Statistics of eigenvectors in non-invariant random matrix ensembles

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Abstract

In this thesis we begin by presenting an introduction on random matrices, their different classes and applications in quantum mechanics to study the characteristics of the eigenvectors of a particular random matrix model. The focus of this work is on one of the oldest and most well-known symmetry classes of random matrices - the Gaussian unitary ensemble. We look at how the different possible deformations of the Gaussian unitary ensemble could have an impact on the nature of the eigenvectors, and back up our results by numerical simulations to confirm validity.

We will begin exploring the structure of the eigenvectors by employing the supersymmetry technique, a method for studying eigenvectors of complex quantum systems. In particular, we can analyse the moments of the eigenvectors, a quantity used in the classification of eigenvectors, in different random matrix models. Eigenvectors can either be extended, localised or critical and the scaling of the moments of the eigenvectors with matrix size N is used to determine the exact type. This enables one to study the transition of the eigenvectors from extended to localised and the intermediate stages.

We consider different classes of random matrices, such as random matrices with an external source and structured random matrices. In particular, we studied the Rosenzweig-Porter model by generalising our previous results from a deterministic potential to a random one and study the impact of such an alteration to the model.

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is an experience that I will never forget.

Publications

This thesis is revolved around the work in the publications below:

1. K. Truong and A. Ossipov, J. Phys. A: Math. Theor. **49**, 145005 (2016), "Statistics of eigenvectors in the deformed Gaussian unitary ensemble of random matrices".

2. K. Truong and A. Ossipov, Europhys. Lett. **116** 37002 (2016), "Eigenvectors under a generic perturbation: Non-perturbative results from the random matrix approach".

3. K. Truong and A. Ossipov, J. Phys. A: Math. Theor. **51** 065001 (2018), "Statistical properties of eigenvectors and eigenvalues of structured random matrices".

Contents

1 Introduction

Quantum mechanics has revolutionised physics in the past century and shaped our understanding of nature; it underpins research in many areas including condensed matter physics, quantum optics, quantum field theory and string theory. In the microscopic world the laws of classical physics, as we know it, ceases to be useful and so inspired the development of new physical laws. In that time many experiments were performed, such as the black-body radiation and double slit experiment, which contradicted the predictions from classical physics, this lead notable physicists to develop a new set of theories which could explain the phenomena observed in their experiments.

One of the most famous experiments was perhaps the experiment on the black-body radiation at thermal equilibrium. A black-body is an object that absorbs all incoming light and does not reflect any. Classical physics failed to accurately predict the total energy emitted from the black-body, which called into question the laws of physics used to reach this prediction. The formulae from classical physics can quite accurately predict the energy emitted from an ideal black-body at lower frequencies but fails to predict the experimental data accurately at larger frequencies, particularly around the ultraviolet region. According to the laws of classical physics each mode contains the same amount of energy and the energy emitted is proportional to the number of modes at a given frequency so it stands to reason that most of the energy emitted will come from the higher frequencies and even diverges with frequency; this did not happen in the experiments. By calculating the total radiated energy from the black-body at each frequency, the classical laws predict that the black-body would release an infinite amount of energy, thereby violating the principle of conservation of energy, contrary to their observations in the experimental data, this was later dubbed the ultraviolet catastrophe. Max Planck had studied the black-body radiation extensively and after much work he was able to empirically describe the spectral-energy distribution from a black-body that was in complete agreement with the experimental data. The

idea behind his theory was that energy is quantised so atoms can only absorb energy in discrete amounts, known as quanta, rather than taking the traditional approach that energy should be continuous. This meant that each wave has its own associated energy given by the equation $E = hf$, where h is Planck's constant and f is the frequency of the wave. By taking this approach Max Planck had found a formula that fitted the spectrum discovered experimentally.

This is the starting point of the development of quantum mechanics, considered to be one of the most important theories of the 20th century, which Einstein would later build upon to explain the photoelectric effect.

Quantum mechanics underlies many areas of research, one of which is random matrix theory which we shall introduce in this thesis from the basics to our current research topics. We will begin by providing a full structure of the thesis in the next paragraph.

This thesis will introduce the reader to the topic of random matrices in a simple manner, slowly delving into invariant and non-invariant ensembles which are fundamental to our work and have many applications in physics. We will begin by giving an introduction on the background of the topic of random matrix theory, introduce the well-known Gaussian unitary ensemble and the prominent Wigner's semi-circle law. In this thesis we will mainly be studying the statistical properties of two random matrix models: 1) $W\tilde{H}W$, 2) $\tilde{H} + D$ where $\tilde{H}$ is part of the GUE, as we refer back to these models frequently we shall name them model I and model II, respectively.

In the first chapter we will give the definitions of the most important quantities that we are interested in such as the density of states, moments of the eigenvectors, generating function and the Green's function. We will also introduce supervectors and supermatrices as they are used extensively in our work as well as superintegrals and how they are related to the Grassmann variables.

In the second chapter we will be studying a particular deformed ensemble, the first of the two models mentioned previously, where W is considered to

be a deterministic diagonal matrix, this ensemble is known as a deformed Gaussian unitary ensemble. By applying the supersymmetry technique we study the eigenvectors statistics of this system, we also produce an alternative derivation of the zero energy result to confirm our original expressions and generalise this to the case of diagonal W with Gaussian distributed elements.

In the third chapter we study the second model $\tilde{H} + D$, which is another deformation of the Gaussian unitary ensemble, by deriving the moments of the eigenvectors for this particular model for a deterministic diagonal D. Next we are able to generalise this result to random diagonal D. In particular we focus on the special model in which variances of the elements of D become N-dependent. Afterwards we considered the case where $D \gg \tilde{H}$, in which case our original expressions are no longer valid, so we derive the distribution function, and hence the moments of the eigenvectors, from perturbation theory.

In the fourth chapter we introduce a generalised model which comprises the features of both models I and II. We calculate the density of states and the moments of the eigenvectors for this generalised model and verify numerically the validity of these expressions.

The fifth chapter contains unfinished work on the calculation of the density of states of non-Hermitian random matrices, which are random matrices that do not have to conform to the Hermiticity constraint $H_{ij} = H_{ji}^*$.

The sixth chapter contains unfinished work on computing the moments of the eigenvectors for the Anderson type of model with the chiral symmetry. We build on the work in [1] by using the results in the paper to apply to our new model to see how this alteration of the model will affect the moments of the eigenvectors.

1.1 Background on random matrices

Random matrices were first introduced in physics by Eugene Wigner in the 1950s and have applications in many areas of physics. In that time many experiments were performed with heavy nuclei in order to determine their

energy spectra. To approach the problem theoretically we need to solve the eigenvalue problem $\hat{H}|\Psi\rangle = E|\Psi\rangle$, which means we first need to determine the Hamiltonian of the system, but this is a complex task due to the amount of nucleons involved. In addition to this, the Hamiltonians in these complex situations are also typically unsolvable and therefore cannot provide any detailed information on the system.

Wigner and Dyson took a statistical approach to this problem by proposing that random matrices could be used to describe the statistical properties of heavy nuclei. Wigner postulated that the statistics of the energy levels are modelled by the corresponding statistics of eigenvalues of the Gaussian distributed random matrices, this means that the eigenvalues of the random matrices should exhibit the same statistical behaviour as the energy levels of the heavy nucleus as long as the random matrix ensemble is chosen appropriately. This is due to the fact that some important statistical properties are universal because they are only dependent on the physical symmetries of the matrix and therefore will also be present in the random matrix ensemble.

There are 10 different types of symmetry classes, of which 3 are known as the classical symmetry classes, for random matrix ensembles. The classical symmetry classes are Gaussian unitary ensemble (GUE), Gaussian orthogonal ensemble (GOE) and Gaussian symplectic ensemble (GSE). The first of the classical symmetries (GUE) represents a system that is non-invariant under time reversal; the second (GOE) case represents a Hamiltonian that is invariant under time reversal symmetry with the time-reversal operator requirement $T^2 = 1$; the final case (GSE) represents a Hamiltonian that is invariant under time reversal with the time reversal operator $T^2 = -1$.

Random matrix theory can be used in particular to describe various phenomena in disordered quantum systems, including Anderson localisation and the metal-insulator transition. Originally introduced in the 1950s Anderson localisation is prevalent in quantum disordered systems; localisation is a phenomenon that crops up in many areas of physics and is characterised by a lack of diffusion waves in a disordered system. It predicts that an electron

in a disordered lattice under certain conditions can be confined to a specific region, this means that the probability amplitude decreases exponentially as one moves away from the centre of localisation rather than being extended over the whole region; therefore the chance of finding the electron is highest at the centre of localisation. Such states are called localised states. They can also be contrasted to extended states, which are characterised by equal probability of finding an electron in any part of the available space.

The standard ensembles, however, cannot be used for describing the Anderson localisation transition. Indeed, the statistical distribution of the eigenvectors of GUE is very simple because the normalised eigenvectors are distributed uniformly over a unit sphere in $\mathbb{C}^N$. This is closely related to the invariance of the distribution function of GUE under arbitrary unitary transformations which follows from the equivalence of the variances of the matrix elements [2]. The elements of GUE are distributed with mean $\langle H_{ij} \rangle = 0$ and variance $\langle |H_{ij}|^2 \rangle = 1/N$, which satisfy the Hermiticity constraint $H_{ij} = H_{ji}^*$. N in this context corresponds to the system size of a Hamiltonian acting on a one-dimensional lattice with N vertices.

For this reason the non-invariant ensembles were studied in order to describe Anderson localisation such as power-law banded random matrices [3,4], banded random matrices [5,6], almost diagonal random matrices [7,8], ultrametric [9], and Ruijsenaars-Schneider ensembles [10]. Non-invariant ensembles are matrix ensembles for which the variance is no longer a constant but given by a non-trivial matrix $\left\langle |\tilde{H}_{ij}|^2 \right\rangle = F_{ij}$. The breaking of unitary invariance makes it possible for localisation of eigenvectors to occur so we study in detail the statistical properties of some non-invariant random matrix ensembles. There are many applications of non-invariant ensembles, including random reactance networks [11], statistical signal processing [12] and vibration analysis [13].

Before we proceed with any calculations, we will introduce some concepts below which are fundamental to our work, so it is important that the reader will familiarise themselves with these concepts.

1.1.1 Gaussian unitary ensemble

One of the most prominent and oldest random matrix ensembles is the Gaussian unitary ensemble, this is fundamental to our work so we will begin by defining the ensemble in detail.

The GUE is an ensemble of random matrices with particular structures built into them: all of the elements $\tilde{H}_{ij}$ are independently and identically distributed Gaussian random variables, up to the Hermiticity constraint $H_{ij} = H^*_{ji}$, with mean

$$\left\langle \tilde{H}_{ij} \right\rangle = 0, \tag{1}$$

variance

$$\left\langle |\tilde{H}_{ij}|^2 \right\rangle = 1/N, \tag{2}$$

and probability distribution

$$\mathcal{P}(\tilde{H}) \propto e^{-\frac{1}{2N}\mathrm{Tr}\tilde{H}^2}. \tag{3}$$

The matrices are invariant under an arbitrary unitary transformation matrix U:

$$\mathcal{P}(U\tilde{H}U^\dagger) = \mathcal{P}(\tilde{H}), \tag{4}$$

due to this invariance the ensemble describes extended states of a system.

In the next section we look at how the distribution of eigenvalues behave as the system size N tends to infinity.

1.1.2 Wigner's semi-circle law and the density of states

Eugene Wigner first noticed, in 1955, that for certain classes of random matrices, one of which is GUE, the distribution of eigenvalues remained the same as the matrix size N tends to infinity. This distribution is in fact the famous Wigner's semi-circle law and is satisfied by many symmetric random matrices in the large N limit.

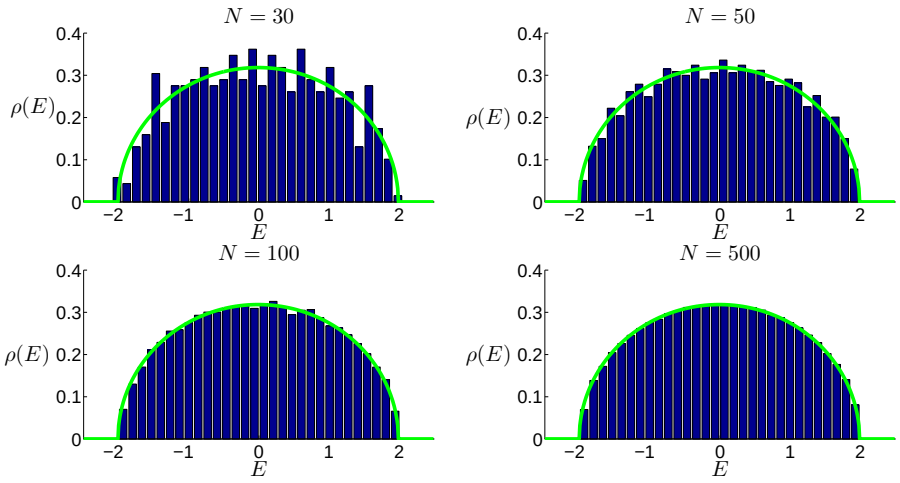

Figure 1: In these figures the Wigner's semi-circle law is given by the solid green line and the histogram consists all eigenvalues from the GUE. We can see that as N increases, the histogram tends to Wigner's semi-circle law.

The distribution of eigenvalues is defined as

$$\rho(E) = \frac{1}{N} \sum_{i}^{N} \langle \delta(E - E_i) \rangle \,, \tag{5}$$

this quantity is also known as the density of states where E_i are the eigenvalues of the matrix at a given energy E and δ is the delta-function.

In all three classical cases (the Gaussian unitary ensemble, Gaussian orthogonal ensemble and Gaussian symplectic ensemble) their density of states tends to Wigner's semi-circle law as the size of the system N tends to infinity. As our work is based on the Gaussian unitary ensemble, we will show the reader the density of states for different values of N in the GUE case, which is shown in fig. 1. As can be seen in the figure the distribution of eigenvalues rapidly tends to Wigner's semicircle law as N tends to infinity.

1.2 Moments of the eigenvectors

Another important quantity in our work is the moment of eigenvectors since it can provide an insight into the characteristics of the eigenvectors of a random

matrix ensemble. It is defined in the following way

$$I_q(n) = \frac{1}{\rho(E)N} \sum_{\alpha} \left\langle |\psi_\alpha(n)|^{2q} \delta(E - E_\alpha) \right\rangle , \tag{6}$$

where E_α is the eigenvalue corresponding to the normalised eigenvector ψ_α at a given energy E and $\rho(E)$ is the density of states as defined previously.

There are three qualitatively different types of eigenvectors: localised, critical and extended. The moments of the eigenvectors are used to differentiate between the three types by measuring the scaling of the moments of the eigenvectors, I_q, with matrix size N we are able to determine the characteristics of the eigenvectors

$$I_q = \sum_{n}^{N} I_q(n) \propto N^{-d_q(q-1)}, \tag{7}$$

where d_q is known as the fractal dimension and this is what characterises the nature of the eigenvectors:

$$d_q = \begin{cases} 0, & \text{Localised} \\ d, & \text{Extended} \\ 0 < f(q) < d, & \text{Multi-fractal} \end{cases} , \tag{8}$$

where d corresponds to the dimension of the lattice. In our case we work on a one-dimensional lattice, therefore $d = 1$.

1.3 Green's function

The generating function is an important quantity that is widely used in statistical physics. In this thesis it is used in order to compute the Green's functions.

The Green's function is defined as [14]

$$G^{R/A}(E) = (E^{\pm} - H)^{-1}, \tag{9}$$

where $E^{\pm} = E \pm \mathrm{i}\epsilon$ and ϵ is an infinitesimally small parameter.

One of the simplest generating functions is defined as a ratio of two determinants

$$Z(E^{\pm}, j) = \frac{\det \mathrm{i}[E^{\pm} - H]}{\det \mathrm{i}[E^{\pm} - H - j]}, \tag{10}$$

it can be used to calculate the trace of the Green's function:

$$\mathrm{Tr} G^{R/A}(E) = \left. \frac{\partial Z(E^{\pm}, j)}{\partial j} \right|_{j=0}. \tag{11}$$

This is an important relation as it is the foundation of the supersymmetry technique, which is a powerful and indispensable method used to study the statistics of eigenvectors of a quantum system. This relation can also be generalised for the product of Green's functions by considering the corresponding products of generating functions.

Many useful physical quantities can be expressed through Green's functions. For example, the density of states is given by

$$\rho(E) = \lim_{\epsilon \to 0^+} \frac{1}{\pi N} \mathrm{Im}\, \mathrm{Tr} \frac{1}{E^- - H}, \tag{12}$$

which can therefore be calculated from the generating function.

1.4 Supervectors and supermatrices

In this section we will introduce supermatrices and supervectors, which are also used extensively in our work as they allow one to conveniently express the superintegrals over commuting and anti-commuting variables in a compact form for ease of manipulation.

A supervector, Φ, is a $(2M)$-component vector with M commuting components $S_1, S_2, ..., S_M$ and M anti-commuting components $\chi_1, \chi_2, ..., \chi_M$,

$$\Phi = \begin{pmatrix} S \\ \chi \end{pmatrix}, \quad S = \begin{pmatrix} S_1 \\ \vdots \\ S_M \end{pmatrix}, \quad \chi = \begin{pmatrix} \chi_1 \\ \vdots \\ \chi_M \end{pmatrix}, \tag{13}$$

where the anti-commuting components contains an odd number of Grassmann

variables, $\chi_i = \sum_j f_{ij}\eta_j + \sum_{jkl} f_{ijkl}\eta_j\eta_k\eta_l + ...$ with commuting coefficients f, and similarly while the commuting component does not prevent the appearance of Grassmann variables, it does restrict it to an even number of them, $S_i = z_i + \sum_{ij} z_{ij}\eta_i\eta_j + ...$, where z are commuting variables. The commuting components are referred to as bosonic elements, the anti-commuting components are referred to as fermionic elements and the vectors ordered in this way are referred to as the Bose-Fermi notation.

In order to introduce linear transformations which preserve the structure of the supervector, we define $2M \times 2M$ supermatrices in the following way:

$$F = \begin{pmatrix} a & \sigma \\ \rho & b \end{pmatrix}, \quad F\Phi = \begin{pmatrix} a & \sigma \\ \rho & b \end{pmatrix} \begin{pmatrix} S \\ \chi \end{pmatrix} = \begin{pmatrix} aS + \sigma\chi \\ \rho S + b\chi \end{pmatrix}. \tag{14}$$

The resulting vector from the matrix multiplication must also contain the same intrinsic structure as the original supervector, as defined above. To ensure this consistency, we require that a and b must contain bosonic elements and ρ and σ must contain fermionic elements. It is standard to refer to a as the Bose-Bose block or Bose block, b as the Fermi-Fermi block or Fermi block, σ as the Bose-Fermi block and ρ as the Fermi-Bose block.

In (14) the supervectors contain the same number of bosonic and fermionic elements; all the blocks in the supermatrices are of the same size, but they can be generalised to have different numbers of bosonic and fermionic elements as well as having different sized blocks in the supermatrices. However, we shall not be elaborating on this any further as this is unnecessary for our work.

The transposition for supermatrices would also need to be defined in a slightly different way to the transposition of a standard matrix, since the rule $(F_1F_2)^T = F_2^T F_1^T$ must be satisfied, we have to modify the familiar rule such that this relation holds

$$F^T = \begin{pmatrix} a & \sigma \\ \rho & b \end{pmatrix}^T = \begin{pmatrix} a^t & \rho^t \\ -\sigma^t & b^t \end{pmatrix}, \tag{15}$$

where the superscript T denotes the transposition of a supermatrix and the lower-case t denotes the more familiar matrix transposition. The reader should notice the addition of a minus sign in the Fermi-Bose block, which is required for the relation to hold. Another cost for such a modification is that the double transposition would no longer yield the original matrix

$$\left(F^T\right)^T = \left(\begin{pmatrix} a & \sigma \\ \rho & b \end{pmatrix}^T\right)^T = \begin{pmatrix} a^t & \rho^t \\ -\sigma^t & b^t \end{pmatrix}^T = \begin{pmatrix} a & -\sigma \\ -\rho & b \end{pmatrix} \neq F. \tag{16}$$

Next, we look at how the supertrace of a supermatrix is computed and how it differs from the standard trace of a matrix. The supertrace, which is denoted by Str must preserve the cyclic nature of the trace operation, is defined as

$$\mathrm{Str}F = \mathrm{Tr}a - \mathrm{Tr}b, \tag{17}$$

where Tr denotes the standard trace; the addition of the minus is required to ensure the cyclic property holds.

Finally, we introduce the superdeterminant which is analogous to the determinant in standard matrices. Similarly, to the determinant of standard matrices, the superdeterminant satisfies the following three properties:

1. $\mathrm{Sdet}F = \mathrm{Sdet}F^T$
2. $\mathrm{Sdet}F_1F_2 = \mathrm{Sdet}F_1\mathrm{Sdet}F_2$
3. $\ln \mathrm{Sdet}F = \mathrm{Str} \ln F$

where F is a supermatrix and F^T is the transposition of the supermatrix.

The superdeterminant for a supermatrix,

$$F = \begin{pmatrix} a & \sigma \\ \rho & b \end{pmatrix}, \tag{18}$$

is defined as the following

$$\mathrm{Sdet}F = \det\left(a - \sigma b^{-1}\rho\right)/\det b \quad \text{if } b_0 \neq 0, \tag{19}$$

the inverse of the superdeterminant of F is defined as

$$\mathrm{Sdet}^{-1}F = \det\left(b - \rho a^{-1}\sigma\right)/\det a \quad \text{if } a_0 \neq 0, \tag{20}$$

where a_0 and b_0 are the numerical part of the Bose and Fermi block, respectively.

If F is block diagonal ($\sigma = \rho = 0$), then the superdeterminant is a ratio of determinants of the Bose and Fermi blocks

$$\mathrm{Sdet}F = \frac{\det a}{\det b} \quad \text{if } \sigma = \rho = 0, \tag{21}$$

this special case is a starting point for the representation of generating functions in the supersymmetry technique.

1.5 Superintegrals and the Grassmann variable

We also introduce the notion of a superintegral for the reader as it is used extensively and essential in our work for expressing the ratio of determinants as a superintegral. Firstly, the reader may be familiar with the ordinary Gaussian integral [14]

$$\int \left(\prod_{i=1}^{N} \frac{d^2 S_i}{\pi}\right) \exp\left(-\sum_{i,j=1}^{N} S_i^* a_{ij} S_j\right) = \frac{1}{\det a}, \tag{22}$$

where we have used the notation $d^2S = d\,\mathrm{Re}\,S d\,\mathrm{Im}\,S$ and S_i are complex commuting variables.

There is also an exact analogue of the previous integral for Grassmann variables

$$\int \left(\prod_{i=1}^{N} d\chi_i^* d\chi_i\right) \exp\left(-\sum_{i,j=1}^{N} \chi_i^* b_{ij} \chi_j\right) = \det b, \tag{23}$$

where Grassmanns are anti-commuting variables defined by the following properties

$$\chi_i\chi_j = -\chi_j\chi_i, \quad \chi_i\chi_j^* = -\chi_j^*\chi_i, \quad \chi_i^*\chi_j^* = -\chi_j^*\chi_i^*. \tag{24}$$

It follows from this definition that $\chi_i^2 = \chi_i^{*2} = 0$.

Integration over Grassmann variables, unlike commuting variables, do not require limits to become definite, is defined in the following way

$$\int d\chi_i \chi_i = \int d\chi_i^* \chi_i^* = 1, \quad \int d\chi_i 1 = \int d\chi_i^* 1 = 0. \tag{25}$$

The differentials, $d\chi_i$ and $d\chi_i^*$, also, like Grassmann variables, anti-commute with themselves.

A superintegral is an integral over both commuting and anti-commuting variables, so we can now use both (22) and (23) to represent the ratio of two determinants as a superintegral:

$$\frac{\det b}{\det a} = \int \left(\prod_{i=1}^{N} \frac{d^2 S_i}{\pi} d\chi_i^* d\chi_i \right) \exp\left(- \sum_{i,j=1}^{N} S_i^* a_{ij} S_j + \chi_i^* b_{ij} \chi_j \right), \tag{26}$$

note the relation between Eq. (26) and F in Eq. (21). This identity is fundamental and the basis of the supersymmetry technique, which shall be explained in the later sections.

Now that we have introduced the basics, we are ready to proceed to the next section where we will begin our first project in studying the characteristics of the eigenvectors of a deformed Gaussian unitary ensemble.

2 Model I: Characteristics of eigenvectors of a deformed Gaussian unitary ensemble

For our first project we considered how the deformation of the GUE would affect the nature of the eigenvectors which are either localised, extended or critical. In applying the supersymmetry technique to the model we followed closely Ref. [11].

In our work we studied random matrices in which the variances of H_{ij} are given by

$$\left\langle |H_{ij}|^2 \right\rangle = \frac{w_i^2 w_j^2}{N}, \tag{27}$$

where all $w_i > 0$.

This model can also be represented as

$$H = W\tilde{H}W, \tag{28}$$

where $\tilde{H} \in$ GUE, and W is a deterministic diagonal matrix with elements w_i, $i = 1, 2, ..., N$.

The presence of W breaks the unitary invariance of the GUE, therefore making it possible to deviate from extended states, which enables one to study the Anderson localisation of eigenvectors. It is worth noting that this particular model is closely related to the well-known generalised eigenvalue problem for a GUE matrix and a positive definite matrix A:

$$\tilde{H}g = \lambda Ag, \tag{29}$$

where λ is the eigenvalue and g is the corresponding eigenvector [15].

The equation can be expressed as

$$A^{-1/2}\tilde{H}A^{-1/2}A^{1/2}g = \lambda A^{1/2}g, \tag{30}$$

where λ is an eigenvalue and $f = A^{1/2}g$ is the corresponding eigenvector of $A^{-1/2}\tilde{H}A^{-1/2}$. Since $\tilde{H}$ is unitary invariant, this enables one to choose a basis in which A is diagonal, thereby reducing the generalised eigenvalue problem down to a standard eigenvalue problem with $W = A^{-1/2}$. The generalised eigenvalue problem is an important and well studied problem since it has applications in many areas of physics. The mean eigenvalue density for the eigenvalue problem, Eq. (30), in the case where $\tilde{H}$ and W are both random, have been well studied [16, 17]. More recently there has been a focus on the correlation properties of the eigenvalues [11], where it was found that the presence of the W matrix does not change the two-point spectral correlation function from the standard GUE case.

To start, we first remind the reader of the definition of the moments of the

eigenvectors, which is fundamental for studying the characteristics of eigenvectors and is the main quantity we are going to calculate:

$$I_q(n) = \frac{1}{N\rho(E)} \sum_\alpha \left\langle |\psi_\alpha(n)|^{2q} \delta(E - E_\alpha) \right\rangle , \tag{31}$$

where E_α is an eigenvalue of H corresponding to a normalised eigenvector ψ_α at a given energy E, n is the site on the lattice and $\rho(E) = \frac{1}{N} \sum_\alpha \langle \delta(E - E_\alpha) \rangle$ is the density of states.

The moments can be calculated from the diagonal matrix elements of the Green's functions (see Appendix B for details) as follows [18]

$$I_q(n) = \frac{\mathrm{i}^{l-m}}{2\pi\rho(E)N} \frac{(l-1)!(m-1)!}{(l+m-2)!} \lim_{\epsilon\to 0} (2\epsilon)^{l+m-1} \left\langle (G^R_{nn})^l (G^A_{nn})^m \right\rangle , \tag{32}$$

l and m are positive integers such that $q = l + m$ and the retarded, G^R, and advanced, G^A, Green's functions are defined as

$$G^{R/A}(E) = (E \pm \mathrm{i}\epsilon - H)^{-1}. \tag{33}$$

Although l and m are considered to be integers, and hence q is also an integer, we believe that the final result will also be valid for non-integer q.

The moments of the eigenvectors in this form provides a convenient foundation for supersymmetry, as the product of diagonal matrix elements of the Green's functions can be expressed through the ratio of determinants; the details of the supersymmetric approach, for this particular model, will be given in the following subsection.

2.1 Moments of the eigenvectors by the supersymmeric approach

In order to calculate the moments of the eigenvectors, according to Eq. (32), we need to calculate the product of two Green's functions, which can in turn be calculated from the product of two generating functions. Thus the moments of

the eigenvectors are computed from the product of two generating functions, $Z(E, j)$, by replacing the scalar parameter j with the matrix $\hat{J}$ and applying the supersymmetry technique.

We introduce the product of two generating functions, a ratio of four determinants, for our particular model:

$$Z(E, \hat{J}_1, \hat{J}_2) = (-1)^N \frac{\det \mathrm{i}(E^- - W\tilde{H}W) \det \mathrm{i}(E^+ - W\tilde{H}W)}{\det \mathrm{i}(E^- - W\tilde{H}W + \hat{J}_1) \det(-\mathrm{i})(E^+ - W\tilde{H}W + \hat{J}_2)}, \tag{34}$$

where $E^{\pm}$ is the energy term with a small imaginary shift above and below the real axis, $\hat{J}_1$ and $\hat{J}_2$ are source terms whose specific form will be chosen later on.

Now we can represent the ratio of determinants as a superintegral by using the identity, Eq. (26), resulting in

$$\begin{aligned} Z(E, \hat{J}_1, \hat{J}_2) = (-1)^N & \int \prod_{i=1}^{N} \frac{d^2 r_i(-)}{\pi} d\eta_i^*(-) d\eta_i(-) \int \prod_{i=1}^{N} \frac{d^2 r_i(+)}{\pi} d\eta_i^*(+) d\eta_i(+) \\ & \times \exp \mathrm{i} \Big[r^\dagger(-)(W\tilde{H}W - E + \mathrm{i}\epsilon - \hat{J}_1) r(-) + \chi^\dagger(-)(W\tilde{H}W - E + \mathrm{i}\epsilon)\chi(-) \Big] \\ & \times \exp \mathrm{i} \Big[-r^\dagger(+)(W\tilde{H}W - E - \mathrm{i}\epsilon - \hat{J}_2) r(+) + \chi^\dagger(+)(W\tilde{H}W - E - \mathrm{i}\epsilon)\chi(+) \Big], \end{aligned} \tag{35}$$

where $r(-)$, $r(+)$ are vectors containing commutative variables, $r_i(-)$, $r_i(+)$, and $\chi(-)$, $\chi(+)$ are vectors containing anti-commutative variables, $\eta_i(-)$, $\eta_i(+)$, $i = 1, 2, ...N$. As can be clearly seen here, sneaking in various factors of i and -1 is to ensure the convergence of the bosonic Gaussian integral, where we require ϵ to be real with a negative sign since $\epsilon > 0$, in the argument with $r(+)$.

For a more compact representation of the superintegral we introduce the supervector, Φ,

$$\Phi = (r(-), r(+), \chi(-), \chi(+))^T = (\Phi_1, \Phi_2, \Phi_3, \Phi_4)^T, \tag{36}$$

which enables one to write the superintegral in terms of the supervectors

$$\begin{aligned} Z(E,\hat{J}_1,\hat{J}_2) &= (-1)^N \int d\Phi^* d\Phi \\ &\times \exp\left\{ \mathrm{i}\sum_{\alpha=1}^{4} \Phi_\alpha^\dagger L_\alpha [W\tilde{H}W - E - \mathrm{i}(-1)^\alpha \epsilon - \tilde{J}_\alpha]\Phi_\alpha \right\}, \end{aligned} \tag{37}$$

where $L_1 = L_3 = L_4 = -L_2 = 1$, $\tilde{J} = \mathrm{diag}(\hat{J}_1, \hat{J}_2, 0, 0)$, $\hat{J}_1 = \mathrm{diag}(0, 0, ..., J_1, ..., 0)$ and $\hat{J}_2 = \mathrm{diag}(0, 0, ..., J_2, ..., 0)$, both J_1 and J_2 are the *nth* element of their respective diagonals, and the integration measure $d\Phi^* d\Phi$ comprises of those in the preceding integral, Eq. (35).

In order to calculate the product of two Green's functions, we must differentiate this expression with respect to J_1 and J_2, m and l times respectively. Once the differentiation is completed we send J_1 and J_2 to zero resulting in

$$\begin{aligned} \left(G_{nn}^R\right)^l \left(G_{nn}^A\right)^m &= (-1)^N \int d\Phi^* d\Phi (r_n^*(-) r_n(-))^m (r_n^*(+) r_n(+))^l \\ &\exp\left\{ \mathrm{i}\sum_{\alpha=1}^{4} \Phi_\alpha^\dagger L_\alpha [W\tilde{H}W - E - \mathrm{i}(-1)^\alpha \epsilon]\Phi_\alpha \right\}, \end{aligned} \tag{38}$$

the whole expression must be averaged over random elements of $\tilde{H}$

$$\begin{aligned} \left\langle \left(G_{nn}^R\right)^l \left(G_{nn}^A\right)^m \right\rangle &= (-1)^N \int d\Phi^* d\Phi (r_n^*(-) r_n(-))^l (r_n^*(+) r_n(+))^m \\ &\times \left\langle \exp\left\{ \mathrm{i}\sum_{\alpha=1}^{4} \Phi_\alpha^\dagger L_\alpha [W\tilde{H}W - E - \mathrm{i}(-1)^\alpha \epsilon]\Phi_\alpha \right\} \right\rangle. \end{aligned} \tag{39}$$

In order to complete the averaging we utilise the identity [14]:

$$\left\langle \exp\left[\mathrm{i}\sum \tilde{H}_{mn} M_{nm} \right] \right\rangle = \exp\left[-\frac{1}{2} \left\langle \left(\sum \tilde{H}_{mn} M_{nm} \right)^2 \right\rangle \right], \tag{40}$$

in our case it is of the form

$$\begin{aligned} &\left\langle \exp\left[\mathrm{i}\sum_\alpha \sum_{m,n} \tilde{H}_{mn} \Phi_\alpha^*(n) L_\alpha w_n \Phi_\alpha(m) w_m \right] \right\rangle \\ &= \exp\left[-\frac{1}{2} \left\langle \left(\sum \tilde{H}_{mn} \Phi_\alpha^*(n) L_\alpha w_n \Phi_\alpha(m) w_m \right)^2 \right\rangle \right], \end{aligned} \tag{41}$$

where $M_{nm} = \sum_\alpha \Phi^*_\alpha(n) L_\alpha w_n \Phi_\alpha(m) w_m$ by comparing coefficients and Φ^* denotes the complex conjugate of Φ.

Therefore to simplify Eq. (41), we must determine the average in the exponential by splitting $\tilde{H}$ into its real and imaginary parts:

$$\begin{aligned}\left\langle \left(\sum_{m,n} \tilde{H}_{mn} M_{nm} \right)^2 \right\rangle &= \left\langle \left(\sum_{m,n} (\mathrm{Re}\, \tilde{H}_{mn} + \mathrm{i}\, \mathrm{Im}\, \tilde{H}_{mn}) M_{nm} \right)^2 \right\rangle \\ &= \Bigg\langle \Bigg(\sum_{m<n} \mathrm{Re}\, \tilde{H}_{mn}(M_{nm} + M_{mn}) + \mathrm{i}\, \mathrm{Im}\, \tilde{H}_{mn}(M_{nm} - M_{mn}) \\ &\qquad + \sum_p \mathrm{Re}\, H_{pp} M_{pp} \Bigg)^2 \Bigg\rangle, \end{aligned} \tag{42}$$

the second equality is from availing the Hermiticity property of $\tilde{H}$; we can also further simplify the expression by expanding out the bracket, thus we arrive at

$$\begin{aligned}&\Bigg\langle \left(\sum_{m<n} \mathrm{Re}\, \tilde{H}_{mn}(M_{nm} + M_{mn}) + \mathrm{i}\, \mathrm{Im}\, \tilde{H}_{mn}(M_{nm} - M_{mn}) + \sum_p \mathrm{Re}\, H_{pp} M_{pp} \right) \\ &\quad \times \left(\sum_{j<k} \mathrm{Re}\, \tilde{H}_{jk}(M_{kj} + M_{jk}) + \mathrm{i}\, \mathrm{Im}\, \tilde{H}_{jk}(M_{kj} - M_{jk}) + \sum_q \mathrm{Re}\, H_{qq} M_{qq} \right) \Bigg\rangle \\ &\qquad = \Bigg\langle \sum_{m<n} \sum_{j<k} \mathrm{Re}\, \tilde{H}_{mn}(M_{nm} + M_{mn})\, \mathrm{Re}\, \tilde{H}_{jk}(M_{kj} + M_{jk}) \\ &\qquad\qquad + \sum_{p,q} (\mathrm{Re}\, \tilde{H}_{pp}) M_{pp} (\mathrm{Re}\, \tilde{H}_{qq}) M_{qq} \\ &\qquad\qquad - \sum_{m<n} \sum_{j<k} \mathrm{Im}\, \tilde{H}_{mn}(M_{nm} - M_{mn})\, \mathrm{Im}\, \tilde{H}_{jk}(M_{kj} - M_{jk}) \Bigg\rangle. \end{aligned} \tag{43}$$

Note that not all the terms are included from the expansion of the brackets, this is due to the fact that those terms do not contribute in the averaging and can hence be neglected for simplicity, since all the random variables are independently distributed with zero mean. We can also reduce these further by using the same reasoning, the only non-zero contribution comes from $m =$

j, $n = k$ and $p = q$, and as a consequence the expression simplifies down to

$$\left\langle \sum_{m<n} (\mathrm{Re}\,\tilde{H}_{mn})^2 (M_{nm} + M_{mn})^2 - (\mathrm{Im}\,\tilde{H}_{mn})^2 (M_{nm} - M_{mn})^2 + \sum_p (\mathrm{Re}\,\tilde{H}_{pp})^2 M_{pp}^2 \right\rangle. \tag{44}$$

As explained previously, the variance of the GUE is given by $\left\langle |\tilde{H}_{mn}|^2 \right\rangle = 1/N$, therefore we can assume that the real and imaginary parts of the complex Gaussian variable are each distributed with variance $\left\langle (\mathrm{Re}\,\tilde{H})^2 \right\rangle = \left\langle (\mathrm{Im}\,\tilde{H})^2 \right\rangle = 1/(2N)$. When we substitute in the variance for both the real and imaginary parts of the complex Gaussian random variables, we are, once the averaging has been completed, left with

$$\frac{1}{N} \sum_{m<n} 2 M_{nm} M_{mn} + \frac{1}{N} \sum_p M_{pp}^2 = \frac{1}{N} \sum_{m,n} M_{nm} M_{mn}, \tag{45}$$

hence after the averaging we arrive at the final expression for the averaged exponential function

$$\exp\left[-\frac{1}{2} \left\langle \left(\sum_{m,n} H_{mn} M_{nm} \right)^2 \right\rangle \right] = \exp\left[-\frac{1}{2N} \sum_{m,n} M_{nm} M_{mn} \right] \tag{46}$$

$$= \exp\left[-\frac{1}{2N} \sum_n (M^2)_{nn} \right] = \exp\left[-\frac{1}{2N} \mathrm{Str}(M^2) \right], \tag{47}$$

where $M_{nm} = \sum_\alpha \Phi_\alpha^*(n) L_\alpha w_n \Phi_\alpha(m) w_m = w_n w_m \sum_\alpha \langle \Phi_\alpha(n) | L_\alpha | \Phi(m)_\alpha \rangle$.

Substituting M_{nm} into the exponential to evaluate the averaging for our case:

$$\exp\left\{ -\frac{1}{2N} \sum_{\alpha,\beta} \sum_{m,n} w_n w_m \langle \Phi_\alpha(n) | L_\alpha | \Phi(m) \rangle \, w_m w_n \langle \Phi_\beta(m) | L_\beta | \Phi_\beta(n) \rangle \right\}$$

$$= \exp\left\{ -\frac{1}{2N} \sum_{\alpha,m} w_m^2 |\Phi_\alpha(m)\rangle\langle\Phi_\beta(m)| L_\beta \sum_{\beta,n} w_n^2 |\Phi_\beta(n)\rangle\langle\Phi_\alpha(n)| L_\alpha \right\} \tag{48}$$

$$= \exp\left\{ -\frac{N}{2} \mathrm{Str}(\tilde{Q}\hat{L})^2 \right\},$$

where $\tilde{Q}_{\alpha\beta} = \frac{1}{N}\sum_m^N w_m^2 |\Phi_\alpha(m)\rangle\langle\Phi_\beta(m)|$ and $\hat{L} = \mathrm{diag}(1,-1,1,1)$.

Substituting this result in the superintegral for the average of the generating function and simplifying the expression yields

$$\left\langle \left(G^R_{nn}\right)^l \left(G^A_{nn}\right)^m \right\rangle = (-1)^N \int d\Phi^* d\Phi (r_n^*(-)r_n(-))^m (r_n^*(+)r_n(+))^l \exp\left\{ -N\mathrm{Str}\left[\frac{1}{2}(\tilde{Q}\hat{L})^2 + \mathrm{i}\hat{E}\frac{1}{N}\sum_m^N |\Phi(m)\rangle\langle\Phi(m)|\hat{L} \right] \right\} \tag{49}$$

where $\hat{E} = \mathrm{diag}(E^-, E^+, E^-, E^+)$ and $\tilde{Q}$ and $\hat{L}$ are the same as defined previously.

As this superintegral is quartic, since $\tilde{Q}$ is quadratic in commutative and anti-commutative variables, we cannot proceed any further as it stands but this obstacle can be overcome by the well-known Hubbard-Stratonovich transformation.

The full details of the derivation of the Hubbard-Stratonovich transformation will not be discussed in this thesis, but for the supersymmetry technique we invoke three variants of the transformation [14]

$$e^{-\frac{1}{2N}\left(\sum |r_i(\pm)|^2\right)^2} = \sqrt{N}\int_{-\infty}^{\infty} \frac{da}{\sqrt{2\pi}} e^{-\frac{N}{2}a^2 + \mathrm{i}a\sum |r_i(\pm)|^2}, \tag{50}$$

$$e^{\frac{1}{2N}\left(\sum \chi_i(\pm)\chi_i^*(\pm)\right)^2} = \sqrt{N}\int_{-\infty}^{\infty} \frac{db}{\sqrt{2\pi}} e^{-\frac{N}{2}b^2 - b\sum \chi_i(\pm)\chi_i^*(\pm)}, \tag{51}$$

$$e^{-\frac{1}{N}\sum r_i(\pm)\chi_i^*(\pm)r_j^*\chi_j(\pm)} = \sqrt{N}\int_{-\infty}^{\infty} d\sigma d\sigma^* e^{-N\sigma^*\sigma + \mathrm{i}\sigma^*\sum r_i^*(\pm)\chi_i(\pm) + \mathrm{i}\sum r_j(\pm)\chi_j^*\sigma}. \tag{52}$$

The first two are elementary Gaussian integrals and the third one is an integral over the Grassmann variables σ and σ^*. This can be generalised for supermatrices, as in our case, and the Hubbard-Stratonovich transformation for supermatrices is given by:

$$\exp\left[-\frac{N}{2}\mathrm{Str}(\tilde{Q}\hat{L})^2 \right] = -\int_{\mathcal{M}_4} dQ \exp\left\{ -N\mathrm{Str}\left[\frac{1}{2}(Q\hat{L})^2 - \mathrm{i}Q\hat{L}\tilde{Q}\hat{L} \right] \right\}, \tag{53}$$

Q is the integration variable introduced by the Hubbard-Stratonovich trans-

formation with the integration over the manifold $\mathcal{M}_4$ [14].

Applying the Hubbard-Stratonovich transformation to our superintegral, and writing $\tilde{Q}$ in terms of Φ, reduces the non-Gaussian character of the integration variables down to their quadratic form

$$\left\langle \left(G^R_{nn}\right)\left(G^A_{nn}\right)^{q-1}\right\rangle = (-1)^{N+1}\int_{\mathcal{M}_4} dQ e^{-\frac{N}{2}\mathrm{Str}(Q\hat{L})^2}\int d\Phi^* d\Phi$$
$$(r_n^*(-)r_n(-))^m (r_n^*(+)r_n(+))^l \exp\left\{\mathrm{i}\sum_i \Phi^\dagger(i)\left(\hat{L}Q\hat{L}w_i^2 - \hat{L}\hat{E}\right)\Phi(i)\right\}, \tag{54}$$

where $\Phi(i)$ is a four-component vector with elements $\Phi_\alpha(i)$. Once we have the integral in this form we can apply Wick's theorem which enables one to integrate over the supervectors Φ and Φ^*.

Wick's theorem can be derived by the change of integration variables $\Phi(i) - \sum_j^N (F^{-1})_{ij}\Psi(j)$ and $\Phi^\dagger(i) - \sum_j^N \Psi^\dagger(j)(F^{-1})_{ji}$, $\Psi(j) = (\Psi_j^-, \Psi_j^+, 0, 0)$, in the Gaussian integral for the superdeterminant:

$$\prod_i^N \mathrm{Sdet}F_i \int d\Phi^* d\Phi \exp\left\{-\mathrm{i}\sum_i^N \Phi^\dagger(i)F_i\Phi(i)\right\} = 1, \tag{55}$$

where $F_i = \hat{E} - w_i^2 Q\hat{L}$ in our case.

Substituting the change of integration variables yields

$$1 = \prod_i^N \mathrm{Sdet}F_i \int d\Phi^* d\Phi \tag{56}$$
$$\exp\left\{-\mathrm{i}\sum_i^N \left(\Phi^\dagger(i) - \sum_j^N \Psi^\dagger(j)(F_i^{-1})_{ji}\right)\hat{L}F_i\left(\Phi(j) - \sum_j^N (F_i^{-1})_{ij}\Psi(j)\right)\right\}, \tag{57}$$

multiplying out the brackets simplifies the expression to

$$\exp\left\{\mathrm{i}\sum_{i,j}^{N}\Psi^{\dagger}(j)(F_i^{-1})_{ji}\hat{L}\Psi(j)\right\}=\prod_{i}^{N}\mathrm{Sdet}F_i\int d\Phi^{*}d\Phi \tag{58}$$

$$\exp\left\{-\mathrm{i}\sum_{i}^{N}\Phi^{\dagger}(i)\hat{L}F_i\Phi(i)\right\}\exp\left\{\mathrm{i}\sum_{i}^{N}\sum_{j}^{N}\Phi^{\dagger}(i)\hat{L}\Psi(j)+\mathrm{i}\sum_{i}^{N}\sum_{j}^{N}\Psi^{\dagger}(j)\hat{L}\Phi(i)\right\}. \tag{59}$$

Differentiating this expression with respect to the auxiliary variables, Ψ_n, one can derive the generalisation of the superanalytic version of the Wick's theorem.

We are going to apply the Wick's theorem for Eq. (54) to integrate over the supervectors Φ and Φ^{*} by expressing the integral in a more convenient notation

$$\begin{aligned}&\overline{(r_n^{*}(-)r_n(-))^{m}(r_n^{*}(+)r_n(+))^{l}}\\&\equiv\overline{\underbrace{r_n^{*}(-)r_n(-)...r_n^{*}(-)r_n(-)}_{m\ \text{times}}\underbrace{r_n^{*}(+)r_n(+)...r_n^{*}(+)r_n(+)}_{l\ \text{times}}},\end{aligned} \tag{60}$$

where

$$\overline{f}=\prod_{i}^{N}\mathrm{Sdet}F_i\int d\Phi^{*}d\Phi f\exp\{-\mathrm{i}\sum_{i}^{N}\Phi^{\dagger}(i)\hat{L}F_i\Phi(i)\}. \tag{61}$$

According to Wick's theorem, we have to group the terms into their distinct pairs and sum them together. If we start by grouping them into pairs of $r_n^{*}(-)r_n(-)$ and $r_n^{*}(+)r_n(+)$ then there is one way of doing this, by keeping the pairs grouped as they are, therefore this is equivalent to

$$\overline{r_n^{*}(-)r_n(-)}^{m}\,\overline{r_n^{*}(+)r_n(+)}^{l}, \tag{62}$$

but we must remember that the pairs do not have to stay in their original position, the first $r_n^{*}(+)$ could be swapped with the second $r_n^{*}(+)$ and so on. If we take this into account then there are $m!l!$ different combinations for $\overline{r_n^{*}(-)r_n(-)}^{m}\,\overline{r_n^{*}(+)r_n(+)}^{l}$.

If we were to mix one pair of $r_n^*(-)r_n(-)$ and $r_n^*(+)r_n(+)$, we obtain

$$\overline{r_n^*(-)r_n(-)}^{m-1}\overline{r_n^*(+)r_n(+)}^{l-1}\overline{r_n^*(-)r_n(+)}\overline{r_n^*(+)r_n(-)}. \tag{63}$$

However, this is only one way, of many, of swapping distinct pairs, if we look at the original expression

$$\overline{r_n^*(-)r_n(-)...r_n^*(-)r_n(-)r_n^*(+)r_n(+)...r_n^*(+)r_n(+))}, \tag{64}$$

the green arrow designates the pairs that were mixed with each other but it could have also been the first of $r_n^*(-)r_n(-)$ and the second of $r_n^*(+)r_n(+)$

$$\overline{r_n^*(-)r_n(-)...r_n^*(-)r_n(-)r_n^*(+)r_n(+)r_n^*(+)r_n(+)...r_n^*(+)r_n(+))}, \tag{65}$$

and so on. This means that since there are m pairs of $r_n^*(-)r_n(-)$ and l pairs of $r_n(+)r_n^*(+)$, there are a total of $C_1^l C_1^m$ combinations:

$$C_1^m C_1^l \overline{r_n^*(-)r_n(-)}^{m-1}\overline{r_n^*(+)r_n(+)}^{l-1}\overline{r_n^*(-)r_n(+)}\overline{r_n^*(+)r_n(-)}, \tag{66}$$

where $C_q^p = p!/((p-q)!q!)$.

By using the same logic we can generalise it to mixing j pairs

$$C_j^m C_j^l \overline{r_n^*(-)r_n(-)}^{m-j}\overline{r_n^*(+)r_n(+)}^{l-j}\overline{r_n^*(-)r_n(+)}^{j}\overline{r_n^*(+)r_n(-)}^{j}, \tag{67}$$

but we must remember that the pairs can be in any order, hence

$$\begin{aligned}
&\overline{(r_n^*(-)r_n(-))^l(r_n^*(+)r_n(+))^m} \\
&= l!m!\sum_{j=0}^{k} C_j^l C_j^m \overline{r_n^*(-)r_n(-)}^{l-j}\overline{r_n^*(+)r_n(+)}^{m-j}\overline{r_n^*(-)r_n(+)}^{j}\overline{r_n^*(+)r_n(-)}^{j} \\
&= l!m!\sum_{j=0}^{k} C_j^l C_j^m (g_{aa}^{BB})^{l-j}(g_{rr}^{BB})^{m-j}(g_{ar}^{BB})^{j}(g_{ra}^{BB})^{j},
\end{aligned} \tag{68}$$

where $k = \min(l, m)$ and

$$\begin{aligned} g_{aa}^{BB} &= \overline{r_n^*(-)r_n(-)}, \quad g_{rr}^{BB} = \overline{r_n^*(+)r_n(+)}, \\ g_{ar}^{BB} &= \overline{r_n^*(-)r_n(+)}, \quad g_{ra}^{BB} = \overline{r_n^*(+)r_n(-)}. \end{aligned} \tag{69}$$

As we are free to set the values of l and m, provided that $q = l + m$, then, for simplicity, we choose $l = 1$ and $m = q - 1$. This greatly simplifies down the pre-exponential factor as it eliminates the summation

$$\begin{aligned} &(q-1)!\left\{(g_{aa}^{BB})^{q-1} g_{rr}^{BB} + (q-1) g_{aa}^{q-2} g_{ar}^{BB} g_{ra}^{BB}\right\} \\ &= (q-1)!(g_{aa}^{BB})^{q-2}\left\{g_{aa}^{BB} g_{rr}^{BB} + (q-1) g_{ar}^{BB} g_{ra}^{BB}\right\}. \end{aligned} \tag{70}$$

Therefore the superintegral is reduced to

$$\begin{aligned} &\left\langle \left(G_{nn}^R\right)\left(G_{nn}^A\right)^{q-1}\right\rangle = (-1)^{N+1}\int_{\mathcal{M}_4} dQ (g_{aa}^{BB})^{q-2}\left\{g_{aa}^{BB} g_{rr}^{BB} + (q-1) g_{ar}^{BB} g_{ra}^{BB}\right\} \\ &\exp\left\{-\frac{N}{2}\mathrm{Str}(Q\hat{L})^2\right\}\prod_i^N \mathrm{Sdet}^{-1}[\hat{L}(\hat{E} - w_i^2 Q\hat{L})]. \end{aligned} \tag{71}$$

For convenience we also permute the rows and columns of the matrices so that

$$\hat{E} \to \mathrm{diag}(E^-, E^-, E^+, E^+), \tag{72}$$

$$\hat{L} \to \mathrm{diag}(1, 1, -1, 1), \tag{73}$$

$$\Phi = (\Phi_1, \Phi_3, \Phi_2, \Phi_4)^T, \tag{74}$$

the notation for the matrices, however, will remain unchanged but the reader should understand that the rows and columns have been permuted. This arrangement is known as the retarded-advanced notation, whereas the former is known as the bosonic-fermionic notation.

The superintegral can be expressed in a more convenient form using the

properties of the superdeterminant

$$\left\langle \left(G^R_{nn}\right)\left(G^A_{nn}\right)^{q-1}\right\rangle = (-1)^{N+1}\int_{\mathcal{M}_4} dQ (g^{BB}_{aa})^{q-2}\{g^{BB}_{aa}g^{BB}_{rr} + (q-1)g^{BB}_{ar}g^{BB}_{ra}\} \exp\left\{-\frac{N}{2}\mathrm{Str}(Q\hat{L})^2 - \sum_i^N \ln \mathrm{Sdet}(\hat{L}\hat{E} - \hat{L}Q\hat{L}w_i^2)\right\}, \tag{75}$$

where $g^{BB}_{\alpha\beta}$ are the matrix elements of the Bose-Bose block of the supermatrix $(\hat{E} - w_n^2 Q_n)^{-1}$, in the retarded-advanced notation.

Now that it is in the desired form, we can apply a property of the superdeterminant to express the superintegral as

$$\begin{aligned}\left\langle \left(G^R_{nn}\right)\left(G^A_{nn}\right)^{q-1}\right\rangle &= (-1)^{N+1}\int_{\mathcal{M}_4} dQ (g^{BB}_{aa})^{q-2}\{g^{BB}_{aa}g^{BB}_{rr} + (q-1)g^{BB}_{ar}g^{BB}_{ra}\} \\ &\times \exp\left\{-\frac{N}{2}\mathrm{Str}(Q\hat{L})^2 - \sum_i^N \mathrm{Str}\ln(\hat{L}\hat{E} - \hat{L}Q\hat{L}w_i^2)\right\} \\ &= (-1)^{N+1}\int_{\mathcal{M}_4} dQ (g^{BB}_{aa})^{q-2}\{g^{BB}_{aa}g^{BB}_{rr} + (q-1)g^{BB}_{ar}g^{BB}_{ra}\} \\ &\times \exp\left\{-N\mathrm{Str}\left[\frac{1}{2}(Q\hat{L})^2 + \frac{1}{N}\sum_i^N \ln \hat{L}(\hat{E} - Q\hat{L}w_i^2)\right]\right\}.\end{aligned} \tag{76}$$

We can further simplify this superintegral by, again, invoking a property of the superdeterminant

$$\begin{aligned}&\mathrm{Str}\ln \hat{L}(\hat{E} - Q\hat{L}w_i^2) = \ln \mathrm{Sdet}\hat{L}(\hat{E} - Q\hat{L}w_i^2) \\ &= \ln \mathrm{Sdet}\hat{L} + \ln \mathrm{Sdet}(\hat{E} - Q\hat{L}w_i^2),\end{aligned} \tag{77}$$

where $\mathrm{Sdet}\hat{L} = -1$, and by absorbing $\hat{L}$ and -1 into the integration measure the superintegral simplifies down to

$$\begin{aligned}\left\langle \left(G^R_{nn}\right)\left(G^A_{nn}\right)^{q-1}\right\rangle &= \int_{\mathcal{M}_4\hat{L}} dQ (g^{BB}_{aa})^{q-2}\{g^{BB}_{aa}g^{BB}_{rr} + (q-1)g^{BB}_{ar}g^{BB}_{ra}\} \\ &\times \exp\left\{-N\mathrm{Str}\left[\frac{1}{2}Q^2 + \frac{1}{N}\sum_i^N \ln(\hat{E} - w_i^2 Q)\right]\right\}.\end{aligned} \tag{78}$$

Recalling definition of $\hat{E} = \mathrm{diag}(E^-, E^-, E^+, E^+)$, this can be expressed as

$E - \mathrm{i}\epsilon\hat{\Lambda}$, where $\hat{\Lambda} = \mathrm{diag}(1, 1, -1, -1)$.

As we are interested in the limit $\epsilon \to 0$, we can expand the logarithm correspondingly

$$\ln(E - w_i^2 Q - \mathrm{i}\epsilon\hat{\Lambda}) = \ln(E - w_i^2 Q) - (E - w_i^2 Q)^{-1}\mathrm{i}\epsilon\hat{\Lambda} + ..., \tag{79}$$

substituting this back into the exponential argument, we arrive at

$$\begin{aligned}\left\langle \left(G_{nn}^R\right)\left(G_{nn}^A\right)^{q-1}\right\rangle = \int_{\mathcal{M}_4\hat{L}} dQ (g_{aa}^{BB})^{q-2}\{g_{aa}^{BB}g_{rr}^{BB} + (q-1)g_{ar}^{BB}g_{ra}^{BB}\} \\ \times \exp\left\{-N\mathcal{L}(Q) + \mathrm{i}\epsilon\sum_i^N \mathrm{Str}(E - w_i^2 Q)^{-1}\hat{\Lambda}\right\},\end{aligned} \tag{80}$$

where $\mathcal{L}(Q) = \frac{1}{2}\mathrm{Str}Q^2 + \sum_i \mathrm{Str}\ln(E - w_i^2 Q)$, the explicit form of dQ is given in Ref. [14].

In the limit $N \to \infty$, the integral over Q is dominated by the saddle point equation that satisfies

$$Q = \frac{1}{N}\sum_i^N \frac{w_i^2}{E - w_i^2 Q}, \tag{81}$$

where relevant solutions are parametrised as [11]

$$Q_{s.p.} = t + \mathrm{i}s\hat{T}^{-1}\hat{\Lambda}\hat{T}, \tag{82}$$

and $s \neq 0$ and t are real parameters satisfying the simultaneous equations

$$t = \frac{1}{N}\sum_i^N \frac{w_i^2(E - w_i^2 t)}{(E - w_i^2 t)^2 + s^2 w_i^4}, \quad \text{and} \quad 1 = \frac{1}{N}\sum_i^N \frac{w_i^4}{(E - w_i^2 t)^2 + s^2 w_i^4}, \tag{83}$$

the matrix $\hat{T}^{-1}\hat{\Lambda}\hat{T}$ belongs to the standard coset space appearing in the GUE case [14, 19].

