Compensated Convex Transforms and Geometric Singularity Extraction from Semiconvex Functions[*]

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Dedicated to Professor Kung-ching Chang on the occasion of his 80th Birthday

Abstract

The upper and lower compensated convex transforms [30, 31, 33] are 'tight' one-sided approximations for a given function. We apply these transforms to the extraction of fine geometric singularities from general semiconvex/semiconcave functions and DC-functions in $\mathbb{R}^n$ (difference of convex functions). Well-known geometric examples of (locally) semiconcave functions include the Euclidean distance function and the Euclidean squared-distance function. For a locally semiconvex function f with general modulus, we show that 'locally' a point is singular (a non-differentiable point) if and only if it is a scale 1-valley point, hence by using our method we can extract all fine singular points from a given semiconvex function. More precisely, if f is a semiconvex function with general modulus and x is a singular point, then locally the limit of the scaled valley transform exists at every point x and can be calculated as $\lim_{\lambda \to +\infty} \lambda V_\lambda(f)(x) = r_x^2/4$, where r_x is the radius of the minimal bounding sphere [18] of the (Fréchet) subdifferential $\partial_- f(x)$ of the locally semiconvex f and $V_\lambda(f)(x)$ is the valley transform at x. Thus the limit function $\mathcal{V}_\infty(f)(x) := \lim_{\lambda \to +\infty} \lambda V_\lambda(f)(x) = r_x^2/4$ provides a 'scale 1-valley landscape function' of the singular set for a locally semiconvex function f. At the same time, the limit also provides an asymptotic expansion of the upper transform $C_\lambda^u(f)(x)$ when λ approaches $+\infty$. For a locally semiconvex function f with linear modulus we show further that the limit of the gradient of the upper compensated convex transform $\lim_{\lambda \to +\infty} \nabla C_\lambda^u(f)(x)$ exists and equals the centre of the minimal bounding sphere of $\partial_- f(x)$. We also show that for a DC-function $f = g - h$, the scale 1-edge transform, when $\lambda \to +\infty$, satisfies $\liminf_{\lambda \to +\infty} \lambda E_\lambda(f)(x) \geq (r_{g,x} - r_{h,x})^2/4$, where $r_{g,x}$ and $r_{h,x}$ are the radii of the minimal bounding spheres of the subdifferentials $\partial_- g$ and $\partial_- h$ of the two convex functions g and h at x, respectively.

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1 Introduction and main results

About ten years ago, the first author submitted the paper [30] dedicated to Professor Kung-ching Chang on the occasion of his 70th birthday. Ten years on, the subject discussed in [30] has seen some further theoretical developments [31, 33, 34, 32]. As a step towards applications, we have been granted a UK patent [35] on image processing methods based on this theory. In the present paper we work along a similar line to that in [30]. We study the approximations and geometric singular extractions for semiconvex and semiconcave functions by using compensated convex transforms introduced in [30].

Semiconcave and semiconvex functions have been extensively studied in the context of Hamilton-Jacobi equations [8]. DC-functions (difference of convex functions) [13] have been used in many optimisation problems [15]. Important classes of such functions include the Euclidean distance function and the squared distance function. Since general DC-functions and semiconvex/semiconcave functions are locally Lipschitz functions in their essential domains ([8, Theorem 2.1.7]), Rademacher's theorem implies that they are therein differentiable almost everywhere. Fine properties for the singular sets of convex/concave and semiconvex/semiconcave functions have been studied extensively [3, 2, 8] showing that the singular set of a semiconvex/semiconcave function is rectifiable. However, from the applied mathematics point of view, natural questions arise, such as how such functions can be effectively approximated by smooth functions, whether all singular points are of the same type, that is, for semiconcave (semiconvex) functions, whether all singular points are geometric 'ridge' ('valley') points, how singular sets can be effectively extracted beyond the definition of differentiability and how the information concerning 'strengths' of different singular points can be effectively measured. Answers to these questions have important applications in image processing and computer-aided geometric design. For example, the singular set of the Euclidean squared-distance function $\text{dist}^2(\cdot, \Omega^c)$ to the complement of a bounded open domain $\Omega \subset \mathbb{R}^n$ (called the medial axis [6] of the domain Ω) carries important 'compact' geometric information of the domain. It is also well known that the squared Euclidean distance function $\text{dist}^2(\cdot, K)$ is 2-semiconcave [8]. An answer to the question of how to extract the medial axis in a 'stable' manner with respect to the domain under consideration has been addressed in [32] and has many applications [26]. In [32] we introduced the notion of the *medial axis map* defined by $M_\lambda(K)(x) = (1 + \lambda)R_\lambda(\text{dist}^2(\cdot, K))(x)$ for a closed set $K \subset \mathbb{R}^n$, where $R_\lambda(f)$ is the ridge transform of f defined in [33], and studied its properties. We showed that $M_\lambda(\Omega)$ defines a Hausdorff stable multiscale representation of the medial axis for finite $\lambda > 0$ and the limit $\lim_{\lambda \to +\infty} M_\lambda(\Omega)(x) = \text{dist}^2(x, K) - \text{dist}^2(x, \text{co}[K(x)])$ exists for all $x \in \mathbb{R}^n$, where $K(x) = \{y \in K, \text{dist}(x, K) = |x - y|\}$ and $\text{co}[K(x)]$ is its convex hull. This provides a 'multiscale landscape' of the medial axis in the sense that higher is the height, higher is the distance between the generating points of the medial axis branch.

The present work is partly motivated by [32]. Our approximation results in the present work are much more general than those in [32]. Simple examples which were not covered in [32] are the Euclidean distance function itself and the weighted squared distance function [23] for a finite set $K = \{x_i, \ i = 1, \ldots, m\}$ defined by $\text{dist}^2_{w,b}(x, K) = \min\{w_i|x - x_i|^2 + b_i, \ x_i \in K, \ w_i > 0, \ b_i \in \mathbb{R}\}$. It is known that the Euclidean distance function $\text{dist}(\cdot, K)$ is locally semiconcave of linear modulus in $\mathbb{R}^n \setminus K$ [8] and its singular set is more difficult to study geometrically than that of the squared Euclidean distance function. It can be easily verified that the weighted squared distance function is globally semiconcave. However, singularities for both of these functions are difficult to study at a 'finite scale'. This is in contrast with the standard Euclidean functions [32].

In [33, 34], we introduced several singularity extraction devices for detecting geometric ridges, valleys, edges for functions and geometric intersections between smooth manifolds defined by their

characteristic functions (point clouds) based on compensated convex transforms. These tools can also be used to measure the strength of singularities of a particular type at a finite scale. In this paper we apply these tools to extract fine geometric singularities from semiconvex/semiconcave functions and from DC-functions. Our results demonstrate that our tight approximations by compensated convex transforms are of very high quality in the sense they can extract geometric information of the original semiconvex/semiconcave functions up to the first order derivative.

We denote by $\mathbb{R}^n$ the standard n-dimensional Euclidean space with standard inner product $x \cdot y$ and norm $|x|$ for $x, y \in \mathbb{R}^n$. We denote by $\bar{A}$ the closure of a set A in $\mathbb{R}^n$ and by $B_r(x)$ and $\bar{B}_r(x)$ the open and closed balls in $\mathbb{R}^n$ centred at $x \in \mathbb{R}^n$ with radius $r > 0$. We also denote by $C^1(\bar{B}_r(x))$ the space of real-valued continuously differentiable functions in an open set containing $\bar{B}_r(x)$ and by $C^{1,1}(\bar{B}_r(x))$ the space of real-valued continuous differentiable functions whose gradients are Lipschitz mappings. Before we state our main results, let us first introduce the notions of compensated convex transforms in $\mathbb{R}^n$. We state the definitions only for functions of linear growth which will cover functions we deal with in this paper. For definitions under more general growth conditions, see [30]. Let $f : \mathbb{R}^n \mapsto \mathbb{R}$ satisfy the linear growth condition $|f(x)| \leq C|x| + C_1$ for some constants $C \geq 0$ and $C_1 > 0$ and for all $x \in \mathbb{R}^n$.

The lower compensated compensated convex transform (lower transform for short) (see [30]) for f is defined for $\lambda > 0$ by

$$C_\lambda^l(f)(x) = \mathsf{co}[f + \lambda| \cdot |^2](x) - \lambda|x|^2, \quad x \in \mathbb{R}^n, \tag{1.1}$$

where $\mathsf{co}[g]$ is the convex envelope [24, 16] of a function $g : \mathbb{R}^n \mapsto (-\infty, +\infty]$, whereas the upper compensated compensated convex transform (upper transform for short) (see [30]) for f is defined for $\lambda > 0$ by

$$C_\lambda^u(f)(x) = \lambda|x|^2 - \mathsf{co}[\lambda| \cdot |^2 - f](x), \quad x \in \mathbb{R}^n. \tag{1.2}$$

The two mixed compensated convex transforms are defined by $C_\tau^u(C_\lambda^l)(f)$ and $C_\tau^l(C_\lambda^u)(f)$ when $\lambda, \tau > 0$.

It is known [33] that the lower and upper transforms are respectively the critical mixed Moreau envelopes [21, 22, 20, 4] and they can be viewed as morphological openings and closings [33] respectively, in mathematical morphology terms [25, 17].

Since our main aim is to describe the behaviour of the ridge, valley and edge transforms for large $\lambda > 0$, we introduce the following local versions of compensated convex transforms. Due to the 'locality property' for compensated convex transforms (see Proposition 2.3 below), it will be obvious later that such definitions do not depend on the choices of domains involved.

Let $\Omega \subset \mathbb{R}^n$ be an open set and let $f : \Omega \mapsto \mathbb{R}$ be a locally Lipschitz function, which is thus bounded on every compact subset of Ω. Assume $x \in \Omega$ and let G be a bounded open subset of Ω such that $x \in G \subset \bar{G} \subset \Omega$. Let $L_G \geq 0$ be the Lipschitz constant of f restricted to $\bar{G}$ denoted by $f|_{\bar{G}} : \bar{G} \mapsto \mathbb{R}$. By Kirszbraun's theorem [11], $f|_{\bar{G}}$ can be extended to $\mathbb{R}^n$ as a Lipschitz continuous function $f_G : \mathbb{R}^n \mapsto \mathbb{R}$ with the same Lipschitz constant L_G. Of course such an extension is not unique. However, due to the locality property of compensated convex transforms, our results are independent of the Lipschitz extensions given by Kirszbraun's theorem and the choices of G.

Now we define the local lower compensated convex transform (local lower transform for short) and the local upper compensated convex transforms (local upper transform for short) for a locally Lipschitz function $f : \Omega \mapsto \mathbb{R}$ at $x \in \Omega$ with respect to G respectively by

$$C_{\lambda,G}^l(f)(x) = C_\lambda^l(f_G)(x) \quad \text{and} \quad C_{\lambda,G}^u(f)(x) = C_\lambda^u(f_G)(x), \quad x \in \mathbb{R}^n. \tag{1.3}$$

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In [33] we introduced the notions of the ridge transform $R_\lambda(f)$, the valley $V_\lambda(f)$ transform and the edge transform $E_\lambda(f)$, respectively, as

$$R_\lambda(f)(x) = f(x) - C_\lambda^l(f)(x), \qquad V_\lambda(f)(x) = C_\lambda^u(f)(x) - f(x),$$

$$E_\lambda(f)(x) = C_\lambda^u(f)(x) - C_\lambda^l(f)(x) = R_\lambda(f)(x) + V_\lambda(f)(x) \tag{1.4}$$

for $x \in \mathbb{R}^n$.

We should point out that our valley transform defined here is always non-negative and there is a sign difference in comparison with the valley transform defined in [33]. Given an open set $\Omega \subset \mathbb{R}^n$ and a locally Lipschitz function $f : \Omega \mapsto \mathbb{R}$, we also define the local versions of ridge, valley and edge transforms as follows.

