Properties of Banach Function Algebras

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Contents

Abstract

This thesis is devoted to the study of various properties of Banach function algebras. We are particularly interested in the study of antisymmetric decompositions for uniform algebras and regularity of Banach function algebras. We are also interested in the study of Swiss cheese sets, essential uniform algebras and characterisations of $C(X)$ among its subalgebras.

The maximal antisymmetric decomposition for uniform algebras is a generalisation of the celebrated Stone-Weierstrass theorem and it is a powerful tool in the study of uniform algebras. However, in the literature, not much attention has been paid to the study of closed antisymmetric subsets. In Section 1.7 we give a characterisation of all the closed antisymmetric subsets for the disc algebra on the unit circle, and we use this characterisation to give a new proof of Wermer's maximality theorem. Then in Section 4.1 we give characterisations of all the closed antisymmetric subsets for normal uniform algebras on the unit interval or the unit circle.

The two types of regularity points, the R-point and the point of regularity, are important concepts in the study of regularity of Banach function algebras. In Section 3.2 we construct two examples of compact plane sets X, such that $R(X)$ has either one R-point while having no points of regularity, or $R(X)$ has one point of continuity while having no R-points. There are the first known examples of natural uniform algebras in the literature which show that R-points and points of continuity can be different. We then use properties of regularity points to study $R(X)$ which is not regular while having no non-trivial Jensen measures. We also use properties of regularity points in Section 4.2 to study small exceptional sets for uniform algebras.

In Chapter 2 we study Swiss cheese sets. Our approach is to regard Swiss cheese sets "abstractly": we study the family of sequences of pairs of numbers, where the numbers represent the centre and radius of discs in the complex plane. We then give a natural topology on the space of abstract Swiss cheeses and give topological proofs of various classicalisation theorems.

It is standard that the study of general uniform algebras can be reduced to the study of essential uniform algebras. In Chapter 5 we study methods to construct essential uniform algebras. In particular, we continue to study the method introduced in [26] to show that some more properties are inherited by the constructed essential uniform algebra from the original one.

We note that the material in Chapter 2 is joint work with J. Feinstein and S. Morley and is published in [28, 27]. The material in Chapter 3 is joint work

with J. Feinstein and is published in [32]. Section 4.2 contains joint work with J. Feinstein.

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Chapter 1

Introduction

In this chapter, we give an outline of the thesis, along with standard definitions, terminologies and some basic results used in this thesis. In the final section of this chapter, we give two well known examples of uniform algebras to illustrate the definitions and results introduced earlier.

In the whole thesis, we reserve the terms *proposition* and *theorem* for prominent results. Usually a proposition is a standard result or a statement that is well known, and we usually do not claim originality. Note that in this thesis we sometimes give new or independent proofs for propositions, and in these cases we will state clearly that the proof is new and original. Usually both the statement of a theorem and its proof are original. We will make it clear when a lemma, an example or a corollary is original, and give appropriate references when possible.

We shall assume the reader is familiar with the notion of associate algebras. See [14] for more information. Unless otherwise stated, throughout this thesis all linear spaces and algebras are over the complex field $\mathbb{C}$, and all algebras are associative, commutative and unital (that is the algebra contains a multiplicative identity).

A number of the definitions and results mentioned in this chapter concerning Banach algebras have generalisations to the non-commutative case. We refer the readers to [2, 8, 14] for treatment of general (commutative and non-commutative) Banach algebra theories.

1.1 Outline of the thesis

In this section we give an outline of the thesis. Please refer to the corresponding chapters and the rest of Chapter 1 for the definitions and terminologies.

1.1.1 Chapter 2

In Chapter 2 we study Swiss cheese sets, abstract Swiss cheeses and various classicalisation theorems. A Swiss cheese set is a compact plane set obtained by deleting a sequence of open discs from a closed disc. A classical Swiss cheese set is one where the open discs and the complement of the closed disc have pairwise disjoint closures. Classical Swiss cheese sets have better topological properties, which make them desirable in the study of uniform algebras $R(X)$. In [25], Feinstein and Heath proved a classicalisation theorem, which asserts that under some mild conditions, every Swiss cheese set contains a classical Swiss cheese set. In this chapter we aim to give a topological proof of this fact. In order to do this, we propose to look at a Swiss cheese set "abstractly": we regard it as the realisation of a sequence of pairs of numbers, where they represent the centre and radius of a disc respectively. We then put a natural topology on the collection of all abstract Swiss cheeses and give our classicalisation theorem. We then study the controlled classicalisation theorem, where we leave some of the deleted open discs unchanged, and we also study the annular Swiss cheeses and their classicalisations, where we delete open discs from an annulus instead of a closed disc. In the final section of this chapter, we apply the classicalisation theorems developed in this chapter to give a classical Swiss cheese set which serves as a counterexample to the conjecture of S. E. Morris. The original counterexample, on which our example builds, is due to Feinstein in [24].

1.1.2 Chapter 3

In Chapter 3 we study regularity of natural uniform algebras in terms of regularity points: points of continuity and R-points. These two terms are coined by Feinstein and Somerset in [31], where they studied the failure of regularities for Banach function algebras. They gave examples of Banach function algebras where points of continuity do not coincide with R-points, but these algebras are not natural. There do not appear to be any examples in the literature of natural Banach function algebras where the two types of regularity point are different. In Section 3.2 we give an example of a compact plane set X where

$R(X)$ has only one point of continuity while having no R-points; and we also give a compact plane set X where $R(X)$ has only one R-point while having no points of continuity. In Section 3.3 we propose a new way to construct Swiss cheese sets X such that $R(X)$ has no non-trivial Jensen measures, but $R(X)$ is not regular. The original method is due to Feinstein ([23]). We also provide a more elementary argument to show the fact that $R(X)$, where X is one of our constructed Swiss cheese sets, is not regular.

1.1.3 Chapter 4

This chapter can be mainly divided into three parts. In the first part we study normal uniform algebras on the unit interval $\mathbb{I}$ or the unit circle $\mathbb{T}$. We manage to characterise all the closed antisymmetric subsets for such uniform algebras. In the second part, we study small exceptional sets for uniform algebras. Let $A \subseteq B$ be two uniform algebras on X such that for each proper closed subset K of X we have $A|_K = B|_K$. We give certain conditions on A or B that will guarantee that $A = B$. The third part of this chapter is related to Wermer's maximality theorem. We give a proper closed linear subspace of $C(\mathbb{T})$ which properly contains the disc algebra.

1.1.4 Chapter 5

In Chapter 5 we study methods to construct essential uniform algebras which share many properties with a given uniform algebra A. In the literature there are two different methods to do so as described in [26, 53]. We will first briefly describe these two methods of constructions and compare their similarities and differences. Then we will further study the method described in [26] to show that some more properties are also shared between the constructed uniform algebra and the original one A.

1.2 Banach algebras

In this section, we give some preliminary definitions and results concerning Banach algebras.

Let A be an algebra, we denote the group of invertible elements of A by $\mathrm{inv}(A)$. We shall denote the multiplicative identity of A by 1 if there is no risk of ambiguity. Let E be a normed space, we shall denote the (topological) dual space of E by E^*. If X is a non-empty topological space, we shall denote the collection of all complex-valued continuous functions from X to $\mathbb{C}$ by $C(X)$. Note that with pointwise addition and multiplication, $C(X)$ is an algebra.

Definition 1.2.1. Let A be an algebra. By an *algebra norm* on A we mean a mapping $a \mapsto \|a\|$ of A into the non-negative real numbers such that $(A; \|\cdot\|)$ is a normed space over $\mathbb{C}$, $\|ab\| \leq \|a\| \cdot \|b\|$ for all $a, b \in A$, and such that $\|1\| = 1$.

A *normed algebra* is a pair $(A; \|\cdot\|)$, where A is an algebra and $\|\cdot\|$ is an algebra norm on A. A normed algebra is a *Banach algebra* if the algebra norm is complete.

Let X be a non-empty compact Hausdorff space. Then it is standard that $(C(X); \|\cdot\|_X)$ is a Banach algebra, where $\|\cdot\|_X$ is the uniform norm on X defined by

$$\|f\|_X = \sup_{x \in X} |f(x)|, \quad f \in C(X).$$

In the rest of this thesis we shall always regard $C(X)$ as a Banach algebra with the norm $\|\cdot\|_X$, whenever X is (non-empty) compact Hausdorff.

Definition 1.2.2. Let A be an algebra. A *character* on A is a non-zero algebra homomorphism $\varphi : A \to \mathbb{C}$.

The following result on characters of a Banach algebra is of fundamental importance.

Proposition 1.2.3 ([2] Theorem 4.43)**.** *Let A be a Banach algebra, and let φ be a character on A. Then φ is continuous, so $\varphi \in A^*$ and $\|\varphi\| = 1$.*

Let A be an algebra. We shall write Φ_A for the set of all characters on the algebra A and call it the *character space of* A. We shall write M_A for the set of all maximal ideals of A and call it the *maximal ideal space of* A. We have the following correspondence between Φ_A and M_A.

Proposition 1.2.4 ([2] Theorem 4.46)**.** *Let A be a Banach algebra. Then $\Phi_A \neq \emptyset$ and the mapping $\varphi \mapsto \ker \varphi$ is a bijection from Φ_A onto M_A.*

In view of the previous result, for a Banach algebra A we usually use the term "character space of A" and the term "maximal ideal space of A" interchangeably. This observation proves the following fact.

Proposition 1.2.5. *Let A be a Banach algebra and let $a \in A$. Then a is an invertible element in A if and only if $\varphi(a) \neq 0$ for all $\varphi \in \Phi_A$.*

Let A be a Banach algebra with character space Φ_A. For each $a \in A$, we define the *Gel'fand transform* function $\hat{a} : \Phi_A \to \mathbb{C}$ by

$$\hat{a}(\varphi) = \varphi(a), \quad \varphi \in \Phi_A.$$

From Proposition 1.2.3, we see that the character space Φ_A of a Banach algebra A is actually a subset of the unit ball of A^*. If we give Φ_A the subspace topology induced by the weak-$*$ topology on A^*, then it is standard (Theorem 4.54 in [2]) that Φ_A is a (non-empty) compact Hausdorff space. This relativization to Φ_A of the weak-$*$ topology on A^* is called the *Gel'fand topology* on Φ_A, and this is the standard topology that we shall consider in this thesis. The above observation (essentially contained in [2, p. 188]) is summarized in the following proposition.

Proposition 1.2.6. *Let A be a Banach algebra. The Gel'fand topology on Φ_A is the weakest topology such that, for each $a \in A$, $\hat{a}$ is continuous.*

Definition 1.2.7. Let A be a Banach algebra. We say that A is *semi-simple* if the intersection of all the maximal ideals of A contains only the zero element.

In view of Proposition 1.2.4, we see that a Banach algebra A is semi-simple if and only if the intersection of the kernels of all characters on A contains only the zero element.

Now we are ready to state the famous *Gel'fand representation theorem.*

Proposition 1.2.8 ([2] Theorem 4.59)**.** *Let A be a Banach algebra. Then the Gel'fand transform*

$$\mathcal{G} : a \mapsto \hat{a}, \quad A \to C(\Phi_A)$$

is a continuous, unital algebra homomorphism. The Gel'fand transform is injective if and only if A is semi-simple.

We sometimes use the notions of Banach A-bimodules and derivations, which we briefly introduce there.

Let A be an algebra, and let E be a vector space. We say that E is an *A-bimodule* if there exist bilinear maps $(a, \xi) \mapsto a \cdot \xi$ and $(a, \xi) \mapsto \xi \cdot a$ from $A \times E$ to E such that for each $a, b \in A$ and $\xi \in E$ the following holds (we temporarily use 1_A to denote the multiplicative unit of A):

(i) $a \cdot (b \cdot \xi) = ab \cdot \xi$;

(ii) $(\xi \cdot a) \cdot b = \xi \cdot ab$;

(iii) $a \cdot (\xi \cdot b) = (a \cdot \xi) \cdot b$;

(iv) $1_A \cdot \xi = \xi \cdot 1_A = \xi$.

We say E is a *commutative A-bimodule* if $a \cdot \xi = \xi \cdot a$, for all $a \in A$ and $\xi \in E$.

Definition 1.2.9. Let $(A, \|\cdot\|)$ be a Banach algebra, and let $(E, |\cdot|)$ be a Banach space such that E is an A-bimodule. Then we say E is a *Banach A-bimodule* if the two bilinear maps $(a, \xi) \mapsto a \cdot \xi$ and $(a, \xi) \mapsto \xi \cdot a$ are continuous.

We note that by changing to an equivalent norm on E we may and shall suppose that if E is a Banach A-bimodule, then

$$|a \cdot \xi| \leq \|a\| \, |\xi| \, , \quad |\xi \cdot a| \leq \|a\| \, |\xi| \, , \quad a \in A, \xi \in E.$$

We give two examples of Banach A-bimodules introduced in [2, p. 229], which will be used later.

Example 1.2.10. Let A be a Banach algebra.

(1) Let E be a Banach A-bimodule. Then the (topological) dual space E^* has a Banach A-bimodule structure with the bilinear maps defined by

$$(a \cdot f)(\xi) := f(\xi \cdot a), \quad (f \cdot a)(\xi) := f(a \cdot \xi), \quad a \in A, \xi \in E, f \in E^*.$$

In this case, we call E^* the *dual module* of E.

(2) Let $\varphi \in \Phi_A$. Then $\mathbb{C}$ is a Banach A-bimodule for the bilinear maps specified by

$$a \cdot z = z \cdot a := \varphi(a)z, \quad a \in A, z \in \mathbb{C}.$$

This A-bimodule is usually denoted by $\mathbb{C}_\varphi$.

Definition 1.2.11. Let A be an algebra, and let E be an A-bimodule. Then a *derivation* from A to E is a linear map $D : A \to E$ such that

$$D(ab) = a \cdot D(b) + D(a) \cdot b, \quad a, b \in A.$$

Again we give two examples of derivations, which will be used later.

Example 1.2.12. (1) Let A be an algebra, and let E be an A-bimodule. Fix a $\xi \in E$, and set

$$\delta_\xi(a) = a \cdot \xi - \xi \cdot a, \quad a \in A.$$

Then $\delta_\xi : A \to E$ is a derivation. We call these derivations *inner derivations.*

(2) Let A be a Banach algebra, and fix a $\varphi \in \Phi_A$. Then a linear map $d : A \to \mathbb{C}_\varphi$ is a derivation if

$$d(ab) = \varphi(a)d(b) + \varphi(b)d(a), \quad a, b \in A.$$

Such derivations are called *point derivations at* φ.

1.3 Banach function algebras

For the rest of Chapter 1, X will always be a non-empty compact Hausdorff space.

Definition 1.3.1. Let A be a subalgebra of $C(X)$. We say that A is *self-adjoint* if the complex conjugate $\bar{f}$ is in A whenever f is in A. We say that A *separates the points of* X if, for each pair of distinct points x, y in X, there exists a function $f \in A$ such that $f(x) \neq f(y)$.

It is clear that $C(X)$ is self-adjoint. By Urysohn's lemma ([2] Theorem 1.26) we see that $C(X)$ separates the points of X. For self-adjoint, point separating subalgebras of $C(X)$ which contain the constant functions, we have the following celebrated Stone-Weierstrass theorem.

Proposition 1.3.2. *Let A be a self-adjoint subalgebra of $C(X)$ that contains the constant functions and separates the points of X. Then A is dense in $(C(X); \|\cdot\|_X)$.*

Definition 1.3.3. A *Banach function algebra* A on X is a subalgebra of $C(X)$ such that A separates the points of X and contains the constant functions, and such that A has a complete algebra norm. A *uniform algebra* A on X is a Banach function algebra on X whose algebra norm is $\|\cdot\|_X$.

Note that $C(X)$ is a uniform algebra on X. Also note that in the literature, some authors use the term *function algebras* to refer to uniform algebras. For the rest of this thesis, we use the convention that whenever we say A is a Banach function algebra (or a uniform algebra) on a space X, then X is assumed to be a non-empty compact Hausdorff space.

In particular, a Banach function algebra is a Banach algebra. We say that a Banach function algebra A on X is *trivial* if $A = C(X)$. A non-trivial Banach function algebra is also called a *proper* Banach function algebra. Note that, by the Stone-Weierstrass theorem ([2, Corollary 2.33]), a uniform algebra A is trivial if and only if A is self-adjoint.

For the rest of this section, A will be a Banach function algebra on X, unless otherwise stated.

For each point x in X, there exists a *point evaluation character* ε_x on A defined by

$$\varepsilon_x : f \mapsto f(x), \quad f \in A.$$

Then it is easy to see that A is a semi-simple Banach algebra, since the only function in the intersection of the kernels of all ε_x is the zero function. We say A is a *natural* Banach function algebra (on X) if the point evaluation characters ε_x are the only characters for A. Here we include three general methods to test if a Banach function algebra is natural or not.

Proposition 1.3.4 (Theorem 4.56 in [2]). *Let A be a Banach function algebra on X. Then A is natural on X if and only if, for each finite subset $\{f_1, \cdots, f_n\}$ of A with no common zero in X, there exist $g_1, \cdots, g_n$ in A such that*

$$\sum_{i=1}^{n} f_i g_i = 1.$$

Proposition 1.3.5 (Corollary 4.58 in [2]). *Let A be a Banach function algebra on X. Suppose that the uniform closure $\overline{A}$ is natural on X. Then A is natural on X if and only if $1/f$ is in A whenever f is in A such that f has no zero in X.*

Proposition 1.3.6 ([41]). *Let A be a Banach function algebra on X with the algebra norm $\|\cdot\|$. Then A is natural on X if and only if the uniform closure $\overline{A}$ is natural on X and*

$$\lim_{n\to\infty} \|f^n\|^{1/n} = 1, \quad \textit{for all } f \in A \textit{ with } \|f\|_X = 1.$$

For an illustration of Propositions 1.3.5 and 1.3.6 we refer the reader to Example 4.2.6.

By Proposition 1.2.8, we see that each semi-simple Banach algebra B is algebraically isomorphic to a subalgebra $\hat{B}$ of $C(\Phi_B)$ via the Gel'fand transform. If we endow $\hat{B}$ with the algebra norm induced by the algebra norm on B via the Gel'fand transform, it is clear that $\hat{B}$ becomes a natural Banach function algebra on Φ_B. Therefore the study of Banach function algebras provides another perspective on the study of semi-simple Banach algebras.

It is elementary to show that the topology on X is the weakest topology for which every function in A is continuous ([9, p. 2]). This is essentially the content of the following proposition, in view of Proposition 1.2.6.

Proposition 1.3.7 ([2] Lemma 4.55)**.** *The map*

$$x \mapsto \varepsilon_x, \quad X \to \Phi_A$$

is a homeomorphism from X onto a closed subset of Φ_A.

In view of the above proposition, we usually regard X as being a closed subset of Φ_A. We also usually use A to denote its Gel'fand transform, and regard A as a Banach function algebra on Φ_A, when there is no danger of ambiguity. However, when we discuss naturality of A, we should always specify on which space the algebra of functions A is defined.

1.4 Uniform algebras and their antisymmetric decompositions

Let A be a uniform algebra on X. If K is a non-empty closed subset of X, we use $A|_K$ to denote the collection of restrictions of functions in A to K, and we use A_K to denote the uniform closure of $A|_K$ in $C(K)$. It is clear that A_K is a uniform algebra on K.

Definition 1.4.1. Let A be a uniform algebra on X. We say A is *pervasive on* X if $A_K = C(K)$ for each non-empty proper closed subset K of X.

By the Tietze extension theorem ([2, Theorem 3.39]) we see that $C(X)$ is a pervasive uniform algebra. See Section 1.7 for an example of a proper pervasive uniform algebra.

By the Stone–Weierstrass theorem we see that A is non-trivial if and only if the real-valued functions in A do not separate points of X. In this section, we introduce an useful generalisation of this observation due to Bishop ([6]).

Definition 1.4.2. Let A be a uniform algebra on X. A subset K of X is called an *antisymmetric subset* for A if, for each $f \in A$ with $f(K) \subseteq \mathbb{R}$, the restriction function $f|_K$ is a constant function. We say A is an *antisymmetric uniform algebra on* X if X is an antisymmetric subset for A.

From the definition we see that if A is an antisymmetric uniform algebra on X, then the only real valued functions in A are the constant functions taking real values. At the other extreme, it is clear that the only antisymmetric subsets for $C(X)$ are the single-point subsets of X. The following proposition from [36] shows that the latter case actually characterises $C(X)$ and generalises the Stone–Weierstrass theorem.

Proposition 1.4.3. *Let A be a uniform algebra on X. Every antisymmetric subset for A is contained in a unique maximal antisymmetric subset for A. The collection $\mathcal{K}$ of maximal antisymmetric subsets for A forms a partition of X into closed subsets satisfying the following properties.*

(i) For each $f \in C(X)$, if $f|_K \in A|_K$ for all K in $\mathcal{K}$, then $f \in A$.

(ii) For each K in $\mathcal{K}$, $A|_K$ is uniformly closed in $C(K)$.

(iii) The uniform algebra A is natural on X if and only if, for each K in $\mathcal{K}$, the uniform algebra $A|_K$ is natural on K.

We note that if A is a natural uniform algebra on X and $K \subseteq X$ is a maximal antisymmetric subset for A, then an application of Shilov's idempotent theorem ([2, Theorem 9.5]) shows that K is connected. From Proposition 1.4.3 (i) we see that a proper pervasive uniform algebra is antisymmetric.

Let A be a uniform algebra on X. If $f \in A$ is a non-constant real-valued function, then e^{if} is a non-constant function in A whose modulus is always 1. Thus if A is antisymmetric on X, then no non-constant unimodular function in A can be obtained in this way. The following proposition shows that if A is natural and antisymmetric on X, then A does not contain any non-constant unimodular functions. This proposition is probably well known, but the author cannot find a direct reference for it. (The author realised the content of this proposition while reading [38, Theorem 2].) We include a simple proof of this proposition for the convenience of the reader.

Proposition 1.4.4. *Let A be a natural antisymmetric uniform algebra on X. Then the only unimodular functions in A are the constant functions of modulus 1.*

Proof. Let f be a unimodular function in A. Since A is natural and f has no zero on X, by Proposition 1.2.5 we see that $1/f$ is also in A. Since f is unimodular, the function $1/f$ is just the complex conjugate of f. Therefore both the real and imaginary parts of f are in A. Since A is antisymmetric, both the real and imaginary parts must be constant functions. Therefore f must be a constant function of modulus 1. □

From Proposition 1.4.3 we see that a uniform algebra A on X is trivial if and only if every maximal antisymmetric subset for A is a single-point set. To see how Proposition 1.4.3 generalises the Stone-Weierstrass theorem, let A be an algebra of continuous functions on X such that A contains all the constant functions, A is self-adjoint, and such that A separates the points of X. Then it is clear that the uniform closure $\bar{A}$ of A is a uniform algebra on X, where the only antisymmetric subsets for $\bar{A}$ are the single-point sets. Therefore by Proposition 1.4.3 we see that $\bar{A}$ is trivial, which is equivalent to A being uniformly dense in $C(X)$.

To help us to give some examples of antisymmetric subsets for uniform algebras, we introduce the following definitions. We refer the reader to [58] for the definition of a regular Borel measure and the support of a measure. We use the convention that all measures discussed in this thesis are regular Borel measures, unless otherwise specified.

Definition 1.4.5. Let A be a uniform algebra on X, and let φ be a character on A. A *representing measure* for φ is a probability measure μ on X such that

$$\varphi(f) = \int_X f \, \mathrm{d}\mu, \quad f \in A.$$

If ε_x is the point evaluation character on A for a point x in X, then the unit point mass at x would be a representing measure for ε_x. The existence of a representing measure for a general character is guaranteed by the Hahn-Banach theorem and the Riesz representation theorem ([58, p. 40]). However, a given character may have several different representing measures.

The following proposition gives a good supply of antisymmetric subsets for uniform algebras. This proposition is mentioned in [10] without a proof. We supply a simple proof here.

Proposition 1.4.6. *Let A be a uniform algebra on X, and let φ be a character on A. Let μ be a representing measure for φ, and let S be the closed support of μ in X. Then S is a set of antisymmetry for A.*

Proof. Assume towards a contradiction that S is not a set of antisymmetry for A. Then we can find a function f in A whose restriction to S is a non-constant real-valued function. By subtracting a real constant, we can assume that

$$\varphi(f) = \int_S f \, \mathrm{d}\mu = 0.$$

On the other hand, by the multiplicativity of φ we have

$$\varphi(f)^2 = \varphi(f^2) = \int_S f^2 \, \mathrm{d}\mu > 0,$$

since $f|_S$ is continuous, real-valued and non-constant. This leads to a contradiction. □

Let μ be a probability measure on X, and let A be a uniform algebra on X. We shall say that μ is *multiplicative* (with respect to A) if

$$\int_X fg \, \mathrm{d}\mu = \int_X f \, \mathrm{d}\mu \int_X g \, \mathrm{d}\mu, \quad f, g \in A.$$

(Note that this terminology does not appear to be standard.) In this case, it is clear that μ induces a character on A. The following proposition adapted from [2, Theorem 4.102] characterizes all probability measures on X that are multiplicative. (For the non-commutative case, see [59, Theorem 10.9].)

Proposition 1.4.7. *Let μ be a probability measure on X, and let A be a uniform algebra on X. Then μ is multiplicative if and only if*

$$\int_X f \, \mathrm{d}\mu \neq 0, \quad f \in \mathrm{inv}(A).$$

The following definition will also be used many times in this thesis.

Definition 1.4.8. Let A be a uniform algebra on X, and let μ be a complex measure on X. We say μ is an *annihilating measure* for A if

$$\int_X f \, \mathrm{d}\mu = 0, \quad f \in A.$$

It is standard that a uniform algebra A on X equals $C(X)$ if and only if the only annihilating measure for A is the zero measure ([9, p. 80]).

Let A be a uniform algebra on X. In [9, Theorem 2.8.1] it is shown that there is a unique minimal (possibly empty) closed subset E of X with the property that A contains every continuous function on X which vanishes on E. This set E is important in the study of A, and it is closely related to the antisymmetric decomposition of A.

Definition 1.4.9. Let A be a uniform algebra on X. The *essential set* for A is the unique minimal closed subset E of X such that A contains every continuous function on X which vanishes on E. We say A is an *essential uniform algebra* on X if X is the essential set for A.

From the definition it is clear that A equals $C(X)$ if and only if the essential set E is empty. An important characterisation of E is the following (see [9, p. 145]).

Proposition 1.4.10. *Let A be a uniform algebra on X, and let E be the essential set for A. Then E is the closure of the union of the supports of all annihilating measures for A.*

The next proposition shows that the essential set for A can be easily recovered from the antisymmetric decomposition of A.

Proposition 1.4.11 ([62] Theorem 3)**.** *Let A be a uniform algebra on X, let E be the essential set for A, and let $\mathcal{K}$ be the collection of all maximal antisymmetric subsets for A. Let*

$$P = \{x \in X : \{x\} \in \mathcal{K}\} .$$

Then

$$E = X \setminus \operatorname{int} P.$$

1.5 Peak sets and boundaries for uniform algebras

In this section, A will always be a uniform algebra on X. For analogous definitions and results for Banach function algebras, see [2, p. 202].

Definition 1.5.1. A non-empty subset K of X is a *peak set* (for A) if there exists f in A such that $f(K) = \{1\}$, and such that $|f(x)| < 1$ for all $x \in X \setminus K$. We call K a *peak set in the weak sense* if K is the non-empty intersection of some collection of peak sets. A point $x \in X$ is a *peak point* or a *peak point in the weak sense* for A if $\{x\}$ is a peak set or a peak set in the weak sense for A, respectively.

Let $K \subseteq X$ be a peak set for A. If a function $f \in A$ has the properties that $K = f^{-1}\{1\}$ and $|f(y)| < 1$ for all $y \in X \setminus K$, then we say that f *peaks on* K, or f is a *peaking function* for K.

It is standard that if K is a peak set in the weak sense for A, then $A|_K$ is uniformly closed ([9, Corollary 2.4.3]). It is also standard (see [9, Lemma 2.3.1]) that if X is metrizable, then each peak set in the weak sense (for A) is also a peak set.

We use $\Gamma_0(A)$ to denote the collection of all peak points in the weak sense for A, and we use $\Gamma(A)$ to denote the closure of $\Gamma_0(A)$ in X. Note that these notations are not standard.

In view of Urysohn's lemma, we see that each closed subset of X is a peak set in the weak sense for $C(X)$. This actually characterises $C(X)$ among uniform algebras on X. The following proposition is discussed in [9, p. 113].

Proposition 1.5.2. *Let A be a uniform algebra on X. If for each closed subset K of X, K is a peak set in the weak sense for A, then $A = C(X)$.*

It is clear from the definition of essential set that if $K \subseteq X$ is a closed subset containing the essential set for A, then K is a peak set in the weak sense for A.

Definition 1.5.3. A subset Y of X is called a *boundary* for A if for each f in A there exists a $y \in Y$ such that $|f(y)| = \|f\|_X$.

From the definition it is clear that X is always a boundary for A, and each boundary for A must contain all peak points for A. The following proposition adapted from [9, Section 2-2] shows that more is true.

Proposition 1.5.4. *The collection of points $\Gamma_0(A)$ is a boundary for A, and $\Gamma(A)$ is the unique minimal closed boundary for A.*

In view of the above proposition, $\Gamma_0(A)$ is usually called the *Choquet boundary* of A, and $\Gamma(A)$ is usually called the *Shilov boundary* of A. We note that when X is metrizable, $\Gamma_0(A)$ is the intersection of a countable collection of open subsets of X and hence (Borel) measurable ([9, Corollary 2.2.7]). In general the set $\Gamma_0(A)$ need not be (Borel) measurable, see [7].

Since $\Gamma(A)$ is a boundary for A, it is clear that the restriction algebra $A|_{\Gamma(A)}$ is also a uniform algebra, and the restriction map

$$r : f \mapsto f|_{\Gamma(A)}, \quad A \to A|_{\Gamma(A)}$$

is an isometric algebra isomorphism. Therefore A can also be viewed as an algebra of functions on $\Gamma(A)$. In particular, each character φ for A induces a character $\tilde{\varphi}$ for $A|_{\Gamma(A)}$. Hence by the Hahn-Banach theorem and the Riesz representation theorem again there exists a probability measure μ on $\Gamma(A)$ such that

$$\varphi(f) = \tilde{\varphi}(f|_{\Gamma(A)}) = \int_{\Gamma(A)} f \, \mathrm{d}\mu, \quad f \in A.$$

This shows that each character for A actually has a representing measure on the Shilov boundary $\Gamma(A)$ of A. See Section 1.7 for an example of this phenomenon.

In the study of uniform algebras, it is sometimes convenient to "identify" A with a uniform algebra on some other compact Hausdorff space. The following proposition is useful in the special case that A is a natural uniform algebra on X. This proposition is probably well known. We supply a simple proof for the convenience of the reader.

Proposition 1.5.5. *Let A be a natural uniform algebra on X, and let $\Lambda : Y \to X$ be a homeomorphism. Set $\tilde{A} = \{f \circ \Lambda : f \in A\}$. Then $\tilde{A}$ is a natural uniform algebra on Y and $\Gamma(\tilde{A}) = \Lambda^{-1}(\Gamma(A))$, and $\tilde{A}$ is isometrically algebra isomorphic to A.*

Proof. Since Λ is a homeomorphism, it is clear that $\tilde{A}$ is a uniform algebra and $\|f \circ \Lambda\|_Y = \|f\|$ for each $f \in A$. It is also clear that a closed subset K of X is a peak set for A if and only if $\Lambda^{-1}(K)$ is a peak set for $\tilde{A}$. This implies that the Shilov boundary of $\tilde{A}$ is $\Lambda^{-1}(\Gamma(A))$. It is also easy to see that the following map

$$\Lambda^* : f \mapsto f \circ \Lambda, \quad A \to \tilde{A}$$

is an isometric algebra isomorphism from A to $\tilde{A}$.

It remains to show that $\tilde{A}$ is natural on Y. Let $\tilde{\varphi} \in \Phi_{\tilde{A}}$, then $\tilde{\varphi}$ induces a character φ on A by

$$\varphi(f) := \tilde{\varphi}(f \circ \Lambda), \quad f \in A.$$

Since A is natural on X, there exists $x \in X$ such that $\varphi = \varepsilon_x$. Thus we have

$$\tilde{\varphi}(f \circ \Lambda) = \varepsilon_x(f) = f(x) = (f \circ \Lambda)(\Lambda^{-1}(x)), \quad f \in A.$$

Thus $\tilde{\varphi} = \varepsilon_y$ with $y = \Lambda^{-1}(x) \in Y$, and this shows that $\tilde{A}$ is natural on Y. □

We finish this section by giving several equivalent conditions that characterise peak points in the weak sense. The *state space* of A is defined by

$$K = \{\varphi \in A^* : \varphi(1) = \|\varphi\| = 1\}.$$

It is standard that K is a convex subset of the closed unit ball of A^*, K contains all characters for A (and hence each ε_x for $x \in X$) and K is weak-* compact. A linear functional φ in K is called an *extreme point* of K if, whenever $\varphi = t\psi + (1-t)\eta$ with $0 \leq t \leq 1$ and $\psi, \eta \in K$, we have $\varphi = \psi = \eta$. The following proposition is adapted from several facts and theorems from Chapter 2 of [9]. Recall that $\Gamma_0(A) \subseteq \Gamma(A) \subseteq X$ and $\Gamma(A) = \overline{\Gamma_0(A)}$.

Proposition 1.5.6. *Let A be a uniform algebra on X, and let $x \in X$. Then the following are equivalent.*

(i) The point x is a peak point in the weak sense for A.

(ii) The evaluation character ε_x admits a unique representing measure on X.

(iii) The evaluation character ε_x is an extreme point of the state space K.

(iv) For each representing measure μ for ε_x, $\mu(\{x\}) > 0$.

(v) For each annihilating measure μ for A, $\mu(\{x\}) = 0$.

(vi) There exist $0 < \alpha < \beta < 1$, such that for any neighbourhood U of x there exists $f \in A$ such that $\|f\|_X \leq 1$, $f(x) > \beta$, and such that $|f(y)| < \alpha$ for all $y \in X \setminus U$.

Moreover, every extreme point of K is an evaluation character at some point of $\Gamma_0(A)$.

1.6 Shilov boundaries of uniform algebras under identifications

Let A be a uniform algebra on X. As discussed in the previous sections, we may also regard A as a uniform algebra on its Shilov boundary $\Gamma(A)$ or as a uniform algebra on its character space Φ_A. In the literature of uniform algebras, people usually make (implicitly) these identifications. But this may lead to some confusions and ambiguities. For example, when we use $\Gamma(A)$ to denote the Shilov boundary of A, does it matter if we regard A as a uniform algebra on X or on Φ_A? In this section, we show that via some identification, the Shilov boundary of a uniform algebra can always be identified with the Shilov boundary of the Gel'fand transform of the uniform algebra. We note that most of this section is standard and probably well known, but we cannot find references that clarify these points.

In the rest of this section, we fix A to be a uniform algebra on X. Recall that the Gel'fand transform of A is denoted by $\hat{A}$, see Proposition 1.2.8.

We temporarily denote the map defined in Proposition 1.3.7 by ι_A. Thus for each $x \in X$ we have $\iota_A(x) = \varepsilon_x$, and ι_A is a homeomorphism from X onto its image $\iota_A(X) \subseteq \Phi_A$. We note that via ι_A there is a one to one correspondence between the probability measures μ on X and the probability measures $\tilde{\mu}$ on $\iota_A(X)$ given by

$$\tilde{\mu}(\iota_A(E)) = \mu(E), \quad E \text{ Borel measurable subset of } X.$$

Proposition 1.6.1. *The set* $\Gamma_0(\hat{A})$ *coincides with* $\iota_A(\Gamma_0(A))$.

Proof. First let $\varphi \in \Gamma_0(\hat{A})$. Then by Proposition 1.5.6 we know that φ admits a unique unit point mass representing measure on Φ_A. But since φ is a character on A it induces a representing measure μ on X. Via ι_A the measure μ induces a measure $\tilde{\mu}$ supported on $\iota_A(X)$, which is an $\hat{A}$-representing measure for φ. Thus $\tilde{\mu}$ is just the unit point mass at φ and φ is in $\iota_A(X)$. So $\varphi = \varepsilon_x$ for some $x \in X$. We claim that x is in $\Gamma_0(A)$, because otherwise by Proposition 1.5.6 ε_x would admits two distinct A-representing measures on X, which via ι_A gives two distinct $\hat{A}$-representing measures on Φ_A, a contradiction.

