

Novel Methods of Measuring the Similarity and Distance between Complex Fuzzy Sets

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Abstract

This thesis develops measures that enable comparisons of subjective information that is represented through fuzzy sets. Many applications rely on information that is subjective and imprecise due to varying contexts and so fuzzy sets were developed as a method of modelling uncertain data. However, making relative comparisons between data-driven fuzzy sets can be challenging. For example, when data sets are ambiguous or contradictory, then the fuzzy set models often become non-normal or non-convex, making them difficult to compare.

This thesis presents methods of comparing data that may be represented by such (complex) non-normal or non-convex fuzzy sets. The developed approaches for calculating relative comparisons also enable fusing methods of measuring similarity and distance between fuzzy sets. By using multiple methods, more meaningful comparisons of fuzzy sets are possible. Whereas if only a single type of measure is used, ambiguous results are more likely to occur.

This thesis provides a series of advances around the measuring of similarity and distance. Based on them, novel applications are possible, such as personalised and crowd-driven product recommendations. To demonstrate the value of the proposed methods, a recommendation system is developed that enables a person to describe their desired product in relation to one or more other known products. Relative comparisons are then used to find and recommend something that matches a person's subjective preferences. Demonstrations illustrate that the proposed method is useful for comparing complex, non-normal and non-convex fuzzy sets. In addition, the recommendation system is effective at using this approach to find products that match a given query.

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Notation	Description	Section
X	the universe of discourse	2.2
A	a type-1 fuzzy set	2.2.1
\tilde{A}	a type-2 fuzzy set	2.2.2
$\mathcal{P}(\mathbb{R})$	the set of all crisp sets in \mathbb{R}	2.4.2
$T1(X)$	the set of all type-1 fuzzy sets in X	2.2.1
$GT2(X)$	the set of all general type-2 fuzzy sets in X	2.2.2
$IT2(X)$	the set of all interval type-2 fuzzy sets in X	2.2.3
A_α	an alpha-cut (α -cut) of A	2.2.1
\tilde{A}_{α_U}	an alpha-cut (α -cut) of the upper membership function of an interval type-2 fuzzy set \tilde{A}	2.2.3
\tilde{A}_{α_W}	an alpha-cut (α -cut) of the lower membership function of an interval type-2 fuzzy set \tilde{A}	2.2.3
\bar{A}	a continuous real-valued interval	2.2.1
$\bar{\bar{A}}$	a discontinuous real-valued interval containing continuous intervals \bar{A}	2.2.1
$\mathcal{L}(W)$	the set formed by the union of the zLevels of W where $W \subseteq GT2(X)$	4.3
$\tau(X)$	difference between the lower and upper boundary of X where X is finite	5.3

Table 1: Descriptions of notations used throughout this paper and the sections in which they are first introduced.

Function	Description	Section
s	a similarity measure	2.4.1
d	a distance measure	2.4.2
f	ordered weighted average operator	2.5
s_j^{T1}	Jaccard similarity measure for type-1 fuzzy sets	2.4.1
s_j^{IT2}	Jaccard similarity measure for interval type-2 fuzzy sets	2.4.1
s_j^{GT2}	Jaccard similarity measure for general type-2 fuzzy sets	4.3
d_p^*	a distance measure proposed as part of this thesis, where * indicates the type of fuzzy sets d compares	3.2
$d_p^{T1:nc}$	directional distance measure for normal, convex type-1 fuzzy sets	3.2
$d_p^{T1:c}$	directional distance measure for convex type-1 fuzzy sets that may be non-normal	3.2
d_p^{T1}	directional distance measure for any type-1 fuzzy sets, including non-normal and non-convex sets	3.2
d_p^{IT2}	distance measure for any interval type-2 fuzzy sets	4.2
d_p^{GT2}	distance measure for any general type-2 fuzzy sets	4.3
m_λ	a function that extends a measure for interval type-2 fuzzy sets to compare general type-2 fuzzy sets	4.3
\bar{d}_p	directional distance measure for intervals	2.4.2
$\dot{\bar{d}}_p$	distance measure for α -cuts where one α -cut may be the empty set	3.3
$\bar{\bar{d}}_p$	distance measure for discontinuous α -cuts $\bar{\bar{A}}$	3.4
c'_p	incompatibility measure (fusion of s and d)	5.3

Table 2: Descriptions of functions used throughout this paper and the sections in which they are first introduced.

Chapter 1

Introduction

Human decision making involves resolving issues that are often described by uncertain and subjective information that cannot be captured through traditional two-valued logic. While some classes of objects have very clear and certain definitions (e.g., all integers greater than 10), others have inherently ambiguous definitions (e.g., all integers around 10). The concept of a fuzzy set was developed to enable one to create mathematical models that capture the vagueness of languages [1]. Fuzzy set theory provides a framework with which imprecise problems can be approached in a natural way, instead of forcing one to redefine unclear terms as precise concepts. By having such models, it is possible to develop methods for mathematically handling uncertain information in a manner that is similar to natural human reasoning.

This thesis develops methods of comparing fuzzy sets by measuring their similarities and their relative, directional distances, in particular for non-normal and non-convex fuzzy sets used to model subjective information. A single measure is proposed which captures both of these concepts and is able to compare any fuzzy set model from type-1 (i.e., two-dimensional), normal, convex fuzzy sets to general type-2 (i.e., three-dimensional), non-normal, non-convex fuzzy sets.

1.1 Background and Motivation

A common problem with fuzzy sets is the task of how to compare them. Two useful measures often used in the literature involve comparing the similarities and distances between fuzzy sets. Similarity measures are commonly used in classification and clustering [2–7] to determine how similar a variable is to different categories, and then classify it into the category with which it is most similar. Distance is a common measure used within ranking and decision making [8–13]. A distance measure is often used to rank the impact and importance of different variables and attributes, where importance may be a subjective term. This often involves determining the relative positions (i.e., is one fuzzy set to the left or right of another) as well as the magnitude of the distance. Many different methods have been developed to compare the similarity or distance between fuzzy sets.

One under-explored area is the ranking of fuzzy sets that are non-normal or non-convex. Though some attention has been put forward to compare non-normal fuzzy sets [11, 14, 15], ranking non-convex information is a problem which has fallen behind. Though some methods in the literature can be applied to non-convex fuzzy sets (e.g., comparing the centres of fuzzy sets), these methods do not always produce expected results. An example of a non-convex fuzzy set is one that models preferences of food. Whilst some people may rate a given food highly, others may give it a low rating, resulting in two distinct descriptions.

A single measure on fuzzy sets often focuses on only one aspect of the model. For example, similarity focuses on comparing degrees of membership whereas distance focuses more on the values within the fuzzy sets and less so on their membership. On their own, these measures provide useful information, but together they are even more informative. Thus, it can be beneficial to consider an evaluation based on both measures instead of just one.

1.2 Aims and Objectives

This thesis focuses on developing methods of measuring the similarities and directional distances between fuzzy sets that model subjective information, and may therefore be non-normal or non-convex. To illustrate the utility of these methods, a recommendation system is developed, which relies on relative comparisons of subjective information. Given this, the key research aims of this thesis are

- How can relative comparisons of subjective information be achieved, where the information is modelled by fuzzy sets that may be non-normal or non-convex?
- How can these comparisons be utilised within recommendation systems which rely on only subjective information?

The remainder of this section discusses the objectives that must be achieved to attain these goals.

Different types of fuzzy sets have been developed in the literature and each type is capable of modelling uncertainty to different levels of detail (discussed further in Section 2.2). Type-1 fuzzy sets offer a two-dimensional representation of uncertainty, whereas type-2 fuzzy sets provide additional information through three-dimensional models. It is important that there are measures of comparing type-1 and type-2 models so that a given application or measure is not restricted to only one type of fuzzy set.

These measures must have the same properties to compare each type of fuzzy set, i.e., the same characteristics of similarity and distance must always be observed, regardless of the type of fuzzy set. This ensures that results of the measures can easily be compared because the fuzzy set type does not affect the nature of the measure or its interpretation.

Additionally, although a single measure between fuzzy sets is useful, decision making often involves observing the outcomes of several different comparisons. To achieve this process using fuzzy sets, a new measure will be developed which fuses the concepts of similarity and distance, enabling the comparison of multiple features of fuzzy sets. The results of this measure should provide information that must typically be captured by both a similarity and a directional distance measure. Its main advantage is the representation of this information without one having to analyse and interpret the results of two separate measures.

These proposed measures can be useful in complex decision making. One such example is the application of recommendation systems. Using a database of fuzzy sets, products can be compared against a person's subjective and uncertain desires and the best fitting product is recommended.

Given the above, the objectives of this thesis are as follows:

1. Elucidate the relationship and differences between similarity and distance.
2. Develop a directional distance measure on fuzzy sets which can be applied to ranking.
3. Extend the distance measure to be able to compare non-normal and non-convex fuzzy sets.
4. Expand these measures to enable the same method comparison for type-1 and type-2 fuzzy sets.
5. Develop a measure which incorporates the concepts of similarity and distance together, providing the information of two measures within a single result.
6. Develop experiments that illustrate the advantages of these measures compared with the current literature.

-
7. Provide demonstrations of the proposed measures applied to decision making on recommendations.

1.3 Organisation of the thesis

The remainder of this thesis is organised as follows. Chapter 2 presents a background on fuzzy sets of type-1 and type-2, discusses the variety of type-2 models and covers methods of generating fuzzy sets that capture subjective information. Following this, a review of similarity and distances measure on fuzzy sets is given. The contrasting approaches taken to measure each concept are examined, and some gaps within the field are highlighted. A brief overview of aggregation operators is also given, which is used to fuse similarity and distance into a single measure. After this, a review of knowledge-based recommendation systems within the literature is given. The ideas and methods used within this field give an example of the application of similarity and distance in an area where fuzzy sets will be of benefit but are yet unexplored.

Chapter 3 examines gaps within distance measures in the literature. A new directional distance measure is developed which can be used to determine not only the difference between two fuzzy concepts but also understand which contains higher or lower values. The measure is developed for fuzzy sets that may be non-normal or non-convex.

Chapter 4 expands the work of Chapter 3 onto type-2 fuzzy sets. A distance measure for interval type-2 fuzzy sets is proposed, followed by a general method of extending interval type-2 measures to compare general type-2 fuzzy sets. This is applied to measure both similarity and distance. Both type-2 distance measures utilise the theory from the previous chapter and can thus measure the directional distance between non-normal and non-convex fuzzy sets; i.e., the same approach (from Chapter 3) may be applied to each type of fuzzy set.

Chapter 5 proposes a novel measure which fuses the comparisons of similar-

ity and distance. The disadvantages of the individual measures - as a result of missing information - are counterbalanced by the advantages of the other measure. This results in a single comparison on fuzzy sets that is more informative than similarity or distance alone.

Chapter 6 illustrates the utility of the proposed measures when used in applications. A demonstration is applied to a recommendation system in which the knowledge base consists of highly subjective information that is replete with ambiguity and contradictions. A person gives a relative description of their desired product (e.g., *something similar to this but with more/less of these attributes*) and relative comparisons - using a fusion of similarity and distance - are used to find the product which best matches the individual's desires.

Chapter 7 demonstrates the proposed recommendation system using data-driven type-1 and type-2 fuzzy sets that have complex, non-normal and non-convex membership functions. These demonstrations discuss how the results of the combined similarity and distance measure affect how well a product is recommended for a given query.

The final chapter discusses the conclusions, contributions and limitations of the research accomplished in this thesis. The scope for future work is also reviewed, addressing the need for methods with which the uncertainty and contradictions in the results of the measures can be better understood.

Note that, for simplicity, initial demonstrations of the new measures proposed in this thesis are given using synthetic fuzzy sets. This is because the properties of the measures are clearer if the fuzzy sets are simple. In later chapters, demonstrations are given on data-driven fuzzy sets, where data has been collected through surveys.

1.4 Contributions to Knowledge

The research in this thesis expands upon similarity and distance measures in the literature and develops new methods of measuring these concepts on complex fuzzy sets. These complexities stem from fuzzy set models that have non-normal or non-convex shapes or are three-dimensional (known as type-2 fuzzy sets), each of which introduces unique challenges. New measures are developed to determine the directional distance between fuzzy sets and to incorporate the concept of similarity and how this affects the perception of distance. New application areas are explored, where fuzzy sets are not typically used but provide a more natural representation of the data.

The contributions resulting from this thesis are as follows:

- A directional distance is proposed to compare fuzzy sets that may be non-normal or non-convex.
- Distance measures (directional and non-directional) are proposed for interval type-2 fuzzy sets.
- A general method of extending interval type-2 measures to general type-2 fuzzy sets is developed. This introduces a new distance measure and a new similarity measure on general type-2 fuzzy sets.
- A new measure based on the combined evaluation of similarity and distance is introduced.
- A recommendation system based on the relative comparisons of fuzzy information is developed and demonstrated.
- A new distance measure is developed that represents the distance between fuzzy sets as a fuzzy set.

This research has contributed to five peer-reviewed conference papers and one journal paper that is under review. These publications are listed below.

-
- McCulloch, Josie; Wagner, Christian; Khaled Bachour, Tom Rodden (2015)
“Give me what I want” - Enabling Complex Queries on Rich Multi-Attribute Data
In: International Conference on Fuzzy Systems 2015 (Fuzz-IEEE 2015), pp. TBC, 2015.
 - McCulloch, Josie; Wagner, Christian; Aickelin, Uwe (2014)
Analysing Fuzzy Sets Through Combining Measures of Similarity and Distance.
In: Proceedings of the 2014 World Congress on Computational Intelligence (WCCI 2014), pp. 155-162, 2014.
 - McCulloch, Josie; Hinde, Chris; Wagner, Christian; Aickelin, Uwe (2014)
A Fuzzy Directional Distance Measure.
In: Proceedings of the 2014 World Congress on Computational Intelligence (WCCI 2014), pp. 141-148, 2014.
 - McCulloch, Josie; Wagner, Christian; Aickelin, Uwe (2013)
Extending Similarity Measures of Interval Type-2 Fuzzy Sets to General Type-2 Fuzzy Sets.
In: International Conference on Fuzzy Systems 2013 (Fuzz-IEEE 2013), pp. 955-962, 2013.
 - McCulloch, Josie; Wagner, Christian; Aickelin, Uwe (2013)
Measuring the Directional Distance Between Fuzzy Sets.
In: UKCI 2013, the 13th Annual Workshop on Computational Intelligence, Surrey University, pp. 38-45, 2013.

Chapter 2

Background

2.1 Introduction

This chapter provides a background of the literature on which this thesis is based. First, Section 2.2 gives a theoretical study of fuzzy sets, after which Section 2.3 presents a review of techniques used to construct fuzzy sets from data. Section 2.4 gives a survey of relative comparisons on fuzzy sets, detailing different techniques that have been developed to calculate the similarities and distances between fuzzy sets. After this, Section 2.5 provides a brief overview of aggregation operators. Additionally, as the theoretical work developed in this thesis is demonstrated and applied to a knowledge-based recommendation system, a survey of such systems is presented in Section 2.6. Finally, Section 2.7 presents some conclusions to the literature survey.

As a variety of mathematical notations and functions are used throughout this thesis, Pages xvii and xviii provide look-up tables for quick reference. Additionally, Appendix A provides definitions for different properties of mathematical functions and indicates which properties are typically found in which measures.

2.2 Fuzzy Sets

Fuzzy sets are a simple yet powerful model of representing uncertainty. A fuzzy set is best described in comparison to a standard set, often referred to as a *crisp* set. In crisp set theory an element or object x completely belongs or does not belong to a set A . The membership of x within A is written as

$$\mu_A(x) = \begin{cases} 1 & \text{iff } x \in A \\ 0 & \text{iff } x \notin A \end{cases} \quad (2.1)$$

where iff is a shorthand for *if and only if*; thus, $\mu_A(x) \in \{0, 1\}$. Fuzzy sets [1] extend upon this idea by representing the membership $\mu_A(x)$ within the interval $[0, 1]$ (i.e., $\mu_A(x) \in [0, 1]$), where the values 0 and 1 have the same meaning as in a crisp set. However, it is now possible to represent the uncertainty that x belongs to A by any value between 0 and 1. A membership value close to 1 indicates that x has a high degree of membership within A and a value close to 0 indicates that x has a low degree of membership.

Fuzzy sets are particularly useful for modelling human perceptions as they are able to capture uncertainty from different points of view. Firstly, people naturally think in terms of fuzzy concepts rather than crisp values. For example, if someone is asked to describe the temperature of a room they are more likely to use a word such as *warm* than state the precise temperature.

However, what is *warm* is unclear as many temperatures may fit this description. More importantly, it is difficult to describe the boundaries of *warm*, i.e., at what temperature does a room cease to be warm and become *cool* or *hot*. Secondly, different people often have different perceptions of the same thing. For example, whilst someone from a cold climate may describe 18°C as warm, someone from a tropical climate may describe it as *cool*.

To model these two types of uncertainty one may use type-1 or type-2 fuzzy sets. Type-1 fuzzy sets are useful for defining an uncertain term from a single point of view (e.g., modelling one person's definition of *warm*). Type-2

fuzzy sets build upon this by representing additional degrees of uncertainty that cannot be captured in a type-1 model (e.g., modelling multiple people’s conflicting opinions on the definition of *warm*).

The remainder of this section presents the theoretical background on type-1 and type-2 fuzzy sets and how these can be used to model people’s perceptions.

2.2.1 Type-1 Fuzzy Sets

Fuzzy Set Notations

A fuzzy set is a concept developed by Zadeh [1] that can model the uncertainty of information. As stated earlier, the membership of an element x in a set A is a value that may lie anywhere in the interval $[0, 1]$.

Definition 1. *Let $T1(X)$ denote the set of all fuzzy sets in the universe of discourse X . The fuzzy set $A \in T1(X)$ may be defined by a set of pairs as*

$$A = \{(x, \mu_A(x)) \mid \forall x \in X\}, \quad (2.2)$$

where x is an element in X and $\mu_A(x) \in [0, 1]$ denotes the membership value of x in A . Where X is a continuous universe of discourse, this is often also expressed as

$$A = \int_{x \in X} \mu_A(x)/x, \quad (2.3)$$

where \int does not denote integration, but instead denotes the union of all admissible x within X with associated membership value $\mu_A(x)$. When X is discrete, this is often written as

$$A = \sum_{i=1}^N \mu_A(x_i)/x_i, \quad (2.4)$$

where \sum also denotes the union of all admissible x in X and N is the total number of discretisations in X .

Unless stated otherwise, throughout this thesis when X is discretised in the interval $[0, 10]$ the value N is fixed as 101, and where $X \in [1, 5]$ the value N is fixed at 41.

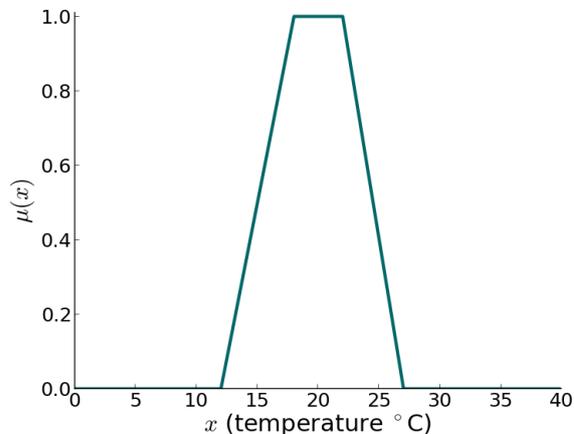


Figure 2.1: A type-1 fuzzy set representing *comfortably warm*.

Referring to the earlier example, the concept *comfortably warm* can be described as a fuzzy set for which the continuous universe of discourse X consists of temperatures and the membership value of each temperature represents the certainty that the temperature may be described as *comfortably warm*. An example of such a fuzzy set is shown in Figure 2.1. In this example, $\mu_{\text{warm}}(5) = 0.0$ indicates that 5°C is not warm, $\mu_{\text{warm}}(15) = 0.5$ suggests that 15°C is by equal amounts both possibly and possibly not warm, and $\mu_{\text{warm}}(20) = 1.0$ shows that 20°C is definitely warm.

Referring back to the mathematical representations, we will refer to equations (2.3) and (2.4) as vertical slices, because one effectively draws a vertical line at a given value x to find out the membership associated with x . Note that these representations are more commonly referred to as vertical slices in type-2 fuzzy sets [16] but, for consistency, will be given the same name for type-1 fuzzy sets.

Another common representation of fuzzy sets is the alpha-cut (α -cut) representation, which is a horizontal sliced approach. This, in contrast, involves drawing a horizontal line at a given membership value (denoted α) and finding which values of x have a membership of α or greater.

Definition 2. An α -cut of $A \in T1(X)$ is written as [17]

$$\overline{A}_\alpha = \{x \mid \mu_A(x) \geq \alpha, \alpha \in [0, 1]\}. \quad (2.5)$$

Definition 3. The fuzzy set A can be represented by its alpha-cuts (α -cuts) as [17]

$$A = \int_0^1 \alpha \overline{A}_\alpha,$$

where \int_0^1 is the union of all A_α within the continuous interval from 0 to 1, and $\alpha \overline{A}_\alpha$ is not multiplication but shows the mapping of pairs α and \overline{A}_α . Using a discrete range of α -cuts, this may be rewritten as

$$A = \sum_{m=1}^M \alpha_m \overline{A}_{\alpha_m},$$

where M is the total number of discretisations on the membership axis; i.e., the total number of α -cuts. Unless stated otherwise, throughout this thesis when α -cuts are discrete M is fixed as 10 such that the coordinates $\alpha \in \{0.05, 0.1, 0.15, \dots, 1.0\}$ are used.

Definition 4. The α -cut of a fuzzy set can be represented as a continuous interval. Thus, an α -cut may be rewritten as

$$\begin{aligned} \overline{A}_\alpha &= [\overline{A}_{\alpha L}, \overline{A}_{\alpha R}] \\ \overline{A}_{\alpha L} &= \min \{x \mid \mu_A(x) \geq \alpha, \alpha \in [0, 1]\} \\ \overline{A}_{\alpha R} &= \max \{x \mid \mu_A(x) \geq \alpha, \alpha \in [0, 1]\} \end{aligned}$$

However, this representation changes when fuzzy sets are non-normal or non-convex, as discussed next.

Non-Normal and Non-Convex Type-1 Fuzzy Sets

A fuzzy set may be normal or non-normal.

Definition 5. The height H_A of a fuzzy set A is its maximum membership value, defined as $\sup_{x \in X} \mu_A(x)$ [1].

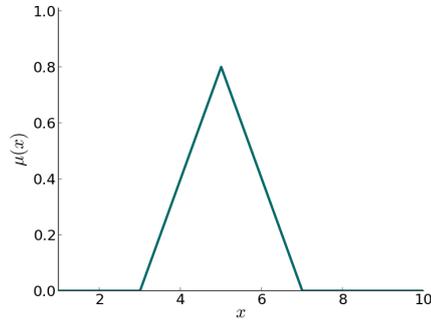


Figure 2.2: A non-normal fuzzy set.

Definition 6. A type-1 fuzzy set $A \in T1(X)$ is normal if there is at least one value of x that has certain membership within A ; i.e., $\exists x \in X, \mu_A(x) = 1.0$ or $H_A = 1.0$. If no such element exists then the fuzzy set is non-normal.

Figure 2.2 shows an example of a non-normal fuzzy set with a height of 0.8.

Whether a fuzzy set is normal depends on a multitude of factors which are essentially a choice by the designer. In some applications it may be necessary for all fuzzy sets to be normal, and in other applications this may not be sensible or possible [18]. Non-normal fuzzy sets can introduce complications if the α -cut representation of fuzzy sets is required. When a fuzzy set is normal and convex (see Definition 7), any given α -cut can be represented as a continuous interval. However, if the fuzzy set is non-normal then any α -cut exceeding its height will be the empty set. For example, in Figure 2.2, $\overline{A}_\alpha = \emptyset$ where $\alpha > 0.8$.

Another design choice when constructing fuzzy sets is that of convexity. Typically, a convex membership function is chosen, but complex data may require a more complex model that is non-convex [18].

Definition 7. A fuzzy set A is convex if and only if all of its α -cuts are

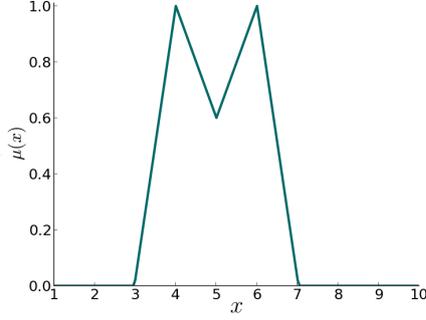


Figure 2.3: A non-convex fuzzy set.

continuous. This is defined as [1]

$$\forall x_1 \in X, \forall x_2 \in X, \forall \lambda \in [0, 1],$$

$$\mu_A(\lambda x_1 + (1 - \lambda)x_2) \geq \min\{\mu_A(x_1), \mu_A(x_2)\}. \quad (2.6)$$

Thus, a fuzzy set that does not satisfy (2.6) is non-convex.

The α -cut representation of non-convex fuzzy sets cannot be represented by a continuous interval (as is possible with convex fuzzy sets). For example, Figure 2.3 shows an example of a non-convex fuzzy set. Any α -cut at $\alpha > 0.6$ can only be represented by a discontinuous interval, for example at $\alpha = 0.5$, $\overline{A}_\alpha = [3.49, 6.51]$, however at $\alpha = 0.7$, $\overline{A}_\alpha = \{[3.69, 4.75], [5.25, 6.31]\}$, which contains two intervals that do not intersect. Such intervals will be referred to as *discontinuous intervals*.

Definition 8. Let a discontinuous interval $\overline{\overline{H}}$ be [19]

$$\overline{\overline{H}} = \bigcup_{i=1}^I [\overline{H}]_i \quad (2.7)$$

where $[\overline{H}]_i$ represents the i^{th} continuous interval within $\overline{\overline{H}}$ and I is the total number of intervals within $\overline{\overline{H}}$.

Set-Theoretic Operations on Type-1 Fuzzy Sets

Set-theoretic operations are commonly used, for example, to join two fuzzy sets together. These operations are used in many applications, such as fuzzy logic systems [20] or when calculating the similarity between fuzzy sets (see Section 2.4.1). The most common set-theoretic operations are union, intersection and complement. To calculate the intersection and union of fuzzy sets, any given membership value is the t-norm or t-conorm of the given fuzzy sets' membership values, respectively.

Definition 9. *The intersection of two fuzzy sets $A, B \in T1(X)$ is*

$$A \cap B = \{(x, t(\{\mu_A(x), \mu_B(x)\})) \mid \forall x \in X\}$$

where t is any t-norm. For any given value x , this may be written as

$$\mu_{A \cap B}(x) = t(\{\mu_A(x), \mu_B(x)\}).$$

Definition 10. *Formally, the union of two fuzzy sets $A, B \in T1(X)$ is*

$$A \cup B = \{(x, t'(\{\mu_A(x), \mu_B(x)\})) \mid \forall x \in X\}$$

where t' is any t-conorm. For any given value x , this is often written as

$$\mu_{A \cup B}(x) = t'(\{\mu_A(x), \mu_B(x)\}).$$

Typically, t and t' are the minimum and maximum t-norm and t-conorms. However, there are many others available in the literature.

Definition 11. *The complement of a fuzzy set is represented by the complement of its membership value for each $x \in X$. The complement of A , denoted A' is*

$$A' = \{(x, 1 - \mu_A(x)) \mid \forall x \in X\}.$$

Another common calculation on fuzzy sets is known as defuzzification. This involves reducing a fuzzy set to a single value by essentially using the average value (often the mean or median) of the fuzzy set. Defuzzification is commonly used to provide an easy to understand output in expert systems [20].

Definition 12. *To reduce a fuzzy set to a single crisp value, the centroid of a fuzzy set $A \in T1(X)$ is [20]*

$$A_c = \frac{\sum_{i=1}^N x_i \mu_A(x_i)}{\sum_{i=1}^N \mu_A(x_i)}, \quad (2.8)$$

where N is the total number of discretisations used on the x -axis.

In this thesis, (2.8) is used to achieve defuzzification. However, the reader should be aware that there are several other methods within the literature [20].

2.2.2 General Type-2 Fuzzy Sets

Ten years after the establishment of fuzzy sets (often referred to as type-1 fuzzy sets), Zadeh [17] introduced the concept of a type-2 fuzzy set as a model of a linguistic variable. The difference between type-1 and type-2 fuzzy sets lies in the representation of a value's membership. In a type-1 fuzzy set, the membership of x in A is represented by a single value within $[0, 1]$. However, the membership of x in a type-2 fuzzy set \tilde{A} is represented by a type-1 fuzzy set with the universe of discourse in $[0, 1]$.

Definition 13. *Let $GT2(X)$ represent the set of all general type-2 fuzzy sets within X , then the fuzzy set $\tilde{A} \in GT2(X)$ is formally written in terms of a set of pairs as [16]*

$$\tilde{A} = \{((x, u), \mu_{\tilde{A}}(x, u)) \mid \forall x \in X, \forall u \in J_x \subseteq [0, 1]\}, \quad (2.9)$$

where x is the primary variable in X , u is the secondary variable which has the domain $J_x \in [0, 1]$, and the amplitude of $\mu_{\tilde{A}}(x, u)$ is known as the secondary

grade. If X is a discrete universe of discourse then \tilde{A} is often rewritten as [16]

$$\tilde{A} = \sum_{x \in X} \sum_{u \in J_x} \mu_{\tilde{A}}(x, u) / (x, u) \quad J_x \subseteq [0, 1], \quad (2.10)$$

where $\sum \sum$ denotes the union over all admissible x and u . In a continuous universe of discourse, $\sum \sum$ is replaced with $\int \int$.

These are referred to as vertical slice representations [16].

Note that the terms for the universe of discourse, primary membership and secondary membership are often labelled x , u , and μ , respectively, within general type-2 literature. However, when describing type-1 fuzzy sets x and μ are used to describe the universe of discourse and the primary membership, respectively.

Additionally, throughout this thesis, when discussing the non-normality or non-convexity of type-2 fuzzy sets, it is only the primary membership (the (x, u) pairs) in which this is explored. Non-normal and non-convex memberships may also be modelled within the secondary membership functions (in $\mu(x, u)$). However, this is outside the scope of this thesis.

The introduction of secondary membership functions is useful as it enables one to more correctly define membership values that are noisy or uncertain by using a less precise representation. This additional uncertainty can be a result of collecting data from noisy devices (e.g., from sensors) or from individuals who have differing opinions (e.g., from a survey).

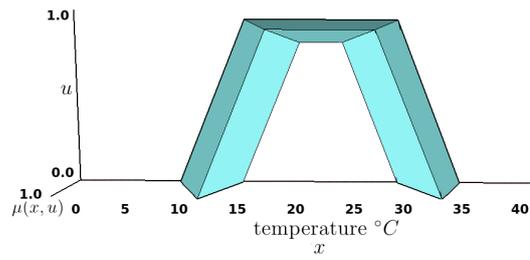
As stated earlier, a type-2 fuzzy set represents uncertainty that cannot be modelled by a type-1 fuzzy set. Using the earlier example in which the concept *warm* is defined, a type-2 fuzzy set can depict multiple people's opinions on the definition of *warm*. Figure 2.4 shows a general type-2 fuzzy set representing the concept *warm*. Figure 2.4 shows a general type-2 fuzzy set representing the concept *warm* using a three-dimensional model. Taking a vertical slice at $x = 21$ produces the type-1 secondary membership function shown in Figure 2.4c. This shows that 21°C is possibly warm with a membership of approximately

0.9. From this, we can see that the majority of people agree that $21^{\circ}C$ is *warm*, but there is no complete consensus.

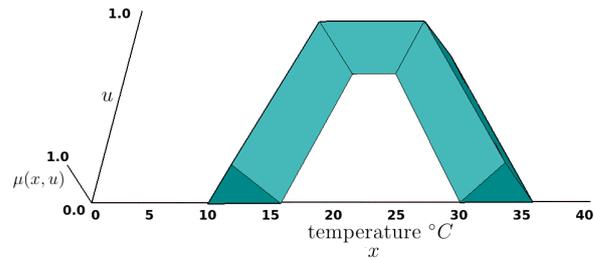
As a result of their three-dimensional nature, the computational complexity of modelling and performing calculations on general type-2 fuzzy sets is significantly higher than that of type-1 fuzzy sets. To handle this increased complexity there have been many different representations of general type-2 fuzzy sets over the years. As well as describing the fuzzy sets as a collection of vertical slices (2.9), the most well known representations include using embedded membership functions and wavy-slices [16], which involve representing a type-2 fuzzy set as a collection of its embedded type-1 fuzzy sets. These methods are well established and have been used in numerous publications in the literature.

Another approach is the geometric representation [21], which uses computational geometry to enable one to model secondary membership functions without the need for discretisation. Other methods include the alpha-plane model [22] and the zSlices approach [23], which use strict discretisations in the secondary membership values. Although these two methods go by different names, the theory is the same [24].

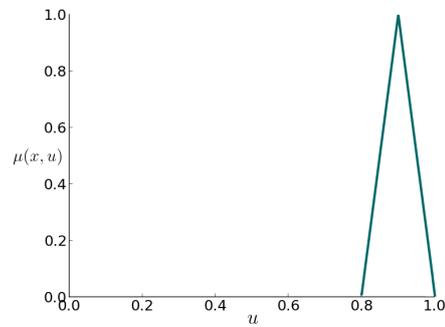
Throughout this thesis, the zSlices approach has been chosen to simplify the representation of general type-2 fuzzy sets and thus the zSlices notations will be used. However, if one wishes, it is also possible to represent all of the given general type-2 theory using the alpha-plane notations [25]. A zSlices type-2 fuzzy set is best represented as a collection of interval type-2 fuzzy sets, thus the next section will introduce interval type-2 fuzzy sets, followed by details of the zSlices representation in the succeeding section.



(a) View of the front



(b) View from below



(c) View of the secondary membership function at $x = 21$.

Figure 2.4: A general type-2 fuzzy set representing *comfortably warm*. x is the universe of discourse, u is the primary membership and $\mu(x, u)$ is the secondary membership at x and u .

2.2.3 Interval Type-2 Fuzzy Sets

Type-2 fuzzy have often been less popular than type-1 fuzzy sets due to the increased complexities that come with modelling them. Firstly, they are more difficult to draw or conceptualise because they can only be fully depicted by a three-dimensional image; unlike type-1 fuzzy sets which are two-dimensional. Additionally, formulae such as union and intersection are less straightforward than for type-1 fuzzy sets and are more computationally complex [16]. To reduce these issues, interval type-2 fuzzy sets have frequently been used in the literature as an alternative to the general type-2 form. This representation involves a drastic simplification of the secondary membership functions.

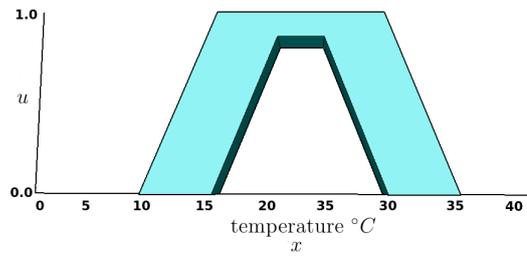
In an interval type-2 fuzzy set, the membership of a value x is not represented by a type-1 fuzzy set, but instead by an interval. The values contained in this interval secondary membership function are the same as contained in the type-1 secondary membership function of a general type-2 fuzzy set, but now all of the membership values that were greater than 0 (in the general type-2 case) are changed to 1.

Definition 14. Let $IT2(X)$ represent the set of all interval type-2 fuzzy sets within X . The fuzzy set $\tilde{A} \in IT2(X)$ is formally written as [26]

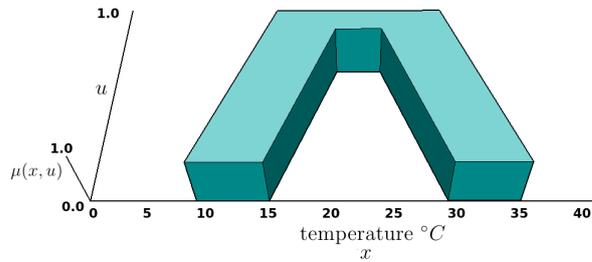
$$\tilde{A} = \{((x, u), \mu_{\tilde{A}}(x, u) = 1) \mid \forall x \in X, \forall u \in J_x \subseteq [0, 1]\} \quad (2.11)$$

Note, this is a vertical slice representation, where J_x denotes each vertical slice.

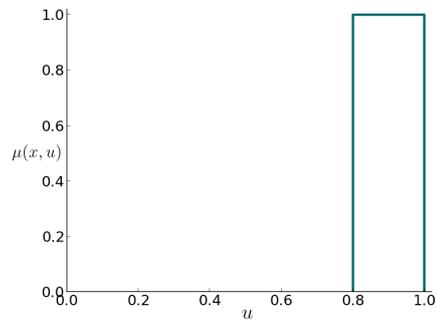
Figure 2.5 shows an interval type-2 model of the general type-2 fuzzy set previously shown in Figure 2.4. From this figure, the reader should be able to see that interval type-2 fuzzy sets are a simplification of general type-2 fuzzy sets. The vertical slice at $x = 21$ in Figure 2.5c shows the possibility that $21^\circ C$ is considered to be *warm*. In this case, there is agreement that there is at least a 0.8 possibility that $21^\circ C$ is *warm*. Note that this is a more simplified interpretation of the secondary membership function of the general type-2 fuzzy set in Figure 2.4c.



(a) View of the front



(b) View from below



(c) View of the secondary membership function at $x = 21$.

Figure 2.5: An interval type-2 fuzzy set representing *comfortably warm*. x is the universe of discourse, u is the primary membership and $\mu(x, u)$ is the secondary membership at x and u

The interval boundaries of an interval type-2 fuzzy set are often referred to as lower and upper membership functions. For example, in Figure 2.5, the lower membership function is the trapezoid bounded within $15 \leq x \leq 30$ and its membership values are denoted $\underline{\mu}_{\tilde{A}}(x), \forall x \in X$. Likewise, the upper membership function is the trapezoid bounded within $10 \leq x \leq 35$ with its membership values denoted $\overline{\mu}_{\tilde{A}}(x), \forall x \in X$. This is described formally, next.

Definition 15. *The vertical slice J_x of $\tilde{A} \in IT2(X)$ is written as [26]*

$$J_x = [\underline{\mu}_{\tilde{A}}(x), \overline{\mu}_{\tilde{A}}(x)], \forall x \in X,$$

where $\underline{\mu}_{\tilde{A}}(x)$ and $\overline{\mu}_{\tilde{A}}(x)$ refer to the lower and upper membership functions, respectively. The union of bounded regions J_x is commonly referred to as the footprint of uncertainty; this is the region where $\mu(x, u) = 1$.

Definition 16. *The α -cut of an interval type-2 fuzzy set may be represented by the α -cuts of the upper and lower membership functions; throughout this thesis this is denoted $\tilde{A}_\alpha = \{\overline{\tilde{A}_{\alpha W}}, \overline{\tilde{A}_{\alpha U}}\}$ for $\tilde{A} \in IT2(X)$ where $\overline{\tilde{A}_{\alpha W}}$ and $\overline{\tilde{A}_{\alpha U}}$ are the α -cuts of the lower and upper membership functions of \tilde{A} , respectively.*

Note that the letters W and U have been used to distinguish between the α -cuts of the lower and upper membership functions, whilst the letters L and R distinguish between the left and right boundaries of the continuous intervals within $\overline{\tilde{A}_{\alpha W}}$ and $\overline{\tilde{A}_{\alpha U}}$.

One should bear in mind that it is common for the lower membership function of a type-2 fuzzy set to be non-normal (as is the case in Figure 2.5), thus any α -cuts above the height of the lower membership function will be empty (e.g., where $\alpha = 0.9$ in Figure 2.5).

Defuzzification of an interval type-2 fuzzy set is typically referred to as type reduction as it reduces the type-2 fuzzy set to a type-1 fuzzy set. This is achieved by calculating the centroid of each embedded type-1 fuzzy set within the type-2 fuzzy set. The Karnik-Mendel algorithms are the most well known

methods of achieving this [27]. The type-reduced centre of an interval type-2 fuzzy set is an interval-bounded type-1 fuzzy set. For $\tilde{A} \in IT2(X)$, the type reduced set of \tilde{A} will be denoted $C(\tilde{A}) = [C_L(\tilde{A}), C_R(\tilde{A})]$.

Note that the union, intersection and centroid of interval type-2 fuzzy sets have been well studied, but are not used within this thesis. One may refer to [26] for such operations.

2.2.4 zSlices-Based General Type-2 Fuzzy Sets

As stated in Section 2.2.2, the zSlices [23] and alpha-plane [22] approaches are identical methods of simplifying the representation of general type-2 fuzzy sets. Throughout this thesis, the zSlices notations will be used.

A zSlices type-2 fuzzy set can be composed by slicing a general type-2 fuzzy set along the z -axis (or the $\mu(x, u)$ axis), effectively breaking the fuzzy set down into many interval type-2 fuzzy sets called zSlices. However, unlike regular interval type-2 fuzzy sets that have a secondary membership grade of 1, each zSlice has a height of z_i , referred to as the zLevel.

Definition 17. *The zSlice Z_i , whose secondary membership grade is z_i , is written as [23]*

$$\tilde{Z}_i = \{((x, u), \mu_{\tilde{Z}_i}(x, u) = z_i) \mid \forall x \in X, \forall u \in J_x \subseteq [0, 1]\}. \quad (2.12)$$

This is the vertical slice representation of an individual zSlice.

Definition 18. *The fuzzy set \tilde{A} is represented as a collection of zSlices [23] as*

$$\tilde{A} = \sum_{i=1}^I \tilde{Z}_i, \quad (2.13)$$

where \sum also denotes the union of all admissible z_i and I is the total number of zSlices.

Note that $\mu_{\tilde{A}_{z_i}}(x)$ is used to denote the primary membership of the i^{th} zSlice of \tilde{A} and, as in interval type-2 fuzzy sets, $\bar{\mu}_{\tilde{A}_{z_i}}(x)$ and $\underline{\mu}_{\tilde{A}_{z_i}}(x)$ represent the upper and lower membership values, respectively, of the zSlice \tilde{A}_{z_i} at x .

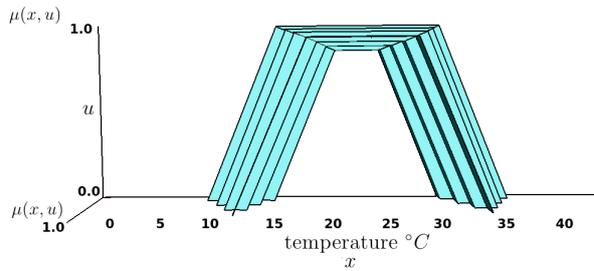
The zSlice Z_0 is disregarded because its secondary grade is 0 and thus it does not contribute to the fuzzy set [23]. As more zSlices are used to represent a general type-2 fuzzy set, the zSlices-based representation of the original set becomes more accurate. Additionally, if only one zSlice is used then the zSlices representation reduces to an interval type-2 fuzzy set.

In addition to simplifying general type-2 fuzzy sets, “Pure” zSlices-based fuzzy sets (i.e., sets that do not simplify general type-2 fuzzy sets but are intended to be only zSlices-based) have also been used in the literature to model agreement shared between individuals [28, 29].

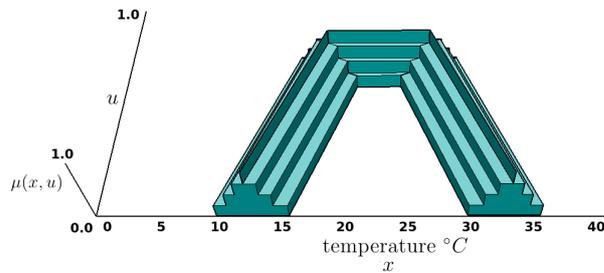
Referring to the earlier example in which *warm* is represented by a fuzzy set, Figure 2.6 depicts the general type-2 example (in Figure 2.4) as a zSlices fuzzy set using four zSlices. One can see in this figure that a zSlices fuzzy set is represented by many interval type-2 fuzzy sets of differing heights. As the height of a zSlice increases, the area of its footprint of uncertainty (where $\mu_{\tilde{Z}_i}(x, u) > 0$) decreases; thus each zSlice represents a higher degree of membership than the previous slice. Although only four zSlices are used in this figure, if this number is increased the model will more closely represent the general type-2 fuzzy set in Figure 2.4. Note also that the vertical slice in Figure 2.6c is a discretised version of the general type-2 case shown in Figure 2.4c.

To calculate operations such as union, intersection and centroid, one can apply the interval type-2 methods to each zSlice and aggregate the results, thereby achieving the operation on a general type-2 fuzzy set [23]. This technique will be used later in the thesis to apply methods of comparing interval type-2 fuzzy sets to enable the same comparison on general type-2 fuzzy sets.

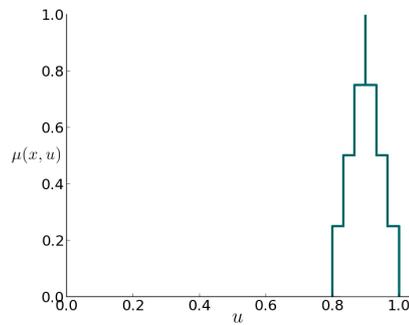
Also using this approach, the α -cuts of a zSlices fuzzy set can be represented by the α -cuts of each zSlice.



(a) View of the front.



(b) View from below.



(c) View of the secondary membership function at $x = 21$.

Figure 2.6: A zSlices general type-2 fuzzy set representing *comfortably warm*. x is the universe of discourse, u is the primary membership and $\mu(x, u)$ is the secondary membership at x and u .

Definition 19. *The α -cut of an interval type-2 fuzzy set \tilde{A} is $\tilde{A}_\alpha = \{\overline{\tilde{A}_{\alpha_W}}, \overline{\tilde{A}_{\alpha_U}}\}$ (described in Definition 16), therefore the α -cut of each zSlice of a zSlices general type-2 fuzzy set \tilde{A} may be represented as $\tilde{A}_{z_i\alpha} = \{\overline{\tilde{A}_{z_i\alpha_W}}, \overline{\tilde{A}_{z_i\alpha_U}}\}$. Following from this, the α -cuts of all zSlices may be represented as a set of pairs as $\tilde{A}_\alpha = \{(z_i, \tilde{A}_{z_i\alpha}), \forall z_i \in Z\}$.*

Note that the union, intersection and centroid of zSlices general type-2 fuzzy sets have been well studied, but are not used within this thesis. One may refer to [23] for such operations.

This section has presented a theoretical background on the mathematical representations of different types of fuzzy sets and their properties. The next section gives an overview of how the membership functions of these fuzzy sets can be constructed from data.

2.3 Constructing Membership Functions

Many techniques have been developed in the literature to generate membership functions of fuzzy sets from data and several surveys of methods have been written [30–32]. Constructing type-1 memberships functions have been most commonly researched. However, several methods of constructing type-2 fuzzy sets have also been developed in recent years. This section gives an overview of different approaches within the literature.

2.3.1 Type-1 Fuzzy Sets

This section describes four unique methods that have often been used to generate the membership function of a type-1 fuzzy set from data.

The *polling technique* [33] involves asking a group of n subjects if “ x belongs to A ” is a true or false statement for some $x \in X$; for example “ x is 18°C warm?” Given a set of subjects $\{s_1, s_2, \dots, s_n\}$, let $s_i(x)$ denote the answer from

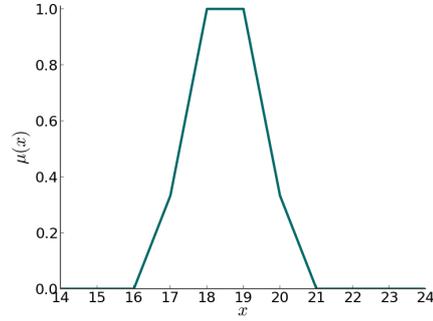


Figure 2.7: A fuzzy set representing the data (2.15) using the polling technique (2.14).

the subject s_i for the value x , where $s_i(x) = 1$ if the statement is true and is 0 otherwise. The membership value of x in the fuzzy set A is then given as

$$\mu_A(x) = \frac{\sum_{i=1}^n s_i(x)}{n}. \quad (2.14)$$

where n is the total number of subjects.

For example, suppose three subjects are asked to state which temperatures from $\{17, 18, 19, 20\}$ describe *room temperature*, the results being

$$\begin{aligned} s_1(17) &= 0, & s_1(18) &= 1, & s_1(19) &= 1, & s_1(20) &= 0 \\ s_2(17) &= 1, & s_2(18) &= 1, & s_2(19) &= 1, & s_2(20) &= 0 \\ s_3(17) &= 0, & s_3(18) &= 1, & s_3(19) &= 1, & s_3(20) &= 1 \end{aligned} \quad (2.15)$$

Using (2.14), these form the fuzzy set

$$\{(17, 0.33), (18, 1.0), (19, 1.0), (20, 0.33)\}.$$

Figure 2.7 represents this fuzzy set, using linear interpolation between integers.

Note, one can also weight $s_i(x)$ based on the expertise of s_i ; i.e., subjects with more knowledge of A are given a higher impact/weight in $\mu_A(x)$ [34].

Reverse rating [35] involves asking subjects what value x has a given membership value μ in the fuzzy set A . To help decide the value of x , surveys

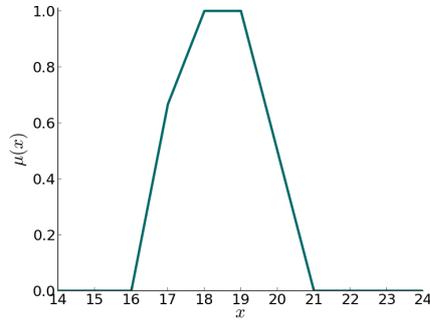


Figure 2.8: A fuzzy set representing the data (2.16) using the reverse rating technique (2.14).

often present a restricted set of x values from which the subject can choose [35]. For example, “given a list of houses, which best represent a *pleasing house* with a membership of 0.8?” The fuzzy set is constructed in the same manner shown in (2.14) where $s_i(x) \in [0, 1]$.

For example, consider if subjects are asked to state from the set of temperatures $\{17, 18, 19, 20\}$ which temperature best describes *room temperature* with certainties 0.5 and 1.0. The results from three subjects are

$$\begin{aligned}
 s_1(17) &= 0.5, & s_1(18) &= 1.0, & s_1(19) &= 1.0 \\
 s_2(17) &= 1.0, & s_2(18) &= 1.0, & s_2(19) &= 1.0, & s_2(20) &= 0.5 \\
 s_3(17) &= 0.5, & s_3(18) &= 1.0, & s_3(19) &= 1.0, & s_3(20) &= 1.0
 \end{aligned} \tag{2.16}$$

Using (2.14), these form the fuzzy set

$$\{(17, 0.67), (18, 1.0), (19, 1.0), (20, 0.5)\}.$$

Figure 2.8 represents this fuzzy set, using linear interpolation between integers.

Note that the polling and reverse rating methods result in highly discretised membership functions. However, it is often necessary to have a continuous function, for example to enable the extraction of α -cuts, as necessary for a number of operations. There are several techniques in the literature that

achieve this. For example, one may use linear interpolation [35], Lagrange interpolation [34], or least-square curve fitting [34]. As this thesis focuses on comparisons between fuzzy sets rather than their construction, linear interpolation is adequate and will be used throughout.

Another method of avoiding discretised data is to assign membership values to interval values rather than singletons. Two interval-based techniques have been developed to construct type-1 membership functions.

The *interval estimation-1* approach [36] is a technique that is similar to *reverse rating*. Subjects give a range of values $[x_l, x_r]$ that have a given membership value μ in the fuzzy set A . For example, “what range of heights best represents a *tall person* with a membership of 0.8?” Multiple subjects answers are joined together in the same manner as the reverse rating method.

For example, consider if subjects are asked to state from a range of temperatures within $[17, 20]$ which intervals best describe *room temperature* with certainties 0.5 and 1.0. Consider three subjects who gave the results

$$\begin{aligned} s_1([17, 18]) &= 0.5, & s_1([18, 19]) &= 1.0 \\ s_2([17, 19]) &= 1.0, & s_2([19, 20]) &= 0.5 \\ s_3([17, 18]) &= 0.5, & s_3([18, 20]) &= 1.0 \end{aligned} \tag{2.17}$$

Using (2.14), these form the fuzzy set

$$\{([17, 20], 0.5), ([17, 19], 0.67), ([18, 19], 0.83), (19, 1.0)\}.$$

Note that if a value appears in multiple intervals, its highest assigned membership is used. Figure 2.9 represents this fuzzy set.

Interval estimation-2 [32] is another method in which subjects give a range of values. However, this range is not associated with a specific membership value; i.e., from the subject’s point of view, everything within the interval has a membership value of 1, and everything outside has zero membership. For example, “what range of heights best represents a *tall person*?” The results are then aggregated to create a type-1 fuzzy set.

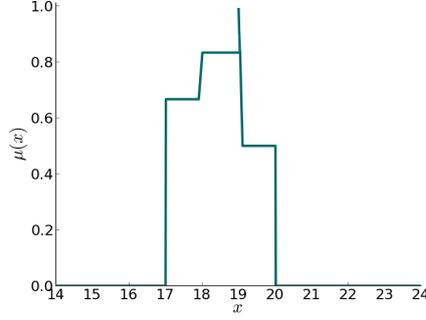


Figure 2.9: A fuzzy set representing the data (2.17) using the interval estimation-1 technique (2.14).