Definition 1.1. *For $x \in \Omega$ and for a fixed open set G whose closure is compact and G satisfies $x \in G \subset \bar{G} \subset \Omega$, we define the local ridge, valley and edge transforms of f at x with respect to G respectively as*

$$R_{\lambda,G}(f)(x) = R_\lambda(f_G)(x), \quad V_{\lambda,G}(f)(x) = V_\lambda(f_G)(x), \quad E_{\lambda,G}(f)(x) = E_\lambda(f_G)(x). \tag{1.5}$$

Suppose $f : \mathbb{R}^n \mapsto \mathbb{R}$ is a Lipschitz function with Lipschitz constant $L \geq 0$. It was established in [33, Theorem 2.12 (iii)] that

$$C_\lambda^l(f)(x) \leq f(x) \leq C_\lambda^l(f)(x) + \frac{L^2}{4\lambda}, \quad C_\lambda^u(f)(x) - \frac{L^2}{4\lambda} \leq f(x) \leq C_\lambda^u(f)(x), \tag{1.6}$$

for $\lambda > 0$. Hence, the following estimates also hold [33]

$$0 \leq R_\lambda(f)(x) \leq \frac{L^2}{4\lambda}, \qquad 0 \leq V_\lambda(f)(x) \leq \frac{L^2}{4\lambda} \tag{1.7}$$

for $\lambda > 0$, and at every point $x_0 \in \mathbb{R}^n$ where f is differentiable, we have

$$\lim_{\lambda \to \infty} \lambda R_\lambda(f)(x_0) = 0 \quad \text{and} \quad \lim_{\lambda \to \infty} \lambda V_\lambda(f)(x_0) = 0, \quad \text{hence} \quad \lim_{\lambda \to \infty} \lambda E_\lambda(f)(x_0) = 0. \tag{1.8}$$

For convenience later we call the quantities $\lambda R_\lambda(f)$, $\lambda V_\lambda(f)$ and $\lambda E_\lambda(f)$ the scale 1-ridge, -valley and -edge transforms, respectively.

We will need also the following result on the minimal bounding sphere for a compact set in $\mathbb{R}^n$. The question was first asked by J. J. Sylvester in a two line statement [27] in 1857 for finite sets in the plane, which he then studied in his 1860 paper [28]. The general result was proved by Jung in 1901 [18]. There are however many later elementary proofs [7, 29, 9] by using Helly's theorem [14].

Lemma 1.2. *([18, 7, 29, 9]) Let $K \subset \mathbb{R}^n$ be a non-empty compact set. Then*

(i) *There is a unique minimal closed ball $\bar{B}_r(y_0)$ containing K in the sense that $\bar{B}_r(y_0)$ is the closed ball containing K with the smallest radius. The sphere $S_r(x_0) := \partial B_r(x_0)$ is called the minimal bounding sphere of K.*

(ii) *Let d be the diameter of K, then $r \leq \sqrt{\frac{n}{2(n+1)}}d$.*

(iii) *The centre of the ball x_0 satisfies $x_0 \in \mathrm{co}[K \cap S_r(x_0)]$, the convex hull of $K \cap S_r(x_0)$.*

The proofs of Lemma 1.2(*i*) and (*ii*) can be found in [7] while for the proof of (*iii*) we refer to [9, 2.6 and 6.1] or [12, Lemma 2].

In this paper we will consider semiconvex and semiconcave functions, which are defined as follows [8, 1]

Definition 1.3. *Let* $\Omega \subset \mathbb{R}^n$ *be a non-empty open convex domain.*

(*i*) *A function* $f : \Omega \mapsto \mathbb{R}$ *is called* **semiconvex** *in* Ω *with modulus* ω *if there is a non-decreasing upper semicontinuous function* $\omega : [0, +\infty) \mapsto [0, +\infty)$ *such that* $\lim_{t \to 0+} \omega(t) = 0$ *and*

$$sf(x) + (1 - s)f(y) - f(sx + (1 - s)y) \geq -s(1 - s)|x - y|\omega(|x - y|) \qquad (1.9)$$

for all x, $y \in \Omega$ *and for all* $0 \leq s \leq 1$.

(*ii*) *A function* $f : \Omega \mapsto \mathbb{R}$ *is* **semiconcave** *in* Ω *with modulus* ω *if* $-f$ *is semiconvex with modulus* ω.

(*iii*) *When* $\omega(r) = \lambda_0 r$ *for* $r \geq 0$ *and for some* $\lambda_0 \geq 0$, *we say that* $f : \Omega \mapsto \mathbb{R}$ *is* $2\lambda_0$-**semiconvex with linear modulus** *[8] (*$2\lambda_0$*-semiconvex for short). In this case, there is a convex function* $g : \Omega \mapsto \mathbb{R}$ *such that* $f(x) = g(x) - \lambda_0|x|^2$ *for all* $x \in \Omega$ *[8, Propostion 1.1.3].*
A function f *is* $2\lambda_0$-**semiconcave with linear modulus** *(*$2\lambda_0$*-semiconcave for short) if* $-f$ *is* $2\lambda_0$*-semiconvex with linear modulus. In this case, there is a concave function* $g : \Omega \mapsto \mathbb{R}$ *such that* $f(x) = g(x) + \lambda_0|x|^2$ *for all* $x \in \Omega$ *[8, Propostion 1.1.3].*

(*iv*) *A function* $f : \Omega \mapsto \mathbb{R}$ *is called* **locally semiconvex** *(respectively,* **locally semiconcave***) in* Ω *if, on every convex compact set* $K \subset \Omega$, f *is semiconvex (respectively, semiconcave) with a modulus* ω_K *depending on* K.

(*v*) *A function* $f : \Omega \mapsto \mathbb{R}$ *is called locally semiconvex (respectively, locally semiconcave) with linear modulus if for every convex compact subset* $K \subset \Omega$, *there is a constant* $\lambda_K \geq 0$ *and a convex function (respectively, concave function)* $g_K : K \mapsto \mathbb{R}$ *such that when* $x \in K$, *we have* $f(x) = g_K(x) - \lambda_K|x|^2$ *(respectively,* $f(x) = g_K(x) + \lambda_K|x|^2$*).*

From Definition 1.3, it can be easily seen that the lower and upper compensated convex transforms with scale $\lambda > 0$ are 2λ-semiconvex and 2λ-semiconcave functions, respectively. In fact, they are 2λ-semiconvex and 2λ-semiconcave 'envelopes' of the given function.

Let $\Omega \subset \mathbb{R}^n$ be a non-empty open convex set. We also recall [1, pag. 221] that a locally semiconvex/semiconcave function $f : \Omega \to \mathbb{R}^n$ is locally Lipschitz continuous in Ω, that is, in every compact subset $K \subset \Omega$, f is a Lipschitz function on K.

The following is our main result on local approximations and geometric singular extraction of semiconvex functions by the upper transform. The result regards the Fréchet subdifferential of semiconvex functions. For its definition, we refer to Definition 2.9 below and to its characterization (2.11).

Theorem 1.4. (*i*) *Let* $\Omega \subset \mathbb{R}^n$ *be a non-empty open convex domain. Suppose* $f : \Omega \mapsto \mathbb{R}$ *is a locally semiconvex function in* Ω. *Let* $x_0 \in \Omega$ *be a non-differentiable (singular) point of* f. *Then for every bounded open set* $G \subset \Omega$ *such that* $x_0 \in G \subset \bar{G} \subset \Omega$,

$$\lim_{\lambda \to +\infty} \lambda V_{\lambda,G}(f)(x_0) = \frac{r_{x_0}^2}{4}, \qquad (1.10)$$

where $r_{x_0} > 0$ is the radius of the minimal bounding sphere of the subdifferential $\partial_- f(x_0)$ of f at x_0.

(ii) *Assume that $f : \Omega \to \mathbb{R}$ is a locally semiconvex function with linear modulus in Ω, i.e. on every convex compact subset K of Ω, there exists $\lambda_K \geq 0$ such that $f(x) = g_K(x) - \lambda_K |x|^2$ for $x \in K$, where $g_K : K \to \mathbb{R}$ is a convex continuous function on K, and let $x_0 \in \Omega$ be a non-differentiable (singular) point of f. Then for every bounded open set $G \subset \Omega$ such that $x_0 \in G \subset \bar{G} \subset \Omega$,*

$$\lim_{\lambda \to +\infty} \nabla C^u_{\lambda,G}(f)(x_0) = y_0, \tag{1.11}$$

where $y_0 \in \partial_- f(x_0)$ is the centre of the minimal bounding sphere of $\partial_- f(x_0)$.

A similar result holds also for locally semiconcave functions, with the differences that we have to replace the valley transform by the ridge transform so that (i) of Theorem 1.4 reads

$$\lim_{\lambda \to +\infty} \lambda R_{\lambda,G}(f)(x_0) = \frac{r^2_{x_0}}{4}, \tag{1.12}$$

with $r_{x_0} > 0$ the radius of the minimal bounding sphere of the (Fréchet) superdifferential $\partial_+ f(x_0)$ of the locally semiconcave function f at x_0 (see Definition 2.10 below), while (ii) becomes

$$\lim_{\lambda \to +\infty} \nabla C^l_{\lambda,G}(f)(x_0) = y_0, \tag{1.13}$$

with $y_0 \in \partial_+ f$ the centre of the minimal bounding sphere of $\partial_+ f(x_0)$.

Since near every point $x \in G$, with G a bounded open subset of Ω such that $x \in G \subset \bar{G} \subset \Omega$, $C^u_\lambda(f_G)$ is a C^1 function in any given neighbourhood $B_r(x) \subset \bar{B}_r(x) \subset G$ for sufficiently large $\lambda > 0$ due to the locality property (see Proposition 2.3 below), $C^u_\lambda(f_G)$ realizes a locally smooth approximation from above and the error of the approximation satisfies

$$\lambda V_\lambda(f_G)(x) = \lambda(C^u_\lambda(f_G)(x) - f_G(x)) \to r^2_x/4 \quad \text{for } \lambda \to +\infty$$

at a singular point $x \in G$.

In order to help readers to have an intuitive view on compensated convex transforms, the ridge/valley transforms and their limit for semiconvex/semiconcave functions, we consider the following simple example first.

Example 1.5. *Let $f(x) = |x|$ for $x \in \mathbb{R}$. Clearly, f is a convex function. For $\lambda > 0$, we have*

$$C^u_\lambda(f)(x) = \begin{cases} \lambda x^2 + \dfrac{1}{4\lambda}, & |x| \leq \dfrac{1}{2\lambda}, \\ |x|, & |x| \geq \dfrac{1}{2\lambda}; \end{cases} \qquad \lambda V_\lambda(f)(x) = \begin{cases} \lambda^2 \left(|x| - \dfrac{1}{2\lambda} \right)^2, & |x| \leq \frac{1}{2\lambda}, \\ 0, & |x| \geq \frac{1}{2\lambda}; \end{cases}$$

$$\lim_{\lambda \to +\infty} \frac{d}{dx} C^u_\lambda(f)(x) = \begin{cases} -1, & x < 0, \\ 0, & x = 0, \\ 1, & x > 0. \end{cases} \qquad \lim_{\lambda \to +\infty} \lambda V_\lambda(f)(x) = \begin{cases} \dfrac{1}{4}, & x = 0, \\ 0, & x \neq 0; \end{cases}$$

$$\tag{1.14}$$

For this example the subdifferential of f at 0 is given by $\partial_- f(0) = [-1, 1]$. Thus the smallest closed interval which contains $\partial_- f(0)$ coincides with $\partial_- f(0)$ itself, with the mid point 0 and radius 1. Note also that Theorem 1.4(i) and (ii) hold in this case. □

There are many examples of locally semiconvex/semicave functions [8]. Suppose $\Omega \subset \mathbb{R}^n$ is open and $K \subset \mathbb{R}^m$ is compact. If $F : K \times \Omega \mapsto \mathbb{R}$ and $\nabla_x F$ are both continuous in $K \times \Omega$, then $f(x) = \sup_{s \in K} F(s, x)$ is locally semiconvex. If $\nabla_x^2 F$ also exists and is continuous in $K \times \Omega$, then f is locally semiconvex with linear modulus (see [8, Proposition 3.4.1]).

