EVALUATING THE IMPACT OF COMPUTERS ON THE LEARNING AND TEACHING OF CALCULUS

by

Behiye Ubuz, B.Sc., M.Sc.

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ABSTRACT

Calculus has held its importance in the core undergraduate mathematics curriculum. The introduction of low cost microcomputers about 1979 has led some people to reconsider the content of the calculus curriculum. This investigation was about how microcomputer technology has been integrated into the teaching and learning of calculus to improve students' understanding. The study mainly focused on whether and how computers could influence students' learning of calculus concepts.

The rationale for using microcomputer technology to enhance calculus learning was explored through interviews with teachers. But the main issue of concern here was whether and how, for first year engineering students, computers could influence their learning of calculus concepts. Following a survey conducted with teachers in sixth form colleges and universities, an experiment in realistic classroom environments was conducted because of the desire to obtain an in-depth insight into the impact of computers on students' learning of calculus concepts.

Four groups in four different universities were involved in this main study; two served as computer groups (the computer was used in teaching); two served as non-computer groups (the computer was not used in teaching). Calculus (Differential and Integral) was taught over two semesters in all universities.

The flow of ideas in this study shows several stages in the design where both quantitative and qualitative approaches were used in the collection and the analysis of data. The reason for combining quantitative and qualitative approaches was to demonstrate convergence in results, and to add scope and

breadth to the study. Furthermore, this mixed approach was expected to bring meaningful information about the process of learning because classrooms as a whole, and also individual students and their idiosyncrasies could be analyzed using these two approaches.

Data was gathered from three sources in the main study. The pre-test was administered prior to the instruction in differentiation and integration and the post-test was given upon the completion of these topics. At the end of the post-test, computer groups were asked to complete a computer attitude questionnaire in order to gain background information. The pre-test and the post-test data were collected using a specially developed diagnostic test. The diagnostic test included 10 questions, some of them having several items dealing with similar content. Each item on the diagnostic test was scored on its own six-point scale, developed by the researcher and checked by a specialist in this field. The data from the pre-test, post-test and computer attitude questionnaire was coded, and analysis of variance and other techniques were used to analyze the results.

In a follow-up post-testing, a few students were interviewed from each group in order to substantiate inferences deduced from the quantitative data.

A pilot study of the test was performed with two separate samples of 14 and 18 A-level students, which served to check its administration and to improve the final version. Some questions were deleted and others were revised using item-analysis information.

Among a complex pattern of results reflecting the different teaching approaches, the computer groups did better than the non-computer groups, particularly on questions requiring drawing the graph of a derivative by looking at its function and recognising the graph of a function by looking at the derivative graph. Thus, the use of computers over an extended period of time and their availability when carrying out graphical tasks seem to make an impact on the performance of the students on graphical interpretations.

Treatments in each group improved understanding much more so for A-level students than non-A-level students because the weaker students lag behind the rest of the class.

In all groups the students showed a remarkable consistency in their errors.

These results are interpreted in the light of previous research on understanding of calculus and recommendations are made to teachers and researchers.

CHAPTER ONE

INTRODUCTION

> We need to know how we learn before we can decide how to teach.
>
> Morgan (1990, p.976)

The primary purpose of this chapter is to provide the overall perspective of the study. In undertaking this task, an attempt will be made to discuss the background, the purpose and the significance of the study.

1.1 The Background of the Study

The last two decades have witnessed a growth in the literature concerning students' conceptual understanding of calculus (e.g., Monaghan, 1986; Orton, 1980; Selden, Mason & Selden, 1989; Tall, 1986). A trend toward rote procedure applications and the lack of understanding of concepts appeared to be happening and was examined in order to improve the quality of higher mathematics teaching and learning. On this, Nemirovsky (1993) comments that:

> most calculus courses do not go beyond the students' acquisition of procedures and notations which are quickly forgotten. As a consequence, the understanding of calculus that students require for science and engineering education is rarely achieved. (p.14)

But, despite Nemirovsky's and similar warnings (Hundhausen, 1992), an emphasis on conceptual understanding is still missing in the calculus curriculum (Beckmann, 1992). Rote learning is based on the memorization of formulas and rules in order to solve computational problems. Conceptual understanding or concept-formation, however, is centered around the development of concepts and their relationships to the fundamental principles of mathematics. It is worth quoting an excerpt from Lovell's (1961) book entitled *The Growth of Basic Mathematical and Scientific Concepts in Children* to illustrate what is meant by concept-formation:

> When the child forms a concept he has to be able to discriminate or differentiate between the properties of the objects or events before him, and to generalize his findings in respect of any common feature he may find. Discrimination demands that the child should recognize and appreciate common relationships, and differentiate between these and unlike properties..... ... In addition to abstraction (some people prefer abstraction rather than discrimination) or discrimination, generalization occurs, by means of which some notion of the concept has been developed..... Concept formation is likely to be aided by memories and images. (p.12)

An image of a concept is a mental representation of the external world known to each individual. This representation is determined by the external environment and its values to each individual. The images in such a way include elements obtained from direct experience, from what one has heard about a concept, and from imagined information. They include impressions about its use and representation. These images can be thought of as guiding so called 'schemata' or 'cognitive mapping'. The term concept image is also described by Tall and Vinner (1981) as:

> .... the total cognitive structure that is associated to the concept, which includes all the mental pictures and associated properties and processes. (p.152)

Calculus is the beginning of advanced mathematics and is universal in higher mathematics and fundamental for the understanding of further concepts. Barnes (1992) drew attention to that fact:

> Calculus can help students to appreciate the power of mathematics and its use in helping us to understand the changing world we live in, to predict outcomes and in some cases to control them. By becoming confident in mathematics at this level, students may be encouraged to question its use by so-called "experts" who use mathematics as propaganda to support their side of an argument. So I think that teaching calculus to this group can be justified as a part of education for the whole of life, rather than solely as a preparation for career or further study. Ultimately, an understanding of the power of mathematics is important, not just for those who will use it in their work, but for those who make policy decisions and, ultimately, for all citizens. (p.73)

and also Ferrini-Mundy and Lauten (1994) stated:

> Calculus is a critical landmark in the mathematical preparation of students intending to pursue nearly all areas of science and, increasingly, the social sciences. (p.120)

As has been stated by Morgan (1990), teaching methods should be changed in order to produce conceptual understanding, rather than mechanical skills with standard problems only. Students must be encouraged to think in a flexible and inferential mode, rather than in a mechanical fashion. That is to say traditional teaching methods should be re-examined and, as pointed out by Morgan,

> consideration should be given to individual, self-paced, self-study, including general reading, programmed learning, computer-aided learning, interactive video, slides, tapes. (p.981)

However, some would say programmed learning only promotes rote learning.

Since the development of microcomputer technology an attempt is being made to eliminate much of the procedural work of calculus such as finding complicated derivatives through the use of it. Thus, the learner can be focused on the products, which would help them gain a conceptual understanding. Parallel to the interest in improving the quality of mathematics teaching and learning there have been widespread efforts to utilize computers in order to improve the teaching and learning process, in calculus as well as in other subjects in higher education. The computer has been incorporated into the mathematics courses in a variety of ways. The introduction of the computer brings a new dimension into the learning and a new factor into the classroom relationship between students, teacher and the mathematical concepts to be considered. Teachers sometimes shift their roles from expositor and drill-master to tasksetter, counselor, information resource, manager, explainer, and fellow-pupil, while students engage in more self-directed exploratory learning activity (Fraser, 1986). A Socratic dialogue between teacher and students, and the enhancement by the addition of the computer may help the students form their own concept images (Tall, 1986a, 1987a).

As many students in calculus have difficulties in understanding concepts, there is a question of whether the use of a computer in a 'realistic' classroom environment can be helpful or whether it merely introduces another source of anxiety. It is also not clear whether the computer could assist conceptual change in a better way. Therefore it was decided to evaluate the impact of computers on the learning of calculus by comparing diagnostic test results for computer and non-computer groups in a 'realistic' classroom environments. Glass (1984) commented that:

> One of the benefits of the computer was computers can teach concepts. Higher-level concepts and skills, especially those involving the interrelationships of various elements, are difficult to teach and learn from a book. Simulated experiences on a computer make them more achievable. The computer is far more effective and efficient than a teacher in helping individuals develop higher level concepts through analyzing a problem, gaming, and perceiving spatial relationships. (p.12)

Research in a 'realistic' classroom environment should be of value in order to acquire some knowledge on the detailed processes of learning and the corresponding conditions in the daily life of mathematics classrooms. There is a shortage of well focused research that has undertaken classroom-based research in a 'realistic' calculus environment, investigating learning or concept formation in calculus.

Most research on students' difficulties or errors in calculus has taken place without consideration of the content and delivery of instruction. Further, they have not addressed questions regarding the knowledge base that students bring to instruction, and how this knowledge influences further learning or how it changes as a result of instruction.

A simplified model for the conceptual and theoretical framework of research is presented in Figure 1.1. This model integrates the perspectives of instructional (external state of a student) and cognitive science (internal state of a student) to study students' learning. Among these external conditions with which the student interacts, I have considered in particular: those linked to the available resources, specifically computers. Among the internal conditions of the student, I have in particular considered: those linked to the knowledge of the students acquired prior to instruction; and those linked to the previous mathematics background of the student (A-level or Non-A-level).

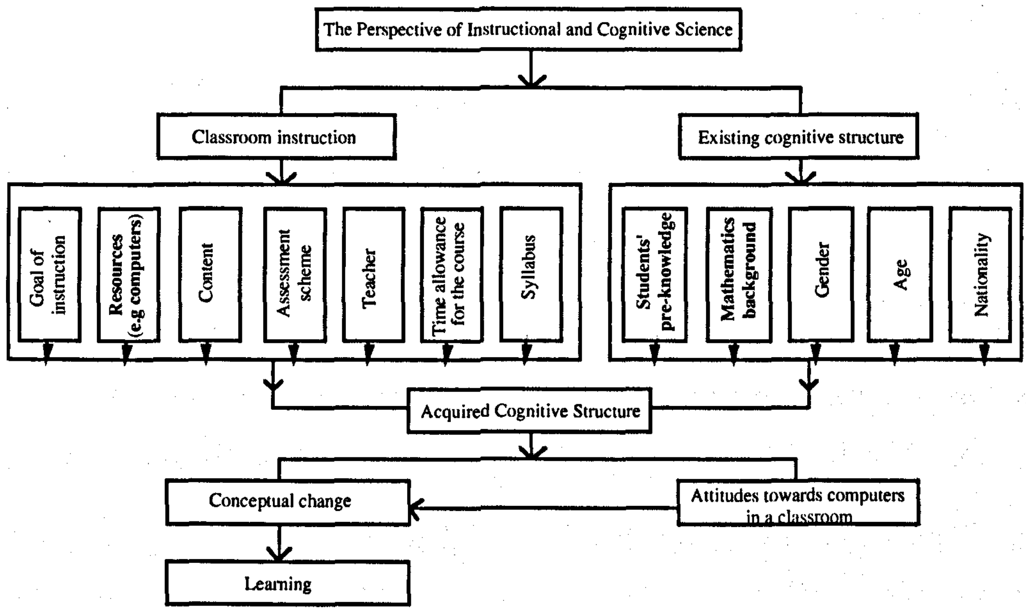


Figure 1.1. Conceptual and theoretical framework of the research plan

Note: • Main focus (written in bold) • Contextual information (written in plain text)

To represent cognitive structure, the intended formal knowledge - the knowledge that exists in textbooks - has been taken as a focus, and so we have concentrated on those parts of cognitive structure that should be shared among the learners who are part of the course. One of the central assumptions underlying most current research on students' thinking is that knowledge is absorbed and constructed by the learner, rather than passively received (Carpenter & Fennema, 1991; Goldin, 1990). Students bring a great deal of knowledge, some of which is correct and some incorrect. Some knowledge facilitates learning, and some hinders it. Nevertheless, learning is influenced by students' pre-instructional knowledge (Carpenter & Fennema, 1991; Ferrini-Mundy &

Lauten, 1994: Ginsburg, 1988; Skemp, 1971), and to understand the effects of instruction, teachers and researchers need to first understand the nature of that knowledge and how it influences what children learn. Skemp also emphasized that:

> inappropriate early schemes (The term, schema, includes not only the complex conceptual structure of mathematics but also relatively simple structures which coordinate sensor-motor activity (p.37)) will make the assimilation of later ideas much more difficult, perhaps, impossible. 'Inappropriate' also includes non-existent.(p.48)

As used here, learning mathematics refers to both replacing students' existing misconceptions with accepted forms of knowledge (Smith, 1992) and concept formation.

The reason for studying cognitive structure and classroom instruction is grounded in the assumption that the evaluation of cognitive structures involves the evaluation of the classroom instruction as well as the evaluation of each student's cognition. The starting point for describing the cognitive structures of the students in a segment of a course is the course itself (West, Fensham & Garrard, 1985). Different types of formal knowledge have been extracted from the syllabus and hand-outs given to the students. Each piece of information was used as a stimulus to explore the depth of a learner's private understandings.

1.2 The Purpose of the Study

The aim here is to present a careful analysis of the progress in learning various concepts of calculus, made by 147 first year engineering students, in a computer or non-computer environment, based on their performance on a diagnostic test. Engineering students are expected to transfer their knowledge of mathematics to contexts outside the mathematics classroom and to use mathematics as a tool (Hundhausen, 1992). Mathematics and especially calculus is an indispensable tool for engineering work.

Four groups in four different universities were involved in the study; two served as computer groups (a computer was used by students as part of a course); and two served as non-computer groups. Although the study has been set up as a pretest-posttest design, it is also an exploratory study intended to collect information in several different ways.

The diagnostic test designed especially for this study (see Appendix A) targets particular areas of difficulty, and was given as a pre-test and a post-test. The items of the test were selected from various references which are mentioned further in Chapter Four and Five. Bolletta (1988) stated that "there are some critical points where students of different classes, with different teachers, and in certain cases of different ages, perform in the same way and make the same kinds of errors" (p.170). For example, Cornu (1991) has described four epistemological obstacles in the history of the limit concept. Thus, I got the impression that the use of items from previous researches might aid the evaluation of the computers' impact on students' learning of calculus. The task variables that have been taken into consideration for the choice of items are presented in Figure 1.2. This was done by considering and embedding Orton's (1980) research chart.

Many of the suggestions regarding the teaching and learning of calculus provided by several authors (Amit and Vinner, 1990; Barnes, 1992; NCTM, 1989; Orton, 1980, 1980a, 1984; Tall, 1990) were taken into consideration while designing the diagnostic test. For example, Barnes and NCTM emphasized that graphical representation of functions needed to be stressed and students needed experience in interpreting graphs, especially the global features of a graph while Barnes, NCTM and Orton mentioned the introduction of the idea of the 'rate of change' of a function.

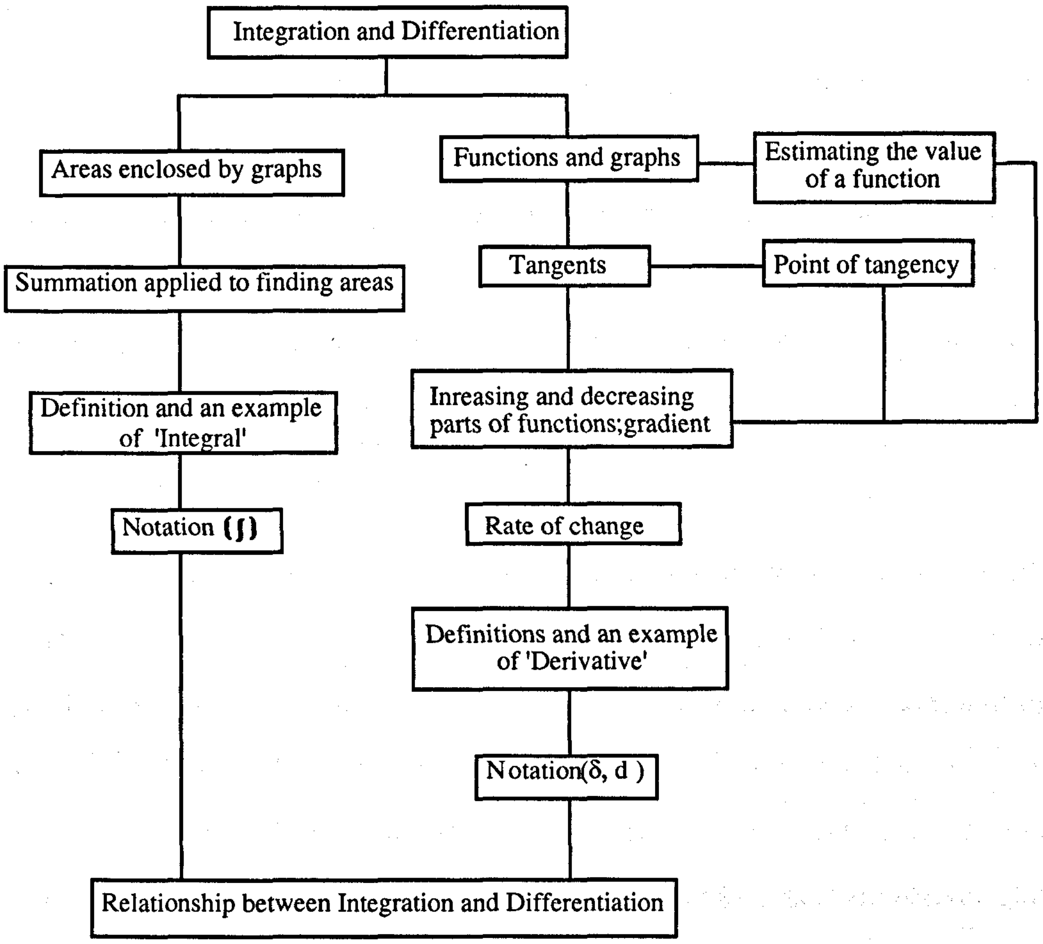


Figure 1.2. Interconnections reflected in the choice of items

Data was gathered from three sources. The pre-test was administered prior to instruction in 'differentiation' and 'integration' but the post-test was given upon the completion of these topics. Following the post-test, the computer groups were asked to complete a computer attitude questionnaire (see Appendix C) in order to get their attitudes toward the use of the computer in a calculus course. The data from these sources was coded, and SPSS-X was used to analyze it. In addition to that 21 students were interviewed after the post-testing.

The flow of ideas in this study shows several stages in the design where both quantitative and qualitative approaches were used in the collection and the analysis of data. The reason for combining quantitative and qualitative approaches was to demonstrate convergence in results, and to add scope and breath to the study. Moreover, it was important to know not only whether a learner can solve a particular problem, but also how the learner solves the problem. It seemed to me that qualitative and quantitative aspects were very closely connected at this stage of the work. The methodology design used was the *Mixed-Methodology Design* (Creswell, 1994). One could clearly see that this brought meaningful information on the process of learning because classrooms as a whole, and also individual students in their idiosyncrasies were analyzed quantitatively and qualitatively, respectively.

Prior to the study discussed above a survey was conducted with teachers in sixth form colleges and universities on the use of technology in calculus in order to get them to tell their own stories about their experiences and achievement in using it. Although the information obtained from teachers and lecturers has proved to be very illuminating, some of their answers were rather surprising. This led me to reflect on some other work which was discussed above. The results gave a very different view of classroom happenings than those reported by mathematics educators who are seeking to change practice by the introduction of technology.

1.2.1 Quantitative Study - Research Questions

The responses of the 147 students to the 26 items were assessed on a six-point scale specific to each item.

It was intended that the investigation would find answers to the following questions.

• Is there any significant difference in the performance of students, who interacted with computers in addition to the lectures in the classroom, compared to those who did not ?

• Is there any significant difference between the performance of A-level students and that of non-A-level students?

In addition it sought to provide answers to the following questions.

• To which extent and in which ways attitudes towards computers can help in the development of calculus concepts?

It was hoped that by exploring the answers to these questions, some insight would be gained into the problems and difficulties associated with the learning and teaching of calculus.

1.2.2 Qualitative Study - Research Questions

The data collected by means of pre-testing, post-testing and interviewing was used to substantiate inferences deduced from the quantitative analysis. Within this investigation I was mainly engaged in finding out which types of errors were produced by the students before and after the teaching units of 'differentiation' and 'integration' and how these errors and changes can be explained. It has also been possible to identify some of the common errors made by students in certain topic areas.

Students' errors were addressed using a cognitive strategy intervention as:

> learning is the creation on internal bonds, then not much else can be said within this conceptual framework. Are errors in performance to be viewed as faulty-bonds or no-bonds, or what? (Bélanger, 1988, p.21)

By exploring students' errors or difficulties in a computer or non-computer environment, one may also gain a new perspective on calculus and on the possible roles of computers in calculus learning.

This part of the study addressed the following questions:

• What are the errors that occur with each item?

• What kind of structure do errors form?

• Are these patterns of errors associated with the four groups?

The presentation of the results consists of four issues under each concept: (i) illustrations of the errors in the various categories; (ii) the distribution of errors by groups; (iii) detection of patterns in each student's pre-test and post-test responses; and (iv) clarification of student's understanding by interviews.

With the error analysis, the origins and the nature of errors as well as many misconceptions came to light, indicating the difficulty most students have in reaching a full understanding of precisely and analytically defined abstract concepts.

1.3 Anticipated Significance of the Study

The anticipated significance of this research is as follows:

(1) The results should be of great value to the teachers in helping them learn more about the depth of understanding of his/her students. They will also be useful to researchers wanting, for various reasons, to obtain measures of cognitive structure as outcome measures.

(2) Curriculum developers could make profitable inferences about the kind of calculus teaching that would match students of different mathematics background.

(3) If one were to understand how students are making sense of the teaching situation, then one would find the results and errors to be reasoned and supportable.

(4) Teachers could use the results to help their students understand the concepts meaningfully and correctly and to identify more effectively inaccurate conceptions about the concepts included in the study.

(5) This research presents the errors and their causes. The list of errors are explicit pieces of knowledge that may be tacit for many teachers. Presumably, this knowledge could help teachers as well as students. Students can solve such problems meaningfully and understand them better. Teachers can widen their knowledge and also help students modify their misconceptions and form more richer concept images. Fennema and Franke (1992) claimed that "when a teacher has a conceptual understanding of mathematics, it influences classroom instruction in a positive way" (p.151).

(6) For decision makers, planning the use of information technology in mathematics courses.

1.4 Summary of the Chapters

This section summarizes the contents of the chapters of this thesis. This chapter presented the basis of the study; considering the background and the purpose of the study, and stated its anticipated significance.

In chapter two, the thesis continues with a discussion of the previous research. Technology in mathematics teaching and learning, and more specifically calculus concerns are discussed. This perspective is then applied to make sense of the different forms that computing environments take, and especially how they change the processes of learning. Moreover, students' difficulties and misconceptions in understanding calculus concepts are examined.

In chapter three, the attitudes of teachers to types of software and the manner in which they use them are explored. The data for this exploratory study were obtained from a series of interviews conducted with calculus teachers in universities and sixth form colleges.

Chapter four presents the framework of the study; defining subjects and instruments; considering the method of data collection and analysis; describing different learning environments, with and without computers; and stating the criteria for scoring items in the instrument called the diagnostic test. The learning environments are described in order to present a comprehensive picture of them and make better explanations of different learning outcomes.

Chapter five focuses on the design of the diagnostic test in a detailed way. It describes the general nature of the questions together with the results of the investigatory work which took place at two sixth form colleges.

In chapter six, data from the pre- and post-test, and computer attitude questionnaire are analyzed quantitatively using analysis of variance. As has been mentioned in this chapter, the fundamental questions in this main study center upon issues of individual learning.

In chapter seven, data from the pre- and post-test, and interviews are analyzed qualitatively. Students' errors and their underlying reasons are reported. This should give the reader a detailed insight into the type of misconceptions typically held by students learning calculus.

The last chapter presents a discussion, summarizing the quantitative and qualitative results, and ends with suggestions for further research, and with conclusions.

The above considerations led to the adoption of the following general research model outlining the course of the action adopted (see Figure 1.3).

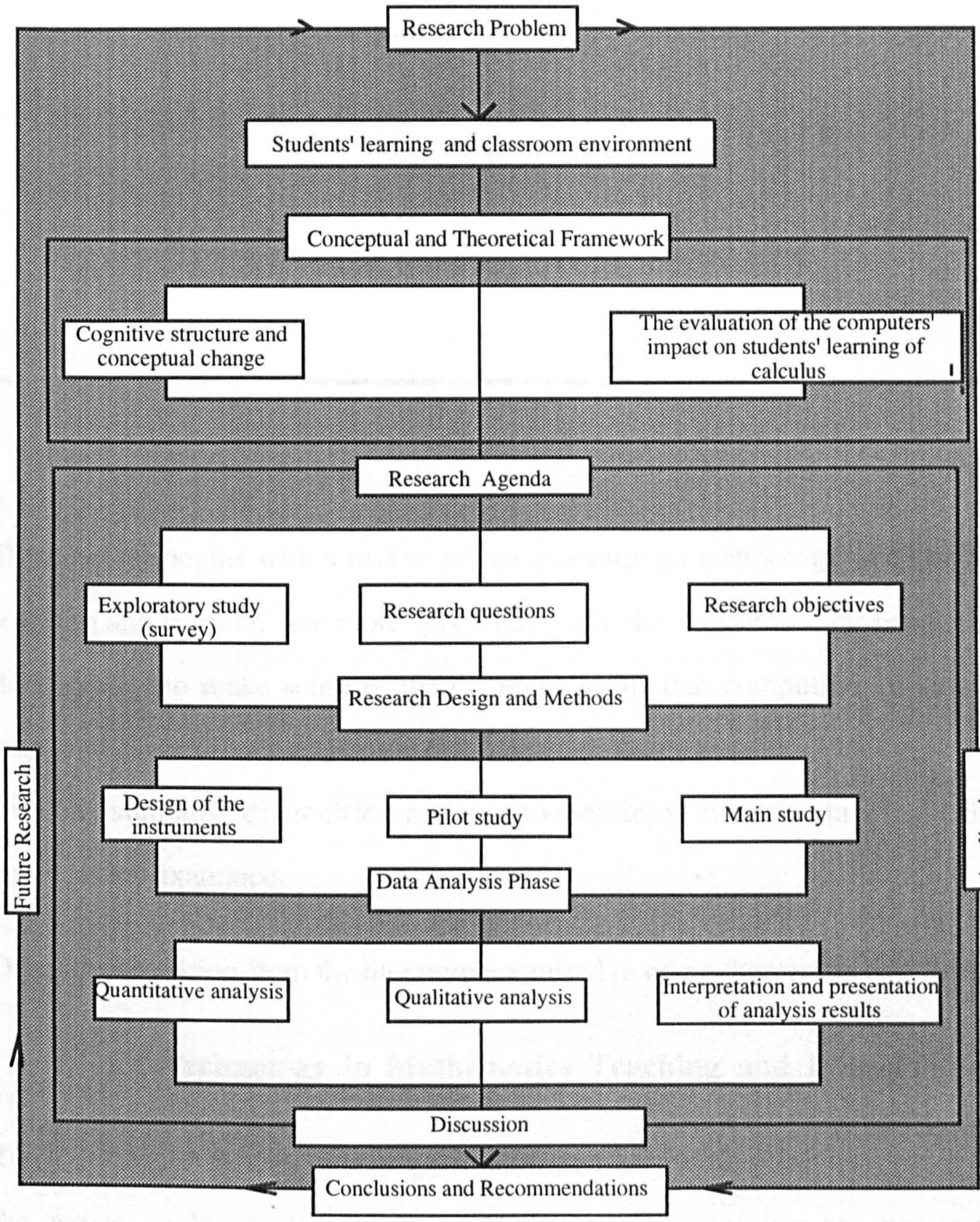


Figure 1.3. The general research model

CHAPTER TWO

REVIEW OF PREVIOUS RESEARCH

This chapter begins with a review of the literature on technology in mathematics teaching and learning, and more specifically calculus concerns. This perspective is then applied to make sense of the different forms that computing environments take, and especially how they change the processes of learning. Finally, in the last section, students' difficulties and misconceptions in understanding calculus concepts are examined.

Other issues arising from the literature are raised in other chapters as appropriate.

2.1 Technology in Mathematics Teaching and Learning

To activate information technology in mathematics teaching and learning is one of the major goals of all countries. Despite this, computing has not changed mathematics teaching very much in the daily work. On this point Cox (1987) writes:

> the successful implementation of computer assisted learning (CAL) in schools over the next decade, will depend upon the adequate support of governments, the willingness of teachers to use computers in education and a conviction that CAL is worth the support and effort required. (p.33)

Literature surveys (Fey, 1989; Kaput, 1992) have pointed out that the numerical, graphic, and symbol manipulation tools provided by computers offer unique kinds of insight and power in mathematical teaching, learning, and problem solving.

Kozma and Johnson (1991) provided a richer picture of how computers can change what students learn in the classroom by examining various software packages and instructional innovations. Their comments can be summarized as follows:

- From reception to engagement. With technology, students are moving away from the passive reception of information to active engagement in the construction of knowledge.
- From text to multiple representations. Technology is expanding our ability to express, understand, and use ideas in other symbol systems.
- From isolation to interconnection. Technology has helped us move from a view of learning as an individual act performed in isolation toward learning as a collaborative activity. And we have moved from the consideration of ideas in isolation to an examination of their meaning in the context of other ideas or events.
- From products to processes. With technology, we are moving past a concern with the products of academic work to the processes that create knowledge. Students learn what is that scholars do: how historians, mathematicians, and authors write, think, and solve problems.
- From mechanics to understanding in the laboratory.

In a closely related work, Wimbish (1992) also pointed out some benefits of technology-rich environment to students:

> students begin to see mathematics as something in which they can actively engage; mathematics is not just for the intellectual giant; the use of the PC's gives their work a focus that may have been lacking before; the software provides a mutually convenient language with which to communicate both among themselves and the PC; they are not as concerned about coming up with just the right "problem solving" formula or answer, they begin to understand there just might be different ways of "solving" the same problem with not necessarily equivalent solutions.(p.82)

Gordon (1983) based on the findings of a three year project suggested the incorporation of computers into mathematics and related courses in physics, chemistry, earth and space sciences, engineering and economics. In this project it has been found that the most valuable effect of the computer was the tremendous stimulus it provides to the imagination. If computers are used to do all the work, the user, whether a student or teacher is free to ask "what happens if..." and to obtain an answer immediately. In this way, the computer adds an extra dimension to many courses and thereby enhance them dramatically.

To achieve all these things, as mentioned above, information technology would be useful and beneficial if one uses it properly to serve his/her curricular goal such as numerically, algebraically or graphically (Tall, 1987). Otherwise, the use of technology cannot replace the thinking process (Hickernell & Proskurowski, 1985).

2.1.1 Calculus Concerns

Ralston (1989) has described principles to give curriculum developers the direction and perspective they need in making many of the decisions which will be required. For example, two of them were:

> Principle 1: - Mathematics education must focus on the development of mathematical power not mathematical skills. By mathematical power is meant the development of the abilities to:
> - understand mathematical concepts and methods
> - discern mathematical relations
> - reason logically
> - apply mathematical concepts, methods and relations to solve a variety of nonroutine problems
>
> Principle 2: - Calculators and computers should be used throughout the K-12 mathematics curriculum. (p.35)

Previous studies have indicated that conceptual understanding is one of the biggest problems in calculus. Orton (1980, 1983, 1983a), for example, reported that most errors students made when carrying out some tasks in differential and integral calculus were the result of their failure to grasp some conceptual principles which were essential to the solution. As indicated in Heid's (1988) paper, computing devices are natural tools for the refocusing of the mathematics curriculum on concepts. In addition to that Dubinsky and Tall (1991) pointed out that the computer is proving itself to be a powerful tool in advanced mathematical thinking.

At the 5th International Congress on Mathematical Education, the Tertiary Level Mathematics Group argued that:

> For the majority of students the concepts should be introduced in a way which is sensitive to the students' intuitions and existing schemas. This can be achieved in many cases by avoiding unnecessary rigour, formalities, and over-subtle mathematical notions. In other cases, special attention should be given to recognising and resolving students' conflicts. (Vinner, 1984, p.161)

Accordingly Tall (1985), for example, pointed out that the micro computer enabled a cognitive approach to calculus without the prerequisites of limiting processes, chords approaching tangents. A computer program capable of magnifying graphs over tiny ranges can help a student to see the gradient of a curved graph without any use of tangents or chords. Tall (1986b) also mentioned that students doing calculus the traditional way have difficulty understanding the reasons for positive and negative areas but using *Graphic Calculus* (Tall, 1986c) they may investigate positives and negative steps as well as positive and negative ordinates.

A growing body of research suggests that, when used wisely, calculators or computers can enhance student conceptual understanding and attitudes toward mathematics - without apparent harm to the acquisition of traditional skills. Those are the conclusions reached in a meta-analysis of 42 studies by Bangert-Drowns et.al (1985), and in a survey reported by Fey (1989).

According to Breidenbach's et al.(1992) theoretical viewpoint, a major requirement for understanding functions that students do not seem to meet is the ability to construct processes in their minds and use them to think about functions. The results indicated that working with computers in certain ways seems to have, in general, a salutary effect on students' abilities to make such constructions.

Zorn (1987) provided an outline of the issues in computer use in undergraduate mathematics, and pointed out that computing can benefit undergraduate mathematics teaching in many ways:

- to make undergraduate mathematics more like real mathematics.
- to illustrate mathematical ideas.
- to help students work examples.

- to study, rather than just perform, algorithms.
- to support more varied, realistic, and illuminating applications.
- to exploit and improve geometric intuition.
- to encourage mathematical experiments.
- to facilitate statistical analysis and enrich probabilistic intuition.
- to teach approximation.
- to prepare students to compute effectively - but skeptically - in careers.
- to show the mathematical significance of the computer revolution.
- to make higher-level mathematics accessible to students.

In a related work, Hickernell and Proskurowski (1985) exposed the benefits of using microcomputers in calculus. They have noted that the microcomputer stimulates an interest in calculus, motivates students to work more and to think in mathematical terms, and allows students to consider and easily solve sophisticated problems, bringing them closer to real life applications. Although these studies have not yet had a major impact on calculus teaching/learning, they bring the techniques of depth investigation to the analysis of effectiveness of technology in mathematics education.

Tall et al. (1986d) referred to "the changes" which would be caused in the calculus curriculum with the arrival of the computer at the Ware Conference in 1985:

> 1) Fast numerical methods will be available on the computer to give numerical solutions of problems previously handled formally by the calculus.
> 2) These numerical techniques can usually be programmed in simple algorithms. The understanding of the process of the algorithm may be aided by the students carrying out their own programming.
> 3) The graphic facilities of computers are providing dynamic ways of viewing the concepts, enabling them to be understood with more profound insight by students and mathematicians at all levels.
> 4) Symbolic mathematical manipulators are becoming available that are able to produce the formulae for differentiating and integrating functions. This may allow more time on theoretical insight and less on specific tricks of integration.
> 5) Both graphical and symbolical modes of operation are becoming available in interactive modes that enable the user to explore the concepts concerned.
> 6) The method of teaching may be modified, with exposition and exercises being enhanced by exploration, conjecture and testing by the pupils.
> 7)Application concerned with differential equations which may not have solutions in closed formulae. Graphic and numerical techniques used in tandem give the possibility of investigating both qualitative and quantitative aspects of the solution.

> 8) The development of insights into the processes will lead to alternative approaches to the subject, e.g. numerical differentiation before symbolic differentiation, allowing the topic to be started without an initial discussion on limits. This will require a rethink of the balance between calculus theory and numerical practice and lead to modification of the syllabus. (p.124)

Many of the studies about the use of technology in calculus teaching have focused on:

- its effect on students' performance;
- its effect on students' achievement;
- its effects on student motivation and attitudes.

Most of them within an earlier paradigm compared technology-based teaching/learning to traditional instructional methods. Some studies comparing technology-based calculus instruction to traditional approaches have found the computer-based approaches to be significantly more effective (Estes, 1990; Heid, 1985, 1988; Palmiter, 1986, 1991). These studies determined that technology played a crucial role in concept development. Others have found no significant difference between experimental and control groups (Hawker, 1987; Judson, 1990a) but nevertheless there have been differences in motivation, interest, and attitude. Pertinent features of these studies will be referred to in succeding sections.

The following part of this section is divided into four major parts considering the impact of:

1. Numerical computation
2. Symbolical computation
3. Graphical computation
4. Tutorial systems

The Impact of Numerical Mathematical Systems

The numerical assistance provided by calculators and microcomputers offers a clear and attractive alternative to traditional paper-and-pencil skills whenever arithmetic computation is required - in elementary arithmetic, algebra, geometry, trigonometry, statistics, or calculus (Fey, 1989). The instructional power of numerical approaches to advanced mathematics lies not only in allowing very complex numerical calculations but also making the transition from arithmetic to algebraic reasoning and, as stated by Tall (1990), to making estimations and approximations more accurately. But, unfortunately, it should be pointed out that there is a growing body of research showing that neglect of numerical investigations deprives students of an important perspective on many mathematical ideas (Fey, 1989).

Numerical programs seem a likely medium for the involvement of the user in interpreting the output. For example, Gordon (1983) stated that a graphic program kept the interest of all the students because of the "action" on the screen but a numerical program must involve the students in thinking about the output in order to interpret it.

The Impact of Symbolic Mathematical Systems

Since microcomputers had symbolic mathematics system, the effects of a computer algebra system have been studied. Fey (1989) has noted that the use of symbol manipulation software could have a significant impact on mathematics education in at least three ways: (i) the software extends the complexity of algebraic expressions that can be handled at any level of instruction; (ii) with computer assistance in routine symbol manipulation it seems quite possible to reorient instruction to focus on the conceptual understanding and procedural planning that remains essential in mathematical problem solving; and (iii) in much of the same way that the *Geometric Supposer* or *Cabri-Geometry* tools facilitate exploratory learning, symbol manipulation utility programs can support rapid exploration of patterns in algebraic reasoning - leading to discovery of important general principles.

The studies found evidence that students using several computer algebra systems (MuMath, Maple, Macsyma) in calculus performed on conceptual exams as well as or better than students in traditional versions of courses (Hawker, 1986; Heid, 1985, 1988; Judson, 1990a; Palmiter, 1986, 1991). In addition to that Heid (1988) found some evidence that "students from the experimental classes spoke about the concepts of calculus in more detail, with greater clarity, and with more flexibility than did students from the comparison group" (p.21). The results of the studies also pointed out that skills acquisition was not a prerequisite to conceptual understanding or the ability to solve application problems (Heid, 1985, 1988; Palmiter, 1986, 1991; Judson, 1990a). In each of these studies, students from calculus courses intended for business majors were used except in the studies of Palmiter in which engineering students were used. All these studies also used the computer as a tool in a concept oriented calculus course. As stated by Heid (1988):

> An introductory calculus course is an ideal testing ground for the notion of using the computer as a tool in concept development. (p.3)

Most of the studies mentioned above give a clear indication that "a student plus manipulation tool" can be more successful in conceptual and computational tasks than a student working in a traditional manner.

Another insight into ways to use computer algebra is offered by Judson (1990). He identified four important ways computer algebra would be used:

- Computer algebra can be used to solve realistic problems which do not have 'nice' solutions.
- Computer algebra can be used to stimulate thinking and provide motivation for future topics.
- Computer algebra routines can be effectively used in homework assignments to promote learning by discovery.
- Computer algebra can promote pattern recognition.

Computer algebra systems can compute all limits, derivatives, and integrals posed in most calculus texts and handle the problem either symbolically, numerically or graphically.

The Impact of Graphical Mathematical Systems

In mathematics, the opportunity provided by graphical mathematical systems such as computers or calculators has excited particular interest, through the display of mathematical expressions and data graphically (Ruthven, 1990; Smart, 1995). Hickernell and Proskurowki (1985) gave the several benefits of using microcomputers in a calculus which were referred to in the first part of this chapter. Another benefit of using microcomputers in calculus was stated by them wherein they write,

> It stimulates geometric intuition and improves conceptual understanding through the use of high resolution graphics. (p.111)

Other studies too have indicated this to be the case. For example, Alkalay (1993) analysed students answers given to the evaluation of units in which computers were used for independent exploration in precalculus. The results indicated that the speed of graphic and visual accuracy helped students understand the concepts presented.

Graphic tools are a powerful medium which would enrich understanding algebraic forms - giving visual images of symbolic information - and construction of rules for functions that fit given graphical patterns (Fey, 1989). Much use of graphical tools has focused on using them to enhance understanding and skills in traditional mathematical topics. For example, once the graphs are produced it is possible to answer a variety of questions equivalent to solving equations or inequalities, determining maxima or minima, and studying rates of change (Fey, 1989).

Fey also stated that the teacher role shifts from demonstration of "how to" produce a graph to explanations and questions of "what the graph is saying" about an algebraic expression or the situation it represents. Students' tasks shift from the plotting of points and drawing curves to writing explanations of key graph points or global features.

At the beginning of 1991 at Swinburne, first year calculus students have been prescribed using graphics calculators. The result of the investigation concerning the students' attitude to the introduction of the graphics calculator and its impact on their mathematical practice indicated that

> Synthesis of feedback from students concerning the positive aspects of using the graphics calculators were: the power to draw and gain information from graphs, its usefulness in enabling quick numeric or graphical checks of algebraically derived answers, its value as an aid for understanding and interpreting graphs, on screen editing which was useful in calculating and checking difficult formulae and, finally, that the calculators increased confidence and enthusiasm. Negative aspects mentioned by the students included the possibility of developing calculator dependence and that the calculator could remove the need to know why. (Boers & Jones 1992, p.85)

Smart (1995) reported on work that took place in a lower secondary school mathematics classroom. This project was based on a case study of a class of 13-14 year old girls observed during their mathematics lessons over a 13-week term. They had free access to graphic calculators during their mathematics lessons. Prior to this, the girls had not used a graph plotting calculator or software. As a result of the experience, the girls started to develop a robust visual image of many algebraic functions. As was pointed out by Ferrini-Mundy and Lauten (1994), thinking visually can be useful in many calculus-related contexts, and activities that promote and encourage visual thinking are likely to help students' understanding.

Moreover, Ruthven (1990) compared the mathematical performance of upper secondary school mathematics students for whom a graphic calculator is a standard mathematical tool, with that of students of a similar background without regular access to graphic technology. The findings illustrate that, under appropriate conditions, access to information technology can have an important influence both on the mathematical approaches employed by students and on their mathematical attainment. On the symbolisation items, calling for an algebraic description of some cartesian graph, the use of graphic calculators was associated not only with markedly superior attainment by all students, but with greatly enhanced relative attainment on the part of female students.

Tall (1986) was concerned with a theoretical building of a cognitive approach to the calculus and an empirical testing of the theory in the classroom where a computer was used as a resource enabling the learner (from age 16 through to university) to explore examples of mathematical processes and concepts, providing cognitive experience to assist in the abstraction of higher order concepts embodied in the organiser. Three experimental classes used the software (The generic organiser *GRADIENT*) as an adjunct to the normal study of the calculus and five other classes acted as controls. Experimantal students were able to sketch derivatives for given graphs and to recognise the graph of a function from its derivative graph significantly better than the controls on the post-test, at a level comparable with more able students reading mathematics at university.

A lot of the issues of using computers to create screen and mental images are discussed in depth in *Exploiting Mental Imagery with Computers in Mathematics Education* (Sutherland & Mason, 1995). This book consists of four parts. The first part (emphasizing the external) focuses on the ways in which external images and discourse interact with students' approaches to solving problems in mathematics. The second part (imagery in support of geometry) focuses on the teaching and learning of geometry. The third part (links between screen and mental imagery) is related to the question "what is the nature of the influence of mental imagery on what is seen and of what is seen mental imagery?" In the fourth and last part (employing imagery) the ways in which images can be used deliberately, and mindful that the user is inevitably drawn into interpretation are examined.

The Impact of Tutorial Tools

Under this heading the impact of tutorial tools will be considered. One important application of which is their use of them in assessment. Tutorial tools can of course show a numerical, algebraic and graphical information as described above.

Tilidetzke (1992) investigated the use of a tutorial package in a course of instruction in college algebra, taught at the university of North Carolina in 1988. This course served as a precalculus course for those students planning to take calculus. The study compared four sections of the college algebra course. Two instructors participated in the study, each teaching a CAI section and a control section. Students in two sections of college algebra studied three topics by means of CAI in an Apple Computer Lab: (1) the multiplication and division of complex numbers; (2) completing the square; and (3) solving the linear equations. The main conclusions of this study were (i) that the two CAI sections as a group performed as well as or better than the two control sections; and (ii) that the two CAI sections as a group performed as well as or better than the two control sections as a group on the delayed posttest, after controlling for instructors.

The availability of assessment is another development in tutorial tools which offers the student an opportunity to get immediate feedback and the teacher to carry out formative evaluation with remediation possibilities, and to monitor homework and laboratory activities (Helgeson and Kumar, 1993). A possible advantage of immediate feedback is that it can assist students in improving their future performance.

2.1.2 Summary

The introduction of low cost microcomputers at around 1979 was a turning point in the application of computers, and it is believed among educators that microcomputers will play an important role in education and especially as a strong tutorial tool in the form of computer-assisted instruction.

The impact of numerical, graphical and symbolical manipulation tools and also tutorial tools described in each preceding section demonstrate the possibilities for dynamic multiple representations of mathematics in microworld environments.

2.2 Students' Difficulties and Misconceptions in Understanding Calculus Concepts

Research on student difficulties and misconceptions in understanding calculus concepts has received considerable attention over the last two decades. Before considering the various difficulties and misconceptions it is necessary to define what is meant by the terms "understanding", and "misconception" in the literature. These two terms have recurred or will recur throughout the thesis.

First, a general discussion of these terms will be provided and then the following paragraphs will outline the difficulties and misconceptions faced by students trying to understand calculus concepts.

According to Skemp (1971), "to understand something means to assimilate it into an appropriate schema" (p.43). More recently, Carpenter and Elizabeth (1991) also defined understanding as interconnected knowledge which is the network analogy suggests the connection of new knowledge to the existing knowledge.

Narode (1987) described misconception as "a person's conceptualization of a problem or phenomenon that generally is reasonable to themselves but at variance from the conceptualization of an 'expert' in the field from which the problem came" (p.322).

Shin (1993) investigated 120 12th grade and 30 college freshmen Korean students' error patterns in limit, derivative and integral during the school year 1991-1992, and found that there was a lack of understanding to present the rate of change including Δx, Δy, the difference $\frac{f(3+h)-f(3)}{h}$ and $\lim_{h\to 0}\frac{f(3+h)-f(3)}{h}$. These results showed that students could not connect f'(x) to a limiting process. Moreover, the following mathematical ideas in integration showed a serious lack of understanding: approximation of the area under f(x) between x = a and x = b by connecting area $\sum_{i=1}^{m} f(x_i)\Delta x$ to $\lim_{\Delta x \to 0}\sum_{i=1}^{m} f(x_i)\Delta x$. Furthermore, Tall (1986d) also showed that students did not derive the gradient of the tangent using a limiting process.

Vinner (1982, 1984) has observed that the early experiences of students about the tangent in a circle geometry introduces a belief that the tangent is a line that touches the graph at one point but does not cross it. The results analyzed also show that some students seemed not to think of a horizontal tangent at a point other than a maximum and minimum. Investigation in connection with the studies of Vinner showed that the term "chord" introduced in a circle geometry to mean the line segment between two points on the curve creates a very common difficulty (Tall, 1985). Some students see the chord as a finite line segment which tends to zero length as the points get close together, so 'the chord tends to the tangent' because the vanishing chord gets closer to the tangent line. Students looking at the expression $\frac{\delta y}{\delta x}$ often imagine δx tending to dx and δy tending to dy, rather than the whole expression tending to $\frac{dy}{dx}$.

Artigue (1986) reported a study in which 122 first year university students were asked the following questions, during exercise sessions in mathematics:

> Do you know the notations $\frac{df}{dx}$, df, dx? If yes, where did you meet them? With which meaning?

It appeared that differentials were known by students as either an indicator to denote the variable of integration, or as a small quantity. Students did not see differentials as approximations, nor do they see them as functions.

Mundy (1984), in analyzing 2196 first semester college calculus students responses to a multiple choice final examination, showed a tendency of calculus students to operate at a rote level of applications of procedures and symbol manipulation which is not supported by the understanding of the concepts involved. For example, 34.1% of the students answered that the derivative of $y = x^{x+1}$ is $y' = (x+1)x^{x}$, clearly having applied a derivative rule which is familiar for a different, but similar-looking, type of function. Another piece of evidence from this research also showed that answer 12 for the problem " $\int_{-3}^{3}|x+2|dx = ...$" was an overwhelmingly popular distractor for the students which of course was obtained by integrating $\int_{-3}^{3}(x+2)dx$.

Selden et al. (1989) administered five non-routine conceptual calculus problems to 17 students who passed a routine calculus test with grade C. The highest score was 35 out of 100. Three of these problems were:

> 1. Find values of a and b so that the line 2x + 3y = a is tangent to the graph of $f(x)=bx^2$ at the point where x = 3.
>
> 3. Let $f(x)=\begin{cases} ax & x \le 1 \\ bx^2+x+1 & x > 1 \end{cases}$
> Find a and b so that f is differentiable
>
> 5. Is there an a so that $\lim_{x \to 3} \frac{2x^2-3ax+x-a-1}{x^2-2x-3}$ exists? Explain your answer.

In their study the following results were found with regard to the problems above.

> The favored method of solution to problem 1, given by seven students, was to solve equations of the line and parabola simultaneously, thereby not using the crucial information that the line was tangent to the parabola. ..... On problem 3, nine students set the two formulas, ax and $bx^2 + x + 1$, equal and guessed what a and b should be. ..... On problem 5, eight students substituted x = 3 in the function, found the denominator was 0 and either didn't continue or concluded the limit could not exist because one can't divide by zero. (p.49)

Morgan (1990) investigated the problem of lack of mathematical expertise demonstrated by first year degree and higher national diploma students in mechanical and electrical engineering courses by means of a multiple-choice diagnostic mathematics test containing 57 items. Some of the main difficulties encountered in differentiation and integration listed by Morgan were as follows:

- Differentiation of the function which is given as an algebraic fraction. Main error - differentiating numerator and denominator separately.
- The chain rule for differentiation. Main error - differentiating with respect to wrong variable.
- Partial differentiation. Main error - realization of which parameters are constant and which are variable.

- Visualization and understanding of the basic principle of integration in terms of elemental areas.
- Understanding that integration must be with respect to a specified variable, e.g. $\int x dx$ makes sense but $\int x dy$ does not.
- Confusion with the rule $\int dx / f(x) = \log(f(x))$. Students think that f(x) can be any function.
- Difficulties with integration problems when an expression has to be re-arranged before being integrable.

Amit (1992) provided some insights based on a systematic analysis of 294 first year college students' solutions to conceptual problems about the derivative and its applications given in a diagnostic questionnaire. In this paper she presented distorted usage of the Intermediate Value of Theorem and a theorem concerning extreme values and critical points. The distortion was classified into three types:

> 1. Distortion of the antecedent: applying the claim within conditions which differ from those stated in the theorem.
> 2. Distortion of the consequent: ignoring or modifying the original claim while keeping the conditions intact.
> 3. Distortion of an inference rule: applying invalid logical inference on a given antecedents and consequents. (p.38)

An example of one problem discussed in terms of the above classification, including four justifications and points for discussion, was:

> Given $f(x) = |x - 6|$. What is the derivative at $x = 0$? at $x = 6$?

Orton (1980, 1980a, 1983, 1983a, 1984) has published a substantial amount of research papers on students' understanding of calculus concepts related to his Ph.D. study.

The study was based on individual interviews with 110 students in the age range 16-22 years. The subjects of the study came from school sixth forms or from colleges concerned with teacher education. All of the students had chosen mathematics as a major subject in their own studies. The different tasks on differentiation and integration proposed in the interview were regrouped after the interviews, and the responses were scored for each item on a scale from 0 to 4.

The difficult group of items found is presented in Table 2.1, together with the means of scores obtained by students on each item.

College students experienced rather more difficulty with the items given in Table 2.1 than did school students, except the items related to the introduction of integration based on the sequence of approximations to areas under the curve.

Without going into further detail, it seems important to note that students have great difficulty in conceptualizing the limiting processes underlying the notion of derivative (rate of change) and integral. As mentioned by Orton (1980a), "Comprehensive treatments of both rate of change and limit are important if they are to form a base on which calculus is to be built" (p.214).

Table 2.1. The items and mean scores

Item	Mean Score (out of 4)	
	School	College
A. The Introduction of integration based on the sequence of approximations to areas under the curve		
• Limit of sequence equals area under graph	0.78	1.00
• Limit of sequence of fractions and from general term	1.67	2.48
• explaining integration	1.57	1.62
B. Graphical items concerning rate of change		
• Rate of change from straight line graph	2.22	2.02
• Rate, average rate and instantaneous rate	0.88	1.18
• Average rate of change from curve	2.22	1.92
• Instantaneous rate of change and tangents	2.02	1.46
C. Some applications of introduction		
• Integral of sum equals sum of integrals	1.10	0.60
• Volume of revolution	0.95	0.88
D. An application of differentiation and integration		
• Relationship between differentiation and integration	1.78	1.20
E. Understanding of differentiation		
• Differentiation as a limit	1.88	0.95
• Use of δ-symbolism	1.52	1.40

This thesis has also been summarized by Tall (1986) and Artigue (1991). Tall's summary included critiques on some of Orton's categorisation. For example, Orton categorised the explanations given for 'dx' such as "the differential of x" as incorrect. As Tall emphasised, the response that "may be regarded as correct is denied and classified with other errors that may be of a different nature"(p.11).

2.2.1 Summary

The literature review on calculus carried by Dreyfus (1990) reaches to similar conclusions to those above.

> 1. Students learn the procedures of calculus (finding limits, differentiation, etc.) on a purely algorithmic level that is often built on a very poor concept images. Difficulties with a process conception can be explained in terms of the students lacking the necessary high level abstraction (objectification) of both of the function concept (as an object) and the approximation processes involved in differentiation and integration.
>
> 2. Visualization is rare, and if it occurs the cognitive link between the visual/graphical and the analytic/algebraic representations is a major point of difficulty.(p.125)

An investigation of these studies has indicated a common message that there is a great concern about conceptual understanding.

It is encouraging that mathematics educators are seriously reflecting on their research endeavors with an emphasis on conceptual developments in a technological environment.

CHAPTER THREE

THE EXPLORATORY STUDY

In this chapter the data gathered in the series of interviews with teachers are examined in detail. The teachers interviewed are from English universities and sixth form colleges and have been using computers or calculators in calculus teaching. At the time of these interviews only a small number of enthusiastic teachers had been using computers or calculators in calculus teaching.

Twelve teachers were interviewed, to provide a detailed description of the problems faced in different technology environments. The teachers interviewed represent users of the different main systems under consideration and volunteered to participate in the study after being invited by telephone. The interviews were carried out between January and June, 1993.

The questions in the interview (see Appendix D) were structured around a flexible set of questions and the Nonschedule-Structured Interview model was used. As Nachmias and Nachmias (1987) said, this form has four characteristics:

> • It takes place with respondents known to have been involved in a particular experience;
> • It refers to situations that have been analysed prior to the interview;
> • It proceeds on the basis of interview guide specifying topics related to the research hypotheses;
> • It is focused on the subjective experiences regarding the situations under study. (p.237)

Even though the meetings between the interviewer and the teachers were structured, teachers were given considerable liberty in expressing their ideas and feelings. The order of questions varied according to the answers given by the teachers. This was done in order to make data collection somewhat systematic for each interviewee, and to keep the interview fairly conversational, thus anticipating and closing logical gaps in the data. The interviews were recorded after the interviewees were approached for their consent.

3.1 Theoretical Background of the Study

In mathematics teaching a wide variety of information technology (IT) tools have been used to aid learners' mathematical development. Calculus is a mathematical subject in which much of the current innovation is centered around the use of IT. Calculus is the beginning of higher mathematics and presents many students with obstacles which they never surmount. It is thus important to address the question of what software and hardware best support the learning of calculus.

Literature surveys (Fey, 1989; Kaput, 1992; Steen, 1987) identify the main new technology tools used in calculus instruction as specifically designed software including numerical, graphical and symbolical computation systems. Fairly new on the scene are electronic notebooks using hypertext (Kaput, 1992).

Computer numerical tools such as a*Spreadsheet* provide attractive opportunities to enrich the teaching of concepts and to extend the reach of problem solving in secondary school and university mathematics. For instance, students' work with a *Spreadsheet* establishes a solid intuitive understanding of variables and functions by investigating relations among variables (Fey, 1989).

Graphical tools such as *Graph Plotter* software and handheld devices such as graphical calculators allow the user to draw the graph of functions by algebraic rules, giving the possibility of investigating the relationship between the algebraic expressions and their graphical representations (Tall & Thomas, 1989; Fey, 1989). Fey also noted that graphic tools will enrich the understanding of algebraic forms - giving visual images of symbolic information - and construction of rules for functions that fit given graphic patterns.

Symbol manipulation tools, for example *Derive* and *Maple*, could have a significant impact on mathematics education: (i) extension of the complexity of algebraic expressions, (ii) reorientation of instruction to focus on the conceptual understanding, and (iii) exploration of patterns in algebraic reasoning - leading to the discovery of important general principles (Fey, 1989). Studies by Heid (1988), Palmiter (1991), and Judson (1990) have shown that the use of symbol manipulation software in teaching calculus has a positive effect on students' conceptual learning without loss in symbolic computational skills.

While it is important that independent evaluations are made for each of these tools it is the teacher who will use them, and his/her role will be crucial to their future success. The attitudes of teachers to software and the manner in which they use them is thus worthy of study. Furthermore, teachers may use a tool in a variety of ways within their courses (Heid, 1988; Judson, 1990a). What are these ways and are they dictated by the tools or are they independent of the tools?; What facilities are needed and/or used?; Does the extent of student use affect students' achievement, performance, or attitudes?; Does use depend on the type of course or institution? This study was designed to address the questions above from the viewpoint of teachers' attitudes and beliefs.

Computers and calculators are available in schools and universities throughout the United Kingdom, but it is not clear what their most effective role is in the process of learning at universities and sixth form colleges. For educational uses of computing, resource requirements such as technical support, institutional support, time, courseware, technical information, and shared experience - are needed (Zorn, 1987) since a growing body of research confirms that using technology as a learning tool can make important differences under the right circumstances (e.g. Ruthven, 1990). The study conducted by Lawrenz and

Thornton (1992) showed that factors such as the numbers and location of computers; the continuing assistance with computer use; the numbers, type, and quality of software; and the time available for the use of computers were perceived as serious problems in the use of computers by the 7-12 science teachers. Moreover, effective computer use is also dependent on the existence of a combination of important conditions: the subject area, characteristics of the student population, the teacher's role, the way in which students are grouped, the design of the software, and the level of access to technology (Bialo, & Sivin, 1990).

As it was difficult to know where we have been, where we are now, and how far it is possible to go in teaching with technology, a survey was needed to look at the factors previously mentioned. Zorn (1987) said:

> Modern computing arises unprecedented opportunities, needs, and issues for undergraduate mathematics. The relation between computing and mathematics is too young, and changing too quickly, to admit definitive positions. (p.917)

It was also argued by Hammond (1994), that the framework for evaluating the impact of information technology needed to consider the aims of the teacher, the principles of the software and an interpretation of what went on in the classroom.

The conclusions drawn from the present study are likely to be of value in demonstrating the prospects for higher mathematics classrooms using calculators and computers, by looking at different things they do. It is important to keep in mind that these interviews are only a small sample of what is going on in this exciting arena today.

3.2 The Analysis

A thorough investigation of interviews with teachers has enabled a categorisation of the current situation in calculus in terms of (I) resources, and (II) policy and classroom environment.

I. Resources

Relevant questions in this area are –

• Hardware - Which computer system or calculator is being used?

• Software - Which program is being used?

• The number of computers - How many computers and calculators are available?

• The location of computers - What is the location of the computer systems?

II. Policy and Classroom Environment

Some questions in this area are –

• Course - In which course or subject is the technology being used?

• Subject of students - Which students use the technology?

• Number of students - How many students are involved with the technology?

• Students' experience with the technology - Have students had previous experience with the technology?

• Time spent with the technology - What proportion of the time do students spend with the technology ?

• Type of technology use - what type of technology use is employed?

The potential effectiveness of technology-based instructional programs is dependent upon the resources. But the key to making effective use of the resources is dependent upon the development of policy to control or design the classroom environment. The main characteristics of each environment in universities and sixth-form colleges are presented in Table 3.1.

Table 3.1. Results of interviews with teachers about the use of technology in university departments and colleges

Universities	CATEGORIES											
	Hardware	Software	Number of computers or calculators (students ratio per computer)	Location of computers and calculators	Course	Subject of students	Number of students	Type of application	Time spent with technology	Students previous experience with technology	Duration of instruction	Materials
I 28.01.1993	PC	Matlab	30x 2 (1:1)	Computer lab	Fundamental Core Course	Mathematics	200	Exploration of Matlab and exercises from print material	Used as needed	Mixed	20 hours lecture and 10 hours example class	Handout • Beginners Guide • Intermediate's Guide • Advanced Person's Guide
I 28. 01. 1993	PC	Maple V	30x2 (1:1)	Computer lab	Core Course	Computer scientists	55	Exercises from print material	6 sessions (50 minutes each)	Almost none at all	15 hours theoretical discussions and exercises work	Handout • Beginners Guide • Sketching and Functions •Differenciation and Integration • Recurrence Relations
II 16. 02. 1993	Apple Macintosh	Mathematics 2 (CD- ROM)	7 (3:1)	Computer lab	Calculus	From all university departments	20	Hypertext Tutorial	Through all class hours	Mixed	5 class hours	Special preapered material •Differentiation and Integration • Mathematical Functions • Geometry and Algebra
III 05. 05. 1993	BBC	Maths Foundation (developed for courses in there)	5	Computer lab	Service Course for Engineering	Manufacture, Civil and Mechanical Engineering	80	Revision (Practice)	At the beginning an hour a week but later in their own time	Mixed	3 hours a week	
IV 20. 05. 1993	PC	• CALMAT • Derive •A Graphic Approach to Calculus	3x20 (1:1)	Computer lab	Service Courses for Engineering	Engineering	200	Practice and concept development	1 hour a week	Mixed	2 hours lecture and 1 hour tutorial a week	Worksheets
V 08.06. 1993	PC	Spreadsheet	150 (1:1 or 2:1)	Computer lab	Calculus	Engineering Science Computer Studies Mathematics Education	400 Engineering 25 Education	Illustration and concept development	Outside class time	Nearly all	3 hours a week	•Worksheets (open-ended tasks) • Handouts (work through)
V 08. 06. 1993	PC	Derive	20 (3:1)	Computer lab	Mathematical Modelling	Computing Mathematics	30	Problem solving	Sometimes 2 hours a week or in their own time	All have	2 hours a week	• Introduction Material to Derive • Some Problem for Modelling
Sixth Form Colleges												
I 02. 02. 1993	BBC master Casio fx-7700	• Graphic Calculus • Function Graph Plotter • Graphic Calculator	8 computers and each student has his/her own calculator (2:1 or 3:1)	Computer lab	Calculus	A-level Maths Students	30	Exercises from print material	Used as needed	Mixed	5 hours a week	Handout • Curve Sketching •Power Functions • Gradient • Logarithms
II 01. 03. 1993	Casio fx -7000	Graphic Calculator	One calculator for each student (1:1)	Classroom	Calculus	A-level Maths Students	20	Exercises from print material and investigation	Used as needed	None at all	5 hours a week	
III 18. 03. 1993	Archimedes	•Mouse Plotter •Derive •Mega-Math	• 8x2 & 9 •1 in each classroom (1:!)	• Computer lab • Classroom	Calculus	A-level Maths Students	4x15	Demonstration and practice	Used as needed	Mixed	5 hours a week (upper-sixth) 6 hours a week (lower-sixth)	Worksheet •Introduction to Derive • Limits, Integration, Sum , and etc.

Based on the interview results, two distinct styles of technology use were identified - tutorial use and software tools. In tutorial use, traditional programmed learning was extended and expanded to include a wide range of illustrations and activities by the student: e.g. *Mathematics 2*. In software tools, students were taught to use powerful software packages, initially through carrying out exercises, and then increasingly for their own independent investigation: e.g. *Graphic Calculators, and Maple.*

As seen in Table 3.1, most of the students who have not had previous experience with computers or graphic calculators used them individually or in pairs. Nevertheless, the extent of the use of technology depended on the enthusiasm of the teacher. The majority of the computers in schools and university departments were kept and used in the laboratory. Unfortunately, technology is still not an integral part of students' learning patterns.

3.3 The Interviews

As mentioned in the previous sections, information technology influences mathematics education by providing tools to assist with mathematical procedures - arithmetic calculations, symbol manipulation, and graphic displays. What is needed here is to show what is different about those various technology uses in classrooms and what those differences mean in terms of educational goals, appropriate pedagogical strategies, the nature of the subject matter, the nature of learners and learning, and the classroom facilities.

The interviews given here were considered to be representative of different environments which included numerical, graphical, symbolical or electronic notebooks.

3.3.1 Interviews in Universities

Teacher I (University I)

He is a lecturer in Applied and Computational Mathematics, and has been using *Matlab* in courses like numerical analysis, optimization and programming courses for three years.

Approximately two hundred first year mathematics students were involved in the Fundamental Core Course with *Matlab* in the first term of the academic year 1992-1993. Of those two hundred students, one hundred and fifty or so were also involved with *Matlab* in the second term. In each term, twenty hours of lectures and ten hours of example sessions were carried out. In the example sessions, sheets of examples were given to students to work through. The computer, however, was put into use when it was needed. For example, one week a computer session was introduced instead of having example or lecture classes. Students worked through the handouts which consisted of the exercises to be done specifically on the computer. One example of the exercises was:

> Write a function m-files for the function
> $f(x) = \frac{\sin(4x)}{4x}$
> Use it to produce the graph of this function for x in the range $[-\pi, \pi]$.

The following excerpt taken from the handout given to the students shows the reason for using *Matlab.*

> Matlab is essentially a rather clever form of the hand-calculator available on a PC. As well as most of the functions that are available on your hand-calculator Matlab includes many more, not least of which are a suite of graphics functions which allow students to picture the results of their calculations. It has, in the background, a simple form of programming language but we will not bother about that just yet; we will concentrate on the basic ideas.

In the mathematics department, there were two computer laboratories with 30 computers each connected to the campus network so that students could have access to *Matlab* directly. One computer specialist was always present at tutorials in the laboratory. He said:

> ..the lecturer goes to the computer laboratory and works with students but it is done more or less on an ad-hoc basis ... it will be up to lecturer....

Talking about the students experience with computers, he said:

> ....well, we have got the whole range. We have got people who have not used a computer at all, people who have used it regularly and even may have their own computer.

In describing the purpose of the first term course he said that the course was to teach students how to use the computer and the software. The focus was purely on how to set up the machines to use *Matlab* and *Maple.*

His opinion about *Matlab* was "... it is very easy to get them use, and to produce graphics for deeper levels of work".

However, there were a few specific difficulties which students generally met with, such as differences between the packages in the use of semicolons or full stops in writing down expressions. In addition to these difficulties mentioned above the teacher also commented:

> ..... students need lecturers to be saying to them: solve this problem in Matlab, go on to look at that, and what happens in Matlab? I don't think we have, as lecturers, fully taken on the board that the facilities are available. This involves extra work as well. Each lecturer has to think about how he or she could use *Matlab* in their lecturers and that's extra work and everybody is feeling over worked and underpaid so there is a resistance doing that...

His answer to the question – "Is there any difference in the content of the course or is it the same as a traditional method?"– was:

> .... that is still experimental but the content of the course should change because *Matlab* and *Maple* open up possibilities for students to go and try things that would have been too difficult to do things in the past....... you can get them to solve open ended questions, for example, solve this problem and now tell me what happens if you change things, what happens if you add something else?.......