Next let $x \in \Gamma_0(A)$, we aim to show that $\varepsilon_x \in \Gamma_0(\hat{A})$. Let $K \subseteq \Phi_A$ be the intersection of all $\hat{A}$ peak sets containing ε_x. We first claim that $K \cap \iota_A(X) = \{\varepsilon_x\}$. Let $\varepsilon_y \in \iota_A(X) \setminus \{\varepsilon_x\}$. Since x is an A peak point in the weak sense, we can find an A peak set $L \subseteq X$ such that $y \notin L$, and a function $f \in A$ peaking on L. So $\hat{f}$ peaks on a set $F \subseteq \Phi_A$ that contains $\iota_A(L)$ such that $\varepsilon_y \notin F$. We have $K \subseteq F$, so $\varepsilon_y \notin K$. This shows that $K \cap \iota_A(X) = \{\varepsilon_x\}$. Now we show that $K = \{\varepsilon_x\}$. Let $\varphi \in K$ be a character on A. Then φ admits an A representing measure μ on X with closed support $S \subseteq X$. Assume towards a contradiction that there exists $y \in S \setminus \{x\}$. Then as in the above proof we can find a function $f \in A$ such that $\hat{f}$ peaks on a closed set $H \subseteq \Phi_A$ containing K such that $\varepsilon_y \notin H$, i.e. $|f(y)| < 1$. Therefore we have

$$1 = \hat{f}(\varphi) = \varphi(f) = \int_S f \,\mathrm{d}\mu = \left| \int_S f \,\mathrm{d}\mu \right| < 1,$$

a contradiction. Thus $S = \{x\}$. So the A representing measure μ for φ is just the unit point mass at x. This shows that $\varphi = \varepsilon_x$ and proves that $K = \{\varepsilon_x\}$. From Definition 1.5.1 we see that ε_x is in $\Gamma_0(\hat{A})$. This finishes the proof. $\square$

Corollary 1.6.2. *The Shilov boundary* $\Gamma(\hat{A})$ *coincides with* $\iota_A(\Gamma(A))$.

Proof. This is an easy consequence of the previous lemma. Note that the Shilov boundary is the closure of the Choquet boundary, and ι_A is a homeomorphism. □

1.7 Properties of $C(X)$ and of the disc algebra

In this section, we discuss some properties of $C(X)$ and the disc algebra (defined below). This will help to illustrate some of the definitions from the previous sections. We classify all closed antisymmetric subsets for the disc algebra on the unit circle. We then apply this classification to give a new proof of the fact that the disc algebra on the unit circle is pervasive, and a new proof of Wermer's maximality theorem.

We note that most of the material in this section are standard facts and are probably well known. The following parts of this section appear to be new: Theorem 1.7.1 and its proof; Proposition 1.7.2 and its proof; Lemmas 1.7.4 and 1.7.5 and their proofs; the proof of Proposition 1.7.6.

We first analyse $C(X)$, where some of these properties were noted before. It is standard that $C(X)$ is a natural uniform algebra on X ([9, Theorem 1.3.1]), and $\Gamma_0(C(X)) = \Gamma(C(X)) = X$. Thus every point of X is a peak point in the weak sense for $C(X)$. Urysohn's lemma implies that each (non-empty) closed subset of X is a peak set in the weak sense for $C(X)$. It is also clear (using Tietze's extension theorem) that $C(X)$ is pervasive on X. The only annihilating measure for $C(X)$ is the zero measure. The only antisymmetric subsets for $C(X)$ are the singleton subsets of X. Therefore each maximal antisymmetric subset for $C(X)$ is a singleton subset of X. The essential set for $C(X)$ is the empty set.

Next we introduce and analyse the disc algebra. For the rest of this section, we use D to denote the open unit disc in $\mathbb{C}$, and we use $\overline{D}$ to denote the closed unit disc. Let $\mathbb{T}$ be the unit circle in $\mathbb{C}$. We use $\mathrm{d}t$ to denote the Lebesgue (arc length) measure on $\mathbb{T}$. We note that, since $\overline{D}$ is metrizable, for each uniform algebra on $\overline{D}$, each peak point in the weak sense is also a peak point.

The *disc algebra* A on $\overline{D}$ is the algebra of all continuous functions on $\overline{D}$ whose restrictions to D are holomorphic. The coordinate function $Z : z \mapsto z$ is in A, so A separates the points of $\overline{D}$. It is standard that A is uniformly closed, and hence A is a uniform algebra on $\overline{D}$. It is standard that each function in A can be uniformly approximated by polynomials in the coordinate function Z, see [2, Theorem 2.86].

For the rest of this section, we reserve the letter A for the disc algebra.

It is standard that A is a natural uniform algebra on $\overline{D}$, see [9, p. 35]. By the maximum modulus principle from complex analysis we know that each $f \in A$ attains its maximum modulus on $\mathbb{T}$, so $\Gamma(A) \subseteq \mathbb{T}$. On the other hand, for each point $e^{i\theta}$ in $\mathbb{T}$ the function

$$f(z) = \frac{ze^{-i\theta} + 1}{2}$$

is in A and peaks at the point $e^{i\theta}$. Therefore we conclude that each point in $\mathbb{T}$ is a peak point for A, and $\Gamma_0(A) = \Gamma(A) = \mathbb{T}$. As noted in Section 1.5, the disc algebra A is isometrically algebra isomorphic to

$$A|_{\Gamma(A)} = A|_{\mathbb{T}} = \{f \in C(\mathbb{T}) : f \text{ can be extended to be a function in } A\}.$$

The character space of $A|_{\mathbb{T}}$ can be identified with $\overline{D}$.

To avoid ambiguity, we always regard the disc algebra A as a natural uniform algebra on $\overline{D}$. We use $A|_{\mathbb{T}}$ to denote the restriction of A to $\mathbb{T}$, which is a uniform algebra on $\mathbb{T}$.

We know that, for each $z \in \overline{D}$, the character ε_z on A admits a representing measure on the Shilov boundary $\mathbb{T}$. If z is in $\mathbb{T}$, then z is a peak point for A, and hence the unit point mass measure at z is the unique representing measure for ε_z. On the other hand, if $z \in D$, then z has a representing measure on $\mathbb{T}$ corresponding to the Poisson integral. Write $z = re^{i\theta}$ with $0 \leq r < 1$ and $\theta \in \mathbb{R}$, and for $t \in \mathbb{R}$ let

$$P_r(\theta - t) = \sum_{-\infty}^{\infty} r^{|n|} e^{in(\theta - t)} = \frac{1 - r^2}{1 - 2r\cos(\theta - t) + r^2}.$$

Then we have

$$\varepsilon_z(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{it}) P_r(\theta - t)\, \mathrm{d}t, \quad f \in A.$$

It is standard that the measure

$$\frac{1}{2\pi} P_r(\theta - t)\, \mathrm{d}t$$

is the *unique* representing measure on $\mathbb{T}$ for ε_z (see [58, p. 111]). Note that on $\overline{D}$, ε_z admits many other representing measures, including the unit point mass at z (cf. Proposition 1.5.6). Note that $z \in D$ is not a peak point for A. This follows from the maximum modulus principle, or using Proposition 1.5.6 (ii).

The famous F. and M. Riesz theorem ([58, Theorem 17.13]) gives a characterisation of annihilating measures on $\mathbb{T}$ for A: a complex measure μ on $\mathbb{T}$ annihilates the disc algebra if and only if $\mu = f\,\mathrm{d}t$, where f is a Lebesgue integrable function on $\mathbb{T}$ with

$$\int_{-\pi}^{\pi} f(e^{it})e^{int}\,\mathrm{d}t = 0, \quad n \in \mathbb{N}.$$

In particular, each annihilating measure for A on $\mathbb{T}$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{T}$. It is standard (see [9, Theorem 2.4.9] and the second corollary on page 52 of [39]) that a proper (non-empty) closed subset K of $\mathbb{T}$ is a peak set for $A|_{\mathbb{T}}$ if and only if K has Lebesgue measure 0 on $\mathbb{T}$.

It follows easily from the open mapping theorem from complex analysis that A is an antisymmetric uniform algebra on $\overline{D}$. Then by Proposition 1.4.11 we see that A is an essential uniform algebra on $\overline{D}$. Since $\mathbb{T}$ is the support of each of the representing measures

$$\frac{1}{2\pi}P_r(\theta - t)\,\mathrm{d}t,$$

by Proposition 1.4.6 we see that $\mathbb{T}$ is a set of antisymmetry for A. In the next theorem, we characterise all the closed antisymmetric subsets for $A|_{\mathbb{T}}$. We are thankful to Prof. J. Langley for helpful discussions relating to the proof of this theorem.

Theorem 1.7.1. *The only closed antisymmetric subsets of* $\mathbb{T}$ *for* $A|_{\mathbb{T}}$ *are* $\mathbb{T}$ *and the singleton subsets.*

Proof. Let $K \subsetneq \mathbb{T}$ be a proper closed subset with at least two elements. Then there exists a rectangle F in $\mathbb{C}$ with one of its edges L contained in the real axis, such that there exists a function f in A which sends $\overline{D}$ bijectively onto F and such that $f(K) \subseteq L$. Such a mapping can be constructed using the Schwarz-Christoffel mapping, see [48]. Then it is clear that $f|_{\mathbb{T}}$ is in $A|_{\mathbb{T}}$ and $f|_K$ is non-constant and real valued. This shows that K is not a set of antisymmetry for A. □

As an application of the classification of all closed antisymmetric subsets for $A|_{\mathbb{T}}$ of $\mathbb{T}$, we give new proofs of two known results. The first result is that $A|_{\mathbb{T}}$ is a pervasive uniform algebra (see, for example, [33, Theorem 8.7]). The second result is a famous theorem of Wermer in [65], known as Wermer's maximality theorem, which states that the only uniform algebras on $\mathbb{T}$ which contain $A|_{\mathbb{T}}$ are $C(\mathbb{T})$ and $A|_{\mathbb{T}}$.

In order to show that $A|_{\mathbb{T}}$ is a pervasive uniform algebra, we first prove a more general result that appears to be new.

Proposition 1.7.2. *Let B be a uniform algebra on X such that the only closed antisymmetric subsets for B are X and the singleton subsets. Then B is a pervasive uniform algebra on X.*

Proof. Let K be a non-empty proper closed subset of X. Since any closed antisymmetric subset for B_K is also a closed antisymmetric subset for B, we see that B_K is a uniform algebra on K whose only closed antisymmetric subsets are the singleton subsets. Therefore by Proposition 1.4.3 we see that $B_K = C(K)$. This shows that B is a pervasive uniform algebra on X. □

We do not know whether the converse of this result is true. See Question 1.8.1.

Corollary 1.7.3. *The disc algebra on the unit circle, $A|_{\mathbb{T}}$, is a pervasive uniform algebra on $\mathbb{T}$.*

For the proof of Wermer's maximality theorem, we first need a lemma. We note that if f is a continuous function between the topological spaces Y_1 and Y_2, then we call f an *embedding* if f is a homeomorphism from Y_1 onto $f(Y_1)$.

Lemma 1.7.4. *Let B be a uniform algebra on $\mathbb{T}$ which contains $A|_{\mathbb{T}}$. Then the map*

$$\Psi : \varphi \mapsto \varphi|_{A|_{\mathbb{T}}}, \quad \Phi_B \to \Phi_{A|_{\mathbb{T}}}$$

is an embedding.

Proof. It is easy to see that Ψ is a continuous function. Since both $\Phi_{A|_{\mathbb{T}}}$ and Φ_B are compact Hausdorff spaces, we only need to show that Ψ is an injection.

We first note that if $\varphi \in \Phi_B$, then φ admits a unique representing measure on $\mathbb{T}$. This is because a representing measure for φ on $\mathbb{T}$ would also be a representing measure for $\varphi|_{A|_{\mathbb{T}}}$, and, as noted earlier, $\varphi|_{A|_{\mathbb{T}}}$ admits a unique representing measure on $\mathbb{T}$. So the unique representing measure for φ on $\mathbb{T}$ is the unique representing measure for $\varphi|_{A|_{\mathbb{T}}}$ on $\mathbb{T}$. From this we see that $\varphi|_{A|_{\mathbb{T}}}$ determines φ, so Ψ is an injection. □

Since we know that the character space of $A|_{\mathbb{T}}$ can be identified with $\overline{D}$, the previous lemma gives the following lemma.

Lemma 1.7.5. *Let B be a uniform algebra on $\mathbb{T}$ which contains $A|_{\mathbb{T}}$. The character space Φ_B of B is homeomorphic to a closed subset K of $\overline{D}$ and $\mathbb{T} \subseteq K$. There exists a natural uniform algebra $\tilde{B}$ on K such that $\Gamma(\tilde{B}) = \mathbb{T}$, $A|_K \subseteq \tilde{B}$, and such that $\tilde{B}$ is isometrically algebra isomorphic to B.*

Proof. The proof of this lemma is nothing but applications of some "identifications".

Let Ψ be the embedding introduced in the previous lemma. Note that each character in $\Phi_{A|_{\mathbb{T}}}$ is just an evaluation ε_z at some point z in $\overline{D}$, and the mapping $\varepsilon_z \mapsto z$ is a homeomorphism (Proposition 1.3.7). Therefore for each $\varphi \in \Phi_B$ there exists some $z \in \overline{D}$ such that

$$\Psi(\varphi) = \varphi|_{A|_{\mathbb{T}}} = \varepsilon_z.$$

We can define an embedding $\tilde{\Psi} : \Phi_B \to \overline{D}$ such that for each $\varphi \in \Phi_B$ we have $\tilde{\Psi}(\varphi) = z$, where $\varepsilon_z = \Psi(\varphi)$. Let K be the image of $\tilde{\Psi}$, and let $\Lambda : K \to \Phi_B$ be the inverse of $\tilde{\Psi}$. Note that since for each $z \in \mathbb{T}$ the point evaluation ε_z is in Φ_B, we conclude that $\mathbb{T}$ is actually contained in K.

Recall that $\hat{B}$ is the Gel'fand transform of B. Note that since each function $f \in B$ attains its maximum modulus at a point z in $\mathbb{T}$, the corresponding Gel'fand transform $\hat{f}$ attains its maximum modulus at ε_z. Let $\tilde{B}$ be the collection of continuous functions on K of the form $\hat{f} \circ \Lambda$. Since Λ is a homeomorphism, by Proposition 1.5.5 we see that $\tilde{B}$ is a natural uniform algebra on K. Since the modulus of each function $\hat{f} \in \hat{B}$ attains its maximum at a point in $\{\varepsilon_z : z \in \mathbb{T}\}$, we conclude that the modulus of each function $\hat{f} \circ \Lambda$ attains its maximum on $\mathbb{T}$. Thus the Shilov boundary of $\tilde{B}$ is contained in $\mathbb{T}$.

Lastly we note that the function $\hat{Z} \circ \Lambda$ is in $\tilde{B}$. But $\hat{Z} \circ \Lambda$ is nothing but the restriction of the coordinate function Z to K. Thus $A|_K$ is contained in $\tilde{B}$, and this shows that the Shilov boundary of $\tilde{B}$ equals $\mathbb{T}$. □

The above lemma shows that B can be identified with $\tilde{B}$ via the isometric algebra isomorphism $f \mapsto \hat{f} \circ \Lambda$. It is with this identification in mind that we say B can be regarded as a natural uniform algebra on K, and Φ_B can be regarded as K, where each $\varphi \in \Phi_B$ is identified with the unique point z of $\overline{D}$ such that, for each $f \in A$, $\varphi(f|_{\mathbb{T}}) = f(z)$. We will use these identifications in the proof of the next proposition.

Proposition 1.7.6. *Let B be a uniform algebra on $\mathbb{T}$ which contains $A|_{\mathbb{T}}$. Then either $B = A|_{\mathbb{T}}$, or $B = C(\mathbb{T})$.*

Proof. First assume that B is not an antisymmetric uniform algebra on $\mathbb{T}$. Since each closed B-antisymmetric subset of $\mathbb{T}$ is also an $A|_{\mathbb{T}}$-antisymmetric subset of $\mathbb{T}$, by Theorem 1.7.1 we see that the only closed B-antisymmetric subsets of $\mathbb{T}$ are the singleton subsets. Then by Proposition 1.4.3 we know that $B = C(\mathbb{T})$.

Next assume that $\mathbb{T}$ is an antisymmetric subset for B. By the previous lemma we know that Φ_B can be regarded as a closed subset K of $\overline{D}$ containing $\mathbb{T}$. We claim that $K = \overline{D}$. Assume towards a contradiction that $\zeta \in \overline{D} \setminus K$. Then $|\zeta| < 1$ since $\mathbb{T}$ is contained in K by the previous lemma. Let

$$f(z) = \frac{z - \zeta}{1 - \overline{\zeta} z}.$$

Then f is a function in A which only takes the value zero at $z = \zeta$, and which maps $\mathbb{T}$ bijectively onto $\mathbb{T}$. Since $f|_{\mathbb{T}} \in B$ and f has no zeros in K, by Proposition 1.2.5 we know that $f|_{\mathbb{T}}$ is invertible in B. But on $\mathbb{T}$ we have $1/f = \overline{f}$, thus $\overline{f}|_{\mathbb{T}}$ is in B. Hence $(f + \overline{f})|_{\mathbb{T}}$, which is a non-constant real valued function on $\mathbb{T}$, is in B. This contradicts our assumption that $\mathbb{T}$ is a set of antisymmetry for B, and proves that $K = \overline{D}$. Via the previous lemma, we may regard B as a collection of continuous functions on $\overline{D}$ such that $A \subseteq B$, and such that the Shilov boundary of B is $\mathbb{T}$. By the converse of the maximum modulus theorem ([58, Theorem 12.13]) we see that actually $B = A$. □

1.8 Open questions

We end this chapter with two open questions.

As discussed in the previous section, the disc algebra on the unit circle, $A|_{\mathbb{T}}$, is a pervasive uniform algebra on $\mathbb{T}$. However we know that $A|_{\mathbb{T}}$ is not natural and the disc algebra on the closed unit disc is not pervasive. It is an open question in the literature of uniform algebras whether there exits a proper, natural and pervasive uniform algebra. Note that since proper pervasive uniform algebras are always antisymmetric, we know at least that A being proper and pervasive on X implies X and singleton subsets are closed antisymmetric subsets. In light of Proposition 1.7.2 we ask the following questions.

Question 1.8.1. *Does there exist a proper and pervasive uniform algebra A on X, such that there exists a closed antisymmetric subset other than X or the* *singleton subsets?*

Question 1.8.2. *Does there exist a proper natural uniform algebra on a compact Hausdorff space X such that the only closed antisymmetric subsets are X and the singleton subsets?*

Chapter 2

Abstract Swiss cheeses

For the rest of this thesis, we use the term *compact plane set* to mean a non-empty, compact subset of the complex plane.

Swiss cheese sets are compact plane sets obtained by deleting a sequence of open discs from a closed disc. We are usually interested in Swiss cheese sets where there are some constraints on the sequence of deleted open discs, since otherwise each compact plane set would be a Swiss cheese set. Swiss cheese sets are important in the theory of uniform algebras, as many important examples of uniform algebras with some desirable properties are constructed using $R(X)$ (defined in Section 2.1) where X are Swiss cheese sets.

In the construction of some known examples of Swiss cheese sets, the deleted open discs may overlap, or the open discs may have non-empty intersection with the complement of the closed disc. As a result, these Swiss cheese sets may have some undesirable topological properties, e.g. they may be disconnected, or they may have isolated points. Thus we are particularly interested in Swiss cheese sets where the open discs and the complement of the closed disc have pairwise disjoint closures. We call such Swiss cheese sets (with some extra conditions) classical Swiss cheese sets (this will be made precise in Definition 2.1.1), which have many desirable topological properties. In [25], Feinstein and Heath proved a classicalisation theorem, which showed that under some mild conditions, every Swiss cheese set contains a Classical Swiss cheese set. Later in [47] Mason gave another proof of the classicalisation theorem.

In this chapter, we aim to give a topological proof of the classicalisation theorem. In order to do this, we propose to look at a Swiss cheese set $X \subseteq \mathbb{C}$ "abstractly": we regard X as the realisation of a sequence of pairs of numbers, which represent the centre and the radius of a disc. We call such sequences abstract Swiss cheeses. This allows us to put some topology on the space of all abstract Swiss cheeses and give a topological proof of the classicalisation theorem. We are also able to obtain a controlled classicalisation theorem and an annulus classicalisation theorem.

In Section 2.1 we give an introduction to the uniform algebras $R(X)$, and we also give a literature review on Swiss cheese sets. In Section 2.2 we introduce the abstract Swiss cheese spaces. In Section 2.4 we give the topological proof of the classicalisation theorem. In Section 2.5 we introduce the controlled classicalisation theorem and in Section 2.6 we introduce the annulus classicalisation theorem. In the last section we give an application of the theories developed in this chapter to construct a classical Swiss cheese set due to J. Feinstein.

We note that all of the work in this chapter is joint work with J. Feinstein and S. Morley. This work has been published in [28, 27].

2.1 Swiss cheese sets and $R(X)$

Let X be a compact plane set. By $R_0(X)$ we denote the collection of all rational functions with no pole in X, and by $R(X)$ we denote the collection of all continuous functions on X which can be uniformly approximated by functions in $R_0(X)$. It is clear that $R(X)$ is a uniform algebra on X.

It is standard that $R(X)$ is natural on X, see [9, p. 43]. It is clear that if $\operatorname{int} X$ is non-empty, then each function in $R(X)$ is holomorphic on $\operatorname{int} X$. Thus in particular $R(X) \neq C(X)$ if $\operatorname{int} X$ is not empty. On the other hand, if the two dimensional Lebesgue measure of X is zero, then the Hartogs-Rosenthal theorem ([9, Theorem 3.2.4]) shows that $R(X) = C(X)$. Therefore to study non-trivial $R(X)$, we are interested in compact plane sets X with no interior, such that the two dimensional Lebesgue measure of X is positive.

Definition 2.1.1. By a *Swiss cheese set*, we mean a compact plane set obtained by deleting a sequence of open discs from a closed disc. By a *semi-classical Swiss cheese set*, we mean a Swiss cheese set where the open discs and the complement of the closed disc are pairwise disjoint, and where the sum of the radii of the deleted open discs is finite. By a *classical Swiss cheese set*, we mean a Swiss cheese set where the open discs and the complement of the closed disc have pairwise disjoint closures, and where the sum of the radii of the deleted open discs is finite.

Note that, in the literature, Swiss cheese sets are often simply called Swiss cheeses, and some authors require additional conditions, or fewer conditions, on these sets. However in this thesis we reserve the term Swiss cheese for another purpose, see Definition 2.1.3.

As mentioned before, classical Swiss cheese sets have many desirable topological properties. For example, Dales and Feinstein [16] proved that given two points x, y in a classical Swiss cheese set there is a rectifiable path connecting x, y and such that the length of this path is no more than $\pi\,|x-y|$; in fact, the constant π can be replaced by $\pi/2$ here. As a consequence of this observation we see that a classical Swiss cheese set is path connected (and hence connected), locally path connected (and hence locally connected), and uniformly regular, as defined in [16]. Also as a consequence of connectedness, we see that a classical Swiss cheese set cannot have any isolated points. In [25] it was noted that every classical Swiss cheese set with empty interior is homeomorphic to the Sierpiński carpet as a consequence of a theorem of Whyburn [68].

Browder notes in [9, p. 162] that if X is a classical Swiss cheese set then $R(X)$ is an essential uniform algebra. In [25] Feinstein and Heath proved that $R(X)$ is essential even for semiclassical Swiss cheese sets X. In particular, $R(X) \neq C(X)$ if X is a semiclassical Swiss cheese set, as originally proved by Roth [56]. It follows from the Hartogs-Rosenthal theorem that X must have positive area. A direct proof that every classical Swiss cheese set has positive area is due to W. Allard, as outlined in [9, p. 163-164].

Swiss cheese sets were first introduced in 1938 by the Swiss mathematician A. Roth in [56]. Roth's examples are actually classical Swiss cheese sets with no interior in our current definition. See [13] for an interesting discussion of Roth's life and work, including her work on Swiss cheese sets.

Roth's Swiss cheese sets were the first known examples of compact plane sets X with empty interior such that $R(X) \neq C(X)$, and her examples were rediscovered by Mergelyan in [50]. Since then there have been numerous applications of Swiss cheese sets in the literature. We now introduce some definitions and give some applications of Swiss cheese sets.

Definition 2.1.2. Let A be a Banach function algebra on X. We say A is *regular* on X if for each $x \in X$ and for each closed subset K of X not containing x, there exists a function $f \in A$ such that $f(x) = 1$ and $f(K) \subseteq \{0\}$. We say A is regular if A is regular on Φ_A.

The first known example of a non-trivial regular uniform algebra is an $R(X)$ on a Swiss cheese set X constructed by McKissick in [49]. For a simpler approach to McKissick's example, see [44] or [61, p. 344].

In the literature of uniform algebras, it had been conjectured that if A is a natural uniform algebra on X such that each point of X is a peak point, then $A = C(X)$. This is called the peak point conjecture. In [12], Cole made use of the properties of McKissick's example in order to construct his famous counterexample to the peak point conjecture. A simpler counterexample to the peak point conjecture was constructed by Basener in [5], again based on McKissick's example.

Let X be a compact plane set, and let $x \in X$. Recall that (see Example 1.2.12) a point derivation at x is a linear map $d : R(X) \to \mathbb{C}_{\varepsilon_x}$ such that

$$d(fg) = f(x)d(g) + g(x)d(f), \quad f, g \in R(X).$$

It was proved by Bishop (see [9, p. 178]) that $R(X)$ admits non-zero point derivations at many points in X whenever $R(X) \neq C(X)$. In [67] Wermer constructed a classical Swiss cheese set X such that $R(X)$ admits no non-zero *bounded* point derivations. Wermer's construction was adapted by O'Farrell in [52] to construct a Swiss cheese set X for which $R(X)$ is non-trivial, and such that $R(X)$ admits a non-zero bounded point derivation at exactly one point of X.

In [60] Steen constructed a Swiss cheese set X for which $R(X)$ is not antisymmetric.

We note that there are several constructions of Swiss cheese sets due to J. Feinstein. We refer the reader to Section 2.4 and Section 3.2 for discussions of two of these examples.

Where existing examples of Swiss cheese sets in the literature are not classical, it is of interest to construct classical Swiss cheese sets which solve the same problems. In [25] Feinstein and Heath proved a classicalisation theorem for Swiss cheese sets, which we briefly discuss below.

Let $a \in \mathbb{C}$ and let $s > 0$. We denote the open disc of radius s and centre a by $\mathrm{B}(a, s)$ and the corresponding closed disc by $\bar{\mathrm{B}}(a, s)$. We also set $\bar{\mathrm{B}}(a, 0) = \{a\}$ and $\mathrm{B}(a, 0) = \emptyset$, and call these degenerate discs. For a non-degenerate open or closed disc D in the plane, let $r(D)$ denote the radius of D; for a degenerate disc D we define $r(D) = 0$.

Definition 2.1.3. Let $\Delta \subseteq \mathbb{C}$ be a non-degenerate open disc and let $\mathcal{D}$ be a countable collection of non-degenerate open discs in the plane. Then we call the ordered pair $E = (\overline{\Delta}, \mathcal{D})$ a *Swiss cheese*. We also define the following.

1. The *Swiss cheese set X_E associated with the Swiss cheese E* is defined by

$$X_E = \overline{\Delta} \setminus \bigcup_{D \in \mathcal{D}} D. \tag{2.1}$$

2. The *discrepancy* $\delta(E)$ of E is defined by

$$\delta(E) = r(\overline{\Delta}) - \sum_{D \in \mathcal{D}} r(D),$$

   where $r(D)$ is the radius of the disc D.

3. The Swiss cheese E is *semiclassical* if $\delta(E) > -\infty$, for each $D \in \mathcal{D}$ we have $D \subseteq \overline{\Delta}$, and for each $D' \in \mathcal{D}$ with $D \neq D'$ we have $D \cap D' = \emptyset$.

4. The Swiss cheese E is *classical* if $\delta(E) > -\infty$, for each $D \in \mathcal{D}$ we have $\overline{D} \subseteq \Delta$, and for each $D' \in \mathcal{D}$ with $D \neq D'$ we have $\overline{D} \cap \overline{D'} = \emptyset$.

5. The Swiss cheese E is *finite* if the collection $\mathcal{D}$ is finite and infinite otherwise.

Note that if $E = (\overline{\Delta}, \mathcal{D})$ is a classical (semiclassical) Swiss cheese, then the associated Swiss cheese set X_E is a classical (semiclassical) Swiss cheese set.

The following proposition is the Feinstein–Heath classicalisation theorem, as stated in [25]. We note that this classicalisation theorem was also proved by Mason in [47].

Proposition 2.1.4 (Feinstein–Heath Classicalisation Theorem [25])**.** *For every Swiss cheese $E = (\overline{\Delta}, \mathcal{D})$ with $\delta(E) > 0$, there is a classical Swiss cheese E' with $X_{E'} \subseteq X_E$ and $\delta(E') \geq \delta(E)$.*

In Section 2.4 we aim to give a topological proof of this theorem.

2.2 Abstract Swiss cheese space

In this section, we consider what we call *abstract Swiss cheeses*, which are sequences of pairs consisting of a complex number and a non-negative real number. Each pair in this sequence corresponds to a centre and radius of a disc in the plane. We give the set of all abstract Swiss cheeses a natural topology and use this topology to give a new proof of the Feinstein–Heath classicalisation theorem.

We first observe a simple fact. Let $(r_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers such that $\lim_{n\to\infty} r_n = 0$. Then there exists a bijection $\sigma : \mathbb{N} \to \mathbb{N}$ such that $r_{\sigma(n)} \geq r_{\sigma(n+1)}$ for all $n \in \mathbb{N}$. Thus we can re-enumerate the terms in (r_n) to make it into a non-increasing sequence.

We denote the set of all non-negative real numbers by $\mathbb{R}^+$, the set of positive integers by $\mathbb{N}$ and the set of all non-negative integers by $\mathbb{N}_0$.

Throughout this chapter, we will work in what we call *abstract Swiss cheese space* $\mathcal{F}$, where $\mathcal{F} = (\mathbb{C} \times \mathbb{R}^+)^{\mathbb{N}_0}$ with the product topology. It is standard that $\mathcal{F}$ is a metrizable space, and a sequence with terms $A^{(m)} = ((a_n^{(m)}, r_n^{(m)}))$ converges to $A = ((a_n, r_n))$ if and only if $a_n^{(m)} \to a_n$ and $r_n^{(m)} \to r_n$, for each $n \in \mathbb{N}_0$.

Definition 2.2.1. Let $A = ((a_n, r_n))_{n=0}^{\infty} \in \mathcal{F}$. We call A an *abstract Swiss cheese*, and we define the following.

1. The *significant index set of* A is $S_A := \{n \in \mathbb{N} : r_n > 0\}$. We say that A is *finite* if S_A is a finite set, otherwise A is *infinite*.

2. The *associated Swiss cheese set* X_A is defined by

$$X_A = \bar{\mathrm{B}}(a_0, r_0) \setminus \left(\bigcup_{n=1}^{\infty} \mathrm{B}(a_n, r_n) \right). \tag{2.2}$$

3. We say that A is *semiclassical* if $\sum_{n=1}^{\infty} r_n < \infty$, $r_0 > 0$ and for all $k \in S_A$ the following hold:

   (a) $\mathrm{B}(a_k, r_k) \subseteq \mathrm{B}(a_0, r_0)$;

   (b) whenever $\ell \in S_A$ has $\ell \neq k$, we have $\mathrm{B}(a_k, r_k) \cap \mathrm{B}(a_\ell, r_\ell) = \emptyset$.

4. We say that A is *classical* if $\sum_{n=1}^{\infty} r_n < \infty$, $r_0 > 0$ and for all $k \in S_A$ the following hold:

   (a) $\bar{\mathrm{B}}(a_k, r_k) \subseteq \mathrm{B}(a_0, r_0)$;

   (b) whenever $\ell \in S_A$ with $\ell \neq k$, we have $\bar{\mathrm{B}}(a_k, r_k) \cap \bar{\mathrm{B}}(a_\ell, r_\ell) = \emptyset$.

For $\alpha \geq 1$ we define the *discrepancy function of order* α , $\delta_\alpha : \mathcal{F} \to [-\infty, \infty)$ by

$$\delta_\alpha(A) = r_0^\alpha - \sum_{n=1}^{\infty} r_n^\alpha, \quad A = ((a_n, r_n))_{n=0}^{\infty} \in \mathcal{F}. \tag{2.3}$$

Note that in (2.2) we could instead write

$$X_A := \bar{\mathrm{B}}(a_0, r_0) \setminus \left(\bigcup_{n \in S_A} \mathrm{B}(a_n, r_n) \right).$$

If A is semiclassical or classical, then $\pi\delta_2(A)$ is the area of the Swiss cheese set X_A. We will usually write $A = ((a_n, r_n))$ for an abstract Swiss cheese. In the rest of this chapter, all sequences, unless otherwise specified, will be indexed by $\mathbb{N}_0$.

We also define the following functions on $\mathcal{F}$.

Definition 2.2.2. The *radius sum function* is the map $\rho : \mathcal{F} \to [0, \infty]$ defined by

$$\rho(A) = \sum_{n=1}^{\infty} r_n, \quad A = ((a_n, r_n)) \in \mathcal{F}.$$

The *centre bound function* is the map $\mu : \mathcal{F} \to [0, \infty]$ defined by

$$\mu(A) = \sup_{n \in \mathbb{N}} |a_n|, \quad A = ((a_n, r_n)) \in \mathcal{F}.$$

Let $E \subseteq \mathbb{C}$. For an abstract Swiss Cheese $A = ((a_n, r_n))$ we define $H_A(E)$ to be the set of those $n \in S_A$ such that $\bar{\mathrm{B}}(a_n, r_n) \cap E \neq \emptyset$. The *local radius sum function on* E is the function $\rho_E : \mathcal{F} \to [0, \infty]$ defined by

$$\rho_E(A) = \sum_{n \in H_A(E)} r_n, \quad A = ((a_n, r_n)) \in \mathcal{F}.$$

Since μ is the supremum of a collection of continuous functions, it is standard (see [58, p. 38]) that μ is lower semicontinuous. We shall show in Lemma 2.2.3 that δ_α is upper semicontinuous, while ρ is lower semicontinuous.

We now explain the connection between Swiss cheeses, as in Definition 2.1.3, and abstract Swiss cheeses. We construct a many-to-one surjection of a subset of $\mathcal{F}$ onto the collection of all Swiss cheeses. Let $A = ((a_n, r_n))$ be an abstract Swiss cheese with $r_0 > 0$. Then we can obtain an associated Swiss cheese E_A by setting

$$E_A := (\bar{\mathrm{B}}(a_0, r_0), \{\mathrm{B}(a_n, r_n) : n \in S_A\}).$$

The associated Swiss cheese sets of A and E_A are equal, and $\delta(E_A) \geq \delta_1(A)$. Moreover, if A is finite then E_A is finite; if A is semiclassical then E_A is semi-classical; and if A is classical then E_A is classical. Conversely, if E is a finite Swiss cheese then there is a finite abstract Swiss cheese A such that $E_A = E$.

Let $E = (\overline{\Delta}, \mathcal{D})$ be a Swiss cheese. If E is classical (semiclassical) then there is an abstract Swiss cheese A with $E_A = E$ such that A is classical (semiclassical). Moreover, when the sum of the radii of the open discs in $\mathcal{D}$ is finite, we can find an abstract Swiss cheese $A = ((a_n, r_n))$ with $\rho(A) < \infty$ and $E = E_A$ such that the sequence $(r_n)_{n=1}^{\infty}$ is non-increasing.

We denote the collection of all abstract Swiss cheeses $A = ((a_n, r_n))$ with $\rho(A) < \infty$ and $(r_n)_{n=1}^{\infty}$ non-increasing by $\mathcal{N}$. In addition, for each $M > 0$ and $R > 0$, we denote the set of all those abstract Swiss cheeses $A = ((a_n, r_n)) \in \mathcal{N}$ such that $\mu(A) \leq M$ and $\rho(A) \leq R$ by $\mathcal{N}(M, R)$. Note that, for $A = ((a_n, r_n)) \in \mathcal{N}(M, R)$ we have $r_n \leq R/n$ for all $n \in \mathbb{N}$. This is because $(r_n)_{n\in\mathbb{N}}$ is non-increasing, so we have

$$R = \sum_{k=1}^{\infty} r_k \geq \sum_{k=1}^{n} r_k \geq n r_n. \tag{2.4}$$

For the rest of this chapter, for each $n \in \mathbb{N}_0$, we let $\pi_n : \mathcal{F} \to (\mathbb{C} \times \mathbb{R}^+)$ denote the projection onto the nth coordinate.

Lemma 2.2.3. *Let $M, R > 0$ and let $\alpha \geq 1$.*

(i) The function $\delta_\alpha : \mathcal{F} \to [-\infty, \infty)$ is upper semicontinuous. The function $\rho : \mathcal{F} \to [0, \infty]$ is lower semicontinuous.

(ii) For $\alpha > 1$, the function $\delta_\alpha|_{\mathcal{N}(M,R)} : \mathcal{N}(M, R) \to \mathbb{R}$ is continuous.

(iii) The set $\mathcal{N}(M, R)$ is a closed subset of $\mathcal{F}$. A closed subset $S \subseteq \mathcal{N}(M, R)$ is compact if and only if $\pi_0(S) \subseteq (\mathbb{C} \times \mathbb{R}^+)$ is compact.

