1}{2}}\left(\operatorname{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3}{2b}}}\left[|v_{1i}|^{\frac{3}{2}}\right]\right)$$

Note that the term

$$r_1(1+b)k_{ai}\left(\frac{\varepsilon_{1\mathrm{i}}}{k_{1i}}\right)^{\frac{3}{2}}\left(\int_0^{v_{1i}}|r|^{\frac{3}{2}}\operatorname{sign}(r)-\operatorname{sat}_{\left(\frac{\varepsilon_{1\mathrm{i}}}{k_{1i}}\right)^{\frac{3}{2b}}}\left[|r|^{\frac{3}{2}}\operatorname{sign}(r)\right]\ dr\right)\ge 0$$

$$r_1\left(\frac{3+3b}{2}\right)\left(\int_0^{v_{1i}} |r|^{\frac{1}{2}} \mathrm{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3}{2}}}[|r|^{\frac{3b}{2}}]\, dr\right) v_{2i} \geq -\begin{pmatrix} r_1(1+b)\mathrm{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3}{2}}}\left[|v_{1i}|^{\frac{3b}{2}}\right]|v_{1i}|^{\frac{3}{2}} \\ -r_1 b\,\mathrm{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3+3b}{2b}}}\left[|v_{1i}|^{\frac{3+3b}{2}}\right] \end{pmatrix}|v_{2i}|$$

$$\geq -\begin{pmatrix} r_1 2^{\frac{1}{2}}(1+b)\mathrm{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3}{2}}}\left[|v_{1i}|^{\frac{3b}{2}}\right]\left(|v_{1i}| - \mathrm{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{1}{b}}}[|v_{1i}|]\right)^{\frac{3}{2}} \\ +r_1\left(2^{\frac{1}{2}} + \left(2^{\frac{1}{2}} - 1\right)b\right)\mathrm{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3+3b}{2b}}}\left[|v_{1i}|^{\frac{3+3b}{2}}\right] \end{pmatrix}|v_{2i}|$$

$$\geq -\begin{pmatrix} r_1 2^{\frac{1}{2}}(1+b)\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3}{2}}\left(|v_{1i}| - \mathrm{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{1}{b}}}[|v_{1i}|]\right)^{\frac{3}{2}} \\ +r_1\left(2^{\frac{1}{2}} + \left(2^{\frac{1}{2}} - 1\right)b\right)\mathrm{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3+3b}{2b}}}\left[|v_{1i}|^{\frac{3+3b}{2}}\right] \end{pmatrix}|v_{2i}|$$

Now, firstly note that the upper bound on the Lyapunov function:

$$V \leq \left(\frac{1}{2}|v_{2i}|^2 + \frac{k_{1i}}{1+b}\mathrm{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{1+b}{b}}}\left[|v_{1i}|^{1+b}\right] + \varepsilon_{1i}\left(|v_{1i}| - \mathrm{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{1}{b}}}[|v_{1i}|]\right)\right)^2$$

$$+ r_1\,\mathrm{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3+3b}{2b}}}\left[|v_{1i}|^{\frac{3+3b}{2}}\right]|v_{2i}| + r_1(1+b)\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3}{2}}\left(|v_{1i}|^{\frac{3}{2}} - \mathrm{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3}{2b}}}\left[|v_{1i}|^{\frac{3}{2}}\right]\right)|v_{2i}|$$

$$+ r_1(1+b)k_{ai}\left(\frac{\varepsilon_{1\mathrm{i}}}{k_{1i}}\right)^{\frac{3}{2}}\left(\int_0^{v_{1i}} |r|^{\frac{3}{2}}\mathrm{sign}(r) - \mathrm{sat}_{\left(\frac{\varepsilon_{1\mathrm{i}}}{k_{1i}}\right)^{\frac{3}{2b}}}\left[|r|^{\frac{3}{2}}\mathrm{sign}(r)\right] dr\right)$$

which is positive definite and radially unbounded.

Next, the lower bound on the Lyapunov function, using the above properties, can be obtained as:

$$V \geq \left(\frac{1}{2} v_{2i}{}^2 + \frac{k_{1i}}{1+b} \operatorname{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{1+b}{b}}} \left[|v_{1i}|^{1+b} \right] + \varepsilon_{1i} \left(|v_{1i}| - \operatorname{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{1}{b}}} [|v_{1i}|] \right) \right)^2$$

$$- r_1 2^{\frac{1}{2}} (1+b) \left(\frac{\varepsilon_{1i}}{k_{1i}} \right)^{\frac{3}{2}} \left(|v_{1i}| - \operatorname{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{1}{b}}} [|v_{1i}|] \right)^{\frac{3}{2}} |v_{2i}|$$

$$- r_1 \left(2^{\frac{1}{2}} + \left(2^{\frac{1}{2}} - 1 \right) b \right) \operatorname{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3+3b}{2b}}} \left[|v_{1i}|^{\frac{3+3b}{2}} \right] |v_{2i}|$$

$$\geq \frac{1}{8} |v_{2i}|^4 + \frac{1}{2} \left(\frac{k_{1i}}{1+b} \right)^2 \operatorname{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{2+2b}{b}}} \left[|v_{1i}|^{2+2b} \right] + \frac{\varepsilon_{1i}{}^2}{2} \left(|v_{1i}| - \operatorname{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{1}{b}}} [|v_{1i}|] \right)^2$$

$$+ \frac{1}{16} |v_{2i}|^4 + \frac{\varepsilon_{1i}{}^2}{2} \left(|v_{1i}| - \operatorname{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{1}{b}}} [|v_{1i}|] \right)^2 - r_1 2^{\frac{1}{2}} (1+b) \left(\frac{\varepsilon_{1i}}{k_{1i}} \right)^{\frac{3}{2}} \left(|v_{1i}| - \operatorname{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{1}{b}}} [|v_{1i}|] \right)^{\frac{3}{2}} |v_{2i}|$$

$$+ \frac{1}{16} |v_{2i}|^4 + \frac{1}{2} \left(\frac{k_{1i}}{1+b} \right)^2 \operatorname{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{2+2b}{b}}} \left[|v_{1i}|^{2+2b} \right] - r_1 \left(2^{\frac{1}{2}} + \left(2^{\frac{1}{2}} - 1 \right) b \right) \operatorname{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3+3b}{2b}}} \left[|v_{1i}|^{\frac{3+3b}{2}} \right] |v_{2i}|$$

Using lemma 2.1,

$$\frac{1}{16} |v_{2i}|^4 + \frac{\varepsilon_{1i}{}^2}{2} \left(|v_{1i}| - \operatorname{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{1}{b}}} [|v_{1i}|] \right)^2 \geq \left(\frac{1}{4} \right)^{\frac{1}{4}} |v_{2i}| \left(\frac{4\varepsilon_{1i}{}^2}{6} \right)^{\frac{3}{4}} \left(|v_{1i}| - \operatorname{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{1}{b}}} [|v_{1i}|] \right)^{\frac{3}{2}}$$

and

$$\frac{1}{16}|v_{2i}|^4+\frac{1}{2}\left(\frac{k_{1i}}{1+b}\right)^2\operatorname{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{2+2b}{b}}}\left[|v_{1i}|^{2+2b}\right]\geq\left(\frac{1}{4}\right)^{\frac{1}{4}}|v_{2i}|\left(\frac{2}{3}\left(\frac{k_{1i}}{1+b}\right)^2\right)^{\frac{3}{4}}\operatorname{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{2+2b}{2b}}}\left[|v_{1i}|^{\frac{3+3b}{2}}\right]$$

Consequently:

$$V\geq\frac{1}{8}|v_{2i}|^4+\frac{1}{2}\left(\frac{k_{1i}}{1+b}\right)^2\operatorname{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{2+2b}{b}}}\left[|v_{1i}|^{2+2b}\right]+\frac{\varepsilon_{1i}{}^2}{2}\left(|v_{1i}|-\operatorname{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{1}{b}}}\left[|v_{1i}|\right]\right)^2$$

with

$$\min\left\{\left(\frac{2^{\frac{1}{2}}}{6^{\frac{3}{4}}}\right)\left(\frac{k_{1i}{}^{\frac{3}{2}}}{1+b}\right),\left(\frac{2^{\frac{1}{2}}}{6^{\frac{3}{4}}}\right)\frac{2^{\frac{1}{2}}}{\left(2^{\frac{1}{2}}+\left(2^{\frac{1}{2}}-1\right)b\right)}\left(\frac{k_{1i}}{1+b}\right)^{\frac{3}{2}}\right\}>r_1$$

Note that such an $r_1 > 0$ always exists for any positive k_{1i} and b. Thus, the Lyapunov function is positive definite and radially unbounded. In accordance to lemma 2.4, the time derivative of the Lyapunov function along the solutions of the system exists almost everywhere:

<u>Time derivative of Lyapunov function $b > 0$:</u>

Note that for $b > 0$, V is continuously differentiable:

$$\dot{\tilde{V}}=\bigcap_{\xi\in\partial V(v(t),t)}\xi^T K\begin{bmatrix}\dot{v}_{1i}\\ \dot{v}_{2i}\end{bmatrix}=\nabla V^T K\begin{bmatrix}\dot{v}_{1i}\\ \dot{v}_{2i}\end{bmatrix}\subset\dot{V}_{1s}+\dot{V}_{2s}+\dot{V}_s$$

with

$$
\dot{V}_{1s} = \left(
\begin{array}{l}
-\left(\frac{1}{2}v_{2i}{}^{2} + \int_{0}^{v_{1i}} \mathrm{sat}_{\varepsilon_{1i}}\left[k_{1i}|r|^{b}\,\mathrm{sign}(r)\right]dr\right) \times \left(v_{2i}\mathrm{sat}_{\varepsilon_{2i}}\left[k_{2i}|v_{2i}|^{\frac{2b}{1+b}}\mathrm{sign}(v_{2i})\right]\right) \\
+ r_1\left(\frac{3+3b}{2}\right)|v_{1i}|^{\frac{1}{2}}\mathrm{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3}{2}}}[|v_{1i}|^{\frac{3b}{2}}]v_{2i}(v_{2i}) \\
- r_1\,\mathrm{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3+3b}{2b}}}\left[|v_{1i}|^{\frac{3+3b}{2}}\right]\mathrm{sign}(v_{1i}) \times \left(\begin{array}{l} k_{ai}v_{2i} - k_{ai}\mathrm{sat}_{\left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+b}{2b}}}[v_{2i}] \\ + \mathrm{sat}_{\varepsilon_{2i}}\left[k_{2i}|v_{2i}|^{\frac{2b}{1+b}}\mathrm{sign}(v_{2i})\right] \end{array}\right) \\
- \frac{r_1}{2}\mathrm{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3+3b}{2b}}}\left[|v_{1i}|^{\frac{3+3b}{2}}\right]\mathrm{sat}_{\varepsilon_{1i}}\left[k_{1i}|v_{1i}|^{b}\right] \\
- \left(\begin{array}{l} \frac{r_1(1+b)}{2}\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3}{2}}\left(|v_{1i}|^{\frac{3}{2}} - \mathrm{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3}{2b}}}\left[|v_{1i}|^{\frac{3}{2}}\right]\right)\mathrm{sign}(v_{1i}) \\ \quad \times\left(k_{ai}\mathrm{sat}_{\mu_i}[\tilde{e}_{2i}] - k_{ai}\mathrm{sat}_{\left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+b}{2b}}}\left[v_{2i} + s_i + \mathrm{sat}_{\mu_i}[\tilde{e}_{2i}]\right]\right) \end{array}\right) \\
- \left(\begin{array}{l} \frac{r_1(1+b)}{2}\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3}{2}}\left(|v_{1i}|^{\frac{3}{2}} - \mathrm{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3}{2b}}}\left[|v_{1i}|^{\frac{3}{2}}\right]\right)\mathrm{sign}(v_{1i}) \\ \quad \times\mathrm{sat}_{\varepsilon_{2i}}\left[k_{2i}|v_{2i} + s_i + \mathrm{sat}_{\mu_i}[\tilde{e}_{2i}]|^{\frac{2b}{1+b}}\mathrm{sign}\left(v_{2i} + s_i + \mathrm{sat}_{\mu_i}[\tilde{e}_{2i}]\right)\right] \end{array}\right) \\
- \frac{r_1(1+b)}{2}\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3}{2}}\left(|v_{1i}|^{\frac{3}{2}} - \mathrm{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3}{2b}}}\left[|v_{1i}|^{\frac{3}{2}}\right]\right)\mathrm{sign}(v_{1i})\left(\mathrm{sat}_{\varepsilon_{1i}}\left[k_{1i}|v_{1i}|^{b}\,\mathrm{sign}(v_{1i})\right]\right)
\end{array}
\right)
$$

$$
\dot{V}_{2s} = \left(
\begin{aligned}
& -\frac{r_1}{2}\operatorname{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3+3b}{2b}}}\left[\left|v_{1i}\right|^{\frac{3+3b}{2}}\right]\operatorname{sat}_{\varepsilon_{1i}}\left[k_{1i}\left|v_{1i}\right|^{b}\right] \\
& -\left(
\begin{aligned}
& \frac{r_1(1+b)}{2}\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3}{2}}\left(\left|v_{1i}\right|^{\frac{3}{2}}-\operatorname{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3}{2b}}}\left[\left|v_{1i}\right|^{\frac{3}{2}}\right]\right)\operatorname{sign}(v_{1i}) \\
& \times\left(k_{ai}\operatorname{sat}_{\mu_i}\left[\tilde{e}_{2i}\right]-k_{ai}\operatorname{sat}_{\left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+b}{2b}}}\left[v_{2i}+s_i+\operatorname{sat}_{\mu_i}\left[\tilde{e}_{2i}\right]\right]\right)
\end{aligned}
\right) \\
& -\left(
\begin{aligned}
& \frac{r_1(1+b)}{2}\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3}{2}}\left(\left|v_{1i}\right|^{\frac{3}{2}}-\operatorname{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3}{2b}}}\left[\left|v_{1i}\right|^{\frac{3}{2}}\right]\right)\operatorname{sign}(v_{1i}) \\
& \times\operatorname{sat}_{\varepsilon_{2i}}\left[k_{2i}\left|v_{2i}+s_i+\operatorname{sat}_{\mu_i}\left[\tilde{e}_{2i}\right]\right|^{\frac{2b}{1+b}}\operatorname{sign}\left(v_{2i}+s_i+\operatorname{sat}_{\mu_i}\left[\tilde{e}_{2i}\right]\right)\right]
\end{aligned}
\right) \\
& -\frac{r_1(1+b)}{2}\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3}{2}}\left(\left|v_{1i}\right|^{\frac{3}{2}}-\operatorname{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3}{2b}}}\left[\left|v_{1i}\right|^{\frac{3}{2}}\right]\right)\operatorname{sign}(v_{1i})\left(\operatorname{sat}_{\varepsilon_{1i}}\left[k_{1i}\left|v_{1i}\right|^{b}\operatorname{sign}(v_{1i})\right]\right) \\
& -\left(\frac{1}{2}{v_{2i}}^{2}+\int_0^{v_{1i}}\operatorname{sat}_{\varepsilon_{1i}}\left[k_{1i}\left|r\right|^{b}\operatorname{sign}(r)\right]dr\right)\times\left(v_{2i}\operatorname{sat}_{\varepsilon_{2i}}\left[k_{2i}\left|v_{2i}\right|^{\frac{2b}{1+b}}\operatorname{sign}(v_{2i})\right]\right) \\
& -2\left(\frac{1}{2}{v_{2i}}^{2}+\int_0^{v_{1i}}\operatorname{sat}_{\varepsilon_{1i}}\left[k_{1i}\left|r\right|^{b}\operatorname{sign}(r)\right]dr\right)\times\left(v_{2i}k_{ai}v_{2i}-v_{2i}k_{ai}\operatorname{sat}_{\left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+b}{2b}}}\left[v_{2i}\right]\right)
\end{aligned}
\right)
$$

$$
\dot{V}_s = \left(
\begin{aligned}
&-\left(
\begin{aligned}
&2\left(\frac{1}{2}v_{2i}{}^{2} + \int_0^{v_{1i}} \mathrm{sat}_{\varepsilon_{1i}}\left[k_{1i}|r|^{b}\,\mathrm{sign}(r)\right]dr\right)\\
&\times\left(
\begin{aligned}
&v_{2i}k_{ai}s_i + v_{2i}k_{ai}\mathrm{sat}_{\mu_i^{\frac{1}{a}}}\left[\tilde{e}_{2i}\right]\\
&- v_{2i}k_{ai}\mathrm{sat}_{\left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+b}{2b}}}\left[v_{2i} + s_i + \mathrm{sat}_{\mu_i}\left[\tilde{e}_{2i}\right]\right] + v_{2i}k_{ai}\mathrm{sat}_{\gamma_i}\left[v_{2i}\right]\\
&+ v_{2i}\mathrm{sat}_{\varepsilon_{2i}}\left[k_{2i}\left|v_{2i} + s_i + \mathrm{sat}_{\mu_i}\left[\tilde{e}_{2i}\right]\right|^{\frac{2b}{1+b}}\mathrm{sign}\left(v_{2i} + s_i + \mathrm{sat}_{\mu_i}\left[\tilde{e}_{2i}\right]\right)\right]\\
&- v_{2i}\mathrm{sat}_{\varepsilon_{2i}}\left[k_{2i}\left|v_{2i}\right|^{\frac{2b}{1+b}}\mathrm{sign}\left(v_{2i}\right)\right] - s_i\ \mathrm{sat}_{\varepsilon_{1i}}\left[k_{1i}\left|v_{1i}\right|^{b}\mathrm{sign}\left(v_{1i}\right)\right]
\end{aligned}
\right)
\end{aligned}
\right)\\
&+ r_1\left(\frac{3+3b}{2}\right)\left|v_{1i}\right|^{\frac{1}{2}}\mathrm{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3}{2}}}\left[\left|v_{1i}\right|^{\frac{3b}{2}}\right]v_{2i}\left(s_i\right)\\
&-\left(
\begin{aligned}
&r_1\ \mathrm{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3+3b}{2b}}}\left[\left|v_{1i}\right|^{\frac{3+3b}{2}}\right]\mathrm{sign}\left(v_{1i}\right)\\
&\times\left(
\begin{aligned}
&k_{ai}\left(s_i + \mathrm{sat}_{\mu_i}\left[\tilde{e}_{2i}\right]\right) + k_{ai}\mathrm{sat}_{\left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+b}{2b}}}\left[v_{2i}\right]\\
&- k_{ai}\mathrm{sat}_{\left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+b}{2b}}}\left[v_{2i} + s_i + \mathrm{sat}_{\mu_i}\left[\tilde{e}_{2i}\right]\right]\\
&+ \mathrm{sat}_{\varepsilon_{2i}}\left[k_{2i}\left|v_{2i} + s_i + \mathrm{sat}_{\mu_i}\left[\tilde{e}_{2i}\right]\right|^{\frac{2b}{1+b}}\mathrm{sign}\left(v_{2i} + s_i + \mathrm{sat}_{\mu_i}\left[\tilde{e}_{2i}\right]\right)\right]\\
&- \mathrm{sat}_{\varepsilon_{2i}}\left[k_{2i}\left|v_{2i}\right|^{\frac{2b}{1+b}}\mathrm{sign}\left(v_{2i}\right)\right]
\end{aligned}
\right)
\end{aligned}
\right)
\end{aligned}
\right)
$$

