Products of Eisenstein series, their L-functions, and Eichler cohomology for arbitrary real weights.

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Abstract

One topic of this thesis are products of two Eisenstein series. First we investigate the subspaces of modular forms of level N that are generated by such products. We show that if the weight k is greater than 2, for many levels, one can obtain the whole of $\mathcal{M}_k(N)$ from Eisenstein series and products of two Eisenstein series. We also provide a result in the case $k = 2$ and treat some spaces of modular forms of non-trivial nebentypus. We then analyse the L-functions of products of Eisenstein series. We reinterpret a method by Rogers–Zudilin and use it in two applications, the first concerning critical L-values of a product of two Eisenstein series, and the second special values of derivatives of L-functions.

The last part of this thesis deals with the theory of Eichler-cohomology for arbitrary real weights, which was first developed by Knopp in 1974. We establish a new approach to Knopp's theory using techniques from the spectral theory of automorphic forms, reprove Knopp's main theorems, and also provide a vector-valued version of the theory.

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Contents

Chapter 1

Introduction

Let $\mathcal{M}_k(N)$ be the space of weight k modular forms for the congruence group

$$\Gamma_0(N) = \{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) | \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod N\}.$$

These are holomorphic functions on the upper half plane $\mathcal{H}$ that satisfy

$$f(\frac{az+b}{cz+d}) = (cz+d)^k f(z), \text{ for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$$

for all $z \in \mathcal{H}$. In addition they are required to be holomorphic at each rational number, a condition that we explain in §1.1. A modular form that vanishes at each rational number is called a cusp form and the space of cusp forms is denoted by $\mathcal{S}_k(N)$. The space $\mathcal{M}_k(N)$ splits into a direct sum

$$\mathcal{M}_k(N) = \mathcal{S}_k(N) \oplus \mathcal{E}_k(N),$$

where $\mathcal{E}_k(N)$ is the Eisenstein subspace, generated by Eisenstein series.

To each modular form f we can associate the L-function $L(f,s)$, a meromorphic function in s. While cusp forms and their L-functions are the subject of many conjectures and open problems in number theory, the Eisenstein subspace is very well understood. The Eisenstein series that form its basis have explicit and rather simple Fourier expansions and their L-functions come from the Riemann zeta function or Dirichlet L-functions.

A central topic of this thesis are products of two Eisenstein series. A product of Eisenstein series is, in general, not an element of $\mathcal{E}_k(N)$. Hence we can generate cusp forms by taking linear combinations of products of Eisenstein series. In Chapter 2 we study the space of functions that is generated by linear combinations of products of two Eisenstein series and show that in many cases this space equals the whole of $\mathcal{M}_k(N)$. This leads to the idea that one can study L-functions of cusp forms by analysing the L-functions of products of Eisenstein series. In the next chapter, Chapter 3 we derive relations between special values of the L-functions of different products of Eisenstein series and also a formula for the special value of a derivative of the L-function of an Eisenstein series in terms of L-values of products.

A classical example of a representation of a cusp form as a linear combination of products of Eisenstein series is the discriminant modular form Δ which can be defined by

$$\Delta = \frac{E_4E_8 - E_6^2}{1728},$$

where, for even k, $E_k = 1 - \frac{2k}{B_k}\sum \sigma_{k-1}(n)q^n$ is the normalised Eisenstein series of weight k. A classical result in the theory of modular forms is that every modular form of even weight k for the group $\mathrm{SL}_2(\mathbb{Z})$ is a linear combination of product of the Eisenstein series E_4 and E_6. Allowing all Eisenstein series as factors it suffices to look at products of at most two of them. The following theorem follows directly from results by Kohnen–Zagier in [KZ84].

Theorem 1.0.1. *Let $k \geq 4$ be an even integer and $\mathcal{Q}_k(1)$ be the space of modular forms generated by the products E_lE_{k-l} for even $l \in \{4 \dots k-4\}$. Then*

$$\mathcal{M}_k(1) = \mathcal{Q}_k(1) + \mathcal{E}_k(1).$$

The main theorem in Chapter 2 is a generalisation of this theorem to modular forms with respect to more general congruence subgroups. The results in this chapter were obtained in collaboration with M. Dickson and will be included in a joint paper [DN15].