As usual, lack of facilities was a problem: in his case, using a computer in the lecture room. For the future, his expectation was to be able to use the computer with some projection facilities, such as an overhead projector. Because of shortage of funds, he did not anticipate this being available for a few years.

Finally, he said that there were two things which came out of the use of *Matlab* and *Maple*:

• students with a less strong mathematics background would have a resource which they can use, which might then make it easier for them to solve problems;

• students would gain a deeper understanding because all teachers would be able to pose harder problems which would be difficult to solve manually. With the use of computer tools it would become relatively straightforward to expand their ideas.

Teacher II (University I)

He is a lecturer in the Mathematics Department, and began to use *Maple* in a core course for computer scientists, in the first term of the 1992-93 academic year. The version of *Maple* used was *Maple V*, running on a PC. The class consisted of fifty-five students. Each student was given a handout describing the course. The syllabus of the course fell into four parts:

• Functions and Differentiation (unit 1);
• Integration (unit 2 and 3);
• Calculus of functions of several variables (unit 4);
• Recurrence relations (unit 5) and curves (unit 6).

The course aimed at teaching the methods which students have done or have not done at A-level because what students have learnt in elementary calculus can vary enormously due to the variety of examination syllabuses. There were three 50 minute classes a week over a period of ten weeks. Two of those classes were used to work through the units. There were mini-lectures (15-20 minutes) followed by tutorial classes. Most of the time was spent on exercises in the units. In the third class, there were six sessions over ten weeks with *Maple*, the remaining four weeks taken up by the short multiple choice tests. In other words, one class each week was either on *Maple* to help students understand the concepts, or a 50 minutes test. Students were supposed to come to *Maple* sessions for an hour between 9 and 11 but there was difficulty when they all came at the same time, as the teacher emphasised "if they come gradually they can all get started...." and so the problem would be solved.

During each laboratory session, students worked through a handout, learning new commands, and applying them to different problems. The first handout contained information to enable students with almost no previous experience in the use of computers to learn to use the 'symbolic algebraic manipulation' package *Maple* on the university's PC network. The purpose of the second session was to use *Maple's* graphics capability to get more insight into functions. The main body of the exercises consisted of plotting polynomial functions in order to decide what is their range and where their zeros lie. In the third session students used *Maple* to get an insight into differentiation and integration. Students plotted a polynomial function and its tangent line simultaneously on an interval to observe how the curve becomes closer to the straight line by replacing this interval by taking smaller intervals. Later, students investigated the definite integral as the limit of the Riemann Sum. This exercise required students to use *Maple* in finding sum and limit. The following session let students investigate in more detail some aspects of integration. The exercises consisted of several improper integrals. The fifth session required students to use *Maple's* graphics facility to get insight into the geometry of the function of two variables. The last session showed how the package can handle recurrence relations and also sketch curves. The following is a portion from the second computer session handout:

> .....Plot the following polynomials. See if you can decide what is their range (approximately) and decide from the graph (approximately) where their zeros lie
> (i) x^2-7x+2 *on* $[0,8]$
> (ii) x^2+5x+3 *on* $[-6,0]$
> (iii) $2+3x-x^2$ *on* $[0,4]$
> (iv) x^3-12x^2+7x+5 *on* $[-3,11]$
> The Maple program also has a facility for sketching a function over its whole domain by 'squashing up' the ends of the x-axis. The command is implemented as
> plot (f(x), x=-infinity..infinity);

The reason for choosing the computer science students was:

> they would presumably be the students who would find it easiest to use the computer , surprisingly they didn't (find it especially easy). They found using a PC quite difficult .... found just operating system, PC, difficult not so much Maple.... the other reason was... the computer science students might be more sympathetic to the idea.

What he said about the reasons for using the computer in teaching can be summed up in the following way:

- The computer helps students understand calculus concepts;
- You can draw pictures of functions of several variables which cannot be drawn on the blackboard practically;
- You can move pictures around and thereby obtain a better understanding of what a maximum and minimum is;
- The computer draws pictures very quickly;
- The computer can help students to concentrate on a particular part of a problem.

In addition to these he said "we don't know how much it is helping them".

He mentioned that *Maple* was easy for students to use, but *Derive* was easier than *Maple*, and added that he would have chosen *Derive* but it was expensive to buy a site license. He prefers *Derive* because of the possibility of seeing graphics and symbolic expressions on the same screen. Furthermore, he said that *Maple* was much more a programming language and was less user-friendly, than *Derive*. Since this interview, the possibility of having graphics and symbolic computations on the same screen was made available for *Maple* as well.

Teacher III (University II)

He is a lecturer in the Mathematics Department. He has been using *Mathematics 2* (CD-ROM) which can be used for self-study and also had supplementary material for lecturers who want to use this software. In this hypertext style CD-ROM, the following subjects were included:

- Preliminary Material: Geometry and Algebra;
- Mathematical Functions;
- Differentiation;
- Integration;
- Solving Differential Equations;
- Mathematical Modeling Using Calculus.

The reason for using it can be summarised as follows:

1. It was appropriate for different groups of students who had different mathematics backgrounds. Students can use *Mathematics 2* at different levels and as he said:

> good students will progress more rapidly; weak students will have more time to work on concepts. They pace themselves.

2. It was very open-ended and let students simply explore the material.

3. It had computer games, graphics, exploration, animation and sound which engage students in mathematics and make difficult concepts clear and attractive.

Students involved in this teaching were foundation year students who were not sure what degree they wanted to read. They were taking lectures from every department on the campus. As part of the foundation year students attended a variety of courses. This course consisted in 5 class hours approximately and was studied by students at their own pace.

A classroom of seven Macintosh computers was available: 2 or 3 students had worked at each machine. He felt that 2 or 3 students to a machine encouraged interaction and discussion of the material.

In order to get feedback, very open-ended questions were given to the students at the beginning of the course, and then students were left to explore the material, thus gaining an association with the given question. The students had given reports after working on the questions. They also had written an essay on whether the software was useful or not, at the end, as this software had been in use for about one year, providing valuable feedback.

The benefit of the software was that students were enjoying mathematics and were more willing to ask questions about mathematics. Furthermore, questions were becoming quite informative and students were beginning to think about the concepts involved. Students, however, had expected to be taught, and instead they had to work at their own pace. As they lacked confidence in using computers, this caused difficulties for many students.

Finally, he said that we had to explore the media, such as graphics, animation, or computer algebra systems, in order to use computers in an effective way. At the same time, he made some comments on other software such as *Mathematica* and *Maple*:

> .. Mathematica and Maple are ideal for doing mathematics, but I am not quite sure they are for teaching mathematics yet in their own right. They need an additional teaching package.

<u>Teacher IV (University III)</u>

She is a part-time lecturer in the Mathematics Department and has been teaching a Service Course for engineering students for 3 years, using the software written by herself. The software is called *Foundation Maths*, running on a BBC, which comprises the following topics:

1. Lines - investigates line graphs
2. Quads - draws quadratic curves
3. Powers - draws power functions of x
4. Inverse - draws functions with inverse
5. Trigonometric functions
6. Add Trig - draws sum of trig functions
7. Hyp - draws hyperbolic functions
8. Plotgr - a general function plotter
9. Sketch - a demo on curve sketching
10. Polar - investigates polar coords

11. Conic - in any position

12. Limits - of a function

13. Quots - estimates derivatives

14. Deriv - illustrates derivatives

15. RSum -illustrates Riemann Sums

16. Taylor - expansions

17. Newton - illustrates law of cooling

18. Vibs - introduces vibrations

19. Numerical programs

20. Exit

There were 5 BBC computers in the computer laboratory. The class consisted of 80 first-year manufacturing, civil, and mechanical engineering students, who have not done A-level Mathematics. The software was used to help students revise the things they have forgotten or needed more practice at. Each student had different gaps in his/her mathematical background. The course constituted 3 hours of lectures and 1 hour of tutorial each week throughout the year. At the same time, the students had the possibility to use the computers outside the normal course time. However, she said, they seldom did this as they did not perceive this of value to them and did not have any extra time. The students usually had used the computers without support from the teacher. Some of them had a lot of experience with them, while some had very little.

She conveyed the reasons for using the computers in her teaching as follows:

• the computers are useful extra resources;

• the computers have graphical power;

• the computers calculate quickly;

• visualization is very important but difficult,

and also emphasized that she would like to have an opportunity to demonstrate things in the classroom environment. She had written this program because BBC computers were very easy to use, and the programs like Mathematica were difficult to get and use.

Teacher V-VII (University IV)

These three teachers are lecturers at the School of Mathematical and Information Sciences. They have been working mainly with engineering students and using computers in calculus teaching in a major way for the last two or three years. In the department, there were three computer laboratories each containing 20 PC's, and they were timetabled for use. They were all available for one afternoon, but in general, the three labs were very heavily timetabled for at least 65 % of the time. Some of the students had their own PC's but some of them had never touched a keyboard. As said by Teacher VII, "If students do not have experience with the computer, they get it very soon. It is no big deal...... ".

Teacher VI had been using *CALMAT*. Students had two lectures plus a tutorial plus an hour in the computer lab each week. She, however, said that it was not enough, especially as the students' capabilities varied - some were brilliant and others were really struggling. She remarked that they might make very good engineers but they had very poor mathematical backgrounds. She did not think that two lectures were enough to keep them going. Some of the students had A-level Mathematics but the rest of the students came through a BTEC or HND (vocational course). She said that "*CALMAT* does not really enable students to do calculus. It just gives them a background and lots and lots of examples to do themselves".

The students had their own disks and chose how to work through the network of subjects in the menu of *CALMAT*. The subjects were interlinked with suggested pathways between them. The students were encouraged to follow these pathways because each unit depended on previous work. When they completed the unit, it changed colour. So, the students had a record of what they have done. They had problems with colours dropping out occasionally, so she had encouraged students to make a copy of the network. Talking about the difficulties experienced, she said:

A lot of students have great difficulty working out where the answer in the computer is different from theirs, and obviously sometimes computers reject a correct answer, and then they worry that they have done something wrong but they have not. They do not have confidence..... The computer does not help them find common mistakes like their algebra. I think they do need somebody helping but they have not found a great difficulty in actually using it as a tool.

She had lectured to the students on each topic, and then students had followed the program to understand it and do examples. She said:

Calculus was the one thing I had to lecture on. The students really needed somebody to explain it to them. Calculus was the one thing they could not follow.

She found that the students' attitudes to the computers changed throughout the year. They were very keen when they came, they had thought it was something that was different from what they had done at school. But by the end of the year, they had felt that they missed the help that the teacher would give and the sort of individual questioning. So, she said that the novelty wore off during the year. Furthermore, she had found that the students had become much more accurate in calculations. They were not embarrassed if they got it wrong on the computer, and were much more careful in what they entered. As she said, when lecturing it would have been difficult to set extra problems but it was easy to set them on the computers.

Teacher V had been using *Derive* and *A Graphic Approach to Calculus*. In Engineering, he had used them very heavily to introduce the concept of the gradients and tangents rather than integration. In his opinion although the graphic calculus package is very user friendly, in terms of teaching calculus it cannot be used on its own. He said that he had to use worksheets to tell them to do such and such an example using the package, and then it was a very easy thing for them to pick up and get into. But, there was a very marked difference in the speed with which they got through the material when they were using *Graphic Approach to Calculus* and *Derive*. Talking about the difficulties, he said:

when I was teaching Derive to my first year students, they seemed to have some difficulty getting into it and they were finding twin learning objectives. They need to learn to use package and they need to learn the Maths. One said ' either we learn the maths or we use the computer, and I found I couldn't do both'. So, they end up pushing the buttons blindly without thinking about the mathematics. We did not have long enough getting used to the package. I think next year I may change the way we teach it.

His comment on whether computing should change what students should know was:

> I think this is a big issue. CALMAT is really just something that slots into a very traditional approach to mathematics and it is not aiming to change at all the philosophy of education. It is just adjusting the practice. Once you get to a computer algebra package like Derive, you actually start to ask questions. For example, if you look at last year's exam papers, there is a question which says ' differentiate Log(x^3Sinx/Cos2x). They need to know all the techniques of differentiation to do this. If you are going to have access to a package like Derive, you are not going to set a question like that on an exam. The students would just type it in and read the answer off the screen. The question comes as whether they need to know those techniques.......... I suspect we are going to change. Particularly with engineers, there are two ways to go. Either the engineering department says we do not need maths teaching anymore, because they just punch buttons and get things done for them, or we are going to argue successfully that what technology does is actually allow us to concentrate far more on giving some genuine understanding of the concepts. So much of our teaching at the moment has been based on teaching them techniques rather than understanding.

Teacher VII's comment on that was:

> If you want to focus on Calculus, Derive does what it does very rapidly, but you can reach a limit of what it does and then you have to know something in order to be able to get to the final answer or in order to be able to cast problems in a sensible form to which you can apply the tool. Some of that work is actually going to be using traditional skills. One fairly obvious example is integration by parts, and trying to integrate $x^n e^x$. For some reasons Derive insists on having $x^n e^x/x$. This is fairly simple example. You cancel off the x on the bottom with x on the top but Derive just doesn't seem to want to do it. So while Derive will get things in standard forms, they may not be the standard forms that you are actually looking for. So, the students need to know the concepts, otherwise they can not effectively use these tools. They still need some traditional techniques.

<u>Teacher VIII (University V)</u>

He is a lecturer in the Mathematics Department and has been teaching Calculus for half of the first year to Engineering, Science, Computer Studies, Mathematics, and Education students. He said that they had around 400 students in the first year engineering program and they taught them in four groups, but the groups in the education area were much smaller; about 20 to 25. A *Spreadsheet*, running on 150 PCs, was used in the course for three years or so; and besides, software like *Maple* was under consideration. The reasons for using computers were:

• Visualization - Computers can immediately give people a picture of something and they have the option of doing more advanced things as the mechanical part is taken out.

• Scope for experimenting the 'what if' question - They give you the scope to do 'What if I change that number?' and 'How will that change the solution?'.

• Concentration on conceptual development - They allow you to get through the concepts and the mechanical side plays a much smaller part.

Worksheets were given to students either to be worked on only in the computer laboratory or to be worked in the computer laboratory to develop concepts there in the first place, enabling them to bring the theory in the classroom.

The majority of the students worked on their own on the computer. He said that he often encouraged them to work in pairs because he thought they did get a lot from it. Some of them, however, still preferred to work alone. The course was three hours a week. The time on the computer was not spent during classroom hours. He said that an increasing number of students had their own machines and some of the programs were either public domain or were shareware. So they used it at home and just came in to print things out. In addition, he mentioned:

> A lot of students get enthusiastic about using the computer and they come in the evening. The labs are all open access so they use them outside of the class time. We do organise just a few special sessions actually in the computer lab but it varies from one course to another. Some courses we have the regular one hour a week on the computers with somebody there as a consultant to help out while they are on the computers. Other courses; you set them tasks and the students go away and find their own time on the computer, so it varies from one group to another.

As most of the students had their own computers, the experience of the students with the computers was not an obstacle anymore.

As he said, students were very keen on computers since they realised that with software like a *Spreadsheet*

> they can get good graphs very easily for their lab reports... They realise that it is much easier to do that than plot the graphs by hand. So once they have caught onto that, they realise the benefits of that and so it is not only the immediate things that we show them they can do with the various packages. They very quickly realise that there are a lot of things they can do.

Students did not have difficulty in using the *spreadsheet*. The main difficulty, however, was, he said:

> how to tell the students clearly what you want them to do on the Spreadsheet without actually effectively leading them through it step by step. It is getting that balance, so you can either give them a ' do this, do that' which is not really much use or you give them such an open-ended thing that they come back and say ' we do not know what we are doing'. Somewhere between those two is the right balance and sometimes we hit it and sometimes we go too far one way or the other......

There was one important question to be answered: whether the computers should change what is being taught. He responded to that question with an example:

> if you are talking about matrix manipulation for instance, and finding matrix inverses. .... If you are teaching mathematics to engineers, for instance, then I believe it is no longer necessary to go through all the business of co-factors and adjoint matrix because you have a computer that can do that mechanical work. Therefore, we can put a lot more emphasis on the concepts of the inverse; the job that the inverse does and how you use it. .... It actually allows you to concentrate on understanding the useful things of mathematics rather than laborious........ We have had a fair bit of debate over the impact on the calculus. .... we used to teach differentiation from first principles and talk about the limiting process and do it algebraically. I think very few of the students really took in very much of what was going on; particularly the engineering students. So recent years we stopped it and we just taught it purely as an algebraic technique. But then now we have got the computer tools, we are actually beginning to talk about the first principles ideas again. You have taken a lot of the heavy mechanical work out of it....... Having that tool has had an impact on the curriculum and I think it is going to have more.

Finally, the following problems were encountered while making use of the computers in teaching:

• political issue - being able to buy the software and being able to store it on the network. ' To get the money from somewhere to do that you have to present very strong cases' he said.

• curriculum issue - development of new materials to make use of the software.

• software issue - being able to find menu driven software because he said that it was so much better for the students who did not have strong mathematics background.

• practical issue - not having enough printers to go around and the speed of development of the software is much greater than the facilities for the students to use.

<u>**Teacher IX (University V)**</u>

He is a lecturer in the Mathematics Department, who used *Derive* on 20 PC machines with students who were doing a degree called Computing Mathematics and the work they did was mathematical modeling. He also used it in their final year of work. The number of students was 30 for the modeling and about 40 for the final year. He has been using it for four years.

In the courses, the students had two hours each week. Modeling students were often in groups looking at a problem and then moving from the room that they were in into the computing room, but the other group had lectures and tutorials, and used *Derive* in their own time.

The computers were included in teaching to save time because they cut through the detail of mathematical analysis. He said:

> ... if you do mathematical modeling, the aim of the situation is to solve a problem. I am interested in getting them to set up the model and to draw some conclusions from the analysis of the model. So the analysis of the model may involve a certain amount of mathematics. I am not too bothered how they do that mathematics......

He had given an introduction to *Derive* to the students at the beginning and then students had worked through a document. He said that he was there to guide them through.

He described the way computers were being used with an example:

> ... a problem where I am trying to determine the location of sleeping policeman. The problem comes down to essentially having to minimise a function which is a quotient. They are relatively awkward functions and the students may or may not be able to struggle through it. Particularly if they want to use the second derivative to determine whether it is a minimum or not. So, consequently, the students would simply set this up on the machine, find its derivative, solve the equation resulting from putting the

> derivative to zero which turns out to be quartic and then finding the second derivative and checking that it is negative at the stationary point we have determined......

But, he mentioned that there were no fundamental changes in the content of the course and in the teaching time.

His comment on whether computing should change what students should know was:

> I think that we need to put less emphasis on the drill exercises for example, that students need to do, and allow us to put emphasis on students to get a better understanding of what is going on. There is no real need to teach a multitude of integration techniques to non-mathematicians. Instead, we should be putting emphasis on what integration is, whether it is possible always to integrate a function.

The primary problem he said was that students took a problem and just loaded it into the machine without giving too much thought to how it could be simplified. So they tended to be less critical of what they were doing when they used *Derive*. He appeared certain that *Derive* did not enhance algebraic skills or mathematical modeling and he did not see *Derive* as a mathematics teaching package but rather as an assistant to help them do mathematics more quickly and accurately. He used it essentially to cut through the details of mathematical analysis.

3.3.2 Interviews in Sixth Form Colleges

Teacher X (Sixth Form College I)

She is the Head of Mathematics department at the college. She said that the use of computers in teaching was up to each member of staff. *Function Graph Plotter* (FGP), *Graphic Calculus* packages, and *Graphic Calculators* were available in the school at the time of the interview. In the computer room there were about eight BBC master computers, thus students worked in groups of two or three around one computer. As far as graphic calculators were concerned there were 15 calculators in the department. Each student usually had one. Handouts were ordinarily given to work in the computer room. Introductory sheets about options had been given on which approximately 15 minutes were spent. While students were in the computer laboratory she discussed things with students if necessary, and afterwards they had a class discussion as well.

Teachers had the opportunity to bring a computer into the classroom environment but it had been mostly used in Statistics teaching. They occasionally used the computer as a demonstration.

FGP, for example, was used in differentiation of trigonometric functions to show what the derivative of a function was, to plot the curve, and then plot the derivative. She said "... it is also useful to demonstrate the difference between using degrees and using radians...".

Later, power functions were introduced by getting students to investigate lots of different power functions using FGP again. In addition to these she said ".......using computers is slightly complicated. We have to book the computer room a week in advance, for one hour, or one and a half hour, sessions......".

In sum, the computer was used as needed. As for the time spent on the computer in calculus teaching she said:

> .... it might well be that we use the computer room twice a week for 3 or 4 weeks at a time. Then we get on to other topics in the syllabus and we may not use computers for a few weeks. It varies......

Graphic Calculators were used in lots of situations, but they did not actually differentiate. Quite a lot of students had bought Casio-7700 which gives numerical values of integration. She, however, said ".. it might be useful for that but we haven't developed material as yet....".

The familiarity of students with computers varied. Some of them had done computer studies in GCSE, and many of them had their own computers. Nevertheless, she said ".... there are significant number of students who haven't done computing before....", and added that:

> ... if they have experience, they feel more confident in the first place. To start with the students who haven't done computing before can feel a little bit inadequate. It takes

them a long time to key things into the computer and they get worried when things go wrong....

She mentioned that FGP was very easy to use but some of the programs were difficult because of the notation.

Her response to whether there is any difference in the content was "... single Maths A level students are doing less formal proof, using the computer. They can use the computer to demonstrate results....".

What she said about students comments using the computer was:

> .... students are enjoying using the computers and graphic calculators, but not for the whole of their time. Most students say it makes a change and that they enjoy a variety of methods of teaching and learning, so they are happy to have them available as part of the course. ... But a few students say they prefer reading books and don't like doing things practically.....

One difficulty experienced by the teacher was the very large group of students. At the beginning the main problem was that students' algebraic background was not very strong. The difficulty experienced by the students, on the other hand, was lack of confidence. They were doubtful about pressing the wrong keys.

Students had a test once every half term. The purpose was to learn how they got on in all subjects and they had a questionnaire on which teaching methods they preferred, at the end of October.

In her opinion, A level mathematics should be rethought, as to whether we should be using more of the software and hardware available, rather than spending a lot of time teaching routine tasks which can be easily done by machines. Then teachers would have time to teach mathematics at a deeper level.

For the future, she wishes to get good software used in GSCE teaching so that students can work by themselves when they have problems with particular topics.

In this session, *FGP* was used with an A-level mathematics class studying the 'power functions' section of their A-level syllabus. The class had done some work on 'differentiation' in earlier lessons. The lesson lasted one and a half hours. The first 15 minutes, and the last 30 minutes were spent in the classroom but the rest of the time was spent in a computer laboratory.

In the first 15 minutes she made an introduction to the 'power functions' in order to prepare students for work on the computer. She told a story (losing money on gambling) in order to introduce '2^x'. The money lost was in this manner:

1, 2, 4, 8,16,...........

and then she asked which kind of relation existed in this sequence. After that the class went to the computer laboratory to investigate the graph and the gradient of this function.

The teacher gave each student one sheet of paper to work on on the computer, and two students were around one computer machine. The contents of the paper are given below:

- Plot the curve $y = 2^x$ and predict what its gradient function will look like. Superimpose the graph of the gradient to check if you are right.
Repeat for $y = (1.5)^x$, $y = 3^x$, $y = 5^x$.
Do you notice anything different about curves?
- Can you find a value of a such that $y = a^x$ and its gradient function are exactly the same?
The value of a for which this is true is called the exponential number, e, and $y = e^x$ is called an exponential function.
- Investigate the graphs of other exponential functions,
e.g. $y = e^{2x}$, $y = e^{-x}$, $y = e^{3x}$.
Look for the relationship between each of these functions and its derivative?
Are your results consistent with those you would have expected, using appropriate rules of differentiation?

While the students were studying this sheet, the teacher went around and discussed some issues, either by asking questions to help the students understand or by answering questions the students had. Lastly, the class went back to the classroom again.

In the classroom, the teacher summed up some properties of 'exponential function' and gave some examples about it in real life, such as radiation, and the oscillation of a pendulum.

Teacher XI (Sixth form College II)

She is the head of the Mathematics Department. She has considerable experience in the use of calculators. But in her college graphic calculators had been used for one year only. Each student has got one Casio fx-7000. If they did not own one there were available Casio fx-7000 to be used in the college.

The syllabus used for teaching was SMP 16-19. But as the calculus SMP 16-19 book starts with notation, it was not used to introduce the topic. Mathematics was taught for 5 hours a week. The reasons for using calculators were as follows:

- students have the picture of a concept.
- calculators help concept building.
- calculators give confidence to the students.
- students feel positive.
- students can find other ways of doing the same thing.
- calculators help doing mathematics experimentally.

The students have not had previous experience with graphic calculators. She said that she first avoided teaching notation until students started to understand the concepts behind differentiation. The reason she gave was that, difficulty came in notation and language, rather than concepts. Thus, students were encouraged to use exploratory methods prior to meeting.

About the use of calculators this teacher stated that difficulties may be experienced by the teacher if he/she does not feel confident in using them.

In addition, she said that the teacher should think carefully about how he/she could use calculators in an advanced way.

In this session, Casio fx-7000s were used with a sixth form mathematics class studying the 'differentiation' section of their A-level syllabus. The class had done some work on 'differentiation' in earlier lessons. This class was observed for half an hour. The lesson was set in a classroom, and every student had one calculator. The teacher wanted students to use their own calculators to answer the following question:

1) $f(x) = x$
Draw the graph of the 'difference quotient':

$$\frac{f(x+0.0001)-f(x)}{0.0001}$$

What is gradient of the function f(x)?

2) Choose f(x) as x^2, and $2x^2$, and then do the same thing as above.

Students can use zoom-in to guess the result. While the students were working on these problems, the teacher went around the tables and asked some questions to help them understand. One difficulty observed by the author while one student was working with the calculator was the use of parenthesis, for example,

$(2x + 0.0001)^2$ instead of $2(x + 0.0001)^2$.

This is probably a consequence of the level of algebraic awareness rather than a difficulty in using the calculators.

Teacher XII (Sixth Form College III)

She is a mathematics teacher in this college and has been using *Derive* and *Mouse Plotter*, in calculus, run on an Archimedes. Before using an Archimedes, she used a BBC computer with her own written program. She had taught one upper-sixth and one lower-sixth A-level students. She had attended a 3 day course on the use of *Derive* to teach maths, in March 1992 at Bradford University. She had TVE project money to extend the use of computers in teaching maths generally. Her approach to teaching was, first let students do some basic things manually, then let the computer take over because otherwise students did not know what was going on. She said that she was using computers as a teaching aid.

The college had 2 computer rooms with 18 computers each. Moreover, there was another computer room in the library with 9 computers approximately and sets of worksheets for the students to go through, such as introduction to 'Derive', 'limits', 'integration', and 'summation' and one additional computer which was available in the classroom for students' use. The intention was to get students familiar with these and then allow them to go and investigate other areas for themselves. Furthermore, there was the *Mega-Math* program on the computer which was a revision package. She said that although it has got a lot of faults, it was useful.

Teachers did not need to book the computer rooms the same way they used to. At the time of the interview, 6 upper-sixth form groups and 5 lower-sixth form groups for which, Cambridge AEB and AS syllabi were used, were present. There were approximately 15 students in the classroom.

Her approach to teaching was first to get students to plot a graph, e.g. $y = x^2$, by themselves and calculate the gradient by drawing tangents (different students took different values), and then to plot the gradient function on the same graph. Later, *Mouse Plotter* was used to draw accurately the graph of a function, and to guess the derivative function and superimpose it on the graph. She said ".... once they have got the idea, then the computer does it quickly. The computer doesn't eliminate what I want to get first".

The students had repeatedly used *Mouse Plotter* for differentiating x^3, x^4, $3x^2$, $4x^3$, x^2+1 etc. in order to guess the derivative function and to plot it on same axes to test its validity. For differentiating trigonometric functions, again students had done a graph first, then used *Mouse Plotter*, and discussed the use of radians/degrees. *Derive* was used, again after students did something like

$(3x+1)^2$ and $(3x+1)^3$ manually, for: chain rule differentiation. She said *Derive* can be used for:

• expand;

• differentiation;

• factorise;

• limit: to get computer to calculate for different values of h then limit;

• partial fractions: to expand and check answers.

In addition, she said "Diagrams and graphs can often convince students that something works when an algebraic proof does not".

Students' algebra was not good and this was true, even for some of the best A-level students. That is why graph plotting had been used so much.

The use of computers did not reduce the teaching time. As she said,

> When used for drawing graphs it does reduce teaching time but for a student to get to know packages like Derive can be very time consuming. If every student is encouraged to use a computer this can either be done by moving the whole class to a computer room or a flexible learning approach is needed in order to use an individual computer in the classroom. Getting students to investigate problems using a computer individually takes much more organising than letting whole class consider the problem together.

The difficulties experienced, as a teacher, were as follows:

• while using one computer in a class, not all students can see the screen;

• getting the students to come and use the computer;

• finding the time to look at the software and make sure how to do it; teachers need to know exactly what they are doing and how to use it;

• willingness of students to use the computer; sometimes students do not want to touch a computer.

Finally, as everyone said, they were still practicing and expanding.

3.4 Discussion

At the time of this exploratory study, the use of technologies in calculus teaching was limited to a fairly small group of enthusiasts in sixth-form colleges and universities. Two distinct styles of technology use were evidenced from the interviews:

- tutorial use
- software tools.

In all cases except one, technology was not a substitute for class instruction but rather a tool for extending learning in a computer laboratory. Therefore the 'software tool' use was dominant.

This study provided further insights into the advantages of technology in teaching, and the inhibitions in using it. Although some advantages could be seen, when the teachers talked about the use of technology in teaching, some problems were also noted. Among the advantages remarked and mentioned were:

- pupils become more independent;
- the teacher's role is shifted from being the provider of information to a supporter or facilitator in the joint search for solutions;
- the computer is a motivational tool;
- learning can be reinforced through a visual medium;
- facilitates cooperative learning;
- increases geometric intuition.

Factors inhibiting the use of computers were:

- problems of infrequency of use because of the problem of finding the time in a crowded timetable to use the computer lab;
- quality and availability of educational software;
- lack of facilities, such as a portable computer or overhead projector.

These findings would suggest that the lack of resources is the major obstacle for the use of technology in the schools. Without adequate resources in a school it is not possible to use technology on a regular basis. Moreover, teachers must be keen on using technology in teaching.

Furthermore, there were some problems with the interaction between students and the computer:

• lack of confidence;

• unfamiliarity with particular systems;

• the time it takes students to learn how to use a software package;

• an unwillingness of some students to spend time on the computer.

Students were active participants in the learning process in a computer laboratory while technology was used for various reasons which include: the building up of intuitions prior to the theory in the classroom; the visualisation; the practice of some exercises; doing mathematical modeling - cutting through the need for lengthy calculation.

While teachers acknowledge that teaching with technology is undergoing change and development, they could not state explicitly what they had achieved through its use. However, many held a strong belief in its value. Finally, another area of research which needs careful examination is the way in which we assess students learning throughout a computer-based learning environment in a classroom. The first large scale comprehensive, longitudinal and in-depth investigation entitled *The ImpacT Report* was carried out by a team of researchers in the Centre for Educational Studies, King's College London. The focus of the work was on pupils' learning and classroom activity involving IT in four school subject areas - mathematics, science, geography, and english - at three age levels 8-10, 12-14, and 14-16 (Watson, 1993). Similar studies need to be conducted in higher education.

It is likely that the range of views presented here is not specific to this particular sample, but it could also be representative of a wide range of teachers and courses.

3.5 Recommendations

To those who are considering the introduction of technology into mathematics teaching, the following points should be taken into consideration:

• Quality of software. We must address the question: In what ways can it be used? For example, according to Hatfield (1984) categorisation, there are eight different types of uses: practicing, tutoring, simulating, gaming, demonstrating, testing, informing, and communicating. Another question is: What are the characteristics of it? Does it ask questions? Does it need teacher support/intervention? Does it give answers? Does it allow students to control the kind of feedback?

• Background of students. Most classes contain students with a greater diversity of mathematical backgrounds than before. Degrees involving mathematics as a service course now take students with varying mathematical backgrounds.

• Pedagogy. How do you use technology in mainstream teaching and how do you use it with students who are less able or need special help?

• Appropriateness of content. Are computing activities matched to the goals and objectives of the course? It would be a serious distortion if they are not relevant.

• Style of teaching and using of technology. We must take the tool and find intellectual uses for it. As David Hill said at the 7th SEFI Conference, "just presenting mathematics on a screen is not enough! and vision is not visualization: to see is not necessarily to understand".

Technology might require a new approach to the teaching/learning procedure.

• Appropriate role models for teachers and pupils in the use of computers and calculators. Mathematics teachers traditionally have been transmitters. Meanwhile, students have been receivers. But technology, especially computers, may be used as a source of information with which a student experiments, discovers, generalizes, and conjectures. The teacher's role must be passive, not active: as a manager of their students' construction of mathematical knowledge.

• Integration of technology with learning. How can we make students confident in using it, and an integral part of their learning activities? Meanwhile, how can we bring the challenge and excitement of exploring the potentials of it into their learning?

3.6 Concluding Facts

Here are some of the points that emerged from the interviews:

• Mathematics is changing because of technology. Computers and calculators support students' work in a variety of ways; by giving access to the necessary knowledge; by providing feedback to the students and by monitoring their progress. They allow and encourage a number of changes in the classroom roles of the teacher and the students.

• Mathematics itself is also changing because of technology.

• Teachers play a key role. The use of technology is dependent on the enthusiasm of the teacher.

• Time and Content have an important impact. In schools and universities greater emphasis is placed on covering the syllabus within a limited time because the time schedule is very tight.

• Software often has an 'Entry fee'. It takes time to learn how to use it before it becomes useful.

• The uncertainty about the contribution of technology to students' performance. During the site visits many people were asked about the evaluation or assessment of the course. Nearly all the answers revealed that they were not sure what they achieved with technology.

• Tactical changes. Technology changes the power structure within the classroom and opens the way to increase student participation, and to phrase questions in a more open-ended and demanding manner. Nearly all the students had 'hands on' experience with computers or calculators.

• Time with technology. It was used mostly as needed and rarely at fixed times. Only in one case it was used throughout all class hours.

• Style of uses. Primarily used as tools and mostly introduced in a hands-on manner. Computer usage was typically with students working individually, or two or three students in one group.

3.7 Looking at the Future

Based upon this analysis, a reasoned estimate was made on the future use of technology in UK mathematics classrooms in the next ten years or so.

The influence and pressures from government itself can be a major contributory factor to the implementation of technology in mathematics classrooms.

The speed of development of the software is much greater than the facilities for the students to use. Also, there is not enough hardware to answer the needs for the time being. But, it is hoped that every student will have their own hardware, and every university and school will have some more to enable them more extensive use of technology in the next ten years.

For the future, all the teachers hope to have the facility to use a laptop computer with projectors so that they can do demonstrations. The next ten years promise to be different in four different ways: (1) more numerous and more powerful technology, (2) more satisfactory software to be used in mathematics teaching, (3) supporting materials to make use of each software efficiently, and (4) more teachers who are willing to use technology in their teaching.

Software Mentioned in Text

CALMAT (1987) , Glasgow Caledonian University.

Graphic Calculus (Tall, 1986), Glentop Publishers Ltd.

Derive (1988) , Soft Warehouse, Hawaii.

Function Graph Plotter (FGP) (1982), Shell Centre, University of Nottingham.

Maple (1980), University of Waterloo, Canada.

Mathematics 2 (1992), The Cambridge and Keele Universities team.

Matlab (1985) The MathWorks Available from

Mouse Plotter (1988), Shell Centre, University of Nottingham.

CHAPTER FOUR

RESEARCH DESIGN AND METHOD

This chapter presents the framework of the study; defining subjects and instruments; considering the method of data collection and analysis; describing different learning environments, with and without computers; and stating the criteria for scoring items in the instrument called the diagnostic test. The learning environments are described in order to present a comprehensive picture of them and make better explanations of different learning outcomes. Briefly, this chapter is a guide to a reader as well as the researcher.

4.1 Subjects

Subjects were first year university students majoring in engineering enrolled in calculus courses. A total of 147 students from four classes of a different English universities were used. Two groups constituted computer groups (91 students), while two groups constituted non-computer groups (56 students). The calculus course taught in each classroom was quite similar in terms of the subjects covered and the teaching method used in the classroom. The main difference was that computer groups interacted with computers as part of their course. The first computer group (ComG1) used software called *A Graphic Approach to Calculus* (Tall, Blokland & Kok, 1990a). The second computer group (ComG2), however, used two tutorial software *CALMAT* and *CALM*. In addition, ComG2 did a diagnostic test on the computer at the outset and *A Graphic Approach to Calculus* was occasionally used as a demonstration aid by the lecturer. The computer laboratories were equipped with file server and networked PC's.

The general criteria for the selection of a sample were as follows:

1. Each student should take the pre-test and the post-test. Every student in the sample of each class completed the pre-test and the post-test, so information on comparable performance by the same student was available.
2. The sample of each class should be representative of three different mathematics backgrounds (A-level, BTEC, and Others such as overseas qualification).
3.The sample of each class should learn differential and integral calculus.

The sample of students tested in each class represented a varied mathematics background in terms of previous education: A-level, BTEC, foundation year and others. In addition to that a small portion of students in each class were from overseas countries. Figure 4.1 shows the different countries from which the total population of students originated.

<u>COUNTRIES REPRESENTED BY THE SAMPLE</u>

Brunei (1)	Canada (1)
Greece (2)	Ireland (3)
Israel (1)	Italy (3)
Kenya (1)	Malaysia (1)
Mauritius (1)	Botswana (1)
Nigeria (1)	Portugal (1)
Sri Lanka (1)	United Kingdom (126)
USA (1)	

Figure 4.1. Countries represented by the sample and the number of students

Table 4.1, below, presents the details of the sample, including the number of students tested in each class, the number of students entering with A-level qualifications and with qualifications other than A-level, the number of students in terms of their nationality, and the median of age in each group.

Table 4.1. Major features of the students in the computer and the non-computer groups

Groups	The number of students	The number of students in terms of their mathematical background		The number of students in terms of their nationality		Median of Age
		A-level	Others	British	Others	
ComG1	35	14(40%)	21(60%)	31(89%)	4(11%)	19
ComG2	56	43(77%)	13(23%)	52(93%)	4(7%)	18.5
N-ComG1	20	7(35%)	13(65%)	14(70%)	6(30%)	21.5
N-ComG2	36	8(22%)	28(78%)	29(81%)	7(19%)	21
Total	147	72(49%)	75(51%)	126(86%)	21(14%)	19

The course done by non-A-level students was mainly BTEC. In total there were only 11 female students.

4.2 Instruments

For the purpose of this investigation, a diagnostic test and a computer attitude questionnaire were designed.

4.2.1 Diagnostic Test

The test (see Appendix A) consisted of 10 questions some with several items to be answered in an open-ended format. The test included 26 items altogether. The two main components of this diagnostic test were 'differentiation' and 'integration'. The topics covered under those were given in Chapter One (see Figure 1.2). Each item was typed with spaces, on which all work was to be done. The diagnostic test set out to measure first year engineering students' understanding of different aspects of calculus in a computer and a non-computer environment and to attempt to categorise their errors in each item of the test.

The test was administered twice: one prior to instruction in differentiation and another after instruction in integration. The students were pre-tested in order to establish whether they already had mastery over the objectives of the test. The pre-testing was necessary since we (a) knew little about the competence of each individual student and (b) were working within an instructional system that can adapt instruction to individual differences in competence. The students were post-tested in order to measure the growth in conceptual development. The pre-test and the post-test were carried out without students' prior knowledge. They were given in a 50 minute regularly scheduled class, with an interval of approximately 4 months between them.

The diagnostic test was specially designed for the study but was based on well-established items which were already used by other researchers. The test will be discussed in more detail in Chapter Five.

4.2.2 Computer Attitude Questionnaire

In order to investigate the attitudes of first year engineering students toward the use of computer in a calculus course, a computer attitude questionnaire was developed, consisting of 16 questions. These were a mixture of Likert-type statements, multiple choice questions, open-ended questions and questions in a Yes-No format. Likert-type scales included 7 statements, six worded positively and one worded negatively, with five possible responses: strongly disagree, disagree, neutral, agree, and strongly agree. The questionnaire was administered after the post-test, but only administered to the computer groups.

The first three open-ended questions dealt with students' orientation towards the use of the computer in calculus. The following three open-ended questions and the seven Likert-type statements dealt with students' perceptions of the use of computers in the calculus course. The last two Yes-No format questions and the open-ended question dealt with students' background in computer use and the way they have used it. The students were also asked to give further comments if necessary.

The data obtained was analyzed for evidence of the differential effects on students' performance.

The first task in analyzing open-ended questions was to impose a structure on the responses by identifying and separating the discreet utterances made. Different categories were identified, depending on the type of response.

4.3 Procedure

The central premise of this study was to evaluate the impact of computers on students' learning of calculus, particularly 'differentiation' and 'integration'. Phase I was a quantitative study that looked statistically at: (i) the performance of first year engineering students, with access to computers, on items in the diagnostic test, in comparison with that of students of similar background, but without access to computers; (ii) the effects of students' interest and involvement with a computer and their perceptions of the use of computers in calculus on their performance. The variable (residual score) used to evaluate the performance of the students was the subtraction of the predicted post-test score from the observed post-test score. The predicted post-test score was obtained from the linear regression of the post-test score on the pre-test score pooling all the results together. This study used a 4 x 2 factorial design: treatment in each group (ComG1, ComG2, N-ComG1, or N-ComG2) x students' mathematics background (A-level or Non-A-level). This design was adapted from that defined by Keppel (1991;p.204).

Following this macro level analysis, Phase II looked at the origins and the nature of errors, within and between each treatment using qualitative methods and analysis.

The diagnostic test was designed during 1993 and the process of design involved:

(i) analysis and selection of items from the tests designed by Amit and Vinner (1990), Cornu (1983), Orton (1980), Selden (1989), and Tufte (1988).

(ii) writing items to be used along with selected ones.

(iii) trial of the designed test. The first draft of the test (see Appendix B) was given to one class involving 14 A-level students, to check on administration, difficulties and any misunderstandings of the items. The items were then subjected to a thorough review, which led to changes in their wording, or deletion of them. The new draft was then administered to another class involving 18 A-level students in another college. Finally, 26 items were constructed to be used in the main study. Further details of the diagnostic test design are included in Chapter Five.

The main study was carried out in England during academic years 1993/1994, and 1994/1995. The sample which consisted of first year engineering students were enrolled for the core module or basic calculus course. During the year 1993 and 1994, more than 40 teachers teaching basic calculus to engineering students in different universities were contacted by mail or phone and invited to participate in the study. Surprisingly, only three teachers in 1993 and one teacher in 1994 in different universities were able to offer their classes to be used in the main study.

Data gathered from the pre-test, the post-test and the computer attitude questionnaire, was coded and analyzed using SPSS-X. For each item of the diagnostic test a six-point scale was assigned and the scoring process was checked by a specialist in this field (Dr. John Monaghan) who also checked the scoring of individual responses. This increased confidence in the face validity and the reliability of scoring. The scoring criteria will be given further on in this chapter.

Following the post-test a series of interviews with a few students in each class were conducted (see Table 4.2). The students were classified into the two categories according to their mathematics background: A-level and Non-A-level. A stratified random sample of interviewees (stratified on these categories) was selected from the list of the students ordered randomly.

Table 3.2. The number of students interviewed in each group in terms of their mathematical background

Groups	Mathematics Background		Total
	A-Level	Non-A-Level	
ComG1	2	4	6
ComG2	2	3	5
N-ComG1	3	3	6
N-ComG2	1	3	4
Total	8	13	21

The ideas that emerged from these interviews were used to provide in-depth information about the students' conceptions by clarifying responses, as part of the investigation obtained from the diagnostic test. The main line of inquiry was to ask the student to explain what he or she could remember about the thinking that was behind the written response - always trying to get the student to relate the response to the answer which he or she had given. The students were encouraged to change previous answers and to modify answers given previously. Interviews were tape recorded and transcribed. The interview took the form of a discussion. The students' answers were used as the stimulus for further questioning. Each response was followed up with further questions. New concepts arising from the students' explanations were explored.

In an effort to clarify students' conceptions, one student's lecture notes was examined from each group.

4.4 The Teaching Treatments for the Computer and Non-Computer Groups

West, Fensham and Garrard in 1985 highlighted the starting point in investigating the cognitive structure of students:

> .... is the public knowledge. We can define that from such things as the syllabus, the lectures, the examination papers, and hand-outs to students, This will yield the knowledge bits, but rarely the intended structure. The public knowledge derived from these sources provides the framework to investigate the learner's private understandings. We can investigate, for example, if the learner has internalized each of the sets of propositions and algorithms, how the learner has related them together, and what has been compressed under each of these new "nodes" by the learner.(p.37)

The course taught to all groups aimed at providing a foundation of mathematical methods and techniques for use in engineering courses. The general entry requirement for all courses was BTEC Level III but besides that there were some other students who came through A-level courses, foundation year courses or some other qualifications. The topics covered included 'functions', 'differential calculus', 'integral calculus', 'ordinary differential equations', 'matrix algebra', and 'complex numbers' (except ComG1). A similar pattern of examining in groups was followed. The assessment was based on: (1) examination at the end of the year; and (2) course work assessments, under one heading in non-computer groups and under two headings in computer groups: (a) two or more phase tests; and (b) self-study assignments or tests which were undertaken in the computer laboratory.

4.4.1 Computer Group 1 (ComG1)

This group, including approximately 50 aeronautical mechanical and automotive engineering students, used *A Graphic Approach to Calculus*. Students in this group completed seven worksheets (self-study assignments) requiring the use of *A Graphic Approach to Calculus* in addition to attending one lecture and one tutorial each week. A computer laboratory including 24 computers was booked for students for two separate hours each week. Students could use either of them. Computers were also available at the weekends and evenings. The seven self-study worksheets dealt with the following topics:

Sheet 1: Graphs of standard functions. This sheet was intended to:

- teach how to use the GCAL package to draw graphs;
- help students to investigate the behavior of the graphs of a range of different functions including: (i) powers; (ii) quadratics; (iii) exponentials; and (iv) natural logarithms.

Sheet 2: Slopes of curves. By the end of this sheet students were expected to be able to:

- know how to determine the gradient of a curve at a point on the curve by looking at the gradients of a sequence of straight lines;
- identify the formula for the gradient at a general point of some standard functions;
- plot a simple approximation to the gradient function of some standard functions.

Sheet 3: Gradients and the shape of curves. By the end of this sheet students were expected to know:

- how to recognise certain features of the gradient of a function from its graph
- the effect of the gradient sign changes on the graph of a function
- the meaning of a turning point and how to recognise it from a graph.

Sheet 4: Partial fractions. By the end of this sheet students were expected to know:

- what a partial fraction expansion of a rational expression is
- the form of the partial fraction expansion for a given expression
- how to find the complete partial fraction expansion of a given expression.

Sheet 5: Numerical integration. By the end of this sheet students were expected to know:

- more about the interpretation of a definite integral as an area
- how to numerically computer approximate values of definite integrals, which cannot be evaluated mathematically, by using the trapezium rule and simpson's rule.

Sheet 6: Properties of trigonometric functions. By the end of this sheet students should know:

- how to determine the values of sin, cos and tan of any angle
- what the basic properties of sinusoidal wave forms are
- how to add sinusoidal wave forms of the same frequency.

Sheet 7: Oscillating systems. By the end of this sheet students should know:

- the behavior of an undamped oscillating system
- the mathematical form of a damped oscillation
- how to use GCAL to solve differential equations
- the effect of over damping.

The students had to hand in several self-study assignments. These assignments came in two blocks: sheets 1-4 and sheets 5-7. The first block should be handed in on the day of the first coursework test. The second block should be handed in on the day of the second coursework test.

The recommended book to the students was, *Engineering Mathematics* by Stroud (1987).

In this class graphs played a far more central role than in the non-computer groups. Students examined a variety of computer-generated graphs, reasoned from these graphs, and studied the key features of a graph and the similarities, differences, and connections between graphs. Students also used paper-pencil procedures in (a) basic algebra, (b) differential calculus, (c) integral calculus, (d) ordinary differential equations, and (e) matrix algebra.

As had been stated by the class teacher, the purpose of using computers was to:

1. do experiments.
2. see the power of visualization.
3. assist conceptual understanding.

4.4.2 Computer Group 2 (ComG2)

This second computer group (ComG2), including 120 mechanical engineering students, had two lectures plus one tutorial session each week in addition to 2 computer workshops. All students had access to computers, not only during fixed hours but also after school, in the evening, and during weekends. In the tutorial session students were split into four groups, and each one of which was taught by the class teacher, a part time lecturer, a post doctorate and a post graduate student. Two computer-based tutorial software packages, *CALM* and *CALMAT*, were used throughout the course in the computer laboratory and *A Graphic Approach to Calculus* was mainly used in the classroom for the purpose of demonstration every two or three weeks. The book recommended was *Essentials of Engineering Mathematics* by Jeffrey (1992).

The computer-based tutorial software was unit-based, with the transition from unit to unit being controlled by computer-based diagnostic tests. The purpose of the diagnostic testing was to monitor the performance of students in learning and using the unit skills. Answers from the tests have been used to provide a *CALM*- or *CALMAT* - based personalised scheme of studies for each student, with the aim of correcting possible gaps in basic mathematical knowledge. To make the use of the computer attractive, students were told that if they achieved more than 60% for assigned *CALM* or *CALMAT* units they would not need to take the semester 1 exam. All but eight did computing as an exam. The computers were used to give feedback regarding students mathematical knowledge to monitor them throughout the course and to experiment with graphical calculus.

4.4.3 Non-Computer Groups (N-ComG1 and N-ComG2)

The non-computer groups were taught only using paper and pencil methods. The course for the first non-computer group (N-ComG1) involved 35 students majoring in Civil Engineering. Instruction consisted in 3 lectures, including some tutorial time around 20% of the course time. The list of books given to the students contained Bajpai (1974), Jeffrey (1974), Stephenson (1961) and Heading (1976). The teacher also emphasized that some students used the book of Stroud (1987).

The course for second non-computer group (N-ComG2) involved around 50 students studying electronic and mechanical engineering. Instruction consisted of 2 lectures and 2 tutorials/practice sessions each week. The book of Stroud (1987) was recommended to the students.

4.4.4 Programs Used by the Computer Groups

To arrive at feasible results from a learning point of view, summarization of a few of the major characteristics of each software used in the classrooms are important and pertinent to issues involving "the impact of computers on learning".

A Graphic Approach to the Calculus Package (GCAL)

This Package was the program used by ComG1 and ComG2. This package was specifically designed to be educational, presenting graphs and solving equations, integrals and ordinary differential equations numerically. It does not contain any symbolic manipulation capabilities. The IBM version is a translation and extension of the Omnigraph and Graphical Calculus packages for BBC. A couple of screen examples are given below.

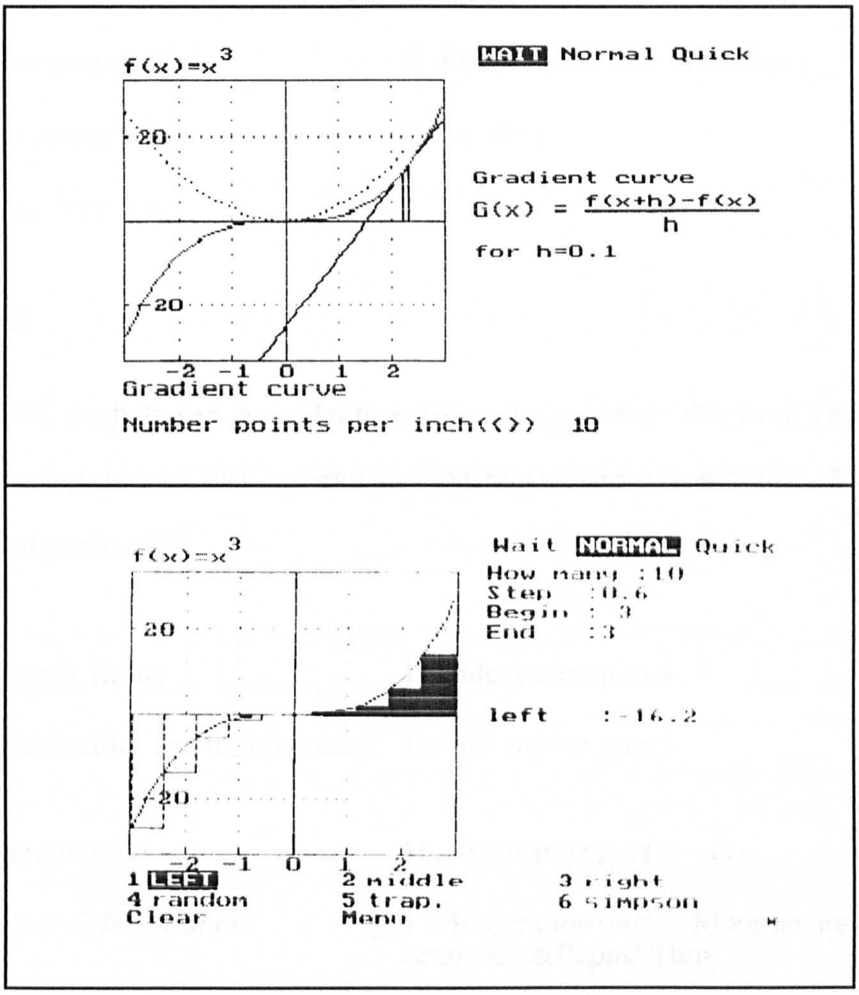


Figure 4.2. Two sample screen examples for Graphic Calculus

(reproduced for Tall, Blokland & Kok, 1990a)

Although the software can be used in a structured way, the design is principally open and exploratory, allowing students to pose and answer their own questions. The main menu of the package is given below.

1. Looking for the formula	A. Parametric Curves
2. Graph	B. Curves in space
3. Magnify	C. Complex functions
4. Gradient (Differentiation	D. Differential equations
5. Area (Integration)	E. Second order differential equations

6. Solving equations

7. Taylor polynomials

8. Blancmange

9. Define function

F. Simultaneous differential equations

G. Functions of two variables

H. Options

CALM

25 units, each based on a lecture topic as given in the first year of an engineering course, are covered in CALM and are given below. The units assigned are in italic.

1. Algebraic limits
2. Discontinuities and trigonometric limits
3. *Differentiation*
4. *Further differentiation*
5. Tangents, normals and related rates
6. Maxima and minima
7. Practical Maxmin and curve sketching
8. *Series, hyperbolic and harder differentiation*
9. L'hopitals rule and parametric differentiation
10. Revision Unit
11. Partial differentiation and simple integration
12. Definite Integrals
13. *Integration part 3*
14. Integration part 4
15. Integration part 5
16. *Integration part 6*
17. Integration part 7: More on areas, centroids &Papus' Thm
18. Integration part 8: Arcs, surfaces and more on volumes
19. Integration part 9
20. Revision Unit
21. Numerical integration
22. Numerical solutions of the equation f(x) =0
23. Ordinary differential equations I
24. Ordinary differential equations II
25. Exam unit - Differentiation

After given a review of the theory covered in the lecture, the student is taken through worked examples based on this, and finally tested using randomly selected questions. A management package is included which stores the student's results for monitoring purposes.

A couple of screen examples of the units that the students might choose to work on are given below (e.g. Unit 3- differentiation). Note that this is not an exact copy of the CALM screen but it is a close transcription.

UNIT 3 - DIFFERENTIATION

1 FIRST PRINCIPLES 2 PRODUCT RULE 3 QUOTIENT RULE 4 CHAIN RULE 5 TEST SECTION 6 QUIT THIS UNIT	INSTRUCTIONS Either: (i) Use space bar and return key, or (ii) Press key required or (iii) Use key pad.

Figure 4.3

If a student chooses option (1) from the above screen, the following screen will appear.

FIRST PRINCIPLES

1 THEORY
2 WORKED EXAMPLE 1 involving
 f(x) = 4x - 7
3 WORKED EXAMPLE 2 involving
 f(x) = 1 - 3x
4 WORKED EXAMPLE 3 involving
 f(x) = Sin(x)
5 RETURN THE MAIN MENU

INSTRUCTIONS
Either:
(i) Use space bar and
 return key, or
(ii) Press key required
 or
(iii) Use key pad.

Figure 4.4

If a student chooses option (2) from the above screen, the following screen will appear.

Let's differentiate from first principles the function f defined by
 f(x) = 4x - 7

STEP 1: You will be asked to calculate f(x + h)
STEP 2: You will be asked to calculate [f(x + h) -f(x)]/h
STEP 3: You will be asked to calculate the limit of the
 ratio you found in STEP 2 as h tends to zero

Write down f(x) = 4x - 7 on your ROUGH PAPER
and attempt steps 1, 2, and 3

Press ANY KEY to continue ....

Figure 4.5

CALMAT

CALMAT is an interactive mathematics tutorial system. It has been designed as a self-paced learning procedure with each topic containing theory as well as test questions. There are fifty different topics in CALMAT all aimed at pre-A level maths students or at anyone with a good basic understanding of GCSE mathematics. The main topics can be summarized as follows.

Arithmetic	Trigonometry
Algebra	Equations
Co-ordinate geometry 1	Co-ordinate geometry 2
Matrices	Functions and their graphs
Complex numbers	Differentiation
Geometry	Integration

Computer Test

The mathematical content of this test was written by two third year students during their work placements at the university where the ComG2 group was chosen from. This test was produced using QuestionMark for Windows Version 1.2. The test was used as a diagnostic test to decide which, if any, CALMAT units the student needs to look at before starting his/her engineering or science degree. The reader should note this test is not used for research purposes and is only described in order to understand the working of CALMAT.

The questions are a mixture of multiple choice, text and hot spot style questions. If students get a question wrong they are asked to answer an easier question on the same topic. Students require a calculator for this test and there is no time limit.

With text questions students need to use the keyboard to type in the answer and then click on the OK key to proceed to the next questions. Hot spot style questions are very similar to multiple choice questions. There are 4 answers listed, one being the correct answer. There is a marker in the bottom left of the screen. The students use the mouse to click on this marker with the left button on the mouse and keep pressing it down to move the marker over the correct, before releasing their finger off the button. When they are satisfied that the marker is on the correct answer they press the OK key to proceed.

When students have answered their question a box containing text appears on the top right of the screen. It tells them whether they have got the answer correct or incorrect. An example of a multiple choice question is given below:

A line passing between the points (9, 15) and (-5, -10) has what values for its slope and intercept.

Slope	Intercept
O 1.786	0.499
O 1.786	-1.071
O 0.560	-1.000
O 0.560	0.499

OK

Figure 4.6

An example of a text question is presented below.

A student estimates that his grade will vary directly with h, the number of hours spent studying and inversely with the number of beers drunk. During the first semester, he studied 500 hours, drunk 200 beers and had a grade 4 average. During the second semester, he studied 300 hours, and drunk 300 beers. What is his average now?

OK

Figure 4.7

An example of a hot spot question is given below.

What is the largest common factor and binomial factors of $5a^2b^2 - 5ab^2 - 10b^2$

[A] $5b^2(a^2 - a + 1)$

[B] $5a^2b^2(3a + 2)$

[C] $5b^2(a + 2)(a - 1)$

[D] $5(a^2b^2 - 3ab^2 - 2b^2)$

OK

This is the marker to move

Figure 4.8

If students get a question wrong they are asked to answer an easier question on the same topic. If they get the answer wrong again in the second trial they get a message such as:

No! Look at CALMAT unit 1

Your answer is wrong

Continue

Figure 4.9

4.5 Items of the Diagnostic Test and Criteria for Scoring

The first step in the preparation of open-ended diagnostic test data for processing was the categorization of the subjects' responses to the individual test items on the basis of whether they were the same or different. Therefore, a key was constructed to characterize students' performance in terms of acquired concepts or skills rather than simply in terms of scores on the test.

The subjects' answers and explanations on each item were classified separately. Subsequently, a thorough comparative study of all answers was carried out, bringing out a six-point scale specific to each item with reference to Okeke (1980). The highest score of 5 was awarded for responses which teachers and examiners regarded as being entirely correct at elementary calculus level, while the lowest score of 0 was reserved for no answer or "don't know". The general specification for scoring is shown in Figure 4.10.

SCORE	RESPONSE
5	Answers including the relevant ideas, relationships, generalizations, applications or explanations considered adequate at the introductory calculus level
4	Answers including relevant ideas with some evidence of knowledge of relationships, generalizations, applications or explanations
3	Answers including important sub ideas without necessary reasoning and explanations
2	Answers containing (an) isolated but relevant fact(s)
1	Totally wrong answers, no understanding or false reasoning
0	"Do not know" or missed

Figure 4.10. General specification for scoring

The preliminary versions of the categorisation for each item were discussed with a specialist in this field before the final versions were determined. The scoring scale was defined in more precise terms for each item as seen in the following figures.

4.5.1 Question 1: Derivative

This question dealt with the 'derivative' concept. The items are given below:

1(a) What is a 'derivative'? Define or explain as you wish.
1(b) What does it mean that the derivative of $f(x)=x^3$ is $3x^2$?

Item 1(a)

Categorisation and examples of responses to item 1(a) on the definition of 'derivative' are presented in Figure 4.11 with an indication of the score awarded.

SCORE	RESPONSE
5	Answers explaining the 'derivative' correctly in terms of function and gradient at a point or in terms of limiting value of the slope of the chord (e.g. "gradient function. You can find the gradient of any point on a curve if you differentiate an equation and then substitute.....")
4	Answers including two important main ideas such as function and gradient (e.g. "a derivative is a function that has been differentiated. The derivative is the derived function and is also gradient of graph")
3	Answers including important sub ideas concerned with function, rate of change or gradient (e.g. "rate of change of one thing in relation to another", "a derivative is a function which has been differentiated from an original one" or "slope of a curve")
2	Answers explaining examples or procedure (e.g. "if you differentiate, say $3x^2$, the derivative of that 6x" or "derivative is obtained by differentiating a function")
1	No understanding of the concept of 'derivative' and false reasoning (e.g. "when an equation is derived, a complete set of rules can be used to obtain the derivative" or "derivative is a function of a function")
0	"Do not know" or missed

Figure 4.11. Item 1(a) with stated objective and examples of responses showing the score awarded based on the scoring criteria

Item 1(b)

Examples of responses to item 1(b) on explaining the meaning of 'the derivative of $f(x) = x^3$ is $3x^2$' are shown in Figure 4.12 with an indication of the score awarded.

SCORE	RESPONSE
5	Answers explaining 'the derivative of $f(x) = x^3$ is $3x^2$' in terms of gradient or rate of change at any point (e.g. " the rate at which f(x) changes in relation to x is given on the graph $3x^2$. This is the gradient function" or "if $y = x^3$ then $\frac{dy}{dx} = 3x^2$. Therefore the gradient can be calculated at any point")
4	Answers explaining 'the derivative of $f(x) = x^3$ is $3x^2$' in terms of rate of change or gradient, with an explanation (e.g. "rate of change of f(x) to x is given by the equation $3x^2$")
3	Answers explaining 'the derivative of $f(x) = x^3$ is $3x^2$' in terms of function, differential or gradient, with an incomplete explanation (e.g. "the differential of $f(x) = x^3$ is $3x^2$ that has been derived by differentiating x^3" or "the gradient of the line $f(x) = x^3$ is $3x^2$")
2	Answers including algebraic procedural explanations (e.g. "differentiation of $f(x) = x^3$ is $3x^2$. $y = ax^n \Rightarrow \frac{dy}{dx} = nax^{n-1}$")
1	No understanding of the concept of 'the derivative of $f(x) = x^3$ is $3x^2$' (e.g. "the function of $f(x) = x^3$ can be broken down to $3x^2$", " area under the graph is the value of the equation $3x^2$" or "the differentiation of x^3 is $3x^2$")
0	"Do not know" or missed

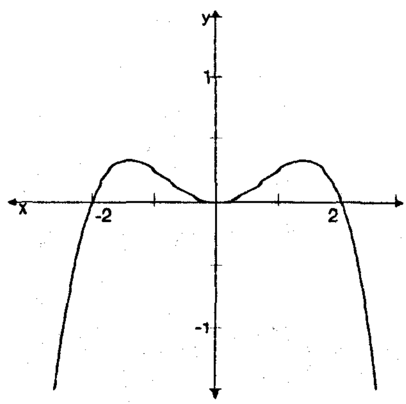
Figure 4.12. Item 1(b) with stated objective and examples of responses showing the score awarded based on the scoring criteria

4.5.2 Question 2: Finding the graph of a derivative from its function

This question dealt with drawing the derivative graph by looking at the original graph. The question is given below. The stated objective and scoring criteria is shown in Figure 4.13.

2. The graph at the right is the graph of a function y = f(x).

Sketch what the derivative looks like. Give the reason(s) for your answer.



SCORE	RESPONSE
5	Correct graph with a correct explanation
4	Correct graph without an explanation or with an explanation but no evidence that the explain relates to the specific curve
3	Incorrect graph but at least one correct feature and some correct explanation(s)
2	Incorrect graph but at least one correct feature
1	Totally wrong graph
0	"Do not know" or missed

Figure 4.13. Question 2 with stated objective and scoring criteria

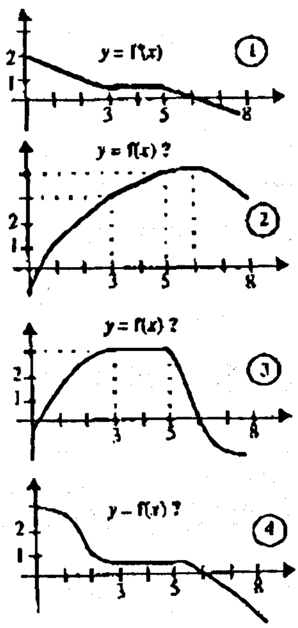
4.5.3 Question 3: Recognising the graph of a function from its derivative graph

In this question students had to choose the correct graph of the original function by looking at the derivative graph and then had to give a justification to support their choices. The stated objective and scoring criteria is given in Figure 4.14. The question is presented below.

3. Graph **1** is the derivative y = f'(x) of a function y = f(x) defined for $0 \leq x \leq 8$.

   Which of the graphs **2, 3, 4** could be the original graph y = f(x)?

   Give you reason(s) for your choice.



SCORE	RESPONSE
5	Correct answer with an explanation including the distinguishing feature of the chosen graph from the other two graphs (e.g. "The graph (2) has a constant gradient between 3 and 5 other than 0")
4	Correct answer with an explanation but no distinguishing feature of the chosen graph from the other two graphs (e.g. "At x = 0 the gradient of graph (2) is positive")
3	Correct answer without an explanation or some attempt at an explanation (e.g. "Graph (2) has the corresponding gradient shown in graph (1)")
2	Incorrect answer with some correct explanation (e.g. "Derivative shows the gradients of the curve")
1	Incorrect answer with or without explanation(s)
0	"Do not know" or missed

Figure 4.14. Question 3 with stated objective and scoring criteria

4.5.4 Question 4: Symbols

In this question students were required to explain the meanings of symbols, such as dx and δx, that are not considered as analogical from the scientific point of view but which are due to similarity in external features students tend to perceive them as analogical. Question 4 is given below.

4. Explain the meaning of each of the following symbols

a) Explain the δx in $\frac{\delta y}{\delta x}$ and in $\sum f(x)\delta x$

b) δy

c) Explain $\frac{\delta y}{\delta x}$

d) Explain the dx in $\frac{d}{dx}(x^2)$ and in $\int x^2\, dx$

e) Explain dy

f) Explain $\frac{dy}{dx}$

g) What is the relationship between $\frac{\delta y}{\delta x}$ and $\frac{dy}{dx}$?

Item 4(a)

Examples of students' responses to item 4(a) explaining the meaning of 'δx in $\frac{\delta y}{\delta x}$ and in $\sum f(x)\delta x$' are shown in Figure 4.15 with an indication of the score awarded.