For example, Wagner et al. [37] introduced the *Interval Agreement Approach* (a method of interval estimation-2), which uses interval-valued data to construct type-1 and general type-2 fuzzy sets for the goal of capturing inter- and intra-source uncertainty. A set of intervals $\bar{A} = \{\bar{A}_1, \bar{A}_2, \dots, \bar{A}_n\}$ is constructed into a type-1 fuzzy set as

$$\begin{aligned}
A = & y_1 / \bigcup_{i_1=1} \bar{A}_{i_1} \\
& + y_2 / \left(\bigcup_{i_1=1}^{n-1} \bigcup_{i_2=i_1+1}^{n-1} (\bar{A}_{i_1} \cap \bar{A}_{i_2}) \right) \\
& + y_3 / \left(\bigcup_{i_1=1}^{n-2} \bigcup_{i_2=i_1+1}^{n-1} \bigcup_{i_3=i_2+1}^n (\bar{A}_{i_1} \cap \bar{A}_{i_2} \cap \bar{A}_{i_3}) \right) \\
& + \dots \\
& + y_n / \left(\bigcup_{i_1=1}^1 \dots \bigcup_{i_n=n}^n (\bar{A}_{i_1} \cap \dots \cap \bar{A}_{i_n}) \right). \tag{2.18}
\end{aligned}$$

Using the same example as (2.15), subjects may choose intervals of temperatures that represent *room temperature* as

$$\begin{aligned}
s_1([18, 19]) &= 1 \\
s_2([17, 19]) &= 1 \\
s_3([18, 20]) &= 1 \tag{2.19}
\end{aligned}$$

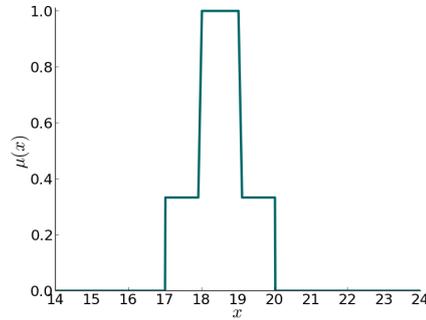


Figure 2.10: A fuzzy set representing the data (2.19) using the interval estimation-2 technique (2.18).

Using (2.18) this is constructed into the fuzzy set shown in Figure 2.10.

This thesis focuses on constructing membership functions using the polling technique and interval estimation-2. These are chosen because they do not require subjects to have an understanding of membership values. This simplifies the data collection process as it distances subjects from needing an understanding of the mathematics behind fuzzy set theory.

2.3.2 Interval Type-2 Fuzzy Sets

Mendel [38] proposed two methods of constructing interval type-2 fuzzy sets from data, referred to as the *person-MF approach* and *interval endpoints approach*.

In the *person-MF approach*, subjects define interval type-2 fuzzy sets to represent their uncertainty in the definition of a given linguistic term. Multiple fuzzy sets are collected and aggregated into a single interval type-2 fuzzy set which gives an overall representation of each subject's uncertainty. This method, however, has the disadvantage that it is not developed to capture the agreement between the fuzzy set given by each subject (a general type-2 model is required to capture this).

In the *interval endpoints approach*, subjects provide an interval of

values to represent a given term. The means and standard deviations of the end points are then used to generate an interval type-2 fuzzy set that models the collection of intervals. However, reducing the data to its mean and standard deviation simplifies the model and does not fully capture disagreement between individuals.

In addition to the above, Liu and Mendel [39] proposed the *Interval Approach*, which was later expanded into the *Enhanced Interval Approach*.

The ***Interval Approach*** maps interval-valued data to an interval type-2 fuzzy set with the goal of modelling linguistic variables. Subjects provide an interval of values that they believe represents a given linguistic term. Each interval is converted into a type-1 membership function and these are treated as embedded type-1 membership functions of an interval type-2 fuzzy set. Any type-1 membership functions that fall outside of a given range are removed and the upper and lower membership functions of the interval type-2 fuzzy set are defined by the union and intersection of the remaining embedded type-1 fuzzy sets, respectively.

The ***Enhanced Interval Approach*** [40] has also been developed which aims to overcome limitations of the *interval approach*. Limitations in the original method include wide footprints of uncertainty and small heights in the lower membership function.

2.3.3 zSlices General Type-2 Fuzzy Sets

The ***Interval Agreement Approach*** [37], as introduced in Section 2.3.1, uses interval-valued data to construct type-1 and general type-2 fuzzy sets. The method of generating type-1 membership functions is described in (2.18). This is expanded to general type-2 fuzzy sets to capture additional information. In this method, a collection of type-1 fuzzy sets are aggregated into general type-2 fuzzy sets to capture their agreement. Given a series of type-1 fuzzy

sets A_n where $n \in \{1, \dots, N\}$ and N is the total number of type-1 fuzzy sets, these are aggregated as [37]

$$\begin{aligned}
\mu(\tilde{A}) &= z_1 / \left(\bigcup_{i_1=1}^N A_{i_1} \right) \\
&+ z_2 / \left(\bigcup_{i_1=1}^{N-1} \bigcup_{i_2=i_1+1}^N (A_{i_1} \cap A_{i_2}) \right) \\
&+ z_3 / \left(\bigcup_{i_1=1}^{N-2} \bigcup_{i_2=i_1+1}^{N-1} \bigcup_{i_3=i_2+1}^N (A_{i_1} \cap A_{i_2} \cap A_{i_3}) \right) \\
&+ \dots \\
&+ z_N / \left(\bigcup_{i_1=1}^1 \dots \bigcup_{i_N=N}^N (A_{i_1} \cap \dots \cap A_{i_N}) \right), \tag{2.20}
\end{aligned}$$

where $z_i = \frac{i}{N}$.

In a similar example, Wagner and Hagraas [28] constructed zSlices fuzzy sets as a collection of interval type-2 fuzzy sets. Multiple interval type-2 fuzzy sets are generated to model a sensor over several days. These fuzzy sets are then aggregated to produce a zSlices general type-2 fuzzy set. Higher secondary membership values (zLevels) occur where there is more agreement between the interval type-2 fuzzy sets.

2.3.4 Considering Non-Normal and Non-Convex Type-1 Membership Functions

Much of the literature focuses on using fuzzy sets that are restricted to normal and convex membership functions. In fact, in many cases, the shape of the membership function is limited to common forms, including triangular, left-shoulder, right-shoulder, trapezoidal and Gaussian distributions. However, these shapes are often a poor representation of linguistic terms because they cannot show irregularities in data [18].

Although non-convex membership functions have gained little attention in

the literature, non-normal fuzzy sets have been utilised many times [11, 14, 15, 41]. One should note that when using the *polling* and *direct rating* techniques (described in Section 2.3.1), it is highly possible that no value x will result with a membership value of 1 within a fuzzy set A ; this will only occur if everybody surveyed agrees that a given x belongs to A . Thus, a non-normal membership function conveys a lack of perfect agreement between people.

Non-convex fuzzy sets are less common in the literature but may be more suitable to represent data than a convex membership function. For example, Garibaldi and John [18] present three cases in which non-convex fuzzy sets may occur. The first is in non-time-related contexts, for example representing how desirable a glass of milk is at varying temperatures. Subjects tend to give high ratings for cold and hot temperatures, but a low rating for warm temperatures, resulting in a non-convex membership function. The second example is in time-related contexts, for example the possibility of eating a meal given the time of day is non-convex, with higher membership values occurring around popular meal times. The final example is of the consequent fuzzy set that results from a fuzzy logic system.

Considering this, it is important to consider that membership functions should not always be restricted to simple, normal, convex shapes. As stated earlier, throughout this thesis the *polling* and *Interval estimation-2* approaches will be used. Using these approaches, it is highly likely that the resulting fuzzy sets will be non-normal or non-convex as no restrictions or pre-processing is applied to simplify the data to stop such membership functions from occurring. Such pre-processing is avoided because simplifying the fuzzy sets results in a model that represents different data to the original data.

Having explored the different types of fuzzy sets and methods of constructing their membership functions, the next section focuses on how relative comparisons can be used to calculate the similarities and distances between fuzzy sets.

2.4 Measures on Fuzzy Sets

This thesis focuses on developing measures of similarity and distance on fuzzy sets. Given this, it is important to make the distinction between these two types of measures clear.

Similarity is a frequently used concept that stems from human thought processes. It involves recognising patterns and making associations which enable one to classify objects and concepts. Similarity involves the highly context-dependent comparison of features that are often qualitative in nature [42]. As a result, it is often difficult to compare objects or concepts because the features of importance may differ from different people's perspectives. Similarity is most commonly applied to solve problems in the domains of categorisation, classification and clustering. In these examples, a new object is classified into a given category if it is more similar to the objects within that category compared to those in other categories.

Distance is also a widely used concept to measure the space or length between two points, sets or objects. Methods of calculating distance, as one would expect, are dependent on the given context. For example, the distance between geometric, numerical data will be calculated using a different approach to measuring the distance between non-numerical sets. Additionally, the properties of the data affect the properties of the measure. For example, in a directional graph, the shortest route from A to B may be different to the shortest route from B to A . In this case, a distance measure should not be symmetrical, even though symmetry is an important property of distance in many other contexts.

The most common measures of distance are metrics. A metric is a function that defines the distance between two points in metric space. A metric space is a set in which the distances between points are clearly defined (by the metric). One of the most common metrics is the Euclidean distance, which measures

the distance between points in Euclidean space. Note that the properties of a metric are important as they strictly correspond to the metric space in which they are used.

The remainder of this section provides a survey of similarity and distance measures on fuzzy sets. Note that Appendix A provides an overview of the properties of each type of measure for quick reference.

2.4.1 Similarity Measures

Within the context of fuzzy sets, the concept of similarity was first introduced by Zadeh in 1971 [43]. After this, due to the complex and context-dependent nature of defining what is similarity, many different methods have been developed. To provide some organisation to these approaches, there have been several comparative studies which shed light on the variety of similarity measures in the literature. Some focus on applications, such as image retrieval [44] and data mining [45], whilst others provide a more general analysis of the literature [46–49].

Measures of comparing similarity on fuzzy sets have been applied to a wide breadth of applications, including linguistic reasoning [50, 51], approximate inference [52, 53], pattern recognition [54] and clustering [2–7]. In recent years, similarity has also become prevalent in computing with words [55, 56]. For a more detailed insight into the use of similarity and compatibility in fuzzy inference and approximate reasoning see [49], and a review of similarity measures used in real-world fuzzy data mining applications is given in [45].

Turning to its mathematical definition, a similarity measure is a function $s : A \times B \rightarrow [0, 1]$, where A and B are both fuzzy sets of type-1, interval type-2, or general type-2. This function evaluates how closely two fuzzy sets share the same membership values across the universe of discourse. A trivial example of similarity is to find out to what degree the fuzzy sets describing

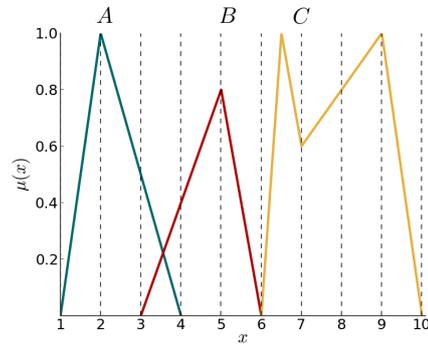


Figure 2.11: A vertical slice approach used to measure similarity.

warm and *hot* share the same meaning.

Measuring the similarity between fuzzy sets involves the comparison of vertical slices, focusing on the membership values of the fuzzy sets. Using the vertical slices approach, as shown in Figure 2.11, any vertical slice intersects a type-1 fuzzy set at only one point. Thus, a similarity measure on type-1 fuzzy sets involves the comparison of two singletons (or two type-1 fuzzy sets for interval and general type-2 fuzzy sets). Thus, the measure of similarity may be the same regardless of the normality or convexity of the fuzzy sets.

Typically, the similarity of two sets is 1 if they are identical, and is 0 if they have nothing in common, i.e., they do not contain any of the same values. In the context of fuzzy sets, they are identical if they both contain the same values with the same degree of membership, and they have nothing in common if their intersection is the empty set. These two properties are referred to as reflexivity and overlapping, respectively. These and two other properties that commonly form the features of a similarity measure are

Reflexivity: $s(A, B) = 1 \iff A = B$

Symmetry: $s(A, B) = s(B, A)$

Overlapping: If $A \cap B \neq \emptyset$, then $s(A, B) > 0$; otherwise, $s(A, B) = 0$

Transitivity: If $A \subseteq B \subseteq C$, then $s(A, B) \geq s(A, C)$

Note that for transitivity, subethood is defined as follows: for $A, B \in T1(X)$, $A \subseteq B$ if $\mu_A(x) \leq \mu_B(x)$, $\forall x \in X$ [1].

It is important to note that the term *similarity* is loosely defined, and thus it is not necessary for a similarity measure to have all of these properties. The properties that are desired are dependent on the context in which the measure will be used. In fact, it has been discussed that there are situations in which symmetry does not need to be satisfied [42], and it has been argued whether transitivity is necessary or even useful in some contexts [57, 58].

Methods of measuring similarity of fuzzy sets can be classified into four categories [49]: 1) proximity-based measures; 2) set-theoretic measures; 3) logic-based measures and 4) fuzzy valued measures. Of these four, the first two are the most common approaches and are discussed next. For a quick overview of some similarity measures in the literature see Table 2.1 on page 58. Note that as the type-1 literature contains many measures on similarity this table has restricted type-1 references to comparative articles which provide an overview of the literature.

Proximity-Based Approaches

One common approach of measuring the similarity between two fuzzy sets is to measure the distance between the membership values for each point in the universe of discourse. This is achieved by using the vertical slices representation. The difference between values is often calculated using some form of the Minkowski distance. The Minkowski distance between two fuzzy sets $A, B \in T1(X)$ is [46]

$$d_r(A, B) = \left(\int_{-\infty}^{+\infty} |\mu_A(x_i) - \mu_B(x_i)|^r dx \right)^{1/r} \quad (2.21)$$

If $r = 1$ this is reduced to

$$d_1(A, B) = \int_{-\infty}^{+\infty} |\mu_A(x_i) - \mu_B(x_i)| \quad (2.22)$$

and when r approaches ∞ this becomes [46],

$$d_\infty(A, B) = \max_i |\mu_A(x_i) - \mu_B(x_i)| \quad (2.23)$$

As the Minkowski r -metric is a measure of distance (such that $d(A, A) = 0$), its complement must be used for it to serve as a measure of similarity. Using d_∞ (2.23), Pappis and Karacapilidis [65] proposed

$$s(A, B) = 1 - \max_i |\mu_A(x_i) - \mu_B(x_i)|, \quad (2.24)$$

which follows reflexivity, symmetry and transitivity. Overlapping is only supported if at least one of the fuzzy sets A or B is normal. Another method using d_1 (2.22) is [48, 49]

$$s(A, B) = 1 - \frac{1}{N} \sum_{i=1}^N (|\mu_A(x_i) - \mu_B(x_i)|), \quad (2.25)$$

where N is the total number of discretisations taken along the universe of discourse and is used to take the average distance among all values of x . This follows the properties of reflexivity and symmetry, but not of overlapping or transitivity.

To compare **interval type-2 fuzzy sets**, Zeng and Li [59] developed a measure of similarity based on the concept of entropy (in the context of fuzzy sets, entropy is a measure of how much a fuzzy set is fuzzy [66]). Their method calculates the Minkowski metric, where $r = 1$, on both the upper and lower membership functions, taking the average of the two results as

$$s(\tilde{A}, \tilde{B}) = 1 - \frac{1}{2N} \sum_{i=1}^N (|\underline{\mu}_{\tilde{A}}(x_i) - \underline{\mu}_{\tilde{B}}(x_i)| + |\overline{\mu}_{\tilde{A}}(x_i) - \overline{\mu}_{\tilde{B}}(x_i)|). \quad (2.26)$$

This is an extension of the type-1 measure (2.25). This method has the properties of reflexivity and symmetry. However, like its type-1 form, it does not reflect overlapping or transitivity. Instead, as two disjoint sets are placed further away in the universe of discourse, thus creating a gap between the fuzzy sets, the value of (2.26) becomes larger because the fuzzy sets become more

similar in the sense that they both have the membership value zero in the space between them. A demonstration of this is shown in [51, 64]. This is also true in the type-1 case (2.25).

To compare **general type-2 fuzzy sets**, Hung and Yang [2] used the proximity based Hausdorff metric to measure the similarity between secondary membership functions as

$$s(\tilde{A}, \tilde{B}) = 1 - d^N(\tilde{A}, \tilde{B}) \quad (2.27a)$$

$$d^N(\tilde{A}, \tilde{B}) = \frac{\sum_{i=1}^N H_f(\tilde{A}(x_i), \tilde{B}(x_i))}{n} \quad (2.27b)$$

$$H_f(A, B) = \frac{\sum_{i=1}^M \alpha_i H(\overline{A_{\alpha_i}}, \overline{B_{\alpha_i}})}{\sum_{i=1}^n \alpha_i} \quad (2.27c)$$

$$H(\overline{A_{\alpha_i}}, \overline{B_{\alpha_i}}) = \max\{L(\overline{A_{\alpha_i}}, \overline{B_{\alpha_i}}), L(\overline{B_{\alpha_i}}, \overline{A_{\alpha_i}})\} \quad (2.27d)$$

$$L(\overline{A_{\alpha_i}}, \overline{B_{\alpha_i}}) = \inf \left\{ \lambda \in [0, \infty] \mid \overline{A_{\alpha_i}}^\lambda \supset \overline{B_{\alpha_i}} \right\}, \quad (2.27e)$$

where $\overline{A_{\alpha_i}}$ is the i^{th} α -cut of the vertical slice A , thus α_i is a secondary membership value. This measure calculates the distance between the secondary membership functions of \tilde{A} and \tilde{B} (within function d^N). This is achieved by calculating the Hausdorff distance between the type-1 fuzzy sets $\mu_{\tilde{A}}(x)$ and $\mu_{\tilde{B}}(x)$ (in functions H and L). The Hausdorff metric is a common method of calculating the distance (rather than similarity) between fuzzy sets, and is explored in more detail in Section 2.4.2.

Note that although this measure uses α -cuts, it is still a measure of similarity rather than distance. This is because the α -cut comparison is made between the vertical slices of the fuzzy sets. Thus, the measure focuses on comparing the membership values of the fuzzy sets at a given x -coordinate; whereas distance compares x -values at a given membership.

This follows all four properties of a similarity measure.

Set-Theoretic Approaches

The most common set-theoretic approach to measuring the similarity between fuzzy sets is the Jaccard index [67]. Let $\mathcal{P}(\mathbb{R})$ be the set of all crisp sets in \mathbb{R} , then for two groups $U, V \in \mathcal{P}(\mathbb{R})$ the Jaccard index is

$$s(U, V) = \frac{|U \cap V|}{|U \cup V|}. \quad (2.28)$$

Using the fuzzy set operations of union and intersection introduced in Section 2.2.1, the Jaccard similarity between two fuzzy sets $A, B \in T1(X)$ is given as

$$s_j^{T1}(A, B) = \frac{\sum_{i=1}^N \min(\mu_A(x_i), \mu_B(x_i))}{\sum_{i=1}^N \max(\mu_A(x_i), \mu_B(x_i))}. \quad (2.29)$$

This measure follows all of the four common properties of a similarity measure. Based on Jaccard's ratio, Tversky [42] proposed the non-fuzzy similarity measure for $A, B \in \mathcal{P}(\mathbb{R})$

$$s_{\alpha\beta}(A, B) = \frac{f(A \cap B)}{f(A \cap B) + \alpha f(A - B) + \beta f(B - A)} \quad (2.30)$$

to be applied in feature matching, where f is typically a cardinality function. Many set-theoretic similarity measures are some form of Tversky's ratio [68–70]. Note that when $\alpha = \beta = 1$ (2.30) reduces to the Jaccard measure (2.29).

For **interval type-2 fuzzy sets**, Wu and Mendel [51], and Nguyen and Kreinovich [60] expanded the Jaccard approach. For $\tilde{A}, \tilde{B} \in IT2(X)$ this is

$$s_j^{IT2}(\tilde{A}, \tilde{B}) = \frac{\sum_{i=1}^N \min(\bar{\mu}_{\tilde{A}}(x_i), \bar{\mu}_{\tilde{B}}(x_i)) + \sum_{i=1}^N \min(\underline{\mu}_{\tilde{A}}(x_i), \underline{\mu}_{\tilde{B}}(x_i))}{\sum_{i=1}^N \max(\bar{\mu}_{\tilde{A}}(x_i), \bar{\mu}_{\tilde{B}}(x_i)) + \sum_{i=1}^N \max(\underline{\mu}_{\tilde{A}}(x_i), \underline{\mu}_{\tilde{B}}(x_i))} \quad (2.31)$$

As with the type-1 approach (2.29), this method has all four properties of a similarity measure. Zheng et al. [61] also proposed a measure related to the Jaccard approach, and Gorzałczany [52] and Bustince [53] developed methods which represent similarity as an interval.

To compare **general type-2 fuzzy sets**, in recent years, several methods of comparing the similarity between type-2 fuzzy sets based on the alpha-plane/zSlices approach were published [62–64, 71]. Hamwari and Coupland

[71] developed a general method of applying any interval type-2 similarity measure to zSlices general type-2 fuzzy sets. As part of this thesis, [64] explores extending a collection of interval type-2 similarity measures to zSlices general type-2 fuzzy sets; this is explored in Section 4.3. Zhao et al. [62] proposed two new measures of similarity on type-2 fuzzy sets. One represents similarity as a fuzzy set (detailed within [62]) and the other represents similarity as a crisp value as

$$s(A, B) = \frac{1}{\Delta+1} \sum_{z=0, \frac{1}{\Delta}, \frac{2}{\Delta}, \dots, \frac{\Delta-1}{\Delta}, 1} \frac{\int_{x \in X} \min(\mu_{\tilde{A}_z}(x), \mu_{\tilde{B}_z}(x)) dx + \int_{x \in X} \min(\bar{\mu}_{\tilde{A}_z}(x), \bar{\mu}_{\tilde{B}_z}(x)) dx}{\int_{x \in X} \max(\mu_{\tilde{A}_z}(x), \mu_{\tilde{B}_z}(x)) dx + \int_{x \in X} \max(\bar{\mu}_{\tilde{A}_z}(x), \bar{\mu}_{\tilde{B}_z}(x)) dx} \quad (2.32)$$

where $\Delta + 1$ denotes the total number of zSlices (alpha-planes) used. Note that, for consistency, the notations within (2.32) have been altered to match the zSlices notations. This approach follows all four properties of similarity. In an alike approach, Hao and Mendel [63] also developed a measure using α -planes to represent similarity as a type-1 fuzzy set. Their result may be reduced to a crisp value by computing the centroid of the resulting fuzzy set. This also follows all four properties of similarity.

Other Approaches

Many other methods have also been developed to measure the similarity between fuzzy sets. For example, Bonissone [50] developed an approach based on the complement of the Bhattacharya distance to compare type-1 fuzzy sets. Wu and Mendel developed a vector similarity measure which uses a combination of both set-theoretic and proximity approaches to measure the similarity between type-1 [72] and interval type-2 [51] fuzzy sets. To compare general type-2 fuzzy sets, Mitchell [54] proposed measuring similarity by comparing the embedded membership functions of the fuzzy sets with any type-1 similarity measure. In addition, Hwang et al. [6] and Li et al. [7] also developed methods based on the Sugeno and Lebesgue integrals, respectively.

2.4.2 Distance Measures

As discussed earlier, distance measures on fuzzy sets often involve comparing α -cuts. Various methods of calculating the distance between α -cuts have been developed, many of which utilise the Hausdorff metric [14, 73] or the Minkowski distance [10, 74, 75]. Just like similarity, the properties of the data affect the properties of a distance measure. For example, in a direction-dependent application, the distance from A to B may be perceived differently to the distance from B to A .

In the context of fuzzy sets, distance has primarily been used in ranking. Typically this is done by measuring the distance between each fuzzy set and a crisp point [8, 12, 76, 77]. However, many methods of ranking use a measure of distance developed to compare two fuzzy sets [10, 11, 74, 76, 78]. Distance has also been developed for many other applications, including decision making [8, 9, 74, 79, 80], linear programming [79], statistical analysis [81] and digital image analysis [82]. As a wide range of applications applying the concept of distance exist within the literature, a great variety of methods to calculate distance have been developed.

A distance measure is a function $d : A \times B \rightarrow \mathbb{R}^+$, where A and B are both fuzzy sets of type-1, interval type-2, or general type-2. The result of this function represents how much difference there is in the values contained within the fuzzy sets. The distance between two fuzzy sets is 0 if they are identical, and increases in value as they become more distant. A trivial use of distance could be to find out how much difference exists between two fuzzy sets describing *warm* and *hot* (e.g., how much hotter is *hot* compared to *warm*).

As distance focuses on the ordering within the x -axis, it is often measured by comparing the α -cuts of the fuzzy sets [46]. Figure 2.12 uses dashed lines along three different fuzzy sets to help visualise this approach. For a given α -cut, the distance tells us how much difference there is between the values

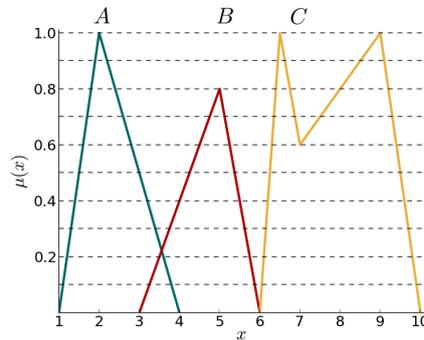


Figure 2.12: A horizontal slice approach used to measure distance.

contained in the fuzzy sets at a given degree of membership. When comparing the whole fuzzy sets (i.e., all α -cuts) it tells us how far their values are from representing the same description.

Figure 2.12 highlights some difficulties that can arise whilst measuring distance. When using α -cuts to measure distance, an α -cut can contain an interval (e.g., every α -cut of A), an empty set (e.g., α -cuts above 0.8 in B) or multiple intervals (e.g., α -cuts above 0.6 in C). When fuzzy sets are automatically generated from data, their normality or convexity may not be known, thus we cannot know in advance what any α -cut will contain. Considering this, it is clear that measuring distance is not as straightforward to calculate as similarity. This has led to gaps in the research which will be highlighted in this section.

Some common properties of distance measures include

Self-Identity: $d(A, B) = 0 \iff A = B$

Symmetry: $d(A, B) = d(B, A)$

Separability (AKA positivity; non-negativity) : $d(A, B) \geq 0$

Triangle inequality : $d(A, C) \leq d(A, B) + d(B, C)$

Transitivity: If $A \leq B \leq C$, then $d(A, B) \leq d(A, C)$

Just as discussed regarding the properties of similarity measures, it is not necessary for a distance measure to have all of the above properties. The context in which it is applied defines the properties that are required for a measure of distance. It is important to note, however, that the properties of a metric are important as they strictly define the distance between any points in the given metric space. A distance function d is a metric if and only if it satisfies self-identity, separability, symmetry and triangle inequality (note that triangle inequality is a form of transitivity). Any distance measure that does not satisfy all of these properties is not a metric. Thus, although a metric is a function of distance, a distance measure is not necessarily a metric.

To account for the ordering in the x -axis, distance is most often calculated by comparing the α -cuts of the fuzzy sets. Usually, multiple comparisons are made at different levels of α and, to reduce these to a single value, the results are aggregated. As discussed in Section 2.2.1, the α -cut of a normal, convex fuzzy set can be represented by a continuous interval. Given this, the distance between each α -cut is often measured using the Pompeiu-Hausdorff metric (also commonly known as the Hausdorff metric [86]) or the Minkowski distance.

This section discusses some of the methods used to measure the distance between different types of fuzzy sets. In addition, Table 2.2 on page 59 provides an overview of some of the literature, displaying gaps where distance measures have not been well explored. Within the table, section numbers highlight where new measures are developed within the thesis.

Pompeiu-Hausdorff Based Approaches

The Pompeiu-Hausdorff (or Hausdorff) distance is a common approach for measuring the distance between the α -cuts of fuzzy sets. Based upon Pompeiu's asymmetric distance measure, the Hausdorff distance between two crisp

sets $A, B \in \mathcal{P}(\mathbb{R})$ is calculated as [46]

$$h(A, B) = \max \left\{ \sup_{b \in B} \inf_{a \in A} d_2(a, b), \sup_{a \in A} \inf_{b \in B} d_2(a, b) \right\}, \quad (2.33)$$

where d_2 is the Euclidean metric. When A and B are intervals, as is the case for the α -cut of a normal, convex membership function, then the distance between $\overline{A_\alpha}$ and $\overline{B_\alpha}$ (2.33) is reduced to [46]

$$h(\overline{A_\alpha}, \overline{B_\alpha}) = \max \{ |\overline{A_{\alpha L}} - \overline{B_{\alpha L}}|, |\overline{A_{\alpha R}} - \overline{B_{\alpha R}}| \} \quad (2.34)$$

where $[\overline{A_{\alpha L}}, \overline{A_{\alpha R}}]$ represents the continuous interval of an α -cut of $A \in T1(X)$.

Based on the Pompeiu-Hausdorff distance, Ralescu and Ralescu [73] proposed two different methods of calculating the distance between two fuzzy sets. For $A, B \in T1(X)$, these measures are

$$d(A, B) = \int_{\alpha=0}^1 h(\overline{A_\alpha}, \overline{B_\alpha}) d\alpha \quad (2.35)$$

and

$$d(A, B) = \sup_{\alpha > 0} h(\overline{A_\alpha}, \overline{B_\alpha}) \quad (2.36)$$

Chaudhuri and Rosenfeld also proposed a new distance based on the Hausdorff metric given as [14]

$$d_{cr}(A, B) = \frac{\sum_{\alpha=1}^M y_\alpha h(\overline{A_\alpha}, \overline{B_\alpha})}{\sum_{\alpha=1}^M y_\alpha}, \quad (2.37)$$

where the y -axis (μ -axis) is discretised into M points (y_1, y_2, \dots, y_M) , $\overline{A_\alpha}$ is the α -cut set of the fuzzy set A at the y -coordinate y_α , and h is the Hausdorff metric in (2.34).

Note that (2.35), (2.36) and (2.37) all have the properties of a metric.

Although (2.37) assumes that the fuzzy sets being measured are normal, Chaudhuri & Rosenfeld [14] also proposed a method of measuring the distance between non-normal fuzzy sets. This method includes two steps. The first step involves normalising the fuzzy sets and applying the distance measure (2.37)

to the resulting fuzzy sets; the normalised forms of A and B are referred to as A' and B' , respectively. After this, the following equation is applied to the original, non-modified fuzzy sets [14]

$$e(A, B) = \varepsilon \frac{\sum_{i=1}^N |\mu_A(x_i) - \mu_B(x_i)|}{\sum_{i=1}^N x_i}, \quad (2.38)$$

where N is the total number of discretisations in X , and ε is a small positive constant, and its value is determined by the importance of the equation (demonstrations later in this thesis set $\varepsilon = 1.0$). Finally, the results of (2.37) and (2.38) are combined as [14]

$$d_{cr}(A, B) = \frac{\sum_{\alpha=1}^M y_\alpha h(\overline{A'_\alpha}, \overline{B'_\alpha})}{\sum_{\alpha=1}^M y_\alpha} + \varepsilon \frac{\sum_{i=1}^N |\mu_A(x_i) - \mu_B(x_i)|}{\sum_{i=1}^N x_i}. \quad (2.39)$$

To compare **interval type-2 fuzzy sets**, Figueroa-García et al. [79] propose measuring the distance between two interval type-2 fuzzy sets \tilde{A} and \tilde{B} by the calculating Pompeiu-Hausdorff distance between the type reduced sets of \tilde{A} and \tilde{B} . Using the Karnik-Mendel type-reduction algorithm, the centre of an interval type-2 fuzzy set $\tilde{A} \in IT2(X)$ is represented as an interval-bounded type-1 fuzzy set $C(\tilde{A}) = [C_L(\tilde{A}), C_R(\tilde{A})]$. The distance between $\tilde{A}, \tilde{B} \in IT2(X)$ after type reduction is then calculated using the Hausdorff distance as [79]

$$d_h(C(\tilde{A}), C(\tilde{B})) = \max \left\{ |C_L(\tilde{A}) - C_L(\tilde{B})|, |C_R(\tilde{A}) - C_R(\tilde{B})| \right\}. \quad (2.40)$$

In addition to this, Figueroa-García et al. [79] also propose measuring the sum of the distances between each boundary as

$$d_s(C(\tilde{A}), C(\tilde{B})) = |C_L(\tilde{A}) - C_L(\tilde{B})| + |C_R(\tilde{A}) - C_R(\tilde{B})|. \quad (2.41)$$

Minkowski Distance Approaches

In contrast to the Hausdorff distance, another common method of comparing α -cuts is by the Minkowski distance (2.21), which for two α -cuts A_α and B_α

where $A, B \in T1(X)$ is

$$\bar{d}_r(A_\alpha, B_\alpha) = \sqrt[r]{1/2((\overline{A_{\alpha L}} - \overline{B_{\alpha L}})^r + (\overline{A_{\alpha R}} - \overline{B_{\alpha R}})^r)}. \quad (2.42)$$

Using (2.42), Grzegorzewski [74] established two methods of measuring the distance between two fuzzy numbers as

$$d_{pq}(A, B) = \begin{cases} \sqrt[p]{(1-q) \int_0^1 |\overline{B_{\alpha L}} - \overline{A_{\alpha L}}|^p d\alpha + q \int_0^1 |\overline{B_{\alpha R}} - \overline{A_{\alpha R}}|^p d\alpha} & \text{if } 1 \leq p < \infty \\ (1-q) \sup_{0 < \alpha \leq 1} (|\overline{B_{\alpha L}} - \overline{A_{\alpha L}}|) + q \sup_{0 < \alpha \leq 1} (|\overline{B_{\alpha R}} - \overline{A_{\alpha R}}|) & \text{if } p = \infty \end{cases} \quad (2.43)$$

and

$$d_p(A, B) = \begin{cases} \max \left\{ \sqrt[p]{\int_0^1 |\overline{B_{\alpha L}} - \overline{A_{\alpha L}}|^p d\alpha}, \sqrt[p]{\int_0^1 |\overline{B_{\alpha R}} - \overline{A_{\alpha R}}|^p d\alpha} \right\} & \text{if } 1 \leq p < \infty \\ \max \left\{ \sup_{0 < \alpha \leq 1} (|\overline{B_{\alpha L}} - \overline{A_{\alpha L}}|), \sup_{0 < \alpha \leq 1} (|\overline{B_{\alpha R}} - \overline{A_{\alpha R}}|) \right\} & \text{if } p = \infty \end{cases} \quad (2.44)$$

where the properties of the above two measures depend on the value of p [74]. The parameter q of (2.43) may be used to weight the sides of the α -cuts (putting more emphasis on the lowest or highest values). However, if there is no reason to weight one side more than the other then q may be set as $1/2$ or (2.44) may be used instead [74]. Based on (2.43), Ban [75] also proposed a similar approach.

Using the Minkowski distance where $r = 1$, the distance between α -cuts is

$$\bar{d}(A_\alpha, B_\alpha) = 1/2(|\overline{A_{\alpha L}} - \overline{B_{\alpha L}}| + |\overline{A_{\alpha R}} - \overline{B_{\alpha R}}|). \quad (2.45)$$

Using (2.45), Wang et al. [15] proposed a measure to compare non-normal fuzzy sets that weights the distance between α -cuts, and weights and compares the angles and heights of the increasing and decreasing functions of trapezoidal membership functions. More details on this are given later in Section 3.3.2.

Yao and Wu [10], and Berkachy and Donz e [87] described a directional distance between fuzzy sets as

$$d_{yw}(A, B) = \frac{1}{2} \int_0^1 [\overline{A_{\alpha L}} + \overline{A_{\alpha R}} - \overline{B_{\alpha L}} - \overline{B_{\alpha R}}] d\alpha. \quad (2.46)$$

A directional distance measure is one that does not follow separability¹ and uses a signed measure to indicate direction; thus changing the distance function to $d : A \times B \rightarrow \mathbb{R}$ instead of in \mathbb{R}^+ . Using (2.46), $d(A, B) \geq 0$ if $A \geq B$ and $d(A, B) < 0$ if $A < B$. Additionally, $d(A, B) = -d(B, A)$. Chapter 3 explores directional distance measures in more detail and develops a new measure based on (2.46).

Yao et al. [80] developed a signed distance measure which calculates the distance between a normal, convex, type-1 fuzzy set and the singleton $x = 0$. The result of the distance measure reduces to the fuzzy set's average value over all α -cuts, essentially providing a centroid of the fuzzy set.

To compare **interval type-2 fuzzy sets**, Figueroa-Garc a et al. [79] developed a distance measure using the Minkowski distance ($r = 1$) to compare α -cuts of the upper and lower membership functions. Note that the notations within [79] have been changed to match the notations used within this thesis. The distance between two interval type-2 fuzzy sets $\tilde{A}, \tilde{B} \in IT2(X)$ is given as [79]

$$d(\tilde{A}, \tilde{B}) = 1/\Lambda \sum_{i=1}^M \alpha_i \left[\left| \overline{\tilde{A}_{\alpha_{iU}L}} - \overline{\tilde{B}_{\alpha_{iU}L}} \right| + \left| \overline{\tilde{A}_{\alpha_{iW}L}} - \overline{\tilde{B}_{\alpha_{iW}L}} \right| + \left| \overline{\tilde{A}_{\alpha_{iW}R}} - \overline{\tilde{B}_{\alpha_{iW}R}} \right| + \left| \overline{\tilde{A}_{\alpha_{iU}R}} - \overline{\tilde{B}_{\alpha_{iU}R}} \right| \right], \quad (2.47)$$

where $\Lambda = \sum_{i=1}^M \alpha_i$ and M is the total number of α -cuts measured. Note that both the upper and lower membership functions of \tilde{A} and \tilde{B} must be normal as (2.47) does not account for non-normality.

¹Separability: $d(A, B) \geq 0$.

Other Approaches

Taking a different approach, Allahviranloo et al. [76] compare the centres and widths of the fuzzy sets. Where $I(A)$ is the centre of A and $D(A)$ is the width of A , the distance between $A, B \in T1(X)$ is

$$d_a(A, B) = \sqrt{[I(A) - I(B)]^2 + [D(A) - D(B)]^2} \quad (2.48a)$$

$$I(A) = \int_0^1 (c\overline{A_{\alpha L}} + (1 - c)\overline{A_{\alpha R}}) d\alpha \quad (2.48b)$$

$$D(A) = \int_0^1 (\overline{A_{\alpha R}} - \overline{A_{\alpha L}})f(\alpha) d\alpha \quad (2.48c)$$

where $0 \leq c \leq 1$ denotes optimism/pessimism in the operation (in demonstrations in this thesis, $c = 0.5$), and $f(\alpha)$ is a function which satisfies $f(0) = 0$, $f(1) = 1$ and $\int_0^1 f(\alpha)d\alpha = 1/2$. In most cases, and in demonstrations in this thesis, $f(\alpha) = \alpha$.

This section has highlighted a concise set of measures on normal and non-normal, type-1 fuzzy sets. However, to the author's knowledge, there have been no α -cut-based distance measures developed to compare non-convex, type-1 fuzzy sets. Although one could compare the centroids of non-convex fuzzy sets, the results are not always what one would expect; this is demonstrated later in Section 3.4. Additionally, some recent research has explored distance on interval type-2 fuzzy sets, but these methods do not account for any non-normality (non-normality is common in the lower membership function) or non-convexity. Additionally, there have been no α -cut-based distance measures developed for general type-2 fuzzy sets.

So far, this chapter has given a review of fuzzy set theory and measures to compare fuzzy sets. The remainder of this chapter moves away from fuzzy set theory and discusses aggregation operators and knowledge based recommendation systems.

2.5 Aggregation

Many aggregation operators exist within the literature, including arithmetic mean and ordered weighted average (OWA) operators. The latter is used in this thesis and will be the focus of this section. The OWA operator [88] was developed as a method of breaking away from classic *and* and *or* aggregation, which both strictly treat all values with equal importance. An OWA operator offers a method of aggregation that lies between the *and* and *or* extremes [88].

An OWA operator assigns weights to objects according to their ordinal position when sorted by magnitude. An ordered set of weights is denoted $w = \langle w_1, \dots, w_n \rangle$, where $w_i \in [0, 1]$ and $\sum_{i=1}^n w_i = 1$. The objects which are to be aggregated are sorted into descending order, and each object is multiplied by the corresponding weight. Thus, for a given set of objects $\{a_1, \dots, a_n\}$ and ordered list of weights $\langle w_1, \dots, w_n \rangle$, the OWA operator is [88]

$$f(\{a_1, \dots, a_n\}, \langle w_1, \dots, w_n \rangle) = w_1 b_1 + w_2 b_2 + \dots + w_n b_n, \quad (2.49)$$

where b_i is the i_{th} largest element in the collection $\{a_1, \dots, a_n\}$.

Using this method, many different types of aggregation can be realised. For example, the weights $w = \langle 0, \dots, 0, 1 \rangle$ and $w = \langle 1, 0, \dots, 0 \rangle$ are the *and* and *or* operators, respectively. Also, if all weights are the same then the result is the equivalent of the arithmetic mean operation. It is clear that the infinite range of possible weights between these examples can lead to the OWA operator's utilisation in a wide range of applications.

Since its introduction, the theory of the OWA operator has been expanded to induced OWAs [89] (used to aggregate tuples) and generalised OWAs [90, 91] (used when the priorities of the inputs are not known). OWAs have been applied to a multitude of applications, the most common of which is multi-criteria decision making.

For example, Canós [92] uses an OWA in decision making applied to the personnel selection problem. Sadiq et al. [93] use an OWA operator to aggregate

different performance indicators to assess the performance of small drinking water utilities, and to aid in the selection of financial products. Other common applications include data mining, image processing and expert systems [94–96]. The OWA operator will be used later in this thesis as a method of aggregating the results of similarity and distance measures on fuzzy sets.

Having given a theoretical background to the literature on which this thesis is based, the next section moves onto applications and gives a brief history and overview of knowledge-based recommendation systems.

2.6 Knowledge-Based Recommendations

To illustrate the measures developed within this thesis, a knowledge-based recommendation system is developed. Recommendation systems have become useful in recent decades as the amount of information on a given topic often exceeds what one can study. Several different approaches have been developed, the most common of which are content-based, collaborative-filtering and knowledge-based systems.

Content-based systems learn what a user likes through the products they have rated or purchased and recommends items that are similar. Collaborative-filtering also includes learning user preferences, but uses this information to find multiple users with common interests. Recommendations are then made based on inter-user comparisons [97]. Many recommendation systems are a hybrid of these [97–101].

For example, Amazon.com[®] use a mixture of content-based and collaborative filtering methods [101]. For a given product a user has bought, the system finds other users that have bought this product and looks through their list of other purchases (collaborative filtering). If an item from this list is similar to the original item then it is recommended (content-based filtering). This process enables Amazon.com[®] to give recommendations in the format “users

who bought this product also bought these similar products.”

However, knowledge-based systems, which will be the focus of this section, do not attempt to learn user preferences. Instead, a user explicitly states their needs and, based on knowledge known about the given topic, the recommender finds items that match the user’s desires. This process often begins by displaying a selection of items to a user and enabling them to describe how a given choice doesn’t match what they want [102–104].

This approach to recommendations is chosen because it is easier for people to express their preferences in relation to an available choice [105]. Describing an ideal product in relation to another is easier on a person because it requires less input than giving a detailed description. It is also easier to design such a recommendation system because the user’s ideal product is represented by a compact description that is formed from the knowledge-base [102].

Burke et al. [103] refer to this method as the FindMe approach and developed several systems that use *assisted browsing* to combine browsing and knowledge-based recommendations. For example, Entree is a system that helps people choose a restaurant in Chicago. An initial restaurant is shown and users can choose from seven different attributes to adjust the restaurants they are shown. These attributes include *less expensive*, *quieter* and a change in cuisine. RentMe is a similar system used to find apartments from a list of classified adverts. It enables a person to browse through adverts and describe changes they would like to see, e.g., *a bigger apartment* or *a more convenient neighbourhood*.

Another example is the Automated Travel Assistant [104] that helps users find an optimal trip by presenting options based on what the user wants. The user can critique the attributes of the plan, e.g., on price and arrival time, and a new suggestion is presented based on these changes. A more recent example is the tag genome project [102] used by MovieLens [106], in which the knowledge of films is represented by user-contributed tags, such as *action*, *romance*, *scary*, etc. Based on this knowledge, a user can browse movies and

request tweaks such as *a film like Star Trek but with more action*.

One possible method of approaching the above examples is to split each goal into two parts. These are finding a similar item (e.g., *a restaurant/ apartment/ movie like this*) and finding differences (e.g., *but quieter/ cheaper/ with more action*). This thesis develops a method of splitting these goals into separate functions. Similarity and distance measures can then be used ascertain how similar or different a given item is compared to what the user desires. Given this, the measures developed within this thesis are driven by their ability to compare fuzzy information according to both their similarities and differences.

When calculating the distance between items, it is important to measure the direction of that distance as well as the magnitude. For example, if a user wants to find a restaurant *at least £10 cheaper* then the direction of the change in price is as important as the magnitude. However, a different approach to direction can sometimes be more beneficial. For example, if a user states they are willing to spend a specific amount of money then anything considerably more expensive should not be recommended, but additionally anything cheaper may also not be preferred. This is because the user has stated that they are willing to spend a certain amount [107].

After evaluating both objectives, the results are joined to find an item that matches all of the user's criteria. For example, the tag genome project calculates the product similarity and distance to assign a score to each film, and Entree [103] aggregates the results of how well a restaurant matches each goal of a given search.

The above recommendation systems use a knowledge base which contains facts about products, yet there are many applications in which information is subjective. For example, the tag genome project builds information about films based on user-descriptions. However, these descriptions are often subjective; a film that one person finds scary may be considered tame by another person. One method of handling this is to introduce personal profiles so that

the relevance of a tag to a film is only from the perspective of the user, but this idea is not used by the Tag Genome project because it requires a more complex system [102].

This thesis uses fuzzy sets to model the subjective and uncertain nature of information. By taking this approach, it is only the methods of measuring similarity and distance that must be changed and the underlying model of a knowledge-based recommendation system remains the same.

2.7 Summary of the Literature

This chapter has presented an overview of fuzzy set theory, including a survey of the different types of fuzzy sets that have been developed, how they are mathematically modelled and some of their properties. For each type of fuzzy set used in this thesis, both their vertical slice and α -cut (horizontal slice) representations have been presented. Using these two representations, it is possible to compare the similarities and distances between fuzzy sets.

A survey is given of how, based on these representations, relative comparisons can be calculated between type-1, interval type-2 and zSlices general type-2 fuzzy sets. This survey shows that, while similarity measures have been well explored, distance measures have gained little attention for type-1 non-convex membership functions and type-2 fuzzy sets.

This chapter has also given an overview of the OWA operator, and a survey of knowledge-based recommendation systems, on which the theoretical work of this thesis is applied. In addition, this chapter has presented methods of constructing membership functions from data sets. Later, this will be used to create fuzzy sets that model subjective product information. With this, relative comparisons using an aggregation (via the OWA operator) of similarity and distance are used to automatically generate knowledge-based recommendations.

The next chapter develops a directional distance measure that can compare type-1 fuzzy sets that may be normal or non-normal and convex or non-convex.

Type-1	Interval Type-2	General Type-2
Zwick et al.* [46]	Zeng & Li [59]	Lin & Yang [3]
Wang et al.* [47]	Wu & Mendel* [51, 55]	Yang & Lin [4]
Chen et al.* [48]	Gorzalczany [52]	Mitchell [54]
Cross and Sudkamp* [49]	Bustince [53]	Hung & Yang [2]
Bouchon-Meunier et al.* [45]	Nguyen & Kreinovich [60]	Hwang et al. [6]
Jain et al.* [44]	Zheng et al. [61]	Zhao et al. [62]
		Hao and Mendel [63]
		<i>Section 4.3</i> [64]

Table 2.1: An overview of existing similarity measures on fuzzy sets where proposed measures in this thesis are highlighted with the corresponding section. An * indicates a comparative paper containing many references.

Type-1	Interval Type-2	General Type-2
Normal, convex		
Chaudhuri & Rosenfeld [14]	Figueroa-García et al. [79]	<i>Section 4.3</i>
Fan [83]	<i>Section 4.2</i>	
Allahviranloo et al. [76]		
Yao & Wu [10]		
Bloch [84]		
Ralescu & Ralescu [73]		
Williams & Steele [9]		
Grzegorzewski [74]		
Ban [75]		
Tran and Duckstein [78]		
Cheng [11]		
Bertoluzza et al. [85]		
Non-normal		
Chaudhuri & Rosenfeld [14]	<i>Section 4.2 & 3.3</i>	<i>Section 4.3 & 3.3</i>
Fan [83]		
Cheng [11]		
Wang et al. [15]		
<i>Section 3.3</i>		
Non-convex		
<i>Section 3.4</i>	<i>Section 4.2 & 3.4</i>	<i>Section 4.3 & 3.4</i>
Directional		
Yao & Wu [10]	<i>Section 3.2</i>	<i>Section 3.2</i>
Yao et al. [80]		
<i>Section 3.2</i>		

Table 2.2: An overview of existing distance measures on fuzzy sets where proposed measures in this thesis are highlighted with the corresponding section.

Chapter 3

Measuring Distance Between Type-1 Fuzzy Sets

3.1 Introduction

This chapter presents a directional distance measure that may be applied to type-1 fuzzy sets that are normal or non-normal and convex or non-convex. As large-scale applications that rely on subjective human perceptions and preferences become popular, measures of comparing the similarities and differences between these perceptions become necessary. Models of such data, including fuzzy set based models, are often non-normal (due to some lack of agreement between people) or may be non-convex (due to contradictory opinions). To accommodate for such models, it is important that any measures comparing them can handle these complexities.

Firstly, this chapter develops a directional distance measure. This measure can account for the *change in direction* between fuzzy sets as well as the magnitude of distance. Thus, it is possible to determine if one fuzzy set is to the left or right of (lower or higher than) another in the given universe of discourse. This is particularly useful when comparing ratings in a recommendation system

because it enables the system to know whether one fuzzy set (or product) has been rated higher or lower than another.

After this, the proposed directional distance measure is extended to enable the comparison of non-normal and non-convex fuzzy sets. As discussed in Section 2.4, similarity measures on fuzzy sets are typically calculated using vertical slices and, as a result, do not require special techniques to compare non-normal or non-convex membership functions. Distance measures, however, commonly use α -cuts and a given α -cut can result in an empty set or a discontinuous interval for non-normal and non-convex membership functions, respectively (see Section 2.2.1). As a result, it is less clear how to measure the distance between such membership functions.

This chapter addresses these issues and proposes methods of measuring the directional distance between non-normal and non-convex type-1 fuzzy sets. To aid in making the chosen methods and their results clear, the proposed measures are compared against some existing measures in the literature.

3.2 Directional Distance

Expanding on the current literature of distance measures on fuzzy sets, this section introduces a new directional distance measure, discusses its properties, and compares it to some existing measures.

3.2.1 Motivation

Most distance measures within the literature are non-directional, i.e., they give a value of distance within \mathbb{R}^+ . However, direction can be important in some contexts. For example, consider fuzzy sets that have been constructed from a survey in which people have been asked to rate, on a scale of 1 to 10, the quality of food of multiple restaurants. Using this data, a distance measure could determine the difference in ratings between two restaurants. However, if

the measure is not directional then the information of distance alone may not be useful.

For example, consider if the distance between the quality of food of two restaurants is calculated as 4. Though this indicates that one of the restaurants was rated 4 points different than the other, it does not show which is the better restaurant. With a non-directional distance measure, the only way to discern which is rated higher is by looking at the fuzzy sets. This is not ideal as it becomes a time-consuming and tedious process if many comparisons have to be made.

This section introduces a directional distance measure that indicates the direction of the results by a signed value. Using the earlier example, for two restaurants denoted A and B , the distance between them ($d(A, B)$) will be 4 if B is rated 4 points higher than A , and it will be -4 if B is rated 4 points lower than A . Essentially, the sign of $d(A, B)$ will indicate which direction is travelled when moving from A to B . By introducing signed values to a distance measure it is now possible, in this example, to see which restaurant is rated better than, rather than *just different* to, the other.

3.2.2 Directional Distance Between Alpha-Cuts

As shown in Section 2.4.2, the distance between two fuzzy sets is commonly calculated by measuring the distance between the α -cuts of the fuzzy sets.

Note that by constructing a distance measure that uses signed-values to indicate direction, the properties of the measure are altered. It is immediately clear that separability and symmetry¹ no longer hold. As a result, such a distance measure is not a metric because a metric must have the properties of separability and symmetry (as well as self-identity and triangle inequality; see Appendix A).