Before we state it we define the Eisenstein series that generate $\mathcal{E}_k(N)$. They are given by the Fourier expansions

$$E_l^{\phi,\psi}(z) = a_l^{\phi,\psi} + 2\sum_{n\geq 1} \sigma_{l-1,\phi,\psi}(n) q^n \in \mathcal{M}_l(M, \phi\psi),$$

where $q = e^{2\pi i z}$, ϕ and ψ are primitive characters of level M_1, M_2 with $M_1 M_2 = M \mid N$, and

$$a_l^{\phi,\psi} = \begin{cases} L(\psi, 1-l) & N_1 = 1, \\ L(\phi, 0) & N_2 = 1 \text{ and } l = 1, \\ 0 & \text{else.} \end{cases}$$

We require not only the functions $E_l^{\phi,\psi}$ but also their image under the operators B_d for $d \in \mathbb{N}$, which act on modular forms of weight l by

$$f|B_d(z) = d^{\frac{l}{2}} f(dz).$$

The main theorem of Chapter 2 shows that products of such Eisenstein series generate $\mathcal{M}_k(N)$ in many cases.

Theorem 1.0.2. *Let $N = N'p^n$ where N' is squarefree or twice a squarefree number and p is prime. Let $\mathcal{Q}_k(N)$ be the subspace of $\mathcal{M}_k(N)$ generated by the products*

$$E_l^{\phi,\psi}|B_{d_1 d} \cdot E_{k-l}^{\overline{\phi},\overline{\psi}}|B_{d_2 d}$$

for $1 \leq l \leq k-1$ and all pairs of primitive characters ϕ, ψ of modulus M_1, M_2 and $d_1, d_2, d \in \mathbb{Z}_{\geq 1}$ such that $\gcd(d_1 M_1, d_2 M_2) = 1$ and $d_1 M_1 d_2 M_2 d \mid N$. We exclude the case $\phi = \psi = \mathbf{1}$ and $l = 2$ or $l = k-2$. Then for even $k \geq 4$

$$\mathcal{M}_k(N) = \mathcal{Q}_k(N) + \mathcal{E}_k(N).$$

The case of weight 2 is different: One sees immediately from the Rankin–Selberg method that products of two Eisenstein series are orthogonal to every newform f with vanishing central L-value, i.e., $L(f,1) = 0$. Accordingly we define the space $\mathcal{S}_k^{\text{rk}=0}(N)$ to be generated by newforms and lifts of newforms with non-zero central L-value. We obtain the analogue of Theorem 1.0.2 subject to this constraint:

Theorem 1.0.3. *Let N and $\mathcal{Q}_2(N)$ be as in Theorem 1.0.2. Then*

$$\mathcal{S}_2^{rk=0}(N) + \mathcal{E}_2(N) = \mathcal{Q}_2(N) + \mathcal{E}_2(N).$$

We also prove this theorem for modular forms of prime level and non-trivial nebentypus.

One of the main ingredients in the proofs of Theorems 1.0.2 and 1.0.3 is a vanishing result that is of independent interest. To state it, let us define twists of modular forms by characters first: if α is a Dirichlet character modulo M and $f = \sum a_n q^n$ is a modular form for the group $\Gamma_0(N)$, then $f_\alpha = \sum \alpha(n) a_n q^n$, the twist of f by α, is again a modular form of the same weight as f and level dividing NM^2. Twisting preserves cusp forms but twisting a newform does not necessarily produce a newform again. Our vanishing theorem follows from the theory of modular symbols and results of Atkin and Li [AL78] on the action of Atkin–Lehner operators on twists of newforms. For detailed definitions of the new subspace $\mathcal{S}_k^{\mathrm{new}}(N)$ and Atkin–Lehner operators see §1.1.3.

Theorem 1.0.4. *Let N be as in Theorem 1.0.2 and $f = \sum a_n q^n \in \mathcal{S}_k^{new}(N)$ be an eigenfunction of all Atkin–Lehner operators. Suppose that*

$$L(f_\alpha, l) = 0$$

for $1 \leq l \leq k-1$ and all primitive characters α modulo $M|N$ such that $\alpha(-1) = (-1)^l$. Then $f = 0$.