SCORE	RESPONSE
5	Answers indicating complete understanding of ' δx ' with reasoning based on context (e.g. "a small change in x. In $\frac{\delta y}{\delta x}$ it is a small increment x with the corresponding change in y to give the slope. In $\sum f(x)\delta x$ a length x is broken into several small increments which are then multiplied by their corresponding y value and then all added up")
4	Answers indicating complete understanding of ' δx ' without reasoning based on context (e.g. "a small difference in x")
3	Answers indicating some evidence of understanding but not sufficiently well explained (e.g. "small value of x", "small part of x")
2	Answers containing an isolated correct fact (e.g. "with respect to x", "differential of x" or "infinitesimally small change in x")
1	Complete misunderstanding of the symbol ' δx ' (e.g. "derivative of x", "the value of x is varying", "rate of change of x" or "with respect to small change in x")
0	"Do not know" or missed

Figure 4.15. Item 4(a) with stated objective and examples of responses showing the score awarded based on the scoring criteria

Item 4(b)

Examples of students' responses to item 4(b) explaining the meaning of ' δy ' are shown in Figure 4.16 with an indication of the score awarded.

SCORE	RESPONSE
5	Answers indicating complete explanation for ' δy ' (e.g. "a small change in y")
4	Answers indicating good explanation for ' δy ' but reasoning not complete
3	Answers including some evidence of understanding but not sufficiently well explained (e.g. "small value of y" or "small part of y" or "difference in y")
2	Answers containing an isolated correct fact (e.g. "infinitesimally small change in y" or "differential of y")
1	Complete misunderstanding of the symbol ' δy '(e.g. "derivative of y", "rate of change of y" or "delta of y")
0	"Do not know" or missed

Figure 4.16. Item 4(b) with stated objective and examples of responses showing the score awarded based on the scoring criteria

Item 4(c)

Examples of students' responses to item 4(c) explaining the meaning of '$\frac{\delta y}{\delta x}$' are shown in Figure 4.17 with an indication of the score awarded.

SCORE	RESPONSE
5	Answers indicating complete explanation for '$\frac{\delta y}{\delta x}$' (e.g. "gradient of a chord" or "$\frac{\text{small change in y}}{\text{small change in x}}$")
4	Answers containing main ideas but insufficiently well explained (e.g. "ratio of the changes")
3	Answers containing important sub ideas concerned with gradient or rate of change but insufficiently well explained (e.g. "rate of change of y with respect to x", "gradient of the line")
2	Answers containing an isolated correct fact (e.g. "derivative of y with respect to x")
1	Complete misunderstanding of the symbol '$\frac{\delta y}{\delta x}$' (e.g. "change in δ with respect to x and y", "a definite differential" or "the rate of small change in y with respect to small change in x")
0	"Do not know" or missed

Figure 4.17. Item 4(c) with stated objective and examples of responses showing the score awarded based on the scoring criteria

Item 4(d)

Examples of students' responses to item 4(d) explaining the meaning of 'dx in $\frac{d}{dx}(x^2)$ and in $\int x^2 dx$' are shown in Figure 4.18 with an indication of the score awarded.

SCORE	RESPONSE
5	Answers indicating important ideas concerned with limit (e.g."the dx is when the difference δx has become so small that it is a point rather than a distance. dx in $\int x^2 dx$ shows integrating with respect to x")
4	Answers indicating important main ideas with reasoning based on context (e.g. "differentiate and integrate with respect to x")
3	Answers indicating important sub ideas (e.g. "differentiate with respect to x", "integrate with respect to x" or "with respect to x")
2	Answers including explanation of whole terms instead of dx (e.g. "$\frac{d}{dx}(x^2)$ is the equation used when differentiating - $\int x^2 dx$ reverse of differentiation")
1	Totally wrong answers for explaining 'the dx in $\frac{d}{dx}(x^2)$ and in $\int x^2 dx$' (e.g. "x times a constant value of d" or "rate of change of dx")
0	"Do not know" or missed

Figure 4.18. Item 4(d) with stated objective and examples of responses showing the score awarded based on the scoring criteria

Item 4(e)

Examples of students' responses to item 4(e) explaining the meaning of 'dy' are shown in Figure 4.19 with an indication of the score awarded.

SCORE	RESPONSE
5	Answers containing an explanation concerned with limit or differential (e.g. "limit of δy", "the differential of y", or "δy becomes dy when the difference between the two points forming the δy is negligible. i.e. the points are so close together they can be assumed as one point")
4	Not applicable (n/a)
3	Answers containing important sub ideas but insufficiently well explained (e.g. "corresponding value when a tangent of a curve is taken")
2	Answers containing an isolated correct fact (e.g. "with respect to y")
1	Totally wrong answers for explaining 'dy' (e.g. "derivative of y", "difference in y" or "rate of change of y")
0	"Do not know" or missed

Figure 4.19. item 4(e) with stated objective and examples of responses showing the score awarded based on the scoring criteria

Item 4(f)

Examples of students' responses to item 4(f) explaining the meaning of '$\frac{dy}{dx}$'

are shown in Figure 4.20 with an indication of the score awarded.

SCORE	RESPONSE
5	Answers containing main ideas with a good explanation (e.g. , "rate of change of y with respect to x" or "gradient at a point")
4	Answers containing important main ideas but insufficiently well explained (e.g. "differentiation of y with respect to x" or "gradient")
3	Answers containing important sub ideas concerned with limits or differentials, but insufficiently well explained (e.g. "differential of y with respect to x", "derivative symbol")
2	Answers do not distinguishing between finite and limit terms (e.g. "gradient between two points, small distance apart" or "$\frac{\text{change in } y}{\text{change in } x}$")
1	Totally wrong answers for explaining $\frac{dy}{dx}$ (e.g. "derivative of x" or "small change in the ratio of $\frac{y}{x}$")
0	"Do not know" or missed

Figure 4.20. Item 4(f) with stated objective and examples of responses showing the score awarded based on the scoring criteria

Item 4(g)

Examples of students' responses to item 4(g) explaining the meaning of the relationship between '$\frac{\delta y}{\delta x}$' and '$\frac{dy}{dx}$' are shown in Figure 4.21 with an indication of the score awarded.

SCORE	RESPONSE
5	Answers containing main ideas with a good explanation (e.g. "$\frac{\delta y}{\delta x}$ - gradient after a given range. $\frac{dy}{dx}$ - instantaneous gradient" or "$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \frac{dy}{dx}$")
4	Answers containing important main ideas but insufficiently well explained (e.g. "$\frac{dy}{dx}$ - instant. $\frac{\delta y}{\delta x} = \frac{y_2 - y_1}{x_2 - x_1}$")
3	Answers containing important sub ideas concerned with limits or differentials, but insufficiently well explained (e.g. "overall small range $\frac{\delta y}{\delta x}$, $\frac{dy}{dx}$ - in general")
2	Answers do not distinguishing between finite and limit terms (e.g. "roughly similar" or "$\frac{dy}{dx}$ is derived from $\frac{\delta y}{\delta x}$" or "$\frac{dy}{dx} \approx \frac{\delta y}{\delta x}$")
1	Complete misunderstanding of the relation between $\frac{dy}{dx}$ and $\frac{\delta y}{\delta x}$ (e.g. "same" or "$\frac{dy}{dx}$ is the sum of all $\frac{\delta y}{\delta x}$")
0	"Do not know" or missed

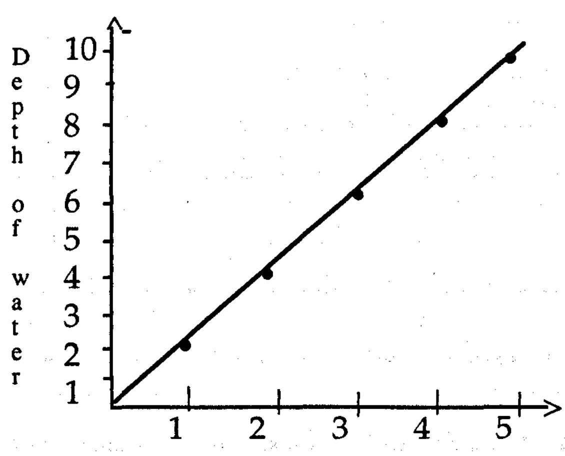
Figure 4.21. Item 4(g) with stated objective and examples of responses showing the score awarded based on the scoring criteria

4.5.5 Question 5: Average rate of change and rate of change of a linear function

This question required a student to apply the quotient formula to find the rate of change over an interval and at a point for a linear function. The question is shown below.

5. Water is flowing into a tank at a constant rate. For each unit increase in the time, the depth of water increases by 2 units. The table and graph illustrate this situation.

Time (x)	0		1		2		3		4		5
Depth (y)	0		2		4		6		8		10
1st difference (depth)		2		2		2		2		2	



a) What is the rate of increase of y as x increases from 3 to 3+h?

b) What is the rate of increase of y at $x = 2\frac{1}{2}$ and at x = X?

Item 5(a)

Examples of students' responses to item 5(a) finding the rate of change over an interval are shown in Figure 4.22 with an indication of the score awarded.

SCORE	RESPONSE
5	Correct answers (e.g. "y = 2x. $\frac{dy}{dx} = 2$", or "the rate of increase of y is constant, and so will be 2")
4	Answers including main ideas with some evidence of knowledge (e.g. "rate of increase= $\frac{\text{increase}}{\text{time}} = \frac{\delta y}{h}$" or "$\delta x = (3+h) - 3 = h$, $\delta y = (6+2h) - 6 = 2h$")
3	Answers including important sub ideas concerned with slope or ratio of quotient, but without necessary reasoning (e.g. "$\frac{dy}{dx} = 2$. rate of increase = 6 from 3 to 3 + h" or "as x increases from 3 to 3+h, y increases from 6 to 6 + 2h. Therefore $\frac{dy}{dx} = 2h$")
2	Partly correct (e.g. "As x increases from 3 to 3+h, y increases from 6 to 6 + 2h" or "rate of change is the gradient ($\frac{dy}{dx}$)")
1	Complete misunderstanding of the concept of 'rate of change' (e.g. "rate of increase = $y = 2 \times h$")
0	"Do not know" or missed

Figure 4.22. Item 5(a) with stated objective and examples of responses showing the score awarded based on the scoring criteria

Item 5(b)

Examples of students' responses to item 5(b) finding the rate of change at a point are shown in Figure 4.23 with an indication of the score awarded.

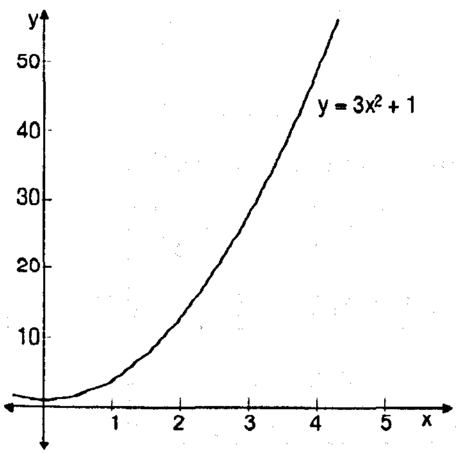
SCORE	RESPONSE
5	Answers including correct answers for both points (e.g. " rate of increase is constant. At $x = 2\frac{1}{2}$ rate of increase = 2 and at $x = X$ rate of increase = 2")
4	Answers including correct answer but only for one point (e.g. " at $x = 2\frac{1}{2}$ rate of change is 2 and at $x = X$ rate of change is 2X")
3	Incorrect answer but correct explanation (e.g. "gradient is the same in both places - $\frac{1}{2}$ unit ")
2	(n/a)
1	Complete misunderstanding of the concept of 'rate of change at a point' (e.g. "at $x = 2\frac{1}{2}$ rate of increase = $2 \times \frac{1}{2} = 1$ and at X rate of increase = $X \times 2 = 2X$")
0	"Do not know" or missed

Figure 4.23. Item 5(b) with stated objective and examples of responses showing the score awarded based on the scoring criteria

4.5.6 Question 6: Average rate of change and rate of change of a quadratic function

This question required a student to apply the quotient formula to find the average rate of change, and the rate of change for a quadratic function. The question is shown below.

6. The graph below represents $y = 3x^2 + 1$, from x=0 to x=4.



a) What are the ratios of changes (average rate of change) of y with respect to x as x changes from: (i) 2 to 2 + 0.1, (ii) 2 to 2 + h, (iii) a to a + h ?

b) What are the rates of change of y with respect to x as x changes from: (i) 2 to 2 + 0.1, (ii) 2 to 2 + h, (iii) a to a + h ?

c) What is the rate of change of y at $x=2\frac{1}{2}$?

Item 6(a)

Examples of students' responses to item 6(a) finding average rate of change are shown in Figure 4.24 with an indication of the score awarded.

SCORE	RESPONSE
5	Correct answer for all intervals using the formula $\frac{\delta y}{\delta x} = \frac{y(x+h) - y(x)}{h}$
4	Correct answer for one or two given intervals using the formula $\frac{\delta y}{\delta x} = \frac{y(x+h) - y(x)}{h}$ (e.g. "$\delta x = 2+0.1-2 = 0.1$, $\delta y = 14.23-13=1.23$ ")
3	Answers including important sub ideas without necessary reasoning (e.g. "$\frac{(3\times 2.1^2+1)-(3\times 2^2+1)}{2} = 0.615$" or " y = 13 when x =2 and y = 14.23 when x = 2.1. Change = 1.23")
2	Partly correct (e.g. "$\frac{dy}{dx} = 6x$. (i) at 2.1 rate of change of y=12.6, at x = 2+h rate of change of y = 6(2+h)=12+6h")
1	Complete misunderstanding of the concept of 'average rate of change' (e.g. "$\frac{dy}{dx} = 6x$. ratio $= \frac{6(2.1)}{6(2)} = 0.6$")
0	"Do not know" or missed

Figure 4.24. Item 6(a) with stated objective and examples of responses showing the score awarded based on the scoring criteria

Item 6(b)

Examples of students' responses to item 6(b) are shown in Figure 4.25 with an indication of the score awarded. The purpose of this item was to examine whether students know that the rate of change is the limiting value of the average rate of change. As you may notice, there is an ambiguity in this item[1].

SCORE	RESPONSE
5	Correct answers concerned with the limiting value of $\frac{\delta y}{\delta x} = \frac{y(x+h)-y(x)}{h}$
4	Answers containing the equation $\frac{\delta y}{\delta x} = \frac{y(x+h)-y(x)}{h}$ but not its limit value.
3	Right with regard to the content but wrong strategy (e.g."$y' = 6x$. (i) at $x = 2$ $y' = 12$ and at $x = 2.1$ $y' = 12.6$")
2	(n/a)
1	Complete misunderstanding of the concept of 'rate of change' (e.g. "$\frac{dy}{dx} = 6x$. $\frac{6\times2.1}{6\times2} = 1.05 \Rightarrow dy = 1.05dx$")
0	"Do not know" or missed

Figure 4.25. Item 6(b) with stated objective and examples of responses showing the score awarded based on the scoring criteria

Item 6(c)

Examples of students' responses to item 6(c) finding the rate of change at a point are shown in Figure 4.26 with an indication of the score awarded.

[1] After analysis, this item was found unsatisfactory. This is discussed on page 300.

SCORE	RESPONSE
5	Correct answer or wrong answer due to trivial arithmetical errors overlooked (e.g."$y' = 6x \Rightarrow y' = 6\times(\frac{5}{2}) = 15$ at $x = 2\frac{1}{2}$" or "$y' = 6x \Rightarrow y' = 6\times(\frac{5}{4}) = 15$ at $x = 2\frac{1}{2}$")
4	Incorrect differentiation but correct logic (e.g. "$y' = 6x+1 \Rightarrow y' = 6\times(\frac{5}{2})+1 = 16$ at $x = 2\frac{1}{2}$")
3	Correct differentiation (e.g. "$\frac{dy}{dx} = 6x$")
2	Correct answer but not related to the question (e.g. "$\frac{dy}{dx} = 6x \Rightarrow dy = 15dx$")
1	Complete misunderstanding of the concept of 'rate of change' at a point (e.g. "$y = 3\left(\frac{5}{2}\right)^2 + 1 = 19.75$")
0	"Do not know" or missed

Figure 4.26. Item 6(c) with stated objective and examples of responses showing the score awarded based on the scoring criteria

4.5.7 Question 7: Integral

This question dealt with the 'integral' concept. The items are given below:

> 7
> (a) What is an integral? Define and explain as you wish.
>
> (b) Function g(t) gives the number of phone calls in time t
>
> What is the meaning of $\int_{5}^{10} g(t)\,dt$?

Item 7(a)

Examples of responses to item 7(a) on the definition of 'integral' are presented in Figure 4.27 with an indication of the score awarded.

SCORE	RESPONSE
5	Answers containing main ideas concerned with derivative and function or derivative and area (e.g. "The integral geometrically is the area under the graph. Mathematically it is a limit of summation as the reduce small rectangular strips to zero width $\lim_{x \to 0} \sum_{a}^{b} y\delta x$ ")
4	Answers containing main ideas concerned with area or derivative (e.g. "Area under a curve between two limits")
3	Answers containing important sub ideas concerned with derivative (e.g. "integration is the opposite of differentiation" or "integral is reverse to a differential")
2	Answers containing an isolated correct fact (e.g. "integral is the results of integration" or "the 'work done' by a function")
1	No understanding of the concept of 'integral' (e.g. "integral is defined as a function of a function")
0	"Do not know" or missed

Figure 4.27. Item 7(a) with stated objective and examples of responses showing the score awarded based on the scoring criteria

Item 7(b)

Examples of students' responses to item 7(b) on explaining the meaning of an example of integral are shown in Figure 4.28 with an indication of the score awarded. As you may notice, there is an ambiguity in this item[2].

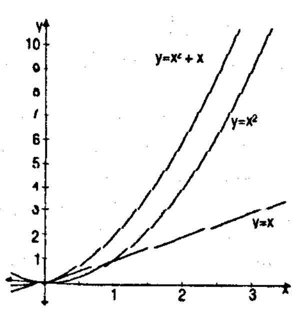
[2] After analysis, this item was found unsatisfactory. This is discussed on page 300.

SCORE	RESPONSE
5	Correct explanation (e.g. "the sum of phone calls between 5 and 10")
4	Correct explanation but incorrect integration (e.g. "the phone calls between 5 and 10 minutes. $\int_{5}^{10} g(t)dt = [g'(t)]_{5}^{10}$ ")
3	Integration fact with an explanation but not related to the question (e.g. "the area under the line the ranges of 10 and 5")
2	Integration fact but not related to the question (e.g. "the integral of g(t) between the limits of 10 and 5")
1	Totally wrong answers (e.g. "total amount of time", "rate of change of phone calls between t=5 and t=10")
0	"Do not know" or missed

Figure 4.28. Item 7(b) with stated objective and examples of responses showing the score awarded based on the scoring criteria

4.5.8 Question 8: Proof of the integral of the sum of two functions equals to the sum of their integrals

The question which is given below required a student to verify that the integral of a sum equals to the sum of integrals.



Explain, by means of the diagram or otherwise, why

$$\int_{o}^{a} (x^2 + x)\, dx = \int_{o}^{a} x^2 dx + \int_{o}^{a} x\, dx$$

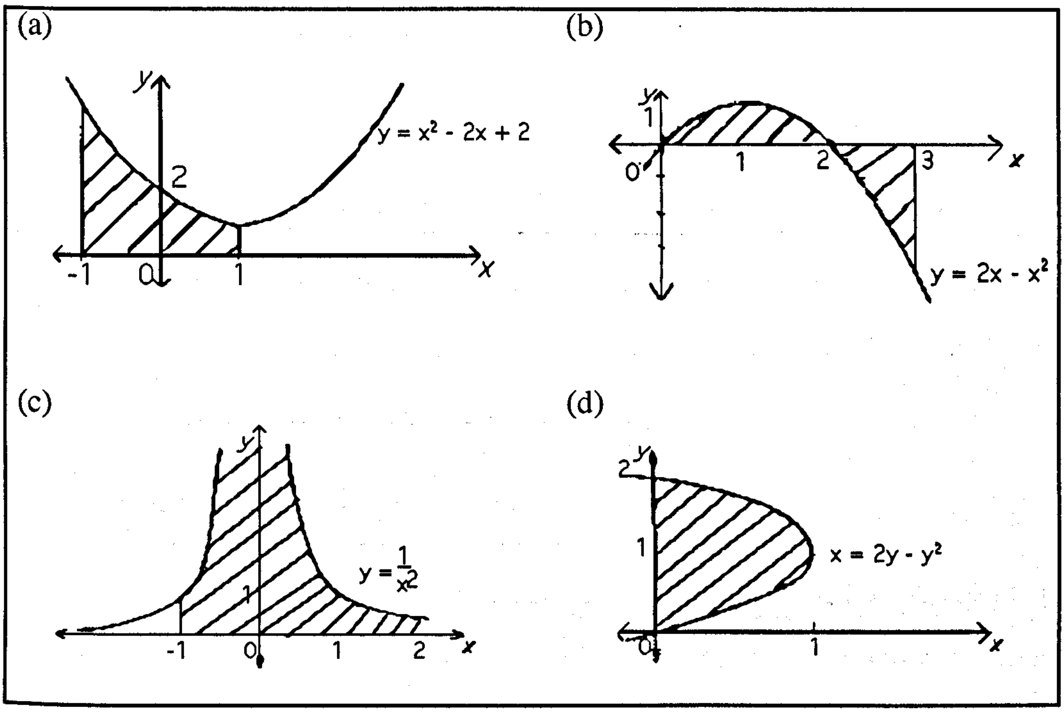
Examples of students' responses to question 8 are shown in Figure 4.29 with an indication of the score awarded.

SCORE	RESPONSE
5	Correct answers including explanations related to narrow strips and their heights (e.g. "Because if $y = x^2$ is taken (from diagram) and then, $y = x$ is added (i.e. sum of individual y values at point along x-axis) then the curve $y = x^2 + x$")
4	Realization that the area under $y = x$ and the area between $y = x^2 + x$ and $y = x^2$ must be equal (e.g. "the area between $y = x^2 + x$ and $y = x^2$ is equal to the area under $y = x$")
3	The explanations concerned with sums of areas under curves but no understanding of the significance of narrow strips or heights, or the explanations gives the algebraic proof (e.g. " the area under $y = x^2 + x$ is the same as the area under $y = x^2$ and $y = x$" or "R.H.S $= \frac{a^3}{3} + \frac{a^2}{2}$, L.H.S $= \frac{a^3}{3} + \frac{a^2}{2}$ therefore L.H.S = R.H.S")
2	Answers including general explanation (e.g. "integral gives area under a graph" or " integral with the same limits can be added")
1	The explanations which shows no understanding (e.g. "because of the law of integration" or "the integral of $(x^2 + x)$ is the same as the integral of $x^2dx + y = x$")
0	"Do not know" or missed

Figure 4.29. Question 8 with stated objective and examples of responses showing the score awarded based on the scoring criteria

4.5.9 Question 9: Area under a curve

The question which is shown below required a student to apply the rules of integration to different functions



Item 9(a)

Examples of students' responses to item 9(a) are shown in Figure 4.30 with an indication of the score awarded.

SCORE	RESPONSE
5	Right answer or wrong answer simply due to trivial arithmetic errors (overlooked) (e.g. "Area $= \int_{-1}^{1}(x^2 - 2x + 2)dx = \left[\frac{1}{3}x^3 - x^2 + 2x\right]_{-1}^{1}$ $= \left(\frac{1}{3} - 1 + 2\right) - \left(-\frac{1}{3} - 1 - 2\right) = 4\frac{2}{3}$")
4	Correct integration formula and correct integration but mistakes in follow up stages or incorrect integration but correct follow up stages (e.g." Area $= \int_{-1}^{1}(x^2 - 2x + 2)dx = \left[\frac{1}{3}x^3 - x^2 + 2x\right]_{-1}^{1}$ $= \left(\frac{1}{3} - 1 + 2\right) + \left(-\frac{1}{3} - 1 - 2\right) = -2$")
3	Answers containing correct integration formula but mistakes in follow up stages or answers including correct answer but integration with respect to wrong variable (e.g. "$\int_{-1}^{1}(x^2 - 2x + 2)dx$", "Area $= \int_{-1}^{1}(x^2 - 2x + 2)dy = \left[\frac{1}{3}x^3 - x^2 + 2x\right]_{-1}^{1}$ $= \left(\frac{1}{3} - 1 + 2\right) - \left(-\frac{1}{3} - 1 - 2\right)$" or "$\int_{-1}^{1}(x^2 - 2x + 2)dx = [2x - 2]_{-1}^{1} = (0 - 4)$")
2	Containing an isolated correct fact (geometrical proof)
1	Incorrect integration formula setting (e.g. "$\int_{-1}^{1} y^2 dx$")
0	"Do not know" or missed

Figure 4.30. Item 9(a) with stated objective and examples of responses showing the score awarded based on the scoring criteria

Item 9(b)

Examples of students' responses to item 9(b) are shown in Figure 4.31 with an indication of the score awarded.

SCORE	RESPONSE
5	Right answer or wrong answer simply due to trivial arithmetic errors (overlooked) (e.g."$\int_0^2 2x - x^2dx + \int_2^3 2x - x^2dx = \left(2^2 - \frac{2^3}{3}\right) + \left(3^2 - \frac{3^3}{3}\right) - \left(2^2 - \frac{2^3}{3}\right) = 0$")
4	Correct integration formula and correct integration but mistakes in follow up stages or incorrect integration but correct follow up stages
3	Correct integration formula but mistakes in follow up stages
2	Containing an isolated correct fact (geometrical proof)
1	Incorrect integration formula or answer including "not possible" (e.g. "This is not possible as part is above the x-axis and part is below" or "$\frac{dy}{dx} = -2x + 2$")
0	"Do not know" or missed

Figure 4.31. Item 9(b) with stated objective and examples of responses showing the score awarded based on the scoring criteria

Item 9(c)

Examples of students' responses to item 9(c) are shown in Figure 4.32 with an indication of the score awarded.

SCORE	RESPONSE
5	Answers showing the impossibility using the symbolical proof of $\int_0^2 \frac{1}{x^2}dx + \int_{-1}^0 \frac{1}{x^2}dx$
4	Correct integration formula (e.g. "$\int_0^2 \frac{1}{x^2}dx + \int_{-1}^0 \frac{1}{x^2}dx$")
3	Answers explaining 'impossibility' for integration in terms of infinity or unboundness (e.g. "graph goes infinity. Therefore area infinite")
2	Answers including the answer "Impossible" but without further explanation or correct integration formula but mistakes in substituting limit values
1	Totally incorrect answer (e.g. "$\int_{-1}^{2} \frac{1}{x^2}dx = \left[-\frac{1}{x}\right]_{-1}^{2} = -\frac{1}{2} - 1 = -\frac{3}{2}$" or "can not do because the graph is in two sections")
0	"Do not know" or missed

Figure 4.32. Item 9(c) with stated objective and examples of responses showing the score awarded based on the scoring criteria

Item 9(d)

Examples of students' responses to item 9(d) are shown in Figure 4.33 with an indication of the score awarded.

SCORE	RESPONSE
5	Right answer or wrong answer due to trivial arithmetic errors overlooked (e.g. " $A=\int_0^2 (2y-y^2)dy=\left[y^2-\frac{y^3}{3}\right]_0^2=4-\frac{8}{3}=\frac{4}{3}$ ")
4	Correct integration formula and correct integration but mistakes in follow up stages or incorrect integration but correct follow up stages. (e.g. " $\int_0^2 (2y-y^2)dy=\left[y^2-\frac{y^3}{3}\right]$ =0")
3	Correct integration formula but mistakes in follow up stages or Correct answer but integration with respect to wrong variable. (e.g. " $\int_0^2 (2y-y^2)dy$ " or " $\int_0^2 (2y-y^2)dy=[2-2y]_0^2$ ")
2	Correct integration formula but with wrong limits (e.g. " $\int_0^1 (2y-y^2)dy-\left[2-\frac{y}{3}\right]_0^1$ ")
1	Incorrect integration formula setting or answer including 'not possible' (e.g. "not possible-graph must be with respect to x")
0	"Do not know" or missed

Figure 4.33. Item 9(d) with stated objective and examples of responses showing the score awarded based on the scoring criteria

4.5.10 Question 10: Point of tangency, numerical calculation of a gradient and estimating the value of a function

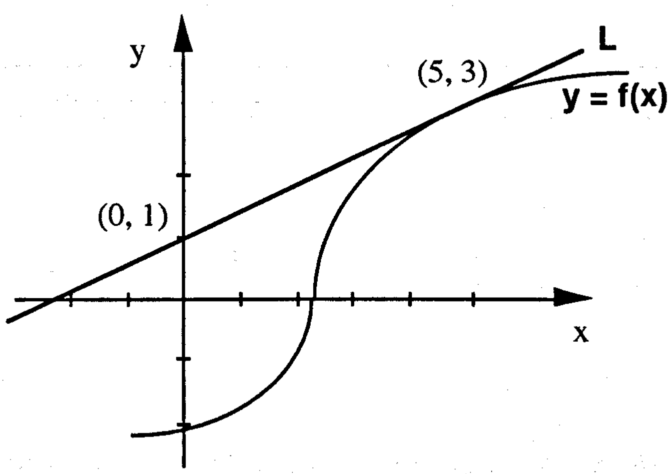
The question which is given below required a student to get the information given in the graph. Having the skills to obtain this information is important both in and out of school.

10. Line L is a tangent to the graph of $y = f(x)$ at the point (5, 3).

a) Find the value of f(x) at x = 5.

b) Find the derivative of f(x) at x = 5.

c) What is the value of the function f(x) at x=5.08?
(Be as accurate as possible)



Item 10(a)

Examples of students' responses to item 10(a) are shown in Figure 4.34 with an indication of the score awarded.

SCORE	RESPONSE
5	Correct answer (e.g. " y =3 at x =5")
4	(n/a)
3	(n/a)
2	(n/a)
1	Wrong answer and false reasoning (e.g. "$m=\frac{5}{3}=1.66$, $y=mx+c$ $y=1.66\times 5+1=9.3$")
0	"Do not know" or missed

Figure 4.34 Item 10(a) with stated objective and examples of responses showing the score awarded based on the scoring criteria

Item 10(b)

Examples of students' responses to item 10(b) are shown in Figure 4.35 with an indication of the score awarded.

SCORE	RESPONSE
5	Correct answer (e.g. "$\frac{dy}{dx}=\frac{2}{5}$" or "slope of the graph at 5=$\frac{2}{5}$")
4	Derivative is the gradient of tangent equation but mistakes in finding it (e.g. "$\frac{dy}{dx}=\frac{3}{5}$")
3	Answers including important ideas but without necessary reasoning (e.g. "$m=\frac{2}{5}$, $y=\frac{2}{5}x+1$, $\frac{dy}{dx}=\frac{2}{5}x+1$")
2	(n/a)
1	Wrong answer and false reasoning (e.g. "$\frac{dy}{dx}=x$")
0	"Do not know" or missed

Figure 4.35. Item 10(b) with stated objective and examples of responses showing the score awarded based on the scoring criteria

Item 10(c)

Examples of students' responses to item 10(c) are shown in Figure 4.36 with an indication of the score awarded.

SCORE	RESPONSE
5	Correct answer or wrong answer due to trivial errors in tangent equation (e.g. "$y = \frac{2}{5}(5 + 0.08) + 1 = 3.032$")
4	Guessing of the approximate value from the graph (e.g. "a little bit bigger than 3")
3	Answers including important sub ideas but without necessary reasoning (e.g. "$y = 2.5x + 1 \Rightarrow$ at $x = 5.08$ $y = (2.5 \times 5.08) + 1 = 13.7$")
2	(n/a)
1	Wrong answer and false reasoning (e.g. "$\frac{dy}{dx} = \frac{2}{5}x + 1, \int (\frac{2}{5}x + 1)dx$")
0	"Do not know" or missed

Figure 4.36. Item 10(c) with stated objective and examples of responses showing the score awarded based on the scoring criteria

4.6 Summary

This chapter has described the methodology of the main study. It has attempted to present enough detail for other researchers to replicate these results or undertake similar studies. This chapter has also described the diagnostic test items and the six-point scoring scheme. The scoring procedure was quite a lengthy process.

The construction of the diagnostic test to probe students understanding and misconceptions in calculus is discussed in the next chapter. It is hoped that this test will be a value to other researchers.

CHAPTER FIVE

THE DESIGN OF THE DIAGNOSTIC TEST

This chapter focuses on the design of the diagnostic test in a detailed way. It describes the general nature of the questions together with the results of the investigatory work which took place at two sixth form colleges. The stages followed for developing the diagnostic test items were: (i) the review of literature, (ii) developing of the test, (iii) piloting of the test, (iv) revision of the test, and (v) piloting of the revised test again.

Tests designed by Amit and Vinner (1990), Cornu (1983), Orton (1983, 1983a), Selden et.al (1989), and Tufte (1988) were referenced. Certain questions from these tests were adapted and used along with specially designed ones. The aim was to design a diagnostic test, in which each question or item tests students' higher level abilities in certain calculus concepts such as, differentiation. As stated by Nitko (1983), to test higher level abilities, the test items should be new or novel to the student. They should not contain any of the specific examples used during instruction. Otherwise, there will be no assurance that a student has understanding. The student's response to the questions may only be due to the particular phrasing, rather than a reflection of his/her comprehension or application of knowledge.

As most of the items selected were validated with students in England, this increased confidence in the test.

In the test, the open-ended question form was used because it offers pupils the opportunity to exhibit their abilities for written production, organization, expression, and interrelationships among ideas. The open-ended questions are different from more structured ones in two major ways:

> First, these questions engage the students in constructive thinking by requiring them to consider the necessary relationships for themselves, and to devise their own strategies for responding to the questions. Second, the questions have more than one possible correct answer. Some students just give one correct response, other might produce many correct answers, and there may be some who will make general statements. (Clark & Sullivan, 1992, p.137)

Each item was designed to assess students' higher level abilities, which are or must be a major intention of classroom instruction. According to Nitko (1983), the acquisition of concepts, principles, and higher-order skills forms the basis for higher-order learning.

5.1 The Pilot Study

Two forms of the test exist: a draft form used in this pilot study and a final form (see Appendix A and B). The pilot study was carried out on A-level students and took place at two colleges. All A-level students taking mathematics in the colleges study differentiation and integration, and simple ideas about differential equations. At the time of the pilot study students had already been taught differentiation and integration. The purpose of the pilot study was:

1. to check the administration time.
2. to check whether items were testing the intended objectives.
3. to decide whether they were of the appropriate level of difficulty. A simple procedure to that end was the tabulation of students responses in terms of their answers.
4. to consider the range of possible levels of answers to each item of a test which can be answered from several levels of thinking.

Student's responses and ideas as well as one calculus teacher's ideas were used to revise the test. The items were checked to see if they match the domain of the calculus course. It would be unfair to judge a student, if that student has had no opportunity to learn the material asked for on the test. While piloting the test some items were revised, discarded or kept for future testing.

Before the test was handed out the students had been told that the results would be used in two ways: to help them evaluate their knowledge and to help us evaluate our teaching.

5.1.1 First Pilot Study

This test consisted of 11 questions, some of them having several items (see Appendix B). The test was administered to 14 second year A-level students at Bilborough Sixth Form College, allocating 50 minutes. The general characteristics of each question assessing higher level abilities are summarized in Table 5.1. The questions and their results are described and discussed separately. An analysis of the written explanations is reported and a selection of explanations is cited.

Some questions in the test were kept, revised or discarded to improve the overall quality of the test.

Question 1: Derivative

This question required a student (i) to produce a correct definition of the 'derivative' concept, and (ii) to identify an exemplar of the concept and to demonstrate what is meant by a defined relation or event - that is, demonstrate an understanding of the concept.

The answers given to this question fell into the following six categories:

I. Derivative is the result of differentiation.

II. Derivative is the gradient function.

III. Procedural explanation: if $y = x^n$ the derivative $\frac{dy}{dx} = nx^{n-1}$.

IV. Derivative is the gradient of the tangent.

V. Derivative is the gradient of a curve at a specific point on a graph (generalization of a specific case).

VI. Derivative is the tangent to a curve.

Table 5.1. General characteristics of the questions used in the first version of the diagnostic test

Questions	General Characteristics of Questions Assessing Higher Level abilities[1]	Type of ability
1	Assessing the acquisition of concepts	•Ability to state a definition of a concept •Ability to identify or explain an example of the particular concept
2®	Assessing the acquisition of higher order abilities	•Ability to draw inferences based on the displayed graph •Ability to identify the relationships between two concepts
3®	Assessing the acquisition of concepts	•Ability to identify the concept exemplar •Ability to explain the meanings of symbols
4°	Assessing the acquisition of higher order abilities	Ability to translate concepts and principles from a verbal form into a symbolic form
5®	Assessing the acquisition of higher order abilities Assessing the acquisition of principles	•Ability to read a graph and a table •Ability to use a principle (quotient formula) in a new situation
6®	Assessing the acquisition of higher order abilities Assessing the acquisition of principles	•Ability to read a graph •Ability to use a principle (quotient formula) in a new situation •Ability to provide or identify the relationship between two concepts
7®	Assessing the acquisition of concepts	•Ability to state a definition of a concept •Ability to identify or explain an example of the particular concept
8	Assessing the acquisition of higher order abilities	Ability to analyze the principle which is given
9	Assessing the acquisition of principles	Ability to apply a principle in a given situation
10°	Assessing the acquisition of higher order abilities Assessing the acquisition of principles	•Ability to translate concepts and principles from a verbal form into a pictorial form •Ability to analyze a new or novel situation and state the principle which it illustrates
11°	Assessing the acquisition of principles	Ability to use a principle in a new situation

Note: ® Revised item ° deleted item

[1]This is categorized according to Nitko (1983) classification.

The analysis of the answers provided us with a better insight into the way students think about the concept of 'derivative'. A coherent explanation, which reflects an understanding, had to refer to the fact that the gradient of the tangent to the graph of the function y = f(x) at any point is given by the value of f'(x) at that point. Two students provided a correct explanation in their statements. Here is an example:

> (a) A derivative is a function gained from another function after differentiation. It can be used to find the gradient of a curve at a particular point.
>
> (b) the gradient of $f(x) = x^3$ is the derivative, $3x^2$. (Student 8)

Five of the others students responded in the same way to both items 1(a) and 1(b): two students included aspect I in their answers; two students aspect VI; and one aspect III.

Three students responded to either item 1(a) or 1(b) including aspect V and missed one of them. Two others responded to item 1(a) including aspect V. While one of them gave the correct answer for item 1(b), the other used an idea similar to the one in item 1(a).

> (b) you can use $3x^2$ to find the gradient of any point on the x^3 curve (graph). (Student 10)
>
> (b) It means that where $3x^2=0$ the gradient of the graph $y = x^3$ is 0. (Student 9)

One student explained item 1(a) literally but 1(b) procedurally. The example is given below:

> (a) Something which is derived or made from something else
> (b) if $y=x^n$ the derivative $\frac{dy}{dx} = nx^{n-1}$. (Student 6)

Another student explained item 1(a) and 1(b) including aspect I and III respectively.

The following remarks were made by a student:

> I do not know how to put in words... explanation is difficult... but I know how to do it..... Teachers never ask us explanations.

Students, however, must be encouraged to use their own language when approaching any theory.

According to the results obtained above and a critique from a calculus lecturer in an engineering department, the question was kept for further testing.

<u>Question 2®: Finding the graph of a function from its derivative graph</u>

(Here and elsewhere ® means a question has been revised).

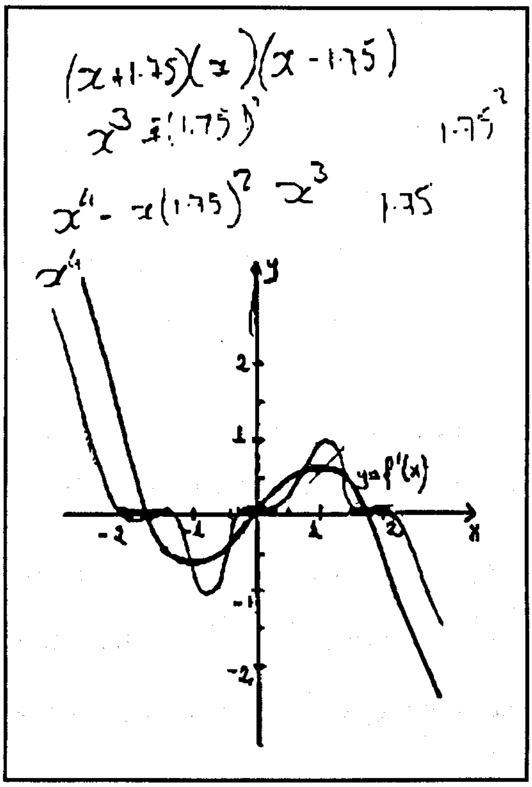
This question required a student to draw inferences based on the displayed graph underlying relationships and facts between the derivative and the integral graph. The skills enabling a student to make an inference from a graph have long been recognized as important. The question required a student to make an inference based on understanding:

- when a derivative function f'(x) is negative, between x = a and x = b, then the function itself f(x) has a negative gradient and is decreasing;
- when f'(x) is positive, between x= a and x = b, then f(x) has a positive gradient and is increasing;
- when f'(x) = 0 at x = a, then the gradient of f(x) at this point is zero;
- when f'(x) has a zero gradient at x =a, then x = a is the inflection point of f(x).

This question dealt with drawing the graph of a function from that of its derivative. Nine of the 18 students could not attempt the question. Only one student drew the right graph without giving any explanation. The others, however, read off the points of intersection and tried to find out the equation of the graph (see the example below).

Four of those students also integrated the equation to find the equation of the original function.

Although the relationship between a function and its derivative is a central theme in calculus, the graphical work included in most traditional syllabuses is arguably inappropriate and inadequate, often consisting of plotting the graph of simple algebraic functions and reading off points of intersection (Bell, Brekke, & Swan, 1987, p.47). However, what needs to be taught and learnt is how to interpret such graphs by concentrating not only on one factor of the relevant data.



Student 1

Since the students cannot answer the question, it was posed in the reverse way for further testing. That is, in the final test students were given the graph of a function and asked to sketch the derivative function. The same problem, from the point of view of its mathematical structure, may elicit very different responses if it is posed as a derivative/its function or as a function/ its derivative.

Question 3®: Symbols

A student was required to (i) explain the meanings of symbols, and (ii) demonstrate or identify the relationship(s) between symbols. Questions requiring explanation of symbols might provide some assurance that a pupil has not responded simply by learning a verbal chain of words that comprise a definition. Also, as pointed out by Tirosh and Stavy (1993), analogies play an important role in the development and acquisition of concepts. For example, $\frac{\delta y}{\delta x}$ and $\frac{dy}{dx}$ are not considered as analogical from the scientific point of view but if they are considered in the same context students tend to perceive them analogical.

This question was designed to show if students had assimilated the meanings of the symbols - δx, δy, dx, dy, $\frac{\delta y}{\delta x}$ - and the relationships between $\frac{\delta y}{\delta x}$ and $\frac{dy}{dx}$.

Twelve of the 14 students were able to explain 'δx' and 'δy' satisfactorily. The answers given can be classified as follows: "small change in x", "small bit of x", or "tiny value of x". One of the remaining missed the items and the other gave an answer such as "a little bit more than x".

Five of those 12 students who answered 'δx' and 'δy' satisfactorily also answered '$\frac{\delta y}{\delta x}$' satisfactorily. Three of them gave an answer involving gradient, such as "gradient of small part of x-y graph" or "gradient" and another two students thought numerator and denominator separately: " small bits of y/ small bits of x". One student gave the response "differential". Eight students did not attempt a response.

The symbols which caused great difficulty for students were 'dx', 'dy', and the relation between '$\frac{\delta y}{\delta x}$' and '$\frac{dy}{dx}$'. One student explained 'dx' and 'dy' as same as 'δx' and 'δy'. Two students gave an answer such as 'derivative of x or y'. Four students pointed out that 'dx' or 'dy' is small x or y value but bigger than 'δx' or 'δy'. Three students answered as "with respect to x" or "with respect to y". Four students could give no response at all for 'dx' and 'dy'.

In item 4(f), 11 students did not attempt any answer and one student thought that they are the same. The remaining two students gave incorrect responses such as "$\frac{dy}{dx}$ - is set $\frac{\delta y}{\delta x}$ is within the limits".

Since the symbol 'dx' as opposed to 'δx' which was found easy caused problems to the students, it was decided that the wording in the second test would be as follows:

- Explain the δx in $\frac{\delta y}{\delta x}$ and in $\sum f(x)\delta x$.
- Explain the dx in $\frac{d}{dx}(x^2)$ and in $\int x^2 dx$.

Furthermore, the meaning of $\frac{dy}{dx}$ was added. Tall (1992) stated that some of the difficulties in calculus were worth extended investigation. One of them is the Leibniz notation $\frac{dy}{dx}$ which proves to be almost indispensable in the calculus and still causes serious conceptual problems. Is it a fraction, or a single indivisible symbol? What is the relationship between the dx in $\frac{dy}{dx}$ and dx in $\int f(x)dx$?

The main reason for not being able to give satisfactory coherent meaning is the perception of these symbols as analogical by students. An important question to be answered is what causes students to perceive symbols or situations as analogical.

<u>Question 4° : Application of differentiation from first principles</u>

(Here and elsewhere (°) means a question has been omitted).

This question required a student to translate a concept or principle from a verbal form to a symbol form. The general method of differentiation is illustrated by this question; to obtain the derivative of s with respect to t, students proceed in exactly the same way as they do to obtain the gradient of a given curve (differentiation from first principles). Specifically, this question required students to find out velocity. In order to do that, students should measure the distance traveled in a small time and work out the average velocity in this interval.

None of the students got the entire question correct. Eight students did not attempt to answer the question. In the six solution attempts, students did not show a substantial progress toward a correct response: three students obtained $\frac{ds}{dt}$ by differentiating; one of the three and one more substituted δs and δt for s and t respectively in $s = 3t^2 + 1$ to find out δs; another two students answered that δs is either $\delta s = 6\delta t$ or $\delta s = 3\delta t^2$.

In discussion with students one of the comments was:

> It is a mechanics.... it is scary...... we have not done mechanics...

In addition to that, a class teacher's comment was:

> .....not possible to solve.

It appears that the students found it substantially more difficult to provide answers to this task. This suggests that it would be inappropriate to ask this question since questions to be included in a test must focus on major rather than minor points of content.

Question 5®: Average rate of change and rate of change of a linear function

This question was explicitly dealing with the rate of change for a straight line graph. A number of students could not answer both items: two and eight for item 5(a) and 5(b) respectively. Twelve students attempted item 5(a) but only five students obtained the correct answer "2"; the rest gave the equation "y = 2x" only or substituted some values for x in 'y =2x'. In item 5(b), no one gave the correct response: four students responded with the y-value, "5", and two gave the response "$\frac{1}{2}$" or "1". The student who gave the answer "1" probably thought that the answer must be half of 2 which had been obtained for item 5(a).

As a number of students got confused with the unknowns, the letter 'a', was given a numerical value instead of being treated as an unknown for further testing.

Question 6®: Average rate of change and rate of change of a quadratic function

The purpose of this question was to examine whether students know the distinction between 'average rate of change' and 'rate of change' in a non-linear situation.

Results show that most of the students could not handle this question with ease. Not all students knew the different meanings of 'average rate of change' and 'rate of change'. Only two students differentiated y with respect to x and substituted $x=2\frac{1}{2}$ to find the rate of change at this point. While five students left out item 6(a), the same number of students and three more left out item 6(b).

There were incorrect responses from seven and five students for item 6(a) and item 6(b) respectively. In item 6(a) the responses were quite striking: two students perceived 'the average rate of change' as a 'mean value', and another two perceived it as a 'definite integral' as evidenced by the following examples:

a) $y = 3(1)^2 + 1 = 4 \quad y = 3(2)^2 + 1 = 13 \quad y = 3(3)^2 + 1 = 28 \quad y = 3(4)^2 + 1 = 37$

$$\frac{4+13+28+37}{4} = 20.25$$ (Student 5)

a) $\int_0^4 3x^2 + 1dx = [x^3 + x]_0^4 = 64 + 4 = 68$

$\text{Average} = \frac{68}{4} = 17$

b) $2\frac{1}{2} \times 17 = 42.5$ (Student 8)

The remaining three gave very interesting responses too which are illustrated by the following examples:

a) $3x^2$ b) 18.75 (Student 1)

a) $x = 4$ $y = 48$ $\frac{48}{4} = 12$ $y = 12x + 1$ (Student 2)

In item 6(b), as can be seen from the above examples, the students who gave the incorrect responses found the required value by substituting $x = 2\frac{1}{2}$ to the equation obtained in item 6(a) or by multiplying $x = 2\frac{1}{2}$ with the value obtained in item 6(a).

In this question, the word 'average rate of change' had been used to indicate relative change. But this was not understood by most of the students. One common misinterpretation was due to fact that 'average' is also used to describe 'average value' or 'mean'. Thus, in order to avoid ambiguity for further testing, both 'average rate of change' and the 'ratio of changes' would have been used. As mentioned in the book *Calculus* (Mulholland, 1976), the ratio of the changes is a more familiar term.

Question 7®: Integral

This question required a student (i) to produce a correct definition of 'integration', and (ii) to identify an exemplar of the concept and to demonstrate what is meant by a defined relation or event - that is, demonstrate an understanding of the concept.

For item 7(a), the expectation was mainly confirmed but three students nevertheless left out this item and two students thought of 'integration' as a product rather than a process. One student gave a procedural explanation in a rather strange way.

$$\int_b^a f(x)^n dx = \frac{x^{n+1}}{n+1}$$ (Student 13)

The content of the explanations given correctly for 'integration' was as follows:

I. Integration is the opposite of differentiation.
II. A way of finding out the original function from the derivative.
III. A way of finding out the area under a curve.

However, the responses for item 7(b) were more problematic: eight students avoided this item and two students explained it generally, for example, "area under graph between $t = 5$ and $t = 10$". The remaining of the students gave the correct answer such as "the total number of phone calls between 5 and 10".

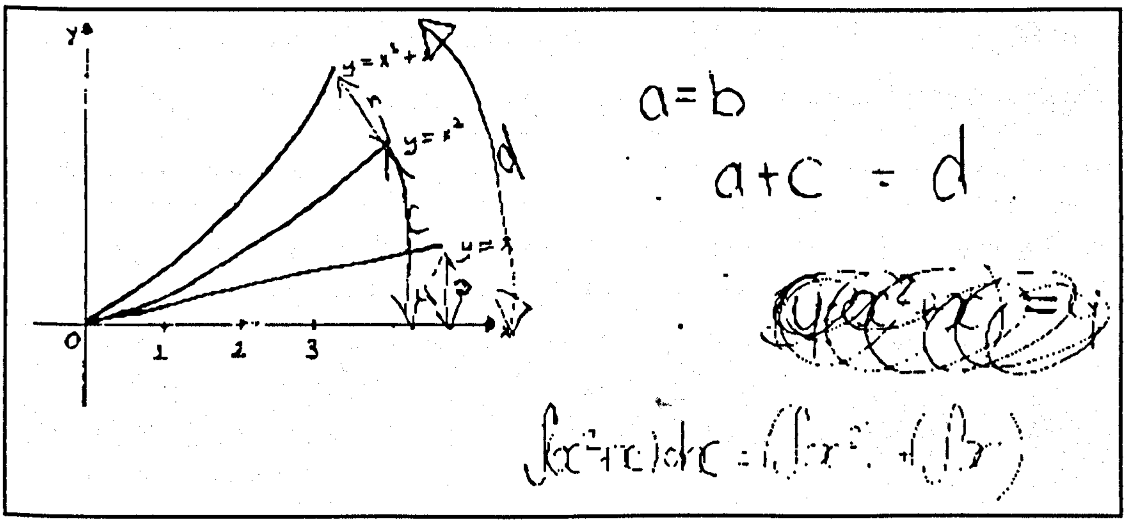
Turning to explanations, some students did not explain that integration is a process or a way to find out the area under a curve to the x-axis or between two curves, but instead described it only as the opposite of differentiation. Thus, it was decided that in order to avoid ambiguity, 'integration' would be replaced by 'integral'.

Question 8: Proof of the integral of the sum of two functions equals to their integrals

This question required students to analyze the principle which is given and to produce a statement of the principle which explains a set of events.

In this question the main interest was to see how students explain the rule - the definite or indefinite integral of the sum of two functions is equal to sum of their separate definite or indefinite integrals - with reference to summing rectangles together with the use of limits.

None of the students gave an explanation as mentioned above since students are usually given this rule on the blackboard and are left to make use of it by rote. Seven students did not attempt to answer the question. A further seven students gave some answers. For example, three of them tried to find out the results of both sides of the integration. Only two of them could think that the area under $y = x$ must be equal to the area between $y = x^2$ and $y = x^2 + x$ without giving any explanation as evidenced by the following example:



Student 9

The question was retained in the test without alteration.

Question 9: Area under a curve

This question required students to apply principles of integration in a given situation. It was designed to test acquaintance with graphical distraction in investigating the required area between the graph of a function and the x-axis or y-axis.

In item 9(a), results showed that students could handle this part with ease. However, five students committed errors in the arithmetic.

In item 9(b), five students failed in arithmetic, sometimes committing more than one error. Four students calculated the integral in the two separate parts, but three of those did not know that the integral was the sum of the positive part from $x = 0$ to $x = 2$ and the negative part from $x = 2$ to $x = 3$. Example:

$$\text{b)} \int_0^2 2x - x^2 dx = [x^2 - \frac{x^3}{3}]_0^2 = (4 - \frac{8}{3}) = \frac{4}{3}$$

$$\int_2^3 2x - x^2 dx = [x^2 - \frac{x^3}{3}]_2^3 = (9-9) - (4 - \frac{8}{3}) = -\frac{4}{3}$$

$$\frac{4}{3} + \frac{4}{3} = \frac{8}{3} \text{ (Student 8)}$$

One of those, however, did not think that the value of $\int_2^3 2x - x^2 dx$ was equal to the value of $\int 2x - x^2 dx$ when $x = 3$, minus the value $\int 2x - x^2 dx$ when $x = 2$ as can be seen from the following example:

$$\text{b)} \int_0^2 2x - x^2 dx = \int_0^2 x^2 - \frac{x^3}{3} = [4 - \frac{8}{3}] + 0 = 1\frac{1}{3}$$

$$\int_2^3 2x - x^2 dx = \int_2^3 x^2 - \frac{x^3}{3} = [9 - \frac{27}{3}] + [4 - \frac{8}{3}] = 0 + 1\frac{1}{3}$$

$$2\frac{2}{3} \text{ (Student 10)}$$

Four students gave a correct response, while one said "I can not do".

In Item 9(c), the majority of the students said that the integral does not exist because, for example, "the graph is tending to infinity along the y-axis". Only two students tried to integrate $\int_{-1}^{2} \frac{1}{x^2} dx$.

In item 9(d), eight students were able to give the correct response but only one of those integrated the equation with respect to $\frac{dx}{dy}$ instead of dy. One student committed errors in the arithmetic and another one found a minus value :

d) $\int_0^2 2y - y^2 dy = [y^2 - \frac{y3}{3}] = [0-0] - [4 - \frac{8}{3}] = -1\frac{1}{3}$.(Student 7)

One student gave an unexpected response. Two students did not attempt the item.

Briefly, the following problems were found:

- Students seem to have difficulty in thinking of some areas as negative.
- There is a confusion with the rule $\int_a^b f(x)dx = F(b) - F(a)$. Students seem to have difficulty in a geometrical argument.
- Students appear to have difficulty in arithmetic.

Question 10°: The volume of revolution

This question required students to translate a concept or principle from a verbal form to a pictorial form. This question was about the volume of revolution. Students had been given the portion of the curve $y = x^2$ between x=1 and x=3. The purpose was to examine how they make use of a limiting process.

Four students sketched the solid of revolution formed but one of those left out the item 10(b). The remaining gave answers to item 10(b) in a right or partial way.

Item 10(a) and 10(b) were omitted by six and four students respectively. Three of the students who omitted item 10(b) also omitted item 10(a). The remaining gave an answer to item 10(b), two of those used the formula $\int \pi y^2 dx$ but one did not give the right upper and lower limit.

Another one, however, gave the following explanation:

b) integrate the function to find the shaded area and then rotate it.(Student 7)

A further four students gave a verbal explanation to item 10(a) as can be seen from the following examples

a) a sort of pot with a hole in the bottom. (Student 8)
a) it would look like a straw. (Student 9)

Moreover, all of these students gave an answer to part 10(b), three of those used the formula $\int \pi y^2 dx$ in a right way and found correct numerical solution. Another one gave a partial explanation as evidenced by the following example:

- b) Trapezium rule. dx is discovered, each section is rotated about the x-axis giving a very thin disc. the volume of this disc is found and dx is taken again, continuing until all the area has been covered then they are added together. (Student 3)

The results obtained here support the results obtained by Orton (1983a) because the majority of students did not have any idea of the procedure dissecting an area or volumes of sections, summing the areas or volumes of the sections.

It can be concluded that although a large proportion of students gave some answers to this question, their understanding was deficient.

A calculus lecturer comment was:

we do not do volumes of revolution.

As a result, this question was rejected for further testing.

<u>Question 11°: Differentiability</u>

A student was required to comprehend differentiability and to apply that principle to the given context to come up with an appropriate answer.

This question was about the existence of the derivative of a function. The function f(x) by piecing together two differentiable functions ax and $bx^2 + x + 1$ was defined. The question that students investigated was the following: if f (x) is differentiable at x = 1, what are the values of a and b?

Thirteen students could not do anything but one student only set the two formulas, ax and $bx^2 + x + 1$, equal and guessed what a and b should be.

One student's comment about this question was quite interesting:

> it is like our tradition we never touch this sort of problems....

Furthermore, the comments of a class teacher and a calculus lecturer in an engineering department were, respectively, as follows:

> ....not possible to solve.

> We do not dwell very much on ideas of differentiability or differentiation from formal principles in formal definition.

As the purpose of testing is to explore whether students have learned major points of content, this question is rejected for further testing.

Summary

As a result of this pilot study, some of the questions were revised(®) or omitted(°). In that way the second test which is discussed in the next section was formed. The revised questions numbers were 2, 3, 5, 6, and 7 and the omitted questions numbers were 4, 10, and 11.

In the second test question 3 was posed as question 4. In addition to that two questions were added in the second test which were questions 3 and 10.

5.1.2 Second Pilot Study

This test contained 10 open-ended questions (see Appendix A) which required students to construct or supply their own answers with the exception of one question which offers choices from pre-established alternative answers. These kinds of questions are called constructed-response questions because they permit the testing of a student's ability to organize ideas and thoughts and allow for creative verbal expression (Nitko, 1983). Note that this test also used in the main study.

Table 5.2 summarizes the general characteristics of the questions assessing higher level abilities. The test was administered to 18 second year A-Level students at the Rushcliffe Comprehensive School.

Question 1: Derivative

This question was not modified after the first pilot study.

Here, four important aspects were raised in relation to this question:

I. Derivative is the gradient function (general statement).
II. Derivative is the result derived from a function (general statement).
III. Derivative is the result of differentiation.
IV. The function of a derivative is equal to its original function.(incorrect statement)

Many of the responses (13) embraced aspect I on both item 1(a) and 1(b). Two students also included aspect I in their responses given for item 1(b) but gave the following answers for item 1(a).

> A derivative is the product of a calculation as it is derived for a certain term. (Student 9)
> A derivative is the product gained from working through a problem. (Student 12)

A more interesting explanation given by a student who included aspect IV in both items is given below:

> A derivative is a genuine way of expressing the same thing $x^3 = 3x^2$. (Student 11)

One of the last two students included aspect II and III for item 1(a) and 1(b) respectively and the other included aspect III for both items.

Table 5.2. General characteristics of the questions used in the second version of the diagnostic test

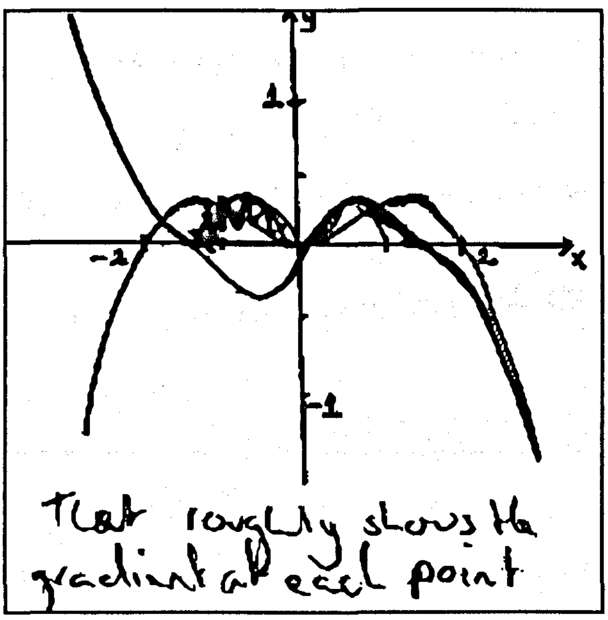
Questions	General Characteristics of Questions Assessing Higher Level abilities[1]	Type of ability
1	Assessing the acquisition of concepts	•Ability to state a definition of a concept •Ability to identify or explain an example of the particular concept
2	Assessing the acquisition of higher order abilities	•Ability to draw inferences based on the displayed graph •Ability to identify the relationships between two concepts
3*	Assessing the acquisition of higher order abilities	•Ability to draw inferences based on the displayed graph •Ability to identify the relationships between two concepts
4	Assessing the acquisition of concepts	•Ability to identify the concept exemplar •Ability to explain the meanings of symbols
5	Assessing the acquisition of higher order abilities Assessing the acquisition of principles	•Ability to read a graph and a table •Ability to use a principle (quotient formula) in a new situation
6	Assessing the acquisition of higher order abilities Assessing the acquisition of principles	•Ability to read a graph •Ability to use a principle (quotient formula) in a new situation •Ability to provide or identify the relationship between two concepts
7	Assessing the acquisition of concepts	•Ability to state a definition of a concept •Ability to identify or explain an example of the particular concept
8	Assessing the acquisition of higher order abilities	Ability to analyze the principle which is given
9	Assessing the acquisition of principles	Ability to apply a principle in a given situation
10*	Assessing the acquisition of higher order abilities	•Ability to form a new rule to solve a problem by combining previously learned rules •Ability to draw inferences based on the displayed graph

Note: * New item

[1] This is categorised according to Nitko (1983) classification

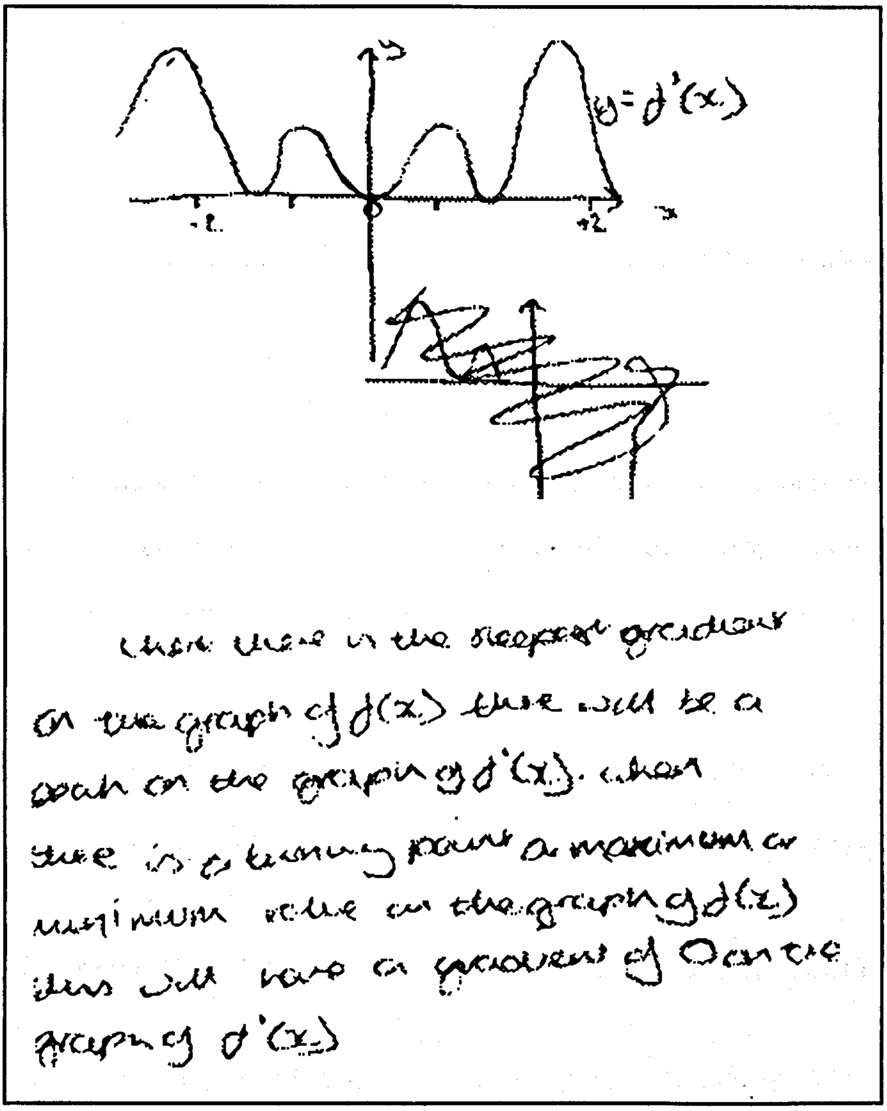
Question 2: Finding the graph of a derivative from the graph of the function given

This question was the reverse form of question 2 which was used in the previous pilot test. Previously it has been asked from derivative function to its original or integral function. The question here required a student to draw the graph of the derivative function from that of its original function. Four students drew the right graph but only two of them gave some reason for their answer as can be seen from the following example:

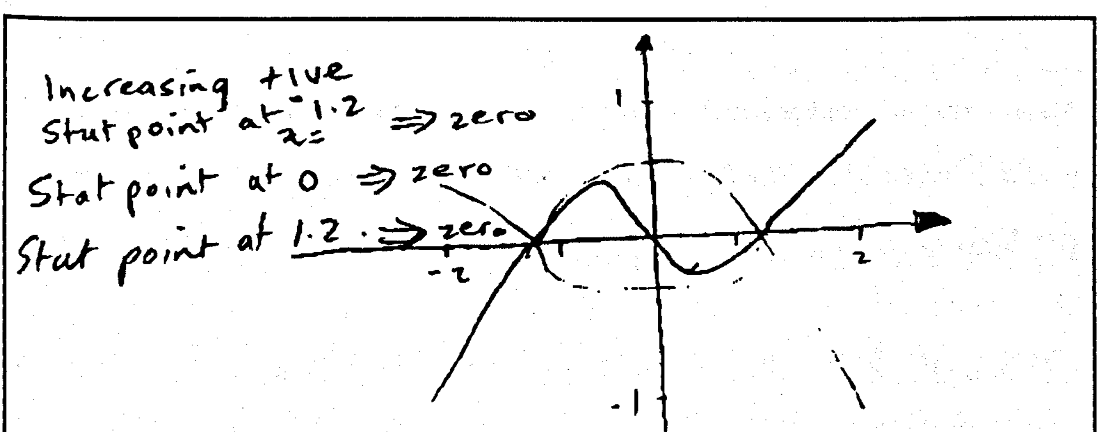


Student 2

While five students omitted the question, eight students sketched incorrect derivative graphs but some with correct stationary points, as can be seen from the examples below.



Student 13



Student 4

Four students sketched the derivative graph like Student 13. Turning to explanations, some students were capable of reading stationary points on the graph but not the positive and negative gradient.