¹Separability: $d(A, B) \geq 0$

Symmetry: $d(A, B) = d(B, A)$

It is clear that separability is not desired in a directional measure because it prevents the information of direction from being conveyed. Thus, one could argue that the loss of this property has a positive impact on a directional distance measure. The property of symmetry, however, is not entirely lost. Instead, an alternative notion of symmetry is adopted. Referring to the restaurant example given earlier, if restaurant B is rated 4 points higher than A then $d(A, B) = 4$ and $d(B, A) = -4$. Although these results do not strictly follow symmetry (i.e., symmetry: $d(A, B) = d(B, A)$), they do follow a looser form of symmetry defined as follows:

Definition 20 (Partial Symmetry). *Let partial symmetry describe the property of a distance measure $d : A \times B \rightarrow \mathbb{R}$ for two points or objects A and B as*

$$d(A, B) = -d(B, A).$$

In addition to this, a directional distance measure has a new form of separability defined as follows:

Definition 21 (Directional Separability). *The sign of the distance indicates the relative positions between the variables.*

$$\begin{aligned} d(A, B) &\geq 0 && \text{if } B \geq A \\ d(A, B) &< 0 && \text{if } B < A, \end{aligned}$$

As in a non-directional measure, the proposed distance will also follow self-identity and transitivity, however the property of triangle-inequality becomes more strict; this is discussed in detail within this section on Page 67. First, it is necessary to introduce the proposed directional distance before its properties can be further explored.

One common approach of calculating the distance between α -cuts is to use the Minkowski distance (2.21) [15, 74, 79]. Using $r = 1$ - commonly referred to as Manhattan distance - will be sufficient to compare two parallel α -cuts. Let

$\mathcal{P}(\mathbb{R})$ denote the set of all crisp sets within \mathbb{R} . For two continuous intervals $\bar{A}, \bar{B} \in \mathcal{P}(\mathbb{R})$, the Manhattan distance is

$$\bar{d}(\bar{A}, \bar{B}) = 1/2(|\bar{A}_L - \bar{B}_L| + |\bar{A}_R - \bar{B}_R|), \quad (3.1)$$

where $\bar{A} = [\bar{A}_L, \bar{A}_R]$ and $\bar{B} = [\bar{B}_L, \bar{B}_R]$.

Definition 22 (Directional Distance between α -cuts.). *To attain a directional distance measure that has the property of partial symmetry, the Manhattan distance may be altered such that it does not take the absolute distance between intervals as follows*

$$\bar{d}_p(\bar{A}, \bar{B}) = \frac{1}{2}(\bar{B}_L - \bar{A}_L + \bar{B}_R - \bar{A}_R) \quad (3.2)$$

Note that, as adopted in \bar{d}_p (3.2), all distance measures proposed as part of this thesis will be denoted with $_p$ to differentiate them from other distance measures in the literature.

Also, note that (3.2) is equivalent to the directional distance between α -cuts used by Yao and Wu (2.46) [10], which focuses on the context of ranking. This chapter explores some of the properties of this measure further and uses a different method of fusing distances calculated at multiple α -cuts.

One can see that \bar{d}_p has the property of partial symmetry given in Definition 20. This directional distance takes the average distance of the left-most and right-most values of the intervals \bar{A} and \bar{B} . This may also be written as

$$\bar{d}_p(\bar{A}, \bar{B}) = \frac{\bar{B}_L + \bar{B}_R}{2} - \frac{\bar{A}_L + \bar{A}_R}{2},$$

given that (3.2) essentially calculates the distance between the centres of the intervals.

To give an example of the directional distance \bar{d}_p (3.2), consider the intervals \bar{A} and \bar{B} in Figure 3.1. The distance $\bar{d}_p(\bar{A}, \bar{B})$ according to (3.2) is 6, and the distance $\bar{d}_p(\bar{B}, \bar{A})$ is -6 . This demonstrates that \bar{d}_p can be used to both calculate the distance between two intervals and discern which interval

contains values larger or smaller than the other. This also demonstrates the property directional separability (i.e., $\bar{d}_p(\bar{A}, \bar{B}) \geq 0$ if $B \geq A$ and $\bar{d}_p(\bar{A}, \bar{B}) < 0$ if $\bar{B} < \bar{A}$). Essentially, the sign of the result represents the direction taken in the universe of discourse to travel from \bar{A} to \bar{B} . Additionally, if \bar{A} and \bar{B} are identical then $\bar{d}_p(\bar{A}, \bar{B}) = 0$.

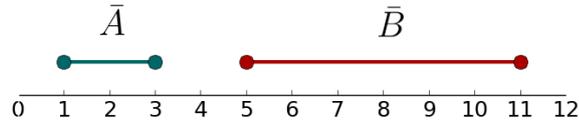


Figure 3.1: Two interval sets, \bar{A} and \bar{B} .

The directional distance \bar{d}_p (3.2) also has an additional property that emerges if one interval is a subset of the other and the distances $|\bar{B}_L - \bar{A}_L|$ and $|\bar{B}_R - \bar{A}_R|$ are equal. This property is defined next as reflectivity.

Definition 23 (Reflectivity). *The distance between two intervals is 0 if the distances between their respective end points are equal to each other and in opposite directions.*

$\bar{d}_p(A, B) = 0$ if $(\bar{B}_L - \bar{A}_L) = -(\bar{B}_R - \bar{A}_R)$, where $\bar{A} = [\bar{A}_L, \bar{A}_R]$ and $\bar{B} = [\bar{B}_L, \bar{B}_R]$

In other words, the distance between two intervals is 0 if one interval contains values lower than another interval by a given amount, and also contains values higher than the other by the same amount. For example, considering the intervals \bar{A} and \bar{B} in Figure 3.2, $\bar{B}_L - \bar{A}_L = 2$ and $\bar{B}_R - \bar{A}_R = -2$, thus \bar{B} can be described as being both to the right and to the left of \bar{A} by equal amounts. Given this, it may not make sense to describe the distance between \bar{A} and \bar{B} as a signed value.

One possible method of representing this information is by introducing a sign that represents the distance as being both positive and negative, e.g., $\bar{d}(\bar{A}, \bar{B}) = \pm 2$. However, such a result may not be practical for many applications because it may lead to an overcomplicated system. Considering this,

the value given by the directional distance \bar{d}_p (3.2) in such cases is 0. This is a reasonable result as the centres of the intervals are the same and their boundaries are both equal distance from the centre in opposing directions.

Note that if it is necessary to distinguish the degree to which sets overlap, such as the intervals in Figure 3.2, then a similarity measure may be more appropriate than distance.

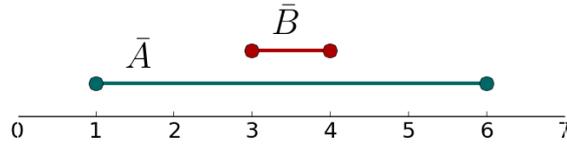


Figure 3.2: Two overlapping interval sets, \bar{A} and \bar{B} .

Theorem 1. *The directional distance measure \bar{d}_p (3.2) follows the properties*

partial symmetry: $\bar{d}(\bar{A}, \bar{B}) = -\bar{d}(\bar{B}, \bar{A})$

reflectivity: $\bar{d}(\bar{A}, \bar{B}) = 0$, where $\bar{A} = [\bar{A}_L, \bar{A}_R]$ and $\bar{B} = [\bar{B}_L, \bar{B}_R]$,

if $(\bar{B}_L - \bar{A}_L) = -(\bar{B}_R - \bar{A}_R)$.

self-identity: $\bar{d}(\bar{A}, \bar{A}) = 0$

transitivity: If $\bar{A} \leq \bar{B} \leq \bar{C}$, then $\bar{d}(\bar{A}, \bar{B}) \leq \bar{d}(\bar{A}, \bar{C})$

Proof. Partial Symmetry:

$$\begin{aligned} \bar{d}_p(\bar{A}, \bar{B}) &= -\bar{d}_p(\bar{B}, \bar{A}) \\ \bar{B}_L - \bar{A}_L + \bar{B}_R - \bar{A}_R &= -(\bar{A}_L - \bar{B}_L + \bar{A}_R - \bar{B}_R) \\ &= -\bar{A}_L + \bar{B}_L - \bar{A}_R + \bar{B}_R \end{aligned}$$

Reflectivity:

$$\begin{aligned} \text{Let } \beta &= \bar{B}_L - \bar{A}_L = -(\bar{B}_R - \bar{A}_R) \\ \bar{d}_p(\bar{A}, \bar{B}) &= \bar{B}_L - \bar{A}_L + \bar{B}_R - \bar{A}_R \\ &= \beta + (-\beta) \\ &= 0 \end{aligned}$$

Self-identity:

If $\bar{A} = \bar{B}$ then $\bar{B}_L - \bar{A}_L = \bar{B}_R - \bar{A}_R = 0$ therefore $\bar{d}_p(\bar{A}, \bar{B}) = 0$

Transitivity:

$$\bar{d}_p(\bar{A}, \bar{B}) \leq \bar{d}_p(\bar{A}, \bar{C})$$

$$\bar{B}_L - \bar{A}_L + \bar{B}_R - \bar{A}_R \leq \bar{C}_L - \bar{A}_L + \bar{C}_R - \bar{A}_R$$

$$\bar{B}_L + \bar{B}_R \leq \bar{C}_L + \bar{C}_R$$

Given that $\bar{B} \leq \bar{C}$ it correctly follows that $\bar{d}_p(\bar{A}, \bar{B}) \leq \bar{d}_p(\bar{A}, \bar{C})$.

□

Note that \bar{d}_p does not have the standard property of triangle inequality. Due to using signed results, the rule of triangle inequality is stricter. In a non-directional distance measure, because $\bar{d}(\bar{A}, \bar{B}) = \bar{d}(\bar{B}, \bar{A})$, the ordering of the given intervals which are measured has no effect on the rule of triangle inequality, e.g., both

$$\bar{d}(\bar{A}, \bar{C}) \leq \bar{d}(\bar{A}, \bar{B}) + \bar{d}(\bar{B}, \bar{C})$$

and

$$\bar{d}(\bar{A}, \bar{C}) \leq \bar{d}(\bar{B}, \bar{A}) + \bar{d}(\bar{B}, \bar{C})$$

are true. The proposed directional distance measure \bar{d}_p does not follow this rule of triangle inequality because it gives a signed result. However, a more strict form of triangle inequality can be used if the sign of the result is taken into consideration. For example, one can conclude that

$$\bar{d}_p(\bar{A}, \bar{C}) \leq \bar{d}_p(\bar{A}, \bar{B}) + \bar{d}_p(\bar{B}, \bar{C})$$

and

$$\bar{d}_p(\bar{A}, \bar{C}) \leq -\bar{d}_p(\bar{B}, \bar{A}) + \bar{d}_p(\bar{B}, \bar{C})$$

are true. These are proven as follows:

$$\begin{aligned}
\bar{d}_p(A, \bar{C}) &\leq \bar{d}_p(A, \bar{B}) + \bar{d}_p(\bar{B}, \bar{C}) \\
\frac{1}{2}(\bar{C}_L - \bar{A}_L + \bar{C}_R - \bar{A}_R) &\leq \frac{1}{2}(\bar{B}_L - \bar{A}_L + \bar{B}_R - \bar{A}_R) + \frac{1}{2}(\bar{C}_L - \bar{B}_L + \bar{C}_R - \bar{B}_R) \\
\bar{C}_L - \bar{A}_L + \bar{C}_R - \bar{A}_R &\leq \bar{B}_L - \bar{A}_L + \bar{B}_R - \bar{A}_R + \bar{C}_L - \bar{B}_L + \bar{C}_R - \bar{B}_R \\
\bar{C}_L - \bar{A}_L + \bar{C}_R - \bar{A}_R &= \bar{B}_L - \bar{A}_L + \bar{B}_R - \bar{A}_R + \bar{C}_L - \bar{B}_L + \bar{C}_R - \bar{B}_R
\end{aligned}$$

and

$$\begin{aligned}
\bar{d}_p(\bar{A}, \bar{C}) &\leq -\bar{d}_p(\bar{B}, \bar{A}) + \bar{d}_p(\bar{B}, \bar{C}) \\
\frac{1}{2}(\bar{C}_L - \bar{A}_L + \bar{C}_R - \bar{A}_R) &\leq -\frac{1}{2}(\bar{A}_L - \bar{B}_L + \bar{A}_R - \bar{B}_R) + \frac{1}{2}(\bar{C}_L - \bar{B}_L + \bar{C}_R - \bar{B}_R) \\
\bar{C}_L - \bar{A}_L + \bar{C}_R - \bar{A}_R &\leq -\bar{A}_L + \bar{B}_L - \bar{A}_R + \bar{B}_R + \bar{C}_L - \bar{B}_L + \bar{C}_R - \bar{B}_R \\
\bar{C}_L - \bar{A}_L + \bar{C}_R - \bar{A}_R &= -\bar{A}_L + \bar{B}_L - \bar{A}_R + \bar{B}_R + \bar{C}_L - \bar{B}_L + \bar{C}_R - \bar{B}_R
\end{aligned}$$

Note that the sign \leq (typically given in triangle inequality) is changed to $=$ to show that both sides of the equation are equal. From this it can be seen that the distance $\bar{d}_p(A, C)$ can be calculated by the sum of the distances between A and B and between B and C if the directions of these pairs are known. Thus, one can infer $\bar{d}_p(A, C)$ if $\bar{d}_p(A, B)$ and $\bar{d}_p(B, C)$ are already known.

This form of triangle inequality may also be written using the absolute values, without concern for direction, e.g., as

$$|\bar{d}_p(\bar{A}, \bar{C})| \leq |\bar{d}_p(\bar{A}, \bar{B})| + |\bar{d}_p(\bar{B}, \bar{C})|. \quad (3.3)$$

However, by taking this approach, the direction of each distance is no longer known. As a result, one could not infer distances; e.g., if $d(A, B)$ and $d(B, C)$ are known, one cannot infer $d(A, C)$ if the absolute results are used as in (3.3). However, if the sign of direction is maintained, it is possible to infer the exact magnitude and direction of $d(A, C)$.

This constraint on triangle inequality restricts the ordering of the parameters when measuring distance. This can be explained with the aid of Table 3.1. The first input of part 1 must appear as the first input where it occurs

in part 2, if not then the negative of the result is used instead. Likewise, the second input of part 1 must appear as the second input where it occurs in part 2, if not then the negative of the result is used instead. Note that the restricted triangle inequality is not affected by the ordering of the fuzzy sets, i.e., $\bar{d}_p(\bar{A}, \bar{C}) \leq \bar{d}_p(\bar{A}, \bar{B}) + \bar{d}_p(\bar{B}, \bar{C})$ is true if $\bar{A} \leq \bar{B} \leq \bar{C}$, or if $\bar{B} \leq \bar{A} \leq \bar{C}$, or any other ordering on \bar{A}, \bar{B} and \bar{C} .

Part 1	Part 2
$\bar{d}_p(\bar{A}, \bar{C})$	$\leq \bar{d}_p(\bar{A}, \bar{B}) + \bar{d}_p(\bar{B}, \bar{C})$
$\bar{d}_p(\bar{A}, \bar{C})$	$\leq -\bar{d}_p(\bar{B}, \bar{A}) + \bar{d}_p(\bar{B}, \bar{C})$
$\bar{d}_p(\bar{A}, \bar{C})$	$\leq \bar{d}_p(\bar{A}, \bar{B}) - \bar{d}_p(\bar{C}, \bar{B})$
$\bar{d}_p(\bar{A}, \bar{C})$	$\leq -\bar{d}_p(\bar{B}, \bar{A}) - \bar{d}_p(\bar{C}, \bar{B})$

Table 3.1: An example of the restricted property of triangle inequality on the directional distance measure \bar{d}_p (3.2).

3.2.3 Directional Distance Between Fuzzy Sets

The proposed directional distance \bar{d}_p (3.2) is so far only developed to compare α -cuts, however it may easily be used to compare fuzzy sets. By modifying an existing distance measure on fuzzy sets, the distance used to compare α -cuts in the respective measure (e.g., the Hausdorff metric or Minkowski distance) may be replaced with the proposed directional distance. For example, the α -cuts of two fuzzy sets could be compared using Ralescu & Ralescu's distance (2.35), replacing h with \bar{d}_p . This, in fact, results in the directional distance proposed by Yao & Wu (2.46) [10].

However, the proposed measure uses a distance between fuzzy sets based on the method by Chaudhuri and Rosenfeld (2.37). This measure is preferable because it weights the distance between α -cuts according to the location of

the α -cuts. Intuitively, the more certainty there is in the membership value of a given point, the more certainty there must also be in the distance between these points. Given this, it makes sense to weight the distance between α -cuts to reflect how certain we are of that distance.

Based on the weighted approach by Chaudhuri and Rosenfeld (2.37) and using the proposed directional distance between α -cuts (3.2) the following definition introduces a new directional distance measure.

Definition 24 (Distance between non-normal fuzzy sets). *The directional distance between two normal, convex fuzzy sets $A, B \in T1(X)$ is*

$$d_p^{T1:nc}(A, B) = \frac{\sum_{\alpha=1}^M y_\alpha \frac{1}{2}(\overline{B_{\alpha L}} - \overline{A_{\alpha L}} + \overline{B_{\alpha R}} - \overline{A_{\alpha R}})}{\sum_{\alpha=1}^M y_\alpha}, \quad (3.4)$$

where y_α is the membership value of the given α -cut, M is the total number of α -cuts, $\overline{A_\alpha}$ is the continuous interval $[\overline{A_{\alpha L}}, \overline{A_{\alpha R}}]$ and, likewise, $\overline{B_\alpha} = [\overline{B_{\alpha L}}, \overline{B_{\alpha R}}]$

Note that the function name $d_p^{T1:nc}$ indicates that the distance measure is for type-1 fuzzy sets that are normal and convex. Also note that $d_p^{T1:nc}$ could in theory be used for non-normal fuzzy sets with equal heights. However, such cases are not explored here but are covered in detail in Section 3.3.

It is clear to see that $d_p^{T1:nc}$ (3.4) follows all of the properties shown for the directional distance between α -cuts $\overline{d_p}$ alone (3.2).

Note that the property of reflectivity (Definition 23) states that for two intervals $[\overline{A_L}, \overline{A_R}]$ and $[\overline{B_L}, \overline{B_R}]$, if $(\overline{B_L} - \overline{A_L}) = -(\overline{B_R} - \overline{A_R})$, then the distance $\overline{d}(\overline{A}, \overline{B})$ is 0. If this is true at every α -cut of A and B , then their distance using $d_p^{T1:nc}$ (3.4) will be 0. In other words, this property states that when one interval is a subset of the other, the distance between the left end points cancels out the distance between the right end points. If these two distances are equal then the result of $d_p^{T1:nc}(A, B)$ is 0. If this is true for every α -cut then the distance between the fuzzy sets is 0.

If the difference between $\overline{A_{\alpha L}}$ and $\overline{B_{\alpha L}}$ is greater than the difference between $\overline{A_{\alpha R}}$ and $\overline{B_{\alpha R}}$ then the resulting distance will be a negative value, and if the opposite is true then the result is positive. Demonstrations of these effects that result from the property of reflectivity are shown in the next section.

Having established a directional distance between fuzzy sets, the next section demonstrates this measure and compares it against others in the literature.

3.2.4 Comparison with the Current Literature

This section compares the proposed directional distance $d_p^{T1:nc}$ (3.4) with some other directional and non-directional distance measures in the literature. Comparisons are given against:

d_{yw} Yao and Wu's [10] directional distance measure (2.46).

d_{cr} Chaudhuri and Rosenfeld's [14] Hausdorff-based distance (2.37).

d_g Grzegorzewski's [74] Minkowski-based distance (2.43), where $p = 2$ and $q = 0.5$.

d_a Allahviranloo et al.'s [76] distance based on the widths and heights of the fuzzy sets (2.48).

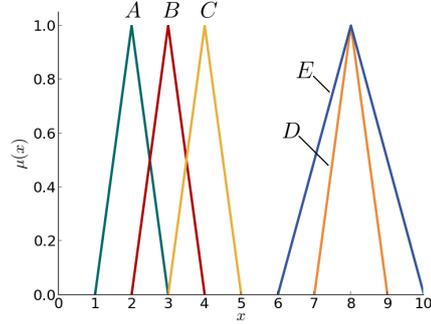
d_{cc} The directional distance between the centroids of the fuzzy sets; given in (3.5).

Section 2.4.2 provides details of d_{yw} , d_{cr} , d_g and d_a . Using the centroids (d_{cc}), the directional distance between two fuzzy sets A and B is given as

$$d_{cc}(A, B) = B_c - A_c \quad (3.5)$$

where A_c and B_c are the centroids of A and B , respectively (given in (2.8)).

Figure 3.3 shows an example of five fuzzy sets with triangular membership functions. The fuzzy sets A , B , C and D have the same membership function

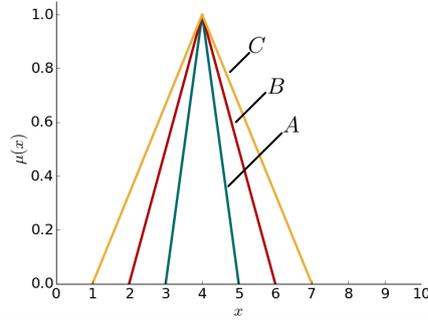


	$d(A, A)$	$d(A, B)$	$d(A, C)$	$d(B, A)$	$d(A, D)$	$d(A, E)$
$d_p^{T1:nc}$ (3.4)	0.0	1.0	2.0	-1.0	6.0	6.0
d_{yw} (2.46)	0.0	-1.0	-2.0	1.0	-6.0	-6.0
d_{cr} (2.37)	0.0	1.0	2.0	1.0	6.0	6.33
d_g (2.43)	0.0	1.0	2.0	1.0	6.0	6.028
d_a (2.48)	0.0	1.0	2.0	1.0	6.0	6.009
d_{cc} (3.5)	0.0	1.0	2.0	-1.0	6.0	6.0

Figure 3.3: Five fuzzy sets and their comparison according to the proposed directional distance measure $d_p^{T1:nc}$ and five other distance measures (d_{yw} , d_{cr} , d_g , d_a and d_{cc}).

shapes and only the position is changed. This example shows that the results of $d_p^{T1:nc}$ are intuitive and do not differ from existing methods when the fuzzy sets are constructed using simple membership functions. Further examples within this chapter, however, show that the results of the proposed method differ when fuzzy sets become more complex.

Within Figure 3.3, as expected, for all of the measures the distance between a fuzzy set and itself is 0. Note that the direction of the distance measure by Yao and Wu measures distance in the opposite direction to the proposed method, i.e., $d_{yw}(A, B) = d_p^{T1:nc}(B, A)$.



	$d(A, A)$	$d(A, B)$	$d(A, C)$	$d(B, C)$
$d_p^{T1:nc}$ (3.4)	0	0	0	0
d_{yw} (2.46)	0	0	0	0
d_{cr} (2.37)	0.0	0.317	0.633	0.317
d_g (2.43)	0.0	0.577	1.155	0.577
d_a (2.48)	0.0	0.333	0.667	0.333
d_{cc} (3.5)	0	0	0	0

Figure 3.4: Three fuzzy sets and their comparison according to the proposed directional distance measure $d_p^{T1:nc}$ and five other distance measures (d_{yw} , d_{cr} , d_g , d_a and d_{cc}).

The distance between A and D and between A and E show that the proposed measure $d_p^{T1:nc}$ is not affected by the width of a symmetrical fuzzy set. This is because while the right side of E is further from A than D , the left side is closer than D ; thus the distance remains the same. The function d_{cr} , however, focuses on the largest distance between α -cuts, and thus gives a noticeably larger result for $d(A, E)$. Additionally, although d_g and d_a give a larger result for (A, E) compared to (A, D) , the change in value is small and may have little effect on one's perception of that distance.

Figure 3.4 demonstrates how the property of reflectivity affects the results of the directional distance measure. In this example, A , B and C share the

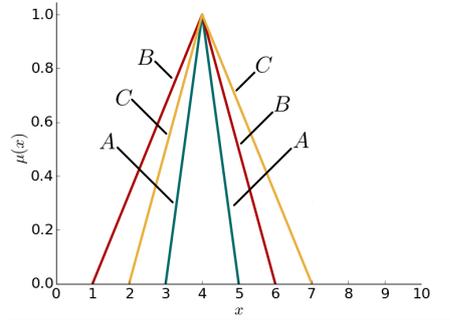
same mean value (centroid) and are all symmetrical. The only difference between the membership functions is a change in width. Therefore, each fuzzy set's membership function can be described as being both to the left and to the right of another fuzzy set by an equal amount. As a result, the directional measures $d_p^{T1:nc}$ and d_{wy} result in a distance of 0 for each comparison. Additionally, as a result of each fuzzy set sharing the same centroid, the distance between the centroids d_{cc} is also always 0. The other non-directional measures, however, convey a different perspective of distance in the fuzzy sets. Though their results vary, they are all non-zero values.

This highlights an important difference between a directional and a non-directional distance measure. Although a non-directional measure will show the overall distance regardless of direction, a directional distance, such as $d_p^{T1:nc}$ (3.4) or d_{yw} (2.46), will subtract the distance in the left direction from the distance in the right direction. In this case, this has caused the distance to reduce to 0; this is the property of reflectivity.

To further highlight this effect, Figure 3.5 shows asymmetric fuzzy sets B and C , where B has the greatest distance to the right of A , and C has the greatest distance to the left of A . The sign of the measures $d_p^{T1:nc}$ (3.4) and d_{yw} (2.46) reflect where the greatest distance between the fuzzy sets lies. Note that this example highlights the difference between the two directional distance measures $d_p^{T1:nc}$ and d_{yw} and the centroid based distance d_{cc} .

The proposed measure $d_p^{T1:nc}$ calculates a smaller value of distance than Yao and Wu's directional measure d_{yw} because the proposed approach weights the distance of each α -cut by its degree of membership. Therefore, the closer an α -cut approaches the value 1 the more possibility there is in the values belonging to the set and thus the more confidence there is regarding the distance at these α -cuts.

The distance between A and B , and between A and C , decreases at higher α -cuts which results in an absolute distance of 0.165. Yao and Wu's method,



	$d(A, B)$	$d(A, C)$
$d_p^{T1:nc}$ (3.4)	-0.159	0.159
d_{yw} (2.46)	0.25	-0.25
d_{cr} (2.37)	0.633	0.633
d_g (2.43)	0.913	0.913
d_a (2.48)	0.559	0.559
d_{cc} (3.5)	-0.333	0.333

Figure 3.5: Three fuzzy sets and their comparison according to the proposed directional distance measure $d_p^{T1:nc}$ and five other distance measures (d_{yw} , d_{cr} , d_g , d_a and d_{cc}).

however, does not weight the distance of each α -cut and instead takes the average of all α -cuts. Therefore, the α -cuts near $\alpha = 0$ are given the same importance as α -cuts near $\alpha = 1$. As a result, in this example the overall value of distance from d_{yw} is larger than $d_p^{T1:nc}$. In addition to this, the centroid-based distance d_{cc} gives an even larger result than $d_p^{T1:nc}$ and d_{yw} . This is because d_{cc} also does not take into account that the fuzzy sets are closer where α is near 1 than where α is near 0.

Note, also, that the distance in Figure 3.5, resulting from either directional distance measure, is considerably smaller than when using any of the non-directional distance measures (i.e., d_{cr} , d_g and d_a). This is because, as

discussed on Page 70, due to the fuzzy sets overlapping, the directional measures essentially subtract the distance to the left of the membership function from the distance to the right. As a result, the distance decreases if fuzzy sets overlap such that one of the membership functions contains values both to the left and right of the other. However, the non-directional distance measures do not share this property.

This section has introduced a directional distance measure for normal, convex, type-1 fuzzy sets. The next two sections expand on this to measure the distance between fuzzy sets that may be non-normal or non-convex.

3.3 Non-Normal Fuzzy Sets

Non-normal fuzzy sets occur when there is no absolute certainty for any value, i.e., $\mu_A(x) < 1, \forall x \in X, A \in T1(X)$ (discussed in Section 2.2.1; Page 13). This may arise, for example, when modelling data taken from a survey in which no one is in agreement with each other, from the output of a fuzzy logic system or from the lower membership function of an interval type-2 fuzzy set.

In practice, it may be difficult to calculate the distance between non-normal fuzzy sets. This is because the α -cut of a non-normal fuzzy set cannot be measured where α exceeds the height of the fuzzy set. This introduces the problem *how can we measure the distance between fuzzy sets with different heights?* For example, consider the two non-normal fuzzy sets $A, B \in T1(X)$ in Figure 3.6. The height of A is 0.5 and the height of B is 0.8. As a result, $\overline{A_\alpha} = \emptyset$ where $\alpha > 0.5$ and $\overline{B_\alpha} = \emptyset$ where $\alpha > 0.8$. To measure the distance between A and B , two problems must be addressed.

1. Firstly, how can the distance be measured at α -cuts where for one fuzzy set the α -cut is the empty set. In this example, this is where $0.5 < \alpha \leq 0.8$.

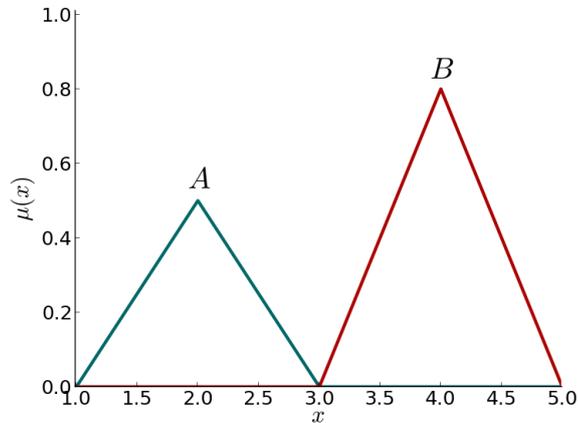


Figure 3.6: Two non-normal fuzzy sets A and B .

2. Secondly, how can the distance be measured where for both fuzzy sets the α -cut is the empty set; in this case where $\alpha > 0.8$.

Regarding these problems, there is no clear definition of how the distance between non-normal fuzzy sets should be measured. Several techniques have been developed in the literature, however these methods may not be ideal; as demonstrated in Section 3.3.2.

For example, Chaudhuri and Rosenfeld [14] use an approach that normalises the membership functions of the fuzzy sets so that there is no α -cut which is represented by the empty set. However, this produces inconsistent results compared to their approach for normal fuzzy sets. Wang et al. [15] compare the distance between the heights of the fuzzy sets, however this results in unexpectedly high values of distance. Cheng [11] compares the mean x and y ($\mu(x)$) values of the fuzzy sets, however this does not give expected results for fuzzy sets with identically shaped membership functions. These are each demonstrated in Section 3.3.2.

Another method is to compare every α -cut of one fuzzy set with every α -cut of the other, making no comparison where an α -cut is empty; an example of this approach is given in Appendix B. However, the computational complexity

of this approach grows exponentially as more α -cuts are used (this is especially noticeable when comparing general type-2 fuzzy sets), and the results of comparing all permutations are consistent with the simpler process developed in this section (this is demonstrated in Appendix B). Given that the proposed method in this section produces similar results and is computationally quicker, it is the more favoured method.

3.3.1 Distance Between Non-Normal Fuzzy Sets

This section introduces a method of obtaining the distance between the α -cuts of fuzzy sets where an α -cut may be the empty set. One simple method of achieving this goal is to compare the closest non-empty α -cuts.

Definition 25 (Distance Between Empty α -cuts). *For two α -cuts \overline{A}_α and \overline{B}_α , if $\overline{A}_\alpha \neq \emptyset \wedge \overline{B}_\alpha = \emptyset$ or $\overline{A}_\alpha = \emptyset \wedge \overline{B}_\alpha \neq \emptyset$ (where \wedge is the logical ‘and’ operation), then the proposed distance is*

$$\overset{\dot{d}}{d}_p(\overline{A}_\alpha, \overline{B}_\alpha) = \begin{cases} \bar{d}(\overline{A}_{\alpha_k}, \overline{B}_\alpha) & \text{if } \overline{A}_\alpha = \emptyset \wedge \overline{B}_\alpha \neq \emptyset \\ \bar{d}(\overline{A}_\alpha, \overline{B}_{\alpha_k}) & \text{if } \overline{A}_\alpha \neq \emptyset \wedge \overline{B}_\alpha = \emptyset, \end{cases} \quad (3.6)$$

where $\overline{A}_{\alpha_k} = \max \{ \overline{A}_\alpha \mid \overline{A}_\alpha \neq \emptyset, \forall \alpha \in [0, 1] \}$ and \bar{d} may be any distance function between two α -cuts.

Thus, where an α -cut exceeds the height of a given fuzzy set, the α -cut at the height of the fuzzy set is used as a substitute. Note that this is one of several possible methods, for example the average distance from all non-empty α -cuts could be used instead. This method has been chosen because it provides the closest possible comparison between respective α -cuts in different sets.

Where an α -cut is the empty set in both fuzzy sets then the distance does not need to be considered. This is because it does not make sense to measure the distance between two empty sets.

Expanding upon the directional distance measure $d_p^{T1:nc}$ (3.4), the proposed distance between fuzzy sets is as follows.

Definition 26 (Distance between non-normal fuzzy sets).

$$d_p^{T1:c}(A, B) = \frac{\sum_{\alpha \in [0, \lambda]} y_\alpha \dot{d}_p(\overline{A}_\alpha, \overline{B}_\alpha)}{\sum_{\alpha \in [0, \lambda]} y_\alpha}, \quad (3.7)$$

where $\lambda = \max \{ \alpha \mid \overline{A}_\alpha \neq \emptyset \vee \overline{B}_\alpha \neq \emptyset, \alpha \in [0, 1] \}$ and \dot{d}_p is

$$\dot{d}_p(\overline{A}_\alpha, \overline{B}_\alpha) = \begin{cases} \bar{d}_p(\overline{A}_\alpha, \overline{B}_\alpha) & \overline{A}_\alpha \neq \emptyset \wedge \overline{B}_\alpha \neq \emptyset \\ \bar{d}_p(\overline{A}_{\alpha_k}, \overline{B}_\alpha) & \overline{A}_\alpha = \emptyset \wedge \overline{B}_\alpha \neq \emptyset \\ \bar{d}_p(\overline{A}_\alpha, \overline{B}_{\alpha_k}) & \overline{A}_\alpha \neq \emptyset \wedge \overline{B}_\alpha = \emptyset \end{cases} \quad (3.8)$$

where $\overline{A}_{\alpha_k} = \max \{ \overline{A}_\alpha \mid \overline{A}_\alpha \neq \emptyset, \forall \alpha \in [0, 1] \}$. This uses the directional distance \bar{d}_p (3.2). However, to attain a non-directional distance measure \bar{d} (3.1) may be used instead of \bar{d}_p .

Note that the function name $d_p^{T1:c}$ in (3.7) indicates that the fuzzy sets must be type-1 and convex, but may be non-normal.

3.3.2 Comparison with the Current Literature

This section demonstrates the results of the proposed measure $d_p^{T1:c}$ (3.7) when comparing type-1 fuzzy sets that may be non-normal. As well as demonstrating the proposed approach, a comparison is also given against

d_{cr} Chaudhuri and Rosenfeld's [14] non-normal distance measure (2.39); detailed in Section 2.4.2.

d_w Wang et al.'s (d_w) non-normal distance measure for only trapezoidal fuzzy sets [15]; given in (3.9).

d_c Cheng's [11] non-normal distance measure; given in (3.11).

d_{cc} The directional distance between the centroids of the fuzzy sets (3.5).

The equations d_w and d_c are given next.

Details of the measures demonstrated within this section

Wang et al.'s [15] (d_w) non-normal distance measure for two trapezoidal fuzzy sets $A, B \in T1(X)$ is

$$d_w(A, B) = (1 - \sigma)d_{w1}(A, B) + \sigma d_{w2}(A, B) \quad (3.9a)$$

$$d_{w1}(A, B) = \frac{1}{4}(|a_1 - b_1| - |a_4 - b_4|) + \frac{3}{4}(|a_2 - b_2| + |a_3 - b_3|) \quad (3.9b)$$

$$d_{w2}(A, B) = \frac{3}{4}|\omega_A - \omega_B| + \frac{1}{4}(|Lk_A - Lk_B| + |Rk_A - Rk_B|) \quad (3.9c)$$

where $\sigma \in [0, 1]$ and is set as 0.25 [15], A is represented by a trapezoidal membership function $(a_1, a_2, a_3, a_4; \omega)$, $a_1 < a_2 < a_3 < a_4$ and ω is the height of the fuzzy set as

$$\mu_A(x) = \begin{cases} \frac{\omega(x-a_1)}{a_2-a_1}, & a_1 \leq x \leq a_2 \\ \omega, & a_2 \leq x \leq a_3 \\ \frac{\omega(a_4-x)}{a_4-a_3}, & a_3 \leq x \leq a_4 \\ 0, & \text{otherwise.} \end{cases} \quad (3.10)$$

Also in d_w (3.9), $Lk_A = \frac{a_2-a_1}{\omega}$ and $Rk_A = \frac{a_3-a_4}{\omega}$.

Cheng's [11] (d_c) distance for non-normal fuzzy sets gives each fuzzy set a rank position defined by the centroid along the x -axis (2.8) and the centroid along the y -axis (μ -axis), denoted x_0 and y_0 , respectively. The rank value of $A \in T1(X)$ is $R(A) = \sqrt{x_0^2 + y_0^2}$. This rank value essentially reduces a fuzzy set to a single centroid (R) based on its centroids along both axes. This can then be used to determine the relative positions between multiple fuzzy sets. In these demonstrations, the distance between $A, B \in T1(X)$ will be given as

$$d(A, B) = |R(A) - R(B)| \quad (3.11)$$

Demonstrations of the measures

This section demonstrates that non-normality itself does not necessarily change the results of $d_p^{T1:nc}$ and it is rather the symmetry, or asymmetry, of the fuzzy

sets that affects the results.

For example, Figure 3.7 shows pairs of fuzzy sets that are symmetrical and have different heights. The results of the proposed method and the centroid-approach are the same if the fuzzy sets are symmetrical regardless of whether the fuzzy sets are normal or non-normal.

The measures d_{cr} , d_w and d_c , however, always result in a different value when the fuzzy sets are non-normal. Note that in Figure 3.7 (and Figure 3.8) d_w is unexpectedly high, giving values around 6 when one would expect values around 4. Additionally, d_c and d_{cr} give unexpected results for Figure 3.7a as one would expect the result to be 4.

The functions d_{cr} , d_w and d_c each give different results where the height of a fuzzy set differs. In Figure 3.7, the distance decreases or increases when the height of B is less than 1.0. However, this makes it impossible to discern between non-normal fuzzy sets that are close, and normal fuzzy sets that are distant. For example, using d_w the distance between A and B in Figure 3.7d would be the same as if B were a normal, symmetrical fuzzy set with the centre at $x = 8.3$.

To give another example, Figure 3.8 shows that the results of the proposed method $d_p^{T1:c}$ change if the height of an asymmetric fuzzy set changes. This is because changing the height affects the gradient of each side of the triangular membership function, which in turn changes the coordinates of the α -cuts. As a result, although the left-most, centre and right-most coordinates of B are the same (at $x = 5, 6, 8$), the x values within each α -cut, and therefore the centre of each α -cut, are different as the height of the fuzzy set is different. This, however, does not occur with the centroid-based approach d_{cc} .

The distance measures d_{cr} , d_w and d_c state that the distance according to the non-normal fuzzy sets changes in Figures 3.7 and 3.8. d_{cr} and d_c give a decreasing distance as the height of B decreases, whereas the distance according to d_w increases. Regarding d_{cr} , this is because it measures the distance

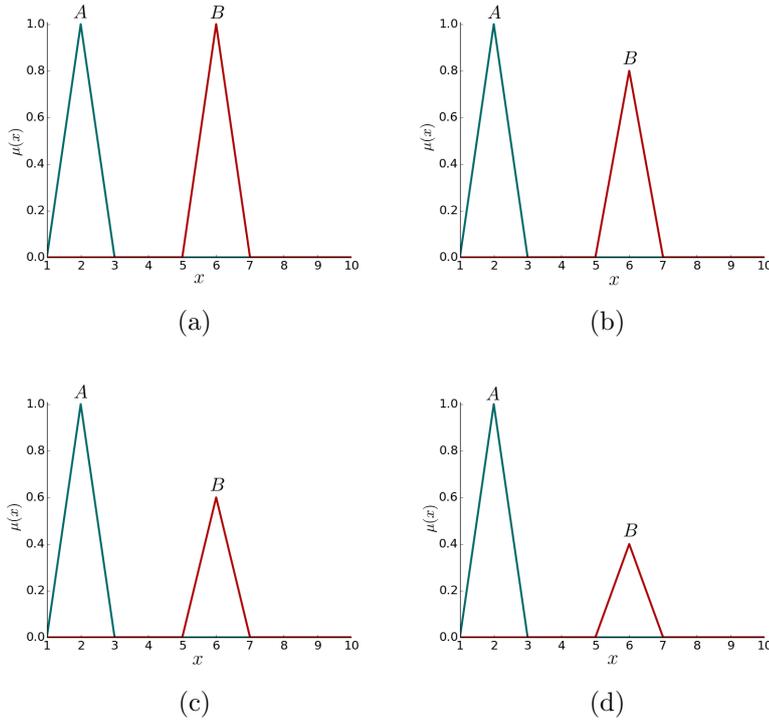


Figure	$d_p^{T1:c}$	d_{cr}	d_w	d_c	d_{cc}
	(3.7)	(2.39)	(3.9)	(3.11)	(3.5)
a	4.0	4.3143	6.0	3.9592	4.0
b	4.0	4.2829	6.0687	3.9517	4.0
c	4.0	4.2514	6.1583	3.9447	4.0
d	4.0	4.22	6.3	3.9417	4.0

Figure 3.7: Four pairs of symmetrical fuzzy sets with differing heights, and the distances according to the proposed measure $d_p^{T1:c}$ and four other approaches (d_{cr} , d_w , d_c and d_{cc}).

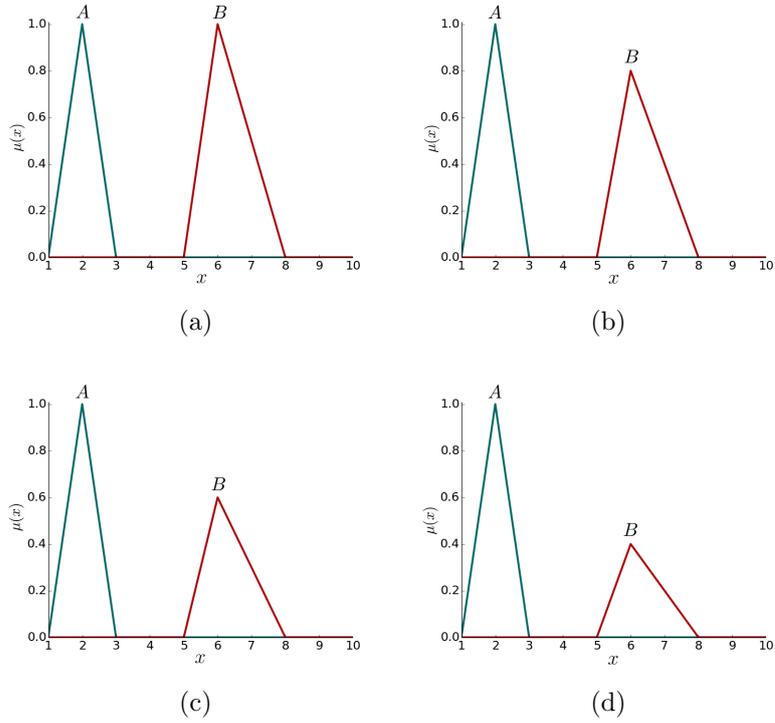


Figure	$d_p^{T1:c}$ (3.7)	d_{cr} (2.39)	d_w (3.9)	d_c (3.11)	d_{cc} (3.5)
a	4.1583	4.656	6.25	4.2908	4.3333
b	4.1012	4.6153	6.3344	4.2839	4.3333
c	4.0569	4.5746	6.45	4.2775	4.3333
d	4.0251	4.5338	6.6437	4.2748	4.3333

Figure 3.8: Four pairs of symmetrical and asymmetric fuzzy sets with differing heights, and the distances according to the proposed measure $d_p^{T1:c}$ and four other approaches (d_{cr} , d_w , d_c and d_{cc}).

between the vertical slices of the fuzzy sets. As B 's height decreases, the difference also decreases between its membership values (at $x \in [5, 7]$) and A 's membership values of the same elements ($\mu_A(x) = 0, \forall x \in [5, 7]$).

The d_c approach uses the average μ value to compare distance, and as the height of a fuzzy set decreases the average μ also decreases in value, resulting in a smaller distance than for normal fuzzy sets. The d_w method directly compares the heights of the fuzzy sets, and so the resulting distance is always affected by the heights of the fuzzy sets regardless of the shape of the membership functions.

Additionally, there is inconsistency between the results from Chaudhuri and Rosenfeld's approaches for normal (2.37) and non-normal (2.39) fuzzy sets. If the non-normal approach is applied to normal fuzzy sets, then the distance is given as a larger value than the normal approach. For example, referring to Table 3.2, in Figure 3.7a, using d_{cr} for normal fuzzy sets (2.37) the result is 4.0, however the non-normal approach (2.39) calculates the distance as 4.3143. Additionally, in Figure 3.8a, using the normal fuzzy set approach the distance is 4.3167. However, using the non-normal approach the distance is 4.656.

In contrast, one of the advantages of the proposed approach is that it does not give inconsistent results between normal and non-normal fuzzy sets. Whether using the standard comparison $d_p^{T1:nc}$ (3.4) or the extension for non-normal fuzzy sets $d_p^{T1:c}$ (3.7), the results are always the same when measuring normal fuzzy sets.

Having introduced a method of measuring the distance between non-normal fuzzy sets, the next section proceeds to extend this measure to compare fuzzy sets that are non-convex.

d_{cr} method	Figure 3.7a	Figure 3.8a
Normal (2.37)	4.0	4.3167
Non-Normal (2.39)	4.3143	4.656

Table 3.2: Distances between two pairs of normal fuzzy sets using the normal and non-normal approaches of d_{cr} .

3.4 Non-Convex Fuzzy Sets

In most cases, fuzzy sets are convex (detailed in Section 2.2.1; Page 14), but it can be useful to instead represent data by a non-convex membership function [18] (as discussed in Section 2.3.4). Any α -cut of a normal, convex fuzzy set can be represented as a continuous interval, but a non-convex α -cut is instead represented by a discontinuous interval (see Definition 8).

For example, consider the non-convex and convex fuzzy sets A and B in Figure 3.9. Any α -cut of B is represented by a continuous interval, however any α -cut of A at $\alpha > 0.6$ consists of two separate intervals. For example, at $\alpha = 0.8$ $\overline{A}_\alpha = \{[1.8, 2.5], [3.5, 4.2]\}$. This introduces the problem *how can the distance between discontinuous α -cuts be measured?*

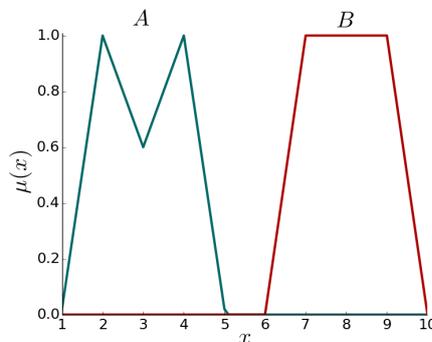


Figure 3.9: A non-convex fuzzy set A and a convex fuzzy set B .

3.4.1 Distance Between Non-Convex Fuzzy Sets

In Section 3.2.2, the distance between α -cuts is represented by the average distance between the boundaries of the α -cuts. Taking the same approach, this section proposes a method of calculating the distance between non-convex fuzzy sets by taking the average distance between the α -cut's continuous regions. This ensures that the distance measure has the same properties for both convex and non-convex fuzzy sets. The proposed measure is as follows:

Definition 27 (Distance between discontinuous intervals). *The directional distance between α -cuts that may be discontinuous is calculated as*

$$\bar{\bar{d}}_p(\bar{\bar{A}}_\alpha, \bar{\bar{B}}_\alpha) = \frac{1}{|\bar{\bar{A}}_\alpha| |\bar{\bar{B}}_\alpha|} \sum_{i=1}^{|\bar{\bar{A}}_\alpha|} \sum_{j=1}^{|\bar{\bar{B}}_\alpha|} \bar{d}_p(A_{\alpha_i}, B_{\alpha_j}) \quad (3.12)$$

where A_{α_i} represents the i^{th} continuous interval within $\bar{\bar{A}}_\alpha$ where $\bar{\bar{A}}_\alpha$ may be discontinuous, and $|\bar{\bar{A}}_\alpha|$ and $|\bar{\bar{B}}_\alpha|$ are the total number of continuous intervals within $\bar{\bar{A}}_\alpha$ and $\bar{\bar{B}}_\alpha$, respectively. This uses the directional distance between continuous α -cuts \bar{d}_p (3.2). However, to attain a non-directional distance measure \bar{d} (3.1) may be used instead of \bar{d}_p .

To demonstrate $\bar{\bar{d}}_p$, Figure 3.9 shows a non-convex and a convex fuzzy set. At $\alpha = 0.8$, $\bar{\bar{A}}_\alpha = \{[1.8, 2.5], [3.5, 4.2]\}$ and $\bar{B}_\alpha = [6.8, 9.2]$, and so the distance between A and B must be calculated using $\bar{\bar{d}}_p$ (3.12). $\bar{\bar{A}}_\alpha$ is split into two intervals, \bar{A}_{α_1} and \bar{A}_{α_2} and the distance is calculated between \bar{A}_{α_1} and \bar{B}_α and between \bar{A}_{α_2} and \bar{B}_α ; using \bar{d}_p (3.2) these are $\bar{d}_p(\bar{A}_{\alpha_1}, \bar{B}_\alpha) = 5.85$ and $\bar{d}_p(\bar{A}_{\alpha_2}, \bar{B}_\alpha) = 4.15$. Finally, the average of these is 5.0 and is used as the result of $\bar{\bar{d}}_p(\bar{\bar{A}}_\alpha, \bar{\bar{B}}_\alpha)$ at $\alpha = 0.8$.

Expanding on the proposed measure for non-normal fuzzy sets $d_p^{T1:c}$ (3.7), the following definition proposes a distance measure for non-convex fuzzy sets.

Definition 28 (Distance between non-convex fuzzy sets). *The directional distance for two fuzzy sets $A, B \in T1(X)$ that may be normal or non-normal and*

convex or non-convex is

$$d_p^{T1}(A, B) = \frac{\sum_{\alpha \in [0, \lambda]} y_\alpha \bar{d}_p(\bar{A}_\alpha, \bar{B}_\alpha)}{\sum_{\alpha \in [0, \lambda]} y_\alpha}, \quad (3.13)$$

where $\lambda = \max \{ \alpha \mid \bar{A}_\alpha \neq \emptyset \vee \bar{B}_\alpha \neq \emptyset, \alpha \in [0, 1] \}$ and \bar{d}_p is

$$\bar{d}_p(\bar{A}_\alpha, \bar{B}_\alpha) = \begin{cases} \bar{d}_p(\bar{A}_\alpha, \bar{B}_\alpha) & \bar{A}_\alpha \neq \emptyset \wedge \bar{B}_\alpha \neq \emptyset \\ \bar{d}_p(\bar{A}_{\alpha_k}, \bar{B}_\alpha) & \bar{A}_\alpha = \emptyset \wedge \bar{B}_\alpha \neq \emptyset \\ \bar{d}_p(\bar{A}_\alpha, \bar{B}_{\alpha_k}) & \bar{A}_\alpha \neq \emptyset \wedge \bar{B}_\alpha = \emptyset \end{cases} \quad (3.14)$$

where $\bar{d}_p(\bar{A}_\alpha, \bar{B}_\alpha)$ is given in (3.12)

and $\bar{A}_{\alpha_k} = \max \{ \bar{A}_\alpha \mid \bar{A}_\alpha \neq \emptyset, \forall \alpha \in [0, 1] \}$.

This may be directional or non-directional according to the chosen function within \bar{d}_p . Note that the α -cuts are denoted \bar{A}_α and \bar{B}_α to show that they are possibly, but not necessarily, discontinuous. Also, note that the function name d_p^{T1} in (3.13) indicates that the measure compares type-1 fuzzy sets with no restrictions on normality or convexity.

3.4.2 Demonstrations

This section demonstrates the results of the centroid-based d_{cc} (3.5) and directional α -cut-based d_p^{T1} (3.13) distance measures on non-convex fuzzy sets. Note that, to the author's knowledge, there are no α -cut-based distance measures for non-convex fuzzy sets within the literature.

As demonstrated in previous examples, if fuzzy sets are symmetrical and disjoint the measurement of distance is the same as the distance between the fuzzy sets' centroids.² This is shown for non-convex fuzzy sets in Figure 3.11 where dashed lines indicates the average point of each α -cut. This shows that the distance against the fuzzy set B is always measured at $x = 8$.

²However, this is not true for overlapping fuzzy sets as demonstrated in Figure 3.5 and later in Figure 3.14.

Next, Figures 3.12 and 3.13 show how skewed (i.e., asymmetric), non-convex regions affect the results of the distance measures. Figures 3.12 and 3.13 show concave regions that are skewed to the left and right, respectively. Note that, in these examples, the fuzzy sets have been given unusual and exaggerated shapes to demonstrate the properties of the measures. This is because the properties of the distance measures are clearer to see if the causes of these properties are exaggerated.

Comparing Figures 3.12 and 3.13, the results of the centroid-based d_{cc} and α -cut-based d_p^{T1} distance measures directly contradict each other. In Figure 3.12, which contains left-skewed concave regions, the distance using the centroid-based method d_{cc} increases as the concave region becomes deeper, whereas the α -cut-based distance d_p^{T1} decreases. Likewise, in Figure 3.13, in which the fuzzy sets contain right-skewed concave regions, the distance according to the centroid-based method d_{cc} decreases, whereas the α -cut-based distance d_p^{T1} increases. The rest of this section discusses these results, firstly of d_{cc} followed by a discussion of d_p^{T1} .