The simultaneous equations can be derived by substituting the parametrisation into the saddle-point equation:

$$t + \mathrm{i}s\hat{T}^{-1}\hat{\Lambda}\hat{T} = \frac{1}{N}\sum_i^N \frac{w_i^2}{E - w_i^2 t - \mathrm{i}s w_i^2\hat{T}^{-1}\hat{\Lambda}\hat{T}}, \tag{84}$$

the matrix in the saddle point equation can be easily inverted

$$(E - w_i^2 t - \mathrm{i}s w_i^2 \hat{T}^{-1}\hat{\Lambda}\hat{T})^{-1} = \hat{T}^{-1}(E - w_i^2 t - \mathrm{i}s w_i^2 \hat{\Lambda})^{-1}\hat{T}. \tag{85}$$

This can be solved by reducing it down to a simple 2×2 matrix problem

$$\begin{aligned} &\begin{pmatrix} E - w_i^2 t - \mathrm{i}s w_i^2 & 0 \\ 0 & E - w_i^2 t + \mathrm{i}s w_i^2 \end{pmatrix}^{-1} \\ &= \frac{1}{(E - w_i^2 t)^2 + s^2 w_i^4} \begin{pmatrix} E - w_i^2 t + \mathrm{i}s w_i^2 & 0 \\ 0 & E - w_i^2 t - \mathrm{i}s w_i^2 \end{pmatrix}, \end{aligned} \tag{86}$$

hence

$$(E - w_i^2 t - \mathrm{i}s w_i^2 \hat{T}^{-1}\hat{\Lambda}\hat{T})^{-1} = \frac{E - w_i^2 t + \mathrm{i}s w_i^2 \hat{T}^{-1}\hat{\Lambda}\hat{T}}{(E - w_i^2 t)^2 + s^2 w_i^4}. \tag{87}$$

By substituting this result back into the saddle-point equation, the equation yields

$$t + \mathrm{i}s\hat{T}^{-1}\hat{\Lambda}\hat{T} = \frac{1}{N}\sum_i^N \frac{w_i^2(E - w_i^2 t + \mathrm{i}s w_i^2 \hat{T}^{-1}\hat{\Lambda}\hat{T})}{(E - w_i^2 t)^2 + s^2 w_i^4}, \tag{88}$$

once we equate the real and imaginary parts of this equation, we obtain the simultaneous equations in Eq. (101).

Now we can focus our attention back to the main calculation. By using the saddle-point approximation, we are able to simplify the argument of the exponential from substituting $Q_{s.p.}$, as the integral is dominated by the saddle-point when $N \to \infty$. Once this step has been completed, we can integrate over the hard modes resulting in the following

$$\begin{aligned} \left\langle \left(G_{nn}^R\right) \left(G_{nn}^A\right)^{q-1} \right\rangle = \int d\mu(T) (g_{aa}^{BB})^{q-2} \{ g_{aa}^{BB} g_{rr}^{BB} + (q-1) g_{ar}^{BB} g_{ra}^{BB} \} \\ \times \exp\left\{ -\epsilon \pi N \rho(E) \mathrm{Str}(\hat{T}^{-1}\hat{\Lambda}\hat{T}\hat{\Lambda}) \right\}, \end{aligned} \tag{89}$$

since the integration over the hard modes of the manifold is dominated at the saddle-point, then the integration is approximately equal to unity as it represents the ratio of determinants, where the superdeterminant of a block-diagonal matrix whose Bose and Fermi blocks are the same is equal to unity.

We are left with $d\mu(T)$ whose explicit expression is given in Appendix A, the integration measure of the coset space spanned by $\hat{T}$, also known as the soft modes which can not be integrated over by the saddle-point approximations, so their contributions must be calculated separately. The matrix elements $g^{BB}_{\alpha\beta}$ are given explicitly in the Appendix A and $\rho(E)$ is the density of states as calculated by Ref. [11] using the supersymmetric approach, which is given by

$$\rho(E) = \frac{s}{\pi N} \sum_{i}^{N} \frac{w_i^2}{(E - w_i^2 t)^2 + s^2 w_i^4}. \tag{90}$$

Explicit evaluation of the integral above requires one to employ Efetov's parametrisation given in Ref. [20], please see Appendix A for further details.

The averaged product of two Green's functions now reads, after integrating out the Grassmannians and both ϕ_1 and ϕ_2,

$$\begin{aligned}
&\left\langle \left(G^R_{nn}\right) \left(G^A_{nn}\right)^{q-1} \right\rangle = -\frac{(E - w_n^2 t + \mathrm{i}s w_n^2)^{q-2}}{[(E - w_n^2 t)^2 + s^2 w_n^4]^{q-1}} + (q-1) \int_1^\infty d\lambda_1 \int_{-1}^1 d\lambda_2 \\
&\times \frac{(E - w_n^2 t + \mathrm{i}s w_n^2 \lambda_1)^{q-2}}{[(E - w_n^2 t)^2 + s^2 w_n^4]^q} e^{-2\epsilon\pi N \rho(E)(\lambda_1 - \lambda_2)} s^2 w_n^4 \\
&\times \left[1 + \frac{\lambda_1 + \lambda_2}{\lambda_1 - \lambda_2} + \frac{\mathrm{i}(q-2) s w_n^2 (\lambda_1^2 - 1)}{(E - w_n^2 t + \mathrm{i}s w_n^2 \lambda_1)(\lambda_1 - \lambda_2)} \right],
\end{aligned} \tag{91}$$

by adding the zeroth order Grassmannian integral result.

Recalling Eq. (32), the definition of the moments of the eigenvectors, the reader will see that one needs to take the limit $\epsilon \to 0$. As λ_1 is unbounded, the integral over λ_1 must be considered carefully as the integral will not be negligible and therefore cannot be ignored.

To take the limit $\epsilon \to 0$, a substitution, $z = \epsilon\lambda_1$, must be made in the integral

$$\begin{aligned}
&\left\langle \left(G^R_{nn}\right) \left(G^A_{nn}\right)^{q-1} \right\rangle = -\frac{(E - w_n^2 t + \mathrm{i}s w_n^2)^{q-2}}{[(E - w_n^2 t)^2 + s^2 w_n^4]^{q-1}} \\
&+ (q-1) \int_\epsilon^\infty \frac{dz}{\epsilon} \int_{-1}^1 d\lambda_2 \frac{(E - w_n^2 t + \mathrm{i}s w_n^2 \frac{z}{\epsilon})^{q-2}}{[(E - w_n^2 t)^2 + s^2 w_n^4]^q} e^{-2\epsilon\pi N \rho(E)(\frac{z}{\epsilon} - \lambda_2)} s^2 w_n^4 \\
&\times \left[1 + \frac{\frac{z}{\epsilon} + \lambda_2}{\frac{z}{\epsilon} - \lambda_2} + \frac{\mathrm{i}(q-2) s w_n^2 ((\frac{z}{\epsilon})^2 - 1)}{(E - w_n^2 t + \mathrm{i}s w_n^2 \frac{z}{\epsilon})(\frac{z}{\epsilon} - \lambda_2)} \right],
\end{aligned} \tag{92}$$

factoring out ϵ, and performing algebraic manipulations, simplifies the integral down to

$$\begin{aligned}\left\langle \left(G^R_{nn}\right)\left(G^A_{nn}\right)^{q-1}\right\rangle &= -\frac{(E-w_n^2t+\mathrm{i}sw_n^2)^{q-2}}{[(E-w_n^2t)^2+s^2w_n^4]^{q-1}} + (q-1)\int_\epsilon^\infty dz \\ &\times \int_{-1}^{1} d\lambda_2 \epsilon^{1-q}\frac{(\epsilon E-\epsilon w_n^2t+\mathrm{i}sw_n^2z)^{q-2}}{[(E-w_n^2t)^2+s^2w_n^4]^{q}} e^{-2\pi N\rho(E)(z-\epsilon\lambda_2)}s^2w_n^4 \\ &\times \left[1+\frac{z+\epsilon\lambda_2}{z-\epsilon\lambda_2}+\frac{\mathrm{i}(q-2)sw_n^2(z^2-\epsilon^2)}{(\epsilon E-\epsilon w_n^2t+\mathrm{i}sw_n^2z)(z-\epsilon\lambda_2)}\right],\end{aligned} \tag{93}$$

this now allows one to take the limit $\epsilon \to 0$,

$$\begin{aligned}&\lim_{\epsilon\to 0}(2\epsilon)^{q-1}\left\langle \left(G^R_{nn}\right)\left(G^A_{nn}\right)^{q-1}\right\rangle = -\lim_{\epsilon\to 0}(2\epsilon)^{q-1}\frac{(E-w_n^2t+\mathrm{i}sw_n^2)^{q-2}}{[(E-w_n^2t)^2+s^2w_n^4]^{q-1}} \\ &+2^{q-1}(q-1)\int_\epsilon^\infty dz\int_{-1}^{1}d\lambda_2\frac{(\epsilon E-\epsilon w_n^2t+\mathrm{i}sw_n^2z)^{q-2}}{[(E-w_n^2t)^2+s^2w_n^4]^{q}}e^{-2\pi N\rho(E)(z-\epsilon\lambda_2)} \\ &\times s^2w_n^4\left[1+\frac{z+\epsilon\lambda_2}{z-\epsilon\lambda_2}+\frac{\mathrm{i}(q-2)sw_n^2(z^2-\epsilon^2)}{(\epsilon E-\epsilon w_n^2t+\mathrm{i}sw_n^2z)(z-\epsilon\lambda_2)}\right] \\ &=2^{q-1}q(q-1)\int_\epsilon^\infty dz\int_{-1}^{1}d\lambda_2\frac{(\mathrm{i}sw_n^2z)^{q-2}}{[(E-w_n^2t)^2+s^2w_n^4]^{q}}e^{-2\pi N\rho(E)z}s^2w_n^4.\end{aligned} \tag{94}$$

Therefore the integral reduces to

$$\begin{aligned}&\lim_{\epsilon\to 0}(2\epsilon)^{q-1}\left\langle \left(G^R_{nn}\right)\left(G^A_{nn}\right)^{q-1}\right\rangle = 2^{q-1}q(q-1)\int_0^\infty dz\int_{-1}^{1}d\lambda_2 \\ &\times\frac{(\mathrm{i}sw_n^2z)^{q-2}}{[(E-w_n^2t)^2+s^2w_n^4]^{q}}\exp\{-2\pi N\rho(E)z\}s^2w_n^4,\end{aligned} \tag{95}$$

as the integrand is independent of λ_2, this part of the integration is simple and just produces a factor of 2

$$\lim_{\epsilon\to 0}(2\epsilon)^{q-1}\left\langle \left(G^R_{nn}\right)\left(G^A_{nn}\right)^{q-1}\right\rangle = \frac{2^q q(q-1)(\mathrm{i}sw_n^2)^{q-2}s^2w_n^4}{[(E-w_n^2t)^2+s^2w_n^4]^{q}}\int_0^\infty dz z^{q-2}e^{-2\pi N\rho(E)z}. \tag{96}$$

Finally, the integral over z is an elementary integral which can be computed

by making a substitution, $y = 2\pi N\rho(E)z$,

$$\lim_{\epsilon\to 0}(2\epsilon)^{q-1}\left\langle \left(G^R_{nn}\right)\left(G^A_{nn}\right)^{q-1}\right\rangle = \frac{2^q q(q-1)(\mathrm{i}sw_n^2)^{q-2}s^2w_n^4}{[(E-w_n^2t)^2+s^2w_n^4]^q}\frac{1}{2\pi N\rho(E)}\int_0^\infty dy \times\left(\frac{y}{2\pi N\rho(E)}\right)^{q-2}e^{-y} = \frac{2^q q(q-1)(\mathrm{i}sw_n^2)^{q-2}s^2w_n^4}{[(E-w_n^2t)^2+s^2w_n^4]^q}\frac{1}{(2\pi N\rho(E))^{q-1}} \times\int_0^\infty dy y^{q-2}e^{-y}, \tag{97}$$

this is a standard gamma function integral, hence

$$\lim_{\epsilon\to 0}(2\epsilon)^{q-1}\left\langle \left(G^R_{nn}\right)\left(G^A_{nn}\right)^{q-1}\right\rangle = \frac{2^q q(q-1)(\mathrm{i}sw_n^2)^{q-2}s^2w_n^4}{[(E-w_n^2t)^2+s^2w_n^4]^q}\frac{1}{(2\pi N\rho(E))^{q-1}}\Gamma(q-1) = \frac{2\mathrm{i}^{q-2}(sw_n^2)^q}{[(E-w_n^2t)^2+s^2w_n^4]^q}\frac{1}{(\pi N\rho(E))^{q-1}}\Gamma(q+1). \tag{98}$$

Substituting this result into the definition of the moments of the eigenvectors, Eq. (32), yields

$$I_q(n) = \frac{1}{(\pi\rho(E)N)^q}\left[\frac{sw_n^2}{[(E-w_n^2t)^2+s^2w_n^4]}\right]^q\Gamma(q+1), \tag{99}$$

where the density of states is given by

$$\rho(E) = \frac{s}{\pi N}\sum_i^N\frac{w_i^2}{(E-w_i^2t)^2+s^2w_i^4} \tag{100}$$

with $s\neq 0$ and t satisfying the simultaneous equations

$$t = \frac{1}{N}\sum_i^N\frac{w_i^2(E-w_i^2t)}{(E-w_i^2t)^2+s^2w_i^4}, \quad \text{and} \quad 1 = \frac{1}{N}\sum_i^N\frac{w_i^4}{(E-w_i^2t)^2+s^2w_i^4}. \tag{101}$$

After a long calculation we have arrived at the main result in the second chapter, the moments of the eigenvectors are derived from the saddle-point approximation and thus only valid for $N\to\infty$.

From Eq. (99), one can see that the moments of the eigenvectors are not independent of n, this means that each component in the eigenvectors is distributed in a different way, which is a natural consequence of breaking

the unitary invariance of the GUE. At first glance it would seem that the moments of the eigenvectors are only dependent of n in a simple way, but one must remember that there is a non-trivial dependence of all w_i from $\rho(E)$, s and t.

Numerical simulations were performed over a total of 5000 realisations to test the validity of the analytical results for $w_n^2 = (N/n)^{-1/2}$ with N ranging from 500 to 5000. The test was carried out for $q = 1.5$ and $q = 2.2$, whose eigenvectors were close to $E = 0$ and the results of the numerical simulations for $I_q = \sum_n^N I_q(n)$ are shown in fig. 2.

2.2 Distribution function

It turns out in this case, since q appears in the moments in a simple way, we can also recover the probability distribution. The moments of the eigenvectors, $I_q(n)$, and the distribution function, $P_n(y)$, are related by the following definition

$$I_q(n) = \int_0^\infty dy y^q P_n(y), \quad y = |\psi(n)|^2, \tag{102}$$

from this we can deduce the form of the distribution function since we have the result for the moments of the eigenvectors:

$$P_n(y) = a(n)e^{-a(n)y}, \tag{103}$$

where $a(n)$ is a function of n with $a(n) > 0$ for all n. This is because we know that the form of the integral must be the same as a gamma function integral as we already know the result for the moments of the eigenvectors.

Substituting the ansatz into the integral enables one to determine the exact form of $a(n)$:

$$\int_0^\infty dy y^q a(n)e^{-a(n)y}, \tag{104}$$

we can proceed with the integration by the substitution $x = a(n)y$, which

yields

$$\int_0^\infty dy y^q a(n) e^{-a(n)y} = \int_0^\infty \frac{dx}{a(n)} a(n) \left(\frac{x}{a(n)}\right)^q e^{-x} \tag{105}$$

$$= \frac{1}{(a(n))^q} \int_0^\infty dx x^q e^{-x}, \tag{106}$$

this is now a standard elementary integral that we can easily compute to receive

$$\frac{1}{(a(n))^q} \Gamma(q+1). \tag{107}$$

As we know from the relation, Eq. (102), the result above must be equal to the moments of the eigenvectors, which we have already calculated

$$\frac{1}{(a(n))^q} \Gamma(q+1) = \frac{1}{(\pi\rho(E)N)^q} \left[\frac{s w_n^2}{[(E - w_n^2 t)^2 + s^2 w_n^4]}\right]^q \Gamma(q+1), \tag{108}$$

by comparing coefficients we can conclude that

$$a(n) = \pi\rho(E) N \left[\frac{(E - w_n^2 t)^2 + s^2 w_n^2}{s w_n^2}\right], \tag{109}$$

hence

$$\begin{aligned} P_n(y) = \pi\rho(E) N & \left[\frac{(E - w_n^2 t)^2 + s^2 w_n^2}{s w_n^2}\right] \\ & \times \exp\left\{-\pi\rho(E) N \left[\frac{(E - w_n^2 t)^2 + s^2 w_n^2}{s w_n^2}\right] y\right\}, \end{aligned} \tag{110}$$

where $y = |\psi(n)|^2$.

The moments of the eigenvectors and the distribution function were the main results of our project in the second chapter. Now that we have the distribution function, this provides one with information on how the elements of the eigenvectors are each distributed. The moments of the eigenvectors are used to characterise eigenvectors into their respective states later.

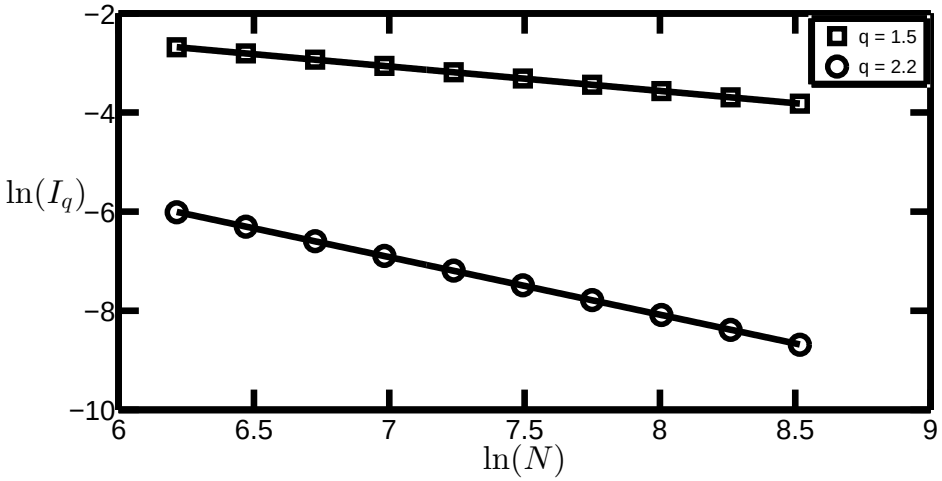

Figure 2: Numerical results (symbols) for $\ln(I_q)$ as a function of $\ln(N)$ for q = 1.5 and q = 2.2. The solid lines represent the best fit to the numerical data.

2.3 Alternative derivation of the zero energy result for the moments of the eigenvectors

In order to verify the result for the moments of the eigenvectors a different approach is taken to calculate the moments of the eigenvectors at $E = 0$, by considering the generalised eigenvalue problem as explained at the beginning of this section:

$$\tilde{H}f = \lambda \hat{A} f, \tag{111}$$

where $\tilde{H} \in$ GUE and $\hat{A}$ is positive definite.

Since $\tilde{H}$ is unitary invariant, $U\tilde{H}U^{\dagger} \in$ GUE, then we can assume $\hat{A}$ is diagonal without loss of generality. Therefore we can express Eq. (111) in the following way

$$\begin{aligned} A^{-1/2}\tilde{H}f = \lambda A^{1/2} f \implies & A^{-1/2}\tilde{H}A^{-1/2}A^{1/2}f = \lambda A^{1/2} f \\ \implies & A^{-1/2}\tilde{H}A^{-1/2}g = \lambda g. \end{aligned} \tag{112}$$

The generalised eigenvalue problem for $\tilde{H}$ is equivalent to the standard eigenvalue problem for $H = A^{-1/2}\tilde{H}A^{-1/2}$, this is the same as our model with $W \equiv A^{-1/2}$ as a diagonal matrix with positive entries.

Focusing on the zero eigenvalue of this model, the problem simplifies down to

$$W\tilde{H}Wg = 0 \implies \tilde{H}Wg = 0, \tag{113}$$

where $u = Wg$ is an eigenvector of $\tilde{H}$ with corresponding eigenvalue $\lambda = 0$. Although we are talking about the zero eigenvalue, the probability of the eigenvalue being exactly zero is 0, since $\tilde{H}$ is random. What we do is take a small window of eigenvalues around zero, so negligible that the eigenvalues are vanishingly small, which we can then effectively equate to zero.

Since the statistical properties of u are known, we can use this to calculate the averaging over u. If we assume u is a normalised eigenvector then g is not normalised, so in order to normalise g we define

$$\hat{g} = \frac{g}{\|g\|} = \frac{W^{-1}u}{\|W^{-1}u\|}, \quad W = \mathrm{diag}(w_1, w_2, w_3, ..., w_N), \tag{114}$$

therefore the nth element of the vector $\hat{g}$ can be written as

$$\hat{g}_n = \frac{w_n^{-1}u_n}{(\sum_i^N |w_i^{-2}u_i^2|)^{1/2}}. \tag{115}$$

Substituting this into the definition of the moments of the eigenvectors, we receive

$$I_q(n) \equiv \left\langle |g_n|^{2q} \right\rangle = \left\langle \frac{|w_n^{-1}u_n|^{2q}}{\left(\sum_i^N |w_i^{-2}u_i^2|\right)^q} \right\rangle_u, \tag{116}$$

from this point we set $x_i \equiv w_i^{-1}$ for ease of notation.

To proceed with the calculation, we first note that the denominator can be expressed in terms in an integral

$$\left(\sum_i^N |x_i u_i|^2\right)^{-q} = \frac{1}{\Gamma(q)} \int_0^\infty d\alpha \alpha^{q-1} e^{-\alpha \sum_i^N x_i |u_i|^2}, \tag{117}$$

once we have it in this form we can compute the average in the moments of the eigenvectors.

Eq. (117) can be shown by first considering the integral

$$\int_0^\infty d\alpha \alpha^{q-1} e^{-\alpha \sum_i^N x_i |u_i|^2}, \tag{118}$$

using the substitution $t = \alpha \sum_i^N x_i |u_i|^2$ simplifies it down to a standard Gamma function integral

$$\int_0^\infty \frac{dt}{\sum_i^\infty x_i |u_i|^2} \left(\frac{t}{\sum_i^N x_i |u_i|^2} \right)^{q-1} e^{-t} \tag{119}$$

$$= \frac{1}{\left(\sum_i^N x_i |u_i|^2 \right)^q} \int_0^\infty dt t^{q-1} e^{-t} = \frac{1}{\left(\sum_i^N x_i |u_i|^2 \right)^q} \Gamma(q). \tag{120}$$

Substituting our original variable α back into the integral yields

$$\frac{1}{\left(\sum_i^N x_i |u_i|^2 \right)^q} \int_0^\infty d\alpha \sum_i^N x_i |u_i|^2 \left(\alpha \sum_i^N x_i |u_i|^2 \right)^{q-1} e^{-\alpha \sum_i^N x_i |u_i|^2} \tag{121}$$

$$= \int_0^\infty d\alpha \alpha^{q-1} e^{-\alpha \sum_i^N x_i |u_i|^2} = \left(\sum_i^N x_i |u_i|^2 \right)^{-q} \Gamma(q), \tag{122}$$

as required.

Hence, the moments of the eigenvectors are equal to

$$I_q(n) = \frac{1}{\Gamma(q)} \int_0^\infty d\alpha \alpha^{q-1} \left\langle |x_n u_n|^{2q} e^{-\alpha \sum_i^N x_i^2 |u_i|^2} \right\rangle, \tag{123}$$

since u_i are the components of an eigenvector of a GUE matrix, the distribution function is widely known

$$y = N|u_i|^2, \quad \mathcal{P}(y) = e^{-y}, \tag{124}$$

which enables one to compute the average

$$\left\langle |x_n u_n|^{2q} e^{-\alpha \sum_i^N x_i^2 |u_i|^2} \right\rangle \equiv \prod_{i \neq n} \left\langle e^{-\alpha x_i^2 |u_i|^2} \right\rangle_{u_i} \left\langle |x_n u_n|^{2q} e^{-\alpha x_n^2 |u_n|^2} \right\rangle_{u_n}. \tag{125}$$

Now we can calculate the average over u_i and u_n separately. Calculating the

first averaging over u_i we receive:

$$\left\langle e^{-\alpha x_i^2|u_i|^2}\right\rangle_{u_i} = \int_0^\infty dy\mathcal{P}(y)e^{-\alpha x_i^2\frac{y}{N}} = \int_0^\infty dye^{-y}e^{-\frac{\alpha x_i^2}{N}y} = \frac{1}{1+\frac{\alpha x_i^2}{N}}, \quad (126)$$

then the averaging over u_n:

$$\left\langle |x_n u_n|^{2q} e^{-\alpha x_n^2|u_n|^2}\right\rangle_{u_n} = \int_0^\infty dye^{-y}x_n^{2q}\left(\frac{y}{N}\right)^q e^{-\frac{\alpha x_i^2}{N}y} \quad (127)$$

$$= \frac{x_n^{2q}}{N^q}\int_0^\infty dyy^q e^{-\left(1+\frac{\alpha x_n^2}{N}\right)y} = \frac{x_n^{2q}}{N^q}\left(1+\frac{\alpha x_n^2}{N}\right)^{-q-1}\Gamma(q+1), \quad (128)$$

which was calculated in the same way as previously shown for Eq. (117), therefore we will not detail the steps in this calculation.

Hence, the moments of the eigenvectors are equivalent to

$$I_q(n) = \frac{\Gamma(q+1)}{\Gamma(q)}\frac{x_n^{2q}}{N^q}\int_0^\infty d\alpha\alpha^{q-1}\left(1+\frac{\alpha x_n^2}{N}\right)^{-q-1}\prod_{i\neq n}\frac{1}{\left(1+\frac{\alpha x_i^2}{N}\right)}, \quad (129)$$

whereby absorbing $\left(1+\frac{\alpha x_n^2}{N}\right)^{-1}$ into the product operator simplifies the integrand down to

$$I_q(n) = \frac{\Gamma(q+1)}{\Gamma(q)}\frac{x_n^{2q}}{N^q}\int_0^\infty d\alpha\alpha^{q-1}\left(1+\frac{\alpha x_n^2}{N}\right)^{-q}\prod_{i}\frac{1}{\left(1+\frac{\alpha x_i^2}{N}\right)}. \quad (130)$$

This can be expressed in a more convenient form by using the identity

$$\prod_i\left(1+\frac{\alpha x_i^2}{N}\right)^{-1} \equiv e^{-\sum_i \ln\left(1+\frac{\alpha x_i^2}{N}\right)}, \quad (131)$$

where we expand the right-hand side using the Taylor series

$$\sum_i^N \ln\left(1+\frac{\alpha x_i^2}{N}\right) = \sum_i^N \frac{\alpha x_i^2}{N} - \frac{\alpha^2 x_i^4}{2N^2} + ..., \quad (132)$$

and use the fact that

$$\rho(0) \propto \frac{1}{N}\sum_i^N x_i^2 \implies \sum_i^N x_i^2 \propto N, \tag{133}$$

since $\rho(0)$ must be independent of N as $N \to \infty$. Also, using the fact that x_i is bounded, Eq. (133) implies that

$$\sum_i^N x_i^4 \propto N, \tag{134}$$

hence

$$\sum_i^N \frac{\alpha x_i^2}{N} - \frac{\alpha^2 x_i^4}{2N^2} = \sum_i^N \frac{\alpha x_i^2}{N} + O(1/N). \tag{135}$$

Therefore the moments of the eigenvectors simplify to

$$I_q(n) = \frac{\Gamma(q+1)}{\Gamma(q)}\frac{x_n^{2q}}{N^q}\int_0^\infty d\alpha \alpha^{q-1}\left(1+\frac{\alpha x_n^2}{N}\right)^{-q} e^{-\frac{\alpha}{N}\sum_i^N x_i^2 + O(1/N)}, \tag{136}$$

keeping all the leading order terms reduces the integral to

$$I_q(n) = \frac{\Gamma(q+1)}{\Gamma(q)}\frac{x_n^{2q}}{N^q}\int_0^\infty d\alpha \alpha^{q-1} e^{-\frac{\alpha}{N}\sum_i^N x_i^2}(1+O(1/N)) \tag{137}$$

$$= \frac{\Gamma(q+1)}{\Gamma(q)}\frac{x_n^{2q}}{N^q}\left[\left(\frac{1}{N}\sum_i^N x_i^2\right)^{-q}\Gamma(q) + ...\right] \approx \frac{x_n^{2q}}{\left(\sum_i^N x_i^2\right)^q}\Gamma(q+1), \tag{138}$$

arriving at our result for the moments of the eigenvectors at $E = 0$

$$I_q(n) = \frac{x_n^{2q}}{\left(\sum_i^N x_i^2\right)^q}\Gamma(q+1). \tag{139}$$

By comparing this result with the zero energy result for the moments of the eigenvectors, Eq. (99), derived from the supersymmetry approach:

$$I_q(n) = \frac{1}{(\pi\rho(0)N)^q}\frac{1}{w_n^{2q}}\Gamma(q+1) = \frac{w_n^{-2q}}{\left(\sum_i^N \frac{1}{w_i^2}\right)^q}\Gamma(q+1), \tag{140}$$

this is the zero energy result from Eq. 292 since $\pi\rho(0)N = \sum_i^N \frac{1}{w_i^2}$, and the

zero energy result derived in this section

$$I_q(n) = \frac{w_n^{-2q}}{\left(\sum_i^N \frac{1}{w_i^2}\right)^q}\Gamma(q+1), \quad x_i = w_i^{-1}, \tag{141}$$

we can see that the two results coincide, validating our result as required. Therefore we can confirm that this is indeed the result for the moments of the eigenvectors.

2.4 Extended, localised and critical eigenvectors

In this section we consider some possible entries for w_i to observe how the deformation affects the characteristics of the eigenvectors. We present our results for arbitrary value of E and verify them by direct diagonalisation. In order to check that the scaling of the moments of the eigenvectors for the zero energy and non-zero energy remains principally the same, we consider the general case of an arbitrary E.

Our focal point is on the power-law dependence w_n on n, as this provides an interesting variety of characteristics of the eigenvectors:

$$w_n^2 = c\left(\frac{1}{n}\right)^p, \tag{142}$$

c is a normalisation constant in which its dependence on N is uniquely determined by requiring the density of states to be independent of N, as $N \to \infty$. We consider three variants of the power-law dependence: $p > 0$, $-1 < p < 0$ and $p < -1$, which will each produce their own unique eigenvector characteristics.

The first case studied was for $p > 0$, which is normalised by first looking at the density of states in the zero energy case for $w_i^2 = ci^{-p}$

$$\rho(0) = \frac{1}{\pi N}\sum_i^N \frac{1}{w_i^2} = \frac{1}{\pi N c}\sum_i^N i^p \propto \frac{1}{\pi N c}N^{p+1} = \frac{1}{\pi c}N^p \implies c \propto N^p, \tag{143}$$

since $s = 1$ and $t = 0$ when $E = 0$.

Now to verify whether this normalisation constant is also valid for $E \neq 0$

$$\rho(E) = \frac{s}{\pi N} \sum_i^N \frac{(N/i)^p}{(E-(N/i)^p t)^2 + s^2 (N/i)^{2p}} \leq \frac{1}{\pi N^{p+1} s} \sum_i^N i^p \tag{144}$$

$$\propto \frac{1}{\pi N^{p+1} s} N^{p+1} \to \text{const}, \tag{145}$$

for the lower bound we have

$$\begin{aligned} &\frac{s}{\pi N} \sum_i^N \frac{(N/i)^p}{(E(N/i)^p + (N/i)^p t)^2 + s^2 (N/i)^{2p}} \\ &\qquad = \frac{s}{\pi N} \frac{1}{(E+t)^2 + s^2} \sum_i^N \left(\frac{i}{N}\right)^p \\ &\propto \frac{s}{\pi N^{p+1}} \frac{1}{(E+t)^2 + s^2} N^{p+1} \to \text{const} \leq \rho(E) \end{aligned} \tag{146}$$

as $N \to \infty$, therefore $\rho(E)$ is independent of N as $N \to \infty$. This is due to the fact that $\rho(E)$ is bounded above and below by the same series that is independent of N as $N \to \infty$.

Next, the scaling of the moments of the eigenvectors, with $w_i^2 = (N/i)^p$, must be calculated in order to determine how the deformation of the GUE has altered the eigenvector characteristics

$$\sum_n^N I_q(n) = \frac{1}{(\pi\rho(E)N)^q} \sum_n^N \left[\frac{s(N/n)^p}{(E-(N/n)^p t)^2 + s^2 (N/n)^{2p}}\right]^q \Gamma(q+1) \tag{147}$$

$$\leq \frac{1}{(\pi\rho(E)N)^q} \frac{1}{s^q N^{pq}} \Gamma(q+1) \sum_n^N n^{pq}, \tag{148}$$

its lower bound can also be determined by

$$\frac{1}{(\pi\rho(E)N)^q} \sum_n^N \left[\frac{s(N/n)^p}{(E(N/n)^p + (N/n)^p t)^2 + s^2 (N/n)^{2p}}\right]^q \Gamma(q+1) \tag{149}$$

$$= \frac{1}{(\pi\rho(E)N)^q} \frac{s^q}{[(E+t)^2 + s^2]^q N^{pq}} \Gamma(q+1) \sum_n^N n^{pq} \leq \sum_n^N I_q(n), \tag{150}$$

hence

$$\sum_n^N I_q(n) \propto \frac{1}{N^{q(p+1)}} \sum_n^N n^{pq} \propto N^{1-q}, \quad p > 0. \tag{151}$$

This scaling corresponds to extended states, the same as the GUE case, so this particular deformation has had no effect on the eigenvectors.

For the next case, $-1 < p < 0$, the normalisation constant is formally the same so the calculation will not be shown as it would follow the same steps as the previous case, with $w_i^2 = (N/i)^p$.

The steps in calculating the scaling of the moments of the eigenvectors are also largely the same and therefore will also not be shown again

$$\sum_n^N I_q(n) \propto \frac{1}{N^{q(p+1)}} \sum_n^N n^{pq}, \quad -1 < p < 0. \tag{152}$$

The main distinction in this case is that the summation can converge or diverge which is conditional on q, so must be considered carefully. There are 3 possibilities in this case: $pq > -1$, $pq = -1$ and $pq < -1$ since p is negative.

In the case for $pq < -1$

$$\sum_n^N n^{pq} \to \text{const as } N \to \infty, \tag{153}$$

by using the well-known result

$$\sum_n^\infty \frac{1}{n^k} \to \text{const, for all } k > 1, \tag{154}$$

for $pq = -1$

$$\sum_n^N \frac{1}{n} \approx \int^N dx \frac{1}{x} \propto \ln(N), \text{ as } N \to \infty, \tag{155}$$

and for the third and final case $pq > -1$

$$\sum_n^N n^{pq} \propto N^{pq+1}. \tag{156}$$

Hence the scaling of the moments of the eigenvectors for all 3 cases

$$\sum_{n}^{N} I_q(n) \propto \begin{cases} N^{-q(p+1)}, & q > -\frac{1}{p}, \\ N^{1-q}, & q < -\frac{1}{p}, \\ \ln(N) N^{1-q}, & q = -\frac{1}{p}. \end{cases} \tag{157}$$

The scaling of the moments of the eigenvectors is dependent on q and it in fact is a critical state, where the fractal dimension can be determined by comparing with Eq. (7),

$$d_q = \begin{cases} \frac{q(p+1)}{q-1}, & q > -\frac{1}{p}, \\ 1, & q < -\frac{1}{p} \end{cases}. \tag{158}$$

This critical state is part of the "frozen" phase, sharing the characteristics of extended and critical states. A frozen state is such that d_q becomes a constant that is q-independent for some values of q.

Finally, for the case of

$$w_i^2 = c\frac{1}{i^p} \quad p < -1, \tag{159}$$

where c is the normalisation constant. In order to normalise this, the zero energy case for the density of states must be considered

$$\rho(0) = \frac{1}{\pi N} \sum_{i}^{N} \frac{1}{w_i^2} = \frac{1}{\pi N c} \sum_{i}^{N} i^p, \tag{160}$$

by using Eq. (154), the summation must converge to a constant since $p < -1$, the normalisation constant $c \propto 1/N$, hence

$$w_i^2 = \frac{1}{N i^p}. \tag{161}$$

Now to verify the normalisation constant for the non-zero energy case

$$\begin{aligned}\rho(E) &= \frac{s}{\pi N}\sum_{i}^{N}\frac{1/(Ni^p)}{(E-1/(Ni^p)t)^2+s^2/(N^2i^{2p})} \leq \frac{1}{\pi s}\sum_{i}^{N} i^p \\ &\to \text{const as } N\to\infty, \quad \text{since } p<-1,\end{aligned} \tag{162}$$

for the lower bound we have

$$\begin{aligned}&\frac{s}{\pi N}\sum_{i\geq i_0}^{N}\frac{1/(Ni^p)}{(E/(Ni^p)+1/(Ni^p)t)^2+s^2/(N^2i^{2p})} \\ &\qquad = \frac{s}{\pi}\frac{1}{(E+t)^2+s^2}\sum_{i\geq i_0}^{N} i^p \leq \rho(E),\end{aligned} \tag{163}$$

where $i_0 = \lceil \sqrt[-p]{N} \rceil$ and $\lceil\ \rceil$ denotes the ceiling function. Although it is not clear why this is the lower bound since $1/(Ni^p) \not> 1$ for all i, we must remember that $p<-1$ and therefore i^p will counteract N as as i increases. To ensure that this remains a lower bound we require an i_0 such that $1/(Ni_0^p) \geq 1$ for all $i \geq i_0$ and delete all the previous terms in the series where this condition is not satisfied. As $\rho(E)$ is bounded above and below by two different series that behave in the same way and are independent of N as $N\to\infty$, we can conclude that $\rho(E)$ will also behave in the same way as $N\to\infty$.

Therefore the normalisation constant remains the same in the non-zero energy case, so we can proceed with computing the scaling of the moments of the eigenvectors to see how this particular deformation has affected the eigenvectors

$$\sum_{n}^{N} I_q(n) = \frac{1}{(\pi\rho(E)N)^q}\sum_{n}^{N}\left[\frac{s/(Nn^p)}{(E-1/(Nn^p)t)^2+s^2/(N^2n^{2p})}\right]^q \Gamma(q+1) \tag{164}$$

$$\leq \frac{1}{(\pi\rho(E))^q}\frac{1}{s^q}\Gamma(q+1)\sum_{n}^{N} n^{pq}, \quad p<-1. \tag{165}$$

The summation can, again, converge or diverge which is dependent on the value of q, so 3 separate cases must be considered: $pq<-1$, $pq>-1$ and

$pq = -1$. As these cases have been considered before in the previous example, we will not detail the same steps and we will just refer to the previous example.

The lower bound can also be calculated by

$$\frac{1}{(\pi\rho(E)N)^q}\sum_{n}^{N}\left[\frac{s/(Nn^p)}{(E/(Nn^p)+1/(Nn^p)t)^2+s^2/(N^2n^{2p})}\right]^q\Gamma(q+1), \quad (166)$$

by using the same reasoning as in the case for the density of states for the lower bound, hence

$$\frac{1}{(\pi\rho(E)N)^q}\sum_{n\geq n_0}^{N}\left[\frac{s/(Nn^p)}{(E/(Nn^p)+1/(Nn^p)t)^2+s^2/(N^2n^{2p})}\right]^q\Gamma(q+1) \quad (167)$$

$$=\frac{1}{(\pi\rho(E))^q}\left[\frac{s^q}{(E+t)^2+s^2}\right]\Gamma(q+1)\sum_{n\geq n_0}^{N}n^{pq}\leq\sum_{n}^{N}I_q(n), \quad p<-1, \quad (168)$$

where $n_0 = \lceil \sqrt[-p]{N} \rceil$.

Since the moments of the eigenvectors are bounded above and below by summations that behaves in the same way as $N \to \infty$, the moments of the eigenvector must converge or diverge in a similar fashion as $N \to \infty$.

Looking at Eqs. (153), (155) and (156) shows how the summation would diverge or converge as $N \to \infty$. Hence, by collecting all the information together, the scaling of the moments of the eigenvectors can be calculated as shown below

$$\sum_{n}^{N}I_q(n)\propto\begin{cases}const, & q>-\frac{1}{p},\\ N^{pq+1}, & q<-\frac{1}{p}\\ \ln(N), & q=-\frac{1}{p}.\end{cases} \quad (169)$$

The scaling of the moments of the eigenvectors is dependent on q and therefore is a critical state, where its fractal dimension can be found by com-

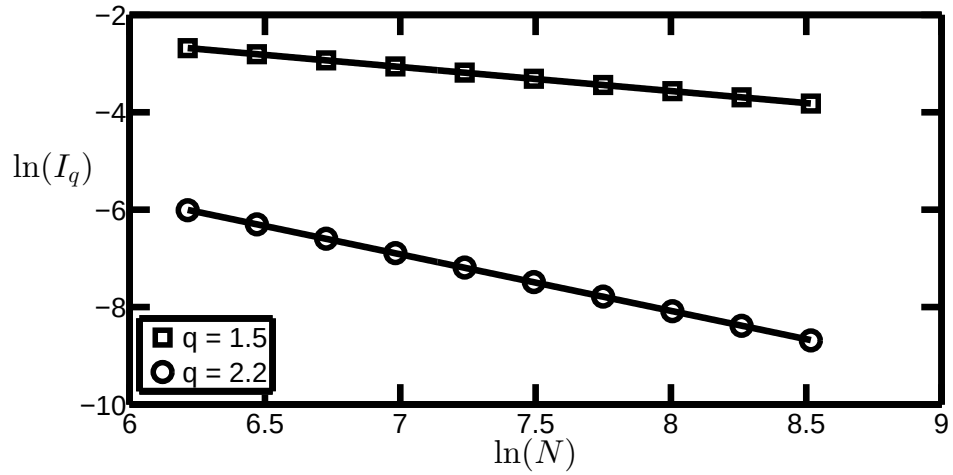

Figure 3: Numerical results (symbols) for $\ln(I_q)$ as a function of $\ln(N)$ for q = 1.5 and q = 2.2 with $p = 1/2$. The solid lines represent the best fit to the numerical data.

paring with Eq. (7), hence

$$d_q = \begin{cases} 0, & q > -\frac{1}{p}, \\ \frac{pq+1}{1-q}, & q < -\frac{1}{p} \end{cases}. \tag{170}$$

This is another "frozen" state, this time sharing the characteristics of a critical state and a localised state.

Various numerical simulations were carried out to verify our analytical results, one such example was for $w_n^2 = (N/n)^{-1/2}$, with N ranging from 500 to 5000. The results of the numerical simulations are shown in fig. 3.

2.5 Model I with random diagonal matrix

We also generalise our first result for the moments of the eigenvectors to a random diagonal W matrix in model I, Eq. (28). As a reminder the moments of the eigenvectors are given by

$$I_q(n) = \frac{1}{(\pi\rho(E)N)^q}\left[\frac{sw_n^2}{(E - w_n^2 t)^2 + s^2 w_n^4}\right]^q \Gamma(q+1), \tag{171}$$

with distribution function

$$P_n(y) = \pi\rho(E)N\left[\frac{(E-w_n^2t)^2+s^2w_n^2}{sw_n^2}\right] \times \exp\left\{-\pi\rho(E)N\left[\frac{(E-w_n^2t)^2+s^2w_n^2}{sw_n^2}\right]y\right\}, \tag{172}$$

where $y = |\psi(n)|^2$, s and t satisfy the simultaneous equations

$$t = \frac{1}{N}\sum_i^N \frac{w_i^2(E-w_i^2t)}{(E-w_i^2t)^2+s^2w_i^4}, \quad \text{and} \quad 1 = \frac{1}{N}\sum_i^N \frac{w_i^4}{(E-w_i^2t)^2+s^2w_i^4}. \tag{173}$$

If w_i are random variables then the parameters s and t must also be random variables since the equations, which define them, involve the random variables w_i. As s and t are determined by a large number of random variables, we postulated that they must therefore satisfy some generalisation of the law of large numbers: the deviation of s and t from their mean values $\langle s\rangle$ and $\langle t\rangle$ become negligible as $N \to \infty$. To corroborate the validity of this conjecture we ran numerical simulations which confirm the validity of our assumption. Therefore, in order to average over the Gaussian distribution of w_i, we can replace s and t by their mean values in the simultaneous equations:

$$\langle t\rangle = \left\langle \frac{x^2(E-x^2\langle t\rangle)}{(E-x^2\langle t\rangle)^2+\langle s\rangle^2 x^4}\right\rangle_x, \quad 1 = \left\langle \frac{x^4}{(E-x^2\langle t\rangle)^2+\langle s\rangle^2 x^4}\right\rangle_x, \tag{174}$$

where x is the Gaussian random variable with $\langle x\rangle = 0$, variance $\langle x^2\rangle = \sigma^2$ and probability distribution function

$$P(x) = \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{x^2}{2\sigma^2}}. \tag{175}$$

We will first focus our attention on averaging the second simultaneous equation by expressing the equation in a different and more convenient form:

$$\begin{aligned} 1 &= \left\langle \frac{x^4}{(E - x^2 \langle t \rangle)^2 + \langle s \rangle^2 x^4} \right\rangle_x \\ &= \frac{1}{\langle t \rangle^2 + \langle s \rangle^2} \left(1 + \frac{1}{\langle t \rangle^2 + \langle s \rangle^2} \left\langle \frac{2Ex^2 - E^2}{\left(x^2 - \frac{E\langle t \rangle}{\langle t \rangle^2 + \langle s \rangle^2} \right)^2 + \frac{E^2 \langle s \rangle^2}{\left(\langle t \rangle^2 + \langle s \rangle^2 \right)^2}} \right\rangle_x \right), \end{aligned} \tag{176}$$

for detailed steps showing the equivalence of the two expressions, see Appendix D.

Therefore, to calculate the averaging over w_i in the second simultaneous equation, we must calculate the quantity

$$\left\langle \frac{2Ex^2 - E^2}{\left(x^2 - \frac{E\langle t \rangle}{\langle t \rangle^2 + \langle s \rangle^2} \right)^2 + \frac{E^2 \langle s \rangle^2}{\left(\langle t \rangle^2 + \langle s \rangle^2 \right)^2}} \right\rangle_x . \tag{177}$$

In order to average over the simultaneous equations, it is beneficial to apply the Fourier transform of

$$P(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\kappa e^{-\mathrm{i}\kappa x} \hat{P}(\kappa), \quad \text{where} \quad \hat{P}(\kappa) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\kappa^2 \sigma^2}. \tag{178}$$

The averaging of Eq. (177) can then be expressed as

$$\begin{aligned} \left\langle \frac{2Ex^2 - E^2}{\left(x^2 - \frac{E\langle t \rangle}{\langle t \rangle^2 + \langle s \rangle^2} \right)^2 + \frac{E^2 \langle s \rangle^2}{\left(\langle t \rangle^2 + \langle s \rangle^2 \right)^2}} \right\rangle_x &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\kappa \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\kappa^2 \sigma^2} \\ \times \int_{-\infty}^{\infty} dx \frac{(2Ex^2 \langle t \rangle - E^2) e^{-\mathrm{i}\kappa x}}{\left(x^2 - \frac{E\langle t \rangle}{\langle t \rangle^2 + \langle s \rangle^2} \right)^2 + \frac{E^2 \langle s \rangle^2}{\left(\langle t \rangle^2 + \langle s \rangle^2 \right)^2}}. \end{aligned} \tag{179}$$

To calculate the integral over x on the real axis, we first extend the integral over the complex plane which enables one to employ the residue theorem.

The integral extended over the complex plane is given by

$$\int_C dz \frac{(2Ez^2 \langle t \rangle - E^2)e^{-\mathrm{i}\kappa z}}{\left(z^2 - \frac{E\langle t \rangle}{\langle t \rangle^2 + \langle s \rangle^2}\right)^2 + \frac{E^2 \langle s \rangle^2}{\left(\langle t \rangle^2 + \langle s \rangle^2\right)^2}} \tag{180}$$

where C is the contour of integration.

The residue theorem can be applied by first finding the poles of the integrand by equating the denominator to 0 and solving for z:

$$z = \pm \sqrt{\frac{E}{\langle t \rangle^2 + \langle s \rangle^2}} \sqrt{\langle t \rangle \pm \mathrm{i} \langle s \rangle}. \tag{181}$$

We define a contour that goes along the real axis from $-R$ to R, then going clockwise along a semi-circle centred on 0, from R to $-R$ below the real axis and by taking $R > |z|$ so that the poles are enclosed within the contour.

The poles that are enclosed within the contour are the ones that contribute to the integral therefore any poles outside of the contour can be neglected, this means that we can ignore the two poles above the real axis and take the two below. The question of which two are above and below the real axis arises, to proceed we must make some assumptions, we assume that $E > 0$, $\langle t \rangle > 0$ and $\langle s \rangle > 0$ for the rest of the calculation. Under these assumptions we can see that the two poles which contribute towards the integral are

$$z = \sqrt{\frac{E}{\langle t \rangle^2 + \langle s \rangle^2}} \sqrt{\langle t \rangle - \mathrm{i} \langle s \rangle}, \quad -\sqrt{\frac{E}{\langle t \rangle^2 + \langle s \rangle^2}} \sqrt{\langle t \rangle + \mathrm{i} \langle s \rangle}. \tag{182}$$

Now to apply the residue theorem, we have to calculate the residues of

$$f(z) = \frac{(2Ez^2 \langle t \rangle - E^2)e^{-\mathrm{i}\kappa z}}{\left(z^2 - \frac{E\langle t \rangle}{\langle t \rangle^2 + \langle s \rangle^2}\right)^2 + \frac{E^2 \langle s \rangle^2}{\left(\langle t \rangle^2 + \langle s \rangle^2\right)^2}}, \tag{183}$$

more precisely if a is a pole of order n, then the residue of $f(z)$ is determined by the formula

$$Res(f(z), a) = \frac{1}{(n-1)!} \lim_{z \to a} \frac{d^{n-1}}{dz^{n-1}}((z-a)^n f(z)), \tag{184}$$

but as all of our poles are of order 1, the formula simplifies greatly and there will be no derivatives involved.

To work out the first residue we let

$$\begin{aligned} z_1 &= \sqrt{\frac{E}{\langle t\rangle^2+\langle s\rangle^2}}\sqrt{\langle t\rangle - \mathrm{i}\,\langle s\rangle}, \\ z_2 &= -\sqrt{\frac{E}{\langle t\rangle^2+\langle s\rangle^2}}\sqrt{\langle t\rangle + \mathrm{i}\,\langle s\rangle}, \\ z_3 &= \sqrt{\frac{E}{\langle t\rangle^2+\langle s\rangle^2}}\sqrt{\langle t\rangle + \mathrm{i}\,\langle s\rangle}, \\ z_4 &= -\sqrt{\frac{E}{\langle t\rangle^2+\langle s\rangle^2}}\sqrt{\langle t\rangle - \mathrm{i}\,\langle s\rangle}, \end{aligned} \tag{185}$$

then by using the formula in Eq. (184), the residue is equal to

$$\begin{aligned} Res(f(z),z_1) &= \lim_{z\to z_1}(z-z_1)\frac{(2Ez^2\langle t\rangle - E^2)e^{-\mathrm{i}\kappa z}}{(z-z_1)(z-z_4)(z-z_3)(z-z_2)} \\ &= \lim_{z\to z_1}\frac{(2Ez^2\langle t\rangle - E^2)e^{-\mathrm{i}\kappa z}}{(z-z_4)(z-z_3)(z-z_2)} \\ &= \frac{(2Ez_1^2\langle t\rangle - E^2)e^{-\mathrm{i}\kappa z_1}}{(z_1-z_4)(z_1-z_3)(z_1-z_2)}. \end{aligned} \tag{186}$$

Inputting the values of z_1, z_2, z_3 and z_4 we find that

$$Res(f(z),z_1) = -\frac{\left(\langle t\rangle^2+\langle s\rangle^2\right)\sqrt{E}\left(\frac{2\langle t\rangle}{\sqrt{\langle t\rangle+\mathrm{i}\langle s\rangle}} - \sqrt{\langle t\rangle+\mathrm{i}\,\langle s\rangle}\right)e^{-\mathrm{i}\kappa\sqrt{\frac{E}{\langle t\rangle+\mathrm{i}\langle s\rangle}}}}{4\mathrm{i}\,\langle s\rangle}. \tag{187}$$

The second residue of $f(z)$ can also be calculated in the same way

$$\begin{aligned} Res(f(z),z_2) &= \lim_{z\to z_2}(z-z_2)\frac{(2Ez^2\langle t\rangle - E^2)e^{-\mathrm{i}\kappa z}}{(z-z_1)(z-z_4)(z-z_3)(z-z_2)} \\ &= \lim_{z\to z_2}\frac{(2Ez^2\langle t\rangle - E^2)e^{-\mathrm{i}\kappa z}}{(z-z_1)(z-z_4)(z-z_3)} \\ &= \frac{(2Ez_2^2\langle t\rangle - E^2)e^{-\mathrm{i}\kappa z_2}}{(z_2-z_4)(z_2-z_3)(z_2-z_2)}. \end{aligned} \tag{188}$$

By inputting the values of z_1, z_2, z_3 and z_4 we receive

$$Res(f(z), z_2) = -\frac{\left(\langle t\rangle^2 + \langle s\rangle^2\right)\sqrt{E}\left(\frac{2\langle t\rangle}{\sqrt{\langle t\rangle - \mathrm{i}\langle s\rangle}} - \sqrt{\langle t\rangle - \mathrm{i}\langle s\rangle}\right) e^{\mathrm{i}\kappa\sqrt{\frac{E}{\langle t\rangle - \langle s\rangle}}}}{4\mathrm{i}\langle s\rangle}. \tag{189}$$

According to the residue theorem the integral over the complex plane is equal to

$$\int_C dz f(z) = -2\pi\mathrm{i}(Res(f(z), z_1) + Res(f(z), z_2)), \tag{190}$$

hence,

$$\begin{aligned}\int_C dz f(z) &= \frac{\pi\left(\langle t\rangle^2 + \langle s\rangle^2\right)\sqrt{E}}{2\langle s\rangle} \\ &\times \left(\frac{\langle t\rangle - \mathrm{i}\langle s\rangle}{\sqrt{\langle t\rangle + \mathrm{i}\langle s\rangle}} e^{-\mathrm{i}|\kappa|\sqrt{\frac{E}{\langle t\rangle + \mathrm{i}\langle s\rangle}}} + \frac{\langle t\rangle + \mathrm{i}\langle s\rangle}{\sqrt{\langle t\rangle - \mathrm{i}\langle s\rangle}} e^{\mathrm{i}|\kappa|\sqrt{\frac{E}{\langle t\rangle - \langle s\rangle}}}\right).\end{aligned} \tag{191}$$

The appearance of the absolute value signs from κ comes from the fact that this integral remains the same regardless of the sign of κ, which is well-known in the residue theorem. The integral over the complex plane converges to the real integral as $R \to \infty$ so it is equivalent to our required integral; see Appendix D for further details.

Now that the integration over x is completed, we can proceed with the integration over κ

$$\begin{aligned}&\left\langle \frac{2Ex^2 - E^2}{\left(x^2 - \frac{E\langle t\rangle}{\langle t\rangle^2 + \langle s\rangle^2}\right)^2 + \frac{E^2\langle s\rangle^2}{\left(\langle t\rangle^2 + \langle s\rangle^2\right)^2}} \right\rangle_x = \\ &\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\kappa \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\kappa^2\sigma^2} \frac{\pi\left(\langle t\rangle^2 + \langle s\rangle^2\right)\sqrt{E}}{2\langle s\rangle} \\ &\times \left(\frac{\langle t\rangle - \mathrm{i}\langle s\rangle}{\sqrt{\langle t\rangle + \mathrm{i}\langle s\rangle}} e^{-\mathrm{i}|\kappa|\sqrt{\frac{E}{\langle t\rangle + \mathrm{i}\langle s\rangle}}} + \frac{\langle t\rangle + \mathrm{i}\langle s\rangle}{\sqrt{\langle t\rangle - \mathrm{i}\langle s\rangle}} e^{\mathrm{i}|\kappa|\sqrt{\frac{E}{\langle t\rangle - \langle s\rangle}}}\right),\end{aligned} \tag{192}$$

substituting this into Eq. (176) and simplifying terms in the expression, we

find

$$\left\langle \frac{x^4}{(E-x^2\langle t\rangle)^2+\langle s\rangle^2 x^4}\right\rangle_x = \frac{1}{\langle t\rangle^2+\langle s\rangle^2}\left(1+\frac{\sqrt{E}}{4\langle s\rangle}\int_{-\infty}^{\infty} d\kappa e^{-\frac{1}{2}\kappa^2\sigma^2} \right.$$
$$\left. \times\left(\frac{\langle t\rangle - \mathrm{i}\langle s\rangle}{\sqrt{\langle t\rangle+\mathrm{i}\langle s\rangle}} e^{-\mathrm{i}|\kappa|\sqrt{\frac{E}{\langle t\rangle+\mathrm{i}\langle s\rangle}}} + \frac{\langle t\rangle+\mathrm{i}\langle s\rangle}{\sqrt{\langle t\rangle-\mathrm{i}\langle s\rangle}} e^{\mathrm{i}|\kappa|\sqrt{\frac{E}{\langle t\rangle-\langle s\rangle}}}\right)\right). \tag{193}$$

The integrand above is an even function in κ, this means we can change the lower integration limit from $-\infty$ to 0 to ensure positive integration limits which permits the dropping of the absolute value signs; see Appendix D for details. Once completed and integrated over, we arrive at

$$1 = \frac{1}{\langle t\rangle^2+\langle s\rangle^2}\left(1+2\langle t\rangle^2 + \frac{\mathrm{i}\sqrt{E}}{2}\sqrt{\frac{\pi}{2}}\frac{1}{\sigma}\left[F_+\left(\langle t\rangle,\langle s\rangle\right) - F_-\left(\langle t\rangle,\langle s\rangle\right)\right]\right), \tag{194}$$

where we introduced the functions

$$F_\pm(x,y) = \frac{e^{-\frac{E(x\pm\mathrm{i}y)}{2(x^2+y^2)\sigma^2}}}{\sqrt{x\mp\mathrm{i}y}}\left(1\pm\mathrm{i}\,\mathrm{erfi}\left[\sqrt{\frac{E(x\pm\mathrm{i}y)}{2(x^2+y^2)\sigma^2}}\right]\right), \tag{195}$$

and $\mathrm{erfi}(z)$ stands for the imaginary error function.