Question 3*: Recognising the graph of a function from its derivative

This question required a student to demonstrate or identify the relationship(s) between 'derivative function' and 'integral function' and to discriminate exemplar from non-exemplars. The question allows the teacher to control (i) familiarity with the stimulus materials, and (ii) the number of discriminations required. An important point is that the student's response must indicate the reasons for identification.

This question (taken from Cornu 1983) shows a graph which is the derivative of one of the three others. The students had to say which, and give reason (s) for their opinion. Tall (1986) showed that 67 % of the experimental group students, who used *Graphic Calculus* Package, gave the correct response, whilst only 8% of the control students were able to do the same.

Students were expected to predict the graph of a function from that of its derivative. The student had a double task: first (s)he had to choose the correct answer among the 3 choices and then give reasons for his/her opinion.

Results of the choices which were made by the students show that nine students (50%) chose the right answer (graph 2). A full correct explanation to the right answer (graph 2) had to refer to the following facts: (i) where the derivative of a function is negative - indicates that the function is decreasing in that interval; (ii) where the derivative of a function is positive - indicates that the function is increasing in that interval; (iii) where the derivative of a function is constant positively or negatively- indicates that the function is a linear function in that interval; (iv) where the value of $f'(x) = 0$, the function has a stationary point. Only one of nine made the correct choice and gave the above explanations.

Although three of those who chose the correct answer included in their explanation a statement regarding the fact (iv) only, one student used the fact (iii). Moreover, two students referred to the fact (i) only and one student referred to facts (i) and (ii).

Three students chose graph 3. These students apparently do not know the meaning of stationary point and(or) the meaning of the positive or negative value of the derivative of a function. Examples:

> It shows the stationary points in the correct areas. (Student 5)

> From 0-3 the graph is rising at a slowing rate. After x=5 the graph y = f′(x) decreases into the negative part and so does y = f(x). (Student 8)

Four students chose graph 4. These students probably thought that increasing derivatives correspond to increasing functions, and decreasing derivatives correspond to decreasing functions. Moreover, constant derivatives correspond to constant functions. Examples:

> • The only when where gradient does not increase at any point. The gradient graph does not increase at any point. (Student 4)

> • Crosses x axis at 6. Level stages between x=3 and x=5. (Student 7)

Two students could not handle the question.

It can be concluded that although a half of the students identified the right answer, their understanding seems unsatisfactory. They fail to realize all the facts which are necessary for the drawing of the original function.

Question 4: Symbols

Question 4 is more or less similar to Question 3 in the first piloting test. Slight changes have been done after the first pilot study. For example, the meaning of δx is in need of explanation in $\frac{\delta y}{\delta x}$ and in $\sum f(x)\delta x$.

Here it was found that both δx and δy were conceived of as:

I. the value of x or y when finding the gradient.

II. the difference in x or y.

III. no of given values or terms.

In addition to that some students gave an explanation for $\frac{\delta y}{\delta x}$ rather than an explanation of δx in $\frac{\delta y}{\delta x}$:

IV. the difference in area beneath the graph.

V. the sum of the numbers of the function.

VI. the difference of y over the difference of x.

and some students gave an explanation for $\sum f(x)\delta x$ rather than an explanation of δx in its:

VII. the sum of the difference in area beneath the graph.

VIII. the sum of the area behind f(x).

However, δy was explained, in addition to the above categorisation, as follows:

IX. with respect to y.

X. the difference in area above the graph.

XI. the differentiated values of y.

The number of students who missed item 4(a) , and item 4(b) were respectively four and six. The students who missed item 4(a) also missed item 4(b). The majority of the remaining students included the category II in their explanations.

The content of the explanations given for $\frac{\delta y}{\delta x}$ presented four sets of aspects (number of responses are indicated in brackets):

I. The gradient of the graph. (8 students)

II. $\frac{\text{difference in y}}{\text{difference in x}}$. (2 students)

III. Differentiated symbol of a function. (1 student)

IV. The function to work out the other graph of speed against time. (1 student)

The remaining six students left out item 4(c).

In item 4(d) although dx in $\frac{d}{dx}(x^2)$ was interpreted as:

I. with respect to x. (2 students)

II. the value of the function. (1 student)

III. is the area to be measured. (1 student)

IV. the differentiated values of x for the function x^2. (1 student)

V. the difference in x-values between two points in the x^2 graph. (1 student)

VI. $\frac{\text{difference}}{\text{difference in x}}(x^2)$. (1 student)

dx in $\int x^2 dx$ was thought of as:

VII. difference in x-value. (4 students)

VIII. the area which has to be measured. (1 student)

IX. the integrated values of x for the function x^2. (1 student)

X. the integration of a gradient slope of $\frac{d}{dx}(x^2)$. (1 student)

XI. integral of x^2. (1 student)

XII. integral between two points of the function $x^2 \Rightarrow$ enables you to calculate area under the graph'.

Moreover, five students omitted this item. It can be seen that some students did not think dx in two situations but instead thought two situations as a whole.

In item 4(e) although eight students perceived dy as 'difference in y-value', five students left out this part. The rest of the responses included 'length of y-axis when finding the gradient', 'the opposite to dx', 'differentiated values of y', 'with respect to y', 'the area above the graph up to y'.

In item 4(f) most of the students gave some acceptable answers while explaining $\frac{dy}{dx}$. The examples of the explanations are given below (numbers of responses are indicated in brackets):

I. gradient function. (10 students)

II. differentiated function. (1 student)

III. differentiation. (1 student)

IV. the equation to work out speed from a distance. (2 students).

V. $\frac{\text{difference in y}}{\text{difference in x}}$. (3 students)

Only one student left out this part.

In item 4(g) $\frac{\delta y}{\delta x}$ and $\frac{dy}{dx}$ were thought of as the same thing by 11 students and as the opposite by one student. Six students missed this item .

This discussion of responses to this question illustrates the great variety of answers given by students.

Question 5: Average rate of change and rate of change of a linear function

As mentioned earlier, this question was designed to ascertain if students could deal with the rate of change of a straight line graph.

Of the 18 students tested only one gave the correct answer, "2", for item 5(a) and five students gave the right answer, "2", for item 5(b).

In item 5(a) four students could give no answer at all and the remaining gave different incorrect answers. Typical responses included "y = 2x", "y + 2h", "2h", and "2(3 + h)". It seems that students need help in coming to understand the concept of rate of change.

In item 5(b) seven students could give no response at all and the remaining six students could not give a correct response. The largest number of these students gave an answer such as "y = 2x" or "(2.5)2".

Question 6: Average rate of change and rate of change of a quadratic function

As can be seen from the previous results, rate of change gives students a great difficulty. This question refers here to the student's perception of rate of change and average rate of change in a non-linear situation.

A number of students did not answer all three parts in item 6(a): 10, 12, and 12 for parts (i), (ii), and (iii) respectively. Item 6(b) and 6(c) were also omitted by thirteen and nine students respectively.

In item 6(a) the rest of the students gave an incorrect answer which was categorised as follows:

I. *Values of the Original Function* Here students substituted values such as 2, 2.1, 2 + h to the equation $y = 3x^2 + 1$ to find the value of y, for example:

(i) $3 \times 2^2 + 1 = 13$. (Student 5)

Some of these students also did some algebraic mistakes as can be seen from the following example:

(ii) $3 \times (2+h)^2 + 1 = 13 + h$. (Student 5)

II. *Values of the Derivative Function.* Here students found the derivative of $y = 3x^2 + 1$ as y = 6x and then substituted values such as 2.1, 2 + h, a + h for x.

III. *Ratio.* Here average rate of change was thought of as a ratio - $\frac{f(a+h)}{f(a)}$.

IV. *Derivative.* Here average rate of change was thought of as the derivative of the function.

In item 6(b) five students gave some incorrect answers. Only one student found correctly the values of the Difference Quotient using $\frac{f(a+h)-f(a)}{h}$. Note that there is a degree of ambiguity in this question as the main purpose of this question is to get students to find the limiting value of the difference quotient. In adopting this item into the final test, it was recognised that the concepts being measured are difficult to assess with a verbal question of this kind. As the discussion in section 7.1.5 shows, this item or this question is open to misunderstandings.

Finally, in item 6(c) typical responses for rate of change at a specific point could be summarised as follows:

I. *Values of the Original Function*. Here students substituted the 2.5 for x in the $y = 3x^2 + 1$.

II. *Gradient Function itself.* This describes responses which are the equation of the gradient function instead of the numerical value of it at $x = 2\frac{1}{2}$.

Three students, however, obtained the correct answer "15".

Question 7: Integral

It was found that the students explained 'integral' in a number of different ways, which are summarised below:

I. *Area.* This describes responses where integral is defined as "... the area under a graph" or " .. the area under a graph between a certain values a and b".

II. *Opposite of Differentiation.* Here integral is regarded as a process rather than a product. For example, "An integral is the opposite of differentiation".

III. *Integer.* Here integral is thought of as an integer. For instance, "An integral is a whole number which is positive".

IV. *Reverse of a Differentiated Function.* Here integral is thought of as an antidifferentiation. For example, "An integral is the reverse of a differentiated function. If f(x) was a function and it was differentiated and then integrated the resultant function would be f(x), still the same".

V. *Result of an Integration.*

Results which were made by the students on item 7(a) show that: 9, 1, 2, 1, and 1 students' explanations included the aspect I, II, III, IV, and V respectively. Furthermore, another two students gave the following explanation:

An integral is when a section of the graph is taken out by picking 2 points on the graph. (Student 11)

The remaining two students left out this item and one of them also left out item 7(b).

Students' explanations for item 7(b) were classified into three groups:

I. *Area.* This example is considered as the area under the graph between 5 and 10.
II. *Number of Calls.* This describes responses where this example is thought of as "the total number of phone calls lasting between 5 and 10".
III. *Length of Calls.* The meaning of this example is seen as 'the length of phone calls between 5 and 10'.

Results obtained by the students on item 7(b) show that: 8, 8, and 1 students' explanation included the aspect I, II, and III respectively.

Results about judgements: in item 7(a) it was expected that students might state the integral as an expression of the area under a curve by means of a limit, whereas in item 7(b) students were expected to state that the expression $\int_5^{10} g(t)dt$ is telling us the sum of the calls received between 5 and 10. The expectation was mainly confirmed in the second part.

<u>Question 8: Proof of the integral of the sum of two functions equals the sum of their integrals</u>

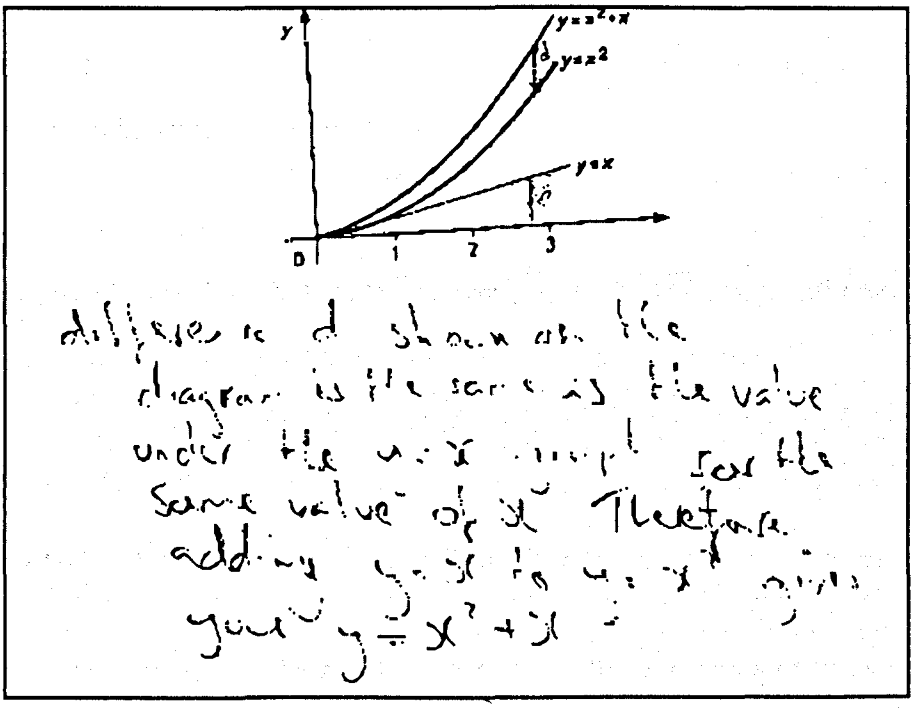
As mentioned earlier, the purpose of this question was to examine the arguments students used for $\int_0^a (x^2 + x)dx = \int_0^a x^2 dx + \int_0^a x dx$.

It was possible to categorise the students' explanations or answers into two main categories:

I. *Verbal Description.* It refers to the explanation of the rule in words, for example, ' the area under the graph y = x added to area under the graph $y = x^2$ equals to the area $y = x^2 + x$ between 0 and a.

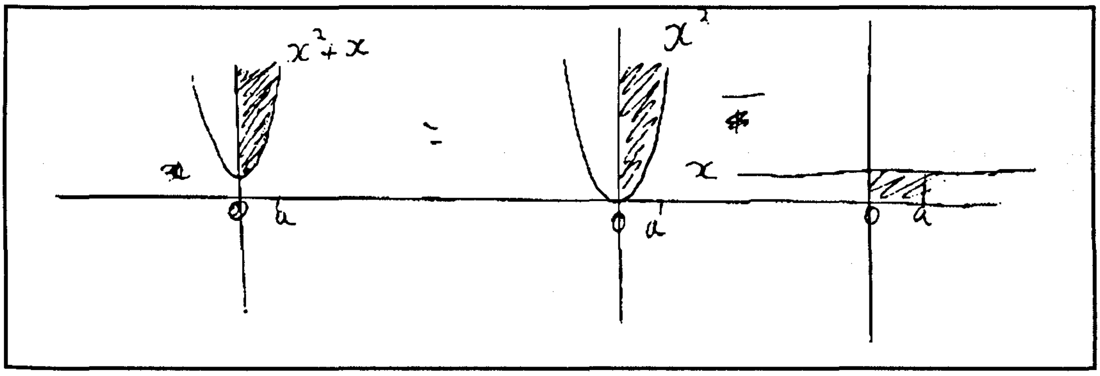
II. *Logic.* It refers to the explanation of the rule with logic, for example, ' this is because when you add x^2 to x you produce x^2+x, so the two components of that formulae go together to make the one $(x^2+x)=x^2+x$', 'this is because $[\frac{x^3}{3}+\frac{x^2}{2}]_0^a=[\frac{x^3}{3}]_0^a+[\frac{x^2}{2}]_0^a$'.

Although category I was used only by one student, the category II was used by eleven students. Furthermore, three students gave an answer by thinking that the area between x^2+x and x^2 must be equal to the area under y = x as evidenced by the following example:



Student 2

However, two students gave no response and one student shaded areas on the graphs in a rather strange way as can be seen from the following example:



Student 6

Question 9: Area under a curve

In item 9(a) six students obtained the right answer but one without showing any attainment. Another six students attempted to find the area geometrically . Only one student missed the item. The remaining students made some mistakes in arithmetic or in using fundamental theorem of calculus.

In item 9(b) six students obtained the right answer but one did so without showing any attainment, one thought that the answer, 0, must be wrong, and one did not give a numerical result as can be seen from the examples that follow:

(a) $= [2\frac{x^2}{2} - \frac{x^3}{3}]_0^3 = [0^2 - \frac{0^3}{3}] - [3^2 - \frac{3^3}{3}] = 0 - 0 = 0$ WRONG. (Student 12)

(a) $\int_0^3 (2x - x^2)dx = [2(\frac{x^2}{2}) - \frac{x^3}{3}]_0^3$. (Student 1)

Three students missed this item of the question and the remaining gave an incorrect answer: four students attempted to find the area geometrically and five students did some mistakes in integration and in arithmetic as can be evidenced by the following example:

- $\int_0^3 (2x - x^2)dx = x^2 - \frac{1}{3}x^3 + 1$. (Student 4)

In item 9(c) the majority of the students gave an explanation such as "cannot be calculated because tends to infinity thus the area tends to infinity". But only two students attempted to find the area but failed and two students left out this part.

In item 9(d) only one student obtained the right answer but used 'x' as the variable of integration instead of 'y'. Four students could not give an answer and nine students seemed to think that it was not possible to find the area as indicated by the following responses "this is not possible because it is on the y-axis", "this cannot be done", or "$x = 2y - y^2$ has to be changed for y". Three students attempted to provide solutions geometrically and one student tried to leave y on the other side of the equality.

<u>Question 10*: Point of tangency, numerical calculation of a gradient and estimating the value of a function</u>

This question required a student to form a new (for the learner) rule to solve a problem, by combining two or more previously learned rules as well as to read the graph and determine what the value of the function at a point is. In this question, much information is condensed in the graph. Having the skills to obtain this information is important for further learning both in and out of school. Item 10(a) and 10(b) were taken from Tufte (1988), and item 10(c) from Amit and Vinner (1990).

This question examined different aspects of the concept of derivative, such as the use of the tangent equation in finding the derivative at a point; and the use of $f'(x) \approx \frac{f(a+h) - f(a)}{h}$ in calculating an approximation f(a+h). As can be seen from the Figure 5.1, a number of students did not answer all three parts: 9, 11, and 15 for items (a), (b) and (c) respectively.

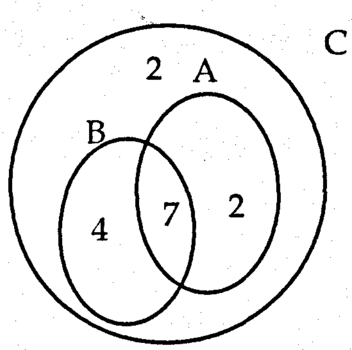


Figure 5.1. The number of students who missed items 10(a), 10(b) and 10(c)

For example, four students missed item 10(b) and 10(c) but not 10(a).

In item 10(a) nine students obtained the answer, "3", from the graph except one, who substituted $x = 5$ to the equation of the tangent to find the value of f(x) at $x = 5$. Why didn't he get it right away from the graph? Did he think that the answer was the same in both situations because the graph of f(x) and its tangent intersect each other at this point?

In item 10(b) four students gave the right answer "$\frac{2}{5}$"; two students thought that the derivative was the equation of the tangent and the one who gave the answer "$\frac{3}{5}$" probably divided the y-coordinate 3 by the x-coordinate 5.

In item 10(c) only two students gave the correct answer by substituting $x = 5.08$ to the equation of the tangent.

Summary

The questions used in the second pilot study were used without modification in the main study. It was also decided that the main data collection would be in the written test format rather than the interview method because (a) interviews are time and labor intensive, and thus difficult to apply to large numbers of students; (b) the potential for generalizing the findings to large groups of students is rather limited (Amir, Frankl, & Tamir, 1987). However, the interview method provides excellent in-depth information about the student's conceptions and enable us to judge the answers in a detailed way. For a few cases, thus, this method will be used to clarify responses in addition to written responses.

The findings of the pilot study formulated my thoughts for constructing (i) a marking scheme for the test and (ii) the hypotheses of the research.

CHAPTER SIX

THE RESULTS OF QUANTITATIVE ANALYSIS

> In some ways, [measuring individual change over time] is the most important topic in educational measurement. The primary object of teaching is to produce learning (that is, change), and the amount and kind of learning that occur can be ascertained only by comparing an individual's or a group's status before the learning period with what it is after the learning period.
>
> Frank B. Davis (1964)

> Investigators who ask questions regarding gain scores would ordinarily be better advised to frame their questions in other ways.
>
> Lee J. Cronbach and Lita Furby (1970)

As has been mentioned in previous chapters, the fundamental questions in this main study center upon issues of individual learning. Data was collected on each student over a period of time and their growth was characterized by the measure of a residual-change score. Data obtained by means of the computer attitude questionnaire were also analyzed for evidence of the role of computers and possible differential effects on students' performance.

6.1 The Results of the Pre-Test

6.1.1 Classroom-Level Analysis

One-way analyses of variance (ANOVA) were carried out for each 26 items of the diagnostic test (see Appendix A) in order to examine whether there are significant differences between the four groups prior to instruction in differentiation. As was the case for the one-way analysis of variance test, the F-test only tells us whether there are significant differences between the four unrelated scores but does not inform us where this difference lies. To look at the difference between any two of the four groups, a Scheffé post hoc test was conducted.

Results of the ANOVA and Scheffé test for each item are summarized in Table 6.1 and their means and standard deviations are presented in Appendix E. Also, the response distribution for six scoring criteria (see Chapter Four) is reported in Appendix F.

All groups did not differ significantly in items 4(e), 4(f), 5(a), 5(b), 6(a), 6(b), 6(c), 7(a), 8, 9(d), 10(a), and 10(c) but the ComG2 group was significantly better than the other groups in item 4(a) and significantly better than the two non-computer groups in items 1(a), 1(b) and 2. Also, the ComG2 was significantly better than the N-ComG1 in items 9(b) and 10(b), and better than N-ComG2 in items 3 and 4(c). The ComG2 did better than both ComG1 and N-ComG2 in items 4(b), 4(g) and 9(c). Although the analysis of variance showed a significant difference at the 0.05 level between the four groups on questions 4(d), 7(b) and 9(a), Scheffé post hoc test did not show a significant difference at the 0.05 level, taking two groups at a time.

The most likely reason for the ComG2 students performing better on some of the items was that they came from a university that could attract higher ability students in mathematics. These differences should not affect the main analysis where the post-test is compared with the pre-test, although some caution should be used. A problem that could occur in this situation is ceiling effects but from examination of the means for the ComG2 students it showed that these were not close to the top of the scale.

Table 6.1. Summary of ANOVA results for the pre-test

(Pairs of means that are significantly different are <u>not</u> underlined by the same line)

Items	ANOVA Results (F-values)	Sheffe-test Results (means ranked from highest to lowest)
1(a)	6.32***	ComG2, ComG1, N-ComG2, N-ComG1
1(b)	5.26**	ComG2, ComG1, N-ComG2, N-ComG1
2	5.34 **	ComG2, ComG1, N-ComG2, N-ComG1
3	4.62**	ComG2, N-ComG1, ComG1, N-ComG2
4(a)	13.34 ***	ComG2, N-ComG1, ComG1, N-ComG2
4(b)	9.64***	ComG2, N-ComG1, N-ComG2, ComG1
4(c)	3.48*	ComG2, N-ComG1, ComG1, N-ComG2
4(d)	2.90 *	
4(e)	0.61 n.s	
4(f)	1.80 n.s	
4(g)	8.44 ***	ComG2, N-ComG1, ComG1, N-ComG2
5(a)	2.22 n.s	
5(b)	0.73 n.s	

Table 6.1 continued

Items	F-values	Sheffe-test Results (means in terms of an order from highest to lowest)
6(a)	0.77 n.s	
6(b)	0.52 n.s	
6(c)	1.08 n.s	
7(a)	2.56 n.s	
7(b)	2.89*	
8	0.007 n.s	
9(a)	2.92*	
9(b)	3.51*	ComG2, N-ComG2, ComG1, N-ComG1
9(c)	4.92**	ComG2, N-ComG1, N-ComG2, ComG1
9(d)	2.56 n.s	
10(a)	2.48 n.s	
10(b)	4.23 **	ComG2, N-ComG2, ComG1, N-ComG1
10(c)	1.01 n.s	

Note. *$p < 0.05$, ** $p < 0.01$, *** $p < 0.001$, n.s : not significant, df= (3, 143).

6.1.2 Student-Level Analysis

The data from the pre-testing on all students was used to explore ways of combining some of the items. Factor analysis was one method employed by which the items on the test could be grouped to form a 'type'. Factor analysis and correlation matrices were used in an exploratory way, to complement common sense and theory based ways of grouping items together.

Two factor analyses were undertaken: (i) a factor analysis of the 26 items; and (ii) a factor analysis of the 10 questions, which are the combined form of the items under each one. Firstly, responses to the 26 items were submitted to factor analysis with varimax rotation. The criterion of eigen values above unity was used to extract nine factors. The nine factors along with factor loadings for each item are presented in Appendix G. The nine-factor solution accounted for 72 percent of the total variance. Grouping decisions, as shown in Appendix G, were based on a minimum factor loading of 0.40. The factor structures show a clear pattern linking the items of the questions. This picture is supported by the simple correlation (see Appendix H) with one or two exceptional cases. The first factor brings together the items of question 9 which were designed to test students' understanding of integral and its relationship with graphs. The second and fourth factors suggest a substantial separation in question 4 between the symbols related to 'δ' and these relating to 'd'. The third factor is composed of the items of question 6 concerning students' understanding of 'rate of change' and 'average rate of change' based on a quadratic graph. The fifth factor brings together question 1 with both items concerning students' understanding of 'derivative' and item 7(a) concerning students' understanding of 'integral'. The sixth factor was clearly dominated by the items of question 10. The seventh factor loaded on two questions, both on graphical interpretation: question 2 and 3. The eighth factor combined the items of question 5 concerning students' understanding of 'average rate of change' and 'rate of change' based on a linear graph. The ninth factor included items 7(b) and 8, and represented the integral questions. The evidence of these groupings is supported in the following chapter.

To examine further the constructs underlying the nine factors, Pearson correlation coefficients between each items of the questions were undertaken. Taking these in conjunction with the earlier factor analyses it is possible to suggest groups of items which hold together fairly consistently. The correlation matrix used to confirm these groupings is presented in Appendix H. There is consistent support for these groupings. The items of each question were moderately or high positively correlated with one another under each question. These intercorrelations warrant the conceptualisation of the items of each question as one question. But within these groups two exceptions are worth noting. There is a substantial separation between the symbols related to 'δ' and 'd' and there is also a substantial overlap between the graphical questions related to either drawing an original graph from a derivative graph or vice versa.

To examine further the constructs underlying the ten questions, another factor analysis was undertaken. A factor analysis with varimax rotation on each question score yielded three factors. These three factors had eigen values above unity and accounted for 58 percent of the total variance. This analysis appeared to relate to the two main components of calculus: Differentiation and Integration. It also provided a basis for grouping questions. Grouping decisions was based on a minimum factor loading of 0.40. The first factor was dominated by differentiation questions. The second factor included integration questions. The third factor had one question on differentiation. The three factors along with factor loadings for each question are presented in Appendix I. To examine further the constructs underlying the three factors Pearson correlation coefficients between each 10 questions were also undertaken. The above results were supported by these Pearson correlation coefficients (see Appendix K).

6.2 The Results of Changes between the Pre-Test and Post-Test Scores

6.2.1 Classroom-Level Analysis

In this part, a 4 x 2 classrooms-by-mathematics background design was employed. Two-way analysis of variance (ANOVA) with the classroom as the unit of analysis was employed. As described in textbooks, this research design is a “nested” or “hierarchical” one in which students are nested within classrooms and classrooms are nested within treatments (Kirk, 1982, Chap. 10). Two-way (ANOVA) using the treatments and mathematics background (A-level or non A-level) as independent variables was employed to compare residual-change scores. A residual change score was used in preference to simply taking the difference between pre-test and post-test.

Cronbach and Furby have defined the residual-change score as “primarily a way of singling out individuals who have changed more (or less) than expected” (1970, p.74), where “expected” in this context has been defined solely in terms of each individual's true initial status. The residual score was obtained by subtracting the predicted post-test score from the corresponding observed post-test score. The predicted post-test score was obtained from the linear regression of post-test score on the pre-test. A residual score is seen as an advantage over a simple difference score because “residuals do not give an advantage to persons with certain values of pre-test scores whereas difference scores do” (Linn & Slinde, 1977, p.125). A residual score is also seen as an advantage with non random assignments because “ ‘post-test only’ scores were found to work reasonably well given random assignments, but not with non random assignments” (Linn & Slinde, 1977, p.132).

The regression analysis method computed residuals for the individual students and then averaged these residuals over a group or classroom system to get the mean residual.

One way analysis of variance on residual scores was also carried out for some items. Where a Scheffé post hoc comparison is carried out these make use of one-way analysis of variance. Scheffé post hoc was conducted to assess differential impacts of treatments.

Appendix E gives the descriptive results for the four groups on each item in the pre-test and the post-test, providing means and standard deviations.

The distribution of the responses to all items is included in Appendix F. The table summarizes the frequencies and percentages in terms of the scoring criteria given in Chapter Four. The frequencies reported also show the difficulty level of the items.

The means of the residual scores are given as each question or item is discussed in this chapter.

The present study addressed several of the hypotheses generated by the research on learning gains and the relationship of mathematics background to treatments. The specific research questions were posed as follows:

1. Are there differences between the four groups of the universities on residual scores for each item and question?

2. Does treatment affect A-level and non-A level students differently with regard to their performances?

3. Is there an interaction between mathematics background and treatment?

Mathematics background was included in the analysis because of its known effect on student's performance (e.g. Edwards, 1995; Morgan, 1988) and because it is possible that effects discovered may be confined to just the more able to the less able students. The number of students with mathematics background for the four groups are shown in Table 6.2. Results for each question are described and discussed separately.

Table 6.2. Number of students broken down by mathematics background for each group

Groups	Mathematics Background	
	A-level	Non-A-level
ComG1	14	21
ComG2	43	13
N-ComG1	7	13
N-ComG2	8	28
Total	72	75

Question 1: Derivative

In this question the 'derivative' concept was tested in two contexts. The items are given below:

1(a) What is a 'derivative'? Define or explain as you wish.

1(b) What does it mean that the derivative of $f(x)=x^3$ is $3x^2$?

Means on residual scores completed for each group and for mathematics background on all students are given in Table 6.3. ANOVA results for this question are summarised in Table 6.4.

Four main points should be noted:

1) For items 1(a) and 1(b), and question 1(the total score), there was no significant difference at the 0.05 level between the residual scores of the four groups. That is, all groups changed similarly from pre-test to post-test, allowing the initial differences in the pre-test.

Table 6.3. Means on residual scores for each group and for mathematics background on question 1

Items	Groups†				Mathematics Background	
	ComG1 (n=35)	ComG2 (n=56)	N-ComG1 n=(20)	N-ComG2 (n=36)	A-level (n=72)	Non-A-level (n=75)
1(a)	-.12	.18	.05	-.19	.26	-.25
1(b)	-.22	.37	-.09	-.32	.46	-.44
Total	-.16	.24	.03	-.23	.33	-.32

Note. † The four groups are described in Chapter Four. ComG1 used GCAL, ComG2 used CALM and CALMAT, and also GCAL was used by the teacher as a demonstration aid.

Table 6.4. Summary of ANOVA results for question 1

Items	Treatment		Mathematics Background		Interaction	
	df	F	df	F	df	F
1(a)	(3, 139)	0.18	(1, 139)	5.55*	(3, 139)	1.01
1(b)	(3, 139)	0.37	(1, 139)	9.47**	(3, 139)	0.65
Total	(3, 139)	0.31	(1, 139)	9.50**	(3, 139)	1.05

* $p < 0.05$, ** $p < 0.01$.

2) For item 1(a), there was a significant difference at the 0.05 level between the A-level and non-A-level students on the residual score. This was because A-level students performed better than non-A-level students.

3) For item 1(b) and the total score, there was a significant difference at the 0.01 level between the A-level and non-A-level students on the residual score. This was because A-level students performed better than non-A-level students.

4) For item 1(a), 1(b) and the total score, there was no interaction at the 0.05 level between mathematics background and groups on the actual post-test score and predicted post-test score.

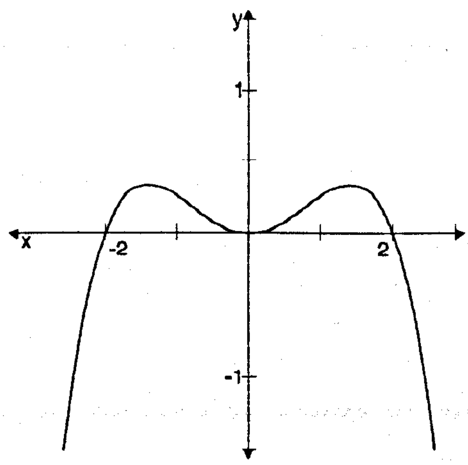
Question 2: Finding the graph of a derivative from the graph of the function given

Question 2 dealt with the drawing of derivative graph from a given original graph. The question is given below:

2. The graph at the right is the graph of a function y = f(x)

Sketch what the derivative looks like

Give the reason(s) for your answer.



The following two tables, Table 6.5 and Table 6.6, present means on residual scores completed for each group and for mathematics background on all students, and the summary of ANOVA results, respectively.

The following points emerged from this analysis:

1) For question 2, there was a significant difference at 0.05 level between the residual scores of the four groups ($p = 0.014$).

2) For question 2, there was a significant main effect of mathematics background at 0.01 level, favoring the students who had A-level mathematics background.

Table 6.5. Means on residual scores for each group and for mathematics background on question 2

Question	Groups				Mathematics Background	
	ComG1 (n=35)	ComG2 (n=56)	N-ComG1 (n=20)	N-ComG2 (n=36)	A-level (n=72)	Non-A-level (n=75)
2	.2 5	.60	-.82	-.72	.58	-.55

Table 6.6. Summary of ANOVA results for question 2

Question	Treatment		Mathematics Background		Interaction	
	df	F	df	F	df	F
2	(3, 139)	3.65*	(1, 139)	8.00**	(3, 139)	1.19

* $p < 0.05$, ** $p < 0.01$.

3) For question 2, there was no significant interaction.

Having found that there was an overall significant difference in the actual post-test score and predicted post-test score between the four groups, there was a need to find out where this difference lies. To meet this point, evidence from a Scheffé post hoc test indicated that students in ComG2 did significantly better at the 0.05 level than students in two non-computer groups. No significant difference was observed between students in ComG1 and the students in two non-computer groups. However, students in ComG1 did much better than students in the two non-computer groups.

To clarify the picture further, a 2 x 2 analysis of variance was carried out to compare the two computer groups with the two non-computer groups. There was a significant difference between these groups, $F(1, 143) = 10.75$, $p<0.01$.

There was also a significant difference between A-level and non-A-level students on the residual score, $F(1, 143) = 8.94$ $p<0.01$, but no interaction between treatment and mathematics background.

The results show that the computer group students developed a higher level of performance than did non-computer group students. In either case, the computer groups scored better than the non-computer groups.

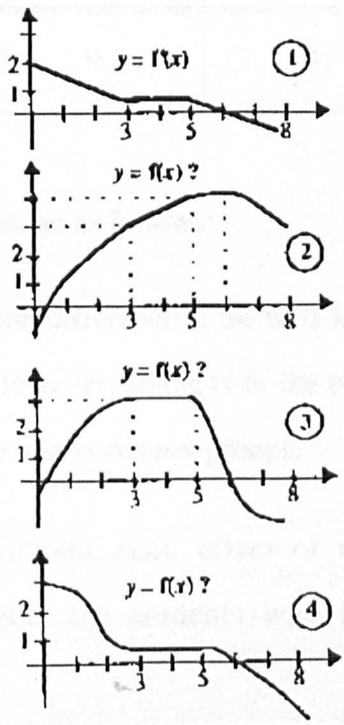
Question 3: Recognising the graph of a function from its derivative graph

This question unlike question 2 dealt with the selection of an original graph from the derivative graph. The question is given below.

3. Graph **1** is the derivative $y = f'(x)$ of a function $y = f(x)$ defined for $0 \leq x \leq 8$.

Which of the graphs **2, 3, 4** could be the original graph $y = f(x)$?

Give you reason(s) for your choice.



The means on residual scores and the summary of ANOVA results are reported in Table 6.7 and 6.8, respectively.

Table 6.7. Means on residual scores for each group and for mathematics background on question 3

Question	Groups				Mathematics Background	
	ComG1 (n=35)	ComG2 (n=56)	N-ComG1 (n=20)	N-ComG2 (n=36)	A-level (n=72)	Non-A-level (n=75)
3	.2 8	.31	-.41	-.54	.51	-.49

Table 6.8. Summary of ANOVA results for question 3

Question	Treatment		Mathematics Background		Interaction	
	df	F	df	F	df	F
3	(3, 139)	1.17	(1, 139)	9.26**	(3, 139)	1.56

** $p < 0.01$.

The points that emerged from this analysis are as follows:

1) For question 3, there was no significant difference at the 0.05 level between the residual scores of the four groups. However, students in the two computer groups did better than students in the two non-computer groups.

2) For question 3, there was a significant main effect of mathematics background at the 0.01 level, favoring the students who had A-level mathematics background.

3) For question 3, the results revealed no significant interaction.

Since factor analysis yielded one factor including question 2 and 3, these questions were combined for further analysis. In the subsequent analysis, the combination of these questions will be used, to allow comparisons between the four groups according to their performance in graphical questions.

Combined Subscale by Question 2 and 3

Two-way analysis of variance was also undertaken on this subscale. A summary of this result is reported in the subsequent two tables.

Table 6.9. Means on residual scores for each group and for mathematics background on combined scores for questions 2 and 3

Question	Groups				Mathematics Background	
	ComG1 (n=35)	ComG2 (n=56)	N-ComG1 (n=20)	N-ComG2 (n=36)	A-level (n=72)	Non-A-level (n=75)
2+3	.2 9	.43	-.62	-.60	.51	-.49

Table 6.10. Summary of 4x2 ANOVA results for combined questions 2 and 3

Question	Treatment		Mathematics Background		Interaction	
	df	F	df	F	df	F
2+3	(3, 139)	3.14*	(1, 139)	11.33***	(3, 139)	1.79

* $p < 0.05$, *** $p \leq 0.001$.

Note that there was a significant difference at the 0.05 level between the residual scores of the four groups and a significant difference at the 0.001 level between the A-level and non-A-level students, but there was no interaction at the 0.05 level between the mathematics background and the groups. As the following Table 6.11 indicates, the same result was also found for computer and non-computer groups.

Table 6.11. Summary of 2 x 2 ANOVA results for combined questions 2 and 3 for both computer and non-computer groups

Question	Treatment		Mathematics Background		Interaction	
	df	F	df	F	df	F
2+3	(1, 143)	8.90**	(1, 139)	11.03***	(3, 139)	0.20

** $p < 0.01$, *** $p \le 0.001$.

The general trend evident regarding questions 2 and 3, either separately or in the combined form, was that each or both computer groups performed better than each or both non-computer groups on graphical questions.

Question 4: Symbols

In this question the definitions or explanations of different kinds of symbols were incorporated as follows:

4. Explain the meaning of each of the following symbol

a) Explain the δx in $\frac{\delta y}{\delta x}$ and in $\sum f(x)\delta x$.

b) δy

c) Explain $\frac{\delta y}{\delta x}$

d) Explain the dx in $\frac{d}{dx}(x^2)$ and in $\int x^2 dx$.

e) Explain dy

f) Explain $\frac{dy}{dx}$

g) What is the relationship between $\frac{\delta y}{\delta x}$ and $\frac{dy}{dx}$?

The following two tables, Table 6.12 and Table 6.13, give the means on residual scores completed for each group and for mathematics background on all students, and the summary of ANOVA results, respectively.

Table 6.12. Means on residual scores for each group and for mathematics background on question 4

Items	Groups				Mathematics Background	
	ComG1 (n=35)	ComG2 (n=56)	N-ComG1 (n=20)	N-ComG2 (n=36)	A-level (n=72)	Non-A-level (n=75)
4(a)	-.73	.36	.06	.12	-.03	.03
4(b)	-.75	.42	.35	-.12	.09	-.09
4(c)	-.92	.55	.51	-.24	.13	-.12
4(d)	-.40	.18	.56	-.19	-.03	.03
4(e)	-.07	.03	.66	-.34	.02	-.02
4(f)	-.02	.33	-.02	-.48	.00	.00
4(g)	.00	.07	.42	-.34	-.02	.02
Total	-.38	.20	.42	-.18	-.07	.07

The results, provided in Table 6.12 and 6.13, can be summarised as follows:

1) For items 4(a), 4(b), 4(c), and the total score, there was a significant overall main effect of treatment on the residual score. Follow-up Sheffe post hoc test was carried out after the data have been initially analysed by one way analysis of variance. This showed the effect of treatment on performance, favoring ComG2 to ComG1 for item 4(a) and 4(b); ComG2 and N-ComG1 to ComG1 for item 4(c) and total score.

2) For items 4(d), 4(e), 4(f), and 4(g), there was no significant overall main effect of treatment on the residual score.

Table 6.13. Summary of ANOVA results for question 4

Items	Treatment		Mathematics Background		Interaction	
	df	F	df	F	df	F
4(a)	(3, 139)	6.06***	(1, 139)	1.64	(3, 139)	2.02
4(b)	(3, 139)	4.50**	(1, 139)	0.10	(3, 139)	1.43
4(c)	(3, 139)	6.83***	(1, 139)	0.12	(3, 139)	0.24
4(d)	(3, 139)	2.31	(1, 139)	0.64	(3, 139)	1.73
4(e)	(3, 139)	2.03	(1, 139)	0.01	(3, 139)	0.26
4(f)	(3, 139)	1.75	(1, 139)	1.01	(3, 139)	0.93
4(g)	(3, 139)	1.49	(1, 139)	0.37	(3, 139)	1.19
Total	(3, 139)	6.44***	(1, 139)	4.37*	(3, 139)	2.91*

* $p<0.05$, ** $p < 0.01$, *** $p \leq 0.001$.

3) For each item of this question, there was no significant overall main effect of mathematics background at 0.05 level on residual score but there was a significant main effect of mathematics background on the total score for question 4.

4) For each item of question 4, the results revealed no significant treatment-by-mathematics background interaction at 0.05 level but the analysis of variance of total score on question 4 revealed a significant treatment-by-mathematics background interaction. The interaction is displayed graphically in Figure 6.1. The interaction suggests that A-level students in ComG1, N-ComG1, and N-ComG2 did not display as great an improvement in performance as did the A-level students in ComG2. A-level students who were taught in ComG2 did better than non-A-level students, but this case was not prevalent in other groups.

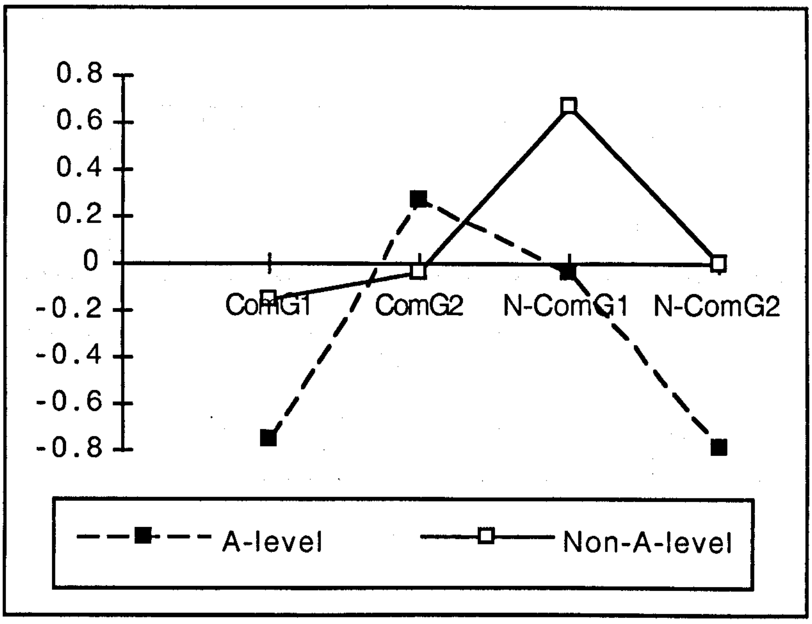


Figure 6.1. Question 4 (total) as a function of treatment condition and mathematics background

The results that emerged from the factor analysis showed that the items of question 4 were grouped into two subscales (see Appendix G). Responses to items 4(a), 4(b) and 4(c) and also 4(d), 4(e), 4(f) and 4(g) were averaged into two single measures. In the subsequent analysis, each subscale was again analysed by two-way analysis of variance.

The analysis for the combined measure of the items related to 'δ' symbol or 'd' symbol indicated that there was a significant main effect of treatment conditions for both combined scores but there were no significant main effects of mathematics background and no interaction between treatment conditions and mathematics background at 0.05 level. The results of two-way analysis of variance, calculated for each of the subscales produced by combining the items of question 4 according to the 'δ' symbol or 'd' symbol, are set out in Table 6.14. The means on residual score are also displayed in Table 6.15.

Table 6.14. Summary of ANOVA results for combined items in question 4

Items	Treatment		Mathematics Background		Interaction	
	df	F	df	F	df	F
4(a)+4(b)+4(c)	(3, 139)	8.34***	(1, 139)	1.22	(3, 139)	1.63
4(d)+4(e)+4(f) +4(g)	(3, 139)	3.42*	(1, 139)	1.88	(3, 139)	1.78

* p<0.05, *** p<0.001.

Table 6.15. Means on residual scores for each group and for mathematics background on combined scores in question 4

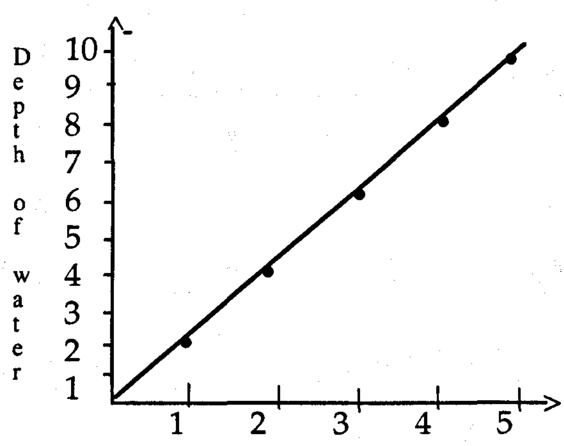
Items	Groups				Mathematics Background	
	ComG1 (n=35)	ComG2 (n=56)	N-ComG1 (n=20)	N-ComG2 (n=36)	A-level (n=72)	Non-A-level (n=75)
4(a)+4(b) +4(c)	-.78	.40	.31	-.04	.02	-.02
4(d)+4(e) +4(f)+4(g)	-.14	.14	.46	-.34	-.05	.05

Question 5: Average rate of change and rate of change of a linear function

The aim of this question was to provide information concerning students' abilities and understanding related to average rate of change and rate of change based on a linear graph. The question is given below.

5. Water is flowing into a tank at a constant rate. For each unit increase in the time, the depth of water increases by 2 units. The table and graph illustrate this situation.

Time (x)	0		1		2		3		4		5
Depth (y)	0		2		4		6		8		10
1st difference (depth)		2		2		2		2		2	



a) What is the rate of increase of y as x increases from 3 to 3+h?

b) What is the rate of increase of y at $x = 2\frac{1}{2}$ and at $x = X$?

The points that emerged from the following two tables, Table 6.16 and 6.17, were :

1) For item 5(a) and the total score on question 5, there was no significant difference at the 0.05 level between the residual scores of the four groups. That is, all groups changed similarly from the pre-test to the post-test, allowing the initial differences in the pre-test.

2) For item 5(b), there was a significant difference at the 0.05 level between the residual scores of the four groups. Comparison of the four groups by Sheffe contrasts on the residual scores indicated that only the ComG2 differed significantly from the ComG1 ($F = 4.08$, $p = 0.0082$).

Table 6.16. Means on residual scores for each group and for Mathematics background on question 5

Items	Groups				Mathematics Background	
	ComG1 (n=35)	ComG2 (n=56)	N-ComG1 (n=20)	N-ComG2 (n=36)	A-level (n=72)	Non-A-level (n=75)
5(a)	-.22	.51	-.55	-.28	.73	-.70
5(b)	-.71	.57	-.62	.14	.69	-.66
Total	-.44	.52	-.55	-.08	.67	-.64

Table 6.17. Summary of ANOVA results for question 5

Items	Treatment		Mathematics Background		Interaction	
	df	F	df	F	df	F
5(a)	(3, 139)	0.44	(1, 139)	18.13***	(3, 139)	1.14
5(b)	(3, 139)	2.69*	(1, 139)	14.95***	(3, 139)	0.17
Total	(3, 139)	1.48	(1, 139)	18.94***	(3, 139)	0.62

* $p < 0.05$, *** $p < 0.001$

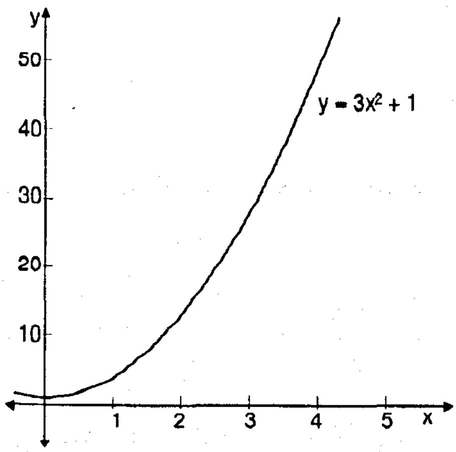
3) For each item and the total score on question 5, there was a significant difference at the 0.001 level between the A-level and non-A-level students on the residual score.

4) For each item and the total score on question 5, there was no significant interaction at the 0.05 level.

Question 6: Average rate of change and rate of change of a quadratic function

The aim of this question was to provide information concerning students' abilities and understanding relating to 'average rate of change' and 'rate of change' based on a quadratic function. The question is given below.

6. The graph below represents $y = 3x^2 + 1$, from $x=0$ to $x=4$.



a) What are the ratios of changes (average rate of change) of y with respect to x as x changes from: (i) 2 to 2 + 0.1, (ii) 2 to 2 + h, (iii) a to a + h ?

b) What are the rates of change of y with respect to x as x changes from: (i) 2 to 2 + 0.1, (ii) 2 to 2 + h, (iii) a to a + h ?

c) What is the rate of change of y at $x=2\frac{1}{2}$?

Means on the residual scores completed for each group and for mathematics background on all students, and the summary of the ANOVA results are presented in Table 6.18 and 6.19, respectively.

According to the two tables below, the main findings can be summarized as follows:

1) For each item and the total score on question 6, there was no significant difference at the 0.05 level between the residual scores of the four groups. That is, all groups changed similarly from the pre-test to the post-test, allowing the initial differences in the pre-test.

Table 6.18. Means on residual scores for each group and for mathematics background on question 6

Items	Groups				Mathematics Background	
	ComG1 (n=35)	ComG2 (n=56)	N-ComG1 (n=20)	N-ComG2 (n=36)	A-level (n=72)	Non-A-level (n=75)
6(a)	-.33	.13	-.22	.24	.12	-.12
6(b)	-.17	-.10	.18	.23	.00	.00
6(c)	.21	-.33	-.25	.44	.14	-.14
Total	-.09	-.12	-.07	.31	.04	-.04

Table 6.19. Summary of ANOVA results for question 6

Items	Treatment		Mathematics Background		Interaction	
	df	F	df	F	df	F
6(a)	(3, 139)	1.27	(1, 139)	0.98	(3, 139)	0.22
6(b)	(3, 139)	1.34	(1, 139)	0.48	(3, 139)	2.64
6(c)	(3, 139)	1.72	(1, 139)	2.64	(3, 139)	1.38
Total	(3, 139)	1.52	(1, 139)	1.38	(3, 139)	1.10

2) For each item and the total score on question 6, there was no significant difference at the 0.05 level between the A-level and non-A-level students on the residual score.

3) For each item and the total score on question 6, there was no interaction at the 0.05 level between treatment and mathematics background. Although no significant interaction occurred between treatment and mathematics background on question 6(b), a better performance for A-level students in ComG2 and especially N-ComG1 was observed (see Figure 6.2) ($p=0.052$ for 6(b)).

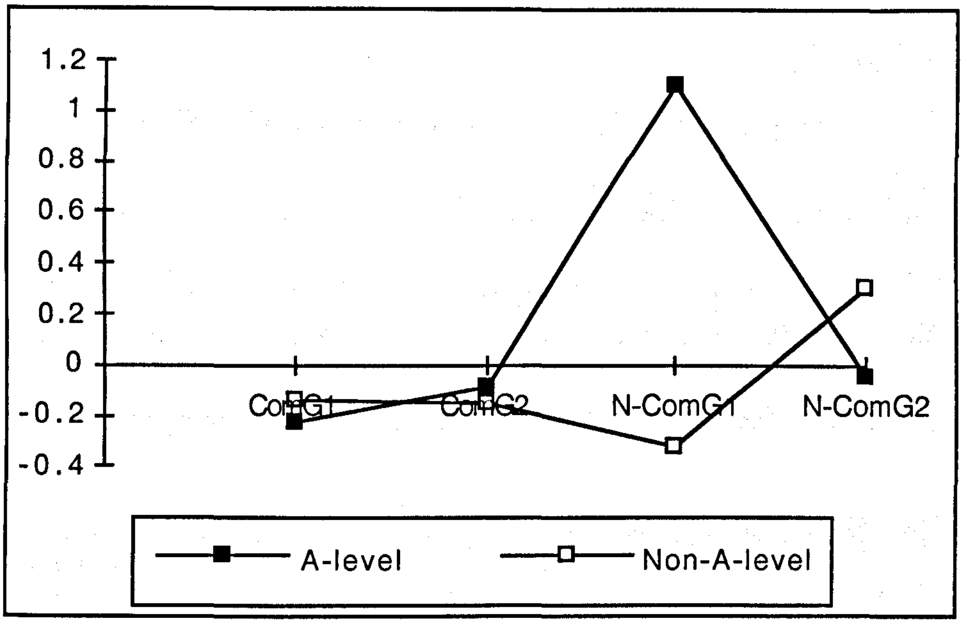


Figure 6.2. Item 6(b) as a function of treatment condition and mathematics background

Question 7: Integral

In this question the integral concept was tested in two contexts. The items were as follows:

> 7 (a) What is an ' integral'? Define and explain as you wish.
>
> 7 (b) Function g(t) gives the number of phone calls in time t
>
> What is the meaning of $\int_{5}^{10} g(t)\,dt$?

Means on the residual scores completed for each group and for mathematics background on all students, and the summary of the ANOVA results are given in Table 6.20 and 6.21, respectively.

Table 6.20. Means on residual scores for each group and for mathematics background on question 7

Items	Groups				Mathematics Background	
	ComG1 (n=35)	ComG2 (n=56)	N-ComG1 (n=20)	N-ComG2 (n=36)	A-level (n=72)	Non-A-level (n=75)
7(a)	-.09	.12	-.57	.21	.12	-.11
7(b)	-.09	.34	-.56	-.13	.07	-.07
Total	-.09	.23	-.53	.03	.08	-.07

The main findings can be summarized as follows:

1) For each item of and the total score on question 7, there was no significant difference at the 0.05 level between the residual scores of the four groups. That is, all groups changed similarly from the pre-test to the post-test, allowing the initial differences in the pre-test.

2) For each item and the total score on question 7, there was no significant difference at the 0.05 level between the A-level and non-A-level students on the residual score.

3) For item 7(a) and the total score on question 7, there was no interaction at the 0.05 level between treatment and mathematics background.

Table 6.21. Summary of ANOVA results for question 7

Items	Treatment		Mathematics Background		Interaction	
	df	F	df	F	df	F
7(a)	(3, 139)	1.43	(1, 139)	0.70	(3, 139)	0.27
7(b)	(3, 139)	1.29	(1, 139)	0.11	(3, 139)	4.90**
Total	(3, 139)	1.53	(1, 139)	0.004	(3, 139)	2.16

4) For item 7(b), there was an interaction at the 0.01 level between treatment and mathematics background. It is attributable to the fact that for A-level students, N-ComG1 was more helpful than N-ComG2, ComG1, and ComG2, and that the opposite was prevalent for non-A-level students. The interaction is displayed graphically in Figure 6.3.

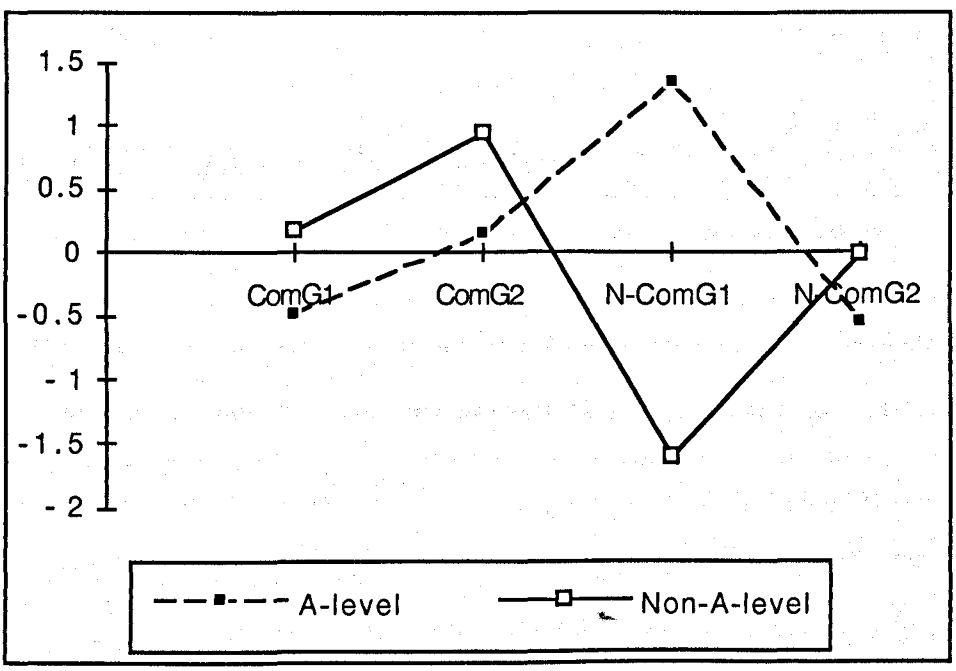
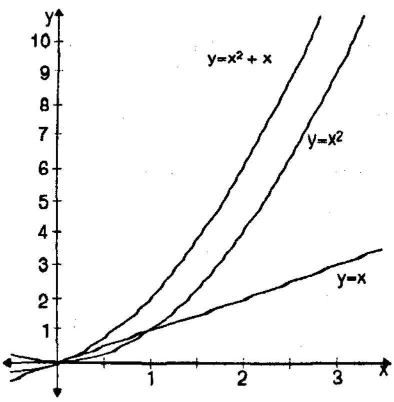


Figure 6.3. Item 7(b) as a function of treatment condition and mathematics background

Question 8: Proof of the integral of the sum of two functions equals the sum of their integrals

The aim of this question was to provide information concerning students' understanding of the fact that the integral of the sum of two functions equals the sum of their integrals.

8.



Explain, by means of the diagram or otherwise, why

$$\int_0^a (x^2 + x)\,dx = \int_0^a x^2 dx + \int_0^a x\,dx$$

Means on the residual scores completed for each group and for mathematics background on all students, and the summary of the ANOVA results are shown in Table 6.22 and 6.23, respectively.

Three main points should be noted:

1) For question 8, there was no significant difference at the 0.05 level between the residual scores of the four groups. That is, all groups changed similarly from the pre-test to the post-test, allowing the initial differences in the pre-test.

2) For question 8, there was no significant difference at the 0.05 level between the A-level and non-A-level students on the residual score.

3) For question 8, there was no interaction at the 0.05 level between treatment and mathematics background.

Table 6.22. Means on residual scores for each group and for mathematics background on question 8

Question	Groups				Mathematics Background	
	ComG1 (n=35)	ComG2 (n=56)	N-ComG1 (n=20)	N-ComG2 (n=36)	A-level (n=72)	Non-A-level (n=75)
8	-.24	.11	-.58	.39	.14	-.14

Table 6.23. Summary of ANOVA results for question 8

Question	Treatment		Mathematics Background		Interaction	
	df	F	df	F	df	F
8	(3, 139)	2.55	(1, 139)	1.71	(3, 139)	1.17

Question 9: Area under a curve

The aim of this question was to test students understanding of integration and its relation to graphs. The question is given overleaf.

The following tables present means on residual scores completed for each group and for mathematics background, and the summary of ANOVA results, respectively.

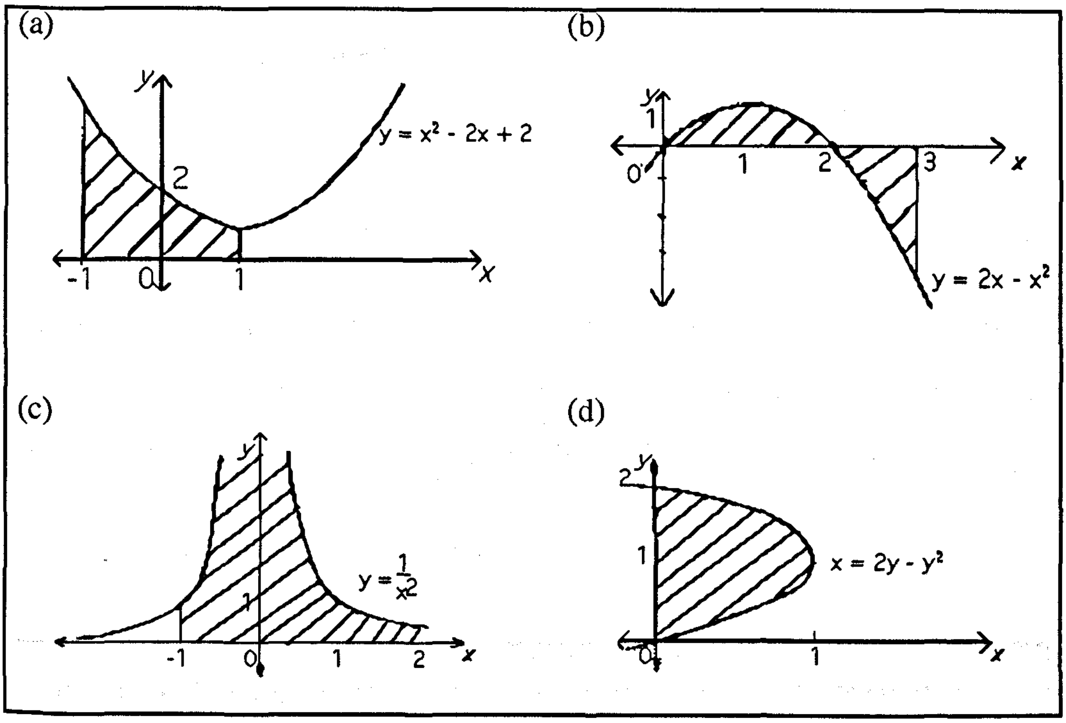


Table 6.24. Means on residual scores for each group and for mathematics background on question 9

Items	Groups				Mathematics Background	
	ComG1 (n=35)	ComG2 (n=56)	N-ComG1 (n=20)	N-ComG2 (n=36)	A-level (n=72)	Non-A-level (n=75)
9(a)	-.01	-.02	.38	-.16	.09	-.08
9(b)	.36	.16	-.18	-.51	.39	-.37
9(c)	-.13	.36	-.11	-.37	.40	-.39
9(d)	.06	.22	-.05	-.37	.39	-.38
Total	.10	.15	.02	-.34	.29	-.28

The main findings can be summarized as follows:

1) For each item and the total score on question 9, there was no significant difference at the 0.05 level between the residual scores of the four groups. That

is, all groups changed similarly from the pre-test to the post-test, allowing the initial differences in the pre-test.

2) For item 9(a), there was no significant difference between the A-level and non-A-level students on the residual score.

3) For item 9(b), 9(c), 9(d), and the total score on question 9, there was a significant difference between the A-level and non-A-level students on the residual score, favoring the students who had A-level mathematics.

Table 6.25. Summary of ANOVA results for question 9

Items	Treatment		Mathematics Background		Interaction	
	df	F	df	F	df	F
9(a)	(3, 139)	0.95	(1, 139)	0.90	(3, 139)	0.31
9(b)	(3, 139)	1.06	(1, 139)	4.69*	(3, 139)	3.48*
9(c)	(3, 139)	0.52	(1, 139)	7.29**	(3, 139)	0.07
9(d)	(3, 139)	0.14	(1, 139)	3.88*	(3, 139)	1.86
Total	(3, 139)	0.65	(1, 139)	6.74**	(3, 139)	1.23

* $p \leq 0.05$, ** $p < 0.01$.

4) For item 9(a), 9(c), 9(d) and the total score on the question 9, there was no interaction at 0.05 level between mathematics background and the university groups.

5) For item 9(b), there was an interaction at 0.05 level between mathematics background and the university groups. The interaction is graphed in Figure 6.4. The significant interaction is attributable to the fact that A-level students in ComG1 and N-ComG2 were more affected by treatment than students in ComG2 and N-ComG1 whereas the treatment in ComG2 and N-ComG1 showed more or less the same effect on A-level and non-A-level students.

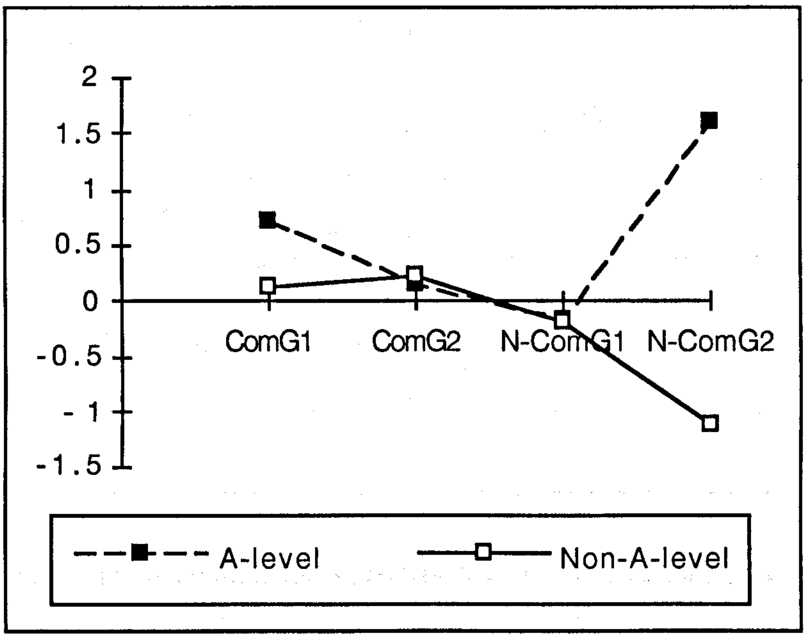


Figure 6.4. Item 9(b) as a function of treatment condition and mathematics background

Question 10: Point of tangency, numerical calculation of a gradient and estimating the value of a function

In this question the derivative concept was tested in two contexts as well as the reading of points from a graph. The question is given below:

Means on residual scores completed for each group and for mathematics background on all students are presented in Table 6.26.

10. Line L is a tangent to the graph of $y = f(x)$ at the point (5, 3).

a) Find the value of $f(x)$ at $x = 5$.

b) Find the derivative of $f(x)$ at $x = 5$.

c) What is the value of the function $f(x)$ at $x=5.08$? (Be as accurate as possible)

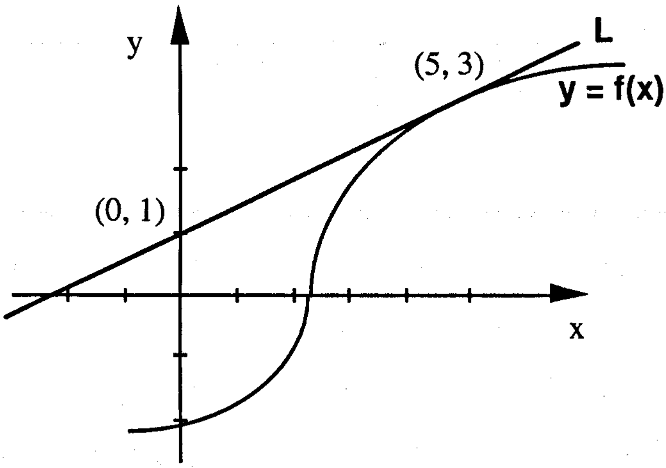


Table 6.26. Means on residual scores for each group and for mathematics background on question 10

Items	Groups				Mathematics Background	
	ComG1 (n=35)	ComG2 (n=56)	N-ComG1 (n=20)	N-ComG2 (n=36)	A-level (n=72)	Non-A-level (n=75)
10(a)	-.07	-.10	.65	-.13	.31	-.29
10(b)	-.08	-.36	1.08	.03	.33	-.32
10(c)	-.02	-.18	.24	.17	.30	-.28
Total	.01	-.26	.70	.01	.28	-.27

The results of ANOVA are given in Table 6.27. Four main points should be noted:

1) For item 10(a) and 10(c), there was no significant difference at the 0.05 level between the residual scores of the four groups. That is, all groups changed similarly from the pre-test to the post-test, allowing the initial differences in the pre-test.

