Pairwise mergers in bipartite matching games with an application in collaborative logistics

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Abstract

Merging among players in a cooperative game can alter the structure of the core. This paper shows that in bipartite matching games, if pairs of players from different sides merge, the structure of the core remains unchanged. This allows us to extend the well-known result regarding the characterization of the core with dual solutions for simple games to their associated pairwise merger games. We introduce the class of vehicle scheduling games as an area of application for our result.

1. Introduction

Matching problems are among the most important problems in operations research and cooperative matching games are well studied in the literature. The players in a matching game are commonly considered to be the nodes of the underlying graph. A particularly interesting observation in matching games over bipartite graphs is that all stable allocations in the core—i.e. allocations that distribute the total value among the players such that no subgroup of players have incentives to break apart from the grand coalition—can be obtained from the solutions to some linear program [14]. This phenomenon is also observable in few other classes of cooperative games, e.g. flow games on simple networks [6], and lane covering games with simple shippers [4].

A merger among a group of players refers to the consolidation of the group into a single, usually more powerful, player. In general, when players merge the structure of the core changes as well. Lehrer [9] is among the first papers to study mergers among pairs of players. Knudsen and Østerdal [7] present some possibility and impossibility results on merging and splitting of players in cooperative games. Although in certain situations it is possible to devise allocations that make merging among players non-profitable (e.g. Moulin [11] and Gómez-Rúa and Vidal-Puga [3]), to the best of the author's knowledge there is no prior study on games wherein merging among the players preserves the structure of their cores.

In this paper we show that the structure of the cores of matching games over bipartite graphs are not affected by pairwise merging, i.e. collusion among pairs of players from different partitions of the graph. As the result, every allocation in the core of a merger game can be obtained from an allocation in the core of the associated game with node players. This

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allows us to extend the characterization result of Shapley and Shubik [14] to the pairwise merger bipartite matching games.

Our result has two areas of application. The first application is in situations where pairwise merging can occur naturally by collusions between individual players, e.g. when couples participate as teams in the marriage model. The second application pertains to situations where the underlying optimization problem can be modelled as a matching problem with each player corresponding to two nodes on each partition of a bipartite graph. We introduce the class of vehicle scheduling games as an example of the later scenario and conclude that the cores of these games are non-empty and can be characterized by the solutions to some linear programs. In this way, this paper contributes to the growing literature on game theoretical study of collaborative logistics (see for example Hezarkhani et al. [5], Özener and Ergun [12], Frisk et al. [2], Lozano et al. [10], Krajewska et al. [8], etc.)

The remainder of this paper is organized as follows. Section 2 outlines the relevant game theoretical concepts. Section 3 presents the matching games on bipartite graphs. Section 4 defines the pairwise merger bipartite matching games and gives our main result. Finally, Section 5 introduces the class of vehicle scheduling games and discusses the application of our result in these games.

2. Cooperative Games, Merger Games, and Their Cores

A cooperative game is a pair (V, z) with a set of players V and a characteristic function $z : 2^N \to \mathbb{R}$ that assigns to every coalition $S \subseteq V$ the value $z(S)$. We call (V, z) a simple game since it does not have any merged players. An allocation for the game (V, z) is a vector $\varphi = (\varphi_i)_{i \in V}$. The core of (V, z) is the set of all allocations φ such that $\sum_{i \in V} \varphi_i = z(V)$ and $\sum_{i \in S} \varphi_i \geq z(S)$ for all coalitions $S \subset V$.

A player $i \in V$ is called a null player if $z(S \cup \{i\}) = z(S)$ for every $S \subseteq V$. In every allocation φ in the core of (V, z) it holds that $\varphi_i = 0$ whenever i is a null player [13]. Therefore, adding or removing null players to a cooperative game does not alter the structure of the core.