Proof. (i) We show that δ_α is upper semicontinuous. The fact that ρ is lower semicontinuous can be proved in a similar way.

For each $m \in \mathbb{N}_0$ let $A^{(m)} = ((a_n^{(m)}, r_n^{(m)})) \in \mathcal{F}$. Assume that $A^{(m)}$ converges to $A = ((a_n, r_n))$ and there exists $a \in \mathbb{R}$ such that for each $m \in \mathbb{N}_0$

we have $\delta_\alpha(A^{(m)}) \geq a$. Then we have

$$\begin{aligned} a &\leq \limsup_{m\to\infty} \left((r_0^{(m)})^\alpha - \sum_{n=1}^{\infty} (r_n^{(m)})^\alpha \right) \\ &= (r_0)^\alpha - \liminf_{m\to\infty} \sum_{n=1}^{\infty} (r_n^{(m)})^\alpha \\ &\leq (r_0)^\alpha - \sum_{n=1}^{\infty} \liminf_{m\to\infty} (r_n^{(m)})^\alpha \\ &= (r_0)^\alpha - \sum_{n=1}^{\infty} (r_n)^\alpha \\ &= \delta_\alpha(A), \end{aligned}$$

where the second inequality comes from Fatou's lemma ([58, p. 23]). This proves that δ_α is upper semicontinuous.

(ii) Fix $\alpha > 1$. For each $m \in \mathbb{N}_0$ let $A^{(m)} = ((a_n^{(m)}, r_n^{(m)})) \in \mathcal{N}(M, R)$ and suppose $A^{(m)} \to A = ((a_n, r_n)) \in \mathcal{N}(M, R)$ as $m \to \infty$. As noted in (2.4) we have $(r_n^{(m)})^\alpha \leq R^\alpha / n^\alpha$ for all $n \in \mathbb{N}$. Since $\sum_{n=1}^{\infty} R^\alpha / n^\alpha < \infty$, by the dominated convergence theorem ([58, Theorem 1.34]), we have

$$\delta_\alpha(A) = r_0^\alpha - \sum_{n=1}^{\infty} r_n^\alpha = \lim_{m\to\infty} \left((r_0^{(m)})^\alpha - \sum_{n=1}^{\infty} (r_n^{(m)})^\alpha \right) = \lim_{m\to\infty} \delta_\alpha(A^{(m)}).$$

So δ_α is continuous from $\mathcal{N}(M, R)$ to $\mathbb{R}$.

(iii) We note that $\mathcal{N}(M, R) = \mu^{-1}([0, M]) \cap \rho^{-1}([0, R])$. Since both μ and ρ are lower semicontinuous functions, we see that $\mathcal{N}(M, R)$ is a closed subset of $\mathcal{F}$.

Now let S be a closed subset of $\mathcal{N}(M, R)$, so in particular S is closed in $\mathcal{F}$. If S is compact, then because π_0 is continuous we see that $\pi_0(S)$ is a compact subset of $\mathbb{C} \times \mathbb{R}^+$. On the other hand, if $\pi_0(S)$ is a compact subset of $\mathbb{C} \times \mathbb{R}^+$, then from the definition of $\mathcal{N}(M, R)$ we see that $\pi_n(S)$ is a compact subset of $\mathbb{C} \times \mathbb{R}^+$ for all $n \in \mathbb{N}_0$. Therefore S is compact by Tychonoff's theorem ([2, Theorem 1.29]).

□

We remark that the following example shows that δ_1 is only upper semicontinuous, but not continuous.

Example 2.2.4. For each $m \in \mathbb{N}_0$ let $A^{(m)} = ((a_n^{(m)}, r_n^{(m)})) \in \mathcal{F}$ such that $a_n^{(m)} = 0$ for all $n \in \mathbb{N}_0$, $r_0^{(m)} = 1$ and such that

$$r_n^{(m)} = \begin{cases} \frac{1}{2m} & 1 \leq n \leq m, \\ 0 & m < n. \end{cases}$$

Then it is clear that $A^{(m)}$ is in $\mathcal{N}(1,1)$ and $A^{(m)}$ converges to $A = ((a_n, r_n))$, where $a_n = 0$ for all $n \in \mathbb{N}_0$, $r_0 = 1$ and $r_n = 0$ for all $n \in \mathbb{N}$. Thus $\delta_1(A) = 1$. On the other hand we see that for each $m \geq 1$

$$\delta_1(A^{(m)}) = 1 - \sum_{n=1}^{m} \frac{1}{2m} = \frac{1}{2}.$$

This shows that δ_1 is not continuous on $\mathcal{N}(1,1)$.

Let $A = ((a_n, r_n)) \in \mathcal{F}$ be an abstract Swiss cheese. We note that there may exist $n \in S_A$ such that $\mathrm{B}(a_n, r_n) \subseteq \mathbb{C} \setminus \bar{\mathrm{B}}(a_0, r_0)$, and there may exist $\ell \in \mathbb{N}$ with $\ell \neq n$ such that $\mathrm{B}(a_n, r_n) \subseteq \mathrm{B}(a_\ell, r_\ell)$. The open disc $\mathrm{B}(a_n, r_n)$ is "redundant", in the sense that it has no effect in forming the associated Swiss cheese set X_A. This inspires the following definition.

Definition 2.2.5. Let $A = ((a_n, r_n))$ be an abstract Swiss cheese. Then A is *redundancy-free* if, for all $k \in S_A$, we have $\mathrm{B}(a_k, r_k) \cap \bar{\mathrm{B}}(a_0, r_0) \neq \emptyset$, and for all $\ell \in S_A$ with $k \neq \ell$ we have $\mathrm{B}(a_k, r_k) \nsubseteq \mathrm{B}(a_\ell, r_\ell)$.

We have the following lemma, which shows that for an abstract Swiss cheese with finite radius sum, we can always eliminate redundant open discs.

Lemma 2.2.6. *Let $A = ((a_n, r_n)) \in \mathcal{F}$ with $\rho(A) < \infty$. Then there exists a redundancy-free abstract Swiss cheese $B = ((b_n, s_n)) \in \mathcal{N}$ with $X_B = X_A$, $\mu(B) < \infty$ and $\bar{\mathrm{B}}(b_0, s_0) = \bar{\mathrm{B}}(a_0, r_0)$ such that $\rho_E(B) \leq \rho_E(A)$ for each subset $E \subseteq \mathbb{C}$. Moreover, there exists an injection $\varphi : S_B \to S_A$ such that for each $n \in S_B$ we have $\mathrm{B}(b_n, s_n) = \mathrm{B}(a_{\varphi(n)}, r_{\varphi(n)})$.*

Proof. We first eliminate all open discs that lie outside $\bar{\mathrm{B}}(a_0, r_0)$. Let

$$S = \{n \in S_A : \mathrm{B}(a_n, r_n) \cap \bar{\mathrm{B}}(a_0, r_0) \neq \emptyset\}.$$

Let $A' = ((a'_n, r'_n)) \in \mathcal{F}$ such that $a'_n = a_n$, $r'_n = r_n$ if $n = 0$ or $n \in S$, and such that $a'_n = 0$, $r'_n = 0$ if $n \in \mathbb{N} \setminus S$. Then we see that $X_A = X_{A'}$, such that S is the significant index set of A', and such that for each $n \in S$ we have $\mathrm{B}(a'_n, r'_n) \cap \bar{\mathrm{B}}(a'_0, r'_0) \neq \emptyset$. From the construction it is clear that $\rho_E(A') \leq \rho_E(A)$

for each $E \subseteq \mathbb{C}$. By taking $A = A'$ if necessary, we may now assume without loss of generality that for each $n \in S_A$ we have $\mathrm{B}(a_n, r_n) \cap \bar{\mathrm{B}}(a_0, r_0) \neq \emptyset$. Note that we still have $\rho(A) < \infty$.

Now assume that S_A is an infinite set. The case S_A is finite or empty can be proved in a similar, but easier way.

Note that since $\rho(A) < \infty$ and S_A is infinite, the set $\{\mathrm{B}(a_n, r_n) : n \in S_A\}$ must be an infinite set - otherwise there would be infinitely many $n \in S_A$ such that $\mathrm{B}(a_n, r_n)$ is the same (non-degenerate) open disc in $\mathbb{C}$, a contradiction to the fact that $\rho(A) < \infty$.

Since $\{\mathrm{B}(a_n, r_n) : n \in S_A\}$ is an infinite collection of non-degenerate open discs in $\mathbb{C}$ such that $\sum_{n \in S_A} r_n < \infty$, we can enumerate these open discs into a sequence $(B_k)_{k \in \mathbb{N}}$ such that $r(B_k) \geq r(B_{k+1})$ for each $k \in \mathbb{N}$. We construct a new sequence of open discs $(C_k)_{k \in \mathbb{N}}$ by induction. Let $C_1 = B_1$. Assume C_k has been defined. Then let $C_{k+1} = \emptyset$ if $B_{k+1} \subseteq B_\ell$ for some $1 \leq \ell \leq k$, and let $C_{k+1} = B_{k+1}$ otherwise.

From this construction, we claim that $C_k \nsubseteq C_\ell$ whenever $C_k \neq \emptyset$ and $k \neq \ell$. When $k > \ell$, the fact that $C_k \neq \emptyset$ implies that $C_k = B_k$ and $B_k \nsubseteq B_\ell$. Therefore $C_k \nsubseteq C_\ell$. If $\ell > k$ then $r(B_\ell) \leq r(B_k)$, so $B_k \subseteq B_\ell$ if and only if $B_k = B_\ell$. This implies that $C_k \nsubseteq C_\ell$.

We also claim that

$$\bigcup_{k=1}^{\infty} C_k = \bigcup_{k=1}^{\infty} B_k. \tag{2.5}$$

From the construction it is clear that $\cup_{k=1}^{\infty} C_k \subseteq \cup_{k=1}^{\infty} B_k$. Now let $x \in \cup_{k=1}^{\infty} B_k$, say $x \in B_k$, where k is the smallest index in $\mathbb{N}$. Then $x \in C_k$, since otherwise from the construction of C_k (note that either $C_k = B_k$ or $C_k = \emptyset$) we know that $B_k \subseteq B_\ell$ for some $1 \leq \ell \leq k-1$, and hence $x \in B_\ell$, a contradiction to the minimality of k.

Now drop all C_k which equals $\emptyset$, and re-enumerate the remaining discs to form a collection $\{C_k\}_{k \in J}$, where J is either $\mathbb{N}$, or $J = \{1, 2, \cdots, \ell\}$ for some $\ell \in \mathbb{N}$, such that the radius of C_k is non-increasing (as a function of k). Let $b_0 = a_0$, $s_0 = r_0$. For each $k \in J$, choose $j_k \in \mathbb{N}$ such that $C_k = \mathrm{B}(a_{j_k}, r_{j_k})$ and let $b_k = a_{j_k}$, $s_k = r_{j_k}$ (note that this $k \mapsto j_k$ is injective); otherwise (the case $k \in \mathbb{N} \setminus J$) just let $b_k = 0$, $s_k = 0$. In this way, we have defined a new abstract Swiss cheese $B = ((b_k, s_k))$ with $S_B = J$. Since the non-degenerate $\mathrm{B}(b_k, s_k,)$ for $k \in J$ are in 1-1 correspondence with the non-degenerate C_k, we see that B is redundancy-free, $\rho(B) < \infty$, and $(s_k)_{k \in \mathbb{N}}$ is non-increasing. Thus $B \in \mathcal{N}$. From (2.5) we see that $X_A = X_B$. From the construction we see that there exists an injection $\varphi : S_B \to S_A$ (the map $k \mapsto j_k$ noted before), such that for each $n \in S_B$ we have $\mathrm{B}(b_n, s_n) = \mathrm{B}(a_{\varphi(n)}, r_{\varphi(n)})$. This shows that $\rho_E(B) \leq \rho_E(A)$ for each $E \subseteq \mathbb{C}$. Lastly, since each $\mathrm{B}(b_n, s_n,)$ with $n \in \mathbb{N}$ either has non-empty intersection with $\bar{\mathrm{B}}(b_0, s_0,)$ or is the open degenerate disc with centre at 0, we see that $\mu(B) < \infty$. This finishes the proof. □

Note that the conditions $\bar{\mathrm{B}}(a_0, r_0) = \bar{\mathrm{B}}(b_0, s_0)$ and $\rho(B) \leq \rho(A)$ imply that $\delta_1(B) \geq \delta_1(A)$.

Lastly in this section, we introduce three pieces of terminology which we will use in the modifications of an abstract Swiss cheese.

Definition 2.2.7. Let $A = ((a_n, r_n))$ be an abstract Swiss cheese.

1. Let $a \in \mathbb{C}$ and $r > 0$ and let $m \in \mathbb{N}_0$. We say an abstract Swiss cheese $B = ((b_n, s_n))$ is obtained from A by *inserting* a disc $\mathrm{B}(a, r)$ at index m if, for $0 \leq n < m$, we have $b_n = a_n$, $s_n = r_n$; for $n > m$ we have $b_n = a_{n-1}$, $s_n = a_{n-1}$; and $b_m = a$, $s_m = r$.

2. Let $m \in \mathbb{N}_0$. We say an abstract Swiss cheese $B = ((b_n, s_n))$ is obtained from A by *deleting* the disc at index m if, for $0 \leq n < m$, we have $b_n = a_n$, $s_n = r_n$ and for all $n \geq m$ we have $b_n = a_{n+1}$, $s_n = r_{n+1}$.

3. Suppose $A \in \mathcal{N}$. Let $a \in \mathbb{C}$ and $r > 0$ and $k, \ell \in \mathbb{N}$ with $k \neq \ell$. We say an abstract Swiss cheese $B = ((b_n, s_n))$ is obtained from A by *replacing* the discs $\mathrm{B}(a_k, r_k), \mathrm{B}(a_\ell, r_\ell)$ by $\mathrm{B}(a, r)$ if B is obtained by deleting the discs at indices k, ℓ and inserting the disc $\mathrm{B}(a, r)$ at the first index in $\mathbb{N}$ such that the sequence $(s_n)_{n=1}^{\infty}$ is non-increasing.

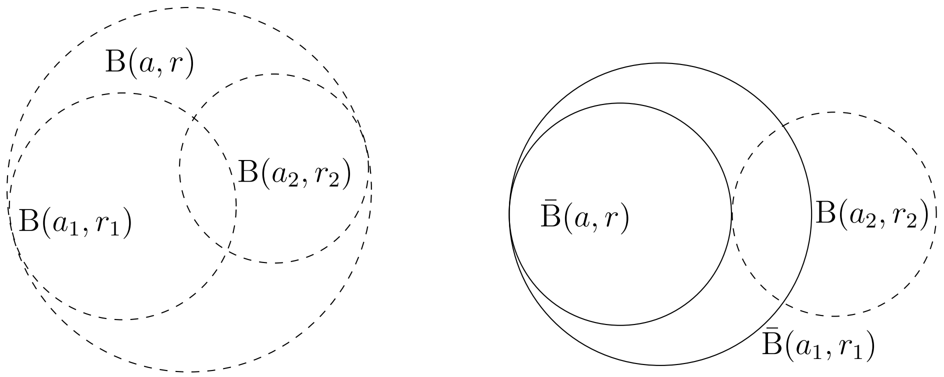
Note that, if $A \in \mathcal{N}$, then the abstract Swiss cheese B obtained by deleting or replacing discs, as defined in Definition 2.2.7, is also in $\mathcal{N}$.

2.3 Some geometric results

We will require some elementary geometric results in order to give our topological proof of the classicalisation theorem. These results are probably well known, and they are all self-evident. Thus we will not provide proof here.

Lemma 2.3.1. *Let $z, w \in \mathbb{C}$ and $r, s \in \mathbb{R}^+$, then $\bar{\mathrm{B}}(z, r) \subseteq \bar{\mathrm{B}}(w, s)$ if and only if $|z - w| \leq s - r$. If $r > 0$, then $\mathrm{B}(z, r) \subseteq \mathbb{C} \setminus \bar{\mathrm{B}}(w, s)$ if and only if $|w - z| \geq s + r$.*

The following two elementary lemmas are essentially those used in [25, 47], but including some additional information distilled from the original proofs. These lemmas are summarised in Figures 2.1 and 2.2. In Lemma 2.3.2, we allow for the line segment to be degenerate.

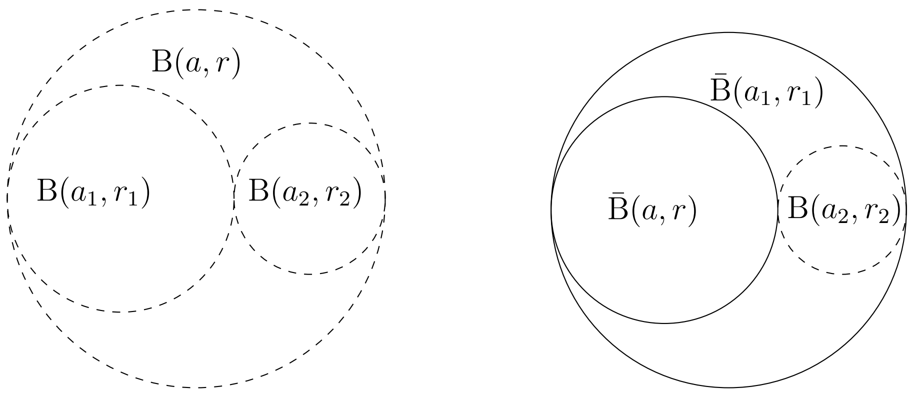


(a) Combining open discs. (b) Pulling in the closed disc.

Figure 2.1: Elementary lemmas for combining and pulling in discs.

Lemma 2.3.2 (Combining open discs)**.** *Let* $a_1, a_2 \in \mathbb{C}$ *and* $r_1, r_2 > 0$*. Then there exists a unique pair* $(a, r) \in \mathbb{C} \times \mathbb{R}^+$ *with* $\mathrm{B}(a_1, r_1) \cup \mathrm{B}(a_2, r_2) \subseteq \mathrm{B}(a, r)$ *such that* r *is minimal. Moreover, the point* a *lies on the line segment joining* a_1 *and* a_2*. Suppose further that* $\bar{\mathrm{B}}(a_1, r_1) \cap \bar{\mathrm{B}}(a_2, r_2) \neq \emptyset$*. Then* $r \leq r_1 + r_2$*, and equality holds if and only if* $\mathrm{B}(a_1, r_1) \cap \mathrm{B}(a_2, r_2) = \emptyset$*.*

Lemma 2.3.3 (Pulling in the closed disc)**.** *Let* $a_1, a_2 \in \mathbb{C}$ *and* $r_1 > r_2 > 0$ *with* $\bar{\mathrm{B}}(a_2, r_2) \not\subseteq \mathrm{B}(a_1, r_1)$*. Then there exists a unique pair* $(a, r) \in \mathbb{C} \times \mathbb{R}^+$ *with* $\bar{\mathrm{B}}(a, r) \subseteq \bar{\mathrm{B}}(a_1, r_1)$ *and* $\mathrm{B}(a_2, r_2) \cap \bar{\mathrm{B}}(a, r) = \emptyset$ *such that* r *is maximal. Moreover,* $r \geq r_1 - r_2$ *and equality holds if and only if* $\mathrm{B}(a_2, r_2) \subseteq \mathrm{B}(a_1, r_1)$*.*



(a) Case where equality holds in Lemma 2.3.2. (b) Case where equality holds in Lemma 2.3.3.

Figure 2.2: Extreme cases in the combining and pulling in lemmas.

2.4 Classicalisation of abstract Swiss cheeses

We now aim to give a topological proof of the Feinstein-Heath classicalisation theorem (Proposition 2.1.4). We first state this proposition below in the language of abstract Swiss cheeses.

Proposition 2.4.1. *Let $B = ((b_n, s_n))$ be an abstract Swiss cheese with $\delta_1(B) > 0$. Then there exists a classical, abstract Swiss cheese $B' \in \mathcal{F}$ such that $X_{B'} \subseteq X_B$ and $\delta_1(B') \geq \delta_1(B)$.*

We now define a relation on $\mathcal{F}$ which will help us to construct a compact subset of $\mathcal{F}$. Then we prove the existence of classical abstract Swiss cheeses with desired properties in this compact subset.

Definition 2.4.2. Let $A = ((a_n, r_n))$ and $B = ((b_n, s_n))$ be abstract Swiss cheeses. We say B is *partially above* A if $\bar{\mathrm{B}}(b_0, s_0) \subseteq \bar{\mathrm{B}}(a_0, r_0)$, and, for each $n \in \mathbb{N}$, either $\mathrm{B}(a_n, r_n) \subseteq \mathbb{C} \setminus \bar{\mathrm{B}}(b_0, s_0)$, or there exists $m \in \mathbb{N}$ such that $\mathrm{B}(a_n, r_n) \subseteq \mathrm{B}(b_m, s_m)$, or both.

It is clear that A is partially above itself and that if B is partially above A, then $X_B \subseteq X_A$.

It is clear, by Lemma 2.2.6, that to prove Proposition 2.4.1 it is enough to consider B such that $\delta_1(B) > 0$ and B is redundancy-free. For the rest of this section, we let $A = ((a_n, r_n)) \in \mathcal{N}$ be a redundancy-free abstract Swiss cheese with $\delta_1(A) > 0$. Note that $\rho(A) < \infty$ and, since A is redundancy-free, $\mu(A) < \infty$. We set $R = \rho(A)$ and $M = \mu(A)$.

Let $\mathcal{S}(A)$ be the collection of all $B = ((b_n, s_n)) \in \mathcal{N}(M, R)$ such that B is partially above A. Recall that, since $B \in \mathcal{N}(M, R)$, we have $s_n \leq R/n$ for all $n \in S_B$ so that

$$n \leq \frac{R}{s_n}, \quad n \in S_B. \tag{2.6}$$

Since A is partially above itself we see that $\mathcal{S}(A)$ is not empty. We now prove that $\mathcal{S}(A)$ is compact.

Lemma 2.4.3. *The set $\mathcal{S}(A)$ is a compact subset of $\mathcal{F}$.*

Proof. As noted in Lemma 2.2.3, it is enough to prove that $\mathcal{S}(A)$ is closed in $\mathcal{N}(M, R)$ and that $\pi_0(\mathcal{S}(A)) \subseteq (\mathbb{C} \times \mathbb{R}^+)$ is compact. Once we proved that $\mathcal{S}(A)$ is closed, the latter is clear from the definition of $\mathcal{S}(A)$, so we prove that $\mathcal{S}(A)$ is closed in $\mathcal{N}(M, R)$.

For each $m \in \mathbb{N}_0$, let $A^{(m)} = ((a_n^{(m)}, r_n^{(m)}))$ be an abstract Swiss cheese in $\mathcal{S}(A)$, and suppose the sequence $(A^{(m)})$ converges to $B = ((b_n, s_n)) \in \mathcal{N}(M, R)$. It remains to show that B is partially above A.

It is easy to see (by Lemma 2.3.1, for example) that $\bar{\mathrm{B}}(b_0, s_0) \subseteq \bar{\mathrm{B}}(a_0, r_0)$. Fix $k \in \mathbb{N}$. We show that either $\mathrm{B}(a_k, r_k) \subseteq \mathbb{C} \setminus \bar{\mathrm{B}}(b_0, s_0)$ or there exists $\ell \in S_B$ with $\mathrm{B}(a_k, r_k) \subseteq \mathrm{B}(b_\ell, s_\ell)$. If $r_k = 0$ then $\mathrm{B}(a_k, r_k) = \emptyset$ and the result is trivial, so we may assume that $k \in S_A$.

First assume that there exists $n_0 \in \mathbb{N}_0$ such that, for all $m \geq n_0$ we have $\mathrm{B}(a_k, r_k) \subseteq \mathbb{C} \setminus \bar{\mathrm{B}}(a_0^{(m)}, r_0^{(m)})$. Then we have $\left|a_k - a_0^{(m)}\right| \geq r_k + r_0^{(m)}$ for all $m \geq n_0$ by Lemma 2.3.1. Letting $m \to \infty$, we obtain $|a_k - b_0| \geq r_k + s_0$, and so, by Lemma 2.3.1 again, $\mathrm{B}(a_k, r_k) \subseteq \mathbb{C} \setminus \bar{\mathrm{B}}(b_0, s_0)$.

Otherwise for each $n_0 \in \mathbb{N}_0$, there exists $m \geq n_0$ and $\ell_m \in \mathbb{N}$ such that

$$\mathrm{B}(a_k, r_k) \subseteq \mathrm{B}(a_{\ell_m}^{(m)}, r_{\ell_m}^{(m)}). \tag{2.7}$$

By passing to a subsequence of $A^{(m)}$ if necessary, we can assume (2.7) holds for all $m \in \mathbb{N}_0$. For each m, since $r_{\ell_m}^{(m)} \geq r_k > 0$, by (2.6) we have $\ell_m \leq R/r_k$. Thus there must be a $p \in \mathbb{N}$ that appears infinitely many times in the sequence (ℓ_m). Passing to a subsequence again if necessary, we may assume $\ell_m = p$ for all m. Since $A^{(m)} \to B$ as $m \to \infty$ and $\mathrm{B}(a_k, r_k) \subseteq \mathrm{B}(a_p^{(m)}, r_p^{(m)})$, it is again easy to show, using Lemma 2.3.1, that $\mathrm{B}(a_k, r_k) \subseteq \mathrm{B}(b_p, s_p)$. Thus B is partially above A and we have proved that $\mathcal{S}(A)$ is closed. □

Since δ_1 is upper semicontinuous and $\mathcal{S}(A)$ is compact and non-empty, δ_1 attains a maximum value on $\mathcal{S}(A)$ and this value is at least $\delta_1(A) > 0$. Let

$$\mathcal{S}_1(A) := \{A' \in \mathcal{S}(A) : \delta_1(A') = \sup_{B \in \mathcal{S}(A)} \delta_1(B)\},$$

which is also compact and non-empty.

Lemma 2.4.4. *Let $B = ((b_n, s_n)) \in \mathcal{S}(A)$.*

(i) Suppose that $k, \ell \in S_B$ with $k \neq \ell$ such that $\bar{\mathrm{B}}(b_k, s_k) \cap \bar{\mathrm{B}}(b_\ell, s_\ell) \neq \emptyset$. If we have $\mathrm{B}(b_k, s_k) \cap \mathrm{B}(b_\ell, s_\ell) \neq \emptyset$ then there exists $B' \in \mathcal{S}(A)$ such that $\delta_1(B') > \delta_1(B)$. Otherwise, there exists $B' \in \mathcal{S}(A)$ with $\delta_1(B') = \delta_1(B)$ and $\delta_2(B') < \delta_2(B)$.

(ii) Suppose that $k \in S_B$ with $s_k < s_0$ such that $\bar{\mathrm{B}}(b_k, s_k) \nsubseteq \mathrm{B}(b_0, s_0)$. If we have $\mathrm{B}(b_k, s_k) \nsubseteq \mathrm{B}(b_0, s_0)$ then there exists $B' \in \mathcal{S}(A)$ such that

$\delta_1(B') > \delta_1(B)$. Otherwise, there exists $B' \in \mathcal{S}(A)$ with $\delta_1(B') = \delta_1(B)$ and $\delta_2(B') < \delta_2(B)$.

Proof. (i) Let $\mathrm{B}(b,s)$ be the open disc obtained by applying Lemma 2.3.2 to the discs $\mathrm{B}(b_k,s_k)$ and $\mathrm{B}(b_\ell,s_\ell)$. Let $B' = ((b'_n,s'_n))$ be obtained by replacing the discs $\mathrm{B}(b_k,s_k)$ and $\mathrm{B}(b_\ell,s_\ell)$ by $\mathrm{B}(b,s)$, as introduced in Definition 2.2.7.

If $\mathrm{B}(b_k,s_k) \cap \mathrm{B}(b_\ell,s_\ell) \neq \emptyset$ then we have $s < s_k + s_\ell$ and so $\delta_1(B') > \delta_1(B)$. Otherwise, we have $s = s_k + s_\ell$ and hence $s^2 > s_k^2 + s_\ell^2$. In this case, we have $\delta_1(B') = \delta_1(B)$ and $\delta_2(B') < \delta_2(B)$.

We now show that $B' \in \mathcal{S}(A)$. Clearly $B' \in \mathcal{N}$ by our definition of replacing discs in an abstract Swiss cheese. Since b lies on the line segment connecting b_k and b_ℓ, it follows that $\mu(B') \leq \mu(B)$ and since $s \leq s_k + s_\ell$ we have $\rho(B') \leq \rho(B)$. Thus $B' \in \mathcal{N}(M,R)$. It remains to show that B' is partially above A.

We have $\bar{\mathrm{B}}(b'_0,s'_0) = \bar{\mathrm{B}}(b_0,s_0)$ so that $\bar{\mathrm{B}}(b'_0,s'_0) \subseteq \bar{\mathrm{B}}(a_0,r_0)$. Fix $p \in \mathbb{N}$. Since B is partially above A, we have $\mathrm{B}(a_p,r_p) \subseteq \mathrm{B}(b_m,s_m)$ for some $m \in S_B$ or $\mathrm{B}(a_p,r_p) \subseteq \mathbb{C} \setminus \bar{\mathrm{B}}(b_0,s_0)$. If $\mathrm{B}(a_p,r_p) \subseteq \mathbb{C} \setminus \bar{\mathrm{B}}(b_0,s_0)$ then we also have $\mathrm{B}(a_p,r_p) \subseteq \mathbb{C} \setminus \bar{\mathrm{B}}(b'_0,s'_0)$. Otherwise, let $m \in S_B$ with $\mathrm{B}(a_p,r_p) \subseteq \mathrm{B}(b_m,s_m)$. If $m = k$ or $m = \ell$, then, with q as the index where $\mathrm{B}(b,s)$ was inserted, we have $\mathrm{B}(a_p,r_p) \subseteq \mathrm{B}(b'_q,s'_q)$. If $m \neq k,\ell$, then there exists $q \in S_{B'}$ such that $\mathrm{B}(b'_q,s'_q) = \mathrm{B}(b_m,s_m)$. Thus $\mathrm{B}(a_p,r_p) \subseteq \mathrm{B}(b'_q,s'_q)$. Hence B' is partially above A, and so $B' \in \mathcal{S}(A)$ as required.

(ii) Let $\bar{\mathrm{B}}(b,s)$ be the closed disc obtained by applying Lemma 2.3.3 to the discs $\bar{\mathrm{B}}(b_0,s_0)$ and $\mathrm{B}(b_k,s_k)$. Let $B' = ((b'_n,s'_n))$ be the abstract Swiss cheese obtained by deleting the discs at indices 0 and k and inserting the disc $\bar{\mathrm{B}}(b,s)$ at index 0.

If $\mathrm{B}(b_k,s_k) \nsubseteq \mathrm{B}(b_0,s_0)$ then we have $s > s_0 - s_k$ so that $\delta_1(B') > \delta_1(B)$. Otherwise, we have $s_0 = s + s_k$ and $s_0^2 > s^2 + s_k^2$ so that $\delta_1(B') = \delta_1(B)$ and $\delta_2(B') < \delta_2(B)$.

The proof that $B' \in \mathcal{S}(A)$ is similar to the proof in part (i).

□

We are now ready to prove the main results of this section.

Theorem 2.4.5. *All abstract Swiss cheeses in $\mathcal{S}_1(A)$ are semiclassical.*

Proof. Let $B = ((b_n, s_n)) \in \mathcal{S}_1(A)$. Suppose for a contradiction that B is not semiclassical. Consider first the case where there are distinct $k, \ell \in S_B$ with $\mathrm{B}(b_k, s_k) \cap \mathrm{B}(b_\ell, s_\ell) \neq \emptyset$. By Lemma 2.4.4 (i) there exists $B' \in \mathcal{S}(A)$ with $\delta_1(B') > \delta_1(B)$, which is a contradiction.

The remaining case is where there is a $k \in S_B$ with $\mathrm{B}(b_k, s_k) \nsubseteq \mathrm{B}(b_0, s_0)$. We have $\delta_1(B) \geq \delta_1(A) > 0$ so that $s_k < s_0$. By Lemma 2.4.4 (ii) there exists $B' \in \mathcal{S}(A)$ with $\delta_1(B') > \delta_1(B)$, which is a contradiction. $\square$

Since $\mathcal{S}_1(A)$ is compact and non-empty, δ_2 attains both maximum and minimum values on $\mathcal{S}_1(A)$. Let

$$\mathcal{S}_2(A) := \{A' \in \mathcal{S}_1(A) : \delta_2(A') = \inf_{B \in \mathcal{S}_1(A)} \delta_2(B)\},$$

which is again non-empty and compact. Since all the abstract Swiss cheeses in $\mathcal{S}_1(A)$ are semiclassical, $\pi\delta_2(B)$ is the area of X_B for all $B \in \mathcal{S}_1(A)$, and hence for all $B \in \mathcal{S}_2(A)$. So the abstract Swiss cheeses in $\mathcal{S}_2(A)$ are obtained by finding those $B \in \mathcal{S}_1(A)$ for which the area of X_B is minimal.

Theorem 2.4.6. *All abstract Swiss cheeses in $\mathcal{S}_2(A)$ are classical.*

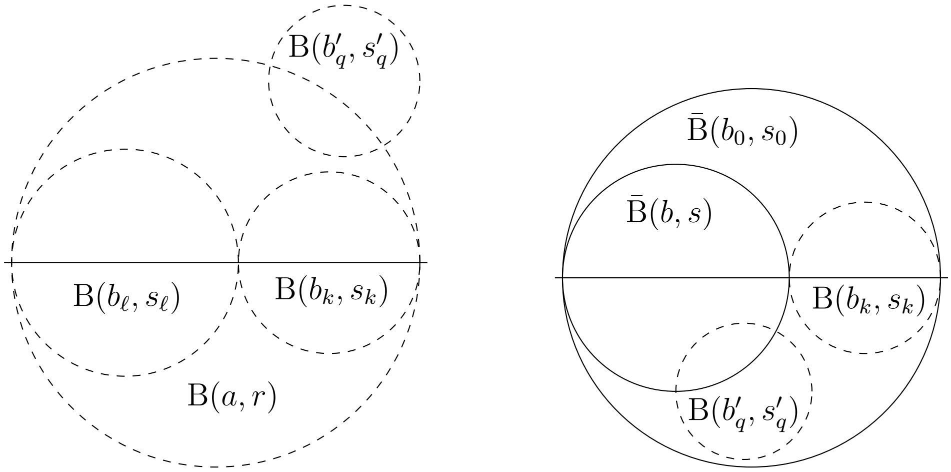
Proof. Let $B = ((b_n, s_n)) \in \mathcal{S}_2(A)$. Suppose for a contradiction that B is not classical. If there are distinct $k, \ell \in S_B$ with $\bar{\mathrm{B}}(b_k, s_k) \cap \bar{\mathrm{B}}(b_\ell, s_\ell) \neq \emptyset$ then, by Lemma 2.4.4 (i), there exists $B' \in \mathcal{S}(A)$ such that either $\delta_1(B') > \delta_1(B)$ or $\delta_1(B') = \delta_1(B)$ and $\delta_2(B') < \delta_2(B)$. In either case we obtain a contradiction since $B \in \mathcal{S}_2(A)$.

Otherwise there exists $k \in S_B$ with $\bar{\mathrm{B}}(b_k, s_k) \nsubseteq \mathrm{B}(b_0, s_0)$. Note that $s_k < s_0$ since $\delta_1(B) > 0$. By Lemma 2.4.4 (ii) there exists $B' \in \mathcal{S}(A)$ such that either $\delta_1(B') > \delta_1(B)$ or $\delta_1(B') = \delta_1(B)$ and $\delta_2(B') < \delta_2(B)$. In either case we obtain a contradiction since $B \in \mathcal{S}_2(A)$. $\square$

In the next theorem, we show that if X_A has empty interior then we do not have to minimise δ_2 on $\mathcal{S}_1(A)$ to find classical abstract Swiss cheeses.

Theorem 2.4.7. *If* $\operatorname{int} X_A = \emptyset$ *then each abstract Swiss cheese in $\mathcal{S}_1(A)$ is classical.*

Proof. Let $B = ((b_n, s_n)) \in \mathcal{S}_1(A)$. Then, by Theorem 2.4.5, B is semiclassical. Suppose for a contradiction that B is not classical. Then there are two cases summarised in Figure 2.3. First suppose there exist distinct $k, \ell \in S_B$ with $\bar{\mathrm{B}}(b_k, s_k) \cap \bar{\mathrm{B}}(b_\ell, s_\ell) \neq \emptyset$. Then by Lemma 2.3.2, since $\mathrm{B}(b_k, s_k) \cap \mathrm{B}(b_\ell, s_\ell) = \emptyset$,



(a) Case 1.

(b) Case 2.