Table 6.27. Summary of ANOVA results for question 10

Items	Treatment		Mathematics Background		Interaction	
	df	F	df	F	df	F
10(a)	(3, 139)	1.10	(1, 139)	4.01*	(3, 139)	0.30
10(b)	(3, 139)	4.37**	(1, 139)	9.14**	(3, 139)	1.12
10(c)	(3, 139)	1.99	(1, 139)	9.28**	(3, 139)	0.17
Total	(3, 139)	4.28**	(1, 139)	11.50***	(3, 139)	0.70

* $p < 0.05$, ** $p < 0.01$, *** $p \leq 0.001$.

2) For item 10(b) and the total score on question 10, there was a significant difference at 0.01 level between the residual scores of the four groups. Follow-up Sheffe post hoc was carried out after the data has been initially analysed by a one-way ANOVA. The results, however, showed that there was no significant differences at the 0.05 level between the four groups on the residual score since the variations between A-level and non-A-level students within the university groups were different. These variations were taken into account when calculating the two-way analysis of variance. The one-way effect of group thus took out the variation due to mathematics background and this is why it was highly significant in this case when it was not significant in the straight one-way analysis of variance of the group effect.

3) For item 10(a), 10(b), 10(c), and the total score, there was a significant difference between the A-level and non-A-level students on the residual score, favoring the students who had A-level mathematics.

4) For each item of question 10 and the total score, there was no interaction at the 0.05 level between mathematics background and the university groups.

6.3 The Results of the Computer Attitude Questionnaire

Of the 35 students in ComG1 and 56 students in ComG2 who completed the pre-test and post-test, 26 in ComG1 and 53 in ComG2 also completed the Computer Attitude Questionnaire. The non-computer group did not receive this.

The students' responses to the two questions concerning "how often did you use the computer...?" and "On average, how much time did you spend on the computer in each session?" are summarized in Figure 6.5 and Figure 6.6 for ComG1 and ComG2, respectively. These responses were defined as "interest and involvement" with the computer.

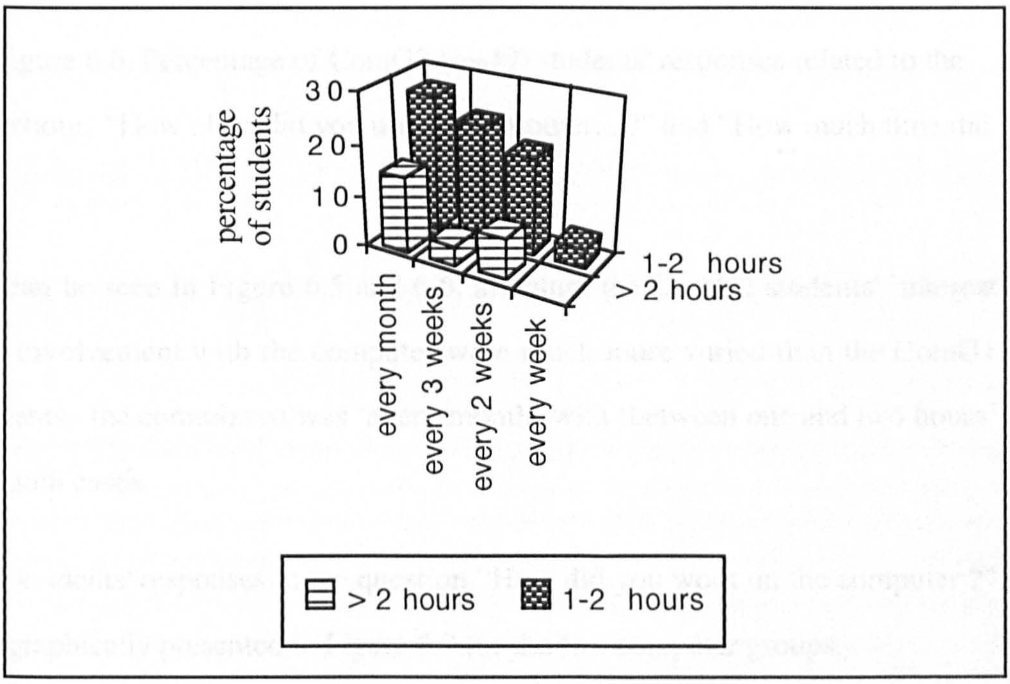


Figure 6.5. Percentage of ComG1(n=26) students' responses related to the questions: "How often did you use the computer....?" and "How much time did you spend...?"

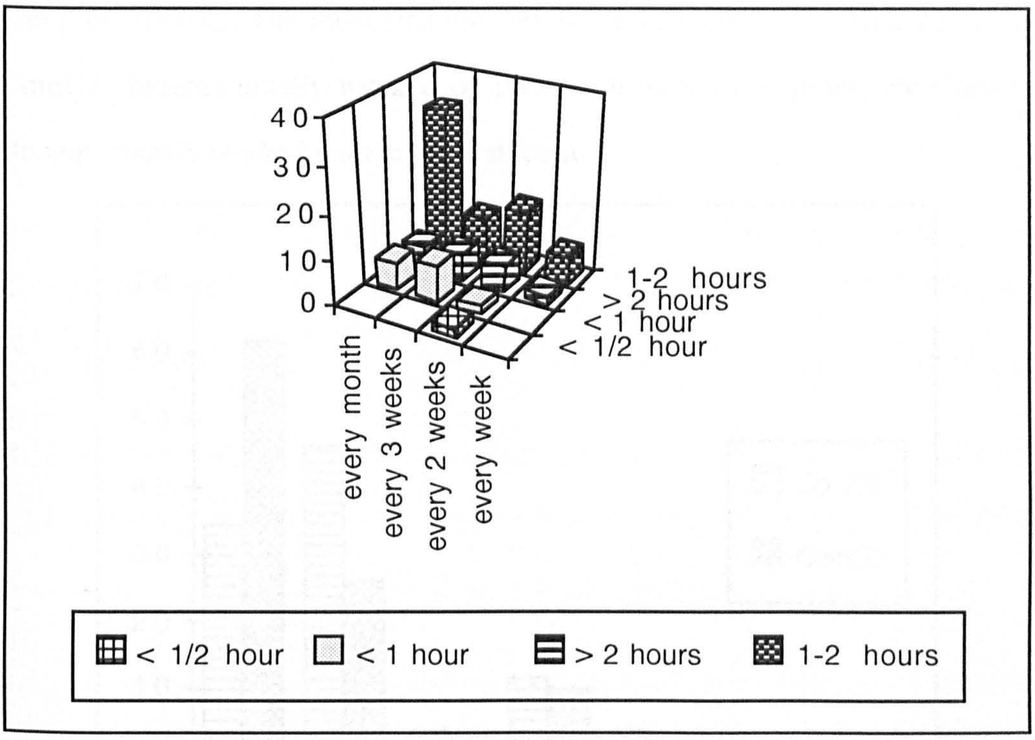


Figure 6.6. Percentage of ComG2 (n=49) students' responses related to the questions: "How often did you use the computer....?" and "How much time did you spend...?"

As can be seen in Figure 6.5 and 6.6, although the ComG2 students' interest and involvement with the computer were much more varied than the ComG1 students, the commonest was 'every month' with 'between one and two hours' for both cases.

The students' responses to the question "How did you work on the computer ?" are graphically presented in Figure 6.7 for the two computer groups.

Of these 50 students in ComG2, 31(62 %) said that they worked on the computer on their own. Of these, 13 (26 %) worked with a student, 1 (2%) worked with a group of students and 5 (10 %) worked both on their own and with a student. Of these 26 students in ComG1, 9 (35 %) responded that they worked on the computer on their own. Of these, 12 (46 %) worked with a student, 1(4 %) worked with a group of students, 3 (12 %) worked both on their own and with a student, and 1 (4 %) worked both on their own and with a group of students. The most striking difference in Figure 6.7 is that while the ComG2 students mostly worked on their own on the computer, the ComG1 students mostly worked with another student.

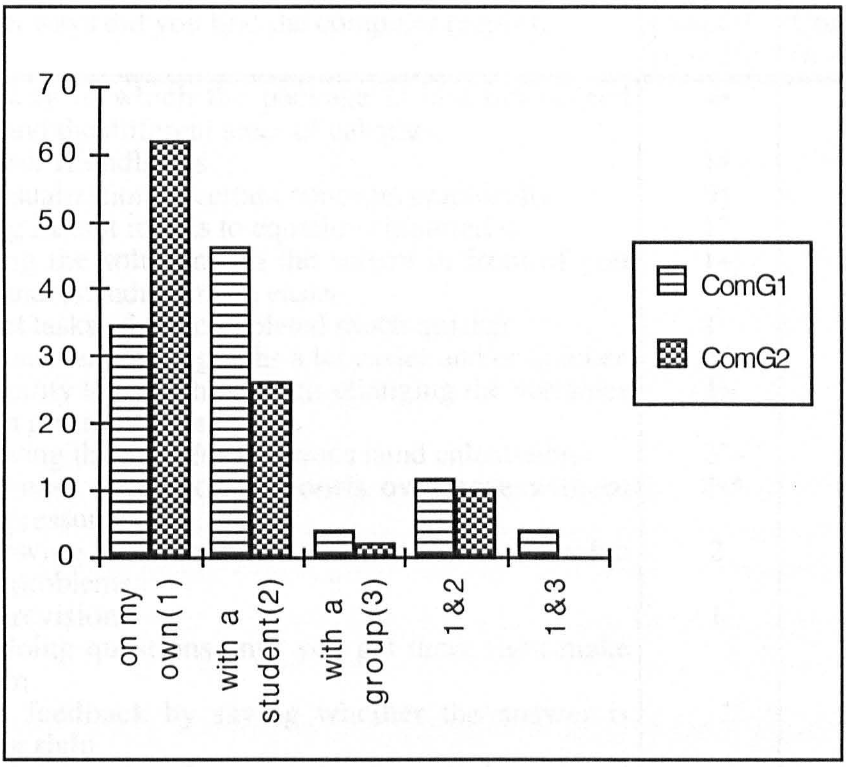


Figure 6.7. Percentage of ComG1(n=26) and ComG2 (n= 50) students responses related to the question: "How did you work on the computer....?"

Students also responded to three open-ended questions. Written comments to the first question, "In which ways did you find the computer was helpful?", are shown in Table 6.28.

In contrast to those who found the computer helpful, there were those who were of different opinion. Comments made on the "none" issue were:

> Personally, I found it difficult to use the computer as a learning/teaching aid, and thus didn't find it helpful (ComG2- S9).
>
> Takes longer to complete work and sometimes confuse me (ComG2- S22).
>
> CALM is annoying and frustrating, not helpful (ComG2-S23).
>
> I didn't not really, the question were repeated so I ended up learning those actual answers parrot fashion (ComG2-S24).
>
> none really, I already had a good understanding of calculus (ComG1-S25).

Table 6.28. Student responses to the question: " In which ways did you find the computer helpful?"

In which ways did you find the computer helpful?	ComG1 (n = 26)	ComG2 (n = 43)
1. the way in which the package is laid out helped understand the different areas of calculus	4•	
2. the user friendliness	1†	
3. the visualization of certain concepts graphically	9†	
4. giving instant results to equations inputted in	1°	
5. having the solutions on the screen in front of you makes understanding much easier	1~	
6. the set tasks where completed much quicker	1	
7. the visualization of graphs a lot easier and/or quicker	5*≈	
8. the ability to see what effects changing the variables has for a given function	1≈	
9. removing the need for laborious hand calculations	2°~	
10. the ease of working at one's own pace without feeling pressurised	2•*	8•
11. showing how calculus could be used to solve various problems	2	
12. for revision	1	3
13. re-doing questions until you get them right make you learn		2
14. the feedback by saying whether the answer is wrong or right		2
15. explaining theories very clearly and giving good examples		1
16. the possibility of repeating things as much as you want		1
17. carrying out equations		1
18. adding variety to teaching or learning methods		3
19. backing up theory taught in lectures		1
20. enjoyable		1
21. checking or testing your own knowledge		2
22. working examples in steps		2•
23. providing more or wider range of practice questions		3
24. giving theory before units		1
25. dealing with the topics in a systematic way		1
26. availability of a lot of information		1
27. giving you the way it wanted you to learn for its questions whereas text books wonder about		1
28. Derive was a useful package		2
29. none	2	8

Note. Each sign shows that one student appeared in two categories

A typical justification given by a student on the "variety to teaching" issue was:

> It was a different method of study and I could get more motivated to go to the computer to do maths than in the library with text books (ComG2-S26).

It is noted that, as one would have expected, the ComG1 students largely agreed that the visualization of graphs was a valuable part of the computer use. The ComG2 students, in contrast, largely agreed that working their own pace was a valuable part of the computer use. This may be due to the fact that the features of the software's used by students were different.

Comments made in response to the second question, " What changes might you suggest for the computer part of this calculus course when it is run again?", could be summarized under four broad areas, including none, computer program, course, and task (see Table 6.29).

The table shows that the ComG1 students suggested some changes for the task given. The ComG2 students, in contrast, did not suggest anything for the task as the two software packages used by the students were tutorial ones, but made more suggestions for the computer program.

A typical justification given by a student who wanted more computer sessions is given below:

> more scheduled time available to work on computers.... (ComG2-S27).

A comment made on the "variability in the marking" was:

> Has the ability to carry on errors so one small mistake does not mean a 0 answer (ComG2-S28).

Concerns have been expressed on the "input values and equations" issue:

> You might get the answer correct but the computer didn't always realise that it was correct (ComG2-S10).

> easier ways to input your data. Symbols i.e. [ ] { } not defined and log, ln.. (ComG2-St29).

Table 6.29. Student responses to the question: "What changes might you suggest for the computer part of this calculus course when it is run again?"

What changes might you suggest for the computer part of this calculus course when it is run again?	ComG1 (n = 23)	ComG2 (n = 33)
I. none	7	4
II. Computer Program		
a few sample calculations at the bottom of the screen while graph is being drawn	1	
an option for a student to start from a basic level	1	
wider range of examples		5*
more variability in the marking		3
making the questions more easy to understand		1
CALMAT better than CALM		1
reduce computer content		1
easier ways to input values and equations		6
not to give out answers		2
help from the computer if you get into trouble		1
III. Course		
more formal lectures	1	
more computer sessions	1	1*
incorporation of extra courses at the beginning of the course about the use of computer (in problem solving)	1	1
tuition in the computer rooms especially for students who did not have A-level	1	
more regular handing of the written work rather than at the end of each term	2	
a summary course before revision	2	
not to use the computers to replace exam		2
more graphical		1
traditional teaching with text book		1
supervised work on the computer		3
find a better program		1
IV. Task		
better layout for the worksheets, reducing confusion in some examples and some worked examples on the worksheet	1	
more understandable questions towards the end of the last booklet	3	
reduce the amount of simple tasks	2	
not drawing graphs on the worksheet	1	

Note. * sign shows that one student appeared in two categories

The third open-ended question was as follows: "What is the biggest worry affecting your work in this calculus?". Table 6.30 illustrates students' responses to this question.

Table 6.30. Student responses to the question: "What is the biggest worry affecting your work in calculus?"

What is the biggest worry affecting your work in calculus?	ComG1 (n = 23)	ComG2 (n = 32)
I. none	6	5
II. Mathematics Factor		
differential equation	2	
integration by parts and harder differentiation	1	
properties of trigonometric functions, oscillating	1	
system		2
integration		3
standard integrals and differentials		3
quantity of methods		1
graphical analysis of the integration of curves		1
III. Computer Factor		
not being computer phobic	1*	
the time spent on the computer	2	
the understanding what is going on on the computer	1	
not gaining an in-depth knowledge of calculus using computers		1
getting headaches using computers		1
variety of problems range from simple to impossible		1
IV. Ability Factor		
reading the questions properly	1	
not being competent in calculus	1*	
transformation of knowledge from theory to practice	3	1
some basic background knowledge		1
no previous experience of calculus	1	
not understanding the material		2
knowing less now than A-level		1
remembering how to do past work		3
poor understanding of fundamentals, although I can integrate and differentiate		1
I can learn more being taught in a traditional way from lectures, seminars and textbooks		1
not being able to understand and interpret the results correctly	3	
V. Affective Factor		
making a small mistake		1
study habits (leaving everything to the last minute)		1
not like calculus	1	
exam		4
global warming		1

In addition to the comments made on three open-ended questions, a few students also made some further comments. Additional statements made by the ComG2 students were on "not having enough computers" and "input answers":

> Not enough computers on campus so it is difficult to use one - waiting etc. (ComG2-S26).

> The computer would often disagree with your answer if you had it in an alternative form to theirs.... (ComG2-S30).

Further comments made by the ComG1 students, however, included one on "hand-holding" and one on "use of computers in applications":

> .... The nature of the course gave me a reason for learning calculus as it explained why and where calculus is needed/useful. The computing part would have been more appreciable if a similar approach had been given to the computers. Students who have used computers before are far more capable of getting on and completing the tasks than "phobic" like myself who would have benefited from a bit of "hand-holding" till more at ease with the equipment... (ComG1-S2).

> I think the computer could be used to show applications of calculus in situations so we students don't only see it in a relatively raw form, we can see how it can be used as a tool in engineering generally. I feel this would be easily done as a set of questions what to use and how to complete the question given, this would be best represented by computer anyway, due to diagrams and visual representation possible with a computer (ComG1-S31).

In summary, comments made on the open-ended questions showed a clear distinction between the students in the two different computer environments. Taken all together, the students' comments nicely support my contention that students' comments and suggestions concerning the use of computers in teaching are related to the type of software used. For example, ComG1 students found computers helpful in the visualization of concepts and graphs. ComG2 students found computers helpful because of the availability of various and the possibility of working on their own pace.

A measure of students perceptions of the use of computers in calculus was obtained from the Computer Attitude Scale on seven likert-type items, each measured on a 5-point rating scale. Figure 6.8 and Figure 6.9 summarize student responses to these statements or items.

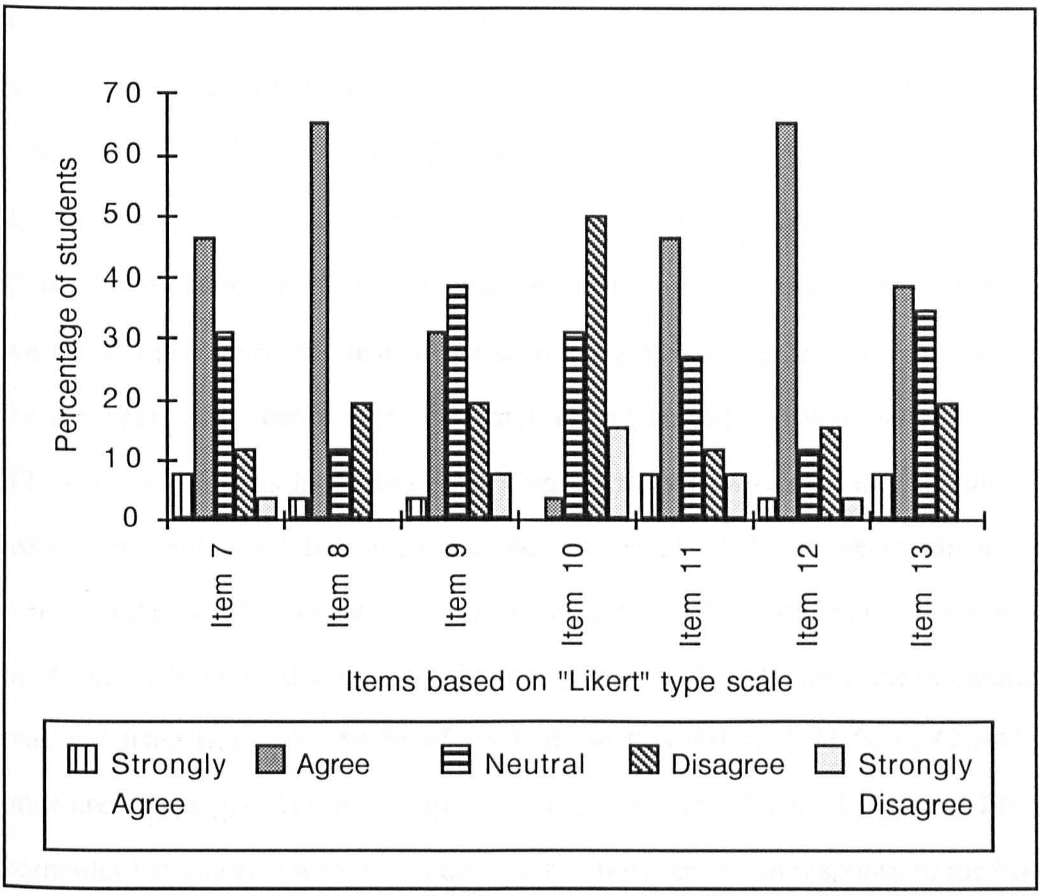


Figure 6.8. ComG1(n=26) students' responses to the items based on a "Likert" type scale

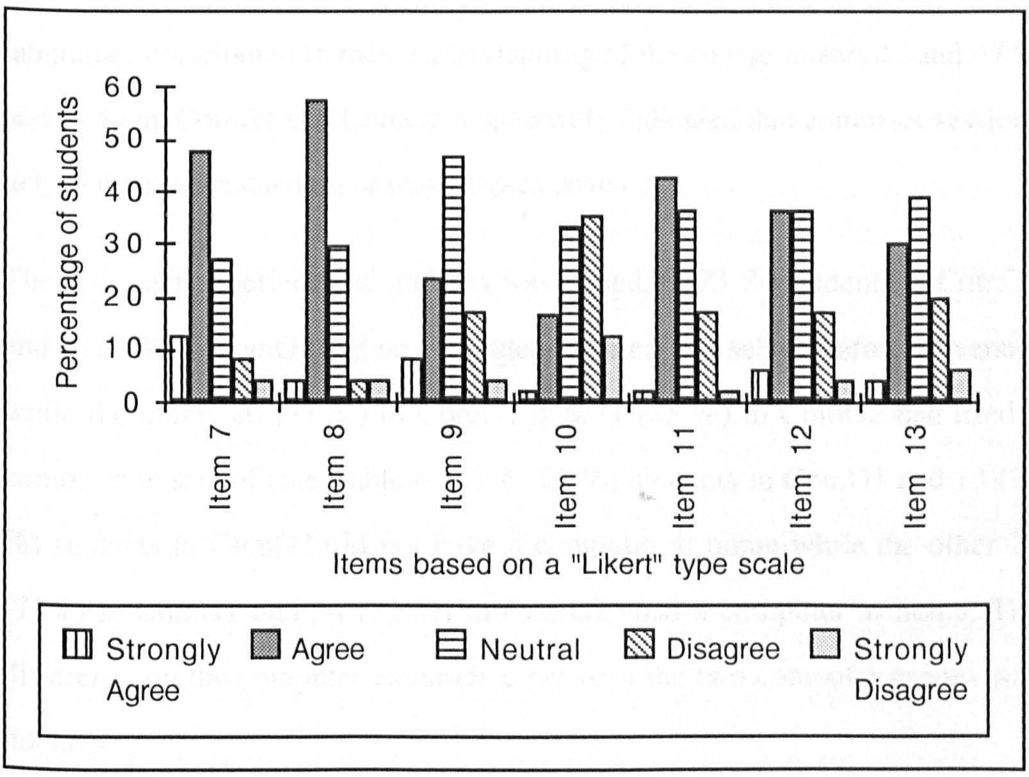


Figure 6.9. ComG2 (n=47) students' responses to the items based on a "Likert" type scale

The first statement (item 7) asked students to determine whether they like working on the computer as part of their calculus course. 54 % and 61 % of the students in ComG1 and ComG2 respectively answered *Strongly Agree* or *Agree*. For the second statement (item 8), 69 % and 61 % of the students in ComG1 and ComG2 respectively indicated that the computer assignments were well integrated with the rest of the course by answering *Strongly Agree* or *Agree*. Analysis of responses to the statement 3 (Item 9) revealed that 35 % and 32 % of the students in ComG1 and ComG2 respectively said that computer assignments enhanced their interest in the course material by answering *Strongly Agree* or *Agree*. 65 % of the students in ComG1 and 48 % in ComG2 *disagreed* or *strongly disagreed* that they would have preferred the calculus if the computer was not used(Item 10). 54 % of students in ComG1 and 45 % in ComG2 answered *Strongly Agree* or *Agree* to the statement (item 11): "overall the computer laboratories were a valuable part of the course". In response to the last two statements (item 12 and item 13) in this part, 69 % and 42 % of the students in ComG1 and ComG2 respectively indicated that using a computer in the laboratory contributed to their understanding of the course material , and 47 % and 34 % in ComG1 and ComG2 respectively indicated that computer sessions helped their understanding of many topics covered.

The computer experience of students was varied. 6 (23 %) students in ComG1 and 7 (15 %) in ComG2 had no computer experience at school before university while the other 20 (77 %) in ComG1 and 41 (85 %) in ComG2 had used a computer at school (see Table 6.31). 6 (23 %) students in ComG1 and 13 (28 %) students in ComG2 did not have a computer at home while the other 20 (77%) in ComG1 and 34 (72 %) in ComG2 had a computer at home. The difference on the computer experience between the two computer groups was not large

15 (58%) students in ComG1 and 31(66%) students in ComG2 had some experience in using computers both at home and at school. More than half of the students in each group had previous experience in using computers both at home and at school.

Table 6.31. Percentage of ComG1 and ComG2 students responses to the question: " Have you ever used a computer in the following situations?"

Computer Groups	At school %	At home %
ComG1(n=26)		
Yes	77	77
No	23	23
ComG2 (n=47)		
Yes	85	72
No	15	28

Table 6.32 summarizes student responses to the question "Have you ever used a computer for....".

Table 6.32. Percentage of ComG1 and ComG2 students responses to the question: " Have you ever used a computer ....?"

Choices	ComG1 (n = 26)		ComG2 (n = 48)	
	Yes %	No %	Yes %	No %
for games	88.5	11.5	93.8	6.3
for writing your own program	80.8	19.2	68.1	31.9
for using a word processor	84.6	15.4	93.8	6.3
in a lesson at school	69.2	30.8	82.6	17.4
in a work situation as part of the university course other than calculus	84.6	15.4	91.5	8.5
in a paid employment	23.1	76.9	23.1	76.9

6.3.1 Correlation between Attitude Scores

To describe the relationship between (1) the students' interest and involvement with computers, (2) the students' perceptions of the use of the computer, and (3) the students' background in computer use, Pearson product-moment correlation coefficients were employed. The correlations between these variables are given in Tables 6.33 and 6.34 for ComG1 and ComG2, respectively.

These tables show that, in the two computer groups, students' interest and involvement with the computer had no relation with their perceptions of the use of computers in calculus. The tables also show that students' perceptions of the use of computers in calculus were not related to their background in computer use.

Table 6.33. Correlations between the students' interest and involvement with the computer, the students' perception of the use of the computer, and the students' background in computer use for ComG1

	Interest and involvement[1]	Perceptions[2]
Interest and involvement		
Perception	-.10	
Background in computer use		
at school	.45**	-.05
at home	.07	.19
games	-.01	.30
program writing	.30	-.01
word processor	.18	-.08
in a lesson at school	.52**	.03
university course other than calculus	.31	.21
paid employment	-.17	.02
period of computer use	.36*	-.07

* $p < 0.05$, ** $p \leq 0.01$

Note. [1] An estimation of time spent using a computer [2] Likert scale

Table 6.34. Correlations between the students' interest and involvement with the computer, the students' perception of the use of the computer, and the students' background in computer use for ComG2

	Interest and involvement[1]	Perceptions[2]
Interest and involvement		
Perception	-.04	
Background in computer use		
at school	-.07	.23
at home	.23	.01
games	-.04	.08
program writing	-.13	-.19
word processor	.02	.14
in a lesson at school	.10	.24
in a university course other than calculus	.14	.03
paid employment	.09	.08
period of computer use	.22	.19

Note. [1] An estimation of time spent using a computer [2] Likert scale

Table 6.33 indicates that, ComG1 students' interest and involvement with a computer was moderately related to their background in computer use such as 'at school', 'in a lesson at school', and 'period of computer use'. In contrast, ComG2 students' interest and involvement with a computer (see Table 6.34) was not related to their background in computer use.

6.4 Attitudes and Performance

This section aimed at addressing the following research questions: (1) What are the effects of students' interest and involvement with a computer and perceptions of the use of computers in calculus on their performance? (2) Are these relationships similar for both computer groups?

To examine the association of the students' performance with students' interest and involvement with the computer, and the students' perceptions of the use of a computer, Pearson product-moment correlation coefficients were calculated in which students' performance was measured on residual score. The correlations are shown in Table 6.35.

Table 6.35. Correlations of residual scores with interest and involvement and perceptions

Items	ComG1		ComG2	
	Interest and involvement[1]	Perceptions[2]	Interest and involvement[1]	Perceptions[2]
1(a)	-.42*	.28*	-.10	.29*
1(b)	-.22	.19	.13	.19
2	-.29	.23	.24*	-.01
3	-.29	.32*	-.05	-.07
4(a)	.06	-.09	.10	.20
4(b)	-.22	.13	.10	.01
4(c)	-.17	-.19	-.06	.12
4(d)	.09	.18	-.01	.15
4(e)	-.15	.18	.00	.38*
4(f)	-.22	-.03	.17	.13
4(g)	-.18	.28*	-.07	.27*
5(a)	-.09	.30*	.20	-.11
5(b)	-.34*	.38*	-.14	-.02
6(a)	-.10	.33*	.17	.20
6(b)	-.15	.05	.08	.01
6(c)	-.05	.19	.01	.13
7(a)	-.11	.32*	-.10	.24*
7(b)	.10	-.03	.00	.12
8	-.43*	.16	-.14	.02
9(a)	.01	.05	.32*	.02
9(b)	.01	.09	.18	.28*
9(c)	-.39*	-.06	.04	.02
9(d)	-.02	-.05	.07	.24*
10(a)	.07	.34*	-.17	.38*
10(b)	.06	.26	-.12	.26*
10(c)	.25	.02	-.16	.22*

* $p \leq 0.05$
Note. [1] An estimation of time spent using a computer [2] Likert scale

The majority of the correlations of performance with interest and involvement, and perception tends to zero. It was worth noting that performance was neither related to the interest and involvement nor the perception although there might be some relative relations in particular cases. For example, the correlations between the performance in items 1(a), 4(g), 7(a), and 10(a), and perception were positive and significant at 0.05 level for both ComG1 and ComG2. Significant negative correlation at 0.05 level between the ComG1 students' performance and interest and involvement with the computer on items 1(a), 5(b), 8, and 9(c) may have been the result of more time spent on the computer by poor students. By contrast, there was a significant positive correlation at 0.05 level with question 2, and item 9(a) between the ComG2 students' performance and interest and involvement with the computer.

6.5 Summary

This chapter has described the statistical analysis results of the main study in which two computer and two non-computer groups were involved.

Computer group 1 students used the *Graphical Calculus* package in addition to the lectures. Computer group 2 students used two tutorial packages called CALMAT and CALM besides lectures. In addition to that they did a computer test at the outset of the course in order to measure their weak points and the *Graphical Calculus* package was also used by the teacher as a demonstration.

On sketching the graph of a derivative by looking at its function, and recognising the graph of a function by looking at its derivative, computer groups show an improvement from pre to post in performance over the non-computer groups. However, on other items or questions, the computer groups did not show a consistent improvement from pre-test to post-test in performance compared to the non-computer groups.

On explaining the symbols relating to ' δ ' and finding rate of change at a point for a linear function computer group 2 students did better than computer group 1 students. This could be attributed to the other packages used by computer group 2 students.

As the data does not satisfy the three conditions for the parametric statistical tests (see Brymann & Cramer, 1990, p.117) especially the fact that the distribution of the data is not normal, the data was also checked by non-parametric tests. The distribution for the total pre-test and post-test was roughly normal but for individual test items this was not the case. The scale used to measure the items was only from 0 to 5. As this was the case, they could not be close to a normal distribution. The Kruskal-Wallis one-way Anova was also carried out to compare pre-test and the difference scores between the pre-test and the post-test in four groups. The results found using the Kruskal-Wallis one-way Anova are consistent with the parametric anova test.

The perceptions of the use of computers in calculus obtained from the Computer Attitude Questionnaire showed similar results for both computer groups. Comments made on these open-ended questions: "In which ways did you find the computer helpful?"; What changes might you suggest for the computer part of this calculus course when it is run again?; What is the biggest worry affecting your work in calculus? showed a clear distinction between the students in two computer groups. For example, while computer group 1 students found computers helpful mainly because of the visualization of certain concepts, computer group 2 students found computers helpful mainly because of the ease of working at their own pace without pressure.

CHAPTER SEVEN

EXPLORATION OF AN UNDERSTANDING OF CALCULUS USING ERROR ANALYSIS

> The list of types of errors in the process of division is most illuminatory, but there is a need for further analysis to show the mental processes involved ... The discovery of types of errors is the necessary foundation for diagnostic analysis, but the real diagnosis consists not in the listing of errors but rather in the detailed analysis of the mental processes which cause the errors.
>
> Uhl (1917, p.120)

> Viewing an error as a "wrong answer" is not sufficient, one wants to know why, explain, account for.
>
> Bélanger (1988, p.34)

In this chapter errors in the students' responses given to the diagnostic test items are examined in detail. Thorough consideration of the errors provided information on some of the deeper common reasons underlying these errors. Errors and their underlying reasons can be used to identify pitfalls in the learning process of a topic and consequently to improve curricula in order to avoid them in the future (Davis &Vinner, 1986).

The interviews, which took place after post-testing, were conducted in order to clarify students' inner states of understanding and the ambiguities in the responses given to the diagnostic test items. Thus, they augmented the analysis. Extracts from these interviews are included in this chapter. The interviews were held with 21 randomly selected students: six from ComG1, five from ComG2, six from N-ComG1, and four from N-ComG2. Each interview lasted approximately 25 minutes.

The following sections in this chapter will summarise the errors encountered under two main headings: differentiation and integration. Each section under these two headings is devoted to a different topic:

- Derivative.
- Finding the graph of the derivative function from the graph of a function.
- Recognition of the graph of a function from its derivative graph.
- Symbols.
- Average rate of change.
- Rate of change.
- Point of tangency.
- Numerical calculation of gradient.
- Estimating the value of a function.
- Integral
- Proof of the integral of the sum of two functions is the sum of their integrals.
- The area under a curve.

The general tendency was (i) to construct detailed categories of errors in differentiation and integration, (ii) to determine the potential causes of errors, and (iii) to detect patterns in each student's pre-test and post-test responses. The classification system also included information on the frequency of errors by groups, in order to assess the similarities and the differences between groups.

The goal of the research reported here was (a) to identify students' conceptualizations of various mathematical concepts; (b) to investigate similarities and differences between students' conceptual structures associated with these concepts; (c) to identify factors that influence this development; and

(d) to identify factors that produce or facilitate transitions from one level of conceptualization to another for individual concepts.

Because of my interest in substantive mathematics content and educational implications, while categorising students responses, an attempt has been made to preserve the student's actual wording. Although several earlier studies were concerned with the strategies students used to solve or give answers to the items (e.g. Orton, 1980), this chapter attempts to give more detailed explanations of the underlying errors shown by the students. Second, students' strategies were described in such a way that a clear connection can be established between different items and how students solve or give answers to them.

In the analysis of errors in each group, the following questions have been taken into consideration:

1. What are the errors that occur with each item?
2. What kind of structure do errors form?
3. Are these patterns of errors associated with the four groups?

Answer sheets of students were examined and every error noticed has been assigned its own coding number. In each item, the errors were coded numbering from one. Although some categories were quite small, each one has been kept separate because it reflects a common tendency of thought. The lack of classification of some answers in each item was due to the fact that students' answers were inexplicit or problematic. Therefore, it was extremely difficult to interpret what the student meant.

7.1 Differentiation

7.1.1 Question 1: Derivative

This question includes the two items, which dealt with the student's ability to define the 'derivative' and to explain the meaning of an example of it.

Item 1(a)

The students' errors to item 1(a) is shown in Table 7.1.

Errors I-II seemed to stem from incomplete understanding of the function concept. Students often appear to confuse 'function' with the word 'equation', 'formula', 'identity', 'value' , 'number', and 'term' (Breidenbach, Dubinsky, Hawks, & Nichols, 1992; Dreyfus & Vinner, 1982; Vinner, 1983; Vinner & Dreyfus, 1989) . These words are used together with the word 'function' in calculus, as well as in algebra, for example, 'value' and 'function' in: "when x=3, the value of the function is 8". It is necessary to present words in pairs like this so that specific contexts of calculus can be given a focus. Students, however, cannot always identify each pair as belonging to a particular context.

Errors III-IV were rooted in a lack of discrimination between concepts like differentiation, differential and derivative. Such a problem occurs because there is a superficial similarity of words and the use of the words in the same context causes a failure to discriminate between them. For example, the concept 'differential coefficient' (denoting the derivative) is likely to be confused with the concept 'differential' (for example, used to denote the differential of y: $dy = y'(x)\, dx$). The concepts of differentiation and differential are in a sense fundamental schemes on which the concept of derivative can be built. The discrimination of these words is important as derivative is a product, whereas differentiation and differential, are a process and an object respectively.

Table 7.1. Student errors on item 1(a): What is the meaning of a 'derivative'?

LIST OF ERRORS AND THEIR EXAMPLES FOR ITEM 1(A)		
CODING	DESCRIPTIVE LABEL	EXAMPLE
I	Derivative defined without reference to function	•A derivative is something which can be derived from something else • The result of differentiating something
II	Derivative as an equation /formula/ identity/ value/ number/ term	• It is the equation that is obtained when differentiation is obtained • Derivative is a derived value which has come from a function.... • Derivative is a value obtained after differentiating •A number found from reducing power
III	Differentiation as Differential	A derivative is the differential of the function
IV	Derivative as process	It is a method used to calculate the gradient on a graph at any point
V	Derivative as tangent	....Geometrically the derivative of a function $y = x^2$ it's the tangent in that point...
VI	Derivative as slope of a curve at a particular point	A point on a curve of graph for a given function.. the derivative representing the slope of the graph at that point.
VII	Derivative as the value of the function	the differentiation of the equation of a curve to find a point on the curve
VIII	Derivative as small change	It is a small change
IX	Derivative as product of integration	• The derivative is a derived answer of a differential function • a derivative is the way to go back to the precedent form of a function
X	Unclassified	

Errors V-VII seemed to stem from replacing concepts with pictures which is a well known tendency in cognition that pictures replace the concepts. Since in the graphical picture only one tangent is drawn at a point, the equation of this tangent line or the slope of this tangent line might incorrectly replace the derivative (Amit & Vinner, 1990). So the derivative is considered as the equation of the tangent line or the slope of the tangent line at a particular point. Error VI also seemed to occur because the slope of the tangent at a particular point, describing a special case, is inappropriately extended to a more general case. Such a confusion is likely when the special case has been encountered first in one's learning experience.

Error VIII was rooted in a lack of intuitive understanding of the process that might be expressed in a calculus textbook as follows: "as a point moves closer and closer to another fixed point, the slope of the secant line between these two points varies by a smaller and smaller amount and, in fact, approaches a constant limiting value". This process was misinterpreted as "the slope of the secant line approaches a smaller value". This showed that students do not conceive of a derivative as an approximation (Dreyfus, 1990). Conflicts experienced by in limiting process were reported by several authors (Orton, 1977; Schewarzenberger & Tall, 1978; Tall, 1977).

Error IX seemed to stem from a confusion of the derivative concept with the integral concept. Such a confusion is caused by a failure to discriminate between two related concepts which occur frequently in the same context.

The distribution of the errors to item 1(a) in four groups is presented in Table 7.2. This table contains a large amount of information which lets you relate the pre-test and the post-test performance of particular students. The correct answer by a student does not always mean that the error made was eradicated or introduced, due to the fact that a single concept may be understood and presented from several points of view. In order to make this aspect clear, the errors which were eradicated or introduced were shown by "*" sign on the "c" (correct) sign.

Table 7.2. Number of students who made each type of error on item 1 (a): What is the meaning of a 'derivative'?

FREQUENCY OF ERRORS FOR ITEM 1(A)										
	Computer Groups				Non-Computer Groups					
	ComG1 ee=8 ec=4 ce=1 em=0 me=3 (n=35)		ComG2 ee=5 ec=10 ce=3 em=0 me=1 (n=56)		N-ComG1 ee=8 ec=3 ce=0 em=0 me=2 (n=20)		N-ComG2 ee=5 ec=7 ce=2 em=0 me=2 (n=36)		Total ee=26 ec= 24 ce=6 em=0 me=8 (n=147)	
Coding	Pre n=12	Post n=12	Pre n=15	Post n=9	Pre n=11	Post n=10	Pre n=12	Post n=9	Pre n=50	Post n=40
I	4 1→I 1→IV 1→II 1→X	3 I→1 X→1	4 3↑c 1→III	1	2 1↑c 1→III	1 II→1	1 1↑c*		11	5
II	2 1↑c 1→X	2 I→1	3 3↑c		2 1→I 1→X		3 1↑c* 2→II	2 II→2	10	4
III	2 2↑c*	1 X→1	4 2↑c 2→III	4 c*↓1 I→1 III→2		4 X→1 IX→1 I→1		1 X→1	6	10
IV		1 I→1		1 c↓1	1 1↑c*				1	2
V		1 IX→1								1
VI			1 1→VI	1 VI→1			1 1→VII		2	1
VII								1 VI→1		1
VIII		1 c↓1			1 1↑c*				1	1
IX	1 1→V				1 1→III		1 1↑c*	1 c*↓1	3	1
X	3 1↑c 1→III 1→I	3 I→1 II→1	3 2↑c 1→X	2 c↓1 X→1	4 1→III 3→X	5 II→1 X→3	6 4↑c 1→III 1→X	4 c↓1 X→1	16	14

Note. **ee** shows the number of students who made errors both in the pre-test and post-test.
ec shows the number students who made errors in the pre-test but gave a correct response in the post-test
ce shows the number of students who gave the correct response in the pre-test but made errors in the post-test
em shows the number students who made errors in the pre-test but missed out the question in the post-test
me shows the number of students who missed the question in the pre-test but made errors in the post-test
→ indicates a similar performance on the pre-test and post-test (e.g. 1→X in the first pre-test column indicates that one student who made error II on the pre-test made error X on the post-test).
↑ indicates an improvement from pre-test to post-test (e.g. 1↑c on pre-test indicates that one student made an error on the pre-test but gave a correct answer on the post-test)
↓ indicates a decrement from pre-test to post-test (e.g. c↓1 on post-test indicates that one student made an error on the post-test but had given a correct answer on the pre-test)
* indicates that the error is either eradicated or introduced

As can be seen from Appendix F, a small number of students missed item 1(a) on the pre-test and the post-test. The above table shows that the majority of the students who made errors on the pre-test did not repeat the same type of errors on the post-test. This might be due to the fact that students either incurred the errors through the teaching or they already had these errors but they did not come out in the pre-test situation. It is then the form of the words that the student uses for his/her own explanation of his/her (evoked) concept image. The *evoked concept image* was defined by Tall and Vinner (1981) as "the portion of the concept image which is activated at a particular time" (p.152). The students' actual correct responses can be summarised as follows:

> Rate of change of a function.
>
> The derivative of a function gives the gradient of the curve of that function at any particular point.
>
> If a function is differentiated, the result can be called the derivative.

Some of the errors reported above may be considered as a failure to use language appropriately. As stated by Fujii (1993),

> Mathematics can be characterised as some sort of language. To understand mathematics as a language, it is crucial for students to grasp the grammar, vocabulary and rhetoric method besides understanding the contents.(p.173)

The correct usage of mathematical terms is clearly one important aspect of mathematical understanding.

Item 1(B)

The students' errors to item 1(b) are shown in Table 7.3.

As can be seen from Table 7.3, the same type of errors as in item 1(a) were observed when students were required to explain the meaning of "the derivative of $f(x)=x^3$ is $3x^2$". Therefore, the results of item 1(a) and 1(b) are in close agreement.

Table 7.3. Student errors on item 1(b): What does it mean that the derivative of

$f(x) = x^3$ is $3x^2$?

LIST OF ERRORS AND THEIR EXAMPLES FOR ITEM 1(B)		
CODING	DESCRIPTIVE LABEL	EXAMPLE
I	Differentiation as Differential[1(A).III]	The differential of x^3 is $3x^2$
II	Derivative as tangent equation[1(A).V]	$3x^2$ is the equation of tangent at any point on $f(x)=x^3$
III	Derivative as slope of a curve at a particular point[1(A).VI]	when a graph of $f(x)=x^3$ is plotted, the slope at a particular point is $3x^2$.
IV	Derivative as product of integration[1(A).IX]	..the integral of x^3 is $3x^2$ The area under x^3 graph is $3x^2$
V	Wrong notation	The derivative of $x^3 = \frac{dx}{dy} = 3x^2$
VI	Unclassified	

Note. [1(A).III] compare Error I in this table with Error III in question 1(A)

The distribution of the errors to item 1(b) in four groups is given in Table 7.4.

In Table 7.4, the important thing to note is that of the five students who made Error I on the post-test, two had given a correct answer on the pre-test either using 'differentiation' instead of 'differential' or using 'differential' in the right context such as 'differential equation'. This is probably because they were treating 'differentiation' and 'differential' as synonyms.

As in item 1(a), not many students missed item 1(b) on the pre-test and the post-test (see Appendix F).

Table 7.4. Number of students who made each type of error on item 1(b): What does it mean that the derivative of $f(x) = x^3$ is $3x^2$?

FREQUENCY OF ERRORS FOR ITEM 1(B)										
	Computer Groups				Non-Computer Groups					
	ComG1 ee=0 ec=4 ce=3 em=0 me=2 (n=35)		ComG2 ee=2 ec=0 ce=2 em=2 me=0 (n=56)		N-ComG1 ee=2 ec=4 ce=1 em=0 me=0 (n=20)		N-ComG2 ee=2 ec=5 ce=2 em=1 me=1 (n=36)		Total ee=6 ec=13 ce=8 em=3 me=3 (n=147)	
Coding	Pre n=4	Post n=5	Pre n=4	Post n=4	Pre n=6	Post n=3	Pre n=8	Post n=5	Pre n=22	Post n=17
I	1 $1\uparrow c$	1 $c^*\downarrow 1$	2 $2\rightarrow I$	2 $I\rightarrow 2$	1 $1\uparrow c^*$		2 $1\uparrow c$ $1\rightarrow I$	2 $c^*\downarrow 1$ $I\rightarrow 1$	6	5
II					1 $1\rightarrow II$	2 $c\downarrow 1$ $II\rightarrow 1$			1	2
III				1 $c\downarrow 1$			1 $1\uparrow c$		1	1
IV	2 $2\uparrow c$				1 $1\uparrow c$		1 $1\rightarrow IV$	1 $IV\rightarrow 1$	4	1
V	1 $1\uparrow c$	1 $c\downarrow 1$							1	1
VI		3 $c\downarrow 1$	2	1 $c\downarrow 1$	3 $2\uparrow c$ $1\rightarrow VI$	1 $VI\rightarrow 1$	4 $3\uparrow c$	2 $c\downarrow 1$	9	7

Note. **ee** shows the number of students who made errors both in the pre-test and post-test.
ec shows the number students who made errors in the pre-test but gave a correct response in the post-test
ce shows the number of students who gave the correct response in the pre-test but made errors in the post-test
em shows the number students who made errors in the pre-test but missed out the question in the post-test
me shows the number of students who missed the question in the pre-test but made errors in the post-test
$\rightarrow$ indicates a similar performance on the pre-test and post-test (e.g. $I\rightarrow 2$ in the second post-test column indicates that two students who made error I on the pre-test also made error I on the post-test).
$\uparrow$ indicates an improvement from pre-test to post-test (e.g. $1\uparrow c$ on pre-test indicates one student made an error on the pre-test but gave a correct answer on the post-test)
$\downarrow$ indicates a decrement from pre-test to post-test (e.g. $c\downarrow 1$ on post-test indicates one student made an error on the post-test but had given a correct answer on the pre-test)
* indicates either the error is eradicated or introduced

Interviews

Students S7 from ComG2, and S2 and S4 from ComG1 had answered item 1(a) as follows: "a derivative is the answer of the differentiation process" or "it is the equation that is obtained when differentiation is performed". They were then, asked why we differentiate a function. S2 and S7 answers were: "Well, for example, if you have got velocity if you differentiate it - you can end up with the acceleration and you can use it for things like that" and "we differentiate a function to find gradients of a line or curve" respectively. S4, however, could not answer the question.

Student S5 from ComG1 whose answer to item 1(a) was "A derivative is the answer you derive at from differentiating a function" was asked again what is a 'derivative'. He replied as:

1st Episode (ComG1):

S5:	I don't know.
Inter:	You said "A derivative is the answer you derive at from differentiating a function" Why do we differentiate?
S5:	To find the answer.
inter:	For what?
S5:	For finding the points on the curve.

The above episode shows that an acceptable answer given by a student cannot be taken as evidence that the student knows the correct idea.

Students S9 from ComG2, and S12 and S13 from N-ComG1 had answered item 1(a) respectively, in the following ways: "The rate of change of a function. i.e. if $y = x^2$ the derivative $\frac{dy}{dx} = 2x$", "A derivative is the reduction in the power of number" and "If you differentiate say $3x^2$ the derivative of that is 6x". Subsequently, their answers were repeated to them and they were asked if they could say a bit more on what a 'derivative' is. Student S9 answer was: "I don't know what it means. You work out the derivative - but I don't know what it means". The following two episodes from the interviews indicate students S12 and S13 answers.

2nd Episode (N-ComG1):

S12:	Not really. I find that in maths the definitions and the theory are a lot more difficult than actually doing the questions. If they ask me to do this question, for example, I find that a lot easier than theory questions. That's why I can't answer that. I can say what you can find out about differentiation and that sort of thing, such as integrations, the area under a curve.

3rd Episode (N-ComG1):

S13: It comes from $6x^2$, it's just ...... I don't know the meaning. It's something that is derived from something; it's origin is from this.

Inter: On 1(b), for example, when I differentiate the function y equals to x^3, I get $3x^2$. What is the meaning of $3x^2$ then?

S13: It's x^2 by 3. So if x is 2, 2^2 by 3.

Inter: Why do we differentiate?

S13: I'm not too sure. I know it's something to do with graphs. I'm not 100% sure.

Student S12 was also asked item 1(b) but he said " I don't know". The results above show that students possess some idea on the techniques of the differentiation but they do not understand what they are attempting to find.

Student S1 from ComG1 who had answered item 1(a) and 1(b) as 'gradient of a curve' was asked again what a 'derivative' is. He replied as follows:

4rd Episode (ComG1):

S1: To find the gradient of a curve, so if you've got velocity, it will be finding acceleration.

Inter: How do we find the gradient?

S1: y over x if it's a straight line.

Inter: If you have a curve, how could you find the gradient?

S1: I don't know.

In the case of a straight line the student had some idea how to find the gradient. By saying the ratio y/x gives the gradient for a linear function he seemed to think that a straight line always passes through the origin. As might be expected, there is a great problem in dealing with gradient or rate of change for a curve. A possible explanation for this phenomena may lie in the role played by the limiting process to find the rate of change at any point for a curve.

Students S14 and S17 from N-ComG1 whose answer to item 1(a) was: "This is the differential of some function of x" were asked again what a 'derivative' is. Their answers were:

5th Episode (N-ComG1):

S14:	A derivative is dy by dx. Let's say you have an equation f(x). The derivative of the equation would be the differential of that equation. dy by dx is a rate kind of thing. It's like a ratio or something else.

6th Episode (N-ComG1):

S17:	The differential of something; if x^2 is given, the derivative would be 2x.
nter:	Can you tell me in more detail?
S17:	It's the slope of the graph - the gradient.
Inter:	On 1(b), I said if I differentiate a function, for example, $f(x) = x^3$ and I get $3x^2$ what is the meaning of this result?
S17:	That would be the gradient of $y = x^3$.

As can be seen in the above episodes, students used 'differential' in the context of 'differentiation'. Student S14 also used 'equation' as a replacement for 'function'. This student did not seem to appreciate the major point in a graphical study of rate of change concerns the difference between straight lines and curves.

Student S15 from N-ComG1 whose answer to item 1(a) was: "The result of differentiating something. The rate of change of y from x" was asked again what a 'derivative' is. The following episode from the interview indicates his rational:

7th Episode (N-ComG1):

Inter:	Can you tell me what is a'derivative'?
S15:	No.
Inter:	But you have written that a derivative is "The result of differentiating something. The rate of change of y from x".

S15: Oh yes. The rate of change. So, it is a gradient. Yeah! rate of change basically.

Inter: When we differentiate a function like $f(x) = x^3$ we get $3x^2$, what is the meaning of this function?

S15: For any value of x, the gradient of the curve at that point is the $3x^2$.

Student S18 from N-ComG2 who had missed item 1(a) was asked again what a 'derivative' is. He gave the following answers:

8th Episode (N-ComG2):

S18: No, What is it?........ It's a function of something.

Inter: Is it function? and How do we get it?

S18: Using differentiation.

Inter: But what is this showing us?

S18: Nothing. You don't need to know that, you just need to know how to do it.

This student did not seem to have a conceptual understanding.

Student S20 from N-ComG2 whose answer to item 1(a) was "a derivative is variable with respect to time. i.e. dy/dx" was asked what a 'derivative' is. His answer was:

9th Episode (N-ComG2):

S20: I thought it was a variable that changes under the influence of another variable, so if you've got dy/dx; that's the change in y with respect to x or if you have ds/dt, it's the change in s with respect to t. I find difficult expressing your questions in words. I don't know precisely what you are looking for?

Also here the student did not seem to grasp the difference between average rate change and rate of change. Furthermore, the student thought a function as a variable. Probably a function must be considered as a process (Dreyfus, 1990).

7.1.2 Question 2: Finding the graph of the derivative function from the graph of a function

This question dealt with the student's ability to draw the derivative graph by looking at the original graph.

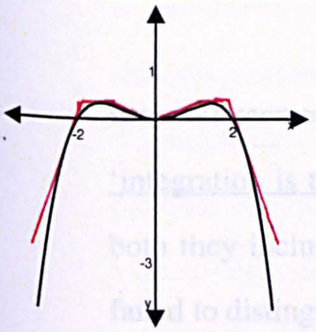
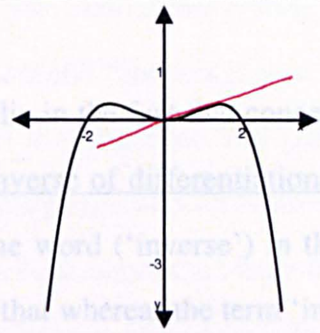
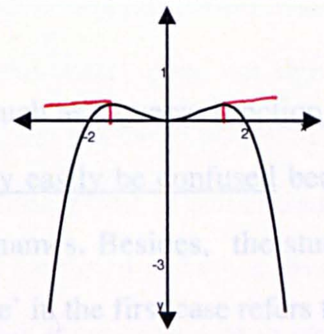
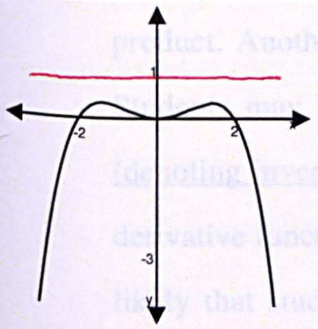
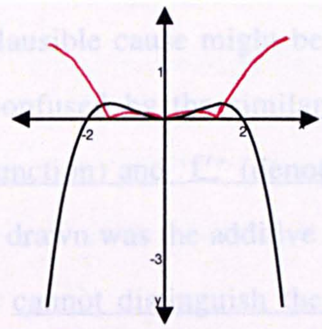
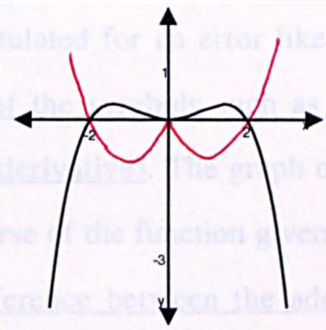
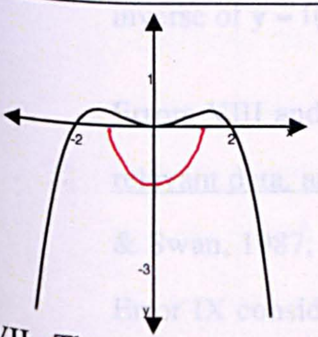
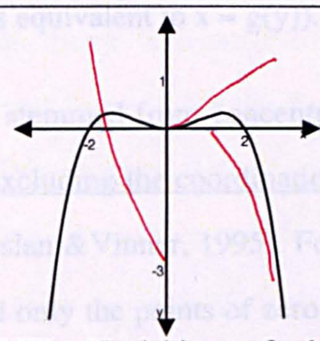
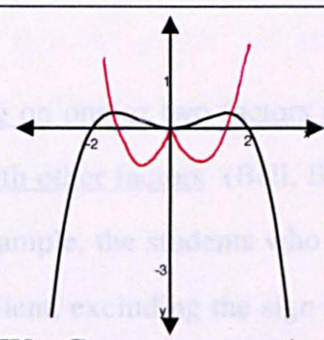
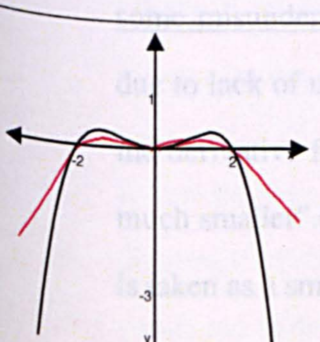
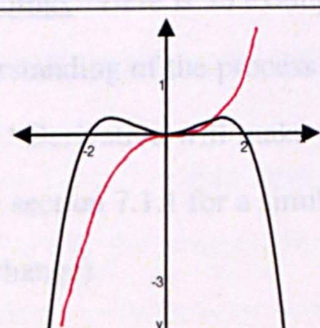
The students' errors to this question are shown below.

The unclassifiable answers in question 2 were due to the fact that students did not show any structural drawing. The analysis of students' errors to question 2 are presented in Table 7.5, together with the students drawings of the derivative function superimposed on the original function. The original function shows the graph of $y = \left(\frac{-x^4}{12}\right) + \left(\frac{x^2}{3}\right)$.

Errors I-III, and may be IV, seemed to stem from replacing pictures with concepts. Students have drawn one or more straight lines which appear to be tangents to the curve. So these tangent lines seemed to replace the derivative function (See previous section 7.1.1 for details of the similar misconception, resulting from the replacement of concepts by pictures).

Students who made Error V seemed to think that if the graph was 'upside down', the power of x (independent variable) should be an odd number and therefore the function with an odd power should be an odd function. The important misconception emerged here was that the power of a function tell us in which direction the curve is bending. The book *Teaching Calculus* defines 'bending to the left' and 'bending to the right' as "If $f''(x) > 0$, then the curve is bending left, and if $f''(x) < 0$, then it is bending right"(p.53). Thus, the sign of $f''(x)$ at a point tells us in which direction the curve is bending at that point.

Table 7.5. Student errors on question 2: Sketch what the derivative looks like.

 I. Differentiation gives the nearest straight lines to the curve	 II. Differentiation gives a tangent	 III. Differentiation gives the tangents of the turning points
 IV. Differentiation will give a single number	 V. Derivative function must be even	 VI. Differentiation gives the inverse function
 VII. The graph given is cubic therefore the gradient is quadrant/ the derivative drops a number	 VIII. Vanishing of the rule that a point at which a curve changes from concavity to convexity	 IX. Correct conceptions of the points of zero gradient but no conception related to the sign of the gradient function
 X. Differentiating gives a shallower slope	 XI. The graph given is $y=x^4$ therefore the derivative function is $y=4x^3$	XII. Unclassified

Error VI seemed to lie in the fact that concepts such as 'inverse function' and 'integration is the inverse of differentiation' may easily be confused because both they include the word ('inverse') in their names. Besides, the students failed to distinguish that whereas the term 'inverse' in the first case refers to the product, in the second case it refers to the process. These failure points at how important it is for students to master the distinction between process and product. Another plausible cause might be postulated for an error like this: Students may be confused by the similarity of the symbols such as 'f^{-1}' (denoting inverse function) and 'f'' (denoting derivative). The graph of the derivative function drawn was the additive inverse of the function given. It is likely that students cannot distinguish the difference between the additive inverse of a function (f(x) +(-f(x))) and the inverse of a function (e.g. the inverse of y = f(x) is equivalent to x = g(y)).

Errors VIII and IX stemmed from concentrating on one or two factors of the relevant data, and excluding the coordination with other factors (Bell, Brekke & Swan, 1987; Rasslan &Vinner, 1995). For example, the students who made Error IX considered only the points of zero gradient, excluding the sign of the derivative function around the point of zero gradient and the point of inflexion.

As is often the case, approaching differentiation from first principles induces some misunderstandings. Here is an example of a student who made Error X due to lack of understanding of the process in calculating an approximation to the derivative f'(a): "Derivative will make y' rate of change with respect to x much smaller".(See section 7.1.1 for a similar misconception where derivative is taken as a small change)

Errors VII and XI were rooted in the students' tendency to know the unknown, which is the equation of the graph. Students' discomfort with the presence of an 'unknown' prevents them from getting into the question. For example, students suggested that the function is $y=x^4$ but the sketch does not show all the major features of that function. The graph given has three points of zero gradient whilst the polynomial $y=x^4$ has only one point of zero gradient. Students also did not think about the behaviour of the function for negative and positive x-values. In the case of a cubic polynomial, there cannot be more than two points at which $f'(x) = 0$, and so there cannot be more than two points of zero gradient. Thus, rather than use the information provided by the graph, students approached the task inventing a likely formula for the function.

Errors V and VII-XI have also shown that students had difficulties in using graphical information to give meaning to symbolic representations or vice versa.

The distribution of the errors to question 2 in four groups is presented in Table 7.6. From this table, there is some evidence that the errors VII and XI (inventing an incorrect formula for the function), and IX (concentrating on only one factor of the relevant data) could be eradicated through teaching especially with the use of computers. These errors also appeared in the post-test but not committed by the same students. Computers seem to help students become more aware of the sign of the gradient around the points of zero gradient as well as these points of zero gradient.

Of the 6 students in ComG2, only 2 students continued to use the same misconception derivative is the tangent lines drawn.

It can be seen from Table 7.6 that more students in the computer groups displayed some errors in the pre-test but corrected them in the post-test, compared to the students in the non-computer groups.

Table 7.6. Number of students who made each type of error on question 2: Sketch what the derivative looks like.

FREQUENCY OF ERRORS FOR QUESTION 2										
	Computer Groups				Non-Computer Groups					
	ComG1 ee=5 ec=6 ce=0 em=2 me=8 (n=35)		ComG2 ee=6 ec=18 ce=3 em=2 me=2 (n=56)		N-ComG1 ee=5 ec=0 ce=0 em=4 me=1 (n=20)		N-ComG2 ee=11 ec=2ce=0 em=6 me=6 (n=36)		Total ee=27 ec=26 ce=3 em=14 me=17 (n=147)	
coding	Pre n=13	Post n=13	Pre n=26	Post n=11	Pre n=9	Post n=6	Pre n=19	Post n=17	Pre n=67	Post n=47
I			4 2↑c 2→I	2 I→2			3 1→I 1→XII	1 I→1	7	3
II			2 2↑c				1		3	
III	1 1→XII							1 XII→1	1	1
IV							1 1→XII		1	
V	2 1↑c 1→IX	1 VI→1							2	1
VI	2 1→V 1→IX	2		2	1				3	4
VII	3 1↑c	2	4 4↑c		2 1→VII	1 VII→1	4 1↑c 1→XII 1→VII	3 VII→1	13	6
VIII	1 1→IX								1	
IX	3 3↑c	5 V→1 VI→1 VIII→1	3 3↑c	2 c↓2			1 1↑c		7	7
X		1			1 1→X	1 X→1			1	2
XI	1 1↑c		3 3↑c	1 c↓1		1 XII→1		1 XII→1	4	3
XII		2 III→1	10 4↑c 4→XII	4 XII→4	5 1→XI 2→XII	3 XII→2	9 1→III 1→XI 4→XII	11 I→1 IV→1 VII→1 XII→4	24	20

In ComG1 13 (37%) students drew the correct graph in the post-test. Six of these also drew the correct graph in the pre-test. These six were the only students who drew the correct graph in the pre-test. No students got the pre-test correct and subsequently got the post-test wrong. Apparently, there is an improvement in the students answers. In ComG2 36 (64%) students drew the correct graph in the post-test. Of these 36 students, 16 (29%) also drew the correct graph in the pre-test. Apart from these 16 students, four more students were able to draw the correct graph in the pre-test but not in the post-test. Apparently, there is more improvement than deterioration in the students answers. In N-ComG1 three (15%) students drew the correct graph in the post-test and two of those also drew the correct graph in the pre-test. No students got the pre-test correct and subsequently got the post-test wrong or vice versa. There is more or less no improvement in the students answers. In N-ComG2 four (11%) students drew the correct graph in the post-test and only one of those also drew the correct graph in the pre-test. Apart from this one student, another student drew the correct graph in the pre-test but missed the question in the post-test. To summarise, it seems reasonable to attribute this improvement to the computer sessions.

Interviews

By considering students' correct answers in question 2 and examining their understanding of the relation of an original graph and its derivative graph, it was possible to clarify the states of their understanding related to the factors considered while drawing the derivative graph. The interview transcripts of one student in ComG2, one student in N-ComG1 and two students in N-ComG2 were analysed. The justification given by a student in N-ComG2 who drew the right graph without an explanation is presented below:

1st Episode (N-ComG2):

Inter: Could you tell me what you thought while you were drawing this graph?

S18: Maximum and minimum.

Inter: How did you decide it should go this way down and up? (pointing out the graph drawn)

S18: Is it right?

Inter: Yes. You said maximum and minimum. Where is the maximum and minimum?

S18: I couldn't tell you.

As can be seen from this episode, the student cannot explain why the graph was drawn like this. He has a hazy idea that maximum and minimum are relevant to the question but cannot explain it.

Students S15 from N-ComG1 and S21 from N-ComG2 drew the correct derivative graph and explained where the gradient is zero, positive and negative. S11 from ComG2 also drew the correct derivative graph and gave the explanation "this is because the sketch shows how the gradient of the graph varies". The students seemed to have no idea about the 'point of inflexion' of a graph at which the graph changes the direction in which it is bending. The following extract illustrates this point.

2nd Episode (N-ComG2):

Inter:	Could you explain to me how you drew this graph on question 2?
S21:	The first thing I did was to find out where the turning points were, where dy by dx equals to zero. If you are drawing a graph x against the derivative, then that will be where the graph cuts the x axis. Also, I see that the gradient here (indicating until the first turning point around x = -1) is positive. This point (x = 0) is where it changes between negative and positive.
Inter:	But how did it become curved like that? (pointing out the turning points on the derivative graph)
S21:	I don't know. I also knew that this is a quartic and when you differentiate a quartic, you get a cubic. That's the way I did it.

Moving on to incorrect or missing answers in question 2, the transcripts of four students in ComG1, two students in ComG2, five students in N-ComG1 and two students in N-ComG2 were analysed. The examples (below) demonstrate the effort of the interviewer to make sure that the incorrect or missing answer did not come about through carelessness or omission due to lack of interest. Students were asked what they had thought or what they thought since, while drawing the derivative graph by looking at the original graph. Most of them replied, "I have no idea what was going on" or "I do not know".