A merger of players $T \subset V$, denoted by $\bar{T}$, represents a structural change in the number of players participating in the game so that the new player set $\bar{V} = V \setminus T \cup \{\bar{T}\}$ contains all individual players in T as a single new player. In general, $\bar{V}$ can contain several merged players. In the merged game $(\bar{V}, \bar{z})$ associated with the simple game (V, z), for every $S \subset \bar{V}$ such that S contains a single merged player $\bar{T} \in S$ we have $\bar{z}(S) = z(S \setminus \{\bar{T}\} \cup T)$. The latter can be easily extended to coalitions with multiple merged players. Each allocation in the core of the simple game can be used to construct an allocation in the core of the associated merger game.

Lemma 1. *Given (V, z), let φ be an allocation in its core. Let $(\bar{V}, \bar{z})$ be a merger game associated with (V, z). Define $\varphi' = (\varphi'_i)_{i \in \bar{V}}$ such that for every $i \in \bar{V} \cap V$ we have $\varphi'_i = \varphi_i$ and for every merged player $\bar{T} \in \bar{V}$ we have $\varphi'_{\bar{T}} = \sum_{i \in T} \varphi_i$. The allocation φ' is in the core of $(\bar{V}, \bar{z})$.*

Proof. By definition we have $\sum_{i \in \bar{V}} \varphi'_i = \sum_{i \in V} \varphi_i = z(V) = \bar{z}(\bar{V})$. For every coalition of individual players $S \subset \bar{V} \cap V$ we have $\sum_{i \in S} \varphi'_i = \sum_{i \in S} \varphi_i \geq z(S) = \bar{z}(S)$. For a subset of players $S \subset \bar{V}$

containing a single merged player $\bar{T}$ we also have $\sum_{i\in S}\varphi_i' = \sum_{i\in S\setminus\{\bar{T}\}\cup T}\varphi_i \geq z(S\setminus\{\bar{T}\}\cup T) = \bar{z}(S)$. The latter argument can be extended over coalitions with more than one merged player. Therefore, if φ is an allocation in the core of (V,z), then φ' is an allocation in the core of $(\bar{V},\bar{z})$. □

However, the reverse of the latter observation does not hold necessarily. That is, a merger game can contain allocations in its core which does not correspond to any allocation in the core of its associated simple game. An example is given below.

Example 1. *Consider the game (V,z) with $V=\{1,2,3,4\}$ and the characteristics function z such that $z(S)=0$ if $|S|=1$, $z(S)=4$ if $|S|=2$, $z(S)=10$ if $|S|=3$, and $z(V)=12$. Observe that the core of (V,z) is empty. Now consider the merged player set $\bar{V}=\{\bar{21},\bar{34}\}$ and the merger game $(\bar{V},\bar{z})$. By definition we have $\bar{z}(\{\bar{21}\})=\bar{z}(\{\bar{43}\})=4$ and $\bar{z}(\{\bar{21},\bar{43}\})=12$. The core of $(\bar{V},\bar{z})$ includes any allocation of the form $\varphi_{\bar{21}}=4+\epsilon$ and $\varphi_{\bar{43}}=8-\epsilon$ with $0\leq\epsilon\leq 4$.*

3. Bipartite Matching Games

Let $G=(V,E,w)$ be a weighted undirected graph with node set V, edge set $E\subseteq V\times V$, and weight function $w: E\to\mathbb{R}^+$. In the simple matching game defined on G, the node set V corresponds to a set of $|V|$ node players. We denote an edge with end points $k,h\in V$ as $kh\in E$. The graph G is bipartite if it can be partitioned into two disjoint sets V^A and V^B such that no edge in E has its both ends in either V^A or V^B. The graph G is called a balanced bipartite graph if $|A|=|B|$. Moreover, G is a complete bipartite graph if there is an edge from each node on one partition to every node in the other partition. We assume hereafter that G is a bipartite, balanced, and complete graph. The last two assumptions are without loss of generality for our purpose. An incomplete graph can be completed by adding edges with weight zero. Also, an unbalanced graph can be turned into a balanced one by adding a set of nodes to either V^A and V^B along with edges with zero weights from these nodes to all the nodes in the other partition of the graph. These additional nodes correspond to null players which, as stated in the previous section, do not alter the structure of the core.