The results of the centroid-based method d_{cc} change according to where there is more membership in the fuzzy set. For example, in Figure 3.12, there is a dip in membership (the concave region) to the left and so, comparatively, there is more membership to the right. As a result, the centroid becomes more skewed to the right from Figure 3.12a to (d), and so the distance between A and B increases. The same effect occurs in Figure 3.13, resulting in the centroid becoming skewed to the left and the distance becoming smaller from Figure 3.13a to (d).

The α -cut-based measure d_p^{T1} uses a different approach to determine the distance between fuzzy sets and thus produces different results. As this method uses α -cuts to compare fuzzy sets, for simplification this next demonstration compares the distance between discontinuous α -cuts rather than fuzzy sets as a whole. Figure 3.10 shows three pairs of α -cuts derived from Figures 3.11d,

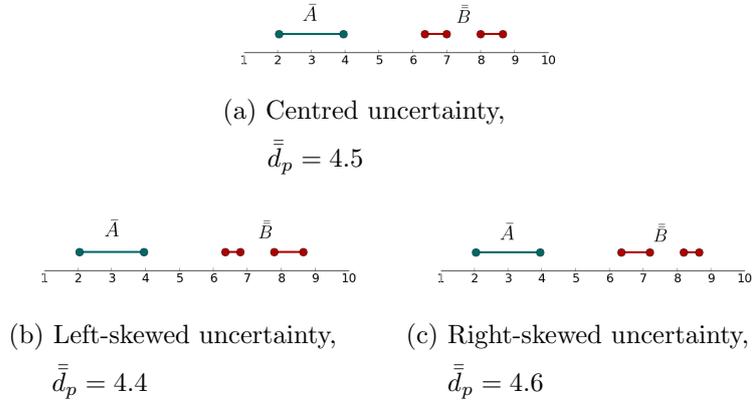


Figure 3.10: Centred, left-skewed and right-skewed discontinuous intervals derived from Figures 3.11d, 3.12d, 3.13d, respectively, at $\alpha = 0.85$ and their distances according to the measure \bar{d}_p in (3.12).

3.12d and 3.13d, respectively, at $\alpha = 0.85$ showing that \bar{B} is a discontinuous interval with a gap centred, skewed to the left, and skewed to the right. Looking at these examples, one can see how the locations of the end points of \bar{B} affect the resulting distance.

For each example in Figure 3.10, $\bar{A} = [2.05, 3.95]$, the centre of which is 3. In Figure 3.10a (containing a centred break in \bar{B}), $\bar{B} = ([6.3469, 7.0], [8.0, 8.6531])$. The centres of these two continuous intervals are (6.67345, 8.32655), the average of which is 7.5; this is used as the average value of \bar{B} . Thus the distance $\bar{d}(\bar{A}, \bar{B})$ in Figure 3.10a is 4.5.

In Figure 3.12, B has a concave region to the left of its centroid, thus the α -cut at $\alpha = 0.85$ (shown in Figure 3.10b) is a discontinuous pair of intervals where the gap is skewed to the left. Specifically, $\bar{B} = ([6.3469, 6.8], [7.8, 8.6531])$. The centres of these continuous intervals are (6.57345, 8.22655) and their average is 7.4. Thus the distance $\bar{d}(\bar{A}, \bar{B})$ in Figure 3.10b is 4.4. This is smaller than in Figure 3.10a where the gap in \bar{B} is centred. This is because the end of the first interval of \bar{B} and the beginning of the second interval are each a smaller values in Figure 3.10b than in Figure 3.10a (i.e. $6.8 < 7$ and $7.8 < 8$).

Thus, $\bar{\bar{B}}$ is closer to A in Figure 3.10b, resulting in a smaller value of distance. The reverse can also be seen when comparing Figures 3.10a and 3.10c.

The proposed method $\bar{\bar{d}}$ provides a suitable comparison between discontinuous α -cuts, though one may explore other methods such as weighting the continuous intervals according to their length. For example, one could place a higher weight on the widest interval, deeming the widest interval as the most significant. Such alternative methods are left for future work.

To determine which of d_p^{T1} and d_{cc} give the most appropriate results for non-convex fuzzy sets, the next demonstration (after the figures) uses data-driven fuzzy sets to further compare and discuss these methods.

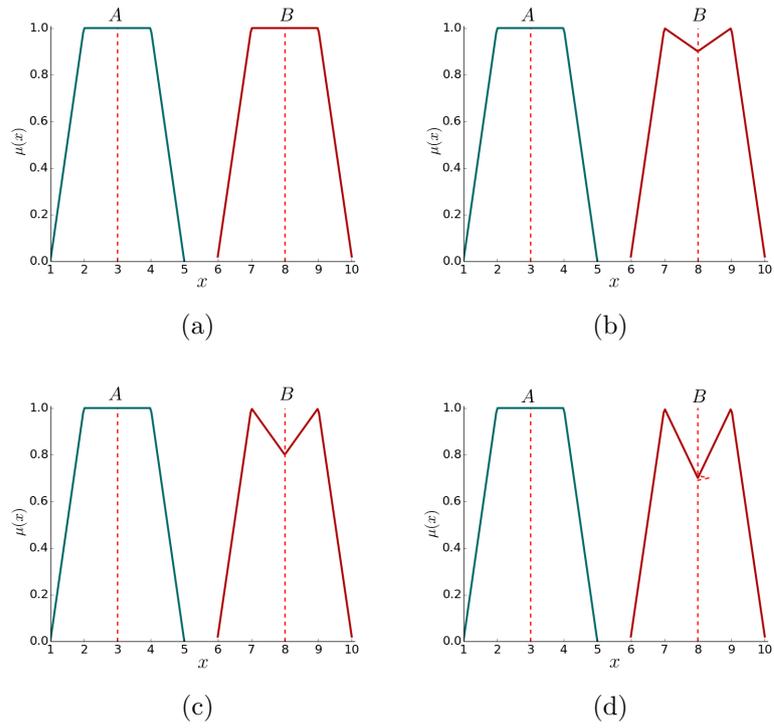


Figure	d_p^{T1}	d_{cc}
a	5.0	5.0
b	5.0	5.0
c	5.0	5.0
d	5.0	5.0

Figure 3.11: Four pairs of fuzzy sets with convex and non-convex membership functions, and the distances according to the proposed measure d_p^{T1} (3.13) and a centroid based approach d_{cc} (3.5).

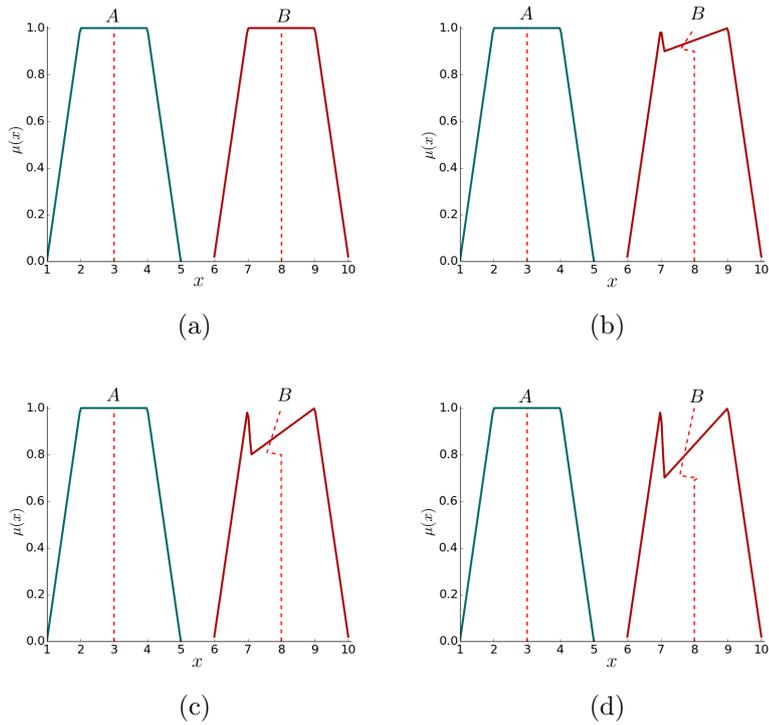


Figure	d_p^{T1}	d_{cc}
a	5.0	5.0
b	4.9796	5.0103
c	4.9431	5.0213
d	4.9161	5.0331

Figure 3.12: Four pairs of fuzzy sets with convex and non-convex membership functions, and the distances according to the proposed measure d_p^{T1} (3.13) and a centroid based approach d_{cc} (3.5).

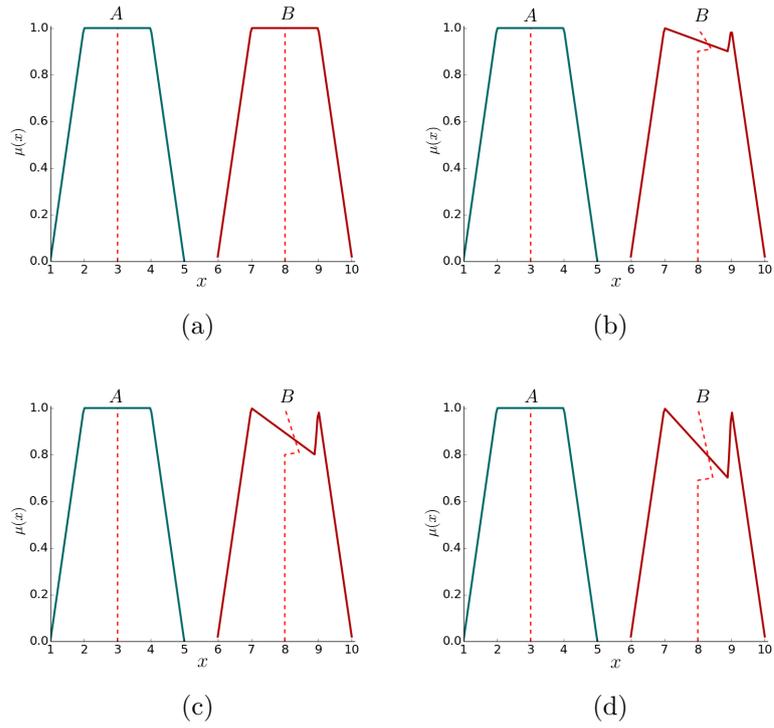


Figure	d_p^{T1}	d_{cc}
a	5.0	5.0
b	5.0205	4.9897
c	5.057	4.9787
d	5.084	4.9669

Figure 3.13: Comparing distance between an asymmetric, right-skewed, non-convex fuzzy set and a convex fuzzy set proposed measure d_p^{T1} (3.13) and a centroid based approach d_{cc} (3.5).

Different results often occur when using the α -cut-based d_p^{T1} or centroid based d_{cc} distance measures to compare non-convex fuzzy sets. In such cases, d_p^{T1} often produces results closer to what one might expect.

To demonstrate this, Figure 3.14 shows four different pairs of highly non-convex, spiky fuzzy sets constructed from a survey in which participants rated attributes of different cakes; more details on this data set are given in Section 7.3. The distances according to d_p^{T1} and d_{cc} are given under each figure and are listed in Table 3.3. Within d_p^{T1} , 40 α -cuts were measured to improve the accuracy of the results.

In these examples, the proposed α -cut based distance d_p^{T1} is always larger than d_{cc} . In Figures 3.14a, b and c the peaks (x value with highest membership) of the two fuzzy sets are at $x = 2$ and $x = 6$. The function d_p^{T1} reflects this degree of distance by giving values between 3.4 and 3.6. Note, the membership values at all other points are nearly equal. The function d_{cc} , however, gives a much lower value for each pair and so does not reflect the difference between the peaks of the fuzzy sets as clearly as d_p^{T1} .

It is because d_p^{T1} uses α -cuts that it picks up on changes in the peaks of the fuzzy sets more effectively than d_{cc} , which looks at the overall shape. When d_p^{T1} measures an α -cut around $\alpha = 0.2$ it calculates the distance between the peaks of the fuzzy sets (at $x = 2$ and $x = 6$) and accounts for this in the overall value result. The centroid based approach does not use α -cuts, however, and as a result it does not pick up this difference between the sets leading to a less accurate result. Thus, the final result of d_{cc} is smaller than d_p^{T1} .

These demonstrations show that the proposed α -cut method is better suited to non-convex fuzzy sets as it produces results closer to what one would expect. The next section provides a summary of the distance measures proposed within this chapter.

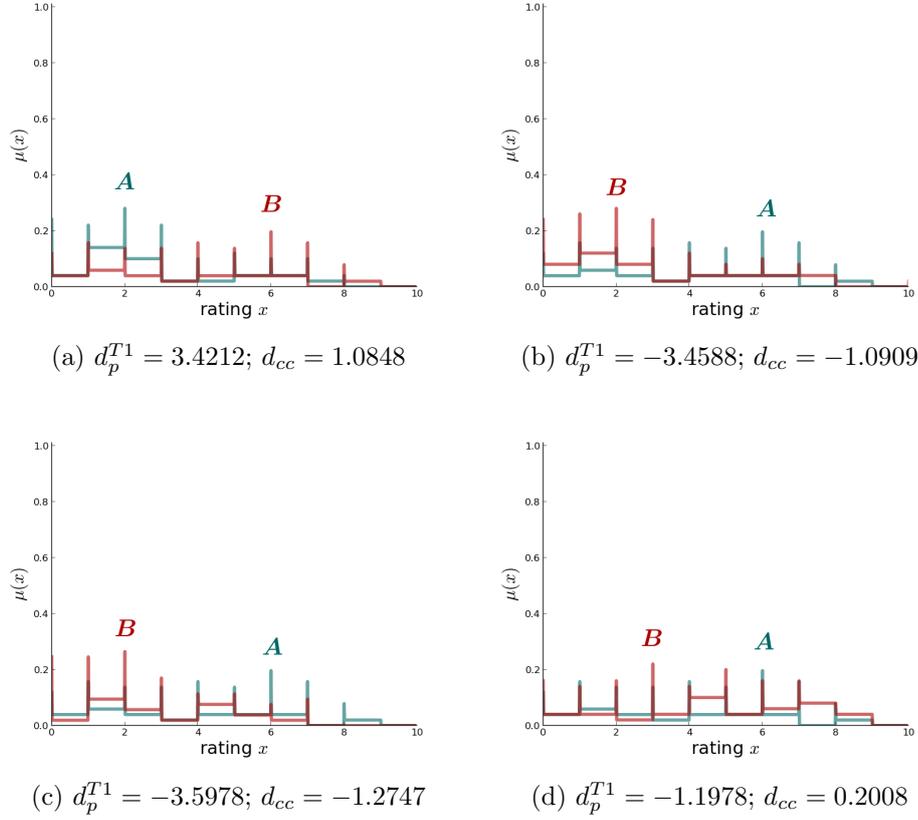


Figure 3.14: Data-driven fuzzy sets representing the distributions of ratings for how crumbly two different cakes have been rated. The α -cut-based d_p^{T1} and centroid-based d_{cc}^{T1} distances are shown.

Figure 3.14	$d_p^{T1}(A, B)$	$d_{cc}(A, B)$
a	3.4212	1.0848
b	-3.4588	-1.0909
c	-3.5978	-1.2747
d	-1.1978	0.2008

Table 3.3: Distances between the fuzzy sets in Figure 3.14 using the α -cut based d_p^{T1} and centroid-based d_{cc} distance measures.

3.5 Summary

This chapter has developed and introduced

- a directional distance measure to compare type-1 fuzzy sets (3.4),
- a distance measure (directional and non-directional) that can compare non-normal fuzzy sets (3.7),
- a distance measure (directional and non-directional) that can compare non-convex fuzzy sets (3.13), and
- a method of joining the above three points to attain a directional or non-directional distance measure between fuzzy sets that may be normal or non-normal and convex or non-convex.

The **directional distance measure** produces a signed value, where the sign indicates the relative positions of the fuzzy sets in the universe of discourse. More specifically, it enables one to know which fuzzy set is to the left or right (i.e., contains lower or higher values) than the other. The absolute value resulting from this measure indicates the magnitude of the distance between the fuzzy sets.

The **distance measure on non-normal fuzzy sets** uses the α -cut at the height of the fuzzy set as a substitute where α -cuts are empty. Another approach that avoids empty α -cuts is to instead compare every non-empty α -cut of one fuzzy set with every non-empty α -cut of the other, ignoring any empty α -cuts. An example of this is given in Appendix B. However, comparing all α -cuts is more computationally complex and the results show no noticeable benefits when using this method. Instead, the proposed method will be suitable for most applications.

The **distance measure on non-convex fuzzy sets** calculates the average distance between each continuous interval of a discontinuous α -cut and it was

demonstrated that the results better match the expected value compared to a centroid-based measure.

Having developed a directional distance measure for non-normal and non-convex type-1 fuzzy sets, the next chapter expands this measure to interval and general type-2 fuzzy sets.

Chapter 4

Measuring Distance and Similarity Between Type-2 Fuzzy Sets

4.1 Introduction

This chapter presents methods of calculating the similarity and distance between type-2 fuzzy sets. First, based on the techniques developed in the previous chapter, the type-2 distance measure d_p^{T1} is extended to compare interval type-2 fuzzy sets. To compare general type-2 fuzzy sets, a method of extending any interval type-2 measure to general type-2 fuzzy sets is proposed, and it is shown that any properties of an interval type-2 approach (distance, similarity or otherwise) are also present when extended to general type-2 fuzzy sets.

It is commonly stated that “when all uncertainties disappear a type-2 membership function must reduce to a type-1 membership function” [16]. With this in mind, when all uncertainties disappear in a type-2 membership function, the measure comparing type-2 fuzzy sets should reduce to the equivalent measure on type-1 fuzzy sets. Thus, measures on type-2 fuzzy sets should ideally have

the same properties as their type-1 forms, i.e., the same characteristics of similarity and distance should always be observed, regardless of the type of fuzzy set. This ensures that the results of the measures can be easily compared because the fuzzy set type does not affect the measure or its interpretation.

Section 4.2 presents a directional and non-directional distance measure for interval type-2 fuzzy sets that may have non-normal or non-convex membership functions. After this, Section 4.3 presents a method of extending interval type-2 measures onto general type-2 fuzzy sets.

4.2 Distance on Interval Type-2 Fuzzy Sets

This section extends the directional distance measure on type-1 non-normal and non-convex fuzzy sets d_p^{T1} (3.14) to interval type-2 fuzzy sets, demonstrates the new measure, and shows that its results are consistent with those of the type-1 measure d_p^{T1} .

4.2.1 Distance Measure

As stated in Section 2.2.3, the α -cut of an interval type-2 fuzzy set is represented by the α -cuts of the lower and upper membership functions; this is denoted $\tilde{A}_\alpha = \{\overline{\tilde{A}_{\alpha W}}, \overline{\tilde{A}_{\alpha U}}\}$ for $\tilde{A} \in IT2(X)$ where $\overline{\tilde{A}_{\alpha W}}$ and $\overline{\tilde{A}_{\alpha U}}$ are the α -cuts of the lower and upper membership functions of \tilde{A} , respectively.

Based on this α -cut representation and the distance measures developed in Chapter 3, the following definition introduces the proposed directional distance measure for interval type-2 fuzzy sets:

Definition 29 (Interval Type-2 Distance Measure). *The directional distance between two fuzzy sets $\tilde{A}, \tilde{B} \in IT2(X)$ may be measured by comparing the upper and lower membership functions as*

$$d_p^{IT2}(\tilde{A}, \tilde{B}) = \frac{\sum_{\alpha \in [0, \gamma_U]} y_\alpha \check{d}_p(\overline{\tilde{A}_{\alpha U}}, \overline{\tilde{B}_{\alpha U}}) + \sum_{\alpha \in [0, \gamma_W]} y_\alpha \check{d}_p(\overline{\tilde{A}_{\alpha W}}, \overline{\tilde{B}_{\alpha W}})}{\sum_{\alpha \in [0, \gamma_U]} y_\alpha \sum_{\alpha \in [0, \gamma_W]} y_\alpha}, \quad (4.1)$$

where y_α is the y -coordinate (or u value) for the given α -cut, \tilde{d}_p is described in (3.14), $\gamma_U = \max \left\{ \alpha \mid \overline{\tilde{A}_{\alpha_U}} \neq \emptyset \vee \overline{\tilde{B}_{\alpha_U}} \neq \emptyset, \alpha \in [0, 1] \right\}$, and γ_W is the same for the lower membership functions of \tilde{A} and \tilde{B} .

By using γ_U and γ_W , d_p^{IT2} compares the upper and lower membership functions of \tilde{A} and \tilde{B} up to the maximum height of the respective membership functions.

Using the distance between α -cuts \tilde{d}_p (3.14) within d_p^{IT2} (4.1) enables one to compare non-normal fuzzy sets and non-convex fuzzy sets. Additionally, d_p^{IT2} may be either directional or non-directional. Within d_p^{IT2} , the function \tilde{d}_p is used to calculate the distance between α -cuts that may be non-normal (i.e., non-existent) or non-convex. This may be directional or non-directional by using \bar{d}_p (3.2) or \bar{d} (3.1), respectively, to compare continuous, non-empty α -cuts.

Theorem 2. *The interval type-2 distance measure d_p^{IT2} (4.1) has the same properties as the directional distance measure d_p^{T1} (3.4) with \bar{d}_p (3.2) for type-1 fuzzy sets. These are self-identity, partial-symmetry, transitivity, triangle-inequality, directional separability and reflective distance.*

Proof. These properties were proven for type-1 membership functions in Chapter 3, so it follows that these properties are also present when comparing the upper and lower membership functions of an interval type-2 fuzzy set. It is then trivial to see that d_p^{IT2} (4.1) therefore has all of these properties. \square

Theorem 3. *The interval type-2 distance measure d_p^{IT2} (4.1) is a metric when using the metric distance measure d_p^{T1} (3.4) with \bar{d} (3.1) for type-1 fuzzy sets. Therefore, it has the properties self-identity, separability, symmetry and triangle inequality.*

Proof. As proven in Theorem 2, as these properties are present when comparing type-1 membership functions, it follows that they are also present when comparing interval type-2 fuzzy sets using d_p^{IT2} (4.1). \square

4.2.2 Demonstrations

The remainder of this section demonstrates the proposed directional distance measure d_p^{IT2} compared against three different measures proposed by Figueroa-García et al. [79]. These are

d_α The directional distance between α -cuts (2.47).

d_h The Hausdorff distance between the centroids of the fuzzy sets (2.40).

d_s The sum of the centroids of the fuzzy sets (2.41).

These equations are detailed in Section 2.4.2.

This demonstration compares the fuzzy sets shown in Figure 4.1, in which the membership functions have been created based on the type-1 examples shown from Figures 3.3 to 3.13. The distances $d(\tilde{A}, \tilde{B})$ for each pair of fuzzy sets are shown in Table 4.1 for the four given methods. Note that the d_α (2.47) can only compare normal, convex fuzzy sets, and so no results can be given where fuzzy sets contain non-normal or non-convex membership functions using this method.

From these results, one can see that the proposed distance produces similar results to the type-1 distance measure demonstrations in Figures 3.3 to 3.13. For example, in Figure 4.1a, the distance is as expected. The α -cut based d_α and sum of centroids d_s measures by Figueroa-García et al., however, produce double what one would expect. This is because, unlike the proposed approach, they do not take the average distance between the two membership functions and instead sum the results.

Additionally, the proposed distance d_p^{IT2} in Figure 4.1b is 0.0 due to the property of reflectivity; this also occurred in the type-1 results in Figure 3.7. Note that the two centroid based approaches also give values close to 0.0. However due to the differences in the gradients of the fuzzy set membership functions, the values are slightly greater than 0 (note that the differences in

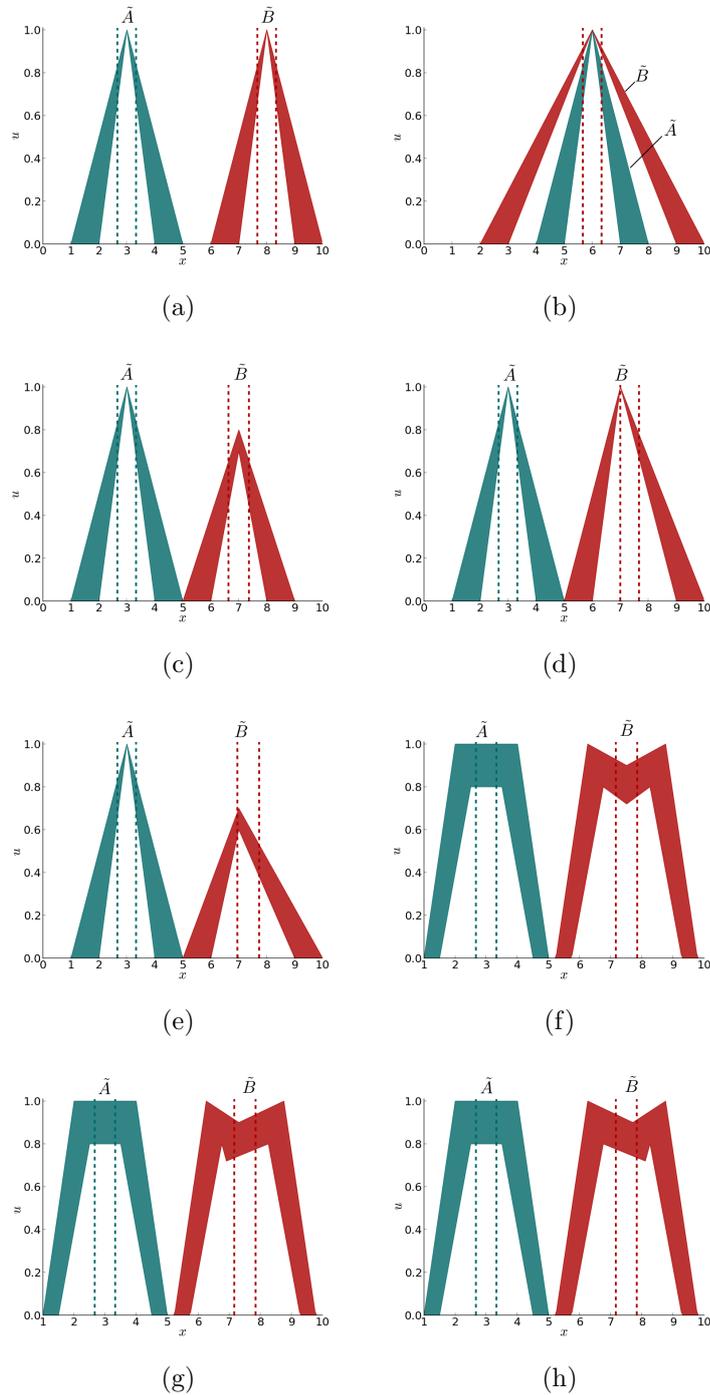


Figure 4.1: A collection of interval type-2 fuzzy set pairs which are a mixture of normal, non-normal, convex and non-convex, used to demonstrate d_p^{IT2} . Dashed lines highlight the boundaries of the type-reduced sets.

Figure 4.1	d_p^{IT2} (4.1)	d_α (2.47)	d_h (2.40)	d_s (2.41)
a	5.0	10.0	5.0	10.0
b	0.0	1.267	0.002	0.004
c	4.0	-	4.035	8.0
d	4.159	8.317	4.335	8.671
e	4.067	-	4.391	8.675
f	4.5	-	4.518	9.0
g	4.489	-	4.568	9.103
h	4.511	-	4.557	9.086

Table 4.1: The distance between the interval type-2 fuzzy set pairs shown in Figure 4.1 using the proposed measure d_p^{IT2} and three other measures (d_α , d_h and d_s) in the literature.

their type-reduced sets are too small to see within Figure 4.1b). The measure d_α , however, gives a noticeably higher value of distance.

Figures 4.1c, (d) and (e) demonstrate the effects of non-normality on distance. As shown with type-1 fuzzy sets (in Figure 3.7), the distance between symmetrical, disjoint fuzzy sets is the same regardless of normality using the proposed method d_p^{IT2} as well as the sum of centroids approach d_s (2.41); as indicated by Figure 4.1c. However, a non-symmetrical fuzzy set has a more noticeable change in membership value if its height is changed. Using the proposed approach, the distance decreases from Figure 4.1d to (e) as the height of \tilde{B} decreases; note that this effect was also demonstrated for type-1 fuzzy sets in Figure 3.8. The centroid based approaches, however, have the opposite effect.

The comparison of non-convex fuzzy sets also produces similar results to the type-1 demonstrations (shown in Figures 3.11, 3.12 and 3.13). For the

proposed method d_p^{IT2} , the distance between symmetrical, disjoint fuzzy sets is the same, regardless of convexity.

Additionally, the distance changes if the concave region of a non-convex fuzzy set is skewed. In Figure 4.1g, the proposed distance decreases whilst the centroid distance increases, and in Figure 4.1h the opposite effect is observed. Both of these effects were also shown in the type-1 results in Figures 3.12 and 3.13, where a detailed discussion of these results is given in Section 3.4.

4.2.3 Comparing Interval Type-2 and Type-1 Distance Measures

It is important that the interval type-2 directional distance measure d_p^{IT2} (4.1) gives consistent results with the type-1 measure d_p^{T1} (3.13) so that the results can be evaluated in the same way regardless of the type of fuzzy set. In this section, a brief demonstration is given to show that d_p^{IT2} and d_p^{T1} both produce the same results for type-1 fuzzy sets.

Figure 4.2 shows pairs of type-1 fuzzy sets based on the interval type-2 pairs in Figure 4.1. When using the interval type-2 measure d_p^{IT2} , the type-1 fuzzy sets are treated as interval type-2 fuzzy sets with identical upper and lower membership functions. The results of both the type-1 d_p^{T1} (3.13) and interval type-2 d_p^{IT2} (4.1) directional distance measures are shown in Table 4.2. It is clear from these identical results and the demonstration in Table 4.1 that the interval type-2 distance measure produces consistent results compared to the type-1 distance measure.

Also note that Table 4.1 and Table 4.2 have the same results for Figures 4.1a to (f) and Figures 4.2a to (f), respectively because the centres of each α -cut in these figures are the same. This is not the case for Figures 4.2g and (h) and Figures 4.1g to (h), however, because the interval type-2 fuzzy sets have different shaped lower and upper membership functions.

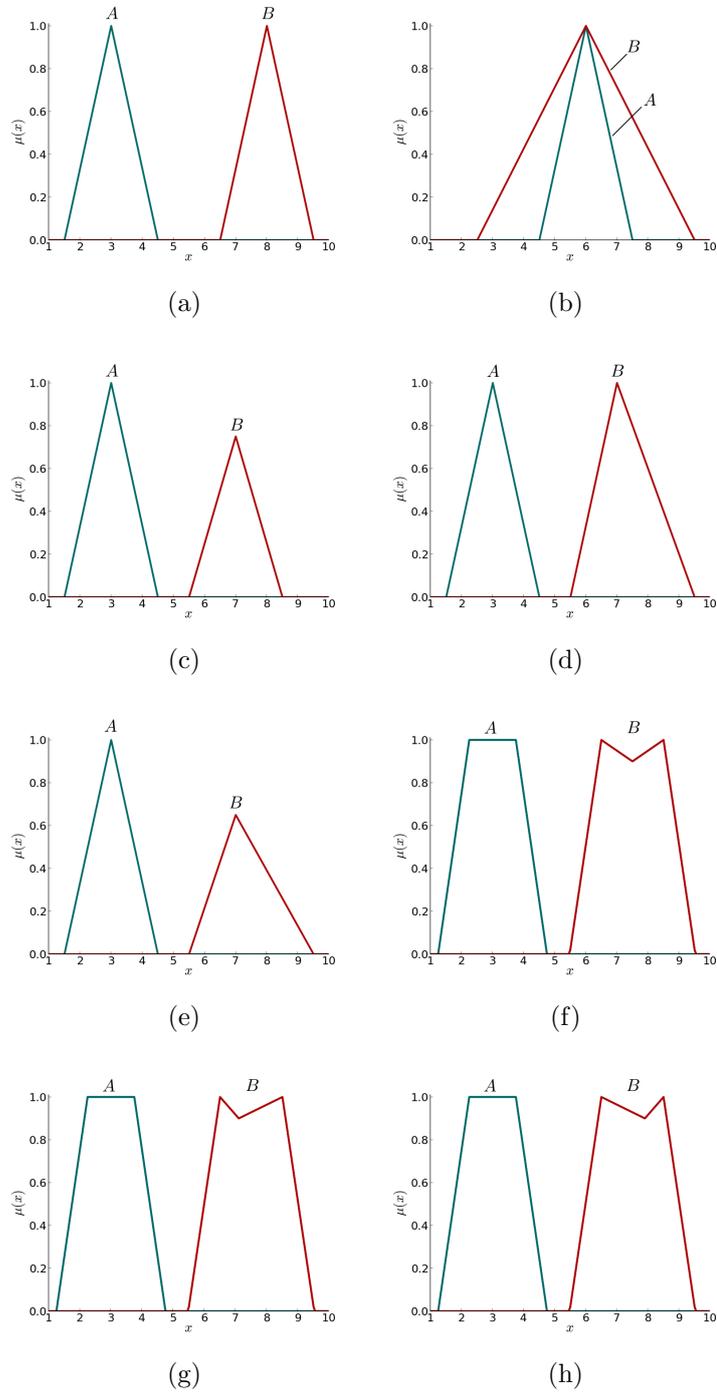


Figure 4.2: A collection of type-1 fuzzy set pairs based on the interval type-2 fuzzy sets in Figure 4.1.

Figure 4.2	Type-1: d_p^{T1}	Interval Type-2: d_p^{IT2}
a	5.0	5.0
b	0.0	0.0
c	4.0	4.0
d	4.159	4.159
e	4.067	4.067
f	4.5	4.5
g	4.491	4.491
h	4.509	4.509

Table 4.2: The distance between the type-1 fuzzy set pairs shown in Figure 4.2 using the proposed type-1 d_p^{T1} (3.13) and interval type-2 d_p^{IT2} (4.1) directional distance measures.

This section has presented a method of measuring the distance between interval type-2 fuzzy sets that may be normal or non-normal and convex or non-convex. The next section focuses on extending this to general type-2 fuzzy sets, by proposing a method that is not restricted to only distance measures, but may be used to apply any interval type-2 measure on general type-2 fuzzy sets.

4.3 Extending Interval Type-2 Measures to General Type-2 Fuzzy Sets

Section 2.2.4 gave a background on the zSlices representation of fuzzy sets which simplifies general type-2 fuzzy sets by representing them as a collection of zSlices. These zSlices are equivalent to interval type-2 fuzzy sets but with a secondary membership value of a given value z instead of 1. The zSlices approach to general type-2 fuzzy sets is valuable because it has made it possible to extend any theoretical work of interval type-2 fuzzy sets to general type-2 fuzzy sets. Using the zSlices approach, this section develops a general method of taking any measure on interval type-2 fuzzy sets and utilising it on general type-2 fuzzy sets. This method was first introduced in [64].

4.3.1 A General Function to Extend Measures

As an individual zSlice is akin to an interval type-2 fuzzy set, the following proposes a method of extending interval type-2 measures onto general type-2 fuzzy sets. First, consider the zLevels used by a given fuzzy set.

Definition 30. *Let \tilde{A}_Z denote the set of all zLevels of the zSlices in \tilde{A} . This is defined as*

$$\tilde{A}_Z = \{z_i \mid \forall i \in \{1, 2, \dots, I\}\}, \quad (4.2)$$

where I is the total number of zLevels in \tilde{A} .

Now, zSlices general type-2 fuzzy sets may be compared by measuring the zSlices at the given zLevels used by each fuzzy set.

Definition 31. *An interval type-2 measure can be used on each individual zSlice of a general type-2 fuzzy set and weighted as [64]*

$$m_\lambda(\{\tilde{A}_1, \dots, \tilde{A}_N\}) = \frac{\sum_{i \in \mathcal{L}(\{\tilde{A}_1, \dots, \tilde{A}_N\})} z_i \lambda(\{\tilde{A}_1, \dots, \tilde{A}_N\})}{\sum_{i \in \mathcal{L}(\{\tilde{A}_1, \dots, \tilde{A}_N\})} z_i}, \quad (4.3)$$

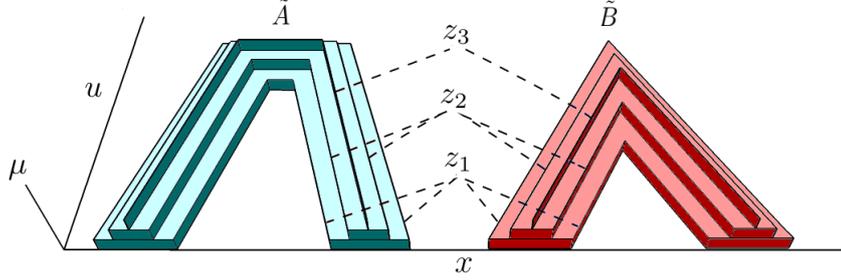


Figure 4.3: Two zSlices based fuzzy sets \tilde{A} and \tilde{B} represented by three zSlices, where x , u and μ are the universe of discourse, and the primary and secondary membership values, respectively.

where $\{A_1, \dots, A_N\} \subseteq GT2(X)$ is the set of zSlices fuzzy sets that are being measured, λ is any measure for interval type-2 fuzzy sets, and $\mathcal{L}(\{A_1, \dots, A_N\})$ is the set of zLevels used to represent the fuzzy sets in $\{A_1, \dots, A_N\}$ as

$$\mathcal{L}(\{\tilde{A}_1, \dots, \tilde{A}_N\}) = \bigcup_{n=1}^N \tilde{A}_{nz} \quad (4.4)$$

where N is the total number of fuzzy sets and \tilde{A}_{nz} (as defined in (4.2)) is the set of zLevels of all zSlices within the general type-2 fuzzy set \tilde{A}_n .

For example, consider the two zSlices-based general type-2 fuzzy sets \tilde{A} and \tilde{B} shown in Figure 4.3. To compare \tilde{A} and \tilde{B} , the interval type-2 measure λ will be used three times, once for each zSlice; i.e., $\lambda(\tilde{A}_{z_1}, \tilde{B}_{z_1})$, $\lambda(\tilde{A}_{z_2}, \tilde{B}_{z_2})$ and $\lambda(\tilde{A}_{z_3}, \tilde{B}_{z_3})$.

As stated in Section 2.2.4 and [23], when using the zSlices representation to simplify general type-2 fuzzy sets, as more zSlices are used the representation of the original set becomes more accurate. Likewise, as the representation becomes more accurate, the result of the extended measure λ also becomes more accurate.

Using \mathcal{L} ensures that all the zSlices of the given fuzzy sets are compared, even if the fuzzy sets have a different number of zSlices and/or their zSlices are at different z -coordinates. Generally, it is most likely that when creating

zSlices-based fuzzy sets one will prefer all fuzzy sets to have the same number of zLevels. However, it is possible that fuzzy sets may be constructed using different numbers of zLevels, requiring that the union of their zLevels be used to compare them, as given by \mathcal{L} in (4.4).

Consider, for example, the general type-2 fuzzy set \tilde{A} in Figure 4.4a. Figures 4.4b and 4.4c represent \tilde{A} with four and three zSlices, respectively and are referred to as \tilde{B} and \tilde{C} . The zLevels belonging to \tilde{B} are $\{0.25, 0.5, 0.75, 1.0\}$ and the zLevels of \tilde{C} are $\{0.33, 0.67, 1.0\}$. The zLevels can be seen more clearly by \tilde{B} and \tilde{C} 's vertical slices at $x = 3$, which are shown in Figure 4.4d. To compare \tilde{B} and \tilde{C} using m_λ (4.3), the union of the zLevels is compared which is $\mathcal{L}(\{\tilde{A}, \tilde{B}\}) = \{0.25, 0.33, 0.5, 0.66, 0.75, 1.0\}$. Figure 4.4e shows, using dashed lines, where the comparisons would be made for all of the zLevels of \tilde{B} and \tilde{C} .

Using this approach, m_λ may be used to compare not only general type-2 fuzzy sets with different zLevels, but also different types of fuzzy sets. For example, one may compare general and interval type-2 fuzzy sets, or general and type-1 fuzzy sets. In the latter case, the type-1 fuzzy set is treated as a zSlices type-2 fuzzy set with one zSlice and identical upper and lower membership functions.

In this thesis, the zSlices extension (4.3) will only be used as part of similarity and distance measures, which involve the comparison of two fuzzy sets. Given this, m_λ (4.3) may be simplified as follows

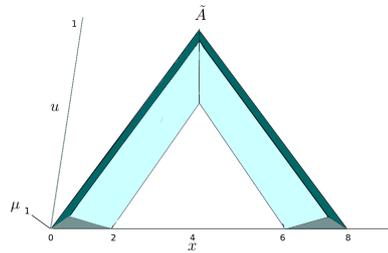
Definition 32. *Two general type-2 fuzzy sets $\tilde{A}, \tilde{B} \in GT2(X)$ may be measured as*

$$m_\lambda(\tilde{A}, \tilde{B}) = \frac{\sum_{i \in \mathcal{L}(A, B)} z_i \lambda(\tilde{A}_{z_i}, \tilde{B}_{z_i})}{\sum_{i \in \mathcal{L}(A, B)} z_i}, \quad (4.5)$$

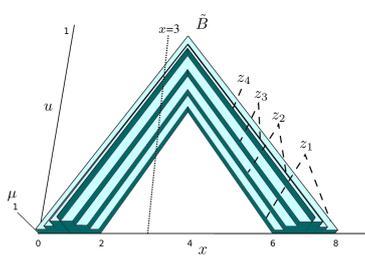
where λ is any measure on two interval type-2 fuzzy sets and

$$\mathcal{L}(\tilde{A}, \tilde{B}) = \tilde{A}_Z \cup \tilde{B}_Z \quad (4.6)$$

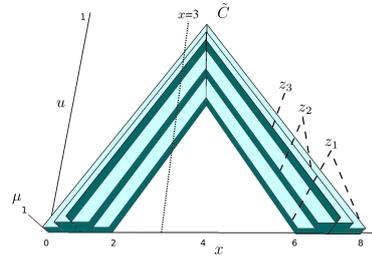
where \tilde{A}_Z is the set of zLevels in the zSlices general type-2 fuzzy set \tilde{A} as defined in (4.2).



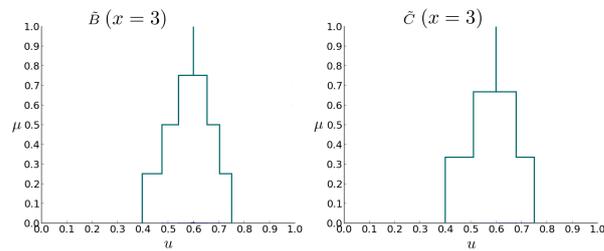
(a) A general T2 fuzzy set \tilde{A}



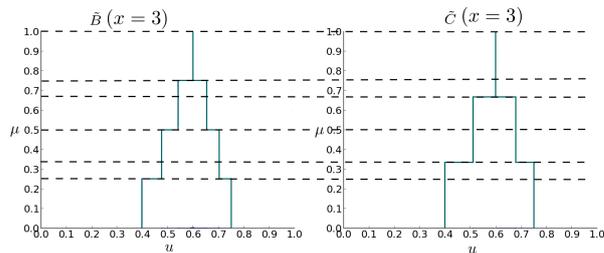
(b) A zSlices-based model of \tilde{A} with four zLevels (referred to as \tilde{B}).



(c) A zSlices-based model of \tilde{A} with three zLevels (referred to as \tilde{C}).



(d) Vertical slices of \tilde{B} (left) and \tilde{C} (right) at $x = 3$.



(e) Vertical slices of \tilde{B} (left) and \tilde{C} (right) at $x = 3$ with dashed lines marking their shared zLevels.

Figure 4.4: Comparing general type-2 fuzzy sets with different numbers of zSlices, where x is the universe of discourse, and u and μ are the primary and secondary membership values, respectively.

Within this thesis, m_λ is only used for similarity and distance measures on two fuzzy sets and so, for clarity, equations (4.5) and (4.6) will be referred to throughout, instead of (4.3) and (4.4).

Note that this is similar to the approach by Hamwari and Coupland [71], which uses the zSlices representation to extend interval type-2 methods of measuring containment, cardinality, similarity and subthood to general type-2 fuzzy sets. The distinction between m_λ and their method is that m_λ weights the calculation at each zSlice by the given zLevel. Intuitively, the more certainty there is in the secondary membership value of two given zSlices, the more certainty there must also be in their comparison. Given this, it makes sense to weight their comparison to reflect how certain we are of their membership. Thus, m_λ weights each comparison of zSlices by their zLevel.

Theorem 4. *A similarity, distance metric or directional distance measure, denoted λ , that is extended by m_λ (4.5) has the same properties as the original measure λ ; a list of each measure's properties are given in Appendix A.*

Proof. The extension m_λ does not change the ordering of the fuzzy sets measured by λ , thus the properties transitivity, triangle inequality, separability and symmetry are all maintained through the extension.

The extension does not alter the sign of the results, thus the proofs of the signed based properties of the directional distance (directional separability, partial symmetry and reflectivity) are trivial.

If the fuzzy sets are identical then every zSlice will be identical, thus reflexivity and self-identity will be kept through the extension. If two fuzzy sets are disjoint then every zSlice is disjoint, thus overlapping is also kept through the extension. □

4.3.2 Using the zSlices-based Extension Over Other Approaches

There are several approaches that may be taken when developing a measure to compare general type-2 fuzzy sets. These include using vertical slices to compare the secondary membership functions, comparing embedded membership functions, and using a zSlices approach.

Comparing embedded membership functions has been used to evaluate the similarity between general type-2 fuzzy sets [54]. However, this is computationally complex because there are an infinite number of embedded fuzzy sets, thus it is difficult to ensure an accurate comparison between fuzzy sets. Additionally, for any general type-2 fuzzy set, many of its embedded fuzzy sets will be so irregularly shaped that one may argue they don't adequately represent the type-2 fuzzy set.

Another approach to measuring general type-2 fuzzy sets is by comparing their vertical slices. In many cases, this is an effective method. For example, Yang and Lin [4] measure similarity by using a Jaccard based approach on the vertical slices of fuzzy sets. However, taking a vertical slice approach to compare fuzzy sets is not ideal for calculating distance. This is because distance focuses on the difference between values (by α -cuts) instead of between their membership (by vertical slices). Thus, a different approach should be taken to enable a measure of distance between general type-2 fuzzy sets.

The zSlices approach is ideal as it facilitates the extension of any interval type-2 measure on general type-2 fuzzy sets. Thus, it is not necessary to develop a new function for each variety needed (e.g., similarity, distance, etc.) to attain measures on general type-2 fuzzy sets. By taking the zSlices approach, the fundamental method of the original measure is maintained.

For example, if the directional distance on interval type-2 fuzzy sets d_p^{IT2} (4.1) is extended with m_λ (4.5), the result still uses an α -cut approach to

compare fuzzy sets as in the original approach. It is also clear that by using m_λ (4.5) with any interval type-2 measure, its properties are maintained through the extension. In addition, the demonstrations within the next section show that the results of the extended measures are consistent with those of their original form.

4.3.3 Demonstrations

As stated earlier, this chapter builds upon methods of measuring distance and similarity, thus the zSlices extension to measure interval type-2 fuzzy sets will be demonstrated for distance and similarity only. In both demonstrations, a comparison is made against the original interval type-2 measures and their extension to general type-2 fuzzy sets, first for distance then for similarity. Additional demonstrations also show that the extended measures produce the same results as interval type-2 and type-1 measures when comparing interval type-2 and type-1 fuzzy sets, respectively.

Distance

Using the zSlices extension (4.5) and the interval type-2 distance measure d_p^{IT2} (4.1) the following definition presents the proposed directional distance measure on general type-2 fuzzy sets.

Definition 33 (General Type-2 Distance Measure). *The directional or non-directional distance between two fuzzy sets $\tilde{A}, \tilde{B} \in GT2(X)$ may be calculated as*

$$d_p^{GT2}(\tilde{A}, \tilde{B}) = \frac{1}{\sum_{i \in \mathcal{L}(\tilde{A}, \tilde{B})} z_i} \sum_{i \in \mathcal{L}(\tilde{A}, \tilde{B})} z_i \frac{\sum_{\alpha \in [0, \gamma_{z_{iU}}]} y_\alpha \check{d}_p(\overline{\tilde{A}_{z_{i\alpha U}}}, \overline{\tilde{B}_{z_{i\alpha U}}}) + \sum_{\alpha \in [0, \gamma_{z_{iW}}]} y_\alpha \check{d}_p(\overline{\tilde{A}_{z_{i\alpha W}}}, \overline{\tilde{B}_{z_{i\alpha W}}})}{\sum_{\alpha \in [0, \gamma_{z_{iU}}]} y_\alpha \sum_{\alpha \in [0, \gamma_{z_{iW}}]} y_\alpha}, \quad (4.7)$$

where $\mathcal{L}(\tilde{A}, \tilde{B})$ is given in (4.6), $\overset{\circ}{d}_p$ is described in (3.14),

$\gamma_{z_{iU}} = \max \left\{ \alpha \mid \overline{\tilde{A}_{z_{i\alpha U}}} \neq \emptyset \vee \overline{\tilde{B}_{z_{i\alpha U}}} \neq \emptyset, \alpha \in [0, 1] \right\}$ (i.e., the maximum α -cut where at least one of the upper membership functions of the zSlices \tilde{A}_{z_i} or \tilde{B}_{z_i} is non-empty), and $\gamma_{z_{iW}}$ is the same for the lower membership functions of \tilde{A}_{z_i} and \tilde{B}_{z_i} .

Note that within $\overset{\circ}{d}_p$ (3.14), one may choose the directional distance \bar{d}_p (3.2) or non-directional distance \bar{d} (3.1) functions to compare continuous α -cuts. In this demonstration, the directional distance \bar{d}_p is used so that d_p^{GT2} is a directional distance measure.

This demonstration also compares d_p^{GT2} against the distance between the centroids of the fuzzy sets. The centroids are derived by calculating the centres of each zSlice and averaging the results as

$$\tilde{A}_c = \frac{\sum_{z=1}^I z_i \tilde{A}_{z_{ic}}}{\sum_{z=1}^I z_i}, \quad (4.8)$$

where z_i is the i^{th} zSlice, I is the total number of zSlices, and $\tilde{A}_{z_{ic}}$ is the centre of the Karnik-Mendel type-reduction [108] on A_{z_i} . Using this, the centroid based distance between $\tilde{A}, \tilde{B} \in GT2(X)$ is

$$d_{cc}(\tilde{A}, \tilde{B}) = \tilde{B}_c - \tilde{A}_c \quad (4.9)$$

Note that, to the author's knowledge, there are no other general type-2 distance measures in the literature that compare fuzzy sets horizontally along the x -axis.

Figure 4.5 shows general type-2 fuzzy sets that have been constructed using the same footprint of uncertainty (FOU) as the interval type-2 fuzzy sets demonstrated in Figure 4.1. In the general type-2 case, a secondary membership value of 1 is at the centre of the FOU, and the membership value decreases linearly towards the edge of the FOU; this is shown by the shading within Figure 4.5, where darker shades indicate a higher secondary grade. Figure 4.6 depicts this with a three-dimensional model of Figure 4.5a.

Table 4.3 shows the distances between the pairs of fuzzy sets in Figure 4.5. These are calculated using d_p^{GT2} (4.7) and d_{cc} (4.9). From these results, one can see that the distances between the general type-2 fuzzy sets are the same as the interval type-2 results in Table 4.1 from Figure 4.5a to (f). This is because the secondary membership functions are symmetrical at the centre of the FOU, which results in any given α -cuts having the same mean value at every zLevel.

Note, however, that due to the complexity of modelling general type-2 fuzzy sets, this is not the case for the non-convex examples. This is because generating non-convex general type-2 membership functions where the α -cuts at different zLevels have the same mean values is a computationally challenging task, and so a more general model has been created instead. As a result Figures 4.5g and (h) do not have the same centre value for a given α -cut at each zLevel, and so the distances in the general type-2 case are different, but close, to those in the interval type-2 case.

These results show that the proposed extension of the interval type-2 distance d_p^{IT2} (4.1) to general type-2 fuzzy sets (d_p^{GT2}) gives consistent results compared to the proposed distance for type-1 and interval type-2 fuzzy sets. Additionally, d_p^{GT2} gives the same results when measuring the interval type-2 fuzzy sets as the distance measure on interval type-2 fuzzy sets d_p^{IT2} . As well as this, d_p^{GT2} produces the same results on type-1 fuzzy sets as the distance measure for type-1 fuzzy sets d_p^{T1} (3.13).

Next, a demonstration of the zSlices extension on similarity measures is given; this will be applied to recommendation systems in Chapter 6.