A justification given by a student in N-ComG1 who missed the question is presented below:

3rd Episode (N-ComG1):

Inter:	What is the relation between the original and the derivative graph?
S12:	The second graph will be the value of the gradient of the first graph at different points. The turning points of the second graph would be 0.
Inter:	Where should it be on the graph then if the turning point is 0?
S12:	At this point on the second graph. (pointing out one of the turning points)
Inter:	Is there anything else we should consider?
S12:	With the turning points?
Inter:	Or any other things.
S12:	Yes, the turning points. I don't know whether it's a maximum or a minimum or just a turning point.
Inter:	And how many turning points are there on this graph?
S12:	3.
Inter:	Can you show me?
S12	There and there. (pointing out the turning points on the graph)
Inter:	Okay. Is there anything else we can think about on this graph to help us to draw the derivative graph?
S12:	The two points that are 0. If it's y =, you could put y as 0 and then the new equation and differentiate it. You can decide what the derivative of the new graph is going to cross the x -axis in the second graph.
Inter:	But, do you know the connection between, for example, if the original graph is increasing, how does it affect the derivative graph?
S12	The derivative graph decreasing as that is increasing. It's a mirror image.
Inter:	If the original graph increases, the derivative graph is....?
S12:	I'd guess that it was a mirror image similar to that, but I couldn't say.

In this example, the student explained his answer by pointing out turning points only, ignoring the other points about the shape of the original graph. He seemed to think that finding the turning points of a function involves the process of differentiation of the equation of the function after it has been equated to zero. He, however, explained correctly the meaning of a 'derivative'.

Student S10 from ComG2 who missed the question and S4 from ComG1 who made Error IX explained correctly where the gradients are zero. The student from ComG2 also clarified where the original function is increasing and decreasing. They, however, were unable to express their ideas graphically because as student S10 said "when it comes to actually plotting it again, I cannot quite visualise it". Visualisation of the graph of the derivative function includes two directions: the interpretation and understanding of an original graph and the ability to translate it into another visual information form (the graph of the derivative function). It appears that students have more difficulties in the second part of the visualisation.

The following extract is a typical example of someone who tends to replace concepts with pictures. Asking him, what he thought while drawing the derivative graph, he replied as follows:

4rd Episode (ComG2):

S7:	I don't know. I've never had to do anything like that before. Draw a derivative over a curve. But, a way of finding the derivative may be to draw an approximate straight line through the curve. I don't know.
Inter:	Can you tell me how the function is behaving in different areas? And then by looking at this, can you draw the derivative graph?
S7:	What do you mean?
Inter:	For example, in here, the function is increasing. How does the derivative function behave if it's increasing? And in here, for example, how does the derivative function behave at a turning point?
S7:	I'm not sure. I just drew straight lines.

7.1.3 Question 3: Recognising the graph of a function from its derivative graph

This question dealt with the student's ability to choose an original graph by looking at the derivative graph. It was asked to examine the criteria students use for the choice of the original graph by looking at the derivative graph. It was possible to categorise the students' wrong choices with their explanations into three main categories: (i) graph 3, (ii) graph 4, and (iii) graph 3 or graph 4. The reasons and their examples in the three main categories provide us with the students' views of problems associated with derivative or curve sketching.

Table 7.7 gives the analysis of the student errors to question 3.

As can be seen in Table 7.7, Error I.1 and Error II.1 exhibited two misconceptions contrary to one another. For example, students' knowledge of curve sketching, who made Error I.1, seemed to be governed by their experience in this area of polynomial functions. This faulty generalisation may lead to problems when conceptualizing the effects of differentiation on other functions, particularly exponential functions. For example, the differentiation of $y = e^x$ with respect to x gives the derivative function, $y' = e^x$, which equals to the original function.

The reason for Error I.3 may be that the students' concept image of a turning point did not include a point on a curve other than the point which intersects the x-axis. In other words, the student's point of view was too narrow and exclusive, and thus insufficient for dealing with a given situation or for solving a given problem.

Table 7.7. Student errors on question 3: Which of the graphs 2, 3, 4 could be the original graph y =f(x)?

LIST OF ERRORS AND THEIR EXAMPLES FOR QUESTION 3		
CODING	DESCRIPTIVE LABEL	EXAMPLE
I	**Graph 3**	**Graph 3 selected**
I.1	Original graph cannot be similar to derivative graph	Graph (4) is too similar to f' (x)
I.2	Derivative is the inverse function of an original function 2.VI	Derivative is the inverse function of the original function
I.3	The original graph and the derivative graph should cross the x-axis at the same point	Graphs (3) and (1) cross x-axis at x=6
I.4	Turning points are the points at which the graph changes direction (compare II.6 below)	Graph (3) changes direction at the same points
I.5	Negative gradient means negative function.	Graphs (1) and (3) both go beneath the x-axis after 8
I.6	Zero gradient means positive constant derivative function	Gradient of Graph(3) between 1 and 3 is positive but getting less. Between 3 and 5 gradient is 0.
II	**Graph 4**	**Graph 4 selected**
II.1	The original graph is similar to the derivative graph (compare II.3 below)	Similar shape
II.2	The integrated function (original function) is steeper than the function (derivative function) itself	After integrating the derivative the graph would become much steeper
II.3	Gradients are tangents to the original graph 1(a).V & 2.I-IV	Straight lines of the gradient function are tangents to the original graph.
II.4	The original graph and the derivative graph must have the same values (compare II.3 above)	The values of y for graphs (1) and (4) are the same
II.5	The areas under both graphs are the same	The areas under the graphs (1) and (4) are similar
II.6	Turning points are the points at which the graph changes direction (compare I.4 above)	•f' (x) has two turning points therefore f(x) has 3 turning points •Turning points remain the same

Table 7.7 continued

LIST OF ERRORS AND THEIR EXAMPLES FOR QUESTION 3		
CODING	DESCRIPTIVE LABEL	EXAMPLE
II.7	If the gradient is constant, the derivativative function must be a straight line	The gradient changes are constant
II.8	Incorrect graphical visualization of x^2	The first part of the graph (1) is straight therefore it would have to be the derivative of x^2
II.9	Decreasing derivative function means negative gradient	Graph(1)shows a negative gradient between 0 and 3, and between 5 and 8
II.10	When the constant function is differentiated, the result must be constant	Straight line region would not have changed when the function was differentiated
IIII	Zero gradient means straight derivative graph	Graph 3 or Graph 4

Note. 2.VI compare Error I.2 in this table with Error VI in question 2

It has been noticed that Error I.5 occurred because students believed that the sign of a function and its derivative function must be the same at all points along the curves. It appears that students misunderstood the idea: "when a function has decreasing values as the independent variable increases, the sign of derivative function is negative; when a function has increasing values as the independent variable increases, the sign of derivative function is positive". This phenomenon may be explained as a misconception of sign: 'decreasing function' is taken to mean 'negative function'. Error II.9 also seemed to occur because of the reason given above.

Errors I.6, II.5 and III showed that students cannot be successful in curve sketching unless they have good understanding of the concept of function, linked to visual ideas. In particular, the visualisation of 'constant', 'linear' and 'zero' function. As the graph of a function consists of straight lines, students fail to distinguish the difference between them.

From the table above, it is easy to see that both Error I.4 and II.6 were due to a misconception about turning points: "turning point is a point on a curve at which the ordinates of the curve cease to increase and begin to decrease, or vice versa" was taken to mean "a point on a curve at which the curve moves from upside-down to left-right' or vice versa".

Error II.2 may be explained as a misconception of steepness: "the larger (numerically) the value of the derivative function at a point, the steeper (numerically) the tangent to the graph at that point" was taken to mean "the greater the power of a function is, the steeper is the function".

Table 7.8 shows the number of students' errors falling within each category while choosing an original graph by looking at the derivative graph.

The results are given in Table 7.8, from which we can see, for example, that approximately 70% of the students who made errors chose graph 4 as the original graph. It was also noted that not all students gave some reasons for choosing a particular graph. Graph 4 was chosen by the majority of students for two main reasons: "gradients are tangents to the original graph" and "decreasing derivative function means negative gradient".

Although Error I.6 was the most frequent category for the choice of graph 3 on the pre-test and the post-test, this error was more frequent on the post-test than on the pre-test.

Table 7.8. Number of students who made each type of error on question 3:

Which of the graphs 2, 3, 4 could be the original graph y =f(x)?

FREQUENCY OF ERRORS FOR QUESTION 3										
	Computer Groups				Non-Computer Groups					
	ComG1 ee=5 ec=5 ce=2 em=0 me=4 (n=35)		ComG2 ee=5 ec=11 ce=4 em=1 me=4 (n=56)		N-ComG1 ee=2 ec=2 ce=3 em=1 me=0 (n=20)		N-ComG2 ee=13 ec=6ce=4 em=3 me=4 (n=36)		Total ee=25 ec=24 ce=13 em=5 me=12 (n=147)	
coding	Pre n=10	Post n=11	Pre n=17	Post n=13	Pre n=5	Post n=5	Pre n=22	Post n=21	Pre n=54	Post n=50
I	2 1→II.1 1→II	2	2 1↑c 1→II	3 c↓1	3 1↑c 1→II.2		6 1↑c 1→II.10 2→II	2 II.3→1 II→1	13	7
I.1	1 1→I.3								1	
I.2					1 1↑c				1	
I.3		1 I.1→1								1
I.4								1 II→1		1
I.5			1 1↑c	2 c↓1					1	2
I.6		1 c↓1	1 1↑c	4 c↓1 II.9→1 II→1 II.6→1			1 1→II.1	3 c↓2 II→1	2	8
II	3 2↑c 1→II	5 c↓1 I→1 II.5→1 II→1	2 1↑c 1→I.6	2 I→1	1 1→II	3 c↓2 II→1	8 2↑c 1→I.6 1→I.4 1→II.3 1→I 1→II.4	6 c↓1 I→2 II.9→1	14	16
II.1	1 1↑c	1 I→1	1 1↑c			1*	1 1→II.6	2 I.6→1	3	4
II.2						1* I→1				1
II.3	1 1↑c		3 2↑c 1→II.3	1 II.3→1			1 1→I	2 II→1	5	3
II.4							1 1↑c	1 II→1	1	1
II.5	2 1↑c 1→II								2	
II.6			1 1→I.6					1 II.1→1	1	1

Table 7.8 continued

FREQUENCY OF ERRORS FOR QUESTION 3										
	Computer Groups				Non-Computer Groups					
	ComG1 ee=5 ec=5 ce=2 em=0 me=4 (n=35)		ComG2 ee=5 ec=11 ce=4 em=1 me=4 (n=56)		N-ComG1 ee=2 ec=2 ce=3 em=1 me=0 (n=20)		N-ComG2 ee=13 ec=6ce=4 em=3 me=4 (n=36)		Total ee=25 ec=24 ce=13 em=5 me=12 (n=147)	
coding	Pre n=10	Post n=11	Pre n=17	Post n=13	Pre n=5	Post n=5	Pre n=22	Post n=21	Pre n=54	Post n=50
II.7							1 1↑c		1	
II.8			1 1↑c						1	
II.9		1	4 2↑c 1→I.6	1 c↓1			3 1↑c 1→II.9 1→II	2 c↓1 II.9→1	7	4
II.10								1 1→1		1
IIII			1 1↑c			1 c↓1			1	1

Note. **ee** shows the number of students who made errors both in the pre-test and post-test.
ec shows the number students who made errors in the pre-test but gave a correct response in the post-test
ce shows the number of students who gave the correct response in the pre-test but made errors in the post-test
em shows the number students who made errors in the pre-test but missed the question in the post-test
me shows the number of students who missed the question in the pre-test but made errors in the post-test
→ indicates a similar performance on the pre-test and post-test (e.g. I→II on pre-test indicates that one student who made an error on the pre-test made error II on the post-test).
↑ indicates an improvement from pre-test to post-test (e.g. 1↑c on pre-test indicates that one student made an error on the pre-test but gave a correct answer on the post-test)
↓ indicates a decrement from pre-test to post-test (e.g. c↓1 on post-test indicates that one student made an error on the post-test but had given a correct answer on the pre-test)
* the student appeared in two categories

In ComG1 35% (11 students) of the students improved their performances from the pre-test to the post-test, compared to 6% (2 students) of the students whose performances deteriorated. By improvement it is meant that students who displayed an error in the question or missed the question in the pre-test gave a correct answer in the post-test. By contrast, deterioration indicates that students who gave a correct answer in the pre-test made an error or missed the question in the post-test. In ComG2 21% (12 students) of the students improved their performances from the pre-test to the post-test, compared to 9%(5 students) of the students whose performances got worse. In N-ComG1 15%(3 students) of the students improved their performances from the pre-test to the post-test, compared to another 15% of the students whose performances got worse. In N-ComG2 22% (8 students) of the students performances got better from the pre-test to the post-test while 14% (5 students) of the students performances got worse. Apparently, there is much more improvement than deterioration in the students responses in the computer groups, compared to the students' responses in the non-computer groups.

Interviews

An analysis of the kinds of factors a student considers when choosing the correct original graph by looking at the derivative graph shows clearly her/his understanding of curve sketching. The transcripts of two students from ComG2, four students in N-ComG1 and one student from N-ComG2 were analysed. Justification given by a student from N-ComG2 who chose the right graph with the explanation "Graph (2) has the same physical factor" is presented below:

1st Episode (N-ComG2):

Inter:	What is the meaning of physical factor?
S18:	It has the same gradient position as between 3 and 5?
Inter:	What do you mean by gradient position?
S18::	I don't know how to explain that, but it has ....
Inter:	You looked between 3 and 5. Have you thought of any other things when you were choosing graph (2)?
S18:	No, I didn't think of anything else.

In the above situation the student had some idea related to the gradient of a straight line but was unable to discuss it.

The following episode illustrates that a student from N-ComG1 was in conflict. He did not realise that no acceleration means zero gradient rather than positive or negative gradient. As in the above situation, the attempt to choose graph (2) was based on the straight line between 3 and 5.

2nd Episode (N-ComG1):

Inter:	On question 3, I've given you a derivative graph, and then I wanted you choose the original graph from here. You chose graph (2). You gave an explanation " because the section between 3 and 5 is straight line. Therefore there is no change in the derivative". Could you tell me anything else that made you choose graph (2)? And why did you not choose graph (3) and (4)?
S12:	I considered them as I consider a velocity graph turning into an acceleration graph. That's how I thought of them. Or just the distance - velocity - that sort of thing. I would have said that if this was the velocity (indicating graph (2)), the velocity stays constant at this point between 3 and 5. So there is no acceleration. I thought that no acceleration meant that the gradient was constant, not that there was no change in the derivative.
Inter:	What is the derivative of graph (3) and (4) between 3 and 5?
S12:	I don't know.
Inter:	Are there any other characteristics that made you choose graph (2)?
S12:	I didn't choose (4) because it had a sudden drop there which didn't really correspond to anything in this graph at all.

Another student, S10, from ComG2 chose graph (2) and gave the explanation "at $3 \leq x \leq 5$ gradient is 1. The gradient at $3 \leq x \leq 5$ for graph (3) and (4) are zero". During the interview he showed a very good understanding of positive and zero gradient. However, he was confused about the meaning of negative gradient and said: "I'm a little confused about the last point after 6 (pointing graph (2)). Looking at it now, maybe it is that one (pointing graph (3)) because the gradient goes negative after 6 which is what it does on graph (3) there. So I think if that middle point hadn't been chosen I would have gone for graph (3)". Upon further questioning he replied

3rd Episode (ComG2):

Inter:	If the derivative is negative, what does this mean to you?
S10:	That means that the gradient is negative. It means that the slope must be in the negative region.

This illustrates that students, having some misconceptions, can give a correct answer to a question.

Student S15 from N-ComG1 and S11 from ComG2 chose the correct graph (2) by pointing out the turning point at 6 and the gradient between 3 and 5. In the interviews they also mentioned correctly that: (i) as the derivative function is positive between 0 and 6, the original function must be increasing at this domain; (ii) for graph (3) and (4) the gradient between 3 and 5 is zero.

Two students, S16 and S17, from N-ComG1 chose the correct graph (2) without an explanation. In the interview they were unable to explain their choice. They said "that was just a guess" and "I cannot remember".

Moving on to an incorrect graph choice, the transcripts of two students from ComG1, one student from ComG2, two students in N-ComG1 and two students in N-ComG2 were analysed.

A student from N-ComG2 chose the wrong graph but gave a correct explanation " $f'(x) = 0$ for $3 \leq x \leq 5$ & $f'(x) \approx 2$ when $x = 0$" related to graph (3). The following episode shows that the wrong graph choice seemed to come through carelessness.

4rd Episode (N-ComG2):

Inter: On question 3, you chose graph (3). What made you choose graph (3)?

S21: It was a choice between this one or this one (between graph (3) and (4), because ...... the gradient there (between 3 and 5) is zero. Why did I do that? I don't know. That should have been that one (graph (2))?

Inter: Why?

S21: I'm not sure now. This is a derivative graph. That's the derivative graph so it should have a turning point there which means it's probably this one. I would say that should be graph (2).

Inter: You said there is a turning point at 6 and it should be graph (2), but can you see anything else that made you choose graph (2)?

S21: The gradient should be 2 where $x = 0$. That could be 2. The gradient is constant between 3 and 5. So definitely I would go for (2) now.

A justification given by S20 who chose the incorrect graph (4) with the explanation "turning points remain the same" is given below. Thinking of tangent lines as a derivative function is exemplified in the following episode:

5th Episode (N-ComG2):

Inter:	On question 3, you've chosen graph (4). What is the reason for choosing graph (4)?
S20:	I thought the points of change remained the same. If you differentiate a curve, you get a straight line where you are looking for a gradient. But I thought the points at which it could have changed, remain the same.

A similar tendency was revealed in the interviews with another two students who chose graph (4) as an original graph. S5 from ComG1 and S7 from ComG2 gave the statement "it is basically the same type of shape" during the interviews.

S14 from N-ComG1 and S2 from ComG1 who chose graph (4) could not provide an explanation in the interviews. S13 from N-ComG1 who missed the question said "I do not know".

7.1.4 Question 4: Symbols

This question includes 7 items, each requesting an interpretation of symbols either related to 'δ' or 'd'. The specification of a concept implies a corresponding specification of its symbols. The use of an explicit symbolic expression is an important aid to ensure correct usage of a concept.

<u>Item 4(a)</u>

This item required a student to explain the meaning of 'δx' in two expressions '$\frac{\delta y}{\delta x}$' and '$\sum f(x)\delta x$'.

Considering all the explanations given on the pre-test and the post-test, none of the students explained correctly the meaning of 'δx' separately in the two expressions given above. Instead most of them gave a general answer such a, "(small) change in x". A full answer about 'δx' in context would probably contain a geometrical explanation. For example, 'δx' in '$\sum f(x)\delta x$' denotes the width of a typical strip, which is taken by dividing the region, between any two points, into equal intervals.

Table 7.9 provides the errors made by students explaining 'δx'.

The students who defined '$\frac{\delta y}{\delta x}$' as "differentiation with respect to x" (see Error I) confirmed that they do not distinguish '$\frac{\delta y}{\delta x}$', by which we mean the change in y in comparison to change in x or the slope of a secant line, from '$\frac{dy}{dx}$' which refers to the limiting value of '$\frac{\delta y}{\delta x}$' as δx tends to zero or the slope of the curve at any point on it. Also, the students who defined '$\sum f(x)\delta x$' as "integral" (see Error II) showed that they do not distinguish between '$\sum f(x)\delta x$', which refers to the approximating to the area under the graph of a function or the sum of the areas of the inscribed rectangles, and '$\lim \sum f(x)\delta x$' which refers to the area under the graph of a function. These errors might be due to the fact that both '$\frac{\delta y}{\delta x}$'and '$\frac{dy}{dx}$', and both '$\sum f(x)\delta x$' and '$\lim \sum f(x)\delta x$' refer to the same thing, in the special case of a linear function. However, the distinction between those symbols should be established even when considering a linear function. Moreover, some students who made Error II seemed not to realise that '$\sum f(x)\delta x$' gives the sum of a number of thin strips whose height is f(x) and whose width is δx.

Error III and V seemed to occur because of the consideration of 'δ' as the symbol of or place holder for 'rate of change' and 'derivative'. These results also show that the students have probably not acquired the meaning of 'derivative' and 'rate of change'.

Error VI seemed to occur because of the similarity between symbols '∂' and delta 'δ'. Whereas 'δx' denotes a number (positive or negative) to be added to the number x, '$\frac{\partial f(x,y)}{\partial x}$' shows the ordinary derivative of a function of two variables with respect to x variable, considering the other variable as constant. This similar appearance is causing confusion between them. Some students also considered '∂' as the symbol of or place holder for the concept 'partial derivative'.

Errors IV and VIII show a lack of clarity in understanding dependent or independent variables. This is because when they said "with respect to x" or "with respect to a small change in x", they did not mention exactly what is changing in each case. These mistakes (the misuse of 'with respect to') illustrate that students do not realise the necessity of mentioning the dependent variable in connection with the independent one.

Error VII shows that the geometrical images such as 'point' seem to interfere with the conception of delta x that should be free of this constraint. A variable could represent a point on coordinate axes as well as a length or change between any two points. Students, however, often regard a variable as representing a point on coordinate axes.

Students who made Error IX appeared not to perceive 'δx' as an expression but instead as an equation related to two expressions (delta) and x. This error is also related to the lack of understanding of the concept of variable which is a prerequisite to understanding the systematic structure of algebraic expressions and equations (Kieran, 1988).

Table 7.9. Student errors on item 4(a): Explain the δx in $\frac{\delta y}{\delta x}$ and in $\sum f(x)\delta x$

LIST OF ERRORS AND THEIR EXAMPLES FOR ITEM 4(A)		
CODING	DESCRIPTIVE LABEL	EXAMPLE
I	Inaccurate conception of $\frac{\delta y}{\delta x}$	• $\frac{\delta y}{\delta x}$ - second derivative with respect to x • $\frac{\delta y}{\delta x}$ is saying that you are differentiating with respect to x
II	Inaccurate conception of $\sum f(x)\delta x$	• $\sum f(x)\delta x$ - integral of x with respect to x • $\sum f(x)\delta x$ -The sum of the function f(x) with respect to x • $\sum f(x)\delta x$ is the sum of all these small changes in x
III	δx as derivative of x	$\sum f(x)\delta x$ -the sum of f(x) with respect to the derivative of x
IV	δx as with respect to x	• with respect to x • In $\sum f(x)\delta x$ - δx: with respect to x
V	δx as rate of change of x	
VI	δ as ∂	• the partial derivative of x • partial derivative with respect to x
VII	δx as a point	δx means a small value of x
VIII	δx as with respect to a small change in x	δx stands for "with respect to a small change in x"
IX	δ as variable	change in δ with respect to x
X	Unclassified	

Most of the errors here show that students have difficulties in recognising the surface structure of an expression or equation. The surface structure refers to the given form or disposition of the terms and operations (Kieran, 1988). Results also confirm that students do not understand the nature of mathematical explanations. Probably one must provide students with the appropriate mathematics vocabulary and the appropriate stimulus for the use of language correctly.

Table 7.10 shows the number of students' errors falling within each category while explaining the 'δx' symbol.

Table 7.10. Number of students who made each type of error on item 4(a): Explain the δx in $\frac{\delta y}{\delta x}$ and in $\sum f(x)\delta x$

FREQUENCY OF ERRORS FOR ITEM 4(A)										
	Computer Groups				Non-Computer Groups					
	ComG1 ee=7 ec=3 ce=5 em=2 me=7 (n=35)		ComG2 ee=3 ec=5 ce=6 em=1 me=1 (n=56)		N-ComG1 ee=0 ec=2 ce=1 em=1 me=3 (n=20)		N-ComG2 ee=8 ec=8 ce=4 em=3 me=4 (n=36)		Total ee=18 ec=18 ce=16 em=7 me=15 (n=147)	
coding	Pre n=12	Post n=19	Pre n=9	Post n=10	Pre n=3	Post n=4	Pre n=19	Post n=16	Pre n=43	Post n=49
I	2 1↑c 1→X	3 c↓1	2 1↑c 1→II				1	1 X→1	5	4
II	1 1↑c	1	1 1↑c	3 c↓2 I→1		1	4 2↑c 1→IV	2 V→1	6	7
III		2	1 1→IV		1 1↑c	1			2	3
IV	1			1 III→1			1 1↑c	4 II→1 VIII→1	2	5
V	2 2→X		1 1↑c	1 c↓1			3 2↑c 1→II		6	1
VI		6 c↓3 X→1								6
VII	2 1→X			1 c↓1			1 1→VII	2 c↓1 VII→1	3	3
VIII		1 c↓1	2 1↑c 1→VIII	2 VIII→1			3 1↑c 1→IV 1→VIII	1 VIII→1	5	4
IX								1 c↓1		1
X	4 1↑c 1→VI 2→X	6 VII→1 V→2 X→2 I→1	2 1↑c	2 c↓2	2 1↑c	2 c↓1	6 2↑c 1→I 2→X	5 c↓2 X→2	14	15

Table 7.10 shows that Error VI (confusion of '∂' with delta 'δ') made only by ComG1 student on the post-test. It should be acknowledged that, in this case, students had already started to learn the topic on 'partial differentiation'. A number of students who gave the correct answer such as "small change in x" or "change in x" both on the pre-test and the post-test was 11% (4 students), 70%(39 students), 55% (11 students), and 8% (3 students) for ComG1, ComG2, N-ComG1, and N-ComG2, respectively.

Item 4(B)

This item required a student to explain the meaning of 'δy'. Table 7.11 provides the errors made by students while attempting to do so.

As the two items 4(b) and 4(a) are apparently similar, the results obtained are also similar: The same type of errors have been obtained with item 4(b) as compared with item 4(a), with the exception of the first two classifications given in both Tables 7.11 and 7.9.

Table 7.11. Student errors on item 4(b): Explain 'δy'

LIST OF ERRORS AND THEIR EXAMPLES FOR ITEM 4(B)		
CODING	DESCRIPTIVE LABEL	EXAMPLE
I	δy as infinitesimally small change	
II	δy as differential of y 1(a).III	•Differential of y with respect to x • Differential of y
III	δy as derivative of y	• dependent variable, i.e could be velocity
IV	δy means with respect to y	differentiating with respect to y
V	δy as rate of change of y	
VI	δ as ∂	the partial derivative of y
VII	δy as a point	δy means a small value of y
VIII	δy as with respect to small change in y	• with respect to small changes of y
IX	δ as variable	change in δ with respect to y
X	Unclassified	

Table 7.12 shows the number of students' errors falling within each category while explaining 'δy'.

Table 7.12. Number of students who made each type of error on item 4(b): Explain δy

FREQUENCY OF ERRORS FOR ITEM 4(B)										
	Computer Groups				Non-Computer Groups					
	ComG1 ee=6 ec=5 ce=2 em=0 me=5 (n=35)		ComG2 ee=2 ec=6 ce=3 em=0 me=1 (n=56)		N-ComG1 ee=0 ec=3 ce=0 em=0 me=1 (n=20)		N-ComG2 ee=3 ec=3 ce=6 em=1 me=2 (n=36)		Total ee=11 ec=17 ce=11 em=1 me=9 (n=147)	
coding	Pre n=11	Post n=13	Pre n=8	Post n=6	Pre n=3	Post n=1	Pre n=7	Post n=11	Pre n=29	Post n=31
I	1 1↑c		2 2↑c						3	
II	2 1↑c 1→VIII		1 1↑c		1 1↑c				4	
III	2 2↑c	2 II→1	1 1↑c				1 1↑c	1	4	3
IV		2 X→1		1						3
V	1 1→X	2	2 1↑c 1→X	1 c↓1			2 1↑c 1→X		5	3
VI		2 c↓1 X→1								2
VII	2 2→X		1 1→VIII	1 c↓1			3 1↑c 1→VII	2 c↓1 VII→1	6	3
VIII				1 VII→1				1 c↓1		2
IX								1 c↓1		1
X	3 1↑c 1→VI 1→IV	5 c↓1 VII→2 V→1	1 1↑c	2 c↓1 V→1	2 2↑c	1	1 1→X	6 c↓3 X→1 V→1	7	14

Also Table 7.12 showed that Error VI (confusion of '∂' with delta 'δ') made only by ComG1 student on the post-test. As can be seen from the table above, there is a diminution of errors of types I-III and V and their replacement by a correct response.

A number of students who gave the correct answer such as "small change in y" or "change in y" both on the pre-test and the post-test was 20% (7 students), 77% (43 students), 65% (13 students), and 36% (12 students) for ComG1, ComG2, N-ComG1, and N-ComG2, respectively.

Item 4(c)

This item required a student to explain the meaning of '$\frac{\delta y}{\delta x}$'. Table 7.13 presents the errors made by students while explaining it.

Table 7.13. Student errors on item 4(c): Explain '$\frac{\delta y}{\delta x}$'

LIST OF ERRORS AND THEIR EXAMPLES FOR ITEM 4(C)		
CODING	DESCRIPTIVE LABEL	EXAMPLE
I	$\frac{\delta y}{\delta x}$ as the differentiation of y with respect to x[4(a).I]	•The derivative of y with respect to x •The differentiation of y with respect to x
II	$\frac{\delta y}{\delta x}$ as the differential of y with respect to x [4(a).I & 1(a).III]	Differential of y in terms of x
III	$\frac{\delta y}{\delta x}$ as rate of change[4(a).I]	Rate of change of y with respect to x
IV	$\frac{\delta y}{\delta x}$ as gradient of a curve[4(a).I]	
V	$\frac{\delta y}{\delta x}$ as partial derivative[4(a).VI]	Partial differentiation of a function of y with respect to x
VI	$\frac{\delta y}{\delta x}$ as small change[1(a).VIII]	• A very small change in the value of the ratio of $\frac{y}{x}$ • Small change in gradient •$\frac{\delta y}{\delta x}$ is a small change in y with respect to small value in x
VII	δx or δy as point[4(a).VII]	'$\frac{\delta y}{\delta x}$' is a small value of y divided by small value of x
VIII	Unclassified	

Error I-IV can be interpreted as occurring because of the same reason. Taken together they do imply that students cannot distinguish '$\frac{\delta y}{\delta x}$' from '$\frac{dy}{dx}$', that is they do not appreciate the relationship between them.

Table 7.14 shows the number of students' errors falling within each category while explaining '$\frac{\delta y}{\delta x}$'.

Table 7.14. Number of students who made each type of error on item 4(c): Explain '$\frac{\delta y}{\delta x}$'

FREQUENCY OF ERRORS FOR ITEM 4(C)										
	Computer Groups				Non-Computer Groups					
	ComG1 ee=6 ec=1 ce=5 em=4 me=8 (n=35)		ComG2 ee=15 ec=16 ce=9 em=0 me=3 (n=56)		N-ComG1 ee=4 ec=4 ce=1 em=0 me=1 (n=20)		N-ComG2 ee=11 ec=6 ce=4 em=2 me=8 (n=36)		Total ee=36 ec=27 ce=19 em=6 me=20 (n=147)	
coding	Pre n=11	Post n=19	Pre n=31	Post n=27	Pre n=8	Post n=6	Pre n=19	Post n=23	Pre n=69	Post n=75
I	1	2 c↓1	3 1↑c 2→III	1 c↓1	2 1→IV 1→VIII	1 VIII→1	2 1↑c	5 c↓1 VIII→1 VI→1	8	9
II	3 1→V 1→III 1→VI	1 c↓1	4 3↑c 1→III		1 1↑c		2 1↑c		10	1
III	3 1↑c 1→VIII	4 VIII→1 II→1	10 4↑c 3→III 2→IV 1→VIII	8 c↓1 I→2 II→1 III→3 VIII→1	1 1↑c 1→VIII	2 c↓1	1 1→VII	4 c↓1 VIII→2	15	18
IV	2	1 c↓1	3 2↑c 1→IV	4 c↓1 III→2 IV→1		1 I→1	1 1→VIII	1	6	7
V		3 c↓1 II→1			1 1↑c				1	3
VI		2 VIII→1 II→1	5 2↑c 3→VI	10 c↓3 VIII→1 VI→3			7 3↑c 1→I 3→VI	6 c↓1 VI→3 VIII→1	12	18
VII			1 1↑c	1 c↓1			1 1→VII	2 III→1 VII→1	2	3
VIII	2 1→III 1→VI	6 c↓1 III→1	5 3↑c 1→III 1→VI	3 c↓2 III→1	3 1↑c 1→I	2 I→1 III→1	5 1↑c 1→I 1→VI 2→III	5 c↓1 IV→1	15	16

As shown in Table 7.14, the most common underlying error was that of seeing '$\frac{\delta y}{\delta x}$' as being the same as '$\frac{dy}{dx}$'. Note that Error VI (seeing $\frac{\delta y}{\delta x}$ as a small change) was also quite common. Six of the twelve students who made Error VI on the first occasion made the same error on the second, and five gave an acceptable correct answer. The answers given were as follows:

- small (change) in y divided by (small) change in x;
- the gradient of a secant line or chord.

<u>Item 4(d)</u>

The purpose of this item was to examine how students explain the meaning of 'dx' in two expressions '$\frac{d}{dx}(x^2)$' and '$\int x^2 dx$'. In the first traditional viewpoint, which is historically linked with Leibniz, differentials are infinitely small quantities. In the second traditional view point, 'dx' is regarded as a small, non zero approximation. The answers such as (a) "differentiate and integrate with respect to x" or (b) "a change or increment of x which is so small. It is almost zero" were accepted as a correct answer. Table 7.15 gives the errors made by students while explaining 'dx' in the two expressions.

Errors I and VI were rooted in a <u>lack of mastery and understanding of the limiting process involved in differentiation</u>.

It is clear that Error II was rooted in a <u>lack of mastery and understanding of function concept</u>. The expression 'dx' was seen as being the same as the function notion, say, f(x). Additionally, students seemed to see f(x) "the f function of x" as being the same as $f(x^2)$ "the f function of x^2".

Error III seemed to occur because of the consideration of 'd' as the symbol of or place holder for 'rate of change'. These results also show that the students have probably not acquired the meaning of 'rate of change'.

Error IV showed the importance of understanding algebraic language related to expressions which means understanding from the context, which letters are used as expressions, and understanding the role of expressions as opposed to the role of variables. Here students seemed to see 'dx' as the variable of integration rather 'x'.

Table 7.15. Student errors on item 4(d): Explain the dx in $\frac{d}{dx}(x^2)$ and $\int x^2 dx$

LIST OF ERRORS AND THEIR EXAMPLES FOR ITEM 4(D)		
CODING	DESCRIPTIVE LABEL	EXAMPLE
I	(Small) Change in x	
II	dx as function	• dx is the function which differentiation and integration is going to be applied too • Differentiate $d = x^2$ with respect to x
III	dx as rate of change of x	
IV	dx as the variable of integration	• $\int x^2 dx$ is the integral of x^2 in terms of dx • $\int x^2 dx$ mean integrate with respect to a small change in x • with respect to dx
V	d as variable[4(a).IX]	dx represents the small change in the value of d
VI	dx as the limit of $\frac{dy}{dx}$ [1(a).VIII]	The dx is when $\frac{dy}{dx} \to 0$
VII	Unclassified	

Table 7.16 shows the number of students' errors falling within each category while explaining dx in $\frac{d}{dx}(x^2)$ and in $\int x^2dx$.

Table 7.16. Number of students who made each type of error on item 4(d): Explain dx in $\frac{d}{dx}(x^2)$ and in $\int x^2dx$

FREQUENCY OF ERRORS FOR ITEM 4(D)										
	Computer Groups				Non-Computer Groups					
	ComG1 ee=5 ec=1 ce=0 em=0 me=2 (n=35)		ComG2 ee=2 ec=6 ce=6 em=3 me=4 (n=56)		N-ComG1 ee=1 ec=1 ce=1 em=0 me=2 (n=20)		N-ComG2 ee=5 ec=6 ce=1 em=2 me=2 (n=36)		Total ee=13 ec=14 ce=8 em=5 me=10 (n=147)	
acronym	Pre n=6	Post n=7	Pre n=11	Post n=12	Pre n=2	Post n=4	Pre n=13	Post n=8	Pre n=32	Post n=31
I	1 I→I	2 I→I	6 3↑c I→IV	2		1 VII→I	2 1↑c I→I	1 I→I	9	6
II		1	1 1↑c		1 1↑c*	1 c↓1	2 2↑c	1 V→I	4	3
III				2 c*↓1			1	1	1	3
IV				5 c↓3 I→I VII→I			2 1↑c		2	5
V							1 I→II		1	
VI				1						1
VII	5 1↑c 4→VII	4 VII→4	4 2↑c^{1}* I→IV	2 c^{1}*↓2	1 I→I	2	5 2↑c 3→VII	5 c↓1 VII→3	15	13

Note: * the correct answer given was that "limiting value of δx" or "infinitely small change in x".

The table above shows that the predominant type of error was that of seeing 'dx' as being the same as 'δx'. Four of the nine students who made Error I in the first occasion gave the correct answer such as "integrate or differentiate with respect to x" in the second occasion. Two of those students made the same Error I on both the pre-test and the post-test.

All four students who made Error II in the first occasion gave the correct answer mainly referring to the correct answer given above. There of the five 'second-occasion' occurrences of Error IV were by ComG2 students who gave the correct answer such as "with respect to".

The most common correct answer was that of (a) type which is given above.

Item 4(e)

This item required a student to explain the meaning of 'dy'. Table 7.17 provides the errors made by students while attempting to do so.

As the two items 4(e) and 4(d) are apparently similar, the results obtained are also similar: The same type of errors have been obtained with item 4(e) as compared with item 4(d).

In the traditional view point, dy is regarded as being equal to the product f'(x)dx; or it is regarded as an approximation to the change in y that results from a small change in x.

Table 7.17. Student errors on item 4(e): Explain 'dy'

LIST OF ERRORS AND THEIR EXAMPLES ABOUT ITEM 4(E)		
CODING	DESCRIPTIVE LABEL	EXAMPLE
I	dy as (small) change in y	
II	dy as function	dy means the function of y
III	dy as rate of change of y	
IV	dy as dy/dx	differentiate y derivative of y
V	Unclassified	

Error IV might occur because, as shown by Katz (1986), differential notation can be regarded as a brief notation or a shorthand notation for the derivatives outputs.

Table 7.18 shows the number of students' errors falling within each category while explaining 'dy'.

Table 7.18. Number of students who made each type of error on item 4(e): Explain dy

FREQUENCY OF ERRORS FOR ITEM 4(E)										
	Computer Groups				Non-Computer Groups					
	ComG1 ee=5 ec=5 ce=4 em=1 me=7 (n=35)		ComG2 ee=15 ec=7 ce=2 em=9 me=8 (n=56)		N-ComG1 ee=2 ec=2 ce=2 em=1 me=5 (n=20)		N-ComG2 ee=12 ec=1 ce=1 em=6 me=4 (n=36)		Total ee=34 ec=15 ce=9 em=17 me=24 (n=147)	
coding	Pre n=11	Post n=16	Pre n=31	Post n=25	Pre n=5	Post n=9	Pre n=19	Post n=17	Pre n=65	Post n=67
I	6 2↑c 3→I	7 c↓2 I→3	20 3↑c 9→I 1→V	18 I→9 III→1 IV→1 V→1	1 1→I	3 c↓1 I→1	5 1↑c 3→I	7 I→3 IV→2 III→1	32	35
II								1 III→1		1
III		2 V→1	1 1→I	4 IV→2			6 1→I 1→II	1 V→1	7	7
IV	2 1↑c 1→V	3 c↓1	3 1→I 2→III	2 c↓2	1 1↑c	2 V→1	4 1→IV 2→I 1→V	2 c↓1 IV→1	10	9
V	3 2↑c 1→III	4 c↓1 IV→1	7 4↑c 1→I	1 I→1	3 1↑c 1→IV	4 c↓1	4 2→V 1→III	6 V→2 IV→1	17	15

Table 7.18 shows that the predominant type of error is that of seeing 'dy' as being the same as 'δy'. 16 of the 32 students who made either <u>Error I</u> continued to make the error, but six of those have ceased to do so.

Item 4(f)

This item required a student to explain the meaning of '$\frac{dy}{dx}$'. Table 7.19 provides the errors made by students while attempting to do so.

Table 7.19. Student errors on item 4(f): Explain $\frac{dy}{dx}$

LIST OF ERRORS AND THEIR EXAMPLES ABOUT ITEM 4(F)		
CODING	DESCRIPTIVE LABEL	EXAMPLE
I	$\frac{dy}{dx}$ regarded as same as $\frac{\delta y}{\delta x}$ 4(a).I	•Gradient between two points, a small distance a part • $\frac{\text{(small) change in y}}{\text{(small) change in x}}$
II	$\frac{dy}{dx}$ as Differential 1(a).III	$\frac{dy}{dx}$ is the differential of a function with respect to x
III	$\frac{dy}{dx}$ is a derivative of $\frac{\delta y}{\delta x}$	
IV	$\frac{dy}{dx}$ as tangent 1(a).I	
V	$\frac{dy}{dx}$ as (small) change 1(a).VIII	• The small change in the ratio of $\frac{y}{x}$ • Change in gradient
VI	$\frac{dy}{dx}$ as $\lim_{x\to 0}\frac{\delta y}{\delta x}$ 4(d). VI	An indication of the gradient of a line as $x \to 0$ we obtain an exact point
VII	$\frac{dy}{dx}$ as integral 1(a).IX	Integrate y with respect to x
VIII	dx as variable 4(d). III	$\frac{dy}{dx} = \frac{\text{rate of change of y}}{\text{rate of change of x}}$ •The change in y with respect to the change in x
IX	Confusion in dependent and independent variables	Rate of change of x with respect to y
X	Unclassified	

Table 7.20 shows the number of students' errors falling within each category while explaining '$\frac{dy}{dx}$'.

Table 7.20. Number of students who made each type of error on item 4(f): Explain '$\frac{dy}{dx}$'

FREQUENCY OF ERRORS FOR ITEM 4(F)										
	Computer Groups				Non-Computer Groups					
	ComG1 ee=4 ec=3 ce=2 em=1 me=2 (n=35)		ComG2 ee=1 ec=11 ce=6 em=2 me=5 (n=56)		N-ComG1 ee=1 ec=4 ce=3 em=0 me=3 (n=20)		N-ComG2 ee=4 ec=7 ce=4 em=2 me=2 (n=36)		Total ee=10 ec=25 ce=15 em=5 me=12 (n=147)	
coding	Pre n=8	Post n=8	Pre n=14	Post n=12	Pre n=5	Post n=7	Pre n=13	Post n=10	Pre n=40	Post n=37
I	3 1↑c I→VIII I→II	3 II→2	6 6↑c	7 c↓3	2 1↑c I→I	2 I→1	1 1↑c	4 c↓3 VIII→1	13	16
II	4 1↑c 2→I	2 c↓1 I→1	3 2↑c			1 c↓1	3 1↑c I→II	2 II→1	10	5
III						1 c↓1				1
IV						1				1
V			1 1↑c				1 1↑c		2	
VI							1 1↑c		1	
VII			1 1↑c						1	
VIII		2 c↓1	2 1↑c I→VIII	4 c↓2 VIII→1			3 1↑c I→I	3 c↓1 X→1	5	9
IX				1 c↓1	1 1↑c		1 1↑c		2	1
X	1 1↑c	1 I→1	1		2 2↑c	2 c↓1	3 1↑c I→X I→VIII	1 X→1	7	4

The most common underlying error was that of seeing $\frac{dy}{dx}$ as being the same as $\frac{\delta y}{\delta x}$. Five of the 23 students who made either Error I or Error II continued to make the error, but thirteen of those have ceased to do so.

The students who made Errors V-VIII on the first occasion have ceased to do so on the second occasion. On the contrary, four of the five students who made Error VIII on the second occasion gave a correct answer on the first occasion.

Item 4(g)

As $\frac{dy}{dx}$ and $\frac{\delta y}{\delta x}$ are used together frequently in calculus, I wanted to see whether students can identify each one as belonging to a particular context. So students were presented with the following item " What is the relation between $\frac{dy}{dx}$ and $\frac{\delta y}{\delta x}$?".

When using a difference quotient to approximate to a derivative, students need to realise that the gradient of the chord could be made to approximate to a limiting value by bringing h close enough to zero.

Table 7.21 provides the errors made by students while explaining the relation between $\frac{dy}{dx}$ and $\frac{\delta y}{\delta x}$.

Table 7.21. Student errors on item 4(g): What is the relation between $\frac{\delta y}{\delta x}$ and $\frac{dy}{dx}$?

LIST OF ERRORS AND THEIR EXAMPLES ABOUT ITEM 4(G)		
CODING	DESCRIPTIVE LABEL	EXAMPLE
I	$\frac{dy}{dx}=\frac{\delta y}{\delta x}$ 4(a).I	•They both represent rate of change/derivative
II	δ as ∂ 1(a).VI	•δ is used when there are more than one variable •$\frac{\delta y}{\delta x}$ partial differentiation $\frac{dy}{dx}$ as full differentiation
III	$\frac{\delta y}{\delta x}$ as small change/small rate of change $\frac{dy}{dx}$ as any increment/fixed rate of change/ total change 1(a).VIII	• $\frac{\delta y}{\delta x}$ is very small and is cancelled out to leave the form $\frac{dy}{dx}$ • $\frac{\delta y}{\delta x}$ is small $\therefore \frac{dy}{dx} \approx \frac{\delta y}{\delta x}$ • $\frac{\delta y}{\delta x} \to 0$ $\frac{\delta y}{\delta x} \cong \frac{dy}{dx}$
IV	$\frac{dy}{dx}$ as sum of all the $\frac{\delta y}{\delta x}$	$\frac{dy}{dx}$ as sum of all the infinitely small changes contained in $\frac{\delta y}{\delta x}$
V	$\frac{dy}{dx}$ and $\frac{\delta y}{\delta x}$ both as area 1(a).IX	$\frac{dy}{dx}$ and $\frac{\delta y}{\delta x}$ both are representing area
VI	$\frac{dy}{dx}$ is a derivative of $\frac{\delta y}{\delta x}$ or vice versa	
VII	$\frac{\delta y}{\delta x}=\frac{dy}{dx}$ as $x \to 0$ 4(f).VI	
VIII	δx and dx are the same. δy and dy are approximately the same	
IX	No relationship	
X	Unclassified	

Table 7.22 shows the number of students' errors falling within each category while explaining the relation between "$\frac{\delta y}{\delta x}$" and "$\frac{dy}{dx}$".

Table 7.22. Number of students who made each type of error on item 4(g): What is the relation between $\frac{\delta y}{\delta x}$ and $\frac{dy}{dx}$?

FREQUENCY OF ERRORS FOR ITEM 4(G)										
	Computer Groups				Non-Computer Groups					
	ComG1 ee=6 ec=2 ce=0 em=2 me=6 (n=35)		ComG2 ee=23 ec=3 ce=2 em=5 me=13 (n=56)		N-ComG1 ee=9 ec=1 ce=0 em=1 me=5 (n=20)		N-ComG2 ee=14 ec=0 ce=0 em=3 me=10 (n=36)		Total ee=14 ec=0 ce=0 em=3 me=10 (n=147)	
coding	Pre n=10	Post n=12	Pre n=31	Post n=38	Pre n=11	Post n=14	Pre n=17	Post n=24	Pre n=69	Post n=88
I	4 1→I 2→X	1 I→1	13 1↑c 5→I 2→X 1→VII	12 I→5 X→1	4 2→I 1→III	3 I→2 III→1	8 5→I 1→X	12 I→5 X→2	29	28
II		4								4
III	2 1→III	1 III→1	3 2→III 1→X	4 c↓1 III→2	1 1→I	3 I→1 X→1	2 1→III 1→X	1 III→1	8	9
IV	1 1↑c		1 1→IV	3 c↓1 IV→1			2 2→IV	2 IV→2	4	5
V					1 1→VIII				1	
VI		2 X→1						1 X→1		1
VII				5 X→3 I→1		1 X→1		1		7
VIII						1 V→1				1
IX				1 X→1					1	
X	3 1↑c 1→VI 1→X	4 I→2 X→1	14 1↑2 1→I 3→VII 6→X 1→IX	13 I→2 III→1 X→6	5 1↑c 1→III 2→X 1→VII	6 X→2	5 2→I 1→VI 1→X	7 I→1 X→1 III→1	27	30

The number of students who missed the item both on the pre-test and the post-test was 16(46%), 3 (5%), 3 (15%), and 9 (25%) for ComG1, ComG2, N-ComG1 and N-ComG2 respectively.

Error I is the one made by as many as 29 of the students on the first occasion of testing. 13 of the 29 continued to make the error on the second occasion, and the remaining made different errors especially the unclassified one except one student who has ceased to do so. Error III is the second common error made by as many as eight students on the first occasion. Four of the eight continued to make the error on the second occasion. Three of the four students who made Error IV on the first occasion continued to make the error on the second, but the remaining one has ceased to make the error.

Interviews

The individual interviews confirmed the evidence received from the written assessments. It seemed that translation from one representation (δ) to another (d) is difficult for students to understand. The following episodes are from the transcripts of the five students who gave some correct or incorrect answers for question 4 that indicated no understanding of limiting process involved in differentiation. In order to understand differentiation, students need to have a strong intuitive understanding of limit. The interviews also confirmed that it may even be possible to respond with the correct formal definition whilst having an inappropriate concept image.

Apart of these five students another seven students (S3 and S5 from ComG1, S8 and S9 from ComG2, S13 and S17 from N-ComG1, and S21 from N-ComG2) could not state the relation between $\frac{\delta y}{\delta x}$ and $\frac{dy}{dx}$.

1st Episode (N-ComG1)

Inter: In explaining δx, you said it was an increment in x. Can you tell me a little bit more about it? What is the meaning of increment?

S12: So this is where you split the graph into separate bars and it's increasing because you take such a thin section that the increase is .... you take 0- in the centre. So an increment in x probably meant that there was a very small section of x along the x axis it is really thin. So, there was no change in y for the change in x because the change in x was so small.

Inter: What about here? (pointing out item 4(b)) (The answer given was "a corresponding increment in y")

S12: That's the sum of the increments in x. So this is saying that y does change for x because you are taking the sum of the small increments.

Inter: And for δy over δx, you said "ratio the increment in y to the increment in x". What else can you say about it?

S12: Sorry I don't know.

Inter: Could you tell me the meaning of dy over dx ? (the answer was missing)

S12: dy over dx is the gradient of the curve. This is in first principles with the delta. When you actual solve the problem it is dy over dx.

Inter: Why do we use different symbols?

S12: I guess, to show that this is first principles, and this is actually someone actually solving a problem. I've never really understood first principles for δy. I've never understood the differences.

2nd Episode (N-ComG1)

Inter: Would you tell me the meaning of δx in $\frac{\delta y}{\delta x}$ and in $\sum f(x)\delta x$?

S14: A small change or small increment.

Inter: It's the same in both situations. Is it?

S14: That's like a ratio.

Inter: No. If I give you delta y over delta x, what is the meaning of δx?

S14: With respect to delta x probably. A smaller increment of y with respect to delta x.

Inter: And this one?(pointing out $\sum f(x)\delta x$)

S14: That would be the sum of f(x) with respect to delta x.

Inter: What did you mean here, about delta y over delta x as a ratio to length?

S14: It doesn't have to be length I suppose. It's just the ratio of whatever delta y or delta x are. It doesn't have to be length, it can be height.

Inter: What is the meaning of dx?

S14: That's when you've taken it to infinity and I don't think it's a small change anymore. It's supposed to be something else. When you've taken it to infinity, I don't think it counts.

Inter: What are you taking to infinity?

S14: When you're taking x to infinity, delta x doesn't exist as a small change. It's supposed to be more significant. So, it becomes dy over dx.

3rd Episode (N-ComG2)

Inter: Could you explain to me what you meant by "the very small part of x"? (pointing out item 4(a))

S19: You divide a graph into strips then delta x would be one of those very small strips of the graph.

Inter: What about delta y?

S19: That's just a very small strip of the graph on the opposite axis. On the y axis.

Inter: What is the meaning of "dx"?

S19: That is exactly the same as delta x.

Inter: Do you mean they are the same?

S19: Yes, that is the one I thought.

4rd Episode (N-ComG2)

Inter: What is the meaning of delta y over delta x?

S20: The change in y with respect to x.

Inter: What about the meaning of dy over dx?

S20: I've got this wrong actually.

Inter: Which one?

S20: That should have been there -

Inter: Part (c)

S20: Yes, part (c) should be in there. It's dy by dx. If you have the summation .... no I can't explain that.

Inter: You said part (c) must be part (f) but you can't explain part (c)?

S20: I can't think of the difference between the delta symbol and d symbol.

5th Episode (N-ComG1)

Inter:	Could you tell me what the relation between delta y over delta x and dy over dx?
S17:	Delta y over delta x is a small increment of y over small increment in x. dy over dx is the differential of y with respect to x.
Inter:	But what is the connection between these two things - how are they related to each other?
S17:	I couldn't tell you.

Student S1 from ComG1 who had missed the question seemed to have a sense that if $\frac{\delta y}{\delta x}$ approaches a limit as δx approaches to zero, this limit is the derivative of f at the point x. The following episode exemplifies that.

6th Episode (ComG1)

Inter:	Would it be possible for you to tell me the meaning of delta x?
S1:	A small increment in x.
Inter:	What about delta y?
S1:	The same for y. A small increment.
Inter:	What is the meaning of delta y over delta x?
S1:	The gradient at a point. No, a small section.
Inter:	What is the meaning of dx?
S1:	Just the point of x.
Inter:	What about dy?
S1:	I suppose it's just the point of y.
Inter:	What about dy over dx.
S1:	The gradient at a point.
Inter:	What is the relation between those two? (pointing item 4(g))
S1:	As the delta y's and x's get smaller, it's getting closer and closer to the gradient at that point.
Inter:	Why didn't you write that before?
S1:	I don't think I quite understood it.

7.1.5 Questions 5 and 6: Average rate of change and rate of change

Question 5 and 6 contain 2 and 3 items respectively. Question 5 presents the graph of a linear function, which is likely to be highly familiar to students. By contrast, question 6 presents the graph of a quadratic function. The items were designed with the aim of finding out whether students were able to use the quotient formula in finding the rate of change for a linear and quadratic function. Comparing the errors to both the linear and the quadratic function, provides an indication of the stability of the students' understanding of the average rate of change and rate of change.

Item 5(a)

In this item, students were expected to either say that the graph shows a linear function and therefore the rate of change is constant at any point or calculate the rate of change using the quotient formula: $\frac{f(x+h)-f(x)}{h}$. In Table 7.23, the categories of errors for item 5(a) are presented.

Error IV showed that the answer given as the rate of change (Rate of change=2x+2h) is the image of y=2x after being translated 2h units upwards.

Errors V and VI stemmed from the inclusion of the word 'change' in the phrase 'rate of change'. Namely, the phrase 'rate of change' seems to be misleading.

Table 7.23. Student errors on item 5(a): Water is flowing into a tank at a constant rate. For each unit increase in time, the water's depth increases by 2 units. The table and graph illustrate this situation. What is the rate of change of y with respect to x as x increases from 3 to 3+h?

LIST OF ERRORS AND THEIR EXAMPLES FOR ITEM 5(A)		
CODING	DESCRIPTIVE LABEL	EXAMPLE
I	Rate of change over an interval is the value of the function either at the start $y(x_1)$ or at the end of the interval $y(x_2)$	y =2x $\frac{dy}{dx}=(3+h)\times 2=6+2h$
II	Rate of change over an interval is the value of y(h) (h is the difference between two values)4(a).VII	2h
III	Rate of change over an interval is the function itself	$\frac{dy}{dx}=2x$
IV	Rate of change over an interval is the function itself plus the difference in the y value	y goes from 6 to 6 + 2h Rate of change = 2x + 2h
V	Rate of change over an interval is the value of $y(x_2)-y(x_1)$	Rate of change= (6+2h) - 6 = 2h
VI	Rate of change over an interval is the value of δy	$\frac{\delta y}{\delta x}=\frac{dy}{dx}\Rightarrow \delta y=\frac{6}{3}h$
VII	Incorrect quotient formula	$\frac{dy}{dx}=\frac{f(x+h)-f(x)}{f(x)}$ $=\frac{3+h-3}{3}$
VIII	Rate of change over an interval is the ratio of the slope to the small difference in x	$m=\frac{4-2}{2-1}=2$ the rate of change is constant but the rate of change from 3 to 3+h = $\frac{2}{(3+h)-3}$
IX	Rate of change over an interval is the integral of the function between the start and the end points of the interval	depth y = 2x $y=\int_3^{3+h} 2x\,dx=[2]_3^{3+h}=$ $[(6+2h)-6]=2h$
X	Rate of change over an interval is the integral of the derivative between the start and the end points of the interval	$y=2x$ and $\frac{dy}{dx}=2$ ∴ $\int_3^{3+h} 2dx=[2(3+h)]-[6]$
XI	Wrong formula setting	2y = x
XII	Unclassified	

Error VII seemed to be the result from the failure to apply the 'quotient formula'. One possible interpretation of the failure to apply the formula is that students give precedence to the memorisation of a formula over the capturing of the ideas in it and build a coherent model between them. It makes no sense, except as a sort of formula that they should remember. It seems that visualisation and mental reasoning play an important role in acquiring mathematical meaning. A formula captured by 'rote memorisation' is likely to lead to many errors.

The most common underlying reason for Errors VIII-X was that of the interval given to find the rate of change over an interval. Thus, the ability to find the rate of change over an interval is fundamental for the understanding of the rate of change at a point as the limiting value of rate of change over an interval.

The distribution of errors across the various types mentioned above is given for the four groups in Table 7.24.

The most common underlying error was that of seeing the rate of change over an interval either as the value of a function at a point or the change in the y values. While four of the 14 students corrected Error I or II, rate of change at a point is the value of the function at that point, nine of the 19 students introduced this error.

Table 7.24. Number of students who made each type of error on item 5(a): Water is flowing into a tank at a constant rate. For each unit increase in time, the depth of the water increases by 2 units. The table and graph illustrate this situation. What is the rate of change of y as x increases from 3 to 3+h?

FREQUENCY OF ERRORS FOR ITEM 5(A)										
	Computer Groups				Non-Computer Groups					
	ComG1 ee=8 ec=4 ce=3 em=2 me=3 (n=35)		ComG2 ee=11 ec=6 ce=7 em=0 me=4 (n=56)		N-ComG1 ee=11 ec=2 ce=0 em=1 me=3 (n=20)		N-ComG2 ee=10 ec=6 ce=5 em=0 me=2 (n=36)		Total ee=40 ec=18 ce=15 em=3 me=12 (n=147)	
coding	Pre n=14	Post n=14	Pre n=17	Post n=22	Pre n=14	Post n=14	Pre n=16	Post n=17	Pre n=61	Post n=67
I	4 1↑c 1→VII 1→IV 1→III	3 c↓1 XII→1	1 1→V	3 c↓2 III→1	3 1→III 2→XII	1	3 1↑c 1→XII 1→I	5 c↓3 V→1 I→1	11	12
II	1 1→III			2 c↓2		1 III→1	3 2↑c 1→V	4 c↓1 III→1 V→1 XII→1	4	7
III		4 XII→1 II→1 I→1	3 1↑c I→1 1→V	2 XII→2	1 1→II	2 I→1	1 1→II		5	8
IV		1 I→1				1 XII→1				2
V	1 1→VII	1 c↓1	2 2↑c	7 c↓1 III→1 I→1 X→1 VIII→1	4 1↑c 1→XII 1→V	3 V→1 XII→2	3 1→XII 1→II 1→I	1 II→1	10	12
VI			2 1↑c 1→VI	1 VI→1	1				3	1
VII		2 V→1 I→1								2
VIII			1 1→V						1	
IX		1 XII→1	2 2→XII						2	1
X			1 1→V						1	
XI	2 2↑c						1 1→XII	1	3	1
XII	6 1↑c 1→IX 1→III 1→I	2 c↓1	5 2↑c 1→XII 2→III	7 c↓2 XII→1 IX→2	5 2→XII 2→V 1→IV	6 XII→2 V→1 I→2	5 3↑c 1→XII 1→II	6 c↓1 XII→1 XI→1 V→1 I→1	21	21

Two of the 12 students who made Error V on the post-test gave the correct answer on the pre-test. Each one of these students was from a computer group.

The results showed a success rate of 31% (11 students), 50% (28 students), 15% (3 students), and 39% (14 students) prior to instruction compared to 40% (8+6 students), 54% (20+10 students), 20% (2+2 students), and 42% (8+7 students) respectively for ComG1, ComG2, N-ComG1, and N-ComG2 after instruction. The first addend shows the number of students who gave the correct answer on both the pre-test and the post-test. The results also indicated that 14% (5 students), 5% (3 students), 6% (2 students) of ComG1, ComG2, and N-ComG2 missed item 5(a) on both the pre-test and the post-test.

Item 5(b)

In Table 7.25, the categories of errors for item 5(b) are presented. As can be seen in the table most of the errors which occurred in this item also occurred in item 5(a).

Students who made error II showed that they failed to respond correctly to the rate of change at x = X. A possible explanation for this phenomenon may lie in the role played by x = X. Students did not seem to interpret X as an object denoting a value (Küchemann, 1981).

Table 7.25. Student errors on item 5(b): Water is flowing into a tank at a constant rate. For each unit increase in time, the depth of the water increases by 2 units. The table and graph illustrate this situation. What is the rate of change of y with respect to x at $x=2\frac{1}{2}$ and at x=X?

LIST OF ERRORS AND THEIR EXAMPLES FOR ITEM 5(B)		
CODING	DESCRIPTIVE LABEL	EXAMPLE
I	Rate of change at a point is the value of the function at this point 5(a).I	y=2x at x = X y =2X at $x = 2\frac{1}{2}$ y =5
II	Rate of change at a point X is the value of the function at X	at $x = 2\frac{1}{2}$ rate of inrease = $\frac{5}{2.5} = 2$ at x = X y = 2X
III	Rate of change at a point is the function itself plus the difference in the y value 5(a).IV	Rate of increase = 2x+2h
IV	The rate of change at two given points is the difference in y-values 5(a).V	Rate of increase = 2X - 5
V	Rate of change at a point is the function itself 5(a).III	Rate of increase at x = X and $x = 2\frac{1}{2}$ is 2x
VI	Rate of change at a point is the integral of the function between two given points 5(a).IX	$\int_{2½}^{X} 2x\,dx$
VII	Rate of change at a point is the integral of the derivative between two given points 5(a).X	$\int_{2½}^{X} 2\,dx$
VIII	Rate of change at a point is the derivative of the value of the function at this point.	at $x = 2\frac{1}{2}$ y =5 x = X y =2X $\therefore \frac{dy}{dx} = 0$
IX	Wrong formula setting 5(a).XI	gradients are $\frac{1}{2}$
X	Wrong symbolization	Rate of change $=\delta y = 2$
XI	Points taken as an interval	at $x = 2\frac{1}{2}$ y =5 at x = X y =2X $\frac{5-2X}{X-2½}$
XII	Unclassified	

The distribution of errors across the various types mentioned above is given for the four groups in Table 7.26.

Table 7.26. Number of students who made each type of error on item 5(b): Water is flowing into a tank at a constant rate. For each unit increase in time, the depth of the water increases by 2 units. The table and graph illustrate this situation. What is the rate of change of y at x = $2\frac{1}{2}$ and at x=X?

FREQUENCY OF ERRORS FOR ITEM 5(B)										
	Computer Groups				Non-Computer Groups					
	ComG1 ee=4 ec=1 ce=0 em=5 me=5 (n=35)		ComG2 ee=11 ec=6 ce=2 em=0 me=5 (n=56)		N-ComG1 ee=5 ec=2 ce=2 em=4 me=2 (n=20)		N-ComG2 ee=5 ec=6 ce=2 em=2 me=7 (n=36)		Total ee=25 ec=15 ce=6 em=11 me=19 (n=147)	
coding	Pre n=10	Post n=9	Pre n=17	Post n=18	Pre n=11	Post n=9	Pre n=13	Post n=13	Pre n=51	Post n=49
I	5 I→I I→XII	4 I→I XII→I	6 2↑c I→XI 2→I I→XII	8 c↓1 XII→I VIII→I I→2 IX→I	6 1↑c 2→I I→II	4 I→2 c↓1	6 3↑c 3→I	7 c↓1 I→3	23	23
II	1 1↑c				1 1↑c	1 I→I			2	1
III	1 I→III	1 III→I		1 VI→I					1	2
IV				1 VII→I				1		2
V			1 I→XII	1	1	1 c↓1	1 I→V	1 V→I	3	3
VI			1 I→III						1	
VII			1 I→IV						1	
VIII			1 I→I					1	1	1
IX			2 1↑c I→I						2	
X			1 1↑c						1	
XI				2 I→I XII→I						2
XII Unclass	3 I→I	4 I→I	4 2↑c I→I I→XI	5 c↓1 V→I I→I	3 2→XII	3 XII→2	6 3↑c I→XII	3 c↓1 XII→I	16	15

The most common underlying error was that of seeing the rate of change at a point as the value of the function at that point. While six students corrected the error, rate of change at a point is the value of the function at that point, three students introduced this error. Furthermore, students who made this error only for the x = X in the pre-test corrected it in the post-test.

The results showed a success rate of 29% (10 students), 39% (22 students), 20% (4 students), and 36% (13 students) prior to instruction compared to 34% (9+3 students), 61% (19+15 students), 25% (2+3 students), and 47% (9+8 students) respectively for ComG1, ComG2, N-ComG1, and N-ComG2 after instruction. The results also indicated that 23% (8 students), 5% (3 students), 10% (2 students), and 6% (2 students) missed item 5(b) on both the pre-test and the post-test.

Item 6(a)

This item was concerned with the understanding of the quotient formula. Errors made on this item were summarised in Table 7.27.

As has been noted here, students experience considerable difficulty in conceptualizing 'average rate of change'. Errors IV and VII seemed to occur due to the inclusion of the words 'change' and 'average' in the phrase 'average rate of change'. That is, the phrase 'average rate of change' seems to be misleading.

It seems likely that errors I-II were due to the conception of 'average rate of change' as being the same as the derivative at a point or rate of change at a point. Error III was due to the over-generalisation of the concept that the gradient is constant at any point which is true for linear functions but not quadratics.

Table 7.27. Student errors on item 6(a): The graph given represents $y = 3x^2 + 1$, from x=0 to x=4. What are the ratios of changes (average rate of change) of y with respect to x as x changes from: (i) 2 to 2+0.1, (ii) 2 to 2+h, (iii) a to a + h.

