On a Family of Quotients of the von Dyck Groups

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Finally I would like to dedicate this thesis (and place on record my thanks) to my close friends, and especially to my parents, without whose love, support and patience this document would most certainly not exist.

Abstract

In this publication, we investigate the groups $H(m, n, p, q|k)$ defined by the presentation $\langle x, y|x^m, y^n, (xy)^p, (xy^k)^q\rangle$ and determine finiteness and infiniteness for these parameters as far as possible using geometric arguments, principally through pictures and curvature. We state and subsequently prove theorems relating to infiniteness, and also discuss troublesome cases where spherical pictures may arise. We also provide a list of known finite groups and unresolved cases in Appendix A.

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Chapter 1

Introduction

1.1 The von Dyck groups and $H(m, n, p, q|k)$

In group theory, the von Dyck group $D(m, n, p)$ is defined as the subgroup of index 2 of the triangle group $\Delta(m, n, p)$ and has presentation

$$D(m, n, p) := \langle x, y | x^m, y^n, (xy)^p \rangle$$

where $m, n, p \in \mathbb{N}$. This family of groups is well understood and it is a well-known result that $D(m, n, p)$ is finite if and only if m, n and p satisfy the inequality $\frac{1}{m} + \frac{1}{n} + \frac{1}{p} > 1$. Quotients of the von Dyck groups have been widely studied, for example such as by Coxeter [2] and two of the ensuing von Dyck quotient groups $(l, m|n, k)$ which has presentation $\langle x, y | x^l, y^m, (xy)^n, (xy^{-1})^k \rangle$ and $(l, m, n; q)$ which has presentation $\langle x, y | x^l, y^m, (xy)^n, [x, y]^q \rangle$ where by $[x, y]$ we mean the usual commutator $x^{-1}y^{-1}xy$.

In 1992 Holt and Plesken [15] decided finiteness for three families of groups, two of which were the families $(3, 3|4, n)$ and $(2, 3, 7; n)$ for all $n \in \mathbb{N}$. The third was of a different structure and had presentation

$$\langle x, y | x^2, y^4, (xy)^5, (xy^2)^n \rangle$$

which seemed a rather different quotient to investigate (though it is actually isomorphic to $(5, 4|n, 2)$ as we will discuss shortly). We therefore consider quotients where the extra relator looks to be of similar form, and define the group $H(m,n,p,q|k)$ as having presentation

$$H(m,n,p,q|k) = \langle x, y | x^m, y^n, (xy)^p, (xy^k)^q \rangle$$

which we may refer to as H where appropriate and unambiguous, and call this presentation P_H similarly. This can be considered as not only a one-relator quotient of $D(m,n,p)$ but as a two-relator quotient of the free product $C_m * C_n$ by the normal closure of the words $(xy)^p$ and $(xy^k)^q$. Our leading question in this work is to determine for what values of our parameters m, n, p, q and k whether or not H is finite or infinite, and whether such an elegant governing inequality as for $D(m,n,p)$ exists. We begin with some common terminology; we say that the group H *collapses* if at least one of the orders of x, y, xy and xy^k is strictly less than m, n, p or q respectively (with *total collapse* if all these orders are equal to 1, and *partial collapse* otherwise); if all these orders are exact, then we say H does not collapse. We may also use collapse in relation to a specific exponent (e.g. "suppose m collapses") to mean the same idea.

In this work, we will prove two main theorems relating to proving groups infinite, which we detail below.

Main Theorem A: *Let H be the group $H(2,n,p,q|k)$, with presentation $\langle x, y | x^2, y^n, (xy)^p, (xy^k)^q \rangle$ and with $p, q, k \geq 3$, $n \geq 2k$. Suppose H satisfies one of the following sets of conditions:*

- $p, q \geq 6$
- $p, q \geq 4$ *and* $n \neq 3k, n \neq 2k+1$
- $p = 3, q \geq 12$, $n \neq k+3$ *and* $k > 3$
- $p = 3$, $6 \leq q \leq 11$, $n \neq 2k+1$, $n \neq k+3$ *and* $k > 3$
- $q = 3$, $p \geq 6$ *and* $n \neq z$, *where* $z \in \{2k+1, 3k, 3k \pm 1, 4k, 5k\}$
- $n = 2k$ *and* $(p,q) \in \{(4,3),(5,3),(3,5)\}$
- $n = 2k+1, p = 3$ *and* $q = 11$

- $p = 3, q = 3$ *and* $\exists z \geq 6$ *such that* $z \mid n$ *and* $k \equiv \pm 1 \mod z$
- $p = 4, q = 4$ *and* $\exists z \geq 4$ *such that* $z \mid n$ *and* $k \equiv \pm 1 \mod z$
- $p = 5, q = 5$ *and* $\exists z \geq 4$ *such that* $z \mid n$ *and* $k \equiv \pm 1 \mod z$

then H *is an infinite group.*

Main Theorem B: *Let* H *be the group* $H(m, n, p, q|k)$*, with presentation* $\langle x, y | x^m, y^n, (xy)^p, (xy^k)^q \rangle$ *and with* $p, q, k \geq 3$*,* $n \geq k+2$ *and* $m \geq 4$*. Suppose* H *satisfies one of the following sets of conditions, or is one of the following groups:*

- $q \geq 11$
- $p \geq 10$ *and* $q \geq 4$*, or* $p \geq 6$ *and* $q \geq 5$*, or* $q \geq 6$ *and* $p \geq 5$
- $p, q \geq 4$ *and* $n \neq 2k-1$ *or* $q = 3$ *and* $n \neq 2k-1$*,*
- $p = 3$ *and* $q \geq 6$*,*
- $H(m, 2k+1, 3, 4|k)$ *or* $H(m, 2k+1, p, 3|k)$ *for* $3 \leq p \leq 7$*,*
- $H(m, k+2, 4, 3|k)$ *or* $H(m, k+2, 3, q|k)$ *for* $3 \leq q \leq 7$*,*
- $H(m, 3k-1, 3, 3|k)$*,*
- $H(m, n, 3, 3|3)$*, with* $n \notin \{5, 7, 8, 9\}$

then H *is an infinite group.*

These shall be proved piecemeal; Main Theorem A's results come from Chapter 2 and 3, whereas Main Theorem B's results come from Chapters 4 and 5. Generally, our method for building these theorems is that given a group H with presentation P_H, we wish to prove that the presentation is quasi-aspherical (the meaning of which we discuss in Section 1.3) and that H does not collapse; from this, we can then construct a pushout with leads us to be able to calculate the order of H through Euler characteristics, as in [9]. This will be expanded upon in Section 2.4 when relevant.

1.2 Small value observations for H

To begin with, we can conclude the following groups are finite of calculable order:

- $H(1, n, p, q|k)$ is a cyclic group of order $\gcd(n, p, kq)$ since $x = 1$ where by 1 we denote the identity element
- $H(m, 1, p, q|k)$ is a cyclic group of order $\gcd(m, p, q)$ since $y = 1$
- $H(m, n, 1, q|k)$ implies that $x = y^{-1}$ and is therefore cyclic of order $\gcd(m, n, q(k-1))$
- $H(m, n, p, 1|k)$ implies that $x = y^{-k}$ and is therefore cyclic of order $\gcd(km, n, p(k-1))$

Of course, if m, n and p are sufficiently small enough to satisfy $\frac{1}{m} + \frac{1}{n} + \frac{1}{p} > 1$ then H is immediately finite as a quotient of the finite group $D(m, n, p)$, if not of immediately calculable order.

For $k = 1$ or 2, we can find simple isomorphisms to known groups. For instance, in the case $k = 1$ it can be readily seen that P_H devolves to the presentation of $D(m, n, \gcd(p, q))$ and thus $H(m, n, p, q|1)$ is finite if and only if $\frac{1}{m} + \frac{1}{n} + \frac{1}{\gcd(p,q)} > 1$. In the case of $k = 2$, if we apply the Tietze transformation $a = xy$ to P_H and then remove x as a generator we obtain the following presentation for H:

$$\langle a, y | a^p, y^n, (ay)^q, (ay^{-1})^m \rangle$$

i.e. $H \cong (p, n|q, m)$ in the notation of [2]. These groups were completely categorised for finite/infiniteness by Edjvet and Thomas in [11], and as such we are satisfied that these are dealt with. However, for $k \geq 3$, this trick fails; still, we may assume for all subsequent work that $k \geq 3$. In turn, since we clearly may assume $n > k$ (else we reduce the relator β modulo n to $(xy^{k'})^q$ where k' is the least positive integer with $k' \equiv k$ modulo n) and if $k = n - 1$ then β becomes $(xy^{-1})^q$ and $H \cong (m, n|p, q)$ so this is resolved in [11]. Therefore, we may assume $n > k + 1$ and in particular $n \geq 5$.

In general, the parameters may not be permuted; however, if n and k are co-prime, then we can say a little more.

Lemma 1.1: *Let* $n,\ k \in \mathbb{N}$ *satisfy* $\gcd(n,k) = 1$*, and let* f *be the inverse of* k *mod* n *where we choose* $0 < f < n$*. Then* $H(m,n,p,q|k)$ *is isomorphic to* $H(m,n,q,p|f)$*.*

Proof: From the standard presentation P_H, set $b = y^k$; then $b^f = y$ since $k.f \equiv 1 \bmod n$. Add b to the presentation and eliminate y. Since $\gcd(n,k) = 1$, b also has order n. Thus our relators are now x^m, b^n, $(xb)^q$ and $(xb^f)^p$ hence the result. □

In the case that the greatest common divisors of the pairs (m,q) and (n,k) are sufficiently large and that p is also large enough, we can immediately find some infinite groups.

Theorem 1.2: *Let* H *be the group* $H(m,n,p,q|k)$*. Then if* $\frac{1}{\gcd(m,q)} + \frac{1}{\gcd(n,k)} + \frac{1}{p} \leq 1$*,* H *is infinite.*

Proof: Let $g := \gcd(n,k)$, and let K be the quotient group of H with presentation $\langle x,y|x^m, y^n, (xy)^p, (xy^k)^q, y^g\rangle$ i.e. the quotient found by adding the relator $y^g = 1$. Since g divides n and k, $y^k = 1$ and K's presentation is then identical to $\langle x,y|x^m, y^g, (xy)^p, x^q\rangle$ which after combining $x^m = x^q = 1$ we find is the presentation of the group $D(\gcd(m,q), g, p)$ which is infinite precisely when $\frac{1}{\gcd(m,q)} + \frac{1}{\gcd(n,k)} + \frac{1}{p} \leq 1$. But K is a quotient of H and thus $|K| \leq |H|$ therefore in these cases H must also be infinite. □

1.3 Geometric arguments

1.3.1 Pictures

Our considerations for the most part centre on using diagrams known as *pictures*, and run similarly to those discussed in [10]; the ideas of which are also based on arguments by Howie and Thomas in [17] and [18] (one can also compare with the well-known van Kampen diagrams and pictures over relative presentations e.g. as discussed in [1]). For the sake of convenience and continuity, we shall largely stick to the notation set down in [10] adapting it for our own needs where necessary.

To begin with, let us take the earlier definition of H with presentation P_H as a two-relator quotient of the free product $C_m * C_n$ by the normal closure of the words $\alpha := (xy)^p$ and $\beta := (xy^k)^q$. For convenience below, we will set $A := C_m = \langle x|x^m\rangle$ and $B := C_n = \langle y|y^n\rangle$. Then, we consider a picture Π over P_H as constituting the following objects:

- A disc D^2, with associated boundary denoted by ∂D^2.
- A set V of closed discs called *vertices*, which are pairwise disjoint and lie in the interior of the disc D^2.
- A finite set E of arcs called *edges*, which again are pairwise disjoint and lie in the interior of D^2 and either are 1) simple closed curves which do not contact any vertex, 2) an arc adjoining a vertex to the boundary ∂D^2, 3) an arc adjoining two vertices (which may not necessarily be distinct) or 4) an arc adjoining ∂D^2 to ∂D^2.
- Consequently we define *regions*, which are 2-cells bounded by edges, vertices and ∂D^2 omitting any element of E or V.
- A set of *labels*, which are applied to each corner of each region (in a manner to be described below) and if a region is incident to a component(s) of ∂D^2, then a label is ascribed to each such component.

We now begin to define special classes of pictures. The picture Π is considered *connected* if all its edges are incident to at least one vertex (i.e. no edges of category 1) or 4) apply aside from the uninteresting case where the picture consists of just that edge). We shall also define a *spherical* picture as a connected picture in which no edge meets ∂D^2; we will return to these in due course.

With regards to labelling, each label is either x, y, or y^k or inverses of these (so either the label belongs to A or B; the exception to this is that a label on a segment of ∂D^2, which may be a word in both x and powers of y). Labels not on a segment of ∂D^2 must satisfy the requirement that if we read anti-clockwise around a vertex, the word spelt by these labels must be (up to cyclic permutation) either α or β, or again their inverses. As such, we define α-vertices and β-vertices with their counterpart $\bar{\alpha}$ and $\bar{\beta}$. Further requirements on the labels of regions will be mentioned a little later once we have a few more concepts set.

We now discuss regions. If we let Δ be a region of Π, then we call Δ an *interior* region if none of its edges contact ∂D^2; if it does, then call it a *boundary* region. If Π is a spherical picture, then all of its regions can be considered as interior, separated from ∂D^2 by an annular area which we will refer to as the *distinguished region.* In this case, there will be no labelling on ∂D^2. In all cases, we insist that all labels of interior and boundary regions either belong to A or B; as such, we will refer to *x-regions* and *y-regions*, and that the product of the labels of any such region gives the identity element in whichever of A or B the labels apply. We have a similar restriction for those labels on ∂D^2; we define the *boundary label* of Π to be the word given by reading the labels around ∂D^2 in an anti-clockwise fashion. This word should equal the identity element in H. In the case where Π is spherical, then the boundary label can simply be read as the product of the other labels of the distinguished region.

Now, we may amend pictures by the use of *bridge moves*, which are detailed in further depth in [17]. The gist is thus: if the word w given by a region's label contains a subword w' equal to the identity, we may amend the edges to enclose this w' in its own region. An example of this manoeuvre is given in Fig. 1.1 below in the case $m = 2$. If two vertices contrive to be connected by an edge and their labels (read from the endpoints of this edge) are mutually inverse words in the free product $A * B$, then we call these *cancelling vertices* and we may perform bridge moves in order to create 2-gons between these; in Fig. 1.2 we show a pair of cancelling α and $\bar{\alpha}$-vertices. If we continue to apply bridge moves eventually we will end up with a connected subpicture of Π, fully disconnected from Π, on two vertices; we call this a *dipole.* As detailed in [17], we can actually remove a dipole without altering the boundary label of Π, thereby leaving us a picture on the same boundary word with two fewer vertices. If a picture Π admits no such simplification through bridge moves, we call the picture *reduced.* Again, in a parallel to that of van Kampen diagrams, if some cyclically reduced word w in $A * B$ is equal to the identity in H then there exists a reduced picture whose boundary label is w.

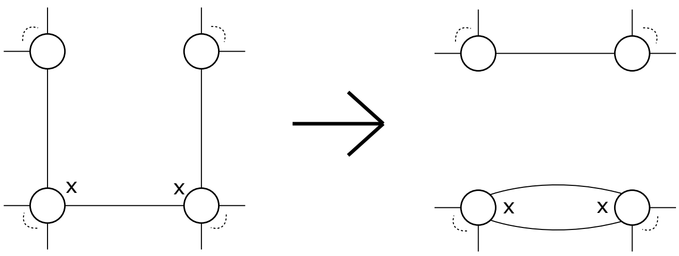


Fig. 1.1

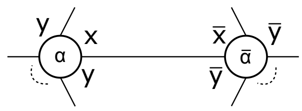


Fig. 1.2

If every spherical picture over P_H fails to be reduced, then we call P_H *quasi-aspherical* over $A * B$; what this means for us, in accordance with [18], is that should we consider P_H as a two-dimensional CW-complex C then its fundamental group $\pi_1(C)$ is isomorphic to our H, and each spherical picture represents an element of the second homotopy group $\pi_2(C)$. Then, P_H being quasi-aspherical implies that as a $\mathbb{Z}\pi_1(C)$-module, $\pi_2(C)$ is generated by dipoles. In practice, we will work with *minimal* pictures, where by this we mean that if we consider the set S of all pictures Π over a presentation P_H we work over the picture $\Pi_i \in S$ where, if $V(\Pi_j)$ denotes the vertex set of Π_j, $\min\{|V(\Pi_j)|; \Pi_j \in S\} = |V(\Pi_i)|$.

If two edges of Π enclose a 2-gon, we consider them to be *parallel* edges and *∂-parallel* if they also both connect to ∂D^2. Should more than two 2-gons be adjacent around a vertex, we call all of the edges of the adjacent 2-gons in question parallel; for example, in creating a dipole the cancelling vertices in Fig. 1.2 form $2p$ adjacent 2-gons and thus $2p$ adjacent parallel edges. This notion extends similarly to ∂-parallel edges. We seek now to ascertain the maximal number of parallel edges in any one set of adjacent 2-gons; in the case where no 2-gons are adjacent, this number is two. In the general case, by considering combinations of α-vertices and β-vertices (plus their inverses) for a reduced picture Π we can only obtain parallel edges between an α-vertex and a β-vertex of opposite sign, which enclose the word xx^{-1}. Indeed, this means that the maximum number of parallel edges is two; to see this, consider a pair of adjacent non-cancelling vertices v_1 and v_2. Up to inversion and permutation, the only possible pairings we may have are that both v_1 and v_2 are α-vertices, both are β-vertices, v_1 is a β-vertex and v_2 an α-vertex and that v_1 is a β-vertex and v_2 an $\bar{\alpha}$-vertex. In turn, to create adjacent 2-gons we require in the first case that both $m = 2$ and $y^2 = 1$, in the second that $m = 2$ and $y^{2k} = 1$, in the third that $m = 2$ and $y^{k+1} = 1$ and in the fourth that $y^{k-1} = 1$. However the first, third and fourth of these mean in turn contradict our earlier assumptions that $n > 5$ and $n > k + 1$, and the second is a special case which we deal with for $m = 2$ but is otherwise problematic, and we will in general assume this does not hold. Observe then that no 2-gon arising from parallel edges can be a y-region and consequently that in all cases outside this exception, an α-vertex has degree $p \leq d(\alpha) \leq 2p$, and similarly a β-vertex has degree $q \leq d(\beta) \leq 2q$ where for a vertex v by the degree $d(v)$ we mean the number of edges incident to v in the usual graph-theoretical way. As in [11], our interest in ∂-parallel edges stems from studying collapse; in particular, when the boundary label of Π is of the form $(xy)^u$ or $(xy^k)^v$ (or the inverse of these) where $0 < u < p$ or $0 < v < q$ as appropriate. Since all regions must be x-regions or y-regions, the segments of ∂D^2 forming ∂-parallel edges must also be labelled with words either from A or B only. In the case we are interested in, those of pictures with boundary label $(xy)^u$ or $(xy^k)^v$ up to inversion, we claim that the maximum number of ∂-parallel edges is two. To see this, suppose that we have more than two on a minimal picture Π whose boundary label is $(xy)^u$ where $0 < u < p$ or $(xy^k)^v$ where $0 < v < q$ as appropriate, and let w be the vertex incident to these ∂-parallel edges; we will demonstrate for the former case, where $(xy)^u = 1$. As can be seen in Fig. 1.3, this forces w to be an $\bar{\alpha}$-vertex ($\bar{\beta}$-vertex in the case $(xy^k)^v = 1$).

Now through bridge moves we can create $2(p - u)$ ∂-parallel edges, and the boundary of Π now connects to w alone; however if we now delete these edges,

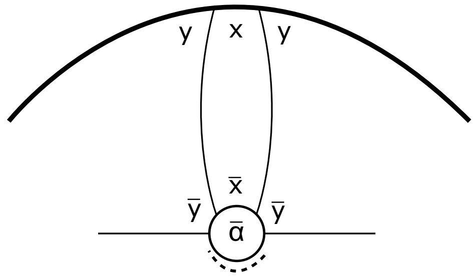


Fig. 1.3

cut the remaining edges of w and then delete the vertex w, the remaining edges form a picture Π' whose boundary label is $(x^{-1}y^{-1})^{p-u}$ (i.e. implying $(xy)^{p-u} = 1$ in H) and with one fewer vertex, which violates the minimality of the number of vertices. This operation is shown in Fig. 1.4 below for the case $u = 2$ with the deleted parts in grey. Thus, we assume that no such manoeuvre can be performed, and that the maximum number of ∂-parallel edges is two.

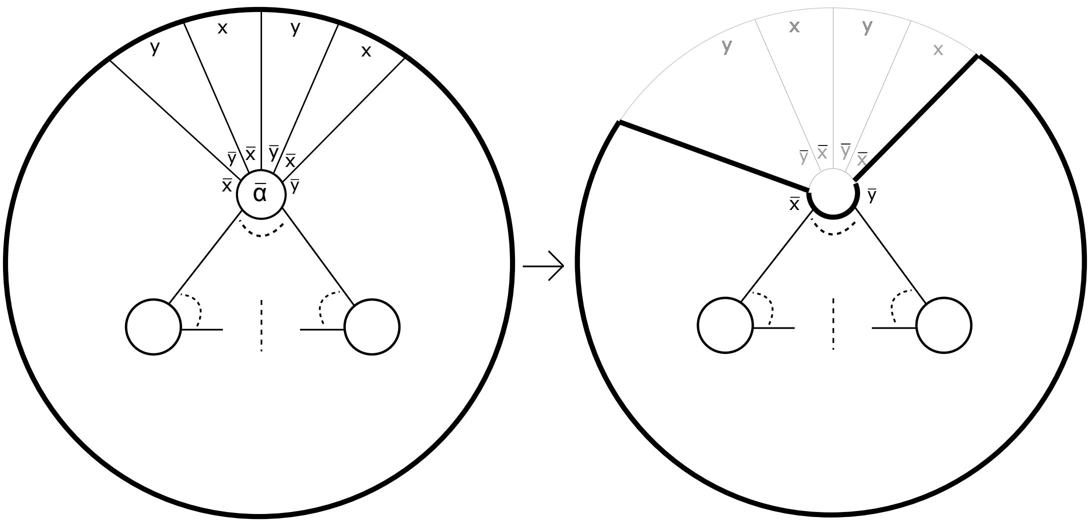


Fig. 1.4

We close this discussion of pictures for now with one final definition which we mentioned briefly above and will use generally in all cases- we consider the *minimal reduced spherical* picture Π over P_H, in which the minimality condition is on the number of vertices. Therefore, it is to be assumed that no more bridge moves creating cancelling vertices may be performed, nor may any dipoles be created. We may also insist on a similar maximality condition on the number of certain region types, which restricts the use of bridge moves further and also may place restrictions on possible regions existing, but in general this changes from case to case and we will specify this as appropriate.

1.3.2 Curvature

As before, we proceed in the same vein as in [10], and will largely use the same notation. Let us assume that Π is a reduced and connected picture over the presentation P_H. From this we can form a planar graph, denoted by $\hat{\Gamma}$, by the following means; first, for each pair of parallel edges and ∂-parallel edges, we remove the 2-gon (or pseudo-2-gon in the case of ∂-parallel) by identifying its two edges into one. Therefore, from this point on, all regions Δ have degree $d(\Delta) \geq 3$, where by the degree of a region we mean the number of vertices along its edges. Now, we have a tessellation of D^2 with an underlying graph $\hat{\Gamma}$; its vertices are those of Π, and its edges are the remaining edges after identification. We maintain the idea of boundary and interior regions as with Π, including when Π is spherical, though in this case we may deform the boundary to a point and remove it, so that our graph is truly a spherical tessellation of S^2. We call this refined graph without boundary Γ.

Now, we define a *curvature function* c in the same vein as in [5], which in turn draws from earlier work by Howie and Edjvet in [8] and [16]. Given a graph Γ from a spherical picture Π, we define that a vertex v with associated degree $d(v)$ has curvature allocated by giving each of its d_v corners the angle $\frac{2\pi}{d_v}$. The total curvature of a vertex is then considered to be 2π minus all allocated angles, i.e. $c(v) = 0$. Thus, all curvature concentrates in regions; given a region Δ, we ascribe $c(\Delta)$ to be equal to 2π minus the sum of the external angles at each vertex bounding Δ, i.e. for a region of degree l

$$c(\Delta) = 2\pi - \sum_{i=1}^{l} (\pi - \frac{2\pi}{d_i}) = \left(2 - l + \sum_{i=1}^{l} \frac{2}{d_i} \right) \pi.$$

The reason for choosing the function c to behave this way is that we obtain a very simple criterion for any spherical picture over P_H, namely the following:

Lemma 1.3: *Let* Π *be a connected, spherical picture with derived graph* Γ *as above. Let* F_Γ *be the set of regions of* Γ. *Then* $\sum_{\Delta \in F_\Gamma} c(\Delta) = 4\pi$.

Proof: If we sum over all regions of F_Γ, we find that

$$\sum_{\Delta \in F_\Gamma} c(\Delta) = \sum_{\Delta \in F_\Gamma} [2\pi - d(\Delta)\pi + 2\pi \left(\frac{1}{d_1} + \ldots + \frac{1}{d_{d(\Delta)}} \right)]$$

$$= 2\pi \sum_{\Delta \in F_\Gamma} \left(\frac{1}{d_1} + \ldots + \frac{1}{d_{d(\Delta)}} \right) + 2\pi |F_\Gamma| - \pi \sum_{\Delta \in F_\Gamma} d(\Delta).$$

Let us set V_Γ to be the set of vertices of Γ and likewise E_Γ be the set of edges of Γ. Now let $v \in V_\Gamma$ be a vertex of degree d. Then v is incident to d different regions, for each of which $\frac{1}{d}$ is allocated to the first summand. Thus each vertex of degree d_v contributes $d_v \frac{1}{d_v} = 1$ to this sum and thus the first summand may be replaced by $|V_\Gamma|$. For the last summand, consider a region Δ with degree $d(\Delta)$ and incident vertices v_1, v_2 and so on to $v_{d(\Delta)}$. Thus in this sum the vertex v_1 appears d_{v_1} times, and counts for d_{v_1} in total. Applying this to every vertex in V_Γ and considering that every edge adjoins two vertices in a spherical picture means we can replace the last summand by $\sum_{v \in V_\Gamma} d(v) = 2|E_\Gamma|$ and we now find $\sum_{\Delta \in F_\Gamma} c(\Delta) = 2\pi(|V_\Gamma| - |E_\Gamma| + |F_\Gamma|)$. Since Γ is planar and connected, by Euler's relation for planar graphs $|V_\Gamma| - |E_\Gamma| + |F_\Gamma| = 2$ and hence the curvature sum equals 4π. $\square$

From this our method is clear. We endeavour to find all spherical pictures that exist; for all H where we can find no such picture, we prove that P_H is quasi-aspherical by showing that any reduced spherical picture over P_H necessarily has total curvature strictly less than 4π.

In general, we will discuss curvature of a region Δ in terms of the degrees of Δ's vertices; so for Δ of degree a with vertices having degrees d_1, d_2, $\ldots$, d_a where we arrange these in ascending order i.e. $d_i \leq d_{i+1}$, we may say $(d_1, d_2, \ldots d_a)$ to describe this region instead and similarly use $c(d_1, d_2, \ldots d_a)$ to mean $c(\Delta)$.

Observe that if we therefore replace a vertex of degree d_i with one of degree d'_i where $d_i < d'_i$ then the net change to Δ's curvature is $\left(\frac{2}{d'} - \frac{2}{d}\right) < 0$. We use this to verify some basic facts about curvature of regions.

Lemma 1.4: *Let H be defined as above with associated presentation P_H, Π be a picture over P_H and Δ be an interior region of Π with degree a. Then the following hold:*

i. If $p, q \geq 2$ then $c(d_1, d_2, \ldots, d_a) \geq c(d_1, d_2, \ldots, d_a, d_{a+1})$ where d_{a+1} is the degree of an extra vertex added to Δ's boundary.

ii. If $p, q \geq 3$ and $a \geq 6$, then $c(\Delta) \leq 0$.

iii. If $p, q \geq 6$ and $a \geq 3$, then $c(\Delta) \leq 0$.

Proof:

i. If we compare the curvature sums, after cancelling we obtain $-\pi + \dfrac{2\pi}{d_{a+1}}$; but since any vertex γ ($\gamma \in \{\alpha, \bar{\alpha}, \beta, \bar{\beta}\}$) necessarily has degree at least p or q clearly $-\pi + \dfrac{2\pi}{d_{a+1}} \leq 0$.

ii. By the above, if we let $a \geq 6$ then $c(\Delta) \leq c(3, 3, 3, 3, 3, 3)$ which when evaluated gives $\left((2-6) + \frac{2.6}{3}\right)\pi = 0$.

iii. Since the minimal degree of an $\alpha/\bar{\alpha}$-vertex is p and the minimal degree of a $\beta/\bar{\beta}$-vertex is q, by *i.* $c(\Delta) \leq c(6, 6, 6) = 0$. □

As a result, if $p, q \geq 3$ our search for positively curved regions needs only to be limited to regions of degree 3, 4 and 5. As a consequence, we will look to show that the sum of the curvature over all interior regions is non-positive (or at least sufficiently small enough that the distinguished region cannot be positively curved enough to create a sum of 4π); we do this by dispersing positive curvature into negatively curved neighbouring regions. To this end, we define the *amended curvature function* c^* for a region Δ as $c^*(\Delta) = c(\Delta) + E$ where E is an erratum term, equal to the total positive curvature received minus any possible dispersion from Δ.

1.4 Cases to consider

Firstly, we will consider the case $m = 2$. In this setting with a picture Π over P_H, every non-distinguished $x-$region has label x^{2a} for some $a \in \mathbb{Z}$; we can therefore apply bridge moves to such an x-region in order to divide it into a 2-gons, which will then be identified in the formation of the graph $\hat{\Gamma}$; consequently, all regions will be y-regions, and the degrees of α-vertices and β-vertices will be exactly p and q respectively. Within there exist families, hereafter referred to as the *Platonics* or *Platonic families*, which are the following families of groups:

i. $H(2, 3k, p, 3|k)$, the *tetrahedra*

ii. $H(2, 4k, p, 3|k)$, the *cubes*

iii. $H(2, 3k, p, 4|k)$, the *octahedra*

iv. $H(2, 3k, p, 5|k)$, the *icosahedra*

v. $H(2, 5k, p, 3|k)$, the *dodecahedra*.

These presentations all yield spherical pictures whose planar graph Γ is analogous to that of the Platonic solids when stretched over S^2. Below in Fig. 1.5 we show graphs of these solids, where in each diagram each vertex is a β-vertex and each region gives the relator y^{3k}, y^{4k} or y^{5k} as appropriate. We will dedicate a special chapter (Chapter 3) to discussing these since the presentations of these can never be quasi-aspherical.

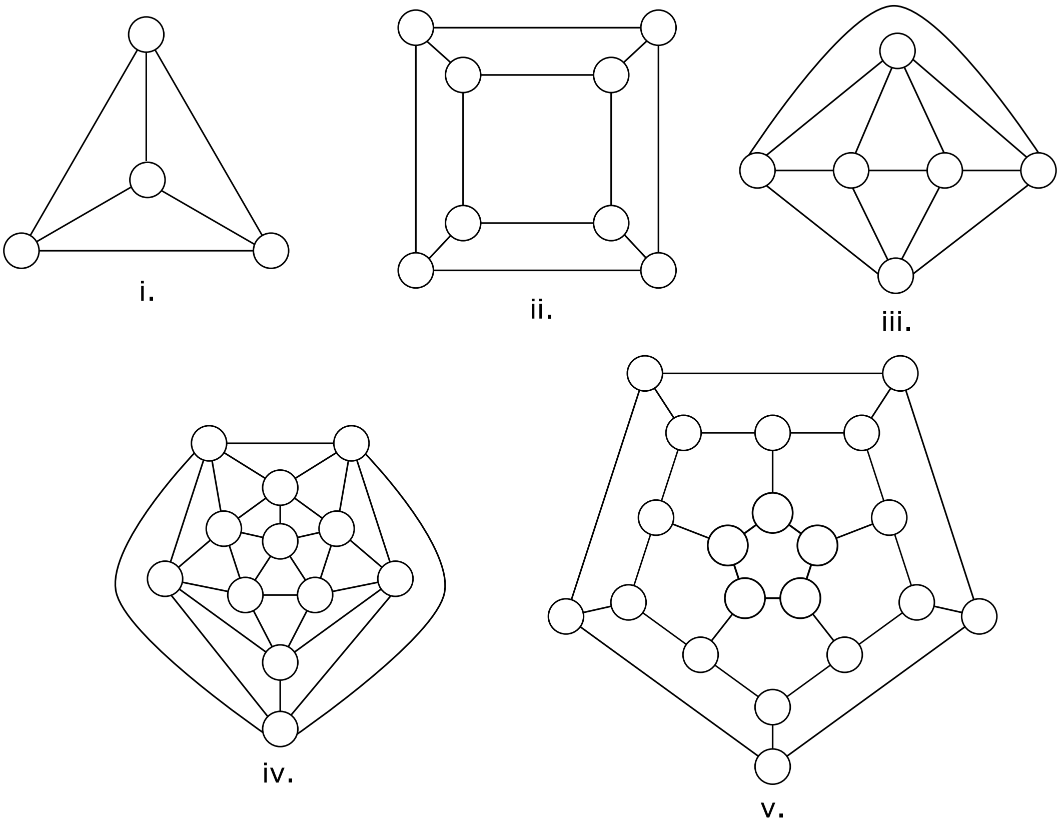


Fig. 1.5

Another subfamily within the $m = 2$ case occurs when $n = 2k$. Since this implies $y^k = y^{-k}$, we find as a consequence that $\beta = \bar{\beta}$; therefore it suffices to consider pictures over only α-, $\bar{\alpha}$- and β-vertices, with the added proviso that no two β-vertices may be adjacent (since these are now cancelling vertices). Again, we will take a look at this special case separately. After this, in Chapter 4 we begin the general investigation for $m \geq 4$; we choose this in order to better control positively curved regions (in particular, for p, $q \geq 3$ this guarantees that any positively curved region must necessarily be a y-region, but we clarify this more in Chapter 4). The difficult case $m = 3$ is largely omitted but some results for small values will be discussed in passing where appropriate.