Before we describe possible applications of Theorems 1.0.2 and 1.0.3 and give a review of related results in the literature, we give several examples. The examples were computed with the Sage Mathematics Software [Sage] (for more of them see §2.7):

1. $N = 1, k = 12$: The most well-known example is of course the discriminant modular form, which, in our normalisation, becomes

$$\Delta = \frac{50}{3} E_4^{\mathbf{1},\mathbf{1}} E_8^{\mathbf{1},\mathbf{1}} - \frac{147}{4} (E_6^{\mathbf{1},\mathbf{1}})^2.$$

2. $N = 11, k = 2$: Let ϕ be the character modulo 11 that maps 2 to ζ_{10} and ψ the character that maps 2 to ζ_{10}^3. Let f be the only newform of level 11. Then

$$f = \frac{1}{5}(-2\zeta_{10}^3 + 2\zeta_{10}^2 - \frac{1}{4}) E_1^{\mathbf{1},\phi} E_1^{\mathbf{1},\overline{\phi}} + \frac{1}{5}(2\zeta_{10}^3 - 2\zeta_{10}^2 - \frac{9}{4}) E_1^{\mathbf{1},\psi} E_1^{\mathbf{1},\overline{\psi}}.$$

3. $N = 32, k = 2$: Let χ_4 be the primitive character modulo 4 and α the primitive character modulo 32 that maps 31 to 1 and 5 to ζ_8. Let f be the only newform of level 32. Then

$$f = \frac{1}{8}(\zeta_8^3 - \zeta_8^2 + \zeta_8 - 1) E_1^{\mathbf{1},\chi_4\alpha} E_1^{\mathbf{1},\chi_4\overline{\alpha}} + \frac{1}{4}(\zeta_8^3 + \zeta_8^2) E_1^{\mathbf{1},\chi_4\alpha^2} \cdot E_1^{\mathbf{1},\chi_4\overline{\alpha}^2} | B_2.$$

A representation of a newform f as a linear combination of products of Eisenstein series has several applications. Of course we can tell, directly from Theorem 1.0.3, that the newforms in examples 2 and 3 have non-vanishing central L-value without the need of calculating $L(f, 1)$.

Also, as remarked in [Rau14], one can use an expression for a modular form as a sum of products of Eisenstein series to compute Fourier expansions at every cusp. This is particularly simple in our case: using results of [Wei77] we know the expansion of an Eisenstein series at any cusp of $\Gamma_0(N)$, so given a newform of level $N = N'p^n$ as above, one can provide an algorithm for calculating the expansion of a newform of $\mathcal{S}_k(N)$ for

$k \geq 4$ (resp. $\mathcal{S}_2^{\mathrm{rk}=0}(N)$ if $k = 2$) at any cusp of $\Gamma_0(N)$. When $N = N'$ is squarefree one can obtain the expansions at other cusps more directly from the expansion at infinity by use of Atkin–Lehner operators (c.f. [Asa76]), but the Fourier expansions at other cusps are much more mysterious and less accessible when the level is not squarefree.

Similarly, [Wei77] also describes the action of the Atkin–Lehner operators on Eisenstein series, so once one has an explicit representation of a newform f as a linear combination of products of Eisenstein series it is straightforward to compute the Atkin–Lehner eigenvalues and the root number of f.

The result in Chapter 2 generalise previous results by Kohnen–Imamoḡlu [IK05], where the case $N = 2$ is studied, and Kohnen–Martin [KM08], where Theorems 1.0.2 and 1.0.3 are proved for odd prime levels.

Raum [Rau14] proves a different, rather general result for vector-valued modular forms: Let $k \geq 12$ be an integer, let ρ be a representation of $\mathrm{SL}_2(\mathbb{Z})$ on a complex vector space V such that $\ker(\rho)$ contains a congruence subgroup, and define $\mathcal{M}_k(\rho)$ to be the space of V-valued functions transforming as modular forms for the automorphy factor $\gamma \mapsto (cz+d)^{-k}\rho(\gamma^{-1})$. Then

$$\mathcal{M}_k(\rho) = \mathcal{E}_k(\rho) + \mathrm{span}_{\phi:\rho_M \otimes \rho_{M'} \to \rho} \left(T_M E_l \otimes T_{M'} E_{k-l} \right),$$

where $4 \leq l \leq k-4$, ρ_M is the permutation representation on $\Gamma_0(M) \backslash \mathrm{SL}_2(\mathbb{Z})$, the E_k are corresponding vector-valued Eisenstein series, and the T_M are certain natural vector-valued Hecke operators. Apart from the inclusion of low weights, our results differ from those of [Rau14] since our generating set does not involve Hecke operators.