Note that the following property has been employed:

$$
r_1\left(\frac{3+3b}{2}\right)\int_0^{v_{1i}}|r|^{\frac{1}{2}}\mathrm{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3}{2}}}\left[|r|^{\frac{3b}{2}}\right]dr = \left(
\begin{aligned}
&r_1\ \mathrm{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3+3b}{2b}}}\left[\left|v_{1i}\right|^{\frac{3+3b}{2}}\right]\\
&+ r_1(1+b)\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3}{2}}\left(\left|v_{1i}\right|^{\frac{3}{2}} - \mathrm{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3}{2b}}}\left[\left|v_{1i}\right|^{\frac{3}{2}}\right]\right)
\end{aligned}
\right)\mathrm{sign}\left(v_{1i}\right)
$$

Let us define the following terms $\dot{V}_1$ and $\dot{V}_2$:

$$
\dot{V}_1 = \begin{pmatrix}
-\left(\frac{1}{2}|v_{2i}|^2 + \int_0^{v_{1i}} \operatorname{sat}_{\varepsilon_{1i}}\left[k_{1i}|r|^b \operatorname{sign}(r)\right] dr\right) \times \left(|v_{2i}| \operatorname{sat}_{\varepsilon_{2i}}\left[k_{2i}|v_{2i}|^{\frac{2b}{1+b}}\right]\right) \\
+ r_1\left(\frac{3+3b}{2}\right)|v_{1i}|^{\frac{1}{2}} \operatorname{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3}{2}}}[|v_{1i}|^{\frac{3b}{2}}]|v_{2i}|^2 \\
+\left(r_1 \operatorname{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3+3b}{2b}}}\left[|v_{1i}|^{\frac{3+3b}{2}}\right]\right) \times \left(k_{ai}|v_{2i}| - k_{ai} \operatorname{sat}_{\left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+b}{2b}}}\left[|v_{2i}|\right] + \operatorname{sat}_{\varepsilon_{2i}}\left[k_{2i}|v_{2i}|^{\frac{2b}{1+b}}\right]\right) \\
-\frac{r_1}{2} \operatorname{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3+3b}{2b}}}\left[|v_{1i}|^{\frac{3+3b}{2}}\right] \operatorname{sat}_{\varepsilon_{1i}}\left[k_{1i}|v_{1i}|^b\right] \\
-\left(\frac{r_1(1+b)}{2}\right)\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3}{2}} \varepsilon_{3i}\left(|v_{1i}|^{\frac{3}{2}} - \operatorname{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3}{2b}}}\left[|v_{1i}|^{\frac{3}{2}}\right]\right)
\end{pmatrix}
$$

$$
\dot{V}_2 = \begin{pmatrix}
-\frac{r_1}{2} \operatorname{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3+3b}{2b}}}\left[|v_{1i}|^{\frac{3+3b}{2}}\right] \operatorname{sat}_{\varepsilon_{1i}}\left[k_{1i}|v_{1i}|^b\right] \\
-\left(\frac{r_1(1+b)}{2}\right)\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3}{2}} \varepsilon_{3i}\left(|v_{1i}|^{\frac{3}{2}} - \operatorname{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3}{2b}}}\left[|v_{1i}|^{\frac{3}{2}}\right]\right) \\
-\left(\frac{1}{2}|v_{2i}|^2 + \int_0^{v_{1i}} \operatorname{sat}_{\varepsilon_{1i}}\left[k_{1i}|r|^b \operatorname{sign}(r)\right] dr\right) \times \left(|v_{2i}| \operatorname{sat}_{\varepsilon_{2i}}\left[k_{2i}|v_{2i}|^{\frac{2b}{1+b}}\right]\right) \\
-2\left(\frac{1}{2}|v_{2i}|^2 + \int_0^{v_{1i}} \operatorname{sat}_{\varepsilon_{1i}}\left[k_{1i}|r|^b \operatorname{sign}(r)\right] dr\right) \times \left(|v_{2i}|k_{ai}|v_{2i}| - |v_{2i}|k_{ai} \operatorname{sat}_{\left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+b}{2b}}}\left[|v_{2i}|\right]\right)
\end{pmatrix}
$$

with

$$\varepsilon_{3i} := \begin{cases} \varepsilon_{1i} - k_{ai}\mu_i - k_{ai}\left(\dfrac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+b}{2b}} - \varepsilon_{2i}, & \text{for } \left|\tilde{e}_{2i}\right| \neq 0, \\ \varepsilon_{1i} - k_{ai}\left(\dfrac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+b}{2b}} - \varepsilon_{2i}, & \text{for } \tilde{e}_{2i} = 0 \quad \text{(for the case of full state feedback)} \end{cases}$$

Note that $\dot{V}_{1s} \leq \dot{V}_1$ and $\dot{V}_{2s} \leq \dot{V}_2$, thus,

$$\dot{\tilde{V}} \leq \dot{V}_1 + \dot{V}_2 + \dot{V}_s$$

Note that $\dot{V}_2$ is negative definite for $\varepsilon_{3i} > 0$. $\dot{V}_s$ is considered as perturbation term.

We are going to show that an $r_1 > 0$ exists such that the $\dot{V}_1$ is negative definite.

<u>Sign definiteness of $\dot{V}_1$:</u>

Let us divide the entire state space into three regions:

$$\Omega_1 = \left\{ \left(v_{1i}, v_{2i}\right) \in \mathbb{R}^2 : \left|v_{1i}\right| \in \mathbb{R},\ \left|v_{2i}\right| \geq \left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+b}{2b}} \right\},$$

$$\Omega_2 = \left\{ \left(v_{1i}, v_{2i}\right) \in \mathbb{R}^2 : \left|v_{1i}\right| \geq \left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{1}{b}},\ \left|v_{2i}\right| \leq \left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+b}{2b}} \right\},$$

$$\Omega_3 = \left\{ \left(v_{1i}, v_{2i}\right) \in \mathbb{R}^2 : v_{1i} \leq \left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{1}{b}},\ v_{2i} \leq \left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+b}{2b}} \right\}$$

For region: $\Omega_1 = \left\{ \left(v_{1i}, v_{2i}\right) \in \mathbb{R}^2 : \left|v_{1i}\right| \in \mathbb{R},\ \left|v_{2i}\right| \geq \left(\dfrac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+b}{2b}} \right\}$

Note that:

$$\int_0^{v_{1i}} \operatorname{sat}_{\varepsilon_{1i}}\left[k_{1i}|r|^b \operatorname{sign}(r)\right] dr = \frac{k_{1i}}{1+b} \operatorname{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{1+b}{b}}}\left[|v_{1i}|^{1+b}\right] + \varepsilon_{1i}\left(|v_{1i}| - \operatorname{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{1}{b}}}(|v_{1i}|)\right)$$

and

$$|v_{2i}| \operatorname{sat}_{\varepsilon_{2i}}\left[k_{2i}|v_{2i}|^{\frac{2b}{1+b}}\right] = \varepsilon_{2i}\left(|v_{2i}| - \operatorname{sat}_{\left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+b}{2b}}}\left[|v_{2i}|\right]\right) + \operatorname{sat}_{\varepsilon_{2i}}\left[k_{2i}|v_{2i}|^{\frac{2b}{1+b}}\right] \operatorname{sat}_{\left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+b}{2b}}}\left[|v_{2i}|\right]$$

and

$$\operatorname{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3+3b}{2b}}}\left[|v_{1i}|^{\frac{3+3b}{2}}\right] = \operatorname{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{1+b}{b}}}\left[|v_{1i}|^{1+b}\right] \operatorname{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{1+b}{2b}}}\left[|v_{1i}|^{\frac{1+b}{2}}\right] \le \operatorname{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{1+b}{b}}}\left[|v_{1i}|^{1+b}\right]\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{1+b}{2b}}$$

and using lemma 2 of chapter 2, we have:

$$\begin{aligned}
|v_{1i}|^{\frac{1}{2}} &= \left(|v_{1i}| - \operatorname{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{1}{b}}}(|v_{1i}|) + \operatorname{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{1}{b}}}(|v_{1i}|)\right)^{\frac{1}{2}} \le \left(|v_{1i}| - \operatorname{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{1}{b}}}(|v_{1i}|)\right)^{\frac{1}{2}} + \left(\operatorname{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{1}{b}}}(|v_{1i}|)\right)^{\frac{1}{2}} \\
&\le \left(|v_{1i}| - \operatorname{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{1}{b}}}(|v_{1i}|)\right)^{\frac{1}{2}} + \left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{1}{2b}}
\end{aligned}$$

Then,

$$
\begin{aligned}
\dot{V}_1 \le & -\left(\frac{1}{2}|v_{2i}| - \varepsilon_{1i}^{\frac{1}{2}}\left(|v_{1i}| - \operatorname{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{1}{b}}}\left(|v_{1i}|\right)\right)^{\frac{1}{2}}\right)^2 \times \left(|v_{2i}| \operatorname{sat}_{\varepsilon_{2i}}\left[k_{2i}|v_{2i}|^{\frac{2b}{1+b}}\right]\right) \\
& -\left(|v_{1i}| - \operatorname{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{1}{b}}}\left(|v_{1i}|\right)\right)^{\frac{1}{2}} |v_{2i}|^2 \left(\varepsilon_{1i}^{\frac{1}{2}} \operatorname{sat}_{\varepsilon_{2i}}\left[k_{2i}|v_{2i}|^{\frac{2b}{1+b}}\right] - r_1\left(\frac{3+3b}{2}\right)\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3}{2}}\right) \\
& -|v_{2i}|^2\left(\frac{1}{4}|v_{2i}| \operatorname{sat}_{\varepsilon_{2i}}\left[k_{2i}|v_{2i}|^{\frac{2b}{1+b}}\right] - r_1\left(\frac{3+3b}{2}\right)\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{1+3b}{2b}}\right) \\
& -\operatorname{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{1+b}{b}}}\left[|v_{1i}|^{1+b}\right]\left(|v_{2i}| - \operatorname{sat}_{\left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+b}{2b}}}\left[|v_{2i}|\right]\right)\left(\frac{k_{1i}}{1+b}\varepsilon_{2i} - r_1\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{1+b}{2b}} k_{ai}\right) \\
& -\operatorname{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{1+b}{b}}}\left[|v_{1i}|^{1+b}\right] \operatorname{sat}_{\varepsilon_{2i}}\left[k_{2i}|v_{2i}|^{\frac{2b}{1+b}}\right]\left(\frac{k_{1i}}{1+b} \operatorname{sat}_{\left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+b}{2b}}}\left[|v_{2i}|\right] - r_1\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{1+b}{2b}}\right)
\end{aligned}
$$

Thus, $\dot{V}_1 < 0$ for:

$$
\min\left\{
\begin{array}{l}
\left(\frac{2}{3+3b}\right) k_{1i}^{\frac{3}{2}}\left(\frac{\varepsilon_{2i}}{\varepsilon_{1i}}\right), \quad \left(\frac{1}{6+6b}\right)\left(\frac{k_{1i}^{\frac{1+3b}{2b}}}{k_{2i}^{\frac{1+b}{2b}}}\right)\left(\frac{\varepsilon_{2i}}{\varepsilon_{1i}}\right)^{\frac{1+3b}{2b}}, \\
\left(\frac{k_{1i}^{\frac{1+3b}{2b}}}{k_{ai}(1+b)}\right)\left(\frac{\varepsilon_{2i}}{\varepsilon_{1i}^{\frac{1+b}{2b}}}\right), \quad \left(\frac{1}{1+b}\right)\left(\frac{k_{1i}^{\frac{1+3b}{2b}}}{k_{2i}^{\frac{1+b}{2b}}}\right)\left(\frac{\varepsilon_{2i}}{\varepsilon_{1i}}\right)^{\frac{1+b}{2b}}
\end{array}
\right\} > r_1
$$

For region: $\Omega_2 = \left\{ (v_{1i}, v_{2i}) \in \mathbb{R}^2 : |v_{1i}| \ge \left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{1}{b}},\ |v_{2i}| \le \left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+b}{2b}} \right\}$

Note that in this region the term $|v_{1i}|$ is lower bounded, thus

$$\int_0^{v_{1i}} \mathrm{sat}_{\varepsilon_{1i}}\left[k_{1i}|r|^b \operatorname{sign}(r)\right] dr = \frac{k_{1i}}{1+b}\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{1+b}{b}} + \varepsilon_{1i}\left(|v_{1i}| - \left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{1}{b}}\right)$$

also in this region the term $|v_{2i}|$ is upper bounded, thus we have,

$$k_{ai}|v_{2i}| - k_{ai}\mathrm{sat}_{\left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+b}{2b}}}\left[|v_{2i}|\right] = 0 \quad \text{and}$$

$$\begin{aligned}|v_{2i}|^2 = \mathrm{sat}_{\left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+b}{b}}}\left[|v_{2i}|^2\right] &= \mathrm{sat}_{\left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+3b}{3b}}}\left[|v_{2i}|^{\frac{2+6b}{3+3b}}\right]\mathrm{sat}_{\left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{2}{3b}}}\left[|v_{2i}|^{\frac{4}{3+3b}}\right] \\ &\le \mathrm{sat}_{\left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+3b}{3b}}}\left[|v_{2i}|^{\frac{2+6b}{3+3b}}\right]\left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{2}{3b}}\end{aligned}$$

or

$$\begin{aligned}|v_{2i}|^2 = \mathrm{sat}_{\left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+b}{b}}}\left[|v_{2i}|^2\right] &= \mathrm{sat}_{\left(\frac{\varepsilon_{2i}}{k_{2i}}\right)}\left[|v_{2i}|^{\frac{2b}{1+b}}\right]\mathrm{sat}_{\left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1}{b}}}\left[|v_{2i}|^{\frac{2}{1+b}}\right] \\ &\le \mathrm{sat}_{\left(\frac{\varepsilon_{2i}}{k_{2i}}\right)}\left[|v_{2i}|^{\frac{2b}{1+b}}\right]\left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1}{b}}\end{aligned}$$

and using lemma 2.2:

$$|v_{1i}|^{\frac{1}{2}} = \left(|v_{1i}|^{\frac{3}{2}} - \left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3}{2b}} + \left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3}{2b}}\right)^{\frac{1}{3}} \le \left(|v_{1i}|^{\frac{3}{2}} - \left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3}{2b}}\right)^{\frac{1}{3}} + \left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{1}{2b}}$$