The first simultaneous equation can also be averaged in the same way:

$$\begin{aligned}\langle t\rangle &= \left\langle \frac{x^2(E-x^2\langle t\rangle)}{(E-x^2\langle t\rangle)^2+\langle s\rangle^2 x^4}\right\rangle_x \\ &= E\left\langle \frac{x^2}{(E-x^2\langle t\rangle)^2+\langle s\rangle^2 x^4}\right\rangle_x - \langle t\rangle\left\langle \frac{x^4}{(E-x^2\langle t\rangle)^2+\langle s\rangle^2 x^4}\right\rangle_x,\end{aligned} \tag{196}$$

but we must remember that

$$1 = \left\langle \frac{x^4}{(E-x^2\langle t\rangle)^2+\langle s\rangle^2 x^4}\right\rangle_x, \tag{197}$$

therefore, Eq. (196) simplifies to

$$\langle t \rangle = \frac{E}{2} \left\langle \frac{x^2}{(E - x^2 \langle t \rangle)^2 + \langle s \rangle^2 x^4} \right\rangle_x . \tag{198}$$

Using the Fourier transform, we can express the average as an integral

$$\begin{aligned} \left\langle \frac{x^2}{(E - x^2 \langle t \rangle)^2 + \langle s \rangle^2 x^4} \right\rangle_x &= \frac{E}{2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\kappa \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\kappa^2\sigma^2} \frac{1}{\langle t \rangle^2 + \langle s \rangle^2} \\ &\times \int_{-\infty}^{\infty} dx \frac{x^2 e^{-\mathrm{i}\kappa x}}{\left(x^2 - \frac{E\langle t \rangle}{\langle t \rangle^2 + \langle s \rangle^2}\right)^2 + \frac{E^2 \langle s \rangle^2}{(\langle t \rangle^2 + \langle s \rangle^2)^2}}, \end{aligned} \tag{199}$$

to integrate over x, we extend the integration over the complex plane and apply the residue theorem in the exact same way as before

$$\int_C dz \frac{z^2 e^{-\mathrm{i}\kappa z}}{\left(z^2 - \frac{E\langle t \rangle}{\langle t \rangle^2 + \langle s \rangle^2}\right)^2 + \frac{E^2 \langle s \rangle^2}{(\langle t \rangle^2 + \langle s \rangle^2)^2}}. \tag{200}$$

As this is a similar integral, we apply the same method therefore the previous notation remains in place: z_1, z_2, z_3 and z_4; to work out the residues where we traverse around a semi-circular contour in a clock-wise direction. Calculating the first residue, we find

$$\begin{aligned} Res(f(z), z_1) &= \lim_{z \to z_1} (z - z_1) \frac{z^2 e^{-\mathrm{i}\kappa z}}{(z - z_1)(z - z_4)(z - z_3)(z - z_2)} \\ &= \lim_{z \to z_1} \frac{z^2 e^{-\mathrm{i}\kappa z}}{(z - z_4)(z - z_3)(z - z_2)} \\ &= \frac{z_1^2 e^{-\mathrm{i}\kappa z_1}}{(z_1 - z_4)(z_1 - z_3)(z_1 - z_2)}, \end{aligned} \tag{201}$$

which can be explicitly calculated by substituting the values for z_1, z_2, z_3, and z_4; once completed, we find

$$Res(f(z), z_1) = -\frac{(\langle t \rangle - \mathrm{i} \langle s \rangle) \sqrt{\langle t \rangle + \mathrm{i} \langle s \rangle} e^{-\mathrm{i}\kappa \sqrt{\frac{E}{\langle t \rangle + \mathrm{i}\langle s \rangle}}}}{4\mathrm{i}\sqrt{E} \langle s \rangle}. \tag{202}$$

The second residue is also calculated, we obtain

$$\begin{aligned} Res(f(z), z_2) &= \lim_{z \to z_2} (z - z_2) \frac{z^2 e^{-\mathrm{i}\kappa z}}{(z - z_1)(z - z_4)(z - z_3)(z - z_2)} \\ &= \lim_{z \to z_2} \frac{z^2 e^{-\mathrm{i}\kappa z}}{(z - z_1)(z - z_4)(z - z_3)} \\ &= \frac{z_2^2 e^{-\mathrm{i}\kappa z_2}}{(z_2 - z_4)(z_2 - z_3)(z_2 - z_2)}, \end{aligned} \tag{203}$$

by substituting the values, we receive

$$Res(f(z), z_2) = -\frac{(\langle t \rangle + \mathrm{i} \langle s \rangle) \sqrt{\langle t \rangle - \mathrm{i} \langle s \rangle} e^{\mathrm{i}\kappa \sqrt{\frac{E}{\langle t \rangle - \langle s \rangle}}}}{4\mathrm{i}\sqrt{E} \langle s \rangle}. \tag{204}$$

Following Eq. (190), the integral over the complex plane on a semi-circular contour is then equal to

$$\begin{aligned} \int_C dz f(z) = -2\pi\mathrm{i} \Bigg(& -\frac{(\langle t \rangle - \mathrm{i} \langle s \rangle) \sqrt{\langle t \rangle + \mathrm{i} \langle s \rangle} e^{-\mathrm{i}\kappa \sqrt{\frac{E}{\langle t \rangle + \mathrm{i} \langle s \rangle}}}}{4\mathrm{i}\sqrt{E} \langle s \rangle} \\ & -\frac{(\langle t \rangle + \mathrm{i} \langle s \rangle) \sqrt{\langle t \rangle - \mathrm{i} \langle s \rangle} e^{\mathrm{i}\kappa \sqrt{\frac{E}{\langle t \rangle - \langle s \rangle}}}}{4\mathrm{i}\sqrt{E} \langle s \rangle} \Bigg) \\ = \frac{\pi}{2\sqrt{E} \langle s \rangle} \Bigg(& (\langle t \rangle - \mathrm{i} \langle s \rangle) \sqrt{\langle t \rangle + \mathrm{i} \langle s \rangle} e^{-\mathrm{i}\kappa \sqrt{\frac{E}{\langle t \rangle + \mathrm{i} \langle s \rangle}}} \\ & + (\langle t \rangle + \mathrm{i} \langle s \rangle) \sqrt{\langle t \rangle - \mathrm{i} \langle s \rangle} e^{\mathrm{i}\kappa \sqrt{\frac{E}{\langle t \rangle - \langle s \rangle}}} \Bigg), \end{aligned} \tag{205}$$

this tends to the real integral over x since

$$\int_{arc} dz f(z) \to 0, \tag{206}$$

as $R \to \infty$. Hence,

$$\begin{aligned} & \int_{-\infty}^{\infty} dx \frac{x^2 e^{-\mathrm{i}\kappa x}}{\left(x^2 - \frac{E\langle t \rangle}{\langle t \rangle^2 + \langle s \rangle^2}\right)^2 + \frac{E^2 \langle s \rangle^2}{(\langle t \rangle^2 + \langle s \rangle^2)^2}} = \frac{\pi}{2\sqrt{E} \langle s \rangle} \\ & \left((\langle t \rangle - \mathrm{i} \langle s \rangle) \sqrt{\langle t \rangle + \mathrm{i} \langle s \rangle} e^{-\mathrm{i}|\kappa| \sqrt{\frac{E}{\langle t \rangle + \mathrm{i} \langle s \rangle}}} + (\langle t \rangle + \mathrm{i} \langle s \rangle) \sqrt{\langle t \rangle - \mathrm{i} \langle s \rangle} e^{\mathrm{i}|\kappa| \sqrt{\frac{E}{\langle t \rangle - \langle s \rangle}}} \right). \end{aligned} \tag{207}$$

Now that we have integrated over x, the first averaged simultaneous equation simplifies to

$$\begin{aligned}
&\left\langle \frac{x^2}{(E-x^2\langle t\rangle)^2+\langle s\rangle^2 x^4}\right\rangle_x \\
&= \frac{E}{2}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\kappa \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\kappa^2\sigma^2}\frac{1}{\langle t\rangle^2+\langle s\rangle^2}\frac{\pi}{2\sqrt{E}\langle s\rangle} \\
&\left((\langle t\rangle - \mathrm{i}\langle s\rangle)\sqrt{\langle t\rangle + \mathrm{i}\langle s\rangle} e^{-\mathrm{i}|\kappa|\sqrt{\frac{E}{\langle t\rangle+\mathrm{i}\langle s\rangle}}} + (\langle t\rangle + \mathrm{i}\langle s\rangle)\sqrt{\langle t\rangle - \mathrm{i}\langle s\rangle} e^{\mathrm{i}|\kappa|\sqrt{\frac{E}{\langle t\rangle-\langle s\rangle}}}\right) \\
&= \frac{\sqrt{E}}{8\langle s\rangle}\frac{1}{\langle t\rangle^2+\langle s\rangle^2}\int_{-\infty}^{\infty} d\kappa e^{-\frac{1}{2}\kappa^2\sigma^2}\Bigg((\langle t\rangle - \mathrm{i}\langle s\rangle)\sqrt{\langle t\rangle + \mathrm{i}\langle s\rangle} e^{-\mathrm{i}|\kappa|\sqrt{\frac{E}{\langle t\rangle+\mathrm{i}\langle s\rangle}}} \\
&+ (\langle t\rangle + \mathrm{i}\langle s\rangle)\sqrt{\langle t\rangle - \mathrm{i}\langle s\rangle} e^{\mathrm{i}|\kappa|\sqrt{\frac{E}{\langle t\rangle-\langle s\rangle}}}\Bigg),
\end{aligned} \tag{208}$$

calculating the integral over κ, we arrive at our final expression for the first equation

$$\langle t\rangle = \frac{\sqrt{E}}{2\langle s\rangle}\sqrt{\frac{\pi}{8}}\frac{1}{\sigma}\big(F_-(\langle t\rangle,\langle s\rangle) + F_+(\langle t\rangle,\langle s\rangle)\big). \tag{209}$$

Therefore the two simultaneous equations derived from the averaging are found to be:

$$1 = \frac{1}{\langle t\rangle^2+\langle s\rangle^2}\left(1 + 2\langle t\rangle^2 + \frac{\mathrm{i}\sqrt{E}}{2}\sqrt{\frac{\pi}{2}}\frac{1}{\sigma}\big[F_+(\langle t\rangle,\langle s\rangle) - F_-(\langle t\rangle,\langle s\rangle)\big]\right), \tag{210}$$

and

$$\langle t\rangle = \frac{\sqrt{E}}{2\langle s\rangle}\sqrt{\frac{\pi}{8}}\frac{1}{\sigma}\big(F_-(\langle t\rangle,\langle s\rangle) + F_+(\langle t\rangle,\langle s\rangle)\big), \tag{211}$$

where

$$F_\pm(x,y) = \frac{e^{-\frac{E(x\pm\mathrm{i}y)}{2(x^2+y^2)\sigma^2}}}{\sqrt{x\mp\mathrm{i}y}}\left(1 \pm \mathrm{i}\,\mathrm{erfi}\left[\sqrt{\frac{E(x\pm\mathrm{i}y)}{2(x^2+y^2)\sigma^2}}\right]\right). \tag{212}$$

These equations can be solved numerically for both $\langle s\rangle$ and $\langle t\rangle$ which can then be used in order to determine the density of states and the moments of the eigenvectors.

Looking back at the density of states, Eq. (90), we see that in order to

average over the density of states, we need to calculate

$$\hat{\rho}(E) \equiv \langle \rho(E) \rangle = \frac{\langle s \rangle}{\pi} \left\langle \frac{x^2}{(E - x^2 \langle t \rangle)^2 + \langle s \rangle^2 x^4} \right\rangle_x . \tag{213}$$

The lengthy calculation can be avoided by simply looking at the first of the two simultaneous equations

$$\begin{aligned} \langle t \rangle = & \left\langle \frac{x^2(E - x^2 \langle t \rangle)}{(E - x^2 \langle t \rangle)^2 + \langle s \rangle^2 x^4} \right\rangle_x = E \left\langle \frac{x^2}{(E - x^2 \langle t \rangle)^2 + \langle s \rangle^2 x^4} \right\rangle_x \\ & - \langle t \rangle \left\langle \frac{x^4}{(E - x^2 \langle t \rangle)^2 + \langle s \rangle^2 x^4} \right\rangle_x , \end{aligned} \tag{214}$$

it follows from the second equation that

$$1 = \left\langle \frac{x^4}{(E - x^2 \langle t \rangle)^2 + \langle s \rangle^2 x^4} \right\rangle_x , \tag{215}$$

hence,

$$\frac{2 \langle t \rangle}{E} = \left\langle \frac{x^2}{(E - x^2 \langle t \rangle)^2 + \langle s \rangle^2 x^4} \right\rangle_x , \tag{216}$$

therefore the averaged density of states simplifies to

$$\hat{\rho}(E) = \frac{2 \langle s \rangle \langle t \rangle}{\pi E}, \tag{217}$$

solving equations (210) and (211) we are able to find $\langle s \rangle$ and $\langle t \rangle$ and hence the density of states as given above.

Finally, we turn our attention to averaging the moments of the eigenvectors and apply the same method to average over the result:

$$\begin{aligned} \hat{I}_q \equiv \sum_n^N \langle I_q(n) \rangle &= \frac{N \langle s \rangle^q \Gamma(q+1)}{(\pi \hat{\rho}(E) N)^q} \left\langle \frac{x^{2q}}{[(E - x^2 \langle t \rangle)^2 + \langle s \rangle^2 x^4]^q} \right\rangle_x \\ &= \frac{E^q \Gamma(q+1)}{2^q \langle t \rangle^q N^{q-1}} \left\langle \frac{x^{2q}}{[(E - x^2 \langle t \rangle)^2 + \langle s \rangle^2 x^4]^q} \right\rangle_x \\ &= \frac{q E^q}{2^q \langle t \rangle^q N^{q-1}} \left[\left(-\frac{1}{2y} \frac{d}{dy} \right)^{q-1} \left\langle \frac{x^{4-2q}}{(E - x^2 \langle t \rangle)^2 + y^2 x^4} \right\rangle_x \right]_{y = \langle s \rangle} , \end{aligned} \tag{218}$$

see Appendix D for details on the equivalence of the final equality.

The averaging over x can then be represented in terms of the Fourier transform

$$\begin{aligned}\left\langle \frac{x^{4-2q}}{(E-x^2\langle t\rangle)^2+y^2x^4}\right\rangle_x &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\kappa \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\kappa^2\sigma^2} \\ \int_{-\infty}^{\infty} dx \frac{(x^2)^{2-q}e^{-\mathrm{i}\kappa x}}{(E-x^2\langle t\rangle)^2+y^2x^4} &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\kappa \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\kappa^2\sigma^2} \\ \frac{1}{\langle t\rangle^2+y^2}\int_{-\infty}^{\infty} dx &\frac{(x^2)^{2-q}e^{-\mathrm{i}\kappa x}}{\left(x^2-\frac{E\langle t\rangle}{\langle t\rangle^2+y^2}\right)^2+\frac{E^2y^2}{(\langle t\rangle^2+y^2)^2}},\end{aligned} \tag{219}$$

we apply the residue theorem on the integral over x by extending the integral over the complex plane

$$\int_C dz \frac{(z^2)^{2-q}e^{-\mathrm{i}\kappa z}}{\left(z^2-\frac{E\langle t\rangle}{\langle t\rangle^2+y^2}\right)^2+\frac{E^2y^2}{(\langle t\rangle^2+y^2)^2}}, \tag{220}$$

where C is the contour of integration as discussed previously.

Since our assumptions are the same: $E>0$, $\langle t\rangle>0$ and $\langle s\rangle>0$; the poles are also the same. This means that we can calculate the residue straight away using the previous notation: z_1, z_2, z_3 and z_4. Hence

$$\begin{aligned}Res(f(z),z_1) &= \lim_{z\to z_1}(z-z_1)\frac{(z^2)^{2-q}e^{-\mathrm{i}\kappa z}}{(z-z_1)(z-z_4)(z-z_3)(z-z_2)} \\ &= \lim_{z\to z_1}\frac{(z^2)^{2-q}e^{-\mathrm{i}\kappa z}}{(z-z_4)(z-z_3)(z-z_2)} \\ &= \frac{(z_1^2)^{2-q}e^{-\mathrm{i}\kappa z_1}}{(z_1-z_4)(z_1-z_3)(z_1-z_2)},\end{aligned} \tag{221}$$

substituting the values in, we receive

$$Res(f(z),z_1) = -\frac{E^{\frac{1}{2}-q}\left(\langle t\rangle^2+y^2\right)(\langle t\rangle+\mathrm{i}y)^{q-\frac{3}{2}}e^{-\mathrm{i}\kappa\sqrt{\frac{E}{\langle t\rangle+\mathrm{i}y}}}}{4\mathrm{i}y}. \tag{222}$$

The second residue is calculated by

$$\begin{aligned} Res(f(z), z_2) &= \lim_{z \to z_2} (z - z_2) \frac{(z^2)^{2-q} e^{-\mathrm{i}\kappa z}}{(z - z_1)(z - z_4)(z - z_3)(z - z_2)} \\ &= \lim_{z \to z_2} \frac{(z^2)^{2-q} e^{-\mathrm{i}\kappa z}}{(z - z_1)(z - z_4)(z - z_3)} \\ &= \frac{(z_2^2)^{2-q} e^{-\mathrm{i}\kappa z_2}}{(z_2 - z_4)(z_2 - z_3)(z_2 - z_2)}, \end{aligned} \tag{223}$$

substituting in the values, we obtain

$$Res(f(z), z_2) = -\frac{E^{\frac{1}{2}-q} \left(\langle t \rangle^2 + y^2 \right) (\langle t \rangle - \mathrm{i}y)^{q-\frac{3}{2}} e^{\mathrm{i}\kappa \sqrt{\frac{E}{\langle t \rangle - \mathrm{i}y}}}}{4\mathrm{i}y}. \tag{224}$$

Therefore the integral over the complex plane is equal to

$$\begin{aligned} \int_C dz f(z) = -2\pi\mathrm{i} \Bigg(& -\frac{E^{\frac{1}{2}-q} \left(\langle t \rangle^2 + y^2 \right) (\langle t \rangle + \mathrm{i}y)^{q-\frac{3}{2}} e^{-\mathrm{i}|\kappa| \sqrt{\frac{E}{\langle t \rangle + \mathrm{i}y}}}}{4\mathrm{i}y} \\ & - \frac{E^{\frac{1}{2}-q} \left(\langle t \rangle^2 + y^2 \right) (\langle t \rangle - \mathrm{i}y)^{q-\frac{3}{2}} e^{\mathrm{i}|\kappa| \sqrt{\frac{E}{\langle t \rangle - \mathrm{i}y}}}}{4\mathrm{i}y} \Bigg) \\ = \frac{\pi E^{\frac{1}{2}-q} (\langle t \rangle^2 + y^2)}{2y} & \left((\langle t \rangle + \mathrm{i}y)^{q-\frac{3}{2}} e^{-\mathrm{i}|\kappa| \sqrt{\frac{E}{\langle t \rangle + \mathrm{i}y}}} + (\langle t \rangle - \mathrm{i}y)^{q-\frac{3}{2}} e^{\mathrm{i}|\kappa| \sqrt{\frac{E}{\langle t \rangle - \mathrm{i}y}}} \right), \end{aligned} \tag{225}$$

which tends to the real integral in the limit $R \to \infty$, where R is the radius of the semi-circular contour. Hence the original integral over x is equal to

$$\begin{aligned} & \int_{-\infty}^{\infty} dx \frac{(x^2)^{2-q} e^{-\mathrm{i}\kappa x}}{\left(x^2 - \frac{E\langle t \rangle}{\langle t \rangle^2 + y^2} \right)^2 + \frac{E^2 y^2}{(\langle t \rangle^2 + y^2)^2}} \\ = \frac{\pi E^{\frac{1}{2}-q} (\langle t \rangle^2 + y^2)}{2y} & \left((\langle t \rangle + \mathrm{i}y)^{q-\frac{3}{2}} e^{-\mathrm{i}|\kappa| \sqrt{\frac{E}{\langle t \rangle + \mathrm{i}y}}} + (\langle t \rangle - \mathrm{i}y)^{q-\frac{3}{2}} e^{\mathrm{i}|\kappa| \sqrt{\frac{E}{\langle t \rangle - \mathrm{i}y}}} \right), \end{aligned} \tag{226}$$

therefore, Eq. (219) is equivalent to

$$\begin{aligned}
&\left\langle \frac{x^{4-2q}}{(E-x^2\langle t\rangle)^2+y^2x^4}\right\rangle_x = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\kappa \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\kappa^2\sigma^2} \\
&\int_{-\infty}^{\infty} dx \frac{(x^2)^{2-q}e^{-\mathrm{i}\kappa x}}{(E-x^2\langle t\rangle)^2+y^2x^4} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\kappa \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\kappa^2\sigma^2} \frac{1}{\langle t\rangle^2+y^2} \\
&\frac{\pi E^{\frac{1}{2}-q}(\langle t\rangle^2+y^2)}{2y}\left((\langle t\rangle+\mathrm{i}y)^{q-\frac{3}{2}} e^{-\mathrm{i}|\kappa|\sqrt{\frac{E}{\langle t\rangle+\mathrm{i}y}}} + (\langle t\rangle-\mathrm{i}y)^{q-\frac{3}{2}} e^{\mathrm{i}|\kappa|\sqrt{\frac{E}{\langle t\rangle-\mathrm{i}y}}}\right) \\
&= \frac{E^{\frac{1}{2}-q}}{4y}\int_{-\infty}^{\infty} d\kappa e^{-\frac{1}{2}\kappa^2\sigma^2}\left((\langle t\rangle+\mathrm{i}y)^{q-\frac{3}{2}} e^{-\mathrm{i}|\kappa|\sqrt{\frac{E}{\langle t\rangle+\mathrm{i}y}}} \right. \\
&\left. + (\langle t\rangle-\mathrm{i}y)^{q-\frac{3}{2}} e^{\mathrm{i}|\kappa|\sqrt{\frac{E}{\langle t\rangle-\mathrm{i}y}}}\right),
\end{aligned} \tag{227}$$

which simplifies to

$$\begin{aligned}
&\left\langle \frac{x^{4-2q}}{(E-x^2\langle t\rangle)^2+y^2x^4}\right\rangle_x = \frac{E^{\frac{1}{2}-q}}{2y}\int_{0}^{\infty} d\kappa e^{-\frac{1}{2}\kappa^2\sigma^2} \\
&\left((\langle t\rangle+\mathrm{i}y)^{q-\frac{3}{2}} e^{-\mathrm{i}\kappa\sqrt{\frac{E}{\langle t\rangle+\mathrm{i}y}}} + (\langle t\rangle-\mathrm{i}y)^{q-\frac{3}{2}} e^{\mathrm{i}\kappa\sqrt{\frac{E}{\langle t\rangle-\mathrm{i}y}}}\right),
\end{aligned} \tag{228}$$

the steps undertaken to bring the integral into this form are shown in Appendix D.

The integral over κ can be calculated, once completed we receive

$$\begin{aligned}
&\left\langle \frac{x^{4-2q}}{(E-x^2\langle t\rangle)^2+y^2x^4}\right\rangle_x = \frac{E^{\frac{1}{2}-q}}{2y\sigma}\sqrt{\frac{\pi}{2}}\left\{ e^{-\frac{E}{2(\langle t\rangle+\mathrm{i}y)\sigma^2}} (\langle t\rangle+\mathrm{i}y)^{q-\frac{3}{2}} \right. \\
&\left(1-\frac{\mathrm{i}\sqrt{-\frac{\mathrm{i}E}{y-\mathrm{i}\langle t\rangle}}\,\mathrm{ierfi}\left[\sqrt{\frac{E}{2(\langle t\rangle+\mathrm{i}y)\sigma^2}}\right]}{\sqrt{\frac{\mathrm{i}E}{y-\mathrm{i}\langle t\rangle}}}\right) + e^{-\frac{E}{2(\langle t\rangle-\mathrm{i}y)\sigma^2}} (\langle t\rangle-\mathrm{i}y)^{q-\frac{3}{2}} \\
&\left.\left(1+\frac{\mathrm{i}\sqrt{\frac{\mathrm{i}E}{y+\mathrm{i}\langle t\rangle}}\,\mathrm{ierfi}\left[\sqrt{\frac{E}{2(\langle t\rangle-\mathrm{i}y)\sigma^2}}\right]}{\sqrt{-\frac{\mathrm{i}E}{y+\mathrm{i}\langle t\rangle}}}\right)\right\},
\end{aligned} \tag{229}$$

simplifying the terms further we arrive at

$$\left\langle \frac{x^{4-2q}}{(E-x^2\langle t\rangle)^2+y^2x^4}\right\rangle_x = \frac{E^{\frac{1}{2}-q}}{y\sigma}\sqrt{\frac{\pi}{8}}\Bigg\{ e^{-\frac{E}{2(\langle t\rangle+\mathrm{i}y)\sigma^2}}(\langle t\rangle+\mathrm{i}y)^{q-\frac{3}{2}} \left(1-\mathrm{ierfi}\left[\sqrt{\frac{E}{2(\langle t\rangle+\mathrm{i}y)\sigma^2}}\right]\right) + e^{-\frac{E}{2(\langle t\rangle-\mathrm{i}y)\sigma^2}}(\langle t\rangle-\mathrm{i}y)^{q-\frac{3}{2}} \left(1+\mathrm{ierfi}\left[\sqrt{\frac{E}{2(\langle t\rangle-\mathrm{i}y)\sigma^2}}\right]\right)\Bigg\}. \tag{230}$$

Finally, the moments of the eigenvectors averaged over w_i are equal to

$$\hat{I}_q = \frac{q\sqrt{E}}{2^q\langle t\rangle^q N^{q-1}}\Bigg[\left(-\frac{1}{2y}\frac{d}{dy}\right)^{q-1}\frac{1}{\sigma y}\sqrt{\frac{\pi}{8}}\Bigg\{ e^{-\frac{E}{2(\langle t\rangle+\mathrm{i}y)\sigma^2}}(\langle t\rangle+\mathrm{i}y)^{q-\frac{3}{2}} \left(1-\mathrm{ierfi}\left[\sqrt{\frac{E}{2(\langle t\rangle+\mathrm{i}y)\sigma^2}}\right]\right) + e^{-\frac{E}{2(\langle t\rangle-\mathrm{i}y)\sigma^2}}(\langle t\rangle-\mathrm{i}y)^{q-\frac{3}{2}} \left(1+\mathrm{ierfi}\left[\sqrt{\frac{E}{2(\langle t\rangle-\mathrm{i}y)\sigma^2}}\right]\right)\Bigg\}\Bigg]_{y=\langle s\rangle}, \tag{231}$$

with density of states

$$\hat{\rho}(E) = \frac{2\langle s\rangle\langle t\rangle}{\pi E}, \tag{232}$$

the variables $\langle s\rangle$ and $\langle t\rangle$ can be determined by the system of equations, which we previously calculated, given below

$$1 = \frac{1}{\langle t\rangle^2+\langle s\rangle^2}\left(1+2\langle t\rangle^2+\frac{\mathrm{i}\sqrt{E}}{2}\sqrt{\frac{\pi}{2}}\frac{1}{\sigma}\left[F_+(\langle t\rangle,\langle s\rangle)-F_-(\langle t\rangle,\langle s\rangle)\right]\right), \tag{233}$$

and

$$\langle t\rangle = \frac{\sqrt{E}}{2\langle s\rangle}\sqrt{\frac{\pi}{8}}\frac{1}{\sigma}\big(F_-(\langle t\rangle,\langle s\rangle)+F_+(\langle t\rangle,\langle s\rangle)\big), \tag{234}$$

where

$$F_\pm(x,y) = \frac{e^{-\frac{E(x\pm\mathrm{i}y)}{2(x^2+y^2)\sigma^2}}}{\sqrt{x\mp\mathrm{i}y}}\left(1\pm\mathrm{i}\,\mathrm{erfi}\left[\sqrt{\frac{E(x\pm\mathrm{i}y)}{2(x^2+y^2)\sigma^2}}\right]\right). \tag{235}$$

These are the main results of this chapter, valid for the case $N\to\infty$.

Later, we will consider the limiting case where $\sigma \to \infty$ for the moments of the eigenvectors to determine its fractal dimension in this limit.

The derivatives can be calculated explicitly for any integer q, and in particular for $q = 2$ one obtains

$$\hat{I}_2 = \frac{\sqrt{E}}{8N \langle t\rangle^2 \langle s\rangle^2 \sigma}\sqrt{\frac{\pi}{2}}\Bigg[\left(\frac{\langle t\rangle - \mathrm{i}\langle s\rangle}{\langle s\rangle} + \frac{\mathrm{i}}{2}\left(1 + \frac{E(\langle t\rangle + \mathrm{i}\langle s\rangle)}{(\langle t\rangle^2 + \langle s\rangle^2)\sigma^2}\right)\right) F_+(\langle t\rangle, \langle s\rangle)$$
$$\left(\frac{\langle t\rangle + \mathrm{i}\langle s\rangle}{\langle s\rangle} - \frac{\mathrm{i}}{2}\left(1 + \frac{E(\langle t\rangle - \mathrm{i}\langle s\rangle)}{(\langle t\rangle^2 + \langle s\rangle^2)\sigma^2}\right)\right) F_-(\langle t\rangle, \langle s\rangle) + \frac{\sqrt{2E}\langle t\rangle}{\sqrt{\pi}\sigma(\langle t\rangle^2 + \langle s\rangle^2)}\Bigg]. \tag{236}$$

We computed numerically $\hat{I}_2$ for $\sigma = 10$ for the eigenvectors, whose eigenvalues were close to 1, and confirm that the numerical results are in agreement with our analytics. The corresponding results are given in fig. 4. According to Eq. (231) the scaling of $\hat{I}_q \propto N^{1-q}$, which is the same as the GUE case, indicating that the eigenvectors of this model are qualitatively similar to GUE eigenvectors.

In the limit $\sigma \to \infty$, we compute the asymptotic behaviour of the simultaneous equations to compare against the numerical result:

$$\langle t\rangle \approx \frac{\sqrt{\pi}}{4\sigma}, \quad \langle s\rangle \approx 1, \tag{237}$$

the explicit steps are shown in the Appendix D.

We are also able to calculate the asymptotic behaviour for the moments of the eigenvectors with $\sigma \gg 1$. In this limit one can show that $\langle s\rangle \gg \langle t\rangle$, the dominant terms of the expression are then given by

$$\begin{aligned}
\hat{I}_q &= \frac{q}{N^{q-1}2^q\langle t\rangle^q}\frac{1}{\sigma}\sqrt{\frac{\pi}{8}}\left[\left|\left(-\frac{1}{2y}\frac{d}{dy}\right)^{q-1}\frac{1}{y}\left(\mathrm{i}^{q-3/2}y^{q-3/2} + (-\mathrm{i})^{q-3/2}y^{q-3/2}\right)\right|\right]_{y=\langle s\rangle} \\
&= \frac{q}{N^{q-1}2^q\langle t\rangle^q}\frac{1}{\sigma}\sqrt{\frac{\pi}{8}}\left|\mathrm{i}^{q-3/2} + (-\mathrm{i})^{q-3/2}\left[\left(-\frac{1}{2y}\frac{d}{dy}\right)^{q-1} y^{q-5/2}\right]_{y=\langle s\rangle}\right| \\
&= \frac{q}{N^{q-1}2^q\langle t\rangle^q\langle s\rangle^{q+\frac{1}{2}}}\frac{1}{\sigma}\sqrt{\frac{\pi}{8}}\left|\left(\mathrm{i}^{q-3/2} + (-\mathrm{i})^{q-3/2}\right)\left(-\frac{1}{2}\right)^{q-1}\prod_{k=0}^{q-2}\left(q - \frac{5+4k}{2}\right)\right|,
\end{aligned} \tag{238}$$

by evaluating the derivatives $q-1$ times, substituting $y = \langle s \rangle$ and setting $E = 1$. Hence,

$$\begin{aligned}
\hat{I}_q &= \frac{q}{N^{q-1} 2^q \langle t \rangle^q \langle s \rangle^{q+\frac{1}{2}}} \frac{1}{\sigma} \sqrt{\frac{\pi}{8}} \left| \mathrm{i}^{q-3/2} \left(1 + (-1)^{q-3/2}\right) \left(-\frac{1}{2}\right)^{q-1} \prod_{k=0}^{q-2} \left(q - \frac{5+4k}{2}\right) \right| \\
&= \frac{q}{N^{q-1} 2^{2q-1} \langle t \rangle^q \langle s \rangle^{q+\frac{1}{2}}} \frac{1}{\sigma} \sqrt{\frac{\pi}{8}} \left| \mathrm{i}^{q-3/2} \left((-1)^{q-1} + \mathrm{i}\right) \prod_{k=0}^{q-2} \left(q - \frac{5+4k}{2}\right) \right| \\
&= \frac{q}{N^{q-1} 2^{2q-1} \langle t \rangle^q \langle s \rangle^{q+\frac{1}{2}}} \frac{1}{\sigma} \sqrt{\frac{\pi}{8}} \sqrt{2} \prod_{k=0}^{q-2} \left| q - \frac{5+4k}{2} \right| \\
&= \frac{q\sqrt{\pi}}{N^{q-1} 2^{2q} \langle t \rangle^q \langle s \rangle^{q+\frac{1}{2}} \sigma} \prod_{k=0}^{q-2} \left| q - \frac{5+4k}{2} \right|,
\end{aligned} \tag{239}$$

substituting the values of $\langle t \rangle$ and $\langle s \rangle$ calculated previously, we arrive at our final expression of the moments of the eigenvectors in this limit:

$$\hat{I}_q \approx q \left(\frac{\sigma}{N\sqrt{\pi}}\right)^{q-1} \prod_{k=0}^{q-2} \left| q - \frac{5+4k}{2} \right|. \tag{240}$$

This is another main result for this chapter and is valid for any $\sigma \gg 1$. In particular we explore the possibility for an N-dependent variance, specifically $\sigma = N^\gamma$ where $\gamma > 0$, we find that $\hat{I}_q \propto N^{(\gamma-1)(q-1)}$. This implies that the eigenvectors are of a fractal nature in this case from the scaling of the moments of the eigenvectors, with the fractal dimension given by $D_q = 1 - \gamma$. This is similar to recent results in [21, 22] for non-erogdic states found in the Rosenzweig-Porter model [23]. As the exponent must remain negative, $(\gamma - 1)(q-1)$, we conclude that this result breaks down for $\gamma > 1$. The results of this numerical simulation are given in fig. 5. In this random case we see that the eigenvectors are also fractal but do not share the characteristics of fractal and localised or extended states which were shown to be the case in the deterministic W model in the previous section, where W contained elements with power-law dependence.

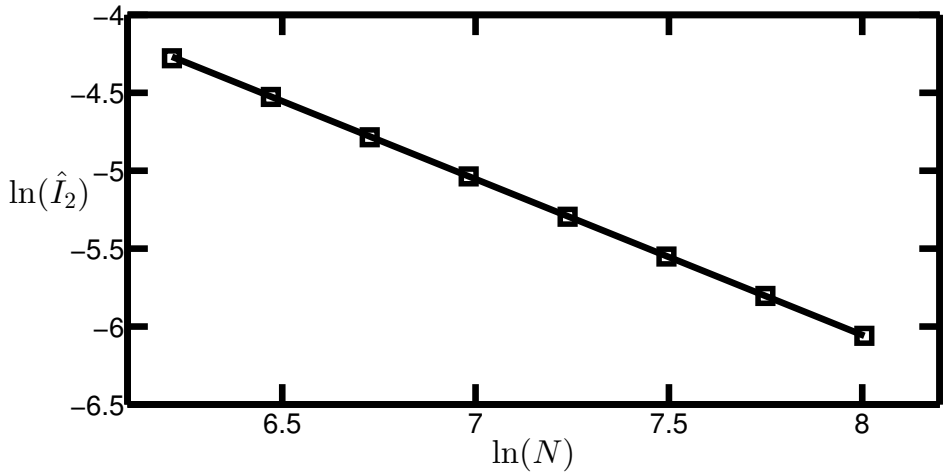

Figure 4: The numerical results are given by the symbols and the solid line depicts our analytical result. In this numerical simulation over a total of 1000 realisations of random matrices, we used $\sigma = 10$.

2.6 Conclusion

By employing the supersymmetry technique we derived an expression for the moments of the eigenvectors for the non-invariant matrix ensemble in which the variances are given by $\langle |H_{ij}|^2 \rangle = w_i^2 w_j^2 / N$, the validity of the expression was also confirmed by an alternative derivation of the moments of the eigenvectors at $E = 0$ where the two results coincided. We studied the eigenvectors statistics of such a model in the limit $N \to \infty$ and it turns out the that two parameters s and t, which enter in many of our expressions, are determined by the same system of equations as studied by Pastur and Girko, who studied the eigenvalue density for deformed random matrix ensembles. We also found the distribution function of the components of the eigenvectors.

The general result can be applied to any case but we consider, in particular, the power-law dependence $w_n^2 \propto 1/n^p$ which exhibited an interesting behaviour as we varied the parameter p. The eigenvectors are changing from extended, $p > 0$, to critical quasi-extended, $-1 < p < 0$, and later to quasi-localised, for $p < -1$. Other choices of w_n may lead to completely different characteristics such as being completely localised but we mainly focused on the power-law dependence.

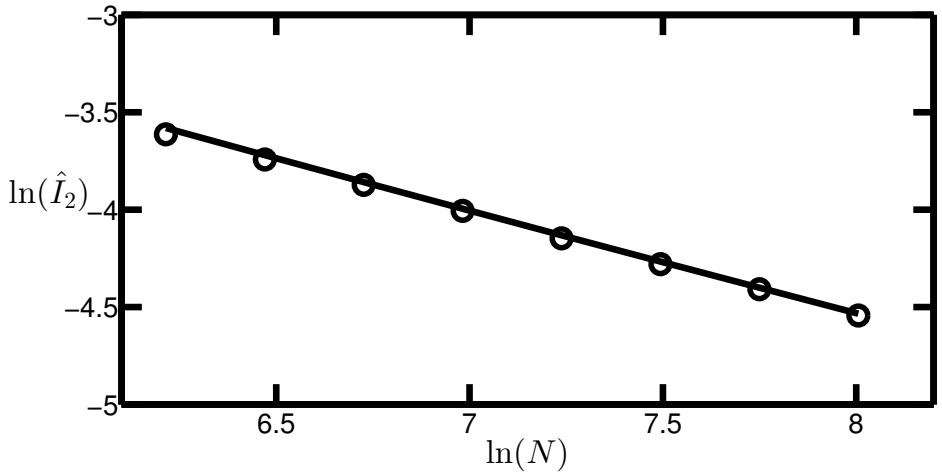

Figure 5: This figure shows the results of numerical simulations (symbols) for $\hat{I}_2$ at $\sigma = N^{1/2}$. The solid line represents our analytical result. The numerical simulations were performed over a total of 1000 realisations of random matrices.

We then generalise the result derived from the supersymmetry technique to the case of random W, as it turns out that the results derived previously were a suitable foundation for this problem. We consider the cases of constant variance and N-dependent variance where we were able to average over the simultaneous equations as well as finding general expressions for the density of states and the moments of the eigenvectors. Studying the asymptotic behaviour of the model in the limit $\sigma \to \infty$, we find an expression for the moments of the eigenvectors in this limit. By considering an N-dependent variance, $\sigma = N^{\gamma}$, we find that the eigenvectors are critical with fractal dimension given by $D_q = 1 - \gamma$.

3 Model II: Random matrices with an external source

One of the most fundamental problems in quantum mechanics is to find the eigenvectors of a Hamiltonian D perturbed by another Hamiltonian $\hat{H}$, so that the total Hamiltonian is $\hat{H} + D$. A particular approach to this problem is by applying perturbation theory, but since the results of the perturbation theory

are only applicable to models where the matrix elements of the perturbed matrix are sufficiently small compared to the level spacing on the Hamiltonian D, one also needs to search for non-perturbative solutions. This is due to the fact that in many applications, the conditions required to satisfy the perturbative approach are not fulfilled and other approaches must be considered.

Non-perturbative results are usually only available for exactly solvable models but there are situations where an unsolvable Hamiltonian will arise, so it is also important to study these in detail. In this project we consider a generic perturbation $\tilde{H}$, in place of the Hamiltonian $\hat{H}$, modelled by an $N \times N$ random matrix drawn from the GUE [24]. Random matrix theory has been proven to correctly capture many universal features of complex quantum systems, including examples such as chaotic and disordered systems [25].

Our model is known as the random matrix model with an external source, which have many applications in network models, telecommunications, neuroscience and other areas. The spectral properties of these types of quantum systems have been studied extensively [16, 26–34] but we are not aware of any analytical results presented on their eigenvectors.

Interest in properties of eigenvectors of random matrix models has been revived recently in the context of Anderson localisation and many-body localisation [35, 36], and in particular the existence of non-ergodic delocalised states which is an intermediate phase of the Anderson transition [22, 37–44]. In this context it is natural to assume D to be random which plays the role of on-site disorder, whereas $\tilde{H}$ contains information on the structural disorder. These two types of disorder are also present in random regular graphs and supposedly essential for many-body localisation.

At first we consider the model where D is a deterministic diagonal matrix with elements d_i and derive an expression for the moments of the eigenvectors using the supersymmetry technique, which also turns out to be a convenient starting point for the case of random D, where each element is Gaussian distributed random variable. It turns that that these results are applicable to

the Rosenzweig-Porter model, given by

$$H = \tilde{H} + D, \quad \frac{\langle d_i^2 \rangle}{\left\langle |\tilde{H}_{ij}|^2 \right\rangle} = N^{\gamma}, \quad D = \mathrm{diag}(d_1, d_2, ...d_N), \quad \gamma > 0 \tag{241}$$

$\tilde{H}$ and D are gaussian distributed matrices where we set $\left\langle |\tilde{H}_{ij}|^2 \right\rangle = N^{-\gamma}$ and $\langle d_i^2 \rangle = 1$ to ensure a bounded spectrum in the limit $N \to \infty$. Using the results from the supersymmetry approach, we find closed analytical expressions for the case of random D and study in particular the Rosenzweig-Porter model. We show that for $1 < \gamma < 2$, the eigenvectors exhibit non-ergodic behaviour characterised by non-trivial fractal dimensions.

3.1 The density of states

In order for one to calculate the density of states we first define the model

$$H = \tilde{H} + D, \quad D = \mathrm{diag}(d_1, d_2, ...d_N), \tag{242}$$

where D is a deterministic diagonal matrix and $\tilde{H} \in$ GUE. The density of states can be expressed in terms of the Green's function, Eq. (12), and since the density of states only involves one Green's function, therefore only the ratio of two determinants is required

$$Z(E, j) = \frac{\det \mathrm{i}(E^- - H)}{\det \mathrm{i}(E^- - H - j)}, \tag{243}$$

where $H = \tilde{H} + D$, $E^- = E - \mathrm{i}\epsilon$ and $\epsilon > 0$.

Since the calculations are similar to the cases presented in section 2, only the main steps will be shown below.

The ratio of determinants above can be represented in its superintegral

form

$$Z(E,j) = \int \frac{d^2 r_i(-)}{\pi} d\eta_i^*(-) d\eta(-)$$
$$\exp \mathrm{i} \left[r^\dagger(-)(\tilde{H} + D - E + \mathrm{i}\epsilon + j) r(-) + \chi^\dagger(-)(\tilde{H} + D - E + \mathrm{i}\epsilon)\chi(-) \right], \tag{244}$$

and in a more convenient notation

$$Z(E,j) = \int d\Phi d\Phi^* \exp \left\{ \mathrm{i} \sum_{\alpha=1}^{2} \Phi_\alpha^\dagger [\tilde{H} + D - E + \mathrm{i}\epsilon + \tilde{J}_\alpha] \Phi_\alpha \right\}, \tag{245}$$

where $\tilde{J} = \mathrm{diag}(j, 0)$, $\Phi = (r(-), \chi(-))^T = (\Phi_1, \Phi_2)^T$ and $d\Phi d\Phi^*$ comprises of those in the former integral.

Averaging over the generating function is simple since D is deterministic so we only need to average over the GUE by using an identity in Ref. [14]:

$$\begin{aligned} \left\langle \exp \left(\mathrm{i} \sum_{\alpha=1}^{2} \Phi_\alpha^\dagger \tilde{H} \Phi_\alpha \right) \right\rangle &= \exp \left(-\frac{1}{2} \left\langle \left(\sum_{\alpha=1}^{2} \Phi_\alpha^\dagger \tilde{H} \Phi_\alpha \right)^2 \right\rangle \right) \\ &= \exp \left(-\frac{1}{2N} \sum_{\alpha,\beta} \sum_{m,n} \Phi_{\alpha m}^* \Phi_{\alpha n} \Phi_{\beta n}^* \Phi_{\beta m} \right), \end{aligned} \tag{246}$$

after applying the averaging identity as shown in the previous sections, the averaged expression over the disorder reads

$$\langle Z(E,j) \rangle = \int d\Phi d\Phi^* \exp \left\{ -N \mathrm{Str} \left[\frac{1}{2} \tilde{Q}^2 + \mathrm{i}(E^- - d_i - \tilde{J}) \tilde{Q} \right] \right\}, \tag{247}$$

where $\tilde{Q}_{\alpha\beta} = \frac{1}{N} \sum_m^N |\Phi_\alpha(m)\rangle \langle \Phi_\beta(m)|$ and d_i represents the ith element on the diagonal of matrix D.

Now that we have averaged over the disorder, the next step is to apply the Hubbard-Stratonovich transformation to reduce the quartic integral down to its quadratic form, Eq. (53), once completed we receive

$$\langle Z(E,j) \rangle = \int dQ e^{-\frac{N}{2} \mathrm{Str} Q^2} \int d\Phi d\Phi^* \exp \left\{ -N \mathrm{Str} \left[\mathrm{i}(E^- - d_i - Q - \tilde{J}) \tilde{Q} \right] \right\}, \tag{248}$$

which simplifies further to

$$\langle Z(E,j)\rangle = \int dQ e^{-\frac{N}{2}\mathrm{Str}Q^2} \prod_i^N \mathrm{Sdet}^{-1}(E^- - d_i - Q - \tilde{J}). \tag{249}$$

Manipulating the integrand we are able to express it in a more convenient form

$$\langle Z(E,j)\rangle = \int dQ \exp\left\{-\frac{N}{2}\mathrm{Str}Q^2 - \sum_i^N \ln \mathrm{Sdet}(E^- - d_i - Q - \tilde{J})\right\}, \tag{250}$$

then by using a property of the superdeterminant the integral is equivalent to

$$\langle Z(E,j)\rangle = \int dQ \exp\left\{-N\mathrm{Str}\left(\frac{1}{2}Q^2 + \frac{1}{N}\sum_i^N \ln(E^- - d_i - Q - \tilde{J})\right)\right\}. \tag{251}$$

As we are interested in the limit $\epsilon \to 0$ and $\tilde{J} \to 0$, it is useful to expand the logarithm function in powers of ϵ and $\tilde{J}$ to first order

$$\langle Z(E,j)\rangle = \int dQ \exp\left\{-N\mathcal{L}(Q) + \sum_i^N \mathrm{Str}(E - d_i - Q)^{-1}(\mathrm{i}\epsilon + \tilde{J})\right\}, \tag{252}$$

where $\mathcal{L}(Q) = \frac{1}{2}\mathrm{Str}Q^2 + \frac{1}{N}\sum_i^N \ln(E - d_i - Q)$.

Differentiating the integrand with respect to j and then taking the limit $j \to 0$, yields

$$\begin{aligned}\mathrm{Tr}\left\langle G^A(E)\right\rangle = \int dQ \sum_i^N \mathrm{Str}\left(\frac{\tilde{J}}{j}(E - d_i - Q)^{-1}\right) \\ \times \exp\left\{-N\mathcal{L}(Q) + \mathrm{i}\epsilon \sum_i^N \mathrm{Str}(E - d_i - Q)^{-1}\right\},\end{aligned} \tag{253}$$

where N is assumed to be large, $N \to \infty$, so we apply the saddle-point approximation on the integral. In the limit $N \to \infty$ the integral over Q is dominated by the saddle-point equation satisfying

$$Q = \frac{1}{N}\sum_i^N \frac{1}{E - d_i - Q}, \tag{254}$$

the solutions of the saddle-point equation can be parametrised by

$$Q_{s.p.} = t + \mathrm{i}s, \tag{255}$$

where $s \neq 0$ and t are real parameters satisfying the two simultaneous equations

$$t = \frac{1}{N}\sum_i^N \frac{E - t - d_i}{(E - t - d_i)^2 + s^2}, \quad 1 = \frac{1}{N}\sum_i^N \frac{1}{(E - t - d_i)^2 + s^2}. \tag{256}$$

Substituting the parametrisation into the integrand, the integral simplifies to

$$\begin{aligned} \mathrm{Tr}\left\langle G^A(E)\right\rangle = \int dQ \sum_i^N \mathrm{Str}\left(\frac{\tilde{J}}{j}\frac{E - t - d_i + \mathrm{i}s}{(E - t - d_i)^2 + s^2}\right) \\ \times \exp\left\{-N\mathcal{L}(Q) + \mathrm{i}\epsilon\sum_i^N \mathrm{Str}(E - d_i - Q)^{-1}\right\}, \end{aligned} \tag{257}$$

taking the limit $\epsilon \to 0^+$, pulling out the pre-exponential factor and simplifying the expression, we arrive at

$$\lim_{\epsilon\to 0^+} \mathrm{Tr}\left\langle G^A(E)\right\rangle = \sum_i^N \frac{E - t - d_i + \mathrm{i}s}{(E - t - d_i)^2 + s^2}\int dQ \exp\left\{-N\mathcal{L}(Q)\right\}, \tag{258}$$

where the integral over Q is unity since the superdeterminant of a block diagonal matrix whose Bose and Fermi blocks are the same is equal to unity. Hence,

$$\lim_{\epsilon\to 0^+} \mathrm{Tr}\left\langle G^A(E)\right\rangle = \sum_i^N \frac{E - t - d_i + \mathrm{i}s}{(E - t - d_i)^2 + s^2}, \tag{259}$$

taking the imaginary part of this expression and multiplying it by a factor of $\frac{1}{\pi N}$ yields the density of states

$$\rho(E) = \frac{s}{\pi N}\sum_i^N \frac{1}{(E - t - d_i)^2 + s^2}, \tag{260}$$

but we must remember from one of the simultaneous equations for s and t

that

$$1 = \frac{1}{N}\sum_{i}^{N}\frac{1}{(E-t-d_i)^2+s^2}. \tag{261}$$

Therefore, we arrive at the final form of the density of states

$$\rho(E) = \frac{s}{\pi}, \quad s > 0, \tag{262}$$

the density of states is in a very simple form dependent only on s, which is especially helpful when we generalise the problem to random diagonal D in the later sections.

3.2 Moments of the eigenvectors by the supersymmetric approach

In this section we will be calculating the moments of the eigenvectors of the model shown previously. Therefore the product of two Green's functions is needed, which means we will need the product of two generating functions, the ratio of four determinants:

$$Z(E,\hat{J}_1,\hat{J}_2) = (-1)^N \frac{\det \mathrm{i}(E^- - \tilde{H} - D)\det \mathrm{i}(E^+ - \tilde{H} - D)}{\det \mathrm{i}(E^- - \tilde{H} - D - \hat{J}_1)\det(-\mathrm{i})(E^+ - \tilde{H} - D - \hat{J}_2)}, \tag{263}$$

where $E^{\pm}$, $\hat{J}_1$ and $\hat{J}_2$ are the same as defined previously.

We can convert the ratio of determinants into their superintegral form

$$\begin{aligned} Z(E,\hat{J}_1,\hat{J}_2) = (-1)^N &\int\prod_{i=1}^{N}\frac{d^2r_i(-)}{\pi}d\eta_i^*(-)d\eta_i(-)\int\prod_{i=1}^{N}\frac{d^2r_i(+)}{\pi}d\eta_i^*(+)d\eta_i(+) \\ &\times \exp \mathrm{i}\left[r^\dagger(-)(\tilde{H}+D-E+\mathrm{i}\epsilon-\hat{J}_1)r(-)+\chi^\dagger(-)(\tilde{H}+D-E+\mathrm{i}\epsilon)\chi(-)\right] \\ &\times \exp \mathrm{i}\left[-r^\dagger(+)(\tilde{H}+D-E-\mathrm{i}\epsilon-\hat{J}_2)r(+)+\chi^\dagger(+)(\tilde{H}+D-E-\mathrm{i}\epsilon)\chi(+)\right], \end{aligned} \tag{264}$$

where $r(-)$, $r(+)$ are vectors containing commutative variables, $r_i(-)$, $r_i(+)$, and $\chi(-)$, $\chi(+)$ are vectors containing anti-commutative variables, $\eta_i(-)$, $\eta_i(+)$ with $i = 1, 2, ...N$.

This can be expressed in a much more compact form using supervectors

$$Z(E,\hat{J}_1,\hat{J}_2) = (-1)^N \int d\Phi^* d\Phi \times \exp\left\{ \mathrm{i}\sum_{\alpha=1}^{4} \Phi_\alpha^\dagger L_\alpha [\tilde{H} + D - E - \mathrm{i}(-1)^\alpha \epsilon - \tilde{J}_\alpha]\Phi_\alpha \right\}, \tag{265}$$

$L_1 = L_3 = L_4 = -L_2 = 1$, $\tilde{J}_\alpha = \mathrm{diag}(\hat{J}_1, \hat{J}_2, 0, 0)$ and
$\Phi = (r(-), r(+), \chi(-), \chi(+))^T = (\Phi_1, \Phi_2, \Phi_3, \Phi_4)^T$.

Similarly, we have to differentiate the expression with respect to J_1 and J_2, m and l times respectively. In doing so, we receive

$$\left(G^R_{nn}\right)^l \left(G^A_{nn}\right)^m = (-1)^N \int d\Phi^* d\Phi (r_n^*(-) r_n(-))^m (r_n^*(+) r_n(+))^l \exp\left\{ \mathrm{i}\sum_{\alpha=1}^{4} \Phi_\alpha^\dagger L_\alpha [\tilde{H} + D - E - \mathrm{i}(-1)^\alpha \epsilon]\Phi_\alpha \right\}. \tag{266}$$

Averaging over the disorder is simple since D is deterministic, then we only need to average over the GUE using the identity in Ref. [14]:

$$\begin{aligned} \left\langle \exp\left(\mathrm{i}\sum_{\alpha=1}^{4} \Phi_\alpha^\dagger L_\alpha \tilde{H} \Phi_\alpha \right) \right\rangle &= \exp\left(-\frac{1}{2} \left\langle \left(\sum_{\alpha=1}^{4} \Phi_\alpha^\dagger L_\alpha \tilde{H} \Phi_\alpha \right)^2 \right\rangle \right) \\ &= \exp\left(-\frac{1}{2N} \sum_{\alpha,\beta} L_\alpha L_\beta \sum_{m,n} \Phi_\alpha^*(m)\Phi_\alpha(n)\Phi_\beta^*(n)\Phi_\beta(m) \right). \end{aligned} \tag{267}$$

Once the expression has been averaged over, the product of two Green's functions reads

$$\left\langle \left(G^R_{nn}\right)^l \left(G^A_{nn}\right)^m \right\rangle = (-1)^N \int d\Phi^* d\Phi (r_n^*(-) r_n(-))^m (r_n^*(+) r_n(+))^l \times \exp\left\{ -N\mathrm{Str}\left[\frac{1}{2}(\tilde{Q}\hat{L})^2 - \mathrm{i}d_i \tilde{Q}\hat{L} + \mathrm{i}\hat{E}\tilde{Q}\hat{L} \right] \right\}, \tag{268}$$

where $\tilde{Q}_{\alpha\beta} = \frac{1}{N}\sum_m^N |\Phi_\alpha(m)\rangle \langle \Phi_\beta(m)|$.

The non-Gaussian character of the averaged elements of the Green's functions can be overcome by the Hubbard-Stratonovich transformation, Eq. (53),

to reduce the integrand down to its quadratic form

$$\left\langle \left(G^R_{nn}\right)^l \left(G^A_{nn}\right)^m \right\rangle = (-1)^{N+1} \int_{\mathcal{M}_4} dQ e^{-\frac{N}{2}\mathrm{Str}(Q\hat{L})^2} \int d\Phi^* d\Phi (r^*_n(-) r_n(-))^m \\ \times (r^*_n(+) r_n(+))^l \exp\left\{ -N\mathrm{Str}\left[\mathrm{i}\hat{E}\tilde{Q}\hat{L} - \mathrm{i}Q\hat{L}\tilde{Q}\hat{L} - \mathrm{i}d_i\tilde{Q}\hat{L} \right] \right\}, \tag{269}$$

the integration over Φ and Φ^* can easily be calculated by the application of Wick's theorem, resulting in

$$\left\langle \left(G^R_{nn}\right) \left(G^A_{nn}\right)^{q-1} \right\rangle = (-1)^{N+1} \int_{\mathcal{M}_4} dQ e^{-\frac{N}{2}\mathrm{Str}(Q\hat{L})^2} (g^{BB}_{aa})^{q-2} \\ \left\{ g^{BB}_{aa} g^{BB}_{rr} + (q-1) g^{BB}_{ar} g^{BB}_{ra} \right\} \prod_i^N \mathrm{Sdet}^{-1} \left[\hat{L}\hat{E} - \hat{L}Q\hat{L} - \hat{L}d_i \right], \tag{270}$$

after setting $l = 1$ and $m = q - 1$ for simplification purposes.

This can, by manipulating the expression, be expressed as

$$\left\langle \left(G^R_{nn}\right) \left(G^A_{nn}\right)^{q-1} \right\rangle = (-1)^{N+1} \int_{\mathcal{M}_4} dQ (g^{BB}_{aa})^{q-2} \left\{ g^{BB}_{aa} g^{BB}_{rr} + (q-1) g^{BB}_{ar} g^{BB}_{ra} \right\} \\ \times \exp\left\{ -\frac{N}{2}\mathrm{Str}(Q\hat{L})^2 - \sum_i^N \ln \mathrm{Sdet}\left[\hat{L}\hat{E} - \hat{L}Q\hat{L} - \hat{L}d_i \right] \right\}, \tag{271}$$

the factor $(-1)^N$ will cancel out, with the factor $\left(e^{\mathrm{i}\pi}\right)^N$, in the same way as explained in the previous case, therefore it simplifies to

$$\left\langle \left(G^R_{nn}\right) \left(G^A_{nn}\right)^{q-1} \right\rangle = \int_{\mathcal{M}_4 \hat{L}} dQ (g^{BB}_{aa})^{q-2} \left\{ g^{BB}_{aa} g^{BB}_{rr} + (q-1) g^{BB}_{ar} g^{BB}_{ra} \right\} \\ \times \exp\left\{ -\frac{N}{2}\mathrm{Str}Q^2 - \sum_i^N \mathrm{Str}\ln\left[\hat{E} - Q - d_i \right] \right\}, \tag{272}$$

by absorbing -1 into the integration measure, using the property $\ln \mathrm{Sdet} F = \mathrm{Str}\ln F$ and integrating over $Q\hat{L}$ instead of Q.

We may recall that $\hat{E} = E - \mathrm{i}\epsilon\hat{\Lambda}$ and an expansion of the logarithm in powers of ϵ is convenient as we are interested in the limit $\epsilon \to 0$. After this

expansion, we arrive at the integral

$$\left\langle \left(G^R_{nn}\right)\left(G^A_{nn}\right)^{q-1}\right\rangle = \int_{\mathcal{M}_4\hat{L}} dQ (g^{BB}_{aa})^{q-2}\left\{g^{BB}_{aa}g^{BB}_{rr} + (q-1)g^{BB}_{ar}g^{BB}_{ra}\right\} \\ \times \exp\left\{-N\mathcal{L}(Q) + \mathrm{i}\epsilon\sum_i^N \mathrm{Str}(E-Q-d_i)^{-1}\hat{\Lambda}\right\}, \tag{273}$$

where $\mathcal{L}(Q) = \frac{1}{2}\mathrm{Str}Q^2 + \sum_i^N \mathrm{Str}\ln(E-Q-d_i)$.

In the limit $N\to\infty$, the integral over Q is dominated by the saddle-point equation that satisfies

$$Q = \frac{1}{N}\sum_i^N \frac{1}{E-d_i-Q}, \tag{274}$$

where the solutions Q can be parametrised as follows [11]

$$Q_{s.p.} = t + \mathrm{i}s\hat{T}^{-1}\hat{\Lambda}\hat{T}. \tag{275}$$

Substituting the parametrisation into the saddle-point equation, we are able to derive two simultaneous equations that s and t must satisfy:

$$t = \frac{1}{N}\sum_i^N \frac{E-t-d_i}{(E-t-d_i)^2+s^2}, \quad 1 = \frac{1}{N}\sum_i^N \frac{1}{(E-t-d_i)^2+s^2}. \tag{276}$$

The integral over the bosonic variables and Grassmannian "angles" of Q in this limit is equal to unity by the saddle-point approximation, hence the integral will reduce down to

$$\left\langle \left(G^R_{nn}\right)\left(G^A_{nn}\right)^{q-1}\right\rangle = \int d\mu(T) (g^{BB}_{aa})^{q-2}\left\{g^{BB}_{aa}g^{BB}_{rr} + (q-1)g^{BB}_{ar}g^{BB}_{ra}\right\} \\ \times \exp\left\{\mathrm{i}\epsilon\sum_i^N \mathrm{Str}(E-t-d_i-\mathrm{i}s\hat{T}^{-1}\hat{\Lambda}\hat{T})^{-1}\hat{\Lambda}\right\}, \tag{277}$$

where the argument of the exponential can be simplified even further down

to

$$\left\langle \left(G^R_{nn}\right)\left(G^A_{nn}\right)^{q-1}\right\rangle = \int d\mu(T)(g^{BB}_{aa})^{q-2}\left\{g^{BB}_{aa}g^{BB}_{rr} + (q-1)g^{BB}_{ar}g^{BB}_{ra}\right\} \times \exp\left\{\mathrm{i}\epsilon\sum_i^N \mathrm{Str}\left(\frac{E-t-d_i+\mathrm{i}s\hat{T}^{-1}\hat{\Lambda}\hat{T}}{(E-t-d_i)^2+s^2}\right)\hat{\Lambda}\right\}. \tag{278}$$

The supertrace of the term proportional to the identity is zero, hence the integral can be expressed as

$$\left\langle \left(G^R_{nn}\right)\left(G^A_{nn}\right)^{q-1}\right\rangle = \int d\mu(T)(g^{BB}_{aa})^{q-2}\left\{g^{BB}_{aa}g^{BB}_{rr} + (q-1)g^{BB}_{ar}g^{BB}_{ra}\right\} \times \exp\left\{-\epsilon s\sum_i^N \frac{1}{(E-t-d_i)^2+s^2}\mathrm{Str}(\hat{T}^{-1}\hat{\Lambda}\hat{T}\hat{\Lambda})\right\}. \tag{279}$$

From one of the simultaneous equations, we can deduce that

$$N = \sum_i^N \frac{1}{(E-t-d_i)^2+s^2}, \tag{280}$$

therefore the integral simplifies down to

$$\left\langle \left(G^R_{nn}\right)\left(G^A_{nn}\right)^{q-1}\right\rangle = \int d\mu(T)(g^{BB}_{aa})^{q-2}\left\{g^{BB}_{aa}g^{BB}_{rr} + (q-1)g^{BB}_{ar}g^{BB}_{ra}\right\} \times \exp\left\{-\epsilon\pi\rho(E)N\mathrm{Str}(\hat{T}^{-1}\hat{\Lambda}\hat{T}\hat{\Lambda})\right\}. \tag{281}$$

since $\rho(E) = s/\pi$.