In [BG01] and [BG03] Borisov–Gunnells use the theory of toric varieties to show that certain spaces of toric modular forms are generated by products of toric Eisenstein series. One of their results is that for any N and $k > 2$ the space of modular forms of weight k with respect to the congruence group

$$\Gamma_1(N) = \{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) | \gamma \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod{N}\}$$

can be spanned by products of toric Eisenstein series, while for $k = 2$ they only obtain a subspace of $\mathcal{M}_2(\Gamma_1(N))$.

Since the main theorems of [BG01] and [BG03] apply in greater generality than Theorem 1.0.2 and Theorem 1.0.3, it is important to point out some differences between the two results. The generating sets for $\mathcal{M}_k(N)$ that we give for $k > 2$ have size $\mathcal{O}(kN^{1+\epsilon})$ for any $\epsilon > 0$, while the generating sets for $\mathcal{M}_k(\Gamma_1(N))$ obtained in [BG03] have size $\mathcal{O}(kN^2)$. As we mention in the applications below, an advantage of working with the well-studied Eisenstein series $E_l^{\phi,\psi}$ is that their Fourier expansions at every cusp of $\Gamma_0(N)$ are known and also the action of the Atkin–Lehner operators on them. Lastly the proofs of our Theorems are shorter than the proofs of the main theorems of [BG01] and [BG03] and do not make use of the theory of toric varieties.

In Chapter 3 we discuss another application of representations of f as above, that was recently found by Rogers and Zudilin [RZ12] in connection with Boyd's conjectures and special values of the L-function of f. Before we describe the Rogers–Zudilin method we give a brief overview of Boyd's beautiful conjectures. The logarithmic Mahler measure of a Laurent polynomial $P(t_1, \dots, t_n) \in \mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm}]$ is defined by

$$m(P) = \int_0^1 \dots \int_0^1 \log |P(e^{2\pi i \theta_1}, \dots, e^{2\pi i \theta_n})| d\theta_1 \dots d\theta_n.$$

It was first noticed by Deninger [Den97] that often $m(P)$ can be interpreted as a Deligne period of a mixed motive. For $P = X^2Y + Y^2X + XY + X + Y$ the expected value of this Deligne period, according to the Bloch–Beilinson conjectures, is a rational multiple of $L'(E, 0)$, where E is the elliptic curve that is the projective closure of the zero locus of P. Motivated by Deninger's findings, Boyd performed computer calculations that indicated that indeed

$$m(P) = L'(E, 0), \tag{1.0.1}$$

and that similar formulas hold for many more elliptic curves. He went on to produce a big list of conjectural relations between Mahler measures and special values of L-functions of elliptic curves or their derivatives in [Boy98]. In 2012 Rogers–Zudilin [RZ12] gave a proof of some of these identities, e.g.,

$$m(X^2Y + Y^2X + 2XY + X + Y) = L'(E_{24}, 0), \tag{1.0.2}$$

where E_{24} is the elliptic curve of conductor 24; the projective closure of the polynomial on the left. Let $f \in \mathcal{S}_2(24)$ be the unique newform with the same L-function as E_{24}. By the functional equation of $L(f, s)$ the right hand side in (1.0.2) can be written as

$$L'(E_{24}, 0) = \frac{6}{\pi^2} L(E_{24}, 2) = -24 \int_0^\infty f(z) z dz. \tag{1.0.3}$$

Rogers–Zudilin start by writing f as a linear combination of products of two weight 1 Eisenstein series. They then swap integration and summation over the Fourier coefficients in (1.0.3) and apply a simple but ingenious change of variables to the integrals in the sum. Swapping summation and integration back, (1.0.3) becomes an integral over elementary functions. Finally they use properties of hypergeometric functions to finish the proof of (1.0.2).

Using the same method, Rogers–Zudilin proved (1.0.1) in 2014, and many other cases of Boyd's conjectures were settled similarly in [Bru] and [Zud14].

In Chapter 3 we reinterpret the Rogers–Zudilin method in terms of a correspondence between modular forms. Most of the work presented in that paper was done in collaboration with N. Diamantis and F. Strömberg and appeared in a joint article [DNS15].