Then,

$$
\begin{aligned}
\dot{V}_1 \le & -\varepsilon_{1i}\left(\left|v_{1i}\right| - \left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{1}{b}}\right)\operatorname{sat}_{\left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+b}{2b}}\varepsilon_{2i}}\left[k_{2i}\left|v_{2i}\right|^{\frac{1+3b}{1+b}}\right] \\
& -\frac{r_1(1+b)}{2}\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3}{2}}\varepsilon_{3i}\left(\left|v_{1i}\right|^{\frac{3}{2}} - \left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3}{2b}}\right) - \left(\frac{k_{1i}}{1+b}\right)\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{1+b}{b}} k_{2i}\operatorname{sat}_{\left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+3b}{2b}}}\left[\left|v_{2i}\right|^{\frac{1+3b}{1+b}}\right] \\
& + r_1\left(\frac{3+3b}{2}\right)\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3}{2}}\left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{2}{3b}}\left(\left|v_{1i}\right|^{\frac{3}{2}} - \left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3}{2b}}\right)^{\frac{1}{3}}\operatorname{sat}_{\left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+3b}{3b}}}\left[\left|v_{2i}\right|^{\frac{2+6b}{3+3b}}\right] \\
& -\frac{r_1}{4}\left(\frac{\varepsilon_{1i}^{\frac{3+5b}{2b}}}{k_{1i}^{\frac{3+3b}{2b}}}\right) - \frac{r_1}{4}\left(\frac{\varepsilon_{1i}^{\frac{3+5b}{2b}}}{k_{1i}^{\frac{3+3b}{2b}}}\right) - \frac{1}{2}k_{2i}\operatorname{sat}_{\left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{3+5b}{2b}}}\left[\left|v_{2i}\right|^{\frac{3+5b}{1+b}}\right] \\
& + r_1\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{1+3b}{2b}}\left(\left(\frac{3+3b}{2}\right)\left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1}{b}} + \left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{1}{b}}k_{2i}\right)\operatorname{sat}_{\left(\frac{\varepsilon_{2i}}{k_{2i}}\right)}\left[\left|v_{2i}\right|^{\frac{2b}{1+b}}\right]
\end{aligned}
$$

From lemma 2.1,

$$
\begin{aligned}
& -\frac{r_1(1+b)}{2}\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3}{2}}\varepsilon_{3i}\left(\left|v_{1i}\right|^{\frac{3}{2}} - \left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3}{2b}}\right) - \left(\frac{k_{1i}}{1+b}\right)\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{1+b}{b}} k_{2i}\operatorname{sat}_{\left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+3b}{2b}}}\left[\left|v_{2i}\right|^{\frac{1+3b}{1+b}}\right] \\
& \le -r_1^{\frac{1}{3}}\left(\left(\frac{3+3b}{2}\right)\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3}{2}}\varepsilon_{3i}\left(\left|v_{1i}\right|^{\frac{3}{2}} - \left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3}{2b}}\right)\right)^{\frac{1}{3}}\left(\frac{3k_{1i}k_{2i}}{2+2b}\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{1+b}{b}}\operatorname{sat}_{\left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+3b}{2b}}}\left[\left|v_{2i}\right|^{\frac{1+3b}{1+b}}\right]\right)^{\frac{2}{3}}
\end{aligned}
$$

and

$$-\frac{r_1}{4}\left(\frac{\varepsilon_{1i}{}^{\frac{3+5b}{2b}}}{k_{1i}{}^{\frac{3+3b}{2b}}}\right)-\frac{1}{2}k_{2i}\operatorname{sat}_{\left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{3+5b}{2b}}}\left[\left|v_{2i}\right|^{\frac{3+5b}{1+b}}\right]$$

$$\leq -r_1^{\frac{3+3b}{3+5b}}\left(\left(\frac{3+5b}{3+3b}\right)\left(\frac{1}{4}\right)\left(\frac{\varepsilon_{1i}{}^{\frac{3+5b}{2b}}}{k_{1i}{}^{\frac{3+3b}{2b}}}\right)\right)^{\frac{3+3b}{3+5b}}\left(\left(\frac{3+5b}{4b}\right)k_{2i}\operatorname{sat}_{\left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{3+5b}{2b}}}\left[\left|v_{2i}\right|^{\frac{3+5b}{1+b}}\right]\right)^{\frac{2b}{3+5b}}$$

Thus, $\dot{V}_1 < 0$ in this region for:

$$r_1^{\frac{1}{3}}\left(\left(\frac{3+3b}{2}\right)\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3}{2}}\varepsilon_{3i}\left(\left|v_{1i}\right|^{\frac{3}{2}}-\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3}{2b}}\right)\right)^{\frac{1}{3}}\left(\frac{3k_{1i}k_{2i}}{2+2b}\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{1+b}{b}}\operatorname{sat}_{\left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+3b}{2b}}}\left[\left|v_{2i}\right|^{\frac{1+3b}{1+b}}\right]\right)^{\frac{2}{3}}$$

$$> r_1\left(\frac{3+3b}{2}\right)\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3}{2}}\left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{2}{3b}}\left(\left|v_{1i}\right|^{\frac{3}{2}}-\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3}{2b}}\right)^{\frac{1}{3}}\operatorname{sat}_{\left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+3b}{3b}}}\left[\left|v_{2i}\right|^{\frac{2+6b}{3+3b}}\right]$$

$$\Leftrightarrow \varepsilon_{3i}{}^{\frac{1}{2}}\left(\frac{2}{3+3b}\right)\left(\frac{3k_{1i}k_{2i}}{2+2b}\right)\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{1+b}{b}}\left(\frac{k_{1i}}{\varepsilon_{1i}}\right)^{\frac{3}{2}}\left(\frac{k_{2i}}{\varepsilon_{2i}}\right)^{\frac{1}{b}} > r_1$$

Also,

$$r_1^{\frac{3+3b}{3+5b}}\left(\left(\frac{3+5b}{3+3b}\right)\left(\frac{1}{4}\right)\left(\frac{\varepsilon_{1i}{}^{\frac{3+5b}{2b}}}{k_{1i}{}^{\frac{3+3b}{2b}}}\right)\right)^{\frac{3+3b}{3+5b}}\left(\left(\frac{3+5b}{4b}\right)k_{2i}\operatorname{sat}_{\left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{3+5b}{2b}}}\left[\left|v_{2i}\right|^{\frac{3+5b}{1+b}}\right]\right)^{\frac{2b}{3+5b}}$$

$$> r_1\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{1+3b}{2b}}\left(\left(\frac{3+3b}{2}\right)\left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1}{b}}+\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{1}{b}}k_{2i}\right)\operatorname{sat}_{\left(\frac{\varepsilon_{2i}}{k_{2i}}\right)}\left[\left|v_{2i}\right|^{\frac{2b}{1+b}}\right]$$

$$\Leftrightarrow \frac{\left(\frac{3+5b}{12+12b}\right)^{\frac{3+3b}{2b}}\left(\frac{3+5b}{4b}\right)\left(\frac{\varepsilon_{1i}^{\frac{3+5b}{2b}}}{k_{1i}^{\frac{3+3b}{2b}}}\right)^{\frac{3+3b}{2b}}\left(\left(\frac{k_{1i}}{\varepsilon_{1i}}\right)^{\frac{1+3b}{2b}}\right)^{\frac{3+5b}{2b}} k_{2i}}{\left(\left(\frac{3+3b}{2}\right)\left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1}{b}}+\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{1}{b}} k_{2i}\right)^{\frac{3+5b}{2b}}} > r_1$$

For region: $\Omega_3 = \left\{(v_{1i}, v_{2i}) \in \mathbb{R}^2 : v_{1i} \le \left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{1}{b}},\ v_{2i} \le \left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+b}{2b}}\right\}$

Note that in this region:

$$\begin{aligned}\dot{V}_1 = &-\left(\frac{1}{2}|v_{2i}|^2 + \frac{k_{1i}}{1+b}|v_{1i}|^{1+b}\right)\times\left(k_{2i}|v_{2i}|^{\frac{1+3b}{1+b}}\right) \\ &+ r_1\left(\frac{3+3b}{2}\right)|v_{1i}|^{\frac{1+3b}{2}}|v_{2i}|^2 + r_1|v_{1i}|^{\frac{3+3b}{2}} k_{2i}|v_{2i}|^{\frac{2b}{1+b}} - \frac{r_1}{2}k_{1i}|v_{1i}|^{\frac{3+5b}{2}}\end{aligned}$$

It can be easily verified that, for any $c > 0$ and $(v_{1i}, v_{2i}) \in \Omega_3$:

$$\dot{V}_1(c^{\frac{2}{1+b}}v_{1i}, cv_{2i}) = c^{\frac{3+5b}{1+b}}\dot{V}_1(v_{1i}, v_{2i}) \qquad \text{(B1 - 1)}$$

Let us define the boundary of this region, bd Ω_3, which encircles the origin that can be written as the union of the sets

$\mathrm{bd}\,\Omega_3 = \Omega_{3,1} \cup \Omega_{3,2}$ where the sets

$$\Omega_{3,1}=\left\{(v_{1i},v_{2i}):|v_{1i}|\le\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{1}{b}},|v_{2i}|=\left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+b}{2b}}\right\},$$

$$\Omega_{3,2}=\left\{(v_{1i},v_{2i}):|v_{1i}|=\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{1}{b}},|v_{2i}|\le\left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+b}{2b}}\right\}.$$

Due to this homogeneity property (B1 - 1), to show sign definiteness, it suffices to prove sign definiteness on the set bd Ω_3, which encircles the origin, since for every $(v_{1i}, v_{2i}) \in \Omega_3\backslash\{(0,0)\}$ there exists a $c > 0$ such that $(c^{2/(1+b)}\, v_{1i}, cv_{2i}) \in$ bd Ω_3. Note that the sign definiteness of $\dot{V}_1$ has been shown to be negative definite on bd Ω_3 already since $\Omega_{3,1} \in \Omega_1$ and $\Omega_{3,2} \in \Omega_2$. Thus, by the homogeneity property (B1 - 1), we have $\dot{V}_1 < 0$ in this region as well.

As a result, $\dot{V}_1 < 0$, negative definite for all $(v_{1i}, v_{2i}) \in \mathbb{R}^2$ with $r_1 > 0$ chosen as:

$$\min\left\{\begin{array}{l}\left(\frac{2}{3+3b}\right)k_{1i}^{\frac{3}{2}}\left(\frac{\varepsilon_{2i}}{\varepsilon_{1i}}\right),\quad \left(\frac{1}{6+6b}\right)\left(\frac{k_{1i}^{\frac{1+3b}{2b}}}{k_{2i}^{\frac{1+b}{2b}}}\right)\left(\frac{\varepsilon_{2i}}{\varepsilon_{1i}}\right)^{\frac{1+3b}{2b}},\\ \left(\frac{k_{1i}^{\frac{1+3b}{2b}}}{k_{ai}(1+b)}\right)\left(\frac{\varepsilon_{2i}}{\varepsilon_{1i}^{\frac{1+b}{2b}}}\right),\quad \left(\frac{1}{1+b}\right)\left(\frac{k_{1i}^{\frac{1+3b}{2b}}}{k_{2i}^{\frac{1+b}{2b}}}\right)\left(\frac{\varepsilon_{2i}}{\varepsilon_{1i}}\right)^{\frac{1+b}{2b}},\\ \varepsilon_{3i}^{\frac{1}{2}}\left(\frac{2}{3+3b}\right)\left(\frac{3k_{1i}k_{2i}}{2+2b}\right)\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{1+b}{b}}\left(\frac{k_{1i}}{\varepsilon_{1i}}\right)^{\frac{3}{2}}\left(\frac{k_{2i}}{\varepsilon_{2i}}\right)^{\frac{1}{b}},\\ \dfrac{\left(\frac{3+5b}{12+12b}\right)^{\frac{3+3b}{2b}}\left(\frac{3+5b}{4b}\right)\left(\frac{\varepsilon_{1i}^{\frac{3+5b}{2b}}}{k_{1i}^{\frac{3+3b}{2b}}}\right)^{\frac{3+3b}{2b}}\left(\left(\frac{k_{1i}}{\varepsilon_{1i}}\right)^{\frac{1+3b}{2b}}\right)^{\frac{3+5b}{2b}}k_{2i}}{\left(\left(\frac{3+3b}{2}\right)\left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1}{b}}+\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{1}{b}}k_{2i}\right)^{\frac{3+5b}{2b}}}\end{array}\right\}>r_1$$

Observe that such an $r_1 > 0$, always exists for any $k_{1i} > 0$, $k_{2i} > 0$, $\varepsilon_{1i} > 0$, $\varepsilon_{2i} > 0$, and $b > 0$.

Sign definiteness of $\dot{V}_2 + \dot{V}_s$:

From the previous development, we have shown that $\dot{V}_1$ is negative definite and since $\dot{V}_2$ is negative definite, the idea is to dominate the $\dot{V}_s$ term with it. Before we proceed, from Appendix B.2-proposition 1, we have the following inequalities

$$\left| \mathrm{sat}_{\left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+b}{2b}}}\left[v_{2i}\right] - \mathrm{sat}_{\left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+b}{2b}}}\left[v_{2i} + s_i + \mathrm{sat}_{\mu_i}\left[\tilde{e}_{2i}\right]\right] \right| \leq \mathrm{sat}_{2\left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+b}{2b}}}\left[\left|s_i + \mathrm{sat}_{\mu_i}\left[\tilde{e}_{2i}\right]\right|\right]$$

$$\begin{aligned} &\left| \begin{array}{l} \mathrm{sat}_{\varepsilon_{2i}}\left[k_{2i}\left|v_{2i} + s_i + \mathrm{sat}_{\mu_i}\left[\tilde{e}_{2i}\right]\right|^{\frac{2b}{1+b}} \mathrm{sign}\left(v_{2i} + s_i + \mathrm{sat}_{\mu_i}\left[\tilde{e}_{2i}\right]\right)\right] \\ \qquad - \mathrm{sat}_{\varepsilon_{2i}}\left[k_{2i}\left|v_{2i}\right|^{\frac{2b}{1+b}} \mathrm{sign}\left(v_{2i}\right)\right] \end{array} \right| \leq \mathrm{sat}_{2\varepsilon_{2i}}\left[\chi\left(\left|s_i + \mathrm{sat}_{\mu_i}\left[\tilde{e}_{2i}\right]\right|\right)\right] \\ &\qquad\qquad \leq \mathrm{sat}_{2\varepsilon_{2i}}\left[\beta\left(\left|s_i + \mathrm{sat}_{\mu_i}\left[\tilde{e}_{2i}\right]\right|\right)\right] \end{aligned}$$

where β is a class $\mathcal{K}$ function. (Refer to the definition of class $\mathcal{K}$ function in [71]). Hence, after simple algebraic manipulation:

$$
\begin{aligned}
\dot{V}_2+\dot{V}_s \le & -\frac{r_1}{2}\operatorname{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3+3b}{2b}}}\left[\left|v_{1i}\right|^{\frac{3+3b}{2}}\right]\operatorname{sat}_{\varepsilon_{1i}}\left[k_{1i}\left|v_{1i}\right|^{b}\right] \\
& -\left(\frac{r_1(1+b)}{2}\right)\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3}{2}}\varepsilon_{3i}\left(\left|v_{1i}\right|^{\frac{3}{2}}-\operatorname{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3}{2b}}}\left[\left|v_{1i}\right|^{\frac{3}{2}}\right]\right) \\
& -\left(\frac{1}{2}\left|v_{2i}\right|^{2}+\int_0^{v_{1i}}\operatorname{sat}_{\varepsilon_{1i}}\left[k_{1i}|r|^{b}\operatorname{sign}(r)\right]dr\right)\times\left(\left|v_{2i}\right|\operatorname{sat}_{\varepsilon_{2i}}\left[k_{2i}\left|v_{2i}\right|^{\frac{2b}{1+b}}\right]\right) \\
& -2\left(\frac{1}{2}\left|v_{2i}\right|^{2}+\int_0^{v_{1i}}\operatorname{sat}_{\varepsilon_{1i}}\left[k_{1i}|r|^{b}\operatorname{sign}(r)\right]dr\right)\times\left(\left|v_{2i}\right|k_{ai}\left|v_{2i}\right|-\left|v_{2i}\right|k_{ai}\operatorname{sat}_{\left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+b}{2b}}}\left[\left|v_{2i}\right|\right]\right) \\
& +\left(\begin{array}{l}
2\left(\frac{1}{2}\left|v_{2i}\right|^{2}+\int_0^{v_{1i}}\operatorname{sat}_{\varepsilon_{1i}}\left[k_{1i}|r|^{b}\operatorname{sign}(r)\right]dr\right) \\
\times\left(\begin{array}{l}
\left|v_{2i}\right|k_{ai}\left|s_i+\operatorname{sat}_{\mu_i}\left[\tilde{e}_{2i}\right]\right|+\left|v_{2i}\right|k_{ai}\operatorname{sat}_{2\left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+b}{2b}}}\left[\left|s_i+\operatorname{sat}_{\mu_i}\left[\tilde{e}_{2i}\right]\right|\right] \\
+\left|v_{2i}\right|\operatorname{sat}_{2\varepsilon_{2i}}\left[\beta\left(\left|s_i+\operatorname{sat}_{\mu_i}\left[\tilde{e}_{2i}\right]\right|\right)\right]
\end{array}\right)
\end{array}\right) \\
& +2\left(\frac{1}{2}\left|v_{2i}\right|^{2}+\int_0^{v_{1i}}\operatorname{sat}_{\varepsilon_{1i}}\left[k_{1i}|r|^{b}\operatorname{sign}(r)\right]dr\right)\varepsilon_{1i}\left|s_i\right| \\
& +r_1\left(\frac{3+3b}{2}\right)\left|v_{1i}\right|^{\frac{1}{2}}\operatorname{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3}{2}}}\left[\left|v_{1i}\right|^{\frac{3b}{2}}\right]\left|v_{2i}\right|\left|s_i\right| \\
& +\left(\begin{array}{l}
r_1\operatorname{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3+3b}{2b}}}\left[\left|v_{1i}\right|^{\frac{3+3b}{2}}\right] \\
\times\left(\begin{array}{l}
k_{ai}\left|s_i+\operatorname{sat}_{\mu_i}\left[\tilde{e}_{2i}\right]\right|+k_{ai}\operatorname{sat}_{2\left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+b}{2b}}}\left[\left|s_i+\operatorname{sat}_{\mu_i}\left[\tilde{e}_{2i}\right]\right|\right] \\
+\operatorname{sat}_{2\varepsilon_{2i}}\left[\beta\left(\left|s_i+\operatorname{sat}_{\mu_i}\left[\tilde{e}_{2i}\right]\right|\right)\right]
\end{array}\right)
\end{array}\right)
\end{aligned}
$$