The following are two important examples on extraction of geometric singular points arising from applications. They refer to the square distance function and to the distance function to a closed set $K \subset \mathbb{R}^n$.

Example 1.6. *Let $K \subset \mathbb{R}^n$ be a non-empty closed set, satisfying $K \neq \mathbb{R}^n$ and denote by $\mathrm{dist}^2(\cdot, K)$ the squared Euclidean distance function to K. Let $M_K := \{y \in \mathbb{R}^n, \exists z_1, z_2 \in K, z_1 \neq z_2 \, \mathrm{dist}(x, K) = |y - z_1| = |y - z_2|\}$ be the medial axis of K. It is known that M_K is the singular set of $\mathrm{dist}^2(\cdot, K)$. In [32] we have the following Luzin type theorem. Let $\lambda > 0$. If we define*

$$V_{\lambda, K} := \{x \in \mathbb{R}^n, \quad \lambda \mathrm{dist}(x, M_K) \leq \mathrm{dist}(x, K)\},$$

then

$$\mathrm{dist}^2(x, K) = C_\lambda^l(\mathrm{dist}(\cdot, K))(x)$$

for $x \in \mathbb{R}^n \setminus V_{\lambda, K}$ and

$$\bar{M}_K = \cap_{\lambda > 0}^\infty V_{\lambda, K}.$$

As a result, we have [32]

$$\mathrm{dist}^2(\cdot, K) \in C^{1,1}(\mathbb{R}^n \setminus V_{\lambda, K})$$

and

$$|\nabla \mathrm{dist}^2(x, K) - \nabla \mathrm{dist}^2(y, K)| \leq 2\max\{1, \lambda\}|x - y|, \quad x, y \in \mathbb{R}^n \setminus V_{\lambda, K}.$$

Since the proof of this result relies on the special geometric features of the squared Euclidean distance function, in [32] we have not been able to extend this result to more general semiconcave functions. We have therefore defined the (quadratic) medial axis map as $M_\lambda(x, K) = (1 + \lambda)R_\lambda(\mathrm{dist}^2(\cdot, K)(x)$ and proved that

$$\lim_{\lambda \to +\infty} M_\lambda(x, K) = \mathrm{dist}^2(x, K) - \mathrm{dist}^2(x, \mathsf{co}[K(x)]), \tag{1.15}$$

where $\mathsf{co}[K(x)]$ is the convex hull of the compact set $K(x) = \{y \in K, \mathrm{dist}(x, K) = |x - y|\}$.

We can now interpret the limit (1.15) by applying Theorem 1.4(i). Since $\lim_{\lambda \to +\infty} R_\lambda(\mathrm{dist}^2(\cdot, K)(x) = 0$, by the definition of $M_\lambda(x, K)$ we have

$$\lim_{\lambda \to +\infty} M_\lambda(x, K) = \lim_{\lambda \to +\infty} \lambda R_\lambda(\mathrm{dist}^2(\cdot, K)(x),$$

where $\lambda R_\lambda(\mathrm{dist}^2(\cdot, K)(x)$ is our scale 1-ridge transform. Now, for $x \in M_K$, the superdifferential of $\mathrm{dist}^2(\cdot, K)$ at x is given by $\partial_+ \mathrm{dist}^2(x, K) = \mathsf{co}\{2(x - y), y \in K(x)\}$ so that the square r_x^2 of the radius of the minimum bounding sphere of $\partial_+ \mathrm{dist}^2(x, K)$ is $4\left(\mathrm{dist}^2(x, K) - \mathrm{dist}^2(x, \mathsf{co}[K(x)])\right)$. Thus $r_x^2/4$, which is the limit of the scale 1-ridge transform (see (1.12)), is the same as $\mathrm{dist}^2(x, K) - \mathrm{dist}^2(x, \mathsf{co}[K(x)])$ (see (1.15), [32, Theorem 3.23]).

Example 1.7. *In this example, we consider the case of the Euclidean distance function $\mathrm{dist}(x, K)$ itself. It is then known [8, Proposition 2.2.2] that $\mathrm{dist}(\cdot, K)$ is locally semiconcave with linear modulus in $\mathbb{R}^n \setminus K$. Therefore if we consider the limit of the scale 1-ridge transform, by Theorem*

1.4 applied to semiconcave functions, we have (see (1.12))

$$\lim_{\lambda\to+\infty} \lambda R_\lambda(\mathrm{dist}(\cdot,K))(x) = r_x^2/4 \quad and \quad \lim_{\lambda\to+\infty} \nabla C_\lambda^l(\mathrm{dist}(\cdot,K))(x) = y_x, \quad x \notin K,$$

where r_x is the radius of the minimal bounding sphere of the superdifferential $\partial_+\mathrm{dist}(x,K)$ and y_x is the centre of the minimal bounding sphere. Since $\partial_+\mathrm{dist}(x,K) = \mathsf{co}\{(x-y)/|x-y|, \ \mathrm{dist}(x,K) = |x-y|\}$, if we let $p_x \in \mathsf{co}[K(x)]$ be the unique closest point from x to $\mathsf{co}[K(x)]$, then we have

$$r_x^2 = \frac{\mathrm{dist}^2(x,\,K) - \mathrm{dist}^2(x,\,\mathsf{co}[K(x)])}{\mathrm{dist}^2(x,\,K)}, \quad y_x = \frac{p_x}{\mathrm{dist}(x,\,K)}. \tag{1.16}$$

By comparing (1.15) and (1.16) we find that for $x \notin K$

$$\lim_{\lambda\to+\infty} \lambda R_\lambda(\mathrm{dist}(\cdot,K))(x) = \frac{\displaystyle\lim_{\lambda\to+\infty} \lambda R_\lambda(\mathrm{dist}^2(\cdot,K))(x)}{4\mathrm{dist}^2(x,\,K)},$$

and

$$\lim_{\lambda\to+\infty} \nabla C_\lambda^l(\mathrm{dist}(\cdot,K))(x) = \frac{\displaystyle\lim_{\lambda\to+\infty} \nabla C_\lambda^l(\mathrm{dist}^2(\cdot,K))(x)}{2\mathrm{dist}(x,\,K)}$$

whereas for $x \in K$, we have that $R_\lambda(\mathrm{dist}(\cdot,K))(x) = R_\lambda(\mathrm{dist}^2(\cdot,K))(x) = 0$ as points in K are minimum points of both the distance function and the squared distance function [30]. We can conclude therefore that Theorem 1.4 links the asymptotic behaviours of $C_\lambda^l(\mathrm{dist}(\cdot,K))(x)$ and $C_\lambda^l(\mathrm{dist}^2(\cdot,K))(x)$, with the latter which is much easier to analyse [32]. □

For DC-functions, that is, functions that can be represented as difference between two convex functions, we have the following sufficient condition for extracting edges.

Corollary 1.8. *Let $\Omega \subset \mathbb{R}^n$ be a non-empty open convex set. Assume $g, h : \Omega \mapsto \mathbb{R}$ are finite continuous convex functions in Ω and let $f(x) = g(x) - h(x)$ for $x \in \Omega$. Take $x_0 \in \Omega$ and $G \subset \Omega$ an open bounded set such that $x_0 \in G \subset \bar{G} \subset \Omega$. Let r_{g,x_0} and r_{h,x_0} be the radii of the minimal bounding spheres of $\partial_-g(x_0)$ and $\partial_-h(x_0)$, respectively. Then,*

$$\liminf_{\lambda\to+\infty} \lambda E_\lambda f_G(x_0) \geq \frac{(r_{g,x_0} - r_{h,x_0})^2}{4}. \tag{1.17}$$

Remark 1.9. *It is easy to see that the lower bound in (1.17) is sharp. If we set $g(x) = h(x) = |x|$ for $x \in \mathbb{R}$, $f \equiv 0$, thus $r_{g,0} = r_{h,0} = 1$ while $E_\lambda(f)(0) = 0$ for all $\lambda > 0$. However, when $r_{g,x_0} = r_{h,x_0}$, there are simple examples that show that the left hand side of (1.17) may be strictly positive. For example, if we let $F(x,y) = |x| - |y|$ in $\mathbb{R}^2$ and let $f(x) = |x|$, it is easy to see that $E_\lambda(F)(x,y) = V_\lambda(f)(x) + V_\lambda(f)(y)$, hence by (1.14), we have $\lim_{\lambda\to+\infty} \lambda E_\lambda(F)(0,0) = 1/2 > 0$. Note that if we write $f_1(x,y) = f(x)$ and $f_2(x,y) = f(y)$, we have $\partial_-f_1(0,0) = [-1,1] \times \{0\}$ while $\partial_-f_2(0,0) = \{0\} \times [-1,1]$. The minimal bounding sphere for both $\partial_-f_1(0,0)$ and $\partial_-f_2(0,0)$ is the unit sphere in $\mathbb{R}^2$, thus $r_{f_1,0} = r_{f_2,0}$. In general, it would be rather technical to analyse the left-hand side of (1.17) based on the subdifferentials $\partial_-g(x_0)$ and $\partial_-h(x_0)$ [15]. We will not consider this case here.*

We say that compensated convex transforms are 'tight approximations' for a given function. Roughly speaking for functions that are locally of class $C^{1,1}$ near x_0, then there is a finite $\Lambda > 0$,

such that $C_\lambda^u(f)(x_0) = f(x_0) = C_\lambda^l(f)(x_0)$ whenever $\lambda \geq \Lambda$ [30, Theorem 2.3(iv)]. This implies that at a smooth point, the graph of the upper/lower transform is tightly attached to that of the original function from above/below. If $f : \Omega \mapsto \mathbb{R}$ is locally a semiconvex/semiconcave function with linear modulus, where $\Omega \subset \mathbb{R}^n$ is a non-empty convex open set, then according to the well-known Alexandrov's theorem [10, 8], f is twice differentiable almost everywhere in Ω, that is, for almost every $x_0 \in \Omega$, there is some $p \in \mathbb{R}^n$ and an $n \times n$ symmetric matrix B such that

$$\lim_{x \to x_0} \frac{f(x) - f(x_0) - p \cdot (x - x_0) - (x - x_0) \cdot B(x - x_0)}{|x - x_0|^2} = 0 \,. \tag{1.18}$$

We say that $x_0 \in \Omega$ is an Alexandrov point if (1.18) holds.

Proposition 1.10. *Let $\Omega \subset \mathbb{R}^n$ be a non-empty open convex set. Suppose $f : \Omega \mapsto \mathbb{R}$ is a locally semiconvex/semiconcave function of linear modulus. Assume $x_0 \in \Omega$ and G a bounded open subset of Ω such that $x_0 \in G \subset \bar{G} \subset \Omega$. If x_0 is an Alexandrov point, there is a constant $\Lambda > 0$, such that when $\lambda \geq \Lambda$, we have*

$$f(x_0) = C_\lambda^u(f_G)(x_0) = C_\lambda^l(f_G)(x_0), \tag{1.19}$$

and

$$\nabla f(x_0) = \nabla C_\lambda^u(f_G)(x_0) = \nabla C_\lambda^l(f_G)(x_0). \tag{1.20}$$

Remark 1.11. *(i) For a locally semiconvex function f with linear modulus, it is not difficult to show that by the locality property, for every fixed $x \in G$, when $\lambda > 0$ is sufficiently large, $f(x) = C_\lambda^l(f_G)(x)$. The slightly more involved part is to show that also the upper transform $C_\lambda^u(f_G)(x)$ attains the value $f(x)$ for a finite $\lambda > 0$ at an Alexandrov point.*

(ii) Theorem 1.4, Proposition 1.10 and (1.8) provide a clearer picture on how compensated convex transforms approach a locally semiconvex function with linear modulus.

