A note on "Anisotropic Total Variation Regularized L^1-Approximation and Denoising/Deblurring of 2D Bar Codes"

Nils Dabrock* Yves van Gennip†

Abstract

This note addresses an error in [1].

In this short note, we address an error in [1, Lemma 4.2 and Theorem 6.4] which was pointed out to YvG by ND in April 2016. In this note we assume familiarity with the notation from [1]. That paper erroneously argues that the only binary signals F_1 and F_3 are faithful to are clean 2D bar codes.

Lemma 4.2 stated that: if $f \in BV(\mathbb{R}^2; \{0,1\})$ is both the measured signal in F_1 and a minimizer of F_1 over $BV(\mathbb{R}^2)$, then $f \in \mathcal{B}$. This statement is false, as can be seen from the following counterexample, which was provided in a private communication by ND.

Let $h \in (1/\sqrt{2}, 1)$ and let $\Omega := B(0,1) \cap [-h,h]^2$ be a truncated circle. Let $f = \chi_\Omega \in L^1(\mathbb{R})$ be the characteristic function of Ω. Define $s := \sqrt{1-h^2}$ and let[1] $\lambda > \frac{2}{s}$. Define, for $r \in \mathbb{R}$, $w(r) := \min\{1, \max\{-1, r/s\}\}$ and let, for $(x,y) \in \mathbb{R}^2$,

$$v(x,y) := \begin{pmatrix} w(x) \\ w(y) \end{pmatrix}.$$

We will now show that $v \in \mathcal{V}(f)$ and hence, by [1, Theorem 3.2], F_1 is faithful to f.

It can be verified by direct computation that, for $x \in \mathbb{R}^2$, $|v(x)|_\infty \leq 1$ and $\|\operatorname{div} v\|_{L^\infty(\mathbb{R}^2)} \leq \frac{2}{s}$. Furthermore, we have for all $z \in \partial\Omega$ that $v(z) \cdot n_{\partial\Omega}(z) = |n_{\partial\Omega}(z)|_1$, where $n_{\partial\Omega}$ is the outward normal vector to the boundary $\partial\Omega$. By the definition of the anisotropic total

*Fakultät für Mathematik, Technische Universität Dortmund, Vogelpothsweg 87, 44227 Dortmund, Germany

†School of Mathematical Sciences, University of Nottingham, University Park, Nottingham, NG7 2RD, UK

[1] Numerical simulations by Nils Dabrock suggest this condition can be weakened to $\lambda > \frac{1}{s}$.

variation in [1, Formula (1)] and [1, Appendix A, Corollary 3] we then find

$$\begin{aligned}
\int_{\mathbb{R}^2} |f_x| + |f_y| &= \sup\left\{\int_{\mathbb{R}^2} f \operatorname{div}\varphi : \varphi \in C_c^1(\mathbb{R}^2;\mathbb{R}^2), \forall z\ |\varphi(z)|_\infty \le 1\right\} \\
&= \sup\left\{\int_{\Omega} \operatorname{div}\varphi : \varphi \in C_c^1(\mathbb{R}^2;\mathbb{R}^2), \forall z\ |\varphi(z)|_\infty \le 1\right\} \\
&= \sup\left\{\int_{\partial\Omega} \varphi \cdot n_{\partial\Omega} : \varphi \in C_c^1(\mathbb{R}^2;\mathbb{R}^2), \forall z\ |\varphi(z)|_\infty \le 1\right\} \\
&\le \int_{\partial\Omega} |n_{\partial\Omega}(z)|_1 = \int_{\partial\Omega} v \cdot n_{\partial\Omega} = \int_{\mathbb{R}^2} f \operatorname{div} v \\
&\le \sup\left\{\int_{\mathbb{R}^2} f \operatorname{div}\varphi : \varphi \in L^\infty(\mathbb{R}^2;\mathbb{R}^2), \operatorname{div}\varphi \in L^\infty(\mathbb{R}^2),\ |\varphi(z)|_\infty \le 1 \text{ a.e.}\right\} \\
&= \int_{\mathbb{R}^2} |f_x| + |f_y|.
\end{aligned}$$

Since the second part of Theorem 6.4 was based directly on Lemma 4.2, that result is also incorrect (part 1 of Theorem 6.4 is unaffected).

For a more general treatment of this topic by ND, including the abovementioned counterexample, we refer to [2].

References

[1] Choksi, R. and van Gennip, Y. and Oberman, A. Anisotropic total variation regularized L^1 approximation and denoising/deblurring of 2D bar codes *Inverse Probl. Imaging 5*, 3 (2011), 591–617.

[2] Dabrock, N. Characterization of minimizers of an anisotropic variant of the Rudin-Osher-Fatemi functional with L^1 fidelity term arXiv preprint arXiv:1704.00451