The $z_{1i}-z_{2i}$ plane is divided into the following three regions for analysis:

$$\Psi_{1i}(|\zeta_i|) = \{(v_{1i}, v_{2i}) \in \mathbb{R}^2 : |v_{1i}| \in \mathbb{R},\ |v_{2i}| \geq \alpha_2(|\zeta_i|)\},$$

$$\Psi_{2i}(|\zeta_i|) = \{(v_{1i}, v_{2i}) \in \mathbb{R}^2 : |v_{1i}| \geq \alpha_1(|\zeta_i|),\ |v_{2i}| \leq \alpha_2(|\zeta_i|)\},$$

$$\Psi_{3i}(|\zeta_i|) = \{(v_{1i}, v_{2i}) \in \mathbb{R}^2 : |v_{1i}| \leq \alpha_1(|\zeta_i|),\ |v_{2i}| \leq \alpha_2(|\zeta_i|)\},$$

where $|\zeta_i|$ is defined as $|\zeta_i| = \max\{|s_i|, \mathrm{sat}_{\mu_i}[|\tilde{e}_{2i}|]\}$,

while $\alpha_1(|\zeta_i|)$ and $\alpha_2(|\zeta_i|)$ are class $\mathcal{K}$ functions defined by

$$\alpha_1(|\zeta_i|) := \max\left\{
\begin{array}{l}
\left(\dfrac{\mathrm{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{1+3b}{2b}}\varepsilon_{1i}}\left[\alpha_7(|\zeta_i|)\right]}{k_{1i}}\right)^{\frac{2}{1+5b}} + \left(\dfrac{\alpha_7(|\zeta_i|)}{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3}{2}}\varepsilon_{1i}}\right)^2 - \left(\dfrac{\mathrm{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{1+3b}{2b}}\varepsilon_{1i}}\left[\alpha_7(|\zeta_i|)\right]}{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3}{2}}\varepsilon_{1i}}\right)^2, \\[2ex]
\left(\dfrac{\mathrm{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{2+3b}{2b}}\varepsilon_{1i}}\left[\alpha_8(|\zeta_i|)\right]}{k_{1i}}\right)^{\frac{2}{2+5b}} + \left(\dfrac{\alpha_8(|\zeta_i|) - \mathrm{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{2+3b}{2b}}\varepsilon_{1i}}\left[\alpha_8(|\zeta_i|)\right]}{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3}{2}}\varepsilon_{1i}}\right), \\[2ex]
\left(\dfrac{\mathrm{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3+3b}{2b}}\varepsilon_{1i}}\left[\alpha_9(|\zeta_i|)\right]}{k_{1i}}\right)^{\frac{2}{3+5b}} + \left(\dfrac{\alpha_9(|\zeta_i|)}{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3}{2}}\varepsilon_{1i}}\right)^{\frac{2}{3}} - \left(\dfrac{\mathrm{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3+3b}{2b}}\varepsilon_{1i}}\left[\alpha_9(|\zeta_i|)\right]}{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3}{2}}\varepsilon_{1i}}\right)^{\frac{2}{3}}
\end{array}
\right\}$$

$$\alpha_2(|\zeta_i|) =: \max\left\{\begin{array}{l} \left(\dfrac{\mathrm{sat}_{\varepsilon_{2i}}\left[6\alpha_3(2|\zeta_i|)\right]}{k_{2i}}\right)^{\frac{1+b}{2b}} + \left(\dfrac{6\alpha_3(2|\zeta_i|) - \mathrm{sat}_{\varepsilon_{2i}}\left[6\alpha_3(2|\zeta_i|)\right]}{6k_{ai}}\right), \\ \left(\dfrac{\mathrm{sat}_{\left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+b}{2b}}\varepsilon_{2i}}\left[6\varepsilon_{1i}|\zeta_i|\right]}{k_{2i}}\right)^{\frac{1+b}{1+3b}} + \left(\dfrac{6\varepsilon_{1i}|\zeta_i| - \mathrm{sat}_{\left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+b}{2b}}\varepsilon_{2i}}\left[6\varepsilon_{1i}|\zeta_i|\right]}{\varepsilon_{2i}}\right), \\ \left(\dfrac{\mathrm{sat}_{\left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+b}{2b}}\varepsilon_{2i}}\left[\alpha_4(|\zeta_i|)\right]}{k_{2i}}\right)^{\frac{1+b}{1+3b}} + \left(\dfrac{\alpha_4(|\zeta_i|) - \mathrm{sat}_{\left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+b}{2b}}\varepsilon_{2i}}\left[\alpha_4(|\zeta_i|)\right]}{\varepsilon_{2i}}\right), \\ \left(\dfrac{\mathrm{sat}_{\left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+b}{b}}\varepsilon_{2i}}\left[\alpha_5(|\zeta_i|)\right]}{k_{2i}}\right)^{\frac{1+b}{2+4b}} + \left(\dfrac{\alpha_5(|\zeta_i|)}{\varepsilon_{2i}}\right)^{\frac{1}{2}} - \left(\dfrac{\mathrm{sat}_{\left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+b}{b}}\varepsilon_{2i}}\left[\alpha_5(|\zeta_i|)\right]}{\varepsilon_{2i}}\right)^{\frac{1}{2}}, \\ \left(\dfrac{\mathrm{sat}_{\left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+b}{2b}}\varepsilon_{2i}}\left[\alpha_6(2|\zeta_i|)\right]}{k_{2i}}\right)^{\frac{1+b}{1+3b}} + \left(\dfrac{\alpha_6(2|\zeta_i|) - \mathrm{sat}_{\left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+b}{2b}}\varepsilon_{2i}}\left[\alpha_6(2|\zeta_i|)\right]}{\varepsilon_{2i}}\right) \end{array}\right\}$$

with the additional class $\mathcal{K}$ functions defined as:

$$\alpha_3(2|\zeta_i|) := k_{ai}(2|\zeta_i|) + k_{ai}\,\mathrm{sat}_{2\left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+b}{2b}}}[2|\zeta_i|] + \mathrm{sat}_{2\varepsilon_{2i}}[\beta(2|\zeta_i|)]$$

$$\alpha_4(|\zeta_i|) := r_1\varepsilon_{1i}\left(\frac{9+9b}{2}\right)\left(\frac{1}{k_{1i}}\right)^{\frac{3}{2}}|\zeta_i|$$

$$\alpha_5(|\zeta_i|) := r_1(18+18b)\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{1+3b}{2b}}|\zeta_i|$$

$$\alpha_6(2|\zeta_i|) := r_1\left(\frac{3+3b}{k_{1i}}\right)\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{1+b}{2b}}\alpha_3(2|\zeta_i|)$$

$$\alpha_7(|\zeta_i|) := \left(\frac{12}{r_1}\right)\left(\frac{\varepsilon_{1i}^{\ 2}}{\varepsilon_{3i}}\right)\left(\alpha_2(|\zeta_i|)\cdot\alpha_3(2|\zeta_i|) + |\zeta_i|\varepsilon_{1i}\right)$$

$$\alpha_8(|\zeta_i|) := (9+9b)\left(\frac{\varepsilon_{1i}}{\varepsilon_{3i}}\right)\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3}{2}}\left(\alpha_2(|\zeta_i|)\cdot|\zeta_i|\right)$$

$$\alpha_9(|\zeta_i|) := \left(\frac{6}{r_1}\right)\left(\frac{\varepsilon_{1i}}{\varepsilon_{3i}}\right)\left([\alpha_2(|\zeta_i|)]^2\left(\alpha_2(|\zeta_i|)\cdot\alpha_3(2|\zeta_i|) + |\zeta_i|\varepsilon_{1i}\right) + r_1\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3+3b}{2b}}\alpha_3(2|\zeta_i|)\right)$$

Note that, $2|\zeta_i| \geq \left|s_i + \mathrm{sat}_{\mu_i}[\tilde{e}_{2i}]\right|$.

For region: $\Psi_{1i}(|\zeta_i|) = \{(v_{1i}, v_{2i}) \in \mathbb{R}^2 : |v_{1i}| \in \mathbb{R},\ |v_{2i}| \geq \alpha_2(|\zeta_i|)\}$,

Consider the following properties,

$$\int_0^{v_{1i}} \mathrm{sat}_{\varepsilon_{1i}}\left[k_{1i}|r|^b\,\mathrm{sign}(r)\right]dr = \frac{k_{1i}}{1+b}\,\mathrm{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{1+b}{b}}}\left[|v_{1i}|^{1+b}\right] + \varepsilon_{1i}\left(|v_{1i}| - \mathrm{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{1}{b}}}(|v_{1i}|)\right),$$

$$\mathrm{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3+3b}{2b}}}\left[|v_{1i}|^{\frac{3+3b}{2}}\right] = \mathrm{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{1+b}{b}}}\left[|v_{1i}|^{1+b}\right]\mathrm{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{1+b}{2b}}}\left[|v_{1i}|^{\frac{1+b}{2}}\right] \leq \mathrm{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{1+b}{b}}}\left[|v_{1i}|^{1+b}\right]\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{1+b}{2b}},$$

and using lemma 2.2, we have:

$$|v_{1i}|^{\frac{1}{2}} = \left(|v_{1i}| - \operatorname{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{1}{b}}}(|v_{1i}|) + \operatorname{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{1}{b}}}(|v_{1i}|) \right)^{\frac{1}{2}} \leq \left(|v_{1i}| - \operatorname{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{1}{b}}}(|v_{1i}|) \right)^{\frac{1}{2}} + \left(\operatorname{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{1}{b}}}(|v_{1i}|) \right)^{\frac{1}{2}}$$

$$\leq \left(|v_{1i}| - \operatorname{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{1}{b}}}(|v_{1i}|) \right)^{\frac{1}{2}} + \left(\frac{\varepsilon_{1i}}{k_{1i}} \right)^{\frac{1}{2b}}$$

Then, after rearrangement:

$$\dot{V}_2+\dot{V}_s \le -\frac{1}{3}|v_{2i}|\begin{pmatrix}\frac{1}{2}|v_{2i}|^2\\ +\int_0^{v_{1i}}\mathrm{sat}_{\varepsilon_{1i}}\left[k_{1i}|r|^b\,\mathrm{sign}(r)\right]dr\end{pmatrix}\begin{pmatrix}\mathrm{sat}_{\varepsilon_{2i}}\left[k_{2i}|v_{2i}|^{\frac{2b}{1+b}}\right]\\ +6k_{ai}\left(|v_{2i}|-\mathrm{sat}_{\left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+b}{2b}}}\left[|v_{2i}|\right]\right)\\ -6\alpha_3\left(|s_i+\mathrm{sat}_{\mu_i}\left[\tilde{e}_{2i}\right]|\right)\end{pmatrix}$$

$$-\frac{1}{3}\left(\frac{1}{2}|v_{2i}|^2+\int_0^{v_{1i}}\mathrm{sat}_{\varepsilon_{1i}}\left[k_{1i}|r|^b\,\mathrm{sign}(r)\right]dr\right)\left(|v_{2i}|\mathrm{sat}_{\varepsilon_{2i}}\left[k_{2i}|v_{2i}|^{\frac{2b}{1+b}}\right]-6\,\varepsilon_{1i}|s_i|\right)$$

$$-\frac{1}{3}\left(\frac{1}{2}|v_{2i}|-\varepsilon_{1i}^{\frac{1}{2}}\left(|v_{1i}|-\mathrm{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{1}{b}}}\left(|v_{1i}|\right)\right)^{\frac{1}{2}}\right)^2\times\left(|v_{2i}|\mathrm{sat}_{\varepsilon_{2i}}\left[k_{2i}|v_{2i}|^{\frac{2b}{1+b}}\right]\right)$$

$$-\left(\frac{\varepsilon_{1i}^{\frac{1}{2}}}{3}\right)|v_{2i}|\left(|v_{1i}|-\mathrm{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{1}{b}}}\left(|v_{1i}|\right)\right)^{\frac{1}{2}}\begin{pmatrix}|v_{2i}|\mathrm{sat}_{\varepsilon_{2i}}\left[k_{2i}|v_{2i}|^{\frac{2b}{1+b}}\right]\\ -r_1\varepsilon_{1i}\left(\frac{9+9b}{2}\right)\left(\frac{1}{k_{1i}}\right)^{\frac{3}{2}}|s_i|\end{pmatrix}$$

$$-\frac{1}{12}|v_{2i}|\left(|v_{2i}|^2\,\mathrm{sat}_{\varepsilon_{2i}}\left[k_{2i}|v_{2i}|^{\frac{2b}{1+b}}\right]-r_1(18+18b)\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{1+3b}{2b}}|s_i|\right)$$

$$-\left(\frac{k_{1i}}{3+3b}\right)\mathrm{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{1+b}{b}}}\left[|v_{1i}|^{1+b}\right]\begin{pmatrix}|v_{2i}|\mathrm{sat}_{\varepsilon_{2i}}\left[k_{2i}|v_{2i}|^{\frac{2b}{1+b}}\right]\\ -r_1\left(\frac{3+3b}{k_{1i}}\right)\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{1+b}{2b}}\times\alpha_3\left(|s_i+\mathrm{sat}_{\mu_i}\left[\tilde{e}_{2i}\right]|\right)\end{pmatrix}$$

Thus, $\dot{V}_2+\dot{V}_s\le 0$ in this region.