For coalition $S\subseteq V$, let $G_S=(S,E_S,w)$ be the subgraph of S where $E_S=\{kh\in E|k,h\in S\}$. Although G_S is complete, it is not necessarily balanced. The cooperative matching game defined on G is (V,z) where the characteristics function z assigns to each coalition $S\subseteq V$ the value of maximum weighted matching on G_S that can be obtained via the following linear program:

$$z(S) = \max \sum_{kh\in E_S} w_{kh}x_{kh} \tag{1}$$

$$s.t. \quad \sum_{h\in V^B: kh\in E_S} x_{kh} \leq 1 \qquad \forall k\in V^A: kh\in E_S \tag{2}$$

$$\sum_{k\in V^A: kh\in E_S} x_{kh} \leq 1 \qquad \forall h\in V^B: kh\in E_S \tag{3}$$

$$x_{kh}\in\{0,1\} \qquad \forall kh\in E_S \tag{4}$$

The constraint matrix in the above program is unimodular thus the integrality constraints can be relaxed without affecting the optimal solution. The dual to the relaxed program for $S \subseteq V$ is:

$$u(S) = \min \sum_{k \in V^A, h \in V^B : kh \in E_S} \lambda_k + \gamma_h \tag{5}$$

$$s.t. \quad \lambda_k + \gamma_h \geq w_{kh} \qquad \forall k \in V^A, h \in V^B : kh \in E_S \tag{6}$$

$$\lambda_k, \gamma_h \geq 0 \qquad \forall k \in V^A, h \in V^B : kh \in E_S \tag{7}$$

Let (λ^*, γ^*) with $\lambda^* = (\lambda_k^*)_{k \in V^A}$ and $\gamma^* = (\gamma_h^*)_{h \in V^B}$ be an optimal solution to $u(V)$ and let $\mathbf{\Omega}$ be the set of all such optimal solutions. The following result, which we provide without proof, is well-known in operations research literature.

Theorem 1 (Shapley and Shubik [14]). *The allocation φ is in the core of (V, z) if and only if $\varphi_k = \lambda_k^*$ for every $k \in V^A$ and $\varphi_h = \lambda_h^*$ for every $h \in V^B$ where $(\lambda^*, \gamma^*) \in \mathbf{\Omega}$. Hence, the set of dual solutions $\mathbf{\Omega}$ completely characterizes the core of (V, z).*

Therefore, not only dual solutions provide allocations in the core of bipartite matching games, they completely characterize the core of these games.

4. Pairwise Merger Bipartite Matching Games

In this section we analyze the games obtained from pairwise mergers of node players in bipartite matching games. We call a merger between two players pairwise if they belong to different partitions of the bipartite graph. We investigate the cores of the pairwise merger games on bipartite graphs and their relationship to the cores of corresponding simple games. To facilitate the analysis, we focus on merger games with the maximum number of pairwise merged players. In light of Lemma 1, our result would also hold for merger games with any possible number of pairwise merged players.

Let $G = (V, E, w)$ be the underlying graph. For node players $k \in V^A$ and $h \in V^B$ define the pairwise merged player $\bar{hk}$. Assume all players in V^A and V^B have formed arbitrary pairwise mergers and let $\bar{V}$ be a set of such pairwise merged players. Thus, $\bar{V}$ is a partitioning of V into pairs. The pairwise merger bipartite matching game associated with G and the player set $\bar{V}$ is $(\bar{V}, \bar{z})$ such that for every $T \subseteq \bar{V}$ we have $\bar{z}(T) = z(S_T)$ where $S_T = \{k, h \in V | \bar{hk} \in T\}$ is the set of node players corresponding to T.

In order to investigate the cores of pairwise merger games, we use an alternative graph associated with G as well as the player set $\bar{V}$, and define a characteristics function in terms of a maximum circulation problem. We construct the directed graph $\tilde{G}^{\bar{V}} = (V, \tilde{E}, \tilde{w})$ in the following manner. For each $kh \in E$ with $k \in V^A$ and $h \in V^B$ a directed edge kh from k to h exists in $\tilde{E}$ and $\tilde{w}_{kh} = w_{kh}$. For each merged player $\bar{hk} \in \bar{V}$ with $k \in V^A$ and $h \in V^B$ a directed edge hk from h to k exists in $\tilde{E}$ and $\tilde{w}_{hk} = 0$ (an example is depicted in Figure 1). For coalition $T \subseteq \bar{V}$, let $\tilde{G}_T^{\bar{V}} = (S_T, \tilde{E}_T, \tilde{w})$ be the subgraph of T with $\tilde{E}_T = \{kh \in \tilde{E} | k, h \in S_T\}$.