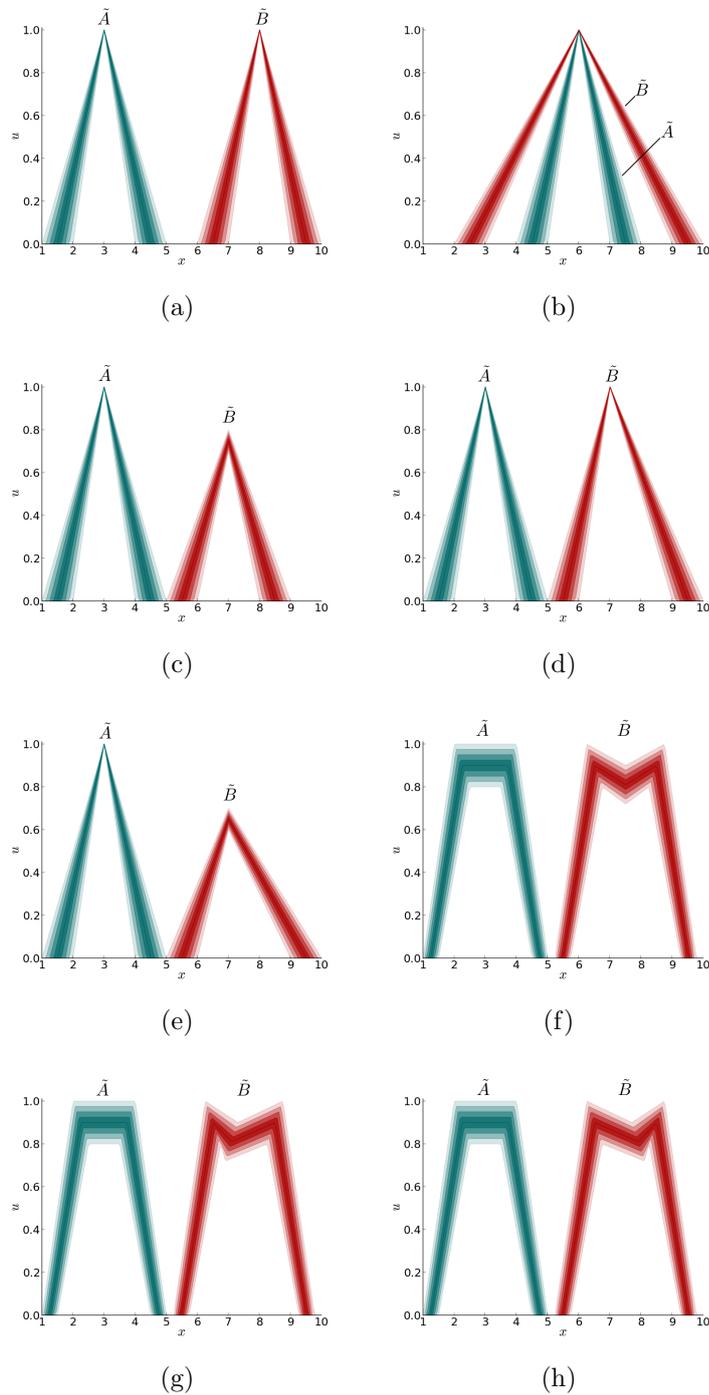


Figure 4.5: A collection of general type-2 fuzzy set pairs which are a mixture of normal, non-normal, convex and non-convex used to demonstrate d_p^{GT2} . x is the universe of discourse and u is the primary membership.

Figure 4.5	d_p^{GT2} (4.7)	d_{cc} (4.9)
a	5.0	5.0
b	0.0	0.0
c	4.0	4.0
d	4.159	4.334
e	4.067	4.334
f	4.5	4.5
g	4.484	4.505
h	4.516	4.496

Table 4.3: The distance between the general type-2 fuzzy set pairs shown in Figure 4.5.

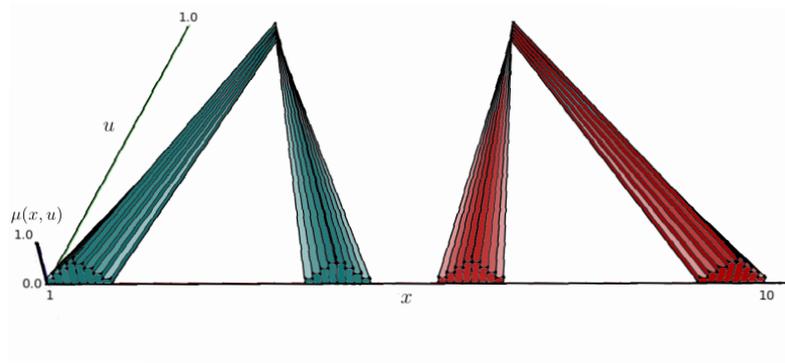


Figure 4.6: A three-dimensional model of Figure 4.5a. x is the universe of discourse, u is the primary membership and $\mu(x, u)$ is the secondary membership at x and u .

Similarity

Throughout this thesis the Jaccard measure will be used to compare the similarity between two fuzzy sets. This method is chosen because it satisfies all four properties of a similarity measure and the interval type-2 approach s_j^{IT2} (2.31) provides results that are consistent with the type-1 method s_j^{T1} (2.29).

More specifically, consider two type-1 (A and B) and interval type-2 (\tilde{C} and \tilde{D}) fuzzy sets, where \tilde{C} and \tilde{D} have the same lower and upper membership functions as A and B , respectively. Comparing A and B with the Jaccard measure for type-1 fuzzy sets s_j^{T1} (2.29) will produce the same results as using the Jaccard measure for interval type-2 fuzzy sets s_j^{IT2} (2.31) to compare \tilde{C} and \tilde{D} . This will be demonstrated within this section.

This section also shows that, in the same manner, the interval type-2 approach extended to zSlices type-2 fuzzy sets produces consistent results compared to the original interval type-2 and the type-1 measures. Note, this extension of similarity to general type-2 fuzzy sets was first introduced in [64].

Definition 34 (General Type-2 Similarity Measure). *The Jaccard interval type-2 similarity measure s_j^{IT2} (2.31) extended to zSlices type-2 fuzzy sets is*

$$s_j^{GT2}(\tilde{A}, \tilde{B}) = \frac{1}{\sum_{i \in \mathcal{L}(\tilde{A}, \tilde{B})} z_i} \sum_{i \in \mathcal{L}(\tilde{A}, \tilde{B})} z_i \frac{\sum_{j=1}^N \min(\bar{\mu}_{\tilde{A}_{z_i}}(x_j), \bar{\mu}_{\tilde{B}_{z_i}}(x_j)) + \sum_{j=1}^N \min(\underline{\mu}_{\tilde{A}_{z_i}}(x_j), \underline{\mu}_{\tilde{B}_{z_i}}(x_j))}{\sum_{j=1}^N \max(\bar{\mu}_{\tilde{A}_{z_i}}(x_j), \bar{\mu}_{\tilde{B}_{z_i}}(x_j)) + \sum_{j=1}^N \max(\underline{\mu}_{\tilde{A}_{z_i}}(x_j), \underline{\mu}_{\tilde{B}_{z_i}}(x_j))} \quad (4.10)$$

where $\tilde{A}, \tilde{B} \in GT2(X)$ and $\mathcal{L}(\tilde{A}, \tilde{B})$ (4.6) is the union of the zLevels used by \tilde{A} and \tilde{B} .

Figure 4.7 shows seven zSlices type-2 fuzzy sets in Figures (a) and (b). Their interval type-2 counterparts are shown in Figures (c) and (d), and type-1 counterparts are shown in (e) and (f). Table 4.4 shows the results of the similarity measures on these figures using the zSlices general type-2 (GT2)

(4.10), interval type-2 (IT2) (2.31) and type-1 (T1) (2.29) Jaccard similarity measures. When using the interval and zSlices measures, the type-1 fuzzy sets are treated as interval type-2 fuzzy sets with identical upper and lower membership functions.

Firstly, from the results in Table 4.4, one can see that the zSlices measure produces similar results for the zSlices fuzzy sets as it does for the interval type-2 fuzzy sets. Differences in the results are due to the membership functions varying at different zLevels.

For example, Figure 4.8 and Table 4.5 demonstrate this for fuzzy sets \tilde{A} and \tilde{C} . Within the figure, the fuzzy sets are shown at each of their zLevels. Due to having different membership values, the similarity between the fuzzy sets changes at each zLevel. Table 4.5 shows the vertical slices at $x = 5.5$ for \tilde{A} and \tilde{C} at each zLevel, and their similarity according to the Jaccard measure at these given points. From these results, one can see that the change in membership functions at each zSlice results in a different value of similarity at each slice. Thus, overall, the zSlices fuzzy set pairs have different similarity results compared to their interval type-2 equivalents.

Note, also, that the results change between the type-1 and interval type-2 demonstrations because their membership functions are different.

It is clear that the outcomes of the zSlices and interval type-2 results are very close. Additionally, the results show that the zSlices method produces the same results for interval type-2 fuzzy sets as the original interval type-2 measure. Likewise, all three approaches (zSlices type-2, interval type-2 and type-1) give the same results when comparing type-1 fuzzy sets. This demonstrates that the methodology behind each approach is consistent.

Note, the extension m_λ has also been demonstrated on extending other interval type-2 similarity measures in the literature within [64].

Having introduced and demonstrated distance and similarity measures on type-1, interval type-2 and general type-2 fuzzy sets, the next section presents

a summary of this chapter.

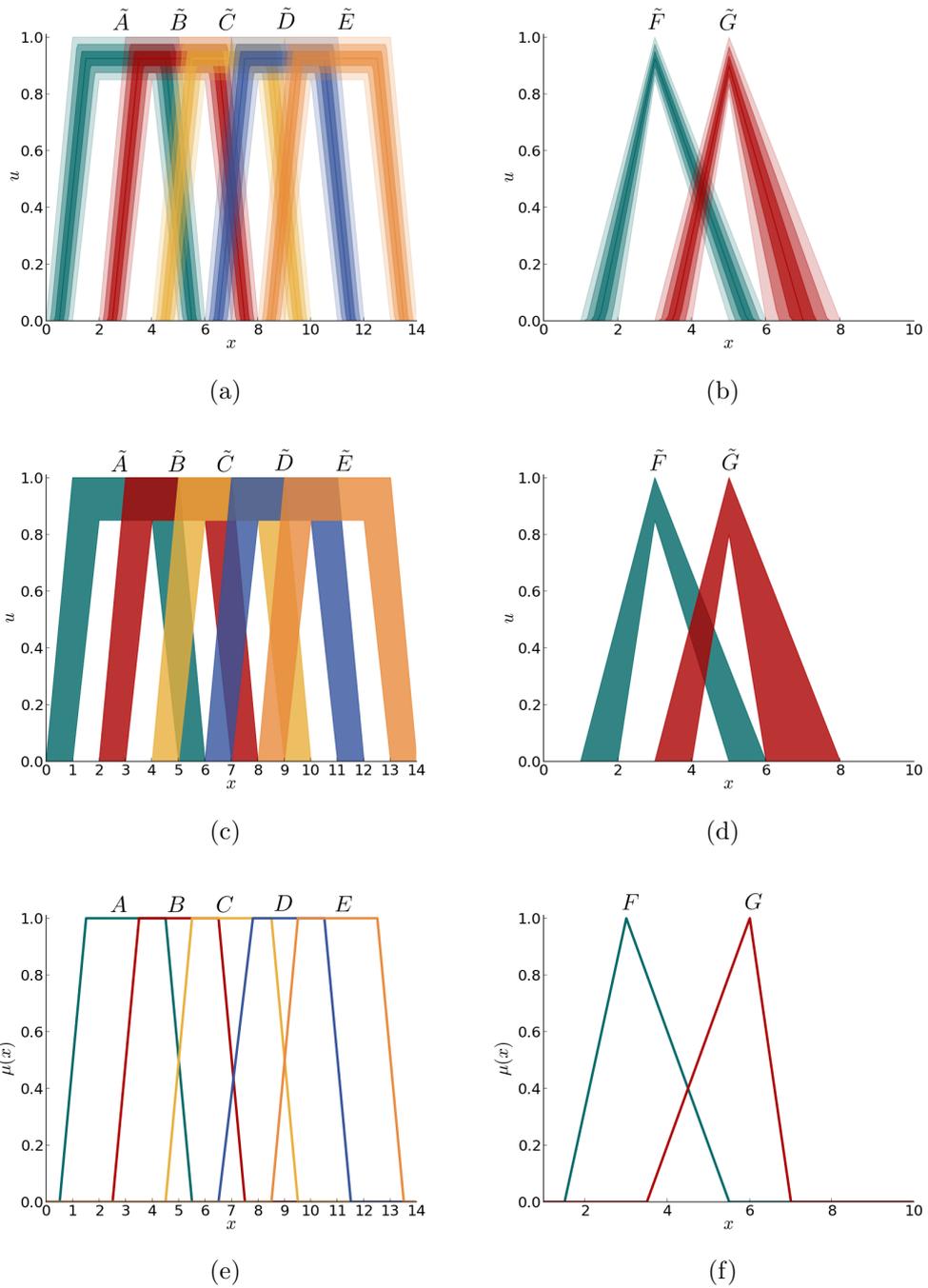


Figure 4.7: General and interval type-2 fuzzy sets used to demonstrate s_j^{GT2} .

Figure	Measure	$s(A, A)$	$s(A, B)$	$s(A, C)$	$s(A, D)$	$s(A, E)$	$s(F, G)$	$s(G, F)$
(a) & (b)	GT2	1.0	0.3353	0.041	0.0	0.0	0.1581	0.1581
(c) & (d)	GT2	1.0	0.3422	0.0709	0.0	0.0	0.1719	0.1719
	IT2	1.0	0.3422	0.0709	0.0	0.0	0.1719	0.1719
(e) & (f)	GT2	1.0	0.3333	0.323	0.0	0.0	0.1534	0.1534
	IT2	1.0	0.3333	0.323	0.0	0.0	0.1534	0.1534
	T1	1.0	0.3333	0.323	0.0	0.0	0.1534	0.1534

Table 4.4: Comparisons of the fuzzy sets in Figure 4.7 using the zSlices general type-2 (GT2) (4.10), interval type-2 (IT2) (2.31) and type-1 (T1) (2.29) Jaccard similarity measures.

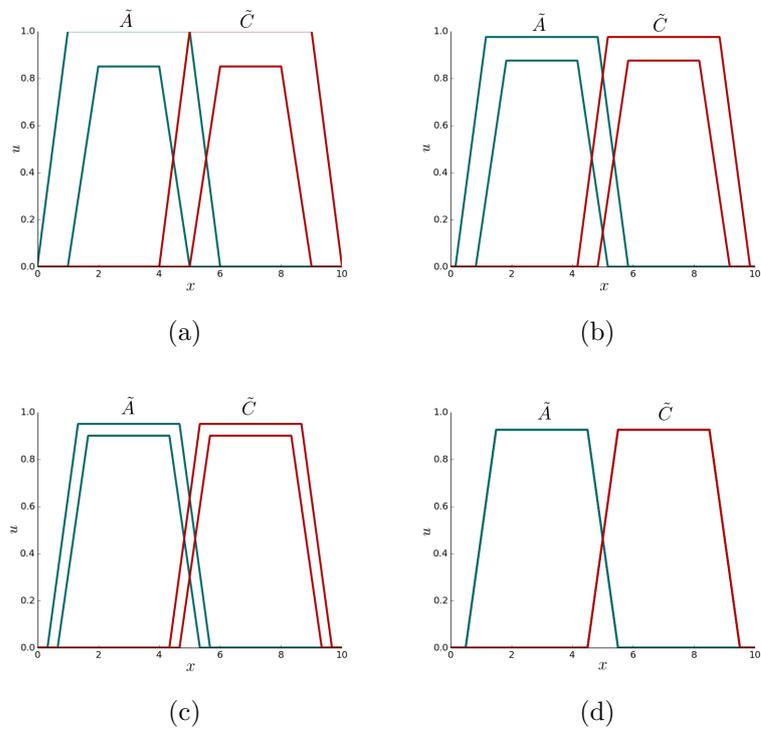


Figure 4.8: Fuzzy sets \tilde{A} and \tilde{C} from Figure 4.7a at different zLevels

zlevel	$\mu_{\tilde{A}_z}(5.5)$	$\mu_{\tilde{C}_z}(5.5)$	s_u	s_l	$\frac{s_u}{s_l}$
0.25	[0.0, 0.5]	[0.425, 1.0]	0.5	1.425	0.3509
0.5	[0.0, 0.325]	[0.5834, 0.975]	0.325	1.5584	0.2085
0.75	[0.0, 0.1584]	[0.75, 0.95]	0.1584	1.7	0.0932
1.0	[0.0, 0.0]	[0.925, 0.925]	0.0	1.85	0.0

Table 4.5: The similarity between the vertical slice $x = 5.5$ of fuzzy sets \tilde{A} and \tilde{C} at different zLevels as shown in Figure 4.8. s_u is the upper of the fraction of the Jaccard similarity as $\min\{\bar{\mu}_{\tilde{A}_z}(x), \bar{\mu}_{\tilde{C}_z}(x)\} + \min\{\underline{\mu}_{\tilde{A}_z}(x), \underline{\mu}_{\tilde{C}_z}(x)\}$, and s_l is, likewise, the lower of the fraction $\max\{\bar{\mu}_{\tilde{A}_z}(x), \bar{\mu}_{\tilde{C}_z}(x)\} + \max\{\underline{\mu}_{\tilde{A}_z}(x), \underline{\mu}_{\tilde{C}_z}(x)\}$.

4.4 Summary

This chapter has focused on developing methods to compare the distance and similarity between two interval or general type-2 fuzzy sets. First, a distance measure was established to compare interval type-2 fuzzy sets. This measure uses the distance between α -cuts developed in Chapter 3, and can compare fuzzy sets that are non-normal or non-convex. One may use either the directional or non-directional distance between α -cuts (as discussed in Chapter 3) to attain a directional or non-directional distance measure between interval type-2 fuzzy sets.

Demonstrations have shown that the distance between interval type-2 fuzzy sets d_p^{IT2} produces consistent results compared to the distance measure between type-1 fuzzy sets d_p^{T1} developed in Chapter 3. Additionally, it has been demonstrated that the interval type-2 distance measure d_p^{IT2} produces the same results on type-1 fuzzy sets as the type-1 distance measure d_p^{T1} developed in Chapter 3.

This chapter has also introduced a method of extending any measure for

interval type-2 fuzzy sets onto general type-2 fuzzy sets. This method ensures that the properties of the interval type-2 measure are still present when extended.

This extension has been demonstrated to measure distance and similarity. In both examples, the comparison between zSlices type-2 fuzzy sets has produced consistent results compared to the original interval type-2 measure. It has also been demonstrated that when comparing interval type-2 fuzzy sets, the extended measure gives the same results as the original interval type-2 measure. In addition, the extended distance d_p^{GT2} and similarity s_j^{GT2} measures give the same results for type-1 fuzzy sets as the type-1 measures d_p^{T1} and s_j^{T1} , respectively.

The next chapter develops a measure of fusing similarity and distance together to enable one to gain information from both measures through a single value.

Chapter 5

An Incompatibility Measure for Fuzzy Sets

5.1 Introduction

This chapter presents a new incompatibility measure on fuzzy sets that aggregates similarity and distance to determine how distinct fuzzy sets are from each other. This is referred to as an *incompatibility measure* because compatibility has been described as *a broad concept that typically encompasses both similarity and proximity* [49]. It is referred to as *incompatibility* instead of *compatibility* as this description best fits its mathematical properties (which are discussed in Section 5.3).

Although a single measure between fuzzy sets is useful, decision making often involves observing the outcomes of several different comparisons. Thus, one comprehensive measure that fuses the concepts of similarity and distance can be more useful than a single measure alone. It also eliminates the necessity for one to decide whether similarity or distance would be most appropriate for a given application, as this fusion of both measures may be used instead.

This chapter introduces a new measure that weights and combines the Jac-

card similarity measure (s_j^{T1} (2.29), s_j^{IT2} (2.31), s_j^{GT2} (4.10)) and the proposed directional distance measure (d_p^{T1} (3.13), d_p^{IT2} (4.1), d_p^{GT2} (4.7)) to produce a single measure that benefits from the properties of both of these approaches. Also note that it was demonstrated in Chapter 4 that the Jaccard similarity and proposed directional distance measures give consistent results between type-1, interval type-2 and general type-2 fuzzy sets. Therefore, for conciseness and simplicity, this chapter primarily focuses on type-1 fuzzy sets. However, a brief demonstration on type-2 fuzzy sets is given in Section 5.5.2.

5.2 Motivation

This section demonstrates the motivation for using a combination of similarity and distance within a single measure. First, the limitations of using only a single measure are discussed, then the advantages of jointly using both measures are highlighted.

5.2.1 Limitations of a Single Measure

On its own, a similarity or distance measure provides a useful relative comparison of fuzzy sets. Similarity shows how much two fuzzy sets share the same values, but it does not indicate what values they do not share. For example, in Figure 5.1, B and C share the same degree of similarity with A ; using the Jaccard measure (2.29) $s_j^{T1}(A, B) = 0.142$ and $s_j^{T1}(A, C) = 0.142$. Given that the similarities of both pairs are identical, it cannot be determined from the measure alone that B and C are distinct from A in different ways.

A distance measure shows how much space there is between two fuzzy sets in their universe of discourse. This is helpful for understanding how much fuzzy sets are distinct from each other. However, a distance measure does not always give a full picture of the fuzzy sets. For example, using the directional distance (3.13) in Figure 5.2, pairs (a) (A and B) and (b) (C and D) give the

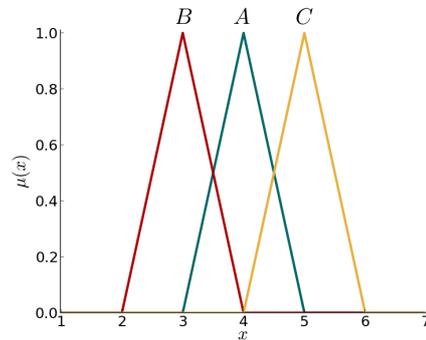


Figure 5.1: A demonstration of identical similarities between different pairs of fuzzy sets.

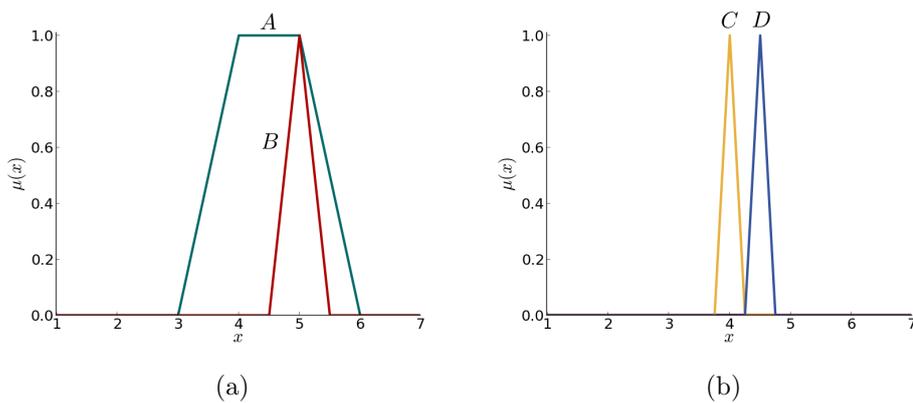


Figure 5.2: A demonstration of identical distances between different pairs of fuzzy sets.

same values of distance ($d_p^{T1}(A, B) = 0.5$; $d_p^{T1}(C, D) = 0.5$), yet C and D could be considered more distinct from each other than A and B because they do not contain any of the same values.

It is important to be clear that similarity cannot be effectively used as a substitute for distance and vice versa. Focusing on the application of recommendation systems as introduced in Section 2.6, consider the query *a film like Star Trek but with more action*. One would assume from this query that a similarity measure would be better suited to find *a film like Star Trek* than a distance measure, because the term *like* is generally understood as *similar*

to. However, one would assume that a directional distance measure would be better suited to find a film with *more action than Star Trek* than a similarity measure.

Figures 5.3 and 5.4¹ demonstrate why it is important to use both similarity and distance to resolve this query instead of using a single measure to accomplish both tasks.

Though a distance measure provides an indication of proximity between fuzzy sets, it is not always suitable when differentiating between overlapping fuzzy sets. For example, Figure 5.3 shows three fuzzy sets A , B and C . The similarity and distance between pairs (A, B) and (A, C) are shown in Table 5.1. The distances between both pairs are identical, yet their similarities differ. These results show that one can't always use the shortest distance to determine which fuzzy sets are the most similar.

Additionally, Figure 5.4 also shows three fuzzy sets labelled A , B and C , the similarities and distances of which are shown in Table 5.2. It is clear from these results that similarity is not a substitute for a distance measure because one cannot tell by vertical slices how much space there is between fuzzy sets. Though similarity shows that the fuzzy sets are disjoint and therefore somewhat distant, it does not indicate the magnitude of this distance. Additionally, it is necessary to use some measure of distance to find out the direction between fuzzy sets; the similarity measure does not indicate if B contains values higher or lower than A .

These demonstrations show that similarity and distance measures each have unique strengths and limitations. Though this chapter focuses on type-1 fuzzy sets, Section 5.5.2 also shows examples of Figures 5.3 and 5.4 using interval and general type-2 fuzzy sets.

¹To measure similarity, in Figure 5.3 X is discretised into 81 equidistant x points, and in Figure 5.4 X is discretised into 91 equidistant points.

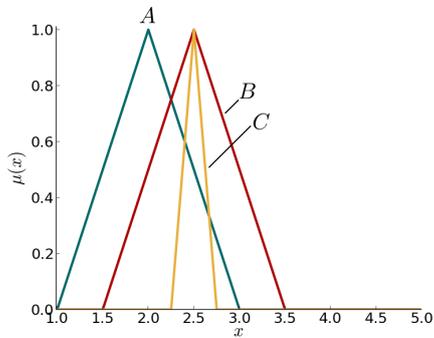


Figure 5.3: Three overlapping fuzzy sets A , B and C

Measure	(A, B)	(A, C)
s_j^{T1} (2.29)	0.389	0.178
d_p^{T1} (3.13)	0.5	0.5

Table 5.1: Results of similarity s_j^{T1} (2.29) and distance d_p^{T1} (3.13) measures on the fuzzy sets in Figure 5.3.

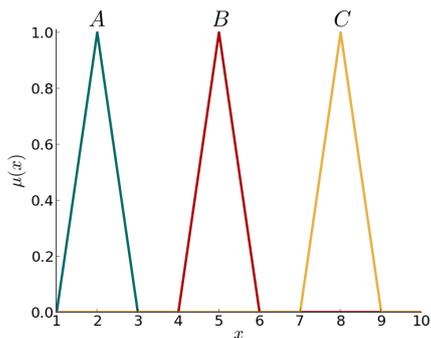


Figure 5.4: Three disjoint fuzzy sets A , B and C

Measure	(A, B)	(A, C)
s_j^{T1} (2.29)	0.0	0.0
d_p^{T1} (3.13)	3.0	6.0

Table 5.2: Results of similarity s_j^{T1} (2.29) and distance d_p^{T1} (3.13) measures on the fuzzy sets in Figure 5.4.

5.2.2 Benefits of Independently Using Multiple Measures

This section demonstrates that similarity and distance measures can be used together to gain a much better understanding of the fuzzy sets without having to visually observe them. In this demonstration, fuzzy sets are constructed from the Movie Lens data set [109], which contains 100,000 ratings from 943 users on 1682 movies. Each user rates how much they enjoyed a film using a value in $\{1, 2, 3, 4, 5\}$ where 1 is a poor rating and 5 is good. This data is modelled by fuzzy sets using the polling technique (2.14) (detailed in Section 2.3) and linear interpolation is used to find degrees of membership between the given integers.

Figure 5.5 shows six pairs of fuzzy sets that have been constructed from the MovieLens data set. Black vertical and horizontal lines indicate the degree of similarity and distance, respectively, between the fuzzy sets. The larger the given line, the greater the similarity or distance.

In this example, a similarity measure can show the amount of agreement between user ratings and a directional distance will indicate the difference between ratings. Table 5.3 shows the results of the Jaccard similarity s_j^{T1} and the directional distance d_p^{T1} measures on the pairs of fuzzy sets in Figure 5.5. For each pair, the fuzzy set A is given as the first parameter of the measure, and B is given as the second parameter. The following discusses the similarities and distances between each pair of fuzzy sets, highlighting where both measures contribute useful information.

Figures 5.5 a & b The low value of similarity indicates that both pairs of fuzzy sets are distinct and almost disjoint. There is some small overlap in the fuzzy sets, but from the similarity alone it is impossible to discern where this overlap lies. The distances, however, show that in (a) B is to the right of A and in (b) B is to the left of A .

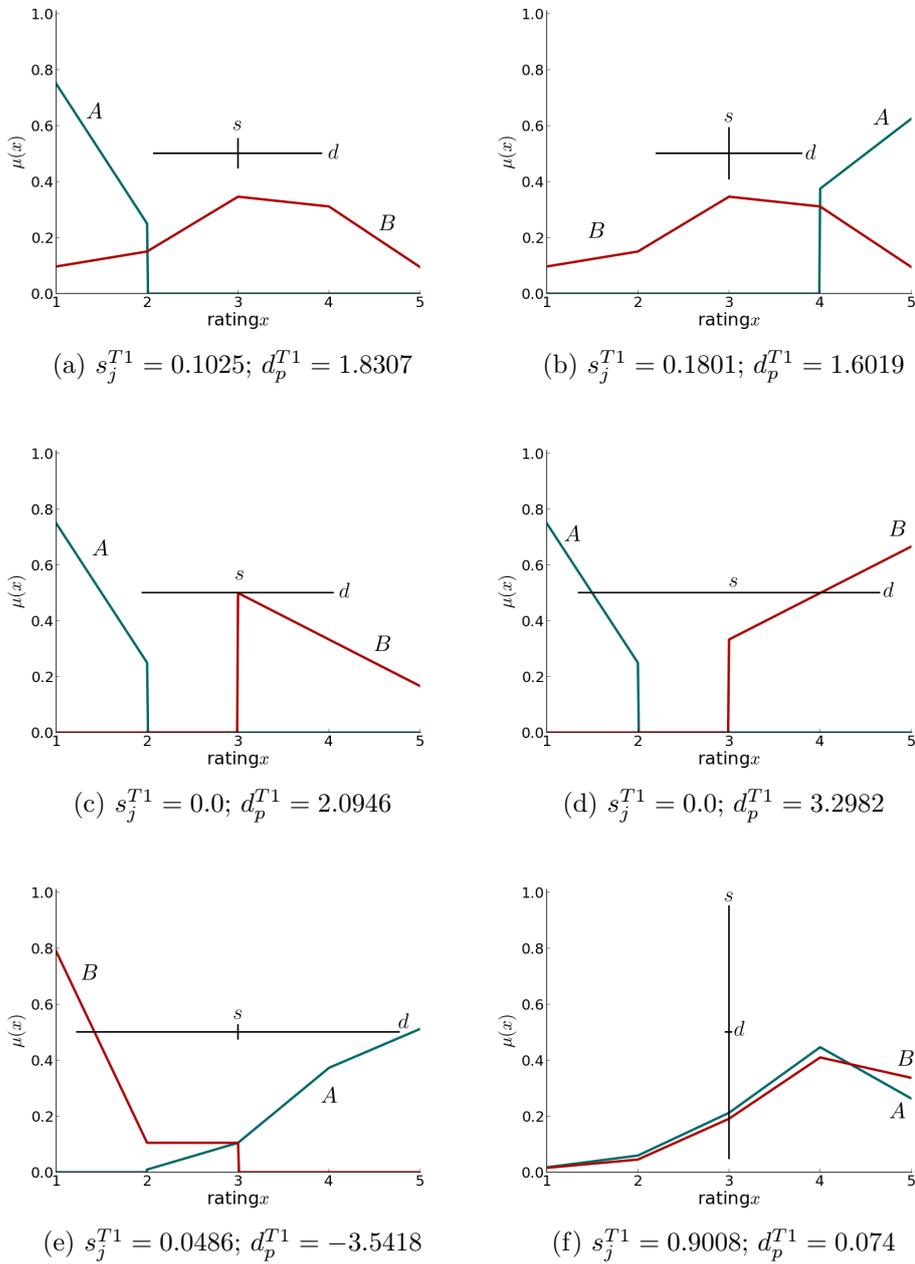


Figure 5.5: Fuzzy sets representing the distributions of ratings for different films in the MovieLens data set. Vertical and horizontal lines represent the degree of similarity (s_j^{T1}) and distance (d_p^{T1}), respectively. Movie IDs titles corresponding to each fuzzy set are listed in Appendix D.

Figure 5.5 - part:	a	b	c	d	e	f
$s_j^{T1}(A, B)$ (2.29)	0.1025	0.1801	0.0	0.0	0.0486	0.9008
$d_p^{T1}(A, B)$ (3.13)	1.8307	-1.6019	2.0946	3.2982	-3.5418	0.074

Table 5.3: Results of similarity s_j^{T1} (2.29) and distance d_p^{T1} (3.13) measures on the fuzzy sets in Figure 5.5

Figures 5.5 c & d The similarity shows that both pair of fuzzy sets are disjoint. The distance shows that in (d) B is further from A than in (c). This is because in (c) the greatest membership in B is in the fuzzy set's closest region to A . In (d), however, the highest membership of B is in its furthest region from A .

Figure 5.5 e Like (b), the similarity indicates the fuzzy sets are almost disjoint but there is some small overlap. However, it is not clear where this overlap lies. From the distance measure it is clear that B is to the left of A .

Figure 5.5 f Both measures indicate that the fuzzy sets are almost identical.

These results show that measuring both similarity and distance is more informative than either measure alone. However, interpreting the results of two distinct outputs can be challenging and time consuming when many fuzzy sets are to be compared. A function that provides information of both similarity and distance would simplify this process.

The next section introduces a combined measure of similarity and distance that evaluates the incompatibility of fuzzy sets.

5.3 Fusing Similarity and Distance

To fuse similarity and distance into a single measure, their unique properties must be addressed. Two problems arise from the nature of these measures.

Firstly, similarity indicates how similarly or closely two fuzzy sets model the same data, whereas distance indicates how much difference exists between these sets. As a result, similarity gives a high value for identical fuzzy sets, whilst distance gives a high value for different fuzzy sets. Thus, to fuse these two measures, one must be altered so that both measures represent closeness in the same manner; i.e., both measures show similarity/closeness or both show dissimilarity/distance.

The second issue is in regard to the range of values calculated by the measures. Similarity gives a result in the interval $[0, 1]$, whereas the directional distance measure gives a value in U , the universe $\mathbb{R} \in U$. To combine these two into a single result, the values of similarity and distance must be altered so they are easily comparable and can thus be fused.

To resolve the first problem, the complement of the similarity measure is used to represent dissimilarity/difference. By doing this, both the dissimilarity and distance measures give the value 0 for identical fuzzy sets. This approach has been chosen because it enables the measure to represent direction. Specifically, negative and positive values will occur for lower and higher relative positions, respectively, and the value 0 indicates identical sets.

If the complement of the directional distance is used instead then only the direction of the signed value would change and the value 0 would still have a different meaning for both measures. The complement of similarity $1 - s$ is often used as dissimilarity in the literature [46, 110] and will be referred to as dissimilarity throughout the remainder of this chapter.

To resolve the second issue, one of the measures must be changed so that both give the same range of values (in $[-1, 1]$ or \mathbb{R}). Which values are used may

be left to personal choice as different applications may benefit from different values. If it is necessary to easily differentiate between small and large distances then a value in \mathbb{R} may be best. However, results in $[-1, 1]$ may be more easily interpreted because similarity and dissimilarity are not typically expressed in \mathbb{R} and therefore may be less well understood in this way.

Normalising the distance in $[-1, 1]$ also enables one to treat results in the same manner regardless of the original data. For example, in different recommendation systems, products may be rated on different scales; for example, 1 to 5, or 1 to 10. In the former example, a difference of 1 point is more significant than in the latter case. Thus, if a measure's results are in \mathbb{R} then there must be an understanding of the universe of discourse in order to realise the significance of any given distance.

However, if the results are normalised in $[-1, 1]$ then a given value of distance is equally significant regardless of the original non-normalised distances because it is in relation to the greatest possible distance. Thus, any underlying application can be the same regardless of the range of values used by the data.

Given these two points, the dissimilarity measure will be joined with the normalised directional distance to produce a combined measure that results in a value within $[-1, 1]$. A value of 0 indicates identical fuzzy sets and -1 or 1 indicates the maximum possible distance between the sets.

To normalise the distance, its result is divided by the largest possible distance. In a finite² universe of discourse $X = [X_L, X_R]$, let $\tau(X)$ denote $X_R - X_L$, then the distance $d(A, B)$ is normalised as $\frac{d(A, B)}{\tau(X)}$.

Given this, the following definition introduces the proposed incompatibility measure for two fuzzy sets A and B where A and B may be type-1, interval type-2 or general type-2 fuzzy sets.

Definition 35 (Incompatibility Measure). *The dissimilarity and distance be-*

²A different approach would be required for an infinite universe of discourse. However, this is left for future work.

tween fuzzy sets A and B may be joined into a single value as

$$c'_p(A, B) = \begin{cases} f\left(\left\{\left(1-s(A, B)\right), \left(\frac{d(A, B)}{\tau(X)}\right)\right\}, \langle w_0, w_1 \rangle\right), & d(A, B) \geq 0 \\ f\left(\left\{-\left(1-s(A, B)\right), \left(\frac{d(A, B)}{\tau(X)}\right)\right\}, \langle w_0, w_1 \rangle\right), & \text{otherwise} \end{cases} \quad (5.1)$$

where f is the ordered weighted average (OWA) operator (2.49), s is a similarity measure in $[0, 1]$, d is a distance measure in \mathbb{R} , $\tau(X)$ is $X_R - X_L$ for a universe of discourse $X = [X_L, X_R]$, and $\langle w_0, w_1 \rangle$ are the weights used by the OWA f .

Note that the function within (5.1) is given as c'_p to be consistent with other function notations. Throughout this thesis, similarity is referred as s and in the next section dissimilarity (the complement of s) is given as s' . To maintain this style, the incompatibility measure (5.1) is labelled as c'_p as one would expect c_p to denote compatibility.

The choice of weights for the OWA operator depends on the application and the nature of the fuzzy sets (e.g., highly overlapping or mostly disjoint). A discussion of the effects of different weights in c'_p (5.1) is given in the next section, in which the weights used within this thesis are chosen. Note that the absolute values of the measures are used when assigning the weights. For example, if the dissimilarity is 0.3 and the distance is -0.45, then the distance will be given the first weight because it has the largest magnitude.

Theorem 5. *Where s and d have the properties of a similarity and directional distance measure, respectively (see Appendix A), the incompatibility measure c'_p ((5.1)) has the properties*

- i) *Self-Identity:* $c'_p(A, A) = 0$
- ii) *Partial Symmetry:* $c'_p(A, B) = -c'_p(B, A)$
- iii) *Directional Separability:* $c'_p(A, B) \geq 0$ if $B \geq A$ and $c'_p(A, B) < 0$ if $B < A$
- iv) *Transitivity:* $c'_p(A, B) \leq c'_p(A, C)$ if $A \leq B \leq C$
- v) *Triangle-Inequality:* $c'_p(A, C) \leq c'_p(A, B) + c'_p(B, C)$

Proof.

- i) For any weights, $1 - s$ and d are 0, thus $c'_p(A, A) = 0$.
- ii) and iii) If $d(A, B) < 0$ then the negative dissimilarity $-1 - s(A, B)$ is used to ensure partial symmetry and directional separability.
- iv) Both dissimilarity and distance follow transitivity, thus the proof is trivial.
- v) Both dissimilarity and distance follow triangle inequality where s is the Jaccard measure [110], thus the incompatibility measure also follows triangle inequality. \square

One can use c'_p to join similarity with a distance metric instead of a directional distance measure to attain a non-directional incompatibility metric. Note, however, that only the directional distance measure d_p^* will be used throughout this thesis, where $*$ denotes the type of fuzzy sets compared.

Theorem 6. *If the similarity s and (non-directional) distance d measures are both metrics then the incompatibility measure c'_p is also a metric with the properties*

- i) *self-identity:* $c'_p(A, A) = 0$
- ii) *separability:* $c'_p(A, B) > 0$
- iii) *symmetry:* $c'_p(A, B) = c'_p(B, A)$
- iv) *triangle inequality:* $c'_p(A, C) \leq c'_p(A, B) + c'_p(B, C)$

Proof. i), ii) and iii) The proofs are trivial.

- iv) Both dissimilarity and a distance metric follow triangle inequality where s is the Jaccard measure [110], thus the incompatibility measure also follows triangle inequality. \square

Table 5.4 gives a summary of the properties of the new directional incompatibility measure compared to the properties of similarity, distance metrics and directional distance measures. Appendix A provides details of these properties.

Symmetry and partial symmetry are contradictory properties where the former is a property of a metric and the latter is only in a signed directional distance. Thus, no measure can have both of these properties. The same is also true for separability and directional separability. Reflexivity and self-identity are also directly contradictory properties and a measure cannot follow both.

Additionally, overlapping (or disjointness) is a property of similarity (or dissimilarity) alone and will always be lost for any weights other than $\langle 1.0, 0.0 \rangle$ as distance effects the results such that this property is removed.

Reflectivity is a property of only the directional distance, giving zero distance for symmetrical fuzzy sets that share the same centroid. However, by adding dissimilarity, the incompatibility measure results in a non-zero value for such fuzzy sets. Thus, for any weights other than $\langle 0.0, 1.0 \rangle$ this property is not followed. Additionally, as a result of the directional distance, the stricter directional form of triangle inequality applies, as described in Section 3.2.2.

Note that the name *incompatibility measure* stems from these properties as the result of the measure indicates the degree to which two fuzzy sets are incompatible. If the comparison of fuzzy sets is 0, then they have no incompatibility, i.e., they are completely compatible. If their comparison is 1, then they are entirely incompatible.

Also note that since the incompatibility measure c'_p (5.1) fuses the *results* of the similarity and distance measures, it may be used on any type of fuzzy set (type-1, interval type-2 and general type-2) where s and d are similarity and distance measures for the given type of fuzzy sets.

Note that throughout this thesis, according to the magnitude and direction of the results, the incompatibility between fuzzy sets will be described as *high negative*, *low negative*, *low positive*, or *high positive*, as shown in Figure 5.6. This is to enable an easier discussion of the results.

Having introduced the proposed combined measure of (dis)similarity and distance, the next section discusses the effects that different weights have on

Properties of Both Similarity and Distance Metrics	
Symmetry	X
Transitivity	✓
Properties of Similarity	
Reflexivity	X
Overlapping	X
Properties of Metric Distance	
Self-identity	✓
Separability	X
Triangle-inequality	✓
Properties of Directional Distance	
Partial Symmetry	✓
Directional Separability	✓
Reflectivity	X

Table 5.4: A summary of the properties of the proposed incompatibility measure compared with those of similarity, distance metrics and directional distance measures.

this measure, and chooses an ideal pair of weights based on experiments.

5.4 Choosing Weights

This section discusses how the weights of the OWA operator are chosen by using an empirical strategy for identifying both a generally applicable set of weights as well as set specific considerations. This discussion is driven by examples of comparing fuzzy sets and selecting the weights that best fit the expected results. Note that the choice of the most ideal weights is subjective as there is no choice that can be universally described as the best. To provide a succinct discussion, only the weight w_0 will be specified throughout, as w_1 can be inferred from w_0 ; i.e., $w_0 + w_1 = 1.0$.

Using different weights, demonstrations of the proposed incompatibility measure are shown using six figures. Each figure contains three fuzzy sets $A, B, C \in T1(X)$ with which the values of incompatibility of $c'_p(A, B)$ and $c'_p(A, C)$ are measured. For simplification, the first four figures use synthetic fuzzy sets that highlight some of the main properties of the measure. The last two examples show the effects of the measure based on data-driven fuzzy sets. In each demonstration, the results of the dissimilarity (s'_j) and normalised distance (d_n) between fuzzy sets are highlighted. Additionally, each figure uses a graph to visualise the changes in the results of $c'_p(A, B)$ and $c'_p(A, C)$ when

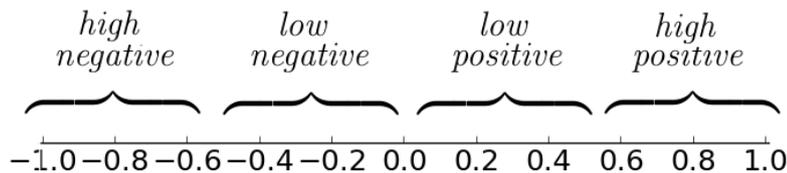


Figure 5.6: Descriptions used in this thesis to describe the values of incompatibility according to magnitude and direction.

using different weights.

Figure 5.7 demonstrates that as long as the weights are not $\langle 1.0, 0.0 \rangle$ it is always possible to distinguish between different pairs of disjoint fuzzy sets where the fuzzy set centroids differ. As the dissimilarity between disjoint fuzzy sets is always 1.0, it will always be assigned the first weight for such fuzzy sets. As w_0 increases the difference between $c'_p(A, B)$ and $c'_p(A, C)$ becomes smaller, so it is advisable if w_0 is not too large so that the distance measure still has a noticeable impact on the results. In this case, $w_0 \leq 0.8$ is appropriate.

Figure 5.8 demonstrates the incompatibility measure on convex and non-convex symmetrical fuzzy sets. With a distance measure alone, the results of $c'_p(A, B)$ and $c'_p(A, C)$ are both 0. However, the dissimilarity shows that these fuzzy sets are, in fact, not the same. As w_0 increases this difference becomes more apparent. However, as the result of c'_p becomes larger one may assume the fuzzy sets are further apart rather than actually subsets. In this case, $0.2 \leq w_0 \leq 0.8$ appears reasonable because these weights show there is a small difference in the fuzzy set pairs.

When fuzzy sets overlap, the choice of weights becomes more subjective and potentially restricted. The next four examples show cases in which the dissimilarity and distance measures give contradicting results for pairs of fuzzy sets. In each of these figures, $d_n(A, B) < d_n(A, C)$ but $s'(A, B) > s'(A, C)$. In Figure 5.9, where $w_0 < 0.3$, $c'_p(A, B) < c'_p(A, C)$, but where $w_0 \geq 0.3$, $c'_p(A, B) > c'_p(A, C)$. As C overlaps A and B does not, it can be argued that C should have a closer compatibility than B (i.e., lower incompatibility) so the most ideal weights are where $w_0 \geq 0.3$.

Figure 5.10 shows a similar example where B now overlaps A . However, the degree to which B overlaps A is much smaller than the overlap from C , so C could still be described as closer to A than B . This is true in the results where $w_0 \geq 0.6$; note that this is the greatest restriction given on the weights thus far.

The final two examples use fuzzy sets from real data. In a survey, participants were asked to rate different attributes of cakes, such as sweetness and fruitiness. Answers were given in intervals and the results are modelled by fuzzy sets using the Interval Agreement Approach (a form of the interval estimation-2 approach; see Section 2.3). Much of the data from this data set is heavily overlapping and the choice of weights must take this into consideration.

In Figure 5.11, A and C have similarly shaped distributions, whereas B diverges from this distribution between $6 \leq x \leq 7$ where there is a large peak in membership. Considering this, the results should ideally be $c'_p(A, C) < c'_p(A, B)$ and this occurs where $w_0 \geq 0.3$.

Figure 5.12 shows another example of highly overlapping fuzzy sets. As in the previous case, A and C have similar distributions, whereas B is noticeably different. Given this, the results should show $c'_p(A, C) < c'_p(A, B)$; this is true where $w_0 \geq 0.6$.

Table 5.5 summarises the demonstrations in Figures 5.7 to 5.12, showing which values of w_0 give expected results for each figure. Note that the weights $\langle 0.0, 1.0 \rangle$ and $\langle 1.0, 0.0 \rangle$ are not suggested because this results in using a single measure, negating the advantages of a combined measure. Given these demonstrations, the most ideal weights are where $0.6 \leq w_0 \leq 0.8$, and so the middle ground $\langle w_0 = 0.7, w_1 = 0.3 \rangle$ will be used throughout this thesis.

Ideally, if the fuzzy sets are known beforehand, one could tune the weights as different weights may be more appropriate if fuzzy sets are highly overlapping or mostly disjoint. They may be tuned so that the greatest diversity of results occurs for the given data set. This would even further alleviate issues where significantly different pairs of fuzzy sets result in the same value (such as the examples shown in Figures 5.3 and 5.4).

The average weights $\langle 0.5, 0.5 \rangle$ can be used as a general case for comparing fuzzy sets. However, if there is a lot of overlap between fuzzy sets then increasing the first weight can yield more useful results. This is because the

Figure	Ideal Weights
5.7	$0.2 \leq w_0 \leq 0.8$
5.8	$0.2 \leq w_0 \leq 0.8$
5.9	$0.3 \leq w_0 \leq 0.8$
5.10	$0.6 \leq w_0 \leq 0.8$
5.11	$0.3 \leq w_0 \leq 0.8$
5.12	$0.6 \leq w_0 \leq 0.8$

Table 5.5: Ideal weights selected for the incompatibility measure considering different examples shown in the listed figures.

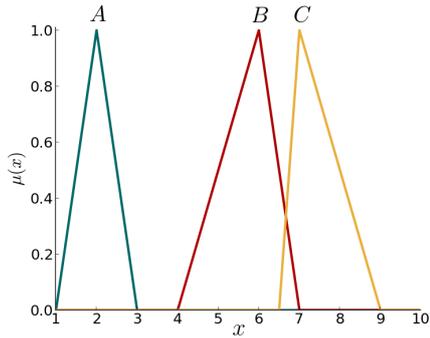
value of similarity is assigned the first weight and similarity is best for differentiating between highly overlapping fuzzy sets because it compares fuzzy sets by measuring vertical slices. Distance, however, is not so effective at making a distinction between highly overlapping sets. Examples of this are shown in Figures 5.11 and 5.12, in which the dissimilarity result is what one would expect (stating that the pair (A, C) is closer than (A, B)), whereas the normalised distance does not give the expected results. For such heavily overlapping fuzzy sets, the weights $\langle 0.7, 0.3 \rangle$ are preferred. As many of the data sets used within this thesis feature such overlapping fuzzy sets, the weights $\langle 0.7, 0.3 \rangle$ are used throughout in the remaining chapters.

Note that by choosing the weights $\langle 0.7, 0.3 \rangle$, the incompatibility measure always gives a value in $(0.7, 1.0]$ if two fuzzy sets are disjoint. This is because the dissimilarity between two disjoint sets is 1.0 and is given the weight 0.7. Note that disjoint sets will never have a value of 0.7 as this would require them to be both disjoint (according to s') and identical (according to d_n). Therefore, a value of 0.7 or lower shows that there is some degree of overlap between the fuzzy sets and a value greater than 0.7 signifies a small or zero overlap between

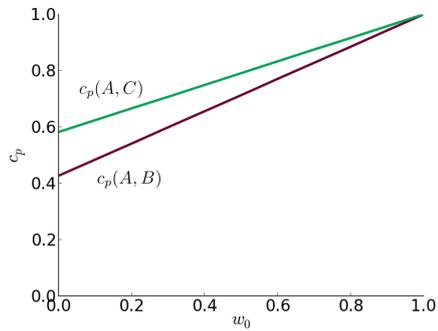
fuzzy sets.

Note, also, that although disjoint fuzzy sets always result in a value higher than 0.7, such a value does not mean that the fuzzy sets are necessarily disjoint. It does, however, indicate that the overlap is very low; an example of this is shown in Figure 5.10.

This section has demonstrated the effects of different weights on the proposed combined (dis)similarity and distance measures. Based on these experiments, the weights $\langle w_0 = 0.7, w_1 = 0.3 \rangle$ have been chosen. The next section provides examples of this measure, demonstrating its advantages over using similarity or distance alone.



(a) Three fuzzy sets A , B and C .



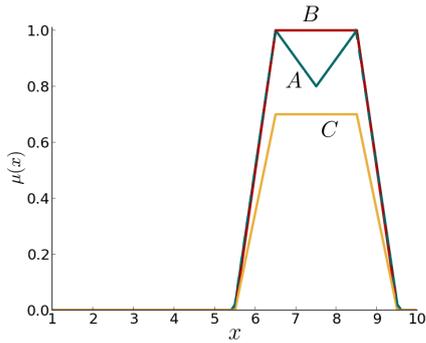
(b) The incompatibilities $c'_p(A, B)$ and $c'_p(A, C)$ for different values of w_0 .

Figure 5.7: Three fuzzy sets A , B and C and the incompatibility between the disjoint pairs (A, B) and (A, C) using different weights.

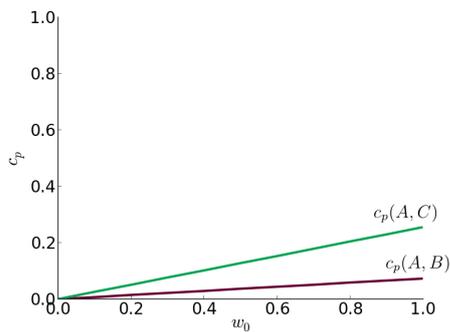
w_0	w_1	$c'_p(A, B)$	$c'_p(A, C)$
0.0	1.0	0.427 (d_n)	0.582 (d_n)
0.1	0.9	0.484	0.624
0.2	0.8	0.541	0.666
0.3	0.7	0.599	0.707
0.4	0.6	0.656	0.749
0.5	0.5	0.713	0.791
0.6	0.4	0.771	0.833
0.7	0.3	0.828	0.875
0.8	0.2	0.885	0.916
0.9	0.1	0.943	0.958
1.0	0.0	1.0 (s')	1.0 (s')

(a)

Table 5.6: The incompatibility between two pairs of disjoint fuzzy sets from Figure 5.7. The normalised distance d_n and dissimilarity s'_j results have been highlighted.



(a) Three fuzzy sets A , B and C .