We must recall from previously, in Appendix A, that $\mathrm{Str}(\hat{T}^{-1}\hat{\Lambda}\hat{T}\hat{\Lambda}) = 2(\lambda_1 - \lambda_2)$, hence,

$$\left\langle \left(G^R_{nn}\right)\left(G^A_{nn}\right)^{q-1}\right\rangle = \int d\mu(T)(g^{BB}_{aa})^{q-2}\left\{g^{BB}_{aa}g^{BB}_{rr} + (q-1)g^{BB}_{ar}g^{BB}_{ra}\right\} \times \exp\left\{-2\epsilon\pi\rho(E)N(\lambda_1-\lambda_2)\right\}. \tag{282}$$

To continue with the integration we must know the explicit expression for $g^{BB}_{\alpha\beta}$, which can calculated by Efetov's parametrisation. This was calculated previously in the application of supersymmetry, so we are able to substitute

the expressions directly into g^{BB}:

$$\begin{aligned} g^{BB} &= [(E - d_n - Q)^{-1}]^{BB} = [(E - t - d_n - \mathrm{i}s\hat{T}^{-1}\hat{\Lambda}\hat{T})^{-1}]^{BB} \\ &= \left[\frac{(E - t - d_n + \mathrm{i}s\hat{T}^{-1}\hat{\Lambda}\hat{T})}{(E - t - d_n)^2 + s^2}\right]^{BB}. \end{aligned} \tag{283}$$

As we are only interested in the Bose-block of the matrix, the calculations can be greatly simplified by reducing it down to a 2×2 matrix problem if we consider only the Bose-block of the parametrisation (see Appendix C for further details).

Once we have integrated over the Grassmanns, as shown in Appendix C, the integral simplifies down to

$$\begin{aligned} &\left\langle \left(G^R_{nn}\right)\left(G^A_{nn}\right)^{q-1}\right\rangle = -\frac{(E - t - d_n + \mathrm{i}s)^{q-2}}{[(E - t - d_n)^2 + s^2]^{q-1}} + (q-1)\int_1^\infty \mathrm{d}\lambda_1 \int_{-1}^1 \mathrm{d}\lambda_2 \\ &\times \frac{(E - t - d_n + \mathrm{i}s\lambda_1)^{q-2}}{[(E - t - d_n)^2 + s^2]^q} e^{-2\epsilon\pi\rho(E)N(\lambda_1 - \lambda_2)} \\ &\times s^2\left[1 + \frac{\lambda_1 + \lambda_2}{\lambda_1 - \lambda_2} + \frac{\mathrm{i}s(q-2)(\lambda_1^2 - 1)}{(E - t - d_n + \mathrm{i}s\lambda_1)(\lambda_1 - \lambda_2)}\right]. \end{aligned} \tag{284}$$

By recalling the definition of the moments of the eigenvectors we are interested in the limit $\epsilon \to 0$ but as λ_1 is unbounded, a substitution, $z = \epsilon\lambda_1$, must be made in order to proceed with the integration. Applying this substitution in the integral gives the following

$$\begin{aligned} &\left\langle \left(G^R_{nn}\right)\left(G^A_{nn}\right)^{q-1}\right\rangle = -\frac{(E - t - d_n + \mathrm{i}s)^{q-2}}{[(E - t - d_n)^2 + s^2]^{q-1}} + (q-1)\int_\epsilon^\infty \mathrm{d}\frac{z}{\epsilon} \int_{-1}^1 \mathrm{d}\lambda_2 \\ &\times \frac{(E - t - d_n + \mathrm{i}s\frac{z}{\epsilon})^{q-2}}{[(E - t - d_n)^2 + s^2]^q} e^{-2\epsilon\pi\rho(E)N(\frac{z}{\epsilon} - \lambda_2)} \\ &\times s^2\left[1 + \frac{\frac{z}{\epsilon} + \lambda_2}{\frac{z}{\epsilon} - \lambda_2} + \frac{\mathrm{i}s(q-2)(\left(\frac{z}{\epsilon}\right)^2 - 1)}{(E - t - d_n + \mathrm{i}s\frac{z}{\epsilon})(\frac{z}{\epsilon} - \lambda_2)}\right], \end{aligned} \tag{285}$$

by performing algebraic manipulations, we can simplify the integrand into a

much more convenient form

$$\left\langle \left(G^R_{nn}\right)\left(G^A_{nn}\right)^{q-1}\right\rangle = -\frac{(E-t-d_n+\mathrm{i}s)^{q-2}}{[(E-t-d_n)^2+s^2]^{q-1}} + (q-1)\int_\epsilon^\infty \mathrm{d}z \int_{-1}^1 \mathrm{d}\lambda_2$$
$$\times \frac{(\epsilon E-\epsilon t-\epsilon d_n+\mathrm{i}sz)^{q-2}}{[(E-t-d_n)^2+s^2]^q}\epsilon^{1-q}e^{-2\pi\rho(E)N(z-\epsilon\lambda_2)}$$
$$\times s^2\left[1+\frac{z+\epsilon\lambda_2}{z-\epsilon\lambda_2}+\frac{\mathrm{i}s(q-2)(z^2-\epsilon^2)}{(\epsilon E-\epsilon t-\epsilon d_n+\mathrm{i}sz)(z-\epsilon\lambda_2)}\right], \tag{286}$$

whereby taking the limit $\epsilon \to 0$, we are left with

$$\lim_{\epsilon\to 0}(2\epsilon)^{q-1}\left\langle \left(G^R_{nn}\right)\left(G^A_{nn}\right)^{q-1}\right\rangle = -\lim_{\epsilon\to 0}(2\epsilon)^{q-1}\frac{(E-t-d_n+\mathrm{i}s)^{q-2}}{[(E-t-d_n)^2+s^2]^{q-1}}$$
$$+(q-1)\lim_{\epsilon\to 0}(2\epsilon)^{q-1}\int_\epsilon^\infty \mathrm{d}z\int_{-1}^1 \mathrm{d}\lambda_2\frac{(\epsilon E-\epsilon t-\epsilon d_n+\mathrm{i}sz)^{q-2}}{[(E-t-d_n)^2+s^2]^q}e^{-2\pi\rho(E)N(z-\epsilon\lambda_2)}$$
$$\times s^2\left[1+\frac{z+\epsilon\lambda_2}{z-\epsilon\lambda_2}+\frac{\mathrm{i}s(q-2)(z^2-\epsilon^2)}{(\epsilon E-\epsilon t-\epsilon d_n+\mathrm{i}sz)(z-\epsilon\lambda_2)}\right]. \tag{287}$$

Therefore, the integral greatly simplifies down to

$$\lim_{\epsilon\to 0}(2\epsilon)^{q-1}\left\langle \left(G^R_{nn}\right)\left(G^A_{nn}\right)^{q-1}\right\rangle = q(q-1)2^{q-1}\int_0^\infty \mathrm{d}z\int_{-1}^1 \mathrm{d}\lambda_2$$
$$\frac{(\mathrm{i}sz)^{q-2}}{[(E-t-d_n)^2+s^2]^q}\exp\{-2\pi\rho(E)Nz\}s^2, \tag{288}$$

as stated previously, this integrand is independent of λ_2, therefore integrating over λ_2 just results in a factor of 2. Hence,

$$\lim_{\epsilon\to 0}(2\epsilon)^{q-1}\left\langle \left(G^R_{nn}\right)\left(G^A_{nn}\right)^{q-1}\right\rangle = q(q-1)2^q$$
$$\times\int_0^\infty \mathrm{d}z\frac{(\mathrm{i}sz)^{q-2}}{[(E-t-d_n)^2+s^2]^q}\exp\{-2\pi\rho(E)Nz\}s^2. \tag{289}$$

This is an elementary integral that can be computed by making the substitu-

tion $y = 2\pi\rho(E)Nz$:

$$\begin{aligned}
&\lim_{\epsilon\to 0}(2\epsilon)^{q-1}\left\langle \left(G^R_{nn}\right)\left(G^A_{nn}\right)^{q-1}\right\rangle = q(q-1)2^q\frac{(\mathrm{i}s)^{q-2}s^2}{[(E-t-d_n)^2+s^2]^q}\\
&\times\int_0^\infty \frac{\mathrm{d}y}{2\pi\rho(E)N}\left(\frac{y}{2\pi\rho(E)N}\right)^{q-2}e^{-y}\\
&= q(q-1)2^q\frac{(\mathrm{i}s)^{q-2}s^2}{[(E-t-d_n)^2+s^2]^q}\frac{1}{(2\pi\rho(E)N)^{q-1}}\int_0^\infty \mathrm{d}y y^{q-2}e^{-y},
\end{aligned} \tag{290}$$

which is now in the form of a standard gamma function integral, hence,

$$\begin{aligned}
&\lim_{\epsilon\to 0}(2\epsilon)^{q-1}\left\langle \left(G^R_{nn}\right)\left(G^A_{nn}\right)^{q-1}\right\rangle =\\
&q(q-1)2^q\frac{(\mathrm{i}s)^{q-2}s^2}{[(E-t-d_n)^2+s^2]^q}\frac{1}{(2\pi\rho(E)N)^{q-1}}\Gamma(q-1)\\
&= 2\mathrm{i}^{q-2}\frac{s^q}{[(E-t-d_n)^2+s^2]^q}\frac{1}{(\pi\rho(E)N)^{q-1}}\Gamma(q+1).
\end{aligned} \tag{291}$$

Substituting this result into the formula for the moments of the eigenvectors, Eq. (32), yields

$$I_q(n) = \frac{1}{(\pi\rho(E)N)^q}\left[\frac{s}{(E-t-d_n)^2+s^2}\right]^q\Gamma(q+1), \tag{292}$$

where

$$\rho(E) = \frac{s}{\pi}, \tag{293}$$

and $s \neq 0$ and t satisfy

$$t = \frac{1}{N}\sum_i^N \frac{E-t-d_i}{(E-t-d_i)^2+s^2}, \quad 1 = \frac{1}{N}\sum_i^N \frac{1}{(E-t-d_i)^2+s^2}. \tag{294}$$

This is one of our main results for the moments of the eigenvectors with a deterministic potential, in the next section we will generalise the result to the case of a random potential using this as a starting point.

3.3 Distribution function

Similar to previously, we are also able to calculate the distribution function in this case, due to the simple appearance of q in the moments of the eigen-

vectors, using the relation between the moments of the eigenvectors and the distribution function, Eq. (102). The distribution function will be of the same form, since we know that it must be of the form of a gamma function integral:

$$P_n(y) = a(n)e^{-a(n)y}, \quad y = |\psi(n)|^2, \tag{295}$$

where $a(n)$ is a function of n such that $a(n) > 0$ for all n.

By substituting this into the relation, it allows one to determine the exact form of $a(n)$ but the steps will not be detailed as it was already shown previously. Referring back to Eq. (107) and by comparing coefficients we deduce that

$$a(n) = \pi\rho(E)N\frac{(E - t - d_n)^2 + s^2}{s}, \tag{296}$$

therefore, the distribution function is given by

$$\begin{aligned} P_n(y) = \pi\rho(E)N &\left[\frac{(E - t - d_n)^2 + s^2}{s}\right] \\ &\times \exp\left\{-\pi\rho(E)N\left[\frac{(E - t - d_n)^2 + s^2}{s}\right]y\right\}, \end{aligned} \tag{297}$$

where $y = |\psi(n)|^2$. Now that we have the distribution function we know not only the qualitative nature of the eigenvectors, but how each of the individual elements of the eigenvectors are distributed.

Testing the validity of the result, Eq. (292), by direct diagonalisation for $d_i = -1 + \frac{2}{N}(i - 1)$ where the moments of the eigenvectors are calculated for the eigenvectors whose eigenvalues were close to $E = 0$. Fig.6 shows the result of the numerical simulation for $q = 2$ and $q = 3$, both are in excellent agreement with the predictions given by the analytical expression.

3.4 Model II with random diagonal matrix

In this section we consider the case where the diagonal matrix D is random and assume that the elements d_i are independently distributed Gaussian random variables with mean $\langle d_i \rangle = 0$ and variance $\langle d_i^2 \rangle = \sigma^2$. It turns out that when

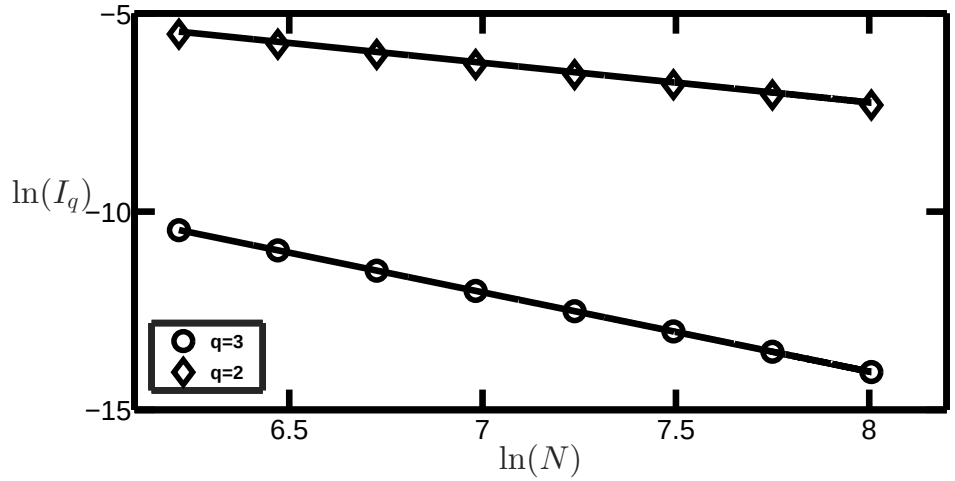

Figure 6: The moments of the eigenvectors $I_q = \sum_{n=1}^{N} I_q(n)$ for N ranging from 500 to 3000, where the symbols represent the numerical data and the straight lines represent the analytical results for $d_i = -1 + \frac{2}{N}(i-1)$.

we are considering the case of random D, the results from the previous section for non-random D provides a good foundation for our problem.

As a reminder, the equations for $s \neq 0$ and t are given by

$$t = \frac{1}{N}\sum_{i=1}^{N} \frac{E - t - d_i}{(E - t - d_i)^2 + s^2}, \quad 1 = \frac{1}{N}\sum_{i=1}^{N} \frac{1}{(E - t - d_i)^2 + s^2}, \tag{298}$$

where the parameter s is related to the density of states in a simple way: $\rho(E) = s/\pi$; the moments of the eigenvectors are given by

$$I_q(n) = \frac{\Gamma(q+1)}{(\pi\rho(E)N)^q}\left[\frac{s}{(E - t - d_n)^2 + s^2}\right]^q. \tag{299}$$

Similar to the previous case, to calculate the average of the simultaneous equations, by applying the same reasoning we have to replace s and t by their average values due to the central limit theorem, as we are averaging over a large number of realisations:

$$\langle t\rangle = \frac{1}{N}\sum_{i=1}^{N}\left\langle \frac{E - \langle t\rangle - d_i}{(E - \langle t\rangle - d_i)^2 + \langle s\rangle^2}\right\rangle, \quad 1 = \frac{1}{N}\sum_{i=1}^{N}\left\langle \frac{1}{(E - \langle t\rangle - d_i)^2 + \langle s\rangle^2}\right\rangle, \tag{300}$$

but we must remember, however, that all d_i are identically distributed, there-

fore the system of simultaneous equations reduces down to

$$\langle t \rangle = \left\langle \frac{\tilde{E} - x}{(\tilde{E} - x)^2 + \langle s \rangle^2} \right\rangle_x, \quad 1 = \left\langle \frac{1}{(\tilde{E} - x)^2 + \langle s \rangle^2} \right\rangle_x, \tag{301}$$

where we introduce $\tilde{E} = E - \langle t \rangle$ for convenience and x is the Gaussian random variable with distribution function

$$P(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma}}. \tag{302}$$

As before, in order to calculate the average, we introduce the Fourier transform, Eq. (178), to express the average in the following form

$$\left\langle \frac{1}{(\tilde{E} - x)^2 + \langle s \rangle^2} \right\rangle_x = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\kappa \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\kappa^2\sigma^2} \int_{-\infty}^{\infty} dx \frac{e^{-\mathrm{i}\kappa x}}{(\tilde{E} - x)^2 + \langle s \rangle^2}. \tag{303}$$

If we focus on the integral over x, the variable of integration can first be changed by the substitution $y = \tilde{E} - x$:

$$\int_{-\infty}^{\infty} dx \frac{e^{-\mathrm{i}\kappa x}}{(\tilde{E} - x)^2 + \langle s \rangle^2} = e^{-\mathrm{i}\kappa\tilde{E}} \int_{-\infty}^{\infty} dy \frac{e^{-\mathrm{i}\kappa y}}{y^2 + \langle s \rangle^2}, \tag{304}$$

then by the residue theorem, the integration yields

$$e^{-\mathrm{i}\kappa\tilde{E}} \int_{-\infty}^{\infty} dy \frac{e^{-\mathrm{i}\kappa y}}{y^2 + \langle s \rangle^2} = e^{-\mathrm{i}\kappa\tilde{E}} \frac{\pi}{\langle s \rangle} e^{-\langle s \rangle |\kappa|}, \tag{305}$$

where we made the assumption $\langle s \rangle > 0$, which holds because $\rho(E)\pi = s$ and the density of states must always be positive. Substituting this result back into Eq. (303) we find

$$\left\langle \frac{1}{(\tilde{E} - x)^2 + \langle s \rangle^2} \right\rangle_x = \frac{1}{2\langle s \rangle} \int_{-\infty}^{\infty} d\kappa e^{-\frac{1}{2}\kappa^2\sigma^2 - \mathrm{i}\kappa\tilde{E} - \langle s \rangle |\kappa|}, \tag{306}$$

this can be simplified even further by splitting the integral into two parts:

$\kappa < 0$ and $\kappa > 0$, which gives

$$\int_{-\infty}^{\infty} d\kappa e^{-\frac{1}{2}\kappa^2\sigma^2 - \mathrm{i}\kappa\tilde{E} - \langle s\rangle|\kappa|} = \int_{-\infty}^{0} d\kappa e^{-\frac{1}{2}\kappa^2\sigma^2 - \mathrm{i}\kappa\tilde{E} - \langle s\rangle|\kappa|} + \int_{0}^{\infty} d\kappa e^{-\frac{1}{2}\kappa^2\sigma^2 - \mathrm{i}\kappa\tilde{E} - \langle s\rangle|\kappa|}. \tag{307}$$

The absolute values signs from κ can be dropped from the integral involving positive integration limits

$$\int_{-\infty}^{\infty} d\kappa e^{-\frac{1}{2}\kappa^2\sigma^2 - \mathrm{i}\kappa\tilde{E} - \langle s\rangle|\kappa|} = -\int_{\infty}^{0} d\kappa e^{-\frac{1}{2}\kappa^2\sigma^2 + \mathrm{i}\kappa\tilde{E} - \langle s\rangle|\kappa|} + \int_{0}^{\infty} d\kappa e^{-\frac{1}{2}\kappa^2\sigma^2 - \mathrm{i}\kappa\tilde{E} - \langle s\rangle\kappa}, \tag{308}$$

we applied the change of variables $\kappa \to -\kappa$ above in the first integral, therefore we are able to drop the absolute value signs in the first integral since $\kappa > 0$, and then by swapping the limits of integration it changes the sign of the whole integral:

$$\begin{aligned}
&\int_{-\infty}^{\infty} d\kappa e^{-\frac{1}{2}\kappa^2\sigma^2 - \mathrm{i}\kappa\tilde{E} - \langle s\rangle|\kappa|} = \int_{0}^{\infty} d\kappa e^{-\frac{1}{2}\kappa^2\sigma^2 + \mathrm{i}\kappa\tilde{E} - \langle s\rangle\kappa} + \int_{0}^{\infty} d\kappa e^{-\frac{1}{2}\kappa^2\sigma^2 - \mathrm{i}\kappa\tilde{E} - \langle s\rangle\kappa} \\
&= \frac{1}{\sigma}\sqrt{\frac{\pi}{2}} e^{-\frac{(\tilde{E}+\mathrm{i}\langle s\rangle)^2}{2\sigma^2}} \left\{ \left(1 + \mathrm{ierfi}\left[\frac{\tilde{E} + \mathrm{i}\,\langle s\rangle}{\sqrt{2}\sigma}\right]\right) \right. \\
&\left. + e^{-\frac{4\mathrm{i}\tilde{E}\langle s\rangle}{2\sigma^2}} \left(1 - \mathrm{ierfi}\left[\frac{\tilde{E} - \mathrm{i}\,\langle s\rangle}{\sqrt{2}\sigma}\right]\right) \right\},
\end{aligned} \tag{309}$$

Combining all of this together, we find that

$$1 = \sqrt{\frac{\pi}{8}} \frac{1}{\sigma\,\langle s\rangle} e^{-\frac{(\tilde{E}+\mathrm{i}\langle s\rangle)^2}{2\sigma^2}} F_{+}\left(\frac{\tilde{E}}{\sqrt{2}\sigma}, \frac{\langle s\rangle}{\sqrt{2}\sigma}\right) \tag{310}$$

where

$$F_{\pm}(x, y) = \left(1 \pm e^{4\mathrm{i}xy}(1 - \mathrm{ierfi}(x - y)) + \mathrm{ierfi}(x + \mathrm{i}y)\right), \tag{311}$$

erfi is the imaginary error function and $\tilde{E} = E - \langle t\rangle$.

We also follow the same method to average over the other simultaneous

equation by first applying the Fourier transform

$$\langle t\rangle = \left\langle \frac{\tilde{E}-x}{(\tilde{E}-x)^2+\langle s\rangle^2}\right\rangle_x = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\kappa \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\kappa^2\sigma^2}\int_{-\infty}^{\infty} dx \frac{(\tilde{E}-x)e^{-\mathrm{i}\kappa x}}{(\tilde{E}-x)^2+\langle s\rangle^2}, \tag{312}$$

the integration variable x can be changed by making the substitution $y = \tilde{E}-x$ to express the integral in a more convenient form

$$\int_{-\infty}^{\infty} dx \frac{(\tilde{E}-x)e^{-\mathrm{i}\kappa x}}{(\tilde{E}-x)^2+\langle s\rangle^2} = -e^{-\mathrm{i}\kappa\tilde{E}}\int_{\infty}^{-\infty} dy \frac{ye^{\mathrm{i}\kappa y}}{y^2+\langle s\rangle^2}, \tag{313}$$

then by changing the integration variable $y \to -y$, we find

$$-e^{-\mathrm{i}\kappa\tilde{E}}\int_{\infty}^{-\infty} dy \frac{ye^{\mathrm{i}\kappa y}}{y^2+\langle s\rangle^2} = e^{-\mathrm{i}\kappa\tilde{E}}\int_{-\infty}^{\infty} dy \frac{(-y)e^{-\mathrm{i}\kappa y}}{y^2+\langle s\rangle^2}. \tag{314}$$

By using the residue theorem, this integral is equal to

$$e^{-\mathrm{i}\kappa\tilde{E}}\int_{-\infty}^{\infty} dy \frac{(-y)e^{-\mathrm{i}\kappa y}}{y^2+\langle s\rangle^2} = e^{-\mathrm{i}\kappa\tilde{E}}\mathrm{i}e^{-\langle s\rangle|\kappa|}\pi\mathrm{sign}(\kappa), \tag{315}$$

therefore, the second averaged simultaneous equation can be expressed as

$$\langle t\rangle = \frac{1}{2}\int_{-\infty}^{\infty} d\kappa \mathrm{i}e^{-\frac{1}{2}\kappa^2\sigma^2-\mathrm{i}\kappa\tilde{E}-\langle s\rangle|\kappa|}\mathrm{sign}(\kappa). \tag{316}$$

This integral can be split in the same way as before between $k<0$ and $k>0$:

$$\frac{1}{2}\int_{-\infty}^{0} d\kappa \mathrm{i}e^{-\frac{1}{2}\kappa^2\sigma^2-\mathrm{i}\kappa\tilde{E}-\langle s\rangle|\kappa|}\mathrm{sign}(\kappa)+\frac{1}{2}\int_{0}^{\infty} d\kappa \mathrm{i}e^{-\frac{1}{2}\kappa^2\sigma^2-\mathrm{i}\kappa\tilde{E}-\langle s\rangle|\kappa|}\mathrm{sign}(\kappa), \tag{317}$$

this is convenient since we can now drop $\mathrm{sign}(\kappa)$ for $k>0$ and also for $k<0$ but we must multiply that integral by -1, hence,

$$-\frac{1}{2}\int_{-\infty}^{0} d\kappa \mathrm{i}e^{-\frac{1}{2}\kappa^2\sigma^2-\mathrm{i}\kappa\tilde{E}-\langle s\rangle|\kappa|}+\frac{1}{2}\int_{0}^{\infty} d\kappa \mathrm{i}e^{-\frac{1}{2}\kappa^2\sigma^2-\mathrm{i}\kappa\tilde{E}-\langle s\rangle|\kappa|}, \tag{318}$$

changing the variables for the first integral from $\kappa \to -\kappa$ will also ensure that

both integrals will have positive integration limits, which yields

$$\frac{1}{2}\int_{\infty}^{0} d\kappa \mathrm{i} e^{-\frac{1}{2}\kappa^2\sigma^2+\mathrm{i}\kappa\tilde{E}-\langle s\rangle|\kappa|} + \frac{1}{2}\int_{0}^{\infty} d\kappa \mathrm{i} e^{-\frac{1}{2}\kappa^2\sigma^2-\mathrm{i}\kappa\tilde{E}-\langle s\rangle|\kappa|}. \tag{319}$$

Reversing the integration limits of the first integral will introduce a minus sign and since both integrals have positive integration limits we can drop the absolute value signs, hence,

$$-\frac{1}{2}\int_{0}^{\infty} d\kappa \mathrm{i} e^{-\frac{1}{2}\kappa^2\sigma^2+\mathrm{i}\kappa\tilde{E}-\langle s\rangle\kappa} + \frac{1}{2}\int_{0}^{\infty} d\kappa \mathrm{i} e^{-\frac{1}{2}\kappa^2\sigma^2-\mathrm{i}\kappa\tilde{E}-\langle s\rangle\kappa}, \tag{320}$$

once the integration is evaluated, we receive

$$-\frac{1}{\sigma}\sqrt{\frac{\pi}{2}}\mathrm{i} e^{-\frac{(\tilde{E}+\mathrm{i}\langle s\rangle)^2}{2\sigma^2}} \left\{1+\mathrm{ierfi}\left[\frac{\tilde{E}+\mathrm{i}\langle s\rangle}{\sqrt{2}\sigma}\right] - e^{\frac{4\tilde{E}\langle s\rangle}{2\sigma^2}}\left(1-\mathrm{ierfi}\left[\frac{\tilde{E}-\mathrm{i}\langle s\rangle}{\sqrt{2}\sigma}\right]\right)\right\}. \tag{321}$$

By combining all of this together, we receive the second averaged simultaneous equation for s and t, hence,

$$\langle t\rangle = -\mathrm{i}\sqrt{\frac{\pi}{8}}\frac{1}{\sigma} e^{\frac{(\tilde{E}+\mathrm{i}\langle s\rangle)}{2\sigma^2}} F_{-}\left(\frac{\tilde{E}}{\sqrt{2}\sigma}, \frac{\langle s\rangle}{\sqrt{2}\sigma}\right). \tag{322}$$

We are then able to find the density of states by solving the system of simultaneous equations numerically for $\langle s\rangle$ and $\langle t\rangle$:

$$\hat{\rho}(E) \equiv \langle \rho(E)\rangle = \frac{\langle s\rangle}{\pi}. \tag{323}$$

To confirm the validity of the equations we consider two limiting cases of the density of states: $\sigma \to 0$ and $\sigma \to \infty$. Taking the first case of $\sigma \to 0$ we use the expansion of $\mathrm{erfi}(z)$ in the limit $z \to \infty$

$$\mathrm{erfi}(z) = \mathrm{sign}(\mathrm{Im}\, z)\mathrm{i} + \frac{e^{z^2}}{\sqrt{\pi}z} + ..., \tag{324}$$

by substituting the expansion into Eq. (310)

$$
\begin{aligned}
1 \approx \sqrt{\frac{\pi}{8}}\frac{1}{\sigma\left\langle s\right\rangle}e^{\frac{(\tilde{E}+\mathrm{i}\langle s\rangle)^2}{2\sigma^2}}\left(1+e^{\frac{2\mathrm{i}\tilde{E}\langle s\rangle}{\sigma^2}}\left(1-\mathrm{i}\left(-\mathrm{i}+\frac{e^{\frac{(\tilde{E}-\mathrm{i}\langle s\rangle)^2}{2\sigma^2}}}{\sqrt{\pi}\frac{(\tilde{E}-\mathrm{i}\langle s\rangle)}{\sqrt{2}\sigma}}\right)\right)\right. \\
\left.+\mathrm{i}\left(\mathrm{i}+\frac{e^{\frac{(\tilde{E}+\mathrm{i}\langle s\rangle)^2}{2\sigma^2}}}{\sqrt{\pi}\frac{(\tilde{E}+\mathrm{i}\langle s\rangle)}{\sqrt{2}\sigma}}\right)\right) = \sqrt{\frac{\pi}{8}}\frac{1}{\sigma\left\langle s\right\rangle}e^{-\frac{(\tilde{E}+\mathrm{i}\langle s\rangle)^2}{2\sigma^2}}\left(1-\mathrm{i}\frac{e^{\frac{(\tilde{E}+\mathrm{i}\langle s\rangle)^2}{2\sigma^2}}}{\sqrt{\pi}\frac{(\tilde{E}-\mathrm{i}\langle s\rangle)}{\sqrt{2}\sigma}}\right. \\
\left.+\mathrm{i}\frac{e^{\frac{(\tilde{E}+\mathrm{i}\langle s\rangle)^2}{2\sigma^2}}}{\sqrt{\pi}\frac{(\tilde{E}+\mathrm{i}\langle s\rangle)}{\sqrt{2}\sigma}}-1\right) = \sqrt{\frac{\pi}{8}}\frac{1}{\sigma\left\langle s\right\rangle}\frac{\mathrm{i}\sqrt{2}\sigma}{\sqrt{\pi}}\left(\frac{1}{\tilde{E}+\mathrm{i}\left\langle s\right\rangle}-\mathrm{i}\frac{1}{\tilde{E}-\mathrm{i}\left\langle s\right\rangle}\right) \\
=\frac{\mathrm{i}}{2\left\langle s\right\rangle}\frac{(-2\mathrm{i})\left\langle s\right\rangle}{\tilde{E}^2+\left\langle s\right\rangle^2}.
\end{aligned}
\tag{325}
$$

Hence, we obtain

$$
(E-\langle t\rangle)^2+\langle s\rangle^2 \approx 1. \tag{326}
$$

The same approach is taken for the second equation for $\langle t\rangle$ by using the expansion for erfi(z)

$$
\begin{aligned}
\langle t\rangle \approx -\mathrm{i}\sqrt{\frac{\pi}{8}}\frac{1}{\sigma}e^{-\frac{(\tilde{E}+\mathrm{i}\langle s\rangle)^2}{2\sigma^2}}\left(1-e^{\frac{2\mathrm{i}\tilde{E}\langle s\rangle}{\sigma^2}}\left(1-\mathrm{i}\left(-\mathrm{i}+\frac{e^{\frac{(\tilde{E}-\mathrm{i}\langle s\rangle)^2}{2\sigma^2}}}{\sqrt{\pi}\frac{(\tilde{E}-\mathrm{i}\langle s\rangle)}{\sqrt{2}\sigma}}\right)\right)\right. \\
\left.+\mathrm{i}\left(\mathrm{i}+\frac{e^{\frac{(\tilde{E}+\mathrm{i}\langle s\rangle)^2}{2\sigma^2}}}{\sqrt{\pi}\frac{(\tilde{E}+\mathrm{i}\langle s\rangle)}{\sqrt{2}\sigma}}\right)\right) = -\mathrm{i}\sqrt{\frac{\pi}{8}}\frac{1}{\sigma}e^{-\frac{(\tilde{E}+\mathrm{i}\langle s\rangle)^2}{2\sigma^2}}\left(1+\mathrm{i}\frac{e^{\frac{(\tilde{E}+\mathrm{i}\langle s\rangle)^2}{2\sigma^2}}}{\sqrt{\pi}\frac{(\tilde{E}-\mathrm{i}\langle s\rangle)}{\sqrt{2}\sigma}}\right. \\
\left.+\mathrm{i}\frac{e^{\frac{(\tilde{E}+\mathrm{i}\langle s\rangle)^2}{2\sigma^2}}}{\sqrt{\pi}\frac{(\tilde{E}+\mathrm{i}\langle s\rangle)}{\sqrt{2}\sigma}}-1\right) = -\mathrm{i}\sqrt{\frac{\pi}{8}}\frac{1}{\sigma}\frac{\mathrm{i}\sqrt{2}\sigma}{\sqrt{\pi}}\left(\frac{1}{\tilde{E}+\mathrm{i}\left\langle s\right\rangle}+\frac{1}{\tilde{E}-\mathrm{i}\left\langle s\right\rangle}\right) \\
=\frac{\tilde{E}}{\tilde{E}^2+\left\langle s\right\rangle^2}=\tilde{E}.
\end{aligned}
\tag{327}
$$

Therefore, we obtain from the simultaneous equations:

$$
\langle t\rangle \approx \frac{E}{2}, \quad \langle s\rangle \approx \sqrt{1-\frac{E^2}{4}}, \quad \hat{\rho}(E) \approx \frac{1}{\pi}\sqrt{1-\frac{E^2}{4}}, \tag{328}
$$

the density of states in this case is given by Wigner's semicircle as expected.

If we consider the opposite limiting case $\sigma \to \infty$, we use the fact that $\langle s\rangle/\sigma$ is a small parameter so we are able to simplify the following functions

to their leading order terms

$$\begin{aligned}
\operatorname{erfi}\left(\frac{\tilde{E}\pm \mathrm{i}\,\langle s\rangle}{\sqrt{2}\sigma}\right) &= \operatorname{erfi}\left(\frac{\tilde{E}}{\sqrt{2}\sigma}\right)+O\left(\frac{\langle s\rangle}{\sigma}\right),\\
e^{-\frac{(\tilde{E}+\mathrm{i}\langle s\rangle)^2}{2\sigma^2}} &= e^{-\frac{\tilde{E}^2}{2\sigma^2}}\left(1+O\left(\frac{\langle s\rangle}{\sigma}\right)\right),\\
e^{\frac{2\mathrm{i}\tilde{E}\langle s\rangle}{\sigma^2}} &= 1+O\left(\frac{\langle s\rangle}{\sigma}\right).
\end{aligned} \tag{329}$$

Substituting this into Eq. (310):

$$\begin{aligned}
&1\approx\sqrt{\frac{\pi}{8}}\frac{1}{\sigma\,\langle s\rangle}e^{-\frac{(\tilde{E}+\mathrm{i}\langle s\rangle)^2}{2\sigma^2}}\left(1+e^{\frac{2\mathrm{i}\tilde{E}\langle s\rangle}{\sigma^2}}\left(1-\mathrm{i}\left[\operatorname{erfi}\left(\frac{\tilde{E}}{\sqrt{2}\sigma}\right)+O\left(\frac{\langle s\rangle}{\sigma}\right)\right]\right)\right.\\
&\left.+\,\mathrm{i}\left[\operatorname{erfi}\left(\frac{\tilde{E}}{\sqrt{2}\sigma}\right)+O\left(\frac{\langle s\rangle}{\sigma}\right)\right]\right)=\sqrt{\frac{\pi}{8}}\frac{1}{\sigma\,\langle s\rangle}e^{-\frac{\tilde{E}^2}{2\sigma^2}}\left(2+O\left(\frac{\langle s\rangle}{\sigma}\right)\right)\\
&\Longrightarrow\ \langle s\rangle\approx\sqrt{\frac{\pi}{2}}\frac{1}{\sigma}e^{-\frac{\tilde{E}^2}{2\sigma^2}},
\end{aligned} \tag{330}$$

by using the same method for Eq. (322), we receive

$$\begin{aligned}
&\langle t\rangle\approx-\mathrm{i}\sqrt{\frac{\pi}{8}}\frac{1}{\sigma}e^{-\frac{(\tilde{E}+\mathrm{i}\langle s\rangle)^2}{2\sigma^2}}\left(1-e^{\frac{2\mathrm{i}\tilde{E}\langle s\rangle}{\sigma^2}}\left(1-\mathrm{i}\left[\operatorname{erfi}\left(\frac{\tilde{E}}{\sqrt{2}\sigma}\right)+O\left(\frac{\langle s\rangle}{\sigma}\right)\right]\right)\right.\\
&\left.+\,\mathrm{i}\left[\operatorname{erfi}\left(\frac{\tilde{E}}{\sqrt{2}\sigma}\right)+O\left(\frac{\langle s\rangle}{\sigma}\right)\right]\right)=\sqrt{\frac{\pi}{2}}\frac{1}{\sigma}e^{-\frac{\tilde{E}^2}{2\sigma^2}}\left(\operatorname{erfi}\left(\frac{\tilde{E}}{\sqrt{2}\sigma}\right)\right.\\
&\left.+\,O\left(\frac{\langle s\rangle}{\sigma}\right)\right)\ \Longrightarrow\ \langle t\rangle\approx\sqrt{\frac{\pi}{2}}\frac{1}{\sigma}e^{-\frac{\tilde{E}^2}{2\sigma^2}}\operatorname{erfi}\left(\frac{\tilde{E}}{\sqrt{2}\sigma}\right),
\end{aligned} \tag{331}$$

in the limit $\sigma\to\infty$.

It follows from Eq. (331) that $\tilde{E}\approx E$, hence

$$\langle s\rangle\approx\sqrt{\frac{\pi}{2}}\frac{1}{\sigma}e^{-\frac{E^2}{2\sigma^2}},\quad \hat{\rho}(E)\approx\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{E^2}{2\sigma^2}}, \tag{332}$$

which, as expected, coincides with the density of states generated by the diagonal matrix D.

Now that we have averaged the simultaneous equations over d_i, we can turn our attention to the moments of the eigenvectors. In order to calculate the averaged moments we have to replace s and t by their average values, using the same reasoning as previously to average over all d_i:

$$\begin{aligned}\hat{I}_q \equiv \sum_n^N \left\langle \hat{I}_q(n) \right\rangle &= \frac{N\Gamma(q+1)}{(\pi\hat{\rho}(E)N)^q} \left\langle \frac{\langle s\rangle^q}{\left((E-\langle t\rangle - d_n)^2 + \langle s\rangle\right)^q} \right\rangle \\ &= \frac{\Gamma(q+1)}{N^{q-1}} \left\langle \frac{1}{\left((E-\langle t\rangle - d_n)^2 + \langle s\rangle^2\right)^q} \right\rangle,\end{aligned} \tag{333}$$

where we used the fact that $\hat{\rho}(E) = \langle s\rangle / \pi$.

In order to calculate the averaged moments, we first note that

$$\begin{aligned}&\frac{1}{\left((E-\langle t\rangle - d_n)^2 + \langle s\rangle^2\right)^q} \\ &= \frac{1}{(q-1)!}\left[\left(-\frac{1}{2y}\frac{d}{dy}\right)^{q-1} \frac{1}{(E-\langle t\rangle - d_n)^2 + y^2}\right]_{y=\langle s\rangle}\end{aligned} \tag{334}$$

As we have already calculated $\left\langle 1/\left((E-\langle t\rangle - d_n)^2 + \langle s\rangle^2\right)\right\rangle^q$, therefore no further calculation for the moments of the eigenvectors is necessary, we can immediately write down the result:

$$\hat{I}_q = \frac{q}{N^{q-1}}\left[\left(-\frac{1}{2y}\frac{d}{dy}\right)^{q-1} \sqrt{\frac{\pi}{8}}\frac{1}{\sigma y} e^{-\frac{(E-\langle t\rangle + \mathrm{i}y)^2}{2\sigma^2}} F_+\left(\frac{E-\langle t\rangle}{\sqrt{2}\sigma}, \frac{y}{\sqrt{2}\sigma}\right)\right]_{y=\langle s\rangle}, \tag{335}$$

where $\langle s\rangle$ and $\langle t\rangle$ are determined by the system of simultaneous equations

$$\begin{aligned}1 &= \sqrt{\frac{\pi}{8}}\frac{1}{\sigma\langle s\rangle} e^{-\frac{(\tilde{E}+\mathrm{i}\langle s\rangle)^2}{2\sigma^2}} F_+\left(\frac{\tilde{E}}{\sqrt{2}\sigma}, \frac{\langle s\rangle}{\sqrt{2}\sigma}\right), \\ \langle t\rangle &= -\mathrm{i}\sqrt{\frac{\pi}{8}}\frac{1}{\sigma} e^{\frac{(\tilde{E}+\mathrm{i}\langle s\rangle}{2\sigma^2}} F_-\left(\frac{\tilde{E}}{\sqrt{2}\sigma}, \frac{\langle s\rangle}{\sqrt{2}\sigma}\right),\end{aligned} \tag{336}$$

with

$$F_\pm(x,y) = \left(1 \pm e^{4\mathrm{i}xy}(1 - \mathrm{ierfi}(x-y)) + \mathrm{ierfi}(x+\mathrm{i}y)\right), \tag{337}$$

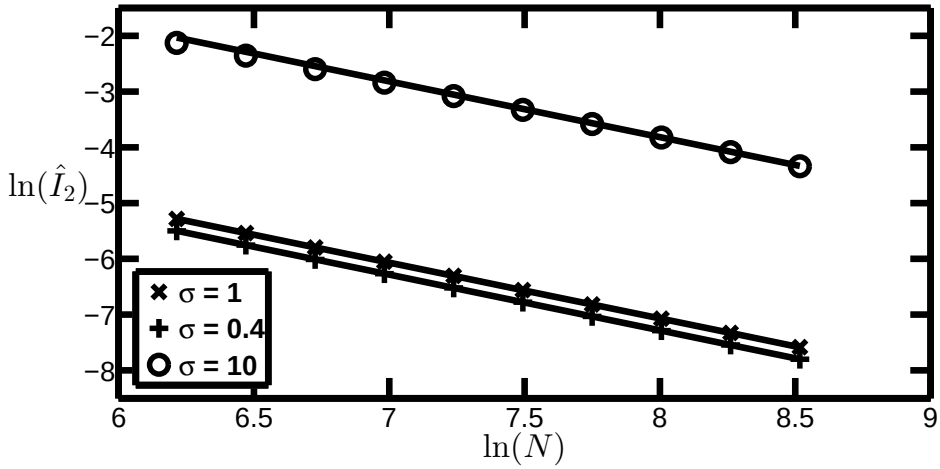

Figure 7: $\hat{I}_2$ calculated for three different values of $\sigma = 0.4$, 1, and 10 with $N = 500$ to $N = 3000$. The symbols represent the numerical result and the solid lines represent the analytical rseult.

and density of states

$$\hat{\rho}(E) \equiv \langle \rho(E) \rangle = \frac{\langle s \rangle}{\pi}. \tag{338}$$

These are the main results of this section, which can be explicitly calculated for any positive integer q, for the case of $q = 2$ we find that

$$\hat{I}_2 = \frac{1}{N \langle s \rangle^2} \left[1 + \frac{1}{\sigma^2} + \mathrm{i}\sqrt{2}\frac{\langle s \rangle}{\sigma} z^* - \mathrm{i}\sqrt{\frac{\pi}{2}} \frac{E - \langle t \rangle}{\sigma^3} e^{-z^2} (1 - \mathrm{ierfi}(z)) \right]_{z = \frac{E - \langle t \rangle - \mathrm{i}\langle s \rangle}{\sqrt{2}\sigma}}. \tag{339}$$

Numerical simulations were carried out to corroborate the result against the direct diagonalisation approach. The tests were performed over a total of 1000 realisations with $\sigma = 0.4$, 1, and 10 where N ranged from 500 to 3000. The numerics were in excellent agreement with our analytical results as shown in fig. 7.

The distribution function, in the case of random D, can also be calculated from our result in the previous section, Eq. (357), by averaging over the random variable x:

$$\hat{\mathcal{P}}(y) \equiv \langle \mathcal{P}(y) \rangle = \int_{-\infty}^{\infty} dx P(x) \left((\tilde{E} - x)^2 + \langle s \rangle^2 \right) e^{-\left((\tilde{E} - x)^2 + \langle s \rangle^2 \right) y}, \tag{340}$$

where $y = xN$, so that $\langle y \rangle = 1$ and

$$P(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}. \tag{341}$$

Calculating the integral, we find

$$\hat{\mathcal{P}}(y) = \frac{\tilde{E}^2 + \sigma^2(1+2y\sigma^2) + \langle s \rangle^2 (1+2y\sigma^2)^2}{(1+2y\sigma^2)^{5/2}} e^{-\left(\frac{\tilde{E}}{1+2y\sigma^2} + \langle s \rangle^2\right) y}. \tag{342}$$

By considering two separate cases, $y\sigma^2 \ll 1$ and $y\sigma^2 \gg 1$, we find that

$$\hat{\mathcal{P}}(y) \approx (\tilde{E}^2 + \sigma^2 + \langle s \rangle^2) e^{-(\tilde{E}^2 + \langle s \rangle^2) y}, \quad y\sigma^2 \ll 1, \tag{343}$$

and

$$\hat{\mathcal{P}}(y) \approx \frac{\langle s \rangle^2}{\sqrt{2y}\sigma} e^{-\langle s \rangle^2 y}, \quad y\sigma^2 \gg 1. \tag{344}$$

Therefore, in the case of $\sigma \to 0$ we have

$$\hat{\mathcal{P}}(y) \approx (\tilde{E}^2 + \langle s \rangle^2) e^{-(\tilde{E}^2 + \langle s \rangle^2) y}, \quad y\sigma^2 \ll 1, \tag{345}$$

as the distribution function which is as expected for the GUE case. This is one of our results for the distribution function in the case $\sigma \ll 1$, showing how the elements of the eigenvectors are distributed in this particular case.

The moments of the eigenvectors show that the eigenvectors are always extended, even at arbitrary large, but fixed σ.

For non-extended eigenvectors to manifest we must consider an N-dependent variance, specifically the Rosenzweig-Porter model where $\sigma^2 = N^{\gamma-1}$. The properties of the Rosenzweig-Porter model was studied extensively [45–48] and was found that the two-point spectral correlation undergoes a transition from Wigner-Dyson to the Poisson form at $\gamma = 2$. Motivated by new developments, a recent study [22] also shows that there is an extra transition from non-ergodic ($1 < \gamma < 2$) to ergodic ($\gamma < 1$) extended states at $\gamma = 1$. The ergodic extended states are similar to GUE states with the same scaling in the moments of the eigenvectors but the non-ergodic extended states have the

characteristics of multi-fractal critical states, usually appearing in the Anderson transition [49]. It turns out that the expressions derived previously can be directly applied to the Rosenzweig-Porter model for $\gamma < 2$, as was originally suggested by our numerical simulations.

If $1 < \gamma < 2$, then substituting $\sigma = N^{\frac{\gamma-1}{2}}$ into Eq. (339) and keeping the leading order terms, we receive

$$\hat{I}_2 \approx \frac{2}{\pi} N^{\gamma-2}, \tag{346}$$

which implies that the fractal dimension $D_2 = 2 - \gamma$. The result can be further generalised for any q by using Eq. (335). The leading order term for $\sigma = N^{\frac{\gamma-1}{2}}$ can be obtained by reducing the term under the differential operator down to $1/y$ from Eq. (335)

$$\left(-\frac{1}{2y}\frac{d}{dy}\right)^{q-1} \frac{1}{y} = \left(-\frac{d}{dy^2}\right)^{q-1} \frac{1}{\sqrt{y^2}}, \tag{347}$$

using the substitution $v = y^2$ we find that

$$\begin{aligned} \left(-\frac{d}{dv}\right)^{q-1} \frac{1}{\sqrt{v}} &= \left(\frac{1}{2}\right)\frac{3}{2} \times ... \times \frac{2(q-2)+1}{2} v^{-\frac{2(q-2)+1}{2}} \\ &= 2^{-(q-1)}(2(q-2)+1)!! v^{-\frac{2(q-2)+1}{2}}, \end{aligned} \tag{348}$$

substituting the original variable y back into the expression yields

$$\left(-\frac{1}{2y}\frac{d}{dy}\right)^{q-1} \frac{1}{y} = 2^{-(q-1)}(2q-3)!! y^{-(2q-1)}. \tag{349}$$

We can then substitute this result into the expression for the moments of the eigenvectors

$$\begin{aligned} \hat{I}_q &= \frac{q}{N^{q-1}} 2^{-(q-1)}(2q-3)!! y^{-(2q-1)+1}\Big|_{y=\langle s\rangle} \\ &= \frac{q(2q-3)!!}{N^{q-1}2^{q-1}} \langle s\rangle^{-2(q-1)}, \end{aligned} \tag{350}$$

but we know that

$$\langle s\rangle \approx \sqrt{\frac{\pi}{2}}\frac{1}{\sigma} e^{-\frac{E^2}{2\sigma^2}} \to \sqrt{\frac{\pi}{2}}\frac{1}{\sigma}, \tag{351}$$

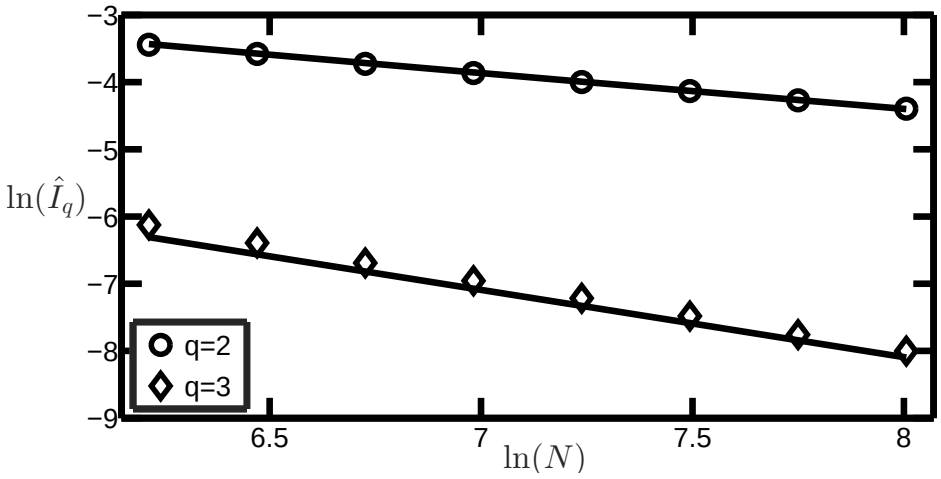

Figure 8: Numerical simulation for the Rosenzweig-Porter model for $\gamma = 1.5$ and N ranging from 500 to 3000. The symbols represent the numerical results and the solid lines represent the analytical prediction.

since $\sigma \to \infty$.

Substituting this into the moments of the eigenvectors, we receive

$$\hat{I}_q = \frac{q(2q-3)!!}{N^{q-1}2^{q-1}} \left(\frac{2}{\pi}\sigma^2\right)^{q-1} = \frac{q(2q-3)!!}{\pi^{q-1}} N^{-(q-1)} N^{(\gamma-1)(q-1)}, \tag{352}$$

hence

$$\hat{I}_q \approx \frac{q(2q-3)!!}{\pi^{q-1}} N^{(\gamma-2)(q-1)}, \tag{353}$$

which confirms, again, the fractal dimension $D_q = 2 - \gamma$ for $q > 1/2$.

Numerical simulations were also carried out under this case and the results show an excellent agreement between our analytical result against the direct diagonalisation approach as shown in fig. 8.

We also studied the asymptotic behaviour of the expressions in the limit $\sigma \to 0$ where we recover the GUE result $\hat{I}_q \approx \hat{I}_q^{GUE}$ but as the steps are similar to those shown previously, we will not detail the steps in this case.

3.5 Conclusion

We studied a total of three different cases where the GUE matrix is perturbed by a diagonal deterministic matrix, then we considered the case where the

diagonal matrix was Gaussian distributed with constant variance, and finally the diagonal matrix is Gaussian distributed with variance $\sigma^2 = N^{\gamma-1}$. By employing the supersymmetry technique we found non-perturbative analytical results for the density of states and the moments of the eigenvectors which are valid in the limit $N \to \infty$. In the first two cases the eigenvectors remain extended even at arbitrary large but fixed σ. In the third case where we consider an N-dependent variance, the eigenvectors can become completely localised, extended or critical depending on the parameter γ. We found that in the region $1 < \gamma < 2$, the eigenvectors are non-ergodic characterised by non-trivial fractal dimensions coinciding with the results in Ref. [22].

For $\gamma > 2$, the moments of the eigenvectors become divergent, signalling a breakdown in our approach so the problem must be tackled perturbatively which is shown in the next section. We consider the case $D \gg \tilde{H}$ which is where the σ-model breaks down, therefore we take a perturbative approach to the problem to derive the distribution function for this model.

3.6 The distribution function of the eigenvector components for almost diagonal random matrices

As $D \gg \tilde{H}$ the σ-model breaks down, this means that the results from the previous section are no longer valid. In order to remedy this we follow the approach developed in the context of so-called almost diagonal random matrices in Ref. [9, 50].

According to Ref. [9], one has to look at the sub-matrices of the original system and calculate the distribution function for each of the sub-matrices:

$$H(n,m) = \begin{pmatrix} H_{nn} & H_{nm} \\ H_{mn} & H_{mm} \end{pmatrix}, \tag{354}$$

where H_{nn} and H_{mm} are on the main diagonal of the original matrix. We first introduce the notation $E_1 = H_{nn}$, $E_2 = H_{mm}$ and $h = H_{nm}$ for convenience

which will be helpful later. The eigenvalues λ of $H(n,m)$ satisfy the equation

$$\lambda^2 - \lambda(E_1 + E_2) + E_1 E_2 = 0, \tag{355}$$

solving this, we obtain

$$\lambda_{1,2} = \frac{1}{2}(E_1 + E_2 \pm \sqrt{(E_1 - E_2)^2 + 4|h|^2}), \tag{356}$$

although unclear at the moment, this will be useful in the averaging of the density of states later.

The definition of the distribution function of the eigenvector components is the starting point to begin our calculations

$$\mathcal{P}(x) = \frac{1}{2\rho(0)}\bigg(\big\langle \delta(x - |\Psi_1(1)|^2)\delta(\lambda_1)\big\rangle + \big\langle \delta(x - |\Psi_2(1)|^2)\delta(\lambda_2)\big\rangle \bigg), \tag{357}$$

where λ_1 and λ_2 are eigenvalues corresponding to eigenvectors Ψ_1 and Ψ_2 of the sub-matrices, respectively, and our focus is on calculating the distribution function for the first component of the eigenvector corresponding to the zero eigenvalue. The form of the distribution function above comes from the definition of the distribution function, where we only consider the first element of the eigenvectors $\Psi_1(1)$ and $\Psi_2(1)$ corresponding to the zero eigenvalue of λ_1 and λ_2.

In order to calculate the distribution function, we must first calculate the density of states at $E = 0$:

$$\rho(0) = \frac{1}{2}\iint_{-\infty}^{\infty} \frac{dE_1 dE_2}{2\pi} \sum_{i=1}^{2} e^{-\frac{E_1^2 + E_2^2}{2}} \delta(\lambda_i), \tag{358}$$

by averaging over E_1 and E_2.

From Eq. (356), we can deduce that $\lambda_1 = 0$ if $E_1 + E_2 \leq 0$ and $E_1 E_2 = |h|^2$, while $\lambda_2 = 0$ if $E_1 + E_2 \geq 0$ and $E_1 E_2 = |h|^2$. This means that we must consider two separate domains in the integral: $E_1 + E_2 \leq 0$ and $E_1 + E_2 \geq 0$.

If we first consider the domain $E_1 + E_2 \leq 0$, $|h|^2 = E_1 E_2$, $\delta(\lambda_2)$ vanishes so

we only have to consider $\delta(\lambda_1)$ in the integral. By applying a property of the δ-function, we are able to change the argument to a more convenient form. In general the δ-function satisfies

$$\delta(x)dx = \delta(F(x))dF(x), \tag{359}$$

applying this to our case, we receive

$$\delta(\lambda_1)d\lambda_1 = \delta\left(E_1 - \frac{|h|^2}{E_2}\right)dE_1, \tag{360}$$

therefore the integral (358) is equivalent to

$$\rho(0) = \frac{1}{2}\iint_{-\infty}^{\infty} \frac{dE_1 dE_2}{2\pi} e^{-\frac{E_1^2+E_2^2}{2}} \delta\left(E_1 - \frac{|h|^2}{E_2}\right)\frac{dE_1}{d\lambda_1}. \tag{361}$$

In order to calculate $\frac{dE_1}{d\lambda_1}$, we refer back to Eq. (355) to differentiate it implicitly

$$2\lambda_1\frac{\lambda_1}{dE_1} - \lambda - E_1\frac{d\lambda_1}{dE_1} + E_2 - E_2\frac{d\lambda_1}{dE_1} = 0, \tag{362}$$

re-arranging for $\frac{d\lambda_1}{dE_1}$, we arrive at

$$\frac{d\lambda_1}{dE_1} = \frac{\lambda_1 - E_2}{2\lambda_1 - E_1 - E_2}, \tag{363}$$

inverting this gives the derivative as required.

Hence, the integral can be expressed as

$$\rho(0) = \frac{1}{2}\iint_{-\infty}^{\infty} \frac{dE_1 dE_2}{2\pi} e^{-\frac{E_1^2+E_2^2}{2}} \delta\left(E_1 - \frac{|h|^2}{E_2}\right)\frac{E_1 + E_2}{E_2}, \tag{364}$$

integrating over E_1 is simple due to the δ-function, once completed we are left with

$$\rho(0) = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dE_2}{2\pi} e^{-\frac{|h|^4/E_2^2+E_2^2}{2}} \frac{\frac{|h|^2}{E_2} + E_2}{E_2} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dE_2}{2\pi} e^{-\frac{|h|^4+E_2^4}{2E_2^2}} \frac{|h|^2 + E_2^2}{E_2^2}. \tag{365}$$

To integrate over E_2, we introduce a new variable $E_2 = x|h|$:

$$\rho(0) = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{2\pi}|h| e^{-\frac{|h|^2/x^2 + x^2|h|^2}{2}} \frac{1+x^2}{x^2}, \tag{366}$$

an averaging over h is performed before proceeding with the integration over the new variable $E_2 = x|h|$

$$\rho(0) = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{2\pi}\frac{1+x^2}{x^2}\int_0^{2\pi} d\phi \int_0^{\infty} d|h||h|^2 e^{-\frac{|h|^2/x^2 + x^2|h|^2}{2}} \frac{e^{-|h|^2/\left\langle |h|^2 \right\rangle}}{\pi \left\langle |h|^2 \right\rangle}, \tag{367}$$

where ϕ is the argument of h.