(iii) Since at every point $x \in G \subset \bar{G} \subset \Omega$, $\lim_{\lambda \to +\infty} \lambda V_{\lambda,G}(f)(x)$ and $\lim_{\lambda \to +\infty} \lambda R_{\lambda,G}(f)(x)$ exist, we can define the 'valley landscape map' and the 'ridge landscape map' for locally semiconcovex and locally semiconcave functions with general modulus, respectively, by

$$\mathcal{V}_\infty(f)(x) = \lim_{\lambda \to +\infty} \lambda V_{\lambda,G}(f)(x), \quad \mathcal{R}_\infty(f)(x) = \lim_{\lambda \to +\infty} \lambda R_{\lambda,G}(f)(x), \tag{1.21}$$

Due to the locality property, the limits (1.21) are independent of the choice of G.

(iv) From the definition of the 'valley landscape map' of a semiconvex function f, we can identify at least three distinct features:

(a) $\lambda V_{\lambda,G}(f)(x) = 0$ in finite time $\lambda > 0$ if x is an Alexandrov point;

(b) If f is differentiable at x and $\lambda V_{\lambda,G}(f)(x) > 0$ for all $\lambda > 0$, then $\lim_{\lambda \to +\infty} \lambda V_{\lambda,G}(f)(x) = 0$;

(c) If f is not differentiable at x, then $\lim_{\lambda \to +\infty} \lambda V_{\lambda,G}(f)(x) = r_x^2/4 > 0$.

Therefore, for large $\lambda > 0$, subject to the boundary effect for points near ∂G, the set $\{x \in G, \lambda V_{\lambda,G}(f)(x) > \epsilon\}$ for a fixed $\epsilon > 0$ contains both singular points of f in G and points of high curvature, that is, either $\nabla^2 f(x)$ does not exist or the largest eigenvalue of $\nabla^2 f(x)$ is very large.

In Section 2, we introduce some further preliminary results which are needed for the proofs of our main results Theorem 1.4 and Corollary 1.8. We prove our results in Section 3.

2   Some preliminary results

In this section, we collect some basic properties of compensated convex transforms which will be needed in the following, and refer to [30, 33, 34] for proofs and details.

The ordering property of compensated convex transforms holds for $x \in \mathbb{R}^n$ and reads as

$$C_\lambda^l(f)(x) \leq C_\tau^l(f)(x) \leq f(x) \leq C_\tau^u(f)(x) \leq C_\lambda^u(f)(x), \quad \tau \geq \lambda.$$

The upper and lower transform for functions $f : \mathbb{R}^n \mapsto \mathbb{R}$ with quadratic growth, i.e. $|f(x)| \leq C(1 + |x|^2)$ for $x \in \mathbb{R}^n$ and for a constant $C \geq 0$, are related to each other when $\lambda > 0$ is large enough by the following relation

$$C_\lambda^l(f)(x) = -C_\lambda^u(-f)(x).$$

If f is a continuous function with quadratic growth,

$$\lim_{\lambda \to \infty} C_\lambda^l(f)(x) = f(x), \quad \lim_{\lambda \to \infty} C_\lambda^u(f)(x) = f(x), \quad x \in \mathbb{R}^n.$$

If f and g are both Lipschitz functions, then for $\lambda > 0$ and $\tau > 0$, we have

$$C_{\lambda+\tau}^l(f + g) \geq C_\lambda^l(f) + C_\tau^l(g), \quad C_{\lambda+\tau}^u(f + g) \leq C_\lambda^u(f) + C_\tau^u(g). \tag{2.1}$$

We recall from [5] the following definition.

Definition 2.1. *We say that $f : \mathbb{R}^n \mapsto \mathbb{R}$ is upper semi-differentiable at $x_0 \in \mathbb{R}^n$ if there is some $u \in \mathbb{R}^n$ such that*

$$\limsup_{y \to 0} \frac{f(x_0 + y) - f(x_0) - u \cdot y}{|y|} \leq 0.$$

The following differentiability property [19, pag 726] and more generally [5, Corollary 2.5] is useful in the proofs of our results.

Lemma 2.2. *Suppose $g : B_r(x_0) \mapsto \mathbb{R}$ is convex and $f : B_r(x_0) \mapsto \mathbb{R}$ is upper semi-differentiable at x_0, such that $g \leq f$ on $B_r(x_0)$ and $g(x_0) = f(x_0)$. Then f and g are both differentiable at x_0 and $\nabla f(x_0) = \nabla g(x_0)$.*

Note that concave functions are upper semi-differentiable.

We recall the following locality property of the compensated convex transforms for Lipschitz continuous functions. A similar result for bounded functions was established in [33].

Proposition 2.3. *Suppose $f : \mathbb{R}^n \mapsto \mathbb{R}$ is Lipschitz continuous with Lipschitz constant $L > 0$. Let $\lambda > 0$ and $x \in \mathbb{R}^n$. Then there exist $(\tau_i, y_i) \in \mathbb{R} \times \mathbb{R}^n$, $i = 1, \dots, n+1$, such that*

$$\mathsf{co}[f + \lambda|(\cdot) - x|^2](x) = \mathsf{co}_{\bar{B}_{r_\lambda}(x)}[f + \lambda|(\cdot) - x|^2](x)$$

$$:= \inf \left\{ \sum_{i=1}^{n+1} \tau_i [f(y_i) + \lambda|y_i - x|^2] : y_i \in \mathbb{R}^n, \, \tau_i \geq 0, \, |y_i - x| \leq r_\lambda, \right. \tag{2.2}$$

$$\left. \sum_{i=1}^{n+1} \tau_i = 1, \, \sum_{i=1}^{n+1} \tau_i y_i = x \right\}$$

where $r_\lambda = (2 + \sqrt{2})L/\lambda$.
Furthermore, there is an affine function $y \mapsto \ell(y) = a \cdot (y - x) + b$ for $y \in \mathbb{R}^n$ with $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$ such that

   (i) $\ell(y) \leq f(y) + \lambda|y - x|^2$ for all $y \in \mathbb{R}^n$;

   (ii) $\ell(x_i) = f(x_i) + \lambda|x_i - x|^2$ for $i = 1, \dots, n+1$;

   (iii) $b = \ell(x) = \mathsf{co}[f + \lambda|(\cdot) - x|^2](x)$.

We call $\mathsf{co}_{\bar{B}_{r_\lambda}(x)}[\lambda|(\cdot) - x|^2 + f](x)$ defined in (2.2) the *local convex envelope* of $y \in \mathbb{R}^n \mapsto \lambda|y - x|^2 + f(y)$ at x in $\bar{B}_{r_\lambda}(x)$.

Remark 2.4. *(i) The locality property given in Proposition 2.3 also applies to the compensated convex transforms. Due to the translation invariance property [33], for every fixed $x_0 \in \mathbb{R}^n$, we have*

$$\begin{aligned} C_\lambda^l(f)(x) &= \mathsf{co}[f + \lambda|(\cdot) - x_0|^2](x) - \lambda|x - x_0|^2, \\ C_\lambda^u(f)(x) &= \lambda|x - x_0|^2 - \mathsf{co}[\lambda|(\cdot) - x_0|^2 - f](x), \end{aligned} \tag{2.3}$$

thus, if we take $x_0 = x$, we obtain

$$C_\lambda^l(f)(x) = \mathsf{co}[f + \lambda|(\cdot) - x|^2](x), \quad C_\lambda^u(f)(x) = -\mathsf{co}[\lambda|(\cdot) - x|^2 - f](x), \tag{2.4}$$

and (2.2) can be used.

   (ii) A consequence of [30, Remark 2.1] is that if f is continuous and with linear growth, then the infimun in the definition of the convex envelope of the function $y \in \mathbb{R}^n \mapsto \lambda|y - x|^2 + f(y)$ at $y = x$ is attained by some $\lambda_i > 0$, $x_i \in \mathbb{R}^n$, $i = 1, \dots, k$ with $2 \leq k \leq n+1$ (see [16, 24]), that is,

$$\mathsf{co}_{\bar{B}_{r_\lambda}(x)}[f + \lambda|(\cdot) - x|^2](x) = \sum_{i=1}^{k} \lambda_i [f(x_i) + \lambda|x_i - x|^2]$$

with $|x_i - x| < r_\lambda$, $i = 1, \dots, k$ with $2 \leq k \leq n+1$, and $\sum_{i=1}^k \lambda_i = 1$, $\sum_{i=1}^{n+1} \lambda_i x_i = x$.

The following lemma can be considered a special case of Theorem 1.4.

Lemma 2.5. *Let $S \subset \mathbb{R}^n$ be a non-empty compact convex set, containing more that one element, and denote by $S_r(-a)$ the minimal bounding sphere of S with radius $r > 0$ and centre $-a \in \mathbb{R}^n$.*

Consider the sublinear function $\sigma : x \in \mathbb{R}^n \to \sigma(x) = \max\{p \cdot x, \ p \in S\}$. Then for a fixed $0 \leq \epsilon < \min\{1, r\}$ and for $\lambda > 0$, we have

$$C^u_\lambda(\sigma - \epsilon| \cdot |)(0) = \frac{(r - \epsilon)^2}{4\lambda}, \tag{2.5}$$

$$\nabla C^u_\lambda(\sigma)(0) = -a; \tag{2.6}$$

and for a fixed $0 < \epsilon < \min\{1, r\}$

$$C^u_\lambda(\sigma + \epsilon| \cdot |)(0) \leq C^u_{(1-\epsilon)\lambda}(\sigma)(0) + C^u_{\epsilon\lambda}(\epsilon| \cdot |)(0) = \frac{r^2}{4(1 - \epsilon)\lambda} + \frac{\epsilon}{4\lambda}, \tag{2.7}$$

where

$$C^u_{\epsilon\lambda}(\epsilon| \cdot |)(x) = \begin{cases} \epsilon\lambda|x|^2 + \dfrac{\epsilon}{4\lambda}, & |x| \leq \dfrac{1}{2\lambda}, \\[2mm] \epsilon|x|, & |x| \geq \dfrac{1}{2\lambda}. \end{cases} \tag{2.8}$$

We have also the following local $C^{1,1}$ result for the upper transform of locally semiconvex functions with linear modulus.

Proposition 2.6. *Let $f : \mathbb{R}^n \mapsto \mathbb{R}$ be a Lipschitz continuous function with Lipschitz constant $L \geq 0$. Assume that for some $r > 0$, f is $2\lambda_0$-semiconvex in the closed ball $B_{2r}(0)$, that is, $f(x) = g(x) - \lambda_0|x|^2$ for $x \in \bar{B}_{2r}(0)$, where $\lambda_0 \geq 0$ is a constant and $g : \bar{B}_{2r}(0) \mapsto \mathbb{R}$ is convex. Then for $\lambda \geq \lambda_0$ sufficiently large, $C^u_\lambda(f) \in C^{1,1}(\bar{B}_r(0))$ and*

$$|\nabla C^u_\lambda(f)(x) - \nabla C^u_\lambda(f)(y)| \leq 2\lambda|x - y|, \quad x, y \in \bar{B}_r(0). \tag{2.9}$$

Remark 2.7. *From the proof of Proposition 2.6 (and [30, Theorem 4.1] with a Lipschitz constant less as sharp) we can derive that if $f : \mathbb{R}^n \mapsto \mathbb{R}$ is both Lipschitz continuous and convex, for example if $f(x) = \sigma(x)$ is the sublinear function [16] defined by*

$$\sigma : x \in \mathbb{R}^n \to \sigma(x) = \max\{x \cdot p, \ p \in S\},$$

where S is compact and convex, the estimate (2.9) holds globally in $\mathbb{R}^n$ with $\lambda_0 = 0$.

We conclude this section by recalling the definition and some properties of the subdifferential of convex and semiconvex functions we need in our proofs.

Definition 2.8. *Let $\Omega \subset \mathbb{R}^n$ be a non-empty open convex set. Assume $f : \Omega \mapsto \mathbb{R}$ is convex and let $x \in \Omega$. The subdifferential of f at x, denoted by $\partial_- f(x)$, is the set of $u \in \mathbb{R}^n$ satisfying [16]*

$$f(y) - f(x) - u \cdot (y - x) \geq 0, \quad \text{for all } y \in \Omega.$$

The subdifferential $\partial_- f(x)$ is a non-empty, compact and convex subset of $\mathbb{R}^n$. If we define the sublinear function [16, Chapter D] $y \in \mathbb{R}^n \to \sigma_x(y) := \max\{u \cdot y, \ u \in \partial_- f(x)\}$ then

$$\lim_{h \to 0} \frac{f(x + h) - f(x) - \sigma_x(h)}{|h|} = 0, \tag{2.10}$$

where $\sigma_x(h)$ defines the directional derivative of f at x along $h \in \mathbb{R}^n$.