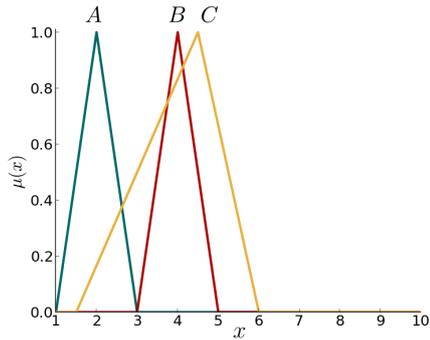
(b) The incompatibilities $c'_p(A, B)$ and $c'_p(A, C)$ for different values of w_0 .

w_0	w_1	$c'_p(A, B)$	$c'_p(A, C)$
0.0	1.0	0.0 (d_n)	0.0 (d_n)
0.1	0.9	0.007	0.026
0.2	0.8	0.015	0.051
0.3	0.7	0.022	0.077
0.4	0.6	0.029	0.102
0.5	0.5	0.037	0.128
0.6	0.4	0.044	0.153
0.7	0.3	0.051	0.179
0.8	0.2	0.059	0.205
0.9	0.1	0.066	0.23
1.0	0.0	0.073 (s')	0.256 (s')

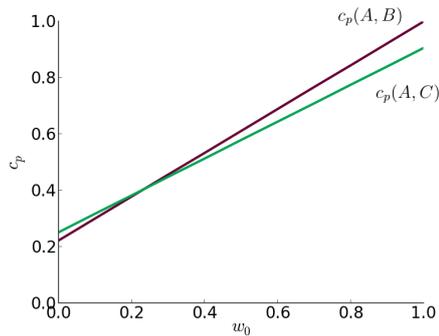
(a)

Figure 5.8: Three convex and non-convex fuzzy sets A , B and C and the incompatibility between the pairs (A, B) and (A, C) using different weights.

Table 5.7: The incompatibility between two pairs of disjoint fuzzy sets from Figure 5.8. The normalised distance d_n and dissimilarity s'_j results have been highlighted.



(a) Three fuzzy sets A , B and C .



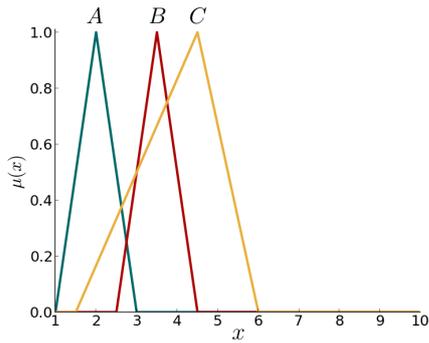
(b) The incompatibilities $c'_p(A, B)$ and $c'_p(A, C)$ for different values of w_0 .

Figure 5.9: Three overlapping fuzzy sets A , B and C and the incompatibility between the pairs (A, B) and (A, C) using different weights.

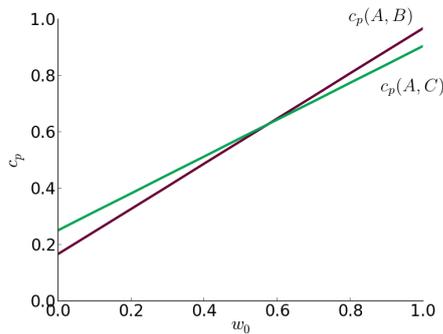
w_0	w_1	$c'_p(A, B)$	$c'_p(A, C)$
0.0	1.0	0.222 (d_n)	0.251 (d_n)
0.1	0.9	0.3	0.317
0.2	0.8	0.378	0.382
0.3	0.7	0.456	0.448
0.4	0.6	0.533	0.513
0.5	0.5	0.611	0.579
0.6	0.4	0.689	0.644
0.7	0.3	0.767	0.709
0.8	0.2	0.844	0.775
0.9	0.1	0.922	0.84
1.0	0.0	1.0 (s')	0.906 (s')

(a)

Table 5.8: The incompatibility between two pairs of disjoint fuzzy sets from Figure 5.9. The normalised distance d_n and dissimilarity s'_j results have been highlighted.



(a) Three fuzzy sets A , B and C .



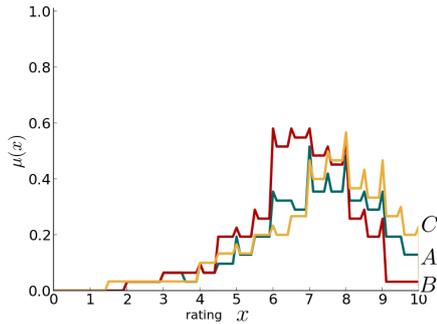
(b) The incompatibilities $c'_p(A, B)$ and $c'_p(A, C)$ for different values of w_0 .

Figure 5.10: Three overlapping fuzzy sets A , B and C and the incompatibility between the pairs (A, B) and (A, C) using different weights.

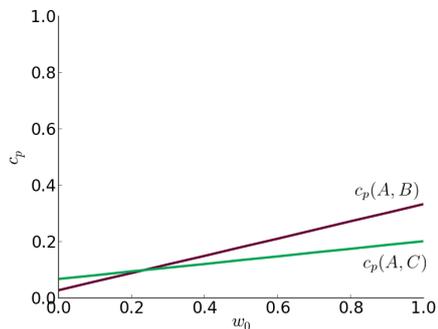
w_0	w_1	$c'_p(A, B)$	$c'_p(A, C)$
0.0	1.0	0.167 (d_n)	0.251 (d_n)
0.1	0.9	0.247	0.317
0.2	0.8	0.327	0.382
0.3	0.7	0.407	0.448
0.4	0.6	0.488	0.513
0.5	0.5	0.568	0.579
0.6	0.4	0.648	0.644
0.7	0.3	0.728	0.709
0.8	0.2	0.809	0.775
0.9	0.1	0.889	0.84
1.0	0.0	0.969 (s')	0.906 (s')

(a)

Table 5.9: The incompatibility between two pairs of disjoint fuzzy sets from Figure 5.10. The normalised distance d_n and dissimilarity s'_j results have been highlighted.



(a) Three fuzzy sets A , B and C .



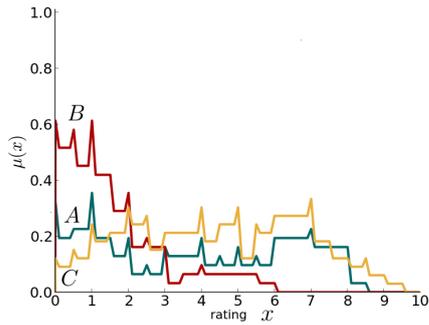
(b) The incompatibilities $c'_p(A, B)$ and $c'_p(A, C)$ for different values of w_0 .

Figure 5.11: Three fuzzy sets A , B and C describing the sweetness of different cakes and the incompatibility between the pairs (A, B) and (A, C) using different weights.

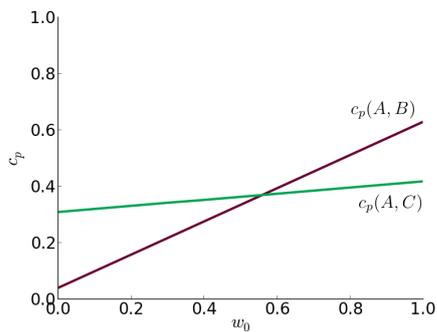
w_0	w_1	$c'_p(A, B)$	$c'_p(A, C)$
0.0	1.0	-0.028 (d_n)	0.068 (d_n)
0.1	0.9	-0.059	0.081
0.2	0.8	-0.089	0.095
0.3	0.7	-0.12	0.108
0.4	0.6	-0.15	0.121
0.5	0.5	-0.181	0.135
0.6	0.4	-0.211	0.148
0.7	0.3	-0.242	0.162
0.8	0.2	-0.273	0.175
0.9	0.1	-0.303	0.189
1.0	0.0	-0.334 (s')	0.202 (s')

(a)

Table 5.10: The incompatibility between fuzzy sets from Figure 5.11 describing the sweetness of different cakes. The normalised distance d_n and dissimilarity s'_j results have been highlighted.



(a) Three fuzzy sets A , B and C .



(b) The incompatibilities $c'_p(A, B)$ and $c'_p(A, C)$ for different values of w_0 .

Figure 5.12: Three fuzzy sets A , B and C describing the fruitiness of different cakes and the incompatibility between the pairs (A, B) and (A, C) using different weights.

w_0	w_1	$c'_p(A, B)$	$c'_p(A, C)$
0.0	1.0	-0.04 (d_n)	0.309 (d_n)
0.1	0.9	-0.099	0.32
0.2	0.8	-0.158	0.331
0.3	0.7	-0.217	0.342
0.4	0.6	-0.276	0.352
0.5	0.5	-0.335	0.363
0.6	0.4	-0.394	0.374
0.7	0.3	-0.453	0.385
0.8	0.2	-0.512	0.396
0.9	0.1	-0.571	0.407
1.0	0.0	-0.63 (s')	0.418 (s')

(a)

Table 5.11: The incompatibility between fuzzy sets from Figure 5.12 describing the fruitiness of different cakes. The normalised distance d_n and dissimilarity s'_j results have been highlighted.

5.5 Demonstrations

This section demonstrates the incompatibility measure c'_p (5.1) based on the examples given in Section 5.2 that demonstrate the utility of observing both similarity and distance. Note that, as stated in Section 5.3 the incompatibility measure c'_p (5.1) may be used on any type of fuzzy set (type-1, interval type-2 and general type-2) where s and d are similarity and distance measures for the given type of fuzzy sets.

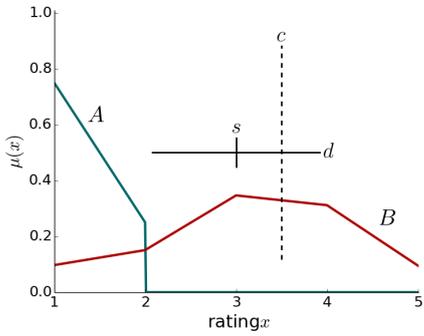
5.5.1 Type-1 Fuzzy Sets

Section 5.2 gave examples of the advantages of viewing both similarity and distance to gain a full comparison of fuzzy sets. This was demonstrated with the MovieLens data set in Figure 5.13 (formally in Figure 5.5) and Table 5.3. To demonstrate the advantages of the combined incompatibility measure, Table 5.12 shows the results of the Jaccard similarity s_j^{T1} , directional distance d_p^{T1} and combined incompatibility c'_p measures on the fuzzy sets in Figure 5.13. Note that, once again, in Figure 5.13 black solid vertical and horizontal lines represent the degree of similarity and distance between the fuzzy sets, respectively. In addition, a dashed vertical line is given to indicate the degree of incompatibility.

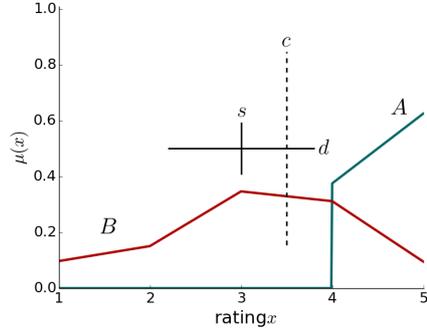
For each pair of fuzzy sets, the information from both the similarity and distance measures can be discerned from the single value given by the incompatibility measure. A discussion of these results is next.

Sets a & b The high values from the incompatibility measure show that (a) and (b) have little overlap/similarity and there is less similarity in (a) than in (b). The sign of the measure also shows that in (a) $A < B$ and in (b) $B < A$.

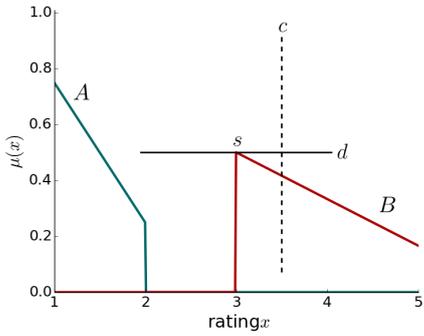
Sets c & d Both values are greater than 0.7 by a considerable amount and



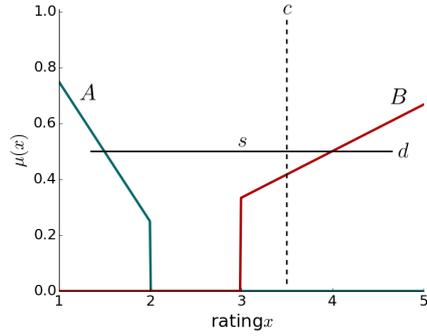
(a) $s_j^{T1} = 0.1025$; $d_p^{T1} = 1.8307$; $c'_p = 0.7656$



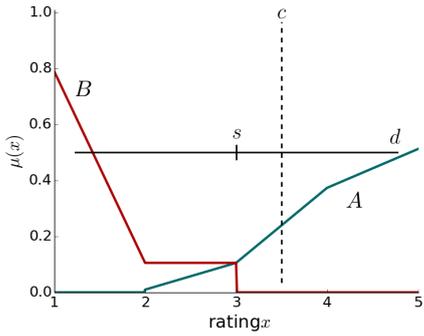
(b) $s_j^{T1} = 0.1801$; $d_p^{T1} = 1.6019$; $c'_p = -0.6941$



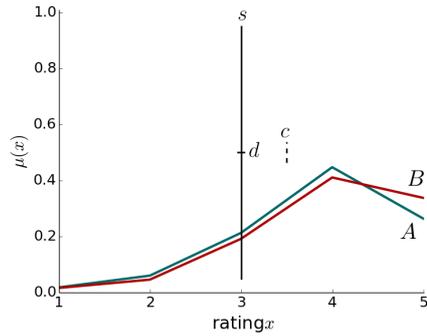
(c) $s_j^{T1} = 0.0$; $d_p^{T1} = 2.0946$; $c'_p = 0.8571$



(d) $s_j^{T1} = 0.0$; $d_p^{T1} = 3.2982$; $c'_p = 0.9474$



(e) $s_j^{T1} = 0.0486$; $d_p^{T1} = -3.5418$; $c'_p = -0.9316$



(f) $s_j^{T1} = 0.9008$; $d_p^{T1} = 0.074$; $c'_p = 0.075$

Figure 5.13: Fuzzy sets representing the distributions of ratings for different films in the MovieLens data set. Solid vertical and horizontal lines represent the degree of similarity (s_j^{T1}) and distance (d_p^{T1}), respectively, and the dashed line represents incompatibility (c'_p).

are therefore practically disjoint (i.e., if there is any overlap it will only be to a small degree). Additionally, the difference in (d) is greater than the difference in (c). This can also be seen in the figures. In (c) the greatest membership in B is in the fuzzy set's closest region to A . In (d), however, the highest membership of B is in the furthest region from A . As a result, the incompatibility of the pair in (d) is greater than that of the pair in (c).

Set e The combined measure shows that A and B are practically disjoint (i.e., if there is any overlap it is only to a small degree). The figure shows that the overlap between the fuzzy sets is very small and there is a large distance between the fuzzy sets at high membership values.

Also note that according to the distance measure alone, the pair (e) is more distance than the pair (d). However, because of the small overlap in (e), the incompatibility measure shows that (d) is in fact a more distinct pair than (e). This is a reasonable result given that in (d) the fuzzy sets are disjoint, whereas in (e) they are not.

Set f As the similarity and distance measures are both equally informative for nearly identical fuzzy sets, nothing is gained by the incompatibility in this case.

In Section 5.2, it was discussed that neither similarity nor distance can be used as a substitute for each other, thus for a given application a choice must be made as to which measure to use. However, by using a combined measure of similarity and distance, one may not have to make this choice. This helps enable the automatic evaluation/reasoning with fuzzy sets without one having to decide which single measure is best. Continuing from the earlier demonstration (in Figures 5.3 and 5.4), Figures 5.14 and 5.15 present the results of the incompatibility measure, showing how the results can be used to ascertain the similarity and distance between pairs of fuzzy sets.

Figure 5.13 - part:	a	b	c	d	e	f
s_j^{T1} (2.29)	0.1025	0.1801	0.0	0.0	0.0486	0.9008
d_p^{T1} (3.13)	1.8307	-1.6019	2.0946	3.2982	-3.5418	0.074
c'_p (5.1)	0.7656	-0.6941	0.8571	0.9474	-0.9316	0.075

Table 5.12: Results of similarity s_j^{T1} (2.29), distance d_p^{T1} (3.13) and incompatibility c'_p (5.1) measures on the fuzzy sets in Figure 5.13.

In Figure 5.14, the distance between A and B and A and C are the same, however the similarity measure shows that A and B share more similarity than A and C . This is also clear from the incompatibility measure, which shows that A and C are more distinct/less compatible than A and B .

Additionally, in Figure 5.15, both pairs of fuzzy sets are disjoint but have different distances. It is clear, however, from c'_p that the distance in each pair is different. According to c'_p , it is also likely, though not definite, that both pairs of fuzzy sets are disjoint.

From Figures 5.14 and 5.15, one can see that if it is not clear whether a given problem is best solved using similarity or distance then the incompatibility measure can be used to gain the perspective of both measures. Figures 5.14 and 5.15 demonstrate how two different pairs of fuzzy sets can result in the same value from a given measure, when ideally a distinction between these pairs is preferred. The incompatibility measure provides a richer comparison of the fuzzy sets and, in both examples, one can distinguish between $c'_p(A, B)$ and $c'_p(A, C)$ where this was not possible with only similarity or only distance.

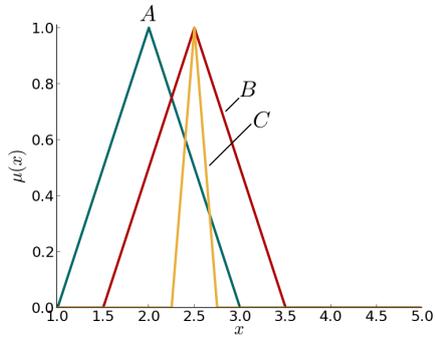


Figure 5.14: Three overlapping fuzzy sets A , B and C

Measure	(A, B)	(A, C)
s_j^{T1} (2.29)	0.389	0.178
d_p^{T1} (3.13)	0.5	0.5
c'_p (5.1)	0.465	0.613

Table 5.13: Results of similarity s_j^{T1} (2.29), distance d_p^{T1} (3.13) and incompatibility c'_p (5.1) measures on the fuzzy sets in Figure 5.14.

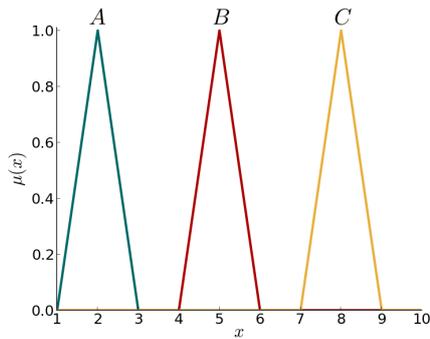


Figure 5.15: Three disjoint fuzzy sets A , B and C

Measure	(A, B)	(A, C)
s_j^{T1} (2.29)	0.0	0.0
d_p^{T1} (3.13)	3.0	6.0
c'_p (5.1)	0.8	0.9

Table 5.14: Results of similarity s_j^{T1} (2.29), distance d_p^{T1} (3.13) and incompatibility c'_p (5.1) measures on the fuzzy sets in Figure 5.15.

5.5.2 Type-2 Fuzzy Sets

The examples shown thus far indicate that neither similarity nor distance can be used as a substitute for each other, however the incompatibility measure can be used to capture both. Though this chapter has focused on only type-1 fuzzy sets, Figures 5.16 and 5.17 demonstrate this with interval type-2 fuzzy sets, and Figures 5.18 and 5.19 show this for general type-2 fuzzy sets.

Figures 5.16 and 5.17 show the uses of similarity and distance in interval type-2 fuzzy sets. Figure 5.16 shows that similarity can distinguish between overlapping fuzzy sets where distance is not always so effective. Figure 5.17 shows how distance is a better measure to analyse the differences between disjoint pairs of fuzzy sets. The incompatibility measure fuses these results and is advantageous because it can distinguish between the different pairs of fuzzy sets in both examples.

The same is also demonstrated for general type-2 fuzzy sets in Figures 5.18 and 5.19. In these examples, the FOUs of the zSlices based general type-2 fuzzy sets are the same as the interval type-2 examples, and four zSlices have been used to represent the secondary membership functions. The centre of the FOU has the highest secondary membership values, and the membership decreases linearly towards the edge of the FOU. Shading is used to highlight this, where darker shades indicate higher secondary memberships.

Note that the same results can be seen in this general type-2 example as seen in the type-1 examples (Figures 5.14 and 5.15) and the interval type-2 examples (Figures 5.16 and 5.17).

More examples of the incompatibility measure being used to capture both similarity and distance are demonstrated on the recommender system proposed in the next chapter. Before this, the next section provides a summary of this chapter.

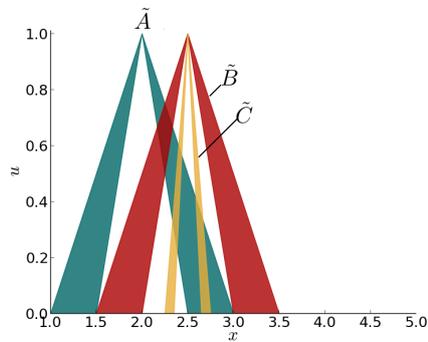


Figure 5.16: Three overlapping fuzzy sets \tilde{A} , \tilde{B} and \tilde{C}

Measure	(\tilde{A}, \tilde{B})	(\tilde{A}, \tilde{C})
$s_j^{IT^2}$ (2.29)	0.293	0.122
$d_p^{IT^2}$ (3.13)	0.5	0.5
c'_p (5.1)	0.532	0.652

Table 5.15: Results of the similarity $s_j^{IT^2}$ (2.29), distance $d_p^{IT^2}$ (3.13) and incompatibility c'_p (5.1) measures on the fuzzy sets in Figure 5.16.

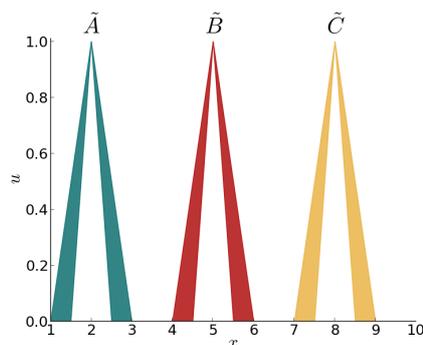


Figure 5.17: Three disjoint fuzzy sets \tilde{A} , \tilde{B} and \tilde{C}

Measure	(\tilde{A}, \tilde{B})	(\tilde{A}, \tilde{C})
$s_j^{IT^2}$ (2.29)	0.0	0.0
$d_p^{IT^2}$ (3.13)	3.0	6.0
c'_p (5.1)	0.8	0.9

Table 5.16: Results of the similarity $s_j^{IT^2}$ (2.29), distance $d_p^{IT^2}$ (3.13) and incompatibility c'_p (5.1) measures on the fuzzy sets in Figure 5.17.

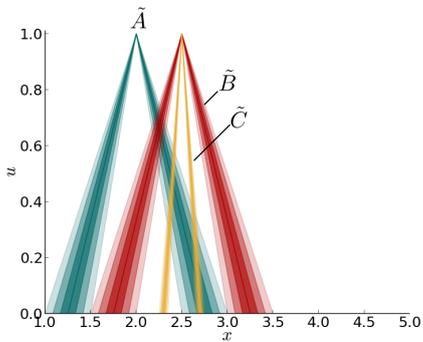


Figure 5.18: Three overlapping fuzzy sets \tilde{A} , \tilde{B} and \tilde{C}

Measure	(\tilde{A}, \tilde{B})	(\tilde{A}, \tilde{C})
s_j^{GT2} (2.29)	0.286	0.121
d_p^{GT2} (3.13)	0.5	0.5
c'_p (5.1)	0.537	0.653

Table 5.17: Results of similarity s_j^{GT2} (2.29), distance d_p^{GT2} (3.13) and incompatibility c'_p (5.1) measures on the fuzzy sets in Figure 5.18.

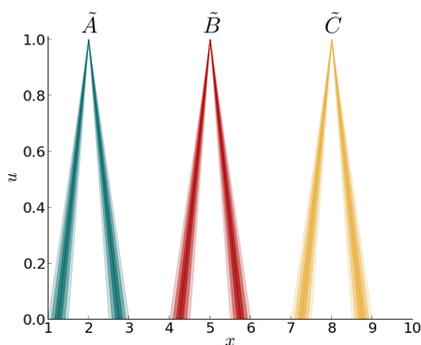


Figure 5.19: Three disjoint fuzzy sets \tilde{A} , \tilde{B} and \tilde{C}

Measure	(\tilde{A}, \tilde{B})	(\tilde{A}, \tilde{C})
s_j^{GT2} (2.29)	0.0	0.0
d_p^{GT2} (3.13)	3.0	6.0
c'_p (5.1)	0.8	0.9

Table 5.18: Results of similarity s_j^{GT2} (2.29), distance d_p^{GT2} (3.13) and incompatibility c'_p (5.1) measures on the fuzzy sets in Figure 5.19.

5.6 Summary

This chapter has introduced an incompatibility measure that is a combination of a similarity and a directional distance measure on fuzzy sets. Using this, one does not have to choose if similarity or distance is best for a given application, and may instead use both to gain the advantages of both. As this measure fuses the *results* of similarity and distance measures, it can be used to compare type-1, interval type-2 or general type-2 fuzzy sets, where the original similarity and distance measures are for the corresponding type of fuzzy set.

In addition, using multiple measures helps alleviate ambiguity in comparing fuzzy sets. It is common for a measure to give identical results for different pairs of fuzzy sets, where different results may be preferred (see Section 5.2). Joining together the outputs of multiple measures helps to ensure that unique and useful results are calculated in such cases (demonstrated in Section 5.5). Although the incompatibility measure can also potentially produce identical values for different pairs of fuzzy sets, the likelihood is much lower than if only a single measure is used.

An ordered weighted average (OWA) operator is used to join similarity and distance together. An OWA is used instead of the standard average operator because using the same weights for both measures may produce unexpected results for heavily overlapping fuzzy sets; this was demonstrated in Section 5.4. Additionally, by using an OWA operator, one can tune the weights to best fit the given fuzzy sets. Whilst the selection of ideal weights is often narrow when comparing overlapping fuzzy sets, there is a wider selection of appropriate weights that may be used to compare disjoint, or nearly disjoint, fuzzy sets. This was also demonstrated in Section 5.4.

By joining similarity and distance, the proposed measure can be suitable to applications that would typically use only similarity or only distance. By tuning or learning ideal weights, it can be utilised for many different applications

and data sets.

Note, however, that the incompatibility measure is not suitable for all applications that would use similarity or distance. For example, overlapping is a common property of similarity that is useful in clustering and classification. The incompatibility measure, however, does not have this property and thus may not be suitable for such applications. Also, though demonstrations have focused on the application of a directional incompatibility measure, one can use a standard distance metric to gain an incompatibility metric that is not directional.

The concept of measuring *compatibility* between fuzzy sets has appeared many times in the literature [49, 52, 53] and *incompatibility* has been used to compare intuitionistic fuzzy sets [111]. However, the current literature does not measure incompatibility as a concept of fusing unique similarity and distance measures. This is a key contribution of this thesis, providing a unique comparison of fuzzy sets where a single measure of comparison may not be sufficient. In addition, this method of fusing different measures may be utilised outside of the field of fuzzy sets as distance and similarity are used within a multitude of non-fuzzy contexts.

The next chapter presents the use of the incompatibility measure applied to find knowledge based recommendations.

Chapter 6

Finding Recommendations for Subjective Information

6.1 Introduction

This chapter develops a fuzzy knowledge based recommendation system that uses the proposed incompatibility measure c'_p (5.1) to find recommendations for products based on subjective information.

Consider a person who is looking for a particular type of cake. They know a number of cakes a shop sells but would like something different that they have not tried. For example, the person may ask for *a cake sweet and nutty like this one but more fruity and crumbly*. This description is detailed (many attributes are described) and subjective (e.g., different people may have different perceptions on how well a cake can be described as *crumbly*). Generally, some human interaction would be required to find a product that matches this detailed description. This type of recommendation is known as a knowledge based recommendation, as it requires detailed knowledge about the products. This chapter introduces a method of automating such recommendations.

A knowledge based recommendation system (described in Section 2.6) is a

type of system that attempts to find products based on the parameters set by the user. For example, a person may search for *a film like Star Trek but with more action*. A list of recommended films can then be generated by comparing films to Star Trek, and selecting ones that have been described in a similar way for all attributes except *action* and have been described as being an action film with more certainty than Star Trek [102].

Such descriptions of products are often uncertain in nature, for example, one person may find a film very funny whilst another person may think it is not at all funny. Such subjectivity can be captured using fuzzy sets. Based on this idea, this chapter develops a fuzzy knowledge based recommendation system that represents product attributes (e.g., how much a film is funny or how much a cake is fruity) using fuzzy sets. Users describe what they want based on a relative comparison with another known product, and the incompatibility measure c'_p is used to compare products and then rank them according to how well they fit the user's description.

The remainder of this section first introduces the structure of the proposed recommendation system followed by methods of finding products that match what a user wants. After this, demonstrations of the proposed system are given using synthetic data with ground truth.

6.2 The Structure of the Proposed System

This section discusses the structure of the proposed knowledge based recommendation system where the knowledge of products is represented by a set of attributes. The descriptions of each product's attributes are modelled using fuzzy sets which may be any type (type-1 or type-2) and may be non-normal and/or non-convex. Using examples, this section shows how queries can be broken down into sub-queries, and discusses how the context of the data affects the interpretations of the queries.

A ranked list of recommended products is given as the result of a consumer's query. Each item is not only represented by a rank position (e.g., first, second, etc.), but also has a real value that indicates its score within the range $[-1, 1]$. A positive value in $[0, 1]$ is given when a product is a good recommendation where 1 is the highest score and a negative value occurs when a product should not be recommended.

Using a real value in $[-1, 1]$ provides more information than only giving rank positions. For example, consider the ranked products A , B , C , D and E in Figure 6.1. Product A is scored 0.9, B is scored 0.85 and C is scored 0.2, thus it is clear that A and B are almost equally good recommendations, and there is much less confidence in recommending product C . If, however, only a rank order of recommendations is used, i.e., first: A , second: B and third: C , then this information is lost.

Additionally, D is ranked -0.1 and E is ranked -0.7 which shows that not only are D and E both poor recommendations, E is much worse than D . With this information, if one wishes, product D may be recommended if there are few positive recommendations given that it is only slightly worse, however, E is unlikely to be recommended because it has such a low score. Note that utilising negative rank values is not covered in this chapter but may be considered for future work.

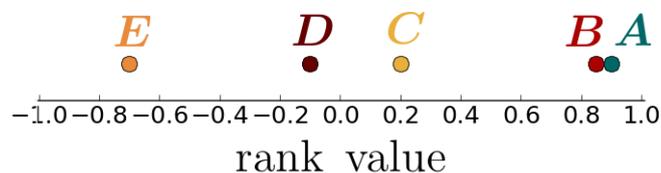


Figure 6.1: Example of products given rank values on the scale $[-1, 1]$.

When designing a recommendation system, it is important to take into account the context of what is being recommended. For example, consider two different recommendation systems, one enables people to find their ideal

hotel, and the other is used to find a person's preferred cake.

When searching for an ideal hotel, the preferred ratings may be implicitly known. The best hotel is the one in which the rooms, value, service, etc. are rated the highest. Also, people generally prefer to find the cheapest hotel that matches their needs. Thus, in this example, the preferred direction of a given attribute when comparing products can be implied without the user having to explicitly state that, for example, higher rated rooms are always preferred. Note that this is a simple example in which more subjective attributes, such as hotel style, have been excluded. This type of recommendation system will be referred to as an *implicit preference recommender*.

However, the direction of preferred ratings cannot be implied when comparing cakes. For example, one person may ask for *a cake like this but more fruity* and another person may ask for *a cake like this but less fruity*. It is clear from this example that the preferred direction when comparing attribute ratings cannot be implied and must be stated explicitly. Therefore, this type of recommendation system will be referred to as an *explicit preference recommender*.

Following on from these examples, the method of choosing which products to recommend should be tailored according to the type of product. First, consider the explicit preference recommender with the example query *find a cake as soft and sweet as this one but more fruity and less salty*. This can be split into two sub-queries. The first sub-query details what the consumer likes about a given product and the second details what they want to be changed.

Sub-query 1: A cake as soft and sweet as this: A product with considerably higher or lower ratings in these attributes does not fit the sub-query and is not desired; i.e., a cake that is rated more/less soft or more/less sweet would deviate from the consumer's preferences. For example, consider a cake where the attribute *sweet* is rated 3 out of 5. Figure 6.2 shows an example of selecting approximately this sweet (i.e., approximately 3) as a crisp range

of values, and a cross indicates the most preferred value (at 3). The further a product's rating is from this value, the less it is preferred as a recommendation.



Figure 6.2: Pictorial description of *roughly 3*.

Sub-query 2: A cake more fruity and less salty than this: In this case, only higher ratings for fruitiness and lower ratings for saltiness fit what the user wants. For example, consider if each attribute is rated 3 out of 5. Figures 6.3 and 6.4 show the crisp range of ratings desired when looking for an attribute that is rated higher/lower, respectively. In both figures, a cross indicates where the most preferred value is (at 5 and 1, respectively). The closer a product's rating is to this preferred value, the more it is preferred as a recommendation.

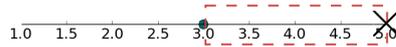


Figure 6.3: Pictorial description of *higher than 3*.



Figure 6.4: Pictorial description of *lower than 3*.

Next, turning to the implicit preference recommender, consider the query *I want a hotel as cheap as this place with rooms about this good, and in a better location*. In this case the user will always desire the attributes of a product to be rated the lowest (e.g., cheapest price) or highest (e.g., best quality) possible. This may also be split into two sub-queries.

Sub-query 1: A hotel as cheap as this place with rooms about this good: In this case, the user likes the rooms of a given hotel and would like to view another that has rooms just as good. Additionally, because higher rated rooms are implicitly preferred, the user would also be happy with a hotel that has better rooms. This sub-query can also be interpreted as *a hotel with rooms about this good or better and about this price or cheaper*.

Consider a hotel where the rooms and price (e.g., how expensive it is considered) are both rated 3 out of 5. Figures 6.5 and 6.6 show the range of ratings desired when looking for an attribute rated *about 3 or higher* and *about 3 or lower*, respectively. In both figures, a cross indicates where the most preferred value is (at 5 and 1, respectively) and the closer a product's rating is to this preferred value, the more preferred it is as a recommendation.



Figure 6.5: Pictorial description of *approximately 3 or higher*.



Figure 6.6: Pictorial description of *approximately 3 or lower*.

A threshold will be used to indicate how much a user is OK with having a product rated slightly worse. For example, consider if an attribute where higher ratings are best is rated 3 out of 5 and a threshold of 0.5 is chosen. This threshold indicates that the user is willing to accept a product rated within the range $[2.5, 5.0]$ for the given attribute.

Note, however, that when a user asks for *approximately this* (as shown in Figure 6.2), a threshold is not required because the interpretation of the sub-query and the position of the most desired value (marked by the cross) affects

how the comparison between products is evaluated. More details on how the interpretation affects the calculation is covered in the next section.

Sub-query 2: A hotel with a better location than this: In this case, the consumer wants location to be rated higher. Lower and similar ratings do not match this. Another example of this type of sub-query is *a hotel cheaper than this*, in which similar and higher ratings for price are not desired and only lower ratings should be recommended. This is the same as sub-query 2 in the explicit preference recommender. Figures 6.3 and 6.4 show the crisp range of ratings desired when looking for an attribute that is rated higher and one that is rated lower, respectively.

In the next section, individual sub-queries are evaluated and the results of the sub-queries are fused, giving each product a score of how well it matches the consumer's query. To calculate how well a product matches these queries, the incompatibility measure c'_p (5.1) can be used to compare products and determine if ratings are close or distant to another (i.e., to find products that are rated similarly or differently), and to determine if a product is rated higher or lower than another.

6.3 Evaluating Queries

This section proposes methods of evaluating the sub-queries in the previous section and then fuses these results to evaluate a query as a whole.

6.3.1 Evaluating Sub-Queries

For each sub-query, the method of comparing products may be approached using the incompatibility measure c'_p . The key difference in each sub-query is how the results of c'_p are interpreted. This section first introduces a general method of comparing products given a list of attributes. After this, this approach is further developed to evaluate the more specific sub-queries. As a

variety of notations and functions are used to describe the proposed system, Table 6.1 provides summary descriptions for quick reference.

A General Method of Comparing Products

Consider a consumer who has found a product, denoted j , which is described by a list of attributes Q , and the consumer wishes to find alternative products by comparing them against j . A measure of how well a product i matches j according to Q can be given as

$$g(j, i, Q) = \frac{\sum_{q \in Q} w_q c'_p(j_q, i_q)}{\sum_{q \in Q} w_q}, \quad (6.1)$$

where i is the new product being evaluated, i_q is the fuzzy set describing attribute q of product i , and c'_p is the incompatibility measure (5.1). The value w_q is the weight given to the attribute q to indicate the relative importance of that attribute where $w_q \geq 0 \forall q \in Q$. If all attributes are equally important then w_q is the same value for each attribute. Note that the result of g (6.1) is always in $[-1, 1]$.

Using weights enables the consumer to describe more specific preferences. For example, if a consumer wants to find *a hotel with better rooms and service than another*, they may specify that better rooms is more important by assigning it a higher weight.

Using g (6.1), a result near -1 indicates that i is rated lower than j (where -1 is the worst rating), a value near 0 indicates that i is similar to j , and a value near 1 illustrates that i is rated higher than j (where 1 is the best rating). Given this, the result of g determines how much product i is rated on average similar, higher or lower than product j when comparing the attributes Q .

The function g can be adjusted to provide results according to more specific sub-queries. The remainder of this section addresses how the sub-queries from the previous section can be evaluated. For each equation, negative or positive

Notation	Description
j	the product being compared against
i	the product being ranked and recommended
Q	the set of attributes for sub-query 1
P	the set of attributes for sub-query 2
V	directions for sub-queries g_{orbetter}^i and g_{diff}^{ie}
w_p	weight assigned to the attribute p
Function	Comparison between products
$*^e$	a function used by the explicit preference recommender
$*^i$	a function used by the implicit preference recommender
g (6.1)	general comparison
g_{orbetter}^i (6.2)	about this good or better
g_{approx}^e (6.3)	similar to this
g_{diff}^{ie} (6.4)	different to this
r^e (6.6)	explicit preference recommendation
r^i (6.5)	implicit preference recommendation

Table 6.1: Descriptions of functions and notations used in the proposed recommendation system.

results indicate that a product respectively slightly fails or slightly meets the given sub-query.

A Product Approximately This Good or Better

Consider an implicit preference recommender in which a consumer is happy with the attributes of a product, but higher or lower ratings are always preferred. For example, a user may ask for *a hotel with rooms and price about this good or better*.

In this case, another hotel with equal ratings for rooms and price can be recommended. Additionally, because the user asks for a hotel *about this good or approximately this good*, ratings that slightly deviate may still be recommended. For example, if a hotel's rooms have been given a rating of 3 then another hotel with rooms rated 2.9 may still be recommended because, although it is lower, it is *around the same value*. A threshold $t \in [0, 1]$ will be used to denote the degree to which slightly worse ratings are acceptable. The value 0 indicates that worse ratings are not wanted and the higher the threshold the more worse ratings are allowed.

Additionally, higher or lower ratings are always implicitly preferred, and so hotels with higher rated rooms and a lower rated price would be even more preferred to hotels rated approximately the same.

Let $V \in \mathcal{P}$ be the set of values $V = \{V_1, \dots, V_{||Q||}\}$ where $||Q||$ is the total number of attributes in Q and each value V_q denotes the assumed direction in which a consumer will want an attribute to change (i.e., higher or lower). V_q is -1 if the consumer will want the attribute q to be rated lower and V_q is 1 if the attribute is to be rated higher. The degree to which a product i is preferred over a product j , where the consumer wants the attributes Q to be *about the same or higher*, or *about the same or lower* as indicated by V can

be evaluated as

$$g_{\text{orbetter}}^i(j, i, Q, V, t) = \begin{cases} g_{\text{orbetter}_{nn}}^i(j, i, Q, V, t) & \text{if } g_{\text{orbetter}_{nn}}^i(j, i, Q, V, t) \geq 0 \\ \frac{g_{\text{orbetter}_{nn}}^i(j, i, Q, V, t)}{1-t} & \text{otherwise} \end{cases} \quad (6.2a)$$

$$g_{\text{orbetter}_{nn}}^i(j, i, Q, V, t) = \frac{\sum_{q \in Q} w_q \min \{V_q(c'_p(j_q, i_q) + t), 1\}}{\sum_{q \in Q} w_q}. \quad (6.2b)$$

Note that $g_{\text{orbetter}_{nn}}^i$ (6.2b) gives a non-normal result in $[-1 + t, 1]$ and (6.2a) normalises this so that one does not have to know the value t to be able to interpret the results.

Within (6.2b), the value V_q is multiplied by the result of c'_p so that c'_p is a positive result if it is in the correct direction (as defined by V_q) and is a negative result otherwise.

A Product Similar to This

Consider an explicit preference recommender in which a consumer has asked for *a product similar to this*; e.g., *a cake as fruity as this one*. In this case, it is desired that the result of g (6.1) be as close to 0 as possible. Although the incompatibility function c'_p , and by extension g , indicates if an item is rated higher or lower than another item, this information is not necessary for this sub-query. Items that are greatly different, whether higher or lower, do not fit the sub-query. Given this, it is ideal to change g to use the absolute result of c'_p , as this makes it easier to interpret the results.

Thus, to determine if item i is similar to item j according to the attributes Q , they may be compared as

$$g_{\text{approx}}^e(j, i, Q) = \frac{\sum_{q \in Q} w_q (-|c'_p(j_q, i_q)|)}{\sum_{q \in Q} w_q}. \quad (6.3)$$

The result is in $[-1, 0]$ where 0 indicates perfect similarity, and -1 indicates no similarity. The value is given within $[-1, 0]$ instead of $[0, 1]$ so that the result

of the sub-query is consistent with all other sub-query results; i.e., a negative value always indicates that a product does not match the sub-query.

A Product Different to This

Here, the sub-query *a product different to this* is addressed. The previous two sections evaluated the first sub-query (in which a person likes something about a product) for implicit and explicit preference recommenders, respectively. This section focuses on the second sub-query (in which a person doesn't like something about a product) for both types of recommendation systems.

To distinguish the attributes of sub-query 1 from those of sub-query 2, let P denote the list of attributes the consumer wishes to be different. Also, let $V \in \mathcal{P}$ be the set of values $V = \{V_1, \dots, V_{||P||}\}$ where $||P||$ is the total number of attributes in P and each value V_p denotes the direction in which a consumer wishes an attribute p to change (i.e., higher or lower). V_p is -1 if the consumer wants the attribute q to be rated lower and V_p is 1 if the attribute is to be rated higher.

The degree to which a product i is preferred over a product j , where the consumer wants the attributes P to be higher or lower according to V can be evaluated as

$$g_{\text{diff}}^{ie}(j, i, P, V) = \frac{\sum_{p \in P} V_p w_p c_p^d(j_p, i_p)}{\sum_{p \in P} w_p} \quad (6.4)$$

This results in a value within $[-1, 1]$. A result close to -1 indicates that the attributes of j are not different to i according the consumer's desires. For example, if a consumer wants a cake less salty than j , then a negative value from g_{diff}^{ie} (6.4) indicates that i is more salty. A result close to 1 , however, indicates that i matches the consumer's sub-query and is, in this case, less salty. A result near 0 indicates that the products i and j are similar in terms of the attributes P .

Note that if $V_p = 1 \forall p \in P$ then g_{diff}^{ie} (6.4) is the same as the basic com-

Query	Eq.
I want a hotel	
with rooms approximately as good as this or better	g_{orbetter}^i (6.2)
and in a better location	g_{diff}^{ie} (6.4)
I want a cake	
as soft and sweet as this	g_{approx}^e (6.3)
and more fruity and less salty	g_{diff}^{ie} (6.4)

Table 6.2: Examples of sub-queries for different categories of recommendation systems and the equations that can be used to evaluate them.

parison g (6.1).

6.3.2 Joining Sub-Queries

Using the equations given in the previous section, it is possible to assign a score of each product according to each of the given sub-queries. Using the examples given in Section 6.2, Table 6.2 shows which equation is used for each sub-query. This section discusses how these can be used together to solve whole queries. As earlier, both *implicit preference* and *explicit preference* recommenders are considered.

Implicit Preference Recommendations

The functions g_{orbetter}^i and g_{diff}^{ie} each give a value in $[-1, 1]$ that describes how much a product matches two given sub-queries. Negative and positive values represent poor and good recommendations, respectively. Using these two sub-

queries, the average recommendation score may be given as

$$r^i(j, i, P, Q, V, t) = \frac{1}{2}(g_{\text{orbetter}}^i(j, i, Q) + g_{\text{diff}}^{ie}(j, i, P, V, t)). \quad (6.5)$$

This gives a value within $[-1, 1]$ and indicates how well a product fits both of the consumer's sub-queries. A positive value indicates a good recommendation, where the value 1 is the best rating and a negative value occurs when a product does not match one or both sub-queries.

By using this method to recommend products, a product may result in a positive value even if one of the sub-queries (from g_{orbetter}^i (6.2) or g_{diff}^{ie} (6.4)) results in a negative value. This is treated as an acceptable compromise in which a product only has a negative result if its undesired change in ratings from one of the sub-queries outweighs the desired changes from the other. Additionally, using r^i (6.5), the result is generally, though not necessarily, higher if both g_{orbetter}^i and g_{diff}^{ie} give positive results.

If one wishes to change the results such that negative values have a higher impact then weights could be applied to r^i to achieve this effect. For example, an ordered weighted average operator may be used to apply higher weights to lower results, causing negative results to have a higher impact on the final score than positive results.

Explicit Preference Recommendations

In an *explicit preference* recommender, different people have different preferred ratings, and as such the results of the sub-queries (g_{approx}^e (6.3) and g_{diff}^{ie} (6.4)) must be joined differently than for an *implicit preference* recommender. g_{approx}^e (6.3) shows how well a product meets the first sub-query in $[-1, 0]$ and g_{diff}^{ie} results in the interval $[-1, 1]$. To evaluate a recommendation score, the value of sub-query 1 is added to sub-query 2 as

$$r^e(j, i, P, Q, V) = \max \{-1, g_{\text{approx}}^e(j, i, Q) + g_{\text{diff}}^{ie}(j, i, P, V)\}. \quad (6.6)$$

Calculating the final result in this way ensures that a positive recommendation only occurs when the desired improvements of a product (according to g_{diff}^{ie}) outweigh any undesired differences in the attributes which the consumer wishes to be similar (according to g_{approx}^e). This ensures that a product is only recommended if the greatest change in attributes between products is where the consumer desires change.

The result from r^e (6.6) is within $[-1, 1]$ where a positive value indicates a good recommendation and a negative value indicates a poor recommendation. Although $g_{\text{approx}}^e + g_{\text{diff}}^{ie}$ would be within $[-2, 1]$, information about rank values in $[-2, -1)$ are not necessarily useful because negative values are simply used to determine what is not worth recommending and it may be sufficient to simplify the results by normalising them. For this reason, r^e restricts $g_{\text{approx}}^e + g_{\text{diff}}^{ie}$ to the interval $[-1, 1]$.

One could also argue that any negative result is not worth recommending and that restricting the values to $[0, 1]$ and disregarding all others may be sufficient. However, negative values are shown in the proposed system so that they may be used for decision making in future recommendations.

Having developed the proposed recommendation systems, the next section presents demonstrations using a synthetic data set that has ground truth. First an example of the explicit preference recommender is given, followed by an example of the implicit preference recommender.

6.4 Synthetic, Ground Truth Demonstrations

This section provides simple demonstrations of the proposed system where the knowledge base contains normal, convex, type-1 fuzzy sets that describe polygons. By using a simple example such as this, there is ground truth to the examples because there is no subjectivity. Thus, the recommendation process is easier to follow and one can judge what should be the expected results. Note that Chapter 7, however, demonstrates the proposed system using subjective data collected from surveys.

As well as demonstrating the recommendation process, this section shows why it is important to use the incompatibility measure by demonstrating that similarity or distance alone cannot provide useful recommendations.

6.4.1 Explicit Preference Ground Truth Examples

Using The Proposed System with the Incompatibility Measure

Figure 6.7 shows six polygons with three to eight sides, all of which have a perimeter of 10cm. It is well known that given the same perimeter, as the number of sides of a polygon increases, its area increases. Table 6.3 shows the areas of each polygon in Figure 6.7.

In order to demonstrate the proposed recommendation system on fuzzy sets, Figure 6.8 shows fuzzy sets that approximate the number of sides and areas of the polygons. Note that whilst the number of sides and the size of the areas are of course precise, fuzzy values have been used to demonstrate that the results are as expected in such a straightforward example. Each fuzzy set has a Gaussian membership function. In Figure 6.8a, the mean value is at the number of sides and the standard deviation is 1. In Figure 6.8b, the mean value is at the area of the polygon, and the standard deviation is 0.1.

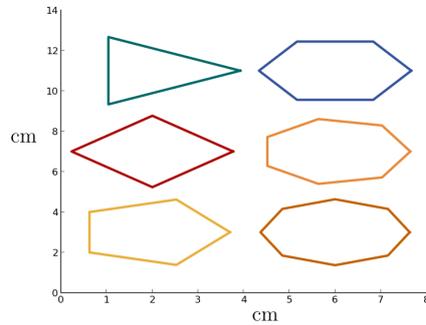


Figure 6.7: Regular polygons from 3 to 8 sides with a perimeter length of 10cm.

shape	area cm ²
triangle	4.8113
square	6.25
pentagon	6.8819
hexagon	7.2169
heptagon	7.4161
octagon	7.5444

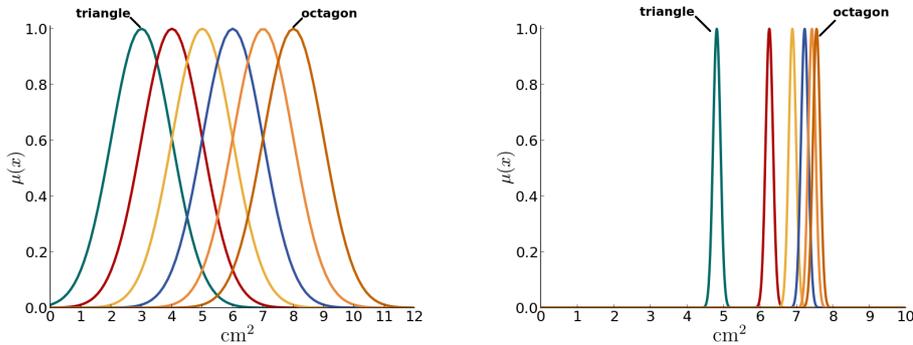
Table 6.3: Area of regular polygons with a perimeter of length 10cm.

These standard deviations have been chosen because they provide some small overlap between the fuzzy sets whilst ensuring they are still distinguishable.

The remainder of this section demonstrates the process of finding

*a polygon with a similar number of sides to a hexagon
but with a smaller area.*

First, the incompatibility of each shape's attribute is compared with the hexagon using c'_p (5.1) with weights $\langle w_0 = 0.7, w_1 = 0.3 \rangle$; Table 6.4 shows these results. Note that when comparing the number of sides and area, 121 and 1001



(a) The number of sides of each shape represented as an approximation. (b) The area of each shape represented as an approximation.

Figure 6.8: The number of sides and area of shapes represented as an approximation using fuzzy sets. In each sub-figure, the left-most fuzzy set represents a triangle, and the total number of sides of the polygons increases towards the right, where the right-most fuzzy set represents an octagon.

discretisations, respectively, were used in the x -axis to calculate dissimilarity. A negative result from c'_p denotes that a shape has fewer sides or a smaller area than the hexagon.

To find which shape best matches *a polygon with a similar number of sides to a hexagon, and with a smaller area*, the task is split into two queries

Sub-query 1 A polygon with a similar number of sides to a hexagon.

Sub-query 2 A polygon with a smaller area than a hexagon.

The first sub-query is calculated using g_{approx}^e (6.3) which gives a result in $[-1, 0]$. The value 0 is the best result and the value -1 is the worst result. The second query is evaluated using g_{diff}^{ie} (6.4) where $V_q = -1$ (q represents the attribute *area*) because the query specifies a shape with a *smaller* area and thus negative results from c'_p are desired. This gives a result in $[-1, 1]$ where 1 is the best result and -1 is the worst result.

Table 6.5 shows these results. Note that as only one attribute is used within

shape	sides	area
triangle	-0.725	-0.772
square	-0.618	-0.729
pentagon	-0.413	-0.676
hexagon	0.0	0.0
heptagon	0.413	0.573
octagon	0.618	0.672

Table 6.4: The incompatibility between the number of sides and area of different polygons against a hexagon using c'_p .

each sub-query, the absolute results are the same as those in Table 6.4.

In the case of sub-query 1, all results are negative, representing the degree to which the shape does not have a similar number of sides to a hexagon. Note that in the second sub-query, because $V_q = -1$, the results from c'_p (in Table 6.4) are reversed. Values that had a positive result from c'_p because they are higher valued are now negative because higher values are not desired.