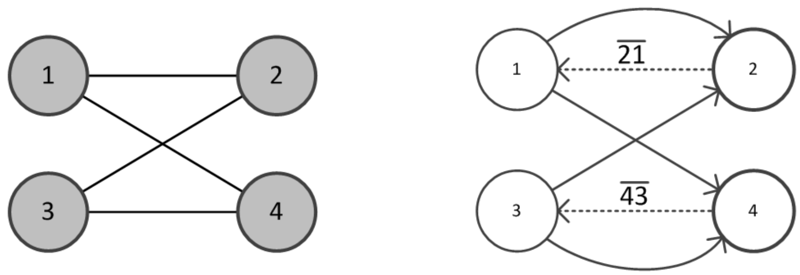

Figure 1: An example of a matching graph (left), and its associated circulation graph (right)

For $T \subseteq \bar{V}$ define the following program:

$$\tilde{z}(T) = \max \sum_{kh \in \tilde{E}_T} \tilde{w}_{kh} y_{kh} \tag{8}$$

$$s.t. \quad \sum_{h \in V^B : kh \in \tilde{E}_T} y_{kh} - \sum_{h \in V^B : hk \in \tilde{E}_T} y_{hk} = 0 \qquad \forall k \in V^A : kh \in \tilde{E}_T \tag{9}$$

$$\sum_{k \in V^A : hk \in \tilde{E}_T} y_{hk} - \sum_{k \in V^A : kh \in \tilde{E}_T} y_{kh} = 0 \qquad \forall h \in V^B : kh \in \tilde{E}_T \tag{10}$$

$$y_{hk} \leq 1 \qquad \forall k \in V^A, h \in V^B : \bar{hk} \in T \tag{11}$$

$$y_{kh} \in \{0, 1\} \qquad \forall kh \in \tilde{E}_T \tag{12}$$

The above program is a maximum circulation problem with every merged player in T corresponding to an edge. To obtain our result, we study the core of the cooperative game $(\bar{V}, \tilde{z})$ as formulated above. Later in this section (Lemma 4) we show that $(\bar{V}, \tilde{z})$ is identical to the pairwise merger game $(\bar{V}, \bar{z})$.

The constraint matrix in the program (8)–(12) is unimodular so the integrality constraints can be relaxed. The dual of the relaxed program for $T \subseteq \bar{V}$ is:

$$\tilde{u}(T) = \min \sum_{\bar{hk} \in T} I_{\bar{hk}} \tag{13}$$

$$s.t. \quad \mu_k - \eta_h \geq \tilde{w}_{kh} \qquad \forall k \in V^A, h \in V^B : kh \in \tilde{E}_T, \bar{hk} \notin T \tag{14}$$

$$I_{\bar{hk}} + \mu_k - \eta_h \geq 0 \qquad \forall k \in V^A, h \in V^B : \bar{hk} \in T \tag{15}$$

$$I_{\bar{hk}} \geq 0 \qquad \forall k \in V^A, h \in V^B : \bar{hk} \in T \tag{16}$$

Although the dual problem contains three groups of variables (I, μ, and η), only I variables are represented in the objective function. Let $I^* = (I^*_{\bar{hk}})_{\bar{hk} \in \bar{V}}$ be an optimal partial solution to the dual $\tilde{u}(\bar{V})$. As we show in the next lemma, the set of optimal partial solutions characterize the core of $(\bar{V}, \tilde{z})$.