Chapter 2

The case $m = 2$; non-Platonic cases

2.1 Parameter constraints and positively curved regions

Here, we discuss the groups $H(2, n, p, q|k)$ with presentation $\langle x, y | x^2, y^n, (xy)^p, (xy^k)^q \rangle$, where as stated in Chapter 1 we assume initially that $p, q \geq 2$, $k \geq 3$ and $n \geq 5$. With $m = 2$, we can actually refine these assumptions further; it follows from the discussion of $D(m, n, p)$ in Section 1.2 that H is finite if $\frac{1}{n} + \frac{1}{p} > \frac{1}{2}$. Therefore we will consider throughout this chapter that $p \geq 3$. Also, by virtue of assuming $k \geq 3$ (from Section 1.2), we may assume $n \geq 6$ due to the following simple result. Lastly, we will assume for the most part that $q \geq 3$; we will discuss the case $q = 2$ and why it is problematic in certain settings later.

Lemma 2.1: $H(2, n, p, q|k) \cong H(2, n, p, q|n - k)$. *In particular, when* $m = 2$ *we may assume* $k \leq \frac{n}{2}$.

Proof: The relators $y^n = 1$ and $(xy^k)^q = 1$ imply that $(xy^{k-n})^q = 1$, and consequently $((xy^{k-n})^{-1})^q = (y^{n-k}x^{-1})^q = 1$. Since $x = x^{-1}$ as $m = 2$, this can be cyclically rewritten to $(xy^{n-k})^q = 1$ hence the result. As a consequence, if $k > \frac{n}{2}$ then $k + \frac{n}{2} > n$ implying $n - k < \frac{n}{2}$. □

As discussed in Section 1.3.2, we will search for positively curved interior regions, where by interior regions we are always referring to the non-distinguished regions of a spherical picture Π (unless specified otherwise). We begin with the following observation which is a direct corollary of Lemma 1.4.

Lemma 2.2: *Let $m = 2$ and $k > 4$. If $q \geq 3$ and $n > 5k$, then no positively curved interior regions with label y^a ($a \neq 0$) exist.*

Proof: Let Δ be a positively curved region of Π. By Lemma 1.4 ii., $3 \leq d(\Delta) \leq 5$; therefore the product of Δ's labels is y^a where a is at most $5k$. But this implies that $a \equiv 0 \mod n$, and therefore $n \leq 5k$ which is a contradiction. □

In discussing region types, we will use j_1 to denote the number of α-vertices on a region's boundary and $\overline{j_1}$ to denote the number of $\bar{\alpha}$-vertices. Similarly we define j_2 and $\overline{j_2}$ for the number of β- and $\bar{\beta}$-vertices. Finally, we introduce $J_i := j_i + \overline{j_i}$ where we wish to discuss the number of vertices of a given type irrespective of sign. Therefore, for a region Δ of degree $3 \leq d(\Delta) \leq 5$ we require $J_1 + J_2 = d(\Delta)$ and $j_1 + kj_2 - \overline{j_1} - k\overline{j_2} \equiv 0 \bmod n$. We list the possibilities below in Table 2.1 for so-called triangular, square and pentagonal regions up to inversion, along with the resultant boundary label; where the label gives y^a, for $|a| \leq 5$, we discount it. We also discount choices whose label is $y^{k\pm2}$; these imply $y^k = y^{\pm2}$. If 2, then $H \cong (p, n|q, 2)$ as discussed in Section 1.2; if -2, then applying Lemma 2.1 $H(2, n, p, q|k) \cong H(2, n, p, q|n + 2) \cong H(2, n, p, q|2)$ which resolves the same way.

As was mentioned in Section 1.4, we will be considering the Platonic cases separately (i.e. where the regions T_2, S_4, P_5 and P_9 theoretically exist) so will not discuss these regions in detail here. Likewise $n = 2k$ will have its own later section so we ignore S_8 for now.

Since we assume that $n \geq 2k$, we may assume that a region having boundary

Name	$(j_1, \overline{j_1}, j_2, \overline{j_2})$	Label	Name	$(j_1, \overline{j_1}, j_2, \overline{j_2})$	Label
T_1	(1, 0, 2, 0)	y^{2k+1}	P_1	(4, 0, 1, 0)	y^{k+4}
T_2	(0, 0, 3, 0)	y^{3k}	P_2	(3, 0, 2, 0)	y^{2k+3}
S_1	(3, 0, 1, 0)	y^{k+3}	P_3	(2, 0, 3, 0)	y^{3k+2}
S_2	(2, 0, 2, 0)	y^{2k+2}	P_4	(1, 0, 4, 0)	y^{4k+1}
S_3	(1, 0, 3, 0)	y^{3k+1}	P_5	(0, 0, 5, 0)	y^{5k}
S_4	(0, 0, 4, 0)	y^{4k}	P_6	(0, 4, 1, 0)	y^{k-4}
S_5	(1, 1, 1, 1)	y^0	P_7	(0, 2, 3, 0)	y^{3k-2}
S_6	(0, 3, 1, 0)	y^{k-3}	P_8	(0, 1, 4, 0)	y^{4k-1}
S_7	(0, 1, 3, 0)	y^{3k-1}	P_9	(1, 1, 3, 0)	y^{3k}
S_8	(1, 1, 2, 0)	y^{2k}	P_{10}	(2, 1, 2, 0)	y^{2k+1}

Table 2.1: Roster of potential positively curved regions when $m = 2$

label y^a implies that either $a = 0$ or $a = n$; if instead to contradict this $n \mid a$ and $n \neq a$, then certainly $a \geq 4k$. Since we deal with y^{4k} and y^{5k} separately, the only possibility according to Table 2.1 is $a = 4k + 1$, then $n = 2k$ since any smaller choice for n would necessarily imply $n < 2k$. But if $\frac{4k+1}{2k} = z$ for $z \geq 2 \in \mathbb{N}$, $z = 2$ implies $1 = 0$ and $z \geq 3$ implies $k = \frac{1}{2z-4}$ which contradicts k being an integer. As a consequence of $n \geq 2k$, we observe that S_1 only exists if $n = k + 3$ i.e. if $k = 3$ and $n = 6$, which puts us into the $n = 2k$ case and so we will disregard S_1 also for now. Also, this requirement means that S_6 and P_6's boundary labels force $n < k$ unless $k = 3$ or $k = 4$ respectively.

Now, evaluating the curvature of each of the other regions, we find the following:

- The pentagons P_i for $i \in \{2, 3, 7, 10\}$ are positively curved only if $p = q = 3$.
- The pentagons P_1 and P_6 require $p = 3$ and $3 \leq q \leq 5$ in order to be positively curved. Similarly, P_4 and P_8 require $q = 3$ and $3 \leq p \leq 5$.
- The squares S_2 and S_5 require either $p = 3$ or $q = 3$ in order to be positively curved; if $p = 3$ then $3 \leq q \leq 5$, if $q = 3$ then $3 \leq p \leq 5$.
- The squares S_3 and S_7 have positive curvature $\frac{2}{p}$ when $q = 3$. When $q = 4$,

positive curvature is attained only when $p = 3$. S_6 works similarly with p, q reversed everywhere.

- The triangle T_1 can be positively curved for $q \leq 11$, and is definitely positively curved if $q = 3, 4$; if $8 \leq q \leq 11$ this requires $p = 3$, if $q = 7$ then we require $p = 3, 4$; if $q = 6$ then we require $3 \leq p \leq 5$ and if $q = 5$ we require $3 \leq p \leq 11$.

From these observations, we now cut down a large swathe of possible cases with the following result, where we need not even assume $n \leq 2k$.

Theorem 2.3: *Let $H = H(2, n, p, q|k)$ with associated presentation P_H. Then if any of the following sets of conditions hold:*

i. $p, q \geq 6$

ii. $p, q \geq 4$ *and* $n \neq 3k, n \neq 2k + 1$

iii. $p = 3, q \geq 12$ *and* $k > 3$

iv. $p = 3,\ 6 \leq q \leq 11,\ n \neq 2k + 1,\ n \neq k + 3$ *and* $k > 3$

v. $q = 3,\ p \geq 6$ *and* $n \neq z$ *where* $z \in \{2k + 1, 3k, 3k \pm 1\, 4k, 5k\}$,

P_H is quasi-aspherical over $C_2 * C_n$.

Proof: If H completely satisfies any of the conditions i. through v. then we can check against the above list of parameters for positive curvature that none of the regions T_1, S_i (for $i \in \{2, 3, 5, 7\}$) or P_j (for $j \in \{1, 2, 3, 4, 6, 7, 8, 10\}$ are indeed positively curved. Thus any reduced spherical picture Π over a presentation P_H arising from these groups can have no positively curved interior regions, thus its distinguished region Δ_D must satisfy $c(\Delta_D) \geq 4\pi$; however since Δ_D receives no curvature from interior regions, $c(\Delta_D) \leq (2 - 1 + \frac{2}{3})\pi < \frac{4\pi}{3}$. $\square$

2.2 Co-existance of positive regions, spherical families

Firstly, we will consider the possibility of co-existance of positively curved regions; that is to say, groups whose parameters admit more than one of the above regions under consideration as valid regions. Rather than list the calculations for every possible pair, we will show an example of the argument since this is unchanged in each case. For instance, suppose that T_1 and P_1 co-exist. Then $2k+1 = n = k+4$ which implies $n = 7$, and thus $k = 3$. Since both of these regions are positively curved, our previous curvature calculations show we require $p = 3$ and $3 \leq q \leq 5$; therefore the only groups for which both co-exist are $H(2, 7, 3, 3|3)$, $H(2, 7, 3, 4|3)$ and $H(2, 7, 3, 5|3)$. We can readily attempt to check these for finiteness using a computer program; in this work we will use GAP [12] and often in particular the KBMAG package [14]; we may also use MAF [22] which performs a similar role. We may also be able to ascertain infiniteness by looking for subgroups of H which are either infinite or themselves contain infinite subgroups (for instance, whose abelianisations are infinite) in GAP. In Table 2.2, we will list candidate groups which may arise from the method above with an example pair of regions for which it occurs and the group order. Note that the example pair may not necessarily be unique (for instance the group $H(2, 10, 3, 3|4)$ arises from both the pairs S_2/P_6 and S_2/P_7). In the cases with order marked as "Unknown" no method tried was able to ascertain finiteness or otherwise.

Group	Regions	Order	Group	Regions	Order
$H(2,7,3,3\|3)$	T_1, P_1	1	$H(2,10,3,3\|3)$	S_3, S_6	150
$H(2,7,3,4\|3)$	T_1, P_1	168	$H(2,10,3,4\|3)$	S_3, S_6	2160
$H(2,7,3,5\|3)$	T_1, P_1	1	$H(2,10,4,3\|3)$	S_3, S_6	2160
$H(2,7,3,6\|3)$	T_1, S_6	1092	$H(2,10,3,3\|4)$	S_2, P_6	60
$H(2,7,3,7\|3)$	T_1, S_6	1092	$H(2,10,3,4\|4)$	S_2, P_6	3960
$H(2,7,3,8\|3)$	T_1, S_6	10752	$H(2,10,3,5\|4)$	S_2, P_6	Unknown
$H(2,7,3,9\|3)$	T_1, S_6	∞	$H(2,11,3,3\|3)$	S_6, P_3	1
$H(2,7,3,10\|3)$	T_1, S_6	∞	$H(2,11,3,3\|4)$	S_7, P_6	1
$H(2,7,3,11\|3)$	T_1, S_6	Unknown	$H(2,11,3,4\|4)$	S_7, P_6	1
$H(2,7,4,3\|3)$	T_1, S_6	1	$H(2,11,4,3\|3)$	S_6, P_8	1
$H(2,8,3,3\|3)$	S_2, S_6	96	$H(2,13,3,3\|3)$	S_6, P_4	1
$H(2,8,3,4\|3)$	S_2, S_6	336	$H(2,13,4,3\|3)$	S_6, P_4	246480
$H(2,8,3,5\|3)$	S_2, S_6	2160	$H(2,13,3,3\|4)$	S_3, P_6	1
$H(2,8,4,3\|3)$	S_2, S_6	336	$H(2,13,3,4\|4)$	S_3, P_6	246480
$H(2,8,5,3\|3)$	S_2, S_7	2160	$H(2,13,3,3\|5)$	S_2, P_7	1
$H(2,9,3,3\|4)$	T_1, P_6	12	$H(2,14,3,3\|4)$	P_3, P_6	1
$H(2,9,3,4\|4)$	T_1, P_6	2448	$H(2,15,3,3\|4)$	P_6, P_8	720
$H(2,9,3,5\|4)$	T_1, P_6	1	$H(2,17,3,3\|4)$	P_4, P_6	1

Table 2.2: Groups for which multiple positively curved regions could arise, $m = 2$

To clarify, the infinite groups of Table 2.2 were found as follows: $H(2,7,3,9|3)$ was found to contain a subgroup K of index 9 where $K''_{ab} \cong \mathbb{Z}^7 \times C_2$, and $H(2,7,3,10|3)$ was found to contain a subgroup K of index 42 where $K'_{ab} \cong \mathbb{Z}^3 \times C_{41}$. For those finite groups which are non-trivial, we present some properties (structure if computable by GAP's StructureDescription command; otherwise interesting subgroups, whether perfect and so on) including their identification number in GAP's SmallGroups library (if available) in square brackets. These are to be found in Appendix A.

From this, we now will consider so-called *duotype* pictures, by which we mean a picture Π whose positively curved regions are all either S_5-type regions (since its labelling does not restrict the parameters in any way) or one other type from

Table 2.1. In the case where $p, q \geq 4$ and S_5 is not positively curved, or where a picture (typically spherical) can be created over copies of one type, we call the picture *monotype*. This definition will be more useful later in the $m \geq 4$ case but we establish the convention now.

However, in an effort to apply quasi-asphericity arguments we must note which cases give rise to spherical pictures. In a monotype picture Π with associated graph Γ, where all regions have degree d and further where all α-vertices have degree p and β-vertices have degree q, we can readily construct in the normal graph-theoretical way the dual graph Γ' with vertices all of degree d and faces of degrees p and q. This therefore guarantees us spherical pictures for certain pairs (p, q) to be listed later; in certain cases where we can describe Γ itself as the graph of a solid as opposed to the dual of one, we mention this explicitly. In most cases these are the standard Platonic or Archimedean solids, descriptions or pictorial representations of which we omit here but can be readily found in for instance [4] or [3]; with the exception of where $p = q$, one can easily replicate the spherical picture whose graph is represented by the dual of an Archimedean solid A by taking the dual of the graph of A, where each vertex inserted into a face of degree p is an α-vertex (or $\bar{\alpha}$-vertex if appropriate) and likewise each vertex inserted into a face of degree q is a β-vertex. As such, we will not provide exhaustive detail into how to draw these from scratch, but if either $p = q$ or the graph of Π is neither a Platonic or Archimedean solid, an illustration will be provided as to how such a picture may arise. Lastly, there are clear analogues between our regions; for instance, if a spherical monotype picture Π_1 can be found for S_3 regions, then the spherical picture Π_7 can clearly be constructed over S_7 regions by replacing every α-vertex in Π_2 with an $\bar{\alpha}$-vertex. We will therefore describe all such equivalences in the following list, and then describe spherical pictures over one representative from each set. The equivalences are between the following:

- T_1 alone
- S_2 alone
- S_1, S_3, S_6, S_7 (starting always from a picture over S_3, replace α-vertices with $\bar{\alpha}$-

vertices to obtain one over S_7; replace β-vertices with α-vertices and original α-vertices with β-vertices to obtain a picture over S_1; and lastly replace β-vertices with $\bar{\alpha}$-vertices and original α-vertices with β-vertices to obtain a picture over S_6)

- P_1, P_4, P_6, P_8 (starting always from a picture over P_4, replace α-vertices with $\bar{\alpha}$-vertices to obtain one over P_8; replace β-vertices with α-vertices and original α-vertices with β-vertices to obtain a picture over P_1; and lastly replace β-vertices with $\bar{\alpha}$-vertices and original α-vertices with β-vertices to obtain a picture over P_6)

- P_2, P_3, P_7 (starting always from a picture over P_3, replace α-vertices with $\bar{\alpha}$-vertices to obtain one over P_7; replace β-vertices with α-vertices and original α-vertices with β-vertices to obtain a picture over P_2)

As a result, we will discuss spherical monotype pictures over T_1, S_2, S_3, P_2 and P_4. Also note that in transformations where β-vertices are replaced by α- or $\bar{\alpha}$-vertices, the pictures become valid over the pair (q, p) rather than (p, q) (for instance the region P_4 is positively curved for $(p, q) = (4, 3)$ but P_1's analogous picture is for the pair $(3, 4)$).

2.2.1 Spherical pictures

If we begin first with the pentagonal region P_2, then it is positively curved only for the pair $(p, q) = (3, 3)$ and any picture consisting solely of P_2 regions would require 12 regions. It can therefore be drawn as the planar representation of the dodecahedron. Similarly, for P_4 which is positively curved for the pairs $(p, 3)$ for $3 \leq p \leq 5$, for $(3, 3)$ we can draw the same dodecahedron. These are shown below in Fig. 2.1.

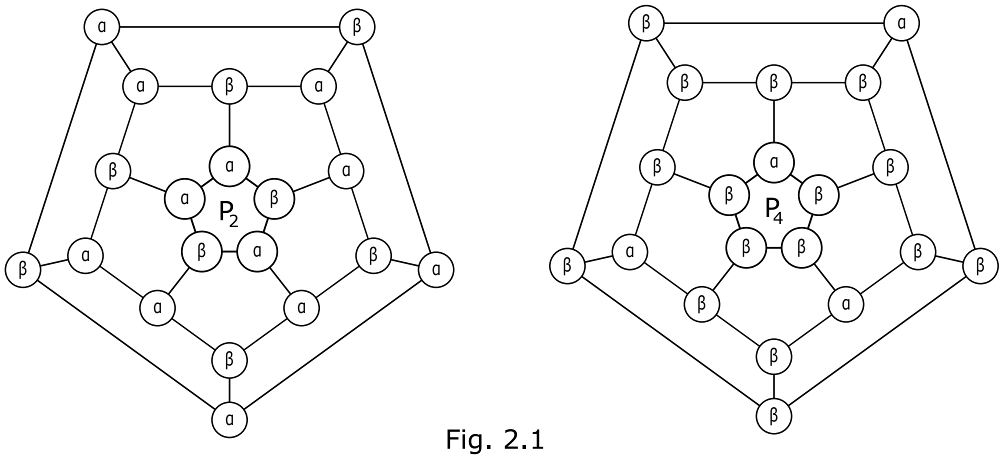


Fig. 2.1

In the case of $(4,3)$ we can draw a graph that is the dual of the planar representation of the Archimedean snub cube, and similarly for the pair $(5,3)$ one can draw a graph whose dual is the planar representation of the Archimedean snub dodecahedron. Therefore, all positively curved regions of degree 5 can admit spherical pictures.

Moving now to the square regions, over S_2 one can construct the following spherical monotype pictures:

(p,q)	No. of S_2 regions	No. of α-vertices	No. of β-vertices	Details of graph
$(3,3)$	6	4	4	Cube
$(3,4)$	12	8	6	Dual of cuboctahedron
$(4,3)$	12	6	8	Dual of cuboctahedron
$(3,5)$	30	20	12	Dual of icosidodecahedron
$(5,3)$	30	12	20	Dual of icosidodecahedron

Table 2.3: Planar representations of spherical pictures over S_2 regions

Therefore spherical pictures exist for all possible positively curved variants of S_2; the diagram for the cube is below in Fig. 2.2. Switching now to S_3, the lone case outside of $q=3$ is $(p,q)=(3,4)$; in this instance, we can produce a picture whose dual graph is the planar representation of the Archimedean small rhombicuboctahedron.

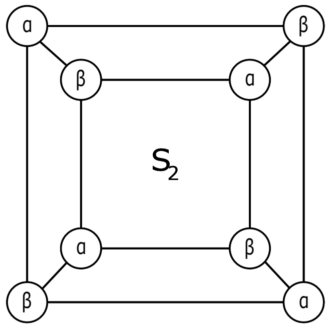
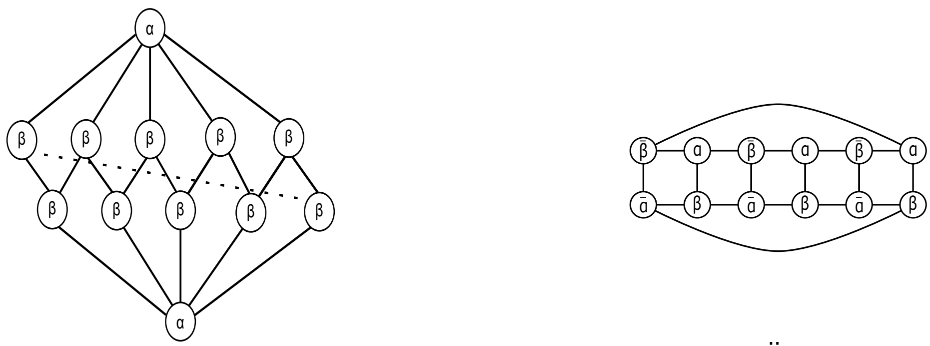


Fig. 2.2

For the remaining cases over S_3, the pairs $(p, 3)$ for any $p \geq 3$, we construct a slightly different picture, with two α-vertices and p β-vertices; from these, we can thus draw the planar representation of the p-trapezohedron, with an example of construction for $p = 5$ given in Fig. 2.3 i. As such, all square regions which are not S_5-regions admit spherical monotype pictures. On a different note, we can construct a somewhat different duotype picture with S_5 regions for $(p, q) = (3, 3)$; by adjoining six S_5 regions, we can form two regions of degree 6 which therefore have curvature $c(3, 3, 3, 3, 3, 3) = 0$. This therefore gives us the planar representation of the hexagonal prism as shown in Fig. 2.3 ii.



i. ii.

Fig. 2.3

Lastly, we discuss spherical pictures over T_1. As with S_3, we can find several spherical pictures related to the Platonic and Archimedean solids, and we list these below. In the case of $(p, q) = (3, 4)$ or $(4, 4)$, the appropriate diagrams are shown in Fig. 2.4 i. and ii.

(p, q)	No. of T_1 regions	No. of α-vertices	No. of β-vertices	Details of graph
$(3, 4)$	6	2	3	Dual of triangular prism
$(3, 6)$	12	4	4	Dual of truncated tetrahedron
$(3, 8)$	24	8	6	Dual of truncated cube
$(3, 10)$	60	20	12	Dual of truncated dodecahedron
$(4, 4)$	8	2	4	Octahedron
$(4, 6)$	24	6	8	Dual of truncated octahedron
$(5, 6)$	60	12	20	Dual of truncated icosahedron

Table 2.4: Planar representations of spherical pictures over T_1 regions

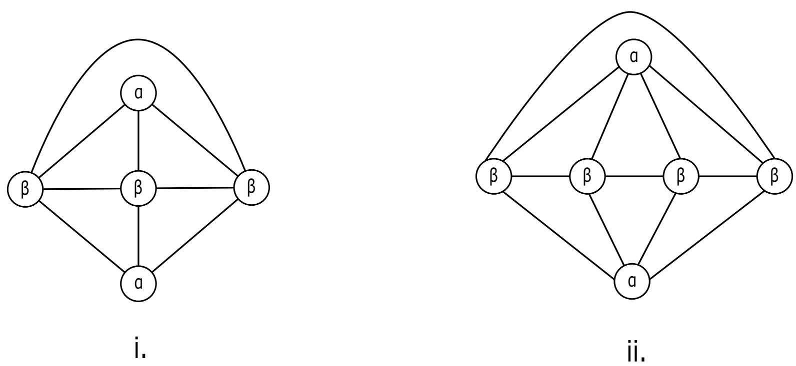


Fig. 2.4

Therefore the only cases where $m = 2$ for which positively curved regions exist, but spherical monotype pictures cannot be obtained are the groups $H(2, 2k + 1, p, q|k)$ where (p, q) is one of $(p, 3)$ for $p \geq 3$, $(p, 4)$ for $p \geq 5$, $(p, 5)$ for $p \geq 3$ or one of the pairs (3,7), (3,9), (3,11), (4,7). Whilst we do not have complete results for all of these, we can at least say something for those pairs for which T_1's positive curvature is relatively small.

2.2.2 Quasi-asphericity for $(p, q) = (3, 11)$

In this section, we will attempt to prove that the presentation P_H is quasi-aspherical over the free product $C_m * C_n$ for the groups $H(2, 2k + 1, 3, 11|k)$. Here, the only positively curved region is the triangle T_1. As discussed in section 1.3.2 we do this in phases- first we prove that curvature from interior regions can be compensated for, then that the distinguished region has amended curvature strictly less than 4π.

To help with this we will set down the following definition. Let Π be some non-empty picture containing at least one vertex. An *α-wheel* W_v is defined to be the sub-picture of Π consisting of an α-vertex v, all vertices which are connected by edges to v, and all regions which have v as an incident vertex plus all their edges. If all these regions are positively curved (i.e. T_1 regions), then we call the α-wheel *complete*, and *incomplete* if not. Of course we can similarly define a β-wheel or $\bar{\alpha}$-wheel by starting with a β-vertex or $\bar{\alpha}$-vertex initially. Similarly to single regions, we call a wheel *interior* if all of its constituent regions are interior regions. An example of a complete α-wheel when $p = 3$ is shown in Fig. 2.5. We define the curvature of an α-wheel in the natural way as the sum of the constituent regions of the wheel, for example a complete α-wheel has curvature $3.c(3, 11, 11)\pi = \frac{1}{11}\pi$.

We now identify where negatively curved regions exist. Incomplete α-wheels contain such a region by definition, and we could also obtain a region formed solely by β-vertices (whose neighbours are potentially complete); we call these latter regions *SB-type*. Since we normally disperse curvature across edges from positive regions to negative ones, this may not be possible when dealing with wheels (for instance, for $q > 5$ one could attach other complete α-wheels to the edges between two β-vertices in Fig. 2.5). As such, we will attempt to move all curvature into incomplete α-wheels or into regions of SB-type, but this may require multiple dispersals; for instance we may have a complete α-wheel W_v with neighbouring α-wheels W_{w_i} $(i = 1, 2, 3)$ which are also complete, but one of these (W_{w_1} say) may have a neighbouring incomplete α-wheel; so we would disperse W_v's curvature into W_{w_1}, and then perform a second dispersal into W_{w_1}'s incomplete neighbour. We create a new definition to clarify this idea; a *complete α-wheel of depth b* is a complete α-wheel for which the minimal number of dispersals required to reach an incomplete α-wheel or SB-type region is b (one may think of it as being b-landlocked by other complete wheels). As intimated above, we permit dispersal of curvature into other positively curved wheels on the proviso that an α-wheel of depth b_1 may only receive curvature from a wheel of depth b_2 if $b_2 > b_1$.

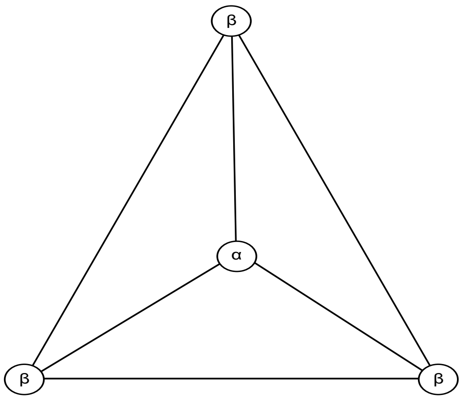


Fig. 2.5

Before we start the proof of quasi-asphericity, we first prove some minor results about α-wheel depths.

Lemma 2.4: *Let P_H be the standard presentation for $H(2, 2k+1, 3, 11|k)$, and further let Π be a reduced spherical picture over P_H. Suppose W_{v_1}, W_{v_2} are complete α-wheels in Π. Then:*

i. The maximal depth of W_{v_1} is 3.

ii. If W_{v_1} and W_{v_2} are neighbouring α-wheels, then at most one of them has depth 3.

Proof: For i., examine one of W_{v_1}'s β-vertices, for instance the central β-vertex in Fig. 2.6 where W_{v_1}'s edges are shaded in grey. Since it has degree 11, it cannot possibly be a part of 11 T_1 regions since it would require its 11 adjacent vertices to alternate between β- and α-vertices; however this requires q to be even. Thus, to maximise the number of T_1 regions, we will assume alternation happens up to creating one negatively curved region, marked on Fig. 2.6 as ε_1. Then if we assume all associated α-wheels are complete in order to calculate the maximal depth, then the dispersal from W_{v_1} can pass into W_{w_1}, then W_{w_2} and then to ε_1, thus W_{v_1} has at most depth 3.

For ii., assume W_{v_1} and W_{v_2} are complete α-wheels which share an edge, and that W_{v_1} has depth 3; then the picture looks as in Fig. 2.6. As then can be seen from Fig. 2.6 it is then apparent that W_{v_2} can have at most depth 2. □

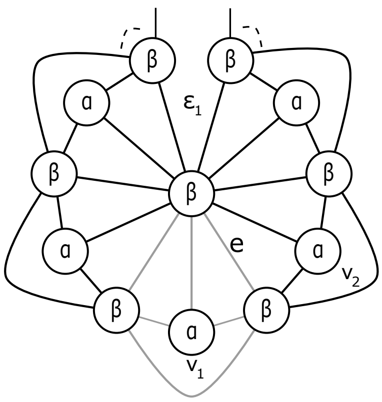


Fig. 2.6

Note also from Lemma 2.4ii. that in the same way one β-vertex cannot be a constituent part of two distinct complete α-wheels of depth 3 (it would be impossible to place two such wheels three dispersals away from ε_1 in Fig. 2.6). From this, we now begin the main proof.

Theorem 2.5: *The standard presentation P_H for $H(2, 2k{+}1, 3, 11|k)$ is quasi-aspherical over $C_2 * C_{2k+1}$.*

Proof: Let Π be a minimal reduced spherical monotype picture over P_H. First, we calculate the maximum possible amended curvature of a complete α-wheel W_v of each possible depth. Starting with W_v being a complete α-wheel of depth 3, it receives no positive curvature as it is of maximal depth (Lemma 2.4i.). Therefore in this case $c^*(W_v) = c(W_v) = \frac{1}{11}\pi$. Now since by Lemma 2.4ii. none of W_v's neighbours can be of depth 3, they must all be complete α-wheels of depth 2, so we disperse $\frac{1}{33}\pi$ into each of these. Now suppose W_v is a complete α-wheel of depth 2. Then, W_v may receive curvature from one complete α-wheel of depth 3, but only one (if two, then those wheels of depth 3 must share a vertex, which is impossible as noted post Lemma 2.4); therefore now $c^*(W_v) \leq \left(\frac{1}{11} + \frac{1}{33}\right)\pi = \frac{4}{33}\pi$. At least one neighbour to W_v must be a complete α-wheel of depth 1, and so we disperse all this curvature into such neighbours. From this, we estimate the amended curvature of a complete α-wheel W_v of depth 1 as $c^*(W_v) \leq \left(\frac{1}{11} + 2.\frac{4}{33}\right)\pi = \frac{1}{3}\pi$. Therefore, the maximum amount of curvature that may pass is $\frac{1}{3}\pi$. We use these estimates in what follows.