Just like the convex case, locally semiconvex functions have a natural notion of generalized gradient given by the subdifferential. This is defined as follows.

Definition 2.9. *Let $f : \Omega \mapsto \mathbb{R}^n$ be a locally semiconvex function in Ω and let $x \in \Omega$. Denote by K an open convex subset of Ω such that $x \in K \subset \bar{K} \subset \Omega$ and by ω_K a semiconvex modulus for f in K. The Fréchet subdifferential $\partial_- f$ of f at x is the set of vectors $p \in \mathbb{R}^n$ satisfying*

$$f(y) - f(x) - p \cdot (y - x) \geq -|y - x|\omega_K(|y - x|) \tag{2.11}$$

for any point y such that the segment of ends y and x is contained in K.

It is not difficult to show that the definition of $\partial_- f(x_0)$ does not depend on K, in fact, condition (2.11) can be expressed in terms of a kind of regularization of the semiconvexity modulus (see [1, Proposition 2.1]). We also have that $\partial_- f(x_0)$ is a non-empty convex compact set. Likewise for convex functions, we can equally define for locally semiconvex functions, the sublinear function $\sigma_x(h) = \max\{p \cdot h, \ p \in \partial_- f(x)\}$. By a similar argument as in the proof of [16, Lemma 2.1.1, Chapter D], we can show that $\sigma_x(h)$ satisfies (2.10) and is therefore referred to as the directional derivative of f along h [8, Theorem 3.36].

In the case of a locally semiconcave function f, we introduce the notion of superdifferential $\partial_+ f$ of f at x as follows.

Definition 2.10. *Let $f : \Omega \mapsto \mathbb{R}^n$ be a locally semiconcave function in Ω and let $x \in \Omega$. Denote by K an open convex subset of Ω such that $x \in K \subset \bar{K} \subset \Omega$ and by ω_K a semiconcave modulus for f in K. The Fréchet superdifferential $\partial_+ f$ of f at x is the set of vectors $p \in \mathbb{R}^n$ satisfying*

$$f(y) - f(x) - p \cdot (y - x) \leq |y - x|\omega_K(|y - x|) \tag{2.12}$$

for any point y such that the segment of ends y and x is contained in K.

Similar observations and properties to $\partial_- f(x)$ can be drawn for $\partial_+ f(x)$.

3 Proofs of results

We first prove the main results Theorem 1.4 and Corollary 1.8 by assuming that other results hold. Then we establish the remaining results.

Proof of Theorem 1.4. *Part (i):* Without loss of generality, we may assume that $x_0 = 0$ is a singular point and $f(0) = 0$. Let G be any bounded open set such that $0 \in G \subset \bar{G} \subset \Omega$ and $r > 0$ be such that $\bar{B}_{2r}(0) \subset G$, and let f be semiconvex in $\bar{B}_{2r}(0)$ with modulus $\omega_r(\cdot)$. Given $x \in \bar{B}_{2r}(0)$, $\partial_- f(x)$ is not empty, thus

$$f(y) - f(x) - p_x \cdot (y - x) \geq -|y - x|\omega_r(|y - x|), \quad y, x \in B_{2r}(0), \ p_x \in \partial_- f(x)$$

hence, $-f$ is upper semi-differentiable in $B_{2r}(0)$. By the locality property (Proposition 2.3) we also have

$$C_\lambda^u(f_G)(x) = \lambda|x|^2 - \mathsf{co}_{\bar{B}_r(0)}[\lambda| \cdot |^2 - f](x)$$

for $x \in \bar{B}_{r/2}(0)$ provided λ is sufficiently large, and

$$\lim_{h \to 0} \frac{f(x) - f(0) - \sigma_0(x)}{|h|} = 0,$$

where $\sigma_0(h) = \max\{p \cdot h, \ p \in \partial_- f(0)\}$. Note that $\partial_- f(0)$ is compact, convex and contains more than one point since we have assumed that 0 is a singular point. Let $r_0 > 0$ be the radius of the minimal bounding sphere of $\partial_- f(0)$. We fix $0 < \epsilon < \min\{1, r_0\}$, then there is $0 < \delta < r/2$ such that $|f(x) - \sigma_0(x)| \le \epsilon|x|$ whenever $x \in \bar{B}_\delta(0)$ as we have assumed that $f(0) = 0$. Thus for $x \in \bar{B}_\delta(0)$,

$$\sigma_0(x) - \epsilon|x| \le f(x) \le \sigma_0(x) + \epsilon|x|.$$

By the locality property, we have, when $\lambda > 0$ is sufficiently large,

$$C^u_\lambda(\sigma_0 - \epsilon|\cdot|)(0) \le C^u_\lambda(f_G)(0) \le C^u_\lambda(\sigma_0 + \epsilon|\cdot|)(0).$$

By (2.7), we have

$$C^u_\lambda(\sigma_0 + \epsilon|\cdot|)(0) \le C^u_{(1-\epsilon)\lambda}(\sigma_0)(0) + C^u_{\epsilon\lambda}(\epsilon|\cdot|)(0) = \frac{r_0^2}{4(1-\epsilon)\lambda} + \frac{\epsilon}{4\lambda}$$

hence we obtain

$$\lambda V_\lambda(f_G)(0) \le \frac{r_0^2}{4(1-\epsilon)} + \frac{\epsilon}{4}.$$

Now by (2.5), we have

$$C^u_\lambda(\sigma_0 - \epsilon|\cdot|)(0) = \frac{(r_0 - \epsilon)^2}{4\lambda},$$

so that

$$\frac{(r_0 - \epsilon)^2}{4} \le \lambda V_\lambda(f_G)(0) \le \frac{r_0^2}{4(1-\epsilon)} + \frac{\epsilon}{4}.$$

Finally we take upper and lower limits first as $\lambda \to +\infty$, then let $\epsilon \to 0+$, we obtain

$$\lim_{\lambda \to +\infty} \lambda V_\lambda(f_G)(0) = r_0^2/4,$$

which completes the proof of *Part (i)*. $\qquad\square$

Part (ii): Let $x_0 \in \Omega$ be a singular point of f and let G be a bounded open convex set such that $x_0 \in G \subset \bar{G} \subset \Omega$. Without loss of generality, we may assume that $x_0 = 0$. Since f is locally semiconvex with linear modulus, we may assume that on $\bar{G}$, $f(x) = g(x) - \lambda_0|x|^2$, where $g : \bar{G} \mapsto \mathbb{R}$ is convex and $\lambda_0 \ge 0$ is a constant. Clearly $\partial_- f(0) = \partial_- g(0)$. As $f(0) = g(0)$, we may further assume that $g(0) = 0$. Let $\sigma(x) = \max\{p \cdot x, \ p \in \partial_- g(0)\}$ be the sublinear function of g at 0.

Now for every fixed $\epsilon > 0$, there is a $\delta > 0$ such that $|g(x) - \sigma(x)| \le \epsilon|x|$ whenever $x \in \bar{B}_\delta(0)$. Therefore we have

$$\sigma(x) - \lambda_0|x|^2 \le f(x) = g(x) - \lambda_0|x|^2 \le \sigma(x) - \lambda_0|x|^2 + \epsilon|x|$$

for $x \in \bar{B}_\delta(0)$. By the locality property, for $x \in \bar{B}_{\delta/2}(0)$, and for sufficiently large $\lambda > 0$, we have

$$C^u_\lambda(\sigma - \lambda_0|\cdot|^2)(x) \le C^u_\lambda(f_G)(x) \le C^u_\lambda(\sigma + \epsilon|\cdot| - \lambda_0|\cdot|^2)(x) \qquad (3.1)$$

Now we apply Proposition 2.6 to $C^u_\lambda(f_G)$, then for large $\lambda > \lambda_0$, $C^u_\lambda(f_G) \in C^{1,1}(\bar{B}_{\delta/2}(0))$. Let $p_\lambda = \nabla C^u_\lambda(f_G)(0)$, we have $|p_\lambda| \le L_G$ and $C^u_\lambda(f_G)$ is an L_G-Lipschitz function (see [33, Theorem 3.12] and [8, Theorem 3.5.3]) and

$$|C^u_\lambda(f_G)(x) - C^u_\lambda(f_G)(0) - p_\lambda \cdot x| \le 2\lambda|x|^2$$

for $x \in \bar{B}_{\delta/2}(0)$. Thus for $x \in \bar{B}_{\delta/2}(0)$, we have

$$
\begin{aligned}
p_\lambda \cdot x &\leq C_\lambda^u(f_G)(x) - C_\lambda^u(f_G)(0) + 2\lambda|x|^2 \\
&\leq C_\lambda^u(\sigma + \epsilon|\cdot| - \lambda_0|\cdot|^2)(x) - C_\lambda^u(\sigma - \lambda_0|\cdot|^2)(0) + 2\lambda|x|^2 \\
&= I - \frac{r_0^2}{4(\lambda + \lambda_0)} + 2\lambda|x|^2.
\end{aligned}
$$

Here we have used the fact that

$$
C_\lambda^u(\sigma - \lambda_0|\cdot|^2)(0) = C_{\lambda+\lambda_0}^u(\sigma)(0) = \frac{r_0^2}{4(\lambda + \lambda_0)}
$$

given by (2.5) with $\epsilon = 0$. By a similar argument to that used to show (2.7), we also have

$$
I = C_\lambda^u(\sigma + \epsilon|\cdot| - \lambda_0|\cdot|^2)(x) \leq C_{(1-\epsilon)\lambda}^u(\sigma - \lambda_0|\cdot|^2)(x) + C_{\epsilon\lambda}^u(\epsilon|\cdot|)(x) = J_1 + J_2
$$

Now

$$
\begin{aligned}
J_1 &= C_{(1-\epsilon)\lambda}^u(\sigma - \lambda_0|\cdot|^2)(x) \\
&= C_{(1-\epsilon)\lambda+\lambda_0}^u(\sigma)(x) - \lambda_0|x|^2 \\
&= \left(C_{(1-\epsilon)\lambda+\lambda_0}^u(\sigma)(x) - C_{(1-\epsilon)\lambda+\lambda_0}^u(\sigma)(0) + a \cdot x \right) + \left(C_{(1-\epsilon)\lambda+\lambda_0}^u(\sigma)(0) - a \cdot x - \lambda_0|x|^2 \right) \\
&\leq 2\left((1-\epsilon)\lambda + \lambda_0 \right)|x|^2 + \frac{r_0^2}{4((1-\epsilon)\lambda + \lambda_0)} - a \cdot x - \lambda_0|x|^2.
\end{aligned}
$$

Here we have used (2.9) and applied Lemma 2.5 to the sublinear function $y \mapsto \sigma(y)$ to obtain that $\nabla C_{(1-\epsilon)\lambda+\lambda_0}^u(\sigma)(0) = -a$, where $-a$ is the centre of the minimal bounding sphere of $\partial_- g(0)$, and $C_{(1-\epsilon)\lambda+\lambda_0}^u(\sigma)(0) = r_0^2/(4((1-\epsilon)\lambda + \lambda_0))$. We will deal with $J_2 = C_{\epsilon\lambda}^u(\epsilon|\cdot|)(x)$ later. Therefore, when $\lambda > \lambda_0$ is sufficiently large, we have

$$
\begin{aligned}
I - \frac{r_0^2}{4(\lambda + \lambda_0)} + 2\lambda|x|^2 &\leq 2((1-\epsilon)\lambda + \lambda_0)|x|^2 + \frac{r_0^2}{4((1-\epsilon)\lambda + \lambda_0)} - a \cdot x + C_{\epsilon\lambda}^u(\epsilon|\cdot|)(x) \\
&\qquad - \frac{r_0^2}{4(\lambda + \lambda_0)} + 2\lambda|x|^2 - \lambda_0|x|^2 \\
&\leq \frac{\epsilon r_0^2}{4(1-\epsilon)\lambda} + 8\lambda|x|^2 + C_{\epsilon\lambda}^u(\epsilon|\cdot|)(x) - a \cdot x,
\end{aligned}
$$

so that

$$
(p_\lambda + a) \cdot x \leq \frac{\epsilon r_0^2}{4(1-\epsilon)\lambda} + 8\lambda|x|^2 + C_{\epsilon\lambda}^u(\epsilon|\cdot|)(x). \tag{3.2}
$$