The integration over ϕ is simple as the integrand is only dependent on $|h|$, which gives a factor of 2π, after some algebraic manipulations we simplify the integral into our desired form

$$\rho(0) = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{\pi \left\langle |h|^2 \right\rangle}\frac{1+x^2}{x^2}\int_0^{\infty} d|h||h|^2 e^{-\frac{|h|^2\left(\frac{1}{x^2} + x^2 - \frac{2}{\left\langle |h|^2 \right\rangle}\right)}{2}}, \tag{368}$$

this allows one to use the result

$$\int_{-\infty}^{\infty} dy y^2 e^{-\frac{1}{2}ay^2} = \left(\frac{2\pi}{a}\right)^{1/2}\frac{1}{a}. \tag{369}$$

Since the integrand is even in $|h|$, one can extend the limit from $-\infty$ to ∞ by halving the integral

$$\rho(0) = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{2\pi \left\langle |h|^2 \right\rangle}\frac{1+x^2}{x^2}\int_{-\infty}^{\infty} d|h||h|^2 e^{-\frac{|h|^2\left(\frac{1}{x^2} + x^2 + \frac{2}{\left\langle |h|^2 \right\rangle}\right)}{2}}, \tag{370}$$

using Eq. (369) to evaluate the integral over $|h|$, resulting in

$$\rho(0) = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{2\pi \left\langle |h|^2 \right\rangle}\frac{1+x^2}{x^2}\left(\frac{2\pi}{\frac{1}{x^2} + x^2 + \frac{2}{\left\langle |h|^2 \right\rangle}}\right)^{1/2}\frac{1}{\frac{1}{x^2} + x^2 + \frac{2}{\left\langle |h|^2 \right\rangle}}, \tag{371}$$

which simplifies further to

$$\rho(0) = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}\left\langle |h|^2 \right\rangle} \frac{1+x^2}{x^2} \left(\frac{\left\langle |h|^2 \right\rangle}{2}\right)^{3/2} \left(1 + \frac{\left\langle |h|^2 \right\rangle}{2}\left(x^{-2}+x^2\right)\right)^{-3/2} \tag{372}$$

$$= \frac{\left\langle |h| \right\rangle^{1/2}}{8\sqrt{\pi}} \int dx \frac{1+x^2}{x^2} \left(1 + \frac{\left\langle |h|^2 \right\rangle}{2}\left(x^{-2}+x^2\right)\right)^{-3/2} \tag{373}$$

$$= \frac{\alpha}{4\sqrt{2\pi}} \int dx \frac{1+x^2}{x^2} \left(1 + \alpha^2\left(x^{-2}+x^2\right)\right)^{-3/2}, \tag{374}$$

where we set $\alpha = \sqrt{\left\langle |h|^2 \right\rangle / 2}$.

Computing this integral, we receive

$$\rho(0) = \frac{\alpha}{4\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \frac{1+x^2}{x^2} \left(1 + \alpha\left(x^{-2}+x^2\right)\right)^{-3/2} \tag{375}$$

$$= \frac{\alpha}{4\sqrt{2\pi}} \times \frac{4}{\alpha(1+2\alpha^2)} = \frac{1}{\sqrt{2\pi}(1+2\alpha^2)}. \tag{376}$$

Now that we have an expression for $\rho(0)$ for a general α we can proceed to calculate the distribution function. Eq. (357) shows that the distribution function should be of the form:

$$\left\langle \mathcal{P}(x) \right\rangle_E = \sqrt{\frac{\pi}{2}}(1+2\alpha^2) \iint_{-\infty}^{\infty} \frac{dE_1 dE_2}{2\pi} \sum_{i=1}^{2} e^{-\frac{E_1^2+E_2^2}{2}} \delta(x - |\Psi_i(1)|^2)\delta(\lambda_i). \tag{377}$$

The first element of the eigenvectors for λ_1 and λ_2 can be calculated by considering the eigenvalue problem

$$\begin{pmatrix} E_1 - \lambda & h \\ h^* & E_2 - \lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0. \tag{378}$$

If we focus on the first equation

$$(E_1 - \lambda)x + hy = 0, \tag{379}$$

we can see that this is zero when $y = \lambda - E_1$ and $x = h$, hence the eigenvector

is found to be

$$\Psi_{1,2} = \begin{pmatrix} h \\ \lambda_{1,2} - E_1 \end{pmatrix}, \tag{380}$$

where the absolute value of the first component of the normalised eigenvector is

$$|\Psi_{1,2}(1)| = \frac{|h|}{\sqrt{|h|^2 + (\lambda_{1,2} - E_1)^2}}. \tag{381}$$

Similarly to the case for the density of states, two separate domains must be considered but only one δ-function survives in each of the domains. We take the same domain as previously: $E_1 + E_2 \leq 0$, $E_1 E_2 = |h|^2$. In this domain only $\delta(\lambda_1)$ survives, hence,

$$\langle \mathcal{P}(x) \rangle_E = \sqrt{\frac{\pi}{2}}(1+2\alpha^2) \iint_{-\infty}^{\infty} \frac{dE_1 dE_2}{2\pi} e^{-\frac{E_1^2+E_2^2}{2}} \delta\left(x - \frac{|h|^2}{|h|^2 + (\lambda_1 - E_1)^2}\right) \delta(\lambda_1), \tag{382}$$

using the property of the δ-function, Eq. (359), we can change the argument

$$\begin{aligned} \langle \mathcal{P}(x) \rangle_E = \sqrt{\frac{\pi}{2}}(1+2\alpha^2) \iint_{-\infty}^{\infty} \frac{dE_1 dE_2}{2\pi} e^{-\frac{E_1^2+E_2^2}{2}} \delta\left(x - \frac{|h|^2}{|h|^2 + E_1^2}\right) \\ \times \delta\left(E_1 - \frac{|h|^2}{E_2}\right) \frac{E_1 + E_2}{E_2}, \end{aligned} \tag{383}$$

since $\lambda_1 = 0$ in this domain.

Integrating over E_1, we obtain

$$\begin{aligned} \langle \mathcal{P}(x) \rangle_E = \sqrt{\frac{\pi}{2}}(1+2\alpha^2) \int_{-\infty}^{\infty} \frac{dE_2}{2\pi} e^{-\frac{|h|^4/E_2^2+E_2^2}{2}} \delta\left(x - \frac{|h|^2}{|h|^2 + |h|^4/E_2^2}\right) \\ \times \frac{|h|^2/E_2 + E_2}{E_2}, \end{aligned} \tag{384}$$

which can be expressed as

$$\langle \mathcal{P}(x) \rangle_E = \sqrt{\frac{\pi}{2}}(1+2\alpha^2) \int_{-\infty}^{\infty} \frac{dE_2}{2\pi} e^{-\frac{|h|^4+E_2^4}{2E_2^2}} \delta\left(x - \frac{E_2^2}{E_2^2 + |h|^2}\right) \frac{|h|^2 + E_2^2}{E_2^2}. \tag{385}$$

To complete the integration over E_2, we use the substitution

$$y = \frac{E_2^2}{E_2^2 + |h|^2} \implies E_2 = \pm|h|\sqrt{\frac{y}{1-y}}, \tag{386}$$

taking the positive square root and differentiating it yields

$$\frac{dE_2}{dy} = \frac{1}{2}|h|y^{-1/2}(1-y)^{-1/2} + \frac{1}{2}|h|y^{1/2}(1-y)^{-3/2}. \tag{387}$$

As we are taking the positive values of E_2, the integral needs to be expressed in a way so that we are only integrating over positive values of E_2. Therefore, using the fact that the integrand is an even function, the integral is equal to

$$\langle \mathcal{P}(x) \rangle_E = \frac{1}{\sqrt{2\pi}}(1+2\alpha^2)\int_0^\infty dE_2 e^{-\frac{|h|^4+E_2^4}{2E_2^2}} \delta\left(x - \frac{E_2^2}{E_2^2+|h|^2}\right)\frac{|h|^2+E_2^2}{E_2^2}, \tag{388}$$

by changing the limit from $-\infty$ to 0 and multiplying the integral by 2.

Then by changing the variable of integration, the integral is equivalent to

$$\begin{aligned}\langle \mathcal{P}(x) \rangle_E = \frac{1}{2\sqrt{2\pi}}(1+2\alpha^2)\int_0^1 dy e^{-\frac{|h|^2\left(\frac{1-y}{y}+\frac{y}{1-y}\right)}{2}} \delta(x-y)\frac{1}{y} \\ \times|h|(y^{-1/2}(1-y)^{-1/2} + y^{1/2}(1-y)^{-3/2}),\end{aligned} \tag{389}$$

integrating over y is now very simple and is equal to

$$\begin{aligned}\langle \mathcal{P}(x) \rangle_E = \frac{1}{2\sqrt{2\pi}}(1+2\alpha^2)e^{-\frac{|h|^2\left(\frac{1-x}{x}+\frac{x}{1-x}\right)}{2}}\frac{1}{x}|h| \\ \times(x^{-1/2}(1-x)^{-1/2} + x^{1/2}(1-x)^{-3/2}),\end{aligned} \tag{390}$$

where $0 < x < 1$.

Finally, we can now average over $|h|$, as the integrand is only dependent on $|h|$ which means the integration over the argument of h produces a factor

of 2π,

$$\langle \mathcal{P}(x) \rangle_{E,h} = \frac{1}{2\sqrt{2\pi}}(1+2\alpha^2)\frac{2\pi}{x}(x^{-1/2}(1-x)^{-1/2} + x^{1/2}(1-x)^{-3/2}) \times \int_0^\infty d|h||h|^2 e^{-\frac{|h|^2\left(\frac{1-x}{x}+\frac{x}{1-x}\right)}{2}} \frac{e^{-|h|^2/\langle |h|^2 \rangle}}{\pi \langle |h|^2 \rangle}, \tag{391}$$

the integral can be expressed in a much more convenient form, as given below

$$\langle \mathcal{P}(x) \rangle_{E,h} = \frac{1}{\sqrt{2\pi}\langle |h|^2 \rangle}(1+2\alpha^2)\frac{1}{x}(x^{-1/2}(1-x)^{-1/2} + x^{1/2}(1-x)^{-3/2}) \times \int_0^\infty d|h||h|^2 e^{-\frac{|h|^2\left(\frac{1-x}{x}+\frac{x}{1-x}+\frac{2}{\langle |h|^2 \rangle}\right)}{2}}. \tag{392}$$

Now that it is in the desired form, we are able to avail the result from Eq. (369) to arrive at

$$\langle \mathcal{P}(x) \rangle_{E,h} = \frac{1}{\sqrt{2\pi}\langle |h|^2 \rangle}(1+2\alpha^2)\frac{1}{x}(x^{-1/2}(1-x)^{-1/2} + x^{1/2}(1-x)^{-3/2}) \times \left(\frac{2\pi}{\frac{1-x}{x}+\frac{x}{1-x}+\frac{2}{\langle |h|^2 \rangle}}\right)^{1/2} \frac{1}{\frac{1-x}{x}+\frac{x}{1-x}+\frac{2}{\langle |h|^2 \rangle}}, \tag{393}$$

which simplifies to

$$\langle \mathcal{P}(x) \rangle_{E,h} = \frac{1}{\langle |h|^2 \rangle}(1+2\alpha^2)\frac{1}{x}(x^{-1/2}(1-x)^{-1/2} + x^{1/2}(1-x)^{-3/2}) \times \left(\frac{\langle |h|^2 \rangle}{2}\right)^{3/2} \left(1 + \frac{\langle |h|^2 \rangle}{2}\left(\frac{1-x}{x}+\frac{x}{1-x}\right)\right)^{-3/2}, \tag{394}$$

this can be reduced further to

$$\langle \mathcal{P}(x) \rangle_{E,h} = \frac{\alpha}{2}(1+2\alpha^2)\frac{x^{-1/2}(1-x)^{-1/2}(x^{-1}+(1-x)^{-1})}{\left(1+\alpha^2\left(\frac{1-x}{x}+\frac{x}{1-x}\right)\right)^{3/2}}, \tag{395}$$

where $\alpha = \sqrt{\frac{\langle |h|^2 \rangle}{2}}$ and $0 < x < 1$.

This is the main result of this section and to corroborate this result we shall derive the expression for the moments of the eigenvectors as shown in Ref. [9] by using the relation between the moments of the eigenvectors and

the distribution function, Eq. (102). The moments of the eigenvectors in this case are therefore given by

$$I_q = \frac{\alpha}{2}\int_0^1 dx \frac{x^{q-1/2}(1-x)^{-1/2}(x^{-1}+(1-x)^{-1})}{\left(1+\alpha^2\left(\frac{1-x}{x}+\frac{x}{1-x}\right)\right)^{3/2}}, \quad \alpha \ll 1, \tag{396}$$

by taking the leading order term in the distribution function.

To proceed, we will change the variables of integration by the substitution $y^2 = \frac{x}{1-x}$, this converts the integration variables to the one used in Ref. [9].

Since $y^2 = \frac{x}{1-x}$ then $2ydy = \frac{dx}{(1-x)^2}$. We can substitute this into the integral once we have it in the following form

$$I_q = \frac{\alpha}{2}\int_0^1 \frac{dx}{(1-x)^2}(1-x)^2 \frac{x^{q-1/2}(1-x)^{-1/2}(x^{-1}+(1-x)^{-1})}{\left(1+\alpha^2\left(\frac{1-x}{x}+\frac{x}{1-x}\right)\right)^{3/2}}, \tag{397}$$

where, after performing algebraic manipulations, we arrive at

$$I_q = \frac{\alpha}{2}\int_0^1 \frac{dx}{(1-x)^2} \frac{x^q\left(\left(\frac{x}{1-x}\right)^{-3/2}+\left(\frac{1-x}{x}\right)^{-1/2}\right)}{\left(1+\alpha^2\left(\frac{1-x}{x}+\frac{x}{1-x}\right)\right)^{3/2}}, \tag{398}$$

substituting in the new variable, the integral becomes

$$I_q = \frac{\alpha}{2}\int_0^\infty dy 2y \frac{(1+y^{-2})^{-q}\left(\left(y^2\right)^{-3/2}+\left(y^{-2}\right)^{-1/2}\right)}{\left(1+\alpha^2\left(y^{-2}+y^2\right)\right)^{3/2}}, \tag{399}$$

which simplifies to

$$I_q = \alpha\int_0^\infty dy \frac{(1+y^{-2})^{-q}(1+y^{-2})}{\left(1+\alpha^2\left(y^{-2}+y^2\right)\right)^{3/2}} = \alpha\int_0^\infty dy \frac{(1+y^{-2})^{1-q}}{\left(1+\alpha^2\left(y^{-2}+y^2\right)\right)^{3/2}}. \tag{400}$$

The integrand is an even function, this means that we can extend the range of integration to $-\infty$ and multiply by a factor of $1/2$ to obtain

$$I_q = \frac{\alpha}{2}\int_{-\infty}^\infty dy \frac{(1+y^{-2})^{1-q}}{\left(1+\alpha^2\left(y^{-2}+y^2\right)\right)^{3/2}}, \tag{401}$$

which is the exact integral given in Ref. [9] and according to the paper the

integration yields

$$\frac{\alpha}{2}\int_{-\infty}^{\infty} dy \frac{(1+y^{-2})^{1-q}}{(1+\alpha^2\,(y^{-2}+y^2))^{3/2}} = 1 - \sqrt{\pi}\frac{\Gamma(q-1/2)}{\Gamma(q-1)}\alpha + O(\alpha^2), \qquad (402)$$

in the limit $\alpha \to 0$.

We have now recovered the moments of the eigenvectors from the distribution function

$$\langle I_q \rangle = 1 - \sqrt{\frac{\pi}{2}}\frac{\Gamma(q-1/2)}{\Gamma(q-1)}\frac{1}{N}\sum_{n\neq m}^{N}\sqrt{\langle |H_{nm}| \rangle}, \qquad (403)$$

where $h = H_{nm}$ and we had set, according to the notation in the paper, $W = 1$.

3.7 Conclusion

In this section we focused on calculating the distribution function for the case $D \gg \tilde{H}$, where the σ-model breaks down and therefore we had to tackle the problem perturbatively. The idea behind the method was that we can break the matrices down to 2×2 sub-matrices and calculate the distribution function for each of those sub-matrices and then by averaging over all the elements we have the distribution function of the original matrix.

One of the original intentions were to calculate the distribution function so that we can use the relation between the distribution function and the moments of the eigenvectors to derive the moments of the eigenvectors. As it turns out this problem was not as simple as it seems, due to the form of the distribution function there was no easy way to derive the moments of the eigenvectors so there was nothing further we could do from this point.

Our efforts, however, have not been in vain as we wanted to calculate the distribution function, not only for the moments of the eigenvectors, but because the distribution function contains more information than the moments of the eigenvectors, since it shows exactly how the elements of the eigenvectors are distributed. It is worth noting that restoring the distribution function

from the moments of the eigenvectors is usually a complex task, but finding the moments from the distribution function is quite simple in most cases.

4 Structured random matrices: generalisations of models I and II

The main idea of RMT is that many features of complex systems are universal and therefore will not depend on system specific information, this means that the Hamiltonian can be replaced by a random matrix which shares the same symmetries as the Hamiltonian. An example of such a random matrix is a GUE matrix, where the only restriction is the Hermiticity property.

RMT has shown to be a great success in studying statistical properties of complex quantum systems, however there has been interest in new ensembles of random matrices where we retain some of the structural information of the original system.

In this section we study one such matrix model, defined as

$$H = W\tilde{H}W + D, \tag{404}$$

where $\tilde{H}$ is an $N \times N$ matrix from GUE and W, D are deterministic diagonal matrices with elements w_i and d_i, $i = 1, ..., N$; respectively. Random matrices of this form have many applications and appears naturally in neural networks, Ref. [33], wireless communication, Ref. [51], signal processing, Ref. [12], and vibration analysis, Ref. [13]. The spectral properties of random matrices have been well studied in Ref. [33, 34], however much less is known about the eigenvectors, Ref. [52].

In this section we generalise the results derived previously from models I and II. The result is a non-perturbative, asymptotic expression for the moments of the eigenvectors for our model H, valid for any w_i and d_i.

4.1 The density of states for the generalised model

In this section we will be deriving the density of states but as the steps have already been shown previously in detail, we will only show the main steps in the calculation. Since we are working on the density of states we only need to consider the ratio of two determinants

$$Z(E,j) = \frac{\det \mathrm{i}(E^- - W\tilde{H}W - D)}{\det \mathrm{i}(E^- - W\tilde{H}W - D - j)}, \tag{405}$$

where $E^- = E - \mathrm{i}\epsilon$.

The ratio of determinants can be represented in its superintegral form

$$\begin{aligned} Z(E,j) &= \int \frac{d^2 r_i(-)}{\pi} d\eta_i^*(-) d\eta(-) \\ &\exp \mathrm{i}\Big[r^\dagger(-)(W\tilde{H}W + D - E + \mathrm{i}\epsilon + j) r(-) \\ &+ \chi^\dagger(-)(W\tilde{H}W + D - E + \mathrm{i}\epsilon)\chi(-) \Big], \end{aligned} \tag{406}$$

which can be expressed in a compact form

$$Z(E,j) = \int d\Phi d\Phi^* \exp\left\{ \mathrm{i} \sum_{\alpha=1}^{2} \Phi_\alpha^\dagger [W\tilde{H}W + D - E + \mathrm{i}\epsilon + \tilde{J}_\alpha] \Phi_\alpha \right\}, \tag{407}$$

the notation is the same as previously in calculating the density of states, but as a reminder: $\tilde{J} = \mathrm{diag}(j,0)$, $\Phi = (r(-), \chi(-))^T = (\Phi_1, \Phi_2)^T$ and $d\Phi d\Phi^*$ consists of those in the preceding integral.

As we have already averaged over the GUE previously, we can just apply the previous results directly:

$$\begin{aligned} \left\langle \exp\left(\mathrm{i} \sum_{\alpha}^{2} \Phi_\alpha^\dagger W\tilde{H}W\Phi_\alpha \right) \right\rangle &= \exp\left(-\frac{1}{2} \left\langle \left(\sum_{\alpha}^{2} \Phi_\alpha^\dagger W\tilde{H}W\Phi_\alpha \right)^2 \right\rangle \right) \\ &= \exp\left(-\frac{N}{2} \mathrm{Str}\tilde{Q}^2 \right), \end{aligned} \tag{408}$$

where $\tilde{Q}_{\alpha\beta} = \frac{1}{N}\sum_m^N w_m^2 |\Phi_\alpha(m)\rangle\langle\Phi_\beta(m)|$. Although we are now working with

2×2 matrices instead of 4×4, it will not change the average with the exception of the $\hat{L}$ matrix in the structure as it is the identity in this case.

Averaging over the generating function using the above result gives the following

$$\langle Z(E,j)\rangle = \int d\Phi d\Phi^* e^{-\frac{N}{2}\mathrm{Str}\tilde{Q}^2} \exp\left\{ \mathrm{i}\sum_{\alpha=1}^{2} \Phi_\alpha^\dagger [D - E^- + \tilde{J}_\alpha]\Phi_\alpha \right\}, \tag{409}$$

the Hubbard-Stratonovich transformation must be performed to decouple Φ and reduce the integrand from a quartic into a quadratic, Eq. (53):

$$\begin{aligned} \langle Z(E,j)\rangle = &\int dQ e^{-\frac{N}{2}\mathrm{Str}Q^2} \int d\Phi d\Phi^* \\ &\exp\left\{ -\mathrm{i}\sum_{i=1}^{N} \Phi^\dagger(i)[E^- - d_i - w_i^2 Q - \tilde{J}]\Phi(i) \right\}. \end{aligned} \tag{410}$$

We can then integrate over Φ as it is equal to the superdeterminant

$$\langle Z(E,j)\rangle = \int dQ e^{-\frac{N}{2}\mathrm{Str}Q^2} \prod_i^N \mathrm{Sdet}^{-1}(E^- - d_i - w_i^2 Q - \tilde{J}), \tag{411}$$

then by using the identity

$$x \equiv e^{\ln x}, \tag{412}$$

we are able to express the integrand into a much more convenient form

$$\langle Z(E,j)\rangle = \int dQ \exp\left(-\frac{N}{2}\mathrm{Str}Q^2 - \sum_i^N \ln \mathrm{Sdet}(E - \mathrm{i}\epsilon - d_i - w_i^2 Q - \tilde{J}) \right). \tag{413}$$

Since we are taking the limit $\epsilon \to 0$ and $j \to 0$, it is beneficial to expand the logarithm in orders of ϵ and $\tilde{J}$. Doing so, we receive

$$\langle Z(E,j)\rangle = \int dQ \exp\left(-N\mathcal{L}(Q) - \sum_i^N \mathrm{Str}(E - d_i - w_i^2 Q)^{-1}(-\mathrm{i}\epsilon - \tilde{J}) \right). \tag{414}$$

where $\mathcal{L}(Q) = \frac{1}{2}\mathrm{Str}Q^2 + \frac{1}{N}\sum_i^N \mathrm{Str}\ln(E - d_i - w_i^2 Q)$.

Differentiating with respect to j and setting $j \to 0$, we receive the trace

of the averaged Green's function

$$\begin{aligned}\mathrm{Tr}\left\langle G(E^-)\right\rangle = \int dQ \sum_i^N \mathrm{Str}\left(\frac{\tilde{J}}{j}\frac{1}{E-d_i-w_i^2Q}\right)\\ \exp\left\{-N\mathcal{L}(Q)+\mathrm{i}\epsilon\sum_i^N \mathrm{Str}(E-d_i-w_i^2Q)^{-1}\right\},\end{aligned} \tag{415}$$

where $\mathcal{L}(Q) = \frac{1}{2}\mathrm{Str}Q^2 + \frac{1}{N}\sum_i^N \mathrm{Str}\ln(E^- - d_i - w_i^2Q)$.

Taking the limit $\epsilon \to 0^+$ and assuming $N \to \infty$, we can apply the saddle-point approximation to this integral and pull out the pre-exponential factor from the integral by evaluating it at the saddle-point

$$\lim_{\epsilon\to 0^+}\mathrm{Tr}\left\langle G(E^-)\right\rangle = \sum_i^N \mathrm{Str}\left(\frac{\tilde{J}}{j}\frac{1}{E-d_i-w_i^2Q_{s.p.}}\right)\int dQ e^{-N\mathcal{L}(Q)}, \tag{416}$$

but we know that

$$\int dQ e^{-N\mathcal{L}(Q)} = 1, \tag{417}$$

hence,

$$\lim_{\epsilon\to 0^+}\mathrm{Tr}\left\langle G(E^-)\right\rangle = \sum_i^N \mathrm{Str}\left(\frac{\tilde{J}}{j}\frac{1}{E-d_i-w_i^2Q_{s.p.}}\right). \tag{418}$$

To proceed we must solve the saddle-point equation

$$Q = \frac{1}{N}\sum_i^N \frac{w_i^2}{E-d_i-w_i^2Q}, \tag{419}$$

the solutions to this equation are parametrised by

$$Q_{s.p.} = t + \mathrm{i}s, \tag{420}$$

where $s \neq 0$ and t satisfy the simultaneous equations

$$t = \frac{1}{N}\sum_i^N \frac{w_i^2(E-d_i-w_i^2t)}{(E-d_i-w_i^2t)^2+w_i^4s^2}, \quad 1 = \frac{1}{N}\sum_i^N \frac{w_i^4}{(E-d_i-w_i^2t)^2+w_i^4s^2}. \tag{421}$$

Substituting the parametrisation into Eq. (418) yields

$$\begin{aligned}\lim_{\epsilon\to 0^+}\operatorname{Tr}\left\langle G(E^-)\right\rangle &= \sum_i^N \operatorname{Str}\left(\frac{\tilde{J}}{j}\frac{1}{E-d_i-w_i^2(t+\mathrm{i}s)}\right)\\ &= \sum_i^N\left(\frac{E-d_i-w_i^2t+\mathrm{i}w_i^2s}{(E-d_i-w_i^2t)^2+w_i^4s^2}\right),\end{aligned} \tag{422}$$

according to Eq. (12), we have to take the imaginary part of this expression and multiply it by a factor of $\frac{1}{\pi N}$ to receive the final form of the density of states, hence,

$$\begin{aligned}\rho(E) &= \frac{1}{\pi N}\operatorname{Im}\lim_{\epsilon\to 0^+}\operatorname{Tr}\left\langle G(E^-)\right\rangle\\ &= \frac{1}{\pi N}\operatorname{Im}\sum_i^N\frac{E-d_i-w_i^2t+\mathrm{i}w_i^2s}{(E-d_i-w_i^2t)^2+w_i^4s^2}\\ &= \frac{s}{\pi N}\sum_i^N\frac{w_i^2}{(E-d_i-w_i^2t)^2+w_i^4s^2}.\end{aligned} \tag{423}$$

This is the generalised density of states for the two previous models, in the next section we will derive the moments of the eigenvectors for this model.

4.2 The moments of the eigenvectors for model I and II

In this section we will generalise the moments of the eigenvectors for model I and II, but as the steps have already been shown in detail for the previous two models, only the main steps will be shown in this case.

We start off by representing this model as a product of two generating functions

$$\begin{aligned}Z(E,\hat{J}_1,\hat{J}_2) = (-1)^N&\frac{\det\mathrm{i}(E^- - W\tilde{H}W - D)}{\det\mathrm{i}(E^- - W\tilde{H}W - D + \hat{J}_1)}\\ \times&\frac{\det\mathrm{i}(E^+ - W\tilde{H}W - D)}{\det(-\mathrm{i})(E^+ - W\tilde{H}W - D + \hat{J}_2)},\end{aligned} \tag{424}$$

which can be expressed in its superintegral form

$$Z(E,\hat{J}_1,\hat{J}_2) = (-1)^N \int d\Phi^* d\Phi \times \exp\left\{ \mathrm{i}\sum_{\alpha=1}^{4} \Phi_\alpha^\dagger L_\alpha [W\tilde{H}W + D - E - \mathrm{i}(-1)^\alpha \epsilon - \tilde{J}_\alpha]\Phi_\alpha \right\}, \tag{425}$$

$L_1 = L_3 = L_4 = -L_2 = 1$, $\tilde{J}_\alpha = \mathrm{diag}(\hat{J}_1, \hat{J}_2, 0, 0)$ and
$\Phi = (r(-), r(+), \chi(-), \chi(+))^T = (\Phi_1, \Phi_2, \Phi_3, \Phi_4)^T$. Differentiating with respect to J_1 and J_2, m and l times, we receive the product of two Green's functions

$$\left(G^R_{nn}\right)^l \left(G^A_{nn}\right)^m = (-1)^N \int d\Phi^* d\Phi (r_n^*(-) r_n(-))^m (r_n^*(+) r_n(+))^l \times \exp\left\{ \mathrm{i}\sum_{\alpha=1}^{4} \Phi_\alpha^\dagger L_\alpha [W\tilde{H}W + D - E - \mathrm{i}(-1)^\alpha \epsilon]\Phi_\alpha \right\}. \tag{426}$$

Averaging over this expression is very similar to the steps taken in the previous cases, so we will not show the details in this case. Once the averaging has been completed, we receive

$$\left\langle \left(G^R_{nn}\right)^l \left(G^A_{nn}\right)^m \right\rangle = (-1)^N \int d\Phi^* d\Phi (r_n^*(-) r_n(-))^m (r_n^*(+) r_n(+))^l e^{-\frac{N}{2}\mathrm{Str}(\tilde{Q}\hat{L})^2} \exp\left\{ \sum_{i}^{N} \Phi^\dagger(i) [\mathrm{i}d_i\hat{L} - \mathrm{i}\hat{L}\hat{E}]\Phi(i) \right\} \tag{427}$$

where $\tilde{Q}_{\alpha\beta} = \frac{1}{N}\sum_m^N w_m^2 \left|\Phi_\alpha(m)\right\rangle \left\langle\Phi_\beta(m)\right|$.

This integrand is quartic in Φ, so we must apply the Hubbard-Stratonovich transformation to decouple the terms and reduce it down into its quadratic form

$$\left\langle \left(G^R_{nn}\right)^l \left(G^A_{nn}\right)^m \right\rangle = (-1)^{N+1} \int_{\mathcal{M}_4} dQ e^{-\frac{N}{2}\mathrm{Str}(QL)^2} \int d\Phi^* d\Phi (r_n^*(-) r_n(-))^m (r_n^*(+) r_n(+))^l \exp\left\{ \sum_{i}^{N} \Phi^\dagger(i) \left(\mathrm{i}w_i^2 \hat{L}Q\hat{L} + \mathrm{i}\hat{L}d_i - \mathrm{i}\hat{L}\hat{E} \right) \Phi(i) \right\}, \tag{428}$$

integrating over Φ can be computed by the application of Wick's theorem,

which has been presented in the previous chapters, resulting in

$$\left\langle \left(G^R_{nn}\right)\left(G^A_{nn}\right)^{q-1}\right\rangle = \int_{\mathcal{M}_4\hat{L}} dQ (g^{BB}_{aa})^{q-2}\left\{g^{BB}_{aa}g^{BB}_{rr} + (q-1)g^{BB}_{ar}g^{BB}_{ra}\right\} \times \exp\left\{-N\mathrm{Str}\left(\frac{1}{2}Q^2 + \frac{1}{N}\sum_i^N \ln(E - \mathrm{i}\epsilon\hat{\Lambda} - d_i - w_i^2 Q)\right)\right\}, \tag{429}$$

after setting $l = 1$ and $m = q - 1$ for simplification purposes as discussed in Section 2. The explicit expression for g^{BB} is given in Appendix E.

Since we are interested in the limit $\epsilon \to 0$, it is useful to expand the logarithm in orders of ϵ:

$$\left\langle \left(G^R_{nn}\right)\left(G^A_{nn}\right)^{q-1}\right\rangle = \int_{\mathcal{M}_4\hat{L}} dQ (g^{BB}_{aa})^{q-2}\left\{g^{BB}_{aa}g^{BB}_{rr} + (q-1)g^{BB}_{ar}g^{BB}_{ra}\right\} \times \exp\left\{-N\mathcal{L}(Q) + \mathrm{i}\epsilon\sum_i^N \mathrm{Str}(E - d_i - w_i^2 Q)^{-1}\hat{\Lambda}\right\}, \tag{430}$$

where $\mathcal{L}(Q) = \frac{1}{2}\mathrm{Str}Q^2 + \frac{1}{N}\sum_i^N \mathrm{Str}\ln(E - d_i - w_i^2 Q)$.

In the limit $N \to \infty$, the integral over Q is dominated by the saddle-points which satisfy the equation

$$Q = \frac{1}{N}\sum_i^N \frac{w_i^2}{E - d_i - w_i^2 Q}, \tag{431}$$

the solutions of the equation can be parametrised as given in Ref. [11]

$$Q_{s.p.} = t + \mathrm{i}s\hat{T}^{-1}\hat{\Lambda}\hat{T}, \tag{432}$$

where $s \neq 0$ and t satisfy the simultaneous equations

$$t = \frac{1}{N}\sum_i^N \frac{w_i^2(E - d_i - w_i^2 t)}{(E - d_i - w_i^2 t)^2 + w_i^4 s^2}, \quad 1 = \frac{1}{N}\sum_i^N \frac{w_i^4}{(E - d_i - w_i^2 t)^2 + w_i^4 s^2}. \tag{433}$$

The integral over Q can be computed by the saddle-point approximation and is dominated by the term involving $\mathcal{L}(Q)$, which is approximately equal to unity. We are left with an integration measure of the coset space spanned by

$\hat{T}$ as shown in Ref. [14]:

$$\left\langle \left(G^R_{nn}\right)\left(G^A_{nn}\right)^{q-1}\right\rangle = \int_{\mathcal{M}_4\hat{L}} d\mu(T)(g^{BB}_{aa})^{q-2}\left\{g^{BB}_{aa}g^{BB}_{rr} + (q-1)g^{BB}_{ar}g^{BB}_{ra}\right\} \times \exp\left\{\mathrm{i}\epsilon\sum_i^N \mathrm{Str}(E - d_i - w_i^2(t + \mathrm{i}s\hat{T}^{-1}\hat{\Lambda}\hat{T}))^{-1}\hat{\Lambda}\right\}, \tag{434}$$

applying Efetov's parametrisation will simplify the integrand greatly. Performing them, we receive

$$\left\langle \left(G^R_{nn}\right)\left(G^A_{nn}\right)^{q-1}\right\rangle = \int_{\mathcal{M}_4\hat{L}} d\mu(T)(g^{BB}_{aa})^{q-2}\left\{g^{BB}_{aa}g^{BB}_{rr} + (q-1)g^{BB}_{ar}g^{BB}_{ra}\right\} \times \exp\left\{-2\epsilon\pi N\rho(E)(\lambda_1 - \lambda_2)\right\}, \tag{435}$$

where

$$\rho(E) = \frac{s}{\pi N}\sum_i^N \frac{w_i^2}{(E - d_i - w_i^2 t)^2 + w_i^4 s^2}, \tag{436}$$

and the explicit expression for g^{BB} is given in Appendix E.

Once the integration over the Grassmannians has been completed, we arrive at

$$\begin{aligned}&\left\langle \left(G^R_{nn}\right)\left(G^A_{nn}\right)^{q-1}\right\rangle = \\ &(q-1)\int_1^\infty d\lambda_1 \int_{-1}^1 d\lambda_2 \frac{(E - d_n - w_n^2 t + \mathrm{i}sw_n^2\lambda_1)^{q-2}}{[(E - d_n - w_n^2 t)^2 + w_n^4 s^2]^q} e^{-2\epsilon\pi N\rho(E)(\lambda_1-\lambda_2)} \\ &s^2 w_n^4\left[1 + \frac{\lambda_1 + \lambda_2}{\lambda_1 - \lambda_2} + \frac{\mathrm{i}sw_n^2(q-2)(\lambda_1^2 - 1)}{(\lambda_1 - \lambda_2)(E - d_n - w_n^2 t + \mathrm{i}sw_n^2\lambda_1)}\right],\end{aligned} \tag{437}$$

in order to proceed, a substitution must be made, as described previously, $z = \epsilon\lambda_1$,

$$\begin{aligned}&\left\langle \left(G^R_{nn}\right)\left(G^A_{nn}\right)^{q-1}\right\rangle = \\ &(q-1)\int_1^\infty \frac{dz}{\epsilon} \int_{-1}^1 d\lambda_2 \frac{(E - d_n - w_n^2 t + \mathrm{i}sw_n^2\frac{z}{\epsilon})^{q-2}}{[(E - d_n - w_n^2 t)^2 + w_n^4 s^2]^q} e^{-2\epsilon\pi N\rho(E)(\frac{z}{\epsilon}-\lambda_2)} \\ &s^2 w_n^4\left[1 + \frac{\frac{z}{\epsilon} + \lambda_2}{\frac{z}{\epsilon} - \lambda_2} + \frac{\mathrm{i}sw_n^2(q-2)(\left(\frac{z}{\epsilon}\right)^2 - 1)}{(\frac{z}{\epsilon} - \lambda_2)(E - d_n - w_n^2 t + \mathrm{i}sw_n^2\frac{z}{\epsilon})}\right].\end{aligned} \tag{438}$$

This integral simplifies down to a standard Gamma function integral as shown

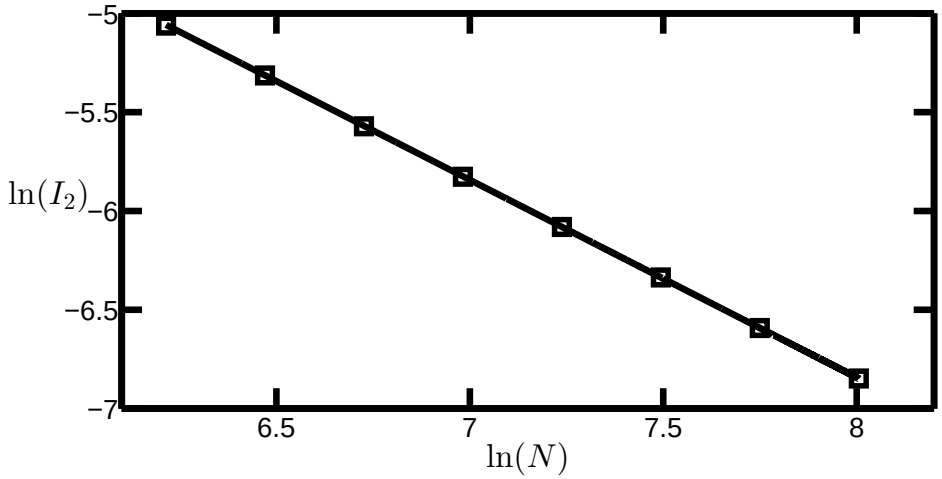

Figure 9: The symbols represent the numerical simulation and the solid line is our analytical result. The numerical simulation is performed over a total of 1000 realizations for $q = 2$ with $w_i = d_i = N/i$.

previously. Once it is completed, we can take the limit $\epsilon \to 0$, hence

$$\lim_{\epsilon\to 0^+} (2\epsilon)^{q-1} \left\langle \left(G^R_{nn}\right) \left(G^A_{nn}\right)^{q-1} \right\rangle = \frac{2^{q-1}}{\pi\rho(E)N)^q} \frac{(sw_n^2)^q}{[(E-d_n-w_n^2 t)^2 + w_n^4 s^2]^q} \\ q(q-1)\int_0^\infty \frac{dx}{2\pi N\rho(E)} \left(\frac{x}{2\pi N\rho(E)}\right)^{q-2} e^{-x}, \tag{439}$$

where $x = 2\pi N\rho(E)z$.

Performing the integration over x, we arrive at the final form for the generalised moments of the eigenvectors

$$I_q(n) = \frac{1}{(\pi\rho(E)N)^q} \left[\frac{sw_n^2}{(E-d_n-w_n^2 t)^2 + w_n^4 s^2}\right]^q \Gamma(q+1). \tag{440}$$

This is our result for the generalised case of the two previous models. We performed various numerical simulations to verify the validity of our analytical expression with the direct diagonalisation, which were in agreement as shown in fig. 9.

4.3 Conclusion

We generalised models I and II where we studied for the moments of the eigenvectors and the density of states. This result enables one to not only determine qualitatively the nature of the eigenvectors, such as the degree of ergodicity, but it reveals how individual elements of the eigenvectors are affected by W and D. As shown in the numerical simulations our analytical expressions are in excellent agreement with the direct diagonalisation. From the result it follows that the nature of the eigenvectors remain qualitative the same as the GUE case for generic choices of w_i and d_i, which means that the extended nature of the GUE are very robust under a variety of deformations. If d_i and w_i become N-dependent however then the eigenvectors can diverge from their extended nature and cross into the critical and localised states.

After completion of this work we became aware of Ref. [53] and Ref. [54] in which the Green's function and the eigenvectors of the Rosenzweig-Porter model are investigated by alternative methods.

5 The density of states for non-Hermitian matrices

In this section we will be calculating the density of states for non-Hermitian random matrices by following closely to the work in Ref. [55]. Non-Hermitian random matrices are random matrices such that all the their elements are completely independent of each other, this means that the relation $H_{ij} = H^*_{ji}$ is no longer valid and hence gives rise to complex eigenvalues.

Non-Hermitian Hamiltonians were introduced in order to solve problems that were not solvable with the standard Hermitian framework. They appear naturally in PT-symmetric quantum mechanics [56–60] and have various applications including optics where the refractive index is complex [61–64], open quantum systems [66–72], quantum systems subject to an external laser field [73–75] and is also a powerful tool to study resonance [76, 77].

For this calculation we will be working on a particular class of random matrices, namely the deformed Ginibre ensemble

$$H = W\tilde{H}_G W \tag{441}$$

where each element is Gaussian distributed with zero mean and variance

$$\left\langle |H_{ij}|^2 \right\rangle = \frac{w_i^2 w_j^2}{N}, \tag{442}$$

with probability distribution

$$\mathcal{P}(H) \propto \exp\left(-\frac{N}{2}\mathrm{Tr}H^2\right), \tag{443}$$

since $\tilde{H}_G$ is drawn from the Ginibre ensemble.

Before we proceed with the calculation of the density of states, we first note that any matrix can be decomposed into a sum of its Hermitian and anti-Hermitian parts. This means that we can split the deformed Ginibre ensemble into two parts:

$$H = \hat{J} + \mathrm{i}\hat{A}, \tag{444}$$

where $\hat{J}$ is the Hermitian part and $\mathrm{i}\hat{A}$ is the anti-Hermitian part, each distributed with zero mean and variance given by

$$\left\langle |\hat{J}_{ij}|^2 \right\rangle = \left\langle |\hat{A}_{ij}|^2 \right\rangle = \frac{w_i^2 w_j^2}{2N}. \tag{445}$$

$\hat{H}$ and $\hat{A}$ contain identical and independently distributed Gaussian random variables with probability distribution

$$\mathcal{P}(\hat{J}) \propto \exp\left(-\frac{N}{2}\mathrm{Tr}\hat{J}^2\right), \quad \mathcal{P}(\hat{A}) \propto \exp\left(-\frac{N}{2}\mathrm{Tr}\hat{A}^2\right). \tag{446}$$

In the case of non-Hermitian random matrices a different approach must be taken as the standard approach is only valid for the Hermitian case.

The density of states provides a full statistical distribution of eigenvalues

of quantum systems with a Hamiltonian of size N, where $N \to \infty$. In the conventional case the Hamiltonian is required to be Hermitian, therefore we would have real eigenvalues but in the non-Hermitian case we would expect to observe complex eigenvalues $E = X + \mathrm{i}Y$.

As opposed to the standard case of taking the imaginary part of the trace of the Green's function, in the non-Hermitian case the density of states is related to a potential given in Ref. [55]:

$$\Phi(X, Y, \kappa) = \frac{1}{2\pi N} \left\langle \left\langle \det[(E - H)(E - H)^\dagger + \kappa^2] \right\rangle_{\hat{A}} \right\rangle_H , \tag{447}$$

where the density of states in the complex plane is defined to be

$$\rho(X, Y) = \lim_{\kappa \to 0} \partial^2 \Phi(X, Y, \kappa), \tag{448}$$

∂^2 stands for the two-dimensional Laplacian $\partial_X^2 + \partial_Y^2$.

In order to calculate the potential, we use the relation

$$\frac{\partial^2 \Phi}{\partial \kappa^2} = \frac{1}{2\pi N} \frac{d}{d\kappa} \lim_{\kappa_b \to \kappa} \frac{\partial}{\partial \kappa} \left\langle \left\langle Z(\kappa_b, \kappa) \right\rangle_{\hat{A}} \right\rangle_{\hat{J}} , \tag{449}$$

where $Z(\kappa_b, \kappa)$, the generating function for non-Hermitian matrices, is defined as

$$Z(\kappa_b, \kappa) = \frac{\det[(E - H)(E - H)^\dagger + \kappa^2]}{\det[(E - H)(E - H)^\dagger + \kappa_b^2]}. \tag{450}$$

For the purpose of the calculation, we represent the generating function in the following form

$$\begin{aligned} \det[(E - H)(E - H)^\dagger + \kappa^2] &= (-1)^N \det \left[\hat{M}^{-1} \begin{pmatrix} \mathrm{i}(E - H)^\dagger & \kappa \\ \kappa & \mathrm{i}(E - H) \end{pmatrix} \hat{M} \right], \\ \det[(E - H)(E - H)^\dagger + \kappa_b^2] &= \det \left[\hat{M}_b^{-1} \begin{pmatrix} \kappa_b & \mathrm{i}(E - H) \\ \mathrm{i}(E - H)^\dagger & \kappa_b \end{pmatrix} \hat{M}_b \right], \end{aligned} \tag{451}$$

where $\hat{M}$ and $\hat{M}_b$ are defined to be

$$\hat{M} = \frac{1}{\sqrt{2}} \begin{pmatrix} \hat{I} & -\hat{I} \\ \hat{I} & \hat{I} \end{pmatrix}, \quad \hat{M}_b = \frac{1}{\sqrt{2}} \begin{pmatrix} \hat{I} & -\mathrm{i}\hat{I} \\ \hat{I} & \mathrm{i}\hat{I} \end{pmatrix}. \tag{452}$$

We have represented the determinants in this form to ensure the convergence of the superintegral as explained previously.

This can then, similarly to previous cases, be expressed in its superintegral form

$$\begin{aligned} Z(\kappa_b, \kappa) &= (-1)^N \int d\Phi d\Phi^* \\ &\times \exp\left\{-\Phi^\dagger \left(\kappa_b \hat{\Lambda}\hat{L} + \mathrm{i}X\hat{L} - Y\hat{\sigma}_0 + (\kappa - \kappa_b)\hat{K} - \mathrm{i}\hat{J} \otimes \hat{L} + \hat{A} \otimes \hat{\sigma}_0\right) \Phi\right\}, \end{aligned} \tag{453}$$

where $\Phi^\dagger = (S_1^\dagger, S_2^\dagger, \chi_1^\dagger, \chi_2^\dagger)$, $d\Phi d\Phi^* = \prod_{i=1}^{2} dS_p dS_p^\dagger d\chi_p d\chi_p^\dagger$ with S_p and χ_p as N-component vectors containing complex commuting and Grassmannian variables, respectively. The matrices introduced in the superintegral above have the following form

$$\begin{aligned} \hat{\Lambda} = \mathrm{diag}(1, -1, 1, -1), \quad \hat{L} = \mathrm{diag}(1, -1, 1, 1), \\ \hat{K} = \mathrm{diag}(0, 0, 1, -1), \quad \hat{\sigma}_0 = \mathrm{diag}(\mathrm{i}\Sigma_x, \Sigma_x), \end{aligned} \tag{454}$$

with

$$\Sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{455}$$

Next, we have to average over the disorder of $\hat{J}$ and $\hat{A}$ in order to proceed with the calculation:

$$\begin{aligned} \langle Z(\kappa_b, \kappa) \rangle = (-1)^N \int d\Phi d\Phi^* \\ \exp\left\{-\Phi^\dagger \left(\kappa_b \hat{\Lambda}\hat{L} + \mathrm{i}X\hat{L} - Y\hat{\sigma}_0 + (\kappa - \kappa_b)\hat{K}\right) \Phi\right\} \\ \left\langle \exp\left\{-\Phi^\dagger \left(-\mathrm{i}\hat{J} \otimes \hat{L} + \hat{A} \otimes \hat{\sigma}_0\right) \Phi\right\} \right\rangle, \end{aligned} \tag{456}$$

averaging over $\hat{J}$ is simple as it is Hermitian and therefore will be very similar to the form previously used, with the exception of the variance being halved

which results in

$$\left\langle \exp\left\{ \mathrm{i}\Phi^\dagger \hat{J} \otimes \hat{L}\Phi \right\} \right\rangle = \left\langle \exp\left\{ \mathrm{i}\sum_{\alpha}^{4} \Phi_\alpha^\dagger L_\alpha \hat{J} \Phi_\alpha \right\} \right\rangle \\ = \exp\left\{ -\frac{N}{4} \mathrm{Str}(\tilde{Q}\hat{L})^2 \right\}, \tag{457}$$

where $\tilde{Q}_{\alpha\beta} = \frac{1}{N}\Phi_\alpha^\dagger W^2 \Phi_\beta$ and $W = \mathrm{diag}(w_1, w_2, ..., w_N)$.

The next step is to average over the non-Hermitian matrix $\mathrm{i}\hat{A}$, which can be computed by first noting that $\hat{A}$ is a random Hermitian matrix, therefore we can apply the technique used previously to calculate the average:

$$\left\langle \exp\left\{ \mathrm{i}\Phi^\dagger \mathrm{i}\hat{A} \otimes \hat{\sigma}_0 \Phi \right\} \right\rangle = \left\langle \exp\left\{ \mathrm{i}\sum_{\alpha,\alpha'}^{4}\sum_{m,n}^{N} \Phi_\alpha^*(n) \mathrm{i}\hat{\sigma}_0(\alpha,\alpha')\hat{A}_{mn}\Phi_{\alpha'}(m) \right\} \right\rangle \\ = \exp\left\{ -\frac{1}{2}\left\langle \left(\sum_{\alpha,\alpha'}^{4}\sum_{m,n}^{N} \mathrm{i}\hat{A}_{mn}\Phi_\alpha^*(n)\hat{\sigma}_0(\alpha,\alpha')\Phi_{\alpha'}(m) \right)^2 \right\rangle \right\}, \tag{458}$$

where $\hat{\sigma}_0(\alpha,\alpha')$ refers to the matrix elements of $\hat{\sigma}_0$, therefore we have to calculate

$$\left\langle \left(\sum_{\alpha,\alpha'}^{4}\sum_{m,n}^{N} \mathrm{i}\hat{A}_{mn}\Phi_\alpha^*(n)\hat{\sigma}_0(\alpha,\alpha')\Phi_{\alpha'}(m) \right)^2 \right\rangle, \tag{459}$$

by using the identity in Eq. (40), comparing coefficients we find that

$$M_{nm} = \sum_{\alpha,\alpha'}^{4} \Phi_\alpha^*(n)\hat{\sigma}_0(\alpha,\alpha')\Phi_{\alpha'}(m). \tag{460}$$

We can now reduce the averaging into a simpler form

$$\left\langle \left(\sum_{m,n}^{N} \mathrm{i}\hat{A}_{mn}M_{nm} \right)^2 \right\rangle = \left\langle \left(\sum_{m,n}^{N} (\mathrm{i}\,\mathrm{Re}\,\hat{A}_{mn} - \mathrm{Im}\,\hat{A}_{mn})M_{nm} \right)^2 \right\rangle, \tag{461}$$

then by using the Hermiticity property of $\hat{A}$ we can express the above as

$$\begin{aligned}
&\left\langle \left(\sum_{m,n}^{N} \mathrm{i}\hat{A}_{mn} M_{nm} \right)^2 \right\rangle = \left\langle \left(\sum_{m,n}^{N} \mathrm{i}\,\mathrm{Re}\,\hat{A}_{mn} M_{nm} - \mathrm{Im}\,\hat{A}_{mn} M_{nm} \right)^2 \right\rangle \\
&= \left\langle \left(\sum_{m\leq n}^{N} \Big(\mathrm{i}\,\mathrm{Re}\,\hat{A}_{mn}(M_{nm} + M_{mn}) - \mathrm{Im}\,\hat{A}_{mn}(M_{nm} - M_{mn}) \Big) \right)^2 \right\rangle \\
&= \Bigg\langle \Bigg(\sum_{m\leq n}^{N} \Big(\mathrm{i}\,\mathrm{Re}\,\hat{A}_{mn}(M_{nm} + M_{mn}) - \mathrm{Im}\,\hat{A}_{mn}(M_{nm} - M_{mn}) \Big) \\
&\times \sum_{m\leq n}^{N} \Big(\mathrm{i}\,\mathrm{Re}\,\hat{A}_{mn}(M_{nm} + M_{mn}) - \mathrm{Im}\,\hat{A}_{mn}(M_{nm} - M_{mn}) \Big) \Bigg) \Bigg\rangle,
\end{aligned} \tag{462}$$

but since $\mathrm{Re}\,\hat{A}_{mn}$ and $\mathrm{Im}\,\hat{A}_{mn}$ are independent some of the terms in the multiplication are neglected as they are equal to zero, since the random variables are distributed with zero mean

$$\begin{aligned}
&\left\langle \left(\sum_{m,n}^{N} \mathrm{i}\hat{A}_{mn} M_{nm} \right)^2 \right\rangle \\
&= \left\langle \sum_{m\leq n}^{N} \Bigg(\Big(-(\mathrm{Re}\,\hat{A}_{mn})^2 (M_{nm} + M_{mn})^2 + (\mathrm{Im}\,\hat{A}_{mn})^2 (M_{nm} - M_{mn})^2 \Big) \Bigg) \right\rangle.
\end{aligned} \tag{463}$$

Since $\left\langle |\hat{A}_{mn}|^2 \right\rangle = \frac{w_m^2 w_n^2}{2N}$ then it is reasonable to assume that

$$\left\langle |\,\mathrm{Re}\,\hat{A}_{mn}|^2 \right\rangle = \frac{w_m^2 w_n^2}{4N}, \quad \left\langle |\,\mathrm{Im}\,\hat{A}_{mn}|^2 \right\rangle = \frac{w_m^2 w_n^2}{4N}, \tag{464}$$

by substituting this into the expression for the averaging we can simplify it

even further

$$\begin{aligned}
&\left\langle \left(\sum_{m,n}^{N} \mathrm{i}\hat{A}_{mn} M_{nm} \right)^2 \right\rangle \\
&= \left\langle \sum_{m\leq n}^{N} -(\mathrm{Re}\,\hat{A}_{mn})^2 (M_{nm}+M_{mn})^2 + (\mathrm{Im}\,\hat{A}_{mn})^2 (M_{nm}-M_{mn})^2 \right\rangle \\
&= \sum_{m\leq n}^{N} -\left\langle (\mathrm{Re}\,\hat{A}_{mn})^2 \right\rangle (M_{nm}+M_{mn})^2 + \left\langle (\mathrm{Im}\,\hat{A}_{mn})^2 \right\rangle (M_{nm}-M_{mn})^2 \quad (465) \\
&= \sum_{m\leq n}^{N} -\frac{w_m^2 w_n^2}{4N} (M_{nm}+M_{mn})^2 + \frac{w_m^2 w_n^2}{4N} (M_{nm}-M_{mn})^2 \\
&= -\sum_{m\leq n}^{N} \frac{w_m^2 w_n^2}{N} M_{nm} M_{mn} = -\sum_{m,n}^{N} \frac{w_m^2 w_n^2}{2N} M_{nm} M_{mn}.
\end{aligned}$$

We can now substitute this average into the original expression Eq. (458)

$$\begin{aligned}
&\left\langle \exp\left\{ \mathrm{i}\Phi^\dagger \mathrm{i}\hat{A} \otimes \hat{\sigma}_0 \Phi \right\} \right\rangle \\
&= \exp\left\{ -\frac{1}{2} \left\langle \left(\sum_{\alpha,\alpha'}^{4} \sum_{m,n}^{N} \mathrm{i}\hat{A}_{mn} \Phi_\alpha^*(n) \hat{\sigma}_0(\alpha,\alpha') \Phi_{\alpha'}(m) \right)^2 \right\rangle \right\} \\
&= \exp\left\{ \frac{1}{4N} \sum_{n,m}^{N} w_n w_m M_{nm} w_m w_n M_{mn} \right\} = \exp\left\{ \frac{1}{4N} \sum_{n}^{N} w_n (WM)_{nn}^2 \right\} \\
&= \exp\left\{ \frac{1}{4N} \mathrm{Str}(W^2 M)^2 \right\} = \exp\left\{ \frac{1}{4N} \mathrm{Str}\left(W^2 \Phi \hat{\sigma}_0 \Phi^\dagger W^2 \Phi \hat{\sigma}_0 \Phi^\dagger \right) \right\} \\
&= \exp\left\{ \frac{1}{4N} \mathrm{Str}\left(W^2 \Phi \hat{\sigma}_0 \Phi^\dagger \right)^2 \right\} \\
&= \exp\left\{ \frac{1}{4N} \mathrm{Str}\left(\Phi^\dagger W^2 \Phi \hat{\sigma}_0 \right)^2 \right\} = \exp\left\{ \frac{N}{4} \mathrm{Str}\left(\tilde{Q}\hat{\sigma}_0 \right)^2 \right\},
\end{aligned} \quad (466)$$

where $\tilde{Q}_{\alpha\beta} = \frac{1}{N} \Phi_\alpha^\dagger W^2 \Phi_\beta$ and $W = \mathrm{diag}(w_1, w_2, ..., w_N)$.

Hence, the averaged density of states for the non-Hermitian case is equivalent to

$$\begin{aligned}
\langle Z(\kappa_b, \kappa) \rangle &= (-1)^N \int d\Phi d\Phi^* \\
&\times \exp\left\{ -\Phi^\dagger \left(\kappa_b \hat{\Lambda} \hat{L} + \mathrm{i} X \hat{L} - Y \hat{\sigma}_0 + (\kappa - \kappa_b) \hat{K} \right) \Phi \right\} \quad (467) \\
&\times \exp\left\{ -\frac{N}{4} \mathrm{Str}(\tilde{Q}\hat{L})^2 \right\} \exp\left\{ -\frac{N}{4} \mathrm{Str}\left(\tilde{Q}\hat{\sigma} \right)^2 \right\},
\end{aligned}$$

where $\hat{\sigma} = -\mathrm{i}\hat{\sigma}_0$.

The next step is to apply the Hubbard-Stratonovich transformation that decouples the quartic Φ terms into quadratics which makes integrating over Φ possible. As we have to decouple two separate terms we have to apply two different Hubbard-Stratonovich transformations, which introduces two new variables that arises from the transformation, rather than the usual one variable. The first one is simple as it is in the usual form so we can use the previous Hubbard-Stratonovich transformation, Eq. (53), the second one, however, requires a bit more work to apply the transformation to the term:

$$\exp\left\{-\frac{N}{4}\mathrm{Str}\left(\tilde{Q}\hat{\sigma}\right)^2\right\}. \tag{468}$$

This work in progress had been stopped since we discovered later that the density of states for non-Hermitian matrices of this type was already calculated in Ref. [34].

5.1 Conclusion

We began this section with the intention to study the form and characteristics of the density of states for non-Hermitian matrices, namely the deformed Ginibre ensemble introduced by Jean Ginibre in 1965.

Density of states of Hermitian matrices have been well studied in the past but much less emphasis is placed on non-Hermitian random matrices, which also have many useful and significant applications in physics. We focused on a particular deformed Ginibre ensemble by using the method developed in Ref. [55], but during the calculations we had seen a newly released article at the time that focused on a more general model. This generalised model covers our model as a subcase of their random matrix noise-plus-structure model given by Ref. [34]:

$$M = S + LXR, \tag{469}$$

where S is a fixed matrix and L, $R > 0$ are diagonal positive definite covariance matrices. X is the source of noise, typically drawn from the Gaussian

ensembles, and in this case is drawn from the Girko-Ginibre ensemble. This is an interesting case as it is not only valid for $N \to \infty$ but also for any finite N, so it is an exact expression for the density of states made possible by circumventing the saddle-point approximation in the supersymmetry technique. If we set $S = 0$ and $L = R$, we can see that this is exactly equivalent to the model that we were studying.