Firstly, in order for a spherical picture to arise on T_1 regions, Π must necessarily contain β-vertices. As argued in Lemma 2.4, this necessarily means that a non-T_1 region must exist; to begin with, assume that a negatively curved region of SB-type exists in Π; we call it ε. Suppose that $d(\varepsilon) = B$, and that each edge is adjacent to complete α-wheels of depth 1 and so receives the maximum possible curvature of $\frac{1}{3}\pi$. Then we calculate the amended curvature as

$$\begin{aligned} c^*(\varepsilon) &\leq \left(2 - B + \frac{2B}{11} + \frac{B}{3}\right)\pi \\ &= \left(\frac{66 - 16B}{33}\right)\pi. \end{aligned}$$

Thus, $c^*(\varepsilon) \leq 0$ unless $B = 3$ or $B = 4$; however, if the former is true then it implies that $y^{3k} = 1$ in H. This combined with $y^{2k+1} = 1$ leads to the conclusion that $y^{6k+3} = y^{6k} = 1 \Rightarrow y^3 = 1$, which contradicts the initial assumptions that $n \geq 6$. Likewise if $B = 4$ then $y^{4k} = 1$ and $y^{2k+1} = 1$ implies that $y^{4k} = y^{4k+2} = 1 \Rightarrow y^2 = 1$, again a contradiction.

Now, suppose that ε is a negatively curved region arising from an incomplete α-wheel. Assume for the maximising of curvature that the incomplete α-wheel consists of two T_1 regions (which we factor into the amended curvature sum) and ε, and suppose further in maximising amended curvature that all of ε's vertices are either α- or β-vertices; so, we can consider ε to be constructed from strings of α- and β-vertices. Again, to maximise amended curvature, assume that all strings of α-vertices are of length one since no curvature is received across $\alpha\alpha$ edges; then if A is the total number of α-vertices, $d(\varepsilon) \geq 2A$. Now, suppose that $A \geq 3$, so $r \geq 6$. Suppose for the crudest possible estimate that every possible edge receives $\frac{1}{3}\pi$. Now, if $B \geq 3$ is the total number of β-vertices, the amended curvature is given by

$$
\begin{aligned}
c^*(\varepsilon) &\leq \left(2 - A - B + \frac{2A}{3} + \frac{2B}{11} + \frac{2}{33} + \frac{B+A}{3}\right)\pi \\
&= \left(\frac{68 - 16B}{33}\right)\pi.
\end{aligned}
$$

So again $c^*(\varepsilon) \leq 0$ unless $B = 3$ or $B = 4$; in the former case, this forces $A = 3$, and as before this implies that $y^{3k+3} = 1$, which when compared with $y^{2k+1} = 1$ implies $y^{6k+6} = y^{6k+3} = 1 \Rightarrow y^3 = 1$, a contradiction. If $B = 4$, $A = 3$ or $A = 4$; in either case, these imply $y^{4k+3} = 1$ and $y^{4k+4} = 1$ respectively, which in turn imply $y^{12k+9} = y^{12k+6} = 1$ (thus $y^3 = 1$) and $y^{8k+8} = y^{8k+4} = 1$ (thus $y^4 = 1$), again both contradictions. If we extend this further, then if $A = 1$ we still need only to check $B = 3$ and $B = 4$ (since $B = 1$ forces $d(\varepsilon) = 2$ and $B = 2$ means ε is a T_1 region, both of which are contradictions). As before, these respectively imply $y^{3k+1} = 1$ and $y^{4k+1} = 1$, which leads to $y^{6k+2} = y^{6k+3} = 1$ (thus $y = 1$) and $y^{8k+2} = y^{8k+4} = 1$ (thus $y^2 = 1$), again both contradictions. Lastly, we check $A = 2$; if then $B = 2$ we have $y^{2k+2} = y^{2k+1} \Rightarrow y = 1$, and if $B = 3$ we have $y^{3k+2} = 1$ and thus $y^{6k+4} = y^{6k+3} = 1 \Rightarrow y = 1$, thus both contradictions. The only possibility then is that $B = 4$, and $c^*(\varepsilon) = \frac{4}{33}\pi$. We draw this possibility in Fig. 2.7, and consider one of the β-vertices denoted by w.

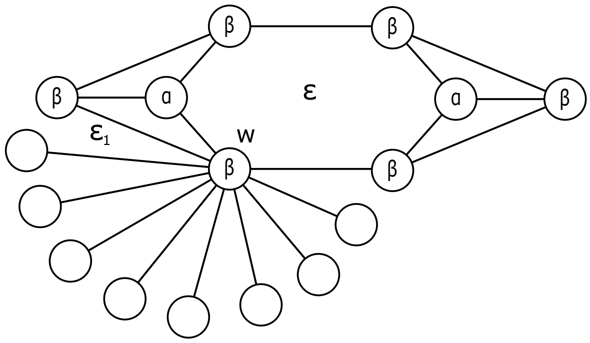


Fig. 2.7

From Fig. 2.7, we can attempt to fill the vertices incident to w in order to create T_1 regions; however, it is not possible to fill in all vertices alternately with α- and β-vertices. Thus, one of the other incident regions to w is not positively curved. Suppose ε_1 in Fig. 2.7 is this region. Then, w can no longer be part of a complete α-wheel of depth 3, even with alternate labelling on the remainder of the vertices; it can at most be a constituent vertex of a complete α-wheel of depth 2. Therefore we assume that instead of all edges of ε receiving $\frac{1}{3}\pi$, one edge receives only $\frac{4}{33}\pi$; then $c^*(\varepsilon) = \left(\frac{4}{33} + \frac{4}{33} - \frac{11}{33}\right)\pi < 0$, and so all positive curvature is accounted for.

If we now consider the distinguished region, which we will call Δ_D, the maximum possible curvature of Δ_D is $(2 - 1 + \frac{2}{3} + \frac{1}{3})\pi = 0$. Since $c(\Delta_D) < 4\pi$, no reduced spherical picture over P_H can exist. $\Box$

2.3 The groups $H(2, 2k, p, q \mid k)$

In this section, we assume that $p, q, k \geq 3$ as at the beginning of Chapter 2. Firstly, note that since $n = 2k$, $y^{-k} = y^k$ and thus by inverting the relator β and re-arranging we find $\beta^{-1} = \beta$. Thus we need only consider α-, $\overline{\alpha}$- and β-vertices; also we may not have adjacent β vertices since these would deform to a dipole. As such, if we let a region Δ of degree d consist of j_1 α-vertices, $\overline{j_1}$ $\overline{\alpha}$- vertices and j_2 β-vertices, we have $j_2 \leq \frac{d}{2}$. Further to this, the regions S_5 and S_8 are now identical since all $\bar{\beta}$-vertices may be freely replaced with β-vertices; we will work with S_8 in these cases.

With this considered, the only possible regions are S_1, S_8 and P_1. Since $n = 2k$, we see that S_1 can only arise for $k = 3$ and P_1 only for $k = 4$, so we shall deal with these cases first and then attempt to deal with $k \geq 5$ separately; in the case $k \geq 5$, we only consider $3 \leq p, q \leq 5$ since quasi-asphericity is otherwise automatic by Lemma 1.4iii. We start with $k = 3$.

Lemma 2.6: *The group $H(2, 6, p, q|3)$ is quasi-aspherical for all pairs (p, q) except for (4, 3), (5,3) or when* $p = 3$.

Proof: $c(S_8) = -2\pi + 2\pi(\frac{2}{p} + \frac{2}{q})$, so we know that an S_8 region is non-positively curved if and only if $(\frac{2}{p} + \frac{2}{q}) \leq 1$. Hence S_8 type regions are non-positive if and only if $p \geq \frac{2q}{q-2}$. By a similar argument, an S_1 region has non-positive curvature if and only if $c(S_1) = -2\pi + (\frac{3}{p} + \frac{1}{q})2\pi \leq 0 \Leftrightarrow p \geq \frac{3q}{q-1}$. By considering $3 \leq q \leq 5$ the only cases for which at least one of these inequalities fails (and therefore positively curved regions can exist) are $(4, 3)$, $(5, 3)$ and $(3, q)$ which completes the result. $\square$

The survey of $k = 3$ is continued using GAP, which ascertains both that for the pair $(p, q) = (4, 3)$ H is finite of order 336 and that for $(p, q) = (5, 3)$ H is infinite, as it finds a subgroup K such that $|H : K| = 5$. If we then take C to be the core of K in H, $C_{ab} \cong \mathbb{Z}^5$. Using results from GAP we can also in fact conjecture precisely the orders of $H(2, 6, 3, q|k)$, which we do so below.

Conjecture 2.7: *The group $H(2, 6, 3, q|k)$ is finite of order* $6q^2$. *Further, if $q \geq 3$ it is the semi-direct product of $C_q \times C_q$ with C_3, which is then taken in a further semi-direct product with C_2.*

Evidence for conjecture: As verified in GAP, for $1 \leq q \leq 10$ we find that $H(2, 6, 3, q|3)$ is a finite group of order 6, 24, 54, 96, 150, 216, 294, 384, 486 and 600 respectively. From this we predict that if $q = 20$ we have $|H| = 2400$, and that if $q = 100$ we have $|H| = 10000$; both of these are verified to be true by

GAP. Further, for $3 \leq q \leq 10$ GAP's StructureDescription command verifies the group as the semi-direct product of the semi-direct product of $C_q \times C_q$ with C_3.

For $k = 4$ the argument runs similarly, except with P_1 in place of S_1.

Lemma 2.8: *The group $H(2, 8, p, q|4)$ is quasi-aspherical for all pairs (p, q) except for possibly (4, 3), (5,3) or (3, q) where $q \in \{3, 4, 5\}$.*

Proof: Runs similarly to Lemma 2.6 but uses instead

$$c(P_1) = -3\pi + \left(\frac{4}{p} + \frac{1}{q}\right) 2\pi \leq 0 \Leftrightarrow p \geq \frac{8q}{3q-2}.\Box$$

This leaves five groups to check again with GAP; $H(2, 8, 3, 5|4)$ is finite of order 14880, $H(2, 8, 3, 4|4)$ is finite of order 768, $H(2, 8, 3, 3|4)$ is the trivial group, whilst $H(2, 8, 5, 3|4)$ is infinite via KBMAG. $H(2, 8, 4, 3|4)$ could not be ascertained as either finite or infinite through our usual methods.

We now turn to $k \geq 5$. Since the only possible region of positive curvature is S_8, we know that if $p, q \geq 4$ H is quasi-aspherical since $c(S_8) \leq c(4, 4, 4, 4) = 0$. Since $p = 3 \Rightarrow c(S_8) \leq 0$ for $q \geq 6$ and the reverse holds for $q = 3$, we need only consider the (p, q) pairs (3, 3), (3, 4), (4, 3), (3, 5) and (5, 3). We begin with (3, 3):

Theorem 2.9: *Let k be even. Then the group $H(2, 2k, 3, 3 \mid k)$ is finite for all $k \geq 5$, and satisfies*

$$|H| = \begin{cases} 1 & \text{if } 6 \nmid k, \\ 3 & \text{if } 6 \mid k. \end{cases}$$

Proof: We prove the result for even k divisible by 6 first using covering space arguments such as those in [6]. First we compute the abelianisation H_{ab} which has presentation

$$\langle x, y | x^2, y^{2k}, (xy)^3, (xy^k)^3, [x, y] \rangle$$

which after applying commutation yields the relators x^2, y^{2k}, x^3y^3 and x^3y^{3k}. After simplification the third relator yields $x = y^{-3}$ which implies $x = y^3$, so making this substitution leaves us with the presentation

$$\langle y | y^6, y^{2k}, y^{3(k+3)} \rangle.$$

Since $6 \mid k$, $y^6 = 1$ renders y^{12} redundant so we remove it. $y^{3(k+3)} = y^{3k+9} = y^{3k+3}$ then implies that $y^3 = 1$ since $3k+3 \equiv 3 \bmod 6$, which then renders y^6 redundant. Thus, H_{ab} has presentation $\langle y \mid y^3 \rangle$.

Now we can draw a diagrammatic representation of H_{ab}; this is in Fig. 2.5. If we now take lifts of our original relators to give us relators for H', we obtain

- **From** x^2**:** x_1^2, x_2^2, x_3^2
- **From** y^{2k}**:** $(y_1y_2y_3)^{4a}$ where $a = \frac{k}{6}$
- **From** $(xy)^3$**:** $x_1y_1x_2y_2x_3y_3$
- **From** $(xy^k)^3$**:** $(x_1(y_1y_2y_3)^{2a})^3$, $(x_2(y_2y_3y_1)^{2a})^3$, $(x_3(y_3y_1y_2)^{2a})^3$

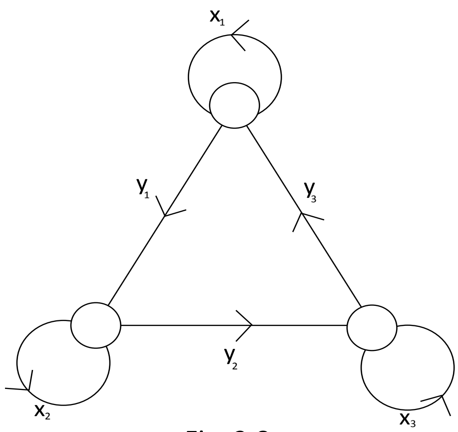


Fig. 2.8

Let us now choose y_1, y_2 as the edges of our maximal tree to contract. Then $y_3 = 1$ as a consequence (consider that we can replace y_3 by $\overline{y_2y_1}$). Then our $(xy^k)^3$ lifts read $x_i^3 = 1$ for $i = 1, 2, 3$. But since we already know $x_i^2 = 1$ for all such i, we conclude that $x_i = 1$ for each i and thus all six generators are equal to 1; hence H' is trivial, H is isomorphic to H_{ab} and is thus of order 3.

We now move to proving H trivial for all other even k. Consider first that the relators $(xy)^3$ and $(xy^k)^3$ imply, respectively, that $yxy = xy^{-1}x$ and $xy^kx =$

y^kxy^k. Then, given $(xy)^3 = (xy^k)^3$ we find $xyx = y^{k-1}xy^kxy^{k-1}$ and hence $xy^{-1}xy^{k-1}xy^kxy^{k-1} = 1$. A similar consideration for $(xy^{-1})^3 = (xy^k)^3$ yields $xyxy^{k+1}xy^kxy^{k+1} = 1$. Equating these and manipulating we find

$$xy^{k-1}xy^kx = y^2xy^{k+1}xy^kxy^2.$$

Using $xy^kx = y^kxy^k$ on the right hand side we can rewrite to $xy^{k-1}xy^kx = y^2xy^{2k+1}xy^{k+2}$ and the right side cancels to y^2xyxy^{k+2}. Squaring both sides and tidying the cancellation then leads to

$$\begin{aligned}
xy^{k-1}xy^{-1}xy^kx &= y^2xyxy^{k+4}xyxy^{k+2} \\
&= y^2xyxy^{k+3}xy^{k+1} \; (yxy = xy^{-1}x) \\
&= y^2xyxy^3xy^kxy \; (y^kxy^k = xy^kx) \\
\Rightarrow y^{k-1}xy^{-1}xy^kx &= (y^{-1}xy^{-2}xy^{-1})yxy^3xy^kxy
\end{aligned}$$

where the last step moves the leading x to the right hand side and utilises the fact that $xy^2x = (xyx)^2 = (y^{-1}xy^{-1})^2$. Tidying up yields $y^kxy^{-1}xy^kx = xyxy^kxy$. If we now post-multiply by x on both sides, we can substitute both the leading and trailing subwords of xyx on the right side for $y^{-1}xy^{-1}$, from which we find

$$\begin{aligned}
y^kxy^{-1}xy^k &= y^{-1}xy^{-1}y^ky^{-1}xy^{-1} \\
\Rightarrow y^{k+1}xy^{-1}xy^{k+1} &= xy^{k-2}x \\
\Rightarrow y^{k+2}xy^{k+2} &= xy^{k-2}x \\
\Rightarrow (xy^{k-2})^3 &= 1.
\end{aligned}$$

If we now return to $xy^{k-1}xy^{-1}xy^kx = y^2xyxy^{k+4}xyxy^{k+2}$, pre-multiplying by x and using that $xy^2x = y^{-1}xy^{-2}xy^{-1}$ we find

$$\begin{aligned}
y^{k-1}xy^{-1}xy^kx &= y^{-1}xy^{-2}xy^{-1}yxy^{k+2}xyxy^{k+2} \\
\Rightarrow y^{k-1}xy^{-1}xy^kx &= y^{-1}xy^{k+2}xyxy^{k+2} \\
\Rightarrow y^{k-1}xy^{k-1}x &= y^{-1}xy^{k+2}xyxy^2.
\end{aligned}$$

We now know $(xy^{k-2})^3 = 1 \Rightarrow xy^{-(k-2)}x = y^{k-2}xy^{k-2}$, and using this turns the left hand side, $y(y^{k-2}xy^{k-2})yx$, into $yxy^{-(k-2)}xyx$. Using this and substituting $y^{-1}xy^{-1}$ for xyx once on each side yields

$$xy^{-(k-1)}xy^{-1} = y^{-2}xy^{k+1}xy$$
$$\Rightarrow y^2xy^{k+1}x = xy^{k+1}xy^2.$$

Thus we know that y^2 and $xy^{k+1}x$ commute. However this implies any powers of these must also commute; since $\gcd(2k, k+1) = 1$ we know that the word $xy^{k+1}x$ has order $2k$, and we can thus express xy^rx as some power of $xy^{k+1}x$ for all $r \in \{1, 2, ..., 2k\}$. Likewise we can find any even power of y as some power of y^2. Then we have that xy^rx and $y^{2a}, r \in \{1, 2, ..., 2k\}$ and $a \in \mathbb{N}$, also commute.

Now examine the relator $xy^kxy^kxy^k = 1$; since k is even we know that xy^kx and y^k commute, giving $y^kx = 1$ hence $x = y^k$.

Substituting this into the presentation and removing superfluous relators gives the presentation for H as

$$\langle y | y^{2k}, y^{k+3} \rangle$$

Now this implies $y^{k-3} = 1$, and thus $y^6 = 1$. But $6 \nmid k \Rightarrow k+3$ is odd and not divisible by 3, so $\gcd(6, k+3) = 1$ and thus we have $y = 1$. Hence H is trivial (and note that if $6 \mid k$, this devolves instead to $y^3 = 1$ and we recoup H being of order 3 as required). □

We complete the survey of $H(2, 2k, 3, 3 \mid k)$ with the result for k odd.

Theorem 2.10: *If k is odd, the group $H(2, 2k, 3, 3 \mid k)$ is finite of order $6k^2$.*

Proof: We deal first with the case that $3 \nmid k$. We will use the Reidemeister-Schreier rewriting process to find a presentation for H', then repeat the process to find a presentation for H'' which shows it to have order k^2. As above the relators of H_{ab} are seen to be x^2, y^{2k}, x^3y^3 and x^3y^{3k}, and again using $x = y^3$ we find the relators change to y^6, y^{2k} and y^{3k+3}. Observe that since $k \equiv 1$ or 2 mod 3, y^{2k} becomes y^{6a+2} or y^{6a+4} $(a \in \mathbb{N})$ from which we deduce $y^2 = 1$. Thus $H_{ab} \cong C_2$. We take a Schreier transversal $\{e, y\}$ and compute the generators of H' of form $uv\overline{uv}^{-1}$, where $u \in \{e, y\}$ and $v \in \{x, y\}$.

u	v	$\overline{uv}$	$uv\overline{uv}^{-1}$
e	x	y	$xy^{-1} =: b_1$
e	y	y	e
y	x	e	$yx =: b_2$
y	y	e	$y^2 =: b_3$

We now have three generators for H', and now we find the relators from the following table (rewritten in the b_i)

u	ux^2u^{-1}	$uy^{2k}u^{-1}$	$u(xy)^3u^{-1}$	$u(xy^k)^3u^{-1}$
e	b_1b_2	b_3^k	$(b_1b_3)^3$	$(b_1b_3^{\frac{k+1}{2}})^3$
y	b_2b_1	b_3^k	b_2^3	$(b_2b_3^{\frac{k-1}{2}})^3$

If we now replace b_1 everywhere by b_2^{-1}, and re-label b_2 by a and b_3 by b, our presentation for H' is

$$\langle a, b | a^3, b^k, (a^{-1}b)^3, (a^{-1}b^{\frac{k+1}{2}})^3, (ab^{\frac{k-1}{2}})^3 \rangle.$$

Now abelianising this group, if we add the commutator $[a, b]$ then $(a^{-1}b)^3 = 1$ implies $b^3 = 1$; but $\gcd(3, k) = 1 \Rightarrow b = 1$, so H'_{ab} is simply the cyclic group of order 3 and a Schreier transversal for it is $\{e, a, a^2\}$. We find generators and relators for H'' in the same fashion as before:

u	v	$\overline{uv}$	$uv\overline{uv}^{-1}$
e	a	a	e
e	b	e	$b =: g_1$
a	a	a^2	e
a	b	a	$aba^{-1} =: g_2$
a^2	a	e	$a^3 =: g_3$
a^2	b	a^2	$a^2ba^{-2} =: g_4$

With this, we now construct as for H' the relator table seen below, rewritten in the generators g_i. The result of this is that H'' has the presentation

$$\langle g_1, g_2, g_3, g_4 | g_3, g_1^k, g_2^k, g_4^k, g_3^{-1}g_4g_2g_1, g_3^{-1}g_4^{\frac{k+1}{2}}g_2^{\frac{k+1}{2}}g_1^{\frac{k+1}{2}}, g_3g_1^{\frac{k-1}{2}}g_2^{\frac{k-1}{2}}g_4^{\frac{k-1}{2}}\rangle.$$

u	ua^3u^{-1}	ub^ku^{-1}	$u(a^{-1}b)^3u^{-1}$	$u(a^{-1}b^{\frac{k+1}{2}})^3u^{-1}$	$u(ab^{\frac{k-1}{2}})^3u^{-1}$
e	g_3	g_1^k	$g_3^{-1}g_4g_2g_1$	$g_3^{-1}g_4^{\frac{k+1}{2}}g_2^{\frac{k+1}{2}}g_1^{\frac{k+1}{2}}$	$g_2^{\frac{k-1}{2}}g_4^{\frac{k-1}{2}}g_3g_1^{\frac{k-1}{2}}$
a	g_3	g_2^k	$g_1g_3^{-1}g_4g_2$	$g_1^{\frac{k+1}{2}}g_3^{-1}g_4^{\frac{k+1}{2}}g_2^{\frac{k+1}{2}}$	$g_4^{\frac{k-1}{2}}g_3g_1^{\frac{k-1}{2}}g_2^{\frac{k-1}{2}}$
a^2	g_3	g_4^k	$g_2g_1g_3^{-1}g_4$	$g_2^{\frac{k+1}{2}}g_1^{\frac{k+1}{2}}g_3^{-1}g_4^{\frac{k+1}{2}}$	$g_3g_1^{\frac{k-1}{2}}g_2^{\frac{k-1}{2}}g_4^{\frac{k-1}{2}}$

Returning to the presentation for H'', clearly g_3 is trivial so we can immediately delete it, and substitute g_4 everywhere with $(g_1^{-1}g_2^{-1})$. Also $g_1^{\frac{k-1}{2}}g_2^{\frac{k-1}{2}}(g_1^{-1}g_2^{-1})^{\frac{k-1}{2}} = 1$ implies that (as a cyclic rewrite) $(g_1^{-1}g_2^{-1})^{\frac{k-1}{2}}g_1^{\frac{k-1}{2}}g_2^{\frac{k-1}{2}} = 1$ too.

Then the product of these last two relators must also equal the identity, i.e.

$$g_1^{\frac{k-1}{2}}g_2^{\frac{k-1}{2}}(g_1^{-1}g_2^{-1})^{k-1}g_1^{\frac{k-1}{2}}g_2^{\frac{k-1}{2}} = 1$$

but this subsequently means that $(g_1^{-1}g_2^{-1})^{k-1} = (g_1^{-1}g_2^{-1})^{-1} = ((g_2g_1)^{-1})^{-1} = g_2g_1$, so we find

$$g_1^{\frac{k-1}{2}}g_2^{\frac{k+1}{2}}g_1^{\frac{k+1}{2}}g_2^{\frac{k-1}{2}} = 1$$
$$g_2^{\frac{k+1}{2}}g_1^{\frac{k+1}{2}} = g_1^{\frac{k+1}{2}}g_2^{\frac{k+1}{2}}$$

Hence the $(k+1)$th powers of the generators commute. If we now define new generators $u := g_1^{\frac{k+1}{2}}$ and $v := g_2^{\frac{k+1}{2}}$, u and v are both of order k since $\gcd(k, \frac{k+1}{2}) = 1$. Now, if we add these generators to the presentation for H'' and delete g_1 and g_2, H'' now has the presentation

$$\langle u, v | u^k, v^k, [u, v]\rangle$$

which is the presentation of the direct product $C_k \times C_k$. Thus $|H| = |H : H'||H' : H''||H''| = 2.3.k^2 = 6k^2$ as desired.

Now, suppose $3 \mid k$. Then $k \equiv 3 \bmod 6$. As above we find H_{ab}'s relators to be y^6, y^{2k} and y^{3k+3}. This implies $2k \equiv 0 \bmod 6$ and $3k+3 \equiv 0 \bmod 6$, so $y^6 = 1$ renders these redundant and $H_{ab} \cong C_6$. If we now draw a covering space diagram (alloting for $x = y^3$) we obtain the graph in Fig. 2.6a. We choose to contract a maximal tree consisting of the edges y_i for $1 \leq i \leq 5$, and replace x_4, x_5, x_6 by $\overline{x_1}, \overline{x_2}, \overline{x_3}$ respectively (these would come from lifting the relator x^2 later). If we set $k = 6a+3$ for some $a \in \mathbb{N}$, then our relators for H' are (after lifting, simplifying and omitting cyclic rewrites)

$$\{y_6^{2a+1}, x_1\overline{x_2}x_3y_6, x_2\overline{x_3}\overline{x_1}, (x_jy_6^{a+1})^3[j = 1, 2, 3]\}.$$

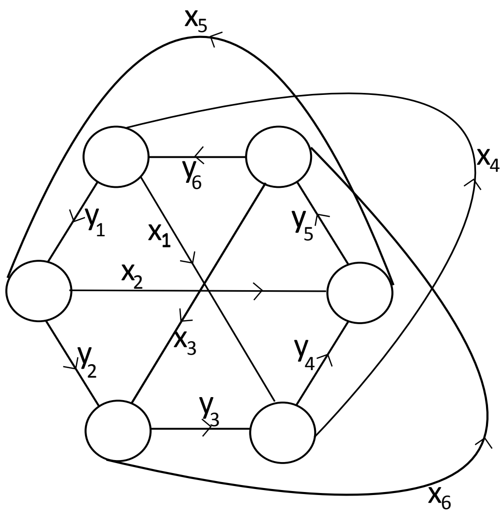


Fig.2.9a

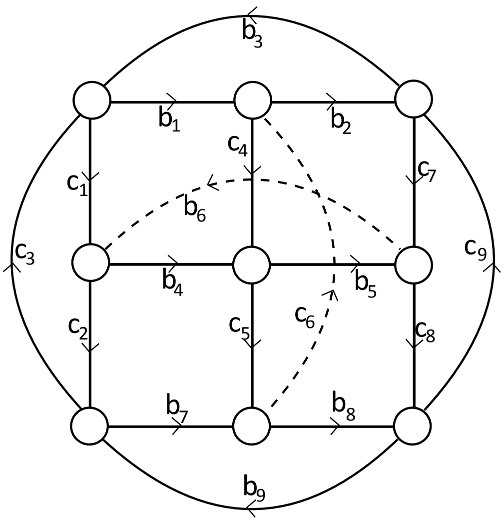


Fig.2.9b

We can now use the third relator to write $x_2 = x_1x_3$, and from this the second gives us $y_6 = \overline{x_3}x_1x_3\overline{x_1}$. Replacing these everywhere in the relators, our finalised presentation for H' is $\langle x_1, x_3 | (\overline{x_3}x_1x_3\overline{x_1})^{2a+1}, (x_1(\overline{x_3}x_1x_3\overline{x_1})^{a+1})^3,$
$(x_3(\overline{x_3}x_1x_3\overline{x_1})^{a+1})^3, (x_1x_3(\overline{x_3}x_1x_3\overline{x_1})^{a+1})^3\rangle$.

It is quickly apparent that H'_{ab} is simply the group with presentation $\langle x_1, x_3 | x_1^3, x_3^3, [x, y]\rangle$, i.e. $H'_{ab} \cong C_3 \times C_3$. The corresponding covering space diagram is seen in Fig. 2.6b. Each relator of H' now has nine lifts to relators in H''; these are presented below (with cyclic rewrites omitted).

Lifts of first relator: $(\overline{c_3}b_7c_6\overline{b_1})^{2a+1}$, $(\overline{c_6}b_8c_9\overline{b_2})^{2a+1}$, $(\overline{c_9}b_9c_3\overline{b_3})^{2a+1}$,

$(\overline{c_1}b_1c_4\overline{b_4})^{2a+1}$, $(\overline{c_4}b_2c_7\overline{b_5})^{2a+1}$, $(\overline{c_7}b_3c_1\overline{b_6})^{2a+1}$, $(\overline{c_2}b_4c_5\overline{b_7})^{2a+1}$, $(\overline{c_5}b_5c_8\overline{b_8})^{2a+1}$, $(\overline{c_8}b_6c_2\overline{b_9})^{2a+1}$

Lifts of second relator: • $b_1(\overline{c_6}b_8c_9\overline{b_2})^{a+1}b_2(\overline{c_9}b_9c_3\overline{b_3})^{a+1}b_3(\overline{c_3}b_7c_6\overline{b_1})^{a+1}$

- $b_4(\overline{c_4}b_2c_7\overline{b_5})^{a+1}b_5(\overline{c_7}b_3c_1\overline{b_6})^{a+1}b_6(\overline{c_1}b_1c_4\overline{b_4})^{a+1}$
- $b_7(\overline{c_5}b_5c_8\overline{b_8})^{a+1}b_8(\overline{c_8}b_6c_2\overline{b_9})^{a+1}b_9(\overline{c_2}b_4c_5\overline{b_7})^{a+1}$

Lifts of third relator: • $c_1(\overline{c_1}b_1c_4\overline{b_4})^{a+1}c_2(\overline{c_2}b_4c_5\overline{b_7})^{a+1}c_3(\overline{c_3}b_7c_6\overline{b_1})^{a+1}$

- $c_4(\overline{c_4}b_2c_7\overline{b_5})^{a+1}c_5(\overline{c_5}b_5c_8\overline{b_8})^{a+1}c_6(\overline{c_6}b_8c_9\overline{b_2})^{a+1}$
- $c_7(\overline{c_7}b_3c_1\overline{b_6})^{a+1}c_8(\overline{c_8}b_6c_2\overline{b_9})^{a+1}c_9(\overline{c_9}b_9c_3\overline{b_3})^{a+1}$

Lifts of fourth relator: • $b_1c_4(\overline{c_4}b_2c_7\overline{b_5})^{a+1}b_5c_8(\overline{c_8}b_6c_2\overline{b_9})^{a+1}b_9c_3(\overline{c_3}b_7c_6\overline{b_1})^{a+1}$

- $b_2c_7(\overline{c_7}b_3c_1\overline{b_6})^{a+1}b_6c_2(\overline{c_2}b_4c_5\overline{b_7})^{a+1}b_7c_6(\overline{c_6}b_8c_9\overline{b_2})^{a+1}$
- $b_3c_1(\overline{c_1}b_1c_4\overline{b_4})^{a+1}b_4c_5(\overline{c_5}b_5c_8\overline{b_8})^{a+1}b_8c_9(\overline{c_9}b_9c_3\overline{b_3})^{a+1}$

Now we choose a tree to contract. We choose to set the following equal to the identity; c_i for $i \in \{1,2,4,5\}$ and b_j for $j \in \{1,2,5,8\}$. We also make the following substitutions; $b_3 = \overline{b_2b_1} = 1$, $b_6 = \overline{b_5b_4} = \overline{b_4}$, $b_9 = \overline{b_8b_7} = \overline{b_7}$, $c_3 = \overline{c_2c_1} = 1$, $c_6 = \overline{c_5c_4} = 1$, $c_9 = \overline{c_8c_7}$. We thus substitute out all b_6, b_9 and c_9 references.