Figure 2.3: The two cases in the proof of Theorem 2.4.7

there exists an open disc $\mathrm{B}(a,r) \supseteq \mathrm{B}(b_k,s_k) \cup \mathrm{B}(b_\ell,s_\ell)$ with $r = s_k + s_\ell$. By replacing the discs $\mathrm{B}(b_k,s_k)$ and $\mathrm{B}(b_\ell,s_\ell)$ with $\mathrm{B}(a,r)$ we obtain a new abstract Swiss cheese $B' = ((b'_n, s'_n))$ such that $B' \in \mathcal{S}_1(A)$ (following the proof of Lemma 2.4.4). Let p be the index at which the disc $\mathrm{B}(a,r)$ was inserted. Since X_B has empty interior, there exists $m \in S_B$ with $m \neq p$ such that $\mathrm{B}(a,r) \cap \mathrm{B}(b_m,s_m) \neq \emptyset$. Let $q \in S_{B'}$ be such that $\mathrm{B}(b'_q,s'_q) = \mathrm{B}(b_m,s_m)$. Note that $p \neq q$. Applying Lemma 2.4.4 (i) to $p, q \in S_{B'}$ and B', we obtain an abstract Swiss cheese $B'' \in \mathcal{S}(A)$ which has $\delta_1(B'') > \delta_1(B')$. But this is a contradiction.

Now suppose there exists $k \in S_B$ with $\bar{\mathrm{B}}(b_k,s_k) \nsubseteq \mathrm{B}(b_0,s_0)$. Let $\bar{\mathrm{B}}(b,s)$ be the closed disc obtained by applying Lemma 2.3.3 to the discs $\bar{\mathrm{B}}(b_0,s_0)$ and $\mathrm{B}(b_k,s_k)$. Since B is semiclassical, we have $s = s_0 - s_k$ (as in Figure 2.2b). By deleting the discs at indices 0 and k and inserting $\mathrm{B}(b,s)$ at index 0, we obtain a new abstract Swiss cheese $B' = ((b'_n, s'_n)) \in \mathcal{S}_1(A)$ (again following the proof of Lemma 2.4.4) such that $\delta_1(B') = \delta_1(B)$. Since X_B has empty interior, there exists $q \in S_{B'}$ such that $\mathrm{B}(b'_q,s'_q) \nsubseteq \bar{\mathrm{B}}(b,s)$. Applying Lemma 2.4.4 (ii) to q and B', we obtain an abstract Swiss cheese $B'' \in \mathcal{S}(A)$ which has $\delta_1(B'') > \delta_1(B')$. But this is a contradiction. □

Finally we give a proof of the Feinstein-Heath classicalisation theorem, in the abstract Swiss cheese form.

Proof of Proposition 2.4.1. We can apply Lemma 2.2.6 to obtain a redundancy-free abstract Swiss cheese $A' \in \mathcal{N}$ with $X_{A'} = X_B$ and such that $\delta_1(A') \geq \delta_1(B) > 0$. Applying the constructions discussed above we get a non-empty collection $\mathcal{S}_2(A')$ of abstract Swiss cheeses. Let $B' \in \mathcal{S}_2(A')$, then by Theorem 2.4.6 we know that B' is classical and B' is partially above A'. Therefore we have $X_{B'} \subseteq X_{A'} = X_B$ and $\delta_1(B') \geq \delta_1(A') \geq \delta_1(B)$. □

2.5 Controlled classicalisation

In this section we discuss some situations in which it is possible to make a Swiss cheese classical without changing certain discs. This process we call "controlled classicalisation".

Recall that, for $E \subseteq \mathbb{C}$ and an abstract Swiss cheese $A = ((a_n, r_n))$, the set $H_A(E)$ is the set of all $n \in S_A$ such that $\bar{\mathrm{B}}(a_n, r_n) \cap E \neq \emptyset$.

Lemma 2.5.1. *Let U be a non-empty open subset of $\mathbb{C}$. For each $m \in \mathbb{N}_0$, let $A^{(m)} = ((a_n^{(m)}, r_n^{(m)})) \in \mathcal{F}$ and suppose that $A^{(m)} \to A = ((a_n, r_n)) \in \mathcal{F}$ as $m \to \infty$. Then $\rho_U(A) \leq \liminf_{m\to\infty} \rho_U(A^{(m)})$.*

Proof. Since U is open and $A^{(m)} \to A$ as $m \to \infty$, for each $k \in H_A(U)$ there exists $m_0 \in \mathbb{N}_0$ such that, for all $m \geq m_0$, we have $k \in S_{A^{(m)}}$ and

$$\bar{\mathrm{B}}(a_k^{(m)}, r_k^{(m)}) \cap U \neq \emptyset\,.$$

Let $\chi_m : \mathbb{N}_0 \to \{0,1\}$ denote the characteristic function of $H_{A^{(m)}}(U) \cap H_A(U)$. Then χ_m converges pointwise to $\chi := \chi_{H_A(U)}$ as $m \to \infty$. Since $r_k^{(m)} \to r_k$ as $m \to \infty$ for each k, by Fatou's lemma, we have

$$\rho_U(A) = \sum_{n=1}^{\infty} \chi(n) r_n \leq \liminf_{m\to\infty} \sum_{n=1}^{\infty} \chi_m(n) r_n^{(m)} \leq \liminf_{m\to\infty} \rho_U(A^{(m)}),$$

as required. □

For the rest of this section $A = ((a_n, r_n)) \in \mathcal{N}$ will be a fixed redundancy-free abstract Swiss cheese. Note that both $\rho(A)$ and $\mu(A)$ are finite and $r_n \leq \rho(A)/n$ for all $n \in \mathbb{N}$. We define the (classical) *error set of* A to be

$$E(A) := \bigcup_{\substack{m,n \in S_A \\ m \neq n}} \left(\bar{\mathrm{B}}(a_m, r_m) \cap \bar{\mathrm{B}}(a_n, r_n)\right) \cup \bigcup_{n \in S_A} ((\mathbb{C} \setminus \mathrm{B}(a_0, r_0)) \cap \bar{\mathrm{B}}(a_n, r_n)).$$

Note that if $E(A) \subseteq \mathrm{B}(a_0, r_0)$ then $\bar{\mathrm{B}}(a_n, r_n) \subseteq \mathrm{B}(a_0, r_0)$ for all $n \in S_A$. Also note that $A \in \mathcal{N}$ is a classical abstract Swiss cheese if and only if $E(A) = \emptyset$. We aim to prove that, under suitable conditions, we can classicalise A while leaving many of the open discs unchanged.

As in Section 2.4, we seek to construct a compact subset of $\mathcal{F}$ on which the function δ_1 can be maximised and then the function δ_2 minimised to give a suitable classical abstract Swiss cheese.

In the rest of this chapter, we will frequently need to consider indexed collections of pairs of sets of the following form. Let $I \subseteq \mathbb{N}$ be non-empty. Let $\mathcal{C} = ((K_n, U_n))_{n\in I}$, where each K_n is a compact plane set and each U_n is an open set with $K_n \subseteq U_n$. We call such an indexed collection a *controlling collection of pairs*. In the special case where I has only one member, we say $\mathcal{C}$ is a *controlling pair* and write $\mathcal{C} = (K, U)$.

Definition 2.5.2. Let $\mathcal{C} = ((K_n, U_n))_{n\in I}$ be a controlling collection of pairs. Define

$$V(\mathcal{C}) := \bigcup_{n\in I} U_n, \quad F(\mathcal{C}) := \bigcup_{n\in I} K_n.$$

Let $\mathcal{L}_A(\mathcal{C})$ denote the set of all $B = ((b_n, s_n)) \in \mathcal{N}(\mu(A), \rho(A))$ such that:

(i) for each $(K, U) \in \mathcal{C}$ we have $\rho_U(B) \leq \rho_U(A)$;

(ii) $\bar{\mathrm{B}}(b_0, s_0) = \bar{\mathrm{B}}(a_0, r_0)$;

(iii) for all $k \in S_A$ with $\bar{\mathrm{B}}(a_k, r_k) \cap V(\mathcal{C}) = \emptyset$ there exists $\ell \in S_B$ such that $\mathrm{B}(b_\ell, s_\ell) = \mathrm{B}(a_k, r_k)$;

(iv) for each $n \in I$ and for all $k \in S_A$ with $\bar{\mathrm{B}}(a_k, r_k) \cap U_n \neq \emptyset$:

   (a) there exists $\ell \in S_B$ with $\mathrm{B}(b_\ell, s_\ell) = \mathrm{B}(a_k, r_k)$; or

   (b) there exists $\ell \in H_B(K_n)$ with $\mathrm{B}(a_k, r_k) \subseteq \mathrm{B}(b_\ell, s_\ell)$.

Note that $A \in \mathcal{L}_A(\mathcal{C})$, and if $B \in \mathcal{L}_A(\mathcal{C})$ then B is partially above A. Thus if $B \in \mathcal{L}_A(\mathcal{C})$ then $X_B \subseteq X_A$. The properties (i)-(iv) reflect the properties we desire for the final abstract Swiss cheese. We will use the open sets U to bound the error set $E(A)$. Under some technical assumptions, conditions (iii) and (iv) ensure that abstract Swiss cheeses maximising δ_1 in $\mathcal{L}_A(\mathcal{C})$ have the property that any open disc which lies outside $V(\mathcal{C})$ is the same as an open disc from A.

We first require some preliminary lemmas. The following lemma is probably well-known and can be proved using a Hausdorff metric argument, but we include an elementary proof for the convenience of the reader.

Lemma 2.5.3. *Let K be a compact plane set. Let (z_n) be a sequence in $\mathbb{C}$, and let (t_n) be a sequence in $\mathbb{R}^+$. Suppose that $\bar{\mathrm{B}}(z_n, t_n) \cap K \neq \emptyset$ for all n, and that $z_n \to z$ and $t_n \to t$ as $n \to \infty$. Then $\bar{\mathrm{B}}(z, t) \cap K \neq \emptyset$.*

Proof. For each $n \in \mathbb{N}_0$ there exists a point $w_n \in \bar{\mathrm{B}}(z_n, t_n) \cap K$. Now since (w_n) is a sequence in K there is a convergent subsequence (w_{n_k}) converging to a point $w \in K$. For each $k \in \mathbb{N}_0$, we have $w_{n_k} \in \bar{\mathrm{B}}(z_{n_k}, t_{n_k})$ so that $|w_{n_k} - z_{n_k}| \leq t_{n_k}$. Hence, taking the limit as $k \to \infty$, we have $|w - z| \leq t$ so that $w \in \bar{\mathrm{B}}(z, t) \cap K$ as required. □

For the next lemma, we show that for any controlling collection of pairs $\mathcal{C}$, the space $\mathcal{L}_A(\mathcal{C})$ is a compact subspace of $\mathcal{F}$.

Lemma 2.5.4. *Let $\mathcal{C} := ((K_n, U_n))_{n \in I}$ be a controlling collection of pairs. Then the set $\mathcal{L}_A(\mathcal{C}) \subseteq \mathcal{F}$ is compact.*

Proof. From the definition of $\mathcal{L}_A(\mathcal{C})$ we see that $\pi_0(\mathcal{L}_A(\mathcal{C}))$ is bounded in $(\mathbb{C} \times \mathbb{R}^+)$. Thus by Lemma 2.2.3 it is sufficient to show that $\mathcal{L}_A(\mathcal{C})$ is closed in $\mathcal{N}(\mu(A), \rho(A))$. For each $m \in \mathbb{N}_0$, let $A^{(m)} = ((a_n^{(m)}, r_n^{(m)})) \in \mathcal{L}_A(\mathcal{C})$. Let $B = ((b_n, s_n)) \in \mathcal{N}(\mu(A), \rho(A))$, and suppose that $A^{(m)} \to B$. We need to show that $B \in \mathcal{L}_A(\mathcal{C})$.

By Lemma 2.5.1 we see that B also satisfies Definition 2.5.2 (i), and it is immediate that Definition 2.5.2 (ii) is also satisfied.

Now we show that Definition 2.5.2 (iii) holds for B. Suppose $k \in S_A$ such that $\bar{\mathrm{B}}(a_k, r_k) \cap V(\mathcal{C}) = \emptyset$. Since, for each $m \in \mathbb{N}_0$, we have $A^{(m)} \in \mathcal{L}_A(\mathcal{C})$ it follows that for each m there exists an integer ℓ_m such that $\mathrm{B}(a_k, r_k) = \mathrm{B}(a_{\ell_m}^{(m)}, r_{\ell_m}^{(m)})$. Now since $r_{\ell_m}^{(m)} = r_k$ for each m we have $1 \leq \ell_m \leq \rho(A)/r_k$ for all m. But then there must exist an integer $1 \leq p \leq \rho(A)/r_k$ such that $\ell_m = p$ infinitely often so by passing to a subsequence if necessary we may assume that $\ell_m = p$ for all m. Since $\mathrm{B}(a_k, r_k) = \mathrm{B}(a_p^{(m)}, r_p^{(m)})$ for all m and $A^{(m)} \to B$ as $m \to \infty$, it follows that $\mathrm{B}(a_k, r_k) = \mathrm{B}(b_p, s_p)$. This proves that Definition 2.5.2 (iii) holds for B.

Now we show that Definition 2.5.2 (iv) holds for B. Suppose $k \in S_A$ such that $\bar{\mathrm{B}}(a_k, r_k) \cap U \neq \emptyset$ for some (K, U) in $\mathcal{C}$. As above, for each $m \in \mathbb{N}_0$ there exists an integer ℓ_m such that $\mathrm{B}(a_k, r_k) \subseteq \mathrm{B}(a_{\ell_m}^{(m)}, r_{\ell_m}^{(m)})$ and $r_{\ell_m}^{(m)} \geq r_k$. We choose ℓ_m as follows: if in $A^{(m)}$ there is an open disc $\mathrm{B}(a, r) = \mathrm{B}(a_k, r_k)$ then we pick ℓ_m to be the index of that open disc, otherwise we choose ℓ_m to be the index of an open disc $\mathrm{B}(a, r)$ in $A^{(m)}$ that properly contains $\mathrm{B}(a_k, r_k)$ and satisfies $\bar{\mathrm{B}}(a, r) \cap K \neq \emptyset$. Note that we have $1 \leq \ell_m \leq \rho(A)/r_k$ for all m and so there exists an integer $1 \leq p \leq \rho(A)/r_k$ such that $\ell_m = p$ infinitely often. By considering a subsequence if necessary we can assume that $\ell_m = p$ for all m. If $\mathrm{B}(a_p^{(m)}, r_p^{(m)}) = \mathrm{B}(a_k, r_k)$ holds for infinitely many m then there is a subsequence $(A^{(m_j)})_j$ such that $\mathrm{B}(a_k, r_k) = \mathrm{B}(a_p^{(m_j)}, r_p^{(m_j)})$ for all j. Since $A^{(m_j)} \to B$ as

$j \to \infty$ it follows that $\mathrm{B}(a_k, r_k) = \mathrm{B}(b_p, s_p)$. If $\mathrm{B}(a_k, r_k) = \mathrm{B}(a_p^{(m)}, r_p^{(m)})$ for only finitely many m then we must have

$$\mathrm{B}(a_k, r_k) \subsetneq \mathrm{B}(a_p^{(m)}, r_p^{(m)}) \quad \text{and} \quad \bar{\mathrm{B}}(a_p^{(m)}, r_p^{(m)}) \cap K \neq \emptyset$$

for sufficiently large m. But then $\mathrm{B}(a_k, r_k) \subsetneq \mathrm{B}(b_p, s_p)$ and, by Lemma 2.5.3, we have $\bar{\mathrm{B}}(b_p, s_p) \cap K \neq \emptyset$. This proves that Definition 2.5.2 (iv) holds for B.

Thus we have proved that $B \in \mathcal{L}_A(\mathcal{C})$ and hence $\mathcal{L}_A(\mathcal{C})$ is closed. This finishes the proof. □

We are interested in those abstract Swiss cheeses B in $\mathcal{L}_A(\mathcal{C})$ on which the discrepancy function δ_1 is maximised. These abstract Swiss cheeses have some desirable properties. Let $\mathcal{L}_A^*(\mathcal{C})$ denote the subset of $\mathcal{L}_A(\mathcal{C})$ of all abstract Swiss cheeses where δ_1 achieves its maximum. Since $\mathcal{L}_A(\mathcal{C})$ is non-empty and compact and δ_1 is upper semicontinuous, $\mathcal{L}_A^*(\mathcal{C})$ is non-empty and compact. Recall that $A \in \mathcal{N}$ is assumed to be redundancy-free.

Lemma 2.5.5. *Let $\mathcal{C} = ((K_n, U_n))_{n \in I}$ be a controlling collection of pairs. Let $B = ((b_n, s_n)) \in \mathcal{L}_A^*(\mathcal{C})$. Then B has the following properties.*

(i) For all $k, \ell \in S_B$ with $k \neq \ell$, we have $\mathrm{B}(b_k, s_k) \neq \mathrm{B}(b_\ell, s_\ell)$.

(ii) For each $k \in S_B$, there exists $\ell \in S_A$ such that $\mathrm{B}(a_\ell, r_\ell) \subseteq \mathrm{B}(b_k, s_k)$. Moreover, if $\bar{\mathrm{B}}(b_k, s_k) \cap F(\mathcal{C}) = \emptyset$ then this $\ell \in S_A$ is unique, and we have $\mathrm{B}(b_k, s_k) = \mathrm{B}(a_\ell, r_\ell)$.

(iii) Let E be a fixed subset of $\mathbb{C}$. Let $H_1 := H_B(E) \setminus H_B(V(\mathcal{C}))$ and let $H_2 := H_A(E) \setminus H_A(V(\mathcal{C}))$. There exists a bijection $\sigma : H_1 \to H_2$ satisfying the following condition: for each $k \in H_1$ and $\ell \in H_2$, we have $\sigma(k) = \ell$ if and only if $\mathrm{B}(b_k, s_k) = \mathrm{B}(a_\ell, s_\ell)$. In particular,

$$\sum_{n \in H_1} s_n = \sum_{n \in H_2} r_n.$$

Proof. (i) If $k, \ell \in S_B$ with $k \neq \ell$ such that $\mathrm{B}(b_k, s_k) = \mathrm{B}(b_\ell, s_\ell)$ then we can obtain an abstract Swiss cheese B' by deleting the disc at index ℓ. Then $B' \in \mathcal{L}_A(\mathcal{C})$ and $\delta_1(B') > \delta_1(B)$, which is a contradiction.

(ii) Let $k \in S_B$. Assume for a contradiction that there does not exist $\ell \in S_A$ such that $\mathrm{B}(a_\ell, r_\ell) \subseteq \mathrm{B}(b_k, s_k)$. Then we can delete the disc at index k from B to obtain an abstract Swiss cheese B' which has $\delta_1(B') > \delta_1(B)$. It is clear that $B' \in \mathcal{L}_A(\mathcal{C})$, which contradicts the maximality of $\delta_1(B)$. Thus there exists $\ell \in S_A$ such that $\mathrm{B}(a_\ell, r_\ell) \subseteq \mathrm{B}(b_k, s_k)$.

Now suppose, in addition, that $\bar{\mathrm{B}}(b_k, s_k) \cap F(\mathcal{C}) = \emptyset$. We show that the $\ell \in S_A$ found above with $\mathrm{B}(a_\ell, r_\ell) \subseteq \mathrm{B}(b_k, s_k)$ is unique and that we have $\mathrm{B}(a_\ell, r_\ell) = \mathrm{B}(b_k, s_k)$. Assume, for a contradiction, that $\mathrm{B}(a_\ell, r_\ell)$ is properly contained in $\mathrm{B}(b_k, s_k)$. Then, since A is redundancy-free, we must have $\mathrm{B}(a_m, r_m) \neq \mathrm{B}(b_k, s_k)$ for all $m \in S_A$. We claim that the abstract Swiss cheese B' obtained by deleting the disc at index k from B has $B' \in \mathcal{L}_A(\mathcal{C})$; this will lead to a contradiction.

Clearly $B' \in \mathcal{N}(\mu(A), \rho(A))$ and it is also clear that B' satisfies conditions (i) and (ii) of Definition 2.5.2. Since $\mathrm{B}(a_m, r_m) \neq \mathrm{B}(b_k, s_k)$ for all $m \in S_A$, it follows that Definition 2.5.2 (iii) remains true for B'. Similarly, since $\bar{\mathrm{B}}(b_k, s_k) \cap F(\mathcal{C}) = \emptyset$, Definition 2.5.2 (iv) remains true for B'. This proves our claim.

But now $\delta_1(B') > \delta_1(B)$, which contradicts the maximality of $\delta_1(B)$. Thus we must have $\mathrm{B}(a_\ell, r_\ell) = \mathrm{B}(b_k, s_k)$. The uniqueness of ℓ follows from the fact that A is redundancy-free.

(iii) Note that if for some $k \in S_B$ and $\ell \in S_A$ we have $\mathrm{B}(b_k, s_k) = \mathrm{B}(a_\ell, r_\ell)$, then $k \in H_1$ if and only if $\ell \in H_2$. Combining this with (ii), for each $k \in H_1$ there exists a unique $\ell \in H_2$ such that $\mathrm{B}(b_k, s_k) = \mathrm{B}(a_\ell, r_\ell)$. Thus we may define $\sigma(k) = \ell$ for such k, ℓ. We must show that σ is a bijection. By (i), σ is injective. Let $\ell \in H_2$. By Definition 2.5.2 (iii), there exists $k \in S_B$ with $\mathrm{B}(b_k, s_k) = \mathrm{B}(a_\ell, r_\ell)$. By the remark above, $k \in H_1$, and so $\sigma(k) = \ell$. This proves that σ is surjective. It is now immediate that $\sum_{n \in H_1} s_n = \sum_{n \in H_2} r_n$. This completes the proof. □

In order to obtain a controlled classicalisation theorem, we need to impose some technical conditions on $\mathcal{C}$. If $E \subseteq \mathbb{C}$ is non-empty and $z \in \mathbb{C}$ then we define the distance of z to E by $\mathrm{dist}(z, E) := \inf\{|z - x| : x \in E\}$. For a non-empty compact set $K \subseteq \mathbb{C}$ and positive real number M we define $U(K, M) := \{z \in \mathbb{C} : \mathrm{dist}(z, K) < M\}$.

The next lemma is analogous to Lemma 2.4.4.

Lemma 2.5.6. *Let $I \subseteq \mathbb{N}$ be non-empty. Let $(K_n)_{n\in I}$ be a collection of compact plane sets and let $(M_n)_{n\in I}$ be a collection of positive real numbers. For each $n \in I$, let $U_n := U(K_n, M_n)$. Suppose that $\rho_{U_k}(A) < M_k/2$ and $U_k \subseteq \mathrm{B}(a_0, r_0)$ for all $k \in I$ and suppose that $U_k \cap U_\ell = \emptyset$ for all distinct $k, \ell \in I$. Let $\mathcal{C}$ be the controlling collection $((K_n, U_n))_{n\in I}$. Let $B = ((b_n, s_n)) \in \mathcal{L}_A(\mathcal{C})$ and fix $m \in I$. Suppose there exist $k, \ell \in S_B$ with $k \neq \ell$ such that $\bar{\mathrm{B}}(b_k, s_k) \cap K_m \neq \emptyset$ and $\bar{\mathrm{B}}(b_k, s_k) \cap \bar{\mathrm{B}}(b_\ell, s_\ell) \neq \emptyset$. Then there exists $B' \in \mathcal{L}_A(\mathcal{C})$ such that either $\delta_1(B') > \delta_1(B)$ or $\delta_1(B') = \delta_1(B)$ and $\delta_2(B') < \delta_2(B)$.*

Proof. Let $\mathrm{B}(b, s)$ be the disc obtained by the application of Lemma 2.3.2 to the discs $\mathrm{B}(b_k, s_k)$ and $\mathrm{B}(b_\ell, s_\ell)$. Let $B' = ((b'_n, s'_n))$ be the abstract Swiss cheese obtained from B by replacing the discs at indices k, ℓ with the disc $\mathrm{B}(b, s)$. Since $B \in \mathcal{L}_A(\mathcal{C})$ we have $\rho_{U_m}(B) \leq \rho_{U_m}(A) < M_m/2$, so that $s \leq s_k + s_\ell < M_m/2$. Since $\bar{\mathrm{B}}(b_k, s_k) \cap K_m \neq \emptyset$, we must have $\bar{\mathrm{B}}(b, s) \subseteq U_m$ and hence $\bar{\mathrm{B}}(b, s) \cap U_n = \emptyset$ for all $n \in I$ with $n \neq m$.

It is clear now that either $\delta_1(B') > \delta_1(B)$, when $s < s_k + s_\ell$, or we have $\delta_1(B') = \delta_1(B)$ and $\delta_2(B') < \delta_2(B)$, when $s = s_k + s_\ell$, so it remains to show that $B' \in \mathcal{L}_A(\mathcal{C})$. By construction, and since $\bar{\mathrm{B}}(b, s) \subseteq U_m$ and $\bar{\mathrm{B}}(b, s) \cap U_n = \emptyset$ for $n \in I$ with $n \neq m$, we have $B' \in \mathcal{N}(\mu(A), \rho(A))$ and B' satisfies (i) and (ii) in Definition 2.5.2.

Fix $j \in S_A$. If $\bar{\mathrm{B}}(a_j, r_j) \cap V(\mathcal{C}) = \emptyset$, then there exists $p \in S_B$ with $p \neq k, \ell$ and $\mathrm{B}(b_p, s_p) = \mathrm{B}(a_j, r_j)$. Hence there is a $p' \in S_{B'}$ such that $\mathrm{B}(b'_{p'}, s'_{p'}) = \mathrm{B}(a_j, r_j)$. Therefore B' satisfies (iii) in Definition 2.5.2.

Suppose that $\bar{\mathrm{B}}(a_j, r_j) \cap V(\mathcal{C}) \neq \emptyset$. Let $n \in I$ such that $\bar{\mathrm{B}}(a_j, r_j) \cap U_n \neq \emptyset$. Since $B \in \mathcal{L}_A(\mathcal{C})$, there exists $p \in S_B$ such that $\mathrm{B}(a_j, r_j) \subseteq \mathrm{B}(b_p, s_p)$, where equality may only fail when $\bar{\mathrm{B}}(b_p, s_p) \cap K_n \neq \emptyset$. If $p \neq k, \ell$, then there exists $q \in S_{B'}$ such that $\mathrm{B}(b'_q, s'_q) = \mathrm{B}(b_p, s_p)$. Thus $\mathrm{B}(a_j, r_j) \subseteq \mathrm{B}(b'_q, s'_q)$ and equality may only fail when $\bar{\mathrm{B}}(b'_q, b'_q) \cap K_n \neq \emptyset$. If $n \neq m$ then we cannot have $p = k$ or $p = \ell$ since $\bar{\mathrm{B}}(b, s) \subseteq U_m$ and $U_n \cap U_m = \emptyset$. If $n = m$ and either $p = k$ or $p = \ell$, then there exists $q \in S_{B'}$ such that $\mathrm{B}(b'_q, s'_q) = \mathrm{B}(b, s)$, so that $\mathrm{B}(a_j, r_j) \subseteq \mathrm{B}(b'_q, s'_q)$ and $\bar{\mathrm{B}}(b'_q, s'_q) \cap K_n \neq \emptyset$. It follows that B' satisfies Definition 2.5.2 (iv) and hence $B' \in \mathcal{L}_A(\mathcal{C})$. This completes the proof. □

We are now ready to prove the controlled classicalisation theorem. We remind the reader that the error set $E(A)$ is defined on page 48. We note that in applications of the following theorem, one usually has an abstract Swiss cheese at hand, and then the compact plane sets K_n and positive numbers M_n are chosen accordingly so as to satisfy the conditions in the statement of the theorem. Please see Theorem 2.7.10 for an application of the following theorem.

Theorem 2.5.7. *Let $I \subseteq \mathbb{N}$ be non-empty. Let $(K_n)_{n \in I}$ be a collection of compact plane sets and let $(M_n)_{n \in I}$ be a collection of positive real numbers. For each $n \in I$, let $U_n := U(K_n, M_n)$. Suppose that $U_k \subseteq \mathrm{B}(a_0, r_0)$ and $\rho_{U_k}(A) < M_k/2$ for all $k \in I$ and suppose that $U_k \cap U_\ell = \emptyset$ for all distinct $k, \ell \in I$. Let $\mathcal{C}$ be the controlling collection $((K_n, U_n))_{n \in I}$ and suppose the error set of A, $E(A)$, is contained in $F(\mathcal{C})$. Then there exists $B = ((b_n, s_n)) \in \mathcal{L}^*_A(\mathcal{C})$ such that $X_B \setminus V(\mathcal{C}) = X_A \setminus V(\mathcal{C})$ and B is classical.*

Proof. We know that $\mathcal{L}^*_A(\mathcal{C})$ is non-empty and compact so δ_2 attains its minimum on $\mathcal{L}_A^*(\mathcal{C})$. Let $B \in \mathcal{L}_A^*(\mathcal{C})$ be such that δ_2 is minimised on $\mathcal{L}_A^*(\mathcal{C})$ at B. We first show that $\bar{\mathrm{B}}(b_k, s_k) \subseteq \mathrm{B}(b_0, s_0)$ for all $k \in S_B$. Let C be the complement of the disc $\mathrm{B}(a_0, r_0) = \mathrm{B}(b_0, s_0)$. Let $k \in S_B$ and assume, for a contradiction, that $C \cap \bar{\mathrm{B}}(b_k, s_k) \neq \emptyset$. If there exists $u \in S_A$ such that $\mathrm{B}(a_u, r_u) = \mathrm{B}(b_k, s_k)$ then

$$\emptyset \neq \bar{\mathrm{B}}(b_k, s_k) \cap C = \bar{\mathrm{B}}(a_u, r_u) \cap C \subseteq C \cap E(A) = \emptyset,$$

which is impossible. Otherwise, by Lemma 2.5.5, there exists $u \in S_A$ with $\mathrm{B}(a_u, r_u) \subseteq \mathrm{B}(b_k, s_k)$. Since $B \in \mathcal{L}_A^*(\mathcal{C})$, it follows by Lemma 2.5.5 (ii) that there exists $m \in I$ such that $\bar{\mathrm{B}}(b_k, s_k) \cap K_m \neq \emptyset$, and so $\bar{\mathrm{B}}(b_k, s_k) \subseteq U_m \subseteq \mathrm{B}(a_0, r_0) = \mathrm{B}(b_0, s_0)$, where the first inclusion follows from the fact that $\rho_{U_m}(B) \leq \rho_{U_m}(A) < M_m/2$. This contradicts the fact that $C \cap \bar{\mathrm{B}}(b_k, s_k) \neq \emptyset$.

We must now show that there do not exist distinct $k, \ell \in S_B$ such that $\bar{\mathrm{B}}(b_k, s_k) \cap \bar{\mathrm{B}}(b_\ell, s_\ell) \neq \emptyset$. Suppose, for a contradiction, that such a pair exists. By Lemma 2.5.5 (i) we know that $\bar{\mathrm{B}}(b_k, s_k) \neq \bar{\mathrm{B}}(b_\ell, s_\ell)$. If $\bar{\mathrm{B}}(b_k, s_k) \cap F(\mathcal{C}) = \emptyset$ and $\bar{\mathrm{B}}(b_\ell, s_\ell) \cap F(\mathcal{C}) = \emptyset$ then by Lemma 2.5.5 (ii) there exist distinct $u, v \in S_A$ with $\mathrm{B}(a_u, r_u) = \mathrm{B}(b_k, s_k)$ and $\mathrm{B}(a_v, r_v) = \mathrm{B}(b_\ell, s_\ell)$, which is a contradiction since $E(A) \subseteq F(\mathcal{C})$. Thus at least one of these discs has non-empty intersection with at least one compact set K_m.

We may assume, without loss of generality, that $\mathrm{B}(b_k, s_k) \cap K_m \neq \emptyset$ for some $m \in I$. It follows that $\bar{\mathrm{B}}(b_k, s_k) \subseteq U_m$. Let $\mathrm{B}(b, s)$ be the open disc obtained by an application of Lemma 2.3.2 to the discs $\mathrm{B}(b_k, s_k)$ and $\mathrm{B}(b_\ell, s_\ell)$. Then, by Lemma 2.5.6, the abstract Swiss cheese $B' \in \mathcal{L}_A(\mathcal{C})$ obtained by replacing the discs $\mathrm{B}(b_k, s_k)$ and $\mathrm{B}(b_\ell, s_\ell)$ with $\mathrm{B}(b, s)$ has either $\delta_1(B') > \delta_1(B)$ or $\delta_1(B') = \delta_1(B)$ and $\delta_2(B') < \delta_2(B)$. Both of these cases are impossible since we assumed that δ_1 was maximised at B and δ_2 was minimised at B. It follows that no such pair k, ℓ can exist and hence B is classical.

It remains to show that $X_B \setminus V(\mathcal{C}) = X_A \setminus V(\mathcal{C})$. Note that $B \in \mathcal{L}_A(\mathcal{C})$ so $X_B \subseteq X_A$, thus $X_B \setminus V(\mathcal{C}) \subseteq X_A \setminus V(\mathcal{C})$. To prove the reverse inclusion, let $x \in X_A \setminus V(\mathcal{C})$. So we have $x \in \bar{\mathrm{B}}(a_0, r_0) = \bar{\mathrm{B}}(b_0, s_0)$. Assume towards a contradiction that $x \notin X_B \setminus V(\mathcal{C})$. Then there exists some $n \in \mathbb{N}$ such that $x \in \mathrm{B}(b_n, s_n)$, and such that there is no $m \in \mathbb{N}$ with $\mathrm{B}(a_m, r_m) = \mathrm{B}(b_n, s_n)$. This shows, by Lemma 2.5.5 (ii) that $\bar{\mathrm{B}}(b_n, s_n) \cap F(\mathcal{C}) \neq \emptyset$. So there exists $m \in I$ such that $\bar{\mathrm{B}}(b_n, s_n) \cap K_m \neq \emptyset$. But since $\rho_{U_m}(B) \leq \rho_{U_m}(A) < M_m/2$ we see that $x \in \bar{\mathrm{B}}(b_n, s_n) \subseteq U_m \subseteq V(\mathcal{C})$, a contradiction. Therefore we have $X_B \setminus V(\mathcal{C}) = X_A \setminus V(\mathcal{C})$ and this finishes the proof. □

Note here that the classical, abstract Swiss cheese B obtained from this theorem is an element of $\mathcal{L}_A^*(\mathcal{C})$ and therefore satisfies properties (i)-(iv) of Definition 2.5.2, and the conclusion of Lemma 2.5.5 holds for B. Note also that, in contrast to the Feinstein-Heath classicalisation theorem, $\delta_1(B)$ may be negative here. We can obtain similar results using transfinite induction.

2.6 Annular classicalisation

In this section we give some results about Swiss cheese like sets obtained by deleting open discs from a closed annulus, rather than a closed disc. If K is a closed annulus in the plane, we can write $K = \bar{\mathrm{B}}(a_0, r_0) \setminus \mathrm{B}(a_1, r_1)$ for some $a_0 = a_1 \in \mathbb{C}$ and $r_0 > r_1 > 0$ real. We say an abstract Swiss cheese $A = ((a_n, r_n))$ is *annular* if $a_0 = a_1$ and $0 < r_1 < r_0$ and let K_A denote the annulus $\bar{\mathrm{B}}(a_0, r_0) \setminus \mathrm{B}(a_1, r_1)$. We shall usually omit 'abstract' from the statement *A is an annular abstract Swiss cheese.*

Lemma 2.6.1. *Let $a \in \mathbb{C}$ and $r_0 > r_1 > 0$ and let $K := \bar{\mathrm{B}}(a, r_0) \setminus \mathrm{B}(a, r_1)$. Let $b \in \mathbb{C}$ and $0 < s < (r_0 - r_1)/2$ be chosen such that $\bar{\mathrm{B}}(b, s) \cap \overline{\mathbb{C} \setminus K} \neq \emptyset$. Then there exist $r_0', r_1' > 0$ such that $K' := \bar{\mathrm{B}}(a, r_0') \setminus \mathrm{B}(a, r_1') \subseteq K$ with $K' \cap \mathrm{B}(b, s) = \emptyset$ and $r_0' - r_1' \geq r_0 - r_1 - 2s$.*

Proof. Set $D = \mathrm{B}(b, s)$. If $D \subseteq \mathbb{C} \setminus K$ then we just let $r_i' = r_i$, $i = 0, 1$. So now assume that $D \nsubseteq \mathbb{C} \setminus K$. Since $s < (r_0 - r_1)/2$ there are only two possible cases. We must have either $\overline{D} \cap \bar{\mathrm{B}}(a, r_1) \neq \emptyset$ or $\overline{D} \setminus \mathrm{B}(a, r_0) \neq \emptyset$.

In the first case, where $\bar{D} \cap \bar{\mathrm{B}}(a, r_1) \neq \emptyset$, let $r_0' = r_0$ and $r_1' = |b - a| + s$. We have $|b - a| \geq r_1 - s$ and $|b - a| \leq r_1 + s$. Hence $r_1' \geq r_1$ and $r_1' \leq r_1 + 2s < r_1 + r_0 - r_1 = r_0$ and

$$r_0' - r_1' = r_0 - (|b - a| + s) \geq r_0 - s - r_1 - s = r_0 - r_1 - 2s.$$

Since for each $z \in D$ we have $|z - a| < |b - a| + s = r_1'$ it follows immediately that $D \subseteq \mathbb{C} \setminus K'$.