6 Moments of the eigenvectors for the chiral simplex model

Anderson localisation has remained as one of the most active and important problems to study in the field of disordered quantum systems. The most prominent ones that can be studied analytically are the one-dimensional Anderson model [78] and its continuous variant [79–81], the quasi-one-dimensional model [5, 82, 83] and the models on tree-like graphs [84–90].

In this section we aimed to calculate the moments of the eigenvectors of a chiral simplex model where we have two separate systems, A and B, each node in system A is linked to every node in system B but is not linked to the other nodes in system A and vice versa. The symmetry of this structure is known as chiral symmetry and we will be following the method developed in Ref. [1] for a generalised Anderson model in order to calculate the moments of the eigenvectors.

The Hamiltonian for our model is given by

$$H = V + T, \tag{470}$$

where

$$V_{ij} = v_i \delta_{ij}, \quad T_{ij} = \frac{1}{N}\left(|f_1\rangle_i \langle g_1|_j + |g_1\rangle_i \langle f_1|_j\right), \quad i, j = 1, ..., N, \tag{471}$$

f_1 and g_1 are vectors where the first half of the elements of f_1 are equal to 1, the

second half are equal to zero and vice versa for g_1, v_i are independent Gaussian distributed random variables with mean $\langle v_i \rangle = 0$ and variance $\langle v_i^2 \rangle = w^2$.

According to Ref. [1] the moments of the eigenvectors are given by

$$I_q(n) = \frac{1}{\pi\rho(E)(q-2)!} \int_0^\infty dt_n t_n^{2q-3} Y(t_n), \tag{472}$$

where $\rho(E)$ is the density of states and

$$Y(t_n) = \frac{\sqrt{2\pi}}{w} \prod_{p \neq n} \int_{-\infty}^{\infty} \frac{dt_p}{\sqrt{2\pi} w t_p} \det B e^{-\sum_p \left(\frac{(\sum_q T_{pq} \frac{t_q}{t_p} - E)^2}{2w^2} + t_p^2 \right)}, \tag{473}$$

the matrix B in the preceding integral is defined by

$$B_{pq} = -T_{pq} + \delta_{pq} \sum_r T_{pr} \frac{t_r}{t_p}, p = 1, ..., N;\ p, q \neq n, \tag{474}$$

which is still applicable to our model. The moments of the eigenvectors in its current form have been calculated for a generic matrix T and therefore applicable in a variety of models in which the form of the matrix T may differ. Please note that the integration variables t_n that appears above are different from the variables s and t appearing in the previous sections. Eqs. (465-466) give the moments of the eigenvectors in terms of an N-fold integral, our goal is to calculate the large N asymptotic behaviour of this integral.

To begin our calculation of the moments of the eigenvectors we must first calculate $\det B$, which is done by following the approach taken in Ref. [1]. The matrix with elements B_{pq} is the $(N-1) \times (N-1)$ matrix which can be obtained from the matrix $\hat{B}$ of size $N \times N$:

$$\hat{B}_{pq} = -T_{pq} + \delta_{pq} \sum_r T_{pr} \frac{t_r}{t_p}, p = 1, ..., N, \tag{475}$$

the only difference from the previous is that in this definition we keep the nth row and column.

According to Ref. [1], $\hat{B}$ and B are related in the following way

$$\det B = |z_n|^2 \overline{\det} \hat{B}, \tag{476}$$

where z_n is the nth element of the normalised eigenvector of $\hat{B}$ corresponding to the zero eigenvalue

$$z = \left(\sum_{p=1}^{N} t_p^2 \right)^{-1/2} (t_1, t_2, ..., t_N)^T, \tag{477}$$

and $\overline{\det} \hat{B}$ is the product of all non-zero eigenvalues of $\hat{B}$, which can also be expressed as

$$\overline{\det} \hat{B} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \det (\hat{B} + \epsilon). \tag{478}$$

This applies for only one vanishing eigenvalue, however the chance of having two or more zero eigenvalues is effectively zero so we do not need to take that into consideration. The reason we consider the zero eigenvalue is because we are interested in the moments of the eigenvectors with $E = 0$, where the eigenvectors are initially degenerate at $w = 0$.

For our particular model, the matrix with elements $\hat{B}_{pq}$ is given by

$$\hat{B}_{pq} = \begin{cases} \frac{1}{Nt_p} \sum_r (|f_1\rangle_p \langle g_1|_r + |g_1\rangle_p \langle f_1|_r) t_r, & p = q \\ -\frac{1}{N} (|f_1\rangle_p \langle g_1|_q + |g_1\rangle_p \langle f_1|_q), & p \neq q \end{cases}, \tag{479}$$

which is equivalent to

$$\hat{B} = \frac{1}{N} (- |f_1\rangle \langle g_1| - |g_1\rangle \langle f_1| + \alpha_1 D_1 + \alpha_2 D_2), \tag{480}$$

where

$$\begin{aligned} \alpha_1 = \sum_{r=1}^{N/2} t_r, \quad \alpha_2 = \sum_{r=\frac{N}{2}+1}^{N} t_r, \quad D_1 = \mathrm{diag}(0, 0, ..., t_{\frac{N}{2}+1}^{-1}, ..., t_N^{-1}), \\ D_2 = \mathrm{diag}(t_1^{-1}, t_2^{-1}, ..., t_{\frac{N}{2}}^{-1}, 0, ..., 0), \end{aligned} \tag{481}$$

D_1 and D_2 are $N \times N$ diagonal matrices.

Now to calculate $\det(\hat{B} + \epsilon)$:

$$\begin{aligned}
&\det(\hat{B} + \epsilon) = \det\left(\frac{1}{N}(-|f_1\rangle\langle g_1| - |g_1\rangle\langle f_1| + \alpha_1 D_1 + \alpha_2 D_2) + \epsilon\right) \\
&= \frac{1}{N^N}\det\left(-|f_1\rangle\langle g_1| - |g_1\rangle\langle f_1| + \alpha_1 D_1 + \alpha_2 D_2 + N\epsilon\right) \\
&= \frac{1}{N^N}\det\left(\epsilon N + \alpha_1 D_1 + \alpha_2 D_2\right)\det\left(1 - G_0|f_1\rangle\langle g_1| - G_0|g_1\rangle\langle f_1|\right),
\end{aligned} \tag{482}$$

where $G_0 = (\epsilon N + \alpha_1 D_1 + \alpha_2 D_2)^{-1}$.

Since the argument in $\det(\epsilon N + \alpha_1 D_1 + \alpha_2 D_2)$ is purely diagonal we can easily compute the determinant by first writing it in its matrix form

$$\begin{pmatrix}
\frac{\alpha_2}{t_1} + \epsilon N & & & & & & & \\
& \frac{\alpha_2}{t_2} + \epsilon N & & & & & & \\
& & \ddots & & & & & \\
& & & \frac{\alpha_2}{t_{\frac{N}{2}}} + \epsilon N & & & & \\
& & & & \frac{\alpha_1}{t_{\frac{N}{2}+1}} + \epsilon N & & & \\
& & & & & \ddots & & \\
& & & & & & \frac{\alpha_1}{t_N} + \epsilon N
\end{pmatrix}, \tag{483}$$

it is a well-known fact that the determinant of a diagonal matrix is equal to the product of all the diagonal elements, hence,

$$\det\left(\epsilon N + \alpha_1 D_1 + \alpha_2 D_2\right) = \prod_{p=1}^{N/2}\left(\frac{\alpha_2}{t_p} + \epsilon N\right)\prod_{k=\frac{N}{2}+1}^{N}\left(\frac{\alpha_1}{t_k} + \epsilon N\right). \tag{484}$$

Then, to work out the second determinant, we note that

$$\begin{aligned}
&\ln\det\left(1 - G_0|f_1\rangle\langle g_1| - G_0|g_1\rangle\langle f_1|\right) = \mathrm{Tr}\ln\left(1 - G_0|f_1\rangle\langle g_1| - G_0|g_1\rangle\langle f_1|\right) \\
&= -\sum_{n\geq 1}\frac{1}{n}\mathrm{Tr}(G_0|f_1\rangle\langle g_1| + G_0|g_1\rangle\langle f_1|)^n,
\end{aligned} \tag{485}$$

but recalling the definition of $|f_1\rangle$ and $|g_1\rangle$, we find that

$$\langle g_1|G_0|f_1\rangle = \langle f_1|G_0|g_1\rangle = 0, \tag{486}$$

all terms in the summation involving odd n are equal to zero as they are proportional to $\langle g_1|G_0|f_1\rangle$ or $\langle f_1|G_0|g_1\rangle$, hence we can simplify the summation as follows

$$\begin{aligned}
&- \sum_{n\ even} \frac{1}{n} \mathrm{Tr}(G_0 \left|f_1\right\rangle \left\langle g_1\right| + G_0 \left|g_1\right\rangle \left\langle f_1\right|)^n \\
&= - \sum_{n\ even} \frac{1}{n} \left(2 \left\langle f_1\right| G_0 \left|f_1\right\rangle^{n/2} \left\langle g_1\right| G_0 \left|g_1\right\rangle^{n/2}\right) \\
&= - \sum_{m\geq 1} \frac{1}{m} (\left\langle f_1\right| G_0 \left|f_1\right\rangle \left\langle g_1\right| G_0 \left|g_1\right\rangle)^m,\ n = 2m \\
&= \ln \left(1 - \sum_{p=1}^{N/2} \left(\frac{\alpha_2}{t_p} + \epsilon N\right)^{-1} \sum_{k=\frac{N}{2}+1}^{N} \left(\frac{\alpha_1}{t_k} + \epsilon N\right)^{-1}\right),
\end{aligned} \tag{487}$$

The last equality comes from the fact that since $G_0 = (\epsilon N + \alpha_1 D_1 + \alpha_2 D_2)^{-1}$ is a diagonal matrix, then to invert it we simply take the inverse of all elements on the diagonal and the previous equality is in the form of a logarithm expansion, which enables one to convert it to its logarithmic form.

Combining all of this together, we see that

$$\begin{aligned}
\det(\hat{B} + \epsilon) = \frac{1}{N^N} &\left(1 - \sum_{p=1}^{N/2} \left(\frac{\alpha_2}{t_p} + \epsilon N\right)^{-1} \sum_{k=\frac{N}{2}+1}^{N} \left(\frac{\alpha_1}{t_k} + \epsilon N\right)^{-1}\right) \\
&\times \prod_{p=1}^{N/2} \left(\frac{\alpha_2}{t_p} + \epsilon N\right) \prod_{k=\frac{N}{2}+1}^{N} \left(\frac{\alpha_1}{t_k} + \epsilon N\right),
\end{aligned} \tag{488}$$

substituting this into Eq. (478), we receive

$$\begin{aligned}
\overline{\det}\hat{B} = \lim_{\epsilon\to 0} \frac{1}{\epsilon} \frac{1}{N^N} &\left(1 - \sum_{p=1}^{N/2} \left(\frac{\alpha_2}{t_p} + \epsilon N\right)^{-1} \sum_{k=\frac{N}{2}+1}^{N} \left(\frac{\alpha_1}{t_k} + \epsilon N\right)^{-1}\right) \\
&\times \prod_{p=1}^{N/2} \left(\frac{\alpha_2}{t_p} + \epsilon N\right) \prod_{k=\frac{N}{2}+1}^{N} \left(\frac{\alpha_1}{t_k} + \epsilon N\right).
\end{aligned} \tag{489}$$

In order to take the limit $\epsilon \to 0$ we first define two functions

$$f_1(\epsilon) = \sum_{p=1}^{N/2} \left(\frac{\alpha_2}{t_p} + \epsilon N \right)^{-1}, \quad f_2(\epsilon) = \sum_{k=\frac{N}{2}+1}^{N} \left(\frac{\alpha_1}{t_k} + \epsilon N \right)^{-1}, \tag{490}$$

and compute the Taylor series of both of these functions

$$f_1(\epsilon) = \frac{\alpha_1}{\alpha_2} - \epsilon N \sum_{p=1}^{N/2} t_p^2 \alpha_2^{-2} + O(\epsilon^2), \quad f_2(\epsilon) = \frac{\alpha_2}{\alpha_1} - \epsilon N \sum_{p=\frac{N}{2}+1}^{N} t_p^2 \alpha_1^{-2} + O(\epsilon^2), \tag{491}$$

multiplying these functions together, we receive

$$\begin{aligned} f_1(\epsilon) f_2(\epsilon) &= 1 - \frac{\alpha_1}{\alpha_2} \epsilon N \sum_{p=\frac{N}{2}+1}^{N} t_p^2 \alpha_1^{-2} - \frac{\alpha_2}{\alpha_1} \epsilon N \sum_{p=1}^{N/2} t_p^2 \alpha_2^{-2} + O(\epsilon^2) \\ &= 1 - \frac{1}{\alpha_1 \alpha_2} \epsilon N \sum_{p=\frac{N}{2}+1}^{N} t_p^2 - \frac{1}{\alpha_1 \alpha_2} \epsilon N \sum_{p=1}^{N/2} t_p^2 + O(\epsilon^2) \\ &= 1 - \frac{1}{\alpha_1 \alpha_2} \epsilon N \sum_{p=1}^{N} t_p^2 + O(\epsilon^2), \end{aligned} \tag{492}$$

substituting this into Eq. (489) yields

$$\begin{aligned} \overline{\det} \hat{B} &= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \frac{1}{N^N} \left(\frac{1}{\alpha_1 \alpha_2} \epsilon N \sum_{p=1}^{N} t_p^2 + O(\epsilon^2) \right) \prod_{p=1}^{N/2} \left(\frac{\alpha_2}{t_p} + \epsilon N \right) \prod_{k=\frac{N}{2}+1}^{N} \left(\frac{\alpha_1}{t_k} + \epsilon N \right) \\ &= \lim_{\epsilon \to 0} \frac{1}{N^N} \left(\frac{1}{\alpha_1 \alpha_2} N \sum_{p=1}^{N} t_p^2 + O(\epsilon) \right) \prod_{p=1}^{N/2} \left(\frac{\alpha_2}{t_p} + \epsilon N \right) \prod_{k=\frac{N}{2}+1}^{N} \left(\frac{\alpha_1}{t_k} + \epsilon N \right) \\ &= \frac{1}{N^{N-1}} \frac{1}{\alpha_1 \alpha_2} \sum_{p=1}^{N} t_p^2 \prod_{p=1}^{N/2} \frac{\alpha_2}{t_p} \prod_{k=\frac{N}{2}+1}^{N} \frac{\alpha_1}{t_k} = \frac{1}{N^{N-1}} \sum_{p=1}^{N} t_p^2 \left(\prod_{k=1}^{N} \frac{1}{t_k} \right) \alpha_1^{\frac{N}{2}-1} \alpha_2^{\frac{N}{2}-1} \\ &= \frac{1}{N^{N-1}} \left(\sum_{r=1}^{N/2} t_r \right)^{\frac{N}{2}-1} \left(\sum_{r=\frac{N}{2}+1}^{N} t_r \right)^{\frac{N}{2}-1} \sum_{p=1}^{N} t_p^2 \prod_{k=1}^{N} \frac{1}{t_k}. \end{aligned} \tag{493}$$

Substituting the above into Eq. (476), we receive our desired result for $\det B$:

$$\begin{aligned}\det B &= \left(\sum_{p=1}^{N} t_p^2\right)^{-1} \frac{t_n^2}{N^{N-1}} \left(\sum_{r=1}^{N/2} t_r\right)^{\frac{N}{2}-1} \left(\sum_{r=\frac{N}{2}+1}^{N} t_r\right)^{\frac{N}{2}-1} \sum_{p=1}^{N} t_p^2 \prod_{k=1}^{N} \frac{1}{t_k} \\ &= \frac{t_n^2}{N^{N-1}} \left(\sum_{r=1}^{N/2} t_r\right)^{\frac{N}{2}-1} \left(\sum_{r=\frac{N}{2}+1}^{N} t_r\right)^{\frac{N}{2}-1} \prod_{k=1}^{N} \frac{1}{t_k},\end{aligned} \tag{494}$$

as required.

Back to the original integral for the moments of the eigenvectors, we can substitute the expression for $\det B$ into the integral to obtain

$$\begin{aligned}Y(t_n) = \frac{\sqrt{2\pi}}{w} \prod_{p\neq n} \int_{-\infty}^{\infty} \frac{dt_p}{\sqrt{2\pi} w t_p} \frac{t_n^2}{N^{N-1}} \left(\sum_{r=1}^{N/2} t_r\right)^{\frac{N}{2}-1} \left(\sum_{r=\frac{N}{2}+1}^{N} t_r\right)^{\frac{N}{2}-1} \\ \times \prod_{k=1}^{N} \frac{1}{t_k} \exp\left\{ -\sum_{\substack{p>\frac{N}{2} \\ p\leq\frac{N}{2}}} \left(\frac{1}{2w^2} \left(\frac{1}{N} \sum_{\substack{q\leq\frac{N}{2} \\ q>\frac{N}{2}}} \frac{t_q}{t_p} - E \right)^2 + t_p^2 \right) \right\},\end{aligned} \tag{495}$$

where the subscripts in the summations for $p > \frac{N}{2}$ and $q \leq \frac{N}{2}$ correspond with one another; similarly for $p \leq \frac{N}{2}$ and $q > \frac{N}{2}$.

As we are looking at the moments of the eigenvectors for the eigenvectors that are initially degenerate at $w = 0$, this corresponds with $E = 0$, so this is the value we are going to consider below. Therefore the moments of the eigenvectors, according to Eq. (472), are equivalent to

$$\begin{aligned}I_q(N) = \frac{1}{\pi\rho(0)(q-2)!} \int_0^{\infty} dt_n t_n^{2q-3} \frac{\sqrt{2\pi}}{w} \prod_{p\neq n} \int_{-\infty}^{\infty} \frac{dt_p}{\sqrt{2\pi} w t_p} \frac{t_n^2}{N^{N-1}} \\ \times \left(\sum_{r=1}^{N/2} t_r\right)^{\frac{N}{2}-1} \left(\sum_{r=\frac{N}{2}+1}^{N} t_r\right)^{\frac{N}{2}-1} \prod_{k=1}^{N} \frac{1}{t_k} \\ \times \exp\left\{ -\sum_{\substack{p>\frac{N}{2} \\ p\leq\frac{N}{2}}} \left(\frac{1}{2w^2} \left(\frac{1}{N} \sum_{\substack{q\leq\frac{N}{2} \\ q>\frac{N}{2}}} \frac{t_q}{t_p} \right)^2 + t_p^2 \right) \right\},\end{aligned} \tag{496}$$

in this model all sites are equal, therefore we will not put so much emphasis on the n-dependence and instead focus on N - the total number of sites.

Collecting all the constant terms in the integral above, we can simplify the integral further to

$$\begin{aligned} I_q(N) = & \frac{2}{\rho(0)(q-2)!(\sqrt{2\pi})^N w^N} \int_0^\infty dt_n t_n^{2q-3} \prod_{p\neq n} \int_{-\infty}^\infty \frac{dt_p}{t_p} \frac{t_n^2}{N^{N-1}} \\ & \times \left(\sum_{r=1}^{N/2} t_r \right)^{\frac{N}{2}-1} \left(\sum_{r=\frac{N}{2}+1}^{N} t_r \right)^{\frac{N}{2}-1} \prod_{k=1}^{N} \frac{1}{t_k} \\ & \times \exp\left\{ -\sum_{\substack{p>\frac{N}{2} \\ p\leq\frac{N}{2}}} \left(\frac{1}{2N^2 w^2 t_p^2} \left(\sum_{\substack{q\leq\frac{N}{2} \\ q>\frac{N}{2}}} t_q \right)^2 + t_p^2 \right) \right\}, \end{aligned} \tag{497}$$

then we note that

$$\prod_{p=1}^{N} \frac{1}{t_p} = \frac{1}{t_n} \prod_{p\neq n}^{N} \frac{1}{t_p}, \tag{498}$$

substituting this into the integral yields

$$\begin{aligned} I_q(N) = & \frac{2}{\rho(0)(q-2)!(\sqrt{2\pi})^N w^N} \int_0^\infty dt_n t_n^{2q-2} \prod_{p\neq n} \int_{-\infty}^\infty \frac{dt_p}{t_p^2} \frac{1}{N^{N-1}} \\ & \times \left(\sum_{r=1}^{N/2} t_r \right)^{\frac{N}{2}-1} \left(\sum_{r=\frac{N}{2}+1}^{N} t_r \right)^{\frac{N}{2}-1} \\ & \times \exp\left\{ -\sum_{\substack{p>\frac{N}{2} \\ p\leq\frac{N}{2}}} \left(\frac{1}{2N^2 w^2 t_p^2} \left(\sum_{\substack{q\leq\frac{N}{2} \\ q>\frac{N}{2}}} t_q \right)^2 + t_p^2 \right) \right\}. \end{aligned} \tag{499}$$

It is convenient to introduce two new integration variables s_1 and s_2 by inserting the decomposition of unity

$$1 = \int_{-\infty}^\infty ds_1 \delta\left(s_1 - \sum_{q=1}^{N/2} \frac{t_q}{wN} \right), \quad 1 = \int_{-\infty}^\infty ds_2 \delta\left(s_2 - \sum_{q=\frac{N}{2}+1}^{N} \frac{t_q}{wN} \right), \tag{500}$$

into the integral, in which we obtain

$$I_q(N) = \frac{1}{\rho(0)(q-2)!N(\sqrt{2\pi})^N w^2} \int_{-\infty}^{\infty} dt_n t_n^{2q-2} \prod_{p\neq n} \int_{-\infty}^{\infty} \frac{dt_p}{t_p^2}$$
$$\times \int_{-\infty}^{\infty} ds_1 \delta\left(s_1 - \sum_{q=1}^{N/2} \frac{t_q}{wN}\right) s_1^{\frac{N}{2}-1} \int_{-\infty}^{\infty} ds_2 \delta\left(s_2 - \sum_{q=\frac{N}{2}+1}^{N} \frac{t_q}{wN}\right) s_2^{\frac{N}{2}-1}$$
$$\times \exp\left\{ - \sum_{\substack{p>\frac{N}{2} \\ p\leq\frac{N}{2}}} \left(\frac{1}{2N^2 w^2 t_p^2} \left(\sum_{\substack{q\leq\frac{N}{2} \\ q>\frac{N}{2}}} t_q \right)^2 + t_p^2 \right) \right\}, \tag{501}$$

by assuming that N is of even parity and then extending the integration limits for t_n to $-\infty$.

The argument of the exponential can also be simplified by splitting the summation into separate parts since

$$\sum_{\substack{p>\frac{N}{2} \\ p\leq\frac{N}{2}}} \left(\frac{1}{2N^2 w^2 t_p^2} \left(\sum_{\substack{q\leq\frac{N}{2} \\ q>\frac{N}{2}}} t_q \right)^2 + t_p^2 \right) = \sum_{p\leq\frac{N}{2}} \frac{s_2^2}{2t_p^2} + \sum_{p>\frac{N}{2}} \frac{s_1^2}{2t_p^2} + \sum_{p=1}^{N} t_p^2, \tag{502}$$

hence,

$$I_q(N) = \frac{1}{\rho(0)(q-2)!N(\sqrt{2\pi})^N w^2} \int_{-\infty}^{\infty} dt_n t_n^{2q-2} \prod_{p\neq n} \int_{-\infty}^{\infty} \frac{dt_p}{t_p^2}$$
$$\times \int_{-\infty}^{\infty} ds_1 \delta\left(s_1 - \sum_{q=1}^{N/2} \frac{t_q}{wN}\right) s_1^{\frac{N}{2}-1} \int_{-\infty}^{\infty} ds_2 \delta\left(s_2 - \sum_{q=\frac{N}{2}+1}^{N} \frac{t_q}{wN}\right) s_2^{\frac{N}{2}-1}$$
$$\times \exp\left\{ - \sum_{p=1}^{N/2} \frac{s_2^2}{2t_p^2} - \sum_{p=\frac{N}{2}+1}^{N} \frac{s_1^2}{2t_p^2} - \sum_{p=1}^{N} t_p^2 \right\}. \tag{503}$$

In order to proceed with the integral we represent the δ-functions by their

respective integrals

$$\begin{aligned}\delta\left(s_1-\sum_{q=1}^{N/2}\frac{t_q}{wN}\right)&=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\theta_1 e^{-\mathrm{i}\left(s_1-\sum_{q=1}^{N/2}\frac{t_q}{wN}\right)\theta_1},\\ \delta\left(s_2-\sum_{q=\frac{N}{2}+1}^{N}\frac{t_q}{wN}\right)&=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\theta_2 e^{-\mathrm{i}\left(s_2-\sum_{q=\frac{N}{2}+1}^{N}\frac{t_q}{wN}\right)\theta_2},\end{aligned}\tag{504}$$

it is more convenient, however, to scale θ_1 and θ_2 in the following way: $\theta_1 = \frac{wN}{s_1}\phi_1$ and $\theta_2 = \frac{wN}{s_2}\phi_2$. The δ-function integrals are then found to be of the form

$$\begin{aligned}\delta\left(s_1-\sum_{q=1}^{N/2}\frac{t_q}{wN}\right)&=\frac{wN}{2\pi|s_1|}\int_{-\infty}^{\infty}d\phi_1 e^{-\mathrm{i}\left(wN-\sum_{q=1}^{N/2}\frac{t_q}{s_1}\right)\phi_1},\\ \delta\left(s_2-\sum_{q=\frac{N}{2}+1}^{N}\frac{t_q}{wN}\right)&=\frac{wN}{2\pi|s_2|}\int_{-\infty}^{\infty}d\phi_2 e^{-\mathrm{i}\left(wN-\sum_{q=\frac{N}{2}+1}^{N}\frac{t_q}{s_2}\right)\phi_2},\end{aligned}\tag{505}$$

we also make an assumption on n, whether $n \leq N/2$ or $n > N/2$, in our case we take $n \leq N/2$ to arrive at the following integral

$$\begin{aligned}I_q(N) = c_N &\int_{-\infty}^{\infty}ds_2\frac{wN}{2\pi|s_2|}s_2^{\frac{N}{2}-1}\int_{-\infty}^{\infty}|s_2|dx_n s_2^{2q-2}x_n^{2q-2}\\ &\times\prod_{p\neq n}^{N/2}\int_{-\infty}^{\infty}ds_1\frac{wN}{2\pi|s_1|}s_1^{\frac{N}{2}-1}\int_{-\infty}^{\infty}\frac{|s_2|dx_p}{s_2^2x_p^2}\int_{-\infty}^{\infty}d\phi_1 e^{-\mathrm{i}\left(wN-\sum_{q=1}^{N/2}\frac{s_2x_q}{s_1}\right)\phi_1}\\ &\times\exp\left\{-\sum_{p=1}^{N/2}\frac{1}{2x_p^2}-s_2^2x_p^2\right\}\prod_{p=\frac{N}{2}+1}^{N}\int_{-\infty}^{\infty}\frac{|s_1|dx_p}{s_1^2x_p^2}\\ &\times\int_{-\infty}^{\infty}d\phi_2 e^{-\mathrm{i}\left(wN-\sum_{q=\frac{N}{2}+1}^{N}\frac{s_1x_q}{s_2}\right)\phi_2}\exp\left\{-\sum_{p=\frac{N}{2}+1}^{N}\frac{1}{2x_p^2}-s_1^2x_p^2\right\},\end{aligned}\tag{506}$$

where $c_N = \frac{1}{\rho(0)(q-2)!N(\sqrt{2\pi})^N w^2}$, scaling the variable t_p by $t_p = s_2x_p$ for $p \leq N/2$ and $t_p = s_1x_p$ for $p > N/2$.

We can simplify the integral by collecting common terms together

$$
\begin{aligned}
I_q(N) = & \frac{N}{\rho(0)(q-2)!(\sqrt{2\pi})^N} \int_{-\infty}^{\infty} dx_n \int_{-\infty}^{\infty} ds_2 \frac{1}{2\pi} |s_2|^{2q-2} x_n^{2q-2} \\
& \times \prod_{p \neq n}^{N/2} \int_{-\infty}^{\infty} ds_1 \frac{1}{2\pi} |s_1|^{-2} \int_{-\infty}^{\infty} \frac{dx_p}{x_p^2} \int_{-\infty}^{\infty} d\phi_1 e^{-\mathrm{i}\left(wN - \sum_{q=1}^{N/2} \frac{s_2 x_q}{s_1}\right)\phi_1} \\
& \times \exp\left\{ -\sum_{p=1}^{N/2} \frac{1}{2x_p^2} - s_2^2 x_p^2 \right\} \prod_{p=\frac{N}{2}+1}^{N} \int_{-\infty}^{\infty} \frac{dx_p}{x_p^2} \\
& \times \int_{-\infty}^{\infty} d\phi_2 e^{-\mathrm{i}\left(wN - \sum_{q=\frac{N}{2}+1}^{N} \frac{s_1 x_q}{s_2}\right)\phi_2} \exp\left\{ -\sum_{p=\frac{N}{2}+1}^{N} \frac{1}{2x_p^2} - s_1^2 x_p^2 \right\},
\end{aligned}
\tag{507}
$$

by re-ordering the integrals we arrive at

$$
\begin{aligned}
I_q(N) = & \frac{N}{\rho(0)(q-2)!(\sqrt{2\pi})^N} \frac{1}{(2\pi)^2} \int d\phi d\phi_2 e^{-\mathrm{i}(\phi_1+\phi_2)wN} \int_{-\infty}^{\infty} ds_1 ds_2 |s_2|^{2q-2} |s_1|^{-2} \\
& \times \prod_{p \neq n}^{N/2} \int_{-\infty}^{\infty} dx_p x_p^{-2} \exp\left\{ -\sum_{p \neq n}^{N/2} \frac{1}{2x_p^2} - s_2^2 x_p^2 + \mathrm{i} \sum_{q \neq n}^{N/2} \frac{s_2}{s_1} \phi_1 x_q \right\} \\
& \times \prod_{p=\frac{N}{2}+1}^{N} \int_{-\infty}^{\infty} dx_p x_p^{-2} \exp\left\{ -\sum_{p=\frac{N}{2}+1}^{N/2} \frac{1}{2x_p^2} - s_1^2 x_p^2 + \mathrm{i} \sum_{q=1}^{N/2} \frac{s_1}{s_2} \phi_2 x_q \right\} \\
& \times \int_{-\infty}^{\infty} dx_n x_n^{2q-2} \exp\left\{ -\frac{1}{2x_n^2} - s_2^2 x_n^2 + \mathrm{i} \frac{s_2}{s_1} \phi_1 x_n \right\},
\end{aligned}
\tag{508}
$$

the integration over all x_p are the same, hence we can simplify the expression down to

$$
\begin{aligned}
I_q(N) = & \frac{N}{(2\pi)^{5/2} \rho(0)(q-2)!} \int d\phi d\phi_2 e^{-\mathrm{i}(\phi_1+\phi_2)wN} \int_{-\infty}^{\infty} ds_1 ds_2 |s_2|^{2q-2} |s_1|^{-2} \\
& \times \left(\int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} x^{-2} \exp\left\{ -\frac{1}{2x^2} - s_2^2 x^2 + \mathrm{i} \frac{s_2}{s_1} \phi_1 x \right\} \right)^{\frac{N}{2}-1} \\
& \times \left(\int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} x^{-2} \exp\left\{ -\frac{1}{2x^2} - s_1^2 x^2 + \mathrm{i} \frac{s_1}{s_2} \phi_2 x \right\} \right)^{\frac{N}{2}} \\
& \times \int_{-\infty}^{\infty} dx x^{2q-2} \exp\left\{ -\frac{1}{2x^2} - s_2^2 x^2 + \mathrm{i} \frac{s_2}{s_1} \phi_1 x \right\}.
\end{aligned}
\tag{509}
$$

This work was left incomplete as we were running out of time and had to focus on completing as much as possible on other more complete work.

6.1 Conclusion

In this chapter our focus was to build on the work from Ref. [1] where they studied the Anderson on a d-simplex model. Since the expressions for the moments of the eigenvectors derived in Ref. [1] are very general and applicable to any model, we are able to apply them to the chiral simplex model directly, which will allow one to study the characteristics of the eigenvectors of the model. The density of states of the chiral simplex model has been studied previously, the density of states remains finite at every point of E, except at $E = 0$ where it diverges off to infinity. Due to this divergence at $E = 0$, this region remained an interesting point to study on this particular model.

In the chiral simplex model we expect to observe localised eigenvectors, at every point except from $E = 0$ where we should see the delocalisation of eigenvectors corresponding to the divergence seen in the density of states at $E = 0$. As we were unable to complete the work we have no results to back up this assumption, but nevertheless this remains as the most logical inference with the information we have at hand.

7 Conclusion

In chapter one we introduced all the basics required for the reader to understand before going into any details on our work. The second chapter is where our work officially begins and is solely focused on studying the statistical properties. This is done by applying the supersymmetry technique on a random matrix model (model I) to calculate the moments of the eigenvectors, which we can then use to determine the characteristics of the eigenvectors under this type of deformation. We confirmed the validity of the moments of the eigenvectors by numerical simulations and by deriving the zero energy result by an alternative approach.

In general since the GUE is unitary invariant, its eigenvectors are known to be extended and by looking at different deformations we are able to see

how this affects the eigenvectors compared to the original non-deformed counterpart. We can also see qualitatively, by altering the control parameter, the transition of one type of eigenvector to another. In this particular model we used carefully chosen power-law dependent entries as the deformation matrix and see that there is a transition from extended to a critical state where it shares the characteristics of extended and critical states, then to another critical state, this time sharing the characteristics of localised and critical states. Later, we generalised the model to Gaussian distributed elements for the deformation matrix by using the results calculated previously as a foundation to this problem. The results previously provides a convenient starting point as we are able to average the results over the Gaussian elements, then we verified numerically that the new results in this framework were in agreement with the numerical results. Afterwards we considered the case where $\sigma \to \infty$ and derived an asymptotic expression for the moments of the eigenvectors, which enables one to explore the possibility of an N-dependent variance, where we find that the eigenvectors are of a fractal nature.

In the third chapter we studied a different random matrix model with an additive deterministic potential (model II) instead of the multiplicative deformation matrix. In much the same way, we calculate the moments of the eigenvectors for this particular model and derive the distribution function. Then, we test the validity of such expressions numerically to confirm the accuracy of the analytical results which were found to be in excellent agreement with one another. Next, we generalised the model to a random potential by using the results derived previously and confirmed the analytical results by taking two limiting cases: $\sigma \to 0$ and $\sigma \to \infty$. We find that in the limit $\sigma \to 0$ the density of states tends to the Wigner's semi-circle law and in the opposite limit we find that the density of states coincides with the density of states generated by the random potential. We also carry out numerical simulations against our analytical results to compare the accuracy and both were found to be consistent. Using the new results we see that the eigenvectors are always extended, even at arbitrary large but fixed σ. For non-extended eigenvectors

to appear the variance must be chosen appropriately with an N-dependence where ergodic and non-ergodic extended states can arise. Applying the results directly to the Rosenzweig-Porter model we find the fractal dimension of this model and confirm the critical nature of the eigenvectors. The model, however, breaks down under certain conditions ($\gamma > 2$), so an alternative perturbative approach is taken to calculate the distribution function which can then be used to derive the moments of the eigenvectors for this model.

In the fourth chapter we generalise model I and model II by calculating the density of states and the moments of the eigenvectors. Then we compare the analytical expressions with the numerical data by direct diagonalisation and found to be in agreement.

In the fifth chapter we were calculating the density of states for non-Hermitian matrices, in particular the deformed Ginibre ensemble, which required a different approach as the standard method is only applicable for Hermitian matrices. The density of states is expressed in terms of a potential, where the potential is related to the generating function and is similar to the Hermitian case as it is a ratio of determinants. Although different, the essence of the method is still quite similar as we are still using the supersymmetric approach with some alterations for the non-Hermitian case. This work was left incomplete as we had come across a recent paper which covered the density of states of non-Hermitian matrices that also covered the deformed Ginibre ensemble case. The result is also an exact expression for the density of states since they circumvented the saddle-point approximation in the supersymmetric approach.

In the sixth chapter we were expanding on the work in Ref. [1] since the expressions for the moments of the eigenvectors derived in the paper for the simplex model is also applicable for the chiral simplex model. The chiral simplex model is split into two sets where each node in the first set is connected to every node in the second set and vice versa, represented graphically it can also be called a complete bipartite graph. The moments calculated in the paper, up to a certain stage, remained very general with no system specific

information in deriving the moments of the eigenvectors, this meant that we could use the equations in the paper to build on from the paper and derive the moments of the eigenvectors for this case. Unfortunately, as we were running out of time, we could not complete this work as we had to focus our efforts elsewhere, therefore this work has still remained incomplete.

A Efetov's parametrisation and the evaluation of the superintegral over Grassmannians

$$Q = \hat{T}^{-1}\hat{\Lambda}\hat{T} = \begin{pmatrix} U_1 & \\ & U_2 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 & \mathrm{i}\mu_1 & 0 \\ 0 & \lambda_2 & 0 & \mu_2^* \\ \mathrm{i}\mu_1^* & 0 & -\lambda_1 & 0 \\ 0 & \mu_2 & 0 & -\lambda_2 \end{pmatrix} \begin{pmatrix} U_1^{-1} & \\ & U_2^{-1} \end{pmatrix}, \quad (510)$$

where

$$U_1 = \exp \begin{pmatrix} 0 & -\alpha^* \\ \alpha & 0 \end{pmatrix}, \quad \exp \mathrm{i} \begin{pmatrix} 0 & -\beta^* \\ \beta & 0 \end{pmatrix}, \quad (511)$$

$1 \leq \lambda_1 < \infty$, $-1 \leq \lambda_2 \leq 1$, $|\mu_1|^2 = \lambda_1^2 - 1$ and $|\mu_2|^2 = 1 - \lambda_2^2$.

Evaluating this, the final form of the parametrisation is given by

$$Q^{AA} = \begin{pmatrix} \lambda_1 - (\lambda_1 - \lambda_2)\alpha^*\alpha & (\lambda_1 - \lambda_2)\alpha^* \\ (\lambda_1 - \lambda_2)\alpha & \lambda_2 + (\lambda_1 - \lambda_2)\alpha\alpha^* \end{pmatrix},$$

$$Q^{AR} = \begin{pmatrix} \mathrm{i}\mu_1(1 - \frac{\alpha^*\alpha}{2})(1 + \frac{\beta^*\beta}{2}) + \mathrm{i}\mu_2^*\alpha^*\beta & -\mu_1(1 - \frac{\alpha^*\alpha}{2})\beta^* - \mu_2^*\alpha^*(1 + \frac{\beta\beta^*}{2}) \\ \mathrm{i}\mu_1\alpha(1 + \frac{\beta^*\beta}{2}) - \mathrm{i}\mu_2^*(1 - \frac{\alpha\alpha^*}{2})\beta & -\mu_1\alpha\beta^* + \mu_2^*(1 - \frac{\alpha\alpha^*}{2})(1 + \frac{\beta\beta^*}{2}) \end{pmatrix},$$

$$Q^{RA} = \begin{pmatrix} \mathrm{i}\mu_1^*(1 + \frac{\beta^*\beta}{2})(1 - \frac{\alpha^*\alpha}{2}) + \mathrm{i}\mu_2\beta^*\alpha & \mathrm{i}\mu_1^*(1 + \frac{\beta^*\beta}{2})\alpha^* - \mathrm{i}\mu_2\beta^*(1 - \frac{\alpha\alpha^*}{2}) \\ -\mu_1^*\beta(1 - \frac{\alpha^*\alpha}{2}) - \mu_2(1 + \frac{\beta\beta^*}{2})\alpha & -\mu_1^*\beta\alpha^* + \mu_2(1 + \frac{\beta\beta^*}{2})(1 - \frac{\alpha\alpha^*}{2}) \end{pmatrix},$$

$$Q^{RR} = \begin{pmatrix} -\lambda_1 - (\lambda_1 - \lambda_2)\beta^*\beta & -\mathrm{i}(\lambda_1 - \lambda_2)\beta^* \\ -\mathrm{i}(\lambda_1 - \lambda_2)\beta & -\lambda_2 + (\lambda_1 - \lambda_2)\beta\beta^* \end{pmatrix},$$

and $\mathrm{Str}(\hat{T}^{-1}\hat{\Lambda}\hat{T}\hat{\Lambda}) = \mathrm{Str}(Q\hat{\Lambda}) = 2(\lambda_1 - \lambda_2)$.

Recalling back to the definition of $g^{BB}_{\alpha\beta} = [(E - \mathrm{i}\epsilon\hat{\Lambda} - w_n^2 Q_n)^{-1}]^{BB}_{\alpha\beta} \rightarrow [(E - w_n^2 Q_n)^{-1}]^{BB}_{\alpha\beta}$ as $\epsilon \rightarrow 0$, where $[...]^{BB}$ denotes the bose-block of the matrix being evaluated.

The explicit expression for $g^{BB}_{\alpha\beta}$ can be calculated by substituting the parametrisation into g^{BB}

$$\begin{aligned} g^{BB} &= [(E - w_n^2 Q_n)^{-1}]^{BB} = [(E - w_n^2 t - \mathrm{i}s w_n^2 \hat{T}^{-1}\hat{\Lambda}\hat{T})^{-1}]^{BB} \\ &= \left[\frac{(E - w_n^2 t + \mathrm{i}s w_n^2 \hat{T}^{-1}\hat{\Lambda}\hat{T})}{(E - w_n^2)^2 + s^2 w_n^4}\right]^{BB}, \end{aligned} \tag{512}$$

since we are only interested in the bose-block of the matrix our calculations can be greatly simplified by reducing it down to a 2×2 matrix consisting of only the bose-block of Q. The matrix Q, however, is currently in advanced-retarded notation, so we must consider carefully the elements of the bose-block by swapping the second and third row and column to convert it back to the bosonic-fermionic notation. After applying the row and column operations, the bose-block is found to be

$$Q^{BB} = \begin{pmatrix} \lambda_1 - (\lambda_1 - \lambda_2)\alpha^*\alpha & \mathrm{i}\mu_1(1 - \frac{\alpha^*\alpha}{2})(1 + \frac{\beta^*\beta}{2}) + \mathrm{i}\mu_2^*\alpha^*\beta \\ \mathrm{i}\mu_1^*(1 + \frac{\beta^*\beta}{2})(1 - \frac{\alpha^*\alpha}{2}) + \mathrm{i}\mu_2\beta^*\alpha & -\lambda_1 - (\lambda_1 - \lambda_2)\beta^*\beta \end{pmatrix}, \tag{513}$$

therefore, the required matrix g^{BB} is equal to

$$g^{BB} = \begin{pmatrix} \frac{E - w_n^2 t + \mathrm{i}s w_n^2 \lambda_1 - \mathrm{i}s w_n^2(\lambda_1 - \lambda_2)\alpha^*\alpha}{(E - w_n^2)^2 + s^2 w_n^4} & -\frac{s w_n^2 \mu_1 (1 - \frac{\alpha^*\alpha}{2})(1 + \frac{\beta^*\beta}{2}) + s w_n^2 \mu_2^* \alpha^*\beta}{(E - w_n^2)^2 + s^2 w_n^4} \\ -\frac{s w_n^2 \mu_1^* (1 + \frac{\beta^*\beta}{2})(1 - \frac{\alpha^*\alpha}{2}) + s w_n^2 \mu_2 \beta^*\alpha}{(E - w_n^2)^2 + s^2 w_n^4} & \frac{E - w_n^2 t - \mathrm{i}s w_n^2 \lambda_1 - \mathrm{i}s w_n^2(\lambda_1 - \lambda_2)\beta^*\beta}{(E - w_n^2)^2 + s^2 w_n^4} \end{pmatrix}, \tag{514}$$

and hence

$$g_{aa}^{BB} = \frac{E - w_n^2 t + \mathrm{i}sw_n^2\lambda_1 - \mathrm{i}sw_n^2(\lambda_1 - \lambda_2)\alpha^*\alpha}{(E - w_n^2)^2 + s^2 w_n^4}, \tag{515}$$

$$g_{ar}^{B}B = -\frac{sw_n^2\mu_1(1 - \frac{\alpha^*\alpha}{2})(1 + \frac{\beta^*\beta}{2}) + sw_n^2\mu_2^*\alpha^*\beta}{(E - w_n^2)^2 + s^2 w_n^4}, \tag{516}$$

$$g_{ra}^{BB} = -\frac{sw_n^2\mu_1^*(1 + \frac{\beta^*\beta}{2})(1 - \frac{\alpha^*\alpha}{2}) + sw_n^2\mu_2\beta^*\alpha}{(E - w_n^2)^2 + s^2 w_n^4}, \tag{517}$$

$$g_{rr}^{BB} = \frac{E - w_n^2 t - \mathrm{i}sw_n^2\lambda_1 - \mathrm{i}sw_n^2(\lambda_1 - \lambda_2)\beta^*\beta}{(E - w_n^2)^2 + s^2 w_n^4}. \tag{518}$$

As can be seen in the pre-exponential factor, we are interested in the products $g_{aa}^{BB}g_{rr}^{BB}$ and $g_{ar}^{BB}g_{ra}^{BB}$:

$$\begin{aligned} g_{aa}^{BB}g_{rr}^{BB} &= \left[\frac{E - w_n^2 t + \mathrm{i}sw_n^2\lambda_1 - \mathrm{i}sw_n^2(\lambda_1 - \lambda_2)\alpha^*\alpha}{(E - w_n^2 t)^2 + s^2 w_n^4}\right] \\ &\times \left[\frac{E - w_n^2 t - \mathrm{i}sw_n^2\lambda_1 - \mathrm{i}sw_n^2(\lambda_1 - \lambda_2)\beta^*\beta}{(E - w_n^2 t)^2 + s^2 w_n^4}\right] = \frac{1}{[(E - w_n^2 t)^2 + s^2 w_n^4]^2} \\ &\times \Big[(E - w_n^2 t)^2 + s^2 w_n^4\lambda_1^2 - \mathrm{i}sw_n^2(\lambda_1 - \lambda_2)(E - w_n^2 t)(\alpha^*\alpha + \beta^*\beta) \\ &+ s^2 w_n^4\lambda_1(\lambda_1 - \lambda_2)(\beta^*\beta - \alpha^*\alpha) - s^2 w_n^4(\lambda_1 - \lambda_2)^2\alpha^*\alpha\beta^*\beta\Big], \end{aligned} \tag{519}$$

$$\begin{aligned} g_{ar}^{BB}g_{ra}^{BB} &= \left[-\frac{sw_n^2\mu_1(1 - \frac{\alpha^*\alpha}{2})(1 + \frac{\beta^*\beta}{2}) + sw_n^2\mu_2^*\alpha^*\beta}{(E - w_n^2)^2 + s^2 w_n^4}\right] \\ &\times \left[-\frac{sw_n^2\mu_1^*(1 + \frac{\beta^*\beta}{2})(1 - \frac{\alpha^*\alpha}{2}) + sw_n^2\mu_2\beta^*\alpha}{(E - w_n^2)^2 + s^2 w_n^4}\right] = \frac{1}{[(E - w_n^2 t)^2 + s^2 w_n^4]^2} \\ &\times \Big[s^2 w_n^4|\mu_1|^2(1 - \alpha^*\alpha + \beta^*\beta) + s^2 w_n^4\mu_1\mu_2\beta^*\alpha \\ &+ s^2 w_n^4\mu_1^*\mu_2^*\alpha^*\beta - s^2 w_n^4(|\mu_1|^2 + |\mu_2|^2)\alpha^*\alpha\beta^*\beta\Big], \end{aligned} \tag{520}$$

as well as the expansion of

$$\begin{aligned} (g_{aa}^{BB})^{q-2} &= \left[\frac{E - w_n^2 t + \mathrm{i}sw_n^2\lambda_1 - \mathrm{i}sw_n^2(\lambda_1 - \lambda_2)\alpha^*\alpha}{(E - w_n^2 t)^2 + s^2 w_n^4}\right]^{q-2} \\ &= \frac{(E - w_n^2 t + \mathrm{i}sw_n^2\lambda_1)^{q-3}}{[(E - w_n^2 t)^2 + s^2 w_n^4]^{q-2}} \\ &\times \Big[E - w_n^2 t + \mathrm{i}sw_n^2\lambda_1 - (q-2)\mathrm{i}sw_n^2(\lambda_1 - \lambda_2)\alpha^*\alpha\Big], \end{aligned} \tag{521}$$

by the binomial expansion which terminates on the first order of $\alpha^*\alpha$ since α and α^* are Grassmann variables, therefore higher order terms are zero.

The expression for the integration measure reads

$$\mathrm{d}\mu(T) = -\frac{\mathrm{d}\lambda_1\mathrm{d}\lambda_2}{(2\pi)^2(\lambda_1-\lambda_2)^2}\mathrm{d}\phi_1\mathrm{d}\phi_2\mathrm{d}\alpha\mathrm{d}\alpha^*\mathrm{d}\beta\mathrm{d}\beta^*. \tag{522}$$

where $\lambda_1 \in [1,\infty),\ \lambda_2 \in [-1,1],\ \phi_2,\phi_2 \in [0,2\pi]$, and $\alpha,\alpha^*,\beta,\beta^*$ are Grassmann variables.

This now enables one to simplify the superintegral which makes integrating out all of the Grassmann variables possible

$$\begin{aligned}
&\left\langle \left(G^R_{nn}\right)\left(G^A_{nn}\right)^{q-1}\right\rangle = -\int \frac{\mathrm{d}\lambda_1\mathrm{d}\lambda_2}{(2\pi)^2(\lambda_1-\lambda_2)^2}\mathrm{d}\phi_1\mathrm{d}\phi_2\mathrm{d}\alpha\mathrm{d}\alpha^*\mathrm{d}\beta\mathrm{d}\beta^* \\
&\frac{(E-w_n^2t+\mathrm{i}sw_n^2\lambda_1)^{q-3}}{[(E-w_n^2t)^2+s^2w_n^4]^q}e^{-2\epsilon\pi N\rho(E)(\lambda_1-\lambda_2)} \\
&\Big[E-w_n^2t+\mathrm{i}sw_n^2\lambda_1-(q-2)\mathrm{i}sw_n^2(\lambda_1-\lambda_2)\alpha^*\alpha\Big] \\
&\Big\{(E-w_n^2t)^2+s^2w_n^4\lambda_1^2+(q-1)s^2w_n^4\mu_1\mu_2\beta^*\alpha \\
&-\mathrm{i}sw_n^2(\lambda_1-\lambda_2)(E-w_n^2t)(\alpha^*\alpha+\beta^*\beta)+s^2w_n^4\lambda_1(\lambda_1-\lambda_2)(\beta^*\beta-\alpha^*\alpha) \\
&-s^2w_n^4(\lambda_1-\lambda_2)^2\alpha^*\alpha\beta^*\beta+(q-1)s^2w_n^4|\mu_1|^2(1-\alpha^*\alpha+\beta^*\beta) \\
&+(q-1)s^2w_n^4\mu_1^*\mu_2^*\alpha^*\beta-(q-1)s^2w_n^4(\lambda_1^2-\lambda_2^2)\alpha^*\alpha\beta^*\beta\Big\}.
\end{aligned} \tag{523}$$

In order to proceed with the superintegral, we expand out the brackets to

simplify some of the terms in the pre-exponential factor

$$
\begin{aligned}
&\left[E - w_n^2 t + \mathrm{i}s w_n^2 \lambda_1 - (q-2)\mathrm{i}s w_n^2(\lambda_1 - \lambda_2)\alpha^*\alpha\right] \\
&\Big\{(E - w_n^2 t)^2 + s^2 w_n^4 \lambda_1^2 + (q-1)s^2 w_n^4 \mu_1 \mu_2 \beta^*\alpha \\
&- \mathrm{i}s w_n^2(\lambda_1 - \lambda_2)(E - w_n^2 t)(\alpha^*\alpha + \beta^*\beta) + s^2 w_n^4 \lambda_1(\lambda_1 - \lambda_2)(\beta^*\beta - \alpha^*\alpha) \\
&- s^2 w_n^4(\lambda_1 - \lambda_2)^2 \alpha^*\alpha\beta^*\beta + (q-1)s^2 w_n^4 |\mu_1|^2(1 - \alpha^*\alpha + \beta^*\beta) \\
&+ (q-1)s^2 w_n^4 \mu_1^* \mu_2^* \alpha^*\beta - (q-1)s^2 w_n^4(\lambda_1^2 - \lambda_2^2)\alpha^*\alpha\beta^*\beta\Big\} \\
&= (E - w_n^2 t + \mathrm{i}s w_n^2 \lambda_1)\Big\{(E - w_n^2 t)^2 + s^2 w_n^4 \lambda_1^2 + (q-1)s^2 w_n^4 \mu_1 \mu_2 \beta^*\alpha \\
&- \mathrm{i}s w_n^2(\lambda_1 - \lambda_2)(E - w_n^2 t)(\alpha^*\alpha + \beta^*\beta) + s^2 w_n^4 \lambda_1(\lambda_1 - \lambda_2)(\beta^*\beta - \alpha^*\alpha) \\
&- s^2 w_n^4(\lambda_1 - \lambda_2)^2 \alpha^*\alpha\beta^*\beta + (q-1)s^2 w_n^4 |\mu_1|^2(1 - \alpha^*\alpha + \beta^*\beta) \\
&+ (q-1)s^2 w_n^4 \mu_1^* \mu_2^* \alpha^*\beta - (q-1)s^2 w_n^4(\lambda_1^2 - \lambda_2^2)\alpha^*\alpha\beta^*\beta\Big\} \\
&- \mathrm{i}s w_n^2(q-2)(\lambda_1 - \lambda_2)(E - w_n^2 t)^2 \alpha^*\alpha - \mathrm{i}s^3 w_n^6(q-2)(\lambda_1 - \lambda_2)\lambda_1^2 \alpha^*\alpha \\
&- s^2 w_n^4(q-2)(\lambda_1 - \lambda_2)^2(E - w_n^2 t)\alpha^*\alpha\beta^*\beta \\
&- \mathrm{i}s^3 w_n^6 \lambda_1(q-2)(\lambda_1 - \lambda_2)^2 \alpha^*\alpha\beta^*\beta \\
&- \mathrm{i}(q-1)(q-2)s^3 w_n^6 |\mu_1|^2(\lambda_1 - \lambda_2)(\alpha^*\alpha + \alpha^*\alpha\beta^*\beta),
\end{aligned}
\tag{524}
$$

where terms involving second order α and α^* have been neglected due to the nature of the Grassmann variables.

Substituting this expansion into (523) we are able to proceed with the integration, by first integrating over ϕ_1 and ϕ_2 which results in a factor of

$(2\pi)^2$, and then integrating over the Grassmanns

$$\begin{aligned}
&\left\langle \left(G^R_{nn}\right)\left(G^A_{nn}\right)^{q-1}\right\rangle = -\int \frac{\mathrm{d}\lambda_1\mathrm{d}\lambda_2}{(\lambda_1-\lambda_2)^2}\mathrm{d}\alpha\mathrm{d}\alpha^*\mathrm{d}\beta\mathrm{d}\beta^* \\
&\frac{(E-w_n^2t+\mathrm{i}sw_n^2\lambda_1)^{q-3}}{[(E-w_n^2t)^2+s^2w_n^4]^q}e^{-2\epsilon\pi N\rho(E)(\lambda_1-\lambda_2)}\Bigg[(E-w_n^2t+\mathrm{i}sw_n^2\lambda_1) \\
&\Bigg\{(E-w_n^2t)^2+s^2w_n^4\lambda_1^2+(q-1)s^2w_n^4\mu_1\mu_2\beta^*\alpha \\
&-s^2w_n^4(\lambda_1-\lambda_2)^2\alpha^*\alpha\beta^*\beta-\mathrm{i}sw_n^2(\lambda_1-\lambda_2)(E-w_n^2t)(\alpha^*\alpha+\beta^*\beta) \\
&+s^2w_n^4\lambda_1(\lambda_1-\lambda_2)(\beta^*\beta-\alpha^*\alpha)+(q-1)s^2w_n^4\mu_1^*\mu_2^*\alpha^*\beta \\
&+(q-1)s^2w_n^4|\mu_1|^2(1-\alpha^*\alpha+\beta^*\beta)-(q-1)s^2w_n^4(\lambda_1^2-\lambda_2^2)\alpha^*\alpha\beta^*\beta\Bigg\} \\
&-\mathrm{i}sw_n^2(q-2)(\lambda_1-\lambda_2)(E-w_n^2t)^2\alpha^*\alpha-\mathrm{i}s^3w_n^6(q-2)(\lambda_1-\lambda_2)\lambda_1^2\alpha^*\alpha \\
&-s^2w_n^4(q-2)(\lambda_1-\lambda_2)^2(E-w_n^2t)\alpha^*\alpha\beta^*\beta \\
&-\mathrm{i}s^3w_n^6\lambda_1(q-2)(\lambda_1-\lambda_2)^2\alpha^*\alpha\beta^*\beta \\
&-\mathrm{i}(q-1)(q-2)s^3w_n^6|\mu_1|^2(\lambda_1-\lambda_2)(\alpha^*\alpha+\alpha^*\alpha\beta^*\beta)\Bigg].
\end{aligned} \tag{525}$$

Great care must be taken with the integration over the zeroth order Grassmannians, the reader may expect the result to be zero after integrating over the Grassmanns, but we must remember that in the vicinity $\lambda_1 = \lambda_2 = 1$ the term $1/(\lambda_1-\lambda_2)^2$ will diverge, which means that there is a singularity in the integral over λ_1 and λ_2. The exact value of the integral over the zeroth order Grassmanians can be evaluated by a special case of the Parisi-Sourlas-Efetov-Wegner theorem, which states [20]

$$\int d\mu(T)F(\lambda_1,\lambda_2) = F(1,1), \tag{526}$$

where F is an arbitrary function that vanishes at $\lambda_1 \to \infty$.