Our eighteen relators are now written solely in four generators, b_4, b_7, c_7 and c_8. We label these as follows:

- $r_1 := b_7^{2a+1}$, $r_2 := (\overline{c_8c_7})^{2a+1}$, $r_3 := (c_7c_8\overline{b_7})^{2a+1}$, $r_4 := \overline{b_4}^{2a+1}$, $r_5 := c_7^{2a+1}$, $r_6 := (\overline{c_7}b_4)^{2a+1}$, $r_7 := (b_4\overline{b_7})^{2a+1}$, $r_8 := c_8^{2a+1}$, $r_9 := (\overline{c_8}\overline{b_4}b_7)^{2a+1}$
- $r_{10} := (\overline{c_8c_7})^{a+1}(c_7c_8\overline{b_7})^{a+1}b_7^{a+1}$, $r_{11} := b_4c_7^{a+1}(\overline{c_7}b_4)^{a+1}\overline{b_4}(\overline{b_4})^{a+1}$, $r_{12} := b_7c_8^{a+1}(\overline{c_8}\overline{b_4}b_7)^{a+1}\overline{b_7}(b_4\overline{b_7})^{a+1}$
- $r_{13} := (\overline{b_4})^{a+1}(b_4\overline{b_7})^{a+1}b_7^{a+1}$, $r_{14} := c_7^{a+1}c_8^{a+1}(\overline{c_8c_7})^{a+1}$, $r_{15} := c_7(\overline{c_7}b_4)^{a+1}c_8(\overline{c_8}\overline{b_4}b_7)^{a+1}\overline{c_8c_7}(c_7c_8\overline{b_7})^{a+1}$
- $r_{16} := c_7^{a+1}c_8(\overline{c_8}\overline{b_4}b_7)^{a+1}\overline{b_7}b_7^{a+1}$, $r_{17} := c_7(\overline{c_7}b_4)^{a+1}\overline{b_4}(b_4\overline{b_7})^{a+1}b_7(\overline{c_8c_7})^{a+1}$, $r_{18} := \overline{b_4}^{a+1}b_4c_8^{a+1}\overline{c_8c_7}(c_7c_8\overline{b_7})^{a+1}$

With some thought, one can see from r_{14} that $(\overline{c_8}\overline{c_7})^{a+1} = c_8^a c_7^a$ $= (\overline{c_8})^{a+1}(\overline{c_7})^{a+1}$. If we apply similar logic to r_{11}, r_{13}, r_{16} and r_{18} we find respectively that $(\overline{c_7}b_4)^{a+1} = \overline{c_7}^{a+1}b_4^{a+1}$, $(b_4\overline{b_7})^{a+1} = b_4^{a+1}\overline{b_7}^{a+1}$, $(\overline{c_8}\overline{b_4}b_7)^{a+1} = \overline{c_8}(\overline{c_7}^{a+1})\overline{b_7}^a$ and $(c_7c_8\overline{b_7})^{a+1} = c_7\overline{c_8}^a b_4^a$. Rewriting r_{17} with these and simplifying leaves us with $b_7^{a+1} = c_8^{a+1}$, which after squaring implies $b_7 = c_8$.

Rewriting again with this r_{10} now says $b_4^a = c_8^a c_7^a$; but the relation we obtained from r_{16}, $(\overline{c_8}\overline{b_4}b_7)^{a+1} = \overline{c_8}(\overline{c_7}^{a+1})\overline{b_7}^a$, now implies that $\overline{c_8}\overline{b_4}^{a+1}c_8 = \overline{c_8}(\overline{c_7})^{a+1}\overline{c_8}^a$. Cancelling the leading $\overline{c_8}$s and post-multiplying both sides by $\overline{c_8}$ leads us to $\overline{b_4}^{a+1} = \overline{c_7}^{a+1}\overline{c_8}^{a+1}$. Since b_4, c_7 and c_8 all have order $2a+1$, this last equation gives us $b_4^a = c_7^a c_8^a$; thus we conclude that a-th powers of c_7 and c_8 commute.

Since $\gcd(a, 2a+1) = 1$ we will introduce new generators, $D_7 := c_7^a$ and $D_8 := c_8^a$ which are both of order $2a+1$. It is not difficult (but is tedious, and we omit the details) to then show that the only relators which are not then made superfluous in rewriting to D_7 and D_8 are D_7^{2a+1}, D_8^{2a+1} and $[D_7, D_8]$, so $H'' \cong C_{2a+1} \times C_{2a+1}$. Thus $|H| = |H : H'||H' : H''||H''| = 6.9.(2a+1)^2 = 6.[3(2a+1)]^2 = 6k^2$ as desired. $\square$

With the case $p = 3, q = 3$ done, we now turn to our other cases. In the case $p = 3$, $q = 4$, a spherical diagram can be created on 12 S_8 regions whose planar representation is that of the dual of the cuboctahedron (as in Section 2.2.1). For the other three, we can show quasi-asphericity with similar methods.

Theorem 2.11: *P_H is quasi-aspherical over $C_m * C_n$ when H is one of the following, with $k \geq 5$:*

i. $H(2, 2k, 4, 3|k)$,

ii. $H(2, 2k, 3, 5|k)$,

iii. $H(2, 2k, 5, 3|k)$.

Proof: In *i.* the curvature of an S_8 region is $\frac{1}{3}\pi$. Suppose that a minimal reduced spherical picture Π exists with a maximal number of S_8 regions. Since all positively curved regions are S_8 regions they contain β-vertices, it suffices to study the neighbourhoods of these. Then, let us examine any β-vertex which has at least one positively curved neighbouring region. This looks like Fig. 2.10, up to inversion (if all three adjacent vertices have the same sign, $d(\varepsilon_i) \geq 6$ for all i or if $k = 4$ could be P_1 regions, but $c(P_1) < c(4, 4, 4, 4) = 0$). Let us suppose for worst case scenario that ε_1 and ε_2 are S_8 regions; then each has two β-vertices, and we will allocate $\frac{\pi}{6}$ to each β-wheel. If we now examine ε_3, there are two possibilities for the remaining vertices of minimal degree; that either there are an additional $k - 2\alpha$-vertices giving ε_3 degree k, or there are two $\bar{\alpha}$-vertices and three β-vertices giving ε_3 degree 8.

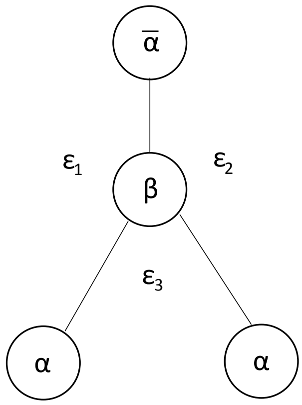


Fig. 2.10

In the first case,

$$c(\varepsilon_3) = (1 - k + \frac{2}{3} + \frac{k}{2})\pi \leq (-4 + \frac{2}{3} + \frac{5}{2})\pi = \frac{-5}{6}\pi.$$

Since ε_3 has just one β-vertex, we may allocate all of this to our β-neighbourhood, and the total curvature of the neighbourhood is $\frac{-\pi}{2}$. Thus the positive curvature is accounted for. If we assume the second case, then $c(\varepsilon_3) = (-6 + \frac{8}{3} + \frac{8}{4})\pi = \frac{-8}{6}\pi$. Splitting this between the four β-neighbourhoods we send $\frac{2}{6}\pi$ to each, in which case our original β-neighbourhood has zero curvature. Thus there are no possible positively curved β-neighbourhoods and there are no positively curved interior S_8 regions which are unaccounted for. Now examine the distinguished region Δ_D. If Δ_D has A α-vertices (including $\bar{\alpha}$-vertices) and

B β-vertices on its boundary, the worst case is that $d(\Delta_D)$ is even alternating between α- and β-vertices (i.e. $A = B$), so that each edge can potentially send across $\frac{1}{6}\pi$. Then

$$\begin{aligned} c^*(\Delta_D) &\leq \left(2 - A - B + \frac{2A}{4} + \frac{2B}{3} + \frac{A+B}{6}\right)\pi \\ &= \frac{12 - 2A - B}{6}\pi \\ &= \left(2 - \frac{A}{2}\right)\pi < 4\pi \end{aligned}$$

and therefore the total curvature over all regions of Π is less than 4π. In *ii.*, consider similarly to *i.* the β-wheel with five adjacent vertices and five incident regions which we call ε_i where $1 \leq i \leq 5$. This then looks something like Fig. 2.11. Since in this case $c(S_8) = \frac{2}{15}\pi$, we split positive curvature from each such region two ways into each β-wheel that region is part of.

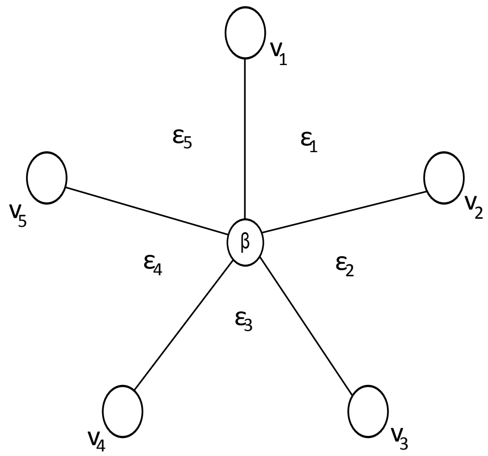


Fig. 2.11

From this picture, we see that for example the region ε_1 can be S_8-type if v_1 and v_2 are α-vertices of opposite sign. Therefore, by consistent alternation of the signs of the v_i, we can at most have four S_8 regions in a β-wheel, contributing $\frac{4}{15}\pi$ total. Let us assume without loss of generality that ε_1-ε_4 are S_8 regions and that ε_5 is not. Then $d(\varepsilon_5) > 4$. Let us also assume that both v_1, v_5 are α-vertices (the argument works similarly for both being $\bar{\alpha}$). Then as in *i.* to complete ε_5 with as small a degree as possible we require the remaining vertices to be (up to permutation where appropriate) either $k - 2\alpha$-vertices giving ε_5 degree $k+1$, or two $\bar{\alpha}$-vertices and three β-vertices giving ε_5 degree 8. In the former case ε_5 is part of only one β-wheel, and thus

$$\begin{aligned} c^*(\varepsilon_5) &\leq \left(-k + 1 + \frac{2k}{3} + \frac{2}{5} + \frac{4}{15}\right)\pi \\ &= \left(\frac{-5k+25}{15}\right)\pi \leq 0 \quad (k \geq 5) \end{aligned}$$

therefore positive interior curvature can be dealt with. If we now consider Δ_D; consider that for the β-wheel argument $\frac{4}{15}\pi$ is dispersed across a substring of $\alpha\beta\alpha$ for each β-wheel, let us suppose then for worst case scenario that $\frac{2}{15}\pi$ is distributed into Δ_D across each $\alpha\beta$ edge. If we assume Δ_D alternates α- and β-vertices, we can maximise this number of edges as $d(\Delta_D)$. Then, using the same language of A denoting the number of α-vertices and B the number of β-vertices as in $i.$,

$$\begin{aligned} c^*(\Delta_D) &\leq \left(2 - A - B + \frac{2A}{3} + \frac{2B}{5} + \frac{(A+B)}{15}\right)\pi \\ &= \frac{30 - 4A - 4B}{15}\pi \\ &= \left(2 - \frac{4(A+B)}{15}\right)\pi < 4\pi \end{aligned}$$

and thus again we conclude no such spherical picture exists. Lastly, $iii.$ follows as a consequence of $i.$ and the fact that $c(3,3,5,5) < c(3,3,4,4)$. □

2.4 Collapse and consequences

We now discuss the concept of collapse of the parameters m, n, p and q. Suppose that the order of one of the words x, y, (xy) or (xy^k) is not exactly m, n, p or q; say instead of p we have $|(xy)| = p'$ for $p' < p$. Then there should exist a picture Π_{red} over P_H where the boundary label is $(xy)^{p'}$. We attempt to determine things about this picture. For instance, in the case where either m or n collapses, Π_{red} should be spherical. To see this, suppose that m collapses but Π_{red} is not spherical; then segments of ∂D^2 are labelled with x^a for some $a \in \mathbb{Z}$, with edges adjoining to intersperse these as in Fig. 2.12; in this setting both γ_1 and γ_2 are x-regions. But since the labels of the regions γ_i should be entirely x-powers or y-powers, this is impossible by the labelling of all possible vertices, so this situation cannot arise. Similarly, if n collapses, we would need two adjacent y-regions prior to alteration of Π_{red}, so in these cases Π_{red} be spherical. Since in our proofs for quasi-asphericity we prove no spherical pictures can exist in general, this proves that if we have quasi-aspherical presentations neither m nor n can collapse.

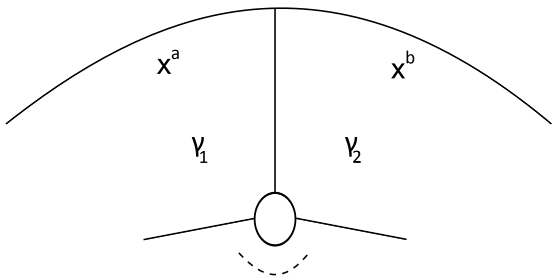


Fig. 2.12

Instead, now consider when p and q collapse. If in this case Π_{red} is spherical, then we would require adjacent vertices (say v_1 and v_2) of the distinguished region to contribute x and y terms to the labelling; but then the interior region formed between v_1 and v_2 would contain both x and y labels, an impossibility. Thus in these cases we will consider Π_{red} to be reduced but not spherical.

This leads us to our key theorem in this section.

Theorem 2.12: *Suppose that H is one of the following groups:*

i. $H(2, 2k+1, 3, 11|k)$, $k \geq 3$

ii. $H(2, 2k, 4, 3|k)$, $k \geq 5$

iii. $H(2, 2k, 3, 5|k)$, $k \geq 5$

iv. $H(2, 2k, 5, 3|k)$. $k \geq 5$

Then H does not collapse.

Proof: Suppose first that H is $H(2,2k+1,3,11|k)$. As ascertained in Theorem 2.5, no reduced spherical picture can exist, so neither m nor n certainly do not collapse. Suppose then Π_{red} is a reduced picture with boundary label either $(xy)^{p'}$ or $(xy^k)^{q'}$ where $1 \le p' < p$ and $1 \le q' < q$. Form a tessellation over the 2-sphere by contracting ∂D_2 to a point, creating a vertex whose degree we shall call d_0 and therefore d_0 boundary regions which we shall call $\hat{\Delta}$ and examine now. In Theorem 2.5, it was shown that if wheels are interior and their negatively curved regions are interior, then their curvature can be compensated for. In the worst case scenario, the picture has boundary label $(xy^k)^{q'}$ and an α-wheel (which we call $\hat{w}$) may contain two $\hat{\Delta}$-regions. Thus we find $c^*(\hat{w}) \le 2.c(3,11,d_0) + c(3,11,11)$, i.e.

$$\begin{aligned} c^*(\hat{w}) &\le \frac{1}{33}\pi + 2.\left(\frac{2}{d_0} - \frac{5}{33}\right)\pi \\ &= \frac{4\pi}{d_0} - \frac{9\pi}{33} < \frac{4\pi}{d_0}. \end{aligned}$$

Thus summing over all boundary regions/wheels we find $\sum c(\hat{\Delta}) < 4\pi$ and the tessellation cannot exist. Let us now look at the groups of Theorem 2.11, beginning with $H(2,2k,4,3|k)$ and with p', q' as before. Again, from Theorem 2.11 we can assume all positive curvature is attained in $\hat{\Delta}$-regions. Now looking at a single $\hat{\Delta}$, we conclude that $c(\hat{\Delta}) \le c(3,3,4,d_0) = \frac{2\pi}{d_0} - \frac{\pi}{6} < \frac{4\pi}{d_0}$ and thus again $c(\Pi_{red}) < 4\pi$. Finally, in the two cases $H(2,2k,p,q|k)$ where $(p,q) = (3,5)$ or $(5,3)$ we proceed exactly as for $(p,q) = (4,3)$, observing that $c(\hat{\Delta}) \le c(3,3,5,d_0) = \frac{2\pi}{d_0} - \frac{4\pi}{15} < \frac{4\pi}{d_0}$, thus sufficing. □

Before stating the main result, we first add some background as in [10] with some further explanation in [13]. Suppose we have a pushout of groups as in Fig. 2.13. Then, if K_A, K_B and K_C are all Eilenberg-Maclane spaces of types $K(A,1)$, $K(B,1)$ and $K(C,1)$ respectively with $\hat{i} : K_A \to K_B$ and $\hat{j} : K_A \to K_C$ being continuous maps which recreate i and j at the fundamental group level. We wish to create a space X such that $\pi_1(X) = H$; this can be done by putting

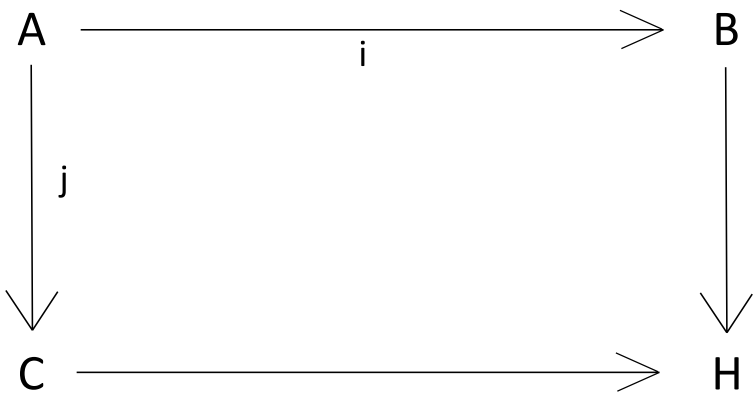


Fig. 2.13

$X = M(\hat{i}) \cup_{K_A} M(\hat{j})$ where M is the normal mapping cylinder. Further, if X is aspherical (and a $K(H, 1)$ space) then the pushout is called geometrically Mayer-Vietoris, from which it can be deduced (as in [18]) that if A, B and C are all of type $FP_{\mathbb{Q}}$ then so is H, and most importantly we find that if $\chi_{\mathbb{Q}}$ is the rational Euler characteristic, then in fact

$$\chi_{\mathbb{Q}}(H) = \chi_{\mathbb{Q}}(B) + \chi_{\mathbb{Q}}(C) - \chi_{\mathbb{Q}}(A). \tag{2.4.1}$$

As a result, we can argue (for example borrowing from [7]) that if our groups do not collapse (as in Section 1.1), then we can create a pushout that is geometrically Mayer-Vietoris and as a result apply the Euler characteristic result. We will do this in the following lemma.

Lemma 2.13: *Suppose H has standard presentation P_H, which is quasi-aspherical, and that H does not collapse. Suppose further that H is finite of order N. Then*

$$\frac{1}{N} = \frac{1}{m} + \frac{1}{n} + \frac{1}{p} + \frac{1}{q} - 1.$$

Proof: We construct a pushout to satisfy Fig. 2.13. Referring to that diagram, let A be the free group of rank two with presentation $\langle s, t | \rangle$, B be $C_p \times C_q$ with presentation $\langle s, t | s^p, t^q \rangle$ and C be $C_m \times C_n$ with presentation $\langle x, y | x^m, y^n \rangle$. Then, set $i(s) = s, i(t) = t$ and $j(s) = xy, j(t) = xy^k$; now under amalgamation we find $s = xy$ and $t = xy^k$, so H in the diagram truly is isomorphic to

$H(m,n,p,q|k)$. Since H does not collapse, as above in [7] we have that our pushout is indeed geometrically Mayer-Vietoris. Then, if H is finite of order N $\chi_{\mathbb{Q}}(H) = \frac{1}{N}$ and thus if we apply (2.4.1) we find $\frac{1}{N} = (\frac{1}{p}+\frac{1}{q}-1)+(\frac{1}{m}+\frac{1}{n}-1)-(-1)$, i.e. $\frac{1}{N} = \frac{1}{m}+\frac{1}{n}+\frac{1}{p}+\frac{1}{q}-1$. □

With this requirement in mind, we can now state our major resulting theorem.

Theorem 2.14: *Suppose H belongs to one of the following families of groups:*

i. $H(2,2k\pm1,3,11|k)$ *for* $k \geq 3$

ii. $H(2,2k,4,3|k)$ *with* $k \geq 5$

iii. $H(2,2k,3,5|k)$ *with* $k \geq 5$, *or*

iv. $H(2,2k,5,3|k)$ *with* $k \geq 5$.

Then H is infinite.

Proof: We go case by case. Beginning with case $i.$, begin by assuming $n = 2k+1$. Then from Lemma 2.13 we find $\frac{1}{N}+1 = \frac{61}{66}+\frac{1}{2k+1}$ and hence $N = \frac{66.(2k+1)}{66-10k-5}$, thus for $k \geq 7$ we have $N < 0$ and hence H is infinite in all these cases. We now check $3 \leq k \leq 6$; if $k = 3$ or $k = 4$, then N is respectively either $\frac{66.7}{31}$ or $\frac{66.9}{21}$, neither of which are integers thus H is infinite. If $k = 5$, then $N = 66$; but we can find through GAP that $H(2,13,3,11|5)$ contains the subgroup K of index 11 where $K'_{ab} \cong \mathbb{Z}^6 \times C_7$, and if $k = 6$ $N = 858$ but we can find a mapping ϕ from $H(2,13,3,11|6)$ to $PSL(2,131)$ where $\ker(\phi)_{ab} \cong \mathbb{Z} \times C_2 \times C_3 \times C_5 \times C_{13}$. Thus, all such H are infinite. In the case $n = 2k-1$, we find $N = \frac{66(2k-1)}{66-10k-5}$ so $N < 0$ for $k \geq 8$. Of course though since $m = 2$ we have from Lemma 2.1 $H(2,2k-1,3,11|k) \cong H(2,2k-1,3,11|k-1)$ and so we can verify for $4 \leq k \leq 7$ that H is infinite from the $2k+1$ case. Thus lastly we check $k = 3$, for which we find $N = \frac{66.5}{41} \notin \mathbb{Z}$, and so all $H(2,2k-1,3,11|k)$ are also infinite for $k \geq 3$.

For $ii.$, this yields that H is finite of order N if $\frac{1}{N} = \frac{1}{12}+\frac{1}{2k}$ and therefore satisfies $N = \frac{12k}{k+6}$. Note then that $\lim_{k\to\infty}(N) = 12$ since $\frac{12k}{k+6} = \frac{12}{1+\frac{6}{k}}$, hence $N < 12$;

consequently since for $k \geq 6$ we have $n \geq 12$ and there is no collapse, clearly $|H| \geq 12$, contradicting $|H| = N$. Then the only remaining case is $k = 5$ for which $N = \frac{60}{11}$ is clearly not an integer, so we are done. Similarly, cases *iii.* and *iv.* lead to $N = \frac{30k}{k+15} = \frac{30}{1+\frac{15}{k}}$ and so for all k we find $|H| < 30$. Thus, since when $k \geq 15$ we have $n \geq 30$ we are done in these cases. For $5 \leq k \leq 14$, one can readily test each value of k to find that $N \notin \mathbb{N}$ unless $k = 10$, in which case N should be 12; but then y has order 20, thus $|H| > 12$ and hence the result. □

This also allows us to classify the more general result started in Theorem 2.3, which we do now with the groundwork laid out.

Theorem 2.15: *Let H be the group $H(2, n, p, q|k)$. If H satisfies any of the following sets of conditions:*

i. $p, q \geq 6$

ii. $p, q \geq 4$ *and* $n \neq 3k, n \neq 2k + 1$

iii. $p = 3, q \geq 12$ *and* $k > 3$

iv. $p = 3,\ 6 \leq q \leq 11,\ n \neq 2k + 1,\ n \neq k + 3$ *and* $k > 3$

v. $q = 3,\ p \geq 6$ *and* $n \neq z$ *where* $z \in \{2k + 1, 3k, 3k \pm 1\, 4k, 5k\}$,

then H is infinite, with the possible exception of $H(2, 39, 7, 3|12)$.

Proof: If H is one of the groups of Theorem 2.3, we know P_H is quasi-aspherical. Equally, collapse cannot occur since this requires a reduced picture Π_{red} with boundary label $(xy)^{p'}$ or $(xy^k)^{q'}$ and distinguished vertex of degree d_0 to exist with curvature 4π as in Theorem 2.12, but the maximal curvature possible of a boundary region $\hat{\Delta}$ of such a picture necessarily satisfies $c^(\hat{\Delta}) \leq c(3, 6, d_0) = \frac{2\pi}{d_0} < \frac{4\pi}{d_0}$ so H in such a case does not collapse. Then, assuming that H is finite of order N and applying (2.4.1), we check the cases *i.* through *v.* In *i.* we find $\frac{1}{2} + \frac{1}{n} + \frac{2}{6} \leq 1$, so $\frac{1}{N} \leq 0$. In *ii.*, if $p = q = 4$ then $N = n$; thus H is the cyclic group of order n with elements $\{1, y, y^2, \ldots, y^{n-1}\}$. Thus, $x = y^z$ for some $0 \leq z \leq n - 1$. Then $y^{2z} = y^n = y^{4(z+1)} = y^{4(z+k)} = 1$, and in particular the first and third of these imply that $y^{4z+4} = y^4 = 1$, a contradiction since $n \geq 6$. In all other instances of *ii*., $\frac{1}{2} + \frac{1}{p} + \frac{1}{q} < 1$, thus $\frac{1}{N} = \frac{1}{n} - \frac{a}{b}$ for some $a \in \mathbb{N}$ and $b \in \mathbb{Z}$, and so $\frac{1}{N} < \frac{1}{n}$ hence $N < n$. We therefore have a finite number of cases to check; for instance, if $p = 4$, then $5 \leq q \leq 11$ otherwise if $q \geq 12$ and N is positive then it forces $n \leq 5$, which contradicts H not collapsing. Subsequently, for each valid triple (n, p, q) there will be then be valid choices for k where $3 \leq k \leq \frac{n}{2}$. Where the triple (n, p, q) yields either that N is not an integer or one of n, p or q does not divide N, we immediately discount these as infinite. In the cases where neither of the above happen, we check through computing; these groups are listed in Appendix B.

Similar working for cases *iii*., *iv*. and *v*. necessarily leads to the conclusion each time that $N > n$ (which again are dealt with in Appendix B, barring $H(2, 39, 7, 3|12)$), aside only for when $(p, q), (q, p) = (3, 6)$ in cases iv. and v. respectively, whence again we have $N = n$ and proceed again saying that H is C_n; but in case *iv*. we have $p = 3$ which implies $y^{3z+3} = y^{z+3} = 1$, but that implies $y^{2z+6} = 1$ and subsequently $y^{2z} = 1$ forces $y^6 = 1$. But $k > 3$, thus $n > 6$ and this contradicts H not collapsing. In the same way in case *v*. with $p = 6$, $q = 3$ we find $y^{6z+6} = 1 \Rightarrow y^6 = 1$, and thus we require $k = 3$; but a check with KBMAG finds $H(2, 6, 6, 3|3)$ to be infinite. Also in case v., if $n = 6$ and hence $k = 3$ then we may find $N = p$ and so $H \cong C_p$; however, if $6 \nmid p$ then either $2 \nmid p$ or $3 \nmid p$ and therefore $|H| = p$ violates Lagrange's Theorem, whereas if $6 \mid p$ then we can add the relator $(xy)^6 = 1$ and map from $H(2, 6, p, 3|3)$ onto $H(2, 6, 6, 3|3)$ which is infinite as noted above. Thus, all candidate groups H must be infinite. □

2.5 A note on $H(2, n, p, p|k)$ for $3 \leq p \leq 5$

We now make some observations about the interesting case $H(2, n, 3, 3|k)$. The subcase where $n = 2k$ has already been discussed and proven finite of given order in Theorems 2.9 and 2.10, but through testing in GAP we have reasonable enough faith to state the following conjecture regarding these groups, which we can prove one implication of readily.

Conjecture 2.16: *The groups $H(2,n,3,3|k)$ are infinite if and only if $\exists r \geq 6$ such that $r \mid n$ and $k \equiv \pm 1 \mod z$.*

Proof of converse: Suppose there is such a r that r divides n and $k \equiv 1 \mod r$ (we return to $k \equiv -1$ in due course). Then we may map from H onto the group K with presentation $\langle x, y | x^2, y^r, (xy)^3, (xy^k)^3 \rangle$ by adding the relator $y^r = 1$; but since $k \equiv 1 \mod z$ K can have the equivalent presentation $\langle x, y | x^2, y^r, (xy)^3, (xy)^3 \rangle$ and consequently $K \cong D(2,r,3)$. Since $r \geq 6$ K is infinite due to the von Dyck inequality and thus so is H. In the case that $k \equiv -1 \mod r$, note that as in Lemma 2.1 $(xy^k)^3 \equiv (xy^{-1})^3$, but then $(xy^{-1})^{-1}$ has order 3 implies (xy) has order 3 and thus we are back to the case $k \equiv 1$. □

In support of this claim we have computation of the orders of the groups $H(2,n,3,3|k)$ for $3 \leq n \leq 25$ and $3 \leq k \leq 13$; a table of these is found in Appendix A.

Whilst we do not have a similar conjecture regarding an absolute criterion for infiniteness for $p = q = 4$ and $p = q = 5$, we can certainly prove a similar result to the converse of Conjecture 2.16 in these cases.

Theorem 2.17: *Let H be the group $H(2,n,4,4|k)$. Then if $\exists r \geq 4$ such that $r \mid n$ and $k \equiv \pm 1 \mod r$, H is infinite. Similarly if H is the group $H(2,n,5,5|k)$ and $\exists r \geq 4$ such that $r \mid n$ and $k \equiv \pm 1 \mod r$, then H is infinite.*

Proof: If $H = H(2,n,4,4|k)$ and such an r exists, then as in the proof above we can kill y^r to map onto $D(2,r,4)$ which is infinite precisely when $r \geq 4$. Likewise, for $H(2,n,5,5|k)$ the mapping is onto $D(2,r,5)$ which is again infinite for $r \geq 4$. □

2.6 The case $q = 2$

The final part of this chapter is devoted to the case where $q = 2$. Recall that we assume for this section that $p \geq 3$. By Theorem 1.2, we can readily see that when $m = q = 2$ H is infinite if $\frac{1}{p} + \frac{1}{\gcd(n,k)} \leq \frac{1}{2}$; therefore if $\gcd(n,k) \geq 6$ H is infinite for all $p \geq 3$, if $\gcd(n,k) = 4, 5$ H is infinite for all $p \geq 4$ and if $\gcd(n,k) = 3$ H is infinite for all $p \geq 6$. However, if $\gcd(n,k) = 1$, then we have a definite conclusion from Lemma 1.1.

Corollary 2.18: *If* $\gcd(n,k) = 1$, *then the group* $H(2,n,p,2|k)$ *is of finite order.*

Proof: By Lemma 1.1, if $\gcd(n,k) = 1$ then $H(2,n,p,2|k) \cong H(2,n,2,p|f)$ where $f \in \mathbb{N}$ satisfies $k.f \equiv 1 \mod n$. Thus H is a quotient of $D(2,2,n)$ which is known to be finite since $\frac{1}{2} + \frac{1}{2} + \frac{1}{n} > 1$. $\square$

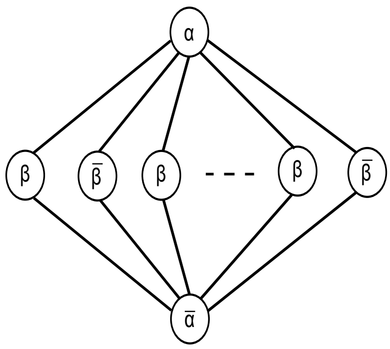


Fig. 2.14

Thus, the problematic cases are when $2 \leq \gcd(n,k) \leq 5$, in particular when $\gcd(n,k) = 2$. Part of the problem arises from the fact that in general when $q = 2$ S_5 regions have curvature $\frac{4\pi}{p}$ and so a spherical picture can be readily created on p S_5 regions, where p is even, by setting a line of alternating β- and $\bar{\beta}$-vertices and connecting each to "polar" α- and $\bar{\alpha}$-vertices, as for example in Fig. 2.14. In the case where $n = 2k$, as discussed before S_5 regions are identical to S_8 regions and in fact we replace the strip of alternating $\beta/\bar{\beta}$-vertices with a line of p β-vertices; in this case we achieve a spherical diagram for all p.

Chapter 3

$m = 2$: The Platonic families

As mentioned when introduced in Section 1.4, the Platonic families cannot be dealt with by curvature arguments since spherical pictures exist over their respective presentations. Therefore, we shall attempt to solve the problem for families of each case where possible, and offer an introductory analysis for relatively small values of the parameters in each case otherwise; roughly speaking, by virtue of Thomas' result in [21], we should expect finite groups to appear for small values of the parameters unless there is either total or substantial partial collapse. Thus we attempt to compute the point where finite passes to infinite, which while not a formal proof gives a reasonable indication.