In the second case, where $\overline{D} \cap \mathbb{C} \setminus \mathrm{B}(a, r_0) \neq \emptyset$, let $r_0' = |b - a| - s$ and $r_1' = r_1$. We have $|b - a| \leq r_0 + s$ and $|b - a| \geq r_0 - s$. Hence $r_0' \leq r_0 + s - s = r_0$,

$$r_0' \geq r_0 - s - s > r_0 - (r_0 - r_1) = r_1$$

and

$$r_0' - r_1' = |b - a| - s - r_1 \geq r_0 - r_1 - 2s.$$

Similarly, for all $z \in D$ we have $r_0' = |b-a| - s < |z-a|$ and so $D \subseteq \mathbb{C} \setminus K$. This completes the proof. $\square$

Definition 2.6.2. The *annular radius sum function* $\rho_{\mathrm{ann}} : \mathcal{F} \to [0, \infty]$ is defined by

$$\rho_{\mathrm{ann}}(A) := \sum_{n=2}^{\infty} r_n, \quad A = ((a_n, r_n)) \in \mathcal{F},$$

and the *annular discrepancy function* $\delta_{\mathrm{ann}} : \mathcal{F} \to [-\infty, \infty)$ is given by

$$\delta_{\mathrm{ann}}(A) = r_0 - r_1 - 2\rho_{\mathrm{ann}}(A), \quad A = ((a_n, r_n)) \in \mathcal{F}.$$

Note that if $\delta_{\mathrm{ann}}(B) > 0$ then $r_0 > r_1$. We aim to prove an analogue of the Feinstein-Heath classicalisation theorem (Proposition 2.4.1) for annular Swiss cheeses by constructing a suitable compact subset of $\mathcal{F}$.

It is easy for the reader to check that the following analogue of Lemma 2.2.6 holds for annular Swiss cheeses.

Lemma 2.6.3. *Let A be an annular Swiss cheese with $\rho_{\mathrm{ann}}(A) < \infty$. Then there exists an annular Swiss cheese $B = ((b_n, s_n))$ with the following properties:*

(i) $X_B = X_A$ *and* $K_B = K_A$*;*

(ii) $\rho_{\mathrm{ann}}(B) \leq \rho_{\mathrm{ann}}(A)$ *and* $\mu(B) < \infty$*;*

(iii) the sequence $(s_n)_{n \geq 2}$ *is non-increasing;*

(iv) for each $j \in S_B \setminus \{1\}$*, we have* $\mathrm{B}(b_j, s_j) \cap K_B \neq \emptyset$ *and* $\mathrm{B}(b_j, s_j) \nsubseteq \mathrm{B}(b_k, s_k)$ *for all* $k \in S_B \setminus \{1, j\}$*;*

(v) for each $E \subseteq \mathbb{C}$ *we have* $\rho_E(B) \leq \rho_E(A)$*.*

Note that, in the previous lemma, $K_B = K_A$ and $\rho_{\mathrm{ann}}(B) \leq \rho_{\mathrm{ann}}(A)$ together imply that $\delta_{\mathrm{ann}}(B) \geq \delta_{\mathrm{ann}}(A)$.

For the rest of this section, let $A = ((a_n, r_n))$ be an annular Swiss cheese with $\delta_{\mathrm{ann}}(A) > 0$, such that $\mu(A) < \infty$ and $(r_n)_{n=2}^{\infty}$ is non-increasing.

Lemma 2.6.4. *Let $\mathcal{A}$ be the family of all $B = ((b_n, s_n)) \in \mathcal{F}$ such that*

(i) the sequence $(s_n)_{n \geq 2}$ *is non-increasing,*

(ii) $\rho_{\mathrm{ann}}(B) \leq \rho_{\mathrm{ann}}(A)$*,*

(iii) $\mu(B) \leq \mu(A)$*,*

(iv) B is partially above A, and

(v) $b_0 = b_1 = a_0$, and $r_0 \geq s_0 \geq s_1 \geq r_1$.

Then $\mathcal{A}$ is compact in $\mathcal{F}$, and each abstract Swiss cheese $B \in \mathcal{A}$ with $\delta_{\mathrm{ann}}(B) > 0$ is annular. Moreover, the function $\delta_{\mathrm{ann}}|_{\mathcal{A}} : \mathcal{A} \to \mathbb{R}$ is upper semicontinuous and the function $\delta_2|_{\mathcal{A}} : \mathcal{A} \to \mathbb{R}$ is continuous.

Proof. It is easy to see that for each $B \in \mathcal{A}$ the coordinate projections $\pi_n(B)$ are bounded for each $n \in \mathbb{N}_0$ by properties (ii), (iii) and (v). So in order to show that $\mathcal{A}$ is compact, by Lemma 2.2.3 it remains only to prove that $\mathcal{A}$ is closed. For each $m \in \mathbb{N}_0$, let $A^{(m)} = ((a_n^{(m)}, r_n^{(m)})) \in \mathcal{A}$ and suppose that $A^{(m)} \to B \in \mathcal{F}$ as $m \to \infty$. It is clear that B satisfies (i)-(iv) (as in the proof of Lemma 2.4.3). Since convergence is pointwise, we have $b_0 = a_0$ and $b_1 = a_1$. Since A is annular, it follows that $b_0 = b_1$. Since for each $A^{(m)} \in \mathcal{A}$ we have $r_0 \geq r_0^{(m)} \geq r_1^{(m)} \geq r_1$, by taking $m \to \infty$ we have

$$r_0 \geq s_0 \geq s_1 \geq r_1.$$

Hence $\mathcal{A}$ is closed, and thus $\mathcal{A}$ is compact.

Let $B = ((b_n, s_n)) \in \mathcal{A}$ with $\delta_{\mathrm{ann}}(B) > 0$. Then we have $b_0 = b_1$ and $s_0 > s_1$, so it follows that B is annular.

The proof that δ_{ann} is upper semicontinuous is an immediate consequence of Fatou's lemma, similar to the upper semicontinuity of δ_1 proved in Lemma 2.2.3.

To prove that the restriction of δ_2 to $\mathcal{A}$ is continuous note that, for $n \in \mathbb{N}$ with $n \geq 2$, we have $s_n^2 \leq \rho_{\mathrm{ann}}(B)^2/n^2$ for each $B = ((b_n, s_n)) \in \mathcal{A}$ (see (2.4)). The result then follows from the dominated convergence theorem as in the proof of Lemma 2.2.3. □

It is clear that $A \in \mathcal{A}$ and so $\mathcal{A}$ is non-empty. For all $B \in \mathcal{A}$ we also have $X_B \subseteq X_A$. We require one additional lemma before we prove the main theorem. In the rest of this section, we reserve the symbol $\mathcal{A}$ to denote the family of abstract Swiss cheeses as described in Lemma 2.6.4.

Lemma 2.6.5. *Let $B = ((b_n, s_n)) \in \mathcal{A}$ be an annular Swiss cheese such that $\delta_{\mathrm{ann}}(B) \geq \delta_{\mathrm{ann}}(A)$. Suppose there exists $k \in S_B \setminus \{1\}$ such that $\bar{\mathbb{B}}(b_k, s_k) \cap \overline{\mathbb{C} \setminus K_B} \neq \emptyset$. Then there exists $B' = ((b'_n, s'_n)) \in \mathcal{A}$ with $\delta_{\mathrm{ann}}(B') \geq \delta_{\mathrm{ann}}(B)$. Moreover, if $\delta_{\mathrm{ann}}(B') = \delta_{\mathrm{ann}}(B)$ then $\delta_2(B') < \delta_2(B)$.*

Proof. Let $b_0' = b_1' = b_0$. As in Lemma 2.6.1, we can find $s_0' > s_1' > 0$ such that $K_{B'} := \bar{\mathrm{B}}(b_0', s_0') \setminus \mathrm{B}(b_1', s_1') \subseteq K_B$, $K_{B'} \cap \mathrm{B}(b_k, s_k) = \emptyset$ and

$$s_0' - s_1' \geq s_0 - s_1 - 2s_k.$$

Let $b_\ell' = b_\ell$ and $s_\ell' = s_\ell$ if $2 \leq \ell < k$, $b_\ell' = b_{\ell+1}$ and $s_\ell' = s_{\ell+1}$ if $k \leq \ell$. Then we obtain an abstract Swiss cheese $B' = ((b_n', s_n'))$, which is annular.

By construction, B' satisfies properties (i), (iii) and (v) in Lemma 2.6.4. We have

$$\delta_{\mathrm{ann}}(B') = s_0' - s_1' - 2\sum_{n=2}^{\infty} s_n' \geq s_0 - s_1 - 2s_k - \left(2\sum_{n=2}^{\infty} s_n - 2s_k\right) = \delta_{\mathrm{ann}}(B).$$

Since $s_0' \leq s_0$ and $s_1' \geq s_1$ we must have $\rho_{\mathrm{ann}}(B') \leq \rho_{\mathrm{ann}}(B) \leq \rho_{\mathrm{ann}}(A)$, so Lemma 2.6.4 (ii) is satisfied.

We now show that B' is partially above A. Fix $j \in S_A$. If $\mathrm{B}(a_j, s_j)$ lies in the complement of $\bar{\mathrm{B}}(b_0, s_0)$, then it lies in the complement of $\bar{\mathrm{B}}(b_0', s_0')$. If $\mathrm{B}(a_j, s_j) \subseteq \mathrm{B}(b_1, s_1)$ then $\mathrm{B}(a_j, s_j) \subseteq \mathrm{B}(b_1', s_1')$. So now suppose there exists $m \in S_B \setminus \{1\}$ such that $\mathrm{B}(a_j, s_j) \subseteq \mathrm{B}(b_m, s_m)$. If $m \neq k$ there exists $\ell \in S_{B'}$ such that $\mathrm{B}(b_\ell', s_\ell') = \mathrm{B}(b_m, s_m)$, and so $\mathrm{B}(a_j, s_j) \subseteq \mathrm{B}(b_\ell', s_\ell')$. If $m = k$ then either $\mathrm{B}(a_j, r_j) \subseteq \mathrm{B}(b_1', s_1')$ or $\mathrm{B}(a_j, s_j)$ lies in the complement of $\bar{\mathrm{B}}(b_0', s_0')$. It follows that B' is partially above A, and hence $B' \in \mathcal{A}$.

It remains to show that if $\delta_{\mathrm{ann}}(B') = \delta_{\mathrm{ann}}(B)$ then $\delta_2(B') < \delta_2(B)$. Assume that $\delta_{\mathrm{ann}}(B') = \delta_{\mathrm{ann}}(B)$. Then either $s_0 = s_0' + 2s_k$ or $s_1' = s_1 + 2s_k$. In the first case we have $s_0^2 > (s_0')^2 + s_k^2$, and in the second case we have $(s_1')^2 > s_1^2 + s_k^2$. In either case, we have $\delta_2(B') < \delta_2(B)$. This completes the proof. ☐

Note that, as for arbitrary abstract Swiss cheeses, if B is a semiclassical Swiss cheese which is also annular, then $\pi\delta_2(B)$ is the area of X_B.

Theorem 2.6.6. *There exists a classical, annular Swiss cheese $B = ((b_n, s_n)) \in \mathcal{A}$ such that $\delta_{\mathrm{ann}}(B) \geq \delta_{\mathrm{ann}}(A)$ and $X_B \subseteq X_A$. Moreover, $r_0 - 2\rho_{\mathrm{ann}}(A) \leq s_0 \leq r_0$ and $r_1 \leq s_1 \leq r_1 + 2\rho_{\mathrm{ann}}(A)$.*

Proof. Since δ_{ann} is upper semicontinuous on $\mathcal{A}$ and $\mathcal{A}$ is compact and non-empty, it follows that δ_{ann} achieves its maximum on $\mathcal{A}$. Let $\mathcal{A}_1$ denote the non-empty, compact subset of $\mathcal{A}$ on which δ_{ann} is maximised. Then δ_2, which is continuous on $\mathcal{A}_1$, achieves its minimum. Let $\mathcal{A}_2$ denote the non-empty, compact subset of $\mathcal{A}_1$ on which δ_2 is minimised and let $B = ((b_n, s_n)) \in \mathcal{A}_2$.

Since $\delta_{\text{ann}}(B) \geq \delta_{\text{ann}}(A) > 0$ and B is partially above A it follows that B is annular and $X_B \subseteq X_A$. Suppose, for a contradiction, that B is not classical. There are two possible cases, which we show cannot happen.

First suppose that there are distinct $k, \ell \in S_B \setminus \{1\}$ with $k > \ell$ such that $\bar{\mathrm{B}}(b_k, s_k) \cap \bar{\mathrm{B}}(b_\ell, s_\ell) \neq \emptyset$. Then, by Lemma 2.3.2 there exists $b \in \mathbb{C}$ and $s > 0$ such that

$$\mathrm{B}(b_k, s_k) \cup \mathrm{B}(b_\ell, s_\ell) \subseteq \mathrm{B}(b, s)$$

and $s \leq s_k + s_\ell$. Let $B' = ((b'_n, s'_n))$ be the abstract Swiss cheese obtained by deleting the discs at indices k, ℓ from B and inserting the disc $\mathrm{B}(b, s)$ at the first index in $\mathbb{N} \setminus \{1\}$ such that the resulting $(s'_n)_{n=2}^{\infty}$ is non-increasing. It is easy to see that $B' \in \mathcal{A}$ and

$$\rho_{\text{ann}}(B) \geq \rho_{\text{ann}}(B) - s_k - s_\ell + s = \rho_{\text{ann}}(B'), \tag{2.8}$$

so that $\delta_{\text{ann}}(B') \geq \delta_{\text{ann}}(B)$. By the maximality of $\delta_{\text{ann}}(B)$, equality must hold here and in (2.8). Thus $s = s_k + s_\ell$ and $s^2 > s_k^2 + s_\ell^2$ so that $\delta_2(B') < \delta_2(B)$. This contradicts the minimality of $\delta_2(B)$. It follows that no such k, ℓ exist.

Now suppose there exists $k \in S_B \setminus \{1\}$ such that $\bar{\mathrm{B}}(b_k, s_k) \cap \overline{\mathbb{C} \setminus K_B} \neq \emptyset$. By Lemma 2.6.5 there exists an annular Swiss cheese $B' \in \mathcal{A}$ with $\delta_{\text{ann}}(B') \geq \delta_{\text{ann}}(B)$ such that, if $\delta_{\text{ann}}(B') = \delta_{\text{ann}}(B)$ then $\delta_2(B') < \delta_2(B)$. This is a contradiction, so no such k can exist. It follows that B is classical.

Since $B \in \mathcal{A}$, we have $r_0 \geq s_0 \geq s_1 \geq r_1$. We also have

$$s_0 - s_1 \geq \delta_{\text{ann}}(B) \geq \delta_{\text{ann}}(A) = r_0 - r_1 - 2\rho_{\text{ann}}(A)$$

so that

$$s_0 \geq r_0 - 2\rho_{\text{ann}}(A) - (r_1 - s_1) \geq r_0 - 2\rho_{\text{ann}}(A)$$

and $s_1 \leq r_1 + 2\rho_{\text{ann}}(A) - (r_0 - s_0) \leq r_1 + 2\rho_{\text{ann}}(A)$. This completes the proof. □

2.7 A classical counterexample to the conjecture of S. E. Morris

In this section, we give an application of the classicalisation theorems developed before. We shall always use $\overline{\Delta}$ to denote the closed unit disc, and use $\mathbb{T}$ to denote the unit circle.

First we introduce a definition. Recall that in this thesis, all Banach algebras are assumed to be commutative.

Definition 2.7.1. Let A be a Banach algebra. We say A is *weakly amenable* if there are no non-zero bounded derivations from A to any commutative Banach A-bimodules.

Recall that if A is a Banach algebra, then A can also be regarded as a Banach A-bimodule as defined in Definition 1.2.9. In Example 1.2.10 we see that the (topological) dual space A^* of A also has a Banach A-bimodule structure, under which we call A^* the dual module of A. It is proved in [4] that in Definition 2.7.1 we do not need to check every commutative Banach A-bimodule: a Banach algebra A is weakly amenable if and only if there are no non-zero bounded derivations from A to A^*.

Now let A be a uniform algebra on X. If A is weakly amenable and $\varphi \in \Phi_A$, then the only bounded point derivation at φ is the zero derivation (note that $\mathbb{C}_\varphi$ can be regarded as the dual module of itself). S. E. Morris conjectured that for $R(X)$, if $R(X)$ has no non-zero bounded point derivations at any character, then $R(X)$ must be weakly amenable.

In [24], Feinstein gave a counterexample to this conjecture by constructing a Swiss cheese set X where $R(X)$ has no non-zero bounded point derivations but $R(X)$ is not weakly amenable. However, the Swiss cheese set constructed in [24] need not be classical. In this section, we aim to classicalise X by methods developed before. We first briefly introduce the construction in [24] and quote some (known) results that we will use in our construction.

In [67], Wermer constructed a classical Swiss cheese set X where $R(X)$ has no non-zero bounded point derivations. From the proofs in [67], we can distill the following proposition.

Proposition 2.7.2. *Let $\varepsilon > 0$, and $s_0 > s_1 \geq 0$. Then there exists a classical abstract Swiss cheese $A = ((a_n, r_n))$ with $a_j = 0$ and $r_j = s_j$ for $j = 0, 1$ and $\rho_{\mathrm{ann}}(A) < \varepsilon$ such that $R(X_A)$ has no non-zero bounded point derivations.*

We note that for a compact plane set X and $x \in X$, the existence of non-zero bounded point derivations at x is a local property for $R(X)$. This is a consequence of a result of Hallstrom (see [37]), which gives a necessary and sufficient condition for the existence of non-zero bounded point derivations for $R(X)$. The following proposition is an easy consequence of Theorem 1 in [37].

Proposition 2.7.3. *Let X be a compact plane set. Let $x \in X$, and let K be a closed neighbourhood of x in X. Then $R(X)$ has no non-zero bounded point* *derivations at x if and only if $R(K)$ has no non-zero bounded point derivations at x.*

In order to eliminate bounded point derivations, in [24] Feinstein proved the following lemma.

Lemma 2.7.4. *Let Y, Z be compact plane sets with $Y \subset Z$. If $R(Z)$ has no non-zero, bounded point derivations, then the same is true for Y.*

Another way to eliminate non-zero bounded point derivations (actually non-zero point derivations) is to note the following proposition, see [9, Corollary 1.6.7].

Proposition 2.7.5. *Let A be a uniform algebra on X. If $x \in X$ is a peak point for A, then there are no non-zero point derivations at x.*

The following two propositions are due to Feinstein (see [24]), who used them to construct non-zero bounded derivations from $R(X)$ to the dual space of $R(X)$ for some suitable compact plane set X. In the following, we use f' to denote the derivative of a function f, and we use $\mathrm{d}t$ to denote the Lebesgue (arc-length) measure on $\mathbb{T}$.

Proposition 2.7.6. *Let X be a compact plane set and let μ be a measure on X. Suppose that the bilinear functional defined on $R_0(X) \times R_0(X)$ by*

$$(f, g) \mapsto \int_X f'(x)g(x)\,\mathrm{d}\mu(x)$$

is bounded. Then we can extend this bilinear functional by continuity to $R(X) \times R(X)$ and obtain a bounded derivation D from $R(X)$ to the dual space of $R(X)$ satisfying

$$D(f)(g) = \int_X f'(x)g(x)\,\mathrm{d}\mu(x), \quad f, g \in R_0(X).$$

Proposition 2.7.7. *Let D_n be a sequence of open discs in $\mathbb{C}$ whose closures are contained in the open unit disc. Set $X = \overline{\Delta} \setminus \bigcup_{n=1}^{\infty} D_n$. Let d_n be the distance from D_n to $\mathbb{T}$ and let r_n be the radius of D_n. Let f and g be in $R_0(X)$. Then*

$$\left| \int_{\mathbb{T}} f'(z)g(z)\,\mathrm{d}t \right| \leq 4\pi \|f\|_X \|g\|_X \sum_{n=1}^{\infty} \frac{r_n}{d_n^2}.$$

An easy application of Propositions 2.7.6 and 2.7.7 gives the following corollary.

Corollary 2.7.8. *Let D_n be a sequence of open discs in $\mathbb{C}$ whose closures are contained in the open unit disc. Set $X = \overline{\Delta} \setminus \bigcup_{n=1}^{\infty} D_n$. Let d_n be the distance from D_n to $\mathbb{T}$ and let r_n be the radius of D_n. If $\sum_{n=1}^{\infty} r_n/d_n^2$ is finite, then there exists a non-zero bounded derivation from $R(X)$ to the dual space of $R(X)$.*

The main theorem in [24] is the following.

Proposition 2.7.9. *For each $C > 0$ there is a Swiss cheese set X such that the unit circle $\mathbb{T}$ is a subset of X, $R(X)$ has no non-zero bounded point derivations, but for all f, g in $R_0(X)$,*

$$\left| \int_{\mathbb{T}} f'(z) g(z) dz \right| \le C \|f\|_X \|g\|_X.$$

In this section we prove a new version of Theorem 2.7.9 where the Swiss cheese set X is classical. The main theorem of this section is the following.

Theorem 2.7.10. *For each $C > 0$, there exists a classical abstract Swiss cheese $B = ((b_n, s_n))$ such that $R(X_B)$ has no non-zero bounded point derivations, $b_0 = 0$ and $s_0 = 1$, the unit circle $\mathbb{T} \subseteq X_B$, and $\sum_{n=1}^{\infty} s_n/d_n^2 \le C$, where d_n is the distance from the disc $\mathrm{B}(b_n, s_n)$ to $\mathbb{T}$ if $s_n > 0$ and $d_n = 1$ if $s_n = 0$.*

Proof. Fix $C > 0$. For each $n \in \mathbb{N}_0$, by Proposition 2.7.2 there exists $A^{(n)} = ((a_m^{(n)}, r_m^{(n)}))$ a classical abstract Swiss cheese with $r_0^{(n)} = (n+1)/(n+2)$, $r_1^{(n)} = n/(n+1)$, $a_0^{(n)} = a_1^{(n)} = 0$,

$$\rho_{\mathrm{ann}}(A^{(n)}) < \min \left\{ \frac{C}{2^{n+4}(n+3)^2}, \frac{1}{24(n+3)^2} \right\}, \tag{2.9}$$

and such that $R(X_{A^{(n)}})$ has no non-zero bounded point derivations.

For each $n \in \mathbb{N}_0$, set

$$K_n = \left\{ z \in \mathbb{C} : |z| \in \left[\frac{n+1}{n+2} - \frac{1}{4(n+3)^2}, \frac{n+1}{n+2} + \frac{1}{4(n+3)^2} \right] \right\}. \tag{2.10}$$

By Proposition 2.7.2 again, we also choose, for each $n \in \mathbb{N}_0$, a sequence of open discs such that the annular Swiss cheese $B^{(n)} = ((b_m^{(n)}, s_m^{(n)}))$ with $b_0^{(n)} = b_1^{(n)} = 0$,

$$s_0^{(n)} = \frac{n+1}{n+2} + \frac{1}{4(n+3)^2},$$

$$s_1^{(n)} = \frac{n+1}{n+2} - \frac{1}{4(n+3)^2},$$

$$\rho_{\rm ann}(B^{(n)}) < \min\left\{\frac{C}{2^{n+4}(n+3)^2}, \frac{1}{24(n+3)^2}\right\}, \tag{2.11}$$

is classical, and such that $R(X_{B^{(n)}})$ has no non-zero bounded point derivations.

We construct an abstract Swiss cheese $A = ((a_m, r_m))$ such that $a_0 = 0$, $r_0 = 1$, (r_m) is an enumeration of $(r_m^{(n)})_{n\geq 0, m\geq 2}$ and $(s_m^{(n)})_{n\geq 0, m\geq 2}$, and (a_m) is the enumeration of $(a_m^{(n)})_{n\geq 0, m\geq 2}$ and $(b_m^{(n)})_{n\geq 0, m\geq 2}$ corresponding to (r_m). By Lemma 2.2.6, there exists a redundancy-free abstract Swiss cheese $A' = ((a'_m, r'_m))$ such that $X_{A'} = X_A$. Notice that for fixed n, both discs $\mathrm{B}(a_m^{(n)}, r_m^{(n)})$ and $\mathrm{B}(b_m^{(n)}, s_m^{(n)})$ are contained in the disc $\mathrm{B}(0, (n+2)/(n+3))$. Let d'_m be the distance from the disc $\mathrm{B}(a'_m, r'_m)$ to $\mathbb{T}$ if $r'_m > 0$ and let $d'_m = 1$ otherwise. We have

$$\sum_{m=1}^{\infty} \frac{r'_m}{(d'_m)^2} \leq \sum_{n=0}^{\infty}\left((n+3)^2 \sum_{m=2}^{\infty}(r_m^{(n)} + s_m^{(n)})\right) \leq \frac{C}{4},$$

where the first inequality follows from Lemma 2.2.6 (note that there is an injection φ from $S_{A'}$ to S_A), and the second inequality follows from equations (2.9) and (2.11).

Set $M_n = 1/(4(n+3)^2)$, and let $U_n = \{z \in \mathbb{C} : \mathrm{dist}(z, K_n) < M_n\}$. (See Figure 2.4 for an illustration of a resulting pair (K_n, U_n)). Then we claim that $\mathcal{C} = ((K_n, U_n))$ is a controlling collection of pairs which satisfies the conditions in Theorem 2.5.7 (with the abstract Swiss cheese in Theorem 2.5.7 replaced by A'). By (2.10) and the choice of M_n we see that $U_n \cap U_m = \emptyset$ whenever $n \neq m$. From the construction and by Lemma 2.2.6 we observe that

$$\begin{aligned} \rho_{U_n}(A') \leq \rho_{U_n}(A) &\leq \rho_{\rm ann}(A^{(n)}) + \rho_{\rm ann}(A^{(n+1)}) + \rho_{\rm ann}(B^{(n)}) \\ &< \min\left\{\frac{1}{8(n+3)^2}, \frac{3C}{2^{n+3}(n+3)^2}\right\} \leq \frac{1}{2}M_n. \end{aligned} \tag{2.12}$$

Note that each $B^{(n)}$ is contained in K_n by construction and each $A^{(n)}$ is a classical abstract Swiss cheese, so $E(A') \subseteq F(\mathcal{C})$. Thus, by Theorem 2.5.7 applied to A' and $\mathcal{C}$, there exists a classical abstract Swiss cheese $B = ((b_n, s_n))$ such that $\delta_1(B) \geq \delta_1(A')$, $X_B \subseteq X_{A'}$, $b_0 = 0$, $s_0 = 1$ and $\rho_{U_n}(B) \leq \rho_{U_n}(A')$ for all $n \in \mathbb{N}$.

Let d_n be the distance from the disc $\mathrm{B}(b_n, s_n)$ to $\mathbb{T}$ if $s_n > 0$ and let $d_n = 1$ if $s_n = 0$. We have

$$\sum_{n=1}^{\infty} \frac{s_n}{d_n^2} = \sum_{n\in S_1} \frac{s_n}{d_n^2} + \sum_{n\in S_2} \frac{s_n}{d_n^2},$$

where $S_2 := \bigcup_{n=1}^{\infty} H_B(U_n)$ and $S_1 = \mathbb{N}\backslash S_2$; note that $H_B(U_m) \cap H_B(U_n) = \emptyset$ for all $m \neq n$. By Lemma 2.5.5, for all $n \in S_1$ we have $B(b_n, s_n) = B(a'_m, r'_m)$ for

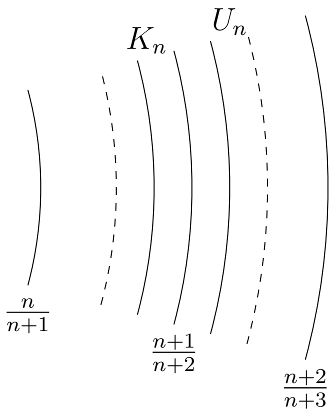


Figure 2.4: A pair (K_n, U_n) as in the proof of Theorem 2.7.10.

some $m \geq 1$. Then we have

$$\sum_{n \in S_1} \frac{s_n}{d_n^2} \leq \sum_{n=1}^{\infty} \frac{r'_n}{(d'_n)^2} \leq \frac{C}{4}. \tag{2.13}$$

On the other hand, from the construction we have

$$\sum_{n \in S_2} \frac{s_n}{d_n^2} = \sum_{n=0}^{\infty} \left(\sum_{m \in S_2^{(n)}} \frac{s_m}{d_m^2} \right),$$

where $S_2^{(n)} := H_B(U_n)$. For each $m \in S_2^{(n)}$, since

$$s_m \leq \rho_{U_n}(B) \leq \rho_{U_n}(A') < \frac{1}{8(n+3)^2}$$

by (2.12), we have $\bar{B}(b_m, s_m) \subseteq B(0, (n+2)/(n+3))$, so $d_m > 1/(n+3)$. Again by (2.12) we observe that

$$\sum_{m \in S_2^{(n)}} s_m = \rho_{U_n}(B) \leq \rho_{U_n}(A') < \frac{3C}{2^{n+3}(n+3)^2}.$$

Therefore we have

$$\sum_{m \in S_2} \frac{s_m}{d_m^2} = \sum_{n=0}^{\infty} \left(\sum_{m \in S_2^{(n)}} \frac{s_m}{d_m^2} \right) \leq \sum_{n=0}^{\infty} (n+3)^2 \sum_{m \in S_2^{(n)}} s_m \leq \frac{3C}{4}.$$

Combining with (2.13) we conclude that

$$\sum_{n=1}^{\infty} \frac{s_n}{d_n^2} \leq C.$$

It remains to show that $R(X_B)$ has no nonzero bounded point derivations. Let $x \in X_B$. If $x \in \mathbb{T}$, then it is clear that x is a peak point for $R(X_B)$, and hence by Proposition 2.7.5 we see that there are no non-zero derivations at x. So now assume $|x| < 1$. Then by choosing $\varepsilon > 0$ small enough, we can make sure that $\bar{\mathrm{B}}(x, \varepsilon)$ is either in $\bar{\mathrm{B}}(a_0^{(n)}, r_0^{(n)}) \setminus \mathrm{B}(a_1^{(n)}, r_1^{(n)})$ for some $n \in \mathbb{N}_0$, or in $\bar{\mathrm{B}}(b_0^{(n)}, s_0^{(n)}) \setminus \mathrm{B}(b_1^{(n)}, s_1^{(n)})$ for some $n \in \mathbb{N}_0$. Assume that the first inclusion holds (the other case can be proved in the same way). So $\bar{\mathrm{B}}(x, \varepsilon) \cap X_{A^{(n)}}$ is a closed neighbourhood of x in $X_{A^{(n)}}$. By Lemma 2.7.4 we see that $R(\bar{\mathrm{B}}(x, \varepsilon) \cap X_{A^{(n)}})$ has no non-zero bounded point derivations at x. Since $\bar{\mathrm{B}}(x, \varepsilon) \cap X_B \subseteq \bar{\mathrm{B}}(x, \varepsilon) \cap X_{A^{(n)}}$, by Lemma 2.7.4 again we see that $R(X_B \cap \bar{\mathrm{B}}(x, \varepsilon))$ has no non-zero bounded point derivations at x. Since $X_B \cap \bar{\mathrm{B}}(x, \varepsilon)$ is a closed neighbourhood of x in X_B, by Proposition 2.7.3 we see that $R(X_B)$ has no non-zero bounded point derivations at x. This finishes the proof. $\square$

Now combining Corollary 2.7.8 and Theorem 2.7.10 we get the following corollary.

Corollary 2.7.11. *There exists a classical Swiss cheese set X such that $R(X)$ has no non-zero bounded point derivations, but $R(X)$ is not weakly amenable.*

Chapter 3

Regularity points for uniform algebras

In [31], Feinstein and Somerset studied the failure of regularity for Banach function algebras in terms of two types of regularity point: points of continuity and R-points. (These technical terms and others are defined in Section 3.1.) They gave examples of Banach function algebras where points of continuity do not coincide with R-points, but these algebras are not natural. There do not appear to be any examples in the literature of natural Banach function algebras where the two types of regularity point are different. In Section 3.2 we give examples to show that they can differ even for natural uniform algebras such as $R(X)$, for suitable compact plane sets X. These examples are inspired by an example in [29], where Feinstein and Mortini also studied these types of regularity points. (In that paper, these points were also described as regularity points of types I and II). We show that it is possible for $R(X)$ to have exactly one point of continuity while having no R-points, and it is also possible for $R(X)$ to have exactly one R-point while having no points of continuity.

In Section 3.3 we give a new way to construct Swiss cheese sets X such that $R(X)$ has no non-trivial Jensen measures, but $R(X)$ is not regular. Such examples were first constructed in [23] by Feinstein. Our new construction is more general, and allows us to obtain additional properties. In particular, we show that in such examples, the "exceptional set" of points where regularity fails (the set of points of discontinuity, and also the set of non-R-points) can have positive area. We also provide a more elementary argument to show that there are no non-trivial Jensen measures for our Swiss cheese sets X, as an alternative to the method used in [23], which was based on the deeper theory of the fine topology. In the next section we describe the background in more detail, and introduce the definitions, terminology and preliminary results we need.

We note that all of the work in this chapter is joint work with J. Feinstein. This work has been published in [32].

3.1 Preliminaries

Let A be a natural Banach function algebra on X, and let $x \in X$. We denote by J_x the ideal of functions f in A such that x is in the interior of the zero set $f^{-1}(\{0\})$ of f. We denote by M_x the ideal of functions f in A such that $f(x) = 0$. For an ideal I in A, the *hull* of I, denoted by $h(I)$, is defined by

$$h(I) = \bigcap_{f \in I} \{z \in X : f(z) = 0\}.$$

The following is the definition of 'regularity point'.

Definition 3.1.1. Let A be a natural Banach function algebra on X, and let $x \in X$. We say that x is a *point of continuity* (for A) if, for all $y \in X \setminus \{x\}$ we have $J_y \not\subseteq M_x$; we say that x is an *R-point* (for A) if, for all $y \in X \setminus \{x\}$ we have $J_x \not\subseteq M_y$.

We note that the name *point of continuity* is related to the *hull kernel topology* on X. Let F be a subset of X. The *kernel* of F, denoted by $k(F)$, is an ideal of A defined by

$$k(F) = \{f \in A : f(F) \subseteq \{0\}\}.$$

The hull kernel topology on X is defined by declaring the closed sets to be subsets $F \subseteq X$ such that $h(k(F)) = F$ (see [33, p. 12]). The hull kernel topology is weaker than the original topology on X, and they coincide if and only if the natural Banach function algebra A is regular as defined in Definition 2.1.2. As shown in [31, Lemma 2.1], a point $x \in X$ is a point of continuity if and only if the identity map from X with the hull kernel topology to X with the original topology is continuous at x, and this holds if and only if all functions in A are hull-kernel-continuous at x.

It is standard (by the above, for example) that a natural Banach function algebra A is regular if and only if every point of X is a point of continuity. This is also equivalent to the condition that every point of X is an R-point ([31, 29]).

Let $x \in X$. Then we note that x is a point of continuity if and only if, for all $y \in X \setminus \{x\}$, we have $x \notin h(J_y)$; x is an R-point if and only if $h(J_x) = \{x\}$.

Let A be a uniform algebra on X, and let φ be a character on A. A *Jensen measure* μ for φ is a representing measure for φ such that

$$\log|\varphi(f)| \leq \int_X \log|f(w)| \, \mathrm{d}\mu(w), \quad f \in A, \tag{3.1}$$

where we follow the usual convention that $\log(0) = -\infty$. It is standard (see, for example, [9, p. 114] or [33, p. 33]) that every character on A has at least one Jensen measure.

Let $x \in X$. We recall that a representing measure for x is trivial if it is the unit point mass measure at x. As noted in [23], if x is a point of continuity, then there are no non-trivial Jensen measures for x. In fact the elementary argument that proves this actually establishes the following lemma. (The first part of this result is essentially stated in [33, page 33].) We include a short proof here for the convenience of the reader.

Lemma 3.1.2. *Let A be a natural uniform algebra on X, and let $x \in X$. Suppose that x has a non-trivial Jensen measure μ, and let F be the closed support of μ. Then, for all $y \in F \setminus \{x\}$, we have $J_y \subseteq M_x$. Thus x is not a point of continuity for A, and no point of $F \setminus \{x\}$ can be an R-point for A.*

Proof. Let $f \in J_y$, so f vanishes on a neighbourhood of y. Since y is in the support of μ we have

$$\int_X \log|f| \, \mathrm{d}\mu = -\infty.$$

Then by (3.1) we see that $\log|f(y)| = -\infty$, that is $f(y) = 0$. Thus $f \in M_x$ and so $J_y \subseteq M_x$. The remaining statements in the lemma follow from the definition of R-points and points of continuity. □

Let X be a compact plane set. By $P(X)$ we denote the set of those functions $f \in C(X)$ which can be uniformly approximated on X by polynomial functions; by $A(X)$ we denote the set of those functions $f \in C(X)$ which are holomorphic on the interior of X. It is standard that when endowed with the uniform norm, both $P(X)$ and $A(X)$ are uniform algebras on X, and $A(X)$ is always natural on X (see [61, p. 274]). It is clear that $P(X) \subseteq R(X) \subseteq A(X)$, and Mergelyan's theorem [33, p.48] asserts that if $\mathbb{C} \setminus X$ is connected, then $P(X) = R(X) = A(X)$.