Next, for region: $\Psi_{2i}(|\zeta_i|)=\{(v_{1i},v_{2i})\in\mathbb{R}^2:|v_{1i}|\ge\alpha_1(|\zeta_i|),\ |v_{2i}|\le\alpha_2(|\zeta_i|)\}$

Consider the following property,

$$-\frac{r_1}{2}\operatorname{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3+3b}{2b}}}\left[|v_{1i}|^{\frac{3+3b}{2}}\right]\operatorname{sat}_{\varepsilon_{1i}}\left[k_{1i}|v_{1i}|^{b}\right]\leq-\left(\frac{r_1}{2}\right)\left(\frac{\varepsilon_{3i}}{\varepsilon_{1i}}\right)\operatorname{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3+3b}{2b}}}\left[|v_{1i}|^{\frac{3+3b}{2}}\right]\operatorname{sat}_{\varepsilon_{1i}}\left[k_{1i}|v_{1i}|^{b}\right],$$

since from the above definition, $\left(\frac{\varepsilon_{3i}}{\varepsilon_{1i}}\right)<1$.

$$-\left(\frac{r_1(1+b)}{2}\right)\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3}{2}}\varepsilon_{3i}\left(|v_{1i}|^{\frac{3}{2}}-\operatorname{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3}{2b}}}\left[|v_{1i}|^{\frac{3}{2}}\right]\right)$$
$$\leq-\left(\frac{r_1}{2}\right)\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3}{2}}\varepsilon_{3i}\left(|v_{1i}|^{\frac{3}{2}}-\operatorname{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3}{2b}}}\left[|v_{1i}|^{\frac{3}{2}}\right]\right),$$

$$\left(\begin{aligned}&-\left(\frac{r_1}{2}\right)\left(\frac{\varepsilon_{3i}}{\varepsilon_{1i}}\right)\operatorname{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3+3b}{2b}}}\left[|v_{1i}|^{\frac{3+3b}{2}}\right]\operatorname{sat}_{\varepsilon_{1i}}\left[k_{1i}|v_{1i}|^{b}\right]\\&\qquad-\left(\frac{r_1}{2}\right)\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3}{2}}\varepsilon_{3i}\left(|v_{1i}|^{\frac{3}{2}}-\operatorname{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3}{2b}}}\left[|v_{1i}|^{\frac{3}{2}}\right]\right)\end{aligned}\right)$$
$$=-\left(\frac{r_1}{2}\right)\left(\frac{\varepsilon_{3i}}{\varepsilon_{1i}}\right)|v_{1i}|^{\frac{3}{2}}\operatorname{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3}{2}}\varepsilon_{1i}}\left[k_{1i}|v_{1i}|^{\frac{5b}{2}}\right],$$

$$\int_0^{v_{1i}} \mathrm{sat}_{\varepsilon_{1i}}\left[k_{1i}|r|^b \operatorname{sign}(r)\right] dr = \frac{k_{1i}}{1+b}\mathrm{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{1+b}{b}}}\left[|v_{1i}|^{1+b}\right] + \varepsilon_{1i}\left(|v_{1i}| - \mathrm{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{1}{b}}}(|v_{1i}|)\right)$$

$$= \frac{1}{1+b}\mathrm{sat}_{\varepsilon_{1i}}\left[k_{1i}|v_{1i}|^b\right]\mathrm{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{1}{b}}}\left[|v_{1i}|\right]$$

$$+ \mathrm{sat}_{\varepsilon_{1i}}\left[k_{1i}|v_{1i}|^b\right]\left(|v_{1i}| - \mathrm{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{1}{b}}}(|v_{1i}|)\right)$$

$$= \mathrm{sat}_{\varepsilon_{1i}}\left[k_{1i}|v_{1i}|^b\right]|v_{1i}| - \left(\frac{b}{1+b}\right)\mathrm{sat}_{\varepsilon_{1i}}\left[k_{1i}|v_{1i}|^b\right]\mathrm{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{1}{b}}}(|v_{1i}|)$$

$$\leq \varepsilon_{1i}|v_{1i}|,$$

and in this region, $|v_{2i}|$ is upper bounded by $|v_{2i}| \leq \alpha_2(|\zeta_i|)$,

Then, after rearrangement:

$$\dot{V}_2 + \dot{V}_s \leq -\left(\frac{r_1}{6}\right)\left(\frac{\varepsilon_{3i}}{\varepsilon_{1i}}\right)|v_{1i}|\left(\begin{array}{l} |v_{1i}|^{\frac{1}{2}}\mathrm{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3}{2}}\varepsilon_{1i}}\left[k_{1i}|v_{1i}|^{\frac{5b}{2}}\right] \\ -\left(\frac{12}{r_1}\right)\left(\frac{\varepsilon_{1i}{}^2}{\varepsilon_{3i}}\right)\left(\alpha_2(|\zeta_i|)\cdot\alpha_3\left(\left|s_i + \mathrm{sat}_{\mu_i}\left[\tilde{e}_{2i}\right]\right|\right) + \varepsilon_{1i}|s_i|\right) \end{array}\right)$$

$$-\left(\frac{r_1}{6}\right)\left(\frac{\varepsilon_{3i}}{\varepsilon_{1i}}\right)|v_{1i}|^{\frac{1}{2}}\left(|v_{1i}|\mathrm{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3}{2}}\varepsilon_{1i}}\left[k_{1i}|v_{1i}|^{\frac{5b}{2}}\right] - (9+9b)\left(\frac{\varepsilon_{1i}}{\varepsilon_{3i}}\right)\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3}{2}}\alpha_2(|\zeta_i|)\cdot|s_i|\right)$$

$$-\left(\frac{r_1}{6}\right)\left(\frac{\varepsilon_{3i}}{\varepsilon_{1i}}\right)|v_{1i}|^{\frac{3}{2}}\mathrm{sat}_{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3}{2}}\varepsilon_{1i}}\left[k_{1i}|v_{1i}|^{\frac{5b}{2}}\right]$$

$$+\left(\left[\alpha_2(|\zeta_i|)\right]^2\left(\alpha_2(|\zeta_i|)\cdot\alpha_3\left(\left|s_i + \mathrm{sat}_{\mu_i}\left[\tilde{e}_{2i}\right]\right|\right) + \varepsilon_{1i}|s_i|\right) + r_1\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{3+3b}{2b}}\alpha_2\left(\left|s_i + \mathrm{sat}_{\mu_i}\left[\tilde{e}_{2i}\right]\right|\right)\right)$$

Thus, $\dot{V}_2 + \dot{V}_s \leq 0$ in this region.

Hence, we have shown that $\dot{\tilde{V}} \leq \dot{V}_1 < 0$, outside the compact set:

$$\Psi_{3i}(|\zeta_i|) = \{(v_{1i}, v_{2i}) \in \mathbb{R}^2 : |v_{1i}| \leq \alpha_1(|\zeta_i|),\ |v_{2i}| \leq \alpha_2(|\zeta_i|)\}$$

$$\dot{\tilde{V}} \leq \dot{V}_1 < 0, \text{ for } (v_{1i}, v_{2i}) \notin \Psi_3(|\zeta_i|)$$

Now using Lyapunov argument, define a Lyapunov level set,

$$\Sigma_v(|\zeta_i|) = \{(v_{1i}, v_{2i}) \in \mathbb{R}^2 : V \leq \rho_{1i}(|\zeta_i|)\}$$

where the class $\mathcal{K}$ function, ρ_{1i} is defined as,

$$\rho_{1i}(|\zeta_i|) = \max_{(v_{1i}, v_{2i}) \in \mathrm{bd}\ \Psi_3(|\zeta_i|)} V$$

which exists since the boundary of the sets ψ_3 are compact. Note that ψ_3 ($|\zeta_i|$) $\subset \Sigma_v$ ($|\zeta_i|$). Consequently,

$$\dot{\tilde{V}} \leq \dot{V}_1 < 0, \quad \text{for} \quad V \geq \rho_{1i}(|\zeta_i|),$$

since ψ_3 ($|\zeta_i|$) $\subset \Sigma_v$ ($|\zeta_i|$) which implies that the trajectories of (v_{1i}, v_{2i}) will enter the set Σ_v ($|\zeta_i|$) in finite time and stay in it once entered.

For sufficiently small $|\zeta_i|$

Consider the compact set $\Psi_{4i} = \left\{ (v_{1i}, v_{2i}) \in \mathbb{R}^2 : |v_{1i}| \le \left(\frac{\varepsilon_{1i}}{k_{1i}} \right)^{\frac{1}{b}}, \ , |v_{2i}| \le \left(\frac{\varepsilon_{2i}}{k_{2i}} \right)^{\frac{1+b}{2b}} \right\}$

and the Lyapunov level set, $\Sigma_{sat} = \{ (v_{1i}, v_{2i}) \in \mathbb{R}^2 : V \le \rho_{i,sat} \}$,

where $\rho_{i,sat} = \min\limits_{(v_{1i}, v_{2i}) \in \mathrm{bd}\ \Psi_{4i}} V$

which exists since the boundary of the sets ψ_{4i} are compact. Note that $\Sigma_{\mathrm{sat}} \subset \psi_{4i}$.

Then, if $|\zeta_i|$ is sufficiently small, such that $\rho_{1i}(|\zeta_i|) \le \rho_{i,sat}$, we have $\Sigma_v(|\zeta_i|) \subset \Sigma_{\mathrm{sat}}$. Note that inside Σ_{sat}, the Lyapunov function becomes,

$$V = \left(\frac{1}{2} v_{2i}{}^2 + \frac{k_{1i}}{1+b} |v_{1i}|^{1+b} \right)^2 + r_1 |v_{1i}|^{\frac{3+3b}{2}} \operatorname{sign}(|v_{1i}|) v_{2i}$$

where from section 2.2, we have $\underline{\pi}_1 \left(|v_{2i}|^4 + |v_{1i}|^{2+2b} \right) \le V$

with $\underline{\pi}_1 := \min\left\{ \frac{1}{8}, \frac{1}{2} \left(\frac{k_{1i}}{1+b} \right)^2 \right\}$

Then, inside the set $\Sigma_v(|\zeta_i|)$, the upper bound on (v_{1i}, v_{2i}) can be found. In particular,

$$\begin{aligned}
&\underline{\pi}_1 \left(|v_{2i}|^4 + |v_{1i}|^{2+2b} \right) \le V \le \rho_{1i}(|\zeta_i|) \le \rho_{i,sat} \\
&\Rightarrow \underline{\pi}_1 |v_{1i}|^{2+2b} \le \rho_{1i}(|\zeta_i|), \text{ and } \quad \underline{\pi}_1 |v_{2i}|^4 \le \rho_{1i}(|\zeta_i|), \\
&\Rightarrow |v_{1i}| \le \left(\frac{\rho_{1i}(|\zeta_i|)}{\underline{\pi}_1} \right)^{\frac{1}{2+2b}}, \text{ and } \quad |v_{2i}| \le \left(\frac{\rho_{1i}(|\zeta_i|)}{\underline{\pi}_1} \right)^{\frac{1}{4}},
\end{aligned}$$

Note that, to ensure that the desired error dynamics is unsaturated, it is necessary that:

$$|v_{1i}| \le \left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{1}{b}}, \text{ and } \left|v_{2i} + s_i + \text{sat}_{\mu_i}[\tilde{e}_{2i}]\right| \le \left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+b}{2b}}$$

Hence, from the above upper bounds on the states, to ensure that no saturation occurs, it is sufficient for the following class $\mathcal{K}$ function to satisfy the inequality,

$$\rho_{1i}(|\zeta_i|) \le \rho_{i,sat}, \text{ and } \alpha_{10}(|\zeta_i|) \le \min\left\{\left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{1}{b}}, \left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+b}{2b}}\right\},$$

where $\alpha_{10}(|\zeta_i|) := \max\left\{\left(\frac{\rho_{1i}(|\zeta_i|)}{\underline{\pi}_1}\right)^{\frac{1}{2+2b}}, \left(\frac{\rho_{1i}(|\zeta_i|)}{\underline{\pi}_1}\right)^{\frac{1}{4}} + 2|\zeta_i|\right\},$

since

$$|v_{1i}| \le \left(\frac{\rho_{1i}(|\zeta_i|)}{\underline{\pi}_1}\right)^{\frac{1}{2+2b}} \le \left(\frac{\varepsilon_{1i}}{k_{1i}}\right)^{\frac{1}{b}}, \text{ and}$$

$$\left|v_{2i} + s_i + \text{sat}_{\mu_i}[\tilde{e}_{2i}]\right| \le |v_{2i}| + \left|s_i + \text{sat}_{\mu_i}[\tilde{e}_{2i}]\right| \le \left(\frac{\rho_{1i}(|\zeta_i|)}{\underline{\pi}_1}\right)^{\frac{1}{4}} + 2|\zeta_i| \le \left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+b}{2b}},$$

which is possible for sufficiently small $|\zeta_i| > 0$. Thus, the control will be unsaturated in finite time and remain so thereafter for sufficiently small $|\zeta_i| > 0$. Once the states are unsaturated, the desired error dynamics become

$$\dot{v}_{1i} = v_{2i} + s_i$$
$$\dot{v}_{2i} = -k_{1i}|v_{1i}|^b \text{sign}(v_{1i}) - k_{2i}\left|v_{2i} + s_i + \text{sat}_{\mu_i}[\tilde{e}_{2i}]\right|^{\frac{2b}{1+b}} \text{sign}\left(v_{2i} + s_i + \text{sat}_{\mu_i}[\tilde{e}_{2i}]\right)$$

which is similar to the one considered in the section where no saturation is considered the control law (see chapter 3, 4).

Special case, when $|\zeta_i| = 0$

When $|\zeta_i| = 0$ and remains so for all future times, the system is asymptotical stable which implies that the trajectories of the system will reach the Lyapunov level set Σ_{sat} in finite time and remain in it for all future times, where we can see that for (v_{1i}, v_{2i}) inside the region Σ_{sat}, the control will be unsaturated. Essentially, it will become homogenous double integrator system and the Lyapunov function is homogeneous as well, where its stability has been studied by the author in [133]. In essence, once the system enters the unsaturated Lyapunov level set in finite time, its convergence properties is dependent on the parameter b, in particular for $0 < b < 1$, we have finite time convergence, for $b = 1$ we have exponential convergence, and for $b > 1$ we have asymptotical stability to the origin.

For the case of $b = 0$:

Consider the following dynamics:

$$\begin{aligned}
\dot{v}_{1i} &= v_{2i} + s_i \\
\dot{v}_{2i} &= -k_{ai}\left(v_{2i} + s_i + \mathrm{sat}_{\mu_i}\left[\tilde{e}_{2i}\right]\right) + k_{ai}\mathrm{sat}\left[v_{2i} + s_i + \mathrm{sat}_{\mu_i}\left[\tilde{e}_{2i}\right]\right] \\
&\quad - k_{2i}\mathrm{sign}\left(v_{2i} + s_i + \mathrm{sat}_{\mu_i}\left[\tilde{e}_{2i}\right]\right) - k_{1i}\mathrm{sign}\left(v_{1i}\right)
\end{aligned}$$

where v_{1i}, v_{2i}, s_i, $\tilde{e}_{2i} \in \mathbb{R}$, are the scalar state variables, k_{1i}, k_{2i}, k_{ai}, μ_i are positive constants, $b \geq 0$ real number with

$$k_{3i} := \begin{cases} k_{1i} - k_{ai}\mu_i - k_{ai} - k_{2i}, & \text{for } |\tilde{e}_{2i}| \neq 0, \\ k_{1i} - k_{ai} - k_{2i}, & \text{for } \tilde{e}_{2i} = 0 \quad \text{(for the case of full state feedback)} \end{cases}$$

The results in this section are applicable to the desired dynamics section of both chapter 5 & 6. In particular, for Chapter 3, let $\tilde{e}_{2i} = 0$, while for Chapter 4, no changes required, and the same differential equations are obtained.

Lyapunov function

$$V = \left(\frac{1}{2} v_{2i}^{\ 2} + k_{1i} \int_0^{v_{1i}} \text{sign}(r)\, dr \right)^2 + r_1 |v_{1i}|^{\frac{3}{2}} \text{sign}(v_{1i})\, v_{2i} + r_1 \left(\frac{2}{5} \right) k_{ai} |v_{1i}|^{\frac{5}{2}}$$
$$- r_1 k_{ai} \int_0^{v_{1i}} \left[\text{sat}_1 [|r|^{\frac{3}{2}}] \right] \text{sign}(r)\, dr \ ,$$

where r_1 is a positive constant scalar, will be shown as a strict Lyapunov function. Note that the above function is locally Lipschitz and regular. It is differentiable everywhere for $b > 0$, and not differentiable on $v_{1i} = 0$ for $b = 0$. In particular, when none of the terms of the dynamics are saturated, the Lyapunov function is indeed identical to that proposed in Section 2.2 for the twisting-based family of algorithms.

<u>Sign definiteness of V:</u>

Firstly, note that the term

$$r_1 k_{ai} \left(\int_0^{v_{1i}} |r|^{\frac{3}{2}} \text{sign}(r) - \text{sat}_1 \left[|r|^{\frac{3}{2}} \text{sign}(r) \right] dr \right) \geq 0,$$

the function is upper bounded by

$$V \le \left(\frac{1}{2} {v_{2i}}^2 + k_{1i} \int_0^{v_{1i}} \operatorname{sign}(r)\, dr \right)^2 + r_1 |v_{1i}|^{\frac{3}{2}} |\, v_{2i} | + r_1 \left(\frac{2}{5} \right) k_{ai} |v_{1i}|^{\frac{5}{2}}$$
$$- r_1 k_{ai} \int_0^{v_{1i}} \left[\operatorname{sat}_1 [|r|^{\frac{3}{2}}] \right] \operatorname{sign}(r)\, dr$$

which is positive definite and radially unbounded, next for the lower bound,

$$V \ge \left(\frac{1}{2} {v_{2i}}^2 + k_{1i} |v_{1i}| \right)^2 - r_1 |v_{1i}|^{\frac{3}{2}} |v_{2i}|$$
$$\ge \frac{1}{8} |v_{2i}|^4 + \frac{{k_{1i}}^2}{2} |v_{1i}|^2 + \frac{1}{8} |v_{2i}|^4 + \frac{{k_{1i}}^2}{2} |v_{1i}|^2 - r_1 |v_{1i}|^{\frac{3}{2}} |v_{2i}|$$

Using lemma 2.1,

$$\frac{1}{8} |v_{2i}|^4 + \frac{{k_{1i}}^2}{2} |v_{1i}|^2 \ge \left(\frac{1}{2} \right)^{\frac{1}{4}} |v_{2i}| \left(\frac{4{k_{1i}}^2}{6} \right)^{\frac{3}{4}} |v_{1i}|^{\frac{3}{2}}$$

Consequently:

$$V \ge \frac{1}{8} |v_{2i}|^4 + \frac{{k_{1i}}^2}{2} |v_{1i}|^2 \text{ with } 2^{\frac{3}{4}} \left(\frac{2^{\frac{1}{2}}}{6^{\frac{3}{4}}} \right) {k_{1i}}^{\frac{3}{2}} > r_1 \ .$$

Note that such an $r_1 > 0$ always exists for any positive k_{1i}. Thus, the Lyapunov function is positive definite and radially unbounded. In accordance to lemma 2.4, the time derivative of the Lyapunov function along the solutions of the system exists almost everywhere.