Lemma 2. *Allocation $(\varphi_{\bar{hk}})_{\bar{hk} \in \bar{V}}$ is in the core of $(\bar{V}, \tilde{z})$ if and only if $\varphi_{\bar{hk}} = I^*_{\bar{hk}}$ for every $\bar{hk} \in \bar{V}$.*

Proof. [If part]. Let $(I^* = (I^*_{\bar{hk}})_{\bar{hk} \in \bar{V}}, \mu^* = (\mu^*_k)_{k \in V^A}, \eta^* = (\eta^*_h)_{h \in V^B})$ be an optimal solution to $\tilde{u}(\bar{V})$. Note that by strong duality theorem we have $\sum_{\bar{hk}\in\bar{V}} I^*_{\bar{hk}} = \tilde{z}(\bar{V})$. Let $T \subset \bar{V}$ and consider $\tilde{u}(T)$. Observe that $(I^*_T = (I^*_{\bar{hk}})_{\bar{hk}\in T}, \mu^* = (\mu^*_k)_{k\in V^A}, \eta^* = (\eta^*_h)_{h\in V^B})$ is a feasible solution to $\tilde{u}(T)$. Subsequently, by weak duality theorem we have $\tilde{z}(T) \leq \sum_{\bar{hk}\in T} I^*_{\bar{hk}}$. We conclude that $(\varphi_{\bar{hk}} = I^*_{\bar{hk}})_{\bar{hk}\in\bar{V}}$ is an allocation in the core of $(\bar{V}, \tilde{z})$.

[Only-if part]. We draw upon the following technical observation. A feasible solution for the program (8)–(12) corresponds to a set of cycles on $\tilde{G}^{\bar{V}}_T$. The total cost of a cycle o, denoted by r_o, is the sum of weights $\tilde{w}_{kh}$ for the edges circulated in o. Let $S_o \subseteq \bar{V}$ be the set of merged players that are circulated in cycle o. Note that $r_o \leq \tilde{z}(S_o)$ since players in S_o can collectively use the cycle o for circulating their edges.

Let $\varphi = (\varphi_{\bar{hk}})_{\bar{hk}\in\bar{V}}$ be an allocation in the core of $(\bar{V}, \tilde{z})$. Consider the following circulation program:

$$\begin{aligned}
\max \sum_{kh\in\tilde{E}} \tilde{w}_{kh} y_{kh} - \sum_{hk\in\tilde{E}:\bar{hk}\in\bar{V}} \varphi_{\bar{hk}} y_{hk} & & \\
s.t. \sum_{h\in V^B:kh\in\tilde{E}} y_{kh} - \sum_{h\in V^B:hk\in\tilde{E}} y_{hk} = 0 & & \forall k \in V^A : kh \in \tilde{E} \\
\sum_{k\in V^A:hk\in\tilde{E}} y_{hk} - \sum_{k\in V^A:kh\in\tilde{E}} y_{kh} = 0 & & \forall h \in V^B : kh \in \tilde{E} \\
y_{kh} \geq 0 & & \forall kh \in \tilde{E}
\end{aligned}$$

A feasible solution to this program also corresponds to a set of cycles on the graph $\hat{G}^{\bar{V}} = (V, \tilde{E}, \hat{w})$ where $\hat{w}_{kh} = \tilde{w}_{kh}$ for every $k \in V^A$ and $h \in V^B$, and $\hat{w}_{hk} = \tilde{w}_{hk} - \varphi_{\bar{hk}}$ for every $k \in V^A$ and $h \in V^B$ such that $\bar{hk} \in \bar{V}$. The total cost of an arbitrary cycle o on this graph would be $r_o - \sum_{\bar{hk}\in S_o} \varphi_{\bar{hk}}$. We already know that $r_o \leq \tilde{z}(S_o)$. Furthermore, since φ is an allocation in the core, we have $\sum_{\bar{hk}\in\bar{V}} \varphi_{\bar{hk}} \geq \tilde{z}(S_o)$, hence $r_o \leq \sum_{\bar{hk}\in\bar{V}} \varphi_{\bar{hk}}$ which means that the total cost of an arbitrary cycle on this graph is non-positive and consequently the program is bounded so the dual program is feasible. The dual to the above program is (with some rearrangements):

$$\begin{aligned}
\mu_k - \eta_h \geq \tilde{w}_{kh} & \qquad \forall k \in V^A, h \in V^B : kh \in \tilde{E}, \bar{hk} \notin \bar{V} \\
\varphi_{\bar{hk}} + \mu_k - \eta_h \geq 0 & \qquad \forall k \in V^A, h \in V^B : \bar{hk} \in \bar{V}
\end{aligned}$$