3.1 Case i), the tetrahedra

As discussed in Section 1.4, we mean by the tetrahedra the sub-family of our groups $H(m, n, p, q|k)$ to be the groups $H(2, 3k, p, 3|k)$ for $p, k \geq 3$. Through using KBMAG, the only finite groups to be found for $3 \leq k \leq 6$ and $3 \leq p \leq 8$ are those where $p = 3$ (to be discussed below) and $H(2, 9, 4, 3|3)$ which KBMAG finds to be of order 2448.

When $p = 3$, we find that in fact H is always finite according to the following result.

Theorem 3.1: *The group* $H := H(2,3k,3,3|k)$ *is finite for all* $k \geq 3$*, and the order satisfies*

$$|H| = \begin{cases} 12 & \text{if } 3 \nmid k, \\ 3 & \text{if } 3 \mid k. \end{cases}$$

Proof: We use covering space arguments as in Theorem 2.9. If we look at the abelianisation H_{ab}, then its presentation devolves to $\langle x, y \mid x^2, y^{3k}, x^3y^3, x^3y^{3k} \rangle$ and consequently to $\langle y \mid y^3 \rangle$. Since the third relator of the former implies $x = y^3$, we again obtain the diagram in Fig. 2.4 for H_{ab}. The relators lift as follows:

- x^2 lifts to x_i^2 for $i \in \{1,2,3\}$
- y^{3k} lifts to $(y_1y_2y_3)^k$ up to cyclic permutation
- $(xy)^3$ lifts to $x_1y_1x_2y_2x_3y_3$ up to cyclic permutation
- $(xy^k)^3$ lifts to $x_1(y_1 \ldots y_\gamma)x_{\gamma+1}(y_{\gamma+1} \ldots y_\delta)x_{\delta+1}(y_{\delta+1} \ldots y_3)$ up to cyclic permutation, where $\gamma, \delta \in \{1,2,3\}$ with $\gamma \equiv k \mod 3$ and $\delta \equiv \gamma + k \mod 3$.

We choose to contract the maximal tree y_1, y_2. As a consequence, $y_3 = \overline{(x_1x_2x_3)} = x_3x_2x_1$. Thus now the relators of H' are x_i^2, $x_1x_2x_3$ and one of the following sets:

- x_i^3 if $k \equiv 0 \mod 3$
- $x_1x_2x_3$ if $k \equiv 1 \mod 3$
- $x_1x_3x_2$ if $k \equiv 2 \mod 3$

In the latter two cases, the existing relator $x_1x_2x_3 = 1$ implies $x_1 = \overline{x_3x_2} = x_3x_2$ and also $\overline{x_1} = x_1 = x_2x_3$, thus $P_{H'}$ becomes

$$\langle x_2, x_3 \mid x_2^2, x_3^2, [x_2, x_3] \rangle.$$

Clearly then H' is $C_2 \times C_2$ and $|H| = |H_{ab}||H'| = 12$ as claimed. In the first case, for each generator x_i we have $x_i^2 = x_i^3 = 1$ which implies $x_i = 1$. Thus H' is the trivial group, $H \cong H_{ab}$ and $|H| = 3$. □

In the investigation with GAP, for the cases $3 \leq k \leq 6$ and $4 \leq p \leq 8$ all groups aside from the known finite group $H(2, 9, 4, 3|3)$ and the unknown groups $H(2, 9, 5, 3|3)$, $H(2, 12, 4, 3|4)$, and $H(2, 15, 4, 3|5)$ were found to be infinite by either using KBMAG, by searching for small index subgroups of H either who directly have infinite abelianisation or whose derived series contains some group with infinite abelianisation, or by using a written program in GAP to find an epimorphism from H to $PSL(2, z)$ for some z. We can then study subgroups of H which arise by considering the preimages of stabilisers, which have index $z \pm 1$ in H. The search program and code used for this method was originally devised and used by Paul [19], to whom credit should go for writing it; this appears in Appendix C. The breakdown in particular is as follows:

- The groups $H(2, 3k, 8, 3|k)$ for $3 \leq k \leq 6$, $H(2, 3k, 7, 3|k)$ for $3 \leq k \leq 6$ and $H(2, 9, 6, 3|3)$, $H(2, 12, 6, 3|4)$ and $H(2, 15, 6, 3|5)$ were all found to be infinite by KBMAG.
- The group $H(2, 18, 6, 3|6)$ has a subgroup K of index 15 where $K'_{ab} \cong \mathbb{Z}^4 \times C_2^{52} \times C_4^4$, $H(2, 18, 5, 3|6)$ has a subgroup K of index 20 where $K_{ab} \cong \mathbb{Z} \times C_2$ and $H(2, 18, 4, 3|6)$ has a subgroup K of index 18 where $K'_{ab} \cong \mathbb{Z}^5 \times C_{17}$.
- The group $H(2, 15, 5, 3|5)$ admits an epimorphism to $PSL(2, 59)$ and $H(2, 12, 5, 3|4)$ admits an epimorphism to $PSL(2, 71)$, both of which yield subgroups with infinite abelianisation.

3.2 Cases ii) and iii), the cubes and octahedra

For the cubes, which comprise the subfamily $H(2, 4k, p, 3|k)$, we perform a similar trick to Theorem 1.2. If we take the quotient of H by adding the relator $y^{2k} = 1$, we arrive at the group $H(2, 2k, p, 3|k)$ as none of the relators barring

$y^{4k} = 1$ are altered. Since for all $p \geq 6$ $H(2, 2k, p, 3|k)$ is infinite, it follows that $H(2, 4k, p, 3|k)$ is also infinite for $p \geq 6$.

In the case of $H(2, 4k, 3, 3|k)$, experimentation in GAP (for $3 \leq k \leq 21$) suggests the following conjecture.

Conjecture 3.2: *The groups $H(2, 4k, 3, 3|k)$ are finite for all $k \geq 3$, and satisfy*

$$|H| = \begin{cases} 24k^2 & \text{if } k \text{ odd,} \\ 1 & \text{if } 6 \nmid k \text{ and } k \text{ is even,} \\ 3 & \text{if } 6 \mid k. \end{cases}$$

Conjecture 3.2 is equivalent to the statement that if we create the standard epimorphism ι_c: $H(2, 4k, 3, 3|k) \to H(2, 2k, 3, 3|k)$ from this quotient (i.e. defined by $\iota_c(y^{2k}) = 1$) $|\ker \iota_c|$ is 1 if k is even and 4 if k odd. It is straightforward enough to see that $\ker \iota_c$ contains y^{2k} and $xy^{2k}x$ and that if we assume $|y^{2k}| \neq 1$ then $|y^{2k}| = |xy^{2k}x| = 2$, so it would appear that $\ker \iota_c \cong C_2 \times C_2$ in this instance; however no proof that for $H(2, 4k, 3, 3|k)$ with k even $y^{4k} = 1 \Rightarrow y^{2k} = 1$ could be found to validate this assertion.

In the case of the octahedra, the subfamily $H(2, 3k, p, 4|k)$, we again look at quotients; this time we can create the quotient of H by adding the relator $y^k = 1$; then $y^{3k} = 1$ is redundant, and $(xy^k)^4 = 1$ implies $x^4 = 1$ which is redundant with $x^2 = 1$, and thus this quotient of H is $D(2, k, p)$. Since $p, k \geq 3$ H is infinite if either p or k are greater than 5 or both are greater than 3, and one can then check readily that the only possible pairs (p, k) which can possibly lead to finite groups are the pairs $(3, 3)$, $(3, 4)$, $(4, 3)$, $(5, 3)$ and $(3, 5)$. Of these, we have no result for the group $H(2, 15, 3, 4|5)$; of the others, $H(2, 9, 3, 4|3)$ is the alternating group A_4 of order 12, $H(2, 12, 3, 4|4)$ is finite of order 15840, $H(2, 9, 5, 4|3)$ is infinite as resolved by KBMAG and $H(2, 9, 4, 4|3)$ is infinite since $H_{ab}^{(3)}$ is isomorphic to $\mathbb{Z}^{108}$.

3.3 Case iv), the icosahedra

As with the tetrahedra, we present findings for an introductory survey of the family of so-called icosahedra $H(2, 3k, p, 5|k)$, in this case for $3 \leq p \leq 10$ and $3 \leq k \leq 7$; unlike the tetrahedral case however, there does not appear to be a subfamily of finite groups. Indeed, only one finite group was found; this was $H(2, 9, 3, 5|3)$ which was found to have order 10260 using GAP, which whilst not perfect has a perfect derived subgroup of order 3420. Of the remainder in our range, only $H(2, 12, 3, 5|4)$, $H(2, 15, 3, 5|5)$ and $H(2, 18, 3, 5|6)$ could not be verified as either finite or infinite.

Following on from before, we subdivide these cases into those solved by KB-MAG, those with useful derived series or low index subgroups with useful derived series, those which admit mappings to $PSL(2, z)$ which lead to subgroups of H with infinite abelianisation as in Section 3.1, and those which utilise the Newman Infinity Criterion. This is covered in Chapter 7 of [19], but we will recap the salient part here. For a general group G, let us define G_1 to be the subgroup of G generated by all commutators and p^{th} powers for some particular prime p. Thus, $G_1 = [G, G]G^p$. Let us create similarly $G_2 = [G_1, G]G^p$, and set $d_p(G)$ and $e_p(G)$ to be the ranks of the subgroups G/G_1 and G/G_2 respectively. Under these circumstances Newman proved in [20] the following theorem:

Theorem 3.3 (Newman 1990): *Let G be a group with a finite presentation on b generators and r relations. For some prime p, let $d = d_p(G)$ and $e = e_p(G)$. If either*

$$r - b + d < \frac{d^2}{2} - \frac{d}{2} - e$$

or

$$r - b + d \leq \frac{d^2}{2} - \frac{d}{2} - e + \frac{d}{2}\left(e + \frac{d}{2} - \frac{d^2}{4}\right)$$

hold, then G has arbitrarily large quotients of p-power order.

As a result of Newman's theorem, we can infer that if either of these criteria hold then G is infinite. This criterion has been built into GAP, and we utilise it whenever the structure of a group's abelianisation contains a large number of copies of cyclic groups whose orders are a particular prime or powers thereof.

Beginning now with classification, the following were all found to be infinite via KBMAG:

- **k = 3:** $H(2,9,p,5|3)$ for all $4 \leq p \leq 10$
- **k = 4:** $H(2,12,p,5|4)$ for all $4 \leq p \leq 10$
- **k = 5:** $H(2,15,p,5|5)$ for all $5 \leq p \leq 10$

Now we describe the groups found to be infinite due to H containing some infinite subgroup:

- $H(2,18,4,5|6)$ contains a subgroup K of index 16 where $K'_{ab} \cong \mathbb{Z}^4 \times C_3$,
- $H(2,18,7,5|6)$ contains a subgroup K of index 10 where $K'_{ab} \cong \mathbb{Z}^{36} \times C_2^{168} \times C_3^{30} \times C_9^{42}$,
- $H(2,18,8,5|6)$ contains a subgroup K of index 6 where $K_{ab} \cong \mathbb{Z}^2$
- $H(2,21,3,5|7)$ contains a subgroup K of index 42 where $K'_{ab} \cong \mathbb{Z} \times C_2 \times C_4 \times C_5 \times C_{41}$
- $H(2,21,4,5|7)$ contains a subgroup K of index 21 where $K'_{ab} \cong \mathbb{Z}^{10} \times C_2^{31} \times C_3^2 \times C_4^2 \times C_5^5$,
- $H(2,21,5,5|7)$ contains a subgroup K of index 5 where $K'_{ab} \cong \mathbb{Z}^{30} \times C_2^{86} \times C_{13}^2 \times C_{43}^2 \times C_{127}^2$,
- $H(2,21,8,5|7)$ contains a subgroup K of index 16 where $K_{ab} \cong \mathbb{Z}^2 \times C_7^2$.

Lastly, in Table 3.1 below we describe the groups which admit mappings to $PSL(2,z)$ for some z, the particular value of z in each case and the abelianisation of the subgroups arising from said maps (again from the code in Appendix C). In the cases $H(2,21,6,5|7)$ and $H(2,21,9,5|7)$, the Newman Infinity Criterion returned true for the primes $p = 2$ and $p = 3$ respectively.

Group	Valid z	Abelianisation of subgroup K
$H(2,15,4,5\|5)$	31	$\mathbb{Z} \times C_2 \times C_3 \times C_5$
$H(2,18,5,5\|6)$	109	$\mathbb{Z}^7 \times C_2^2 \times C_9$
$H(2,18,6,5\|6)$	179	$\mathbb{Z}^{19}$
$H(2,18,9,5\|6)$	109	$\mathbb{Z}^9 \times C_2^{14} \times C_3 \times C_9^2$
$H(2,18,10,5\|6)$	109	$\mathbb{Z}^7 \times C_2^{24} \times C_9$
$H(2,21,3,5\|7)$	41	$\mathbb{Z} \times C_2 \times C_4 \times C_5 \times C_{41}$
$H(2,21,6,5\|7)$	41	$C_2^{11} \times C_5 \times C_8$
$H(2,21,7,5\|7)$	169	$\mathbb{Z}^{20} \times C_2 \times C_3 \times C_7^3$
$H(2,21,9,5\|7)$	41	$C_2 \times C_3^{11} \times C_4 \times C_5$
$H(2,21,10,5\|7)$	41	$\mathbb{Z}^7 \times C_2^2 \times C_4 \times C_5$

Table 3.1: Infinite subgroups of H arising from mappings to $PSL(2,z)$

3.4 Case v), the dodecahedra

Lastly in this chapter, we discuss the dodecahedra $H(2,5k,p,3|k)$. Technically by the assumption of $n \geq 2k$ we could also consider $H(2,5k,p,3|2k)$ but it is clear that this is a subfamily of $H(2,5k,p,3|k)$ and so we merely consider the former. As before, we subdivide the cases into those unresolved and those resolved by KBMAG/MAF, elementary routines for finding infinite subgroups and infinite subgroups from mappings respectively. It is to be noted that there are more unresolved cases in this setting; one can attribute this to the increased order of y making the construction of suitable automata more difficult. The initial survey here was conducted for $3 \leq k \leq 9$ and $4 \leq p \leq 9$; discussion of $p = 3$ will be done separately.

To begin, the unresolved groups were those with (p,k) being one of the following: $(4,4)$, $(5,4)$, $(4,5)$, $(4,6)$, $(4,7)$, $(5,7)$, $(6,7)$, $(8,7)$, $(4,9)$ and $(5,9)$.

Next, the groups discovered to be infinite via KBMAG and MAF were the groups $H(2,15,p,3|3)$ for $6 \leq p \leq 9$, $H(2,20,7,3|4)$ and $H(2,25,p,3|5)$ for $7 \leq p \leq 9$.

Next, we list in Table 3.2 the following groups which contain a subgroup K from which we can find an infinite subgroup using GAP.

Group	$\|H:K\|$	Infinite subgroup found
$H(2,15,4,3\|3)$	32	$K''_{ab} \cong C_2^{75}$ (3.2.1)
$H(2,20,6,3\|4)$	9	$K'_{ab} \cong \mathbb{Z}$
$H(2,20,8,3\|4)$	10	$K_{ab} \cong \mathbb{Z} \times C_2^2$
$H(2,20,9,3\|4)$	14	$K^{(3)}_{ab} \cong \mathbb{Z}^{20}$
$H(2,30,5,3\|6)$	20	$K''_{ab} \cong \mathbb{Z}^4$
$H(2,30,9,3\|6)$	6	$K'_{ab} \cong \mathbb{Z}^6 \times C_2^4 \times C_3^6 \times C_5^3$
$H(2,40,4,3\|8)$	20	$K_{ab} \cong \mathbb{Z} \times C_9$
$H(2,40,6,3\|8)$	7	$K'_{ab} \cong \mathbb{Z}$
$H(2,45,6,3\|9)$	16	$K'_{ab} \cong \mathbb{Z}^2 \times C_3^2$

Table 3.2: Infinite subgroups of H from low index subgroups

In (3.2.1), the Newman Infinity Criterion returned true for $p = 2$. Next, we examine mappings to $PSL(2, z)$ with the results below in Table 3.3. Whilst usually the subgroup itself has infinite abelianisation, occasionally we may have to examine the abelianisation of the derived subgroup; these are also noted below. Again in (3.3.1)-(3.3.4) the Newman Infinity Criterion returns true for $p = 2$ in every case.

Finally, in the case $p = 3$ computation in GAP (performed for $3 \leq k \leq 30$) suggests that the groups $H(2, 5k, 3, 3|k)$ are all finite, with order according to the following conjecture:

Conjecture 3.4: *The groups $H(2, 5k, 3, 3|k)$ are all finite, with order according to the following:*

$$|H| = \begin{cases} 60 & \text{if } k \equiv \pm 1, \pm 4 \mod 15, \\ 180 & \text{if } k \equiv \pm 6 \mod 15, \\ 1 & \text{if } k \equiv \pm 2, \pm 5, \pm 7 \mod 15, \\ 3 & \text{if } k \equiv 0, \pm 3 \mod 15. \end{cases}$$

Group	Valid z	Infinite group found from subgroup K
$H(2,15,5,3\|3)$	29	$K'_{ab} \cong \mathbb{Z}^6 \times C_{29}$
$H(2,25,5,3\|5)$	49	$K'_{ab} \cong \mathbb{Z}^2 \times C_2^2 \times C_7^{10}$
$H(2,25,6,3\|5)$	49	$K \cong \mathbb{Z} \times C_2 \times C_3 \times C_8$
$H(2,30,7,3\|6)$	29	$K'_{ab} \cong \mathbb{Z}^6 \times C_2^{12} \times C_{29}$
$H(2,30,8,3\|6)$	89	$K'_{ab} \cong C_2^{17} \times C_4 \times C_{11}$ (3.3.1)
$H(2,35,7,3\|7)$	71	$\mathbb{Z} \times C_2 \times C_5 \times C_7^2$
$H(2,35,9,3\|7)$	71	$\mathbb{Z}^4 \times C_5 \times C_7$
$H(2,40,5,3\|8)$	41	$K'_{ab} \cong C_2^{18} \times C_5^2 \times C_{11}^2 \times C_{41}$ (3.3.2)
$H(2,40,7,3\|8)$	239	$K_{ab} \cong \mathbb{Z}^4 \times C_2 \times C_7 \times C_{16} \times C_{17}$
$H(2,40,8,3\|8)$	241	$K_{ab} \cong \mathbb{Z}^6 \times C_2 \times C_3 \times C_4 \times C_5 \times C_8^3$
$H(2,40,9,3\|8)$	19	$K'_{ab} \cong C_4^{14} \times C_{19}^3$ (3.3.3)
$H(2,45,7,3\|9)$	29	$K'_{ab} \cong \mathbb{Z}^6 \times C_3^{12} \times C_{29}$
$H(2,45,8,3\|9)$	89	$K_{ab} \cong C_2^{11} \times C_4 \times C_{11}$ (3.3.4)
$H(2,45,9,3\|9)$	171	$K_{ab} \cong \mathbb{Z}^{11}$

Table 3.3: Infinite subgroups of H arising from mappings to $PSL(2,z)$

As in the cube case, the first two lines of Conjecture 3.4 are equivalent to saying that if we take the quotient of H by adding the relator $y^5 = 1$, which renders the relator $y^{5k} = 1$ redundant and means that $(xy^k)^3 = 1$ is equivalent to either $(xy)^3 = 1$ or $(xy^{-1})^3 = 1$ (the inverse of which implies $(xy)^3 = 1$), we arrive at the quotient being the group $D(2,5,3)$; now define the associated epimorphism ι_d from $H(2,5k,3,3|k)$ to $D(2,5,3)$ by setting $\iota_d(y^5) = 1$, then $|\ker \iota_d|$ is of order 3 for $k \equiv \pm 6 \mod 15$ and trivial for $k \equiv \pm 1, \pm 4 \mod 15$. Again, it would suffice then for these parts to show that $k \equiv \pm 1, \pm 4 \mod 15$ implies $y^5 = 1$ in H, but an algebraic proof of this could not be found.

However, we have a partial result for the fourth case which follows.

Theorem 3.5: *The groups $H(2,5k,3,3|k)$ where $15 \mid k$ are isomorphic to C_3 and hence are finite of order 3.*

Proof: We proceed similarly to Theorem 2.10. Firstly, consider the abelianisation H_{ab}; the relator α combined with $x^2 = 1$ implies that $x = y^3$, and then our relators are y^6, y^{5k} and $y^{3k+9} = y^{3(k+3)}$; as $15 \mid k$ since k odd implies $5k$ odd and k even implies $3(k+3)$ is odd, we conclude that $\gcd(6, 5k, 3(k+3)) = 3$. Therefore H_{ab} has presentation $\langle y | y^3 \rangle$. We thus take a Schreier transversal U to be $U = \{1, y, y^2\}$ and compute the relator table below for H':

u	v	$\overline{uv}$	$uv\overline{uv}^{-1}$
1	x	1	$x =: b_1$
1	y	y	1
y	x	y	$yxy^{-1} =: b_2$
y	y	y^2	1
y^2	x	y^2	$y^2xy^{-2} =: b_3$
y^2	y	1	$y^3 =: b_4$

Again, we construct the relators of H' of the form uru^{-1} for $u \in U$ per relator r of H, which are below and rewritten in the new generators b_i. We will also substitute in $k = 15l$ to ease the rewriting a little.

u	ux^2u^{-1}	$uy^{75l}u^{-1}$	$u(xy)^3u^{-1}$	$u(xy^{15l})^3u^{-1}$
e	b_1^2	b_4^{25l}	$b_1b_2b_3b_4$	$(b_1b_4^{5l})^3$
y	b_2^2	b_4^{25l}	$b_2b_3b_4b_1$	$(b_2b_4^{5l})^3$
y^2	b_3^2	b_4^{25l}	$b_3b_4b_1b_2$	$(b_3b_4^{5l})^3$

From $b_3b_4b_1b_2 = 1$ we conclude that $b_3 = b_2^{-1}b_1^{-1}b_4^{-1} = b_2b_1b_4^{-1}$; rewriting the other relators everywhere with this equivalence and removing b_3 from the presentation we arrive at the presentation

$$\langle b_1, b_2, b_4 | b_1^2, b_2^2, b_4^{25l}, (b_2b_1b_4^{-1})^2, (b_1b_4^{5l})^3, (b_2b_4^{5l})^3, (b_2b_1b_4^{5l-1})^3 \rangle$$

for H'. From this we make the further substitution $c := b_4^{5l-1}$; $|c| = 25l$ too since $\gcd(25l, 5l-1) = 1$ because clearly neither 5 nor any divisor of l divide $5l-1$. Observe then that $b_4^{-1} = b_4^{25l^2-1} = b_4^{(5l-1)(5l+1)} = c^{5l+1}$ and $b_4^{5l} = b_4^{100l^2}b_4^{-20l} = b_4^{20l(5l-1)} = c^{20l}$, and we make these substitutions throughout. Consequently, the resultant relators $(b_1c^{20l})^3$ and $(b_2c^{20l})^3$ combined with $c^{25l} = 1$ imply (by taking inverses of these) that $(b_1c^{5l})^3$ and $(b_2c^{5l})^3 = 1$. Now the presentation is

$$\langle b_1, b_2, c | b_1^2, b_2^2, c^{25l}, (b_2b_1c^{5l+1})^2, (b_1c^{5l})^3, (b_2c^{5l})^3, (b_2b_1c)^3 \rangle$$

and we will label these relators as $R_1 := b_1^2$, $R_2 := b_2^2$ through to $R_7 := (b_2b_1c)^3$. Observe then that R_5 and R_6 imply that $b_ic^{-5l}b_i = c^{5l}b_ic^{5l}$ for b_1 and b_2 respectively; we will often use this, and refer to applying this as an R_5-*move*.

Firstly, equating R_4 and R_7 and simplifying we find

$$c^{5l}b_2b_1c^{5l} = b_2b_1cb_2b_1 \tag{3.5.1}$$

and we also rewrite R_4 now as

$$b_2b_1 = c^{-5l-1}b_1b_2c^{-5l-1}. \tag{3.5.2}$$

If we substitute this into R_7 and cyclically rewrite it, we find $(c^{-10l-1}b_1b_2)^3 = 1$. Compare this to the inverse of R_7 and we find $c^{-10l}b_1b_2c^{-10l-1}b_1b_2c^{-10l} = b_1b_2c^{-1}b_1b_2 = c^{-5l}b_1b_2c^{-5l}$ with the last step coming from inverting (3.5.1). Removing the c^{-5l} terms and some manipulation leads to

$$c^{10l}b_1c^{5l} = b_2c^{-5l}b_1c^{10l+1}b_2. \tag{3.5.3}$$

If we look to square both sides of (3.5.3), we find

$$\begin{aligned} c^{10l}b_1c^{15l}b_1c^{5l} &= b_2c^{-5l}b_1c^{5l+1}b_1c^{10l+1}b_2 \\ &= b_2c^{-5l}b_1cb_1c^{-5l}b_1c^{5l+1}b_2 \end{aligned}$$

and the left hand side here simplifies (by splitting into c^{5l} increments and R_5-moves) to $c^{5l}b_1c^{-10l}b_1c^{-10l}b_1$ whence

$$b_1c^{10l}b_1 = b_2c^{5l}b_1cb_1c^{-5l}b_1c^{5l+1}b_2b_1c^{10l}. \tag{3.5.4}$$

Now returning to (3.5.3), premultiplying both sides by c^{-5l}, applying R_5-moves and then squaring gives us

$$\begin{aligned} b_1c^{-10l}b_1 &= b_2c^{5l}b_2b_1c^{15l+1}b_2b_1c^{10l+1}b_2 \\ \Leftrightarrow b_1c^{-10l}b_1 &= b_2c^{-1}b_1b_2c^{10l}b_2b_1c^{10l+1}b_2 \end{aligned} \tag{3.5.5}$$

where the second step comes from (3.5.2) implying $c^{5l}b_2b_1 = c^{-1}b_1b_2c^{-5l-1}$.

Now we equate the right sides of (3.5.4) and (3.5.5); after cancelling on the left, we find

$$\begin{aligned} c^{5l+1}b_2b_1cb_1c^{-5l}b_1c^{5l+1}b_2b_1c^{10l} &= b_1b_2c^{5l-1}c^{5l+1}b_2b_1c^{5l+1}c^{5l}b_2 \\ \Leftrightarrow c^{5l+1}b_2b_1c^{5l+1}b_1c^{10l+1}b_2b_1c^{10l} &= b_1b_2c^{5l-1}b_1b_2c^{5l}b_2 \\ \Leftrightarrow b_1c^{5l}c^{5l+1}b_2b_1c^{5l+1}c^{5l-1} &= c^{5l-1}b_1b_2c^{5l}b_2 \\ \Leftrightarrow c^{-5l}b_1c^{-5l}b_2c^{5l-1} &= c^{5l-1}b_1c^{-5l}b_2c^{-5l} \\ \Leftrightarrow b_1c^{-5l}b_2c^{10l-1} &= c^{10l-1}b_1c^{-5l}b_2 \end{aligned}$$

Thus, the words $b_1c^{5l}b_2$ and c^{10l-1} commute; moreover $b_1c^{5l}b_2$ commutes with any power of c^{10l-1}, and since $\gcd(10l-1, 25l) = 1$ in particular $b_1c^{5l}b_2$ commutes with c and thus all powers of c. Now compare $(b_2b_1c^{5l+1})^2$ and $(b_2c^{5l})^3$, and we find $c^{5l+1}b_2b_1c^{10l+1} = b_1c^{-5l}b_2c^{-5l}b_2$, i.e. $b_1b_2c^{5l} = b_1c^{5l}b_1$ and hence $b_1 = c^{-5l}b_2c^{5l}$. Now naturally $b_1^2 = 1$ is redundant and we replace b_1 everywhere else in the presentation (and remove any duplicates up to cyclic permutation) to obtain the presentation

$$\langle b_2, c | b_2^2, c^{25l}, (b_2c^{5l})^3, (c^{5l}b_2c^{15l+1})^2, (b_2c^{-5l}b_2c^{5l+1})^3 \rangle.$$

Moving the lead b_2 term on the third relator here to the right hand side and then equating it to the fourth relator, we get

$$\begin{aligned} b_2c^{5l}b_2 &= c^{15l+1}b_2c^{15l+1} \\ \Leftrightarrow c^{-5l}b_2c^{-5l} &= c^{15l+1}b_2c^{15l+1} \end{aligned} \tag{3.5.6}$$

Also, $(b_2c^{-5l}b_2c^{5l+1})^3 = (c^{5l}b_2c^{10l+1})^3$ by R_5-move; but this yields $c^{5l}b_2c^{15l+1}b_2c^{15l+1}b_2c^{10l+1} = 1$. By (3.5.6) and subsequent R_5-move we obtain $c^{5l}b_2^2c^{5l}b_2^2c^{10l+1} = 1$, and therefore $c^{20l+1} = 1$. However since $\gcd(20l+1, 25l) = 1$ this gives $|c| = 1$ and therefore for example $(b_2c^{5l})^3 = 1$ now implies $b_2^3 = b_2 = 1$; therefore H' totally collapses, and $H = H_{ab} \cong C_3$. $\square$

Chapter 4

The case $m \geq 4$: preliminaries

4.1 Parameter constraints and positively curved regions

We now consider the groups $H(m, n, p, q|k)$ with $m \geq 4$. Firstly we reconsider our remaining parameters; as discussed in Section 1.2, we insist that $k \geq 3$ and that $p, q > 1$ since otherwise categorisation is complete. However in contrast to the $m = 2$ case we must not only consider $q = 2$ as a valid possibility but also cannot immediately discount $p = 2$ since the group $D(m, n, p)$ is no longer always finite in this case (though we can in fact limit ourselves to $p \geq 3$, as demonstrated later in Lemma 4.1). Also, we cannot conclude that $n \geq 2k$ since $x = x^{-1}$ no longer holds; therefore we are limited to the general $n \geq k + 2$ as described in Section 1.2. Therefore, we assume at all times $n \geq 5$. In terms of regions of pictures over P_H, observe that it is now only possible for 2-gons to exist as x-regions created by the adjacency of an α-vertex with a β-vertex of opposite sign, or as a y-region between two β-vertices when $n = 2k$. However, in this case we can run into difficulties with bridge moves, as can be seen in Fig. 4.1; in this instance the 2-gon y-region can be destroyed by a bridge move to create a new 2-gon x-region. For the sake of simplicity we will disregard this case in our analysis and assume then that $n \neq 2k$ hereon. Since $m > 2$, we may assume that for a minimal picture

Π any x-region which is not a 2-gon is of degree m and is created from m vertices of identical polarity. As we discount $n = 2k$, no 2-gons which are also y-regions can exist, and thus as stated in Chapter 1 we assert that the degree of a vertex v satisfies $p \leq d(v) \leq 2p$ when v is an α-vertex (up to sign) and $q \leq d(v) \leq 2q$ when v is a β-vertex (up to sign). In particular, $d(v)$ is equal to $p + a$ or $q + a$ respectively where v is adjacent to a vertices of identical polarity; we use this idea for finding positively curved regions.

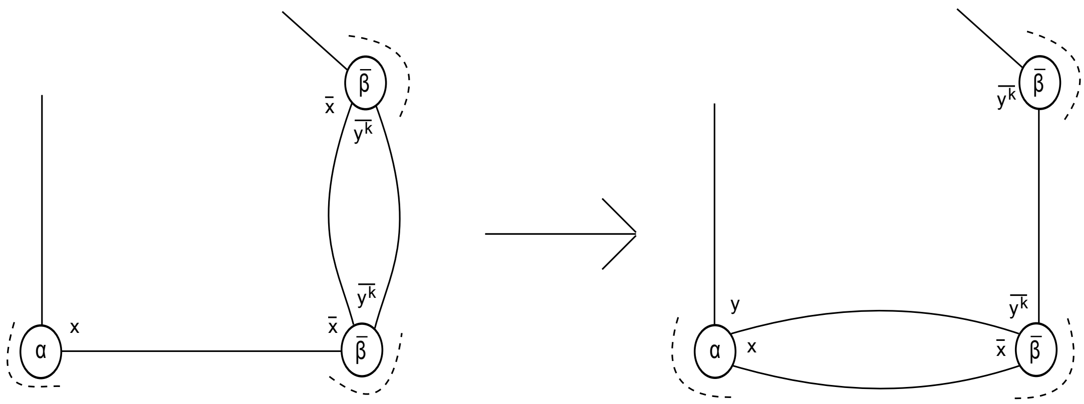


Fig. 4.1

We will also assume here that $p, q \geq 3$. Whilst $q = 2$ may be valid, it also creates difficulties with positively curved regions since it allows for β-vertices of degree 2 to exist, and inserting such a β-vertex into the boundary of a region Δ has the net effect of $\left(-1 + \frac{2}{2}\right)\pi = 0$ on Δ's curvature. As such, we can create arbitrarily large regions of positive curvature provided the labelling suffices; therefore, we omit this possibility here. As for $p = 2$, there is in general a link to $m = 2$ (which was covered in Chapters 2 and 3) with no assumptions about any parameters barring k as we show now.