<u>Time derivative of Lyapunov function $b = 0$:</u>

For $b = 0$, V is not differentiable on $v_{1i} = 0$:

$$\frac{d}{dt}V\left(\left(v_{1i},v_{2i}\right),t\right)\overset{a.e.}{\in}\dot{\tilde{V}}\left(\left(v_{1i},v_{2i}\right),\,t\right)=\bigcap_{\xi\in\partial V(\mathbf{v}(t),t)}\xi^{T}\boldsymbol{K}\begin{bmatrix}\dot{v}_{1i}\\ \dot{v}_{2i}\end{bmatrix}\left(\left(v_{1i},v_{2i}\right),\,t\right)\subset\dot{\tilde{V}}_{1}+\dot{\tilde{V}}_{2},$$

where

$$\dot{\tilde{V}}_{1}=\bigcap_{\xi\in\partial V(\mathbf{v}(t),t)}\xi^{T}K[f]\left(v_{1i},v_{2i}\right),\qquad\dot{\tilde{V}}_{2}=\bigcap_{\xi\in\partial V(\mathbf{v}(t),t)}\xi^{T}K\begin{bmatrix}s_{i}\\ 0\end{bmatrix},$$

$$K[f]\in\begin{cases}\forall v_{1i}\neq 0,v_{2i}\in\mathbb{R}: \\ \begin{bmatrix}v_{2i}\\ \begin{pmatrix}-k_{ai}\left(\left(v_{2i}+s_{i}+\mathrm{sat}_{\mu_{i}}\left[\tilde{e}_{2i}\right]\right)-\mathrm{sat}\left[v_{2i}+s_{i}+\mathrm{sat}_{\mu_{i}}\left[\tilde{e}_{2i}\right]\right]\right)\\ -k_{2i}\mathrm{SGN}\left(v_{2i}+s_{i}+\mathrm{sat}_{\mu_{i}}\left[\tilde{e}_{2i}\right]\right)-k_{1i}\mathrm{sign}\left(v_{1i}\right)\end{pmatrix}\end{bmatrix}, \\ \forall v_{1i}=0,v_{2i}\in\mathbb{R}: \\ \begin{bmatrix}v_{2i}\\ \begin{pmatrix}-k_{ai}\left(\left(v_{2i}+s_{i}+\mathrm{sat}_{\mu_{i}}\left[\tilde{e}_{2i}\right]\right)-\mathrm{sat}\left[v_{2i}+s_{i}+\mathrm{sat}_{\mu_{i}}\left[\tilde{e}_{2i}\right]\right]\right)\\ -k_{2i}\mathrm{SGN}\left(v_{2i}+s+\mathrm{sat}_{\mu_{i}}\left[\tilde{e}_{2i}\right]\right)-k_{1i}[-1,\ 1]\end{pmatrix}\end{bmatrix},\end{cases}$$

$$\partial V=\boldsymbol{K}[\nabla V]=\boldsymbol{K}\begin{bmatrix}\dfrac{\partial V}{\partial v_{1i}}\\ \dfrac{\partial V}{\partial v_{2i}}\end{bmatrix}$$

$$\subset\begin{bmatrix}\boldsymbol{K}\left[\dfrac{\partial V}{\partial v_{1i}}\right]\\ \boldsymbol{K}\left[\dfrac{\partial V}{\partial v_{2i}}\right]\end{bmatrix}=\begin{cases}\begin{bmatrix}\begin{pmatrix}2k_{1i}{}^{2}v_{1i}+\dfrac{3}{2}r_{1}\left|v_{1i}\right|^{\frac{1}{2}}v_{2i}+k_{1i}\mathrm{sign}\left(v_{1i}\right)\left|v_{2i}\right|^{2}\\ +r_{1}k_{ai}\left(\left|v_{1i}\right|^{\frac{3}{2}}-\mathrm{sat}\left[\left|v_{1i}\right|^{\frac{3}{2}}\right]\right)\mathrm{sign}\left(v_{1i}\right)\end{pmatrix}\\ v_{2i}{}^{3}+r_{1}\left|v_{1i}\right|^{\frac{3}{2}}\mathrm{sign}\left(v_{1i}\right)+2k_{1i}\left|v_{1i}\right|v_{2i}\end{bmatrix}, & \forall v_{1i}\neq 0,v_{2i}\in\mathbb{R},\\ \begin{bmatrix}[-1,\ 1]k_{1i}\left|v_{2i}\right|^{2}\\ v_{2i}{}^{3}\end{bmatrix}, & \forall v_{1i}=0,v_{2i}\in\mathbb{R},\end{cases}$$

Computing $\dot{\tilde{V}}_{2}$:

Since s_i is a Filippov solution, it is absolutely continuous, $K\left[s_{i}\right]=s_{i}$:

For $v_{1i} \neq 0$, $\forall\ v_{2i} \in \mathbb{R}$:

$$\dot{\tilde{V}}_2 = 2k_{1i}{}^2 v_{1i} s_i + \frac{3}{2} r_1 |v_{1i}|^{\frac{1}{2}} v_{2i} s_i + k_{1i} \operatorname{sign}(v_{1i}) |v_{2i}|^2 s_i + r_1 k_{ai} \left(|v_{1i}|^{\frac{3}{2}} - \operatorname{sat}\left[|v_{1i}|^{\frac{3}{2}} \right] \right) \operatorname{sign}(v_{1i}) s_i$$

For $v_{1i} = 0$, $\forall\ v_{2i} \in \mathbb{R}$:

Let $(\xi_2 k_{1i} |\ v_{2i}\ |^2,\ v_{2i}{}^3)^{\mathrm{T}}$ with $\xi_2 \in [-1, 1]$ be an arbitrary element of ∂V, then

$$\dot{\tilde{V}}_2 = \bigcap_{\xi_2 \in [-1,\ 1]} \xi_2 k_{1i} |v_{2i}|^2 s_i = \emptyset$$

where by convention $\max \dot{\tilde{V}}_2 = -\infty$, if $\dot{\tilde{V}}_2 = \emptyset$.(see [74])

Thus, for $\forall\ (v_{1i}, v_{2i}) \in \mathbb{R}^2$:

$$\dot{\tilde{V}}_2 \le 2k_{1i}{}^2 |v_{1i}| |s_i| + \frac{3}{2} r_1 |v_{1i}|^{\frac{1}{2}} |v_{2i}| |s_i| + k_{1i} |v_{2i}|^2 |s_i| + r_1 k_{ai} \left(|v_{1i}|^{\frac{3}{2}} - \operatorname{sat}\left[|v_{1i}|^{\frac{3}{2}} \right] \right) \operatorname{sign}(v_{1i}) s_i$$

Computing $\dot{\tilde{V}}_1$ for each case, we have

For $v_{1i} \neq 0$, $\forall\ v_{2i} \in \mathbb{R}$:

$$\begin{aligned}
\dot{\tilde{V}}_1 = {} & \frac{3}{2} r_1 |v_{1i}|^{\frac{1}{2}} |v_{2i}|^2 - 2k_{1i} k_{2i} |v_{1i}| v_{2i} \operatorname{SGN}\left(v_{2i} + s_i + \operatorname{sat}_{\mu_i}[\tilde{e}_{2i}]\right) \\
& - k_{2i} v_{2i}{}^3 \operatorname{SGN}\left(v_{2i} + s_i + \operatorname{sat}_{\mu_i}[\tilde{e}_{2i}]\right) - k_{1i} r_1 |v_{1i}|^{\frac{3}{2}} \\
& - r_1 k_{2i} |v_{1i}|^{\frac{3}{2}} \operatorname{sign}(v_{1i}) \operatorname{SGN}\left(v_{2i} + s_i + \operatorname{sat}_{\mu_i}[\tilde{e}_{2i}]\right) - r_1 k_{ai} \operatorname{sat}\left[|v_{1i}|^{\frac{3}{2}} \right] \operatorname{sign}(v_{1i}) (v_{2i}) \\
& - r_1 k_{ai} |v_{1i}|^{\frac{3}{2}} \operatorname{sign}(v_{1i}) \left(s_i + \operatorname{sat}_{\mu_i}[\tilde{e}_{2i}]\right) + r_1 k_{ai} |v_{1i}|^{\frac{3}{2}} \operatorname{sign}(v_{1i}) \operatorname{sat}\left[v_{2i} + s_i + \operatorname{sat}_{\mu_i}[\tilde{e}_{2i}]\right] \\
& - v_{2i}{}^3 k_{ai} \left(\left(v_{2i} + s_i + \operatorname{sat}_{\mu_i}[\tilde{e}_{2i}]\right) - \operatorname{sat}\left[v_{2i} + s_i + \operatorname{sat}_{\mu_i}[\tilde{e}_{2i}]\right] \right) \\
& - 2k_{1i} |v_{1i}| v_{2i} k_{ai} \left(\left(v_{2i} + s_i + \operatorname{sat}_{\mu_i}[\tilde{e}_{2i}]\right) - \operatorname{sat}\left[v_{2i} + s_i + \operatorname{sat}_{\mu_i}[\tilde{e}_{2i}]\right] \right)
\end{aligned}$$

For $v_{1i} = 0$, $\forall\ v_{2i} \in \mathbb{R}$:

Let $(\xi_2 k_{1i} | v_{2i}|^2, v_{2i}{}^3)^{\mathrm{T}}$ with $\xi_2 \in [-1, 1]$ be an arbitrary element of ∂V, then

$$\begin{aligned}\dot{\tilde{V}}_1 &= \bigcap_{\xi_2\in[-1,\,1]} [\xi_2 - 1, \xi_2 + 1] k_{1i} v_{2i}{}^3 - k_{2i} v_{2i}{}^3 \mathrm{SGN}\left(v_{2i} + s_i + \mathrm{sat}_{\mu_i}[\tilde{e}_{2i}]\right) \\ &\quad - v_{2i}{}^3 k_{ai} \left(\left(v_{2i} + s_i + \mathrm{sat}_{\mu_i}[\tilde{e}_{2i}]\right) - \mathrm{sat}\left[v_{2i} + s_i + \mathrm{sat}_{\mu_i}[\tilde{e}_{2i}]\right]\right) \\ &= -k_{2i} v_{2i}{}^3 \mathrm{SGN}\left(v_{2i} + s_i + \mathrm{sat}_{\mu_i}[\tilde{e}_{2i}]\right) \\ &\quad - v_{2i}{}^3 k_{ai} \left(\left(v_{2i} + s_i + \mathrm{sat}_{\mu_i}[\tilde{e}_{2i}]\right) - \mathrm{sat}\left[v_{2i} + s_i + \mathrm{sat}_{\mu_i}[\tilde{e}_{2i}]\right]\right)\end{aligned}$$

Thus, for $\forall\ (v_{1i}, v_{2i}) \in \mathbb{R}^2$:

$$\begin{aligned}\dot{\tilde{V}}_1 &= \frac{3}{2} r_1 |v_{1i}|^{\frac{1}{2}} |v_{2i}|^2 - 2k_{1i} k_{2i} |v_{1i}| v_{2i} \mathrm{SGN}\left(v_{2i} + s_i + \mathrm{sat}_{\mu_i}[\tilde{e}_{2i}]\right) \\ &\quad - k_{2i} v_{2i}{}^3 \mathrm{SGN}\left(v_{2i} + s_i + \mathrm{sat}_{\mu_i}[\tilde{e}_{2i}]\right) - k_{1i} r_1 |v_{1i}|^{\frac{3}{2}} \\ &\quad - r_1 k_{2i} |v_{1i}|^{\frac{3}{2}} \mathrm{sign}(v_{1i}) \mathrm{SGN}\left(v_{2i} + s_i + \mathrm{sat}_{\mu_i}[\tilde{e}_{2i}]\right) - r_1 k_{ai} \mathrm{sat}\left[|v_{1i}|^{\frac{3}{2}}\right] \mathrm{sign}(v_{1i})(v_{2i}) \\ &\quad - r_1 k_{ai} |v_{1i}|^{\frac{3}{2}} \mathrm{sign}(v_{1i}) \left(s_i + \mathrm{sat}_{\mu_i}[\tilde{e}_{2i}]\right) + r_1 k_{ai} |v_{1i}|^{\frac{3}{2}} \mathrm{sign}(v_{1i}) \mathrm{sat}\left[v_{2i} + s_i + \mathrm{sat}_{\mu_i}[\tilde{e}_{2i}]\right] \\ &\quad - v_{2i}{}^3 k_{ai} \left(\left(v_{2i} + s_i + \mathrm{sat}_{\mu_i}[\tilde{e}_{2i}]\right) - \mathrm{sat}\left[v_{2i} + s_i + \mathrm{sat}_{\mu_i}[\tilde{e}_{2i}]\right]\right) \\ &\quad - 2k_{1i} |v_{1i}| v_{2i} k_{ai} \left(\left(v_{2i} + s_i + \mathrm{sat}_{\mu_i}[\tilde{e}_{2i}]\right) - \mathrm{sat}\left[v_{2i} + s_i + \mathrm{sat}_{\mu_i}[\tilde{e}_{2i}]\right]\right)\end{aligned}$$

Hence, the generalized time derivative of the Lyapunov function, after rearrangement:

$$\begin{aligned}\dot{\tilde{V}} &= \dot{\tilde{V}}_1 + \dot{\tilde{V}}_2 \\ &\le 2k_{1i}{}^2 |v_{1i}| |s_i| + \frac{3}{2} r_1 |v_{1i}|^{\frac{1}{2}} |v_{2i}| |s_i| + k_{1i} |v_{2i}|^2 |s_i| + \frac{3}{2} r_1 |v_{1i}|^{\frac{1}{2}} |v_{2i}|^2 \\ &\quad - 2k_{1i} k_{2i} |v_{1i}| v_{2i} \mathrm{SGN}\left(v_{2i} + s_i + \mathrm{sat}_{\mu_i}[\tilde{e}_{2i}]\right) \\ &\quad - k_{2i} v_{2i}{}^3 \mathrm{SGN}\left(v_{2i} + s_i + \mathrm{sat}_{\mu_i}[\tilde{e}_{2i}]\right) - r_1 |v_{1i}|^{\frac{3}{2}} \left(k_{1i} - k_{2i} - k_{ai} \mathrm{sat}_{\mu_i}[|\tilde{e}_{2i}|] - k_{ai}\right) \qquad \text{(B1 - 2)} \\ &\quad + r_1 k_{ai} \mathrm{sat}\left[|v_{1i}|^{\frac{3}{2}}\right] \left(|v_{2i}| + |s_i|\right) \\ &\quad - v_{2i}{}^3 k_{ai} \left(\left(v_{2i} + s_i + \mathrm{sat}_{\mu_i}[\tilde{e}_{2i}]\right) - \mathrm{sat}\left[v_{2i} + s_i + \mathrm{sat}_{\mu_i}[\tilde{e}_{2i}]\right]\right) \\ &\quad - 2k_{1i} |v_{1i}| v_{2i} k_{ai} \left(\left(v_{2i} + s_i + \mathrm{sat}_{\mu_i}[\tilde{e}_{2i}]\right) - \mathrm{sat}\left[v_{2i} + s_i + \mathrm{sat}_{\mu_i}[\tilde{e}_{2i}]\right]\right)\end{aligned}$$

Next, the state space is divided into three regions for analysis:

$$\phi_{1i}(|\zeta_i|) = \{(v_{1i}, v_{2i}) \in \mathbb{R}^2 : v_{1i} \in \mathbb{R}, |v_{2i}| \geq \alpha_{12}(|\zeta_i|)\},$$

$$\phi_{2i}(|\zeta_i|) = \{(v_{1i}, v_{2i}) \in \mathbb{R}^2 : |v_{1i}| \geq \alpha_{11}(|\zeta_i|), |v_{2i}| \leq \alpha_{12}(|\zeta_i|)\},$$

$$\phi_{3i}(|\zeta_i|) = \{(v_{1i}, v_{2i}) \in \mathbb{R}^2 : |v_{1i}| \leq \alpha_{11}(|\zeta_i|), |v_{2i}| \leq \alpha_{12}(|\zeta_i|)\},$$

where $|\zeta_i|$ is defined as: $|\zeta_i| = \max\{|s_i|, \mathrm{sat}_{\mu_i}[|\tilde{e}_{2i}|]\}$ while $\alpha_{13}(|\zeta_i|)$ and $\alpha_{14}(|\zeta_i|)$ are class $\mathcal{K}$ functions defined by:

$$\alpha_{11}(|\zeta_i|) := \max\left\{\begin{array}{l} \left(\dfrac{4}{r_1 k_{3i}}\right)^2 \left(2k_{1i}{}^2|\zeta_i| + 2k_{1i}k_{2i} \cdot \alpha_{12}(|\zeta_i|) + 2k_{1i}k_{ai} \cdot \alpha_{12}(|\zeta_i|) \cdot (\alpha_{12}(|\zeta_i|) + 2|\zeta_i|)\right)^2, \\ \left(\dfrac{4}{r_1 k_{3i}}\right)\left(\dfrac{3}{2} r_1\, \alpha_{12}(|\zeta_i|) \cdot |\zeta_i| + \dfrac{3}{2} r_1 [\alpha_{12}(|\zeta_i|)]^2\right), \\ \left(\dfrac{4}{r_1 k_{3i}}\right)^{\frac{2}{3}} \left(\begin{array}{l} k_{1i}[\alpha_{12}(|\zeta_i|)]^2 |\zeta_i| + k_{2i}[\alpha_{12}(|\zeta_i|)]^3 \\ + r_1 k_{ai}(\alpha_{12}(|\zeta_i|) + |\zeta_i|) + [\alpha_{12}(|\zeta_i|)]^3 k_{ai}(\alpha_{12}(|\zeta_i|) + 2|\zeta_i|) \end{array}\right)^{\frac{2}{3}} \end{array}\right\},$$

$$\alpha_{12}(|\zeta_i|) := \max\left\{\left(\frac{3k_{1i}}{k_{2i}}\right)|\zeta_i|,\ \left(\frac{3^{\frac{3}{2}} r_1}{2k_{1i}^{\frac{1}{2}} k_{2i}}\right)|\zeta_i|,\ \left(\left(\frac{2}{k_{2i}}\right)^{\frac{1}{3}}\left(\frac{4}{3r_1 k_{3i}}\right)^{\frac{2}{3}} 2r_1 k_{ai}\right)|\zeta_i|\right\},$$