By assumption we have $\varphi_{\bar{hk}} \geq 0$ for every $kh : \bar{hk} \in \bar{V}$ and $\sum_{\bar{hk}\in\bar{V}} \varphi_{\bar{hk}} = \tilde{u}(\bar{V})$ which, in conjunction with the above constraints, imply that $(\varphi_{\bar{hk}})_{\bar{hk}\in\bar{V}}$ is a solution to $\tilde{u}(\bar{V})$. □

Özener and Ergun [12] and Hezarkhani et al. [4] prove statements closely related to the one above. In the next step, we make the connection between the circulation games with pairwise merged players and their associated simple games.

Lemma 3. *For every optimal solution to the dual of simple game $(\lambda^*, \gamma^*) \in \mathbf{\Omega}$, let $I^* = (I^*_{\bar{hk}} = \lambda^*_k + \lambda^*_h)_{\bar{hk}\in\bar{V}}$ and let $\mathbf{I}^*$ be the set of all such I^*. $\mathbf{I}^*$ coincides with the set of all optimal partial solutions to $\tilde{u}(\bar{V})$.*

Proof. Consider the dual for T, $\tilde{u}(T)$, and an optimal partial solution I^*. Note that at optimality, for every $\bar{hk} \in T$ such that $I^*_{\bar{hk}} \neq 0$ we have $I^*_{\bar{hk}} = \eta_h - \mu_k$. Thus the program

(13)–(16) can be written as:

$$\tilde{u}(T) = \min \sum_{k \in V^A, h \in V^B : \bar{kh} \in T} \eta_h - \mu_k$$

$$s.t. \quad \mu_k - \eta_h \geq \tilde{w}_{kh} \qquad \forall k \in V^A, h \in V^B : kh \in \tilde{E}_T, \bar{hk} \notin T$$

$$\eta_h - \mu_k \geq 0 \qquad \forall k \in V^A, h \in V^B : \bar{hk} \in T$$

We change the variables $\mu_k = \lambda_k$ for all $k \in V^A$ and $\eta_h = -\gamma_h$ for all $h \in V^B$. For every $kh \in \tilde{E}_T$ such that $\bar{hk} \notin T$ it holds that $\tilde{w}_{kh} = w_{kh}$. Since all weights are non-negative, the above program can be rewritten as:

$$\tilde{u}(T) = \min \sum_{k \in V^A, h \in V^B : kh \in E_{S_T}} \lambda_k + \gamma_h \tag{17}$$

$$s.t. \quad \lambda_k + \gamma_h \geq w_{kh} \qquad \forall k \in V^A, h \in V^B : kh \in E_{S_T} \tag{18}$$

In comparison with the dual of simple game $u(V)$, it can be observed that the above program is less constrained since $u(V)$ enforces non-negative variables. We continue in three steps.

[Step (i)]. In the program (17)–(18), at optimality there exists at least one binding constraint involving each variable. To see this, consider λ_k (resp. γ_h) and assume that there is no binding constraint involving λ_k (resp. γ_h). Reduce λ_k (resp. γ_h) as far as a constraint becomes binding. In this manner we devised an alternative feasible solution with lower objective function value which contradicts the optimality condition. Therefore, at optimality every variable is part of a binding constraint.

[Step (ii)]. In the program (17)–(18), at optimality only variables in either V^A or V^B can hold negative values. In order to verify this, note that if there exists $\lambda_k < 0$ for some $k \in V^A$ and $\gamma_h < 0$ for some $h \in V^B$, then for edge kh we have $\lambda_k + \gamma_h < 0$ which violates the feasibility condition. Therefore, variables from both partitions cannot take negative values simultaneously in an optimal solution.