Lemma 4.1: *Let H be the group $H(m, n, 2, q|k)$ where $k \geq 2$. Then $H \cong H(2, n, m, q|k-1)$.*

Proof: If H has the standard presentation $P_H = \langle x, y | x^m, y^n, (xy)^2, (xy^k)^q \rangle$ then add the generator $u = xy$; this implies then that $x = uy^{-1}$. Substituting this everywhere in P_H for x gives us the presentation $\langle u, y | u^2, y^n, (uy^{-1})^m, (uy^{k-1})^q \rangle$; but since $|u| = 2$, $uy^{-1} = u^{-1}y^{-1}$ and if this has order m so does its inverse, uy.

Then, replacing $(uy^{-1})^m$ with $(uy)^m$ in the presentation then yields the result. □

As a result, in this work we will limit ourselves to $p, q \geq 3$. Subject to these constraints and the possible degrees of vertices, we find that seven possible positively curved regions can be found up to inversion as listed below:

- Δ_1**:** $\alpha\beta\beta$-region, positively curved for the pairs $(p, q) = (p, 3)$ where $3 \leq p \leq 7$ and $(p, q) = (3, 4)$
- Δ_2**:** $\alpha\alpha\beta$-region, positively curved for $(p, q) = (3, q)$ where $3 \leq q \leq 7$ and $(p, q) = (4, 3)$
- Δ_3**:** $\beta\beta\beta$-region, positively curved for all p where $q = 3$
- Δ_4**:** $\bar{\alpha}\beta\beta$-region, positively curved for $3 \leq q \leq 10$ when $p = 3$, for $3 \leq q \leq 6$ when $p = 4$, for $3 \leq q \leq 5$ when $p = 5$, for $q \in \{3, 4\}$ when $5 \leq p \leq 9$ and $q = 3$ for $p \geq 10$
- Δ_5**:** $\bar{\alpha}\beta\beta\beta$-region, positively curved exactly when $p = 3$ and $q = 3$
- Δ_6**:** $\bar{\alpha}\bar{\alpha}\bar{\alpha}\beta$-region, positively curved exactly when $p = 3$ and $q = 3$, but only exists for $k = 3$;
- Δ_7**:** $\bar{\alpha}\beta\bar{\alpha}\beta$-region, positively curved for the pairs $(p, q) = (3, 3), (3, 4), (3, 5),$ $(4, 3)$ and $(5, 3)$

As in the $m = 2$ case, we first consider groups for which possible co-existence can occur. First we consider pairs, each of which receives special consideration below. In each case, the pair $\{a, b\}$ means the group arises from the co-existence of Δ_a and Δ_b as positively curved regions (though this is not necessarily the only pair which gives rise to this group).

We can also therefore make the following statement, arising naturally from the list of Δ_1 through Δ_7, of the following result.

Group	Regions
$H(m, 7, 3, 3\|3)$	$\{1, 6\}$
$H(m, 6, 3, 3\|4)$	$\{2, 3\}$
$H(m, 6, 4, 3\|4)$	$\{2, 3\}$
$H(m, 5, 3, 3\|3)$	$\{2, 4\}$
$H(m, 5, 3, 4\|3)$	$\{2, 4\}$
$H(m, 5, 3, 5\|3)$	$\{2, 4\}$
$H(m, 5, 3, 6\|3)$	$\{2, 4\}$
$H(m, 5, 3, 7\|3)$	$\{2, 4\}$
$H(m, 5, 4, 3\|3)$	$\{2, 4\}$
$H(m, 7, 3, 3\|5)$	$\{2, 5\}$
$H(m, 6, 3, 4\|4)$	$\{2, 7\}$
$H(m, 6, 3, 5\|4)$	$\{2, 7\}$
$H(m, 9, 3, 3\|3)$	$\{3, 6\}$
$H(m, 6, 5, 3\|4)$	$\{3, 7\}$
$H(m, 8, 3, 3\|3)$	$\{5, 6\}$

Table 4.1: Groups for which multiple positively curved regions could arise, $m > 3$

Theorem 4.2: *Let $H = H(m, n, p, q|k)$ be the group as usual with standard presentation P_H and with $m \geq 4$, $n \geq k + 2$ but $n \neq 2k$ and $p, q \geq 3$. If H does not satisfy any of the following conditions:*

i. $n = 2k + 1$ *and either* $q \in \{3, 4\}$ *with* $p = 3$ *or* $q = 3$ *with* $4 \leq p \leq 7$ *and also with* $k > 3$ *if* $p = q = 3$,

ii. $n = k + 2$ *and either* $3 \leq q \leq 7$ *with* $p = 3$ *or* $q = 3$ *with* $p = 4$ *and also with* $k > 3$ *if* $p = q = 3$,

iii. $n = 3k$ *and* $q = 3$, *and* $k > 3$ *if* $p = 3$

iv. $n = 2k - 1$ *and either* $3 \leq q \leq 10$ *when* $p = 3$ *or* $3 \leq q \leq 6$ *when* $p = 4$ *or* $3 \leq q \leq 5$ *when* $p = 5$ *or* $q \in \{3, 4\}$ *when* $5 \leq p \leq 9$ *or* $q = 3$ *when* $p \geq 10$, *with* $k > 3$ *if* $p = q = 3$,

v. $n = 3k - 1$ *and* $p, q = 3$ *with* $k > 3$, *vi.* $p, q, k = 3$, *or*

vii. $n = 2k - 2$ *and either* $3 \leq q \leq 5$ *with* $p = 3$ *or* $3 \leq p \leq 5$ *with* $q = 3$, *with* $k > 3$ *if* $p = q = 3$

then the presentation P_H *is quasi-aspherical over* $C_m * C_n$.

Proof: If H fails to completely satisfy any set of requirements from *i.* – *vii.* then any picture Π drawn from H cannot contain any of Δ_i as positively curved regions. Therefore by necessity no spherical picture can have positively curved interior regions, so the curvature sum of the interior regions of $\Pi \leq 0$; consequently the distinguished region (assuming for minimal degree it is of degree 1) can have at most curvature $(2 - 1 + \frac{2}{3})\pi < 4\pi$. Thus $c(\Pi) < 4\pi$ so Π cannot be spherical. $\Box$

In fact, if the values of p, q are large enough, we can be definite as to the finiteness of H.

Theorem 4.3: *Let* $H = H(m, n, p, q|k)$ *have the standard presentation* P_H *and let* $m \geq 4$, $n \geq k + 2$ *with* $n \neq 2k$ *and* $p, q \geq 3$. *Suppose* H *does not satisfy any of the conditions i. – vii. of Theorem 4.2, and that* $(p, q) \notin \{(3, 3), (3, 4), (3, 5), (4, 3), (5, 3)\}$. *Then* H *does not collapse, and further to this* H *is infinite.*

Proof: If H does not satisfy any of the conditions of Theorem 4.2 then as noted any picture Π has no positively curved regions, and in particular no spherical pictures. Then m, n cannot collapse. Suppose instead p collapses to some p' with $1 \leq p' < p$ (resp. q' with $1 \leq q' < q$), have Π_{red} be a reduced picture with one of these as its boundary label, contract the boundary to a distinguished vertex of degree d_0 and consider a boundary region $\hat{\Delta}$; however by the restrictions on p and q, plus no possible receipts of positive curvature, $c(\hat{\Delta}) \leq c(3, 6, d_0) = c(4, 4, d_0) = (-1 + 1 + \frac{2}{d_0})$ which must be strictly less than $\frac{4\pi}{d_0}$. Thus, no such reduced picture can occur. Consequently, we can apply the pushout argument of Section 2.4, and in particular note from (2.4.1) that H being finite of order N

forces $\frac{1}{N} \leq \frac{1}{4} + \frac{1}{k+2} + \frac{1}{4} + \frac{1}{4} - 1$ and $k \geq 3$ implies $N < 0$, a contradiction. Thus all such groups H are infinite. $\square$.

4.2 Examination of pairwise-arising groups

In this section, we check these groups for finiteness or infiniteness predominantly using KBMAG. Unless otherwise stated we look for a successful application of KBMAG and in particular note the number of states in the word-acceptor; if from some value of m (say M) we can find an arithmetic progression in terms of m for the number of states for all subsequent m-values, then for all $M' > M$ the automata have similar structures. As such if the group $H(M, n, p, q|k)$ admits an automaton with an infinite structure, then we conjecture that for all $M' > M$ $H(M', n, p, q, |k)$ is also infinite. In all cases, the number of states for $100 \leq m \leq 103$ were first predicted and then verified. Also, in contrast to Chapter 5, we will consider $m = 3$ where the order was able to be computed.

4.2.1 Δ_1, Δ_6

Here, we present the orders of the groups $H(m, 7, 3, 3|3)$. The exception is $H(3, 7, 3, 3|3)$ for which neither KBMAG nor examination of subgroups or mappings to $PSL(2, z)$ were able to draw a conclusion.

4.2.2 Δ_2, Δ_i for $i \neq 4$

For this pairing, we have two potential groups, $H(m, 6, 3, 3|4)$ and $H(m, 6, 4, 3|4)$. As before we list the findings from KBMAG in Table 4.3, with $H(m, 6, 3, 3|4)$ on the left half and $H(m, 6, 4, 3|4)$ on the right, with the exception of $m = 3$ in both cases for which we required subgroup routines.

We close this part by noting that $H(3, 6, 3, 3|4)$ is infinite since GAP finds a

m	Order	Word-acceptor states
4	∞	782
5	∞	406
6	∞	359
7	∞	357
8	∞	359
9	∞	361
10	∞	363
$7 \leq m$	∞	$2m + 343$
100	∞	543
101	∞	545

Table 4.2: Orders of automatic structures for $H(m, 7, 3, 3|3)$

m	Order	Word-acceptor states	m	Order	Word-acceptor states
4	∞	409	4	∞	728
5	∞	312	5	∞	607
6	∞	295	6	∞	541
7	∞	295	7	∞	541
8	∞	297	8	∞	542
9	∞	299	9	∞	544
10	∞	301	10	∞	546
$7 \leq m$	∞	$2m + 281$	$8 \leq m$	∞	$2m + 526$
100	∞	481	100	∞	726
101	∞	483	101	∞	728

Table 4.3: Orders of automatic structures for $H(m, 6, 3, 3|4)$ and $H(m, 6, 4, 3|4)$

subgroup K where $|H : K| = 3$ and $K_{ab} \cong \mathbb{Z}^2$, and similarly $H(3, 6, 4, 3|4)$ is infinite since GAP finds a subgroup K where $|H : K| = 6$ and $K_{ab} \cong \mathbb{Z} \times C_4$.

For co-existence of positive Δ_2 and Δ_5 regions, we are forced to consider the groups $H(m, 7, 3, 3|5)$. Here we have one unknown group as we were unable to determine finiteness for $H(3, 7, 3, 3|5)$ but the remainder are postulated infinite as detailed below in Table 4.4.

m	Order	Word-acceptor states
4	∞	1796
5	∞	1136
6	∞	912
7	∞	893
8	∞	892
9	∞	894
10	∞	896
$8 \leq m$	∞	$2m + 876$
100	∞	1076
101	∞	1078

Table 4.4: Orders of automatic structures for $H(m, 7, 3, 3|5)$

Lastly in this subsection we consider co-existence for Δ_2 and Δ_7. This leads us to $H(m, 6, 3, 4|4)$ and $H(m, 6, 3, 5|4)$. In the former case we find differences in structure depending on whether m is odd or even, as detailed in Table 4.5, but all found were infinite and thus all are conjectured to be infinite.

m	Order	Word-acceptor states	m	Order	Word-acceptor states
3	∞	2385	4	∞	229
5	∞	610	6	∞	194
7	∞	586	8	∞	196
9	∞	590	10	∞	198
11	∞	594	12	∞	200
$7 \leq m$ odd	∞	$2m + 572$	$6 \leq m$ even	∞	$m + 188$
101	∞	774	100	∞	288
103	∞	778	102	∞	290

Table 4.5: Orders of automatic structures for $H(m, 6, 3, 4|4)$

Finally we consider $H(m, 6, 3, 5|4)$ all of which are conjectured to be infinite according to KBMAG as demonstrated below.

m	Order	Word-acceptor states
3	∞	2301
4	∞	1197
5	∞	1104
6	∞	1072
7	∞	1070
8	∞	1072
9	∞	1074
10	∞	1076
$7 \leq m$	∞	$2m + 1056$
100	∞	1256
101	∞	1258

Table 4.6: Orders of automatic structures for $H(m, 6, 3, 5|4)$

4.2.3 Δ_2, Δ_4

Here, there are six candidate groups; $H(m, 5, 3, q|3)$ where $3 \leq q \leq 7$ and $H(m, 5, 4, 3|3)$. Firstly, we present results for $H(m, 5, 3, 7|3)$ on the left and $H(m, 5, 3, 6|3)$ on the right in Table 4.7.

m	Order	Word-acceptor states	m	Order	Word-acceptor states
3	∞	587	3	∞	1885
4	∞	377	4	∞	402
5	∞	335	5	∞	354
6	∞	324	6	∞	343
7	∞	324	7	∞	343
8	∞	326	8	∞	345
9	∞	328	9	∞	347
10	∞	330	10	∞	349
$7 \leq m$	∞	$2m + 310$	$7 \leq m$	∞	$2m + 329$
100	∞	510	100	∞	529
101	∞	512	101	∞	531

Table 4.7: Orders of automatic structures for $H(m, 5, 3, 7|3)$ and $H(m, 5, 3, 6|3)$

Next we display results for $H(m, 5, 3, 5|3)$ and $H(m, 5, 3, 4|3)$ in Table 4.8, again on the left and right respectively. Here we omit $H(3, 5, 3, 5|3)$ which is infinite as it admits a subgroup K of index 10 with $K_{ab} \cong \mathbb{Z}$, $H(3, 5, 3, 4|3)$ which is infinite since it admits a subgroup K with $|H : K| = 5$ and $K_{ab}^{(2)} \cong \mathbb{Z}^3 \times C_2^7 \times C_4$ and $H(4, 5, 3, 4|3)$ which is infinite as it admits a subgroup K of index 6 with $K_{ab} \cong \mathbb{Z}$.

m	Order	Word-acceptor states	m	Order	Word-acceptor states
4	∞	551	4	∞	N/A
5	∞	411	5	∞	4222
6	∞	384	6	∞	460
7	∞	384	7	∞	413
8	∞	386	8	∞	412
9	∞	388	9	∞	414
10	∞	390	10	∞	416
$7 \leq m$	∞	$2m + 370$	$8 \leq m$	∞	$2m + 396$
100	∞	570	100	∞	596
101	∞	572	101	∞	598

Table 4.8: Orders of automatic structures for $H(m, 5, 3, 5|3)$ and $H(m, 5, 3, 4|3)$

Next we deal with $H(m, 5, 3, 3|3)$. Here the pattern for word-acceptor states does not stabilise until m is greater then 20, so we devote the whole of Table 4.9 to it. The only cases not determined by KBMAG are $m = 5$ and $m = 6$; in these cases $H(5, 5, 3, 3|3)$ is infinite since it contains a subgroup K of index 6 where $K_{ab} \cong \mathbb{Z}$ and $H(6, 5, 3, 3|3)$ is infinite since it contains a subgroup K of index 12 where $K_{ab} \cong \mathbb{Z} \times C_5$.

m	Order	Word-acceptor states	m	Order	Word-acceptor states
3	75	N/A	16	∞	1064
4	1080	N/A	17	∞	844
5	∞	N/A	18	∞	730
6	∞	N/A	19	∞	619
7	∞	3100	20	∞	618
8	∞	2390	21	∞	617
9	∞	2245	22	∞	619
10	∞	2103	23	∞	621
11	∞	2030	24	∞	623
12	∞	1891	25	∞	625
13	∞	1739	$21 \leq m$	∞	$2m + 575$
14	∞	1538	100	∞	775
15	∞	1341	101	∞	777

Table 4.9: Orders of automatic structures for $H(m, 5, 3, 3|3)$

Lastly for this subsection we consider the groups $H(m, 5, 4, 3|3)$. Again, these are all conjectured to be infinte from the computations in KBMAG, with stabilisation occurring for $m > 11$.

m	Order	Word-acceptor states
3	1080	N/A
4	∞	4952
5	∞	2838
6	∞	1962
7	∞	1542
8	∞	968
9	∞	808
10	∞	662
11	∞	637
12	∞	630
13	∞	632
14	∞	634
$12 \leq m$	∞	$2m + 606$
100	∞	806
101	∞	808

Table 4.10: Orders of automatic structures for $H(m, 5, 4, 3|3)$

4.2.4 $(\Delta_3,\, \Delta_6)$, $(\Delta_3,\, \Delta_7)$ and $(\Delta_5,\, \Delta_6)$

For the pair Δ_3 and Δ_6 we investigate the groups $H(m, 9, 3, 3|3)$. All of these are conjectured to be infinite for $m \geq 3$. The same applies to the groups $H(m, 6, 5, 3|4)$ which arise from Δ_3 and Δ_7 are also all conjectured to be infinite, as displayed below in Table 4.11 on the left and right respectively.

Finally, our co-existence search ends with Δ_5 and Δ_6 which leads us to consider $H(m, 8, 3, 3|3)$. As for $H(m, 6, 4, 3|4)$ the pattern depends on whether m is odd or even, with the exception of $H(3, 8, 3, 3|3)$ which requires finding a relevant subgroup. In this case, GAP finds a subgroup K of index 13 which satisfies $K'_{ab} \cong \mathbb{Z}^2 \times C_3 \times C_4^2$.

m	Order	Word-acceptor states	m	Order	Word-acceptor states
3	∞	1326	3	∞	3857
4	∞	423	4	∞	1435
5	∞	345	5	∞	1089
6	∞	321	6	∞	976
7	∞	321	7	∞	976
8	∞	323	8	∞	977
9	∞	325	9	∞	979
10	∞	327	10	∞	981
$7 \leq m$	∞	$2m + 307$	$8 \leq m$	∞	$2m + 961$
100	∞	507	100	∞	1161
101	∞	509	101	∞	1163

Table 4.11: Orders of automatic structures for $H(m, 9, 3, 3|3)$ and $H(m, 6, 5, 3|4)$

m	Order	Word-acceptor states	m	Order	Word-acceptor states
3	∞	N/A	4	∞	149
5	∞	355	6	∞	125
7	∞	327	8	∞	127
9	∞	331	10	∞	129
11	∞	335	12	∞	131
$7 \leq m$ odd	∞	$2m + 313$	$6 \leq m$ even	∞	$m + 119$
101	∞	515	100	∞	219
103	∞	519	102	∞	221

Table 4.12: Orders of automatic structures for $H(m, 8, 3, 3|3)$

4.3 Special cases leading to spherical pictures

We now discuss monotype pictures over Δ_i prior to trying to establish quasi-asphericity in Chapter 5. First, we note that in the case of Δ_7 it is possible to construct vertices of exactly degree p or q by insisting that an $\bar{\alpha}$-vertex is adjacent to p β-vertices, and similarly that a β-vertex sits adjacent to q $\bar{\alpha}$-vertices. If we draw a monotype picture Π_7 over Δ_7 regions insisting that every vertex is of minimal degree in this way, then necessarily every x-region is a 2-gon containing $x\bar{x}$ after bridge moves. Thus, all regions of Π_7 are y-regions, and we are analogous to the case $m = 2$; in particular that of S_2 regions. As detailed in Table 2.3, we can therefore find spherical pictures over all possible pairs (p, q) for which Δ_7 is positively curved, and there cannot declare $H(m, 2k-2, p, q|k)$ to be quasi-aspherical in these cases.

Lastly we consider the groups $H(4, 2k-1, 3, 4|k)$, thus considering Δ_4 regions. Precisely when $m = 4$, $p = 3$ and $q = 4$, we can construct complete $\bar{\alpha}$-wheels with curvature 2π by connecting an $\bar{\alpha}$-vertex to three β-vertices, and if we consider the three adjacent x-regions to this wheel we may construct them as $(6, 6, 6, 6)$ regions each on four β-vertices. Then we can create a graph as in Fig. 4.2, where we have omitted the $\bar{\alpha}$-vertices but one can readily add them to the centre of each triangular region and drawing edges from each to the three relevant β-vertices.

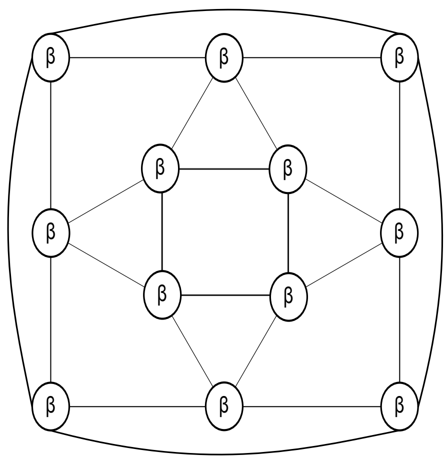


Fig. 4.2

As a one-off presentation, we will conduct an initial investigation into the groups $H(4, 2k-1, 3, 4|k)$ for $k \leq 30$. The conjecture is that $H(4, 2k-1, 3, 4|k)$ is infinite for all $k \geq 3$, since we can find the following:

- For $k = 3, 4$ KBMAG returns an infinite automata with a word-acceptor on 3044 and 4620 states respectively.

- For $k = 5$ GAP finds a subgroup K of index 13 with $K_{ab} \cong \mathbb{Z} \times C_2$, for $k = 6$ GAP finds a subgroup K of index 12 with $K_{ab} \cong \mathbb{Z} \times C_2 \times C_3$, for $k = 11$ GAP finds a subgroup K of index 14 with $K_{ab} \cong \mathbb{Z} \times C_3 \times C_4$ and for $k = 25$ GAP finds a subgroup K of index 14 with $K_{ab} \cong \mathbb{Z} \times C_4 \times C_7$.

- For $k \in \{19, 24, 27, 30\}$, infiniteness or otherwise could not be ascertained.

- Lastly, for all other cases a mapping to $PSL(2, z)$ for some z can be constructed from which a subgroup K of H is found as in Chapter 3; we then either find K or its derived subgroup has infinite abelianisation. As noted in Chapter 3, this code is due to Paul and is found in Appendix C. The one exception to this is $k = 16$, where instead there exists a mapping to $PSL(3, 5)$ with subgroup K for which $K_{ab} \cong C_2^{12} \times C_4^2$. The Newman Infinity Criterion was run in GAP on K and returned true.

k	Valid z	Abelianisation of subgroup K
7	79	$\mathbb{Z}^{10} \times C_3 \times C_{13}$
8	89	$\mathbb{Z}^2 \times C_2^{30} \times C_4$
9	271	$\mathbb{Z}^8 \times C_2^{90} \times C_3$
10	191	$\mathbb{Z}^{30} \times C_{19}$
12	137	$\mathbb{Z}^{23} \times C_4^2$
13	9	$C_4^2 \times C_5$, but $K'_{ab} \cong \mathbb{Z}^{121} \times C_3^{35} \times C_5$
14	271	$\mathbb{Z}^{46} \times C_3 \times C_{27}$
15	463	$\mathbb{Z}^{82} \times C_3$
17	857	$\mathbb{Z}^{193} \times C_4^2$
18	71	$\mathbb{Z}^4 \times C_2^{24} \times C_5 \times C_7$
20	79	$\mathbb{Z}^4 \times C_2^{26} \times C_3^2 \times C_{13}$
21	409	$\mathbb{Z}^{24} \times C_2^{136} \times C_3 \times C_4$
22	257	$\mathbb{Z}^{14} \times C_2^{86} \times C_4$
23	89	$\mathbb{Z}^2 \times C_2^{30} \times C_3^5 \times C_4$
26	271	$\mathbb{Z}^8 \times C_2^{90} \times C_3^{16}$
28	439	$\mathbb{Z}^{30} \times C_2^{146} \times C_3$
29	113	$\mathbb{Z}^{22} \times C_4^2$

Table 4.13: Infinite subgroups of $H(4, 2k-1, 3, 4|k)$ arising from mappings to $PSL(2, z)$

Chapter 5

The case $m \geq 4$; the search for quasi-asphericity

Our method here is to now try and deal with monotype pictures over each potential positively curved region (Δ_1 through Δ_6 in Chapter 4) in turn. The only cases we may need to be mindful of are those with $p = q = k = 3$, since if we consider these cases pictures on Δ_i regions $(1 \leq i \leq 5)$ may not be monotype. We shall begin with Δ_5 and Δ_6 since it has only one possibility, that of $p = q = 3$.

5.1 Δ_5, Δ_6

Recall that since $p = q = 3$, if we take a picture Π over Δ_5 regions then the minimal degree of any vertex of Π is 3 and the maximal degree is 6. Consider first that Δ_5 must necessarily sit adjacent to two x-regions across its $\beta\beta$ edges; then the degrees of the β-vertices are at least 4, 4 and 5 and the degree of the $\bar{\alpha}$-vertex is consequently 3. Since $c(4,4,4,5), c(3,4,5,5)$ and $c(3,4,4,6) \leq 0$, we can in fact confirm that to be positively curved Δ_5 is precisely $(3,4,4,5)$ and has positive curvature $\frac{1}{15}\pi$. This also forces certain labelling for other vertices, as detailed in Fig. 5.1; we also denote by ε_1 and ε_2 the adjacent x-regions.

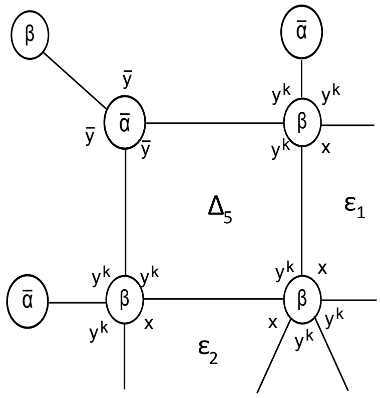


Fig. 5.1

With this opening position in mind we state our theorem.

Theorem 5.1: *Let $H(m, 3k-1, 3, 3|k)$ where $m \geq 4$, $k > 3$ have the standard presentation P_H. Then P_H is quasi-aspherical over $C_m * C_{3k-1}$.*

Proof: Let Π be a reduced spherical monotype picture over Δ_5 regions. Since Π is spherical, there must exist a Δ_5 region, which must look as in Fig. 5.1. From this, suppose that Δ_5 distributes its curvature evenly into its neighbours ε_i. Consider without loss of generality ε_1. Since any vertex of ε_1 contributes $x^{\pm 1}$ to the region and if the region label reads x^a we must have $a \equiv 0 \mod m$, let us assume for worst case scenario that it is of minimal degree (i.e. m) and has A α-vertices and B β-vertices. Since curvature passes only through $\beta\beta$-edges, ε_1 receives at most $\frac{B}{30}\pi$, and all its vertices are of degree at least 4 (aside from one known to be of degree 5 from Δ_5). Then

$$
\begin{aligned}
c^*(\varepsilon_1) &\leq \left(2 - A - B + \frac{2A}{4} + \frac{2(B-1)}{4} + \frac{2}{5} + \frac{B}{30}\right)\pi \\
&= \left(\frac{114 - (30A + 28B)}{60}\right)\pi.
\end{aligned}
$$

If $A+B = m \geq 5$, then $(30A+28B) \geq 28(A+B) \geq 140$ and $c^*(\varepsilon_1) < 0$. Note also that if we assume one more β-vertex has degree 5 instead of 4, $c^*(\varepsilon_1)$ loses $(\frac{2}{4} - \frac{2}{5})\pi = \frac{6}{60}\pi$ curvature, and thus we need $(30A + 28B) < 108$ for $c^*(\varepsilon_1) > 0$, which can be verified is impossible where $A + B \geq 4$. If we then assume $m = 4$, systematic checking finds that $c^*(\varepsilon_1) > 0$ if and only if $B = 4$ and $A = 0$, so we arrive at the position in Fig. 5.2, where the labelling of the starred vertices is forced since all the β-vertices (except one, already in place) must have degree 4. But this means that ε_1's other neighbours, the regions γ_j, cannot be of Δ_5 type; they are thus not positively curved, and do not distribute further curvature into ε_1. Thus $c^*(\varepsilon_1) = c(4,4,4,5) + \frac{1}{30}\pi = \frac{-2}{30}\pi < 0$.

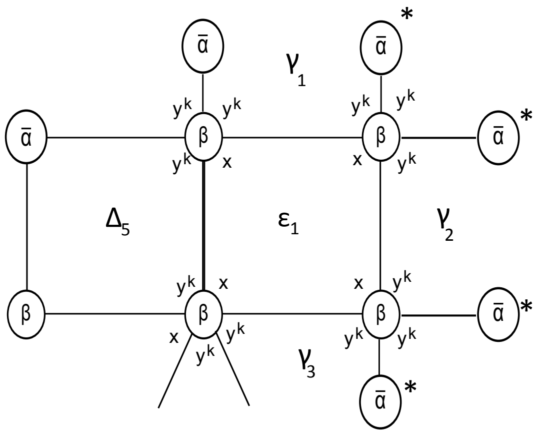


Fig. 5.2

Now consider Δ_D, the distinguished region. If the boundary label is a word in y, then no curvature passes into the distinguished region, and $c(\Delta_D) \leq (2 - 1 + \frac{2}{3})\pi < 4\pi$. Adding a vertex to the boundary of this region gives a net contribution of at most $(-1 + \frac{2}{3})\pi < 0$ and so $c(\Pi) < 0$. Now assume that Δ_D is an x-region of degree d, and assume for worst case that it contains only β-vertices. Then $c^*(\Delta_D) = (2 - d + \frac{2d}{4} + \frac{d}{30})\pi$, i.e. $c^*(\Delta_D) = (2 - \frac{7d}{15})\pi$ which is certainly less than 4π, hence again $c(\Pi) < 4\pi$. □

Now if we examine Δ_6, it looks analogous to the case for Δ_5; it must have adjacent x-regions across $\bar{\alpha}\bar{\alpha}$ edges, and to be positively curved the β-vertex must have degree 3 and the $\bar{\alpha}$-vertices respectively degrees 4, 4 and 5. Thus $c(\Delta_6) = \frac{1}{15}\pi$, as before, and we have the similar position to the Δ_5 case as shown below in Fig. 5.3.