For region: $\phi_{1i}(|\zeta_i|) = \{(v_{1i}, v_{2i}) \in \mathbb{R}^2 : v_{1i} \in \mathbb{R}, |v_{2i}| \geq \alpha_{12}(|\zeta_i|)\}$

Note that in this region, the following properties apply:

$$v_{2i}\mathrm{SGN}(v_{2i} + s_i + \mathrm{sat}_{\mu_i}[\tilde{e}_{2i}]) = |v_{2i}|\mathrm{sign}(v_{2i})\mathrm{SGN}(v_{2i} + s_i + \mathrm{sat}_{\mu_i}[\tilde{e}_{2i}]) = |v_{2i}|$$

when $|\zeta_i| = 0 \Rightarrow (s_i, \widetilde{e}_{2i})^{\mathrm{T}} = (0,\ 0)^{\mathrm{T}}$, the above is trivially satisfied,

when $|\zeta_i| \neq 0$, note that $\left(v_{2i} + s_i + \mathrm{sat}_{\mu_i}[\widetilde{e}_{2i}]\right) \neq 0$, since $|v_{2i}| \geq \dfrac{3k_{1i}}{k_{2i}}|\zeta_i| > \left|s_i + \mathrm{sat}_{\mu_i}[\widetilde{e}_{2i}]\right|$,

Thus, we have

$$\begin{aligned}\mathrm{SGN}\left(v_{2i} + s_i + \mathrm{sat}_{\mu_i}[\widetilde{e}_{2i}]\right) &= \mathrm{sign}\left(v_{2i} + s_i + \mathrm{sat}_{\mu_i}[\widetilde{e}_{2i}]\right),\\ &= \mathrm{sign}(v_{2i}) \text{ since } |v_{2i}| > \left|s_i + \mathrm{sat}_{\mu_i}[\widetilde{e}_{2i}]\right| \text{ in this region}\end{aligned}$$

Similarly, we have

$$\begin{aligned}&v_{2i}\left(v_{2i} + s_i + \mathrm{sat}_{\mu_i}[\widetilde{e}_{2i}] - \mathrm{sat}\left[v_{2i} + s_i + \mathrm{sat}_{\mu_i}[\widetilde{e}_{2i}]\right]\right)\\ &\qquad = |v_{2i}|\left(\left|v_{2i} + s_i + \mathrm{sat}_{\mu_i}[\widetilde{e}_{2i}]\right| - \mathrm{sat}\left[\left|v_{2i} + s_i + \mathrm{sat}_{\mu_i}[\widetilde{e}_{2i}]\right|\right]\right)\\ &\qquad \geq 0\end{aligned}$$

Hence, we have, from (B1 - 2),

$$\begin{aligned}\dot{\widetilde{V}} \leq\ & -k_{1i}|v_{1i}|\left(k_{2i}|v_{2i}| - 2k_{1i}|s_i|\right) - |v_{2i}|^2\left(\frac{k_{2i}}{3}|v_{2i}| - k_{1i}|s_i|\right) - k_{2i}|v_{2i}|\left(\frac{1}{3^{\frac{1}{2}}}|v_{2i}| - k_{1i}^{\frac{1}{2}}|v_{1i}|^{\frac{1}{2}}\right)^2\\ & -|v_{1i}|^{\frac{1}{2}}|v_{2i}|\left(\frac{k_{1i}^{\frac{1}{2}}k_{2i}}{3^{\frac{1}{2}}}|v_{2i}| - \frac{3}{2}r_1|s_i|\right) - |v_{1i}|^{\frac{1}{2}}|v_{2i}|^2\left(\frac{k_{1i}^{\frac{1}{2}}k_{2i}}{3^{\frac{1}{2}}} - \frac{3}{2}r_1\right)\\ & -\frac{k_{2i}}{6}|v_{2i}|^3 - \frac{r_1k_{3i}}{2}|v_{1i}|^{\frac{3}{2}} + r_1k_{ai}\mathrm{sat}\left[|v_{1i}|^{\frac{3}{2}}\right]|v_{2i}| + r_1k_{ai}\mathrm{sat}\left[|v_{1i}|^{\frac{3}{2}}\right]|s_i|\\ & -\frac{k_{2i}}{6}|v_{2i}|^3 - \frac{r_1k_{3i}}{2}|v_{1i}|^{\frac{3}{2}}\\ & -|v_{2i}|^3k_{ai}\left(\left|v_{2i} + s_i + \mathrm{sat}_{\mu_i}[\widetilde{e}_{2i}]\right| - \mathrm{sat}\left[\left|v_{2i} + s_i + \mathrm{sat}_{\mu_i}[\widetilde{e}_{2i}]\right|\right]\right)\\ & -2k_{1i}|v_{1i}||v_{2i}|k_{ai}\left(\left|v_{2i} + s_i + \mathrm{sat}_{\mu_i}[\widetilde{e}_{2i}]\right| - \mathrm{sat}\left[\left|v_{2i} + s_i + \mathrm{sat}_{\mu_i}[\widetilde{e}_{2i}]\right|\right]\right)\end{aligned}$$

From lemma 2.1,

$$-\frac{k_{2i}}{6}|v_{2i}|^3 - \frac{r_1k_{3i}}{2}|v_{1i}|^{\frac{3}{2}} \leq -\left(\frac{k_{2i}}{2}\right)^{\frac{1}{3}}|v_{2i}|\left(\frac{3r_1k_{3i}}{4}\right)^{\frac{2}{3}}|v_{1i}|$$

and note that

$$\mathrm{sat}\left[|v_{1i}|^{\frac{3}{2}}\right] \le \mathrm{sat}[|v_{1i}|] \le |v_{1i}|, \text{ since } \frac{3}{2} > 1$$

Thus,

$$\begin{aligned}
\dot{\tilde{V}} \le & -k_{1i}|v_{1i}|\left(k_{2i}|v_{2i}| - 2k_{1i}|s_i|\right) - |v_{2i}|^2\left(\frac{k_{2i}}{3}|v_{2i}| - k_{1i}|s_i|\right) - k_{2i}|v_{2i}|\left(\frac{1}{3^{\frac{1}{2}}}|v_{2i}| - k_{1i}^{\frac{1}{2}}|v_{1i}|^{\frac{1}{2}}\right)^2 \\
& -|v_{1i}|^{\frac{1}{2}}|v_{2i}|\left(\frac{k_{1i}^{\frac{1}{2}}k_{2i}}{3^{\frac{1}{2}}}|v_{2i}| - \frac{3}{2}r_1|s_i|\right) - |v_{1i}|^{\frac{1}{2}}|v_{2i}|^2\left(\frac{k_{1i}^{\frac{1}{2}}k_{2i}}{3^{\frac{1}{2}}} - \frac{3}{2}r_1\right) \\
& -|v_{1i}||v_{2i}|\left(\frac{1}{2}\left(\frac{k_{2i}}{2}\right)^{\frac{1}{3}}\left(\frac{3r_1k_{3i}}{4}\right)^{\frac{2}{3}} - r_1k_{ai}\right) - |v_{1i}|\left(\frac{1}{2}\left(\frac{k_{2i}}{2}\right)^{\frac{1}{3}}\left(\frac{3r_1k_{3i}}{4}\right)^{\frac{2}{3}}|v_{2i}| - r_1k_{ai}|s_i|\right) \\
& -\frac{k_{2i}}{6}|v_{2i}|^3 - \frac{r_1k_{3i}}{2}|v_{1i}|^{\frac{3}{2}} \\
& -|v_{2i}|^3k_{ai}\left(v_{2i} + s_i + \mathrm{sat}_{\mu_i}[\tilde{e}_{2i}] - \mathrm{sat}\left[v_{2i} + s_i + \mathrm{sat}_{\mu_i}[\tilde{e}_{2i}]\right]\right) \\
& -2k_{1i}|v_{1i}||v_{2i}|k_{ai}\left(v_{2i} + s_i + \mathrm{sat}_{\mu_i}[\tilde{e}_{2i}] - \mathrm{sat}\left[v_{2i} + s_i + \mathrm{sat}_{\mu_i}[\tilde{e}_{2i}]\right]\right)
\end{aligned}$$

where $\dot{\tilde{V}} < 0$ in this region for any positive

$$0 < r_1 < \min\left\{\frac{2k_{1i}^{\frac{1}{2}}k_{2i}}{3^{\frac{3}{2}}},\ \left(\frac{1}{2k_{ai}}\right)\left(\frac{k_{2i}}{2}\right)^{\frac{1}{3}}\left(\frac{3r_1k_{3i}}{4}\right)^{\frac{2}{3}}\right\}$$

For region: $\phi_{2i}(|\zeta_i|) = \{(v_{1i}, v_{2i}) \in \mathbb{R}^2 : |v_{1i}| \ge \alpha_{11}(|\zeta_i|), |v_{2i}| \le \alpha_{12}(|\zeta_i|)\}$

Note that in this region the term $|v_{2i}|$ is upper bounded and recall that

$|\zeta_i| = \max\{|s_i|, \mathrm{sat}_{\mu_i}[|\tilde{e}_{2i}|]\}$.

Also,

$$\left|\left(v_{2i}+s_i+\mathrm{sat}_{\mu_i}\left[\tilde{e}_{2i}\right]\right)-\mathrm{sat}\left[v_{2i}+s_i+\mathrm{sat}_{\mu_i}\left[\tilde{e}_{2i}\right]\right]\right|\le\left|v_{2i}+s_i+\mathrm{sat}_{\mu_i}\left[\tilde{e}_{2i}\right]\right|$$

Then we have, from (B1 - 2),

$$\begin{aligned}\dot{\tilde{V}}\le&-\left|v_{1i}\right|\left(\frac{r_1k_{3i}}{4}\left|v_{1i}\right|^{\frac{1}{2}}-\left(2k_{1i}{}^2\left|\zeta_i\right|+2k_{1i}k_{2i}\cdot\alpha_{12}\left(\left|\zeta_i\right|\right)+2k_{1i}k_{ai}\cdot\alpha_{12}\left(\left|\zeta_i\right|\right)\cdot\left(\alpha_{12}\left(\left|\zeta_i\right|\right)+2\left|\zeta_i\right|\right)\right)\right)\\&-\left|v_{1i}\right|^{\frac{1}{2}}\left(\frac{r_1k_{3i}}{4}\left|v_{1i}\right|-\left(\frac{3}{2}r_1\,\alpha_{12}\left(\left|\zeta_i\right|\right)\cdot\left|\zeta_i\right|+\frac{3}{2}r_1\left[\alpha_{12}\left(\left|\zeta_i\right|\right)\right]^2\right)\right)\\&-\frac{r_1k_{3i}}{4}\left|v_{1i}\right|^{\frac{3}{2}}+\left(\begin{aligned}&k_{1i}\left[\alpha_{12}\left(\left|\zeta_i\right|\right)\right]^2\left|\zeta_i\right|+k_{2i}\left[\alpha_{12}\left(\left|\zeta_i\right|\right)\right]^3+r_1k_{ai}\left(\alpha_{12}\left(\left|\zeta_i\right|\right)+\left|\zeta_i\right|\right)\\&+\left[\alpha_{12}\left(\left|\zeta_i\right|\right)\right]^3k_{ai}\left(\alpha_{12}\left(\left|\zeta_i\right|\right)+2\left|\zeta_i\right|\right)\end{aligned}\right)\\&-\frac{r_1k_{3i}}{4}\left|v_{1i}\right|^{\frac{3}{2}}\end{aligned}$$

where $\dot{\tilde{V}}<0$ in this region.

Now consider the compact region $\phi_{3i}\left(\left|\zeta_i\right|\right)=\left\{\left(v_{1i},v_{2i}\right)\in\mathbb{R}^2:\left|v_{1i}\right|\le\alpha_{11}\left(\left|\zeta_i\right|\right),\left|v_{2i}\right|\le\alpha_{12}\left(\left|\zeta_i\right|\right)\right\}$.

From the analysis in the other regions, we have shown that $\dot{\tilde{V}}<0$ outside of the region ϕ_3. Now, define a Lyapunov level set:

$$\Sigma_{2i}\left(\left|\zeta_i\right|\right)=\left\{\left(v_{1i},v_{2i}\right)\in\mathbb{R}^2:V\le\rho_{2i}\left(\left|\zeta_i\right|\right)\right\}$$

where ρ_{2i} is a class $\mathcal{K}$ function defined as:

$$\rho_{2i}\left(\left|\zeta_i\right|\right)=\max_{\left(v_{1i},v_{2i}\right)\in\mathrm{bd}\,\phi_3\left(\left|\zeta_i\right|\right)}V$$

which exists since the boundary of the set is compact and V is continuous. Then we observe that ϕ_3 ($|\zeta_i|$) $\subset \Sigma_{2i}$ ($|\zeta_i|$). As a result, we have

$$\frac{d}{dt}V(v_{1i}, v_{2i}) \overset{a.e.}{\in} \dot{\tilde{V}}(v_{1i}, v_{2i}) < 0 \quad for \;\; V \geq \rho_{2i}(|\zeta_i|)$$

which implies that the trajectories will enter the compact level set Σ_{2i} ($|\zeta_i|$) in finite time and stay in it once entered.

<u>For sufficiently small $|\zeta_i|$</u>

Consider the compact set $\Omega_{s=0} = \{(v_{1i}, v_{2i}) \in \mathbb{R}^2 : |v_{1i}| \leq 1,\ |v_{2i}| \leq 1\}$

and the Lyapunov level set, $\Sigma_{s=0} = \{(v_{1i}, v_{2i}) \in \mathbb{R}^n : V \leq k_{s=0}\}$,

where $k_{s=0} = \min_{s \in \mathrm{bd}\ \Omega_{s=0}} V$

which exists since the boundary of the sets $\Omega_{s=0}$ are compact. Note that $\Sigma_{s=0} \subset \Omega_{s=0}$.

Then, if $|\zeta_i|$ is sufficiently small, such that $\rho_{2i}(|\zeta_i|) \leq k_{s=0}$, we have Σ_{2v} ($|\zeta_i|$) $\subset \Sigma_{s=0}$,note that inside $\Sigma_{s=0}$, the Lyapunov function becomes,

$$V = \left(\frac{1}{2}v_{2i}{}^2 + k_{1i}|v_{1i}|\right)^2 + r_1|v_{1i}|^{\frac{3}{2}}\,\mathrm{sign}(|v_{1i}|)v_{2i}$$

where from section 2.2, we have $\underline{\pi}_1\left(|v_{2i}|^4 + |v_{1i}|^2\right) \leq V$

with $\underline{\pi}_1 := \min\left\{\frac{1}{8}, \frac{1}{2}k_{1i}{}^2\right\}$

Then, inside the set Σ_{2v} ($|\zeta_i|$), the upper bound on (v_{1i}, v_{2i}) can be found. In particular,

$$\underline{\pi}_1\left(|v_{2i}|^4+|v_{1i}|^2\right)\le V\le\rho_{2i}\left(|\zeta_i|\right)\le k_{s=0}$$
$$\Rightarrow\underline{\pi}_1|v_{1i}|^2\le\rho_{2i}\left(|\zeta_i|\right),\ \text{and}\quad \underline{\pi}_1|v_{2i}|^4\le\rho_{2i}\left(|\zeta_i|\right),$$
$$\Rightarrow|v_{1i}|\le\left(\frac{\rho_{2i}\left(|\zeta_i|\right)}{\underline{\pi}_1}\right)^{\frac{1}{2}},\ \text{and}\quad |v_{2i}|\le\left(\frac{\rho_{2i}\left(|\zeta_i|\right)}{\underline{\pi}_1}\right)^{\frac{1}{4}},$$

Note that, to ensure that the desired error dynamics is unsaturated, it is necessary that:

$$\left|v_{2i}+s_i+\text{sat}_{\mu_i}\left[\tilde{e}_{2i}\right]\right|\le 1,$$

Hence, from the above upper bounds on the states, to ensure that no saturation occurs, it is sufficient for the following class $\mathcal{K}$ function to satisfy the inequality,

$$\rho_{2i}\left(|\zeta_i|\right)\le k_{s=0}\text{, and }\left(\frac{\rho_{2i}\left(|\zeta_i|\right)}{\underline{\pi}_1}\right)^{\frac{1}{4}}+2|\zeta_i|\le 1,$$

since

$$\left|v_{2i}+s_i+\text{sat}_{\mu_i}\left[\tilde{e}_{2i}\right]\right|\le|v_{2i}|+\left|s_i+\text{sat}_{\mu_i}\left[\tilde{e}_{2i}\right]\right|\le\left(\frac{\rho_{2i}\left(|\zeta_i|\right)}{\underline{\pi}_1}\right)^{\frac{1}{4}}+2|\zeta_i|\le 1,$$

which is possible for sufficiently small $|\zeta_i| > 0$. Thus, the control will be unsaturated in finite time and remain so thereafter for sufficiently small $|\zeta_i| > 0$. Once the states are unsaturated, the desired error dynamics become

$$\dot{v}_{1i}=v_{2i}+s_i$$
$$\dot{v}_{2i}=-k_{1i}\text{sign}(v_{1i})-k_{2i}\text{sign}\left(v_{2i}+s_i+\text{sat}_{\mu_i}\left[\tilde{e}_{2i}\right]\right)$$

which is similar to the one considered in the section where no saturation is considered the control law (see chapter 3, 4).