Now we take

$$
x_\lambda = \frac{p_\lambda + a}{2^5(1 + |a| + L_G)\lambda},
$$

Then $|x_\lambda| \leq 1/(2^4\lambda) < \delta/2$ if $\lambda > \lambda_0$ is sufficiently large. Also $|x_\lambda| < 1/(2\lambda)$ so that

$$
C_{\epsilon\lambda}^u(\epsilon|\cdot|)(x_\lambda) = \epsilon\lambda|x_\lambda|^2 + \frac{\epsilon}{4\lambda}
$$

in the explicit formula (2.8). Thus if we substitute x_λ into (3.2), we obtain

$$\frac{|p_\lambda + a|^2}{2^5(1 + |a| + L_G)\lambda} \le \frac{\epsilon r_0^2}{4(1 - \epsilon)\lambda} + \frac{|p_\lambda + a|^2}{2^7(1 + |a| + L_G)^2\lambda} + \frac{\epsilon\lambda|p_\lambda + a|^2}{2^{10}(1 + |a| + L_G)^2\lambda^2} + \frac{\epsilon}{4\lambda}$$

$$\le \frac{|p_\lambda + a|^2}{2^7(1 + |a| + L_G)\lambda} + \frac{\epsilon|p_\lambda + a|^2}{2^{10}(1 + |a| + L_G)\lambda} + \frac{\epsilon r_0^2}{4(1 - \epsilon)\lambda} + \frac{\epsilon}{4\lambda}.$$

As $0 < \epsilon < 1$, we have

$$|p_\lambda + a|^2 \le 2^6(1 + |a| + L_G)(\lambda + \lambda_0)\left(\frac{\epsilon r_0^2}{4(1 - \epsilon)\lambda} + \frac{\epsilon}{4\lambda}\right).$$

Let $\lambda \to +\infty$ in the inequality above, we obtain

$$\limsup_{\lambda \to +\infty} |p_\lambda + a|^2 \le 2^7(1 + |a| + L_G)\left(\frac{\epsilon r_0^2}{4(1 - \epsilon)} + \frac{\epsilon}{4}\right).$$

Finally, we let $\epsilon \to 0+$ and deduce that $p_\lambda \to -a$ as $\lambda \to +\infty$. Thus

$$\lim_{\lambda \to +\infty} \nabla C_\lambda^u(f_G)(0) = -a$$

with $-a$ the centre of the minimal bounding sphere of $\partial_- g(0)$, which completes the proof of Part (ii). $\qquad\square$.

Remark 3.1. *We do not know whether a version of Theorem 1.4(ii) holds for locally semiconvex functions with general modulus. To establish a similar result by following a similar approach, we need to know the regularity properties of $C_\lambda^u(f_G)(x)$ better in order to make the proof work.*

Proof of Corollary 1.8: Again, without loss of generality, we may assume that $x_0 = 0$ and $r_{g,0} < r_{h,0}$. Since $E_\lambda(f_G)(0) = R_\lambda(f_G)(0) + V_\lambda(f_G)(0) \ge 0$, if $r_{g,0} = r_{h,0}$, (1.17) holds. If $r_{g,0} > r_{h,0}$, as $E_\lambda(f_G) = E_\lambda(-f_G)$, we can reduce the problem to the case $r_{g,0} < r_{h,0}$.

Next we prove, under our assumption that $r_{g,0} < r_{h,0}$ that

$$\liminf_{\lambda \to \infty} \lambda R_\lambda(f_G)(0) \ge (r_{g,0} - r_{h,0})^2/4. \tag{3.3}$$

By the locality property (see Proposition 2.3), if $\bar{B}_r(0) \subset G$ for some $r > 0$, we see that for $\lambda > 0$ sufficiently large, we have

$$\mathsf{co}[f_G + \lambda|\cdot|^2](0) = \mathsf{co}_{\bar{B}_r(0)}[g - h + \lambda|\cdot|^2](0).$$

Let $\sigma_g(x) = \max\{p \cdot x, \ p \in \partial_- g(0)\}$ and $\sigma_h(x) = \max\{p \cdot x, \ p \in \partial_- h(0)\}$ for $x \in \mathbb{R}^n$ be the sublinear functions of g and h at 0 respectively, we have, by (2.10) that for $0 < \epsilon < r_{h,0} - r_{g,0}$, there is a $0 < \delta \le r$, such that

$$\left|\left(g(x) - h(x)\right) - \left(g(0) - h(0)\right) - \left(\sigma_g(x) - \sigma_h(x)\right)\right| \le \epsilon|x|$$

whenever $x \in \bar{B}_\delta(0)$, so that

$$g(x) - h(x) \le \left(\sigma_g(x) - \sigma_h(x)\right) + \epsilon|x| + \left(g(0) - h(0)\right)$$

for $x \in \bar{B}_\delta(0)$. Without loss of generality, we may assume that $f(0) = g(0) - h(0) = 0$.

Again by the locality property, if $\lambda > 0$ is sufficiently large, we have

$$\mathsf{co}[\lambda|\cdot|^2 + f_G](0) = \mathsf{co}_{\bar{B}_\delta(0)}[\lambda|\cdot|^2 + g - h](0) \leq \mathsf{co}[\lambda|\cdot|^2 + \sigma_g - \sigma_h + \epsilon|\cdot|](0).$$

Let a_g be the centre of the minimal bounding sphere of $\partial_-g(0)$ and $\ell(x) = a_g \cdot x$ for $x \in \mathbb{R}^n$, we have

$$\sigma_g(x) = \max\{p \cdot x, \ p \in \partial_-g(0)\} - \ell(x) + \ell(x)$$

$$= \max\{(p - a_g) \cdot x, \ p \in \partial_-g(0)\} + \ell(x)$$

$$\leq r_{g,0}|x| + \ell(x).$$

Since the convex envelope is affine co-variant, that is $\mathsf{co}[H + \ell] = \mathsf{co}[H] + \ell$, we see that

$$\mathsf{co}[\lambda|\cdot|^2 + \sigma_g - \sigma_h + \epsilon|\cdot|](0) \leq \mathsf{co}[\lambda|\cdot|^2 + (r_{g,0} + \epsilon)|\cdot| - \sigma_h](0) + \ell(0).$$

Since $\ell(0) = 0$, $C_\lambda^l(H) = -C_\lambda^u(-H)$ for continuous functions H of linear growth, we may use (2.5) in Lemma 2.5 to obtain

$$C_\lambda^l((r_{g,0} + \epsilon)|\cdot| - \sigma_h)(0) = \mathsf{co}[\lambda|\cdot|^2 + (r_{g,0} + \epsilon)|\cdot| - \sigma_h](0)$$

$$= -C_\lambda^u(\sigma_h - (r_{g,0} + \epsilon)|\cdot|)(0)$$

$$= -\frac{(r_{h,0} - r_{g,0} - \epsilon)^2}{4}.$$

Thus

$$C_\lambda^l(f_G)(0) \leq -\frac{(r_{h,0} - r_{g,0} - \epsilon)^2}{4\lambda}$$

when $\lambda > 0$ is sufficiently large. Therefore

$$\lambda R_\lambda(f_G)(0) \geq \frac{(r_{h,0} - r_{g,0} - \epsilon)^2}{4}.$$

If we let $\lambda \to +\infty$, then let $\epsilon \to 0+$, we have

$$\liminf_{\lambda \to +\infty} \lambda E_\lambda(f_G)(0) \geq \liminf_{\lambda \to +\infty} \lambda R_\lambda(f_G)(0) \geq \frac{(r_{h,0} - r_{g,0})^2}{4}.$$

The proof is finished. □

Proof of Proposition 1.10: Suppose that $f : \Omega \mapsto \mathbb{R}$ is locally semiconvex with linear modulus. Without loss of generality, we assume that $x_0 = 0$ is an Alexandrov point. We set $\lambda_0 = \|B\|$, the operator norm of the symmetric matrix B given by (1.18). For $\epsilon = 1$, by (1.18), there is some $\delta > 0$ such that

$$|f_G(x) - f_G(0) - p \cdot x - x^T B x| \leq \epsilon|x|^2 = |x|^2$$

whenever $x \in \bar{B}_\delta(0)$. Now we consider the affine function $\ell(x) = -f_G(0) - p \cdot x$. Clearly $\ell(0) = -f_G(0)$. We show that $\ell(x) \leq \lambda|x|^2 - f(x)$ for all $x \in \mathbb{R}^n$ when $\lambda > 0$ is large enough, so that $-f_G(0) = \mathsf{co}[\lambda|\cdot|^2 - f_G](0)$ hence $f_G(0) = C_\lambda^u(f_G)(0)$.

We have, in $\bar{B}_\delta(0)$ that

$$-f_G(x) \geq -f_G(0) - p \cdot x - x^T B x - |x|^2 \geq \ell(x) - (\lambda_0 + 1)|x|^2$$

so that

$$\lambda|x|^2 - f_G(x) \geq \ell(x) + (\lambda - \lambda_0 - 1)|x|^2 \geq \ell(x)$$

if $x \in \bar{B}_\delta(0)$ and $\lambda \geq \lambda_0 + 1$.

If $|x| > \delta$, note that since f_G is a Lipschitz function with Lipschitz constant $L_G \geq 0$, we then have

$$\lambda|x|^2 - f_G(x) \geq \lambda|x|^2 - L_G|x| - f_G(0),$$

while $\ell(x) = -f_G(0) - p \cdot x \leq -f_G(0) + |p||x|$. Thus $\lambda|x|^2 - f_G(x) \geq \ell(x)$ if $\lambda|x| - L_G \geq |p|$, which holds if $\lambda\delta \geq L_G + |p|$, that is, $\lambda \geq (L_G + |p|)/\delta$. Thus if

$$\lambda \geq \max\left\{\lambda_0 + 1, \ \frac{L_G + |p|}{\delta}\right\}$$

we have $\lambda|x|^2 - f_G(x) \geq \ell(x)$ for all $x \in \mathbb{R}^n$. Therefore $f_G(0) = C_\lambda^u(f_G)(0)$.

Since in G, $f_G(x) = f(x) = g(x) - \lambda_1|x|^2$ for some convex function $g : \bar{G} \mapsto \mathbb{R}$ and for some $\lambda_1 > 0$, if we let $\ell(x) = g(0) + q \cdot x$ for some $q \in \partial_- g(0)$, then clearly $\ell(0) = g(0) = f_G(0)$. We show that $g(0) + q \cdot x \leq f_G(x) + \lambda|x|^2$ for all $x \in \mathbb{R}^n$, hence $f_G(0) = g(0) = \mathsf{co}[f_G + \lambda|\cdot|^2](0) = C_\lambda^l(f_G)(0)$ when $\lambda > 0$ is sufficiently large.

Since $0 \in G$ and G is open, there is a $\delta > 0$ such that $\bar{B}_\delta(0) \subset G$. Thus in $\bar{B}_\delta(0)$, we have

$$f_G(x) + \lambda|x|^2 = g(x) + (\lambda - \lambda_1)|x|^2 \geq g(x) \geq g(0) + q \cdot x$$

if $\lambda \geq \lambda_1$.

If $|x| > 1$, similar to the proof for the upper transform, again we have $f_G(x) + \lambda|x|^2 \geq \ell(x)$ when $\lambda > 0$ is sufficiently large. Thus $f_G(0) = C_\lambda^l(f_G)(0)$ when $\lambda > 0$ is sufficiently large.