For more details concerning the above definitions, see standard texts on uniform algebras [9, 33, 61], and for commutative Banach algebras, see [14].

Note that if $R(X)$ is regular, there are no non-trivial Jensen measures supported on X. In [23], Feinstein constructed Swiss cheese sets X such that $R(X)$ has no non-trivial Jensen measures, but $R(X)$ is not regular. In the examples, the line segment $I = [-1/2, 1/2]$ is contained in X, and only the points in $X \setminus I$ are points of continuity. The proof in [23] that $R(X)$ has no non-trivial Jensen measures uses the theory of fine interior, see [19] or [34, p. 319]. In Section 3.3, we use a more elementary approach based on regularity points.

3.2 R-points and points of continuity for $R(X)$

In this section we construct a compact plane set X such that $R(X)$ has exactly one R-point but has no points of continuity. We also construct a compact plane set X such that $R(X)$ has exactly one point of continuity but has no R-points.

Before we proceed to examples where R-points differ from points of continuity, we give the following lemma, which establishes a relationship between these two types of regularity points.

Lemma 3.2.1. *Let A be a natural Banach function algebra on X, and let $x \in X$. Suppose that there is an open neighbourhood U of x such that each point in $U \setminus \{x\}$ is a point of continuity for A. Then x is an R-point for A.*

Proof. Since each $y \in U \setminus \{x\}$ is a point of continuity, we have $J_x \nsubseteq M_y$ and thus $y \notin h(J_x)$. This shows $h(J_x) \cap (U \setminus \{x\}) = \emptyset$. Since $h(J_x)$ is connected ([31, Theorem 3.2]), we conclude that $h(J_x) = \{x\}$ and x is an R-point. □

In particular, the above lemma holds for $R(X)$.

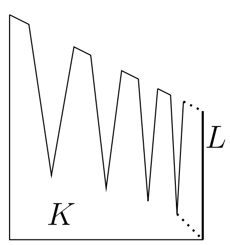


Figure 3.1: A compact plane set K, where the slits accumulate on the right.

The building blocks of our examples are closures of *slit domains*, such as the compact plane set K shown in figure 3.1. The slits cut in K accumulate on the whole of the line segment forming the right edge of K, which we denote by L. Note that since $\mathbb{C} \setminus K$ is connected, we have $P(K) = R(K) = A(K)$.

Note that if $f \in R(K) = A(K)$ and f is constant on the line segment forming the *left* edge of K, then an easy application of the reflection principle (see, for example, [45, Chapter IX, Theorem 2.1]) shows that f is constant on K. On the other hand, as a result of the following proposition, there exists a non-constant $f \in R(K)$ which restricts to a constant function on the *right* edge L of K.

Proposition 3.2.2. *Let K, L be as described above (and shown in figure 3.1). Then the Riemann map ψ from $\operatorname{int} K$ onto the open unit disc can be extended to give a continuous map, still called ψ, from K onto the closed unit disc, such that $\psi(z) \neq \psi(w)$ for all $z \in K \setminus L$ and $w \in K \setminus \{z\}$, and ψ is constant on L.*

We remark that this proposition is a special case of a more general theorem on prime ends explained in [54, Section 2.4]. A similar compact set K was used in [29, Example 4.2].

Let K_n be a sequence of such compact plane sets side by side, touching, tapering, and accumulating on the right at a point x_0, as shown in figure 3.2. Set

$$X = \bigcup_{n=1}^{\infty} K_n \cup \{x_0\}.$$

This gives us our first example. Note that, for the remainder of this chapter, unless otherwise specified, the ideals J_x and M_x considered will always be the ideals in the relevant algebra $R(X)$ under consideration.

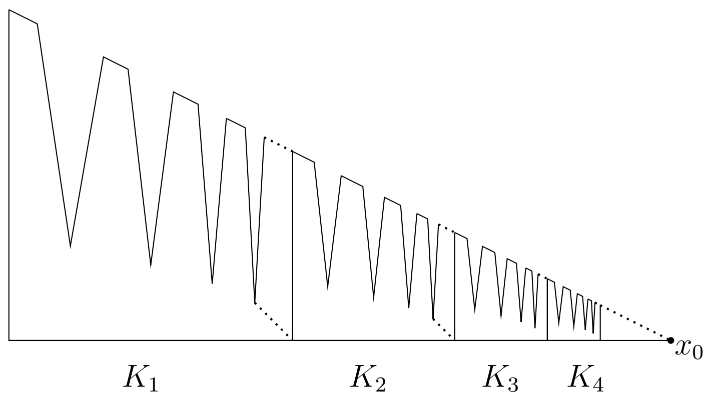


Figure 3.2: A compact set X, where $R(X)$ admits exactly one R-point (the point x_0), but no points of continuity.

Theorem 3.2.3. *Let X be the compact plane set constructed above. Then $\{x_0\}$ is the only R-point for $R(X)$, while $R(X)$ has no points of continuity.*

Proof. For each $n \in \mathbb{N}$, let L_n be the line segment forming the right edge of K_n, and let F_n be the trapezoid-shaped compact domain obtained by filling in the slits in K_n.

Since $\mathbb{C}\backslash X$ is connected, $R(X) = A(X)$ by Mergelyan's theorem. Let $n \in \mathbb{N}$, let $x \in K_n$, and let $f \in J_x$. Then f vanishes identically on K_n, which forces f to vanish identically on K_m for all $m \geq n$, and hence also at x_0. This shows that $J_x \subseteq M_y$ for all $y \in \bigcup_{m \geq n} K_m \cup \{x_0\}$. Since n is arbitrary, it follows that no points in X can be points of continuity for $R(X)$, and no points in $X \setminus \{x_0\}$ can be R-points.

Again let $x \in K_n$ for some $n \in \mathbb{N}$. We show that $J_{x_0} \nsubseteq M_x$. Set

$$F = K_{n+1} \cup \left(\bigcup_{\ell=1}^{n} F_\ell \right).$$

Then $x \in F$, and x is not on L_{n+1}. Note that F is the closure of a slit domain with right hand segment L_{n+1}. By Proposition 3.2.2, the Riemann map ψ from $\operatorname{int} F$ to the open unit disc can be extended to give a continuous map, still denoted by ψ, from F onto the closed unit disc. Note that $\psi(z) \neq \psi(w)$ for $z \in F \setminus L_{n+1}$, and $w \in F \setminus \{z\}$. The map ψ is constant on L_{n+1}: we denote this constant value by c, and set

$$f(z) = \begin{cases} \psi(z) - c & \text{if } z \in F, \\ 0 & \text{if } z \in X \setminus F. \end{cases}$$

Then it is clear that the restriction of f to X is in $A(X) = R(X)$ and $f \in J_{x_0} \setminus M_x$. Thus $J_{x_0} \nsubseteq M_x$. This shows that x_0 is an R-point for $R(X)$. By the above, x_0 is the only R-point. $\square$

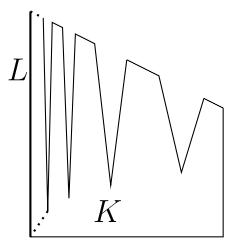


Figure 3.3: A compact plane set K, where the slits accumulate on the left.

Next we modify the example constructed above to show that there exist compact plane sets X, such that $R(X)$ has exactly one point of continuity, but has no R-points. Let K be the closure of a slit domain, with the slits accumulating on the whole of the line segment forming the *left* edge, L, of K, as shown in figure 3.3. As in Proposition 3.2.2, the Riemann mapping ψ from $\operatorname{int} K$ onto the open unit disc can be extended to give a continuous map, still denoted by ψ, from K onto the closed unit disc, such that $\psi(z) \neq \psi(w)$ for all $z \in K \setminus L$ and $w \in K \setminus \{z\}$, and ψ is constant on L. Note that this time, if $f \in A(K)$ and f is constant on the line segment forming the *right* edge of K, then f is constant on K.

Let K_n be a sequence of such compact plane sets side by side, touching, tapering, and accumulating on the right at a point x_0, as shown in figure 3.4.

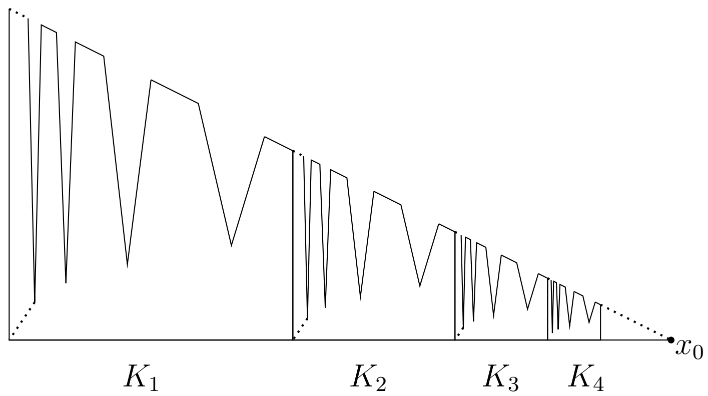


Figure 3.4: A compact set X, where $R(X)$ admits exactly one point of continuity (the point x_0), but no R-points.

Set

$$X = \{x_0\} \cup \left(\bigcup_{n=1}^{\infty} K_n \right).$$

This gives us our second example.

Theorem 3.2.4. *Let X be the compact plane set constructed above. Then $\{x_0\}$ is the only point of continuity for $R(X)$, while $R(X)$ has no R-points.*

Proof. For each $n \in \mathbb{N}$, let L_n be the line segment forming the left edge of K_n, and let F_n be the trapezoid-shaped compact domain obtained by filling in the slits in K_n.

Since $\mathbb{C} \setminus X$ is connected, $R(X) = A(X)$ by Mergelyan's theorem. Let $x \in X$, and let $f \in J_x$. Then f vanishes on an open neighbourhood of a point in some K_n. This forces f to vanish on all K_m with $m \leq n$, and shows that $J_x \subseteq M_y$ for all $y \in \bigcup_{\ell=1}^{n} K_\ell$. Thus x is not an R-point. Let $x \in K_m$ for some m, and let $y \in K_n$ for some $n > m$. If $f \in J_y$, then f vanishes identically on all K_ℓ for $\ell \leq n$. This forces $f \in M_x$, which shows that x is not a point of continuity.

Next we show that x_0 is a point of continuity for $R(X)$. Let $y \in X \setminus \{x_0\}$, then $y \in K_n$ for some n. Set

$$F = K_{n+2} \cup \left(\bigcup_{\ell=n+3}^{\infty} F_\ell \right) \cup \{x_0\},$$

then F is the closure of a slit domain, where the slits accumulate on the left edge L_{n+2}, but where the right edge of F is degenerate (this does not affect the properties that we need here). By Proposition 3.2.2, the Riemann map ψ from $\operatorname{int} F$ onto the open unit disc can be extended to give a continuous map, still denoted by ψ, from F onto the closed unit disc. Note that $\psi(z) \neq \psi(w)$ for $z \in F \setminus L_{n+2}$, and $w \in F \setminus \{z\}$. Also, ψ is constant on L_{n+2}. We denote this constant value by c, and set

$$f(z) = \begin{cases} \psi(z) - c & \text{if } z \in F, \\ 0 & \text{if } z \in X \setminus F. \end{cases}$$

Then it is clear that the restriction of f to X is in $A(X) = R(X)$ and $f \in J_y \setminus M_{x_0}$. This shows that $J_y \not\subseteq M_{x_0}$, and thus x_0 is a point of continuity. By the above, x_0 is the only point of continuity for $R(X)$. □

3.3 Trivial Jensen measures without regularity

In [23], Feinstein constructed a Swiss cheese set X such that $R(X)$ has no non-trivial Jensen measures, but $R(X)$ is not regular. In this section we give a new way to construct such sets X. Our new approach also allows us to obtain some additional properties.

We begin with a lemma concerning representing measures for $R(X)$.

Lemma 3.3.1. *Let X be a compact plane set, let F be a non-empty closed subset of X, and let $x \in F$. Suppose that no bounded component of $\mathbb{C} \setminus F$ is contained in X, and there exists a non-trivial representing measure μ for x with respect to $R(X)$, whose closed support is contained in F. Then μ is also a non-trivial representing measure for x with respect to $R(F)$, and $R(F) \neq C(F)$.*

Proof. Since no bounded component of $\mathbb{C} \setminus F$ is contained in X, an application of Runge's theorem shows that the algebra of functions that are restrictions of functions in $R(X)$ to F is dense in $R(F)$ ([2, Corollary 4.84]). It follows easily that μ is also a non-trivial representing measure for x with respect to

$R(F)$. Since $C(F)$ has no non-trivial representing measures, we must have $R(F) \neq C(F)$. $\square$

We remark that the compact subset F mentioned in the above lemma must have positive area (by the Hartogs-Rosenthal theorem [9, p. 161]), and F cannot have empty interior and connected complement in $\mathbb{C}$ (by Mergelyan's theorem, or Lavrentiev's theorem [33, p. 48]). We also remark that, in the above lemma, the condition that no bounded component of $\mathbb{C} \setminus F$ be contained in X cannot be omitted. To see this, consider the disc algebra A on the closed unit disc $\overline{\Delta}$. Note that we have $A = R(\overline{\Delta})$. Let F be the union of the unit circle and the origin. Then there is a non-trivial representing measure μ supported on F for the origin with respect to A, while $R(F) = C(F)$ by the Hartogs–Rosenthal theorem. Of course this measure μ is *not* a representing measure with respect to $R(F)$.

In our construction below, we will need a lemma of McKissick. The following version of McKissick's lemma was given by Körner in [44]. A detailed explanation of this lemma, when the deleted discs are not required to be pairwise disjoint, is given in [61, p. 344].

Lemma 3.3.2. *Let D be an open disc in $\mathbb{C}$ and let $\varepsilon > 0$. Then there is a sequence Δ_k $(k \in \mathbb{N})$ of pairwise disjoint open discs with each $\Delta_k \subseteq D$ such that the sum of the radii of the Δ_k is less than ε and such that, setting $U = \bigcup_{k\in\mathbb{N}} \Delta_k$, there is a sequence (f_n) of rational functions with poles only in U and such that f_n converges uniformly on $\mathbb{C} \setminus U$ to a function F such that $F(z) = 0$ for all $z \in \mathbb{C} \setminus D$ while $F(z) \neq 0$ for all $z \in D \setminus U$.*

We shall also require the following lemma of Denjoy and Carleman (see [58, Theorem 19.11]). The original lemma is stated for closed intervals in $\mathbb{R}$, but there is no difficulty in replacing a closed interval by a line segment in $\mathbb{C}$. Note that we define derivatives here as limits of quotients using points in the line segment, and all line segments in this section are non-degenerate. We use the usual convention that $1/0 = \infty$ in order to cover the easy special case where the function is a polynomial.

Lemma 3.3.3. *Let f be an infinitely differentiable function on a line segment I in $\mathbb{C}$ such that*

$$\sum_{k=1}^{\infty} \frac{1}{\|f^{(k)}\|_I^{1/k}} = \infty. \tag{3.2}$$

Suppose that there is an $x \in I$ such that $f^{(k)}(x) = 0$ for all $k \geq 0$. Then f is constantly 0 on I.

We also need the following well-known result concerning derivatives of rational functions. (For an explicit proof of this estimate in the case of the first derivative, see the proof of [20, Lemma 2.11].)

Lemma 3.3.4. *Let D_n be a sequence of open discs in $\mathbb{C}$ (not necessarily pairwise disjoint), and set $X = \overline{\Delta} \setminus \bigcup_{n=1}^{\infty} D_n$. Suppose that $z \in X$. Let s_n denote the distance from D_n to z and r_n the radius of D_n. We also set $r_0 = 1$ and $s_0 = 1 - |z|$. Suppose that $s_n > 0$ for all n. Then, for all rational functions f with no poles in X and $k \geq 0$, we have*

$$\left|f^{(k)}(z)\right| \leq k! \sum_{j=0}^{\infty} \frac{r_j}{s_j^{k+1}} \|f\|_X .$$

Let K be a non-empty, compact subset of Δ. Our new approach allows us to construct Swiss cheese sets $X \supseteq K$ such that $R(X)$ has the kind of properties we want. Usually we will be interested in sets K which contain some line segments.

Theorem 3.3.5. *Let K be a non-empty, compact subset of Δ. Then there is a Swiss cheese set X with $K \subseteq X$ such that every point of $X \setminus K$ is both a point of continuity and an R-point for $R(X)$, but such that for every closed line segment I contained in K, every $f \in R(X)$ is infinitely differentiable on I and satisfies condition* (3.2).

Proof. Let $\mathcal{B}$ be the set of all open discs B in the plane such that the centre of B is in $\mathbb{Q} + \mathbb{Q}i$, the radius of B is in $\mathbb{Q} \cap (0, \infty)$, and $\overline{B} \subseteq \overline{\Delta} \setminus K$. Let (B_n) be a sequence enumerating the countable set $\mathcal{B}$. For each $n \in \mathbb{N}$, let d_n be the distance from B_n to K, and choose $\varepsilon_n > 0$ small enough so that the inequality $\varepsilon_n \leq d_n^{k+1}(\log(k+3))^k/2^n$ holds for all $k \in \mathbb{N}$. Note that, for all $k \in \mathbb{N}$, we have

$$\sum_{n=1}^{\infty} \frac{\varepsilon_n}{d_n^{k+1}} \leq (\log(k+3))^k . \tag{3.3}$$

Now we apply Lemma 3.3.2 to each B_n, with ε in the lemma replaced by ε_n. We denote the sequence of open discs obtained from the lemma by $(D_{n,m})_{m\in\mathbb{N}}$, and we denote the radius of $D_{n,m}$ by $r_{n,m}$. Note that for each $n \in \mathbb{N}$ we have $\sum_{m=1}^{\infty} r_{n,m} < \varepsilon_n$. Set

$$X = \overline{\Delta} \setminus \bigcup_{n,m\in\mathbb{N}} D_{n,m} .$$

We have

$$\sum_{n,m\in\mathbb{N}} r_{n,m} < \sum_{n\in\mathbb{N}} \varepsilon_n < \infty ,$$

so the sum of the radii of the deleted discs is finite.

Let $z \in K$, and let f be a rational function with no poles in X. For $n, m \in \mathbb{N}$, let $s_{n,m}$ be the distance from z to $D_{n,m}$. Note that $s_{n,m} \geq d_n$, since $D_{n,m} \subseteq B_n$. Let d_0 be the distance from K to the unit circle. For each non-negative integer k, by Lemma 3.3.4, we have

$$\begin{aligned} \left|f^{(k)}(z)\right| &\leq k!\left(\frac{1}{(1-|z|)^{k+1}} + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{r_{n,m}}{s_{n,m}^{k+1}}\right)\|f\|_X \\ &\leq k!\left(\frac{1}{d_0^{k+1}} + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{r_{n,m}}{d_n^{k+1}}\right)\|f\|_X \\ &\leq k!\left(\frac{1}{d_0^{k+1}} + \sum_{n=1}^{\infty}\frac{\varepsilon_n}{d_n^{k+1}}\right)\|f\|_X \\ &\leq k!\left(\frac{1}{d_0^{k+1}} + (\log(k+3))^k\right)\|f\|_X\,, \end{aligned} \tag{3.4}$$

by (3.3). Let I be a closed line segment in K, and let $f \in R(X)$. Since f can be uniformly approximated by functions in $R_0(X)$, and each function in $R_0(X)$ satisfies inequality (3.4), it follows that f is infinitely differentiable on I and, for each non-negative integer k, we have

$$\|f^{(k)}\|_I \leq k!\left(\frac{1}{d_0^{k+1}} + (\log(k+3))^k\right)\|f\|_X.$$

Now we pick $N \in \mathbb{N}$ large enough such that $(\log(k+3))^k \geq 1/d_0^{k+1}$ for all $k \geq N$. Then we have

$$\begin{aligned} \sum_{k=1}^{\infty}\frac{1}{\|f^{(k)}\|_I^{1/k}} &\geq \sum_{k=N}^{\infty}\frac{1}{\|f^{(k)}\|_I^{1/k}} \\ &\geq \sum_{k=N}^{\infty}\frac{1}{(2\|f\|_X)^{1/k}(k!)^{1/k}\log(k+3)} \\ &\geq \sum_{k=N}^{\infty}\frac{1}{(2\|f\|_X)^{1/k}k\log(k+3)} = \infty. \end{aligned}$$

Thus f satisfies condition (3.2) on I.

Let $z \in X \setminus K$. We show that z is a point of continuity for $R(X)$. Let $w \in X$ with $w \neq z$. Then we can find some B_n with $z \in B_n$ and $w \in X \setminus \bar{B}_n$. By Lemma 3.3.2, we can find $f \in R(X)$ with $f(z) \neq 0$ and such that f vanishes on the whole of $X \setminus \bar{B}_n$. Then $f \in J_w \setminus M_z$, and so $J_w \not\subseteq M_z$. Thus z is a point of continuity for $R(X)$.

Since every point of $X \setminus K$ is a point of continuity for $R(X)$, by Lemma 3.2.1 all of these points are also R-points for $R(X)$. □

Next we give a corollary to show that, for suitable K, we can eliminate all the non-trivial Jensen measures for $R(X)$, where X is as constructed in the above theorem.

Corollary 3.3.6. *Let K be a compact subset of Δ. Suppose that* $\operatorname{int} K = \emptyset$*, $\mathbb{C} \setminus K$ is connected, and K contains at least one closed line segment. Let X be the Swiss cheese set constructed in Theorem 3.3.5, and let I be a line segment contained in K. Then $R(X)$ has no non-trivial Jensen measures, but no point of I is an R-point or a point of continuity for $R(X)$, and so $R(X)$ is not regular.*

Proof. Let I be a closed line segment in K. By Lemma 3.3.3 and Theorem 3.3.5, if $f \in R(X)$ vanishes on a neighbourhood of a point x in I, then f vanishes on I. Thus no point of I can be an R-point or a point of continuity.

By Lemma 3.1.2 there are no non-trivial Jensen measures for points in $X \setminus K$. It remains to show that for each $z \in K$ there are no non-trivial Jensen measures for z. Recall that every point of $X \setminus K$ is an R-point for $R(X)$.

Let $z \in K$. Suppose, for a contradiction, that μ is a non-trivial Jensen measure for z with closed support F. By Mergelyan's theorem, we have $R(K) = C(K)$, and so, by Lemma 3.3.1, F is not contained in K. Let w be a point in $F \setminus K$. Then w is an R-point for $R(X)$. However, this contradicts Lemma 3.1.2. The result follows. □

We remark that [23, Corollary 6] is now the special case of Corollary 3.3.6 where we take K to be the closed line segment $[-1/2, 1/2]$. We also remark that an alternative approach to the last part of our proof is to note that the fine interior of K must be empty, and then use the approach from [23]. Our approach appears to be more elementary, although Lemma 3.2.1 uses [31, Theorem 3.2], and that result uses the Shilov Idempotent Theorem. However, for compact plane sets X with empty interior, the special case of [31, Theorem 3.2] for $R(X)$ can be proved without appealing to the Shilov Idempotent Theorem.

Finally we give a concrete example where K has positive area and empty interior, and such that the Swiss cheese set X constructed in Theorem 3.3.5 has the property that the set of points of continuity for $R(X)$ is precisely $X \setminus K$, and this is also equal to the set of R-points for $R(X)$. For this purpose, we use *fat Cantor* sets. A fat Cantor set, also known as a *Smith-Volterra-Cantor* set, is a compact subset of $\mathbb{R}$ that is nowhere dense and has positive length ([1]).

The Cartesian product of a fat Cantor set and a closed interval can be regarded as a compact subset K of $\mathbb{C}$ with positive area such that $\mathbb{C} \setminus K$ is connected and K has no interior points. Moreover, K is the union of the closed line segments which are contained in K.

The following Theorem is now an immediate consequence of Theorem 3.3.5 and Corollary 3.3.6.

Theorem 3.3.7. *Let $F \subseteq [-1/2, 1/2]$ be a fat Cantor set, and set*

$$K = \{z \in \mathbb{C} : \mathrm{Re}(z) \in F, \mathrm{Im}(z) \in [-1/2, 1/2]\}.$$

Then there is a Swiss cheese set X with $K \subseteq X$ such that every point of $X \setminus K$ is a point of continuity and an R-point for $R(X)$, every point in K is a non R-point and a point of discontinuity, and $R(X)$ has no non-trivial Jensen measures.

3.4 An open question

We end this chapter by asking whether Lemma 3.2.1 remains true if we interchange *R-point* with *point of continuity*.

Question 3.4.1. *Let X be a compact Hausdorff space, let A be a natural Banach function algebra on X, and let $x \in X$. Suppose that there is an open neighbourhood U of x such that each point in $U \setminus \{x\}$ is an R-point for A. Is x necessarily a point of continuity for A?*

We remark that if the algebra A is local or even 2-local (for the definitions, see page 81), then question 3.4.1 has a positive answer. This is because by [31, Proposition 4.1] the set $F_x = \{y \in X : J_y \subseteq M_x\}$ is connected if A is 2-local. If each $y \in U \setminus \{x\}$ is an R-point for A, then $y \notin F_x$, and the connectedness of F_x forces $F_x = \{x\}$, which is equivalent to x being a point of continuity. In particular, the answer to question 3.4.1 is positive for $R(X)$.

Chapter 4

Uniform algebras and small exceptional sets

This chapter can be divided into three main parts. In the first part (Section 4.1), we study normal uniform algebras on the unit interval $\mathbb{I}$ or the unit circle $\mathbb{T}$. We first give a brief survey of the study of uniform algebras on $\mathbb{I}$ or $\mathbb{T}$. Then we focus on the case when the uniform algebra is also assumed to be normal. In Theorems 4.1.5 and 4.1.9 we characterise all the closed antisymmetric subsets for normal uniform algebras on $\mathbb{I}$ or $\mathbb{T}$, respectively.

The second part of this chapter is inspired by the study of "small exception sets" for uniform algebras. Let A be a uniform algebra on X, and let K be a closed subset of X. Assume that for each closed subset $F \subseteq X$ disjoint from K we have $A|_F = C(F)$; what "smallness" condition on K can we impose to guarantee that $A = C(X)$? One such "smallness" condition, as inspired by a result of Rudin [57], is that K has no nonempty perfect subsets (defined later). This is proved in Proposition 4.2.2. We note that the formulation and the proof of this proposition is joint work of the author of this thesis and J. Feinstein. This forms the content of Section 4.2. We also note that after we have given the formulation and proof of this proposition, we realised that H. Ishikawa, J. Tomiyama and J. Wada have given the same formulation of this proposition in [42]. However, their proof uses a different approach. In Section 4.3 we discuss the situation when B is a uniform algebra containing A, such that for each proper closed subset $K \subseteq X$ we have $A|_K = B|_K$. We do not know the full answer to whether B is equal to A, even when we assume that A is natural and B is generated by A and one more function. We will discuss conditions which will guarantee that $A = B$, and we will also discuss some related questions.

The last part of this chapter is related to Wermer's maximality theorem (cf. Section 1.7). In Section 4.4 we give an example of a proper closed linear subspace of $C(\mathbb{T})$ which properly contains the disc algebra on $\mathbb{T}$ and which is $\|\cdot\|_2$ dense in $C(\mathbb{T})$. We end this chapter with two open questions.

4.1 Normal uniform algebras on the unit circle or the unit interval

We first introduce some definitions. Recall that the ideals J_x and M_x are introduced in Section 3.1.

Definition 4.1.1. Let A be a uniform algebra on X. We say A is *normal on X*, if for each pair of disjoint compact subsets $K, F \subseteq X$ there exists a function $f \in A$ with $f(K) \subseteq \{0\}$ and $f(F) \subseteq \{1\}$. We say A is *strongly regular on X*, if for each $x \in X$ we have $\overline{J_x} = M_x$.

It is standard that if A is normal on X, then A is actually natural on X (see [61, Theorem 27.3]). It is clear from the definition that if A is normal on X, then A is also regular on X. On the other hand, if A is natural on X and regular, then A is actually normal on X, see [61, Theorem 27.2]. But there exists a uniform algebra A on a compact Hausdorff space X which is regular on X but which is not natural on X, see [39, Chapter 10]. Thus, in particular, A is regular on X but not normal on X. It is shown by Wilken in [70] that if A is strongly regular on X, then A is normal on X, and hence also natural on X.

Definition 4.1.2. Let A be a uniform algebra on X. If for each $x \in X$, for each closed subset $K \subseteq X \setminus \{x\}$ and for each $\varepsilon > 0$, there exists $f \in A$ such that $|f(x)| < \varepsilon$ and $\|1 - f\|_K < \varepsilon$, then we say A is *approximately regular on X*. If for each pair of disjoint compact subsets $F, K \subseteq X$ and for each $\varepsilon > 0$, there exists $f \in A$ such that $\|f\|_F < \varepsilon$ and $\|1 - f\|_K < \varepsilon$, then we say A is *approximately normal on X*.

Note that unlike normality, a uniform algebra A can be approximately normal on X without being natural on X. For such an example, consider the disc algebra A on the unit circle (note that A is pervasive on the unit circle). If A is a natural uniform algebra on X and A is approximately regular on X, then A is approximately normal on X (see [69, Lemma 2.2]).

Definition 4.1.3. Let A be a uniform algebra on X, and let $f \in C(X)$. For each $x \in X$, we say f is *locally in A on X at x* if there exists a neighbourhood

U of x and a function $g \in A$ such that $f|_U = g|_U$; we say f is *approximately locally in A on X at x* if there exists a neighbourhood U of x such that $f|_U$ can be uniformly approximated by functions in A. We say f is *locally in A on X* if f is locally in A on X at each point of x; we say f is *approximately locally in A on X* if f is approximately locally in A on X at each point of x.

A concept related to locally in (approximately locally in) a uniform algebra is the following.

Definition 4.1.4. Let A be a uniform algebra on X. We say A is a *local uniform algebra on X* if A contains all functions in $C(X)$ which are locally in A on X; we say A is an *approximately local uniform algebra on X* if A contains all functions in $C(X)$ which are approximately locally in A on X. We say A is a *2-local uniform algebra on X* if A contains all functions $f \in C(X)$ for which there are open sets U_1 and U_2 (which may depend on f) in X with $X = U_1 \cup U_2$ and functions g_1 and g_2 in A with $f|_{U_i} = g|_{U_i}$, $i = 1, 2$.

In [69], Wilken proved that normal uniform algebras are approximately local uniform algebras. In the same paper, Wilken also proved that if A is approximately normal on X, then A is a 2-local uniform algebra on X. For any compact plane set X, the uniform algebra $R(X)$ is approximately local on X (see [9, p. 170]). It was once conjectured that every uniform algebra is a local uniform algebra, but Kallin in [43] gave a counter-example to this conjecture.

Let $\mathbb{I}$ be the closed unit interval $[0, 1]$ in $\mathbb{C}$, and let $\mathbb{T}$ be the unit circle in $\mathbb{C}$. A famous open question of Gelfand is the following: does there exist a non-trivial, natural uniform algebra on $\mathbb{I}$ or on $\mathbb{T}$?

An easy example of a uniform algebra on $\mathbb{T}$ is the disc algebra on $\mathbb{T}$ as defined in Section 1.7, whose maximal ideal space can be regarded as the closed unit disc. On the other hand, constructing non-trivial uniform algebras on $\mathbb{I}$ is hard. One way to do so is to use Wermer's example of a non-trivial uniform algebra B on the Cantor set C described in [66]. Now let A be the collection of continuous functions on $\mathbb{I}$ whose restrictions to C are in B. Then A is a non-trivial uniform algebra on $\mathbb{I}$, whose maximal ideal space contains a copy of the Riemann sphere $\mathbb{C} \cup \{\infty\}$. Note that in this example, the uniform algebra A is not essential on $\mathbb{I}$. An example of a non-trivial, essential uniform algebra A on $\mathbb{I}$ with three generators can be found in [46, p. 196], where the character space of A can be identified with the Riemann sphere. We note that for each non-constant function $g \in A$ in this example, the interior of $g(\mathbb{I})$ is non-empty.

One way to try to tackle the Gelfand problem is to study regularity conditions for natural uniform algebras on $\mathbb{I}$ or on $\mathbb{T}$. In [69], Wilken proved that every natural uniform algebra on $\mathbb{I}$ or on $\mathbb{T}$ must be approximately normal. In [63], Wells followed Wilken's approach and proved that every natural uniform algebra on $\mathbb{I}$ is a local uniform algebra. On the other hand, Wilken proved in [70] that if A is a strongly regular uniform algebra on $\mathbb{I}$, then $A = C(\mathbb{I})$. Later Wells proved in [63] that if A is a strongly regular uniform algebra on $\mathbb{T}$, then $A = C(\mathbb{T})$. This result was also obtained by Chalice in [11]. Further results on uniform algebras on $\mathbb{I}$ or $\mathbb{T}$ which are strongly regular except on small exceptional sets can be found in [30]. For other results concerning some more subtle regularity conditions for uniform algebras on $\mathbb{I}$ or $\mathbb{T}$, we refer the reader to [64].

Another approach to try to tackle the Gelfand problem, for the closed unit interval case, is to study the invertible group of a natural uniform algebra on $\mathbb{I}$. It is an interesting observation made in [18] that if A is a natural uniform algebra on $\mathbb{I}$ with dense invertible group, then $A = C(\mathbb{I})$. In [17, Proposition 4.3.2.4] Dawson showed that if A is a normal uniform algebra on $\mathbb{T}$ such that A has dense invertible group, then $A = C(\mathbb{T})$. For further results on Banach function algebras with dense invertible groups, we refer the reader to [15].

We will study natural uniform algebras on $\mathbb{I}$ or $\mathbb{T}$ by imposing regularity conditions on the uniform algebras. Since every natural uniform algebra on $\mathbb{I}$ or $\mathbb{T}$ is approximately normal, and every strongly regular uniform algebra on $\mathbb{I}$ or $\mathbb{T}$ is trivial, a natural way to start our investigation is to assume that the uniform algebra on $\mathbb{I}$ or $\mathbb{T}$ is normal. We will identify in Theorem 4.1.5 and Theorem 4.1.9 all antisymmetric subsets for A when A is a normal uniform algebra on $\mathbb{I}$ or on $\mathbb{T}$.

We say a nonempty closed subinterval of $\mathbb{I}$ or a closed subarc of $\mathbb{T}$ is *degenerate* if the interval or the arc is a one point set (we regard $\mathbb{T}$ as a closed subarc of itself). Let z and w be points in $\mathbb{T}$. By $\operatorname{arc}(z, w)$ we mean the closed subarc of $\mathbb{T}$ from z to w in the counterclockwise direction. We note that if A is a natural uniform algebra on $\mathbb{I}$, then each maximal antisymmetric subset of A is either a single point or a non-degenerate closed subinterval. Similarly, if A is a natural uniform algebra on $\mathbb{T}$, then each maximal antisymmetric subset of A is either a single point or a non-degenerate closed subarc. See Remark 4.1.6 for the meaning of classification of all closed antisymmetric subsets of a normal uniform algebra on $\mathbb{I}$.

Theorem 4.1.5. *Let A be a normal uniform algebra on $\mathbb{I}$, and let $\{F_\alpha\}$ be the collection of maximal antisymmetric subsets for A. For a fixed α, let K be a non-empty closed subinterval of F_α (possibly degenerate). Then A_K is an* *antisymmetric uniform algebra on K. In particular, K is an antisymmetric subset for A.*

Proof. We notice first that since A is normal on $\mathbb{I}$, A is an approximately local uniform algebra on $\mathbb{I}$.

Since A is normal, we see that each closed antisymmetric subset for A is a closed connected subset of $\mathbb{I}$. Therefore F_α is either a single point set, in which case the theorem is trivial, or F_α is a non-degenerate closed subinterval of $\mathbb{I}$. By considering the restriction of A to F_α, we may assume that A is actually an antisymmetric, normal uniform algebra on $\mathbb{I}$. Now let K be a closed subinterval of $\mathbb{I}$.

Assume for a contradiction that A_K is not antisymmetric. Let

$$K = \left(\bigcup_i K_i\right) \cup P$$

be the maximal antisymmetric decomposition for A_K, where P denotes the union of maximal antisymmetric subsets which consist of a single point.