Special case, when $|\zeta_i| = 0$

Consider the special case of $b = 0$ and $|\zeta_i| = 0$, (i.e. $s_i = 0$, and $\tilde{e}_{2i} = 0$), it is possible to show convergence to the origin in finite time.

In particular, from the above development, we obtain,

$$\dot{\tilde{V}} < 0, \quad \forall (v_{1i}, v_{2i}) \in \mathbb{R}^2 \setminus (0,\, 0) \text{ for } |\zeta_i| = 0.$$

since $\alpha_{13}(0) = 0$ and $\alpha_{14}(0) = 0$. Hence, the system is asymptotically stable which implies that the trajectories of the system will reach the Lyapunov level set $\Sigma_{s=0} = \{(v_{1i}, v_{2i}) \in \mathbb{R}^n : V \le k_{s=0}\}$

in finite time and stay in it in all future time,

where $k_{s=0} = \min_{s \in \mathrm{bd}\ \Omega_{s=0}} V$, and

$$\Omega_{s=0} = \{(v_{1i}, v_{2i}) \in \mathbb{R}^2 : |v_{1i}| \le 1,\ |v_{2i}| \le 1\}$$

where $k_{s=0} > 0$ exist since the boundary of the set $\Omega_{s=0}$ is a compact set.

Note that $\Sigma_{s=0} \subset \Omega_{s=0}$. Now note that once inside this region, the system becomes

$$\dot{v}_{1i} = v_{2i},$$
$$\dot{v}_{2i} = -k_{2i}\mathrm{sign}(v_{2i}) - k_{1i}\mathrm{sign}(v_{1i})$$

and the Lyapunov function becomes

$$V=\frac{1}{4}|v_{2i}|^4+k_{1i}{}^2|v_{1i}|^2+k_{1i}|v_{1i}||v_{2i}|^2+r_1|v_{1i}|^{\frac{3}{2}}\operatorname{sign}(v_{1i})\,v_{2i}$$

Essentially, it will become homogenous double integrator system (also known as the twisting algorithm in the literature) and the Lyapunov function is homogeneous as well, where its finite time stability has been studied by the author in [133]. In essence, once the system enters the unsaturated Lyapunov level set in finite time, it will begin to behave like a twisting control that will converge to the origin in finite time (see [133] for the time of convergence estimation). ■

B.2 PROPOSITION 1

For v_{2i}, s_i, $\tilde{e}_{2i} \in \mathbb{R}$, k_{2i}, ε_{2i}, μ_i are positive constants and $b > 0$, the following inequalities are satisfied,

$$\left|\operatorname{sat}_{\left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+b}{2b}}}[v_{2i}]-\operatorname{sat}_{\left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+b}{2b}}}\left[v_{2i}+s_i+\operatorname{sat}_{\mu_i}[\tilde{e}_{2i}]\right]\right|\le \operatorname{sat}_{2\left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+b}{2b}}}\left[\left|s_i+\operatorname{sat}_{\mu_i}[\tilde{e}_{2i}]\right|\right],$$

$$\left|\operatorname{sat}_{\varepsilon_{2i}}\left[k_{2i}\left|v_{2i}+s_i+\operatorname{sat}_{\mu_i}[\tilde{e}_{2i}]\right|^{\frac{2b}{1+b}}\operatorname{sign}\left(v_{2i}+s_i+\operatorname{sat}_{\mu_i}[\tilde{e}_{2i}]\right)\right]-\operatorname{sat}_{\varepsilon_{2i}}\left[k_{2i}|v_{2i}|^{\frac{2b}{1+b}}\operatorname{sign}(v_{2i})\right]\right|$$
$$\le \operatorname{sat}_{2\varepsilon_{2i}}\left[\beta\left(\left|s_i+\operatorname{sat}_{\mu_i}[\tilde{e}_{2i}]\right|\right)\right]$$

where β is a class $\mathcal{K}$ function.

Proof of Proposition 1: Note that from Appendix A.2-proposition 1,

$$\left|\mathrm{sat}_{\left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+b}{2b}}}\left[v_{2i}+s_i+\mathrm{sat}_{\mu_i}\left[\tilde{e}_{2i}\right]\right]-\mathrm{sat}_{\left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+b}{2b}}}\left[v_{2i}\right]\right|$$

$$\leq \max\left\{\begin{array}{l}\mathrm{sat}_{\left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+b}{2b}}}\left[|v_{2i}|+\left|s_i+\mathrm{sat}_{\mu_i}\left[\tilde{e}_{2i}\right]\right|\right]-\mathrm{sat}_{\left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+b}{2b}}}\left[|v_{2i}|\right], \\ \mathrm{sat}_{\left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+b}{2b}}}\left[|v_{2i}|\right]-\mathrm{sat}_{\left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+b}{2b}}}\left[|v_{2i}|-\left|s_i+\mathrm{sat}_{\mu_i}\left[\tilde{e}_{2i}\right]\right|\right]\end{array}\right\}$$

Note that we only need to consider, for the right hand side of the above inequality, v_{2i} on the range

$$|v_{2i}|\in\left[0,\left|s_i+\mathrm{sat}_{\mu_i}\left[\tilde{e}_{2i}\right]\right|+\left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+b}{2b}}\right] \text{ since for } |v_{2i}|\geq\left|s_i+\mathrm{sat}_{\mu_i\frac{1}{a}}\left[\tilde{e}_{2i}\right]\right|+\left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+b}{2b}}$$

$$\max\left\{\begin{array}{l}\mathrm{sat}_{\left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+b}{2b}}}\left[|v_{2i}|+\left|s_i+\mathrm{sat}_{\mu_i}\left[\tilde{e}_{2i}\right]\right|\right]-\mathrm{sat}_{\left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+b}{2b}}}\left[|v_{2i}|\right], \\ \mathrm{sat}_{\left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+b}{2b}}}\left[|v_{2i}|\right]-\mathrm{sat}_{\left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+b}{2b}}}\left[|v_{2i}|-\left|s_i+\mathrm{sat}_{\mu_i}\left[\tilde{e}_{2i}\right]\right|\right]\end{array}\right\}=0$$

Hence, for all $|v_{2i}|\geq 0$:

$$\max\left\{\begin{array}{l}\operatorname{sat}_{\left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+b}{2b}}}\left[|v_{2i}|+\left|s_i+\operatorname{sat}_{\mu_i}[\widetilde{e}_{2i}]\right|\right]-\operatorname{sat}_{\gamma_i}\left[|v_{2i}|\right], \\ \operatorname{sat}_{\left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+b}{2b}}}\left[|v_{2i}|\right]-\operatorname{sat}_{\left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+b}{2b}}}\left[|v_{2i}|-\left|s_i+\operatorname{sat}_{\mu_i}[\widetilde{e}_{2i}]\right|\right]\end{array}\right\}$$

$$=\max_{|v_{2i}|\in\left[0,\left|s_i+\operatorname{sat}_{\mu_i}[\widetilde{e}_{2i}]\right|+\left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+b}{2b}}\right]}\left\{\begin{array}{l}\operatorname{sat}_{\left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+b}{2b}}}\left[|v_{2i}|+\left|s_i+\operatorname{sat}_{\mu_i}[\widetilde{e}_{2i}]\right|\right]-\operatorname{sat}_{\left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+b}{2b}}}\left[|v_{2i}|\right], \\ \operatorname{sat}_{\left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+b}{2b}}}\left[|v_{2i}|\right]-\operatorname{sat}_{\left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+b}{2b}}}\left[|v_{2i}|-\left|s_i+\operatorname{sat}_{\mu_i}[\widetilde{e}_{2i}]\right|\right]\end{array}\right\}$$

$$=\begin{cases}\left|s_i+\operatorname{sat}_{\mu_i}[\widetilde{e}_{2i}]\right|, & \text{for } \left|s_i+\operatorname{sat}_{\mu_i}[\widetilde{e}_{2i}]\right|\le 2\left(\dfrac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+b}{2b}}{}_i, \\ 2\left(\dfrac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+b}{2b}}, & \text{for } \left|s_i+\operatorname{sat}_{\mu_i}[\widetilde{e}_{2i}]\right|> 2\left(\dfrac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+b}{2b}}\end{cases}$$

Thus, the above proposition follows for all $|v_{2i}| \geq 0$ and $\left|s_i+\operatorname{sat}_{\mu_i}[\widetilde{e}_{2i}]\right| \geq 0$.

Next, for the

$$\left|\operatorname{sat}_{\varepsilon_{2i}}\left[k_{2i}|v_{2i}+s_i|^{\frac{2b}{1+b}}\operatorname{sign}(v_{2i}+s_i)\right]-\operatorname{sat}_{\varepsilon_{2i}}\left[k_{2i}|v_{2i}|^{\frac{2b}{1+b}}\operatorname{sign}(v_{2i})\right]\right|$$

Note that from Appendix A.2-proposition 1,

$$\left|\operatorname{sat}_{\varepsilon_{2i}}\left[k_{2i}|v_{2i}+s_i|^{\frac{2b}{1+b}}\operatorname{sign}(v_{2i}+s_i)\right]-\operatorname{sat}_{\varepsilon_{2i}}\left[k_{2i}|v_{2i}|^{\frac{2b}{1+b}}\operatorname{sign}(v_{2i})\right]\right|$$

$$\leq\max\left\{\begin{array}{l}\operatorname{sat}_{\varepsilon_{2i}}\left[k_{2i}\left||v_{2i}|+|s_i|\right|^{\frac{2b}{1+b}}\right]-\operatorname{sat}_{\varepsilon_{2i}}\left[k_{2i}|v_{2i}|^{\frac{2b}{1+b}}\right], \\ \operatorname{sat}_{\varepsilon_{2i}}\left[k_{2i}|v_{2i}|^{\frac{2b}{1+b}}\right]-\operatorname{sat}_{\varepsilon_{2i}}\left[k_{2i}\left||v_{2i}|-|s_i|\right|^{\frac{2b}{1+b}}\operatorname{sign}(|v_{2i}|-|s_i|)\right]\end{array}\right\}$$

Note that we only need to consider, for the right hand side of the above inequality, v_{2i} on the range

$$|v_{2i}| \in \left[0, |s_i| + \left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+b}{2b}}\right] \text{ since for } |v_{2i}| \geq |s_i| + \left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+b}{2b}}$$

$$\max\left\{\begin{array}{l} \mathrm{sat}_{\varepsilon_{2i}}\left[k_{2i}\left\|v_{2i}\right|+\left|s_i\right\|^{\frac{2b}{1+b}}\right]-\mathrm{sat}_{\varepsilon_{2i}}\left[k_{2i}\left|v_{2i}\right|^{\frac{2b}{1+b}}\right], \\ \mathrm{sat}_{\varepsilon_{2i}}\left[k_{2i}\left|v_{2i}\right|^{\frac{2b}{1+b}}\right]-\mathrm{sat}_{\varepsilon_{2i}}\left[k_{2i}\left\|v_{2i}\right|-\left|s_i\right\|^{\frac{2b}{1+b}}\mathrm{sign}\left(\left|v_{2i}\right|-\left|s_i\right|\right)\right] \end{array}\right\}=0$$

Hence, for all $|v_{2i}| \geq 0$:

$$\max\left\{\begin{array}{l} \mathrm{sat}_{\varepsilon_{2i}}\left[k_{2i}\left\|v_{2i}\right|+\left|s_i\right\|^{\frac{2b}{1+b}}\right]-\mathrm{sat}_{\varepsilon_{2i}}\left[k_{2i}\left|v_{2i}\right|^{\frac{2b}{1+b}}\right], \\ \mathrm{sat}_{\varepsilon_{2i}}\left[k_{2i}\left|v_{2i}\right|^{\frac{2b}{1+b}}\right]-\mathrm{sat}_{\varepsilon_{2i}}\left[k_{2i}\left\|v_{2i}\right|-\left|s_i\right\|^{\frac{2b}{1+b}}\mathrm{sign}\left(\left|v_{2i}\right|-\left|s_i\right|\right)\right] \end{array}\right\}=\chi\left(\left|s_i\right|\right)$$

where the function $\chi(s_i)$ is defined as

$$\chi\left(\left|s_i\right|\right):=\max_{|v_{2i}|\in\left[0,|s_i|+\left(\frac{\varepsilon_{2i}}{k_{2i}}\right)^{\frac{1+b}{2b}}\right]}\left\{\begin{array}{l} \mathrm{sat}_{\varepsilon_{2i}}\left[k_{2i}\left\|v_{2i}\right|+\left|s_i\right\|^{\frac{2b}{1+b}}\right]-\mathrm{sat}_{\varepsilon_{2i}}\left[k_{2i}\left|v_{2i}\right|^{\frac{2b}{1+b}}\right], \\ \mathrm{sat}_{\varepsilon_{2i}}\left[k_{2i}\left|v_{2i}\right|^{\frac{2b}{1+b}}\right]-\mathrm{sat}_{\varepsilon_{2i}}\left[k_{2i}\left\|v_{2i}\right|-\left|s_i\right\|^{\frac{2b}{1+b}}\mathrm{sign}\left(\left|v_{2i}\right|-\left|s_i\right|\right)\right] \end{array}\right\}$$

which is a continuous nondecreasing function of $|s_i|$, zero at zero, and strictly positive. Also note that due to the saturation structure, we have

$$\chi\left(\left|s_i\right|\right)=2\varepsilon_{2i} \ \text{ for} \left|s_i\right|>2\varepsilon_{2i}$$

and due to the nondecreasing nature of the function, we have

$$\chi\left(\left|s_i\right|\right)\leq 2\varepsilon_{2i} \ \text{ for} \left|s_i\right|\leq 2\varepsilon_{2i}$$

Thus, for all $|v_{2i}| \geq 0$ and $|s_i| \geq 0$,

$$\left|\mathrm{sat}_{\varepsilon_{2i}}\left[k_{2i}\left|v_{2i}+s_i\right|^{\frac{2b}{1+b}}\mathrm{sign}(v_{2i}+s_i)\right]-\mathrm{sat}_{\varepsilon_{2i}}\left[k_{2i}\left|v_{2i}\right|^{\frac{2b}{1+b}}\mathrm{sign}(v_{2i})\right]\right|\le \mathrm{sat}_{2\varepsilon_{2i}}\left[\chi(\left|s_i\right|)\right]$$

Since the function $\chi(|s_i|)$ is zero at zero, continuous, strictly positive, and nondecreasing, from Lemma 1 of [145], there exists a class $\mathcal{K}$ function $\beta(|s_i|)$ such that

$$\chi(\left|s_i\right|)\le \beta(\left|s_i\right|),\ \forall\left|s_i\right|\ge 0$$

Hence,

$$\begin{aligned}\left|\mathrm{sat}_{\varepsilon_{2i}}\left[k_{2i}\left|v_{2i}+s_i\right|^{\frac{2b}{1+b}}\mathrm{sign}(v_{2i}+s_i)\right]-\mathrm{sat}_{\varepsilon_{2i}}\left[k_{2i}\left|v_{2i}\right|^{\frac{2b}{1+b}}\mathrm{sign}(v_{2i})\right]\right| &\le \mathrm{sat}_{2\varepsilon_{2i}}\left[\chi(\left|s_i\right|)\right]\\ &\le \mathrm{sat}_{2\varepsilon_{2i}}\left[\beta(\left|s_i\right|)\right]\end{aligned}$$

■

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