LIST OF ERRORS AND THEIR EXAMPLES FOR ITEM 6(A)		
CODING	DESCRIPTIVE LABEL	EXAMPLE
I	Average rate of change is the value of y'(h) (h is the difference between two x-values)	$y = 3x^2 + 1 \quad \frac{dy}{dx} = 6x$ $(i)\frac{dy}{dx} = 6 \times 0.1 = 0.6$
II	Average rate of change is the value of $y'(x_2)$ or / and $y'(x_1)$	(i) 6(2)=12
III	Average rate of change is the derivative function itself	$\frac{dy}{dx} = 6x$ (i) 6x (ii) 6x
IV	Average rate of change is the value of $y'(x_2) - y'(x_1)$	(i) 6(2+0.1) - 6(2) = 0.6
V	Average rate of change is the value of $\frac{y'(x_2)}{y'(x_1)}$	$(i)\frac{6(2.1)}{6(2)}$
VI	Average rate of change is the value of $\frac{y(x_2)}{y(x_1)}$ or	$(i)x = 2 \quad x = 2.1,$ $y = 13 \quad y = 14.23$ $\frac{dy}{dx} = \frac{14.23}{13}$
VII	Average rate of change is the value of $\frac{y'(x_2) + y'(x_1)}{2}$	$(i)\frac{6(2) + 6(2 + 0.1)}{2}$
VIII	Average rate of change is the value of $\frac{y(x_2)}{x_2} - \frac{y(x_1)}{x_1}$ **5(a).VII**	(iii) $\frac{3(a+h)^2 + 1}{a+h} - \frac{3(a)^2 + 1}{a}$
IX	Average rate of change is the value of $\frac{y'(x_2) - y'(x_1)}{h}$ **5(a).VII**	$(i)x = 2, \frac{dy}{dx} = 12$ $x = 2.1, \frac{dy}{dx} = 12.6$ $\text{rate of change} = \frac{12.6 - 12}{0.1}$
X	Average rate of change is the value of $\frac{f(x+h) - f(x)}{2}$ **5(a).VII**	$(ii)x = 2, y = 13 \; x = 2 + h$ $y = 3h^2 + 12h + 13$ $\frac{dy}{dx} = \frac{3h^2 + 12h + 13 - 13}{2}$
XI	Average rate of change is the integral of $\int_{x_1}^{x_2} y\,dx$ **5(a).IX**	$(i)y = \int_{2}^{2.1} (3x^2 + 1)dx$

Table 7.27 continued

LIST OF ERRORS AND THEIR EXAMPLES FOR ITEM 6(A)		
ACRONYM	DESCRIPTIVE LABEL	EXAMPLE
XII	Average rate of change is the value of $\frac{y(x_2)}{x_2}$ **5(a).VII**	(iii) $\frac{3(a+h)^2+1}{a+h}$
XIII	Average rate of change is the value of $\frac{y'(x_2)}{x_1}$ **5(a).VII**	(iii) $\frac{6(a+h)}{a}$
XIV	Average rate of change is the value of $\frac{y(h)}{h}$ **5(a).VII**	(iii) $\frac{3(h)^2+1}{h}$
XV	Unclassified	

It seems that errors such as V-VI were rooted in the students' incapability to distinguish 'difference ratio' used in finding 'average rate of change' from other kinds of ratios, such as the ratio of the new area to the original area. The main distinction is that units are needed for specification of the value of 'average rate of change'. Such knowledge can be subtle and needs to be made explicit to avoid likely errors and ambiguities. These errors may also be due to the failure of applying the 'quotient formula'.

Errors VIII-X and XII-XIV seemed to be the result of the failure to apply the 'quotient formula'.

Table 7.28 shows the number of students' errors falling within each category while finding the average rate of change between the intervals given.

Table 7.28. Number of students who made each type of error on item 6(a): The graph given represents $y = 3x^2 + 1$, from x=0 to x=4. What are the ratios of changes (average rate of change) of y with respect to x as x changes from: (i) 2 to 2+0.1, (ii) 2 to 2+h, (iii?) a to a+h.

FREQUENCY OF ERRORS FOR ITEM 6(A)										
	Computer Groups				Non-Computer Groups					
	ComG1 ee=13 ec=1 ce=1 em=2 me=8 (n=35)		ComG2 ee=20 ec=4 ce=4 em=3 me=12 (n=56)		N-ComG1 ee=9 ec=0 ce=0 em=2 me=4 (n=20)		N-ComG2 ee=12 ec=5ce=0 em=4 me=6 (n=36)		Total ee=25 ec=24 ce=13 em=5 me=12 (n=147)	
coding	Pre n=16	Post n=22	Pre n=27	Post n=36	Pre n=11	Post n=13	Pre n=21	Post n=18	Pre n=75	Post n=99
I		2 IV→1	3 2→I	5 XV→1 1→2	1	3 IV→2	4 1↑c 1→V 1→XV 1→I	4 XV→1 1→1	8	14
II	3 1→IV	1 XV→1	1 1↑c	3 V→1	3 2→XV 1→VII	2 XI→1 XV→1	1 1↑c	3 IX→1	8	9
III		1 IV→1		2 XV→2			1 1→III	1 III→1	1	4
IV	3 1→I 1→III 1→VII	2 c↓1 II→1	1 1→IV	2 IV→1 VI→1	2 2→I		3 1→XV	3 V→1	9	7
V	2 1↑c 1→V	4 XV→2 V→1	3 1↑c 1→V 1→II	4 XV→1 V→1 VII→1	1 1→V	1 V→1	5 1↑c 1→IV 1→V 1→XII	2 1→1 V→1	11	11
VI	1 1→VII		2 1↑c 1→IV	1 c↓1		1			3	2
VII		3 XV→1 VI→1 IV→1	5 2→XV 1→V 1→VII	7 c↓3 XV→1 VII→1		1 II→1	1 1→VII	1 VII→1	6	12
VIII				1 XIII→1						1
IX			1 1→IX	1 IX→1					1	1
X		1								1
XI		1	2 1→XV	1 XV→1	1 1→II		1 1→XV		4	2
XII						1 XV→1		1 V→1		2
XIII			1 1→VIII						1	
XIV							1 1→II		1	
XV Unclass	7 2→V 1→II 3→XV 1→VII	7 XV→3	8 1↑c 1→VII 1→V 1→I 2→III 1→XV 1→XI	9 XI→1 VII→2 XV→1	3 1→II 1→XII	4 II→2	4 2↑c 1→I	3 1→1 IV→1 XI→1	22	23

In these results, our attention is mainly attracted by the number of students who overcome errors made on the pre-test as well as the number of students who introduced errors on the post-test. It was only in the responses of those students who participated in the computer groups that errors occurred on the post-test but not on the pre-test. The majority of these errors are the result of the students having confused 'average rate of change' with 'mean'. It would appear that the experience of students in the computer groups, encouraged this confusion.

Another point to notice is that out of the 25 students who made errors on the pre-test and the post-test, 16 made the same type of errors on both tests, whereas other students altered their type of errors.

Overall, the striking feature to notice is that nearly all students, 32 (91%) in ComG1, 42 (75%) in ComG2, 19 (95%) in N-ComG1, and 29 (81%) in N-ComG2, either made errors or missed the item on the pre-test and post-test. This fact shows the difficulty of this question for the majority of students. It appears that students did not gain anything on this concept throughout the courses.

Item 6(b)

In this item, students were expected to take the limit of the average rate of change over an interval including the point as the length of the interval approaches zero. In Table 7.29, the categories of errors for item 6(b) are presented.

Note that the type of errors classified in 6(b) was greatly effected by the immediately preceding task, 6(a). Moreover, Error XIV could result from the misunderstanding of the item.

Table 7.29. Student errors on item 6(b): The graph below represents $y = 3x^2 + 1$, from x=0 to x=4. What are the rates of change of y with respect to x as x changes from: (i) 2 to 2+0.1, (ii) 2 to 2+h, (iii) a to a+h

LIST OF ERRORS AND THEIR EXAMPLES FOR ITEM 6(B)		
CODING	DESCRIPTIVE LABEL	EXAMPLE
I	Rate of change is the value of $y(x_2)$ (error has been done in taking square of the paratheses)[5(a).I]	$y = 3x^2 + 1$ (ii) $3(2 + h^2) + 1$
II	Rate of change is the value of $y'(h)$ (h is the difference between two values)[6(a).I]	$\frac{dy}{dx} = 6x$ (i) $\frac{dy}{dx} = 6 \times 0.1 = 0.6$
III	Rate of change is the derivative function[6(a).III]	$\frac{dy}{dx} = 6x$ (i) 6x (ii) 6x
IV	Rate of change over an interval is the value of $y'(x_2)$ and $y'(x_1)$	$\frac{dy}{dx} = 6x$ (i) 6(2)=12
V	Rate of change over an interval is the value of $y'(x_2) - y'(x_1)$[6(a).IV]	(i) 6(2+0.1) - 6(2) = 0.6
VI	Rate of change over an interval is the value of δy	$\delta y = \frac{dy}{dx}\delta x = (6x)\delta x$ (i) $(6 \times 2) \times 0.1 = 1.2$
VII	Rate of change over an interval is the value of $\frac{y'(x_2) - y'(x_1)}{h}$[6(a).IX]	$\frac{dy}{dx} = 6x$ (i) $x = 2, \frac{dy}{dx} = 12$ $x = 2.1, \frac{dy}{dx} = 12.6$ rate of change $= \frac{12.6 - 12}{0.1}$
VIII	Rate of change over an interval is the value of $\frac{y'(x_2) + y'(x_1)}{h}$[6(a).IX]	$\frac{dy}{dx} = 6x$ (i) $x = 2, \frac{dy}{dx} = 12$ $x = 2.1, \frac{dy}{dx} = 12.6$ rate of change $= \frac{12.6 + 12}{0.1}$
IX	Rate of change over an interval is the value of $\frac{y'(h)}{h}$ (h is the difference between two values)[6(a).XIII]	$\frac{dy}{dx} = 6x$ (i) $\frac{6(0.1)}{0.1} = 6$
X	Rate of change over an interval is the value of $\frac{y'(x)}{h}$[6(a).XIII]	(i) $\frac{dy}{dx} = \frac{(3x^2 + 1)'}{(2 + 0.1) - 2} = \frac{6x}{0.1}$

Table 7.29 continued

LIST OF ERRORS AND THEIR EXAMPLES FOR ITEM 6(B)		
CODING	DESCRIPTIVE LABEL	EXAMPLE
XI	Rate of change over an interval is the value of $\frac{y(x_2)}{y(x_1)}$ 6(a).VI	(i) $x=2 \quad x=2.1,$ $y=13 \quad y=14.23$ $\frac{dy}{dx}=\frac{14.23}{13}$
XII	Rate of change over an interval is the ratio of the derivative values 6(a).V	$\frac{dy}{dx}=6x$ (i) $\frac{6(2.1)}{6(2)}$
XIII	Rate of change over an interval is the value of $\frac{y'(x_2)+y'(x_1)}{2}$ 6(a).VII	$\frac{dy}{dx}=6x$ (i) $\frac{6(2)+6(2+0.1)}{2}$
XIV	Rate of change over an interval is the value of $\frac{f(x+h)-f(x)}{h}$ but not its limit	(ii) $x=2, y=13 \; x=2+h$ $y=3h^2+12h+13$ $\frac{dy}{dx}=\frac{3h^2+12h+13-13}{2+h-2}$
XV	Rate of change over an interval is the integral of $\int_{x_1}^{x_2} y\,dx$ 6(a).IX	(i) $y=\int_2^{2.1}(3x^2+1)dx$
XVI	Rate of change over an interval is the value of the derivative of the inverse function	$x=\sqrt{\left(\frac{y-1}{3}\right)}$ $\frac{dx}{dy}=\frac{1}{\sqrt{3}}\times\frac{1}{2}\times(y-1)^{-½}$ (iii) $\frac{1}{2\sqrt{3(a+h)-3}}$
XVII	Unclassified	

Students do not have the intuition in terms of the move from the gradient of the chord to the gradient of the tangent in their numerical work. It is likely that the distinction between average rate of change and rate of change is blurred by first studying straight lines.

The distribution of errors across the various types mentioned above is given for the four groups in Table 7.30.

Table 7.30. Number of students who made each type of error on item 6(b): The graph below represents $y = 3x^2 + 1$, from x=0 to x=4. What are the rates of change of y with respect to x as x changes from: (i) 2 to 2+0.1, (ii) 2 to 2+h, (iii) a to a+h.

FREQUENCY OF ERRORS FOR ITEM 6(B)										
	Computer Groups				Non-Computer Groups					
	ComG1 ee=5 ec=0 ce=0 em=2 me=6 (n=35)		ComG2 ee=10 ec=0 ce=0 em=12 me=9 (n=56)		N-ComG1 ee=3 ec=0 ce=0 em=4 me=5 (n=20)		N-ComG2 ee=8 ec=0 ce=0 em=3 me=6 (n=36)		Total ee=26 ec=0 ce=0 em=21 me=26 (n=147)	
coding	Pre n=7	Post n=11	Pre n=22	Post n=19	Pre n=7	Post n=8	Pre n=11	Post n=14	Pre n=47	Post n=52
I						1 XVII→1				1
II			3 1→IX 1→II	6 XVII→2 V→1 II→1		1		3 XIV→2	3	9
III				2			1 1→III	1 III→1	1	3
IV	1 1→V		2 1→XVII 1→XII			2 XVI→1	2 1→XIV	1	5	3
V	1 1→XVII	4 XVII→1 IV→1	3 1→II	1			1 1→XVII	3 XVII→1 X→1	5	8
VI				2						2
VII			1 1→VII	1 VII→1					1	1
VIII								1		1
IX				1 II→1						1
X							1 1→V		1	
XI						1				1
XII	1		1	1 IV→1		1			2	2
XIII			1				1		2	
XIV			2 1→XIV	2 XIV→1	1 1→XIV	2 XIV→1	2 2→II	4 XVII→1 IV→1	5	8
XV		1			1				1	1
XVI					2 1→IV				2	
XVII	4 2→XVII 1→V	6 XVII→2 V→1	9 1→XVII 2→II	3 XVII→1 IV→1	3 1→I		3 1→V 1→XIV	1 V→1	19	10

Results showed that students do not have the intuition in terms of the move from the gradient of the chord to the gradient of the tangent in their numerical work.

Overall, it has been noted that none of the students were able to give the correct answer except one student in ComG2 who missed the question on the pre-test but gave the correct answer on the post-test. Rate of change of the quadratic function proved rather more difficult than the rate of change of the linear function. What seemed evident is that students cannot apply the "difference quotient" formulae to find the rate of change and this might account for the difference in these two questions.

Although 50% of the students in ComG1, ComG2 and N-ComG2 and 20% of the students in N-ComG1 gave a correct answer on the post-test when the rate of change was asked in a linear function setting, some did so by differentiating the function or by thinking that the function is linear so the rate of change must be constant. When the question was asked in a quadratic function setting, none of the students gave a correct answer. Students appeared to fail to establish a connection between the quotient formula and its limiting value.

Item 6(c)

In Table 7.31, the categories of errors for item 6(c) are presented.

Table 7.31. Student errors on item 6(c): The graph below represents $y = 3x^2 + 1$, from x=0 to x=4. What is the rate of change of y at $x = 2\frac{1}{2}$?

LIST OF ERRORS AND THEIR EXAMPLES FOR ITEM 6(C)		
CODING	DESCRIPTIVE LABEL	EXAMPLE
I	Rate of change at a point is the value of the function at this point[5(a).I]	$y = 3x^2 + 1 \Rightarrow y = 19.75$ at $x = 2\frac{1}{2}$
II	Rate of change at a point is the value of $\frac{f(x+h) - f(x)}{h}$ [6(b).XIV]	at $x = 2\frac{1}{2}$ y =19.75 x = 2 y =13 $\therefore \frac{dy}{dx} = \frac{19.75 - 13}{0.5} = 13.5$
III	Rate of change at a point is the value of dy	$\frac{dy}{dx} = 6x$ where $x = 2.5$ dy=15 dx
IV	Rate of change at a point is $\frac{y}{x}$ (incorrect y value)	$\frac{20}{2.5}$
V	Rate of change at a point is the value of the integral at the point (incorrect integration)[6(a).XI]	$\int(3x^2 + 1)dx = 6\frac{x^3}{3} + 1$ $2x^3 + 1 = 2(2\frac{1}{2})^3 + 1$
VI	Rate of change at a point is the value of the derivative of the inverse function (incorrect y value)[6(b).XVI]	$x = 2\frac{1}{2}$ y =20 $x = \sqrt{\left(\frac{y-1}{3}\right)}$ $\frac{dx}{dy} = \frac{1}{\sqrt{3}} \times \frac{1}{2} \times (y-1)^{-½}$ $= 0.56$
VII	Incorrect differentiation	$\frac{dy}{dx} = 6x + 1$ or $\frac{dy}{dx} = 6x^2$
VIII	Unclassified	

The distribution of errors across the various types mentioned above is given for the four groups in Table 7.32.

Table 7.32. Number of students who made each type of error on item 6(c): The graph below represents $y = 3x^2 + 1$, from x=0 to x=4. What is the rate of change of y at $x = 2\frac{1}{2}$?

FREQUENCY OF ERRORS FOR ITEM 6(C)										
	Computer Groups				Non-Computer Groups					
	ComG1 ee=0 ec=0 ce=0 em=1 me=4 (n=35)		ComG2 ee=3 ec=1 ce=4 em=1 me=2 (n=56)		N-ComG1 ee=1 ec=2 ce=1 em=1 me=2 (n=20)		N-ComG2 ee=2 ec=2 ce=0 em=1 me=7 (n=36)		Total ee=6 ec=5 ce=5 em=4 me=15 (n=147)	
acronym	Pre n=1	Post n=4	Pre n=5	Post n=9	Pre n=4	Post n=4	Pre n=5	Post n=9	Pre n=15	Post n=26
I		2	3 2→VIII 1→I	4 c↓2 I→1	2 1↑c 1→I	1 I→1	3 1→IV 1→VII	2	8	9
II				1 c↓1			1 1↑c	2	1	3
III							1 1↑c		1	
IV			1			1 c↓1		1 I→1	1	2
V					1				1	
VI		1								1
VII			1 1↑c					2 I→1	1	2
VIII Unclass	1	1		4 c↓1 I→2	1 1↑c	2		2	2	9

The results showed a success rate of 31% (11 students), 39% (22 students), 20% (4 students), and 25% (9 students) prior to instruction compared to 48% (8+9 students), 38% (10+11 students), 35% (3+4 students), and 42% (7+8 students) respectively for ComG1, ComG2, N-ComG1, and N-ComG2 after instruction. The results also indicated that 29% (10 students), 30% (17 students), 40% (8 students), and 25% (9 students) missed the item 6(c).

The items related to rate of change at a point appeared to be easier than the questions about average rate of change and rate of change as the limiting value of average rate of change. Some students seemed able to handle rate of change at a point.

Interviews

Examples of some responses obtained in the interviews are given below. The first example shows that student S12 knew that for a linear function the rate of change is constant. However, he cannot think that the rate of change is also constant over any interval. This student also had a vague idea about average rate of change and rate of change but he cannot use it in the application. As can be seen rate of change at a point for the quadratic function also caused difficulties because he used $\frac{y}{x}$ as a quotient formula which would be used for a linear function passing through the origin.

1st Episode (N- ComG1):

Inter:	On question 5, when I asked you "what is the rate of change of y as x changes from 3 to 3+h?", you said "6 + 2h" How did you find that?
S12:	I think what I did was, I ..... I'm not sure. It looks like I've integrated it.
Inter:	What about on part (b). When I asked you the rate of change of y at $x = 2\frac{1}{2}$ and at x=X, you said "2" for both points. How did you decide the answer was "2"?
S12:	Well I decided that they were the same because the gradients were the same.
Inter:	If I say what is the rate of change of y as x increases from 3 to 3 + h? How could you answer this?
S12:	The time changes from 3 to 3 + h. I don't know (student gets quiet). I think when I did it I considered numbers rather than with h. I considered what would happen if I did it with numbers. Then I tried to substitute h.
Inter:	Do you know the formula to find gradients if I give you any f(x) function?
S12:	y one minus y equals m times x one minus x (Y1 - Y = m (x1 - x)). Or is it the y = mx + c?
Inter:	In here, for example, when I asked you the average rate of change, on question 6, the average rate of change of y with respect to x. You found "dy/dx is 6x". Do you know what the average rate of change means?
S12	So it is a gradient. I would pick a point on the graph and use the y1-y=m(x1-x).

Inter:	What about rate of change?
S12:	With the rate of change, I suppose the average rate of change, I think I have thought the rate of change as being the rate of change at an individual point whereas the average rate is being the average for the whole curve. For the average rate of change I have to consider what point it is actually from.
Inter:	If this is the case, how could you solve it? if x changes from 2 to 2 + 0.1, or from 2 to 2 + h or from a to a + h.
S12:	I'm not sure.
Inter:	On part 6(c), the rate of change of y at $x = 2\frac{1}{2}$ and you found "$\frac{dy}{dx} = \frac{40}{4}$". How did you get this from the graph?
S12:	Two and a half by .... dy over dx is the gradient and the gradient is the y value over the change in y, which is 40. Tangent (student keeps quiet)
Inter:	But I asked you at $x = 2\frac{1}{2}$?
S12:	So (silence)

The following two episodes show that a conflict situation can cause a student to make his/her answer worse as well as better. In question 5, in order to put students in a conflict situation, it has been asked why the rate of change was "2" or why the answer given on the post-test was different. From the analysis of these two episodes, it appears that student S13 changed his answer from correct to incorrect and student S20 from incorrect to correct. For question 6, S13 could not give an answer. Student S20, however, confused the 'average rate of change' with the 'average between two points' and also gave the incorrect explanation for finding the average between two values.

2nd Episode (N- ComG1):

Inter:	On question 5, I asked you what the rate of change of y was, as x increases from 3 to 3 + h. How could we find the rate of change? (The answer was "6+2h")
S13:	x = 1, y = 2. So y is equal to 2x.
Inter:	What is the rate of change?
S13:	2.
Inter:	Why is it 2?
S13:	Sorry, it isn't 2. It's 2x. It goes up 2x every time. If x is 3, y will be 6. If x is 10, y will be 20.

Inter: On question 6, you haven't written anything. Can you try to explain me how to find the average rate of change of y as x changes from 2 to 2+0.1 or .from 2 to 2 + h or from a to a + h?

S13: Not really no. I've done differentiation before like I said BTEC and I got a distinction at maths, but I did what was required but I obviously did not understand how to use graphs. I don't really have a full understanding of graphs. There was quite a bit to learn, even for this exam. I know I passed it but my differentiation and integration, as you can see, are just terrible. I'll be doing some work this summer.

3rd Episode (N-ComG2):

Inter: What is the rate of change of y as x changes from 3 to 3+h?

S20: I didn't really understand the question because you've got a graph here where the $y = 2x$, so $dy/dx = 2$ so your slope is 2. So whatever your x value is, you are going to have the slope times the x value.

Inter: But why have you written here "rate of increase is 6"?

S20: If you're going from 3, if x was 3, the slope of that point would be 6, wouldn't it?

Inter: What about here. You said dy over dx is 2.

S20: Oh! I see. Yeah. I've made a mistake. If that was 2x that would be f.

Inter: What is the rate of change of y ?

S20: It would be 2 constantly.

Inter: What about part (b). What is the rate of change of y?(the answer was "x=X, rate of change is 2X and x=21/2 rate of change is 5)

S20: It would be the same. It is 2 constant all the way up.

Inter: On question 6

S20: Differentiating that, you get 6x, so you have to substitute the values of x to get the slope at any specific value for x.

Inter: But if I ask you to find the average rate of change between these intervals, how would you work it out?

S20: You would find it for 2, which would be 12 and for 2.1 x 6 and take the average between the two values.

Inter: How will you do that?

S20: Subtract one from the other.

Inter: Subtract one from the other.

S20: And take the mid-point I suppose. I can't think of anything else to do.

Inter: What about rate of change of y with respect to x between these intervals?

S20: Well, that's just a question of substituting those values into.

Students S15 from N-ComG1 and S8 from ComG2 who had given correct answers for all items of questions 5 and 6 except item 6(b) were asked whether 'average rate of change' and 'rate of change' are the same thing. S15 replied,

4th Episode (N-ComG1):

S15: To be honest with you, I have never come across average rate of change before, so I assumed it was the same as rate of change. I don't know the difference.

and student S8 said "I would say they are the same".

Although student S19 from N-ComG2 explained correctly that rate of change is constant for a linear function, the answers he gave to question 5 were not correct. The following episode from the interview highlights two important points:

• students can repeat automatically what they hear in the classroom without forming a mental picture related to what have been said.

• the answer given "$\frac{dy}{dx} = 6x = 6(0.1)$" for 6(a) related to average rate of change between 2 and 2 + 0.1 shows that the word 'change' in ' average rate of change' is misleading. In addition to that the student's answer indicates a very serious misconception: $y(x_2) - y(x_1) = y(x_2 - x_1)$.

5th Episode (N-ComG2):

Inter: Could you explain how you found that the rate of increase of y from 3 to 3+h is "2h"?

S19: Everytime x increases by 1, y increases by 2, so I thought it would be the same all the way up, because it is a straight line graph.

Inter: What is the rate of change? Is it 2 or 2h?

S19: 2h because h is in the x -axis. h is the amount it is increased by. So the factors increase by h, then y would have been increased by 2h because y is twice as big as x.

Inter: What about in here? (pointing out 5(b). The answer given was "2x" for both points).

S19: I don't know, I guessed that one.

Inter: On question 6, part (a). When I asked the average rate of change of y with respect to x as x increases from 2 to 2+0.1 or from 2 to 2 +h you found out (dy/dx)=6x, could you tell me that dy/dx is the average rate of change?

S19: I couldn't remember what the average rate of change is. It is just a rate of change.

Inter: Then how did you get his one? (pointing out the answer given to Q.6(a): "(i) dy/dx = 6x = 6(0.1)=0.6").

S19: It's changing by that much, I just timesed one by the other.

Inter: On part (b), what is the rate of change in these given intervals? how did you get these answers? (the answers given to 6(b) are the same as Q.6(a)).

S19: I used those that I got from the previous part.

Inter: Now do you say average rate of change and rate of change is the same thing?

S19: I know it's not but I couldn't remember which one is which.

Inter: Do you remember any formula?

S19: No.

Student S9 from ComG2 who answered item 6(a) as "12/2.1" (the value of the derivative at x=2 divided by x= 2.1) was asked to explain how he found it. The answer he gave was "I did not understand what was to be done". On the post-test he had answered question 5 and item 6(c) correctly.

Student S10 from ComG2 who substituted the values of x into the equation of y=2x in order to find the rate of change gave the correct answer for 5(a) using quotient formula and for 5(b) using differentiation and substituting the values of x. On the test he gave the correct answer for 6(a) and found the derivative of the function for 6(b). Interview results showed that he did not know the connection between average rate of change and rate of change.

In the interview, student S11 from ComG2 and S3 from ComG1 made the distinction between rate of change of change and average rate of change by saying that " average rate of change is over an interval and rate of change is at a point ". When asked what happens as the interval approaches to zero they said "presumably the average of the two values .... then put them into this equation that is being differentiated, i.e., 6x" or "differentiation of the function and input the value of". It appears that students have great difficulty in understanding the limiting process.

Here, the failure to produce a correct solution appears to result from deficiencies in the student's knowledge. It also became clear that although some students explained the difference verbally between the rate of change and the average rate of change, they cannot apply it in the question.

7.1.6 Question 10: Point of tangency, numerical calculation of gradient, and estimating the value of the function

This question includes three items which required a student: (i) to identify the point in which a line tangent to a curve meets the curve; (ii) to calculate the gradient or slope of the tangent numerically; and (iii) to calculate or estimate the approximate value of the function at a given point (see Appendix A for the graph of the question).

Item 10(a)

The errors identified here are listed below in Table 7.33.

From the table below, three misconceptions were identified:

I. A point at which a tangent line is drawn to a curve is not recognised as a common point for both. (See all errors classified in Table 7.33)
II. Derivative at a point is the derivative function. (See Error II in Table 7.33)
III. Tangent equation is the derivative function. (See Error III in Table 7.33)

Table 7.33. Student errors on item 10(a): Line L is a tangent to the graph of y=f(x) at the point (5, 3). Find the value of f(x) at x = 5.

LIST OF ERRORS AND THEIR EXAMPLES FOR ITEM 10(A)		
CODING	DESCRIPTIVE LABEL	EXAMPLE
I	Correct graph reading but incorrect answer as the tangent equation is incorrect	$m = \frac{5}{3} \Rightarrow y = 1.66 \times 5 + 1$ $y = 9.3 \therefore f(x) = 9.3$
II	Correct graph reading but the value of f(x) at x= 5 is the value of the integral function of the slope at x=5 1(a).VI	$\frac{dy}{dx} = \frac{2}{5} \Rightarrow f(x) = \frac{2}{5}x$ $\therefore$ when x = 5, f(x) = 2
III	Correct graph reading but the value of the integral function of the tangent function at x=5 is the value of f(x) at x= 5 1(a).V	$\int \frac{2}{5}x + 1 = \frac{1}{5}x^2 + x + c$ $\therefore f(x) = x(\frac{1}{5}x + 1)$ at x = 5 f(x) = 10
IV	Unclassified	

Note. 1(A).VI Compare Error II in this table with Error VI in item 1(a).

Note that misconceptions 2 and 3 have already occurred in the previous sections (see section 7.1.1. for similar misconceptions where picture replace concepts). Those students who made Error II also failed to realise that the function they found for the curve shows a straight line.

As can be seen from Table 7.33, most of the students were able to read off the point y at x =5 from the graph in order to be able to find the slope of the tangent equation. However, they were not aware that this was the point they were looking for.

From the evidence given above, it was also conjectured that students were drawn by the need of knowing the equation of f(x). They felt uncomfortable not knowing the equation of the function. In fact, there were also mental process breakdowns, such as failure to appreciate the significance of checking the solution. Thus the solution process was the main aim of the item rather than obtaining the value of y which makes sense. The distribution of the errors in the four groups is given in Table 7.34.

As can be seen from Table 7.34, almost all of the students who made <u>Error I</u> (misconception I) in the post-test had answered the item correctly in the pre-test by reading the point from the graph. The results reported in this table also indicate that even though <u>Error II</u> (misconception II) seemed to be eradicated by the students of the computer groups Error III (misconception III) seemed to be introduced by the ComG1 students. Of the two students who introduced <u>Error III</u> (misconception III) in this item, one already had made this error on the pre-test in item 10(b) and 10(c).

Table 7.34. Number of students who made each type of error on item 10(a): Line L is a tangent to the graph of y=f(x) at the point (5, 3). Find the value of f(x) at x = 5.

FREQUENCY OF ERRORS FOR ITEM 10(A)										
	Computer Groups				Non-Computer Groups					
	ComG1 ee=1 ec=1 ce=3 (n=35)		ComG2 ec=2 ce=1 em=2 me=2 (n=56)		N-ComG1 me=1 (n=20)		N-ComG2 ee=1 ec=1 ce=3 em=2, me=2 (n=36)		Total ee=2 ec=4 ce=7 em=4 me=5 (n=147)	
coding	Pre n=2	Post n=4	Pre n=4	Post n=3	Pre n=0	Post n=1	Pre n=4	Post n=6	Pre n=10	Post n=14
I	1 1→I	2 c^r↓1 I→1		1 c^t↓1			2 1↑c^r	2 c^r↓2	3	5
II	1 1↑c^t		2 1↑c^r						3	
III		2 c^{r1}↓2		1		1				4
IV			2 1↑c^r	1			2 1→IV	4 c^r↓1 IV→1	4	5

<u>Note</u>. ^{r} the correct answer was given by reading the value of y from the graph
^{t}the correct answer was given by substituting the value of x into the tangent equation

Overall, the vast majority of correct responses were based on graph reading. 6(17%) students from ComG1, 17 (30%) students from ComG2, 7 (35%) students from N-ComG1, and 12 (33%) students from N-ComG2 gave the correct answer on both the pre-test and the post-test. Moreover, 6 (17%) from ComG1, 6 (11%) from ComG2, 5 (25%) from N-ComG1, and 4 (11%) from

N-ComG2 missed the item on the pre-test but gave the correct answer on the post-test.

The number of students who missed the item on both the pre-test and the post-test were: 18 (51%) from ComG1, 20(36%) from ComG2, 5(25%) from N-ComG1, and 6(17%) from N-ComG2.

Item 10(b)

Here is a list of the errors, by exemplar within the category, of the students to item 10(b) (see Table 7.35).

Table 7.35. Student errors on item 10(b): Line L is a tangent to the graph of y=f(x) at the point (5, 3). Find the derivative of f(x) at x = 5.

LIST OF ERRORS AND THEIR EXAMPLES FOR ITEM 10(B)		
CODING	DESCRIPTIVE LABEL	EXAMPLE
I	Derivative at a point is the tangent equation	$\frac{dy}{dx}=\frac{2}{5}x+1$
II	Derivative at a point is the value of the tangent at that point	$\frac{dy}{dx}=\frac{2}{5}x+1$ at $x = 5 \Rightarrow \frac{dy}{dx}=3$
III	The slope of the tangent is $\frac{y}{x}$	$\frac{dy}{dx}=\frac{3}{5}$
IV	The slope of the tangent is $\frac{x}{y}$	$m=\frac{5}{3}$
V	The slope of the tangent is $\frac{x_2-x_1}{y_2-y_1}$	$m=\frac{5-0}{3-1}=\frac{5}{2}$
VI	Failure in number substitution	$\frac{dy}{dx}=\frac{y_2-y_1}{x_2-x_1}=\frac{5-1}{3-0}$
VII	Derivative at a point is mx	$y=\frac{2}{5}x+1$ then $\frac{dy}{dx}=\frac{2}{5}x$
VIII	Unclassified	

Errors I and II identified here showed two important misconceptions (see section 7.1.1. for a similar misconception where picture replaces concepts):

I. Derivative at a point is the tangent equation

II. Derivative at a point is the value of the tangent equation at that point.

While Errors III-V were due to failure in the application of the formula $\frac{y_2 - y_1}{x_2 - x_1}$, Error V was due to failure in number substitution. Errors III-IV may also be due to the fact that the students can be more familiar with the tangent lines passes through the origin. Error VII, however, was due to the fact that the slope is part of the tangent equation.

The distribution of the errors above in the four groups is given in Table 7.36.

As noted in Table 7.36, three out of six students displayed the misconception, derivative at a point is the tangent equation, in the post-test but not in the pre-test. This was in contrast to the nine out of eleven students, who overcame the above misconception. It seems that by gathering experience through instruction students can incur some misconceptions as well as overcome them. In addition, it is interesting to note that Error III, the slope of the tangent is $\frac{y}{x}$, occurred only in the post-test and three of the six students who displayed this error in the post-test gave the correct answer in the pre-test.

The number of students who missed the item on both the pre-test and the post-test were: 18 (51%) from ComG1, 18 (32%) from ComG2, 7 (35%) from N-ComG1, and 11 (31%) from N-ComG2.

Table 7.36. Number of students who made each type of error on item 10(b): Line L is a tangent to the graph of y=f(x) at the point (5, 3). Find the derivative of f(x) at x = 5.

FREQUENCY OF ERRORS FOR ITEM 10(B)										
	Computer Groups				Non-Computer Groups					
	ComG1 ee=4 ec=4 ce=2 em=1 me=4 (n=35)		ComG2 ee=0 ec=6 ce=2 em=4 me=3 (n=56)		N-ComG1 ee=0 ec=2 ce=0 em=0 me=5 (n=20)		N-ComG2 ee=1 ec=2 ce=5 em=2 me=6 (n=36)		Total ee=5 ec=14 ce=9 em=7 me=18 (n=147)	
coding	Pre n=9	Post n=10	Pre n=10	Post n=5	Pre n=2	Post n=5	Pre n=5	Post n=12	Pre n=26	Post n=32
I	4 3↑c 1→II	3 c↓1 VI→1	5 4↑c		2* 2↑c	1		2 c↓2	11	6
II		1 I→1	2	1				4* c↓2	2	6
III		2 c↓1		1 c↓1		2		1 c↓1		6
IV	1 1→IV	1 IV→1		1	1*				2	2
V	1	1					2 1→VI	1*	3	2
VI	2 1→VIII 1→I		2 2↑c				1 1↑c	1 V→1	5	1
VII								1		1
VIII	1 1↑c	2 VI→1	1	2 c↓1		2	2 1↑c	3	4	9

* One student appears in both classifications.

<u>Item 10(c)</u>

Here is a list of the errors, by exemplar within the category, of the students to item 10(c) (see Table 7.38).

All errors indicated that <u>students had no intuitive understanding as evidenced by their inability to estimate value on the basis of visual perception.</u> Rough estimation can be a powerful weapon in the struggle against wrong procedural reasoning and would help students to concentrate on product rather than process.

Table 7.38. Student errors on item 10(c): Line L is a tangent to the graph of y=f(x) at the point (5, 3). What is the value of the function f(x) at x = 5.08?

LIST OF ERRORS AND THEIR EXAMPLES FOR ITEM 10(C)		
CODING	DESCRIPTIVE LABEL	EXAMPLE
I	The approximate value of the function is the value of the integral of the tangent equation at 5.08.	$f(x) = \int \frac{2}{5}x + 1dx$ $f(x) = \frac{1}{5}x^2 + x$ if x=5.08 $\Rightarrow f(x) = 6.17$
II	A given incorrect value is much bigger than 3 as the tangent equation is incorrect	$y = \frac{5}{2} \times 5.08 + 1 = 13.5$
III	A given incorrect value is smaller than 3	$f(x) = \frac{2}{5} \times 5.08 = 2.032$
IV	Slope itself	$\frac{2}{5}$
V	Less than the value of the slope	A bit less than $\frac{2}{5}$
VI	Unclassified	

Turning to all responses given to this item: One student in ComG1 and one student in ComG2 were able to estimate the answer "little bit bigger than 3" on the pre-test, but they failed to transfer this ability to the post-test and gave an incorrect answer. On the post-test, one student in ComG2, one student in N-ComG1, and two students in N-ComG2 who missed the item on the pre-test were able to estimate the approximate answer. This is no doubt explained in part by their lack of familiarity with carrying out successive estimations, leading to a more vulnerable unawareness of the numerical-graphical link.

The errors occurred because students did not have the ability to make sense out of a situation by constructing a mental process that questions results. The most important misconception that we can also observe here, is the conception of tangent equation as the derivative function. (see section 7.1.1 for a similar misconception where picture replaces concepts)

The frequency of errors classified in each of the categories in the four groups is shown in Table 7.38. The last column of this table shows the frequency of errors in the categories when the data from the four groups are combined.

As can be seen from Appendix F, there were a lot of missing answers to this item in each group. A number of students did miss the item on both pre-test and post-test: 25 (71%), 38 (68%), 14 (70%), and 23 (64%) for ComG1, ComG2, N-ComG1 and N-ComG2 respectively. If lack of response can be interpreted as an expression of difficulty then this item was problematic for most of the students.

Table 7.38. Number of students who made each type of error on item 10(c): Line L is a tangent to the graph of y=f(x) at the point (5, 3). What is the value of the function f(x) at x = 5.08?

FREQUENCY OF ERRORS FOR ITEM 10(C)										
	Computer Groups				Non-Computer Groups				Total	
	ComG1 ee=3 ec=0 ce=2 em=1 me=0 (n=35)		ComG2 ee=2 ec=2 ce=1 em=2 me=1 (n=56)		N-ComG1 ee=0 ec=1 ce=0 em=1 me=1 (n=20)		N-ComG2 ee=1 ec=1 ce=1 em=1 me=4 (n=36)		Total ee=6 ec=4 ce=4 em=5 me=6 (n=147)	
coding	Pre n=4	Post n=5	Pre n=6	Post n=4	Pre n=2	Post n=1	Pre n=3	Post n=6	Pre n=15	Post n=16
I	2 I→I	3 c↓1 I→1 V→1	1	1				2 c↓1	3	6
II			1		1 1↑c		1 1↑c	1	3	1
III			1 1↑c			1		2	1	3
IV		1 c^e↓1								1
V	1 I→I			2 c^e↓1 VI→1					1	2
VI	1 I→VI	1 VI→1	3 1↑c I→VI I→V	1 VI→1	1		2 I→VI	1 VI→1	7	3

Note: c^e shows correct answer given by estimation

The most striking feature that we can observe in comparing the errors of the students on the pre-test with the correct responses of the same students on the post-test, was the movement from Error II or III to the correct response. However, even those who did give the correct answer may not have been constructing a mental process that questions results.

Another most striking feature that we can observe in comparing errors of the students on the post-test with the correct responses of the same students on the pre-test, is the formation of Error I (tangent equation is the derivative function) in ComG1 and N-ComG2.

The students who gave the correct answer showed that they have not recognised the use of the "difference quotient" formula in calculating an approximate value to the function at a given point. Those who gave the correct response substituted the value of x=5.08 into the tangent equation.

Interviews

Students S4 and S5 from ComG1, S7 from ComG2, S13 from N-ComG1, and S19 from N-ComG2 who had missed the question could not answer any of the questions in the interviews.

S3 from ComG1, S8 and S10 from ComG2, and S16 from N-ComG1 who had missed the question were able to answer it totally or partly. For 10(a), Student S3, S10 and S16 provided the correct answer, y equals to 3 at x = 5, by *reading off the point*. S8, however, could not answer it because she said "we do not know the equation of f". Although student S3, S8 and S10 answered item 10(b) correctly either by *finding the gradient using* $y = mx + c$ or *using the quotient formula*, S16 could not answer it. S3, S10 and S16 could not answer item 10(c).

Student S6 from ComG1 who had answered only item 10(c) incorrectly as "0.4", which is the slope of the tangent equation, could not give any reason for his answer.

Student S2 from ComG1 who had answered only item 10(b) as *the equation of the tangent line* was interviewed. The following episode from the interview exemplifies that the student generalises the derivative at a point as a derivative function. He also thought he could only find the value of the function after finding the equation of it which is the integral of the gradient at a point.

1st Episode (ComG1):

Inter:	On question 10, I have given you a function and a tangent line which is drawn to the graph of the function at coordinate (5, 3). What is the value of f(x) at x = 5?
S2:	It's the integral of the gradient of the line L.
Inter:	Why do you integrate?
S2:	At that specific point, that line L represents the gradient of the curve of y is the function of x.
Inter:	This line shows the gradient?
S2:	Yes, at that particular point.
Inter:	If I ask you what is the derivative of f(x) at x = 5 , what would be your answer then?
S2:	It's the gradient of the line.
Inter:	Could you find the answer?
S2:	If I put the right figures in there.
Inter:	What is the result then?
S2:	It's 2 over 5.
Inter:	Now you say the gradient is 2 over 5. If I want you to find the value of f(x) at 5, how would you find it?
S2:	Yes, we integrate 2 over 5.
Inter:	And if I ask you the value of the function at x = 5.08, how could you find the answer?
S2:	You would have to integrate something.

Student S11 from ComG2 had answered item 10(a) correctly but item 10(b) incorrectly as "5/3". In the interview he realised that the answer was wrong because "the tangent does not cross the origin". This time the answer was "5/2" but it was still incorrect. Item 10(c) was missing and in the interview he said he could not do it because "there is no equation of the graph".

Students S17 and S12 from N-ComG1 who had answered items 10(a) and 10(b) correctly and missed item 10(c) were asked again what is the value of the function f(x) at x = 5.08. Student S17 gave the same answer as student S11 for 10(c) but student S12 answered correctly "What I should have done is to use the y = mx + c and use the gradient that I had found here and just put 5.08 as the x value and 1 as the c value".

7.2 INTEGRATION

7. 2.1 Question 7: Integral

This question contains the two items, which dealt with the student' s ability to define the 'integral' and to explain the meaning of an example of it.

Item 7(a)

The students' errors to item 7(a) is shown in Table 7.39.

Table 7.39. Student errors on item 7(a): What is an integral?

LIST OF ERRORS AND THEIR EXAMPLES FOR ITEM 7(A)		
CODING	DESCRIPTIVE LABEL	EXAMPLE
I	Integral as process	Integral is the opposite of differentiation
II	Integral as reverse of a differential1(a).III	An integral is the reverse of a differential.
III	Integral as an identity/a product/ an equation1(a).II	• Integral is an identity that has been integrated. •An integral is an equation relating to the area beneath the curve
IV	integral as average rate of change	
V	Integral means real numbers	
VI	Unclassified	

Error I seemed to result from a lack of discrimination between integral and integration (see section 7.1.1 where similar explanation is given related to derivative and differentiation).

Error IV was rooted in the use of interval in both concepts either to find the area between two intervals or to find the average rate of change between two intervals.

Error V seemed to stem from a confusion of the integral concept with the integer concept. Such a confusion is caused by the similarity of the words sounds.

The distribution of the errors above in the four groups is given in Table 7.40.

Table 7.40. Number of students who made each type of error on item 7(a): What is integral?

FREQUENCY OF ERRORS FOR ITEM 7(A)										
	Computer Groups				Non-Computer Groups					
	ComG1 ee=8 ec=1 ce= 3 em=0 me=3 (n=35)		ComG2 ee=5 ec=11 ce=3 em=1 me=4 (n=56)		N-ComG1 ee=3 ec=4 ce=2 em=1 me=3 (n=20)		N-ComG2 ee= 7 ec=6 ce=3 em=1 me=1 (n=36)		Total ee=23 ec=22 ce=11 em=3 me=11 (n=147)	
coding	Pre n=9	Post n=14	Pre n=17	Post n=12	Pre n=8	Post n=8	Pre n=14	Post n=11	Pre n=48	Post n=45
I	3 3→I	5 I→3 II→1	9 5↑c 2→I 2→II	5 c↓1 I→2	5 3↑c 2→I	4 c↓1 I→2	7 2↑c 4→I 1→II	6 c↓2 I→4	24	20
II	3 2→II 1→I	6 c↓2 II→2 V→1	2 1↑c 1→II	6 c↓1 II→1 I→2		2 V→1	2 1↑c	2 I→1	7	16
III	1 1→III	2 III→1	2 2↑c						3	2
IV							1 1↑c		1	
V			1 1↑c						1	
VI	2 1↑c 1→II	1 c↓1	3 2↑c	1 c↓1	3 1↑c 1→II	2 c↓1	4 2↑c 2→V	3 c↓1 V→2	11	7

As can be from the table above, the majority of the students made Error I (not being able to distinguish the difference between integral and integration) on both occasion. Eleven of those students made this error on both occasions.

Item 7(b)

The students' errors to item 7(b) is shown in Table 7.41.

Table 7.41. Student errors on item 7(b): Function g(t) gives the number of phone calls in time t. What is meaning of $\int_5^{10} g(t)\,dt$?

LIST OF ERRORS AND THEIR EXAMPLES FOR ITEM 7(B)		
CODING	DESCRIPTIVE LABEL	EXAMPLE
I	Distance	for 5-10 sec what distance will have travelled
II	Time	•This is the time difference between t=10 and t=5 •.. we are finding the total time between this time interval
III	Rate of change[7(a).IV]	Rate of change of phone calls between t=10 and t=5
IV	Volume of calls	this means the sum of the area of the graph between t=10 and t=5. so giving the volume of calls in that time
V	Average calls[7(a).IV]	the average number of calls with respect to time between 5 and 10 units
VI	The value of the function	The value of g(t) in the range of 5-10.
VII	Incorrect intuition on functions	It means t is integrated and then the limits are put in
VIII	Incorrect procedure	g is a function of t and we are asked to differentiate with respect to t between the limits 10 and 5
IX	Unclassified	

Errors I-II and IV seemed to occur because of the students' experiences using integration in different purposes. Students just call the idea they can remember.

Error VII shows that the function notion, say g(x), is seen as being same as gx which is the multiplication of the parameter g by x.

The distribution of the errors above in the four groups is given in Table 7.42.

Table 7.42. Number of students who made each type of error on item 7(b): Function g(t) gives the number of phone calls in time t. What is meaning of $\int_{5}^{10} g(t)\,dt$?

FREQUENCY OF ERRORS FOR ITEM 7(B)										
	Computer Groups				Non-Computer Groups				Total	
	ComG1 ee=0 ec=1 ce=0 em=0 me=1 (n=35)		ComG2 ee=1 ec=6 ce=3 em=0 me=0 (n=56)		N-ComG1 ee=3 ec=1 ce=0 em=1 me=2 (n=20)		N-ComG2 ee=1 ec=4 ce=3 em=1 me=0 (n=36)		ee=25 ec=24 ce=13 em=5 me=12 (n=147)	
coding	Pre n=1	Post n=1	Pre n=8	Post n=4	Pre n=5	Post n=5	Pre n=6	Post n=4	Pre n=20	Post n=14
I		1								1
II	1 1↑c					1	1 1↑c		2	1
III			2 1↑c 1→V	1 c↓1	1				3	1
IV				1 c↓1	1 1↑c				1	1
V				1 III→1			1 1↑c		1	1
VI			2 2↑c				1		3	
VII			2 2↑c	1 c↓1	1 1→VIII	1		3 c↓2 IX→1	3	5
VIII						1 VII→1				1
IX Unclass			2 1↑c		2 2→IX	2 IX→2	3 2↑c 1→VII	1 c↓1	7	3

Some of the students ' correct answers were not the expected correct answer. For example, some of them answered the item as "the area under the curve".

Interviews

Students S19 from N-ComG2, S13 from N-ComG1 and S8 from ComG2 who had defined the integral as "the opposite of differentiation" were asked to give the meaning of 'integration' and 'integral' separately in order to create a conflict. It seems that the students were influenced by the conflict and solved it in the following way: e.g. " Integration is the opposite of differentiation" and "An integral is the opposite of a single differentiation function". Since S2, S4, and S5 from ComG1, and S7 from ComG2 defined the integral correctly as 'the result of integration'or 'opposite of derivative'(although it is questionable whether it is correct), they were asked again the meaning of 'integral' in order to get more information. Their answers were "the area under a graph".

S21 from N-ComG2 showed that he formed the conception of approximating the area under the graph of a function both in the post-test and the interview. The answer he had given was "the area under a curve. Sum of area of small elements (rectangles) under a curve, as their number tends to ∞".

1st Episode (N-ComG2):

Inter: Could you explain a little bit more about what is an integral ?

S21: It's an area under a curve. The integration sign is a summation sign and you assume that there are small elements underneath the graph of width delta x and of height y. The smaller they are the closer they are going to be to a rectangle. That's not actually true. You have an upper limit and a lower limit of the area under a curve and the difference between that upper limit and lower limit will be smaller - the smaller the elements get. So if you make delta x really small, then the upper limit will become really close to the lower limit and you have the area in between.

In item 7(b), two students (S2, S3) from ComG1, three students (S7, S8, S10) from ComG2, two students (S14, S17) from N-Com1 and three students (S19, S20, S21) from N-ComG2 were interviewed. Most of these students who had answered the question as "integrate g(t) with respect to t between the limits 5 and 10" and/or "the area under the graph g(t) between the limits 5 and 10" were able to provide the correct answer "the total number of phone calls between 5 and 10 units of time". Only student S14 who had missed the question and student S21 who had answered the question as "total amount of time spent on the phone between t=5 and t=10" were not able to supply the correct answer. The following two episodes from the interviews exemplify this.

2nd Episode (N-ComG1):

Inter: On question 7(b), I gave you a function which shows the number of phone calls in t time, then the function is integrated between 5 and 10. What will be the result after integration? What will it show us?

S14: It will show you the time. I always understood integration to be the reverse of differentiation, so when you've got your rate to integrate the rate would be just the time again. So you probably just get the time.

3rd Episode (N-ComG2):

Inter: What is the meaning of the integration of g(t)dt between 5 and 10 if g(t) shows the number of phone calls?

S21: If it's the area, it's going to be the number of phone calls multiplied by the time taken so it's going to be the total amount of time spent on phone.

7.2.2 Question 8: Proof of the integral of the sum of two functions is the sum of their integrals

Error analysis has not been done on this question because the answers did not show any errors in need of examination. The types of answers given were categorised while doing scoring (see Chapter Four). Most of them tested the correctness of the proposition on a purely algorithmic level.

Interviews

Students' difficulties with graphical and numerical proofs were reflected in the following justifications. When students were asked how they could prove that the area under the graph of $y = x^2 + x$ equals the sum of the areas under $y = x^2$ and $y = x$, student S7 from ComG2, whose muddled answer was "This means that the area under $y = x^2$ curve and $y = x$ line is equal to the area under $y=x^2 + x$ curve" on the post-test, said,

1st Episode (ComG2):

S7:: You could prove it through integration - integrating the two individual curves as in the area and then integrating $x^2 + x$ and then stating the areas are equal.

Similarly, S21 from N-ComG2, whose answer was " *Area under* $y = x^2 + x = \int_0^a (x^2 + x)dx$ *this is made up of area under* $y = x^2 = \int_0^a x^2 dx$ *&* $y = x = \int_0^a x\,dx$ " on the post-test, said,

2nd Episode (N-ComG2):

S21: I don't know really. It looks like the area under this curve here, you add the area under here onto it. You would get something pretty close to that area there. But other than that, I don't know.

It seems that students' difficulties with this question stem from their lack of experience with proofs and proof-based theories. They may be inexperienced in knowing what kinds of arguments are acceptable as a proof.

7.2.3 Question 9: The area 'under' a curve

Error analysis has not been done on this question because the types of answers categorised while doing scoring (see Chapter Four) were quite enough to show the variety of responses.

<u>**Interviews**</u>

For item 9(a), student S8 from ComG2 who left the variable of the integration 'x' was asked "What is the variable of the integration?" and she answered correctly. S16 from N-ComG1 who gave an unclassified answer was able to form the integration formula in the interview.

For item 9(b), student S10 from ComG2 who had missed the item was asked to calculate the area between 0 and 3. In the interview he said that the integral of $2x - x^2$ between 0 and 3 would give the answer. S5 from ComG1 and S15 from N- ComG1 who also missed the item stated correctly that the area should be worked out separately between 0 and 2, and between 2 and 3. S15 proceeded to calculate the area and obtained the correct answer 'zero'. Student S14 from N-ComG1 who missed the item was asked why she did not do anything on this item. Her answer was:

1st Episode (N-ComG1):

S14: It wouldn't be the area under the curve. You could work out that area I suppose, but this area would have to be something else. You would have to work it out in a different fashion.

Inter: You mean, we can't work out the area between 0 and 2, but we can work out the area between 2 and 3?

S14: Apparently integration is supposed to be the area under the curve. So there's the curve, so underneath it would be here wouldn't it? (pointing out the area under the x-axis)

Inter: You mean this is the curve?

S14: This is the curve but underneath it is just here and this is above the curve, so I don't know. Maybe you would have to find the area between 1 and 2 as a rectangle and take away the area under the curve between those two points and then you could find out what that one was. I probably thought it was too difficult.

The above example shows that the student takes the x-axis as a base while thinking whether the area is under the curve or above the curve.

Student S6 from ComG1 who had answered that the area " cannot be worked out as shaded area is on both sides of the graph" was able to give the correct answer working out two separate areas between 0 and 2, and between 2 and 3. Student S12 from N-ComG1 who found the two areas separately said " both are exactly equal $\frac{4}{3}$ each. therefore total $=\frac{8}{3}$ unit^2". In the interview the student's answer to the question, "Can't we get zero area?", was "No, not with an integral because you want to find that area plus that area".

S13 from N-ComG1 who used 'y' as the variable of integration was asked why the integration is with respect to 'y'. The answer given was "because y equals to $2x - x^2$". S16 from N-ComG1 who gave an unclassified answer was able to form the integration formula correctly.

For 9(c), S16 from N-ComG1, and S18 and S19 from N-ComG2 thought that the integration is not possible. An example of the answers given was "it doesn't pass through any point on this line, so you don't have a value of x. If it passes through one of the x or y, so you can pick up the points ". This explanation shows that students need to work on 'improper integral'. An improper integral is "an integral for which the interval (or region) of integration is not bounded, or the integrand is not bounded, or neither is bounded"(James &James, 1992, p.223).

For 9(d), S10 from ComG2 who had answered that the area cannot be calculated because "for every value of x there are two values of y. Also the integral must be between the x axis and the curve" made the following remarks in the interview. The answer given above shows that the student only considers the equation of a standard function in the form $y = f(x)$; where x is the independent variable and y is the dependent variable. In the case of $x = 2y - y^2$, the student did not realize that the equation of the function having the form $x = f(y)$; x is the dependent variable and y is the independent variable. Therefore he thought $x = 2y - y^2$ not a function and integration not possible.

1st Episode (ComG2)

S10: I suppose looking at it afterwards that you can integrate with respect to y, so in effect you could find out the area enclosed within that curve. That would be a better definition of it. Just by integrating the $2y - y^2$ with respect to y by limits 0 and 2.

Inter: Why did you think this way before?(pointing out the answer given previously)

S10: I think I put into my head that it had to be between the x axis - that you had to integrate with respect to x. Obviously you can differentiate or integrate with respect to the y. So obviously if that's the case, then you would find the area between the y axis and the curve as opposed to the x axis and the curve.

Students S4 and S5 from ComG1, S17 from N-ComG1, and S18 from N-ComG2 who missed item 9(d) were able to form the correct integration formula: $\int_0^2 (2y-y^2)dy$. S19 from N-ComG2 who also missed the item tried to make y as a function of x but then he said "I am not sure". S16 from N-ComG1 who gave an unclassified answer was able to form the correct integration formula.

S6 from ComG1said that item 9(d) "cannot be done". In interview he said that it was possible to integrate when he turned it around 90 degrees rather than this part being on the y axis, it will then be on the x axis. The students who think this way do not realize that the area will be the same but $x = 2y - y^2$ will not be $y = 2x - x^2$. It will be either $y = 2x + x^2$ or $y = -2x - x^2$.

7.3 Summary

This chapter has reported the qualitative analysis results in which the data taken from the pre-test and the post-test as well as interviews have been examined.

The detailed categories of errors in each item provided a clear connection between items. For example, the error "derivative as small change" in item 1(a) also occurred in items 2, 4(c), 4(f) in a similar form. The results also showed that there was a consistency in errors made by each group of students.

It seems that the use of computers has advantages with the items related to graphs. More students from computer groups corrected their pre-test errors in items 2 and 3. Those items were concerned with: (i) drawing the graph of the derivative function by looking at the graph of a function, and (ii) recognising the graph of a function by looking at its derivative function.

For the benefit of other researchers who may want to use the diagnostic test, here are some of suggestions for changes to items 6(b) and 7(b). Both items currently have problems in their wordings and meanings.

The purpose of asking item 6(b) was to examine students' understanding of the relationship between average rate of change and rate of change. In redrafting the test, item 6(b) could be asked in the following revised form:

- Show how you can use your answer to 6(a) to find the rate of change of y with respect to x at (i) $x = 2$ (ii) $x = a$.

Item 7(b) does not refer to a realistic situation and also does not include the word "rate". Therefore, it would be better posed as follows:

- If g(t) shows the rate at which water is coming out of the tap in litres per minute. What is the meaning of $\int_{5}^{10} g(t)\,dt$?

I suspect that students will make similar kinds of errors which were classified in items 6(b) and 7(b) even if they are asked in the clear form suggested here. For example, the interview results suggest that students do not know the relationship between the average rate of change and rate of change.

CHAPTER EIGHT

DISCUSSION AND CONCLUSION

The main purpose of this study was to evaluate the impact of computers on the learning of calculus by comparing diagnostic test results for computer and non-computer groups in 'realistic' classroom environments. In order to investigate this issue, the data were analysed quantitatively and qualitatively. In this study two computer groups and two non-computer groups were involved.

Computer group 1 (ComG1) students used the *Graphic Calculus* package in addition to the lectures. Computer group 2 (ComG2) students used two tutorial packages called *CALM* and *CALMAT* besides lectures. In addition to that they did a computer test at the outset of the course in order to measure their weak points. According to the results of this test students followed the units in which they are weak. The *Graphic Calculus* package was also used by the teacher as a demonstration in the classroom. Computers were mainly used by students in the computer laboratories.

The following questions have been taken into consideration while undertaking this study.

1. Are there significant differences between the four groups for each item and question of the diagnostic test?
2. Do the treatments affect A-level students differently with regard to their performance?

3. Is there an interaction between mathematics background and treatment?

In addition it sought to provide answers to the following questions.

4. Is there any difference between the two computer groups in terms of their attitudes to the use of computers in calculus?
5. To what extent and in what ways can attitudes towards computers help in the development of calculus concepts?

The questions above were mainly analysed statistically using analysis of variance. In addition, the study also focused on the following questions:

6. What are the errors that occur with each item?
7. What kind of structure do errors form?
8. Are these patterns of errors associated with the four groups?

This chapter summarizes both quantitative and qualitative results, and draws implications from both types of results.

8.1 Discussion of the Statistical Analyses and Findings

The results of this study must be viewed in the light of its limitations as well as the richness of its data base. The non-computer groups provided a backdrop rather than a tight experimental control. Note that each group has been taught by a different teacher.

In the diagnostic test some of the questions comprised a variety of items based on a single mathematical situation and they therefore incorporated a number of abilities and concepts which needed to be looked at separately. These same skills and concepts may also have appeared in several different items. Each item was analysed by means of factor analysis and correlation coefficients. The factors obtained confirmed that each question together with its items refers to a single mathematical situation, with one or two exceptional cases, although each item might refer to different abilities and concepts. For example, the results show that question 4, associated with symbols, was subdivided into two factors, one reflecting symbols of ' δ ', and the other being associated with the understanding of the symbols of 'd'. Questions 2 and 3 were loaded on one factor.

8.1.1 The Changes of the Pre-test and the Post-test Scores

In this section the first three research questions given above are discussed.

The Effect of Computer and Non-Computer Environments on Students' Performance

The results of this study showed that both computer groups performed better on questions 2 and 3, compared to non-computer groups. The results taken from the other items showed a different picture. The summary table for the analysis results is given in Table 8.1.

The results of this study indicated that students of the both computer groups showed a significant difference on question 2, compared to non-computer groups. This question was concerned with finding the graph of a derivative function by looking at the graph of a function.

The table below did not show a significant difference on question 3 between the computer and the non-computer groups. But the residual mean and the percentages of students responses according to the scoring criteria (see Appendix F) showed that computer groups did much more better than non-computer groups. Question 3 was concerned with recognising the graph of a function by looking at its derivative graph. Computer group students gave much more detailed explanations about their reason for choosing the correct graph. The same result was also find for question 2.

When questions 2 and 3 are combined, computer groups either sepaerately or in combined form did significantly better than non-computer groups.

Table 8.1. Summary of ANOVA results between the pre-test and the post-test scores

(Pairs of means that are significantly different are not underlined by the same line)

Items	ANOVA Results (F-values)	Sheffe-test Results (means in terms of an order from highest to lowest)
1(a)	0.18 n.s	
1(b)	0.37 n.s	
2	3.65*	ComG2, ComG1, N-ComG2, N-ComG1
3	1.17 n.s	
4(a)	6.06***	ComG2, N-ComG2, N-ComG1, ComG1
4(b)	4.50**	ComG2, N-ComG1, N-ComG2, ComG1
4(c)	6.83***	ComG2, N-ComG1, N-ComG2, ComG1
4(d)	2.31 n.s	
4(e)	2.03 n.s	
4(f)	1.75 n.s	
4(g)	1.49 n.s	
5(a)	0.44 n.s	
5(b)	2.69*	ComG2, N-ComG2, N-ComG1, ComG1

Table 8.1 continued

Items	F-values	Sheffe-test Results (means in terms of an order from highest to lowest)
6(a)	1.27 n.s	
6(b)	1.34 n.s	
6(c)	1.72 n.s	
7(a)	1.43 n.s	
7(b)	1.29 n.s	
8	2.55 n.s	
9(a)	0.95 n.s	
9(b)	1.06 n.s	
9(c)	0.52 n.s	
9(d)	0.14 n.s	
10(a)	1.10 n.s	
10(b)	4.37 **	
10(c)	1.99 n.s	

Note. *$p < 0.05$, ** $p < 0.01$, *** $p < 0.001$, n.s : not significant, df= (3, 143).

The finding of better performance by the computer groups over the non-computer groups on these questions suggests that this success can be attributed to the use of computers by students. There seems to be a tendency of computer groups to score somewhat higher on these kind of items (see Tall, 1986). Students seem to learn how to draw inferences from a graph and explain it.

But how might this pattern of performance be explained? Although all of these items in the test are concerned with 'differentiation' and 'integration', they depend, in fact, on very different competences. These two items or questions were concerned with drawing inferences based on the displayed graph and identifing the relationships between the graph of a derivative function and the graph of its integral function. Here, there are two clusters of factors which are likely to have contributed to the better performance of the computer groups. The first cluster relates to the use of computers over an extended period of time. Moreover, reliable access to computers is likely to encourage students and teachers to make more use of graphic approaches in solving problems and developing new mathematical ideas, rather than just strengthening these specific relationships, but rehearsing more general relationships between graphical forms. The second cluster of factors relates to the availability of computers when carrying out graphical tasks. First and foremost, this improves the quality of information avaliable to students, enabling a graphical investigation.

On item 4(a), 4 (b), 4(c), 5(b) and question 4 as a whole ComG2 students performed significantly better than ComG1 students. On item 4(c) and the question 4 as a whole N-ComG1 students also performed significantly better than ComG1 students. Item 4(a), 4(b) and 4(c) dealt with explaining the meaning of "the δx in $\frac{\delta y}{\delta x}$ and in $\sum f(x)\delta x$", "δy" and "$\frac{\delta y}{\delta x}$" and item 5(b) was concerned with the rate of change at a point for a linear function. The better performance of ComG2 students, as opposed to the ComG1 students might be due to the effect of the software used by ComG2.

On item 10(b), numerical calculation of a gradient, there was a significant difference between groups. But the Sheffe test did not show any significant difference taking two groups at a time. Nevertheless, the means show that N-ComG1 students performed better than the other groups.

Answers to other items in the post-test compared to the pre-test, did not show a significant improvement in the computer groups as compared with the non-computer groups. Other significant differences seemed to result from particular aspects of the teaching, rather than computer use specifically. These results were not completely unexpected. Symbols related to 'd', average rate of change and rate of change, proof of the integral of some of two functions equals the sum of their integral, and estimating the value of a function are typically difficult (Amit &Vinner, 1990; Heid, 1988; Orton, 1980) and receive minimal attention in many calculus classrooms. Note that in all of these items a limiting process is involved. This confirms the necessity and mastery of the limit concept involved. This result also show that computers did not work in helping students to understand the limiting process involved in the softwares.

Derivative, integral, the area under the graph of a function, the point of tangency were relatively easy compared to the concepts above.

It is perhaps worth noting that the teachers of the computer groups followed fairly traditional approaches to their teaching of calculus. Thus the experience of the computer group students here may have been little different from that of the non-computer groups. Mastery of the algorithmic skills still forms the basis for courses in calculus.

The Effects of Mathematics Background

The findings suggest that first-year engineering students who have studied A-level mathematics are mainly more successful in each group than those who have not studied A-level.

A-level students showed an improvement, compared to non-A-level students, on items related to 'd' symbols, average rate of change and rate of change for a quadratic function, integral, proof of the integral of some of two functions equals the sum of their integral, and the area under the graph of $x^2 - 2x + 2$ between -1 and 1. But it was not as much as expected. It is likely that these concepts are difficult or easy for both group of students. It seems that integral and the area under the graph of $x^2 - 2x + 2$ between -1 and 1, are much easier for students when compared with other concepts above. Because students can easily remember at least some part of the definition given for a concept in the classroom and also in most of the classes, problems such as the area under the graph of $x^2 - 2x + 2$ between -1 and 1 are studied.

With the other items, A-level students scored mainly higher than non-A-level students.

Given that A-level students were already ahead of the others at the pre-test, it must be concluded that the teaching has the effect of widening the gap in understanding between least able and most able.

Treatments in each group improved understanding much more so for A-level students rather than non-A-level students. There could be several reasons for this. One reason might be that syllabus taught in A-level is quite similar to the one taught in these calculus courses. Therefore, A-level students have been already introduced the ideas of calculus.

8.1.2 The Effects of Different Computer Environments on Students' Attitudes towards Computers and on their Performance

As mentioned above, students in the two computer groups worked on different software. The answers given to the computer attitude questionnaire showed some varition between those groups.

ComG1 students mainly worked with one other student or with a group of students which was approximately 65% of the students. The remainder worked alone. In contrast, ComG2 students worked on at their own pace which was approximately 62% of the students. This is not surprising because the software *Graphical Calculus* is much more open and investigatory compared to the two tutorial packages (CALM and CALMAT). Maybe students who use software which is exploratory need to discuss the things happening on the computer with their friends.

ComG1 students emphasised that computers were helpful because of the visualization of certain concepts graphically. In contrast, ComG2 students emphasised that computers were helpful because of the availability of a lot of examples and the possibility of repeating things as well as the ease of working at their own pace without feeling pressurised.

Students answers to the question " What changes might you suggest for the computer part of this course when it is run again?" have been categorised under four headings: computer program, course, task, and none. While ComG1 students suggested changes for "task", ComG2 students suggested changes for "Computer program". This may be because ComG1 students made much greater use of printed material when working on the computer. A common answer from both groups of students was, to have more easy questions or less confusion.

Students answers to the question " What is the biggest worry affecting your work in calculus" showed that computers are not the students' biggest worry in calculus. This may be because most of the students already had some experience with computers either at home, at school, or both. It appears that their biggest worries relate to their mathematics background and their perceived ability. Sometimes it is difficult to seperate those two factors.