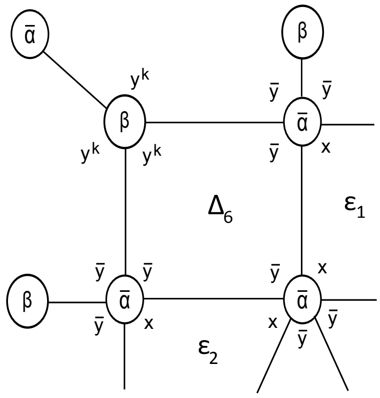


Fig. 5.3

This leads us to state the following:

Corollary 5.2: *Let $H(m, n, 3, 3|3)$, with $m \geq 4$ and $n \notin \{5, 7, 8, 9\}$ have the standard presentation P_H. Then P_H is quasi-aspherical over $C_m * C_n$.*

Proof: The restrictions on n guarantee that the only positively curved regions are Δ_6 regions. We proceed as in Theorem 5.1, starting from a Δ_6 region as in Fig. 5.3 in a reduced spherical monotype picture Π. Suppose that Δ_6 distributes its curvature evenly into its neighbours ε_i. Consider without loss of generality ε_1. As in Theorem 5.1, let us assume for worst case scenario that it is of minimal degree (i.e. m) and has A $\bar{\alpha}$-vertices and B $\bar{\beta}$-vertices. Since curvature passes only through $\bar{\alpha}\bar{\alpha}$ edges, ε_1 receives at most $\frac{A}{30}\pi$, and all its vertices are of degree at least 4 (aside from one known to be of degree 5 from Δ_6). Then

$$\begin{aligned} c^*(\varepsilon_1) &\leq \left(2 - A - B + \frac{2(A-1)}{4} + \frac{2B}{4} + \frac{2}{5} + \frac{A}{30}\right)\pi \\ &= \left(\frac{114 - (28A + 30B)}{60}\right)\pi. \end{aligned}$$

Again as in Theorem 5.1, if $A + B = m \geq 5$ then $(28A + 30B) \geq 28(A + B) \geq 140$ and $c^*(\varepsilon_1) < 0$. Note also that if we assume one more $\bar{\alpha}$-vertex has degree 5 instead of 4, $c^*(\varepsilon_1)$ loses $(\frac{2}{4} - \frac{2}{5})\pi = \frac{6}{60}\pi$ curvature, and thus we need $(28A + 30B) < 108$ for $c^*(\varepsilon_1) > 0$, which can be verified is impossible where $A + B \geq 4$. If we then assume $m = 4$, systematic checking finds that $c^*(\varepsilon_1) > 0$

if and only if $A = 4$ and $B = 0$, so we can extend Fig. 5.3 into something akin to Fig. 5.2, with ε_1 having four $\bar{\alpha}$-vertices and with the starred vertices being β-vertices instead, where the labelling of these starred β-vertices is forced since all the $\bar{\alpha}$-vertices (except one, already in place) must have degree 4. But again this forces ε_1's other neighbours, the regions γ_j, to not be of Δ_6 type; they are thus not positively curved, and do not distribute further curvature into ε_1. Thus $c^*(\varepsilon_1) = c(4,4,4,5) + \frac{1}{30}\pi = \frac{-2}{30}\pi < 0$. If we now consider the distinguished region Δ_D, then the calculation is identical to that in Theorem 5.1 (where if Δ_D is an x-region we assume it is constructed from $\bar{\alpha}$-vertices instead of β-vertices), i.e. $c^*(\Delta_D) = (2 - \frac{7d}{15})\pi < 4\pi$. $\square$

5.2 Δ_1, Δ_2

We start with Δ_1. Since Δ_1's vertices are all positive, any Δ_1 region must necessarily have three x-regions as neighbours. Thus these vertices have degree at least 5. If $q > 4$, then $c(\Delta_1) \leq c(5,7,7) < 0$, so we must have that $q = 3$ or $q = 4$ and in the latter case p must be 3 since $c(6,6,6) = 0$. Then Δ_1 can either be $(5,6,6)$ or $(5,6,7)$. Thus, we formulate the theorem as follows:

Theorem 5.3: *Let $H(m, 2k+1, p, q|k)$ have the standard presentation P_H. If H takes one of the following forms:*

i. $H(m, 2k+1, 3, 4|k)$ with $m \geq 4$ and $k \geq 3$

ii. $H(m, 2k+1, p, 3|k)$ with $m \geq 4$, $k \geq 3$ and $3 \leq p \leq 7$ with $k > 3$ if $p = 3$

then P_H is quasi-aspherical over $C_m * C_{2k+1}$.

Proof: Let Π be a reduced spherical monotype picture over Δ_1 regions. An interior Δ_1 region looks like Fig. 5.4; we start with case i).

Let $q = 4$ and $p = 3$. Assume for the worst case scenario that $c(\Delta_1) = c(5,6,6) = \frac{1}{15}\pi$ since if compensation can be done for all $(5,6,6)$ regions, replacing

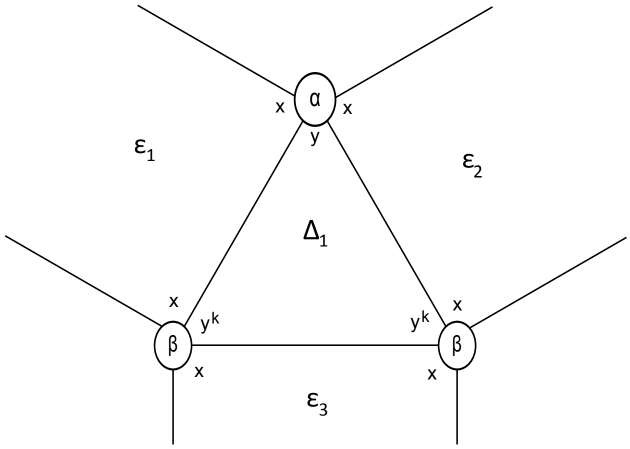


Fig. 5.4

a $(5,6,6)$ region with a $(5,6,7)$ region in Π results in a net loss of curvature. Assume Δ_1 sends $\frac{1}{45}\pi$ into each of its neighbours, and consider without loss of generality the region ε_1. As in Theorem 5.1 assume that it has degree m and consists of A α-vertices, B β-vertices; further, assume aside from those shared with Δ_1 all other β-vertices have degree 5 and other α-vertices have degree 4, the minimum possible, and that each edge receives $\frac{1}{45}\pi$. Then we find

$$
\begin{aligned}
c^*(\varepsilon_1) &\leq \left(2 - A - B + \frac{2(A-1)}{4} + \frac{2}{5} + \frac{2(B-1)}{5} + \frac{2}{6} + \frac{A+B}{45}\right)\pi \\
&= \left(\frac{330 - (86A + 104B)}{180}\right)\pi
\end{aligned}
$$

and since $A + B \geq 4$, $(86A + 104B) \leq 86.4 = 344$ so $c^*(\varepsilon_1) < 0$.

Now we consider $q = 3$. We assume for worst case that $p = 3$ also, since if this can be compensated then for $4 \leq p \leq 7$ all curvatures are less than their counterparts when $p = q = 3$, and that Δ_1 has vertices of minimal degree i.e. Δ_1 is $(5,5,5)$. If we return to Fig. 5.4, then assume that all vertices of ε_1 outside of those shared with Δ_1 have degree 4 (minimal degree), and that Δ_1 has curvature $\frac{1}{5}\pi$. Then all vertices have neighbours which are in turn all negatively signed; therefore ε_1 has no positively curved neighbours outside of Δ_1, and we find

$$c^*(\varepsilon_1) \leq \left(2 - A - B + \frac{2(A-1)}{4} + \frac{2}{5} + \frac{2(B-1)}{4} + \frac{2}{5} + \frac{1}{15}\right)\pi$$
$$= \left(\frac{112 - 30(A+B)}{60}\right)\pi$$

hence $c^*(\varepsilon_1) < 0$. In order to have possible positively curved neighbouring regions, we require adjacent vertices on ε_1 both of degree ≥ 5; therefore, let us assume that all vertices of ε_1 have degree 5, and so we have as many positively curved neighbours as possible. Then since $d(\varepsilon_1) \geq 4$ let us consider a stretch of three consecutive vertices as in Fig. 5.5, where the choice of α is purely illustrative.

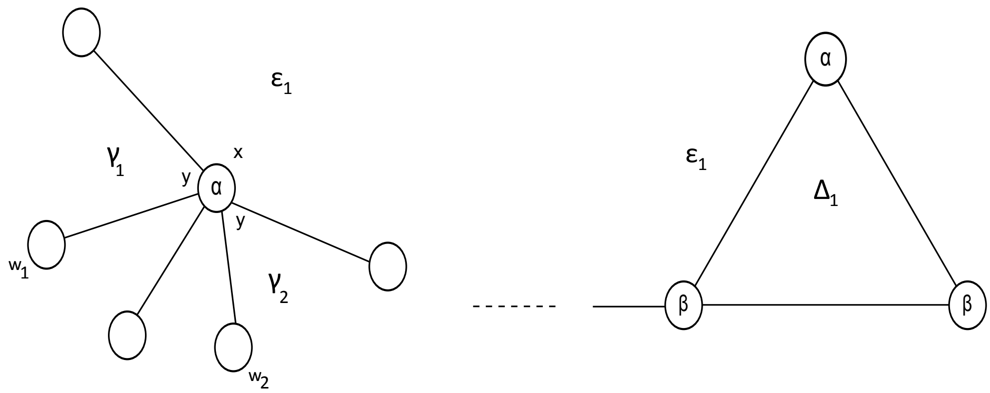


Fig. 5.5

Now if both w_1 and w_2 are positively signed, then neither γ_1 or γ_2's neighbouring x-regions are 2-gons, hence $d(\alpha) > 5$. Thus at least one of w_1 and w_2 is negatively signed and thus if w_i is, then γ_i is not Δ_1-type. Thus, if all vertices have degree 5 curvature does not pass across consecutive edges of ε_1. Therefore if we label ε_1's vertices and their neighbours in order to alternate edges which pass, we can receive at most $\frac{A+B}{2}.\frac{1}{15}\pi$ across such edges, and we find

$$c^*(\varepsilon_1) \leq \left(2 - A - B + \frac{2A}{5} + \frac{2B}{5} + \frac{A+B}{30}\right)\pi$$
$$= \left(\frac{60 - 17(A+B)}{30}\right)\pi < 0.$$

Thus we require at least one vertex of degree 6. But changing a vertex from one of degree 5 to one of degree 6 results in a maximal net effect of $(\frac{-2}{5} + \frac{2}{6} - c(5,5,5) + 2.c(5,5,6))\pi = \frac{-2}{45}\pi$, and adding a vertex of degree 6 to ε_1 results in a maximal net effect of $(-1 + \frac{2}{6} + 2.c(5,5,6))\pi = \frac{-26}{45}\pi$, therefore we can always compensate for positive curvature.

Now for considering the distinguished region, as usual for Δ_D we begin with assuming it is a y-region and receiving no curvature; then $c(\Delta_D) \leq (2 - 1 + \frac{2}{3})\pi < 4\pi$. Again, adding a vertex to the boundary of this region gives a net effect of at most $(-1 + \frac{2}{3})\pi < 0$ and so $c(\Pi) < 0$. If Δ_D is an x-region of degree d, assume all its vertices are of degree 4 and every edge passes $\frac{1}{15}\pi$. Then $c^*(\Delta_D) = (2-d+\frac{2d}{4}+\frac{d}{15})\pi$, i.e. $c^*(\Delta_D) = (2-\frac{26d}{60})\pi < 4\pi$, hence again $c(\Pi) < 4\pi$. $\square$

Following on from this, we can use the similar structure of Δ_2 to similar effect as we did with Δ_5 and Δ_6.

Corollary 5.4: *Let $H(m, k+2, p, q|k)$ have the standard presentation P_H. If H takes one of the following forms:*

i. $H(m, k+2, 4, 3|k)$ with $m \geq 4$ and $k \geq 3$

ii. $H(m, k+2, 3, q|k)$ with $m \geq 4$, $k \geq 3$ and $3 \leq q \leq 7$ with $k > 3$ if $q = 3$

then P_H is quasi-aspherical over $C_m * C_{k+2}$.

Proof: As with the duality between Theorem 5.1 and Corollary 5.2, the argument for this case is very similar to that in Theorem 5.3. Begin first as usual with a reduced spherical monotype picture Π on Δ_2 regions, and consider one such region as in Fig. 5.6. We start with case i. and assume that $(p,q) = (4,3)$.

Assume again for the worst case that $c(\Delta_2) = c(5,6,6) = \frac{1}{15}\pi$ and further assume Δ_2 sends $\frac{1}{45}\pi$ into each of its neighbours, then consider without loss of generality the region ε_1. As in Theorem 5.3 assume that it has degree m and consists of A α-vertices, B β-vertices; further, assume aside from those shared

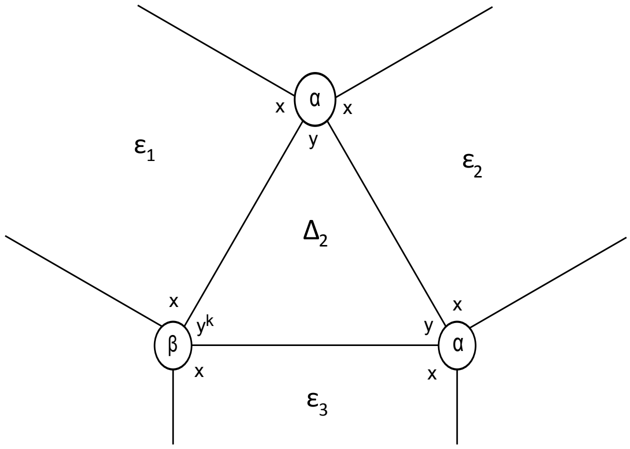


Fig. 5.6

with Δ_2 all other α-vertices have degree 5 and other β-vertices have degree 4, the minimum possible, and that each edge receives $\frac{1}{45}\pi$. Then we find

$$\begin{aligned} c^*(\varepsilon_1) &\leq \left(2 - A - B + \frac{2(A-1)}{5} + \frac{2}{6} + \frac{2(B-1)}{4} + \frac{2}{5} + \frac{A+B}{45}\right)\pi \\ &= \left(\frac{330 - (104A + 186B)}{180}\right)\pi \end{aligned}$$

and since $A + B \geq 4$, $(104A + 86B) \leq 86.4 = 344$ so $c^*(\varepsilon_1) < 0$. Now we consider case ii. where $p = 3$. Again mirroring Theorem 5.3 assume for worst case that $q = 3$ also, since if this can be compensated then for $4 \leq q \leq 7$ all curvatures are less than their counterparts when $p = q = 3$, and that Δ_2 has vertices of minimal degree i.e. Δ_2 is $(5, 5, 5)$. Returning to Fig. 5.6, assume that all vertices of ε_1 outside of those shared with Δ_2 have minimal degree (i.e. degree 4), and that Δ_2 has curvature $\frac{1}{5}\pi$. Then all vertices have neighbours which are in turn all negatively signed; therefore ε_1 has no positively curved neighbours outside of Δ_2, and we find

$$\begin{aligned} c^*(\varepsilon_1) &\leq \left(2 - A - B + \frac{2(A-1)}{4} + \frac{2}{5} + \frac{2(B-1)}{4} + \frac{2}{5} + \frac{1}{15}\right)\pi \\ &= \left(\frac{112 - 30(A+B)}{60}\right)\pi \end{aligned}$$

hence $c^*(\varepsilon_1) < 0$ again. As in Theorem 5.3, we can examine the vertices of ε_1

which are not also part of Δ_2 as in Fig. 5.5 since A, B and the vertex degrees are unchanged to ascertain that $c*(\varepsilon_1) = \left(\frac{60-17(A+B)}{30}\right)\pi < 0$ again, hence all positive curvature can be compensated for. The calculation for the distinguished region Δ_D is also identical, thus no spherical pictures over Δ_2 may exist. $\square$

5.3 Δ_3

The region Δ_3 is positively curved if and only if $q = 3$, regardless of p. Then this looks like Fig. 5.7, similar also to Fig. 5.4. Then, let us assume that all its vertices are of minimal degree and $c(\Delta_3) = c(5,5,5) = \frac{1}{5}\pi$. From this we state the theorem.

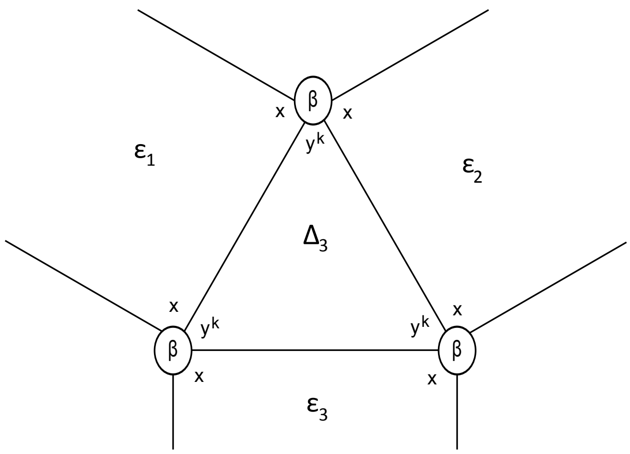


Fig. 5.7

Theorem 5.5: *Let $H(m, 3k, p, 3|k)$ with $m \geq 4$, $p \geq 3$ and $k \geq 3$ with $k > 3$ if $p = 3$ have the standard presentation P_H. Then P_H is quasi-aspherical over $C_m * C_{3k}$.*

Proof: Let Π be a reduced spherical monotype picture on Δ_3 regions and consider a Δ_3 region. It is enough to study the neighbouring regions ε_i. Assume that Δ_3 disperses its curvature evenly into each of the ε_i. Then, we examine without loss of generality ε_1. If it contains an α-vertex v, then no curvature can pass through either of either edge of ε_1 connected to v and its contribution to $c(\varepsilon_1)$ is at most $-1 + \frac{2}{4} < 0$, so we consider for worst case scenario that ε_1

consists solely of β-vertices. Let then $d(\varepsilon_1) = B \geq 4$, and assume that all these vertices have degree 5 as the minimum (again, any β-vertex of degree 4 means that its other two neighbouring vertices outside of ε_1 are necessarily negatively signed and therefore the neighbouring regions do not pass positive curvature). Then all possible neighbours can theoretically pass $\frac{1}{15}\pi$. As such the calculation for amended curvature is

$$\begin{aligned} c^*(\varepsilon_1) &\leq \left(2 - B + \frac{2B}{5} + \frac{B}{15}\right)\pi \\ &= \left(\frac{30 - 8B}{15}\right)\pi < 0. \end{aligned}$$

Again, we consider Δ_D now. As in the case for Δ_1, if it is a y-region and receives no curvature then $c(\Delta_D) \leq (2 - 1 + \frac{2}{3})\pi < 4\pi$. Again, adding a vertex to the boundary of this region gives a net effect of at most $(-1 + \frac{2}{3})\pi < 0$ and so $c(\Pi) < 0$. So now suppose Δ_D is an x-region of degree d; assume all its vertices are of degree 5 and every edge passes $\frac{1}{15}\pi$. Then $c^*(\Delta_D) = (2 - d + \frac{2d}{5} + \frac{d}{15})\pi$ and thus $c^*(\Delta_D) = (2 - \frac{7d}{15})\pi < 4\pi$, hence again $c(\Pi) < 4\pi$. $\square$

5.4 Δ_4

Lastly, we examine monotype pictures over Δ_4. Here we will begin with the largest values of q for which Δ_4 is positively curved and work down as far as we are able. As usual, we begin with one Δ_4 region, and observe that every such region has one non-positive neighbouring region (denoted by ε_1 in Fig. 5.8), through which we will generally disperse Δ_4's total curvature. Observe that $d(\bar{\alpha}) \leq 2p - 2$ after removal of 2-gons, and therefore when $p = 3$ we obtain either a complete $\bar{\alpha}$-wheel or a picture akin to Fig. 5.9; note then in the latter case that the blank vertices must be negatively signed for an $\bar{\alpha}$-vertex of degree 4, the regions ε_2 and ε_3 cannot be of Δ_4-type, and since Π is assumed to be monotype we can conclude that $c(\varepsilon_i) \leq 0$ for $i = 2, 3$. Thus, we assume that curvature passes only through $\beta\beta$ edges. We can of course generalise this picture to one where $d(\bar{\alpha}\text{-vertex}) = p + 1$.

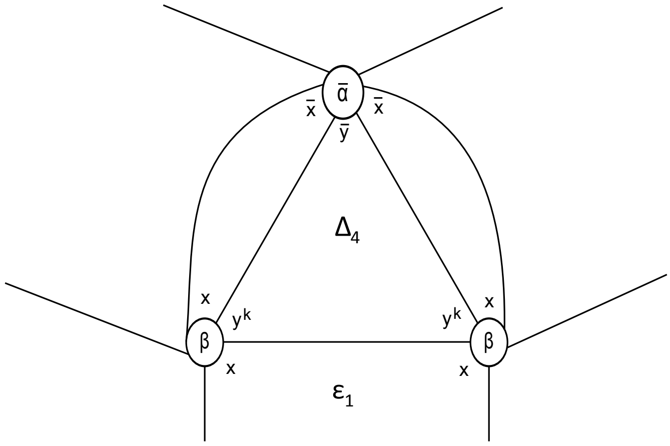


Fig. 5.8

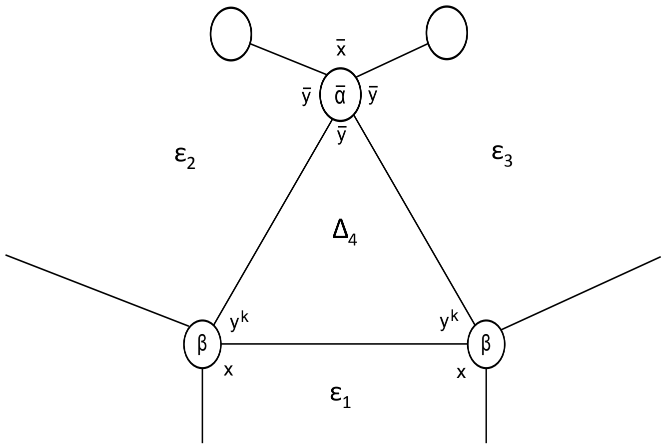


Fig. 5.9

For smaller values, this becomes substantially more difficult (for instance, $p = 3$ and $q = 3$ leads to the region $(3, 4, 4)$ passing $\frac{2}{3}\pi$ across every $\beta\beta$ edge). From this, we offer the following result for some of the smaller pairs.

Theorem 5.6: *Let $H(m, 2k-1, 3, q|k)$ with $6 \leq q \leq 10$ have the standard presentation P_H. Then P_H is quasi-aspherical over $C_m * C_{2k-1}$.*

Proof: As usual, start with a spherical monotype picture Π on Δ_4 regions. We begin by assuming $q = 6$, since from this and $p = 3$ we can create a $(3, 7, 7)$ region and if we can find an argument to compensate for this then we should be able to compensate for all positively curved $(p, q+1, q+1)$ regions where $p \geq 3$ and

$q \geq 6$. Therefore we have the $\bar{\alpha}$-wheel configuration as in Fig. 5.6. If we consider ε_1, we may assume for the worst case scenario that it is of degree B consisting of B β-vertices since all vertices must be positively signed and changing any β-vertex into an α-vertex in such a region removes two potential Δ_4 regions that could disperse curvature into ε_1, thus making a net effect of $(-1 - \frac{2}{7} - \frac{10}{21} + \frac{2}{4})\pi < 0$. Thus, for maximising curvature, we let all vertices of ε_1 have degree 7 and find

$$
\begin{aligned}
c^*(\varepsilon_1) &\leq \left(2 - B + \frac{2B}{7} + \frac{5B}{21}\right)\pi \\
&= 2 - \left(\frac{11B}{21}\right)\pi < 0 \qquad (B \geq 4)
\end{aligned}
$$

thus compensation achieved. Therefore also if we consider the distinguished region Δ_D as an x-region, by the same calculation $c(\Delta_D) < 0$ if all its vertices are β-vertices; however from this position altering one to an α-vertex destroys two possible positive neighbours and hence yields a maximal net effect of $(\frac{-2}{7} + \frac{2}{4} - \frac{2}{6}) < 0$. If instead then Δ_D is a y-region, the best that can be done for maximising curvature is to alternate $\bar{\alpha}$-vertices with β-vertices such that each $\bar{\alpha}$-vertex has degree 3 and each β-vertex degree 7. But even so, letting A and B be the numbers of $\bar{\alpha}$-vertices and β-vertices respectively we find $c(\Delta_D) \leq (2 - A - B + \frac{2A}{3} + \frac{2B}{7})\pi = (2 - \frac{A}{3} - \frac{5B}{7})\pi < 4\pi$. $\square$

5.5 Collapse and infiniteness

As in Chapter 2, we now proceed to show that quasi-asphericity of our presentations leads to the group in question not collapsing, from which we can show infiniteness. First, we investigate possible collapse.

Theorem 5.7: *Let $H = H(m, n, p, q|k)$ with $m \geq 4$ and $k \geq 3$. If H is any one of the following groups:*

- *$H(m, 2k+1, p, q|k)$ where either $q = 3$ and $3 \leq p \leq 7$ (with the possible* *exception of* $p = q = k = 3$*), or* $(p, q) = (3, 4)$

- $H(m, k+2, p, q|k)$ *where either* $p = 3$ *and* $3 \leq q \leq 7$ *(with the possible exception of* $p = q = k = 3$*), or* $(p, q) = (4, 3)$
- $H(m, 3k, 3, 3|k)$
- $H(m, 2k - 1, p, q|k)$ *where* $p = 3$ *and* $6 \leq q \leq 10$
- $H(m, 3k - 1, 3, 3|k)$
- $H(m, n, 3, 3|3)$ *where* $n \notin \{2k \pm 1, k + 2, 3k, 3k - 1\}$

then H *does not collapse.*

Proof: As ascertained prior in Section 2.4, if m or n collapse then there would necessarily be a spherical picture with respectively m' or n' as the boundary label, but we have shown in Theorems 5.1/5.3/5.5/5.6 and Corollaries 5.2/5.4 that no spherical pictures exist. Thus, assume that instead p collapses to some p' with $1 \leq p' < p$ (resp. q' with $1 \leq q' < q$). Then consider a reduced picture Π_{red} with this as its boundary label, contract the boundary to make a distinguished vertex of degree d_0 and consider a boundary region $\hat{\Delta}$. If then $\hat{\Delta}$ is a y-region, the maximal positively curved region possible from the cases discussed in Sections 5.1-5.4 is $(3, 7, d_0)$ and $c(3, 7, d_0) = \frac{2\pi}{d_0} - \frac{\pi}{21} < \frac{4\pi}{d_0}$. If instead $\hat{\Delta}$ is an x-region, let it be of degree m and suppose it is adjacent to one other boundary region. Then it consists at its smallest of $m-1$ vertices of degree 4, the maximal positive curvature it can receive is $(m - 1).\frac{5\pi}{21}$ from $m - 1$ adjacent Δ_4 regions, and thus we find

$$c^*(\hat{\Delta}) \leq \left(2 - m + \frac{2(m-1)}{4} + \frac{2}{d_0} + \frac{5(m-1)}{21}\right)\pi$$
$$= \left(\frac{32 - 11m}{42} + \frac{2}{d_0}\right)\pi < \frac{4\pi}{d_0} \qquad (m \geq 4)$$

Therefore the sum of all boundary regions cannot possibly exceed 4π. □

Now, we use this fact to prove infiniteness for these cases.

Theorem 5.8: *Let $H = H(m, n, p, q|k)$ with $m \geq 4$ and $k \geq 3$. If H belongs to any of the following families of groups:*

i. $H(m, 2k+1, p, q|k)$ *where either* $q = 3$ *and* $3 \leq p \leq 7$ *(with the possible exception of* $p = q = k = 3$*), or* $(p, q) = (3, 4)$

ii. $H(m, k+2, p, q|k)$ *where either* $p = 3$ *and* $3 \leq q \leq 7$ *(with the possible exception of* $p = q = k = 3$*), or* $(p, q) = (4, 3)$

iii. $H(m, 3k, 3, 3|k)$

iv. $H(m, 2k-1, p, q|k)$ *where* $p = 3$ *and* $6 \leq q \leq 10$

v. $H(m, 3k-1, 3, 3|k)$

vi. $H(m, n, 3, 3|3)$ *where* $n \notin \{2k \pm 1, k+2, 3k, 3k-1\}$

then H is infinite.

Proof: Recall that in Section 2.4, we can introduce a pushout of groups leading to the conclusion of statement (2.4.1) therein that if H is finite of order N we have $\frac{1}{N} + 1 = \frac{1}{m} + \frac{1}{n} + \frac{1}{p} + \frac{1}{q}$ (the right hand side of which we will refer to hereon as RHS). Therefore we require the sum of these reciprocals to be greater than one. By (2.4.1), we apply the parameters for each case in turn. In *i.* $k \geq 3$ implies $n \geq 7$ and this in turn implies that $RHS \geq 1$ when $\frac{1}{m} + \frac{1}{p} + \frac{1}{q} \geq \frac{6}{7}$. In the cases where $(p, q) = (7, 3), (6, 3), (5, 3)$ or $(4, 3)$ this inequality only holds if $m < 4$ so these all give N as negative, thus no finite groups can exist here. In the case $(p, q) = (3, 3)$ the requirement becomes $\frac{1}{m} > \frac{4}{21}$ which leads to $m = 4, 5$. If $m = 4$, then inserting this into RHS the exact values of k we can use satisfy $\frac{1}{2k+1} > \frac{1}{12}$ and thus we test $k \in \{3, 4, 5\}$; if $m = 5$, $\frac{1}{2k+1} > \frac{1}{8}$ forces $k = 3$ and we find the following:

- $H(4, 7, 3, 3|3)$ is infinite since if it were finite $N = \frac{84}{5}$.

- $H(4,9,3,3|4)$ is infinite since if finite H ought to have order 36, but GAP finds subgroups of index 7 in H thus violating Lagrange's Theorem,
- $H(4,11,3,3|5)$ and $H(5,7,3,3|3)$ are infinite since KBMAG determined it infinite.

In case *ii*. we check as previously illustrated letting $k = 5$ in our calculations as the minimal value. Working our way through the possible pairs (p, q) we find $(3, 7), (3, 6), (3, 5)$ give us no possible candidates for m, $(3, 4), (4, 3)$ yield $m = 4$ as a possibility and $(3, 3)$ yields $4 \leq m \leq 7$. Dealing with the pairs $(3, 4)$ and $(4, 3)$ first, $m = 4$ forces $k = 3$; but upon checking we find $H(4,5,3,4|3)$ and $H(4,5,4,3|3)$ which are infinite since both forces $N = 30$ but contain elements of order 4. Now, for the pair $(3, 3)$ we find that for $m = 6, 7$ we need $k = 3$, for $m = 5$ we require $3 \leq k \leq 5$ and for $m = 4$ we need $3 \leq k \leq 9$. With the exception of $H(4,5,3,3|3)$, which is finite of order 1080 (as seen in Table 4.9), these are all infinite as categorised below:

- The groups $H(7,5,3,3|3)$, $H(5,6,3,3|4)$, $H(5,7,3,3|5)$, $H(4,11,3,3|9)$, $H(4,10,3,3|8)$, $H(4,9,3,3|7)$, $H(4,8,3,3|6)$ and $H(4,6,3,3|4)$ are all infinite as verified by KBMAG
- The group $H(6,5,3,3|3)$, admits a map to $PSL(2,11)$ whose associated subgroup (see Appendix C) has abelianisation isomorphic to $\mathbb{Z} \times C_5$,
- The group $H(5,5,3,3|3)$ contains a subgroup K of index 6 where $K_{ab} \cong \mathbb{Z}$,
- The group $H(4,7,3,3|5)$ if finite forces $N = \frac{84}{5}$.

Next, in *iii*. we need only check one case since $n \geq 9$ requires $m = 4$; this group, $H(4,9,3,3|3)$ is infinite by KBMAG.

Next, in *iv*. all possible pairs of (p, q) combined with assuming $m = 4$ force $k < 3$ in all cases, so no possible finite groups can be found.

Next, in v. $n \geq 8$ as in iii. forces $m = 4$, from which we can take either $k = 3$ or $k = 4$. However both $H(4, 11, 3, 3|4)$ and $H(4, 8, 3, 3|3)$ are infinite by KBMAG.

Lastly, in vi. for $N > 0$ we require $n \leq 12$ assuming that $m = 4$, $n \leq 7$ for $m = 5$ and $n = 5$ when $m = 6$, where $H(6, 5, 3, 3|3)$, $H(5, 7, 3, 3|3)$, $H(5, 5, 3, 3|3)$ were determined in ii. and $H(4, 7, 3, 3|3)$ was determined in i. For the remainder, we can find that $H(4, n, 3, 3|3)$ is infinite by KBMAG for $8 \leq n \leq 11$, $H(4, 6, 3, 3|3)$ should have order 12 but was found to contain a subgroup of index 5 by GAP (contradicting Lagrange's Theorem) and $H(5, 6, 3, 3|3)$ contains a subgroup K of index 10 where $K'_{ab} \cong \mathbb{Z}^{14} \times C_3^{17} \times C_9^2$, hence all infinite. □

Appendix A

List of miscellaneous found finite groups, orders for $H(2, n, 3, 3|k)$, cases left to solve

In this appendix, we now collate a list of all finite groups found, noted or conjectured in this work. These have all appeared prior, but for ease of reference we will make a list here with a reference to where these appear in the thesis. Perhaps not surprisingly, the majority of these cases occur when $m = 2$. Of course, we do not claim this list to be at all exhaustive, especially since the case $m = 3$ is by and large not mentioned here. First, we note the finite groups which arise in Table 2.2 by pairwise existence of positively curved regions.