For simplicity and convenience we will separate the integral over the zeroth order Grassmannians from the other terms, which means we will have two separate integrals, the first over the zeroth order Grassmannians and the second over terms involving Grassmanns. The first integral over the zeroth

order Grassmannians is given by

$$
\begin{aligned}
&-\int \frac{d\lambda_1 d\lambda_2}{(\lambda_1-\lambda_2)^2} d\alpha d\alpha^* d\beta d\beta^* \frac{(E-w_n^2 t+\mathrm{i}sw_n^2\lambda_1)^{q-2}}{[(E-w_n^2 t)^2+s^2w_n^4]^q} \\
&\left\{(E-w_n^2 t)^2+s^2w_n^4\lambda_1^2+(q-1)s^2w_n^4|\mu_1|^2\right\}\exp\{-2\epsilon\pi N\rho(E)(\lambda_1-\lambda_2)\} \\
&=-\frac{(E-w_n^2 t+\mathrm{i}sw_n^2)^{q-2}}{[(E-w_n^2 t)^2+s^2w_n^4]^q}\left\{(E-w_n^2 t)^2+s^2w_n^4\right\} \\
&=-\frac{(E-w_n^2 t+\mathrm{i}sw_n^2)^{q-2}}{[(E-w_n^2 t)^2+s^2w_n^4]^{q-1}},
\end{aligned}
\tag{527}
$$

where $|\mu_1|^2=\lambda_1^2-1$, and the second integral over all terms containing Grassmanns is given by

$$
\begin{aligned}
&-\int \frac{d\lambda_1 d\lambda_2}{(\lambda_1-\lambda_2)^2} d\alpha d\alpha^* d\beta d\beta^* \frac{(E-w_n^2 t+\mathrm{i}sw_n^2\lambda_1)^{q-3}}{[(E-w_n^2 t)^2+s^2w_n^4]^q} e^{-2\epsilon\pi N\rho(E)(\lambda_1-\lambda_2)} \\
&\Big[(E-w_n^2 t+\mathrm{i}sw_n^2\lambda_1)\Big\{(q-1)s^2w_n^4\mu_1\mu_2\beta^*\alpha+(q-1)s^2w_n^4|\mu_1|^2(\beta^*\beta-\alpha^*\alpha) \\
&-\mathrm{i}sw_n^2(\lambda_1-\lambda_2)(E-w_n^2 t)(\alpha^*\alpha+\beta^*\beta)+s^2w_n^4\lambda_1(\lambda_1-\lambda_2)(\beta^*\beta-\alpha^*\alpha) \\
&-s^2w_n^4(\lambda_1-\lambda_2)^2\alpha^*\alpha\beta^*\beta-(q-1)s^2w_n^4(\lambda_1^2-\lambda_2^2)\alpha^*\alpha\beta^*\beta \\
&+(q-1)s^2w_n^4\mu_1^*\mu_2^*\alpha^*\beta\Big\}-\mathrm{i}s^3w_n^6(q-2)(\lambda_1-\lambda_2)\lambda_1^2\alpha^*\alpha \\
&-\mathrm{i}sw_n^2(q-2)(\lambda_1-\lambda_2)(E-w_n^2 t)^2\alpha^*\alpha-\mathrm{i}s^3w_n^6\lambda_1(q-2)(\lambda_1-\lambda_2)^2\alpha^*\alpha\beta^*\beta \\
&-s^2w_n^4(q-2)(\lambda_1-\lambda_2)^2(E-w_n^2 t)\alpha^*\alpha\beta^*\beta \\
&-\mathrm{i}(q-1)(q-2)s^3w_n^6|\mu_1|^2(\lambda_1-\lambda_2)(\alpha^*\alpha+\alpha^*\alpha\beta^*\beta)\Big].
\end{aligned}
\tag{528}
$$

All terms in this integral either contain factors of $(\lambda_1-\lambda_2)$, $|\mu_1|^2$, $|\mu_2|^2$, $\mu_1\mu_2$ or $\mu_1^*\mu_2^*$ which makes the integral convergent over λ_1 and λ_2, therefore the only non-zero contribution to the integral comes from terms containing the

product $\alpha^*\alpha\beta^*\beta$ and the rest can be neglected so the integral simplifies to

$$
\begin{aligned}
&-\int \frac{d\lambda_1 d\lambda_2}{(\lambda_1-\lambda_2)^2} d\alpha d\alpha^* d\beta d\beta^* \frac{(E-w_n^2 t+\mathrm{i}sw_n^2\lambda_1)^{q-3}}{[(E-w_n^2 t)^2+s^2w_n^4]^q} e^{-2\epsilon\pi N\rho(E)(\lambda_1-\lambda_2)} \\
&s^2w_n^4\Big[(E-w_n^2 t+\mathrm{i}sw_n^2\lambda_1)\Big\{-(\lambda_1-\lambda_2)^2-(q-1)(\lambda_1^2-\lambda_2^2)\Big\} \\
&-(q-2)(\lambda_1-\lambda_2)^2(E-w_n^2 t)-\mathrm{i}sw_n^2\lambda_1(q-2)(\lambda_1-\lambda_2)^2 \\
&-\mathrm{i}(q-1)(q-2)sw_n^2|\mu_1|^2(\lambda_1-\lambda_2)\Big]\alpha^*\alpha\beta^*\beta,
\end{aligned}
\tag{529}
$$

multiplying through by $1/(\lambda_1-\lambda_2)^2$ from the integration measure and further simplifying, we arrive at

$$
\begin{aligned}
&\int d\lambda_1 d\lambda_2 d\alpha d\alpha^* d\beta d\beta^* \frac{(E-w_n^2 t+\mathrm{i}sw_n^2\lambda_1)^{q-2}}{[(E-w_n^2 t)^2+s^2w_n^4]^q} e^{-2\epsilon\pi N\rho(E)(\lambda_1-\lambda_2)} \\
&s^2w_n^4\Big[1+(q-1)\frac{\lambda_1+\lambda_2}{\lambda_1-\lambda_2}+(q-2)+\frac{\mathrm{i}(q-1)(q-2)sw_n^2(\lambda_1^2-1)}{(E-w_n^2 t+\mathrm{i}sw_n^2\lambda_1)(\lambda_1-\lambda_2)}\Big]\alpha^*\alpha\beta^*\beta.
\end{aligned}
\tag{530}
$$

Integrating out the Grassmannians is now very simple, once completed the integral reduces down to

$$
\begin{aligned}
&(q-1)\int_1^\infty d\lambda_1\int_{-1}^1 d\lambda_2 \frac{(E-w_n^2 t+\mathrm{i}sw_n^2\lambda_1)^{q-2}}{[(E-w_n^2 t)^2+s^2w_n^4]^q} e^{-2\epsilon\pi N\rho(E)(\lambda_1-\lambda_2)} s^2w_n^4 \\
&\Big[1+\frac{\lambda_1+\lambda_2}{\lambda_1-\lambda_2}+\frac{\mathrm{i}(q-2)sw_n^2(\lambda_1^2-1)}{(E-w_n^2 t+\mathrm{i}sw_n^2\lambda_1)(\lambda_1-\lambda_2)}\Big],
\end{aligned}
\tag{531}
$$

where $1\leq\lambda_1<\infty$ and $-1\leq\lambda_2\leq 1$.

By adding the zeroth order result onto this integral, we obtain our desired integral for the averaged product of two Green's functions.

B The relation between the moments of the eigenvectors and the Green's functions

In this section we explain how the moments of the eigenvectors are related to the Green's function. The Green's function in terms of the wave-functions is

defined as follows

$$G^{R/A} = \sum_{\alpha} \frac{\psi_\alpha \psi_\alpha^\dagger}{E - E_\alpha + \mathrm{i}\sigma\epsilon},\ \sigma = \pm 1.$$

Focusing on a particular element on the diagonal raised to some power, say l, we arrive at

$$G_{nn}^{R/A} = \left\langle n|G^{R/A}|n\right\rangle^l = \left(\sum_{\alpha} \frac{\psi_\alpha(n)\psi_\alpha^\dagger(n)}{E - E_\alpha + \mathrm{i}\sigma\epsilon}\right)^l.$$

We are however interested in the product of two Green's function, one retarded and the other advanced. In the limit ϵ goes to zero there is a singularity of the form ϵ^{1-m-l}, which we extract by

$$\begin{aligned}
N\rho(E)I_q(n) &\propto \lim_{\epsilon\to 0}(2\epsilon)^{m+l-1}\left\langle \left(G_{nn}^R\right)^l \left(G_{nn}^A\right)^m\right\rangle \\
&= \lim_{\epsilon\to 0}(2\epsilon)^{m+l-1}\left\langle \left\langle n|G^R|n\right\rangle^l \left\langle n|G^A|n\right\rangle^m\right\rangle \\
&= \lim_{\epsilon\to 0}(2\epsilon)^{l+m-1}\left\langle \left(\sum_{\alpha} \frac{\psi_\alpha(n)\psi_\alpha^*(n)}{E - E_\alpha + \mathrm{i}\epsilon}\right)^l \left(\sum_{\beta} \frac{\psi_\beta(n)\psi_\beta^*(n)}{E - E_\beta - \mathrm{i}\epsilon}\right)^m\right\rangle \\
&= \lim_{\epsilon\to 0}(2\epsilon)^{m+l-1}\sum_{\alpha}\left\langle \frac{|\psi_\alpha(n)|^{2l}}{(E - E_\alpha + \mathrm{i}\epsilon)^l}\frac{|\psi_\alpha(n)|^{2m}}{(E - E_\alpha - \mathrm{i}\epsilon)^m}\right\rangle \\
&= \lim_{\epsilon\to 0}(2\epsilon)^{m+l-1}\sum_{\alpha}\left\langle \frac{|\psi_\alpha(n)|^{2q}}{(E - E_\alpha + \mathrm{i}\epsilon)^l(E - E_\alpha - \mathrm{i}\epsilon)^m}\right\rangle \\
&= \sum_{n,\alpha} c\left\langle |\psi_\alpha(n)|^{2q}\delta(E - E_\alpha)\right\rangle,\ q = l + m,
\end{aligned} \tag{532}$$

where c is a constant. The reasoning behind the final equality is shown in the calculation below.

To derive the relation in Eq. (32) explicitly we need to use Cauchy's residue theorem and we begin by making the substitution $a = E - E_\alpha$, then consider the integral

$$\lim_{\epsilon\to 0}(2\epsilon)^{l+m-1}\int_C da\ \frac{1}{(a + i\epsilon)^l(a - i\epsilon)^m} f(a) \tag{533}$$

where C is the contour of integration, $f(a)$ is an arbitrary function. To pro-

ceed with the integration the substitution $a = \epsilon z$ is required. This simplifies the integral to:

$$\begin{aligned}\lim_{\epsilon\to 0}(2\epsilon)^{l+m-1}\int_{C'} d\epsilon z\ \frac{1}{\epsilon^{l+m}}\frac{1}{(z+\mathrm{i})^l(z-\mathrm{i})^m}f(\epsilon z)\\ = 2^{l+m-1}f(0)\int_{C'} dz\ \frac{1}{(z+\mathrm{i})^l(z-\mathrm{i})^m}\end{aligned} \tag{534}$$

where C' is the new contour of integration.

By Cauchy's residue theorem:

$$2^{l+m-1}\int_{C'} dz\ \frac{1}{(z+\mathrm{i})^l(z-\mathrm{i})^m} = (-1)^m\frac{2\pi\mathrm{i}}{\mathrm{i}^{l+m}}\frac{(l+m-2)!}{(l-1)!(m-1)!} \tag{535}$$

Therefore (533) yields

$$\lim_{\epsilon\to 0}(2\epsilon)^{l+m-1}\int_C da\ \frac{1}{(a+\mathrm{i}\epsilon)^l(a-i\epsilon)^m}f(a) = \frac{2\pi}{\mathrm{i}^{l-m}}\frac{(l+m-2)!}{(l-1)!(m-1)!}f(0), \tag{536}$$

which implies that the integrand is equal to

$$\lim_{\epsilon\to 0}(2\epsilon)^{l+m-1}(a+\mathrm{i}\epsilon)^{-l}(a-\mathrm{i}\epsilon)^{-m} = \frac{2\pi}{\mathrm{i}^{l-m}}\frac{(l+m-2)!}{(l-1)!(m-1)!}\delta(a). \tag{537}$$

This is the δ-function that appears in the final equality of Eq. (532).

Therefore

$$N\rho(E)I_q(n) \propto \frac{2\pi}{\mathrm{i}^{l-m}}\frac{(l+m-2)!}{(l-1)!(m-1)!}\sum_{n,\alpha}\left\langle|\psi_\alpha(n)|^{2q}\delta(E-E_\alpha)\right\rangle. \tag{538}$$

Comparing this with the definition of the moments of the eigenvectors, we see that we have to divide by $\frac{2\pi}{\mathrm{i}^{l-m}}\frac{(l+m-2)!}{(l-1)!(m-1)!}$ for $I_q(n)$.

Finally we arrive at

$$I_q(n) = \frac{\mathrm{i}^{l-m}}{2\pi\rho(E)N}\frac{(m-1)!(l-1)!}{(m+l-2)!}\lim_{\epsilon\to 0}(2\epsilon)^{m+l-1}\left\langle\left(G^R_{nn}\right)^l\left(G^A_{nn}\right)^m\right\rangle$$

as required.

C Efetov's parametrisation and evaluation of the superintegrals over the Grassmannians for model II

The Bose-block of the parametrisation, from Eq. (513), substituted into g^{BB} results in

$$g^{BB} = \begin{pmatrix} \frac{E-t-d_n+\mathrm{i}s\lambda_1-\mathrm{i}s(\lambda_1-\lambda_2)\alpha^*\alpha}{(E-t-d_n)^2+s^2} & -\frac{s\mu_1(1-\frac{\alpha^*\alpha}{2})(1+\frac{\beta^*\beta}{2})+s\mu_2^*\alpha^*\beta}{(E-t-d_n)^2+s^2} \\ -\frac{s\mu_1^*(1+\frac{\beta^*\beta}{2})(1-\frac{\alpha^*\alpha}{2})+s\mu_2\beta^*\alpha}{(E-t-d_n)^2+s^2} & \frac{E-t-d_n-\mathrm{i}s\lambda_1-\mathrm{i}s(\lambda_1-\lambda_2)\beta^*\beta}{(E-t-d_n)^2+s^2}, \end{pmatrix} \tag{539}$$

hence,

$$\begin{aligned} g_{aa}^{BB} &= \frac{E-t-d_n+\mathrm{i}s\lambda_1-\mathrm{i}s(\lambda_1-\lambda_2)\alpha^*\alpha}{(E-t-d_n)^2+s^2}, \\ g_{ar}^{BB} &= -\frac{s\mu_1(1-\frac{\alpha^*\alpha}{2})(1+\frac{\beta^*\beta}{2})+s\mu_2^*\alpha^*\beta}{(E-t-d_n)^2+s^2}, \\ g_{ra}^{BB} &= -\frac{s\mu_1^*(1+\frac{\beta^*\beta}{2})(1-\frac{\alpha^*\alpha}{2})+s\mu_2\beta^*\alpha}{(E-t-d_n)^2+s^2}, \\ g_{rr}^{BB} &= \frac{E-t-d_n-\mathrm{i}s\lambda_1-\mathrm{i}s(\lambda_1-\lambda_2)\beta^*\beta}{(E-t-d_n)^2+s^2}. \end{aligned} \tag{540}$$

Similarly, we are interested in the products $g_{aa}^{BB}g_{rr}^{BB}$ and $g_{ar}^{BB}g_{ra}^{BB}$:

$$\begin{aligned} g_{aa}^{BB}g_{rr}^{BB} &= \left[\frac{E-t-d_n+\mathrm{i}s\lambda_1-\mathrm{i}s(\lambda_1-\lambda_2)\alpha^*\alpha}{(E-t-d_n)^2+s^2}\right] \\ &\times \left[\frac{E-t-d_n-\mathrm{i}s\lambda_1-\mathrm{i}s(\lambda_1-\lambda_2)\beta^*\beta}{(E-t-d_n)^2+s^2}\right] = \frac{1}{[(E-t-d_n)^2+s^2]^2} \\ &\times \Big[(E-t-d_n)^2+s^2\lambda_1^2-\mathrm{i}s(E-t-d_n)(\lambda_1-\lambda_2)(\alpha^*\alpha+\beta^*\beta) \\ &+ s^2\lambda_1(\lambda_1-\lambda_2)(\beta^*\beta-\alpha^*\alpha)-s^2(\lambda_1-\lambda_2)^2\alpha^*\alpha\beta^*\beta\Big], \end{aligned} \tag{541}$$

$$
\begin{aligned}
g_{ar}^{BB} g_{ra}^{BB} &= \left[-\frac{s\mu_1(1-\frac{\alpha^*\alpha}{2})(1+\frac{\beta^*\beta}{2}) + s\mu_2^*\alpha^*\beta}{(E-t-d_n)^2+s^2} \right] \\
&\times \left[-\frac{s\mu_1^*(1+\frac{\beta^*\beta}{2})(1-\frac{\alpha^*\alpha}{2}) + s\mu_2\beta^*\alpha}{(E-t-d_n)^2+s^2} \right] = \frac{1}{[(E-t-d_n)^2+s^2]^2} \\
&\times \Big[s^2|\mu_1|^2(1-\alpha^*\alpha+\beta^*\beta) + s^2\mu_1\mu_2\beta^*\alpha \\
&+ s^2\mu_1^*\mu_2^*\alpha^*\beta - s^2(|\mu_1|^2+|\mu_2|^2)\alpha^*\alpha\beta^*\beta \Big],
\end{aligned}
\tag{542}
$$

and the expansion of

$$
\begin{aligned}
(g_{aa}^{BB})^{q-2} &= \left[\frac{E-t-d_n+\mathrm{i}s\lambda_1 - \mathrm{i}s(\lambda_1-\lambda_2)\alpha^*\alpha}{(E-t-d_n)^2+s^2} \right]^{q-2} \\
&= \frac{(E-t-d_n+\mathrm{i}s\lambda_1)^{q-3}}{[(E-t-d_n)^2+s^2]^{q-2}} \\
&\times \Big[E-t-d_n+\mathrm{i}s\lambda_1 - (q-2)\mathrm{i}s(\lambda_1-\lambda_2)\alpha^*\alpha \Big],
\end{aligned}
\tag{543}
$$

by the binomial expansion which, again, terminates on the first order of $\alpha^*\alpha$.

The averaged product of two Green's functions can, therefore, be written as

$$
\begin{aligned}
&\left\langle \left(G_{nn}^R\right)\left(G_{nn}^A\right)^{q-1} \right\rangle = -\int \frac{\mathrm{d}\lambda_1\mathrm{d}\lambda_2}{(2\pi)^2(\lambda_1-\lambda_2)^2}\mathrm{d}\phi_1\mathrm{d}\phi_2\mathrm{d}\alpha\mathrm{d}\alpha^*\mathrm{d}\beta\mathrm{d}\beta^* \\
&\frac{(E-t-d_n+\mathrm{i}s\lambda_1)^{q-3}}{[(E-t-d_n)^2+s^2]^q} e^{-2\epsilon\pi\rho(E)N(\lambda_1-\lambda_2)} \\
&\Big[E-t-d_n+\mathrm{i}s\lambda_1 - (q-2)\mathrm{i}s(\lambda_1-\lambda_2)\alpha^*\alpha \Big] \\
&\Big\{ (E-t-d_n)^2 + s^2\lambda_1^2 - \mathrm{i}s(E-t-d_n)(\lambda_1-\lambda_2)(\alpha^*\alpha+\beta^*\beta) \\
&+ s^2\lambda_1(\lambda_1-\lambda_2)(\beta^*\beta-\alpha^*\alpha) - s^2(\lambda_1-\lambda_2)^2\alpha^*\alpha\beta^*\beta \\
&+ (q-1)s^2|\mu_1|^2(1-\alpha^*\alpha+\beta^*\beta) + (q-1)s^2\mu_1\mu_2\beta^*\alpha \\
&+ (q-1)s^2\mu_1^*\mu_2^*\alpha^*\beta - (q-1)s^2(|\mu_1|^2+|\mu_2|^2)\alpha^*\alpha\beta^*\beta \Big\},
\end{aligned}
\tag{544}
$$

and to simplify the integral even further so that we can integrate over the

Grassmanns, we expand out the backet:

$$
\begin{aligned}
&\Big[E - t - d_n + \mathrm{i}s\lambda_1 - (q-2)\mathrm{i}s(\lambda_1 - \lambda_2)\alpha^*\alpha\Big] \\
&\times \Big\{(E - t - d_n)^2 + s^2\lambda_1^2 - \mathrm{i}s(E - t - d_n)(\lambda_1 - \lambda_2)(\alpha^*\alpha + \beta^*\beta) \\
&+ s^2\lambda_1(\lambda_1 - \lambda_2)(\beta^*\beta - \alpha^*\alpha) - s^2(\lambda_1 - \lambda_2)^2\alpha^*\alpha\beta^*\beta \\
&+ (q-1)s^2|\mu_1|^2(1 - \alpha^*\alpha + \beta^*\beta) + (q-1)s^2\mu_1\mu_2\beta^*\alpha \\
&+ (q-1)s^2\mu_1^*\mu_2^*\alpha^*\beta - (q-1)s^2(|\mu_1|^2 + |\mu_2|^2)\alpha^*\alpha\beta^*\beta\Big\} \\
&= (E - t - d_n + \mathrm{i}s\lambda_1)\Big\{(E - t - d_n)^2 + s^2\lambda_1^2 + (q-1)s^2\mu_1\mu_2\beta^*\alpha \\
&- \mathrm{i}s(E - t - d_n)(\lambda_1 - \lambda_2)(\alpha^*\alpha + \beta^*\beta) + s^2\lambda_1(\lambda_1 - \lambda_2)(\beta^*\beta - \alpha^*\alpha) \\
&- s^2(\lambda_1 - \lambda_2)^2\alpha^*\alpha\beta^*\beta + (q-1)s^2|\mu_1|^2(1 - \alpha^*\alpha + \beta^*\beta) \\
&+ (q-1)s^2\mu_1^*\mu_2^*\alpha^*\beta - (q-1)s^2(|\mu_1|^2 + |\mu_2|^2)\alpha^*\alpha\beta^*\beta\Big\} \\
&- \mathrm{i}s(q-2)(\lambda_1 - \lambda_2)(E - t - d_n)^2\alpha^*\alpha - \mathrm{i}s^3(q-2)\lambda_1^2(\lambda_1 - \lambda_2)\alpha^*\alpha \\
&- s^2(q-2)(\lambda_1 - \lambda_2)^2(E - t - d_n)\alpha^*\alpha\beta^*\beta \\
&- \mathrm{i}s^3(q-2)\lambda_1(\lambda_1 - \lambda_2)^2\alpha^*\alpha\beta^*\beta \\
&- \mathrm{i}s^3(q-1)(q-2)|\mu_1|^2(\lambda_1 - \lambda_2)(\alpha^*\alpha + \alpha^*\alpha\beta^*\beta),
\end{aligned}
\tag{545}
$$

where higher than first order terms involving α and α^* have been neglected again as the squares and higher order terms vanish.

By substituting this into the integral and integrating over ϕ_1 and ϕ_2, which

results in a factor of $(2\pi)^2$, it simplifies the integral down to

$$\begin{aligned}
&\left\langle \left(G^R_{nn}\right)\left(G^A_{nn}\right)^{q-1}\right\rangle = -\int \frac{\mathrm{d}\lambda_1\mathrm{d}\lambda_2}{(\lambda_1-\lambda_2)^2}\mathrm{d}\alpha\mathrm{d}\alpha^*\mathrm{d}\beta\mathrm{d}\beta^* \\
&\frac{(E-t-d_n+\mathrm{i}s\lambda_1)^{q-3}}{[(E-t-d_n)^2+s^2]^q}e^{-2\epsilon\pi\rho(E)N(\lambda_1-\lambda_2)} \\
&\Bigg[(E-t-d_n+\mathrm{i}s\lambda_1)\Big\{(E-t-d_n)^2+s^2\lambda_1^2+(q-1)s^2\mu_1\mu_2\beta^*\alpha \\
&-\mathrm{i}s(E-t-d_n)(\lambda_1-\lambda_2)(\alpha^*\alpha+\beta^*\beta)+s^2\lambda_1(\lambda_1-\lambda_2)(\beta^*\beta-\alpha^*\alpha) \\
&-s^2(\lambda_1-\lambda_2)^2\alpha^*\alpha\beta^*\beta+(q-1)s^2|\mu_1|^2(1-\alpha^*\alpha+\beta^*\beta) \\
&+(q-1)s^2\mu_1^*\mu_2^*\alpha^*\beta-(q-1)s^2(|\mu_1|^2+|\mu_2|^2)\alpha^*\alpha\beta^*\beta\Big\} \\
&-\mathrm{i}s(q-2)(\lambda_1-\lambda_2)(E-t-d_n)^2\alpha^*\alpha-\mathrm{i}s^3(q-2)\lambda_1^2(\lambda_1-\lambda_2)\alpha^*\alpha \\
&-s^2(q-2)(\lambda_1-\lambda_2)^2(E-t-d_n)\alpha^*\alpha\beta^*\beta \\
&-\mathrm{i}s^3(q-2)\lambda_1(\lambda_1-\lambda_2)^2\alpha^*\alpha\beta^*\beta \\
&-\mathrm{i}s^3(q-1)(q-2)|\mu_1|^2(\lambda_1-\lambda_2)(\alpha^*\alpha+\alpha^*\alpha\beta^*\beta)\Bigg],
\end{aligned} \tag{546}$$

using the same reasoning as previously, we separate the integration over the zeroth order Grassmanians from the other terms for convenience. The integral over the zeroth order Grassmannians is equal to

$$\begin{aligned}
&-\int \frac{\mathrm{d}\lambda_1\mathrm{d}\lambda_2}{(\lambda_1-\lambda_2)^2}\mathrm{d}\alpha\mathrm{d}\alpha^*\mathrm{d}\beta\mathrm{d}\beta^*\frac{(E-t-d_n+\mathrm{i}s\lambda_1)^{q-2}}{[(E-t-d_n)^2+s^2]^q} \\
&\Big\{(E-t-d_n)^2+s^2\lambda_1^2+(q-1)s^2|\mu_1|^2\Big\}e^{-2\epsilon\pi\rho(E)N(\lambda_1-\lambda_2)} \\
&=-\frac{(E-t-d_n+\mathrm{i}s)^{q-2}}{[(E-t-d_n)^2+s^2]^q}\Big\{(E-t-d_n)^2+s^2\Big\} \\
&=-\frac{(E-t-d_n+\mathrm{i}s)^{q-2}}{[(E-t-d_n)^2+s^2]^{q-1}},
\end{aligned} \tag{547}$$

where $|\mu_1|^2=\lambda_1^2-1$, and the second integral containing all the other terms

is given by

$$
\begin{aligned}
&-\int \frac{\mathrm{d}\lambda_1\mathrm{d}\lambda_2}{(\lambda_1-\lambda_2)^2}\mathrm{d}\alpha\mathrm{d}\alpha^*\mathrm{d}\beta\mathrm{d}\beta^*\frac{(E-t-d_n+\mathrm{i}s\lambda_1)^{q-3}}{[(E-t-d_n)^2+s^2]^q}e^{-2\epsilon\pi\rho(E)N(\lambda_1-\lambda_2)}\\
&\Bigg[(E-t-d_n+\mathrm{i}s\lambda_1)\Bigg\{(q-1)s^2\mu_1\mu_2\beta^*\alpha\\
&-\mathrm{i}s(E-t-d_n)(\lambda_1-\lambda_2)(\alpha^*\alpha+\beta^*\beta)+s^2\lambda_1(\lambda_1-\lambda_2)(\beta^*\beta-\alpha^*\alpha)\\
&-s^2(\lambda_1-\lambda_2)^2\alpha^*\alpha\beta^*\beta-(q-1)s^2|\mu_1|^2(\alpha^*\alpha-\beta^*\beta)\\
&+(q-1)s^2\mu_1^*\mu_2^*\alpha^*\beta-(q-1)s^2(|\mu_1|^2+|\mu_2|^2)\alpha^*\alpha\beta^*\beta\Bigg\}\\
&-\mathrm{i}s(q-2)(\lambda_1-\lambda_2)(E-t-d_n)^2\alpha^*\alpha-\mathrm{i}s^3(q-2)\lambda_1^2(\lambda_1-\lambda_2)\alpha^*\alpha\\
&-s^2(q-2)(\lambda_1-\lambda_2)^2(E-t-d_n)\alpha^*\alpha\beta^*\beta\\
&-\mathrm{i}s^3(q-2)\lambda_1(\lambda_1-\lambda_2)^2\alpha^*\alpha\beta^*\beta\\
&-\mathrm{i}s^3(q-1)(q-2)|\mu_1|^2(\lambda_1-\lambda_2)(\alpha^*\alpha+\alpha^*\alpha\beta^*\beta)\Bigg].
\end{aligned}
\tag{548}
$$

The only non-zero contribution comes from the terms proportional to $\alpha^*\alpha\beta^*\beta$, the reasoning is provided in Appendix A, Eq. (528), this means that we can neglect the other terms and reduces the integration to the following form

$$
\begin{aligned}
&-\int \frac{\mathrm{d}\lambda_1\mathrm{d}\lambda_2}{(\lambda_1-\lambda_2)^2}\mathrm{d}\alpha\mathrm{d}\alpha^*\mathrm{d}\beta\mathrm{d}\beta^*\frac{(E-t-d_n+\mathrm{i}s\lambda_1)^{q-3}}{[(E-t-d_n)^2+s^2]^q}e^{-2\epsilon\pi\rho(E)N(\lambda_1-\lambda_2)}\\
&\Bigg[(E-t-d_n+\mathrm{i}s\lambda_1)\Bigg\{-s^2(\lambda_1-\lambda_2)^2-(q-1)s^2(|\mu_1|^2+|\mu_2|^2)\Bigg\}\\
&-s^2(q-2)(\lambda_1-\lambda_2)^2(E-t-d_n)-\mathrm{i}s^3(q-2)\lambda_1(\lambda_1-\lambda_2)^2\\
&-\mathrm{i}s^3(q-1)(q-2)|\mu_1|^2(\lambda_1-\lambda_2)\Bigg]\alpha^*\alpha\beta^*\beta,
\end{aligned}
\tag{549}
$$

multiplying through by a factor of $1/(\lambda_1-\lambda_2)^2$ from the integration measure and simplifying the expression, we arrive at

$$
\begin{aligned}
&\int \mathrm{d}\lambda_1\mathrm{d}\lambda_2\mathrm{d}\alpha\mathrm{d}\alpha^*\mathrm{d}\beta\mathrm{d}\beta^*\frac{(E-t-d_n+\mathrm{i}s\lambda_1)^{q-2}}{[(E-t-d_n)^2+s^2]^q}e^{-2\epsilon\pi\rho(E)N(\lambda_1-\lambda_2)}\\
&s^2\Bigg[1+(q-1)\frac{(|\mu_1|^2+|\mu_2|^2)}{(\lambda_1-\lambda_2)^2}+(q-2)\\
&+\frac{\mathrm{i}s(q-1)(q-2)|\mu_1|^2}{(E-t-d_n+\mathrm{i}s\lambda_1)(\lambda_1-\lambda_2)}\Bigg]\alpha^*\alpha\beta^*\beta.
\end{aligned}
\tag{550}
$$

Once we have integrated over the Grassmanns, the integral simplifies down to

$$(q-1)\int_1^\infty \mathrm{d}\lambda_1 \int_{-1}^1 \mathrm{d}\lambda_2 \frac{(E-t-d_n+\mathrm{i}s\lambda_1)^{q-2}}{[(E-t-d_n)^2+s^2]^q} e^{-2\epsilon\pi\rho(E)N(\lambda_1-\lambda_2)} \\ s^2\left[1+\frac{(|\mu_1|^2+|\mu_2|^2)}{(\lambda_1-\lambda_2)^2}+\frac{\mathrm{i}s(q-2)|\mu_1|^2}{(E-t-d_n+\mathrm{i}s\lambda_1)(\lambda_1-\lambda_2)}\right], \tag{551}$$

substituting $|\mu_1|^2=\lambda_1^2-1$ and $|\mu_2|^2=1-\lambda_2^2$ into the integral, we are able to simplify the expression further, reducing it down to

$$(q-1)\int_1^\infty \mathrm{d}\lambda_1 \int_{-1}^1 \mathrm{d}\lambda_2 \frac{(E-t-d_n+\mathrm{i}s\lambda_1)^{q-2}}{[(E-t-d_n)^2+s^2]^q} e^{-2\epsilon\pi\rho(E)N(\lambda_1-\lambda_2)} \\ s^2\left[1+\frac{\lambda_1+\lambda_2}{\lambda_1-\lambda_2}+\frac{\mathrm{i}s(q-2)(\lambda_1^2-1)}{(E-t-d_n+\mathrm{i}s\lambda_1)(\lambda_1-\lambda_2)}\right], \tag{552}$$

where $1\leq\lambda_1<\infty$ and $-1\leq\lambda_2\leq 1$.

Adding this integral along with Eq. (547), we receive the expression for the averaged product of two Green's functions

$$\left\langle \left(G^R_{nn}\right)\left(G^A_{nn}\right)^{q-1}\right\rangle = -\frac{(E-t-d_n+\mathrm{i}s)^{q-2}}{[(E-t-d_n)^2+s^2]^{q-1}}+(q-1)\int_1^\infty \mathrm{d}\lambda_1\int_{-1}^1 \mathrm{d}\lambda_2 \\ \times\frac{(E-t-d_n+\mathrm{i}s\lambda_1)^{q-2}}{[(E-t-d_n)^2+s^2]^q} e^{-2\epsilon\pi\rho(E)N(\lambda_1-\lambda_2)} \\ \times s^2\left[1+\frac{\lambda_1+\lambda_2}{\lambda_1-\lambda_2}+\frac{\mathrm{i}s(q-2)(\lambda_1^2-1)}{(E-t-d_n+\mathrm{i}s\lambda_1)(\lambda_1-\lambda_2)}\right]. \tag{553}$$

D Generalising model I for random diagonal entries

To proceed with the averaging, it is much more convenient to express the equation in the following form

$$
\begin{aligned}
1 &= \left\langle \frac{x^4}{(E-x^2\langle t\rangle)^2+\langle s\rangle^2 x^4} \right\rangle_x \\
&= \left\langle \frac{x^4}{E^2-2Ex^2\langle t\rangle+x^4\langle t\rangle^2+\langle s\rangle^2 x^4} \right\rangle_x \\
&= \left\langle \frac{x^4}{\left(\langle t\rangle^2+\langle s\rangle^2\right)x^4-2Ex^2\langle t\rangle+E^2} \right\rangle_x \\
&= \frac{1}{\langle t\rangle^2+\langle s\rangle^2}\left\langle \frac{x^4}{x^4-\frac{2Ex^2\langle t\rangle}{\langle t\rangle^2+\langle s\rangle^2}+\frac{E^2}{\langle t\rangle^2+\langle s\rangle^2}} \right\rangle_x \\
&= \frac{1}{\langle t\rangle^2+\langle s\rangle^2}\left\langle \frac{x^4-\frac{2Ex^2\langle t\rangle}{\langle t\rangle^2+\langle s\rangle^2}+\frac{E^2}{\langle t\rangle^2+\langle s\rangle^2}+\frac{2Ex^2\langle t\rangle}{\langle t\rangle^2+\langle s\rangle^2}-\frac{E^2}{\langle t\rangle^2+\langle s\rangle^2}}{x^4-\frac{2Ex^2\langle t\rangle}{\langle t\rangle^2+\langle s\rangle^2}+\frac{E^2}{\langle t\rangle^2+\langle s\rangle^2}} \right\rangle_x \\
&= \frac{1}{\langle t\rangle^2+\langle s\rangle^2}\left\langle 1+\frac{\frac{2Ex^2\langle t\rangle}{\langle t\rangle^2+\langle s\rangle^2}-\frac{E^2}{\langle t\rangle^2+\langle s\rangle^2}}{x^4-\frac{2Ex^2\langle t\rangle}{\langle t\rangle^2+\langle s\rangle^2}+\frac{E^2}{\langle t\rangle^2+\langle s\rangle^2}} \right\rangle_x \\
&= \frac{1}{\langle t\rangle^2+\langle s\rangle^2}\left\langle 1+\frac{\frac{2Ex^2}{\langle t\rangle^2+\langle s\rangle^2}-\frac{E^2}{\langle t\rangle^2+\langle s\rangle^2}}{\left(x^2-\frac{E\langle t\rangle}{\langle t\rangle^2+\langle s\rangle^2}\right)^2+\frac{E^2}{\langle t\rangle^2+\langle s\rangle^2}-\frac{E^2\langle t\rangle^2}{\left(\langle t\rangle^2+\langle s\rangle^2\right)^2}} \right\rangle_x \\
&= \frac{1}{\langle t\rangle^2+\langle s\rangle^2}\left\langle 1+\frac{1}{\langle t\rangle^2+\langle s\rangle^2}\frac{2Ex^2-E^2}{\left(x^2-\frac{E\langle t\rangle}{\langle t\rangle^2+\langle s\rangle^2}\right)^2+\frac{E^2\langle s\rangle^2}{\left(\langle t\rangle^2+\langle s\rangle^2\right)^2}} \right\rangle_x \\
&= \frac{1}{\langle t\rangle^2+\langle s\rangle^2}\left(1+\frac{1}{\langle t\rangle^2+\langle s\rangle^2}\left\langle \frac{2Ex^2-E^2}{\left(x^2-\frac{E\langle t\rangle}{\langle t\rangle^2+\langle s\rangle^2}\right)^2+\frac{E^2\langle s\rangle^2}{\left(\langle t\rangle^2+\langle s\rangle^2\right)^2}} \right\rangle_x\right),
\end{aligned}
\tag{554}
$$

by completing the square in the denominator and pulling constants out of the averaging.

The poles of the integrand are calculated by equating the denominator to

0:

$$\begin{aligned}
&\left(x^2 - \frac{E\langle t\rangle}{\langle t\rangle^2 + \langle s\rangle^2}\right)^2 + \frac{E^2\langle s\rangle^2}{\left(\langle t\rangle^2 + \langle s\rangle^2\right)^2} = 0 \\
&\implies \left(x^2 - \frac{E\langle t\rangle}{\langle t\rangle^2 + \langle s\rangle^2}\right)^2 = -\frac{E^2\langle s\rangle^2}{\left(\langle t\rangle^2 + \langle s\rangle^2\right)^2} \\
&\implies x^2 - \frac{E\langle t\rangle}{\langle t\rangle^2 + \langle s\rangle^2} = \pm \mathrm{i}\frac{E\langle s\rangle}{\langle t\rangle^2 + \langle s\rangle^2} \\
&\implies x^2 = \frac{E\langle t\rangle}{\langle t\rangle^2 + \langle s\rangle^2} \pm \mathrm{i}\frac{E\langle s\rangle}{\langle t\rangle^2 + \langle s\rangle^2} \\
&\implies x = \pm\sqrt{\frac{E\langle t\rangle}{\langle t\rangle^2 + \langle s\rangle^2} \pm \mathrm{i}\frac{E\langle s\rangle}{\langle t\rangle^2 + \langle s\rangle^2}} \\
&\implies x = \pm\sqrt{\frac{E}{\langle t\rangle^2 + \langle s\rangle^2}}\sqrt{\langle t\rangle \pm \mathrm{i}\langle s\rangle}.
\end{aligned} \tag{555}$$

Inputting the values of z_1, z_2, z_3 and z_4 we are able to simplify the expression

for the first residue:

$$
\begin{aligned}
&\frac{\left(\frac{\langle t\rangle^2+\langle s\rangle^2}{E}\right)^{3/2}\left(2E\frac{E}{\langle t\rangle^2+\langle s\rangle^2}(\langle t\rangle-\mathrm{i}\langle s\rangle)\langle t\rangle-E^2\right)e^{-\mathrm{i}\kappa\sqrt{\frac{E}{\langle t\rangle^2+\langle s\rangle^2}}}\sqrt{\langle t\rangle-\mathrm{i}\langle s\rangle}}{2\sqrt{\langle t\rangle-\mathrm{i}\langle s\rangle}\left(\sqrt{\langle t\rangle-\mathrm{i}\langle s\rangle}-\sqrt{\langle t\rangle+\mathrm{i}\langle s\rangle}\right)\left(\sqrt{\langle t\rangle-\mathrm{i}\langle s\rangle}+\sqrt{\langle t\rangle+\mathrm{i}\langle s\rangle}\right)}\\
&=\frac{\left(\frac{\langle t\rangle^2+\langle s\rangle^2}{E}\right)^{3/2}E^2\left(\frac{2\langle t\rangle}{\langle t\rangle+\mathrm{i}\langle s\rangle}-1\right)e^{-\mathrm{i}\kappa\sqrt{\frac{E}{\langle t\rangle+\mathrm{i}\langle s\rangle}}}}{2\sqrt{\langle t\rangle-\mathrm{i}\langle s\rangle}\left(\sqrt{\langle t\rangle-\mathrm{i}\langle s\rangle}-\sqrt{\langle t\rangle+\mathrm{i}\langle s\rangle}\right)\left(\sqrt{\langle t\rangle-\mathrm{i}\langle s\rangle}+\sqrt{\langle t\rangle+\mathrm{i}\langle s\rangle}\right)}\\
&=\frac{\left(\frac{\langle t\rangle^2+\langle s\rangle^2}{E}\right)^{3/2}E^2\left(\frac{2\langle t\rangle}{\langle t\rangle+\mathrm{i}\langle s\rangle}-1\right)e^{-\mathrm{i}\kappa\sqrt{\frac{E}{\langle t\rangle+\mathrm{i}\langle s\rangle}}}}{2\sqrt{\langle t\rangle-\mathrm{i}\langle s\rangle}\left(\langle t\rangle-\mathrm{i}\langle s\rangle-(\langle t\rangle+\mathrm{i}\langle s\rangle)\right)}\\
&=-\frac{\left(\frac{\langle t\rangle^2+\langle s\rangle^2}{E}\right)^{3/2}E^2\left(\frac{2\langle t\rangle}{\langle t\rangle+\mathrm{i}\langle s\rangle}-1\right)e^{-\mathrm{i}\kappa\sqrt{\frac{E}{\langle t\rangle+\mathrm{i}\langle s\rangle}}}}{4\mathrm{i}\langle s\rangle\sqrt{\langle t\rangle-\mathrm{i}\langle s\rangle}}\\
&=-\frac{\left(\langle t\rangle^2+\langle s\rangle^2\right)^{3/2}\sqrt{E}\left(\frac{2\langle t\rangle}{\langle t\rangle+\mathrm{i}\langle s\rangle}-1\right)e^{-\mathrm{i}\kappa\sqrt{\frac{E}{\langle t\rangle+\mathrm{i}\langle s\rangle}}}}{4\mathrm{i}\langle s\rangle\sqrt{\langle t\rangle-\mathrm{i}\langle s\rangle}}\\
&=-\frac{\left(\langle t\rangle^2+\langle s\rangle^2\right)\sqrt{\langle t\rangle^2+\langle s\rangle^2}\sqrt{E}\left(\frac{2\langle t\rangle}{\langle t\rangle+\mathrm{i}\langle s\rangle}-1\right)e^{-\mathrm{i}\kappa\sqrt{\frac{E}{\langle t\rangle+\mathrm{i}\langle s\rangle}}}}{4\mathrm{i}\langle s\rangle\sqrt{\langle t\rangle-\mathrm{i}\langle s\rangle}}\\
&=-\frac{\left(\langle t\rangle^2+\langle s\rangle^2\right)\sqrt{\langle t\rangle+\mathrm{i}\langle s\rangle}\sqrt{\langle t\rangle-\mathrm{i}\langle s\rangle}\sqrt{E}\left(\frac{2\langle t\rangle}{\langle t\rangle+\mathrm{i}\langle s\rangle}-1\right)e^{-\mathrm{i}\kappa\sqrt{\frac{E}{\langle t\rangle+\mathrm{i}\langle s\rangle}}}}{4\mathrm{i}\langle s\rangle\sqrt{\langle t\rangle-\mathrm{i}\langle s\rangle}}\\
&=-\frac{\left(\langle t\rangle^2+\langle s\rangle^2\right)\sqrt{\langle t\rangle+\mathrm{i}\langle s\rangle}\sqrt{E}\left(\frac{2\langle t\rangle}{\langle t\rangle+\mathrm{i}\langle s\rangle}-1\right)e^{-\mathrm{i}\kappa\sqrt{\frac{E}{\langle t\rangle+\mathrm{i}\langle s\rangle}}}}{4\mathrm{i}\langle s\rangle}\\
&=-\frac{\left(\langle t\rangle^2+\langle s\rangle^2\right)\sqrt{E}\left(\frac{2\langle t\rangle}{\sqrt{\langle t\rangle+\mathrm{i}\langle s\rangle}}-\sqrt{\langle t\rangle+\mathrm{i}\langle s\rangle}\right)e^{-\mathrm{i}\kappa\sqrt{\frac{E}{\langle t\rangle+\mathrm{i}\langle s\rangle}}}}{4\mathrm{i}\langle s\rangle}.
\end{aligned}
\tag{556}
$$

By inputting the values of z_1, z_2, z_3 and z_4 we receive and expression for the

second residue

$$
\begin{aligned}
&\frac{\left(\frac{\langle t\rangle^2+\langle s\rangle^2}{E}\right)^{3/2}\left(2E\frac{E}{\langle t\rangle^2+\langle s\rangle^2}(\langle t\rangle+\mathrm{i}\,\langle s\rangle)\,\langle t\rangle-E^2\right)e^{\mathrm{i}\kappa\sqrt{\frac{E}{\langle t\rangle^2+\langle s\rangle^2}}\sqrt{\langle t\rangle+\mathrm{i}\langle s\rangle}}}{-2\sqrt{\langle t\rangle+\mathrm{i}\,\langle s\rangle}\left(-\sqrt{\langle t\rangle+\mathrm{i}\,\langle s\rangle}-\sqrt{\langle t\rangle-\mathrm{i}\,\langle s\rangle}\right)\left(-\sqrt{\langle t\rangle+\mathrm{i}\,\langle s\rangle}+\sqrt{\langle t\rangle-\mathrm{i}\,\langle s\rangle}\right)}\\
&=\frac{\left(\frac{\langle t\rangle^2+\langle s\rangle^2}{E}\right)^{3/2}E^2\left(\frac{2\langle t\rangle}{\langle t\rangle-\mathrm{i}\langle s\rangle}-1\right)e^{\mathrm{i}\kappa\sqrt{\frac{E}{\langle t\rangle-\langle s\rangle}}}}{2\sqrt{\langle t\rangle+\mathrm{i}\,\langle s\rangle}\left(\sqrt{\langle t\rangle+\mathrm{i}\,\langle s\rangle}+\sqrt{\langle t\rangle-\mathrm{i}\,\langle s\rangle}\right)\left(-\sqrt{\langle t\rangle+\mathrm{i}\,\langle s\rangle}+\sqrt{\langle t\rangle-\mathrm{i}\,\langle s\rangle}\right)}\\
&=\frac{\left(\frac{\langle t\rangle^2+\langle s\rangle^2}{E}\right)^{3/2}E^2\left(\frac{2\langle t\rangle}{\langle t\rangle-\mathrm{i}\langle s\rangle}-1\right)e^{\mathrm{i}\kappa\sqrt{\frac{E}{\langle t\rangle-\langle s\rangle}}}}{2\sqrt{\langle t\rangle+\mathrm{i}\,\langle s\rangle}\big(-(\langle t\rangle+\mathrm{i}\,\langle s\rangle)+\langle t\rangle-\mathrm{i}\,\langle s\rangle\big)}\\
&=-\frac{\left(\langle t\rangle^2+\langle s\rangle^2\right)^{3/2}\sqrt{E}\left(\frac{2\langle t\rangle}{\langle t\rangle-\mathrm{i}\langle s\rangle}-1\right)e^{\mathrm{i}\kappa\sqrt{\frac{E}{\langle t\rangle-\langle s\rangle}}}}{4\mathrm{i}\,\langle s\rangle\sqrt{\langle t\rangle+\mathrm{i}\,\langle s\rangle}}\\
&=-\frac{\left(\langle t\rangle^2+\langle s\rangle^2\right)\sqrt{\langle t\rangle^2+\langle s\rangle^2}\sqrt{E}\left(\frac{2\langle t\rangle}{\langle t\rangle-\mathrm{i}\langle s\rangle}-1\right)e^{\mathrm{i}\kappa\sqrt{\frac{E}{\langle t\rangle-\langle s\rangle}}}}{4\mathrm{i}\,\langle s\rangle\sqrt{\langle t\rangle+\mathrm{i}\,\langle s\rangle}}\\
&=-\frac{\left(\langle t\rangle^2+\langle s\rangle^2\right)\sqrt{\langle t\rangle+\mathrm{i}\,\langle s\rangle}\sqrt{\langle t\rangle-\mathrm{i}\,\langle s\rangle}\sqrt{E}\left(\frac{2\langle t\rangle}{\langle t\rangle-\mathrm{i}\langle s\rangle}-1\right)e^{\mathrm{i}\kappa\sqrt{\frac{E}{\langle t\rangle-\langle s\rangle}}}}{4\mathrm{i}\,\langle s\rangle\sqrt{\langle t\rangle+\mathrm{i}\,\langle s\rangle}}\\
&=-\frac{\left(\langle t\rangle^2+\langle s\rangle^2\right)\sqrt{E}\left(\frac{2\langle t\rangle}{\sqrt{\langle t\rangle-\mathrm{i}\langle s\rangle}}-\sqrt{\langle t\rangle-\mathrm{i}\,\langle s\rangle}\right)e^{\mathrm{i}\kappa\sqrt{\frac{E}{\langle t\rangle-\langle s\rangle}}}}{4\mathrm{i}\,\langle s\rangle}.
\end{aligned}
\tag{557}
$$

The integral over the complex plane is equal to

$$\begin{aligned}
&\int_C dz f(z) = -2\pi\mathrm{i} \\
&\times \left(- \frac{\left(\langle t\rangle^2 + \langle s\rangle^2\right)\sqrt{E}\left(\frac{2\langle t\rangle}{\sqrt{\langle t\rangle + \mathrm{i}\langle s\rangle}} - \sqrt{\langle t\rangle + \mathrm{i}\,\langle s\rangle}\right) e^{-\mathrm{i}|\kappa|\sqrt{\frac{E}{\langle t\rangle + \mathrm{i}\langle s\rangle}}}}{4\mathrm{i}\,\langle s\rangle} \right. \\
&\left. - \frac{\left(\langle t\rangle^2 + \langle s\rangle^2\right)\sqrt{E}\left(\frac{2\langle t\rangle}{\sqrt{\langle t\rangle - \mathrm{i}\langle s\rangle}} - \sqrt{\langle t\rangle - \mathrm{i}\,\langle s\rangle}\right) e^{\mathrm{i}|\kappa|\sqrt{\frac{E}{\langle t\rangle - \langle s\rangle}}}}{4\mathrm{i}\,\langle s\rangle} \right) \\
&= \left(\frac{\pi\left(\langle t\rangle^2 + \langle s\rangle^2\right)\sqrt{E}\left(\frac{2\langle t\rangle}{\sqrt{\langle t\rangle + \mathrm{i}\langle s\rangle}} - \sqrt{\langle t\rangle + \mathrm{i}\,\langle s\rangle}\right) e^{-\mathrm{i}|\kappa|\sqrt{\frac{E}{\langle t\rangle + \mathrm{i}\langle s\rangle}}}}{2\,\langle s\rangle} \right. \\
&\left. + \frac{\pi\left(\langle t\rangle^2 + \langle s\rangle^2\right)\sqrt{E}\left(\frac{2\langle t\rangle}{\sqrt{\langle t\rangle - \mathrm{i}\langle s\rangle}} - \sqrt{\langle t\rangle - \mathrm{i}\,\langle s\rangle}\right) e^{\mathrm{i}|\kappa|\sqrt{\frac{E}{\langle t\rangle - \langle s\rangle}}}}{2\,\langle s\rangle} \right) \\
&= \frac{\pi\left(\langle t\rangle^2 + \langle s\rangle^2\right)\sqrt{E}}{2\,\langle s\rangle} \left(\left(\frac{2\,\langle t\rangle}{\sqrt{\langle t\rangle + \mathrm{i}\,\langle s\rangle}} - \sqrt{\langle t\rangle + \mathrm{i}\,\langle s\rangle}\right) e^{-\mathrm{i}|\kappa|\sqrt{\frac{E}{\langle t\rangle + \mathrm{i}\langle s\rangle}}} \right. \\
&\left. + \left(\frac{2\,\langle t\rangle}{\sqrt{\langle t\rangle - \mathrm{i}\,\langle s\rangle}} - \sqrt{\langle t\rangle - \mathrm{i}\,\langle s\rangle}\right) e^{\mathrm{i}|\kappa|\sqrt{\frac{E}{\langle t\rangle - \langle s\rangle}}} \right) \\
&= \frac{\pi\left(\langle t\rangle^2 + \langle s\rangle^2\right)\sqrt{E}}{2\,\langle s\rangle} \left(\frac{\langle t\rangle - \mathrm{i}\,\langle s\rangle}{\sqrt{\langle t\rangle + \mathrm{i}\,\langle s\rangle}} e^{-\mathrm{i}|\kappa|\sqrt{\frac{E}{\langle t\rangle + \mathrm{i}\langle s\rangle}}} + \frac{\langle t\rangle + \mathrm{i}\,\langle s\rangle}{\sqrt{\langle t\rangle - \mathrm{i}\,\langle s\rangle}} e^{\mathrm{i}|\kappa|\sqrt{\frac{E}{\langle t\rangle - \langle s\rangle}}} \right).
\end{aligned} \tag{558}$$

The integral, however, is over the complex plane and we are interested in the integral along the real line, but we can split the contour into two parts: the straight line corresponding to the real axis and the arc of the semi-circle.

Therefore, the integral over the complex plane can be expressed as

$$\begin{aligned}
&\int_{-R}^{R} dz f(z) + \int_{arc} dz f(z) \\
&= \frac{\pi\left(\langle t\rangle^2 + \langle s\rangle^2\right)\sqrt{E}}{2\langle s\rangle}\left(\frac{\langle t\rangle - \mathrm{i}\langle s\rangle}{\sqrt{\langle t\rangle + \mathrm{i}\langle s\rangle}} e^{-\mathrm{i}|\kappa|\sqrt{\frac{E}{\langle t\rangle + \mathrm{i}\langle s\rangle}}} + \frac{\langle t\rangle + \mathrm{i}\langle s\rangle}{\sqrt{\langle t\rangle - \mathrm{i}\langle s\rangle}} e^{\mathrm{i}|\kappa|\sqrt{\frac{E}{\langle t\rangle - \langle s\rangle}}}\right),
\end{aligned} \tag{559}$$

using estimations, we can show that

$$\int_{arc} dz f(z) \to 0 \quad \text{as} \quad R \to \infty. \tag{560}$$

To begin we have to find an upper-bound for the integral

$$\begin{aligned}
&\left|\int_{arc} dz \frac{(2Ez^2\langle t\rangle - E^2)e^{-\mathrm{i}\kappa z}}{\left(z^2 - \frac{E\langle t\rangle}{\langle t\rangle^2+\langle s\rangle^2}\right)^2 + \frac{E^2\langle s\rangle^2}{\left(\langle t\rangle^2+\langle s\rangle^2\right)^2}}\right| \\
&\leq \int_{arc} dz \left|\frac{(2Ez^2\langle t\rangle - E^2)e^{-\mathrm{i}\kappa z}}{\left(z^2 - \frac{E\langle t\rangle}{\langle t\rangle^2+\langle s\rangle^2}\right)^2 + \frac{E^2\langle s\rangle^2}{\left(\langle t\rangle^2+\langle s\rangle^2\right)^2}}\right| \\
&\leq \int_{arc} dz \left|\frac{(2Ez^2\langle t\rangle - E^2)}{\left(z^2 - \frac{E\langle t\rangle}{\langle t\rangle^2+\langle s\rangle^2}\right)^2 + \frac{E^2\langle s\rangle^2}{\left(\langle t\rangle^2+\langle s\rangle^2\right)^2}}\right| \\
&\leq \int_{arc} dz \frac{|(2Ez^2\langle t\rangle)|}{\left|\left(z^2 - \frac{E\langle t\rangle}{\langle t\rangle^2+\langle s\rangle^2}\right)^2\right|},
\end{aligned} \tag{561}$$

by the triangle inequality we can see that

$$\begin{aligned}
&|z|^2 = |z^2| = \left|z^2 - \frac{E\langle t\rangle}{\langle t\rangle^2 + \langle s\rangle^2} + \frac{E\langle t\rangle}{\langle t\rangle^2 + \langle s\rangle^2}\right| \\
&\leq \left|z^2 - \frac{E\langle t\rangle}{\langle t\rangle^2 + \langle s\rangle^2}\right| + \left|\frac{E\langle t\rangle}{\langle t\rangle^2 + \langle s\rangle^2}\right| \\
&\implies \left|z^2 - \frac{E\langle t\rangle}{\langle t\rangle^2 + \langle s\rangle^2}\right| \geq |z|^2 - \left|\frac{E\langle t\rangle}{\langle t\rangle^2 + \langle s\rangle^2}\right| = R^2 - \frac{|E\langle t\rangle|}{\langle t\rangle^2 + \langle s\rangle^2},
\end{aligned} \tag{562}$$

hence,

$$\left|\frac{1}{\left(z^2-\frac{E\langle t\rangle}{\langle t\rangle^2+\langle s\rangle^2}\right)^2}\right| \leq \frac{1}{\left(R^2-\frac{|E\langle t\rangle|}{\langle t\rangle^2+\langle s\rangle^2}\right)^2}. \tag{563}$$

The estimation lemma states

$$\left|\int_C dz f(z)\right| \leq M\gamma(C), \tag{564}$$

where $\gamma(C)$ is the arc length of the contour C and M is defined as

$$M = \max_{z\in C} |f(z)|\,. \tag{565}$$

Therefore, by applying the estimation lemma, we receive

$$\begin{aligned}\left|\int_{arc} dz \frac{(2Ez^2\langle t\rangle - E^2)e^{-\mathrm{i}\kappa z}}{\left(z^2-\frac{E\langle t\rangle}{\langle t\rangle^2+\langle s\rangle^2}\right)^2+\frac{E^2\langle s\rangle^2}{\left(\langle t\rangle^2+\langle s\rangle^2\right)^2}}\right| &\leq \frac{\pi R(2|ER^2\langle t\rangle|)}{\left(R^2-\frac{|E\langle t\rangle|}{\langle t\rangle^2+\langle s\rangle^2}\right)^2} \\ = \frac{2\pi R^3|E\langle t\rangle|}{R^4\left(1-\frac{|E\langle t\rangle|}{R^2(\langle t\rangle^2+\langle s\rangle^2)}\right)^2} &= \frac{2\pi|E\langle t\rangle|}{R\left(1-\frac{|E\langle t\rangle|}{R^2(\langle t\rangle^2+\langle s\rangle^2)}\right)^2} \to 0,\end{aligned} \tag{566}$$

as $R\to\infty$.

This means that the integral over the complex plane tends to the integral over the real axis as $R\to\infty$:

$$\int_{-R}^{R} dz f(z) + \int_{arc} dz f(z) \to \int_{-\infty}^{\infty} dx f(x), \tag{567}$$

as $R\to\infty$.