The equalities in (1.20) are direct consequences of Lemma 2.2. Here we have $C_\lambda^l(f_G) \leq f_G \leq C_\lambda^u(f_G)$ and $C_\lambda^l(f_G)(0) = f_G(0) \leq C_\lambda^u(f_G)(0)$, we may deduce that $\nabla C_\lambda^l(f_G)(0) = \nabla C_\lambda^u(f_G)(0) = -p$, hence $\nabla f_G(0) = -p$. $\qquad\square$

Proof of Lemma 2.5: We establish (2.5) first by calculating

$$C_\lambda^u(\sigma - \epsilon|\cdot|)(0) = -\mathsf{co}[\lambda|\cdot|^2 + \epsilon|\cdot| - \sigma](0).$$

We write

$$f_\lambda(x) = \lambda|x|^2 + \epsilon|x| - \sigma(x)$$

for $x \in \mathbb{R}^n$ and let $S = \partial_- f(0)$. Again let $S_r(-a)$ be the minimal bounding sphere of S given by Lemma 1.2. We set

$$b = -\frac{(r - \epsilon)^2}{4\lambda}$$

and define the affine function $\ell(x) = a \cdot x + b$. We show that (i) for $p^* \in S_r(-a) \cap S$, if we let

$$x^* = \frac{(|p^* + a| - \epsilon)}{2\lambda} \frac{p^* + a}{|p^* + a|}, \tag{3.4}$$

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then $f_\lambda(x^*) - a \cdot x^* = b$; and (ii) if x^* is a minimum point of $f_\lambda(x) - a \cdot x$ then there is some $p^* \in S_r(-a) \cap S$ such that x^* satisfies (3.4) and $f_\lambda(x^*) - a \cdot x^* = b$.

We prove (i) first. Suppose (3.4) holds. We have

$$f_\lambda(x^*) - a \cdot x^* = \lambda|x^*|^2 + \epsilon|x^*| - \sigma(x^*) - a \cdot x^*$$

$$= \frac{(|p^* + a| - \epsilon)^2}{4\lambda} - \max\{(p + a) \cdot x^*, \ p \in S\} + \epsilon\frac{|p^* + a| - \epsilon}{2\lambda}$$

$$= \frac{(|p^* + a| - \epsilon)^2}{4\lambda} + \epsilon\frac{|p^* + a| - \epsilon}{2\lambda} - (p^* + a) \cdot x^*$$

$$= -\frac{(|p^* + a| - \epsilon)^2}{4\lambda} = b.$$

Here we have used the facts that x^* is along the direction of $p^* + a$ and $p^* + a \in \partial(S + a)$ is the maximum point of $\max\{(p+a) \cdot x^*, \ p \in S\}$, where $\partial(S+a)$ is the relative boundary of the bounded closed convex set $S + a := \{p + a, \ p \in S\}$.

Since $b < 0$, clearly $x = 0$ is not a minimum point of $f_\lambda(x) - a \cdot x$. As the function $f_\lambda(x) - a \cdot x$ is coercive, and continuous, it reaches its minimum. Let $x^* \neq 0$ be such a point. Let $b' < 0$ be the minimum value of $f_\lambda(x) - a \cdot x$, that is, $f_\lambda(x^*) - a \cdot x^* = b' < 0$. Then as $-\sigma(x)$ is upper semi-differentiable and $\epsilon|x|$ is differentiable for $x \neq 0$, to follows from Lemma 2.2 that $\nabla(f_\lambda(x^*) - a \cdot x^*) = 0$, that is

$$2\lambda x^* + \epsilon\frac{x^*}{|x^*|} - (p^* + a) = 0$$

where $\max\{p \cdot x^*, \ p \in S\} = p^* \cdot x^*$ and $p^* \in \partial S$, that is, p^* must be a relative boundary point of S. Clearly, x^* is along the same direction as $p^* + a$. It is easy to see that

$$|x^*| = \frac{|p^* + a| - \epsilon}{2\lambda} > 0$$

as $x^* \neq 0$. Therefore x^* is given by (3.4). Thus

$$b_\lambda = f_\lambda(x^*) - a \cdot x^* = -\frac{(|p^* + a| - \epsilon)^2}{4\lambda} \geq -\frac{(r_0 - \epsilon)^2}{4\lambda} = b.$$

Thus $b_\lambda = b$, hence $b = \mathsf{co}[f_\lambda](0)$ which implies that

$$\lambda V_\lambda(\sigma - \epsilon|\cdot|)(0) = \lambda C_\lambda^u(\sigma - \epsilon|\cdot|)(0) = -b = \frac{(r - \epsilon)^2}{4\lambda},$$

and this proves (2.5).

Now we prove (2.6), that is, $\nabla C_\lambda^u(\sigma)(0) = -a$. Let $f_\lambda(x) = \lambda|x|^2 - \sigma(x)$. We have found that $\ell(x) = a \cdot x + b \leq f_\lambda(x)$ for all $x \in \mathbb{R}^n$, including the special case $\epsilon = 0$, where $-a$ is the centre of the minimal bounding sphere of $\partial_- g(0)$ and $b = -r^2/(4\lambda)$. Since $f_\lambda(x) = \lambda|x|^2 - \sigma(x)$ is upper semi-differentiable in $\mathbb{R}^n$, by [19], $\mathsf{co}[f_\lambda] \in C^1(\mathbb{R}^n)$. In particular $\ell(x) \leq \mathsf{co}[f_\lambda](x)$ and $b = \ell(0) = \mathsf{co}[f_\lambda](0)$. By Lemma 2.2, we see that $a = \nabla\ell(0) = \nabla\mathsf{co}[f_\lambda](0)$. Thus by definition, $\nabla C_\lambda^u(\sigma)(0) = -a$.

Next we establish (2.7). By (2.1) we have

$$C_\lambda^u(\sigma_0 + \epsilon|\cdot|)(x) \le C_{(1-\epsilon)\lambda}^u(\sigma_0)(x) + C_{\epsilon\lambda}^u(\epsilon|\cdot|)(x)$$

for $x \in \mathbb{R}^n$. At $x = 0$, we have, by (2.5) with $\epsilon = 0$ that

$$C_{(1-\epsilon)\lambda}^u(\sigma)(0) = \frac{r^2}{4(1-\epsilon)\lambda}.$$

Also it is easy to see by a direct calculation that $C_{\epsilon\lambda}^u(\epsilon|\cdot|)(x)$ is given by (2.8). Thus at $x = 0$,

$$C_{\epsilon\lambda}^u(\epsilon|\cdot|)(0) = \frac{\epsilon}{4\lambda},$$

which completes the proof. □

Proof of Proposition 2.3: Without loss of generality, we may assume that $x = 0$. By [30, Remark 2.1], we have

$$C_\lambda^l(f)(0) = \mathsf{co}[f + \lambda|(\cdot) - x|^2](0) = \sum_{i=1}^k \lambda_i[f(x_i) + \lambda|x_i|^2] \tag{3.5}$$

for some $2 \le k \le n+1$, $\lambda_i > 0$, $x_i \in \mathbb{R}^n$ for $i = 1, 2, \ldots, k$ with $\sum_{i=1}^k \lambda_i = 1$ and $\sum_{i=1}^k \lambda_i x_i = 0$. We define $f_\lambda(y) = f(y) + \lambda|y|^2$ for $y \in \mathbb{R}^n$. Since $(x_i, f_\lambda(x_i))$ with $i = 1, 2, \ldots, k$ lie on a support hyperplane of the epi-graph $\mathrm{epi}(f_\lambda) := \{(y, \alpha), y \in \mathbb{R}^n, \alpha \ge f_\lambda(y)\}$, there is an affine function $\ell(y) = a \cdot y + b$ such that

(i) $\ell(y) \le f_\lambda(y)$ for all $y \in \mathbb{R}^n$ and

(ii) $\ell(x_i) = f_\lambda(x_i)$ for $i = 1, 2, \ldots, k$.

By (ii) and (3.5) we also have $\ell(0) = b = C_\lambda^l(f)(0)$. So (iii) also holds.

To derive the bound r_λ we evaluate (i) at $y = a/(2\lambda)$ to derive a bound of $|a|$ as follows:

$$\frac{a \cdot a}{2\lambda} + b = \ell\left(\frac{a}{2\lambda}\right) \le f\left(\frac{a}{2\lambda}\right) + \lambda\left|\frac{a}{2\lambda}\right|^2,$$

so that

$$\frac{|a|^2}{4\lambda} \le f\left(\frac{a}{2\lambda}\right) - b = f\left(\frac{a}{2\lambda}\right) - f(0) + f(0) - b \le \frac{L|a|}{2\lambda} + \frac{L^2}{4\lambda},$$

hence $|a|^2 \le 2L|a| + L^2$. Here we have used the fact that f is L-Lipschitz and $f(0) - b = R_\lambda(f)(0) \le L^2/(4\lambda)$ by (1.7). Thus we have $|a| \le (1 + \sqrt{2})L$.

Now we use (ii) $a \cdot x_i + b = f(x_i) + \lambda|x_i|^2$ to obtain

$$\lambda|x_i|^2 = b - f(x_i) + a \cdot x_i = b - f(0) + f(0) - f(x_i) + a \cdot x_i \le L|x_i| + |a||x_i|,$$

as $b - f(0) = -R_\lambda(f)(0) \le 0$. Thus we can deduce that for each x_i with $i = 1, 2, \ldots, k$,

$$|x_i| \le \frac{L + |a|}{\lambda} \le \frac{(2 + \sqrt{2})L}{\lambda}.$$

Therefore $r_\lambda = (2 + \sqrt{2})L/\lambda$. □

Proof of Proposition 2.6: We use the locality property (Proposition 2.3) to localise the global $C^{1,1}$ property obtained in [5, Proposition 3.7] and [30, Theorem 4.1]. We show that when $\lambda > 0$ is sufficiently large, $C^u_\lambda(f)$ is continuously differentiable in $\bar{B}_r(0)$ and

$$-\lambda_0|y_0 - x_0|^2 \leq C^u_\lambda(f)(y_0) - C^u_\lambda(f)(x_0) - \nabla C^u_\lambda(f)(x_0) \cdot (y_0 - x_0) \leq \lambda|y_0 - x_0|^2 \qquad (3.6)$$

for x_0, $y_0 \in \bar{B}_r(0)$, where $\lambda_0 \geq 0$ is the non-negative constant used in the definition that f is semiconvex in $\bar{B}_{2r}$ satisfying $f(y) = \lambda_0|y|^2 - g(y)$ with $g : \bar{B}_{2r} \mapsto \mathbb{R}$ convex. From (3.6) we see that if $\lambda \geq \lambda_0$ is sufficiently large, $C^u_\lambda(f)$ is both 2λ-semiconvex and 2λ-semiconcave. Therefore by [8, Corollary 3.3.8], $C^u_\lambda(f) \in C^{1,1}(\bar{B}_r(0))$ and

$$|\nabla C^u_\lambda(f)(y) - \nabla C^u_\lambda(f)(x)| \leq 2\lambda|y - x|$$

for x, $y \in \bar{B}_r(0)$.

Since f is L-Lipschitz, by the locality property, when $\lambda > 0$ is sufficiently large, we have, for $x_0 \in \bar{B}_r(0)$,

$$C^u_\lambda(f)(x_0) = -\operatorname{co}[\lambda|\cdot|^2 - f](x_0) = -\sum_{i=1}^{k^{(0)}} \lambda_i^{(0)}[\lambda|x_i^{(0)} - x_0|^2 - f(x_i^{(0)})]$$

with $1 \leq k^{(0)} \leq n + 1$, $\lambda_i^{(0)} > 0$, $|x_i^{(0)} - x_0| \leq r$.