If P has empty interior, we can find maximal antisymmetric subsets $K_i = [a_i, b_i]$, $i = 1, 2$, with $a_i < b_i$ and $b_1 < a_2$. Set $E = [a_1, b_2]$, then E is not a set of antisymmetry for A_K. Therefore we can find $f \in A_K$, such that f takes non-constant real values on E, and such that $f|_{[a_i,b_i]}$ takes constant values for $i = 1, 2$. Now select $c_i \in (a_i, b_i)$ for $i = 1, 2$, then for each $x \in \mathbb{I}$ either $[0, c_1]$, $[c_2, 1]$ or E is a neighbourhood of x. The function

$$g(x) = \begin{cases} f(x) & \text{if } x \in E, \\ f(c_1) & \text{if } x \in [0, c_1], \\ f(c_2) & \text{if } x \in [c_2, 1], \end{cases}$$

is well defined, continuous and takes non-constant real values. We see that on each of $[0, c_1]$, $[c_2, 1]$ and E, the function g can be uniformly approximated by functions in A. Since A is approximately local we conclude that g is in A. This leads to a contradiction, since g is a non-constant real valued function in A, but we assumed that A is antisymmetric.

The case when P has non-empty interior is handled similarly. Let $[a, b]$ be a non-degenerate closed interval contained in $\operatorname{int} P$. Take real numbers c, d such that $a < c < (a + b)/2$ and $(a + b)/2 < d < b$. Then we can find a function $f \in C(K)$ such that f is real valued, $f((a + b)/2) = 1$, and $f|_{K\setminus[c,d]} = 0$. By

Proposition 1.4.3 we see that f is actually in A_K. The function

$$g(x) = \begin{cases} f(x) & \text{if } x \in [a, b], \\ 0 & \text{if } x \in [0, c], \\ 0 & \text{if } x \in [d, 1], \end{cases}$$

is well defined and continuous on $\mathbb{I}$. We see that g is in A, since on both $[0, c] \cup [d, 1]$ and $[a, b]$ the function g can be uniformly approximated by functions in A. This leads to a contradiction, since g is a non-constant real valued function in A, and we assumed that A is antisymmetric. □

Remark 4.1.6. It is an open question whether there exists a non-trivial normal uniform algebra on $\mathbb{I}$. If there does exist such a uniform algebra A, we can assume without loss of generality that A is antisymmetric on $\mathbb{I}$ by restricting A to a maximal antisymmetric subset containing more than one point (such a maximal antisymmetric subset is necessarily a non-degenerate closed subinterval of $\mathbb{I}$). In this case, the above theorem asserts that closed antisymmetric subsets for A, when A is normal and antisymmetric, are all closed subintervals of $\mathbb{I}$.

Before we proceed to give an analogous result for normal uniform algebras on $\mathbb{T}$, we first introduce a construction that is interesting in its own right. The construction of R, Y and the isomorphism between R and $C(Y)$ described in the following theorem was mentioned briefly in the proof of Theorem 3.2 in [40], but no details were given. In the following proposition, we give full details of the construction, and we introduce a new map φ, which will be used later.

Proposition 4.1.7. *Let A be a natural uniform algebra on X, and let R be the closed subalgebra of A containing all functions f in A whose complex conjugate $\overline{f}$ is also in A. Then via the Gelfand transform $\mathcal{G} : R \to \hat{R}$, R can be regarded as $C(Y)$, for some nonempty compact Hausdorff space Y, and there exists a continuous map $\varphi : X \to Y$ such that $f(x) = \hat{f}(\varphi(x))$ for all $f \in R$.*

Proof. It is clear that with the sup-norm on X, the algebra R is a Banach algebra, and we denote its character space by Y. It is standard that Y is a nonempty compact Hausdorff space, see Proposition 1.2.4. We note that R is also semisimple, for the intersection of the kernels of the evaluation characters on R at points in X contains only the zero element in R. Therefore via the Gelfand transform $\mathcal{G}$, R can be regarded as a Banach function algebra on Y. Since $\|a\|^2 = \|a^2\|$ for all $a \in R$, where $\|\cdot\|$ is the uniform norm on X, by [9, pp. 28] we see that via the Gelfand transform R can be regarded as a uniform algebra on Y.

We claim that $\hat{R}$ is a self adjoint uniform algebra on Y. This fact can be deduced from the Gelfand-Naimark theorem ([2, p. 270]), but here we supply a more elementary argument. Let $f \in R$ be a real valued function. We claim that $\hat{f}$ is real valued. Let ψ be a character on R. Assume towards a contradiction that $\psi(f)$ is not a real number. By subtracting some real constant from some real multiple of f, we may assume that $\psi(f) = i$. Then $\psi(f^2) = \psi(f)^2 = -1$, and so $\psi(f^2+1) = 0$. But f^2+1 is a strictly positive function on X, and it is in A. Since A is natural on X, we conclude that f^2+1 is invertible in A, and its inverse, as a real valued function on X, is in R. Thus f^2+1 is invertible in R. This contradicts the fact that $\psi(f^2+1) = 0$. Since ψ is arbitrary, $\hat{f}$ is real-valued, as claimed. Now let $f = f_1 + if_2 \in R$, with f_1 and f_2 the real and imaginary parts of f respectively. Note that f_1 and f_2 are also in R and thus in A, and they are real-valued. Then we have

$$\overline{\hat{f}} = \overline{\widehat{f_1 + if_2}} = \overline{\hat{f_1} + i\hat{f_2}} = \hat{f_1} - i\hat{f_2} = \hat{\overline{f}},$$

which shows that $\overline{\hat{f}}$ is also in $\hat{R}$ whenever f is in R.

Since $\hat{R}$ is a self adjoint uniform algebra on Y, we conclude that $\hat{R} = C(Y)$. Now we define a map $\varphi : X \to Y$ in the following way. For each $x \in X$, we have the evaluation character ε_x on A, whose restriction to R, denoted by $\varepsilon_x|_R$, is a character on R. Therefore there exists a unique $y \in Y$ such that $\varepsilon_x|_R(f) = \hat{f}(y)$ for all $f \in R$. Then we set $\varphi(x) = y$. The map $\varphi : X \to Y$ is continuous, because the map $x \mapsto \varepsilon_x$ and the map $\varepsilon_x|_R \mapsto y$ are both continuous maps, see for example [2, Lemma 4.55]. Therefore we have obtained a continuous map $\varphi : X \to Y$ satisfying $f(x) = \hat{f}(\varphi(x))$ for all $f \in R$. □

Corollary 4.1.8. *With notation as in Proposition 4.1.7, the map φ has the following property: whenever $x, y \in X$ are in the same maximal antisymmetric subset for A, then $\varphi(x) = \varphi(y)$.*

Proof. First we notice that if $K \subseteq X$ is a maximal antisymmetric subset for A, and if f is a function in R, then the restriction of f to K is a constant function. This is because the real and imaginary parts of f are also in R, and so their restrictions to K are constant functions.

Now whenever x and y are in the same maximal antisymmetric subset for A, we have $f(x) = f(y)$ for all f in R. Therefore the evaluation characters on R induced by x and y are the same. Therefore $\varphi(x) = \varphi(y)$. □

We are now ready to give the analogous result of Theorem 4.1.5 for normal uniform algebras on $\mathbb{T}$.

Theorem 4.1.9. *Let A be a normal uniform algebra on $\mathbb{T}$, and let $\{F_\alpha\}$ be the collection of maximal antisymmetric subsets for A. For a fixed α, let K be a non-empty closed subarc of F_α (possibly degenerate). Then A_K is an antisymmetric uniform algebra on K. In particular, K is an antisymmetric subset for A.*

Proof. If F_α is a proper subarc of $\mathbb{T}$, then F_α is homeomorphic to $\mathbb{I}$. Theorem 4.1.5 then implies that for each closed subarc K in F_α, A_K is an antisymmetric uniform algebra on K.

So now assume that $F_\alpha = \mathbb{T}$, that is, A is an antisymmetric uniform algebra on $\mathbb{T}$. Let $K = \operatorname{arc}(z_1, z_2)$ be a proper non-degenerate closed subarc of $\mathbb{T}$, with endpoints z_1 and z_2. Assume for a contradiction that A_K is not antisymmetric. Let $f \in A_K$ be a non-constant real valued function. Then we can find a non-degenerate proper subarc S of K with one of its end points z_3 such that S is a maximal antisymmetric subset for A_K, and such that $f(z_1) \neq f(z_3)$. If $f(z_1) \neq f(z_2)$, let $g = f$. Otherwise, we define a non-constant real valued continuous function on K by

$$g(z) = \begin{cases} f(z) & \text{if } z \in \operatorname{arc}(z_1, z_3), \\ f(z_3) & \text{if } x \in \operatorname{arc}(z_3, z_2). \end{cases}$$

It is clear that the restriction of g to each maximal antisymmetric subset for A_K is a constant function. Therefore g is actually in A_K, and $g(z_1) \neq g(z_2)$.

Now we apply Proposition 4.1.7 on A_K to construct the corresponding R, Y and $\varphi : K \to Y$. Note that since K is connected and φ is continuous, Y is also connected. Since g is in R and $g(z_1) \neq g(z_2)$, we see that $\varphi(z_1) \neq \varphi(z_2)$. Note that Y is compact Hausdorff. Let U and V be disjoint open neighbourhoods of $\varphi(z_1)$ and $\varphi(z_2)$ in Y. Since $\hat{R} = C(Y)$, we can find a non-constant real valued function h in R, such that $\hat{h}(U) \subseteq \{0\}$ and $\hat{h}(V) \subseteq \{0\}$. Now $\varphi^{-1}(U)$ and $\varphi^{-1}(V)$ are disjoint open neighbourhoods of z_1 and z_2 in K respectively, and the restriction of h to the union of $\varphi^{-1}(U)$ and $\varphi^{-1}(V)$ is the zero function. Define ℓ on $\mathbb{T}$ by

$$\ell(z) = \begin{cases} h(z) & \text{if } z \in K, \\ 0 & \text{otherwise.} \end{cases}$$

Then ℓ is a continuous, non-constant real valued function on $\mathbb{T}$. It can be seen that ℓ is approximately locally in A on $\mathbb{T}$. Since A is normal, we conclude that ℓ is actually in A. This contradicts the assumption that A is an antisymmetric uniform algebra on $\mathbb{T}$. Therefore we conclude that A_K is an antisymmetric uniform algebra on K. □

4.2 Small exceptional sets for uniform algebras

In this section we study uniform algebras on compact Hausdorff spaces which, except on "small" exceptional sets, contain all continuous functions.

Definition 4.2.1. A non-empty subset of a topological space is a *perfect subset* if it is closed and has no isolated points.

It is a famous result of Rudin [57] that if X is a non-empty compact Hausdorff space which has no non-empty perfect subsets, then every uniform algebra on X is trivial.

The following proposition from [42] establishes a condition for the smallness of exceptional sets for a uniform algebra. In [42] the authors studied ω-interpolating sets (see the cited paper for the definition) and essential sets of uniform algebras. In joint work, J. Feinstein and the author of this thesis discovered this proposition independently. However, our proof of this proposition is different from the one given in [42]. Our approach uses the antisymmetric decomposition for uniform algebras and properties of regularity points for uniform algebras.

Proposition 4.2.2. *Let A be a uniform algebra on X and let K be a closed subset of X which has no non-empty perfect subsets. Assume that for each closed subset $F \subseteq X$ disjoint from K we have $A|_F = C(F)$. Then $A = C(X)$.*

Proof. Assume for a contradiction that $A \neq C(X)$. Let

$$X = (\cup_\alpha K_\alpha) \cup P$$

be the antisymmetric decomposition for A, where $\{K_\alpha\}$ is the collection of all maximal antisymmetric subsets for A containing more than one point, and P is the union of all maximal antisymmetric subsets containing a single point. By our assumption, the collection $\{K_\alpha\}$ is not empty. If K is contained in P, then each K_α is disjoint from K, and by the conditions of this proposition we know that $A|_{K_\alpha} = C(K_\alpha)$. By Proposition 1.4.3 we conclude that $A = C(X)$, which is a contradiction. Therefore K is not contained in P.

Now let K_α be a maximal antisymmetric subset for A containing more than one point, such that $K \cap K_\alpha \neq \emptyset$. Then $K \cap K_\alpha$ is a closed subset of K_α with no non-empty perfect subset. Meanwhile, if F is a closed subset in K_α disjoint from $K \cap K_\alpha$, then from the conditions of the proposition we know that $A|_F = C(F)$. By restricting A to K_α, we can assume that A is antisymmetric without loss of generality.

So now assume that A is antisymmetric on X. Suppose $f \in A$ vanishes on an open neighbourhood U of K. Since $A|_{U^c} = C(U^c)$, there exists $g \in A$ with $g|_{U^c} = \overline{f}|_{U^c}$. Then fg is a real-valued function in A that vanishes on U. Since A is antisymmetric, this shows that f is the zero function. Therefore any function in A vanishing on a neighbourhood of K vanishes identically on X.

Let $y \in X \setminus K$, and denote

$$F_y = \{w \in X : M_y \supseteq J_w\}.$$

It is clear that if $w \neq y$ with $w \in X \setminus K$, then $w \notin F_y$. So $\{y\} \cup K \supseteq F_y$. By Theorem 4.6 in [31] we know that either $F_y = \{y\}$, or F_y contains a non-empty perfect subset. Since $\{y\} \cup K$ has no non-empty perfect subset, this forces $F_y = \{y\}$. Let $x \in K$. Since $x \notin F_y$, we can find a function $g \in A$ that vanishes on a neighbourhood of x and $g(y) = 1$. By compactness, there exists $f \in A$ that vanishes on a neighbourhood of K and $f(y) = 1$, which contradicts the fact that f must be the zero function as argued in the previous paragraph. This finishes the proof and shows that $A = C(X)$. □

We now show that the technique used in the proof of Proposition 4.2.2 can also be used to establish a known result concerning pervasive algebras (see [40]). We first introduce a definition.

Definition 4.2.3. Let A be a uniform algebra on X. We say that A is an *analytic uniform algebra* on X if every function in A which vanishes on a non-empty open subset of X vanishes identically on X.

Proposition 4.2.4. *Let A be a proper pervasive uniform algebra on X, then A is analytic on X.*

Proof. As noted on page 15 we know that a proper pervasive uniform algebra is antisymmetric. Let U be a proper non-empty open subset of X, and assume $f \in A$ vanishes on U. Since $A_{U^c} = C(U^c)$, we can find a sequence of functions (g_n) in A such that $\|g_n - \overline{f}\|_{U^c} \to 0$. Thus fg_n is a sequence of functions in A that converges uniformly on X to $f\overline{f}$. Therefore the real valued function $f\overline{f}$ is in A. Since A is antisymmetric, $f\overline{f}$ must be the zero function. This shows that A is analytic. □

For Banach function algebras we have the following analogous proposition, which is an easy consequence of [3, Theorem 2.1], or is an easy consequence of the main theorem in [55].

Proposition 4.2.5. *Let A be a Banach function algebra on X such that, for each proper closed subset K of X, we have $A|_K = C(K)$. Then $A = C(X)$.*

The following example adapted from Example 4.1.46 in [14] (see also [22]) shows that Proposition 4.2.2 does not hold for Banach function algebras, even when the Banach function algebra is natural and the exceptional set consists of only one point. (Note that in Proposition 4.2.2 we do not require that the uniform algebra is natural.)

Example 4.2.6. Let X denote the one point compactification of $\mathbb{N}$. Let A denote the collection of continuous functions on X defined by

$$A = \{f \in C(X) : \sum_{n=1}^{\infty} |f(2n) - f(2n-1)| < \infty\}.$$

It is clear that A is not $C(X)$. Elementary verifications show that A is a Banach space of continuous functions on X with the norm

$$\|f\| = \|f\|_X + \sum_{n=1}^{\infty} |f(2n) - f(2n-1)|, \quad f \in A.$$

In Example 4.1.46 in [14] (and also in [22]) it is stated without detail that A is a natural Banach function algebra. We include here a proof of these facts for the convenience of the reader. To show that A is a Banach function algebra, it is sufficient to show that A is actually an algebra and $\|\cdot\|$ is an algebra norm. Let $f, g \in A$ such that $\|f\| = \|g\| = 1$. Note that then we have $\|f\|_X \leq 1$ and $\|g\|_X \leq 1$. Then we have

$$\begin{aligned}
\|fg\| &= \|fg\|_X + \sum_{n=1}^{\infty} |f(2n)g(2n) - f(2n-1)g(2n-1)| \\
&= \|fg\|_X + \sum_{n=1}^{\infty} |f(2n)(g(2n) - g(2n-1)) + g(2n-1)(f(2n) - f(2n-1))| \\
&\leq \|fg\|_X + \|f\|_X \sum_{n=1}^{\infty} |g(2n) - g(2n-1)| + \|g\|_X \sum_{n=1}^{\infty} |f(2n) - f(2n-1)| \\
&\leq \|f\|_X \left(\|g\|_X + \sum_{n=1}^{\infty} |g(2n) - g(2n-1)| \right) + \|g\|_X \sum_{n=1}^{\infty} |f(2n) - f(2n-1)| \\
&\leq \|f\|_X + \sum_{n=1}^{\infty} |f(2n) - f(2n-1)| \\
&= 1.
\end{aligned}$$

This shows that fg is also in A, and $\|\cdot\|$ is sub-multiplicative on A. Since it is clear that A separates the points of X and A contains the constant functions we conclude that A is a Banach function algebra on X.

Let $K = \{\infty\}$. Then it is clear that for each closed subset F disjoint from K the restriction of A to F contains all continuous functions.

We now give two methods to show that A is actually a natural Banach function algebra on X by invoking Propositions 1.3.5 or 1.3.6. Let $h \in C(X)$ and $\varepsilon > 0$. Then $(h(n))_{n=1}^{\infty}$ is a Cauchy sequence. So we can find $N \in \mathbb{N}$ such that

$$|h(n) - h(m)| < \varepsilon, \quad n, m \geq N.$$

Let $f \in C(X)$ be such that $f(i) = h(i)$ if $1 \leq i \leq N$ and $f(i) = h(N)$ if $i > N$ or $i = \infty$. Then it is clear that $f \in A$ and $\|f - h\|_X \leq \varepsilon$. Therefore the uniform closure of A is $C(X)$, which is a natural uniform algebra on X.

To apply Proposition 1.3.5, let $f \in A$ such that f has no zero on X. Let m be the minimum of the modulus of f. Then we have

$$\sum_{n=1}^{\infty} \left| \frac{1}{f(2n)} - \frac{1}{f(2n-1)} \right| \leq \frac{1}{m^2} \sum_{n=1}^{\infty} |f(2n) - f(2n-1)| < \infty.$$

This shows that $1/f$ is also in A. Thus by Proposition 1.3.5 we know that A is natural on X.

To apply Proposition 1.3.6, let $f \in A$ with $\|f\|_X = 1$. Then it is clear that

$$|f^n(2m) - f^n(2m-1)| \leq n\, |f(2m) - f(2m-1)|\,,$$

and so we have the estimate

$$\|f^n\| = \|f^n\|_X + \sum_{m=1}^{\infty} |f^n(2m) - f^n(2m-1)| \leq 1 + n\|f\|.$$

Thus we have the estimation

$$1 = \|f^n\|_X^{1/n} \leq \|f^n\|^{1/n} \leq (1 + n\|f\|)^{1/n} \to 1, \quad n \to \infty.$$

Then by Proposition 1.3.6 we know that A is natural on X.

Remark 4.2.7. We make the following remarks to Proposition 4.2.2.

1. The existence of proper pervasive uniform algebras (for example, the disc algebra on the unit circle) shows that the condition $A|_F = C(F)$ in Propo-

sition 4.2.2 cannot be weakened to the condition $A_F = C(F)$.

2. Proposition 4.2.2 fails if K contains non-empty perfect subsets. In fact, if K contains non-empty perfect subsets, then K contains a homeomorphic copy of the Cantor middle-third set ([51, Corollary 1A.3]). Denote this homeomorphic copy by L. Then by [66, Theorem 9.3], there exists a non-trivial uniform algebra B on L. Now define

$$A := \{f \in C(X) : f|_L \in B\}.$$

Then A is a non-trivial uniform algebra on X such that for each closed subset F disjoint from K, we have $A|_F = C(F)$.

4.3 Single function considerations

We first give a definition.

Definition 4.3.1. Let A be a natural uniform algebra on X. A closed subset K of X is a *local peak set (for A)* if there exists a function f in A and an open neighbourhood U of K, such that $f(K) = \{1\}$ and $|f(x)| < 1$ for all $x \in U \setminus K$.

It is a remarkable result that local peak sets are actually peak sets, see Theorem 8.1 in [33, Chapter 3]. This is called the *local peak set theorem* (for natural uniform algebras). One may also define local peak sets for general uniform algebras, but the analogous local peak set conjecture for general uniform algebras fails.

Let A be a natural uniform algebra on X, and let f be a continuous function on X. Assume that for each proper closed subset K of X we have $f|_K \in A|_K$. Does it follow that f is actually in A? Or equivalently, if we let B be the uniform algebra on X generated by functions in A and the function f (by which we mean the uniform closure of the polynomials in f with coefficients in A), does it follow that $A = B$? Note that the assumption on f shows that for each proper closed subset K of X we have $A|_K = B|_K$.

It is proved in [35] that the uniform algebras A and B share the same character space and the same Shilov boundary. Thus we know that B is also a natural uniform algebra on X. Let K be a proper peak set for B, and let $g \in B$ be a peaking function for K. Choose an open neighbourhood U of K such that its closure $\overline{U}$ is a proper subset of X. Then $g|_{\overline{U}}$ is in $A|_{\overline{U}}$, say $h \in A$ such that $h|_{\overline{U}} = g|_{\overline{U}}$. Then we see that $h(K) = \{1\}$ and $|h(x)| < 1$ for all $x \in U \setminus K$.

Thus by the local peak set theorem we know that K is also a peak set for A. This shows that A and B share the same peak sets.

Even though we do not know the full answer to the above question, there are some easy additional conditions that would guarantee that $A = B$. We make a list of these conditions here with some brief explanations.

(i) If A is not antisymmetric, then $A = B$. This is because on each maximal antisymmetric subset K for A we have $f|_K \in A|_K$, and this shows that f is in A by Proposition 1.4.3.

(ii) If A has a proper Shilov boundary $K \subsetneq X$, then $A = B$. This is because K is also the Shilov boundary of B and we have $A|_K = B|_K$. Note that each function in a uniform algebra is uniquely determined by its values on the Shilov boundary of the uniform algebra. Since $f|_K$ is in $A|_K$ there exists a function $h \in A$ such that $(f - h)|_K = 0$. Since $f - h$ is in B and it vanishes on the Shilov boundary K of B, we conclude that $h = f$. Therefore f is actually in A, which shows that $A = B$.

(iii) If B is a normal uniform algebra, then $A = B$. This is an easy consequence of the main theorem in [55].

(iv) If X is not connected, then $A = B$. Say X is not connected, and let K_1 and K_2 be a closed partition of X. Since A is natural on X, by Shilov's idempotent theorem ([2, Theorem 9.5]) the characteristic functions χ_{K_1} and χ_{K_2} are in A. It is clear that both $f\chi_{K_1}$ and $f\chi_{K_2}$ are in A, which implies that f is in A.

(v) Assume there exists a proper peak set K for A which has non-empty interior, then $A = B$. To see this, note that A is a subalgebra of B, so $A = B$ if and only if each annihilating measure for A is also an annihilating measure for B. Let μ be an annihilating measure for A. If the support of μ is a proper subset of X, then it is clear that μ is also an annihilating measure for B. Now assume the support of μ is the whole of X. By Theorem 2.4.9 in [9] we know that $\chi_K\mu$ and $(1-\chi_K)\mu$ are both annihilating measures for A, where χ_K is the characteristic function for K. Since K has non-empty interior, both $\chi_K\mu$ and $(1 - \chi_K)\mu$ have supports proper closed subsets of X. Therefore they are both annihilating measures for B, which implies that μ is also an annihilating measure for B.

Inspired by the above discussion, we consider the following question, which we resolve below.

Question 4.3.2. *Let A and B be natural uniform algebras on X with $A \subseteq B$. Suppose that A and B share the same peak sets. Assume that A is antisymmetric, and at least one proper peak set for A has non-empty interior. Is it necessarily true that $A = B$?*

First we note that if $B = C(X)$, then the answer to Question 4.3.2 is *yes*. This fact is observed in [9, p.114]. We give another situation where the answer to Question 4.3.2 is yes.

Theorem 4.3.3. *Let A and B be natural uniform algebras on X with $A \subseteq B$. Suppose that A and B share the same peak sets, and such that the restriction of A to its Shilov boundary is a proper maximal uniform algebra. Then $A = B$.*

Proof. Since A and B share the same peak sets on their character space X, we see that A and B share the same peak sets in the weak sense. In particular, A and B share the same collection of peak points in the weak sense. Thus A and B share the same Shilov boundary, which is the closure of the collection of peak points in the weak sense. Denote the Shilov boundary by S. Since $A|_S$ is a proper uniform algebra on S, by Proposition 1.5.2 we see that there exists a closed subset of S which is not a peak set in the weak sense for A. Therefore $B|_S$ is not $C(S)$, since not every closed subset of S is a peak set in the weak sense for B. By the maximality of $A|_S$ we conclude that $B|_S = A|_S$. Therefore we conclude that $A = B$. □

An example illustrating the above theorem would be the disc algebra. Let A be the disc algebra on $\overline{D}$. Then the Shilov boundary of A is the unit circle $\mathbb{T}$, and by Wermer's maximality theorem we known that $A|_{\mathbb{T}}$ is a maximal uniform algebra on $\mathbb{T}$. Therefore if B is a natural uniform algebra on $\overline{D}$ such that B contains the disc algebra, and such that each peak set for B is also a peak set for the disc algebra, then B is also the disc algebra.

In the rest of this section, we shall give an example to show that the answer to Question 4.3.2 is negative. First we give the following lemma. Note that the uniform algebra introduced in the next lemma is a standard example. This uniform algebra is the kernel of a point derivation on the disc algebra, see [9, Section 1-6]. We include a proof of the following lemma for the convenience of the reader.

Lemma 4.3.4. *Let A be the collection of continuous functions on the closed unit disc, whose restriction to the open unit disc is holomorphic, and whose derivative at 0 is 0. Then A is a natural uniform algebra on the closed unit disc.*

Proof. It is routine to verify that A is a uniform algebra. Let φ be a character on A. First notice that each function in A has the form

$$f(z) = f(0) + a_2 z^2 + a_3 z^3 + \cdots .$$

Therefore the effect of φ on $f \in A$ is totally determined by its effect on z^2 and z^3. Since $\varphi(z^6) = (\varphi(z^2))^3 = (\varphi(z^3))^2$, we conclude that if either $\varphi(z^2) = 0$ or $\varphi(z^3) = 0$, then both of them are 0, and so $\varphi = \varepsilon_0$.

Now assume neither of them is zero, and denote $\varphi(z^2) = r_0 e^{i\theta_0}$ and $\varphi(z^3) = r_1 e^{i\theta_1}$. Since $(\varphi(z^2))^3 = (\varphi(z^3))^2$, we can find $a > 0$, $\theta \in [-\pi, \pi)$ and $k \in \mathbb{N}$ such that $r_0 = a^2$ and $r_1 = a^3$, $\theta_0 = (1/3)\theta$ and $\theta_1 = (1/2)\theta + k\pi$. Set $z_0 = ae^{i\theta/6}$ if k is even, or $z_0 = -ae^{i\theta/6}$ if k is odd. Then $\varphi(z^2) = z_0^2$ and $\varphi(z^3) = z_0^3$. Since φ has norm 1, we see that in both cases z_0 lies in the closed unit disc. Therefore we conclude that the character φ on A is the evaluation of the point z_0 in the closed unit disc. This shows that A is natural on the closed unit disc. □

Lemma 4.3.5. *Let A be the uniform algebra introduced in the statement of Lemma 4.3.4, and let B be the disc algebra. Then A and B are natural uniform algebras on $\overline{D}$, such that $A \subseteq B$ and such that they share the same peak sets.*

Proof. It is clear that A is a subalgebra of B, and by Lemma 4.3.4 we know that A is natural on $\overline{D}$. It remains to show that they share the same peak sets.

Note that all functions in both A and B are holomorphic on D. We see that if E is a peak set for either A or B such that E has non-empty intersection with D, then $E = \overline{D}$. Now let E be a peak set for B contained in $\mathbb{T}$. It is standard (see [39, p. 52]) that E has zero Lebesgue measure on the circle. To show that E is also a peak set for A, by Theorem 2.4.9 in [9] it is sufficient to show that $\chi_E \mu$ is an annihilating measure for A whenever μ is an annihilating measure for A supported on $\mathbb{T}$. So let μ be an annihilating measure for A supported on $\mathbb{T}$. Then $z^2\,\mathrm{d}\mu$ is an annihilating measure for B supported on $\mathbb{T}$, and by the F. and M. Riesz theorem ([39, p. 47]) we know that $z^2\,\mathrm{d}\mu$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{T}$. So in particular μ is absolutely continuous with respect to the Lebesgue measure. Therefore $\chi_E \mu$ is the zero measure, which obviously annihilates A. This shows that E is also a peak set for A and finishes the proof. □

As noted in Section 1.7, a proper closed subset K in $\mathbb{T}$ is a peak set for the disc algebra if and only if K has Lebesgue measure 0. Therefore we have the following corollary.

Corollary 4.3.6. *Let A be the uniform algebra introduced in the statement of Lemma 4.3.4. Then a proper closed subset K of $\mathbb{T}$ is a peak set for A if and only if K has Lebesgue measure* 0.

Finally we are ready to show that the answer to Question 4.3.2 is negative.

Theorem 4.3.7. *Let X be the union of the closed unit disc and the closed disc with centre at* 2 *and with radius* 1 *in $\mathbb{C}$. Then there exists a pair of natural uniform algebras A and B on X sharing the same peak sets, such that A is properly contained in B, both A and B are antisymmetric, and such that one peak set for A has non-empty interior.*

Proof. Let $\overline{D}$ denote the closed unit disc, and let $\bar{\mathrm{B}}(2,1)$ denote the closed disc with centre 2 and radius 1. Let A be the collection of continuous functions on X, whose restriction to $\overline{D}$ consists of continuous functions holomorphic on D with derivative zero at 0, and whose restriction to $\bar{\mathrm{B}}(2,1)$ is the disc algebra. Let B be the collection of continuous functions on X, whose restriction to both $\overline{D}$ and $\bar{\mathrm{B}}(2,1)$ are the disc algebras.

It is clear that both A and B are proper uniform algebras on X, with A properly contained in B. Since the restrictions of A and B to each of the closed discs are natural uniform algebras, by Theorem 1 in [35] we see that both A and B are natural uniform algebras on X. By the open mapping theorem in complex analysis we see that both A and B are antisymmetric uniform algebras. Consider the function f in A whose restriction to $\overline{D}$ equals the function $z \mapsto z^2$ and whose restriction to $\bar{\mathrm{B}}(2,1)$ equals the constant function 1. Then $(1+f)/2$ is a function in A which peaks on $\bar{\mathrm{B}}(2,1) \cup \{-1\}$. Then by the local peak set theorem we see that $\bar{\mathrm{B}}(2,1)$ is a peak set for A whose interior is not empty.

It remains to show that A and B share the same peak sets. It is sufficient to show that each peak set for B is also a peak set for A. Let K be a peak set for B. If either K is contained in $\overline{D}$ or in $\bar{\mathrm{B}}(2,1)$ then by Lemma 4.3.5 we see that K is also a peak set for A. So now assume that K has non-empty intersections K_1 and K_2 with $\overline{D}$ and $\bar{\mathrm{B}}(2,1)$, respectively. It is clear that K_2 is also a peak set for $A|_{\bar{\mathrm{B}}(2,1)}$, and by Lemma 4.3.5 we see that K_1 is a peak set for $A|_{\overline{D}}$. First assume that 1 is in K. Let $f_1 \in A|_{\overline{D}}$ and $f_2 \in A|_{\bar{\mathrm{B}}(2,1)}$ such that f_1 peaks at K_1 and such that f_2 peaks at K_2. Then f_1 and f_2 agree on $K_1 \cap K_2$, and thus they define a continuous function f on X such that $f|_{\overline{D}} = f_1$ and such that $f|_{\bar{\mathrm{B}}(2,1)} = f_2$. From the definition of A we see that f is in A and f peaks on K. Thus K is also a peak set for A. So now assume that $1 \notin K$. Since functions in $A|_{\overline{D}}$ are holomorphic on D and functions in $A|_{\bar{\mathrm{B}}(2,1)}$

are holomorphic on $\mathrm{B}(2,1)$ we see that K_1 is contained in $\mathbb{T}$ and K_2 is contained in $\bar{\mathrm{B}}(2,1) \setminus \mathrm{B}(2,1)$. By Corollary 4.3.6 we see that $K_1 \cup \{1\}$ is a peak set for $A|_{\overline{D}}$ and $K_2 \cup \{1\}$ is a peak set for $A|_{\bar{\mathrm{B}}(2,1)}$. By [9, Theorem 2.4.10] there exists $f_1 \in A|_{\overline{D}}$ such that $f_1|_{K_1} = 1$, $f_1(1) = 1/2$ and such that $|f_1(z)| < 1$ for each $z \in \overline{D} \setminus K_1$. Similarly there exists $f_2 \in A|_{\bar{\mathrm{B}}(2,1)}$ such that $f_2|_{K_2} = 1$, $f_2(1) = 1/2$ and such that $|f_2(z)| < 1$ for each $z \in \bar{\mathrm{B}}(2,1) \setminus K_2$. As argued before, these f_1 and f_2 give a function f in A which peaks on K. Thus K is a peak set for A. This finishes the proof. □

4.4 Linear subspaces of $C(\mathbb{T})$ properly containing the disc algebra

In this section, we use A to denote the disc algebra on the closed unit disc $\overline{D}$, and we use $A|_{\mathbb{T}}$ to denote the disc algebra on the unit circle $\mathbb{T}$. Wermer's maximality theorem (Proposition 1.7.6) asserts that the only uniform algebras B on $\mathbb{T}$ containing $A|_{\mathbb{T}}$ are $C(\mathbb{T})$ and $A|_{\mathbb{T}}$.

In the rest of this chapter, the default norm on $C(X)$, where X is any non-empty compact Hausdorff space, is the uniform norm $\|\cdot\|_X$. We will specify if we shall use other norms on $C(X)$.

As before, we use $\mathrm{d}t$ to denote the Lebesgue (arc length) measure on $\mathbb{T}$. For each measurable function f on $\mathbb{T}$ and $p \in [1, \infty)$ we define

$$\|f\|_p = \left(\frac{1}{2\pi} \int_{\mathbb{T}} |f|^p \, \mathrm{d}t \right)^{\frac{1}{p}},$$

and we use $L^p(\mathrm{d}t)$ to denote the collection of all Lebesgue measurable functions f on $\mathbb{T}$ such that $\|f\|_p < \infty$. It is standard that $(L^p(\mathrm{d}t), \|\cdot\|_p)$ is a Banach space such that $C(\mathbb{T})$ is a dense subspace, and such that $L^p(\mathrm{d}t) \subsetneq L^q(\mathrm{d}t)$ when $1 \le q < p < \infty$. It is also standard that when $p = 2$, $(L^2(\mathrm{d}t), \|\cdot\|_2)$ is a Hilbert space. Thus each continuous linear functional on $(L^2(\mathrm{d}t), \|\cdot\|_2)$ is of the form

$$f \mapsto \frac{1}{2\pi} \int_{\mathbb{T}} f \overline{g} \, \mathrm{d}t, \quad f \in L^2(\mathrm{d}t),$$

where g is in $L^2(\mathrm{d}t)$. See [58] for more details on these L^p spaces.

In this section we show that there exists a proper uniformly closed linear subspace of $C(\mathbb{T})$ which properly contains $A|_{\mathbb{T}}$ and which is $\|\cdot\|_2$ dense in $L^2(\mathrm{d}t)$. Thus the requirement that B is a uniform algebra on $\mathbb{T}$ in Wermer's maximality theorem cannot be omitted.

Before we state and prove the main theorem of this section, we note that there exists a non-zero function $g \in L^1(\mathrm{d}t) \setminus L^2(\mathrm{d}t)$ such that

$$\frac{1}{2\pi}\int_{\mathbb{T}} g(e^{it})e^{int}\,\mathrm{d}t = 0, \quad n \in \mathbb{N}.$$

See [58, Chapter 17] or [39] (we can simply take f to be the radial limit of a function in $H^1 \setminus H^2$, as defined in the given references).

Theorem 4.4.1. *There exists a proper closed linear subspace B of $C(\mathbb{T})$, such that B properly contains $A|_{\mathbb{T}}$, and such that B is $\|\cdot\|_2$ dense in $L^2(\mathrm{d}t)$.*

Proof. Let $g \in L^1(\mathrm{d}t) \setminus L^2(\mathrm{d}t)$ such that

$$\frac{1}{2\pi}\int_{\mathbb{T}} g(e^{it})e^{int}\,\mathrm{d}t = 0, \quad n \in \mathbb{N}.$$

Define a measure λ on $\mathbb{T}$ by

$$\mathrm{d}\lambda = \frac{1}{2\pi} g\,\mathrm{d}t.$$

Then $\varphi(f) := \int_{\mathbb{T}} f\,\mathrm{d}\lambda$ defines a non-zero $\|\cdot\|_{\mathbb{T}}$ continuous linear functional on $C(\mathbb{T})$. Let B be the kernel of φ, then it is clear that $A|_{\mathbb{T}} \subseteq B$, and B is a closed linear subspace properly contained in $C(\mathbb{T})$.