[Step (iii)]. Let (λ, γ) be an optimal solution to (17)–(18) that includes negative variables, without loss of generality, in V^A. Let λ_j be the smallest among all λ_k. By the observation in (ii) we know that for every $h \in V^B$ it holds that $\lambda_j + \gamma_h \geq 0$. Define the alternative solution (λ', γ') by setting $\lambda'_k = \lambda_k - \lambda_j$ for every $k \in V^A$ and $\gamma'_h = \lambda_h + \lambda_j$ for every $h \in V^B$. Clearly in the new solution all variables are non-negative so (λ', γ') is a solution to both $\tilde{u}(T)$ and $u(S_T)$. But with this alternative solution for every $\bar{hk} \in T$ it holds that $I_{\bar{hk}} = \lambda_k + \gamma_h = \lambda'_k + \gamma'_h$ so the values of I variables are the same with (λ, γ) and (λ', γ'). We conclude that for every optimal solution to $\tilde{u}(T)$, even if it contains negative variables, there exists an optimal solution for $u(S_T)$ which results in the same values of $I_{\bar{hk}}$ for every $\bar{hk} \in T$. Therefore the set of all optimal partial solutions $\mathbf{I}^*$ is characterized by the set of dual solutions to $u(V)$, i.e. $\mathbf{\Omega}$. □

The equivalence of two games $(\bar{V}, \tilde{z})$ and $(\bar{V}, \bar{z})$ is formally shown in the next lemma.

Lemma 4. *For every $T \subseteq \bar{V}$ we have $\tilde{z}(T) = \bar{z}(T)$.*

Proof. By proof of Lemma 2, for every $T \subseteq \bar{V}$ we have $\tilde{u}(T) = u(S_T)$. By strong duality theorem we conclude that $\tilde{z}(T) = z(S_T) = \bar{z}(T)$ for every $T \subseteq \bar{V}$ which completes the proof. □

An immediate outcome of the last proof is that the objective functions of the matching and circulation problems are equal for corresponding players. We are ready to present the main result of this paper.

Theorem 2. *The set of solutions to $u(V)$ characterizes the core of merger game $(\bar{V}, \bar{z})$.*

Proof. By Lemma 2, the set of optimal partial solutions $\mathbf{I}^*$ to $\tilde{u}(\bar{V})$ characterizes the core of $(\bar{V}, \tilde{z})$. Lemma 4 shows that the games $(\bar{V}, \tilde{z})$ and $(\bar{V}, \bar{z})$ are identical. On the other hand, Lemma 3 states that the set of all optimal solutions to $u(V)$, i.e. $\mathbf{\Omega}$, provides a complete characterization of optimal partial solutions $\mathbf{I}^*$. The claim follows immediately. □

As a final note, it is worth mentioning that although merging could happen among players within one partition of G, the cores of such cooperative games do not necessarily correspond to the cores of their associated simple games. Below we provide an example regarding non-pairwise mergers.

Example 2. *Consider a graph with $V = \{1, 2, 3\}$, $V^A = \{1, 3\}$, $V^B = \{2\}$, $w_{12} = 1$, and $w_{32} = 2$. In the simple game, every allocation in the core must satisfy $\varphi_1 + \varphi_2 + \varphi_3 = 2$, $\varphi_1 + \varphi_2 \geq 1$, $\varphi_2 + \varphi_3 \geq 2$, and $\varphi_1 + \varphi_3 \geq 0$. Therefore, φ is in the core if $\varphi_1 + \varphi_3 \leq 1$. In the non-pairwise merger game obtained by merging among players 1 and 3, an allocation in the core must satisfy $\varphi'_{\bar{31}} + \varphi'_2 = 2$, $\varphi'_{\bar{31}} \geq 0$ and $\varphi'_2 \geq 0$. Thus the allocation $(\varphi'_{\bar{31}} = 2, \varphi'_2 = 0)$ is in the core of the merger game. However, there is no allocation in the core of the simple game corresponding to the latter allocation.*

5. Application in Vehicle Scheduling Games

In this section we introduce the class of vehicle scheduling games as an area of application for the theory developed in the first part of the paper. Consider a set of players each requiring a delivery at a certain time. The players must organize their deliveries by using the services of a logistics provider. The delivery costs include fixed and variable components such as truck utilization fee and direct travel cost. By collaboration players can reduce their associated logistics costs. Figure 2 illustrates an example of these scenarios. The players need to devise fair allocations to distribute the cost among themselves. Carraresi and Gallo [1] introduces the centralized version of this problem. We provide a brief description of the underlying optimization model before presenting the game.