We define $g_\lambda(y) = \lambda|y - x_0|^2 - f(y)$. By Proposition 2.3, there is an affine function $\ell(y) = a \cdot (y - x_0) + b$ such that (i): $\ell(y) \leq g_\lambda(y)$ for all $y \in \mathbb{R}^n$ and (ii): $\ell(x_i^{(0)}) = g_\lambda(x_i^{(0)})$. Let

$$\Delta_{x_0} = \left\{ \sum_{i=1}^{k^{(0)}} \mu_i x_i^{(0)}, \ \mu_i \geq 0, \ i = 1, \ldots, k^{(0)}, \ \sum_{i=1}^{k^{(0)}} \mu_i = 1 \right\}$$

be the simplex defined by $\{x_1^{(0)}, \ldots, x_{k^{(0)}}^{(0)}\}$, then we see that $\operatorname{co}[g_\lambda](y) = a \cdot (y - x_0) + b$ for $y \in \Delta_{x_0}$ as the set $U := \{(y, a \cdot y + b), \ y \in \Delta_0\}$ is contained in a face of the convex hull of the epi-graph $\operatorname{co}[\operatorname{epi}(g_\lambda)]$ of g_λ and $\{(x_1^{(0)}, g_\lambda(x_1^{(0)})) \ldots, (x_m^{(0)}, g_\lambda(x_m^{(0)}))\} \subset U \cap \operatorname{epi}(g_\lambda)$.

Now we have $\operatorname{co}[g_\lambda](y) \leq g_\lambda(y)$ for $y \in B_{2r}(0)$, and $\operatorname{co}[g_\lambda](x_i^{(0)}) = g_\lambda(x_i^{(0)}) = a \cdot (x_i^{(0)} - x_0) + b$ for $i = 1, \ldots, k^{(0)}$. Furthermore, in $\bar{B}_{2r}(0)$, $g_\lambda(y) = \lambda|y - x_0|^2 - f(y)$ where $f(y) = g(y) - \lambda_0|y - x_0|^2$ is $2\lambda_0$-semicovex in $\bar{B}_{2r}(0)$ with $g : \bar{B}_{2r}(0) \mapsto \mathbb{R}$ convex and $\lambda_0 \geq 0$. Thus $g_\lambda(y) = (\lambda + \lambda_0)|y - x_0|^2 - g(y)$ is upper semi-differentiable in $\bar{B}_{2r}(0)$. Thus by Lemma 2.2, we see that both $\operatorname{co}[g_\lambda]$ and g_λ are differentiable at $x_i^{(0)}$ and

$$\nabla \operatorname{co}[g_\lambda](x_i^{(0)}) = \nabla g_\lambda(x_i^{(0)}) = 2(\lambda + \lambda_0)(x_i^{(0)} - x_0) - \nabla g(x_i^{(0)}),$$

hence $\nabla g(x_i^{(0)})$ exists for $i = 1, \ldots, k^{(0)}$. If we apply Lemma 2.2 to the affine function $\ell(y)$ and the upper semi-differentiable function $g_\lambda(y)$ in $\bar{B}_{2r}(0)$, we also have $\nabla g_\lambda(x_i^{(0)}) = a$ for $i = 1, \ldots, k^{(0)}$.

Now we show that $C^u_\lambda(f)$ is differentiable at x_0 and $\nabla C^u_\lambda(f)(x_0) = -a$. We follow an argument in [19]. We know that $\operatorname{co}[g_\lambda](x_0) = \sum_{i=1}^{k^{(0)}} \lambda_i^{(0)} g_\lambda(x_i^{(0)})$ with $1 \leq k^{(0)} \leq n+1$, and we may further assume that $\lambda_1^{(0)} \geq \cdots \geq \lambda_{k^{(0)}}^{(0)} > 0$, $|x_i^{(0)} - x_0| \leq r$ (by the locality property), satisfying $\sum_{i=1}^{k^{(0)}} \lambda_i^{(0)} = 1$ and

$\sum_{i=1}^{k^{(0)}} \lambda_i^{(0)} x_i^{(0)} = x_0$. We then have $\lambda_1^{(0)} \geq 1/(n+1)$. Now for $y \in \mathbb{R}^n$, we have

$$x_0 + y = \lambda_1^{(0)} \left(x_1^{(0)} + \frac{y}{\lambda_1^{(0)}} \right) + \sum_{i=2}^{k^{(0)}} \lambda_i^{(0)} x_i^{(0)}.$$

By the convexity of $\mathsf{co}[g_\lambda]$, we have

$$\mathsf{co}[g_\lambda](x_0 + y) - \mathsf{co}[g_\lambda](x_0) \leq \lambda_1^{(0)} \left(g_\lambda(x_1^{(0)} + y/\lambda_1^{(0)}) - g_\lambda(x_1^{(0)}) \right) + \left(\sum_{i=2}^{k^{(0)}} \lambda_i^{(0)} [g_\lambda(x_i^{(0)}) - g_\lambda(x_i^{(0)})] \right)$$

$$= \lambda_1^{(0)} \left(g_\lambda(x_1^{(0)} + y/\lambda_1^{(0)}) - g_\lambda(x_1^{(0)}) \right)$$

for $y \in \mathbb{R}^n$. Since the left hand side of the above equation is convex in y and the right hand side is upper semi-differentiable at $y = 0$ and the two terms are equal at $y = 0$, by Lemma 2.2, we see that $\nabla \mathsf{co}[g_\lambda](x_0) = \nabla g_\lambda(x_1^{(0)})$. Thus $\mathsf{co}[g_\lambda]$ is differentiable at x_0.

Furthermore, since $\ell(y) \leq g_\lambda(y)$ for $y \in \mathbb{R}^n$, by the definition of convex envelope, we see that $\ell(y) \leq \mathsf{co}[g_\lambda](y)$ for $y \in \mathbb{R}^n$. We also have $\ell(x_0) = b = \mathsf{co}[g_\lambda](x_0)$. Also since $\mathsf{co}[g_\lambda]$ is differentiable at x_0, by Lemma 2.2, we have $\nabla \mathsf{co}[g_\lambda](x_0) = a$. Thus $\nabla C_\lambda^u(f)(x_0) = -a$. Therefore $C_\lambda^u(f)$ is differentiable in $\bar{B}_r(0)$. The continuity of $\nabla C_\lambda^u(f)$ in $\bar{B}_r(0)$ follows from [19].

Now we prove that for all $x_0, y_0 \in \bar{B}_r(0)$, we have

$$C_\lambda^u(f)(y_0) - C_\lambda^u(f)(x_0) - \nabla C_\lambda^u(f)(x_0) \cdot (y_0 - x_0) \geq -\lambda_0 |y_0 - x_0|^2 \tag{3.7}$$

so that $C_\lambda^u(f)$ is $2\lambda_0$-semiconvex in $\bar{B}_r(0)$. We use the notation associated to $C_\lambda^u(f)(x_0)$ as above. We see that (3.7) is equivalent to

$$\lambda |y_0 - x_0|^2 - \mathsf{co}[g_\lambda](y_0) + \mathsf{co}[g_\lambda](x_0) + \nabla \mathsf{co}[g_\lambda](x_0) \cdot (y_0 - x_0) \geq -\lambda_0 |y_0 - x_0|^2$$

which again is equivalent to

$$\mathsf{co}[g_\lambda](y_0) - \mathsf{co}[g_\lambda](x_0) - \nabla \mathsf{co}[g_\lambda](x_0) \cdot (y_0 - x_0) \leq (\lambda + \lambda_0)|y_0 - x_0|^2.$$

Note that

$$\mathsf{co}[g_\lambda](x_0) = \sum_{i=1}^{k^{(0)}} \lambda_i^{(0)} g_\lambda(x_i^{(0)}), \quad \nabla \mathsf{co}[g_\lambda](x_0) = a, \quad \nabla \mathsf{co}[g_\lambda](x_i^{(0)}) = \nabla g_\lambda(x_i^{(0)}) = a.$$

Since $y_0 \in \bar{B}_r(0)$ and $|x_i^{(0)} - x_0| \leq r$, we see that

$$y_0 + (x_i^{(0)} - x_0) \in \bar{B}_{2r}(0) \quad \text{and} \quad \sum_{i=1}^{k^{(0)}} \lambda_i^{(0)} \left(y_0 + (x_i^{(0)} - x_0) \right) = y_0.$$

Thus,

$$\mathsf{co}[g_\lambda](y_0) \leq \sum_{i=1}^{k^{(0)}} \lambda_i^{(0)} \mathsf{co}[g_\lambda](y_0 + (x_i^{(0)} - x_0)) \leq \sum_{i=1}^{k^{(0)}} \lambda_i^{(0)} g_\lambda(y_0 + (x_i^{(0)} - x_0)).$$

We also have

$$\mathsf{co}[g_\lambda](x_0) = \sum_{i=1}^{k^{(0)}} \lambda_i^{(0)} [g_\lambda(x_0 + (x_i^{(0)} - x_0))]$$

and

$$\nabla \mathsf{co}[g_\lambda](x_0) \cdot (y_0 - x_0) = a \cdot (y_0 - x_0) = \sum_{i=1}^{k^{(0)}} \lambda_i^{(0)} a \cdot (y_0 - x_0)$$

$$= \sum_{i=1}^{k^{(0)}} \lambda_i^{(0)} \nabla g_\lambda(x_0 + (x_i^{(0)} - x_0)) \cdot (y_0 - x_0).$$

We notice that in $\bar{B}_{2r}(0)$, f is semiconvex and $f(y) = \lambda_0|y - x_0|^2 - g(y)$ for a convex function $g : \bar{B}_{2r}(0) \mapsto \mathbb{R}$. Thus

$$\mathsf{co}[g_\lambda](y_0) - \mathsf{co}[g_\lambda](x_0) - \nabla \mathsf{co}[g_\lambda](x_0) \cdot (y_0 - x_0)$$

$$\leq \sum_{i=1}^{k^{(0)}} \lambda_i^{(0)} \Big(g_\lambda(y_0 + (x_i^{(0)} - x_0)) - g_\lambda(x_0 + (x_i^{(0)} - x_0)) - \nabla g_\lambda(x_0 + (x_i^{(0)} - x_0)) \cdot (y_0 - x_0) \Big)$$

$$= \sum_{i=1}^{k^{(0)}} \lambda_i^{(0)} (\lambda + \lambda_0) \Big(|(y_0 - x_0) + (x_i^{(0)} - x_0)|^2 - |(x_i^{(0)} - x_0)|^2 - 2(x_i^{(0)} - x_0) \cdot (y_0 - x_0) \Big)$$

$$- \sum_{i=1}^{k^{(0)}} \lambda_i^{(0)} \Big(g(y_0 + (x_i^{(0)} - x_0)) - g(x_0 + (x_i^{(0)} - x_0)) - \nabla g(x_0 + (x_i^{(0)} - x_0)) \cdot (y_0 - x_0) \Big)$$

$$\leq (\lambda + \lambda_0)|y_0 - x_0|^2.$$

Here we have used the facts that $\sum_{i=1}^{k^{(0)}} \lambda_i^{(0)} (x_i^{(0)} - x_0) = 0$ and that g is convex and differentiable at $x_i^{(0)}$. Thus $C_\lambda^u(f)$ is $2\lambda_0$-semiconvex in $\bar{B}_r(0)$. Also by the definition of the upper transform, $C_\lambda^u(f)$ is 2λ-semiconcave, hence for $x_0, y_0 \in \bar{B}_r(0)$

$$C_\lambda^u(f)(y_0) - C_\lambda^u(f)(x_0) - \nabla C_\lambda^u(f)(x_0) \cdot (y_0 - x_0) \leq \lambda|y_0 - x_0|^2 \tag{3.8}$$

Combining (3.7) and (3.8) we see that $C_\lambda^u(f)$ is $2\lambda_0$-semiconvex and 2λ-semiconvex in $\bar{B}_r(0)$. Therefore by [8, Corollary 3.3.8], we see that $C_\lambda^u(f) \in C^{1,1}(\bar{B}_r(0))$ satisfying

$$|\nabla C_\lambda^u(f)(y) - \nabla C_\lambda^u(f)(x)| \leq 2\lambda|y - x|, \qquad y, \ x \in \bar{B}_r(0)$$

if we choose $\lambda \geq \lambda_0$. $\qquad\qquad\square$

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