Table 6.5 also shows the final results joining the sub-queries together using r^e (6.6). Positive valued results indicate ‘good’ recommendations and negative results are ‘bad’ recommendations. This table shows that the pentagon is the best result. Intuitively, this is expected because it has the most similar number of sides to a hexagon, whilst also having a smaller area.

The square and triangle are, respectively, the next best recommendations. Although they fit the second sub-query better than a pentagon (i.e., have a smaller area) they are a worse fit for the first sub-query (i.e., they have fewer sides). As a result, they are not as well recommended. The heptagon and octagon do not fit the second sub-query at all, and therefore should not be recommended.

shape	sub-query 1 g_{approx}^e	sub-query 2 g_{diff}^{ie}	result r^e	rank
triangle	-0.725	0.772	0.047	3
square	-0.618	0.729	0.111	2
pentagon	-0.413	0.676	0.263	1
hexagon	0.0	0.0	0.0	-
heptagon	-0.413	-0.573	-0.986	4
octagon	-0.618	-0.672	-1.0	5

Table 6.5: Ranking polygons against a hexagon for less area and similar number of sides using the incompatibility measure c'_p .

This synthetic example has shown that the results of the proposed recommendation system using the incompatibility measure c'_p produces values that match what is intuitively expected.

Using Separate Similarity and Distance Measures

This next demonstration shows the importance of using the same measure to evaluate both sub-queries. Judging by the nature of each sub-query, one may assume that a similarity and distance measure could be used instead, i.e.,

Similarity A polygon with a similar number of sides to a hexagon.

Distance A polygon with a smaller area than a hexagon.

However, the following demonstrates that the results cannot be fused if the sub-queries are calculated using different techniques. Table 6.6 shows the results of the recommendation process where c'_p is replaced with the dissimilarity $(1 - s_j^{T1})$ and the normalised directional distance $(\frac{d_p(A,B)}{\tau(X)})$ measures.

One can clearly see from these results that the ranked recommendations are not useful. Each shape has a negative result, indicating that there are no

shape	sub-query 1 g_{approx}^e	sub-query 2 g_{diff}^{ie}	result r^e	rank
triangle	-0.928	0.2406	-0.6874	3
square	-0.811	0.0967	-0.7143	4
pentagon	-0.554	0.0335	-0.5205	1
hexagon	0.0	0.0	0	-
heptagon	-0.554	-0.0199	-0.5739	2
octagon	-0.811	-0.0327	-0.8437	5

Table 6.6: Ranking polygons against a hexagon for less area and a similar number of sides using the dissimilarity s' and normalised distance d_n measures in sub-queries 1 and 2, respectively.

good recommendations. Additionally, although the pentagon is still the highest valued result, the heptagon is now the second highest. This is undesired as it does not fit the second sub-query (i.e., its area is higher than a hexagon).

This demonstration shows that to ensure meaningful, intuitive results, it is important to use the same approach to evaluate each sub-query. Chapter 5 shows that the incompatibility measure c'_p can effectively assess both similarity and distance between fuzzy sets. Thus, c'_p will be used to calculate all sub-queries in the recommendation system.

Using Only Distance or Similarity

As discussed in Chapter 5, distance is not a useful substitute for similarity, nor is similarity a useful substitute for distance. This section provides a clearer example of this by demonstrating using only the directional distance measure d_p^{T1} (3.13) to evaluate recommendations. Using the same query as the previous two sections, Table 6.7 shows the results when only the normalised distance is measured. Note that, just as in previous examples, the results from sub-query

shape	sub-query 1 g_{approx}^e	sub-query 2 g_{diff}^{ie}	result r^e	rank
triangle	-0.25	0.2406	-0.0094	1
square	-0.1667	0.0967	-0.07	3
pentagon	-0.0833	0.0335	-0.0498	2
hexagon	0.0	0.0	0	-
heptagon	-0.0833	-0.0199	-0.1032	4
octagon	-0.1667	-0.0327	-0.1994	5

Table 6.7: Ranking polygons against a hexagon for less area and a similar number of sides using the normalised distance d_n measure.

1 ignore direction and are all negative values.

Just as the example in the previous section (in Table 6.6) each shape has a negative result and it is thus impossible to discern between good and bad recommendations. The rank orders are also not what one would expect. Although the top three ranks contain the same polygons as when using the incompatibility measure (in Table 6.5), the ordering is different and counterintuitive. The triangle fits the first sub-query the least well (i.e., a similar number of sides to a hexagon), yet it is the mostly highly recommended shape.

This demonstration shows that distance cannot be used as a substitute for similarity. Note, also, that similarity is not a substitute for distance because it is non-directional and thus cannot determine the relative positions between pairs of fuzzy sets. It also cannot determine the magnitude of distance between disjoint fuzzy sets.

6.4.2 Implicit Preference Ground Truth

Examples

Using the same fuzzy sets as the previous section, this section presents an example of the proposed implicit preference recommendation system r^i . In this example, the task is to resolve the query *a polygon with an area approximately as large or larger than a hexagon and with a greater number of sides*. This is split into two sub-queries:

Sub-query 1 A polygon with an area approximately as large or larger than a hexagon.

Sub-query 2 A polygon with a greater number of sides than a hexagon.

Table 6.8 shows the results of this query using the incompatibility measure c'_p . Note, the threshold $t = 0.1$ is used for the first sub-query. It is clear that the results are what one would expect. The shapes with larger areas and a greater number of sides than a hexagon have positive values, whereas shapes that fail both of these criteria have negative results. The rank order is also what one would expect.

Table 6.9 also shows the results if the normalised directional distance measure d_p^{T1} is used instead of c'_p . This table shows that while the values of the results have changed, the signs (positive/negative) are the same and the rank order is also the same.

This demonstrates that in implicit preference recommender systems, using only a directional distance measure is sufficient and one may not need to use the incompatibility measure. Note, however, the previous section demonstrated that in explicit preference recommender systems, it is necessary to use the incompatibility measure as directional distance alone cannot provide the correct results.

shape	sub-query 1 g_{orbetter}^i	sub-query 2 g_{diff}^{ie}	result r^i	rank
triangle	-0.7467	-0.725	-0.7359	5
square	-0.6989	-0.618	-0.6584	4
pentagon	-0.64	-0.412	-0.526	3
hexagon	0.1	0.0	0.05	-
heptagon	0.673	0.412	0.5425	2
octagon	0.772	0.618	0.695	1

Table 6.8: Ranking polygons against a hexagon for more area and more sides using the incompatibility measure c'_p .

Having presented the proposed recommender using synthetic, ground-truth examples, the next section provides a summary of this chapter.

6.5 Summary

This chapter has demonstrated a fuzzy knowledge based recommendation system with which a person may describe their ideal product in relation to another product. Two different types of data/recommenders were explored. These have been referred to as *implicit preference* and *explicit preference* recommenders. The former describes data in which the preferred ratings for a product are commonly implicitly known for all attributes. In this case, the subjectivity of a product stems from peoples' perceptions/ratings of that product. In the latter case however, the preferred ratings cannot be assumed. In this case, there is not only subjectivity in the attribute rating of products, but also with regards to what ratings are desired.

This chapter has introduced a clear distinction between implicit preferences (what we can assume the consumer likes) and explicit preferences (what we

shape	sub-query 1 g_{orbetter}^i	sub-query 2 g_{diff}^{ie}	result r^i	rank
triangle	-0.1562	-0.25	-0.2031	5
square	-0.0033	-0.1667	-0.0817	4
pentagon	-0.066	-0.0833	-0.0084	3
hexagon	0.0	0.0	0.05	-
heptagon	0.1199	0.0833	0.1016	2
octagon	0.1327	0.1667	0.1497	1

Table 6.9: Ranking polygons against a hexagon for more area and more sides using the directional distance measure d_p^{T1} .

cannot assume). Explicit preferences have also been explored elsewhere in the literature. Typically, explicit knowledge is acquired by querying the user through questionnaires, for example rating products on a Likert scale [112, 113]. The concept of *implicit* preferences, however, is less well-defined. In some cases, experiments using word associations are used to determine if a user associates a product in a positive or negative manner [112]. Another technique is to monitor user activities to gain implicit preferences. For example, if a user visits a website or listens to a song frequently then it is assumed that the user likes the given website or song [113].

However, this chapter has introduced a different idea of implicit preferences, where no information is collected from users, and instead it is assumed that all consumers will have the same preferences. Care must of course be made in such generalisations, and so this idea of implicit preferences is best used where the given preference is obvious; for example, it's likely that everyone will prefer the restaurant rated as having the most delicious food.

Examples of the recommendation system were given using simple synthetic fuzzy sets with ground truth. The synthetic examples demonstrated that the

recommendation system gives intuitive results when using the proposed incompatibility measure developed in Chapter 5. The next chapter demonstrates the implicit and explicit preference recommenders using real-world based data-driven fuzzy sets that are type-1, interval type-2 and general type-2.

Chapter 7

Demonstrating Recommendations on Subjective Data-Driven Fuzzy Sets

7.1 Introduction

This section demonstrates the proposed fuzzy knowledge based recommendation system (described in Chapter 6) using real-world data-driven type-1 and type-2 fuzzy sets. First, Section 7.2 gives examples of the *implicit preference recommender* using survey data. The goal is to make recommendations based on multiple ratings in a similar domain. Specifically, the section focuses on the example of hotel recommendations as are vital in online booking sites. In this example, the fuzzy sets, constructed from customer ratings, represent the quality of multiple attributes of multiple hotels. Using this recommender, one might, for example, wish to find recommendations based on *a hotel with rooms about as good as this one but with a better location*.

After this, Section 7.3 demonstrates the *explicit preference recommender* in which a person can state how similar a given item is to their preferred

item. Demonstrations are based on collected survey data in which people have described different attributes of cakes by, for example, rating how sweet or fruity a cake is on a scale from 0 to 10. Recommendations are then based on the descriptions of the cakes, for example, find *a cake sweet and nutty like this one but more fruity*.

7.2 Data Driven Implicit Preference Demonstrations

This section demonstrates the implicit preference recommender using a data set in which participants rate various attributes of different hotels. Recall that in an *implicit preference recommender*, the preferred ratings of a product are implicitly known and do not have to be stated by the consumer. For example, if a person is trying to find a hotel, one can assume they would prefer the lowest priced hotel with the best rated rooms, and will never want the highest priced place with low rated rooms. Therefore, the direction of change preferred for each attribute does not need to be stated by the consumer.

7.2.1 Data Set

To demonstrate the proposed implicit preference recommendation approach, a TripAdvisor® data set [114, 115] is used, which contains reviews of many hotels across the world. Within the TripAdvisor® data set, users may rate hotels according to the attributes *service, cleanliness, business service (e.g., Internet access), check in / front desk, value, sleep quality, rooms, location, and overall*.

When a user reviews a hotel they are able to rate it according to how well they felt it performed in each of the given attributes. Each rating is given as a value in $\{1, 2, \dots, 5\}$. For the purpose of this thesis, fuzzy sets describing

the attributes of the hotels are all constructed using the same technique as when data is collected using the polling technique with linear interpolation (described in Section 2.3.1). As a result, all of the fuzzy sets are non-normal (as there is no complete agreement for any hotel attribute) and many of the fuzzy sets are non-convex.

Note that it is not necessary for users to rate each attribute; if they wish they may give only an overall rating. As a result, there are many hotels for which there is no data for some attributes. However, every hotel within the data set has received at least one *overall* rating. Given this, when comparing hotels, if there is no information about a given attribute for a hotel, the *overall* ratings are compared instead as a substitute.

When searching for a hotel, the user typically knows the location they wish to stay and a price range which they are willing to pay. Given this, the demonstrations within this section assume the user is searching for a hotel located within New York City with a price-range of \$100 - \$200 per night. Note that even this subset of hotels is too large to adequately discuss, so only a smaller subset of the data is used within these demonstrations.

Recall that, when assigning scores for each hotel, both sub-queries give a value in $[-1, 1]$ representing how differently an attribute has been rated and whether the change in that attribute is in the desired direction (positive and negative values indicate desired and undesired directions, respectively). The first sub-query also uses a threshold t which represents how much the person is willing to accept slightly worse ratings. Within these demonstrations $t = 0.1$. Note that because of the threshold, any result between 0.0 and 0.05 is only a positive recommendation because this threshold has increased the hotel's score.

Within these demonstrations, due to the very subnormal nature of the fuzzy sets, the sample size of α -cuts is increased to 40 α -cuts in order to gain an adequate amount of comparisons. If, however, only 10 α -cuts are

measured then the results may be inaccurate because using fewer α -cuts may not accurately capture the shapes of the fuzzy sets.

7.2.2 Type-1 Fuzzy Sets

Consider if we search for *a hotel with cleanliness about as good as Park Savoy and with better service*. Table 7.1 shows the results in rank order, breaking this down into its two sub-queries. These are

Sub-query 1 A hotel with cleanliness about as good as Park Savoy

Sub-query 2 A hotel with better service than Park Savoy

For ease of presentation, this section focuses on the results of four highlighted hotels; these are *Park 79 Hotel*, *The Amsterdam Inn*, *Riverside Tower* and *Hotel Carter*. Figure 7.1 shows the fuzzy sets describing each of these four hotels' *service* compared against *Park Savoy*. This will be used as a basis for comparison in the next examples.

Referring to Figure 7.1, *Park 79 Hotel* and *Riverside Tower* both have similar positive incompatibilities with *Park Savoy* because they have higher membership at higher ratings (where $x \geq 4$). *Amsterdam Inn* has a lower incompatibility because the membership of ratings decreases from $x = 4$ to $x = 5$. *Hotel Carter* has a large negative incompatibility because its greatest membership is where $x = 1$.

Next, Table 7.2 shows the rank ordered results of searching for *a hotel with cleanliness about as good as Park Savoy and with better service and rooms*. Figure 7.2 shows the fuzzy sets describing each hotels' rooms and their incompatibility.

Referring to this figure, *Park 79 Hotel* and *Amsterdam Inn* have similarly shaped membership functions to *Park Savoy* and both have low negative incompatibility. *Riverside Tower* has a greater membership for $x = 1$ and so

has a higher negative incompatibility. The value $x = 1$ has an even higher membership for *Hotel Carter*, which, as a result, has an even higher negative incompatibility.

As a result of the negative incompatibility for each hotel, when measuring both their *service* and *rooms* against those of *Park Savoy*, they are all rated worse than in the previous demonstration in Table 7.1 (in which only *service* was considered). Whilst Hotel Carter remains the worst recommendation, all other highlighted hotels have changed rank.

As each hotel has a negative incompatibility when comparing rooms, each hotel has a smaller result in sub-query 2 in Table 7.2. As a result, Park 79 Hotel is now ranked 5th when it was 2nd in Table 7.1. The result of The Amsterdam Inn is now close to 0 and is only a positive recommendation because of the threshold t (where $t = 0.1$ indicates that a negative compatibility of up to -0.1 in sub-query 1 is acceptable). Additionally, Riverside Tower is no longer recommended because the combined difference in its rooms and service no longer outweighs its decreased ratings in cleanliness.

The results from Table 7.1 to Table 7.2 noticeably decreased for most hotels because their rooms are rated lower than the rooms of Park Savoy. However, if the rooms are given a lower weight then they will have a smaller effect on the results. Table 7.3 shows the rank ordered results of weighting service twice as highly as rooms.

Park 79 Hotel is now in 4th rank position instead of 5th (in Table 7.2), regaining a better recommendation than Chelsea Inn (as it had in Table 7.1). However, Park 79 Hotel is still rated worse than Americana Inn, whereas it is rated better if its rooms aren't compared. Amsterdam Inn has slightly improved from Table 7.2 and it is now a positive recommendation even if the threshold t is set as 0.

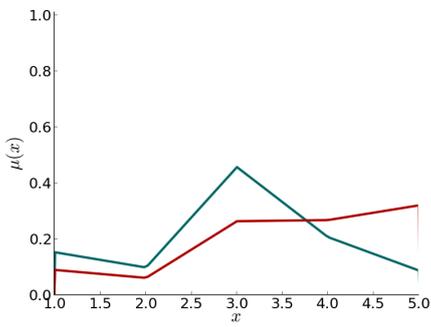
However, the rooms of Riverside Tower are considerably low compared to its service and, as a result, it is still a poor recommendation, though only by

Hotel	Sub-Query 1	Sub-Query 2	Result	Rank
Broadway at Times Sq. Hotel	0.431	0.547	0.489	1
Chelsea Lodge	0.471	0.414	0.4425	2
Park 79 Hotel	0.219	0.385	0.302	3
Americana Inn	0.275	0.321	0.298	4
Chelsea Inn - 17th Street	0.201	0.235	0.218	5
Chelsea Star Hotel	0.019	0.387	0.203	6
The Amsterdam Inn	-0.0011	0.285	0.1419	7
Riverside Tower	-0.2667	0.367	0.0502	8
Latham Hotel	-0.13	-0.176	-0.153	9
Morningside Inn	-0.1489	-0.206	-0.1774	10
Hotel Riverside Studios	-0.59	-0.428	-0.509	11
Hotel Carter	-0.6867	-0.532	-0.6094	12

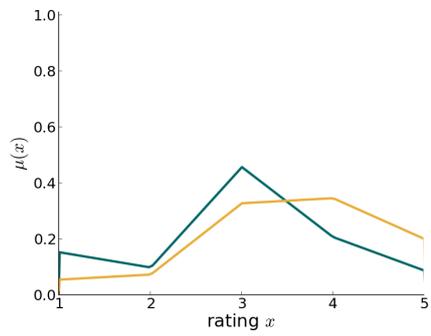
Table 7.1: Query results for finding a hotel with *cleanliness* about as good as Park Savoy and with better *service* listed in rank order.

Hotel	Sub-Query 1	Sub-Query 2	Result	Rank
Broadway at Times Sq. Hotel	0.431	0.4645	0.4477	1
Chelsea Lodge	0.471	0.3995	0.4353	2
Americana Inn	0.275	0.2285	0.2518	3
Chelsea Inn - 17th Street	0.201	0.1855	0.1933	4
Park 79 Hotel	0.219	0.1385	0.1788	5
Chelsea Star Hotel	0.019	0.283	0.151	6
The Amsterdam Inn	-0.0011	0.0715	0.0352	7
Riverside Tower	-0.2667	0.078	-0.0943	8
Latham Hotel	-0.13	-0.1825	-0.1563	9
Morningside Inn	-0.1489	-0.2135	-0.1812	10
Hotel Riverside Studios	-0.59	-0.451	-0.5205	11
Hotel Carter	-0.6867	-0.541	-0.6139	12

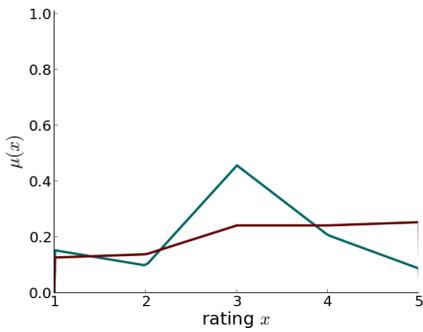
Table 7.2: Query results for finding a hotel with *cleanliness* about as good as Park Savoy but with better *service* and *rooms* listed in rank order.



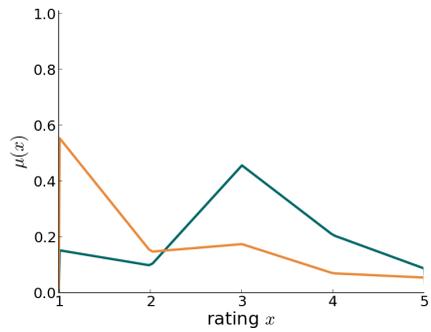
(a) Park 79 $c'_p = 0.385$



(b) The Amsterdam Inn $c'_p = 0.285$

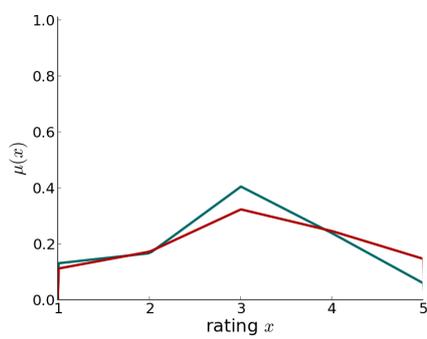


(c) Riverside Tower $c'_p = 0.367$

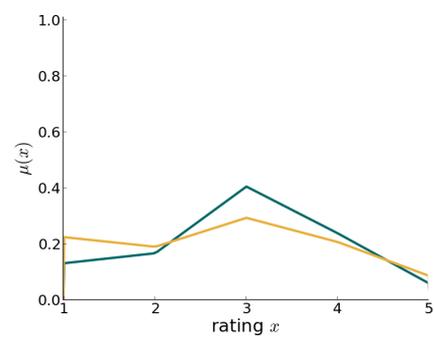


(d) Hotel Carter $c'_p = -0.532$

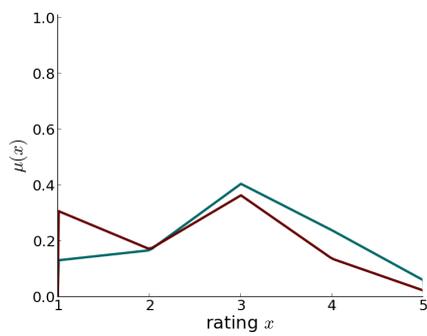
Figure 7.1: Pairs of fuzzy sets representing **Park Savoy** and other hotels (labelled) and their incompatibility (c'_p) for the attribute *service*.



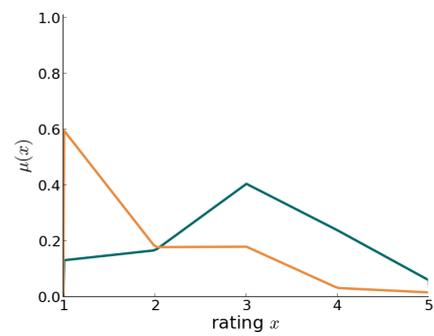
(a) Park 79 $c'_p = -0.108$



(b) The Amsterdam Inn $c'_p = -0.142$



(c) Riverside Tower $c'_p = -0.211$



(d) Hotel Carter $c'_p = -0.55$

Figure 7.2: Pairs of fuzzy sets representing **Park Savoy** and other hotels (labelled) and their incompatibility (c'_p) for the attribute *rooms*.

Hotel	Sub-Query 1	Sub-Query 2	Result	Rank
Broadway at Times Sq. Hotel	0.431	0.4926	0.4618	1
Chelsea Lodge	0.471	0.4044	0.4377	2
Americana Inn	0.275	0.26	0.2675	3
Park 79 Hotel	0.219	0.2223	0.2207	4
Chelsea Inn - 17th Street	0.201	0.2023	0.2016	5
Chelsea Star Hotel	0.019	0.3184	0.1687	6
The Amsterdam Inn	-0.0011	0.1441	0.0715	7
Riverside Tower	-0.2667	0.1763	-0.0452	8
Latham Hotel	-0.13	-0.1803	-0.1552	9
Morningside Inn	-0.1489	-0.211	-0.1799	10
Hotel Riverside Studios	-0.59	-0.4432	-0.5166	11
Hotel Carter	-0.6867	-0.5379	-0.6123	12

Table 7.3: Weighted query results for finding a hotel with at least as good *cleanliness* as Park Savoy but with better *service* ($w_p = 2.0$) and *rooms* ($w_p = 1.0$) listed in rank order.

a small amount. Increasing the threshold from 0.1 to 0.14 would result in Riverside Tower becoming a good, albeit low, recommendation. As expected, Hotel Carter remains the lowest result.

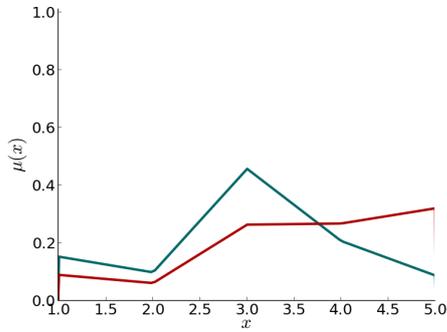
7.2.3 Type-2 Fuzzy Sets

This section provides a brief demonstration of finding hotel recommendations using interval and general type-2 fuzzy sets. One known method of creating type-2 fuzzy sets from type-1 fuzzy sets is to blur (by shifting) the membership function [16, 26]. In these demonstrations, the primary memberships are shifted along the membership axis (vertically) so that each vertical slice is changed from a singleton to a bounded interval centred around the original singleton. For each vertical slice, the interval between the lower and upper membership functions is of the same width. Using this method ensures that the lower and upper membership functions are the same shape as the original type-1 fuzzy sets.

The interval type-2 fuzzy sets have a secondary membership value of 1 throughout this interval. The general type-2 fuzzy sets are given a triangular secondary membership function where the centre of the footprint of uncertainty (where the original type-1 membership function is located) has a secondary membership of 1. Figure 7.3 shows the fuzzy sets representing the service of Park Savoy and Park 79 Hotel together as type-1, interval type-2 and general type-2 fuzzy sets.

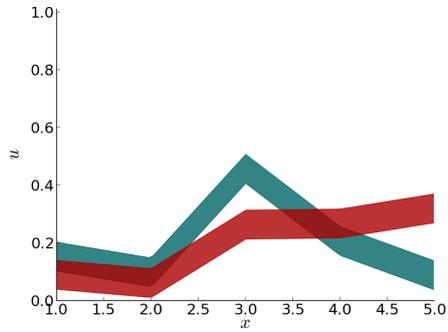
The general type-2 fuzzy set has been split into four zSlices and dark shaded regions within the image represent higher secondary degrees of membership. Four zSlices have been chosen because, as the results show in Table 7.4, the secondary memberships that result from blurring the membership have little effect on the results, and this will be true for any number of zSlices. Increasing the number of zSlices used will not produce significantly different results. This demonstrates that the process of blurring membership functions results in little change within the system that the fuzzy sets are used.

The next demonstrations show the results if one searches for *a hotel with at least as good cleanliness as Park Savoy but with better service*. The rank



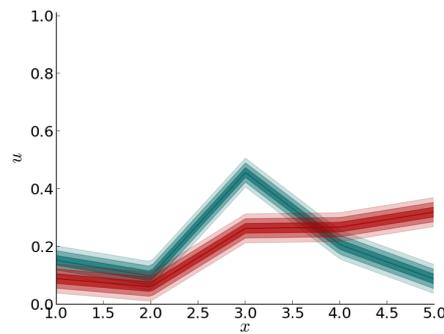
(a) Park Savoy and Park 79; T1;

$$c'_p = 0.385$$



(b) Park Savoy and Park 79; IT2;

$$c'_p = 0.384$$



(c) Park Savoy and Park 79; GT2;

$$c'_p = 0.382$$

Figure 7.3: Pairs of type-1 (T1), interval type-2 (IT2) and general type-2 (GT2) fuzzy sets representing **Park Savoy** and **Park 79 Hotel** and their incompatibility (c'_p) for the attribute *service*.

ordered results are shown for type-1, interval type-2 and general type-2 fuzzy sets in Table 7.4; note that the hotel names have been shortened for space considerations. Each hotel is listed in rank order and so it is clear that each type of fuzzy set produces the same rank order of results. One can also see that the values resulting from each type of fuzzy set are very close. Small changes are due to differing membership values at the chosen discrete points of measurement. An example of this effect was also shown in Chapter 4, Table

4.5, Page 124.

Thus, as one would expect, increasing the uncertainty of the membership values when no new information about the agreement between individuals is known (to create type-2 fuzzy sets) produces similar results to the original type-1 fuzzy sets. The next section, however, demonstrates with a different data set that different membership values can occur between type-1 and type-2 fuzzy sets when the secondary membership functions are constructed differently.

Hotel	Type-1		Interval Type-2		General Type-2		Rank			
	SQ 1	SQ 2	Result	SQ 1	SQ 2	Result		SQ 1	SQ 2	Result
Broadway at TSQ	0.431	0.547	0.489	0.419	0.538	0.4785	0.429	0.544	0.4865	1
Chelsea L.	0.471	0.414	0.4425	0.462	0.409	0.4355	0.47	0.411	0.4405	2
Park 79 H.	0.219	0.385	0.302	0.216	0.384	0.3	0.216	0.382	0.299	3
Americana I.	0.275	0.321	0.298	0.274	0.32	0.297	0.276	0.319	0.2975	4
Chelsea I.	0.201	0.235	0.218	0.193	0.231	0.212	0.2	0.235	0.2175	5
Chelsea S. H.	0.019	0.387	0.203	0.028	0.384	0.206	0.021	0.381	0.201	6
Amsterdam I.	-0.0011	0.285	0.1419	0.007	0.284	0.1455	0.001	0.282	0.1415	7
Riverside Tower	-0.2667	0.367	0.0502	-0.2578	0.364	0.0531	-0.2622	0.36	0.0489	8
Latham	-0.13	-0.176	-0.153	-0.12	-0.176	-0.148	-0.13	-0.178	-0.154	9
Morningside I.	-0.1489	-0.206	-0.1774	-0.1389	-0.205	-0.1719	-0.1456	-0.21	-0.1778	10
H. Riverside S.	-0.59	-0.428	-0.509	-0.5767	-0.427	-0.5019	-0.59	-0.429	-0.5095	11
H. Carter	-0.6867	-0.532	-0.6094	-0.6711	-0.531	-0.6011	-0.6833	-0.531	-0.6072	12

Table 7.4: Query results for finding a hotel with *cleanliness* about as good as Park Savoy but with better *service* using type-1, interval type-2 and general type-2 fuzzy sets. Results are listed in rank order.

7.3 Data-Driven Explicit Preference

Demonstrations

This section demonstrates the explicit preference recommender using a data set in which participants rate various attributes of different cakes. Recall that in an *explicit preference recommender* the consumer's preferences cannot be assumed and must be explicitly stated. For example, if a person is trying to find a cake they like, it cannot be assumed how sweet or fruity they would like that cake to taste. Therefore, the direction of change preferred for each attribute must be explicitly stated by the consumer.

7.3.1 Data Set

In a survey conducted within and ethically approved by the University of Nottingham, six different types of cake were surveyed by participants who were asked to eat a piece of cake whilst answering questions such as “how sweet is this cake” and “how tasty is this cake”. Their answers were given in intervals within the range $[0, 10]$. The aim of this survey was to see how different people perceive the same things differently and how this information can be modelled and utilised. This data has been constructed into type-1 and type-2 fuzzy sets using the interval agreement approach (IAA) (described in Section 2.3). Each cake is referred to by a letter in $\{A, B, C, D, E, F\}$.

To create type-1 fuzzy sets from the data, the type-1 IAA ((2.18) on Page 31) is applied to all of the data. To construct type-2 fuzzy sets, the data is split into four different classes that capture how often the survey participants consume cake; these are

1. On special occasions
2. About once or twice a month

3. About once a week

4. Several times a week

Note that the data set also contains two other categories (*never* and *every day*), but there is so little data in these groups such that only singletons (crisp values) can be built.

Using these four subsets, four type-1 fuzzy sets are built using the IAA (2.18). To generate an interval type-2 fuzzy set, the intersection and union of the four type-1 fuzzy sets are used as the lower and upper membership functions, respectively. This method is chosen so that the footprint of uncertainty represents the range of certainties within the type-1 fuzzy sets. General (zSlices-based) type-2 fuzzy sets are constructed using the general type-2 IAA (2.20), as detailed in Section 2.3.3, Page 34.

Figure 7.4 shows the attribute *tasty* of cake *A* represented by type-1, interval type-2 and general type-2 fuzzy sets using these methods. Note that in Figure 7.4 some of the upper membership values increase in the type-2 fuzzy sets compared to the type-1 fuzzy set. This is because splitting the data into four categories changes the certainties of the type-1 fuzzy sets compared to if the data is all used as one category. Additionally, the lower membership values of the interval and general type-2 fuzzy sets are often 0 throughout the entire universe of discourse as a result of disagreement between survey participants.

As a result of the variety of answers given in the survey, each fuzzy set is non-normal and non-convex. In fact, as shown in Figure 7.4, the fuzzy sets often have spikes at the discrete points 0, 1, ..., 10. This is because although participants gave answers in continuous intervals, they each treated the ends of the intervals in these discrete terms. As a result, where two intervals share the same end point (e.g., [5, 6] and [6, 7]) the value that they share (in this case 6) has a much greater membership than other values within the intervals.

Note that, as with the previous section, to increase the accuracy of the

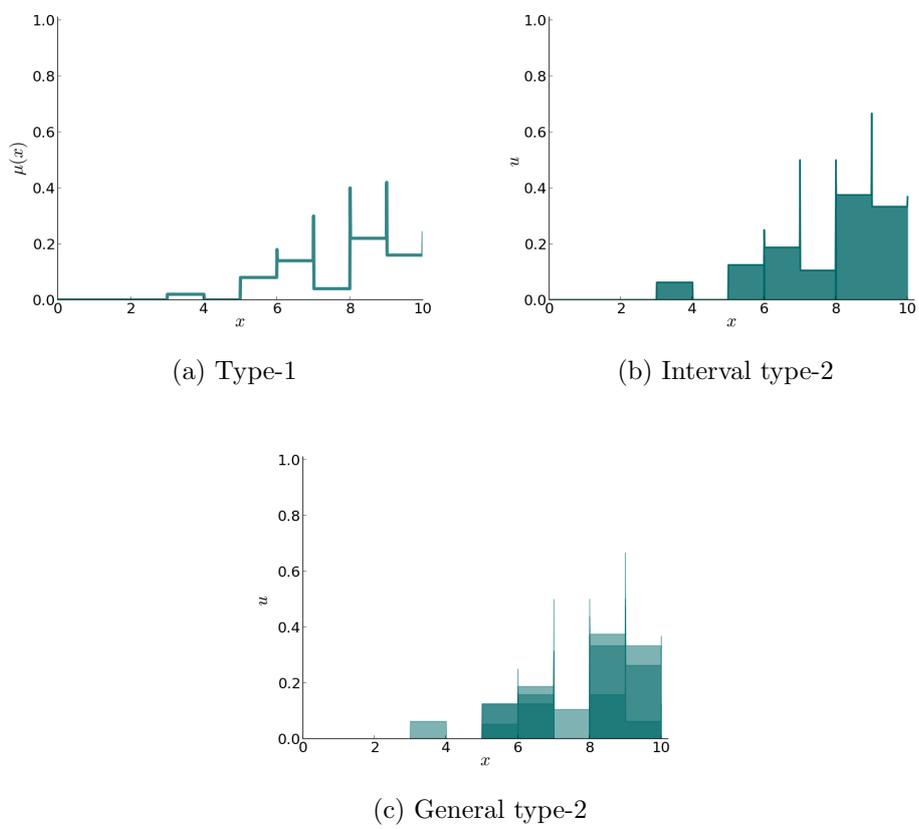


Figure 7.4: Different types of fuzzy sets representing how tasty cake A was rated.

results, 40 α -cuts are used when measuring distance within the incompatibility measure.

The next three sections give examples of the explicit preference recommender applied to this data set using type-1, interval type-2 and general type-2 fuzzy sets, respectively. Recall that, for the first sub-query, this recommender gives values in the range $[-1, 0]$ because it is designed to capture that the consumer wants a similar cake and thus any changes in an attribute should have a negative impact on the results. The second sub-query gives a range within $[-1, 1]$ representing how differently an attribute has been rated and whether the change in that attribute is in the desired direction. Table 7.5 (on Page 209) shows the results of a query split into the following sub-queries

Sub-query 1 (SQ1) a cake similarly crunchy to E

Sub-query 2 (SQ2) a cake less crumbly than E

Results are shown for each type of fuzzy set. To help visualise the changes in values between different types of fuzzy sets, Figure 7.5 shows the values from Table 7.5 represented as bar charts.

Within Appendix E, Figures E.1 to E.6 show the different types of fuzzy sets describing the attribute *crunchy* for cake E and all other cakes in comparison to E . Figures E.7 to E.12 provide the same figures for the attribute *crumbly*. In each figure, the incompatibility c'_p between the fuzzy sets is given.

7.3.2 Type-1 Fuzzy Sets

In Table 7.5, the type-1 results show that cakes D , F and B are worth recommending but cakes C and A are not. Cake C has a negative result because its dissimilarity in *crunchy* outweighs the difference in how much it is less *crumbly*. Cake A is not recommended because it does not match the second sub-query (i.e., it is more crumbly).

Referring to the type-1 fuzzy sets, sub-figure (a) in Figures E.1 to E.6

show that E has been given a low rating for *crunchy* and all other cakes are rated higher. Note, also, that direction does not matter as this attribute is used in SQ1 (to find similar cakes). Cake D (in Figure E.5) has the closest incompatibility to 0 ($c'_p = 0.246$) as it does not include values higher than $x = 5$ (unlike the other four sets) and its membership function is similarly shaped to E 's. All other cakes have higher incompatibility values due to including higher values in the fuzzy set (where $x > 5$).

Figures E.7 to E.12 show the fuzzy sets that represent how *crumbly* cakes are compared to E . These fuzzy sets are much wider, suggesting that participants were more uncertain of this attribute. Although cake A (in Figure E.8) has a positive value of incompatibility ($c'_p = 0.199$), there is little difference between these fuzzy sets. As a result, the value of incompatibility is low. Cake C (in Figure E.10) likewise has a similarly shaped membership function and its incompatibility ($c'_p = -0.315$) shows it is rated somewhat lower.

Cakes B , D and F have a greater change in results ($c'_p < -0.4$) and this can be seen in the fuzzy sets in Figures E.9, E.11 and E.12. Each cake has noticeably higher membership for values $x \in \{1, 2\}$ compared to E and, as such, their incompatibility is a larger negative value compared to cakes A and C .

Given that B , D and F have a large negative incompatibility for *crumbly* (as desired and used in SQ2) and have a comparatively lower incompatibility in the attribute *crunchy* (used in SQ1), as expected, they are positive recommendations in Table 7.5. Additionally, it was discussed that cake D has the lowest incompatibility with E in SQ1 (for the attribute *crunchy*; $c'_p = 0.246$) and so this has become the best recommendation in Table 7.5.

7.3.3 Interval Type-2 Fuzzy Sets

The same query is now demonstrated using interval type-2 fuzzy sets, which are constructed as given on Page 201, the results of which are also in Table 7.5. As a result of how these fuzzy sets are constructed, many fuzzy sets have higher upper membership values in the interval type-2 set than in the type-1 set, and so the recommender results between these types differ.

Referring to the interval type-2 results in Table 7.5, in SQ1, the results of C and D have a higher incompatibility than the type-1 results because their membership functions have changed. Figures E.4 and E.5 shows the fuzzy sets representing how crunchy C and D are, respectively, in comparison to E . C and D have noticeably higher membership values at higher x values and, as an effect, their incompatibility with E has increased. Note that F (in Figure E.6) also has higher membership for higher x values but its incompatibility has remained approximately the same because it also now has higher memberships at low x values (where $x \leq 1$).

For SQ2, Figures E.7 to E.12 show how crumbly each cake has been rated against E . This sub-query demonstrates a stark change in results in the interval type-2 case compared to the type-1 and general type-2 examples; this is noticeable in the results depicted in Figure 7.5. This is because simplifying the general type-2 case results in different distributions of membership. Referring to Figure E.8, the type-1 fuzzy set A has low membership for values $x < 2$ and has a similarly shaped membership function compared to cake E . Also, the highest value within E is $x = 8$, whereas the highest in A is $x = 10$. As a result, the incompatibility $c(A, E)$ is a small positive value.

In the interval type-2 fuzzy sets, within the same figure, due to splitting the data into different sub-groups, the primary membership functions are different to the type-1 case. Whilst increasing the membership of $x = 10$ in A , the membership of values $x < 3$ have also noticeably increased. In E , however,

only the values $3 \leq x \leq 8$ have a noticeable increase in membership. As a result, the incompatibility $c(A, E)$ becomes negative as A is now mostly to the left of E than to the right. The absolute value of incompatibility (0.302) is also higher than the type-1 case (0.199) due to these changes.

In the general type-2 case, the incompatibility has reversed; it is now a positive value (0.346) whereas in the interval type-2 case it is negative (-0.302). This is because the values $x < 2$ in A now have low secondary membership ($\mu = 0.25$) and so have little effect on the incompatibility result. Additionally, the value at $x = 10$ has a higher secondary membership ($\mu = 0.5$) and so has a greater impact on the resulting incompatibility. This results in A being perceived as to the right of E , rather than the left.

This example shows that care must be taken when choosing to use interval type-2 fuzzy sets as a simplification of general type-2 fuzzy sets as it may result in a drastic and undesirable change in the results.

Comparing the type-1 and interval type-2 results in Table 7.5 for the same query, cakes D and F , and cakes A and C have swapped rank positions. As just discussed, D has a higher negative result in SQ1 and F has a higher positive result in SQ2, which has resulted in F fitting the query better than D . Likewise, in SQ1 C has a higher negative value and in SQ2 A now has a positive value, resulting in A fitting the query better than C .

7.3.4 General Type-2 Fuzzy Sets

The same query is now demonstrated using general type-2 fuzzy sets, the results of which are also in Table 7.5. Details of how these fuzzy sets are constructed are given on Page 201.

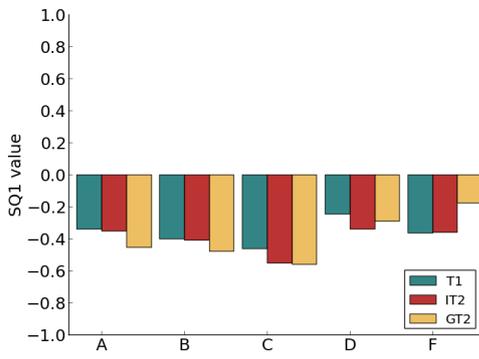
The greatest changes in results from the general type-2 fuzzy sets compared with the type-1 and interval type-2 results are in cakes F and A . Note that the changes regarding cake A were discussed in detail in the previous section. Cake

F has a lower incompatibility with E for the attribute *crunchy* (used in SQ1) when the fuzzy sets are general type-2 (shown in Figure E.6). This is because its biggest difference with E is in the interval $x \in [5, 7]$, however this region has the lowest secondary membership values (all of which are at $\mu_A(x, u) = 0.25$). As a result of the low secondary membership, the difference between the fuzzy sets does not have a large effect on the incompatibility. Cake F also has a higher negative incompatibility in *crumbly* (in Figure E.12) because it has higher secondary membership values for lower ratings; in particular where $x \in \{1, 2\}$.

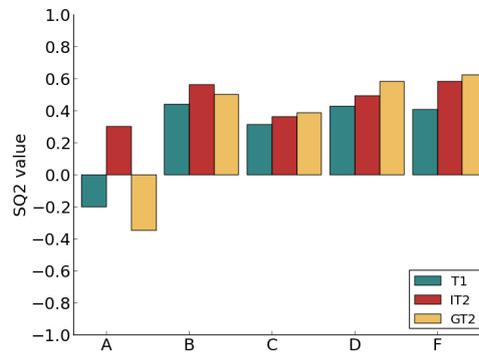
These demonstrations have shown that the proposed fuzzy knowledge based recommendation system with the incompatibility measure c'_p is effective at finding recommendations when the knowledge of products is represented by type-1 or type-2 fuzzy sets that have complex membership function shapes. Having demonstrated the proposed recommendation system on data-driven fuzzy sets, the next section (after the figures) provides a summary of this chapter.

FS Type	Cake	Sub-Query 1	Sub-Query 2	Result	Rank
T1	D	-0.246	0.427	0.181	1
	F	-0.364	0.409	0.045	2
	B	-0.401	0.438	0.037	3
	C	-0.461	0.315	-0.146	4
	A	-0.34	-0.199	-0.539	5
IT2	F	-0.36	0.583	0.223	1
	D	-0.338	0.491	0.153	2
	B	-0.411	0.563	0.152	3
	A	-0.353	0.302	-0.051	4
	C	-0.554	0.364	-0.19	5
GT2	F	-0.177	0.623	0.446	1
	D	-0.29	0.584	0.294	2
	B	-0.48	0.499	0.019	3
	C	-0.561	0.385	-0.176	4
	A	-0.454	-0.346	-0.8	5

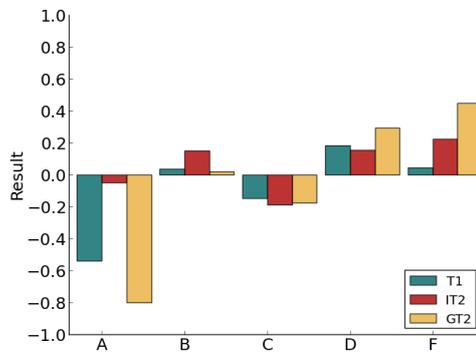
Table 7.5: Query results listed in rank order when finding a cake similarly crunchy to E (sub-query 1) and less crumbly (sub-query 2) using type-1 (T1), interval type-2 (IT2) and general type-2 (GT2) fuzzy sets (FSs).



(a) Sub-query 1



(b) Sub-query 2



(c) Results

Figure 7.5: Results from Table 7.5 represented through bar charts.

7.4 Summary

This chapter has demonstrated the proposed fuzzy knowledge based recommendation system (introduced in Chapter 6) using data-driven fuzzy sets. Both the *implicit* and *explicit preference recommenders* have been demonstrated using type-1, interval type-2 and general type-2 fuzzy sets. In an *implicit preference recommender*, the preferred ratings of a product are naturally known and do not have to be stated by the consumer. For example, if a person is trying to find a hotel, one can assume they would prefer the lowest priced hotel with the best rated rooms. In an *explicit preference recommender*, however, the consumer's preferences cannot be assumed and must be explicitly stated. For example, if a person is trying to find a cake they like, it cannot be assumed how sweet or fruity they would like that cake to taste.

Within the recommendation system, each product is described from surveys in which participants rate its attributes on a numerical scale by providing either single valued or interval valued answers. The results were then modelled by fuzzy sets using the polling and interval agreement approaches. Due to disagreement between survey participants, all of the fuzzy sets were non-normal and many were non-convex. To find recommendations based on these models, the incompatibility between pairs of fuzzy sets was calculated.

Following this, the incompatibility values and resulting recommendations were demonstrated and discussed using data-driven type-1, interval type-2 and general type-2 fuzzy sets. These demonstrations showed that the fuzzy recommendation system using the proposed incompatibility measure can effectively find products based on relative comparisons between known products and a person's desires.

Note that these results have not been compared with pre-existing methods of generating recommendations, and their comparison and validation is left for future work.

This chapter concludes the contributions of this PhD thesis. The next chapter presents conclusions and limitations that have resulted from this thesis and presents directions for future work.

Chapter 8

Conclusions

8.1 Thesis Summary

The core aim of this thesis has been to provide a method of making useful relative comparisons between data-driven fuzzy sets that model subjective information. This is achieved by comparing the similarities and distances of fuzzy sets, accounting for any non-normality or non-convexity in the models.

There are many applications in which such relative comparisons are useful or even necessary, including classification, linguistic reasoning, decision making and ranking. The focus of applications within this thesis has been in knowledge based recommendation systems (KBRS). This has been chosen because the knowledge of products can often be uncertain/fuzzy in nature and recommendation systems generally rely on relative comparisons. To be able to develop a fuzzy KBRS, it is necessary to have i) a method of comprehensively capturing and modelling the uncertain and subjective data; and ii) a method of comparing the complex resulting models. The latter is the main aim of this thesis.

Similarity and distance measures are among the most useful and commonly applied measures to compare fuzzy sets. Chapter 2 gave an overview of the

techniques that have been used in the literature to achieve these comparisons. Many measures of similarity exist and the Jaccard similarity was chosen as the favoured approach because it follows all of the properties of an ‘ideal’ similarity measure and it is not impeded by non-normal or non-convex membership functions.

A variety of distance measures were also explored. However, due to the nature of how distance is measured, common distance measures are hindered by non-normal and non-convex membership functions. Additionally, most measures do not account for the change of direction between fuzzy sets. With this in mind, Chapter 3 developed a directional distance measure for type-1 fuzzy sets where the membership functions may be non-normal or non-convex. Comparisons of the proposed method were given against existing approaches in the literature.

One cannot directly apply a measure for type-1 fuzzy sets to type-2 fuzzy sets and so Chapter 4 then extended the distance measure to compare interval and general type-2 fuzzy sets. Additionally, although the Jaccard similarity between interval type-2 fuzzy sets exists, before this thesis no such method existed for general type-2 fuzzy sets. With this in mind, Chapter 4 also developed a method of measuring the Jaccard similarity between general type-2 fuzzy sets.

Though similarity and distance measures are very useful, they can often produce ambiguous results such that different pairs of fuzzy sets result in the same value when one might have expected different values. If similarity and distance are both observed together, then a more detailed and much less ambiguous interpretation of the results is possible. However, interpreting the results of two unique measures for every comparison can be challenging. To make this process easier and thus more useful, Chapter 5 developed an incompatibility measure, which fuses the results of comparing the similarity and directional distance between fuzzy sets.

This incompatibility measure can then be used in a fuzzy KBRS to make relative comparisons between fuzzy sets in order to find a product that matches a person's complex and uncertain preferences. Chapter 6 developed a fuzzy KBRS and demonstrated that the proposed incompatibility measure is effective for finding products that match an uncertain/fuzzy query. Chapter 7 then demonstrated the recommendation system on data-driven fuzzy sets and discussed how the results of the incompatibility measure affect the rank values of products.

8.2 Contributions

This section summarises the key contributions of this thesis.

A directional distance measure between fuzzy sets

There is no perfect method of calculating the distance between two fuzzy sets and, as a result, several methods have been developed in the literature. Most of these apply the Hausdorff or Minkowski distance between the α -cuts of the fuzzy sets, though other methods also exist. Directional distance measures have received little attention in the literature. Often, direction is inferred by measuring the distance of each fuzzy set from a singleton (crisp value) [8, 12, 76, 77].

This technique, however, is not suitable for making relative comparisons in a recommendation system. This thesis proposed a measure that represents the relative direction between fuzzy sets through a signed value. The absolute value indicates the magnitude of the distance and the sign indicates which fuzzy set is to the left or right of the other.

A distance measure between type-1 fuzzy sets that may be non-normal or non-convex

The proposed directional distance measure has been expanded to compare fuzzy sets that may be non-normal or non-convex. Several methods of calculating the distance between non-normal fuzzy sets exist within the literature [11, 14, 15]. However, these often produce unexpected results as demonstrated in Section 3.3.2. The proposed method gives consistent and intuitive results when comparing normal or non-normal fuzzy sets.

Additionally, the proposed measure has been developed to compare non-convex fuzzy sets. To the author's knowledge, there are no other existing α -cut approaches to calculate the distance between such fuzzy sets to date. Though one could compare the centroids of the fuzzy sets, the proposed method produces more intuitive results for non-convex fuzzy sets, as demonstrated in Section 3.4.2.

A distance measure on type-2 fuzzy sets

There have been few distance measures on interval type-2 fuzzy sets developed within the literature. Figueroa-García et al. [79] developed one α -cut-based and two centroid-based approaches. However, the α -cut method cannot compare non-normal and non-convex membership functions and the centroid-based approaches give inconsistent results for non-convex fuzzy sets compared with the proposed directional distance on type-1 fuzzy sets. Additionally, to the author's knowledge, no α -cut-based distance measure on general type-2 fuzzy sets exist within the literature.

This thesis introduced a new α -cut-based distance measure on interval and general type-2 fuzzy sets. Using the theoretical developments of the previous contributions, this measure may be directional or non-directional and can compare non-normal and non-convex membership functions. In addition, it was demonstrated that when comparing type-1 fuzzy sets, this method gives

the same results as the proposed type-1 distance measure, and gives intuitive results for both interval and general type-2 fuzzy sets.

An incompatibility measure on type-1 and type-2 fuzzy sets

Chapter 5 introduced a new incompatibility measure on fuzzy sets of any type. This measure fuses similarity and distance, providing a comparison based on both vertical and horizontal slices of the fuzzy sets. Chapter 5 demonstrated that similarity and distance measures applied individually can produce ambiguous results where more information is desired. However, the results of one measure can alleviate the ambiguity of the other. It was shown that similarity and distance complement each other well and a measure that combines these concepts can be useful.

In order to address this, the results of these measures are fused using an ordered weighted average (OWA) operator. By using an OWA operator, one can tune the weights to best analyse the given fuzzy sets. Chapter 5 provided some examples of the effects different weights have on different pairs of fuzzy sets and chose the weights that give the most expected results.

Demonstrations showed that the proposed incompatibility measure effectively analyses fuzzy sets such that one can determine both the similarity and distance between them. The results show the relative positions of the fuzzy sets (if one is to the left or right of the other) and indicate if there is a small or large overlap/distance between them.

As this measure fuses the *results* of similarity and distance measures, it can be used to compare type-1, interval type-2 or general type-2 fuzzy sets where the similarity and distance measures are for type-1, interval type-2 or general type-2 fuzzy sets, respectively.

The concept of measuring *compatibility* between fuzzy sets has appeared many times in the literature [49, 52, 53, 111]. However, measuring incompati-

bility as a concept of fusing unique similarity and distance measures is a new contribution of this thesis. This provides a unique comparison of fuzzy sets where a single measure of comparison may not be sufficient. In addition, this method of fusing different measures together may be utilised in other non-fuzzy fields where both similarity and distance provide useful information.

A fuzzy knowledge based recommendation system

Chapter 6 explored real world application scenarios and developed a fuzzy knowledge based recommendation system with which a person can describe their ideal product in relation to another product. For example, a person may ask for *a cake as sweet as this but less fruity*. By representing the subjective attributes of products (e.g., how sweet and fruity cakes are perceived) using fuzzy sets, such comparisons on subjective and uncertain information can be made. These comparisons are achieved using the proposed incompatibility measure, and thus the fuzzy sets may be of any type (type-1 or type-2) and may be non-normal or non-convex.