The results also showed there is a correlation between the interest and involvement of ComG1 students with computers (time spent in using computers) and their background of computer use at school, in a lesson, and the period of computer use. As mentioned above, the *Graphical Calculus* package is quite open to investigatory work. Thus, students who have more experience on a computer are much more willing to spend time on it.

In both computer groups students' performance on items 1(a), 4(g), and 7(a) are correlated positively with their perceptions of the use of computers in calculus. This is difficult to give a simple explanation for it.

8.2 Discussion of the Qualitative Analyses and Findings

This section summarizes the results from the last three research questions (see beginning of this chapter). Here the last two of those are mainly focused upon. The first question has been studied in depth in a previous chapter.

The results showed that most of the errors appear to result from "faulty bonds" which underlie them. Similar error patterns were found across different items. Although errors appear to closely relate to subject-matter, the fact of their appearance seems to be relatively independent of the curriculum and teaching methods. The errors found are in close agreement to the findings of Orton (1980). However, the study here consists of four issues under each concept: (i) illustrations of the errors in the various categories and explaining their underlying reasons; (ii) the distribution of errors by groups; (iii) detection of patterns in each student's pre-test and post-test responses; and (iv) detection of patterns by items.

With the error analysis, the origins and the nature of errors as well as many misconceptions came to light, indicating the difficulty most students have in reaching a full understanding of precisely and analytically defined abstract concepts.

Errors seemed to result from systematic misunderstandings that have sensible origins. These are the distortions in forming a proper concept image. Many errors made by students seemed to be due to the assimilation (Piaget's term) of a new concept to an already familiar one.

To illustrate this, let's think of the new concept and the already known one as two variables. Students think that these variables are equal but in fact they are not equal. We need to show them they are not equal. The following table shows the concepts which cannot be generalized mathematically as equal but which seem equal in terms of the students' point of view. The concepts classified under the existing cognitive structure seem to interfere with the acquired cognitive structure or proper learning.

Students seem to think these inequalities as equal because of:

I. the inappropriate extension of a specific case to a general case: For example, the derivative at a point is thought of as the same as the derivative function. Average rate of change is thought as the same as rate of change.

II. the lack of discrimination of concepts which occur in the same context: For example, derivative, differential and differentiation.

III. the lack of ability to distinguish the meanings of words or symbols in one context and not being able to understand what they are referring to: For example, "inverse" in inverse of differentiation means that integration is the inverse of differentiation but it does not mean that the integral is the inverse of the derivative.

IV. the lack of understanding of a graphical representation: Students seem to remember only what they see without understanding the message which is supposed to be given. For example, the tangent equation is thought as the same as the derivative.

V. the inappropriate connection of a word with another one: For example, negative and decrease.

VI. the lack of understanding of the meaning of a concept in a context and inappropriate connections made with its literal meaning: For example, the meaning of turning point in mathematics is taken to mean "a point at which the curve moves in a different direction".

Table 8.2. Concepts that students incorrectly treat as equivalent

Existed concepts	=	Acquired (new) concepts
Formula, identity, value, number, term		Function
a point at which the curve starts to move in a different direction		a point on a curve at which the ordinates of the curve cease to increase and begin to decrease, or vice versa (turning point)
upside-down function		the function which has an odd power
Negative function		Decreasing function
Function with a greatest power		Steepness
Change		Rate of change or average rate of change
Derivative at a point		Derivative function
Tangent		Derivative function
Differention		Derivative or Differential
Differential		Derivative
Small change		limiting value
Average		Average rate of change
Average rate of change		Rate of change
'δ'		'∂'
'δ' or 'd'		derivative or rate of change
δx		dx
δy		dy
$\frac{\delta y}{\delta x}$		$\frac{dy}{dx}$
Integration		Integral
Inverse of differentiation		Inverse of derivative

Error analysis made it clear that the pattern of errors across items was remarkably consistent regardless of the subject group and context being analysed. Some students, however, were not consistent in their own errors from the pre-test to the post-test. This may be because they do not have a well established general view. Students seem to activate a limited portion of the concept image at a particular time (Tall & Vinner, 1981).

The diagnostic test results together with the interview results showed that: (i) students did not understand the role that an inflexion point plays in graphical representations; (ii) students failed to understand that the rate of change is the limit of the average rate of change as the given interval approaches zero. It seems that this lack of understanding of the notion of limit prevents students from understanding related concepts. It is also clear that students need to be made familiar with using "the average rate of change" and "the rate of change" in calculus courses. Otherwise students would not be able to solve most of the application problems, which require a familiarity with the rate of change. In other words, students need to be familiar with the rate of change as well as the gradient, and to be able to work, on both concepts. Neither of the two will suffice on its own.

It seems that while students in ComG2 confuse with "the average rate of change" and "average", students in ComG1 confuse with "the average rate of change" and "change". This might be due to the different use of computers. For example, ComG2 worked on some questions relating to the concept of average.

With the use of computers, students seem to less likely to try to discover the formula and therefore get further into interpretation of the graph. Moreover, they start focusing on more than one factor related to the graph.

It appears that students who solve an unfamiliar version of a problem rely on their conceptual knowledge rather than on a rote execution of the solving procedure. For example, the students showed serious misconceptions even on a simple item such as 10(a) (point of tangency). Previous studies have also indicated this to be the case (Greeno, 1978; Novak, 1977b; Novak, Gowin, & Johansen, 1983).

A major implication of this study is that the justified errors and their underlying reasons can be of great help to the teacher in learning more about the actual depth of understanding of his or her students and in confronting their misconceptions by bringing them to the level of consciousness. This might be done by the computer technology as well as a teacher.

8.3 Suggestion for Further Research/Recommendations

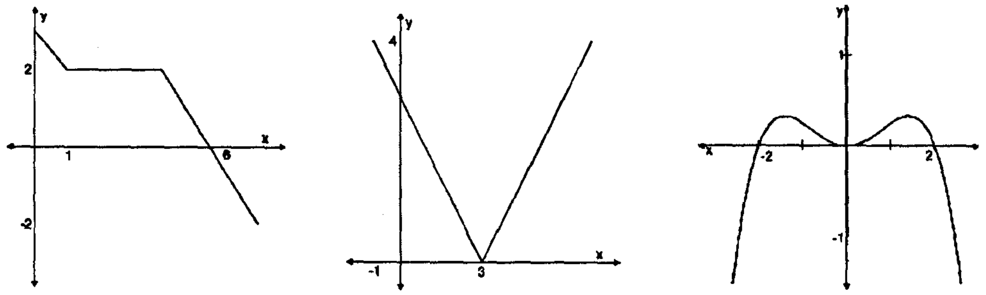
Recommendations based on these results are that more computer sessions should be developed for use in understanding different calculus concepts. Based on the results reported here and on the results reported by Heid (1988), Kulik et al (1980), Skinner (1988), adult students favour computer instruction in combination with teachers. It seems that there is a need for a teacher in order to create an investigatory environment for his/her students while they are working on the computer.

Because of the methodological problems in the study, the interviews did not cover all the interpretations of errors made on the diagnostic test. But for further research, students should be invited to take part in more detailed individual interviews in order to either support or reject the claims brought out in Chapter Seven. For example, interviewing is required in order to investigate the visual estimation or the meaning of inverse function. The explanations given for each error could also be taken as an hypothesis.

In order to gain more evidence on the students problems with in calculus concepts the following questions are offered for further research:

- Could you define the meaning of derivative, differentiation, and differential?
- What is a derivative? and what is a derivative at a point?

- Please show the turning points of the graphs given below, where possible.

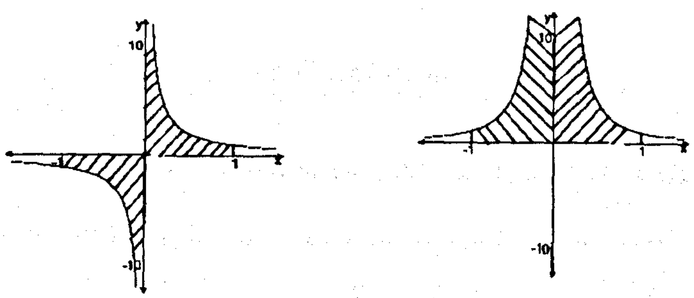


- What is the meaning of $\delta x, \delta y, dx, dy$? Give an explanation graphically and verbally.

- Could you define the meaning of integral and integration?

- Calculate the shaded areas below if it is possible.

A) $y = \frac{1}{x}$ B) $y = \frac{1}{x^2}$



- In question 10, the graph was not used as a frame of reference in estimating the value of the function at x = 5.08 and finding the value of the function at x=5. In order to make students more aware of the quality of their answers, the following item could be added:

Could you show the value you found on the graph?

It is important to use the type of questions mentioned here in the instruction process as well as in further research in order to bring students' misconceptions into their own level of consciousness. The test developed here would be of value to researchers, teachers, and students.

This research could also be carried out in different computer environments such as *Derive*, *Graphic Calculators*, and *Spreadsheets*.

8.4 Limitations of the Study

The main difficulty that could be regarded as a limitation in this study was the impossibility of observing students while using computers. This study is also limited as it draws data from just four teachers.

Two items of the test 6(b) and 7(b) were problematic. Both items have problems in their wordings and meanings. This is discussed in section 7.3 and some suggestions are made for revising these two questions.

8.5 Conclusion

The purpose of this research was to evaluate the impact of computers on first year engineering students' learning of calculus concepts. In order to provide meaningful information about the process of learning, quantitative and qualitative approaches have been combined throughout this study.

Three primary data sources (conceptual comparison diagnostic test results, interview transcripts, and computer attitude questionnaire) were used in order to evaluate the impact of computers on students' learning of calculus.

In this research, three implementation issues related to the use of computers in the teaching and learning of calculus have been examined. These were, (1) the performance of first year engineering students, with access to computers, on items in the diagnostic test, in comparison with that of students of similar background, but without access to computers, (2) students' interest and involvement with the computers and perceived educational value, and (3) the effects of students' interest and involvement with a computer and their perceptions of the use of computers in calculus on their performance.

Conclusion were drawn from the findings related to the use of computers: the availability of computers seemed to encourage students to explore relationships between the graph of a derivative function and the graph of its integral function. Not only statistical analysis, but also the scrutiny (careful and thorough examination) of the students' responses given to the diagnostic test items in the written forms point to the conclusion that there was much more improvement on items concerning drawing the derivative graph or recognising the integral graph in computer groups than non-computer groups.

This study provides a strong evidence of such influence, both on the mathematical performance of the students and on the mathematical approaches that they employ while exploring relationships between the graph of a function, and the graph of its derivative and integral functions.

Findings from this study suggest that the use of computers in calculus teaching has promise both for engendering restructuring of students' existing knowledge and for changing the contents of their knowledge in graphical interpretations.

Appendix A

Diagnostic test used in the second pilot study and the main study

Name and Surname: Sex: F / M

University: Age:

Department: Nationality:

Mathematics Qualifications – Please tick the blank which applies to you and write down your grade.

( ) A-level............ ( ) BTEC.............. ( ) Others ....

Read the following carefully:

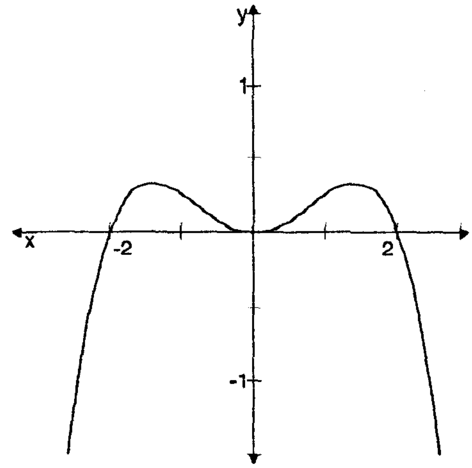
1. The purpose of this test is to gather information about how engineering students think about the mathematical ideas of differentiation and integration.
2. Please try to answer all questions and work as carefully as possible.
3. Your answers will be treated as confidential and will not be used to grade you.

DIAGNOSTIC TEST

1. a) What is the meaning of a derivative? Define or explain as you wish.

b) What does it mean that the derivative of $f(x) = x^3$ is $3x^2$?

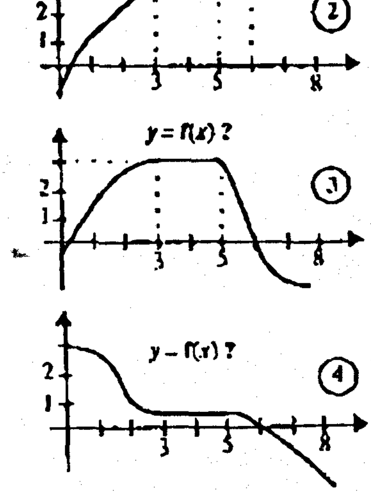
2. The graph at the right is the graph of a function $y = f(x)$.



Sketch what the derivative looks like.

Give the reason(s) for your answer.

3. Graph **1** is the derivative $y = f'(x)$ of a function $y = f(x)$ defined for $0 \le x \le 8$.



Which of the graphs **2**, **3**, **4** could be the original graph $y = f(x)$?

Give reason(s) for your choice.

4. Explain the meaning of each of the following symbols.

a) Explain the δx in $\frac{\delta y}{\delta x}$ and in $\sum f(x)\delta x$.

b) Explain δy

c) Explain $\frac{\delta y}{\delta x}$

d) Explain the dx in $\frac{d}{dx}(x^2)$ and in $\int x^2 dx$

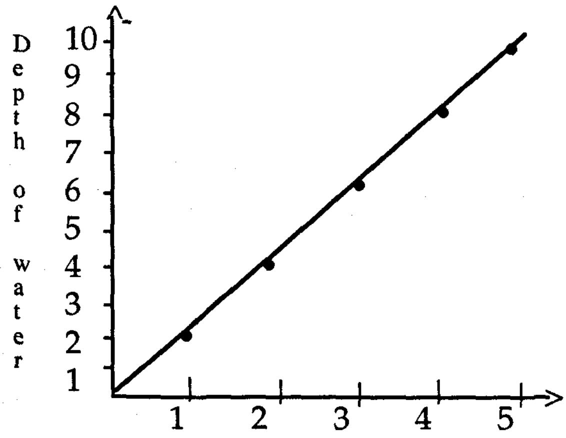
e) Explain dy

f) Explain $\frac{dy}{dx}$

g) What is the relationship between $\frac{\delta y}{\delta x}$ and $\frac{dy}{dx}$?

5. Water is flowing into a tank at a constant rate. For each unit increase in the time, the depth of water increases by 2 units. The table and graph illustrate this situation.

Time (x)	0		1		2		3		4		5
Depth (y)	0		2		4		6		8		10
1st difference (depth)		2		2		2		2		2	

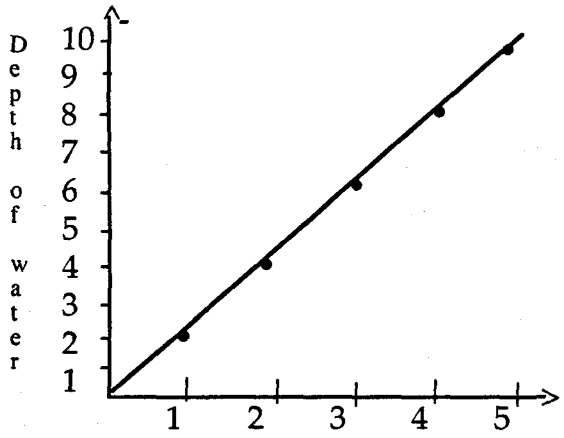


a) What is the rate of increase of y as x increases from 3 to 3+h ?

b) What is the rate of change of y at $x = 2\frac{1}{2}$ and at $x = X$?

5. Water is flowing into a tank at a constant rate. For each unit increase in the time, the depth of water increases by 2 units. The table and graph illustrate this situation.

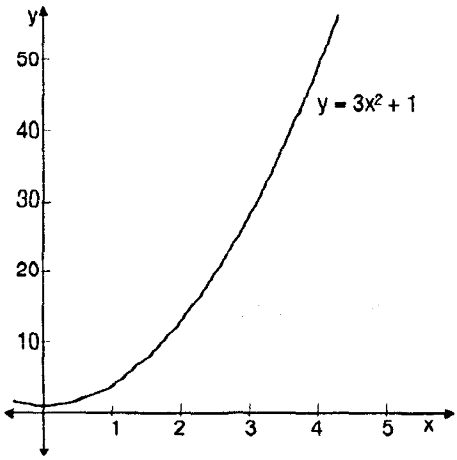
Time (x)	0		1		2		3		4		5
Depth (y)	0		2		4		6		8		10
1st difference (depth)		2		2		2		2		2	



a) What is the rate of increase of y as x increases from 3 to 3+h ?

b) What is the rate of change of y at $x = 2\frac{1}{2}$ and at x = X?

6. The graph below represents $y = 3x^2 + 1$, from $x=0$ to $x=4$.



a) What are the ratios of changes (average rate of change) of y with respect to x as x changes from: (i) 2 to $2 + 0.1$, (ii) 2 to $2 + h$, (iii) a to $a + h$?

b) What are the rates of change of y with respect to x as x changes from: (i) 2 to $2 + 0.1$, (ii) 2 to $2 + h$, (iii) a to $a + h$?

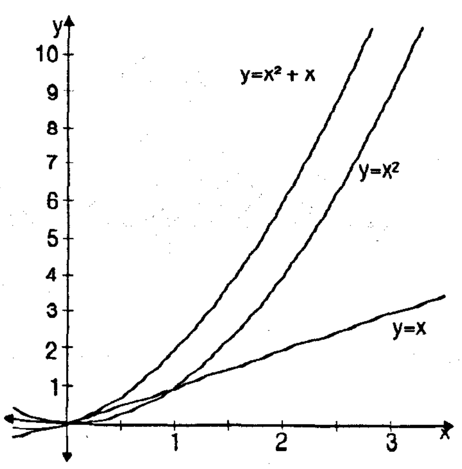
c) What is the rate of change of y at $x=2\frac{1}{2}$?

7. a) What is an integral? Define and explain as you wish.

b) Function g(t) gives the number of phone calls in time t

What is the meaning of $\int_{5}^{10} g(t)\,dt$?

8.

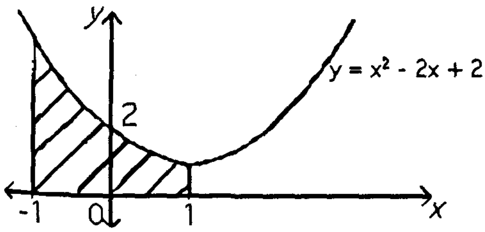


Explain, by means of the diagram or otherwise, why

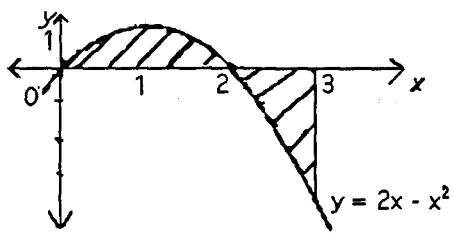
$$\int_0^a (x^2 + x)\,dx = \int_0^a x^2 dx + \int_0^a x\,dx$$

9. Calculate the shaded areas below, where possible. If it is not possible, explain why not.

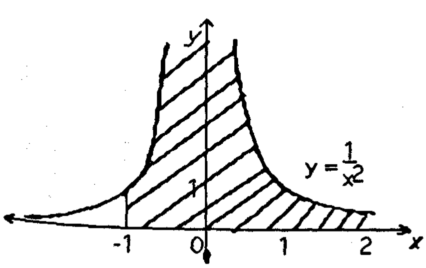
(a)



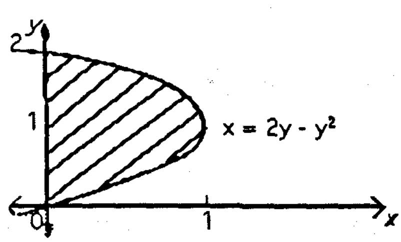
(b)



(c)



(d)

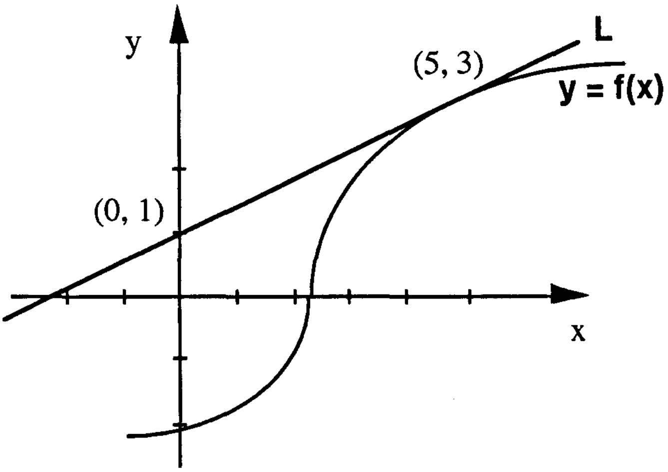


10. Line L is a tangent to the graph of y = f(x) at the point (5, 3).

a) Find the value of f(x) at x = 5.

b) Find the derivative of f(x) at x = 5.

c) What is the value of the function f(x) at x = 5.08? (Be as accurate as possible).



Appendix B

Diagnostic test used in the first pilot study

College:

Name and Surname:

Sex: F / M

Age:

DIAGNOSTIC TEST

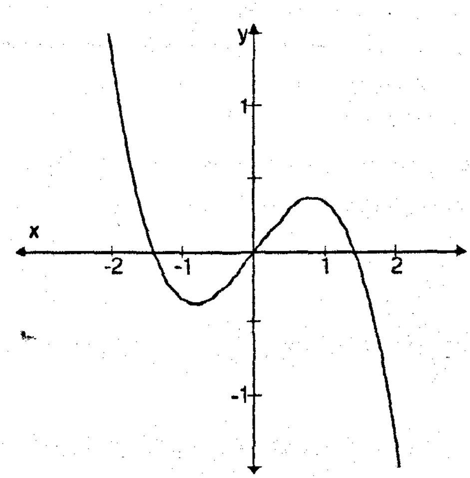
The purpose of this test is to gather information concerning differentiation and integration.

1. a) What is a 'derivative'? Define or explain as you wish.

   b) What does it mean that the derivative of $f(x) = x^3$ is $3x^2$?

2. The graph at the right is the derivative graph $y = f'(x)$ of a function $y = f(x)$.

   Draw the original graph $y = f(x)$



3. Explain the meaning of each of the following symbols

(a) δx

(d) dx

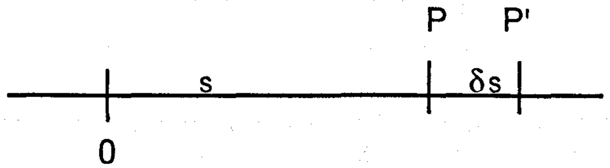
(b) δy

(e) dy

(c) $\dfrac{\delta y}{\delta x}$

(f) what is the relation between $\dfrac{\delta y}{\delta x}$ and $\dfrac{dy}{dx}$?

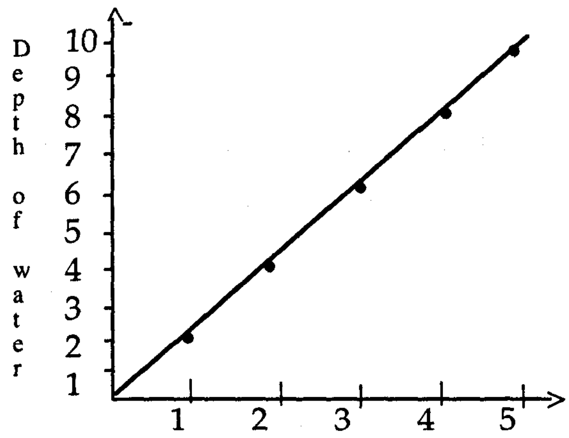
4) A point P moves a long a straight line such that after t seconds it is a distance s metres from the origin 0. In a further small time, δt seconds, the point moves a further distance δs metres to P'. If s and t are connected by the relation $s = 3t^2+1$, find δs in terms of t and δt.



What is the average velocity of the point in the interval PP'?
Obtain the value of $\dfrac{ds}{dt}$. What does $\dfrac{ds}{dt}$ represent?

5) Water is flowing into a tank at a constant rate, such that each unit increase in the time, the depth of water increase by 2 units. The table and graph illustrate this solution.

Time (x)	0		1		2		3		4		5
Depth (y)	0		2		4		6		8		10
1st difference (depth)		2		2		2		2		2	

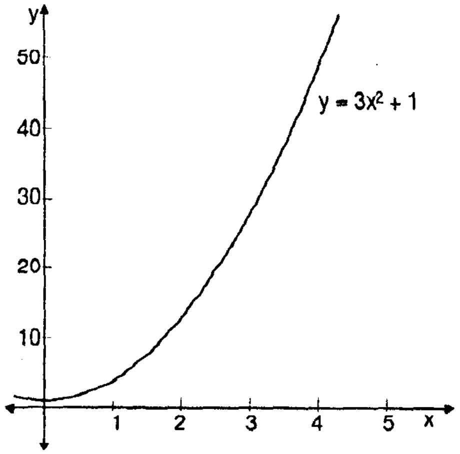


(a) What is the rate of increase of y as x increases from a to a+h?

(b) What is the rate of increase of y at $x = 2\frac{1}{2}$?

at x = X?

6) The graph below represents $y = 3x^2 + 1$, from $x=0$ to $x=4$.



(a) What is the average rate of change of y in the x- interval a to a+h?

(b) Can you use the result of (a) to obtain the rate of change of y at $x=2\frac{1}{2}$?

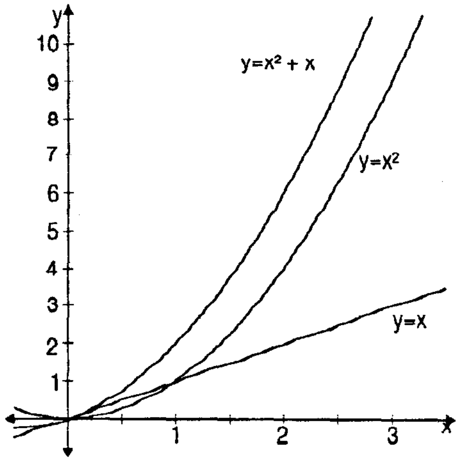
If so, how?

7) a) What is 'integration'? Define and explain as you wish.

b) If g(t) shows the number of phone calls in t time

What is the meaning of $\int_{5}^{10} g(t)\,dt$?

8)

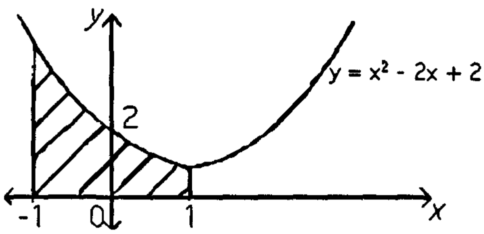


Explain, by means of the diagram or otherwise, why

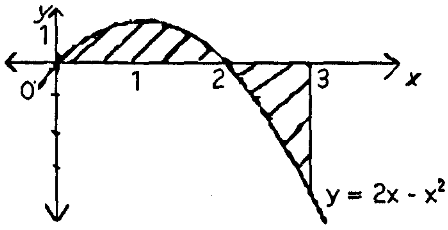
$$\int_o^a (x^2 + x)\,dx = \int_o^a x^2 dx + \int_o^a x\,dx$$

9. Calculate the shaded areas below, where possible. If it is not possible, explain why not.

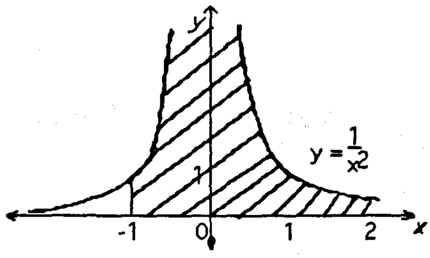
(a)



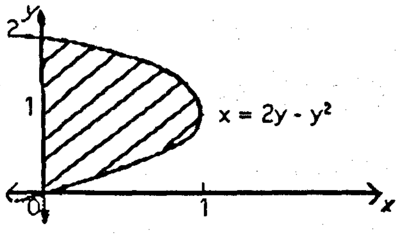
(b)



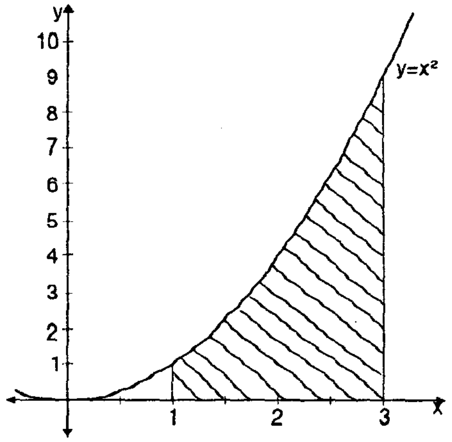
(c)



(d)



10)



When the area under a curve is rotated through 360° about the x- axis a solid is traced out.

(a) Describe the solid traced out when the shaded area above is rotated through 360° about the x-axis.

(b) Explain how to use integration to calculate the volume of this solid. Why does the method work?

11) Let $f(x) = \begin{cases} ax & x \leq 1 \\ bx^2 + x + 1 & x > 1 \end{cases}$ Find a & b so that f is differentiable at 1.

Appendix C

Computer attitude questionnaire

COMPUTER ATTITUDE QUESTIONNAIRE

The purpose of this questionnaire is to gather information concerning your attitudes about using computers in learning your calculus course. This information is for a research project at Nottingham University. All information you provide will remain strictly confidential.

Name:

PLEASE answer the following:

1. How often have you used the computer as part of the calculus course you followed this year?

( ) every week ( ) every two weeks ( ) every three weeks ( ) every month or more

( ) none

If 'none' go to question 5.

2. On average, how much time did you spend on the computer in each session?

( ) less than half an hour ( ) less than an hour ( ) between one hour and two hours

( ) more than two hours

3. How did you work on the computer?

( ) on my own ( ) with one of the students ( ) with a group of students

4. In which ways did you find computer helpful?

5. What changes might you suggest for the computer part of this calculus course when it is run again?

6. What is the biggest worry affecting your work in this calculus ?

	Strongly Agree	Agree	Neutral	Disagree	Strongly Disagree
7. I like working on the computer as part of the calculus course.	( )	( )	( )	( )	( )
8. Computer assignments were well integrated with the rest of the course.	( )	( )	( )	( )	( )
9. Computer assignments enhanced my interest in the course material.	( )	( )	( )	( )	( )
10. I would have prefered the calculus if the computers were not used.	( )	( )	( )	( )	( )
11. Overall the computer labs were a valuable part of the course.	( )	( )	( )	( )	( )
12. Using a computer in the laboratory contributed to my understanding of the course material.	( )	( )	( )	( )	( )
13. Computer sessions helped my understanding of many of the topics covered	( )	( )	( )	( )	( )

	Yes	No
14. Have you ever used a computer in the following situations?		
At school (Before university)	( )	( )
At home	( )	( )
15. Have you ever used a computer for		
games	( )	( )
writing your own programs	( )	( )
using a word processor	( )	( )
in a lesson at school	( )	( )
in a work situation as part of the university course	( )	( )
in a paid employment	( )	( )

16. How long have you been using a computer, for any purpose? ....

Thank you for your help!

Note: If you have any further comments, please write back of the paper.

Appendix D

Survey questionnaire

SURVEY QUESTIONNAIRE

INFORMATION TECHNOLOGY IN CALCULUS

This questionnaire is used as a checklist for face-to-face interview.

Date:

University or College:

Teacher:

How long have you been teaching the course with computers?

RESOURCES

- Which computer system or calculator is being used?
- Which program is being used?
- How many computers or calculators are available?
- Where are the computers located?
- Is the software easy to use?

POLICY AND CLASSROOM ENVIRONMENT

- In which course or subject is the technology being used?
- What are the majors of the students?
- How many students are involved with the technology?
- Have the students had previous experience with computers or programing?
- What proprotion of the time do the students spend with the technology?
- How many class-hours constitute this course?
- Could you describe the way computers are being used?
- What are the reasons for using the computer in teaching?

- What is the syllabus of the course ?
- Is there any difference between this course and one utilizing traditional methods, in terms of content?
- What kinds of testing and evaluating procedures are used?

IMPLEMENTATION

- Do you teach the students how to use the program in advance?
- Do the students have specific tasks to work on, on the computers or calculators?
- Have you done any alteration in the teaching approaches?
- Are you prompted for a response at each step?
- Does software require your intervention?

EVALUATION

- Is there any change in the teaching time?
- What kinds of difficulties do you experience as a teacher ?
- What kinds of difficulties do the students experience?
- What are the students' attitude to the computers or calculators?
- What are the students' comments about the course?
- Does computing change what students should know?
- Does computing help students learn mathematical ideas more deeply, more easily and more quickly?
- Does software offer an alternative in your work?
- Can you suggest some ways in which a teacher can use the computer more effectively in maths classroom?

Appendix E

Means and standard deviations of the pre-test and the post-test scores

Means and standard deviations (in parantheses) of the pre-test and the post-test scores

Items	Computer Groups						Non-Computer Groups					
	ComG1 (n=35)			ComG2 (n=56)			N-ComG1 (n=20)			N-ComG2 (n=36)		
	Pre-test	Post-test	Gain Post-Pre	Pre-test	Post-test	Gain Post-Pre	Pre-test	Post-test	Gain Post-Pre	Pre-test	Post-test	Gain Post-Pre
1(A)	2.05 (1.47)	2.51 (1.17)	0.46	2.71 (1.18)	3.07 (1.26)	0.36	1.45 (1.23)	2.45 (1.27)	1.00	1.83 (1.27)	2.36 (1.17)	0.53
1(B)	2.20 (1.76)	2.57 (1.57)	0.37	3.16 (1.74)	3.55 (1.58)	0.39	1.70 (1.41)	2.50 (1.35)	0.80	2.13 (1.65)	2.44 (1.53)	0.31
2	1.40 (1.78)	2.40 (2.10)	1.00	2.03 (1.74)	3.19 (1.90)	1.16	0.85 (1.18)	0.95 (1.53)	0.10	0.86 (1.12)	1.05 (1.51)	0.19
3	1.62 (2.04)	2.68 (1.99)	1.06	2.58 (1.88)	3.14 (1.83)	0.56	1.75 (1.58)	2.05 (1.73)	0.30	1.30 (1.19)	1.72 (1.61)	0.42
4(A)	1.94 (1.66)	1.51 (1.42)	-0.43	3.42 (1.18)	3.33 (1.14)	-0.09	2.35 (1.66)	2.50 (1.57)	0.15	1.66 (1.49)	2.22 (1.60)	0.56
4(B)	2.25 (1.99)	1.97 (1.96)	-0.28	4.19 (1.50)	4.19 (1.35)	0.00	2.95 (2.01)	3.45 (1.63)	0.50	2.63 (2.09)	2.80 (2.03)	0.15
4(C)	1.97 (2.09)	1.51 (1.66)	-0.46	2.85 (1.72)	3.25 (1.63)	0.40	2.15 (1.89)	3.00 (1.74)	0.85	1.69 (1.63)	2.11 (1.48)	0.42
4(D)	1.62 (1.78)	1.88 (1.67)	0.26	2.44 (1.68)	2.73 (1.54)	0.29	1.80 (1.96)	2.90 (1.61)	1.10	1.50 (1.46)	2.05 (1.68)	0.55
4(E)	1.11 (1.71)	1.17 (1.36)	0.06	1.03 (1.34)	1.28 (1.47)	0.25	0.60 (1.18)	1.95 (1.93)	1.35	0.94 (1.32)	0.91 (1.20)	-0.03
4(F)	1.94 (1.93)	2.57 (1.95)	0.63	2.78 (1.87)	3.12 (1.83)	0.34	2.05 (2.01)	2.60 (1.87)	0.55	2.16 (1.81)	2.16 (1.88)	0.00
4(G)	0.65 (1.28)	1.25 (1.82)	0.60	1.83 (1.71)	1.92 (1.63)	0.09	1.10 (1.37)	1.90 (1.65)	0.80	0.55 (0.69)	0.86 (0.83)	0.31
5(A)	2.05 (2.11)	2.45 (2.16)	0.40	2.94 (2.11)	3.51 (1.79)	0.57	1.85 (1.59)	2.05 (1.73)	0.20	2.69 (2.05)	2.63 (2.08)	-0.06
5(B)	1.85 (2.17)	1.82 (2.16)	-0.03	2.39 (2.24)	3.35 (2.07)	0.96	1.70 (1.89)	1.85 (2.05)	0.15	2.19 (2.16)	2.83 (2.15)	0.64
6(A)	0.94 (1.28)	0.97 (1.04)	0.03	1.30 (1.67)	1.55 (1.61)	0.25	1.00 (1.21)	1.10 (1.16)	0.10	0.91 (1.07)	1.52 (1.71)	0.61
6(B)	0.42 (0.94)	0.42 (0.77)	0.00	0.69 (1.14)	0.57 (1.07)	-0.12	0.50 (0.94)	0.80 (1.32)	0.35	0.52 (1.05)	0.86 (1.41)	0.34
6(C)	1.60 (2.34)	2.51 (2.41)	0.91	2.10 (2.41)	2.10 (2.34)	0.00	1.20 (1.98)	1.95 (2.32)	0.75	1.44 (2.14)	2.69 (2.27)	1.25
7(A)	2.71 (1.72)	3.11 (1.69)	0.40	3.23 (1.48)	3.51 (1.37)	0.28	2.20 (1.76)	2.45 (1.87)	0.25	3.16 (1.48)	3.58 (1.33)	0.42
7(B)	2.51 (2.00)	2.91 (2.04)	0.40	2.83 (1.84)	3.41 (1.79)	0.58	1.55 (1.73)	2.25 (1.97)	0.70	3.00 (1.94)	2.97 (2.09)	-0.03
8	1.45 (1.59)	1.62 (1.62)	0.17	1.46 (1.57)	1.98 (1.51)	0.52	1.50 (1.82)	1.30 (1.49)	-0.20	1.50 (1.44)	2.27 (1.30)	0.77
9(A)	2.77 (2.12)	4.08 (1.29)	1.31	3.85 (1.84)	4.35 (1.29)	0.50	2.85 (1.95)	4.50 (1.23)	1.65	3.11 (1.81)	4.02 (1.27)	0.91
9(B)	2.25 (2.14)	3.48 (1.75)	1.23	3.23 (2.08)	3.53 (1.91)	0.30	1.70 (1.94)	2.80 (2.23)	1.10	2.36 (1.97)	2.63 (1.89)	0.27
9(C)	0.85 (1.08)	1.97 (1.42)	1.12	1.80 (1.41)	2.60 (1.33)	0.80	1.25 (1.29)	2.05 (1.35)	0.80	1.02 (1.15)	1.75 (1.22)	0.73
9(D)	1.82 (2.21)	3.02 (2.18)	1.20	2.58 (2.23)	3.48 (2.13)	0.90	1.25 (1.83)	2.70 (2.36)	0.45	1.72 (1.98)	2.55 (2.15)	0.83
10(A)	1.34 (2.19)	1.97 (2.38)	0.63	2.21 (2.44)	2.28 (2.46)	0.07	2.25 (2.55)	3.05 (2.45)	0.80	2.88 (2.41)	2.52 (2.39)	-0.36
10(B)	1.25 (1.88)	1.77 (2.10)	0.52	2.50 (2.31)	1.98 (2.39)	-0.52	0.80 (1.70)	2.75 (2.29)	1.95	1.69 (2.27)	2.05 (2.16)	0.36
10(C)	0.37 (1.08)	0.71 (1.60)	0.34	0.82 (1.69)	0.67 (1.61)	-0.15	0.45 (1.27)	1.00 (1.94)	0.55	0.41 (1.25)	0.91 (1.67)	0.50

Note: In this table a simple gain score is shown by subtracting pre-test means from post-test means. For analysis of variance a more sophisticated comparison was made using regression to obtain predicted post-test scores.

Appendix F

Frequencies and percentages of students' responses to

the pre-test and the post-test items according to the scoring criteria

Frequencies and percentages of students' responses to the pre-test and the post-test items according to the scoring criteria

Items	Frequencies						Percentages (%)					
	0	1	2	3	4	5	0	1	2	3	4	5
1(A)												
ComG1												
Pre-test	6	7	9	8	2	3	17.10	20.00	25.70	22.90	5.70	8.60
Post-test	3	3	9	13	7	-	8.60	8.60	25.70	37.10	20.00	-
ComG2												
Pre-test	3	5	14	19	13	2	5.40	8.90	25.00	33.90	23.20	3.60
Post-test	4	3	5	21	19	4	7.10	5.40	8.90	37.50	33.90	7.10
N-ComG1												
Pre-test	4	8	5	2	-	1	20.00	40.00	25.00	10.00	-	5.00
Post-test	1	4	5	6	3	1	5.00	20.00	25.00	30.00	15.00	5.00
N-ComG2												
Pre-test	6	9	10	8	2	1	16.70	25.00	27.80	22.20	5.60	2.80
Post-test	4	4	7	17	4	-	11.10	11.10	19.40	47.20	11.10	-
1(B)												
ComG1												
Pre-test	4	14	5	2	3	7	11.40	40.00	14.30	5.70	8.60	20.00
Post-test	2	11	4	5	9	4	5.70	31.40	11.40	14.30	25.70	11.40
ComG2												
Pre-test	5	10	4	5	16	16	8.90	17.90	7.10	8.90	28.60	28.60
Post-test	4	6	2	5	21	18	7.10	10.70	3.60	8.90	37.50	32.10
N-ComG1												
Pre-test	3	8	5	2	-	2	15.00	40.00	25.00	10.00	-	10.00
Post-test	1	4	6	3	5	1	5.00	20.00	30.00	15.00	25.00	5.00
N-ComG2												
Pre-test	6	10	8	-	9	3	16.70	27.80	22.20	-	25.00	8.30
Post-test	3	10	6	5	9	3	8.30	27.80	16.70	13.90	25.00	8.30
2												
ComG1												
Pre-test	16	9	-	4	2	4	45.70	25.70	-	11.40	5.70	11.40
Post-test	9	9	-	4	2	11	25.70	25.70	-	11.40	5.70	31.40
ComG2												
Pre-test	10	23	3	-	15	5	17.90	41.10	5.40	-	26.80	8.90
Post-test	8	9	2	-	19	18	14.30	16.10	3.60	-	33.90	32.10
N-ComG1												
Pre-test	9	9	-	-	2	-	45.00	45.00	-	-	10.00	-
Post-test	11	6	-	-	2	1	55.00	30.00	-	-	10.00	5.00
N-ComG2												
Pre-test	15	17	1	1	1	1	41.70	47.20	2.80	2.80	2.80	2.80
Post-test	15	16	1	-	-	4	41.70	44.40	2.80	-	-	11.10
3												
ComG1												
Pre-test	15	10	-	1	1	8	42.90	28.60	-	2.90	2.90	22.90
Post-test	5	11	-	5	2	12	14.30	31.40	-	14.30	5.70	34.30
ComG2												
Pre-test	8	17	-	11	5	15	14.30	30.40	-	19.60	8.90	26.80
Post-test	5	12	1	12	4	22	8.90	21.40	1.80	21.40	7.10	39.10
N-ComG1												
Pre-test	6	5	-	7	1	1	30.00	25.00	-	35.00	5.00	5.00
Post-test	5	5	-	6	2	2	25.00	25.00	-	30.00	10.00	10.00
N-ComG2												
Pre-test	7	21	1	5	1	1	19.40	58.30	2.80	13.90	2.80	2.80
Post-test	5	21	-	4	1	5	13.90	58.30	-	11.10	2.80	13.90

Items	Frequencies						Percentages (%)					
	0	1	2	3	4	5	0	1	2	3	4	5
4(A)												
ComG1												
Pre-test	11	5	4	5	10	-	31.40	14.30	11.40	14.30	28.60	-
Post-test	9	14	3	3	-	-	25.70	40.00	8.60	8.60	-	-
ComG2												
Pre-test	2	5	3	5	39	2	3.60	8.90	5.40	8.90	69.60	3.60
Post-test	2	5	2	11	35	1	3.60	8.90	3.60	19.60	62.50	1.80
N-ComG1												
Pre-test	5	2	1	5	7	-	25.00	10.00	5.00	25.00	35.00	-
Post-test	4	2	1	6	7	-	20.00	10.00	5.00	30.00	35.00	-
N-ComG2												
Pre-test	10	10	5	4	7	-	27.80	27.80	13.90	11.10	19.40	-
Post-test	8	6	4	6	12	-	22.20	16.70	11.10	16.70	33.30	-
4(B)												
ComG1												
Pre-test	11	4	3	8	-	9	31.40	11.40	8.60	22.90	-	25.70
Post-test	10	11	1	4	1	8	28.60	31.40	2.90	11.40	2.90	22.90
ComG2												
Pre-test	2	4	3	5	-	42	3.60	7.10	5.40	8.90	-	75.00
Post-test	1	4	-	12	-	39	1.80	7.10	-	21.40	-	69.70
N-ComG1												
Pre-test	4	2	1	5	-	8	20.00	10.00	5.00	25.00	-	40.00
Post-test	2	1	-	8	1	8	10.00	5.00	-	40.00	5.00	40.00
N-ComG2												
Pre-test	10	4	1	8	-	13	27.80	11.10	2.80	22.20	-	36.10
Post-test	7	7	-	8	-	14	19.40	19.40	-	22.20	-	38.90
4(C)												
ComG1												
Pre-test	15	3	3	5	-	9	42.90	8.60	8.60	14.30	-	25.70
Post-test	12	12	-	7	-	4	34.30	34.30	-	20.00	-	11.40
ComG2												
Pre-test	6	10	4	18	2	16	10.70	17.90	7.10	32.10	3.60	28.60
Post-test	2	12	-	18	4	20	3.60	21.40	-	32.10	7.10	35.70
N-ComG1												
Pre-test	5	5	1	4	1	4	25.00	25.00	5.00	20.00	5.00	20.00
Post-test	3	1	2	7	1	6	15.00	5.00	10.00	35.00	5.00	30.00
N-ComG2												
Pre-test	11	9	4	8	-	4	30.60	25.00	11.10	22.20	-	11.10
Post-test	5	10	5	12	-	4	13.90	27.80	13.90	33.30	-	11.10
4(D)												
ComG1												
Pre-test	16	4	1	7	5	2	45.70	11.40	2.90	20.00	14.30	5.70
Post-test	12	5	2	7	9	-	34.30	14.30	5.70	20.00	25.70	-
ComG2												
Pre-test	11	11	-	13	18	3	19.60	19.60	-	23.20	32.10	5.40
Post-test	6	11	1	17	16	5	10.70	19.60	1.80	30.40	28.60	8.90
N-ComG1												
Pre-test	9	2	1	2	4	2	45.00	10.00	5.00	10.00	20.00	10.00
Post-test	2	4	-	4	8	2	10.00	20.00	-	20.00	40.00	10.00
N-ComG2												
Pre-test	11	13	-	7	5	-	30.60	36.10	-	19.40	13.90	-
Post-test	10	7	1	8	9	1	27.80	19.40	2.80	22.20	25.00	2.80

Items	Frequencies						Percentages (%)					
	0	1	2	3	4	5	0	1	2	3	4	5
4(E)												
ComG1												
Pre-test	18	10	2	-	-	5	51.40	28.60	5.70	-	-	14.30
Post-test	11	16	5	-	-	3	31.40	45.70	14.30	-	-	8.60
ComG2												
Pre-test	19	31	1	-	-	5	33.90	55.40	1.80	-	-	8.90
Post-test	17	25	7	1	-	6	30.40	44.60	12.50	1.80	-	10.70
N-ComG1												
Pre-test	13	5	1	-	-	1	65.00	25.00	5.00	-	-	5.00
Post-test	4	9	1	1	-	5	20.00	45.00	5.00	5.00	-	25.00
N-ComG2												
Pre-test	14	19	-	-	-	3	38.90	52.80	-	-	-	8.30
Post-test	14	18	1	1	-	2	38.90	50.00	2.80	2.80	-	5.60
4(F)												
ComG1												
Pre-test	15	2	2	6	6	4	42.90	5.70	5.70	17.10	17.10	11.40
Post-test	10	2	4	2	11	6	28.60	5.70	11.40	5.70	31.40	17.10
ComG2												
Pre-test	13	4	4	6	19	10	23.20	7.10	7.10	10.70	33.90	17.90
Post-test	9	4	8	1	18	16	16.10	7.10	14.30	1.80	32.10	28.60
N-ComG1												
Pre-test	8	1	2	4	1	4	40.00	5.00	10.00	20.00	5.00	20.00
Post-test	3	5	1	4	2	5	15.00	25.00	5.00	20.00	10.00	25.00
N-ComG2												
Pre-test	10	7	1	6	9	3	27.80	19.40	2.80	16.70	25.00	8.30
Post-test	12	4	3	3	11	3	33.30	11.10	8.30	8.30	30.60	8.30
4(G)												
ComG1												
Pre-test	23	8	1	1	-	2	65.70	22.90	2.90	2.90	-	5.70
Post-test	18	9	-	2	1	5	51.40	25.70	-	5.70	2.90	14.30
ComG2												
Pre-test	16	14	8	4	9	5	28.60	25.00	14.30	7.10	16.10	8.90
Post-test	9	24	3	9	4	7	16.10	42.90	5.40	16.10	7.10	12.50
N-ComG1												
Pre-test	9	6	1	2	2	-	45.00	30.00	5.00	10.00	10.00	-
Post-test	4	7	2	3	2	2	20.00	35.00	10.00	15.00	10.00	10.00
N-ComG2												
Pre-test	19	15	1	1	-	-	52.80	41.70	2.80	2.80	-	-
Post-test	12	20	1	3	-	-	33.30	55.60	2.80	8.30	-	-
5(A)												
ComG1												
Pre-test	10	12	1	1	-	11	28.60	34.30	2.90	2.90	-	31.40
Post-test	7	12	2	-	-	14	20.00	34.30	5.70	-	-	40.00
ComG2												
Pre-test	11	11	1	5	3	25	19.60	19.60	1.80	8.90	5.40	44.60
Post-test	4	9	1	12	-	30	7.1	16.10	1.80	21.40	-	53.60
N-ComG1												
Pre-test	3	8	4	2	-	3	15.00	40.00	20.00	10.00	-	15.00
Post-test	2	10	1	3	-	4	10.00	50.00	5.00	15.00	-	20.00
N-ComG2												
Pre-test	5	12	1	3	1	11	13.90	33.30	2.80	8.30	2.80	38.90
Post-test	4	15	1	1	-	15	11.10	41.70	2.80	2.80	-	41.70

Items	Frequencies						Percentages (%)					
	0	1	2	3	4	5	0	1	2	3	4	5
5(B)												
ComG1												
Pre-test	15	8	-	-	3	9	42.90	22.90	-	-	8.60	25.70
Post-test	14	10	-	-	1	10	40.00	28.60	-	-	2.90	28.60
ComG2												
Pre-test	17	14	-	2	1	22	30.40	25.00	-	3.60	1.80	39.30
Post-test	4	18	-	-	-	34	7.10	32.10	-	-	-	60.70
N-ComG1												
Pre-test	5	10	-	-	1	4	25.00	50.00	-	-	5.00	20.00
Post-test	6	8	-	-	1	5	30.00	40.00	-	-	5.00	25.00
N-ComG2												
Pre-test	10	12	-	1	1	12	27.80	33.30	-	2.80	2.80	33.30
Post-test	6	11	-	1	2	16	16.70	30.60	-	2.80	5.60	44.40
6(A)												
ComG1												
Pre-test	17	9	7	-	-	2	48.60	25.70	20.00	-	-	5.70
Post-test	12	16	5	1	-	1	34.30	45.70	14.30	2.90	-	2.90
ComG2												
Pre-test	20	25	2	-	1	8	35.70	44.60	3.60	-	1.80	14.30
Post-test	11	30	6	-	-	9	19.60	53.60	10.70	-	-	16.10
N-ComG1												
Pre-test	8	7	4	-	-	1	40.00	35.00	20.00	-	-	5.00
Post-test	6	9	4	-	-	1	30.00	45.00	20.00	-	-	5.00
N-ComG2												
Pre-test	14	16	3	2	-	1	38.90	44.40	8.30	5.60	-	2.80
Post-test	11	14	4	-	2	5	30.60	38.90	11.10	-	5.60	13.90
6(B)												
ComG1												
Pre-test	28	2	2	3	-	-	80.00	5.70	5.70	8.60	-	-
Post-test	24	9	-	2	-	-	68.60	25.70	-	5.70		
ComG2												
Pre-test	34	15	-	4	3	-	60.70	26.80	-	7.10	5.40	-
Post-test	36	16	-	1	2	1	64.30	28.60	-	1.80	3.60	1.80
N-ComG1												
Pre-test	13	6	-	-	1	-	65.00	30.00	-	-	5.00	
Post-test	12	5	-	1	2	-	60.00	25.00	-	5.00	10.00	-
N-ComG2												
Pre-test	25	8	-	1	2	-	69.40	22.20	-	2.80	5.60	-
Post-test	22	8	-	1	5	-	61.10	22.20	-	2.80	13.90	-
6(C)												
ComG1												
Pre-test	23	1	-	-	-	11	65.70	2.90	-	-	-	31.40
Post-test	14	4	-	-	1	16	40.00	11.40	-	-	2.90	45.70
ComG2												
Pre-test	29	4	-	-	1	22	51.80	7.10	-	-	1.80	39.30
Post-test	26	7	-	2	-	21	46.40	12.50	-	3.60	-	37.50
N-ComG1												
Pre-test	12	4	-	-	-	4	60.00	20.00	-	-	-	20.00
Post-test	9	4				7	45.00	20.00				35.00
N-ComG2												
Pre-test	22	3	2	-	-	9	61.10	8.30	5.60	-	-	25.00
Post-test	12	3	2	1	3	15	33.30	8.30	5.60	2.80	8.30	41.70

Items	Frequencies						Percentages (%)					
	0	1	2	3	4	5	0	1	2	3	4	5
7(A)												
ComG1												
Pre-test	8	1	3	7	13	3	22.90	2.90	8.60	20.00	37.10	8.60
Post-test	6	1	2	6	14	6	17.10	2.90	5.70	17.10	40.00	17.10
ComG2												
Pre-test	6	4	1	12	26	7	10.70	7.10	1.80	21.40	46.40	12.50
Post-test	5	1	1	13	25	11	8.90	1.80	1.80	23.20	44.60	19.60
N-ComG1												
Pre-test	6	2	1	5	5	1	30.00	10.00	5.00	25.00	25.00	5.00
Post-test	5	2	2	5	2	4	25.00	10.00	10.00	25.00	10.00	20.00
N-ComG2												
Pre-test	3	4	1	9	14	5	8.30	11.10	2.80	25.00	38.90	13.90
Post-test	2	2	1	7	16	8	5.60	5.60	2.80	19.40	44.40	22.20
7(B)												
ComG1												
Pre-test	9	1	12	1	-	12	25.70	2.90	34.30	2.90	-	34.30
Post-test	8	1	7	4	-	15	22.90	2.90	20.00	11.40	-	42.90
ComG2												
Pre-test	7	8	13	7	1	20	12.50	14.30	23.20	12.50	1.80	35.70
Post-test	5	4	12	5	2	28	8.90	7.10	21.40	8.90	3.60	50.00
N-ComG1												
Pre-test	7	5	4	1	-	3	35.00	25.00	20.00	5.00	-	15.00
Post-test	4	5	5	-	-	6	20.00	25.00	25.00	-	-	30.00
N-ComG2												
Pre-test	4	6	8	2	-	16	11.10	16.70	22.20	5.60	-	44.40
Post-test	7	4	6	2	-	17	19.40	11.10	16.70	5.60	-	47.20
8												
ComG1												
Pre-test	17	3	-	13	1	1	48.60	8.60	-	37.10	2.90	2.90
Post-test	14	5	1	12	1	2	40.00	14.30	2.90	34.30	2.90	5.70
ComG2												
Pre-test	21	15	2	13	1	-	37.50	26.80	3.60	23.20	1.80	-
Post-test	15	9	1	27	1	3	26.80	16.10	1.80	48.20	1.80	5.40
N-ComG1												
Pre-test	10	2	1	4	1	2	50.00	10.00	5.00	20.00	5.00	10.00
Post-test	9	3	3	4	-	1	45.00	15.00	15.00	20.00	-	5.00
N-ComG2												
Pre-test	13	8	1	13	-	1	36.10	22.20	2.80	36.10	-	2.80
Post-test	6	5	-	24	-	1	16.70	13.90	-	66.70	-	2.80
9(A)												
ComG1												
Pre-test	11	1	1	6	4	12	31.40	2.90	2.90	17.10	11.40	34.30
Post-test	1	1	2	5	7	19	2.90	2.90	5.70	14.30	20.00	54.30
ComG2												
Pre-test	9	-	-	8	3	36	16.10	-	-	14.30	5.40	64.30
Post-test	3	1	-	4	9	39	5.40	1.80	-	7.10	16.10	69.60
N-ComG1												
Pre-test	4	-	3	3	5	5	20.00	-	15.00	15.00	25.00	25.00
Post-test	-	2	-	-	2	16	-	10.00	-	-	10.00	80.00
N-ComG2												
Pre-test	4	7	-	6	8	11	11.10	19.40	-	16.70	22.20	30.60
Post-test	1	2	-	6	10	17	2.80	5.60	-	16.70	27.80	47.20

Items	Frequencies						Percentages (%)					
	0	1	2	3	4	5	0	1	2	3	4	5
9(B)												
ComG1												
Pre-test	15	1	-	6	5	8	42.90	2.90	-	17.10	14.30	22.90
Post-test	4	3	1	5	8	14	11.40	8.60	2.90	14.30	22.90	40.00
ComG2												
Pre-test	14	1	1	9	4	27	25.00	1.80	1.80	16.10	7.10	48.20
Post-test	11	-	-	10	7	28	19.60	-	-	17.90	12.50	50.00
N-ComG1												
Pre-test	9	3	-	4	1	3	45.00	15.00	-	20.00	5.00	15.00
Post-test	6	2	-	2	2	8	30.00	10.00	-	10.00	10.00	40.00
N-ComG2												
Pre-test	8	10	-	6	3	9	22.20	27.80	-	16.70	8.30	25.00
Post-test	3	14	1	4	3	11	8.30	38.90	2.80	11.10	8.30	30.60
9(C)												
ComG1												
Pre-test	17	12	-	6	-	-	48.60	34.30	-	17.10	-	-
Post-test	7	9	1	15	2	1	20.00	25.70	2.90	42.90	5.70	2.90
ComG2												
Pre-test	18	6	3	27	2	-	32.10	10.70	5.40	48.20	3.60	-
Post-test	6	8	-	34	4	4	10.70	14.30	-	60.70	7.10	7.10
N-ComG1												
Pre-test	8	5	1	6	-	-	40.00	25.00	5.00	30.00	-	-
Post-test	4	4	-	11	1	-	20.00	20.00	-	55.00	5.00	-
N-ComG2												
Pre-test	16	10	3	7	-	-	44.40	27.80	8.30	19.40	-	-
Post-test	5	15	2	12	2	-	13.90	41.70	5.60	33.30	5.60	-
9(D)												
ComG1												
Pre-test	18	3	1	3	-	10	51.40	8.60	2.90	8.60	-	28.60
Post-test	9	4	-	1	6	15	25.70	11.40	-	2.90	17.10	42.90
ComG2												
Pre-test	21	2	-	10	2	21	37.50	3.60	-	17.90	3.60	37.50
Post-test	13	2	-	5	2	34	23.20	3.60	-	8.90	3.60	60.70
N-ComG1												
Pre-test	12	2	-	3	1	2	60.00	10.00	-	15.00.	5.00	10.00
Post-test	7	2	-	1	1	9	35.00	10.00	-	5.00	5.00	45.00
N-ComG2												
Pre-test	14	10	-	3	2	7	38.90	27.80	-	8.30	5.60	19.40
Post-test	11	5	-	5	3	12	30.60	13.90	-	13.90	8.30	33.30
10(A)												
ComG1												
Pre-test	24	2	-	-	-	9	68.60	5.70	-	-	-	25.70
Post-test	18	4	-	-	-	13	51.40	11.40	-	-	-	37.10
ComG2												
Pre-test	28	4	-	-	-	24	50.00	7.10	-	-	-	42.90
Post-test	28	3	-	-	-	25	50.00	5.40	-	-	-	44.60
N-ComG1												
Pre-test	11	-	-	-	-	9	55.00	-	-	-	-	45.00
Post-test	7	1	-	-	-	12	35.00	5.00	-	-	-	60.00
N-ComG2												
Pre-test	12	4	-	-	-	20	33.30	11.10	-	-	-	55.60
Post-test	13	6	-	-	-	17	36.10	16.70	-	-	-	47.20

Items	Frequencies						Percentages (%)					
	0	1	2	3	4	5	0	1	2	3	4	5
10(B)												
ComG1												
Pre-test	23	1	-	4	4	3	65.70	2.90	-	11.40	11.40	8.60
Post-test	19	1	-	5	4	6	54.30	2.90	-	14.30	11.40	17.10
ComG2												
Pre-test	24	1	-	7	2	22	42.90	1.80	-	12.50	3.60	39.30
Post-test	31	3	-	-	2	20	55.40	5.40	-	-	3.60	35.70
N-ComG1												
Pre-test	16	-	-	2	-	2	80.00	-	-	10.00	-	10.00
Post-test	7	1	-	2	2	8	35.00	5.00	-	10.00	10.00	40.00
N-ComG2												
Pre-test	21	3	-	-	2	10	58.30	8.30	-	-	5.60	27.80
Post-test	14	7	-	3	2	10	38.90	19.40	-	8.30	5.60	27.80
10(C)												
ComG1												
Pre-test	29	4	-	-	1	1	82.90	11.40	-	-	2.90	2.90
Post-test	26	5	-	-	-	4	74.30	14.30	-	-	-	11.40
ComG2												
Pre-test	42	5	-	1	2	6	75.00	8.90	-	1.80	3.60	10.70
Post-test	45	4	-	-	1	6	80.40	7.10	-	-	1.80	10.70
N-ComG1												
Pre-test	17	1	-	1	-	1	85.00	5.00	-	5.00	-	5.00
Post-test	15	1	-	-	1	3	75.00	5.00	-	-	5.00	15.00
N-ComG2												
Pre-test	31	2	-	1	-	2	86.10	5.60	-	2.80	-	5.60
Post-test	25	4	-	2	2	3	69.40	11.10	-	5.60	5.60	8.30

Appendix G

Factor loadings of items measured on the pre-test after factor rotation

Factor loadings of items measured on pre-test after factor rotation

	Factors								
	I	II	III	IV	V	VI	VII	VIII	IX
Eigenvalues	6.30	2.59	1.83	1.68	1.53	1.42	1.20	1.12	1.03
% of variance explained	24.2	10	7	6.5	6	5.5	4.6	4.3	4
Items									
1(a)					81				
1(b)					77				
2							77		
3							83		
4(a)		86							
4(b)		86							
4(c)		80							
4(d)				73					
4(e)				79					
4(f)				67					
4(g)				43					
5(a)								83	
5(b)								74	
6(a)			71						
6(b)			85						
6(c)			69						
7(a)					57				
7(b)									46
8									86
9(a)	77								
9(b)	84								
9(c)	75								
9(d)	81								
10(a)						75			
10(b)						75			
10(c)						78			

Note. '77' means (0.77). Decimal points and values less than 0.40 omitted.

Appendix H

Correlations of items from the pre-test measure

Correlations of items for the pre-test measure

Items	1(a)	1(b)	2	3	4(a)	4(b)	4(c)	4(d)	4(e)	4(f)	4(g)	5(a)	5(b)	6(a)
1(a)														
1(b)	.60													
2	.33	.37												
3	.35	.36	.70											
4(a)	.25	.22	.31	.35										
4(b)	.30	.25	.33	.36	.80									
4(c)	.01	.10	.22	.24	.60	.60								
4(d)	.26	.28	.30	.27	.40	.34	.28							
4(e)	.15	.07	.09	.10	.16	.11	.02	.43						
4(f)	.16	.16	.23	.17	.28	.25	.25	.47	.30					
4(g)	.16	.22	.35	.27	.36	.30	.34	.44	.24	.30				
5(a)	.24	.32	.31	.21	.19	.18	.20	.19	.01	.17	.02			
5(b)	.20	.24	.42	.35	.19	.17	.20	.20	.00	.08	.09	.65		
6(a)	.07	.16	.24	.30	.14	.18	.19	.19	.14	.16	.16	.25	.38	
6(b)	.02	.12	.03	.11	.15	.19	.19	.13	.13	.17	.15	.17	.22	.50
6(c)	.17	.23	.21	.18	.16	.22	.29	.16	.03	.24	.24	.28	.29	.33
7(a)	.32	.35	.22	.12	.23	.30	.18	.12	-.01	-.01	-.02	.16	.16	.13
7(b)	.18	.31	.22	.12	.27	.27	.21	.18	.04	.19	.17	.33	.22	.18
8	.05	.17	.13	.14	.09	.06	.18	.11	-.08	.21	.08	.07	.04	.05
9(a)	.26	.27	.32	.32	.28	.29	.19	.24	.13	.19	.11	.24	.27	.10
9(b)	.20	.25	.27	.22	.26	.19	.20	.19	-.02	.05	.13	.25	.24	.08
9(c)	.15	.20	.20	.15	.18	.10	.06	.13	-.02	-.03	.14	.10	.13	.10
9(d)	.09	.07	.19	.06	.15	.14	.09	.18	.01	.05	.14	.08	.07	.06
10(a)	.01	.09	.19	.06	-.04	.02	-.001	-.005	.01	.05	-.03	.16	.16	.02
10(b)	.20	.31	.36	.21	.18	.20	.18	.07	-.06	.12	.16	.22	.23	.02
10(c)	.03	.11	.12	.05	.10	.05	.11	.07	-.04	.08	.28	.04	.10	.10

Items	6(b)	6(c)	7(a)	7(b)	8	9(a)	9(b)	9(c)	9(d)	10(a)	10(b)	10(c)
1(a)												
1(b)												
2												
3												
4(a)												
4(b)												
4(c)												
4(d)												
4(e)												
4(f)												
4(g)												
5(a)												
5(b)												
6(a)												
6(b)												
6(c)	.49											
7(a)	.14	.22										
7(b)	.12	.29	.32									
8	.01	.12	.29	.27								
9(a)	.12	.24	.39	.30	.29							
9(b)	.06	.22	.39	.26	.30	.74						
9(c)	.07	.16	.32	.18	.14	.48	.57					
9(d)	-.01	.10	.24	.17	.18	.54	.62	.48				
10(a)	-.10	-.01	.12	.12	.05	.21	.18	.22	.21			
10(b)	-.05	.19	.20	.12	.05	.30	.33	.25	.29	.55		
10(c)	.04	.09	.17	.14	.02	.14	.20	.18	.16	.39	.42	

Appendix I

Factor loadings of questions measured on the pre-test after factor rotation

Factor loadings of the questions measured on the pre-test after factor rotation

	Factors		
	I	II	II
Eigenvalues	3.5	1.25	1.04
% of variance explained	35	13	10
Questions			
1	58		
2	80		
3	81		
4	62		
5	60		
6	54		
7		76	
8		72	
9		58	
10			81

Note. Decimal points and values less than 0.40 omitted.

Appendix K

Correlations of the questions from the pre-test measurement

Correlations of the questions from the pre-test measurement

Questions	1	2	3	4	5	6	7	8	9	10
1										
2	.40									
3	.40	.70								
4	.32	.40	.39							
5	.31	.40	.31	.23						
6	.22	.23	.25	.34	.38					
7	.40	.27	.15	.30	.30	.31				
8	.13	.13	.14	.15	.06	.09	.34			
9	.25	.30	.23	.25	.23	.19	.41	.29		
10	.18	.30	.14	.13	.22	.06	.22	.05	.34	

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