We show that $\varphi : (C(\mathbb{T}), \|\cdot\|_2) \to \mathbb{C}$ is not continuous. Assume for a contradiction that φ is continuous with respect to $\|\cdot\|_2$. Since $C(\mathbb{T})$ is $\|\cdot\|_2$ dense in $L^2(\mathrm{d}t)$, φ can be extended to a continuous linear function on $L^2(\mathrm{d}t)$, and we still denote the extended functional by φ. Then there exists $h \in L^2(\mathrm{d}t)$ such that

$$\varphi(f) = \frac{1}{2\pi}\int_{\mathbb{T}} f\overline{h}\,\mathrm{d}t, \quad f \in L^2(\mathrm{d}t).$$

Thus we have

$$\frac{1}{2\pi}\int_{\mathbb{T}} f(g - \overline{h})\,\mathrm{d}t = 0, \quad f \in C(\mathbb{T}),$$

which implies that $g = \overline{h}$ almost everywhere $\mathrm{d}t$ on $\mathbb{T}$. This is a contradiction since $g \notin L^2(\mathrm{d}t)$ while $\overline{h} \in L^2(\mathrm{d}t)$.

Therefore the vector space B, which is the kernel of φ, is $\|\cdot\|_2$ dense in $C(\mathbb{T})$ by [59, Theorem 1.18]. This implies that B is also $\|\cdot\|_2$ dense in $L^2(\mathrm{d}t)$.

It remains to show that B properly contains $A|_{\mathbb{T}}$. Let $u = \overline{Z} \in L^2(\mathrm{d}t)$ where

Z is the coordinate function. Then we have

$$\frac{1}{2\pi}\int_{\mathbb{T}} \overline{u}(e^{it})e^{int}\,\mathrm{d}t = 0, \quad n \in \mathbb{N}.$$

Since B is $\|\cdot\|_2$ dense in $L^2(\mathrm{d}t)$, we see that there exists $f \in B$ such that

$$\frac{1}{2\pi}\int_{\mathbb{T}} f\overline{u}\,\mathrm{d}t \neq 0,$$

while for each $h \in A|_{\mathbb{T}}$ we have

$$\frac{1}{2\pi}\int_{\mathbb{T}} h\overline{u}\,\mathrm{d}t = 0.$$

This finishes the proof. □

4.5 Open questions

We end this chapter with several open questions.

The first question has already been stated in Section 4.3.

Question 4.5.1. *Let A be a natural uniform algebra on a non-empty compact Hausdorff space X. Let f be a function in $C(X)$, such that for each proper closed subset $K \subsetneq X$ we have $f|_K \in A|_K$. Does it follow that f is in A?*

The second question is related to Section 4.4 and Wermer's maximality theorem.

Let $A|_{\mathbb{T}}$ be the disc algebra on the unit circle $\mathbb{T}$. Wermer's maximality theorem asserts that if B is a uniform algebra on T, then either $B = A|_{\mathbb{T}}$ or $B = C(\mathbb{T})$. On the other hand, Theorem 4.4.1 shows that there exists a proper closed linear subspace B of $C(\mathbb{T})$ which properly contains $A|_{\mathbb{T}}$. We have the following open question.

Question 4.5.2. *Does there exist a proper closed linear subspace B of $C(\mathbb{T})$, such that B properly contains the disc algebra $A|_{\mathbb{T}}$, and such that B is a commutative $A|_{\mathbb{T}}$ bimodule, where the module operation is the function multiplications?*

The reader may wish to compare this question with the converse of the maximum modulus theorem [58, Theorem 12.13] stated below.

Proposition 4.5.3. *Let B be a linear space of $C(\overline{D})$ containing the constant functions, such that for each $f \in B$ we have $Zf \in B$. Assume that for each*

$f \in B$ we have

$$\|f\|_{\mathbb{T}} = \|f\|_{\overline{D}},$$

then each function in B is holomorphic on the open unit disc D.

Chapter 5

Construction of essential uniform algebras

In this chapter, we study methods to construct essential uniform algebras. Let A be a uniform algebra on X. We are interested in general methods to construct essential uniform algebras which share many properties with A. In the literature, there are two different methods to do so as described in [26, 53]. We will first briefly describe these two methods of constructions and compare their similarities and differences. Then we will further study the method described in [26] to show that some more properties are shared between the constructed uniform algebra and the original one.

The readers may wish to refer to Chapter 2 for the definitions of Swiss cheese sets and classical Swiss cheese sets.

5.1 Nature and essential uniform algebras with large Shilov boundary

In this section, we fix A to be a natural uniform algebra on a metrizable compact space X. In [53], de Paepe proposed a method to construct from A an essential natural uniform algebra B on Z, where Z is a subset of $X \times \mathbb{C}^2$ and B has a restriction that is isometrically isomorphic to A. The Shilov boundary of B is the whole of Z. We briefly describe this construction in this section.

Let Y be a classical Swiss cheese set with empty interior. For each $0 \leq t \leq 1$

we use tY to denote the set

$$tY = \{tw : w \in Y\}.$$

It is clear that tY is a classical Swiss cheese set with empty interior for each $0 < t \leq 1$, and hence $R(tY)$ is a natural, essential uniform algebra whose Shilov boundary is tY (see Section 2.1).

Choose a dense sequence (x_n) in X, and let (a_n) be a decreasing sequence of positive numbers having 0 as limit. Consider the following subsets of $X \times \mathbb{C}^2$:

$$\tilde{X} = \{(x, 0, 0) : x \in X\},$$

$$Y_n = \{(x_n, t, w) : 0 \leq t \leq a_n, w \in tY\}.$$

Set

$$Z = \tilde{X} \cup \bigcup_{n=1}^{\infty} Y_n,$$

which is a subset of $X \times \mathbb{C}^2$. Let $\Pi_X : Z \to X$ be the continuous surjection defined by

$$\Pi_X(x, t, w) = x, \quad (x, t, w) \in Z.$$

Let B be the algebra of all continuous functions f on Z such that

(i) the function on X defined by $x \mapsto f(x, 0, 0)$ belongs to A,

(ii) the function on αY defined by $y \mapsto f(x_n, \alpha, y)$ belongs to $R(\alpha Y)$ for all fixed n and $0 < \alpha \leq a_n$.

The main result in [53] is the following.

Proposition 5.1.1. *The set Z is a compact subset in $X \times \mathbb{C}^2$, and Z is connected if X is connected. The algebra B is a natural and essential uniform algebra on Z whose Shilov boundary is Z. Moreover, the map $\Pi_X^* : A \hookrightarrow B$ defined by*

$$f \mapsto f \circ \Pi_X, \quad f \in A$$

is an isometric embedding from A into B.

We note that if we choose the classical Swiss cheese set Y properly, we can make sure that certain properties of A are inherited by B. For example, we can take Y to be a classical Swiss cheese set with empty interior such that $R(Y)$ is a normal uniform algebra. Such a Swiss cheese set was first constructed in [25, Example 2.9] (see also [28, Theorem 8.6]). In this case, the following theorem shows that B is a normal uniform algebra whenever A is normal. This result seems to be new.

Theorem 5.1.2. *Let A be a normal uniform algebra on a compact metrizable space X. Let Y be a classical Swiss cheese set with empty interior such that $R(Y)$ is normal. Then B is a normal uniform algebra on Z.*

Proof. Let $z_1 = (\zeta_1, t_1, w_1)$ and $z_2 = (\zeta_2, t_2, w_2)$ be distinct points in Z. It is enough to show that there exists a function f in B which vanishes on a neighbourhood of z_1 but takes a non-zero value at z_2 (see Section 3.1).

First assume $\zeta_1 \neq \zeta_2$. Since A is normal on X, we can find a function g in A which vanishes on a neighbourhood of ζ_1 but takes a non-zero value at ζ_2. The function f in B defined by $(x, t, w) \mapsto g(x)$ has the property that we require.

Next assume $\zeta_1 = \zeta_2$ but $t_1 \neq t_2$. Let $c = (t_1 + t_2)/2$. If $t_1 < t_2$, let f be the continuous function on Z defined by

$$f(x,t,w) = \begin{cases} t - c & \text{if } x = \zeta_1 \text{ and } t \geq c, \\ 0 & \text{otherwise.} \end{cases}$$

Then f is in B and has the property that we require. If $t_1 > t_2$, let f be the continuous function on Z defined by

$$f(x,t,w) = \begin{cases} 0 & \text{if } x = \zeta_1 \text{ and } t \geq c, \\ t - c & \text{if } x = \zeta_1 \text{ and } 0 < t < c, \\ -c & \text{otherwise.} \end{cases}$$

Then f is in B and has the property that we require.

Lastly assume $\zeta_1 = \zeta_2$, $t_1 = t_2$ but $w_1 \neq w_2$. This implies that $t_1 \neq 0$. Since $R(Y)$ is normal, we can find a function h in $R(t_1 Y)$ such that h vanishes on a neighbourhood of w_1 in $t_1 Y$ but takes a non-zero value at w_2. The function f on Z defined by

$$f(x,t,w) = \begin{cases} \frac{t}{t_1} h(\frac{t_1}{t} w) & \text{if } x = \zeta_1 \text{ and } t \neq 0, \\ 0 & \text{otherwise} \end{cases}$$

is in B and has the property that we require. This finishes the proof that B is normal on Z. □

Remark 5.1.3. We have the following remarks concerning the construction in [53].

1. In the original construction of [53], one requires an example of a Swiss cheese set Y with empty interior such that $R(Y)$ is natural and essential on Y. De Paepe suggests that one can use McKissick's Swiss cheese set in [49]. However, it is not known whether McKissick's Swiss cheese set Y produces an essential $R(Y)$. One way to get around this difficulty is to use McKissick's Swiss cheese set Y and to use the restriction of $R(Y)$ to its essential subset. Another way to get around this difficulty is to notice that normality of $R(Y)$ is not used in the construction of de Paepe and one can use classical Swiss cheese sets Y with empty interior instead, as we did in this section.

2. In the construction, de Paepe insists that the original uniform algebra A is natural on X. We note that a slightly more general construction can still produce essential uniform algebras. Now let A be a uniform algebra on a metrizable X with character space Φ_A. Then by using the described construction (note that we again require $\{x_n\}$ in the construction to be a dense subset of X) we can get a uniform algebra B on Z. We claim that B is again an essential uniform algebra. The character space of B can be identified with $Z \cup (\Phi_A \setminus X)$.

3. The uniform algebra B is not antisymmetric. Actually, the function $(x, t, w) \mapsto t$ is in B, which is a non-constant real valued function.

5.2 An alternative method for constructing essential uniform algebras

Let A be a nontrivial uniform algebra on K. Suppose that X is a compact metrizable space such that for each non-empty open subset U of X, U has a nowhere dense subset which is homeomorphic to K. In [26] Feinstein and Izzo proposed a method to construct an essential uniform algebra on X, where a lot of properties hold for A if and only if these properties hold for the new uniform algebra. In this section we briefly discuss this construction and compare it with the construction of de Paepe.

First note that if (X, d) is a compact metric space, and if (K_n) is a sequence of pairwise disjoint closed subspaces of X whose diameters go to zero, then the quotient space Y obtained from X by identifying each K_n to a point is a compact metric space. This fact is proved in [26, Lemma 2.6].

The main construction in [26] is summarised in the following proposition.

Proposition 5.2.1. *Let A be a nontrivial uniform algebra on a compact Hausdorff space K. Let (X, d) be a compact metric space, every non-empty open subset of which contains a nowhere dense subspace homeomorphic to K. Then there exists a sequence (K_n) of pairwise disjoint, nowhere dense subspaces of X, each homeomorphic to K, such that $\bigcup_{n=1}^{\infty} K_n$ is dense in X and $\operatorname{diam}(K_n) \to 0$. If homeomorphisms $h_n : K_n \to K$ are chosen and we set $A_n = \{f \circ h_n : f \in A\}$, then the collection of functions $\tilde{A} = \{f \in C(X) : f|_{K_n} \in A_n, n \in \mathbb{N}\}$ is a uniform algebra on X. The equality $\tilde{A}|_{K_n} = A_n$ holds for all n, and thus $A|_{K_n}$ is uniformly closed for all n. The maximal antisymmetric subsets for $\tilde{A}$ are the single point sets $\{x\}$ with x in $X \setminus \bigcup_{n=1}^{\infty} K_n$ and the maximal antisymmetric subsets for A_n for each $n \in \mathbb{N}$. Furthermore, $\tilde{A}$ is an essential uniform algebra.*

We note that the proof of Proposition 5.2.1 in [26] actually establishes the following.

Corollary 5.2.2. *With notation as in Proposition 5.2.1, the collection of peak points for $\tilde{A}$ has the form*

$$\left(X \setminus \bigcup_{n=1}^{\infty} K_n\right) \cup \left(\bigcup_{n=1}^{\infty} P_n\right),$$

where P_n is the collection of peak points for A_n.

From the antisymmetric decomposition for $\tilde{A}$ it is clear that $\tilde{A}$ is not antisymmetric.

Let A and $\tilde{A}$ be uniform algebras as described in Proposition 5.2.1. Then it is shown in [26] that there are a lot of properties which hold for A if and only if they hold for $\tilde{A}$. We list some of the properties mentioned in [26] which will be relevant to our further study of the construction. We refer the reader to [26, Theorem 1.1] for more details.

Proposition 5.2.3. *With notation as in Proposition 5.2.1, the following relations hold between the properties of A and the properties of $\tilde{A}$:*

(i) $\tilde{A}$ is natural if and only if A is natural;

(ii) $\tilde{A}$ is regular on X if and only if A is regular on K;

(iii) $\tilde{A}$ *is normal if and only if* A *is normal;*

(iv) every point of X *is a peak point for* $\tilde{A}$ *if and only if every point of* K *is a peak point for* A*;*

(v) $\tilde{A}$ *is strongly regular if and only if* A *is strongly regular.*

Now we compare the two methods of constructing essential uniform algebras discussed in this and the previous sections. Let A be a non-trivial natural uniform algebra on a compact metrizable space K. Suppose that X is a compact metrizable space such that for each non-empty open subset U of X, U has a nowhere dense subset which is homeomorphic to K. Let $\tilde{A}$ be the essential uniform algebra on X described in Proposition 5.2.1, and let B be the essential uniform algebra on Z constructed in the previous section using $R(Y)$, where Y is some classical Swiss cheese set with empty interior.

(i) Both $\tilde{A}$ and B are natural and essential uniform algebras.

(ii) Even though both $\tilde{A}$ and B are essential uniform algebras, neither of them is antisymmetric. It would be interesting to come up with methods to construct antisymmetric uniform algebras.

(iii) The construction of an essential $\tilde{A}$ requires that A is a non-trivial uniform algebra - otherwise $\tilde{A}$ would just be $C(X)$. On the other hand, B would still be an essential uniform algebra even when A is $C(K)$.

(iv) Since the construction of B requires a choice of some classical Swiss cheese set, we would usually expect that B has some property if and only if both A and $R(Y)$ have that property. For example, we know that B is normal if and only if both A and $R(Y)$ are normal. On the other hand, a lot of properties hold for $\tilde{A}$ if and only if they hold for A (see Proposition 5.2.3, [26, Theorem 1.1] and also Section 5.4). In particular, we know that $\tilde{A}$ is normal if and only if A is normal.

(v) Since the construction of B uses $R(Y)$, some properties that hold for A may fail, or would be difficult to verify if they hold, for B. For example, it is not possible for each point of Z to be a peak point for B (otherwise each point for $R(Y)$ would be a peak point for Y, which implies that $R(Y) = C(Y)$ by [9, Theorem 3.3.3]). This is in contrast to Proposition 5.2.3 (iv). For another example, if A is strongly regular, we know that $\tilde{A}$ is also strongly regular. On the other hand, it seems to be open whether there exists a nontrivial, strongly regular $R(Y)$ (see [21] for a discussion). Therefore we do not know if we can make B strongly regular when A is strongly regular.

5.3 Preliminary results

In this section, we give a definition and some preliminary results that we need in our further study of the essential uniform algebra $\tilde{A}$ described in Proposition 5.2.1.

We first define the *Gleason parts* of a natural uniform algebra. Recall that for a uniform algebra A on X and a point $x \in X$, ε_x is the evaluation character on A defined by $\varepsilon_x(f) = f(x)$.

Definition 5.3.1. Let A be a natural uniform algebra on X. We define a metric d on X by setting

$$d(x, y) = \|\varepsilon_x - \varepsilon_y\|, \quad (x, y \in X),$$

where $\|\cdot\|$ is the dual norm on A^*. For $x, y \in X$, write $x \sim y$ if and only if $d(x, y) < 2$. Then $\sim$ is an equivalence relation on X. The *Gleason parts* for A are the equivalence classes defined by the relation $\sim$.

The fact that $\sim$ in the above definition is an equivalence relation is proved in [9, Theorem 2.6.3]. Let A be a natural uniform algebra on X. It is proved in [61, Theorem 16.6] that two points x, y in X belong to the same Gleason part if and only if x and y admit mutually absolutely continuous representing measures. In particular, since the only representing measures for peak points for A are the point masses, peak points for A form one point Gleason parts for A.

Lemma 5.3.2. *Let X be a non-empty compact Hausdorff space, and let $\varepsilon > 0$. Let f be a complex-valued continuous function on X. Then we can find a complex-valued continuous function g on X such that $\|f - g\|_X \leq \varepsilon$ on X, and such that $g(x) = 0$ whenever $|f(x)| \leq \varepsilon$.*

Proof. Let ϕ be the complex-valued continuous function on $\mathbb{C}$ defined by

$$\phi(z) = \begin{cases} 0, & \text{if } |z| \leq \varepsilon, \\ z - \varepsilon\frac{z}{|z|}, & \text{if } |z| \geq \varepsilon. \end{cases}$$

Then the complex valued function $g = \phi \circ f$ has the required properties. □

Lemma 5.3.3. *Let A be a uniform algebra on X, and let K be a closed subset of X such that $A|_K$ is uniformly closed. Let $x \in K$. If f is (approximately) locally in A on X at x, then $f|_K$ is (approximately) locally in A_K on K at x. If f is (approximately) locally in A on X, then $f|_K$ is (approximately) locally in A_K on K.*

Proof. We give the proof for the approximately local case. The proof of the other case is similar.

First suppose that f is approximately locally in A on X at x. Then there exists a neighbourhood U of x in X, and a sequence of functions $f_n \in A$, such that

$$\|f_n - f\|_{\overline{U}} \to 0, \quad n \to \infty.$$

We know that $U \cap K$ is a neighbourhood of x in K, and $f_n|_K$ is a sequence of functions in A_K which satisfies

$$\|f_n|_K - f|_K\|_{\overline{U} \cap K} \to 0, \quad n \to \infty.$$

Therefore $f|_K$ can be uniformly approximated on $\overline{U} \cap K$ by functions in $A|_K$, and so $f|_K$ is approximately locally in $A|_K$ on K at x.

Next suppose that f is approximately locally in A on X. Then for each $x \in K$, f is approximately locally in A on X at x, which implies that $f|_K$ is approximately locally in A_K on K at x. This shows that $f|_K$ is approximately locally in A_K on K. $\square$

Lemma 5.3.4. *Let A be a uniform algebra on X. Let K be a closed subset of X, such that the essential set of A is contained in K. If A is an (approximately) local uniform algebra on X, then A_K is an (approximately) local uniform algebra on K.*

Proof. First note that, since K contains the essential set of A, the restriction algebra $A|_K$ is uniformly closed and so $A|_K = A_K$.

We give a proof for the approximately local case. The other case can be proved with a similar, but simpler argument.

Let $f \in C(X)$, such that $f|_K$ is approximately locally in A_K on K. We claim that f is approximately locally in A on X.

Let $x \in X$. If $x \in X \setminus K$, let $\overline{U}$ be a closed neighbourhood of x in X disjoint from K. Since the essential set of A is contained in K, we see that $f|_{\overline{U}} \in A|_{\overline{U}}$. This shows that f is approximately locally in A on X at x. Now assume that

$x \in K$. Then we can find a closed neighbourhood $\overline{U}$ of x in X, and we can find a sequence of functions (g_n) in A, such that

$$\|f - g_n\|_{\overline{U} \cap K} < \frac{1}{n}, \quad (n \in \mathbb{N}).$$

Fix n for the moment. Let $\psi = (f - g_n)|_{\overline{U}}$. By applying Lemma 5.3.2 to the function ψ on $\overline{U}$ with $\varepsilon = 1/n$, we can find a continuous function ϕ on $\overline{U}$ such that

$$\|\psi - \phi\|_{\overline{U}} \leq \frac{1}{n},$$

and such that $\phi(x) = 0$ for each $x \in \overline{U}$ with $|\psi(x)| \leq \frac{1}{n}$. In particular, the restriction of ϕ to $\overline{U} \cap K$ is the zero function. Next we extend ϕ to a continuous function r on $\overline{U} \cup K$ defined by

$$r(x) = \begin{cases} \phi(x), & \text{if } x \in \overline{U} \\ 0, & \text{if } x \in K. \end{cases}$$

Finally, we use Tietze's extension theorem to extend r to a continuous function on X, which we denote by h_n. Since $h_n|_K = 0$, we see that h_n is actually in A. Meanwhile, from the construction it is clear that

$$\|f - g_n - h_n\|_{\overline{U}} \leq \frac{1}{n}.$$

Since $g_n + h_n \in A$, we conclude that f is approximately locally in A on X at x. Since x is arbitrary, we conclude that f is approximately locally in A on X. This proves our claim.

Finally, we show that A_K is an approximately local uniform algebra on K. Let $g \in C(K)$ be a function approximately locally in A_K. Then we can use Tietze's extension theorem to get a continuous function f on X whose restriction to K is g. By the above argument we know that f is approximately locally in X. Since A is an approximately local uniform algebra on X, we see that f is actually in A. Thus $f|_K = g \in A_K$. This finishes the proof. □

Lastly, we prove a fact about representing measures for the uniform algebras $\tilde{A}$ as described in Proposition 5.2.1.

Lemma 5.3.5. *With notation as in Proposition 5.2.1, we have the following: Let ϕ be a character on $\tilde{A}$, and let μ be an $\tilde{A}$-representing measure for ϕ on X. Then either μ is a unit point mass at a point in $X \setminus \cup_{n=1}^{\infty} K_n$, or μ is supported on K_n for some $n \in \mathbb{N}$.*

Proof. By Proposition 1.4.6 we know that the support of any representing measure for ϕ is a set of antisymmetry for $\tilde{A}$. Therefore it follows that the support of μ must lie in a maximal antisymmetric subset of $\tilde{A}$. From the maximal antisymmetric decomposition in Proposition 5.2.1 we conclude that the support of μ is either a point in $X \setminus \cup_{n=1}^{\infty} K_n$, or the support of μ is a subset of K_n for some $n \in \mathbb{N}$. □

5.4 Further properties of $\tilde{A}$

In this section, we continue the study of the properties of the uniform algebras $\tilde{A}$ as described in Proposition 5.2.1. We show in the next theorem that there are some more properties which hold for A if and only if they hold for $\tilde{A}$.

In the rest of this section, we will always adopt notation as in the statement of Proposition 5.2.1.

Theorem 5.4.1. *With notation as in Proposition 5.2.1, the following relations also hold between the properties of A and the properties of $\tilde{A}$:*

(i) Suppose that A is a natural uniform algebra on K. Then $\tilde{A}$ has no non-trivial Jensen measures if and only if A has no non-trivial Jensen measures;

(ii) $\tilde{A}$ is approximately regular on X if and only if A is approximately regular on K;

(iii) $\tilde{A}$ is approximately normal on X if and only if A is approximately normal on K;

(iv) $\tilde{A}$ is an (approximately) local uniform algebra on X if and only if A is an (approximately) local uniform algebra on K.

(v) Suppose A is a natural uniform algebra on K, and let $x, y \in X$ be distinct points. Then x and y are in the same Gleason part for $\tilde{A}$ if and only if there exists $n \in \mathbb{N}$ such that $x, y \in K_n$, and x and y belong to the same Gleason part for $\tilde{A}|_{K_n}$.

We get an immediate corollary from (v) in Theorem 5.4.1.

Corollary 5.4.2. *Suppose A is a natural uniform algebra on K. Every point in X is a one-point Gleason part for $\tilde{A}$ if and only if every point in K is a one-point Gleason part for A.*

In the proof of Theorem 5.4.1, it will be convenient to have some quotient maps and quotient spaces at hand. It is also convenient to construct some new uniform algebras on these quotient spaces. We briefly introduce them here, in order not to interrupt the flow of the proof. These quotient spaces (maps) and uniform algebras on them are as in [26].

Let Y be the quotient space of X obtained by identifying each K_n to a point, and let $q : X \to Y$ be the corresponding quotient map. For each $n \in \mathbb{N}$, let Y_n be the quotient space of X obtained by identifying each K_m with $m \neq n$ to a point, and let $q_n : X \to Y_n$ be the corresponding quotient map. For each $n \in \mathbb{N}$, we denote $K'_n = q_n(K_n)$, which is homeomorphic to K_n. By [26, Lemma 2.6] the spaces Y and Y_n for all $n \in \mathbb{N}$ are compact metric spaces. For each $n \in \mathbb{N}$, we define the collection of functions B_n on Y_n by

$$B_n = \{f \in C(Y_n) : f \circ q_n \in \tilde{A}\}.$$

The fact that, for each $n \in \mathbb{N}$, B_n is a uniform algebra on Y_n was shown in [26]. We establish a little more here.

Lemma 5.4.3. *For each $n \in \mathbb{N}$, B_n is a uniform algebra on Y_n whose essential set is contained in K'_n.*

Proof. From the construction of $\tilde{A}$, it is clear that B_n actually has the form

$$B_n = \{f \in C(Y_n) : (f \circ q_n)|_{K_n} \in A_n\}.$$

Thus we see that B_n is a uniform algebra on Y_n which contains all continuous functions vanishing on K'_n. This shows that the essential set of B_n is contained in K'_n. □

Proof of Theorem 5.4.1. (i) Since A is a natural uniform algebra on K, by Proposition 5.2.3 we know that $\tilde{A}$ is also a natural uniform algebra on X.

First suppose that $\tilde{A}$ has no non-trivial Jensen measures. Fix an $n \in \mathbb{N}$, and let x be a point in K_n. If μ is an $\tilde{A}|_{K_n}$-Jensen measure for x, we can extend μ to be an $\tilde{A}$-Jensen measure for x by setting

$$\mu(U) := \mu(U \cap K_n), \quad U \text{ Borel measurable.} \tag{5.1}$$

Therefore μ is just the unit point mass at x. This implies that $\tilde{A}|_{K_n}$ only has trivial Jensen measures, which is equivalent to saying that A has no non-trivial Jensen measures.

On the other hand, suppose that A has no non-trivial Jensen measures. This is equivalent to saying that each $\tilde{A}|_{K_n}$ has no non-trivial Jensen measures. Let $x \in X$, and let μ be an $\tilde{A}$-Jensen measure for x. By Lemma 5.3.5 we know that μ is either supported on a single point in $X \setminus \bigcup_{n=1}^{\infty} K_n$, or μ is supported on some K_n. In the latter case, we claim that actually x is in K_n. To see this, note that μ is a multiplicative probability measure and thus represents a character for $\tilde{A}|_{K_n}$. Since $\tilde{A}|_{K_n}$ is natural on K_n, we can find an $y \in K_n$ such that μ represents ε_y. Thus for each f in $\tilde{A}$ we have

$$f(y) = \varepsilon_y(f|_{K_n}) = \int_{K_n} f \, \mathrm{d}\mu = \varepsilon_x(f) = f(x).$$

Since $\tilde{A}$ separates the points in X we see that $x = y$. This shows that x is in K_n and proves our claim. Thus μ is an $\tilde{A}|_{K_n}$-Jensen measure for x, and hence must be a unit point mass. Therefore we conclude that $\tilde{A}$ has no non-trivial Jensen measures.

(ii) It is clear that if $\tilde{A}$ is approximately regular on X, then $\tilde{A}|_{K_n}$ is approximately regular on K_n for each $n \in \mathbb{N}$. This is equivalent to saying that A is approximately regular on K.

On the other hand, assume A is approximately regular on K, which is equivalent to saying that $\tilde{A}|_{K_n}$ is approximately regular on K_n for each $n \in \mathbb{N}$. Let $x \in X$, and let F be a closed subset of X disjoint from x. For each $\varepsilon > 0$, we need to find a function $f \in \tilde{A}$ such that $|f(x) - 1| < \varepsilon$ and $|f(y)| < \varepsilon$ for all $y \in F$.

If no K_n meets both $\{x\}$ and F, then $\{q(x)\}$ and $q(F)$ are disjoint compact subsets in Y. Hence we can find a continuous function h in $C(Y)$, such that $h(q(x)) = 1$, and $h(q(y)) = 0$ for all $y \in F$. Then $h \circ q$ is a function in $\tilde{A}$, such that $h \circ q(x) = 1$ and $h \circ q(F) \subseteq \{0\}$.

Now assume that there exists $n \in \mathbb{N}$ such that $x \in K_n$ and $K_n \cap F \neq \emptyset$. Note that there can only be one such n. Fix $\varepsilon > 0$. Since $\tilde{A}|_{K_n}$ is approximately regular on K_n, we see that $B_n|_{K'_n}$ is approximately regular on K'_n. Therefore we can find a function h in B_n such that $|h(q_n(x)) - 1| < \varepsilon$, and such that $|h(q_n(y))| < \varepsilon$ for all $y \in K_n \cap F$. Let $g \in C(Y_n)$ be a real-valued continuous function, such that $g(K'_n) \subseteq \{1\}$, and such that $0 \leq g(y) < 1$ for all $y \in Y_n \setminus K'_n$. Let F_ε be the closed subset of F defined by

$$F_\varepsilon = \{y \in F : |h(q_n(y))| \geq \varepsilon\}.$$

If F_ε is empty, the modulus of $h \circ q_n$ is strictly smaller than ε on F. Then we see that the function $h \circ q_n$ is in $\tilde{A}$, $|h \circ q_n(x) - 1| < \varepsilon$ and $|h \circ q_n(y)| < \varepsilon$ for all $y \in F$. So now assume that $F_\varepsilon \neq \emptyset$. Then F_ε is a compact subset of F disjoint from K_n (note that $|h(q_n(y))| < \varepsilon$ for all $y \in K_n \cap F$). Therefore $\max_{y \in F_\varepsilon} g(q_n(y)) < 1$, and we can find an $m \in \mathbb{N}$ sufficiently large such that

$$\max_{y \in F_\varepsilon} g^m(q_n(y)) < \frac{\varepsilon}{\|h\|_{Y_n}}.$$

Since g^m is constantly 1 on K_n', we have

$$|g^m(q_n(x))h(q_n(x)) - 1| = |h(q_n(x)) - 1| < \varepsilon.$$

For each $y \in F$, if $y \notin F_\varepsilon$, then

$$|g^m(q_n(y))h(q_n(y))| \leq 1 \cdot |h(q_n(y))| < \varepsilon,$$

and if $y \in F_\varepsilon$, then

$$|g^m(q_n(y))h(q_n(y))| < \frac{\varepsilon}{\|h\|_{Y_n}} \|h\|_{Y_n} < \varepsilon.$$

Since the restriction of g to K_n' is a constant function, we see that $g \circ q_n$ is a function in $\tilde{A}$. Therefore we see that $(g^m h) \circ q_n$ is a function in $\tilde{A}$, such that

$$|(g^m h) \circ q_n(x) - 1| < \varepsilon,$$

and such that for all $y \in F$ we have

$$|(g^m h) \circ q_n(y)| < \varepsilon.$$

This shows that $\tilde{A}$ is approximately regular on X.

(iii) It is clear that if $\tilde{A}$ is approximately normal on X, A is approximately normal on K. The other direction can be proved in a similar way to the proof of (ii), but here we illustrate another way to prove this fact. Assume that A is approximately normal on K, which is equivalent to saying that each $\tilde{A}|_{K_n}$ is approximately normal on K_n. By Theorem 2.3 in [10], we know that a uniform algebra $\mathfrak{B}$ on Σ is approximately normal on Σ if and only if each representing measure for $\mathfrak{B}$ on Σ has connected support. Thus each $\tilde{A}|_{K_n}$-representing measure on K_n has connected support. But we know from Lemma 5.3.5 that each $\tilde{A}$-representing measure is either supported on a point of $X \setminus \cup_{n=1}^{\infty} K_n$, or it is supported on some K_n. In the latter case, this $\tilde{A}$-representing measure is also an $\tilde{A}|_{K_n}$-representing measure, and hence it must have connected support.

Therefore each $\tilde{A}$-representing measure on X has connected support, and hence $\tilde{A}$ is approximately normal on X.

(iv) We only give the proof for the approximately local case. The other case can be proved with a similar, but simpler argument.

We first prove the forward implication. Assume that $\tilde{A}$ is an approximately local uniform algebra on X. Fix an $n \in \mathbb{N}$, we shall show that $\tilde{A}|_{K_n}$ is an approximately local uniform algebra on K_n, which is equivalent to saying that A is an approximately local uniform algebra on K. Since $q_n|_{K_n}$ is a homeomorphism, in order to show that $\tilde{A}|_{K_n}$ is an approximately local uniform algebra on K_n it is enough to show that $B_n|_{K'_n}$ is an approximately locally uniform algebra on K'_n. This is what we aim to prove.

First we show that B_n is an approximately local uniform algebra on Y_n. Let $f \in C(Y_n)$ be approximately locally in B_n on Y_n, and let $\tilde{f} = f \circ q_n$. Let $x \in X$, and denote $y = q_n(x)$. Then we can find a closed neighbourhood $\overline{U}$ of y in Y_n, such that there exists a sequence of functions (g_ℓ) in B_n with $\|f - g_\ell\|_{\overline{U}} \to 0$ as $\ell \to \infty$. Then $q_n^{-1}(\overline{U})$ is a closed neighbourhood of x in X, and $(g_\ell \circ q_n)_\ell$ is a sequence of functions in $\tilde{A}$ such that $\|\tilde{f} - g_\ell \circ q_n\|_{q_n^{-1}(\overline{U})} \to 0$ as $\ell \to \infty$. This shows that $\tilde{f}$ is approximately locally in $\tilde{A}$ on X at x. Since x is arbitrary and $\tilde{A}$ is an approximately local uniform algebra on X, we conclude that $\tilde{f} \in \tilde{A}$, and hence $f \in B_n$. Therefore B_n is an approximately local uniform algebra on Y_n.

By Lemma 5.4.3 the essential set for B_n is contained in K'_n. Now if we apply Lemma 5.3.4 to B_n, we conclude that $B_n|_{K'_n}$ is an approximately local uniform algebra on K'_n.

We now prove the converse implication. Suppose that A is an approximately local uniform algebra on K. This is equivalent to saying that $\tilde{A}|_{K_n}$ is an approximately local uniform algebra on K_n for each $n \in \mathbb{N}$. We shall show that $\tilde{A}$ is an approximately local uniform algebra on X.

Let $\tilde{f}$ be a continuous function on X which is approximately locally in $\tilde{A}$ on X. Then by Lemma 5.3.3 we see that $\tilde{f}|_{K_n}$ is approximately locally in $\tilde{A}|_{K_n}$ on K_n for each $n \in \mathbb{N}$, which implies that $\tilde{f}|_{K_n}$ is in $\tilde{A}|_{K_n}$. Therefore $\tilde{f}$ is in $\tilde{A}$, which shows that $\tilde{A}$ is an approximately local uniform algebra on X.

(v) First suppose that x and y are distinct points in the same Gleason part for $\tilde{A}$. Then ε_x and ε_y admit mutually absolutely continuous representing measures μ and λ, respectively. Since peak points for $\tilde{A}$ form one point Gleason parts, we see that $x, y \in \cup K_n$. By Lemma 5.3.5 it is clear that both μ and λ are supported in the same K_n for some $n \in \mathbb{N}$. Since both μ and λ are multiplicative measures, we see that they are representing measures for $\tilde{A}|_{K_n}$. Since $\tilde{A}|_{K_n}$ is natural on K_n, both μ and λ represent point evaluations for points in K_n. Because $\tilde{A}$ separates points, we conclude that both x and y are actually in K_n, and they are in the same Gleason part for $\tilde{A}|_{K_n}$.

Next assume that there exists $n \in \mathbb{N}$, such that $x, y \in K_n$ and they are in the same Gleason part for $\tilde{A}|_{K_n}$. Then x and y admit mutually absolutely continuous representing measures for $\tilde{A}|_{K_n}$. If we extend these two measures on K_n to measures on X as defined in (5.1), then these two measures are also mutually absolutely continuous $\tilde{A}$ representing measures. Therefore x and y are in the same Gleason part for $\tilde{A}$. $\square$

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