- $H(2, 7, 3, 4|3)$ ([168,42]) has the structure of $PSL(3, 2)$
- $H(2, 7, 3, q|3)$ for $q = 6, 7$ ([1092,25]) has the structure of $PSL(2, 13)$
- $H(2, 7, 3, 8|3)$ ([Unavailable]) is perfect, but not soluble
- $H(2, 8, 3, 3|3)$ ([96,64]) is solvable, with derived series of length 3
- $H(2, 8, p, q|3)$ where $(p, q) = (3, 4), (4, 3)$ ([336,208]) has the structure of the semi-direct product of C_2 with $PSL(3, 2)$

- $H(2, n, p, q|3)$ where $(n, p, q) = (8, 3, 5), (8, 5, 3), (10, 3, 4), (10, 4, 3)$ ([Unavailable]) has derived subgroup H' of index 2, which is perfect; H' has GAP identification [1080, 260]

- $H(2, 9, 3, 3|4)$ ([12,3]) has the structure of A_4

- $H(2, 9, 3, 4|4)$ ([Unavailable]) is perfect and not solvable, and has the structure of $PSL(2, 17)$

- $H(2, 10, 3, 3|3)$ ([150,5]) is solvable with derived series of length 3

- $H(2, 10, 3, 3|4)$ ([60,5]) has the structure of A_5

- $H(2, 10, 3, 4|4)$ ([Unavailable]) has a derived series of length 2, ending in a perfect simple subgroup of order 660 which has SmallGroups identification [660,13]

- $H(2, 13, p, q|k)$ where $(p, q, k) = (4, 3, 3), (3, 4, 4)$ ([Unavailable]) are perfect

- $H(2, 15, 3, 3|4)$ ([720,768]) has the structure of $A_5 \times A_4$

Next, we note the finite groups found in the Platonic cases and the finite family $H(2, 2k, 3, 3|k)$.

- $H(2, 2k, 3, 3|k)$ **(Theorems 2.9, 2.10):** finite for all $k \geq 3$, trivial if $k \equiv 2, 4 \mod 6$, of order 3 if $k \equiv 0 \mod 6$ and of order $6k^2$ if k is odd.

- **Octahedra (Section 3.2):** The groups $H(2, 9, 3, 4|3)$ and $H(2, 12, 3, 4|4)$ have order 12 and 15840 respectively.

- **Tetrahedra (Section 3.1):** The group $H(2, 9, 4, 3|3)$ has order 2448, and the groups
  $H(2, 3k, 3, 3|k)$ have order 3 if 3 divides k and 12 otherwise.

- **Cubes (Section 3.2):** When $p = 3$, all of the so-called cube groups
  $H(2, 4k, 3, 3|k)$ are conjectured to be finite; in particular they are conjectured to be trivial if $k \equiv 2, 4 \mod 6$, of order 3 if $k \equiv 0 \mod 6$ and of order $24k^2$ if k is odd.

- **Icosahedra (Section 3.3):** The group $H(2,9,3,5|3)$ has order 10260.

- **Dodecahedra (Section 3.4):** There is conjectured to be a finite family of groups
  $H(2,5k,3,3|k)$ (proven for $k \equiv 0 \mod 15$) according to the following:

$$|H| = \begin{cases} 60 & \text{if } k \equiv \pm 1, \pm 4 \mod 15, \\ 180 & \text{if } k \equiv \pm 6 \mod 15, \\ 1 & \text{if } k \equiv \pm 2, \pm 5, \pm 7 \mod 15, \\ 3 & \text{if } k \equiv 0, \pm 3 \mod 15. \end{cases}$$

- $m \geq 3$**:** The only groups found finite for $m \geq 3$ are $H(3,5,3,3|3)$ (Table 4.9), $H(4,5,3,3|3)$ (Table 4.9) and $H(3,5,4,3|3)$ (Table 4.10) which have orders 75, 1080 and 1080 respectively.

Now, we list in Table A.1 the orders as computed in GAP for the family of groups $H(2,n,3,3|k)$ to support Conjecture 2.16. We have verified these for $3 \leq k \leq 13$ and $3 \leq n \leq 25$. In this table k runs across the top and n down the side.

	3	4	5	6	7	8	9	10	11	12	13
3	3	12	12	3	12	12	3	12	12	3	12
4	24	1	24	1	24	1	24	1	24	1	24
5	1	60	1	60	1	1	60	1	60	1	1
6	54	12	∞	3	∞	12	54	12	∞	3	∞
7	1	1	1	∞	1	∞	1	1	1	1	∞
8	96	1	96	1	∞	1	∞	1	96	1	96
9	3	12	12	3	12	∞	3	∞	12	3	12
10	150	60	150	60	150	1	∞	1	∞	1	150
11	1	1	1	1	1	1	1	∞	1	∞	1
12	216	12	∞	3	∞	12	216	12	∞	3	∞
13	1	1	1	1	1	1	1	1	1	∞	1
14	294	1	294	∞	294	∞	294	1	294	1	∞
15	3	720	12	180	12	12	180	12	720	3	12
16	384	1	384	1	∞	1	∞	1	384	1	384
17	1	1	1	1	1	1	1	1	1	1	1
18	486	12	∞	3	∞	∞	486	∞	∞	3	∞
19	1	1	1	1	1	1	1	1	1	1	1
20	600	60	600	60	600	1	∞	1	∞	1	600
21	3	12	12	∞	12	∞	3	12	12	3	∞
22	726	1	726	1	726	1	726	∞	726	∞	726
23	1	1	1	1	1	1	1	1	1	1	1
24	864	12	∞	3	∞	12	∞	12	∞	3	∞
25	1	60	1	60	1	1	60	1	60	1	1

Table A.1: Orders of $H(2, n, 3, 3|k)$ for $3 \leq n \leq 25$ and $3 \leq k \leq 13$

Lastly, we make a list of those groups where either spherical pictures were not found, but we were unable to ascertain finiteness (or otherwise) through computing or geometric methods, or where spherical pictures exist. We do not include here those groups conjectured to be finite in Conjectures 2.16, 3.2 and 3.4.

- T_1: $H(2, 2k+1, p, q|k)$ where $k \geq 3$ and $(p, q) = (p, 3)$ where $p \geq 3$, $(p, 4)$ where $p \geq 5$, $(p, 5)$ where $p \geq 3$, or any of $(3, 7)$, $(4, 9)$, $(4, 7)$
- $H(2, 7, 3, 11|3)$ and $H(2, 10, 3, 5|4)$ from Table 2.2
- $H(2, 39, 7, 3|12)$ from Theorem 2.15 v.
- Tetrahedra: $H(2, 9, 5, 3|3)$, $H(2, 12, 4, 3|4)$, $H(2, 15, 4, 3|5)$ and $H(2, 3k, p, 3|k)$ where either $k \geq 7$, $p \geq 9$ or both
- Octahedra: $H(2, 15, 3, 4|5)$
- Icosahedra: $H(2, 12, 3, 5|4)$, $H(2, 15, 3, 5|5)$, $H(2, 18, 3, 5|6)$ and $H(2, 3k, p, 5|k)$ where either $k \geq 8$, $p \geq 11$ or both
- Dodecahedra: $H(2, 5k, p, 3|k)$ where either (p, k) is one of $(4, 4)$, $(5, 4)$, $(4, 5)$, $(4, 6)$, $(4, 7)$, $(5, 7)$, $(6, 7)$, $(8, 7)$, $(4, 9)$ or $(5, 9)$, or if either $p \geq 10$ and $k \geq 10$ or both
- $H(4, 2k-1, 3, 4|k)$ where $k = 19, 24, 27$ or $k \geq 30$

Further to these, there are of course the groups $m = 3$ and the groups where $m = 2$ and monotype spherical pictures exist (as in Section 2.2).

Appendix B

Cases to check in Theorem 2.15

Here we list the groups $H(2,n,p,q|k)$ in Theorem 2.15 which could potentially be finite of order N depending on n, p and q (i.e. where n,p,q divide N with N an even positive integer). All of the following were found to be infinite via computation in either GAP, KBMAG or MAF; unless otherwise specified, KBMAG was used.

B.1 Case ii., $p, q \geq 4$ and $n \neq 2k+1, 3k$

In this section, we start with $q = 11$, note the pairs (n,p) necessary to check and which k are required for each, and then proceed to $q = 10$, etc. through to $q = 4$.

- $8 \leq q \leq 11$: $(6,4)$, $k = 3$
- $q = 7$: $(8,4)$, $k = 3,4$; $(6,5)$, $k = 3$
- $q = 6$:
  - $(11,4)$, $k = 3,4$
  - $(10,4)$, $k = 3,4,5$
  - $(8,4)$, $k = 3,4$

- $(6,4)$, $k=3$
- $(6,5)$, $k=3$

- $q=5$:
  - $(19,4)$, $3 \leq k \leq 8$, where for $k=7$ H admits a subgroup K of index 20 with $K'_{ab} \cong \mathbb{Z}^{13} \times C_2^5 \times C_3^2 \times C_4 \times C_{19}^6$ and for $k=8$ H admits a subgroup K of index 20 with $K'_{ab} \cong \mathbb{Z}^{12} \times C_2 \times C_5^3 \times C_{19}^{10}$
  - $(18,4)$, $k=3,4,5,7,8,9$, where for $k=8$ H admits a subgroup K of index 20 with $K'_{ab} \cong \mathbb{Z} \times C_2^2$
  - $(16,4)$, $3 \leq k \leq 8$
  - $(15,4)$, $k=3,4,6$
  - $(10,4)$, $k=3,4,5$
  - $(8,5)$, $k=3,4$
  - $(6,6)$, $k=3$
  - $(6,7)$, $k=3$

- $q=4$:
  - $(6,p)$, $8 \leq p \leq 11$, $k=3$
  - $(8,7)$, $k=3,4$
  - $(n,6)$, $n=6,8,10,11$, where if $n=6$ then $k=3$, if $n=8$ then $k=3,4$, if $n=10$ then $k=3,4,5$, and if $n=11$ then $k=3,4$
  - $(19,5)$, $3 \leq k \leq 8$, where for $k=6,8$ MAF found H infinite and for $k=5$ H admits a subgroup K of index 20 with $K'_{ab} \cong \mathbb{Z}^2 \times C_4^2 \times C_8 \times C_{19}^8$
  - $(18,5)$, $k=3,4,5,7,8,9$
  - $(16,5)$, $3 \leq k \leq 8$
  - $(15,5)$, $k=3,4,6$ where for $k=6$ MAF found H infinite
  - $(10,5)$, $k=3,4,5$.

B.2 Case iii., $p = 3, q \geq 12, k > 3$

In this case, we need only check $8 \leq n \leq 11$, so as in case ii. we begin by examining $n = 11$ and working down to $n = 8$. For each value of n, we note valid possibilities for q and for each of these, valid choices for k.

- $n = 11 : q = 12$ $(k = 3, 4, 5)$ and $q = 13$ $(k = 3, 4, 5)$, where for $(q, k) = (13, 5)$ MAF was used
- $n = 10 : q = 12$ $(k = 3, 4, 5)$ and $q = 14$ $(k = 3, 4, 5)$
- $n = 9$: the pairs $(q, k) = (q, 3)$ and $(q, 4)$ where $q = 12, 15, 16, 17$
- $n = 8$: the pairs $(q, k) = (q, 3)$ and $(q, 4)$ where $q = 12, 16, 18, 20, 21, 22, 23$, and where for $(21, 4)$ and $(23, 4)$ MAF was used.

B.3 Case iv., $p = 3$, $6 \leq q \leq 11$, $n \neq 2k + 1, k + 3$ and $k > 3$

In this case, as there will be in case v., there are many theoretical possibilities for $H(2, n, 3, q|k)$ to be finite (for example, if $q = 7$, then $n = 41$ implies that if finite $|H| = 1722$ and we must check $H(2, 41, 3, 7|k)$ where $4 \leq k \leq 19$). First we will list the triples (n, q, k) for which KBMAG was able to ascertain infiniteness, then we will sort the remainder by q.

- $(n, 11, 4)$ where $n = 11, 12, 13$
- $(n, 11, 5)$ where $n = 12, 13$
- $(10, 10, k)$ where $k = 4, 5$
- $(12, 10, k)$ where $k = 4, 5, 6$
- $(14, 10, k)$ where $4 \leq k \leq 7$

- $(12, 9, k)$ where $k = 4, 5, 6$
- $(15, 9, k)$ where $k = 4, 5, 6$
- $(16, 9, k)$ where $k = 4, 5, 7$
- $(17, 9, k)$ where $k = 4, 5, 6$
- $(12, 8, k)$ where $k = 4, 5, 6$
- $(16, 8, k)$ where $4 \leq k \leq 8$
- $(18, 8, k)$ where $4 \leq k \leq 9$
- $(20, 8, k)$ where $4 \leq k \leq 10$
- $(21, 8, k)$ where $k = 4, 5, 6$
- $(22, 8, k)$ where $4 \leq k \leq 11$
- $(23, 8, k)$ where $k = 4, 5, 6$
- $(n, 7, k)$ where $n = 21, 28, 35, 36, 39, 40, 41$ and $k = 4, 5, 6$
- $(n, 7, 7)$ where $n = 28, 35, 36, 39, 40, 41$
- $(28, 7, k)$ where $k = 9, 11, 13$
- $(36, 7, k)$ where $k = 9, 11, 13, 15$
- $(40, 7, k)$ where $k = 9, 11, 13, 15, 17$

Now we note the remaining cases, again by the triple (n, q, k), which are either determined by MAF, by certain GAP subgroup routines (e.g. containing a subgroup of index K where K does not divide $|H|$, or where the core of K in H has index greater than $|H|$) or by admitting epimorphisms onto groups of larger order than H (were H indeed finite).

- $q = 11$: $(12, 11, 6)$ admits a subgroup of index 24, which does not divide $|H| = 132$

- $q = 9 :$ $(16, 9, 6)$ admits a subgroup of index 21 which does not divide $|H| = 144$, $(16, 9, 8)$ admits a subgroup K of index 18 where $K'_{ab} \cong \mathbb{Z}^2 \times C_8^8$, $(17, 9, 7)$ admits a subgroup K of index 18 where $K_{ab} \cong \mathbb{Z} \times C_2$

- $q = 8 :$

  - $(21, 8, 7)$ admits a subgroup K of index 24 where $K'_{ab} \cong \mathbb{Z}^3 \times C_2^3 \times C_7^3$; $(21, 8, 8)$ admits a subgroup K of index 21 where $K_{ab} \cong \mathbb{Z} \times C_2^2$ and $(21, 8, 9)$ admits a subgroup K of index 24 where $K_{ab} \cong \mathbb{Z}^2 \times C_2^2$

  - $(23, 8, 7)$ admits a subgroup of index 25, which does not divide $|H| = 552$; $(23, 8, 8)$ admits a subgroup K of index 23 where the GAP command IsInfiniteAbelianizationGroup(K'') returns true; $(23, 8, 9)$ admits a subgroup K of index 23 where $K_{ab} \cong \mathbb{Z} \times C_2^2$ and $(23, 8, 10)$ admits a subgroup K of index 24 where $K'_{ab} \cong \mathbb{Z} \times C_2^4 \times C_4$

- $q = 7 :$

  - $(21, 7, 7)$ admits a subgroup of index 22, whilst $(21, 7, 8)$ and $(21, 7, 9)$ both admit subgroups of index 24, all of which contradict $|H| = 42$

  - $(28, 7, 8)$ and $(28, 7, 14)$ admit a subgroup of index 30 which contradicts $|H| = 84$, $(28, 7, 10)$ admits a subgroup K of index 28 with $K_{ab} \cong \mathbb{Z} \times C_2^2$ and $(28, 7, 12)$ admits a subgroup K of index 7 where the core of K in H has index 168

  - $(35, 7, 9)$ and $(35, 7, 12)$ both admit a subgroup K of index 28 where $K_{ab} \cong \mathbb{Z}^2$, $(35, 7, 10)$ and $(35, 7, 11)$ admit a subgroup of index 14 whose core in H has index 1092, $(35, 7, 16)$ admits a subgroup of index 24 which contradicts $|H| = 210$, $(35, 7, 13)$ has a subgroup K of index 35 where the core of K in H has index greater than 210, $(35, 7, 8)$ admits an epimorphism to $PSL(2, 139)$ of order $> 10^6$, whilst $(35, 7, 14)$ and $(35, 7, 15)$ admit epimorphisms to $PSL(2, 71)$ of order $> 10^5$

  - $(36, 7, 8)$ admits a subgroup K of index 28 where $K'_{ab} \cong \mathbb{Z}^2 \times C_2^8 \times C_3^8 \times C_4 \times C_9^2 \times C_{13} \times C_{73}$, $(36, 7, 10)$ admits a subgroup K of index 28 where $K'_{ab} \cong \mathbb{Z}^8 \times C_2^2 \times C_3^{11} \times C_5 \times C_9 \times C_{31} \times C_{103}$, $(36, 7, 12)$ admits a subgroup K of index 9 where $K_{ab} \cong \mathbb{Z} \times C_7$, $(36, 7, 14)$ admits a subgroup of index

9 whose core in H has index 504 and $(36, 7, 16)$ admits an epimorphism to $PSL(2, 71)$

– $(39, 7, 8)$ admits a subgroup of index 14 whose core in H has index 1092 when $|H| = 546$, $(39, 7, 9)$ admits an epimorphism to $H(2, 13, 3, 7|9)$ which in turn admits an epimorphism to $PSL(2, 181)$, $(39, 7, 10)$ admits a subgroup of index 28 which contradicts $|H| = 546$, and $(39, 7, k)$ for $11 \leq k \leq 18$ admits a subgroup of index 39 whose core in H has index greater than 546

– $(40, 7, k)$ for $k = 8, 12, 16$ admits an epimorphism to $PSL(2, 41)$ of order 34440, $(40, 7, 10)$ admits an epimorphism to $PSL(2, 239)$ of order $> 6.10^6$, $(40, 7, 14)$ admits an epimorphism to $H(2, 20, 3, 7|14) \cong H(2, 20, 3, 7|6)$ (Theorem 2.1) which is infinite by MAF, $(40, 7, 18)$ admits a subgroup of index 42 whose core in H has index greater than $|H| = 840$, $(40, 7, 19)$ admits a subgroup of index 22 which contradicts $|H| = 840$ and $(40, 7, 20)$ admits an epimorphism to $H(2, 8, 3, 7|20) \cong H(2, 8, 3, 7|4)$ (Theorem 2.1) which is infinite by MAF

– $(41, 7, k)$, $8 \leq k \leq 19$, all admit an epimorphism to $PSL(2, 83)$ which has order $> 2.10^5$

B.4 Case v., $q = 3$, $p \geq 6$ and $n \neq z$ where $z \in \{2k + 1, 3k, 3k \pm 1\, 4k, 5k\}$

Here we consider the groups $H(2, n, p, 3|k)$ with respect to the above constraints. In the same vein as for case iv., we will first list the triples (n, p, k) for which KBMAG was able to ascertain infiniteness, then we will sort the remainder by p in descending order.

- $(8, p, 4)$ where $p = 12, 16, 18, 20, 21, 22, 23$
- $(10, p, 4)$ where $p = 10, 12, 14$
- $(10, p, 5)$ where $p = 10, 12, 14$

- $(11, p, 3)$ where $p = 11, 12, 13$
- $(12, p, 5)$ where $8 \leq p \leq 11$
- $(12, p, 6)$ where $8 \leq p \leq 11$
- $(13, 11, 3)$
- $(14, 10, k)$ where $k = 3, 4, 7$
- $(16, 8, k)$ where $k = 3, 7, 8$
- $(16, 9, k)$ where $k = 3, 7, 8$
- $(17, 9, 3)$
- $(n, 8, 3)$ where $n = 18, 20, 21, 22, 23$
- $(18, 8, k)$ where $k = 5, 7, 9$
- $(20, 8, 10)$
- $(22, 8, 11)$

As before, continuing to label each group by the triple (n, p, k), we now begin with the remaining case for $p = 11$ and work our way down to $p = 7$ ($p = 6$ is covered in Theorem 2.15 itself).

- $p = 11$: $(13, 11, 5)$ admits a subgroup of index 13 whose core in H has index exceeding $|H| = 858$
- $p = 10$: $(14, 10, 6)$ admits a subgroup K of index 14 where $K'_{ab} \cong \mathbb{Z} \times C_3$
- $p = 9$:
  - $(15, 9, k)$; when $k = 4$ H admits a subgroup K of index 15 where $K'_{ab} \cong \mathbb{Z}^8 \times C_2^2 \times C_3^2$, and when $k = 6$ H admits a subgroup K of index 12 where $K'_{ab} \cong \mathbb{Z}^{274} \times C_5^8$
  - $(16, 9, 6)$; admits a subgroup K of index 18, where the abelianisation of the core of K in H is isomorphic to $\mathbb{Z}^{36} \times C_2^{102}$

- $(17,9,k)$; when $k=4,5$ H admits a subgroup of index 18 where $K_{ab} \cong \mathbb{Z}\times C_2$, and when $k=7$ H admits a subgroup K of index 17 where $K'_a b = \mathbb{Z}^2 \times C_2^{20} \times C_3^5 \times C_4^2 \times C_9^2$

- $p = 8$:
  - $(16,8,6)$; admits a subgroup K of index 8 where $K'_{ab} \cong \mathbb{Z} \times C_3$
  - $(18,8,k)$; when $k=4,8$ then H admits a subgroup of index 14 which contradicts $|H| = 72$
  - $(20,8,k)$; when $k=6,8$ then H admits a subgroup of index 14 which contradicts $|H| = 120$, and when $k=9$ MAF finds H infinite
  - $(21,8,k)$; when $k=4,5,6,8,9$ MAF finds H infinite
  - $(22,8,k)$; when $k=4$ H admits a subgroup K of index 11 where $K'_{ab} \cong \mathbb{Z}^4 \times C_2^9 \times C_4^2 \times C_7$, when $k=5,6,8,9$ MAF finds H infinite and when $k=10$ H admits a subgroup K of index 22 whose core in H has index greater than $|H| = 264$
  - $(23,8,k)$; when $k=4$ MAF finds H infinite, when $k=5,6,9,10$ H admits an epimorphism to $PSL(2,47)$ which has order $> 5.10^4$ and when $k=7$ H admits a subgroup K of index 24 whose core in H has index greater than $|H| = 552$

- $p = 7$:
  - $(21,7,k)$; when $3 \leq k \leq 6$ H admits an epimorphism to $PSL(2,41)$, when $k=8$ H admits an epimorphism to $PSL(2,83)$ and when $k=9$ H admits an epimorphism to $PSL(2,127)$, all of which have orders greater than $|H| = 42$
  - $(28,7,k)$; when $k=5$ H admits a subgroup of index 16 which contradicts $|H| = 84$, when $k=3,4,6,8,10,11$ H admits an epimorphism to $PSL(2,27)$, when $k=12$ H admits an epimorphism to $PSL(2,29)$ and when $k=13,14$ H admits an epimorphism to $PSL(2,167)$
  - $(35,7,k)$; when $k=8$ H admits a subgroup of index 14 whose core in H has index 1092, but $|H| = 210$; when $k=3,4,5,9,10,11,13,14,15,16$ H admits an epimorphism to $PSL(2,71)$ and when $k=6$ H admits an epimorphism to $PSL(2,281)$

- $(36, 7, k)$; when $k = 3, 5, 6, 7, 8, 10, 14, 15, 16$ H admits an epimorphism to $PSL(2, 71)$, when $k = 4$ it admits a subgroup of index 14 whose core in H has index 1092 when $|H| = 252$, when $k = 11$ it admits a subgroup of index 18 whose core in H has index 1008, when $k = 17$ it admits a subgroup of index 14 whose core in H has index 2184 and when $k = 13, 18$ it admits a subgroup of index 15, contradicting $|H| = 252$

- $(39, 7, k)$; when $k = 3, 4, 5, 8, 9, 10, 16, 17, 18$ H admits the epimorphism to $H(2, 13, 7, 3|k)$, (note $H(2, 13, 7, 3|16) \cong H(2, 13, 7, 3|3)$, $H(2, 13, 7, 3|17) \cong H(2, 13, 7, 3|4)$ and $H(2, 13, 7, 3|18) \cong H(2, 13, 7, 3|5)$) and we study these. Then $H(2, 13, 7, 3|10) \cong H(2, 13, 7, 3|3)$ (Lemma 2.1) which has an epimorphism to $PSL(2, 181)$; $H(2, 13, 7, 3|9) \cong H(2, 13, 7, 3|4)$ (Lemma 2.1) which is infinite by MAF; and lastly, $H(2, 13, 7, 3|8) \cong H(2, 13, 7, 3|5)$ (Lemma 2.1) which has an epimorphism to $PSL(2, 27)$. Of the remaining cases, when $k = 6$ MAF finds H infinite, when $k = 7$ $H \cong H(2, 39, 3, 7|28)$ (Lemma 1.1) $\cong H(2, 39, 3, 7|11)$ (Lemma 2.1) which was found infinite in case iv., when $k = 11$ $H \cong H(2, 39, 3, 7|32)$ (Lemma 1.1) $\cong H(2, 39, 3, 7|7)$ (Lemma 2.1) which was found infinite in case iv., when $k = 14$ $H \cong H(2, 39, 3, 7|14)$ (Lemma 1.1) which was found infinite in case iv., and when $k = 15$ H admits an epimorphism to $PSL(2, 547)$

- $(40, 7, k)$; when $k = 3, 5$ MAF finds H infinite, when $k = 4, 6, 7, 9, 11, 12, 14, 16, 18$ H has an epimorphism to $PSL(2, 41)$, when $k = 15, 20$ H has an epimorphism to $PSL(2, 239)$, when $k = 17$ $H \cong H(2, 40, 3, 7|33)$ (Lemma 1.1) $\cong H(2, 40, 3, 7|7)$ (Lemma 2.1) which is infinite from case iv. and when $k = 19$ $H \cong H(2, 40, 3, 7|19)$ (Lemma 1.1) which is infinite from case iv.

- $(41, 7, k)$; when $3 \leq k \leq 19$, $k \neq 7$ or 14, H has an epimorphism to $PSL(2, 83)$ and when $k = 7$ H has an epimorphism to $PSL(2, 549)$.

Appendix C

Sample computer code

In this Appendix we give some sample routines for using GAP, KBMAG (run in GAP itself) and MAF with an example in each case (along with the relevant output) that appears in the thesis. Commentary on each step will not generally be provided.

C.1 GAP

C.1.1 Low index subgroups

For this example, we look for low index subgroups of $H(2, 20, 6, 3|4)$ from Table 3.2 in Section 3.4. The input into GAP is as follows:

```
F:=FreeGroup("x", "y");
x:=F.1;; y:=F.2;;
K:=[x^2, y^20, (x*y)^6, (x*y^4)^3];
G:=F/K;
L:=LowIndexSubgroupsFpGroup(G, 10);
List(L,IndexInWholeGroup);
List(L,AbelianInvariants);
```

```
D3:=DerivedSubgroup(L[3]);
AbelianInvariants(D3);
```

This gives the corresponding output from GAP (re-spaced in order to fit the page):

```
gap> F:=FreeGroup("x", "y");
<free group on the generators [ x, y ]>
gap> x:=F.1;; y:=F.2;;
gap> K:=[x^2, y^20, (x*y)^6, (x*y^4)^3];
[ x^2, y^20, (x*y)^6, (x*y^4)^3 ]
gap> G:=F/K;
<fp group on the generators [ x, y ]>
gap> L:=LowIndexSubgroupsFpGroup(G, 10);
[ Group(<fp, no generators known>),
  Group(<fp, no generators known>),
  Group(<fp, no generators known>),
  Group(<fp, no generators known>),
  Group(<fp, no generators known>),
  Group(<fp, no generators known>),
  Group(<fp, no generators known>) ]
gap> List(L,IndexInWholeGroup);
[ 1, 2, 9, 6, 5, 10, 10 ]
gap> List(L,AbelianInvariants);
[ [ 2 ], [  ], [ 2, 2 ], [ 2, 2, 2 ], [ 2, 3 ], [ 2, 2, 2 ],
  [ 3, 3 ] ]
gap> D3:=DerivedSubgroup(L[3]);
Group(<fp, no generators known>)
gap> AbelianInvariants(D3);
[ 0 ]
```

from which we conclude H contains an infinite subgroup, hence H is infinite.

C.1.2 Code due to Paul; searching for mappings to $PSL(2, z)$, finding subgroups K from these

As mentioned in the thesis, both of the following major pieces of code are due to Paul [19]. In this section we examine $H(2, 45, 7, 3|9)$ from Table 3.3 in Section 3.4 using these. First, we search for mappings using the following routine:

```
for n in [2..2] do
         F := FreeGroup("x", "y");
                 x := F.1;; y:= F.2;;
                 rels := [ x^n,y^45,(x*y)^7,(x*y^9)^3,
];
                 G := F/rels;
                 x := G.1;; y:= G.2;;

                 Print("n = ", n, "\n");

                 for p in [2..300] do

                         if IsPrimePowerInt(p) = false then
                                continue;
                         fi;

                         Print("p = ", p, "\n");

                         A := PSL(2, p);
                         Eps := GQuotients(G, A:findall:=false);
                         if Size(Eps) > 0 then
                                Print(n, " in PSL(2, ", p, ")\n");
                         fi;
                 od;
         od;
```

```
        Print("\n");
```

Once run in GAP (we only show the results for $p < 50$ as necessary rather than through all primes < 300) we obtain the following output:

```
n = 2
p = 2
p = 3
p = 4
p = 5
p = 7
p = 8
p = 9
p = 11
p = 13
p = 16
p = 17
p = 19
p = 23
p = 25
p = 27
p = 29
2 in PSL(2, 29)
p = 31
p = 32
p = 37
p = 41
p = 43
p = 47
p = 49
```

which signifies that a mapping to $PSL(2,29)$ was indeed found. We now use this in the following block of code:

```
F := FreeGroup("x","y");
x:=F.1;;y:=F.2;;
rels:=[ x^2, y^45, (x*y)^7, (x*y^9)^3
];
G:=F/rels;
x:=G.1;;y:=G.2;;
q:=GQuotients(G,PSL(2,29):findall:=false);;
q:=q[1];
p:=Image(q);;
s:=Stabilizer(p,1);;
v:=PreImage(q,s);;
Index(G,v);
AbelianInvariants(v);
```

If we now run this in GAP, we obtain the following (with one extra step to find the derived subgroup of v given v itself does not have infinite abelianisation, and again re-spacing the representation q to fit the page):

```
gap> F := FreeGroup("x","y");
<free group on the generators [ x, y ]>
gap> x:=F.1;;y:=F.2;;
gap> rels:=[ x^2, y^45, (x*y)^7, (x*y^9)^3
> ];
[ x^2, y^45, (x*y)^7, (x*y^9)^3 ]
gap> G:=F/rels;
<fp group on the generators [ x, y ]>
gap> x:=G.1;;y:=G.2;;
gap> q:=GQuotients(G,PSL(2,29):findall:=false);;
gap> q:=q[1];
```

```
[ x, y ] -> [ (2,11)(3,23)(4,29)(5,27)(6,7)(8,20)(9,14)(12,25)
              (13,30)(15,24)(16,26)(17,19)(18,22)(21,28),
  (1,20,8,28,21,10,5,17,29,24,13,6,26,14,2)
  (3,27,25,30,12,11,18,15,19,16,23,22,4,9,7) ]
gap> p:=Image(q);;
gap> s:=Stabilizer(p,1);;
gap> v:=PreImage(q,s);;
gap> Index(G,v);
30
gap> AbelianInvariants(v);
[ 2, 7 ]
gap> DV:=DerivedSubgroup(v);
Group(<fp, no generators known>)
gap> AbelianInvariants(DV);
[ 0, 0, 0, 0, 0, 0, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 29 ]
```

C.1.3 KBMAG

Here we test the group $H(2,15,10,5|5)$ from Section 3.3. Firstly, assuming the KBMAG package is already preloaded, we set up the presentation and the rewriting system KBMAG will use, then amend the options (since the default settings are usually a little low). Then, AutomaticStructure(R) commands KBMAG to begin constructing automata.

```
F:=FreeGroup("x", "y");
x:=F.1;; y:=F.2;;
K:=[x^2, y^15, (x*y)^10, (x*y^5)^5];
G:=F/K;
R:=KBMAGRewritingSystem(G);
O:=OptionsRecordOfKBMAGRewritingSystem(R);
O.maxwdiffs:=20000;
O.maxeqns:=1000000;
```

```
SetInfoLevel(InfoRWS, 2);
AutomaticStructure(R);
```

This particular computation takes about 18 seconds to find a confluent rewriting system and 30 seconds to perform all checks. Once complete, Size(R) asks for the order of the rewriting system, which in this case returns infinity.

C.2 MAF

Assuming it is installed correctly, the standard file read by MAF takes the following form (where here we illustrate $H(2, 11, 3, 13|5)$ from Theorem 2.15, noted in Appendix B):

```
_RWS := rec
(
  isRWS := true,
  ordering := "shortlex",
  generatorOrder := [a,A,b,B],
  inverses := [A,a,B,b],
  equations :=
  [
    [a^2,IdWord],
    [b^11,IdWord],
    [(a*b)^3,IdWord],
    [(a*b^5)^13,IdWord]
  ]
);
```

Running this using MAF's *automata* command, it finds in 518 seconds that the automaton has an accepted language which is infinite.

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