We can simplify the integral by splitting the integral into two parts: $\kappa < 0$

and $\kappa > 0$, which gives

$$\left\langle \frac{x^4}{(E - x^2 \langle t \rangle)^2 + \langle s \rangle^2 x^4} \right\rangle_x = \frac{1}{\langle t \rangle^2 + \langle s \rangle^2} \left(1 + \frac{\sqrt{E}}{4 \langle s \rangle} \left(\int_{-\infty}^{0} d\kappa e^{-\frac{1}{2}\kappa^2\sigma^2} \right. \right.$$
$$\times \left(\frac{\langle t \rangle - \mathrm{i} \langle s \rangle}{\sqrt{\langle t \rangle + \mathrm{i} \langle s \rangle}} e^{-\mathrm{i}|\kappa|\sqrt{\frac{E}{\langle t \rangle + \mathrm{i}\langle s \rangle}}} + \frac{\langle t \rangle + \mathrm{i} \langle s \rangle}{\sqrt{\langle t \rangle - \mathrm{i} \langle s \rangle}} e^{\mathrm{i}|\kappa|\sqrt{\frac{E}{\langle t \rangle - \langle s \rangle}}} \right)$$
$$\left. \left. + \int_0^{\infty} d\kappa e^{-\frac{1}{2}\kappa^2\sigma^2} \left(\frac{\langle t \rangle - \mathrm{i} \langle s \rangle}{\sqrt{\langle t \rangle + \mathrm{i} \langle s \rangle}} e^{-\mathrm{i}|\kappa|\sqrt{\frac{E}{\langle t \rangle + \mathrm{i}\langle s \rangle}}} + \frac{\langle t \rangle + \mathrm{i} \langle s \rangle}{\sqrt{\langle t \rangle - \mathrm{i} \langle s \rangle}} e^{\mathrm{i}|\kappa|\sqrt{\frac{E}{\langle t \rangle - \langle s \rangle}}} \right) \right) \right). \tag{568}$$

In the first integral we make a change of variable $\kappa \to -\kappa$ to ensure positive integration limits and in the second integral since $\kappa > 0$ the absolute value sign can be dropped, hence,

$$\left\langle \frac{x^4}{(E - x^2 \langle t \rangle)^2 + \langle s \rangle^2 x^4} \right\rangle_x = \frac{1}{\langle t \rangle^2 + \langle s \rangle^2} \left(1 + \frac{\sqrt{E}}{4 \langle s \rangle} \left(\int_0^{\infty} d\kappa e^{-\frac{1}{2}\kappa^2\sigma^2} \right. \right.$$
$$\times \left(\frac{\langle t \rangle - \mathrm{i} \langle s \rangle}{\sqrt{\langle t \rangle + \mathrm{i} \langle s \rangle}} e^{-\mathrm{i}|\kappa|\sqrt{\frac{E}{\langle t \rangle + \mathrm{i}\langle s \rangle}}} + \frac{\langle t \rangle + \mathrm{i} \langle s \rangle}{\sqrt{\langle t \rangle - \mathrm{i} \langle s \rangle}} e^{\mathrm{i}|\kappa|\sqrt{\frac{E}{\langle t \rangle - \langle s \rangle}}} \right)$$
$$\left. \left. + \int_0^{\infty} d\kappa e^{-\frac{1}{2}\kappa^2\sigma^2} \left(\frac{\langle t \rangle - \mathrm{i} \langle s \rangle}{\sqrt{\langle t \rangle + \mathrm{i} \langle s \rangle}} e^{-\mathrm{i}|\kappa|\sqrt{\frac{E}{\langle t \rangle + \mathrm{i}\langle s \rangle}}} + \frac{\langle t \rangle + \mathrm{i} \langle s \rangle}{\sqrt{\langle t \rangle - \mathrm{i} \langle s \rangle}} e^{\mathrm{i}|\kappa|\sqrt{\frac{E}{\langle t \rangle - \langle s \rangle}}} \right) \right) \right), \tag{569}$$

the two integrals are equal, hence,

$$\left\langle \frac{x^4}{(E - x^2 \langle t \rangle)^2 + \langle s \rangle^2 x^4} \right\rangle_x = \frac{1}{\langle t \rangle^2 + \langle s \rangle^2} \left(1 + \frac{\sqrt{E}}{2 \langle s \rangle} \int_0^{\infty} d\kappa e^{-\frac{1}{2}\kappa^2\sigma^2} \right.$$
$$\left. \times \left(\frac{\langle t \rangle - \mathrm{i} \langle s \rangle}{\sqrt{\langle t \rangle + \mathrm{i} \langle s \rangle}} e^{-\mathrm{i}\kappa\sqrt{\frac{E}{\langle t \rangle + \mathrm{i}\langle s \rangle}}} + \frac{\langle t \rangle + \mathrm{i} \langle s \rangle}{\sqrt{\langle t \rangle - \mathrm{i} \langle s \rangle}} e^{\mathrm{i}\kappa\sqrt{\frac{E}{\langle t \rangle - \langle s \rangle}}} \right) \right), \tag{570}$$

computing this integral results in the following

$$\begin{aligned}
&\left\langle \frac{x^4}{(E - x^2\left\langle t\right\rangle)^2 + \left\langle s\right\rangle^2 x^4} \right\rangle_x = \frac{1}{\left\langle t\right\rangle^2 + \left\langle s\right\rangle^2} \\
&\times \left(1 + \frac{\sqrt{E}}{2\left\langle s\right\rangle}\sqrt{\frac{\pi}{2}}\frac{e^{-\frac{E\left\langle t\right\rangle}{(\left\langle t\right\rangle^2+\left\langle s\right\rangle^2)\sigma^2}}}{\sqrt{\left\langle t\right\rangle^2 + \left\langle s\right\rangle^2}\sigma} \left(e^{\frac{E}{2(\left\langle t\right\rangle - \mathrm{i}\left\langle s\right\rangle)\sigma^2}} (\left\langle t\right\rangle - \mathrm{i}\left\langle s\right\rangle)^{3/2} \right.\right. \\
&+ e^{\frac{E}{2(\left\langle t\right\rangle + \mathrm{i}\left\langle s\right\rangle)\sigma^2}} (\left\langle t\right\rangle + \mathrm{i}\left\langle s\right\rangle)^{3/2} \\
&- e^{\frac{E}{2(\left\langle t\right\rangle - \mathrm{i}\left\langle s\right\rangle)\sigma^2}} (\left\langle t\right\rangle - \mathrm{i}\left\langle s\right\rangle)^{3/2} \mathrm{erf}\left[\sqrt{-\frac{E}{2(\left\langle t\right\rangle + \mathrm{i}\left\langle s\right\rangle)\sigma^2}} \right] \\
&\left.\left. - e^{\frac{E}{2(\left\langle t\right\rangle + \mathrm{i}\left\langle s\right\rangle)\sigma^2}} (\left\langle t\right\rangle + \mathrm{i}\left\langle s\right\rangle)^{3/2} \mathrm{erf}\left[\sqrt{-\frac{E}{2(\left\langle t\right\rangle - \mathrm{i}\left\langle s\right\rangle)\sigma^2}} \right] \right) \right),
\end{aligned} \tag{571}$$

factoring common terms simplifies the expression down to

$$\begin{aligned}
&\left\langle \frac{x^4}{(E - x^2\left\langle t\right\rangle)^2 + \left\langle s\right\rangle^2 x^4} \right\rangle_x = \frac{1}{\left\langle t\right\rangle^2 + \left\langle s\right\rangle^2} \\
&\times \left(1 + \frac{\sqrt{E}}{2\left\langle s\right\rangle}\sqrt{\frac{\pi}{2}}\frac{e^{-\frac{E\left\langle t\right\rangle}{(\left\langle t\right\rangle^2+\left\langle s\right\rangle^2)\sigma^2}}}{\sqrt{\left\langle t\right\rangle^2 + \left\langle s\right\rangle^2}\sigma} \left(e^{\frac{E}{2(\left\langle t\right\rangle - \mathrm{i}\left\langle s\right\rangle)\sigma^2}} (\left\langle t\right\rangle - \mathrm{i}\left\langle s\right\rangle)^{3/2} \right.\right. \\
&\times \left(1 - \mathrm{erf}\left[\sqrt{-\frac{E}{2(\left\langle t\right\rangle + \mathrm{i}\left\langle s\right\rangle)\sigma^2}} \right] \right) \\
&+ e^{\frac{E}{2(\left\langle t\right\rangle + \mathrm{i}\left\langle s\right\rangle)\sigma^2}} (\left\langle t\right\rangle + \mathrm{i}\left\langle s\right\rangle)^{3/2} \\
&\left.\left. \times \left(1 - \mathrm{erf}\left[\sqrt{-\frac{E}{2(\left\langle t\right\rangle - \mathrm{i}\left\langle s\right\rangle)\sigma^2}} \right] \right) \right) \right),
\end{aligned} \tag{572}$$

where erf is the error function.

This can be expressed in terms the imaginary error functions by first

simplifying the argument of the error function

$$\begin{aligned}
&\left\langle \frac{x^4}{(E-x^2\langle t\rangle)^2+\langle s\rangle^2 x^4}\right\rangle_x = \frac{1}{\langle t\rangle^2+\langle s\rangle^2} \\
&\times\left(1+\frac{\sqrt{E}}{2\langle s\rangle}\sqrt{\frac{\pi}{2}}\frac{e^{-\frac{E\langle t\rangle}{(\langle t\rangle^2+\langle s\rangle^2)\sigma^2}}}{\sqrt{\langle t\rangle^2+\langle s\rangle^2}\sigma}\left(e^{\frac{E}{2(\langle t\rangle-\mathrm{i}\langle s\rangle)\sigma^2}}(\langle t\rangle-\mathrm{i}\langle s\rangle)^{3/2}\right.\right. \\
&\times\left(1-\mathrm{erf}\left[\mathrm{i}\sqrt{\frac{E}{2(\langle t\rangle+\mathrm{i}\langle s\rangle)\sigma^2}}\right]\right) \\
&+e^{\frac{E}{2(\langle t\rangle+\mathrm{i}\langle s\rangle)\sigma^2}}(\langle t\rangle+\mathrm{i}\langle s\rangle)^{3/2} \\
&\left.\left.\times\left(1-\mathrm{erf}\left[-\mathrm{i}\sqrt{\frac{E}{2(\langle t\rangle-\mathrm{i}\langle s\rangle)\sigma^2}}\right]\right)\right)\right),
\end{aligned} \tag{573}$$

where we have taken the square roots in which the argument of the complex number is halved, then by using the relation $\mathrm{ierfi}(z)=\mathrm{erf}(\mathrm{i}z)$, Eq. (573) is equivalent to

$$\begin{aligned}
1 = &\frac{1}{\langle t\rangle^2+\langle s\rangle^2}\left(1+\frac{\sqrt{E}}{2\langle s\rangle}\sqrt{\frac{\pi}{2}}\frac{e^{-\frac{E\langle t\rangle}{(\langle t\rangle^2+\langle s\rangle^2)\sigma^2}}}{\sqrt{\langle t\rangle^2+\langle s\rangle^2}\sigma}\left(e^{\frac{E}{2(\langle t\rangle-\mathrm{i}\langle s\rangle)\sigma^2}}(\langle t\rangle-\mathrm{i}\langle s\rangle)^{3/2}\right.\right. \\
&\times\left(1-\mathrm{ierfi}\left[\sqrt{\frac{E}{2(\langle t\rangle+\mathrm{i}\langle s\rangle)\sigma^2}}\right]\right)+e^{\frac{E}{2(\langle t\rangle+\mathrm{i}\langle s\rangle)\sigma^2}}(\langle t\rangle+\mathrm{i}\langle s\rangle)^{3/2} \\
&\left.\left.\times\left(1+\mathrm{ierfi}\left[\sqrt{\frac{E}{2(\langle t\rangle-\mathrm{i}\langle s\rangle)\sigma^2}}\right]\right)\right)\right),
\end{aligned} \tag{574}$$

since erfi is an odd function, this enables one to pull out the minus sign from the argument of erfi.

The first residue for averaging the second simultaneous equation is calcu-

lated below

$$
\begin{aligned}
&\frac{\left(\frac{\langle t\rangle^2+\langle s\rangle^2}{E}\right)^{3/2}\frac{E}{\langle t\rangle^2+\langle s\rangle^2}(\langle t\rangle-\mathrm{i}\langle s\rangle)e^{-\mathrm{i}\kappa\sqrt{\frac{E}{\langle t\rangle^2+\langle s\rangle^2}}}\sqrt{\langle t\rangle-\mathrm{i}\langle s\rangle}}{2\sqrt{\langle t\rangle-\mathrm{i}\langle s\rangle}\left(\sqrt{\langle t\rangle-\mathrm{i}\langle s\rangle}-\sqrt{\langle t\rangle+\mathrm{i}\langle s\rangle}\right)\left(\sqrt{\langle t\rangle-\mathrm{i}\langle s\rangle}+\sqrt{\langle t\rangle+\mathrm{i}\langle s\rangle}\right)}\\
&=\frac{\left(\frac{\langle t\rangle^2+\langle s\rangle^2}{E}\right)^{3/2}\frac{E}{\langle t\rangle+\mathrm{i}\langle s\rangle}e^{-\mathrm{i}\kappa\sqrt{\frac{E}{\langle t\rangle+\mathrm{i}\langle s\rangle}}}}{2\sqrt{\langle t\rangle-\mathrm{i}\langle s\rangle}\left(\sqrt{\langle t\rangle-\mathrm{i}\langle s\rangle}-\sqrt{\langle t\rangle+\mathrm{i}\langle s\rangle}\right)\left(\sqrt{\langle t\rangle-\mathrm{i}\langle s\rangle}+\sqrt{\langle t\rangle+\mathrm{i}\langle s\rangle}\right)}\\
&=\frac{\left(\frac{\langle t\rangle^2+\langle s\rangle^2}{E}\right)^{3/2}\frac{E}{\langle t\rangle+\mathrm{i}\langle s\rangle}e^{-\mathrm{i}\kappa\sqrt{\frac{E}{\langle t\rangle+\mathrm{i}\langle s\rangle}}}}{2\sqrt{\langle t\rangle-\mathrm{i}\langle s\rangle}\left(\langle t\rangle-\mathrm{i}\langle s\rangle-(\langle t\rangle+\mathrm{i}\langle s\rangle)\right)}\\
&=-\frac{\left(\frac{\langle t\rangle^2+\langle s\rangle^2}{E}\right)^{3/2}\frac{E}{\langle t\rangle+\mathrm{i}\langle s\rangle}e^{-\mathrm{i}\kappa\sqrt{\frac{E}{\langle t\rangle+\mathrm{i}\langle s\rangle}}}}{4\mathrm{i}\langle s\rangle\sqrt{\langle t\rangle-\mathrm{i}\langle s\rangle}}\\
&=-\frac{\left(\langle t\rangle^2+\langle s\rangle^2\right)^{3/2}\frac{1}{\langle t\rangle+\mathrm{i}\langle s\rangle}e^{-\mathrm{i}\kappa\sqrt{\frac{E}{\langle t\rangle+\mathrm{i}\langle s\rangle}}}}{4\mathrm{i}\sqrt{E}\langle s\rangle\sqrt{\langle t\rangle-\mathrm{i}\langle s\rangle}}\\
&=-\frac{\left(\langle t\rangle^2+\langle s\rangle^2\right)\sqrt{\langle t\rangle^2+\langle s\rangle^2}\frac{1}{\langle t\rangle+\mathrm{i}\langle s\rangle}e^{-\mathrm{i}\kappa\sqrt{\frac{E}{\langle t\rangle+\mathrm{i}\langle s\rangle}}}}{4\mathrm{i}\sqrt{E}\langle s\rangle\sqrt{\langle t\rangle-\mathrm{i}\langle s\rangle}}\\
&=-\frac{\left(\langle t\rangle^2+\langle s\rangle^2\right)\sqrt{\langle t\rangle+\mathrm{i}\langle s\rangle}\sqrt{\langle t\rangle-\mathrm{i}\langle s\rangle}\frac{1}{\langle t\rangle+\mathrm{i}\langle s\rangle}e^{-\mathrm{i}\kappa\sqrt{\frac{E}{\langle t\rangle+\mathrm{i}\langle s\rangle}}}}{4\mathrm{i}\sqrt{E}\langle s\rangle\sqrt{\langle t\rangle-\mathrm{i}\langle s\rangle}}\\
&=-\frac{\left(\langle t\rangle^2+\langle s\rangle^2\right)\sqrt{\langle t\rangle+\mathrm{i}\langle s\rangle}\frac{1}{\langle t\rangle+\mathrm{i}\langle s\rangle}e^{-\mathrm{i}\kappa\sqrt{\frac{E}{\langle t\rangle+\mathrm{i}\langle s\rangle}}}}{4\mathrm{i}\sqrt{E}\langle s\rangle}\\
&=-\frac{\left(\langle t\rangle^2+\langle s\rangle^2\right)\frac{1}{\sqrt{\langle t\rangle+\mathrm{i}\langle s\rangle}}e^{-\mathrm{i}\kappa\sqrt{\frac{E}{\langle t\rangle+\mathrm{i}\langle s\rangle}}}}{4\mathrm{i}\sqrt{E}\langle s\rangle}\\
&=-\frac{(\langle t\rangle-\mathrm{i}\langle s\rangle)\sqrt{\langle t\rangle+\mathrm{i}\langle s\rangle}e^{-\mathrm{i}\kappa\sqrt{\frac{E}{\langle t\rangle+\mathrm{i}\langle s\rangle}}}}{4\mathrm{i}\sqrt{E}\langle s\rangle}.
\end{aligned}
\tag{575}
$$

The second residue is also calculated in the same way

$$\begin{aligned}
&\frac{\left(\frac{\langle t\rangle^2+\langle s\rangle^2}{E}\right)^{3/2}\frac{E}{\langle t\rangle^2+\langle s\rangle^2}(\langle t\rangle+\mathrm{i}\langle s\rangle)e^{\mathrm{i}\kappa\sqrt{\frac{E}{\langle t\rangle^2+\langle s\rangle^2}}}\sqrt{\langle t\rangle+\mathrm{i}\langle s\rangle}}{-2\sqrt{\langle t\rangle+\mathrm{i}\langle s\rangle}\left(-\sqrt{\langle t\rangle+\mathrm{i}\langle s\rangle}-\sqrt{\langle t\rangle-\mathrm{i}\langle s\rangle}\right)\left(-\sqrt{\langle t\rangle+\mathrm{i}\langle s\rangle}+\sqrt{\langle t\rangle-\mathrm{i}\langle s\rangle}\right)}\\
&=\frac{\left(\frac{\langle t\rangle^2+\langle s\rangle^2}{E}\right)^{3/2}\frac{E}{\langle t\rangle-\mathrm{i}\langle s\rangle}e^{\mathrm{i}\kappa\sqrt{\frac{E}{\langle t\rangle-\langle s\rangle}}}}{2\sqrt{\langle t\rangle+\mathrm{i}\langle s\rangle}\left(\sqrt{\langle t\rangle+\mathrm{i}\langle s\rangle}+\sqrt{\langle t\rangle-\mathrm{i}\langle s\rangle}\right)\left(-\sqrt{\langle t\rangle+\mathrm{i}\langle s\rangle}+\sqrt{\langle t\rangle-\mathrm{i}\langle s\rangle}\right)}\\
&=\frac{\left(\frac{\langle t\rangle^2+\langle s\rangle^2}{E}\right)^{3/2}\frac{E}{\langle t\rangle-\mathrm{i}\langle s\rangle}e^{\mathrm{i}\kappa\sqrt{\frac{E}{\langle t\rangle-\langle s\rangle}}}}{2\sqrt{\langle t\rangle+\mathrm{i}\langle s\rangle}\left(-\left(\langle t\rangle+\mathrm{i}\langle s\rangle\right)+\langle t\rangle-\mathrm{i}\langle s\rangle\right)}\\
&=-\frac{\left(\langle t\rangle^2+\langle s\rangle^2\right)^{3/2}\frac{1}{\langle t\rangle-\mathrm{i}\langle s\rangle}e^{\mathrm{i}\kappa\sqrt{\frac{E}{\langle t\rangle-\langle s\rangle}}}}{4\mathrm{i}\sqrt{E}\langle s\rangle\sqrt{\langle t\rangle+\mathrm{i}\langle s\rangle}}\\
&=-\frac{\left(\langle t\rangle^2+\langle s\rangle^2\right)\sqrt{\langle t\rangle^2+\langle s\rangle^2}\frac{1}{\langle t\rangle-\mathrm{i}\langle s\rangle}e^{\mathrm{i}\kappa\sqrt{\frac{E}{\langle t\rangle-\langle s\rangle}}}}{4\mathrm{i}\sqrt{E}\langle s\rangle\sqrt{\langle t\rangle+\mathrm{i}\langle s\rangle}}\\
&=-\frac{\left(\langle t\rangle^2+\langle s\rangle^2\right)\sqrt{\langle t\rangle+\mathrm{i}\langle s\rangle}\sqrt{\langle t\rangle-\mathrm{i}\langle s\rangle}\frac{1}{\langle t\rangle-\mathrm{i}\langle s\rangle}e^{\mathrm{i}\kappa\sqrt{\frac{E}{\langle t\rangle-\langle s\rangle}}}}{4\mathrm{i}\sqrt{E}\langle s\rangle\sqrt{\langle t\rangle+\mathrm{i}\langle s\rangle}}\\
&=-\frac{\left(\langle t\rangle^2+\langle s\rangle^2\right)\frac{1}{\sqrt{\langle t\rangle-\mathrm{i}\langle s\rangle}}e^{\mathrm{i}\kappa\sqrt{\frac{E}{\langle t\rangle-\langle s\rangle}}}}{4\mathrm{i}\sqrt{E}\langle s\rangle}\\
&=-\frac{\left(\langle t\rangle+\mathrm{i}\langle s\rangle\right)\sqrt{\langle t\rangle-\mathrm{i}\langle s\rangle}e^{\mathrm{i}\kappa\sqrt{\frac{E}{\langle t\rangle-\langle s\rangle}}}}{4\mathrm{i}\sqrt{E}\langle s\rangle}.
\end{aligned}\tag{576}$$

The integral over the complex plane is therefore equal to

$$\begin{aligned}
\int_C dz f(z) &= -2\pi\mathrm{i}\left(-\frac{\left(\langle t\rangle-\mathrm{i}\langle s\rangle\right)\sqrt{\langle t\rangle+\mathrm{i}\langle s\rangle}e^{-\mathrm{i}\kappa\sqrt{\frac{E}{\langle t\rangle+\mathrm{i}\langle s\rangle}}}}{4\mathrm{i}\sqrt{E}\langle s\rangle}\right.\\
&\qquad\left.-\frac{\left(\langle t\rangle+\mathrm{i}\langle s\rangle\right)\sqrt{\langle t\rangle-\mathrm{i}\langle s\rangle}e^{\mathrm{i}\kappa\sqrt{\frac{E}{\langle t\rangle-\langle s\rangle}}}}{4\mathrm{i}\sqrt{E}\langle s\rangle}\right)\\
&=\frac{\pi}{2\sqrt{E}\langle s\rangle}\left(\left(\langle t\rangle-\mathrm{i}\langle s\rangle\right)\sqrt{\langle t\rangle+\mathrm{i}\langle s\rangle}e^{-\mathrm{i}\kappa\sqrt{\frac{E}{\langle t\rangle+\mathrm{i}\langle s\rangle}}}\right.\\
&\qquad\left.+\left(\langle t\rangle+\mathrm{i}\langle s\rangle\right)\sqrt{\langle t\rangle-\mathrm{i}\langle s\rangle}e^{\mathrm{i}\kappa\sqrt{\frac{E}{\langle t\rangle-\langle s\rangle}}}\right),
\end{aligned}\tag{577}$$

substituting this result back into the equation second simultaneous equation and splitting the integral into two parts: $\kappa<0$ and $\kappa>0$ allows one to sim-

plify down the integral further and remove the absolute value signs. Splitting the integral, we receive

$$\left\langle \frac{x^2}{(E-x^2\langle t\rangle)^2+\langle s\rangle^2 x^4}\right\rangle_x = \frac{\sqrt{E}}{8\langle s\rangle}\frac{1}{\langle t\rangle^2+\langle s\rangle^2}\left(\int_{-\infty}^{0} d\kappa e^{-\frac{1}{2}\kappa^2\sigma^2}\right.$$
$$(\langle t\rangle - \mathrm{i}\langle s\rangle)\sqrt{\langle t\rangle+\mathrm{i}\langle s\rangle}e^{-\mathrm{i}|\kappa|\sqrt{\frac{E}{\langle t\rangle+\mathrm{i}\langle s\rangle}}} + (\langle t\rangle+\mathrm{i}\langle s\rangle)\sqrt{\langle t\rangle-\mathrm{i}\langle s\rangle}e^{\mathrm{i}|\kappa|\sqrt{\frac{E}{\langle t\rangle-\langle s\rangle}}}$$
$$+\int_0^{\infty} d\kappa e^{-\frac{1}{2}\kappa^2\sigma^2}(\langle t\rangle - \mathrm{i}\langle s\rangle)\sqrt{\langle t\rangle+\mathrm{i}\langle s\rangle}e^{-\mathrm{i}|\kappa|\sqrt{\frac{E}{\langle t\rangle+\mathrm{i}\langle s\rangle}}}$$
$$\left.+(\langle t\rangle+\mathrm{i}\langle s\rangle)\sqrt{\langle t\rangle-\mathrm{i}\langle s\rangle}e^{\mathrm{i}|\kappa|\sqrt{\frac{E}{\langle t\rangle-\langle s\rangle}}}\right), \tag{578}$$

changing the variables $\kappa \to -\kappa$ and the swapping the limits of integration in the first integral ensures positive integration limits to allow one to drop the absolute value signs as shown below

$$\left\langle \frac{x^2}{(E-x^2\langle t\rangle)^2+\langle s\rangle^2 x^4}\right\rangle_x = \frac{\sqrt{E}}{8\langle s\rangle}\frac{1}{\langle t\rangle^2+\langle s\rangle^2}\left(\int_{0}^{\infty} d\kappa e^{-\frac{1}{2}\kappa^2\sigma^2}\right.$$
$$(\langle t\rangle - \mathrm{i}\langle s\rangle)\sqrt{\langle t\rangle+\mathrm{i}\langle s\rangle}e^{-\mathrm{i}|\kappa|\sqrt{\frac{E}{\langle t\rangle+\mathrm{i}\langle s\rangle}}} + (\langle t\rangle+\mathrm{i}\langle s\rangle)\sqrt{\langle t\rangle-\mathrm{i}\langle s\rangle}e^{\mathrm{i}|\kappa|\sqrt{\frac{E}{\langle t\rangle-\langle s\rangle}}}$$
$$+\int_0^{\infty} d\kappa e^{-\frac{1}{2}\kappa^2\sigma^2}(\langle t\rangle - \mathrm{i}\langle s\rangle)\sqrt{\langle t\rangle+\mathrm{i}\langle s\rangle}e^{-\mathrm{i}|\kappa|\sqrt{\frac{E}{\langle t\rangle+\mathrm{i}\langle s\rangle}}}$$
$$\left.+(\langle t\rangle+\mathrm{i}\langle s\rangle)\sqrt{\langle t\rangle-\mathrm{i}\langle s\rangle}e^{\mathrm{i}|\kappa|\sqrt{\frac{E}{\langle t\rangle-\langle s\rangle}}}\right) = \frac{\sqrt{E}}{4\langle s\rangle}\frac{1}{\langle t\rangle^2+\langle s\rangle^2}\int_0^{\infty} d\kappa e^{-\frac{1}{2}\kappa^2\sigma^2}$$
$$(\langle t\rangle - \mathrm{i}\langle s\rangle)\sqrt{\langle t\rangle+\mathrm{i}\langle s\rangle}e^{-\mathrm{i}|\kappa|\sqrt{\frac{E}{\langle t\rangle+\mathrm{i}\langle s\rangle}}} + (\langle t\rangle+\mathrm{i}\langle s\rangle)\sqrt{\langle t\rangle-\mathrm{i}\langle s\rangle}e^{\mathrm{i}|\kappa|\sqrt{\frac{E}{\langle t\rangle-\langle s\rangle}}}. \tag{579}$$

The integral over κ is then calculated which results in

$$\left\langle \frac{x^2}{(E-x^2\langle t\rangle)^2+\langle s\rangle^2 x^4}\right\rangle_x = \frac{\sqrt{E}}{4\langle s\rangle}\frac{1}{\langle t\rangle^2+\langle s\rangle^2}\sqrt{\frac{\pi}{2}}\frac{1}{\sigma}e^{-\frac{E\langle t\rangle}{(\langle t\rangle^2+\langle s\rangle^2)\sigma^2}}$$
$$\Bigg(e^{\frac{E}{2(\langle t\rangle-\mathrm{i}\langle s\rangle)\sigma^2}}(\langle t\rangle-\mathrm{i}\langle s\rangle)\sqrt{\langle t\rangle+\mathrm{i}\langle s\rangle}+e^{\frac{E}{2(\langle t\rangle+\mathrm{i}\langle s\rangle)\sigma^2}}(\langle t\rangle+\mathrm{i}\langle s\rangle)\sqrt{\langle t\rangle-\mathrm{i}\langle s\rangle}$$
$$+\mathrm{i}\Bigg(e^{\frac{E}{2(\langle t\rangle-\mathrm{i}\langle s\rangle)\sigma^2}}(\langle s\rangle+\mathrm{i}\langle t\rangle)\sqrt{\langle t\rangle+\mathrm{i}\langle s\rangle}\mathrm{erf}\left[\sqrt{\frac{\mathrm{i}E}{2(\langle s\rangle-\mathrm{i}\langle t\rangle)\sigma^2}}\right]$$
$$-e^{\frac{E}{2(\langle t\rangle+\mathrm{i}\langle s\rangle)\sigma^2}}(\langle s\rangle-\mathrm{i}\langle t\rangle)\sqrt{\langle t\rangle-\mathrm{i}\langle s\rangle}\mathrm{erf}\left[\sqrt{-\frac{\mathrm{i}E}{2(\langle s\rangle+\mathrm{i}\langle t\rangle)\sigma^2}}\right]\Bigg)\Bigg), \tag{580}$$

we are then able, by algebraic manipulation, to express this in a more convenient form

$$\left\langle \frac{x^2}{(E-x^2\langle t\rangle)^2+\langle s\rangle^2 x^4}\right\rangle_x = \frac{\sqrt{E}}{4\langle s\rangle}\sqrt{\frac{\pi}{2}}\frac{1}{\sigma}e^{-\frac{E\langle t\rangle}{(\langle t\rangle^2+\langle s\rangle^2)\sigma^2}}\Bigg(\frac{e^{\frac{E}{2(\langle t\rangle-\mathrm{i}\langle s\rangle)\sigma^2}}}{\sqrt{\langle t\rangle+\mathrm{i}\langle s\rangle}}$$
$$+\frac{e^{\frac{E}{2(\langle t\rangle+\mathrm{i}\langle s\rangle)\sigma^2}}}{\sqrt{\langle t\rangle-\mathrm{i}\langle s\rangle}}+\mathrm{i}\Bigg(\mathrm{i}\frac{e^{\frac{E}{2(\langle t\rangle-\mathrm{i}\langle s\rangle)\sigma^2}}}{\sqrt{\langle t\rangle+\mathrm{i}\langle s\rangle}}\mathrm{erf}\left[\sqrt{-\frac{E}{2(\langle t\rangle+\mathrm{i}\langle s\rangle)\sigma^2}}\right] \tag{581}$$
$$+\mathrm{i}\frac{e^{\frac{E}{2(\langle t\rangle+\mathrm{i}\langle s\rangle)\sigma^2}}}{\sqrt{\langle t\rangle-\mathrm{i}\langle s\rangle}}\mathrm{erf}\left[\sqrt{-\frac{E}{2(\langle t\rangle-\mathrm{i}\langle s\rangle)\sigma^2}}\right]\Bigg)\Bigg).$$

By taking the same convention in which the square root of a complex number halves the argument we can then simplify the expression further

$$\left\langle \frac{x^2}{(E-x^2\langle t\rangle)^2+\langle s\rangle^2 x^4}\right\rangle_x = \frac{\sqrt{E}}{4\langle s\rangle}\sqrt{\frac{\pi}{2}}\frac{1}{\sigma}e^{-\frac{E\langle t\rangle}{(\langle t\rangle^2+\langle s\rangle^2)\sigma^2}}\Bigg(\frac{e^{\frac{E}{2(\langle t\rangle-\mathrm{i}\langle s\rangle)\sigma^2}}}{\sqrt{\langle t\rangle+\mathrm{i}\langle s\rangle}}$$
$$+\frac{e^{\frac{E}{2(\langle t\rangle+\mathrm{i}\langle s\rangle)\sigma^2}}}{\sqrt{\langle t\rangle-\mathrm{i}\langle s\rangle}}+\mathrm{i}\Bigg(\mathrm{i}\frac{e^{\frac{E}{2(\langle t\rangle-\mathrm{i}\langle s\rangle)\sigma^2}}}{\sqrt{\langle t\rangle+\mathrm{i}\langle s\rangle}}\mathrm{erf}\left[\mathrm{i}\sqrt{\frac{E}{2(\langle t\rangle+\mathrm{i}\langle s\rangle)\sigma^2}}\right] \tag{582}$$
$$+\mathrm{i}\frac{e^{\frac{E}{2(\langle t\rangle+\mathrm{i}\langle s\rangle)\sigma^2}}}{\sqrt{\langle t\rangle-\mathrm{i}\langle s\rangle}}\mathrm{erf}\left[-\mathrm{i}\sqrt{\frac{E}{2(\langle t\rangle-\mathrm{i}\langle s\rangle)\sigma^2}}\right]\Bigg)\Bigg),$$

then using the relation $\mathrm{ierfi}(z) = \mathrm{erf}(\mathrm{i}z)$ and the fact that erfi is an odd

function, we arrive at

$$\left\langle \frac{x^2}{(E-x^2\left\langle t\right\rangle)^2+\left\langle s\right\rangle^2 x^4}\right\rangle_x = \frac{\sqrt{E}}{4\left\langle s\right\rangle}\sqrt{\frac{\pi}{2}}\frac{1}{\sigma}e^{-\frac{E\left\langle t\right\rangle}{(\left\langle t\right\rangle^2+\left\langle s\right\rangle^2)\sigma^2}}\left(\frac{e^{\frac{E}{2(\left\langle t\right\rangle-\mathrm{i}\left\langle s\right\rangle)\sigma^2}}}{\sqrt{\left\langle t\right\rangle+\mathrm{i}\left\langle s\right\rangle}}\right.$$
$$+\frac{e^{\frac{E}{2(\left\langle t\right\rangle+\mathrm{i}\left\langle s\right\rangle)\sigma^2}}}{\sqrt{\left\langle t\right\rangle-\mathrm{i}\left\langle s\right\rangle}}+\mathrm{i}\left(-\frac{e^{\frac{E}{2(\left\langle t\right\rangle-\mathrm{i}\left\langle s\right\rangle)\sigma^2}}}{\sqrt{\left\langle t\right\rangle+\mathrm{i}\left\langle s\right\rangle}}\mathrm{erfi}\left[\sqrt{\frac{E}{2(\left\langle t\right\rangle+\mathrm{i}\left\langle s\right\rangle)\sigma^2}}\right]\right. \tag{583}$$
$$\left.\left.+\frac{e^{\frac{E}{2(\left\langle t\right\rangle+\mathrm{i}\left\langle s\right\rangle)\sigma^2}}}{\sqrt{\left\langle t\right\rangle-\mathrm{i}\left\langle s\right\rangle}}\mathrm{erfi}\left[\sqrt{\frac{E}{2(\left\langle t\right\rangle-\mathrm{i}\left\langle s\right\rangle)\sigma^2}}\right]\right)\right),$$

finally collecting common factors together we arrive at the final expression for the second averaged simultaneous equation

$$\left\langle t\right\rangle = \frac{\sqrt{E}}{2\left\langle s\right\rangle}\sqrt{\frac{\pi}{8}}\frac{1}{\sigma}e^{-\frac{E\left\langle t\right\rangle}{(\left\langle t\right\rangle^2+\left\langle s\right\rangle^2)\sigma^2}}\left(\frac{e^{\frac{E}{2(\left\langle t\right\rangle-\mathrm{i}\left\langle s\right\rangle)\sigma^2}}}{\sqrt{\left\langle t\right\rangle+\mathrm{i}\left\langle s\right\rangle}}\right.$$
$$\times\left(1-\mathrm{ierfi}\left[\sqrt{\frac{E}{2(\left\langle t\right\rangle+\mathrm{i}\left\langle s\right\rangle)\sigma^2}}\right]\right)+\frac{e^{\frac{E}{2(\left\langle t\right\rangle+\mathrm{i}\left\langle s\right\rangle)\sigma^2}}}{\sqrt{\left\langle t\right\rangle-\mathrm{i}\left\langle s\right\rangle}} \tag{584}$$
$$\left.\times\left(1+\mathrm{ierfi}\left[\sqrt{\frac{E}{2(\left\langle t\right\rangle-\mathrm{i}\left\langle s\right\rangle)\sigma^2}}\right]\right)\right).$$

In order to proceed with the calculation with the moments of the eigenvectors it is convenient to represent it in another form, we first note that

$$\frac{x^{2q}}{[(E-x^2\left\langle t\right\rangle)^2+\left\langle s\right\rangle^2 x^4]^q}$$
$$=\frac{1}{(q-1)!}\left[\left(-\frac{1}{2x^2y}\frac{d}{dy}\right)^{q-1}\frac{x^2}{(E-x^2\left\langle t\right\rangle)^2+y^2x^4}\right]_{y=\left\langle s\right\rangle} \tag{585}$$
$$=\frac{1}{(q-1)!}\left[\left(-\frac{1}{2y}\frac{d}{dy}\right)^{q-1}\frac{x^{4-2q}}{(E-x^2\left\langle t\right\rangle)^2+y^2x^4}\right]_{y=\left\langle s\right\rangle},$$

therefore the moments of the eigenvectors can be expressed as

$$\begin{aligned}\hat{I}_q &= \frac{E^q\Gamma(q+1)}{2^q\left\langle t\right\rangle^q N^{q-1}}\left\langle \frac{1}{(q-1)!}\left[\left(-\frac{1}{2y}\frac{d}{dy}\right)^{q-1}\frac{x^{4-2q}}{(E-x^2\left\langle t\right\rangle)^2+y^2x^4}\right]_{y=\langle s\rangle}\right\rangle_x \\ &= \frac{qE^q}{2^q\left\langle t\right\rangle^q N^{q-1}}\left[\left(-\frac{1}{2y}\frac{d}{dy}\right)^{q-1}\left\langle\frac{x^{4-2q}}{(E-x^2\left\langle t\right\rangle)^2+y^2x^4}\right\rangle_x\right]_{y=\langle s\rangle},\end{aligned} \tag{586}$$

since $\Gamma(q+1)=q!$ and the derivative over y is independent of x therefore we can pull it out of the averaging.

The first residue for the moments of the eigenvectors is calculated below

$$\begin{aligned}&\frac{\left(\frac{\left\langle t\right\rangle^2+y^2}{E}\right)^{3/2}\left(\frac{E}{\left\langle t\right\rangle^2+y^2}(\left\langle t\right\rangle-\mathrm{i}y)\right)^{2-q}e^{-\mathrm{i}\kappa\sqrt{\frac{E}{\left\langle t\right\rangle^2+y^2}}\sqrt{\left\langle t\right\rangle-\mathrm{i}y}}}{2\sqrt{\left\langle t\right\rangle-\mathrm{i}y}\left(\sqrt{\left\langle t\right\rangle-\mathrm{i}y}-\sqrt{\left\langle t\right\rangle+\mathrm{i}y}\right)\left(\sqrt{\left\langle t\right\rangle-\mathrm{i}y}+\sqrt{\left\langle t\right\rangle+\mathrm{i}y}\right)} \\ &= \frac{\left(\frac{\left\langle t\right\rangle^2+y^2}{E}\right)^{3/2}\left(\frac{E}{\left\langle t\right\rangle+\mathrm{i}y}\right)^{2-q}e^{-\mathrm{i}\kappa\sqrt{\frac{E}{\left\langle t\right\rangle+\mathrm{i}y}}}}{2\sqrt{\left\langle t\right\rangle-\mathrm{i}y}\left(\sqrt{\left\langle t\right\rangle-\mathrm{i}y}-\sqrt{\left\langle t\right\rangle+\mathrm{i}y}\right)\left(\sqrt{\left\langle t\right\rangle-\mathrm{i}y}+\sqrt{\left\langle t\right\rangle+\mathrm{i}y}\right)} \\ &= \frac{\left(\frac{\left\langle t\right\rangle^2+y^2}{E}\right)^{3/2}\left(\frac{E}{\left\langle t\right\rangle+\mathrm{i}y}\right)^{2-q}e^{-\mathrm{i}\kappa\sqrt{\frac{E}{\left\langle t\right\rangle+\mathrm{i}y}}}}{2\sqrt{\left\langle t\right\rangle-\mathrm{i}y}\left(\left\langle t\right\rangle-\mathrm{i}y-(\left\langle t\right\rangle+\mathrm{i}y)\right)} \\ &= -\frac{\left(\frac{\left\langle t\right\rangle^2+y^2}{E}\right)^{3/2}\left(\frac{E}{\left\langle t\right\rangle+\mathrm{i}y}\right)^{2-q}e^{-\mathrm{i}\kappa\sqrt{\frac{E}{\left\langle t\right\rangle+\mathrm{i}y}}}}{4\mathrm{i}y\sqrt{\left\langle t\right\rangle-\mathrm{i}y}} \\ &= -\frac{E^{\frac{1}{2}-q}\left(\left\langle t\right\rangle^2+y^2\right)^{3/2}\frac{1}{(\left\langle t\right\rangle+\mathrm{i}y)^{2-q}}e^{-\mathrm{i}\kappa\sqrt{\frac{E}{\left\langle t\right\rangle+\mathrm{i}y}}}}{4\mathrm{i}y\sqrt{\left\langle t\right\rangle-\mathrm{i}y}} \\ &= -\frac{E^{\frac{1}{2}-q}\left(\left\langle t\right\rangle^2+y^2\right)\sqrt{\left\langle t\right\rangle^2+y^2}\frac{1}{(\left\langle t\right\rangle+\mathrm{i}y)^{2-q}}e^{-\mathrm{i}\kappa\sqrt{\frac{E}{\left\langle t\right\rangle+\mathrm{i}y}}}}{4\mathrm{i}y\sqrt{\left\langle t\right\rangle-\mathrm{i}y}} \\ &= -\frac{E^{\frac{1}{2}-q}\left(\left\langle t\right\rangle^2+y^2\right)\sqrt{\left\langle t\right\rangle+\mathrm{i}y}\sqrt{\left\langle t\right\rangle-\mathrm{i}y}\frac{1}{(\left\langle t\right\rangle+\mathrm{i}y)^{2-q}}e^{-\mathrm{i}\kappa\sqrt{\frac{E}{\left\langle t\right\rangle+\mathrm{i}y}}}}{4\mathrm{i}y\sqrt{\left\langle t\right\rangle-\mathrm{i}y}} \\ &= -\frac{E^{\frac{1}{2}-q}\left(\left\langle t\right\rangle^2+y^2\right)\sqrt{\left\langle t\right\rangle+\mathrm{i}y}\frac{1}{(\left\langle t\right\rangle+\mathrm{i}y)^{2-q}}e^{-\mathrm{i}\kappa\sqrt{\frac{E}{\left\langle t\right\rangle+\mathrm{i}y}}}}{4\mathrm{i}y} \\ &= -\frac{E^{\frac{1}{2}-q}\left(\left\langle t\right\rangle^2+y^2\right)(\left\langle t\right\rangle+\mathrm{i}y)^{q-\frac{3}{2}}e^{-\mathrm{i}\kappa\sqrt{\frac{E}{\left\langle t\right\rangle+\mathrm{i}y}}}}{4\mathrm{i}y},\end{aligned} \tag{587}$$

and the second residue is calculated in the same way

$$
\begin{aligned}
&\frac{\left(\frac{\langle t\rangle^2+y^2}{E}\right)^{3/2}\left(\frac{E}{\langle t\rangle^2+y^2}(\langle t\rangle+\mathrm{i}y)\right)^{2-q}e^{\mathrm{i}\kappa\sqrt{\frac{E}{\langle t\rangle^2+y^2}}\sqrt{\langle t\rangle+\mathrm{i}y}}}{-2\sqrt{\langle t\rangle+\mathrm{i}y}\left(-\sqrt{\langle t\rangle+\mathrm{i}y}-\sqrt{\langle t\rangle-\mathrm{i}y}\right)\left(-\sqrt{\langle t\rangle+\mathrm{i}y}+\sqrt{\langle t\rangle-\mathrm{i}y}\right)}\\
&=\frac{\left(\frac{\langle t\rangle^2+y^2}{E}\right)^{3/2}\left(\frac{E}{\langle t\rangle-\mathrm{i}y}\right)^{2-q}e^{\mathrm{i}\kappa\sqrt{\frac{E}{\langle t\rangle-\mathrm{i}y}}}}{2\sqrt{\langle t\rangle+\mathrm{i}y}\left(\sqrt{\langle t\rangle+\mathrm{i}y}+\sqrt{\langle t\rangle-\mathrm{i}y}\right)\left(-\sqrt{\langle t\rangle+\mathrm{i}y}+\sqrt{\langle t\rangle-\mathrm{i}y}\right)}\\
&=\frac{\left(\frac{\langle t\rangle^2+y^2}{E}\right)^{3/2}\left(\frac{E}{\langle t\rangle-\mathrm{i}y}\right)^{2-q}e^{\mathrm{i}\kappa\sqrt{\frac{E}{\langle t\rangle-\mathrm{i}y}}}}{2\sqrt{\langle t\rangle+\mathrm{i}y}\big(-(\langle t\rangle+\mathrm{i}y)+\langle t\rangle-\mathrm{i}y\big)}\\
&=-\frac{E^{\frac{1}{2}-q}\left(\langle t\rangle^2+y^2\right)^{3/2}\frac{1}{(\langle t\rangle-\mathrm{i}y)^{2-q}}e^{\mathrm{i}\kappa\sqrt{\frac{E}{\langle t\rangle-\mathrm{i}y}}}}{4\mathrm{i}y\sqrt{\langle t\rangle+\mathrm{i}y}}\\
&=-\frac{E^{\frac{1}{2}-q}\left(\langle t\rangle^2+y^2\right)\sqrt{\langle t\rangle^2+y^2}\frac{1}{(\langle t\rangle-\mathrm{i}y)^{2-q}}e^{\mathrm{i}\kappa\sqrt{\frac{E}{\langle t\rangle-\mathrm{i}y}}}}{4\mathrm{i}y\sqrt{\langle t\rangle+\mathrm{i}y}}\\
&=-\frac{E^{\frac{1}{2}-q}\left(\langle t\rangle^2+y^2\right)\sqrt{\langle t\rangle+\mathrm{i}y}\sqrt{\langle t\rangle-\mathrm{i}y}\frac{1}{(\langle t\rangle-\mathrm{i}y)^{2-q}}e^{\mathrm{i}\kappa\sqrt{\frac{E}{\langle t\rangle-\mathrm{i}y}}}}{4\mathrm{i}y\sqrt{\langle t\rangle+\mathrm{i}y}}\\
&=-\frac{E^{\frac{1}{2}-q}\left(\langle t\rangle^2+y^2\right)(\langle t\rangle-\mathrm{i}y)^{q-\frac{3}{2}}e^{\mathrm{i}\kappa\sqrt{\frac{E}{\langle t\rangle-\mathrm{i}y}}}}{4\mathrm{i}y}.
\end{aligned}
\tag{588}
$$

Putting this together, the integral over the complex plane is equal to

$$
\begin{aligned}
\int_C dz f(z) &= -2\pi\mathrm{i}\Bigg(-\frac{E^{\frac{1}{2}-q}\left(\langle t\rangle^2+y^2\right)(\langle t\rangle+\mathrm{i}y)^{q-\frac{3}{2}}e^{-\mathrm{i}|\kappa|\sqrt{\frac{E}{\langle t\rangle+\mathrm{i}y}}}}{4\mathrm{i}y}\\
&\qquad -\frac{E^{\frac{1}{2}-q}\left(\langle t\rangle^2+y^2\right)(\langle t\rangle-\mathrm{i}y)^{q-\frac{3}{2}}e^{\mathrm{i}|\kappa|\sqrt{\frac{E}{\langle t\rangle-\mathrm{i}y}}}}{4\mathrm{i}y}\Bigg)\\
&=\frac{\pi E^{\frac{1}{2}-q}(\langle t\rangle^2+y^2)}{2y}\left((\langle t\rangle+\mathrm{i}y)^{q-\frac{3}{2}}e^{-\mathrm{i}|\kappa|\sqrt{\frac{E}{\langle t\rangle+\mathrm{i}y}}}+(\langle t\rangle-\mathrm{i}y)^{q-\frac{3}{2}}e^{\mathrm{i}|\kappa|\sqrt{\frac{E}{\langle t\rangle-\mathrm{i}y}}}\right),
\end{aligned}
\tag{589}
$$

substituting this back into the averaging for the moments of the eigenvectors

and splitting the integral into two parts, we receive

$$\left\langle \frac{x^{4-2q}}{(E-x^2\left\langle t\right\rangle)^2+y^2x^4}\right\rangle_x = \frac{E^{\frac{1}{2}-q}}{4y}\Bigg(\int_{-\infty}^{0} d\kappa e^{-\frac{1}{2}\kappa^2\sigma^2}\Bigg((\left\langle t\right\rangle+\mathrm{i}y)^{q-\frac{3}{2}}e^{-\mathrm{i}|\kappa|\sqrt{\frac{E}{\left\langle t\right\rangle+\mathrm{i}y}}} + (\left\langle t\right\rangle-\mathrm{i}y)^{q-\frac{3}{2}}e^{\mathrm{i}|\kappa|\sqrt{\frac{E}{\left\langle t\right\rangle-\mathrm{i}y}}}\Bigg) + \int_{0}^{\infty} d\kappa e^{-\frac{1}{2}\kappa^2\sigma^2}\Bigg((\left\langle t\right\rangle+\mathrm{i}y)^{q-\frac{3}{2}}e^{-\mathrm{i}|\kappa|\sqrt{\frac{E}{\left\langle t\right\rangle+\mathrm{i}y}}} + (\left\langle t\right\rangle-\mathrm{i}y)^{q-\frac{3}{2}}e^{\mathrm{i}|\kappa|\sqrt{\frac{E}{\left\langle t\right\rangle-\mathrm{i}y}}}\Bigg)\Bigg). \tag{590}$$

Changing the variables in the first integral $\kappa \to -\kappa$ ensures positive integration limits which enables one to drop the absolute value signs, performing this and manipulating the integral yields

$$\left\langle \frac{x^{4-2q}}{(E-x^2\left\langle t\right\rangle)^2+y^2x^4}\right\rangle_x = \frac{E^{\frac{1}{2}-q}}{2y}\int_{0}^{\infty} d\kappa e^{-\frac{1}{2}\kappa^2\sigma^2} \Bigg((\left\langle t\right\rangle+\mathrm{i}y)^{q-\frac{3}{2}}e^{-\mathrm{i}\kappa\sqrt{\frac{E}{\left\langle t\right\rangle+\mathrm{i}y}}} + (\left\langle t\right\rangle-\mathrm{i}y)^{q-\frac{3}{2}}e^{\mathrm{i}\kappa\sqrt{\frac{E}{\left\langle t\right\rangle-\mathrm{i}y}}}\Bigg). \tag{591}$$

As we are interested in the limit $\sigma \to \infty$, we look at the asymptotic behaviour of the simultaneous equations and the moments of the eigenvectors assuming that $E \propto O(1)$, so for simplicity we set $E = 1$. Starting with the equation below

$$\begin{aligned}\left\langle t\right\rangle = \frac{\sqrt{E}}{2\left\langle s\right\rangle}\sqrt{\frac{\pi}{8}}\frac{1}{\sigma}e^{-\frac{E\left\langle t\right\rangle}{(\left\langle t\right\rangle^2+\left\langle s\right\rangle^2)\sigma^2}}\Bigg(&\frac{e^{\frac{E}{2(\left\langle t\right\rangle-\mathrm{i}\left\langle s\right\rangle)\sigma^2}}}{\sqrt{\left\langle t\right\rangle+\mathrm{i}\left\langle s\right\rangle}} \\ &\times\left(1-\mathrm{ierfi}\left[\sqrt{\frac{E}{2(\left\langle t\right\rangle+\mathrm{i}\left\langle s\right\rangle)\sigma^2}}\right]\right) + \frac{e^{\frac{E}{2(\left\langle t\right\rangle+\mathrm{i}\left\langle s\right\rangle)\sigma^2}}}{\sqrt{\left\langle t\right\rangle-\mathrm{i}\left\langle s\right\rangle}} \\ &\times\left(1+\mathrm{ierfi}\left[\sqrt{\frac{E}{2(\left\langle t\right\rangle-\mathrm{i}\left\langle s\right\rangle)\sigma^2}}\right]\right)\Bigg),\end{aligned} \tag{592}$$

we know that

$$\begin{aligned} e^{-\frac{E\left\langle t\right\rangle}{(\left\langle t\right\rangle^2\pm\left\langle s\right\rangle^2)\sigma^2}} &\approx 1 \\ e^{\frac{E}{2(\left\langle t\right\rangle\pm\mathrm{i}\left\langle s\right\rangle)\sigma^2}} &\approx 1 \\ \mathrm{erfi}\left[\sqrt{\frac{E}{2(\left\langle t\right\rangle\pm\mathrm{i}\left\langle s\right\rangle)\sigma^2}}\right] &\approx 0, \end{aligned} \tag{593}$$

therefore the equations simplifies greatly down to

$$\begin{aligned}\langle t\rangle &= \frac{1}{2\langle s\rangle}\sqrt{\frac{\pi}{8}}\frac{1}{\sigma}\left(\frac{1}{\sqrt{\langle t\rangle+\mathrm{i}\langle s\rangle}}+\frac{1}{\sqrt{\langle t\rangle-\mathrm{i}\langle s\rangle}}\right)\\ &= \frac{1}{2\langle s\rangle}\sqrt{\frac{\pi}{8}}\frac{1}{\sigma}\left(\frac{\sqrt{\langle t\rangle-\mathrm{i}\langle s\rangle}+\sqrt{\langle t\rangle+\mathrm{i}\langle s\rangle}}{\sqrt{\langle t\rangle^2+\langle s\rangle^2}}\right)\\ &= \frac{1}{2\langle s\rangle}\sqrt{\frac{\pi}{8}}\frac{1}{\sigma}\left(\frac{(\langle t\rangle-\mathrm{i}\langle s\rangle)\sqrt{\langle t\rangle+\mathrm{i}\langle s\rangle}+(\langle t\rangle+\mathrm{i}\langle s\rangle)\sqrt{\langle t\rangle-\mathrm{i}\langle s\rangle}}{\langle t\rangle^2+\langle s\rangle^2}\right).\end{aligned} \tag{594}$$

According to numerical simulations, if $\sigma\to\infty$ then $\langle s\rangle \gg \langle t\rangle$, we can drop $\langle t\rangle$. Hence,

$$\begin{aligned}\langle t\rangle &= \frac{1}{2\langle s\rangle}\sqrt{\frac{\pi}{8}}\frac{1}{\sigma}\left(\frac{(-\mathrm{i}\langle s\rangle)\sqrt{\mathrm{i}\langle s\rangle}+(\mathrm{i}\langle s\rangle)\sqrt{-\mathrm{i}\langle s\rangle}}{\langle s\rangle^2}\right)\\ &= \frac{1}{2\langle s\rangle}\sqrt{\frac{\pi}{8}}\frac{1}{\sigma}\left(\frac{-\mathrm{i}\sqrt{\mathrm{i}\langle s\rangle}+\mathrm{i}\sqrt{-\mathrm{i}\langle s\rangle}}{\langle s\rangle}\right)=\frac{1}{2\langle s\rangle}\sqrt{\frac{\pi}{8}}\frac{1}{\sigma}\left(\frac{\sqrt{2}(1-\mathrm{i})\sqrt{\mathrm{i}\langle s\rangle}}{2\langle s\rangle}\right)\\ &= \frac{1}{2\langle s\rangle}\sqrt{\frac{\pi}{8}}\frac{1}{\sigma}\left(\frac{\sqrt{2}(1-\mathrm{i})(1+\mathrm{i})\sqrt{\langle s\rangle}}{2\langle s\rangle}\right)=\frac{\sqrt{\pi}}{4\langle s\rangle^{3/2}\sigma}.\end{aligned} \tag{595}$$

Applying the same approach to the second equation

$$\begin{aligned}1 = \frac{1}{\langle t\rangle^2+\langle s\rangle^2}\Bigg(1+\frac{\sqrt{E}}{2\langle s\rangle}\sqrt{\frac{\pi}{2}}\frac{e^{-\frac{E\langle t\rangle}{(\langle t\rangle^2+\langle s\rangle^2)\sigma^2}}}{\sqrt{\langle t\rangle^2+\langle s\rangle^2}\sigma}\Bigg(e^{\frac{E}{2(\langle t\rangle-\mathrm{i}\langle s\rangle)\sigma^2}}(\langle t\rangle-\mathrm{i}\langle s\rangle)^{3/2}\\ \times\left(1-\mathrm{ierfi}\left[\sqrt{\frac{E}{2(\langle t\rangle+\mathrm{i}\langle s\rangle)\sigma^2}}\right]\right)+e^{\frac{E}{2(\langle t\rangle+\mathrm{i}\langle s\rangle)\sigma^2}}(\langle t\rangle+\mathrm{i}\langle s\rangle)^{3/2}\\ \times\left(1+\mathrm{ierfi}\left[\sqrt{\frac{E}{2(\langle t\rangle-\mathrm{i}\langle s\rangle)\sigma^2}}\right]\right)\Bigg)\Bigg),\end{aligned} \tag{596}$$

with the same assumptions, the equations simplify down to

$$1 = \frac{1}{\langle t\rangle^2 + \langle s\rangle^2}\left(1 + \frac{1}{\langle s\rangle}\sqrt{\frac{\pi}{8}}\frac{1}{\sqrt{\langle t\rangle^2 + \langle s\rangle^2}\sigma}\left((\langle t\rangle - \mathrm{i}\,\langle s\rangle)^{3/2}\right.\right.$$
$$\left.\left. + (\langle t\rangle + \mathrm{i}\,\langle s\rangle)^{3/2}\right)\right) = \frac{1}{\langle s\rangle^2}\left(1 + \frac{1}{\langle s\rangle}\sqrt{\frac{\pi}{8}}\frac{1}{\langle s\rangle\,\sigma}\left((-\mathrm{i}\,\langle s\rangle)^{3/2}\right.\right.$$
$$\left.\left. + (\mathrm{i}\,\langle s\rangle)^{3/2}\right)\right) = \frac{1}{\langle s\rangle^2}\left(1 + \frac{1}{\langle s\rangle}\sqrt{\frac{\pi}{8}}\frac{1}{\langle s\rangle\,\sigma}\left(-\mathrm{i}\,\langle s\rangle\sqrt{-\mathrm{i}\,\langle s\rangle}\right.\right.$$
$$\left.\left. + \mathrm{i}\,\langle s\rangle\sqrt{\mathrm{i}\,\langle s\rangle}\right)\right) = \frac{1}{\langle s\rangle^2}\left(1 + \frac{1}{\langle s\rangle}\sqrt{\frac{\pi}{8}}\frac{1}{\langle s\rangle\,\sigma}\left(-\langle s\rangle\sqrt{\mathrm{i}\,\langle s\rangle}\right.\right.$$
$$\left.\left. + \mathrm{i}\,\langle s\rangle\sqrt{\mathrm{i}\,\langle s\rangle}\right)\right) = \frac{1}{\langle s\rangle^2}\left(1 + \frac{1}{\langle s\rangle}\sqrt{\frac{\pi}{8}}\frac{1}{\langle s\rangle\,\sigma}\left((-1+\mathrm{i})\frac{\sqrt{2}(1+\mathrm{i})}{2}\langle s\rangle\sqrt{\langle s\rangle}\right)\right)$$
$$= \frac{1}{\langle s\rangle^2}\left(1 - \frac{\sqrt{\pi}}{2\sqrt{\langle s\rangle}\sigma}\right). \tag{597}$$

Re-arranging the expression we receive

$$\langle s\rangle^2 - 1 = -\frac{\sqrt{\pi}}{2\sqrt{\langle s\rangle}\sigma} \to 0, \tag{598}$$

as $\sigma \to \infty$, hence, $\langle s\rangle \approx 1$. Therefore, we obtain

$$\langle t\rangle = \frac{\sqrt{\pi}}{4\,\langle s\rangle^{3/2}\,\sigma}, \quad \langle s\rangle \approx 1. \tag{599}$$

E Explicit expressions for Efetov's parametrisation in the generalisation of model I and II

The pre-exponential factors calculated by employing Efetov's parametrisation are given as follows:

$$\begin{aligned}
g_{aa}^{BB} &= \frac{E-d_n-w_n^2t+\mathrm{i}sw_n^2\lambda_1+\mathrm{i}sw_n^2(\lambda_1-\lambda_2)\alpha\alpha^*}{(E-d_n-w_n^2t)^2+s^2w_n^4}, \\
g_{ar}^{BB} &= -\frac{\mu_1 sw_n^2\left(1+\dfrac{\alpha\alpha^*}{2}\right)\left(1-\dfrac{\beta\beta^*}{2}\right)+\mu_2^* sw_n^2\alpha^*\beta}{(E-d_n-w_n^2t)^2+s^2w_n^4}, \\
g_{ra}^{BB} &= -\frac{\mu_1^* sw_n^2\left(1-\dfrac{\beta\beta^*}{2}\right)\left(1+\dfrac{\alpha\alpha^*}{2}\right)+\mu_2 sw_n^2\beta^*\alpha}{(E-d_n-w_n^2t)^2+s^2w_n^4}, \\
g_{rr}^{BB} &= \frac{E-d_n-w_n^2t-\mathrm{i}sw_n^2\lambda_1+\mathrm{i}sw_n^2(\lambda_1-\lambda_2)\beta\beta^*}{(E-d_n-w_n^2t)^2+s^2w_n^4}.
\end{aligned} \tag{600}$$

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