Demonstrations of the proposed system were given using ground truth examples based on simple synthetic fuzzy sets (in Chapter 6), and data-driven examples based on fuzzy sets with complex membership functions (in Chapter 7). The results of the incompatibility measure on these fuzzy sets were discussed, as well as how these results affect the recommendations.

The suitability of each product for a given description is represented by a value within the interval $[-1, 1]$ (to indicate negative and positive recommendations), as well as a rank position, so that the relative preferences of recommendations can be understood. For example, given three products labelled A , B and C , if, for a given recommendation, A is scored 0.9, B is scored 0.8 and C is scored 0.1 then it is clear from these values that A and B are almost equally good recommendations and are much better than C . If only the rank positions (i.e., 1st : A , 2nd : B , 3rd : C) were used, as is more common

in many recommendation systems, then this information would be lost.

Elucidating the Differences between Implicit and Explicit Preferences

Chapter 6 explored the concept of implicit and explicit user preferences. While the concept of explicit preferences is generally well understood, acquiring implicit preferences is less clear. Explicit preferences typically involve querying a user to learn if they like a given product. This is often achieved through written reviews or asking users to review products and product attributes on Likert scales [112, 113].

Implicit preferences are often also acquired by collecting information from users, either directly (for example, through experiments [112]) or indirectly (for example, by monitoring their activities [113]). This thesis, however, has introduced the idea of defining implicit preferences without collecting any information from users. Instead, it is assumed that all consumers will have the same given preference; for example, it's likely that everyone will prefer the restaurant with the tastiest food. This idea could be applied in many recommendation systems where the preferences for a selection of attributes can be safely assumed.

8.3 Limitations

This section presents some of the limitations within the work presented in this thesis.

Using the zSlices extension on fuzzy sets with non-normal secondary membership functions

Chapter 4 presented a method of extending interval type-2 measures for general type-2 fuzzy sets. However, this extension does not account for non-normality

in the secondary membership functions of the general type-2 fuzzy sets. For example, given two fuzzy sets $\tilde{A}, \tilde{B} \in GT2(X)$, if the maximum zLevels of \tilde{A} and \tilde{B} are 1.0 and 0.8, respectively, then it is not possible to compare the fuzzy sets using the union of the zLevels as applied in Chapter 4; this is because \tilde{B}_{z_i} is the empty set where $z_i = 1.0$.

Properties of the incompatibility measure

The incompatibility measure does not have the property of relectivity ¹ as is in the proposed directional distance measure d_p^* (where * indicates the fuzzy set type). Instead, a positive-valued result is given. As a result, it is not possible to determine if two symmetrical fuzzy sets share the same mean but have different widths. In a recommender system, it may be useful to know about such cases because one may perceive such fuzzy sets as being rated the same (which would be clear from d_p^*) rather than one being rated higher than the other (as indicated by c'_p).

Validation of the recommendation system

The proposed recommendation system based on incompatibility shows promising preliminary results. However, this thesis has not explored the validation of these results and how they compare to existing recommendation methods within the literature. Further work is necessary to assess the validity of the results.

¹The direction between two intervals $[\bar{A}_L, \bar{A}_R]$ and $[\bar{B}_L, \bar{B}_R]$ is 0 if $(\bar{B}_L - \bar{A}_L) = -(\bar{B}_R - \bar{A}_R)$

8.4 Future Work

This section presents some potential new directions and future work based on this thesis.

Representing and utilising non-normality and non-convexity in comparisons between fuzzy sets

Chapter 3 developed methods of calculating the distance between non-normal and non-convex membership functions. However, these both involve a simplification of the results by representing the distance between fuzzy sets as a crisp value. Instead, one could argue that the distance between two fuzzy concepts should itself be fuzzy. Some preliminary work has been made as part of this PhD in representing the distance between fuzzy sets as a fuzzy set. Initial results are demonstrated in Appendix C and this idea will be explored further in future work.

This idea then may be employed with the developed recommendation system. For example, if the sub-query results are represented as fuzzy sets and are aggregated using fuzzy set theory then the results may fit the original data better because they have not been simplified by using real values. Thus, there may be potential benefits in using fuzzy sets as the output values and developing a method of making these results easy to interpret.

One possible approach is to use fuzzy sets to model linguistic rank positions (e.g., “very high”, “high”, “low”, etc.) and then each fuzzy result from the recommender is assigned the linguistic term with which it best fits. This can be determined by measuring the incompatibility between the recommender results and the linguistic terms and choosing the term with the lowest incompatibility (i.e., the result closest to 0).

Comparing all permutations of zLevels in the zSlices extension

As discussed in the previous section, the zSlices extension is limited by the dependency that all general type-2 fuzzy sets must have normal secondary membership functions. If one fuzzy set has a higher maximum secondary membership value than the other then a comparison cannot be made between them.

An alternative method of comparing fuzzy sets with different zLevels is to compare all permutations, such that every zSlice of one fuzzy set is compared with every zSlice of the other. With this method, it is not required that fuzzy sets have normal secondary membership functions.

Adapting fuzzy set models depending on the data and application

In Section 7.3, the fuzzy sets describing different types of cakes gave different results depending on what type of fuzzy set (type-1 or type-2) was used to model the data. This demonstration shows that the method of constructing fuzzy sets is important as it has an impact on the results. With this in mind, it will be beneficial to compare and contrast different techniques to understand which, if any, is the most appropriate method of modelling subjective information for the application of recommendations.

Using a threshold on the desired change of an attribute in the recommender

There is much room for future developments within the proposed recommendation system. For example, when a person searches for a product rated higher or lower in a given attribute, the system gives the highest score to the item that is rated the highest or lowest, respectively, in that attribute. However, this may not be what the person wants. For example, if someone asks for *a cake more sweet than this* they will likely not want the *sweetest* cake. Given this, the proposed system may be improved so that it limits such changes by a given

threshold. One possible method of achieving this is through fuzzy hedges. For example, when finding *a cake more sweet than this*, the fuzzy set representing how sweet *this* is may be amended through a hedge, then the incompatibility may then be measured between new cakes and the amended fuzzy set.

Applying fuzzy set weights to the recommender

Additionally, the method of weighting in the proposed recommender system may also be improved by considering other approaches. For example, people do not generally think of weights for attributes as real values, but instead consider them through words such as “high” or “low”. If these words are modelled by fuzzy sets then fuzzy weights can be applied to the recommendations. Experiments will be required to determine if there are advantages to this method when finding recommendations. However, it is likely that using fuzzy words for weights will be easier from the consumer’s point of view.

Appendices

A Properties of Measures on Fuzzy Sets

Measures of similarity and distance each have varying properties. Below lists the properties of the measures used throughout this thesis, grouped by the measures in which they are found. In addition to this, Table A.1 highlights which properties are within which measures used in this thesis. The properties of a function f (where f may be a similarity or a distance measure) referred to within this thesis are as follows; note that the variables given to f may be of any type (e.g., crisp sets, fuzzy sets, etc.).

Used by Similarity and Distance Measures

Symmetry: A function is symmetrical if it returns the same value for any permutation of its variables.

$$f(A, B) = f(B, A)$$

Transitivity: In terms of similarity, transitivity states that as one fuzzy set approaches another in membership values the value of the function increases.

$$\text{If } A \leq B \leq C, \text{ then } f(A, B) \geq f(A, C)$$

In terms of measuring distance, as one fuzzy set approaches another the value of the function decreases.

$$\text{If } A \leq B \leq C, \text{ then } f(A, B) \leq f(A, C)$$

Used by Similarity Measures

Reflexivity: Each fuzzy set is related to itself under the function.

$$f(A, A) = 1$$

Overlapping: Where two fuzzy sets overlap (i.e., their intersection is not the empty set) the value of the function is greater than zero. Otherwise, if the sets are disjoint, the result of the function is zero.

$$\text{If } A \cap B \neq \emptyset, \text{ then } f(A, B) > 0; \text{ otherwise, } f(A, B) = 0$$

Used by Distance Measures

Self-identity: The function between a variable and itself is always zero.

$$f(A, A) = 0$$

Separability (also known as Positivity): The function between two variables always results in a value of zero or greater.

$$f(A, B) \geq 0$$

Triangle inequality: Given three variables on the same universe of discourse, which can be broken into three different pairs, the value of the function for one pair of variables is at most as large as the sum of the function applied to the remaining two pairs.

$$f(A, C) \leq f(A, B) + f(B, C)$$

Used by the Proposed Directional Distance Measure

Partial Symmetry: Partial symmetry is defined within this thesis as a function that is asymmetric, however the absolute values of the function are symmetric.

$$f(A, B) = -f(B, A)$$

Directional Separability: The sign of the function indicates the relative positions between the variables.

$$f(A, B) \geq 0 \text{ if } B \geq A, \text{ and } f(A, B) < 0 \text{ if } B < A.$$

Reflectivity: The function between two intervals is 0 if the distances between their respective end points are equal to each other and in opposite directions.

$$f(\bar{A}, \bar{B}) = 0, \text{ where } \bar{A} = [\bar{A}_L, \bar{A}_R] \text{ and } \bar{B} = [\bar{B}_L, \bar{B}_R], \text{ if } (\bar{B}_L - \bar{A}_L) = -(\bar{B}_R - \bar{A}_R).$$

Distance Measure	Properties	Section
A typical distance measure (that is a metric)	self-identity symmetry separability transitivity triangle inequality	2.4.2
Proposed Directional Distance	self-identity partial symmetry directional separability reflectivity transitivity triangle inequality	3.2
Extension on non-normal fuzzy sets	same as the proposed directional distance	3.3
Extension on non-convex fuzzy sets	same as the proposed directional distance	3.4
Extension on interval type-2 fuzzy sets	same as the proposed directional distance	4.2
Extension on general type-2 fuzzy sets	same as the proposed directional distance	4.3

Table A.1: Properties of a typical distance measure and the proposed approaches.

B Distance by Comparing All α -cuts

Section 3.3 proposes a method of measuring the distance between non-normal fuzzy sets. Parallel α -cuts of the fuzzy sets are measured and their distances are weighted and joined together as shown in (3.7). Where one α -cut is empty, the fuzzy set's closest non-empty α -cut is substituted, and where both α -cuts are empty the distance is not measured.

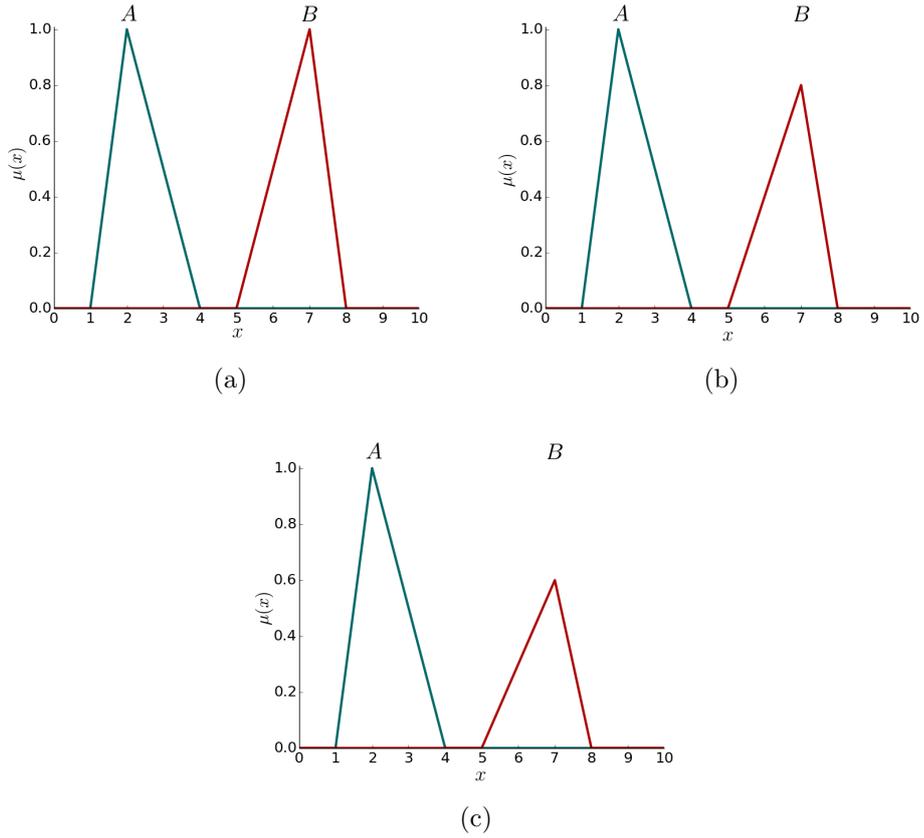
One other method of measuring the distance between two non-normal fuzzy sets $A, B \in T1(X)$ is to compare all of the α -cuts of A with all of the α -cuts belonging to B . For example, the distance between A and B may be weighted and joined as

$$d_{p2}^{T1-c}(A, B) = \frac{\sum_{\alpha_A \in [0, A_\gamma]} \sum_{\alpha_B \in [0, B_\gamma]} \alpha_A \alpha_B \bar{d}_p(\overline{A_{\alpha_A}}, \overline{B_{\alpha_B}})}{\sum_{\alpha_A \in [0, A_\gamma]} \sum_{\alpha_B \in [0, B_\gamma]} \alpha_A \alpha_B} \quad (\text{B.1})$$

where \bar{d}_p is the directional distance between intervals described in (3.2).

This next demonstration compares d_{p2}^{T1-c} (B.1) against the proposed method of comparing parallel α -cuts d_p^{T1-c} in Section 3.3. Figure B.1 shows three pairs of fuzzy sets $A, B \in T1(X)$ with different heights of B , and shows a table of their distances given by measuring parallel α -cuts (3.7) and by comparing all α -cuts (B.1). The methods produce different values, but they are both within the expected range; i.e., by visually observing the fuzzy sets in Figure B.1, one would guess the distance between each (A, B) pair is between 4 and 5.

Additionally, in both methods the value of distance increases slightly as the height of B lowers. Both measures produce similar results such that no method gives results that are clearly preferred over the other. Given that comparing all α -cuts is more computationally expensive than comparing only parallel α -cuts, the latter method is preferred. Additionally, these results show that using the highest available α -cut as a substitute when one fuzzy set's α -cut is empty is a suitable substitute over comparing all α -cuts as in (B.1).



	a	b	c
Compare Parallel α -cuts $d_p^{T^{1-c}}(A, B)$ (3.7)	4.684	4.740	4.785
Compare All α -cuts $d_{p2}^{T^{1-c}}(A, B)$ (B.1)	4.683	4.685	4.689

Figure B.1: Fuzzy sets and results demonstrating the distance $d(A, B)$ between non-normal fuzzy sets by comparing only parallel α -cuts or comparing all permutations of α -cuts.

C Representing the Uncertainty of Distance

It could be reasonably contended that the distance between the fuzzy sets in Figure C.1 should be 4 even though B is non-normal. As A is *around 2* and B is *around 6*, the distance should be *around 4*. Ideally, the normality of the fuzzy sets should affect the certainty of the distance. For example, if both A and B are normal then the distance is *definitely around 4*, whereas if one of the fuzzy sets has a height less than 1 then the distance becomes *possibly around 4*. This is because there is less certainty in the fuzzy set and therefore less certainty in the distance.

Such interpretations are possible by representing distance as a fuzzy value or a fuzzy set [116–118], where [118] was published as part of this PhD and is discussed within this section.

By representing the distance between fuzzy sets as a fuzzy set, the distance between non-normal fuzzy sets is represented by a non-normal fuzzy set, and the distance between non-convex fuzzy sets may also be non-convex. This method is based on using the mass assignment representation of fuzzy sets. The remainder of this section details the mass assignment representation, followed by the method of calculating distance and some examples.

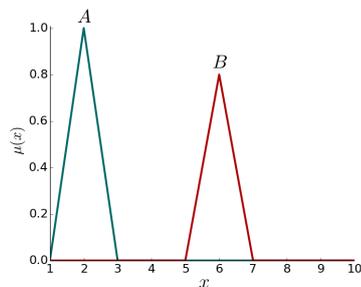


Figure C.1: A normal fuzzy set and a non-normal fuzzy set.

C.1 Mass Assignments of Fuzzy Sets

Calculating mass assignments is a method of breaking down the distribution of memberships within fuzzy sets. The approach is akin to taking alpha-cuts of fuzzy sets, with one distinct difference. Using alpha-cuts to represent a fuzzy set, the membership of a value is defined by the maximum alpha-cut to which it belongs, whereas taking a mass-based approach, the membership of a value is defined by the sum of its masses.

Based on α -cuts, the masses of a fuzzy set $A \in T1(X)$ are

$$m_A = \{A_{\alpha_i} : \alpha_i - \alpha_{i-1} \mid \alpha_i \in \{\alpha_1, \dots, \alpha_n\}, \alpha_0 = 0\}, \quad (C.1)$$

where A_{α_i} is the α -cut of the fuzzy set A at the coordinate α_i , n is the total number α -cuts used, and α_i is the coordinate (membership value) of the i^{th} α -cut.

For example, Figure C.2 shows two fuzzy sets A and B . The α -cuts of A and B given in the format $A = \{([A_{\alpha_L}, A_{\alpha_R}], \alpha) \mid \forall \alpha \in [0.5, 1.0]\}$ are

$$A = \{([1, 4], 0.5), ([2, 3], 1.0)\}$$

$$B = \{([6, 9], 0.5), ([7, 8], 1.0)\}$$

the masses assigned to A and B are $m_A \{A_i : a_i\}$ and $m_B \{B_j : b_j\}$ as follows:

$$m_A = \{[1, 4] : 0.5, [2, 3] : 0.5\}$$

$$m_B = \{[6, 9] : 0.5, [7, 8] : 0.5\}$$

To convert mass assignments to the original fuzzy set, the membership of each value can be obtained by summing the masses which have been assigned to that value. For example, for the fuzzy set A , the value 3 has been assigned the mass 0.5 twice, giving the resulting membership value 1.0.

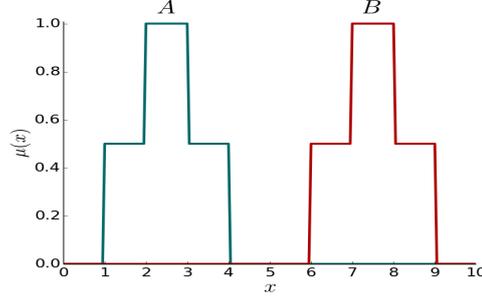


Figure C.2: Two fuzzy sets A and B .

C.2 Fuzzy Distance Between Fuzzy Sets

Once two fuzzy sets have been broken down into their mass assignments, their distance can be calculated by subtracting each mass of one fuzzy set from each mass of the other fuzzy set using interval arithmetic [118]. Each resulting distance is assigned a new mass based on the product of the masses that were compared. Formally this is written as

$$d_m(A, B) = \{ \bar{d}_m(A_i, B_j) : (m_A(A_i)m_B(B_j)), \forall A_i \in m_A, \forall B_j \in m_B \}, \quad (\text{C.2})$$

where $m_A(A_i)$ is the mass assigned to the i^{th} interval in m_A , and $\bar{d}_m(A_i, B_j)$ is

$$\bar{d}_m(A_i, B_j) = \begin{cases} B_j - A_i & \text{if } A_i \neq \emptyset \wedge B_j \neq \emptyset \\ \emptyset & \text{otherwise} \end{cases} \quad (\text{C.3})$$

Using interval arithmetic, $B_j - A_i$ is calculated as

$$[B_{j_L}, B_{j_R}] - [A_{i_L}, A_{i_R}] = [B_{j_L} - A_{i_R}, B_{j_R} - A_{i_L}]$$

where $A_i = [A_{i_L}, A_{i_R}]$ and $B_j = [B_{j_L}, B_{j_R}]$.

For example, given the fuzzy sets A and B in Figure C.2, Table C.1 shows how the distance between A and B is calculated using (C.2). Within the centre of the table, each distance between intervals and the mass assigned to that distance is highlighted. Using the results in Table C.1, the membership values of the final fuzzy set are calculated by summing the masses assigned to

each value. This gives the resulting fuzzy set

$$d_u(A, B) = \{([2, 8], 0.25), ([3, 7], 0.75), ([4, 6], 1.0)\}$$

which is shown in Figure C.3.

		B	
	$d_m(A, B)$	[6.0,9.0]:	[7.0,8.0]:
		0.5	0.5
A	[1.0, 4.0]:	[2.0, 8.0] :	[3.0, 7.0] :
	0.5	0.25	0.25
	[2.0, 3.0]:	[3.0, 7.0] :	[4.0, 6.0] :
0.5	0.25	0.25	

Table C.1: Calculations of the distance between A and B from Figure C.3. Each distance and its mass is highlighted in bold.

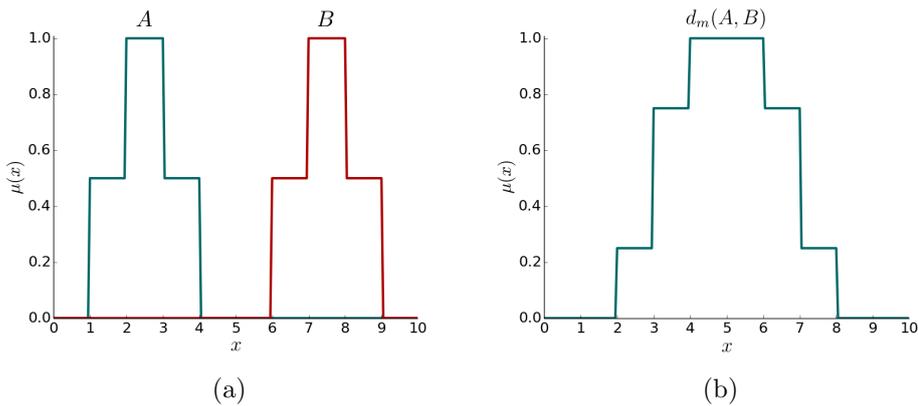


Figure C.3: (a) Normal, convex fuzzy sets A and B ; (b) $d_m(A, B)$ using the mass-based distance measure.

C.3 Distance between Non-Normal Fuzzy Sets

Using (C.3), calculating the distance between empty α -cuts results in assigning mass to the empty set. This produces a non-normal fuzzy set.

Figure C.4a shows non-normal and normal fuzzy sets A and B , respectively. Their α -cuts, given in the format $A = \{([A_{\alpha_L}, A_{\alpha_R}], \alpha) \mid \forall \alpha \in [0.5, 1.0]\}$, are

$$A = \{([1, 8], 0.25), ([2, 7], 0.5)\}$$

$$B = \{([9, 12], 0.5), ([10, 11], 1.0)\}$$

Thus, their masses are

$$A_m = \{[1, 8] : 0.25, [2, 7] : 0.25, \emptyset : 0.5\}$$

$$B_m = \{[9, 12] : 0.5, [10, 11] : 0.5\}$$

The calculations for determining the distance between A and B using d_m (C.2) are shown in Table C.2. The resulting masses of the fuzzy set $d_m(A, B)$ are

$$d_m(A, B) = \{[1, 11] : 0.125, [2, 10] : 0.25, [3, 9] : 0.25, \emptyset : 0.5\}$$

which results in the fuzzy set

$$d_m(A, B) = \{([1, 11], 0.125), ([2, 10], 0.375), ([3, 9], 0.5)\}$$

Figure C.4b shows the resulting fuzzy set $d_m(A, B)$. Thus, one can see when comparing a pair of fuzzy sets of which at least one is non-normal, the resulting distance is a non-normal fuzzy set.

$d_m(A, B)$		B		
		[9.0, 12.0] :	[9.0, 12.0] :	[10.0, 11.0] :
		0.25	0.25	0.5
A	[1.0, 8.0] :	[1.0, 11.0] : 0.0625	[1.0, 11.0] : 0.0625	[2.0, 10.0] : 0.125
	[2.0, 7.0]	[2.0, 10.0] : 0.0625	[2.0, 10.0] : 0.0625	[3.0, 9.0] : 0.125
	\emptyset :	\emptyset :	\emptyset :	\emptyset :
	0.5	0.125	0.125	0.25

Table C.2: Calculations of the distance between A and B from Figure C.4b. Each distance and its mass is highlighted in bold.

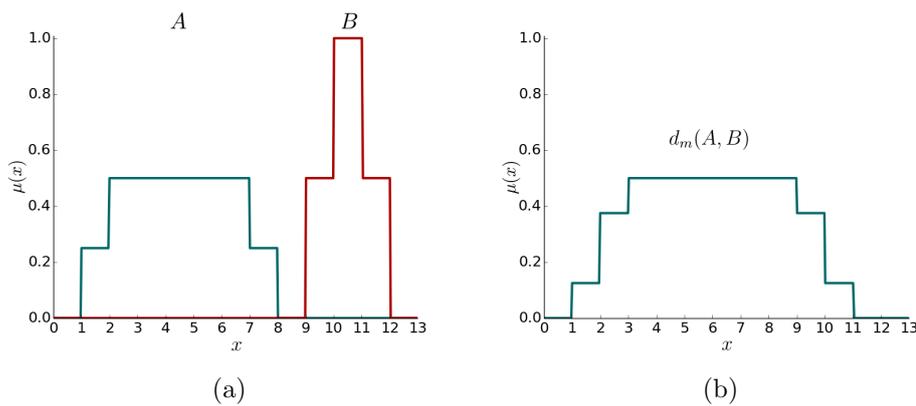


Figure C.4: (a) Non-normal A and normal B fuzzy sets; (b) $d_m(A, B)$ using the mass-based distance measure.

C.4 Distance between Non-Convex Fuzzy Sets

In this section, the difference between a non-convex, asymmetric fuzzy set A is compared with a convex, symmetrical fuzzy set B . The fuzzy sets, shown in Figure C.5a, are distributed as follows

$$A = \{([1, 5], 0.5), ([1, 2], [4, 5]), 0.75), ([1, 2], 1.0)\}$$

$$B = \{([6, 9], 0.5), ([7, 8], 1.0)\}$$

Thus, their masses are

$$A_m = \{[1, 5] : 0.5, ([1, 2], [4, 5]) : 0.25, [1, 2] : 0.25\}$$

$$B_m = \{[6, 9] : 0.5, [7, 8] : 0.5\}$$

The calculations for determining the distance between A and B using d_m (C.2) are shown in Table C.3. Note that where an α -cut of a non-convex fuzzy set results in a discontinuous region with multiple intervals, the subtraction is calculated for each interval. For example, the difference $B_\alpha - A_\alpha$ where $\alpha = 0.75$ is

$$\begin{aligned} & [7, 8] - ([1, 2], [4, 5]) \\ &= ([7, 8] - [1, 2], [7, 8] - [4, 5]) \\ &= ([5, 7], [2, 4]) \end{aligned}$$

Thus, obtaining the difference between this continuous and discontinuous region results in a discontinuous region. Note, however, that the difference between discontinuous intervals does not necessarily result in a discontinuous interval. As shown in Table C.3 the difference $[6.0, 9.0] - ([1.0, 2.0], [4.0, 5.0]) = ([1.0, 5.0], [4.0, 8.0])$ The intervals within this result are reduced to the continuous interval $[1.0, 8.0]$.

Referring to the calculations within Table C.3, the masses of the fuzzy set

$d_m(A, B)$ are

$$\{[1, 8] : 0.375, [2.0, 7.0] : 0.25, [5.0, 7.0] : 0.125, \\ [4.0, 8.0] : 0.0125, ([2.0, 4.0], [5.0, 7.0]) : 0.125\}$$

Adding up the masses, the resulting distance between A and B using (C.2) is

$$d_m(A, B) = \{([1, 8], 0.375), ([2, 8], 0.5), ([2, 7], 0.75), \\ ([4, 4], [5, 7]), 0.875), ([5, 7], 1.0)\}$$

Figure C.5b shows the resulting fuzzy set $d_m(A, B)$.

More details of this fuzzy distance measure are provided in [118].

		B		
$d_m(A, B)$		[6.0, 9.0] :	[7.0, 8.0] :	[7.0, 8.0] :
		0.5	0.25	0.25
[1.0, 5.0] :	0.5	[1.0, 8.0] : 0.25	[2.0, 7.0] : 0.125	[2.0, 7.0] : 0.125
([1.0, 2.0], [4.0, 5.0])	0.25	[4.0, 8.0], [1.0, 5.0] : 0.125	[5.0, 7.0], [2.0, 4.0] : 0.0625	[5.0, 7.0], [2.0, 4.0] : 0.0625
A [1.0, 2.0] :	0.25	[4.0, 8.0] : 0.125	[5.0, 7.0] : 0.0625	[5.0, 7.0] : 0.0625

Table C.3: Calculations of the distance between A and B from Figure C.5b. Each distance and its mass is highlighted in bold.

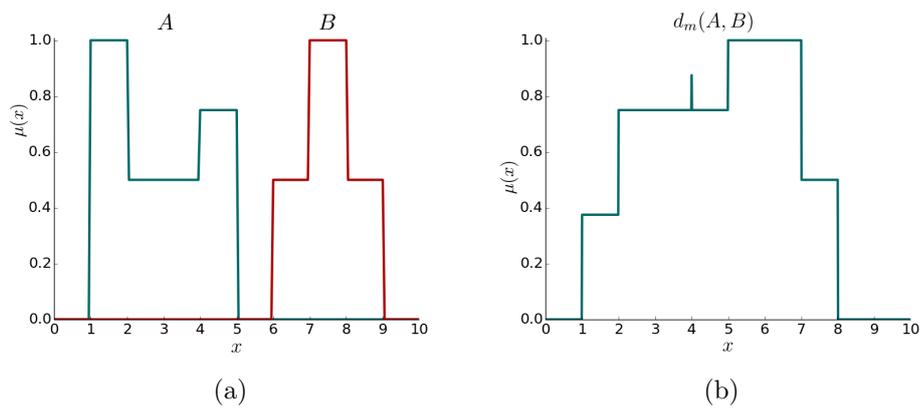


Figure C.5: (a) Non-convex A and convex B fuzzy sets; (b) $d_m(A, B)$ using the mass-based distance measure.

D MovieLens

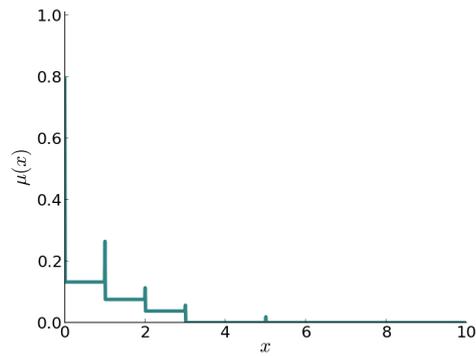
Table D.1 gives the names of the films used within the demonstrations in Chapter 5 in Figures 5.5 and 5.13 on pages 132 and 152, respectively. The ID numbers of the films within the data set [109] are also given.

Figure	<i>A</i>	<i>B</i>
(a)	777: Castle Freak (1995)	294: Liar Liar (1997)
(b)	1449: Pather Panchali (1955)	294: Liar Liar (1997)
(c)	777: Castle Freak (1995)	1428: SubUrbia (1997)
(d)	777: Castle Freak (1995)	1191: Letter From Death Row, A (1998)
(e)	603: Rear Window (1954)	424: Children of the Corn: The Gathering (1996)
(f)	1: Toy Story (1995)	181: Return of the Jedi (1983)

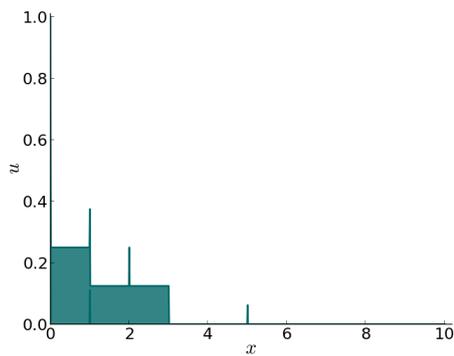
Table D.1: ID numbers and names of the films represented by fuzzy sets in Figures 5.5 and 5.13.

E Fuzzy Sets Depicting Cake Attributes

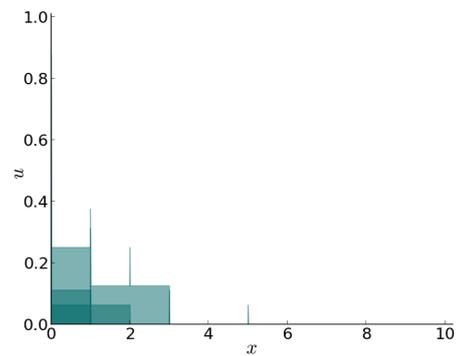
This section presents figures of the fuzzy sets used within the demonstrations in Section 7.3.



(a) Type-1

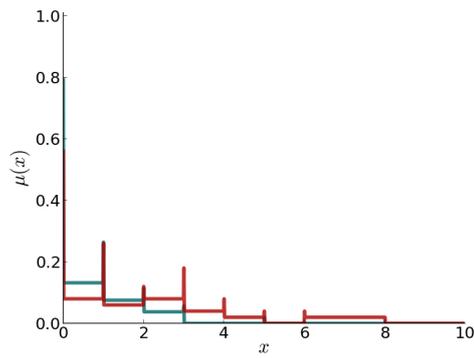


(b) Interval Type-2

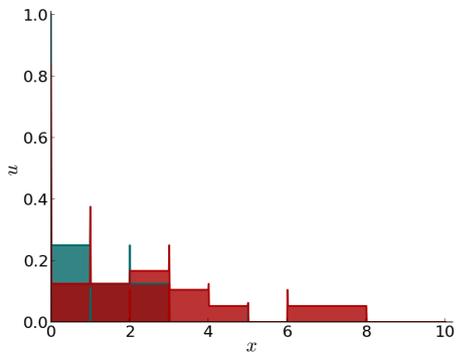


(c) General Type-2

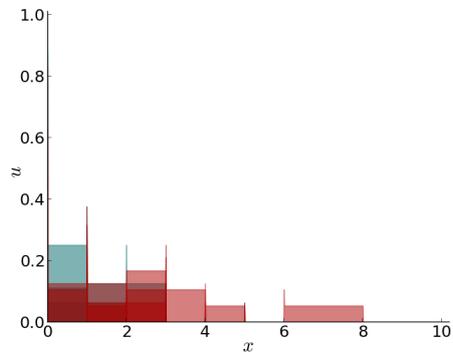
Figure E.1: Fuzzy sets of cake E for the attribute *crunchy*.



(a) Type-1; $c'_p = 0.34$

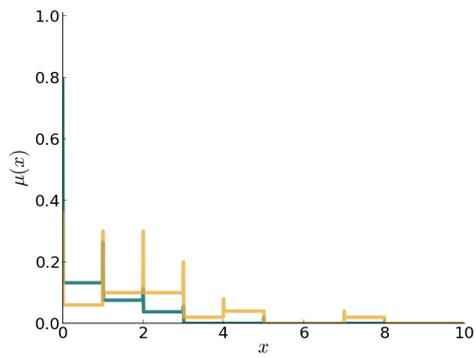


(b) Interval Type-2; $c'_p = 0.353$

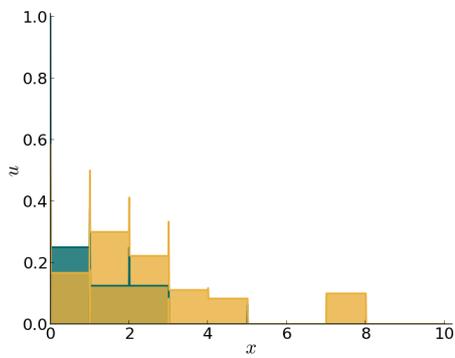


(c) General Type-2; $c'_p = 0.454$

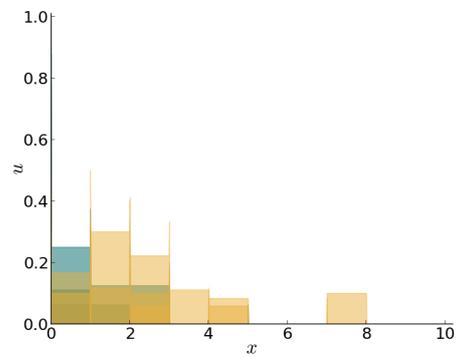
Figure E.2: Fuzzy sets of cakes **A** and **E** for the attribute *crunchy* and their incompatibility (c'_p).



(a) Type-1; $c'_p = 0.401$

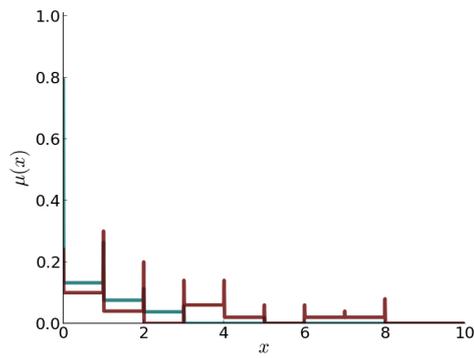


(b) Interval Type-2; $c'_p = 0.411$

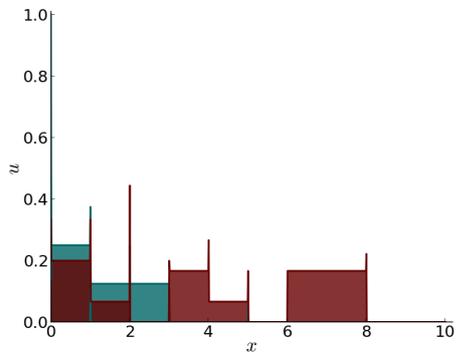


(c) General Type-2; $c'_p = 0.48$

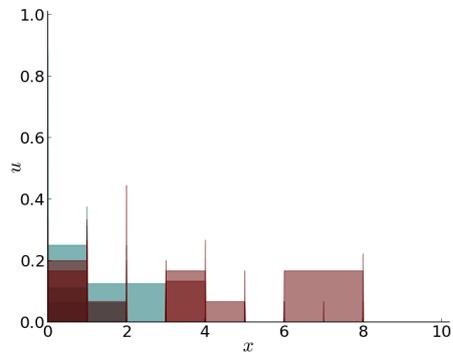
Figure E.3: Fuzzy sets of cakes B and E for the attribute *crunchy* and their incompatibility (c'_p).



(a) Type-1; $c'_p = 0.461$

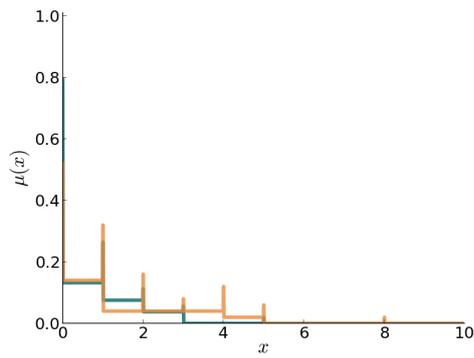


(b) Interval Type-2; $c'_p = 0.554$

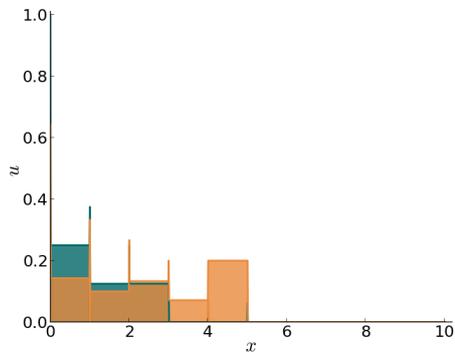


(c) General Type-2; $c'_p = 0.561$

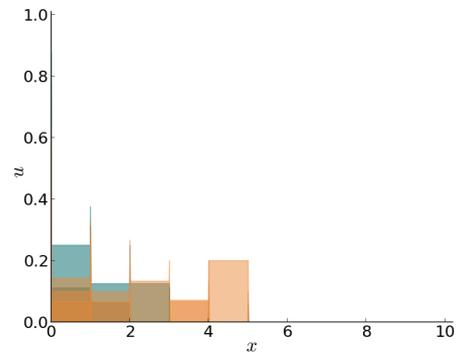
Figure E.4: Fuzzy sets of cakes \mathbf{C} and \mathbf{E} for the attribute *crunchy* and their incompatibility (c'_p).



(a) Type-1 $c'_p = 0.246$

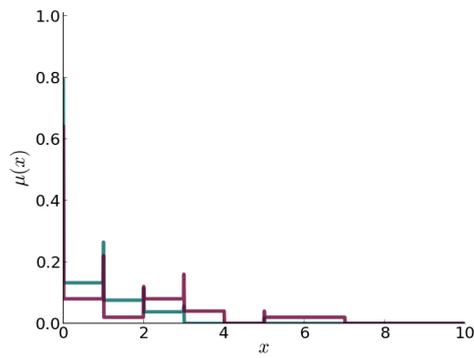


(b) Interval Type-2 $c'_p = 0.338$

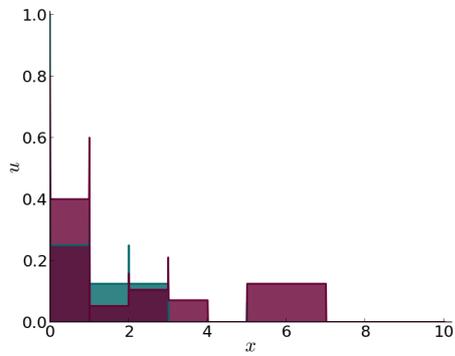


(c) General Type-2 $c'_p = 0.29$

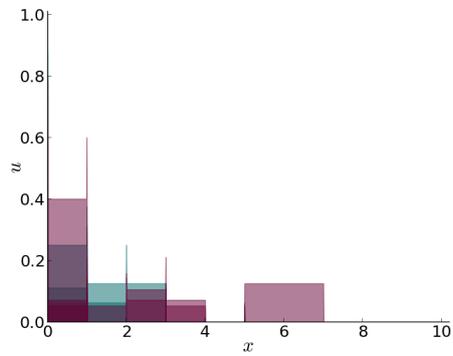
Figure E.5: Fuzzy sets of cakes D and E for the attribute *crunchy* and their incompatibility (c'_p).



(a) Type-1 $c'_p = 0.364$

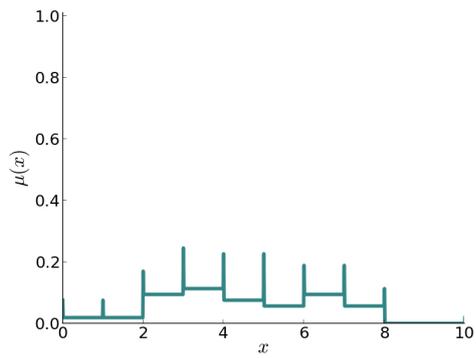


(b) Interval Type-2 $c'_p = 0.36$

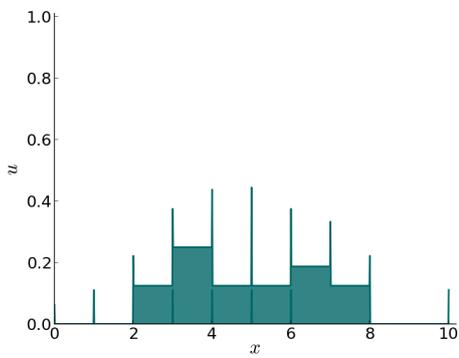


(c) General Type-2 $c'_p = 0.177$

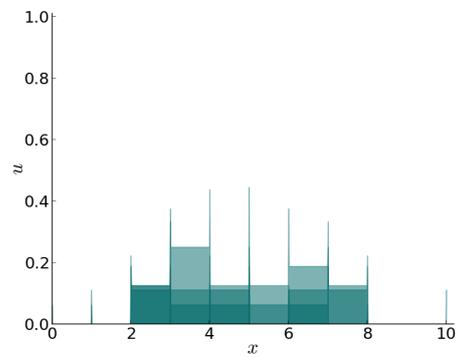
Figure E.6: Fuzzy sets of cakes F and E for the attribute *crunchy* and their incompatibility (c'_p).



(a) Type-1

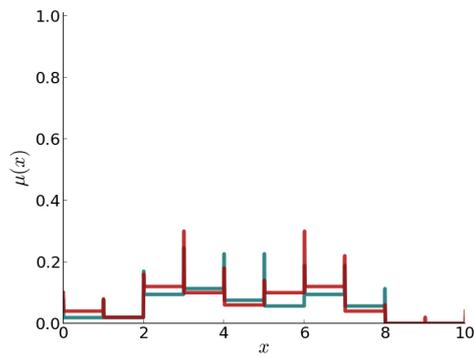


(b) Interval Type-2

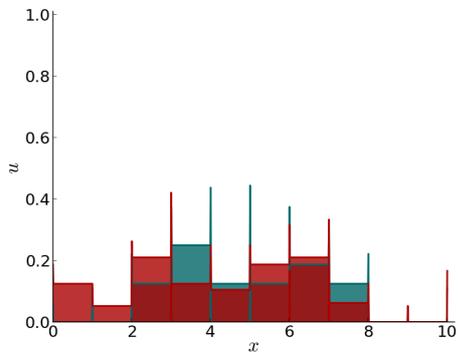


(c) General Type-2

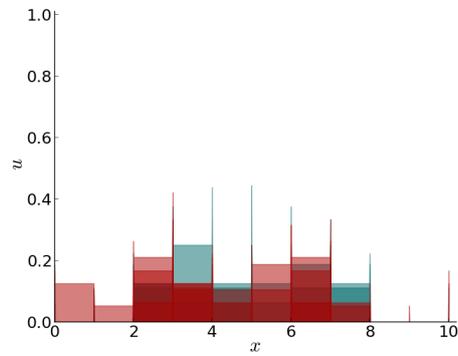
Figure E.7: Fuzzy sets of cake E for the attribute *crumbly*.



(a) Type-1; $c'_p = 0.199$

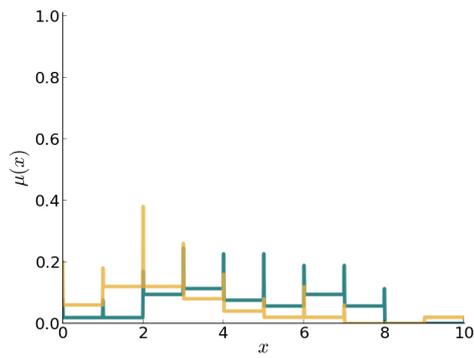


(b) Interval Type-2; $c'_p = -0.302$

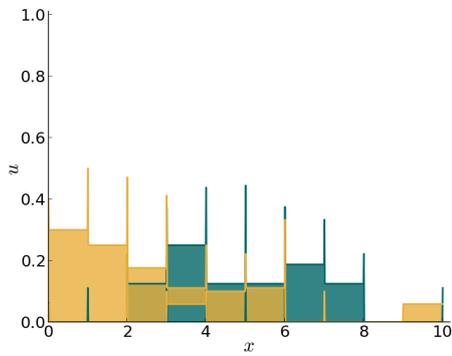


(c) General Type-2; $c'_p = 0.346$

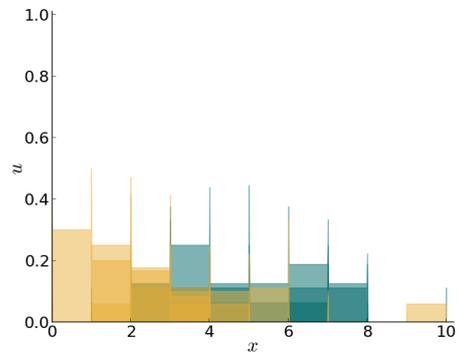
Figure E.8: Fuzzy sets of cakes **A** and **E** for the attribute *crumbly* and their incompatibility (c'_p).



(a) Type-1; $c'_p = -0.438$

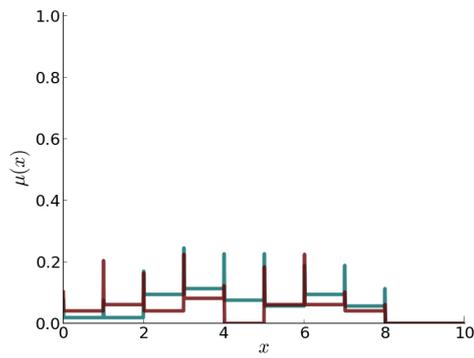


(b) Interval Type-2; $c'_p = -0.563$

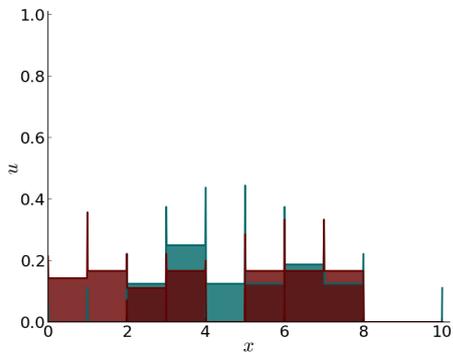


(c) General Type-2; $c'_p = -0.499$

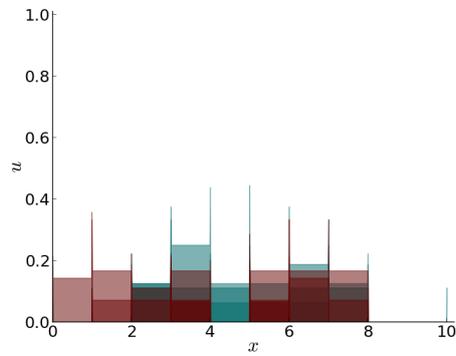
Figure E.9: Fuzzy sets of cakes B and E for the attribute *crumbly* and their incompatibility (c'_p).



(a) Type-1; $c'_p = -0.315$

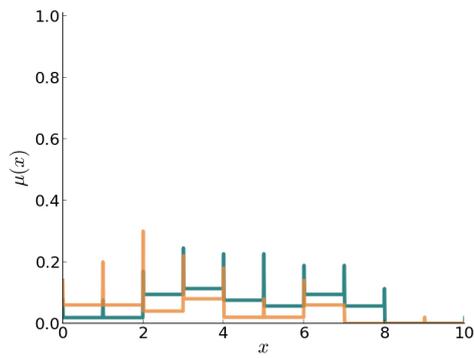


(b) Interval Type-2; $c'_p = -0.364$

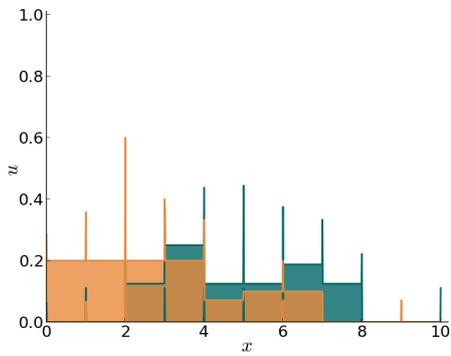


(c) General Type-2; $c'_p = -0.385$

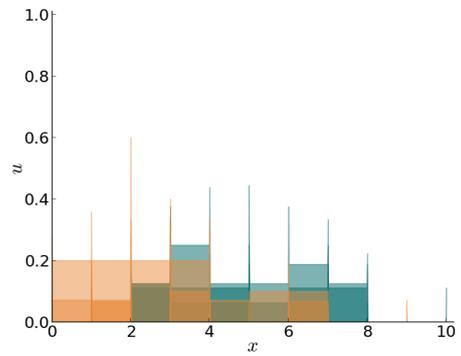
Figure E.10: Fuzzy sets of cakes C and E for the attribute *crumbly* and their incompatibility (c'_p).



(a) Type-1 $c'_p = -0.427$

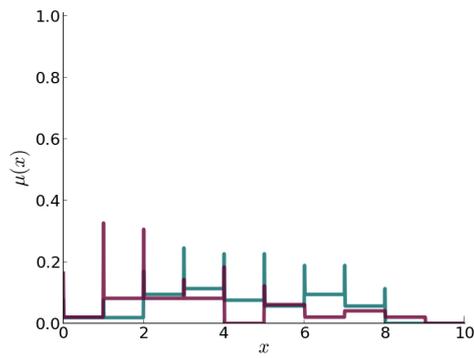


(b) Interval Type-2 $c'_p = -0.491$

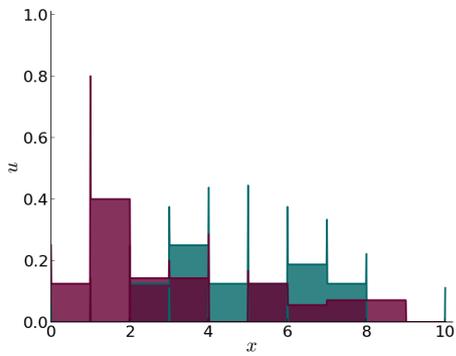


(c) General Type-2 $c'_p = -0.584$

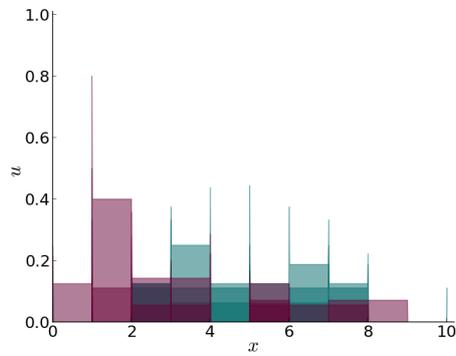
Figure E.11: Fuzzy sets of cakes D and E for the attribute *crumbly* and their incompatibility (c'_p).



(a) Type-1 $c'_p = -0.409$



(b) Interval Type-2 $c'_p = -0.583$



(c) General Type-2 $c'_p = -0.623$

Figure E.12: Fuzzy sets of cakes F and E for the attribute *crumbly* and their incompatibility (c'_p).

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