Let D be a set of nodes corresponding to the players' delivery locations and let o be the location of the depot. Each player/delivery $k \in D$ must be fulfilled at time $t_k > 0$. The function $\theta : D \cup \{o\} \times D \cup \{o\} \to \mathbb{R}^{++}$ gives the travelling times between all pair of locations. Without loss of generality, we assume that the travelling cost is proportional to the travelling time. An ordered pair of delivery requirements kh, $k, h \in D$, are compatible if $t_k + \theta_{kh} \leq t_h$ where θ_{kh} is the time to travel from k to h. In particular $k, h \in D$ are not

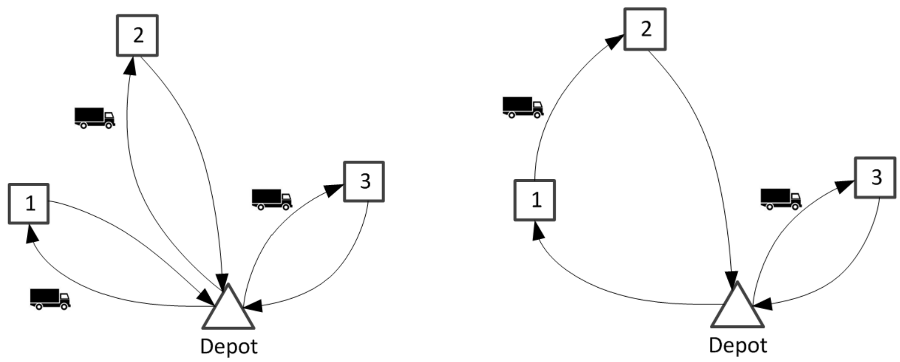

Figure 2: Non-cooperative vehicle scheduling (left), and cooperative vehicle scheduling (right)

compatible if $h = k$. The cost of a trip includes a vehicle utilization cost K, and direct cost of trip obtained from function θ. For obtaining the minimum total travel costs Carraresi and Gallo [1] construct a bipartite graph with each delivery (player) corresponding to two nodes—one node on each partition of the graph. Define $G^c = (V, E, c)$ with $V = D \cup D'$ and let $E = E^1 \cup E^2$ where $E^1 = \{kh | k \in D, h \in D',$ and kh is compatible$\}$, and $E^2 = \{kh | k \in D, h \in D',$ and kh is not compatible$\}$. The weight function c is defined as $c_{kh} = \theta_{kh}$ if $kh \in E_1$, and $c_{kh} = K + \theta_{ok} + \theta_{ho}$ if $kh \in E_2$. Thus, the cost of an edge between two compatible deliveries is the dead-heading cost of travelling between them, and the cost of an edge between two incompatible deliveries is the cost of travelling from the depot to the first one and the return trip to the depot from the other. The minimum cost of delivery trips for coalition $T \subseteq D$ corresponds to the value of minimum perfect matching on G^c_T. Let $M > \max_{kh \in E} c_{kh}$. By setting $w_{kh} = M - c_{kh}$ for every $kh \in E$, the minimum perfect matching problem can be turned into a matching problem on the graph $G = (V, E, w)$ as stated in (1)–(4).

The cooperative vehicle scheduling game can be stated as $(D, \dot{z})$ where for every $T \subseteq D$ we have $\dot{z}(T) = z(T \cup T') - |T|M$. Using our result, we conclude that the core of $(D, \dot{z})$ is non-empty and is characterized by allocations of the form $\varphi = (\varphi_{k'k} = \lambda^*_k + \gamma^*_{k'} - M)_{k \in D, k' \in D'}$ with (λ^*, γ^*) being a solution to the dual of matching game on G.

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