The correspondence associates to a pair of functions F_1, F_2 and $s \in \mathbb{C}$ a new function $\Phi_s(F_1, F_2)$ which, when F_1 and F_2 are connected to modular forms, satisfies properties related to modularity for special values of s. Our main theorem, Theorem 3.2.2, connects the Mellin transform of the product F_1F_2 with the Mellin transform of functions associated to F_1 and F_2 via our correspondence. This is achieved using a simple "duality" relation (Lemma 3.2.1), which reformulates the key change of variables in Rogers–Zudilin's method. The content of the main theorem can be summarised as:

Theorem 1.0.5. *Let F_1 and F_2 be functions on the upper half-plane given by*

$$F_1(z) = \sum_{m_1,n_1 \geq 1} a_1(m_1)b_1(n_1)e^{2\pi i m_1 n_1 z},$$
$$F_2(z) = \sum_{m_2,n_2 \geq 1} a_2(m_2)b_2(n_2)e^{2\pi i m_2 n_2 z},$$

where we assume that the Fourier coefficients grow at most polynomially. For $j = 1, 2$ we set

$$f_j(z) = \sum_{m_j,n_j \geq 1} b_j(n_j)e^{2\pi i m_j n_j z} \quad \text{and} \quad g_j(z) = \sum_{m_j,n_j \geq 1} a_j(m_j)e^{2\pi i m_j n_j z}.$$

Then we have the following relation between Mellin transforms

$$\mathcal{M}(F_1 \cdot F_2|_0 W_N)(s) = \mathcal{M}(\Phi_{s+1}(f_1, f_2) \cdot (\Phi_{-s+1}(g_2, g_1)|_0 W_N))(s) \quad \text{for all } s \in \mathbb{C},$$

where $\Phi_s(f, g)$ is the function associated to f and g as described in Section 3.1.

In the case where F_1 and F_2 are Eisenstein series, the functions that appear in Theorem 1.0.5, $\Phi_{s+1}(f_1, f_2)$ and $\Phi_{-s+1}(g_2, g_1)$, are closely connected to Eisenstein series in many cases. We make use of this fact in two applications. They are stated in terms of completed L-functions which, for a modular form f of level N, are defined as

$$\Lambda(f, s) = \Gamma(s)\left(\frac{\sqrt{N}}{2\pi}\right)^s L(f, s).$$

The first one can be sketched in the following form:

Theorem 1.0.6 (Sketch of Theorem 3.4.2)**.** *If E is in a certain subspace of the weight 2 Eisenstein space on $\Gamma_1(N)$, then*

$$\Lambda'(E, 1) = \Lambda(\tilde{E}, 1) + C$$

for an explicitly determined constant C and an explicit element $\tilde{E}$ in the weight 1 Eisenstein space.

The other application gives a duality between L-values of products of Eisenstein series.

Theorem 1.0.7 (Special case of Theorem 3.3.1)**.** *Let χ_1, χ_2 and ψ_1, ψ_2 be pairs of non-trivial primitive Dirichlet characters modulo M_1, M_2 and N_1, N_2, respectively. Let $k \geq 1$, $l \geq 2$ such that $(\chi_1 \cdot \chi_2)(-1) = (-1)^l$ and $(\psi_1 \cdot \psi_2)(-1) = (-1)^k$. Then for an integer $j \in \{1, \ldots, k + l - 1\}$ such that $(\chi_1 \cdot \psi_1)(-1) = (-1)^{k-j}$ we have*

$$\Lambda(E_l^{\chi_1,\chi_2} \cdot E_k^{\bar{\psi}_2,\bar{\psi}_1}|B_{M_1M_2}, j) = C \cdot \Lambda(E_j^{\chi_1,\psi_2} \cdot E_{k+l-j}^{\bar{\chi}_2,\bar{\psi}_1}|B_{M_1N_2}, l) \tag{1.0.4}$$

where C is an explicit *algebraic number.*

While Theorems 1.0.6 and 1.0.7 have their independent interest, Theorem 1.0.5 was derived with applications to L-functions of newforms in mind. The Rogers–Zudilin method has been successful in proving statements about L-values of newforms of weight 2, like Boyd's conjectures, or the fact that these L-values should be periods in the sense of Kontsevich–Zagier [KZ01] (see [Zud13]). One future goal of the author of this thesis will be to apply Theorem 1.0.5 to the study of L-values of newforms of higher weight.

One crucial fact about L-functions that is used in Chapter 2 is Theorem 1.0.4; if enough L-values associated to a modular form f vanish, then so does f. This follows from one of the main theorems in the theory of modular symbols, which is closely connected to the Eichler–Shimura isomorphism. This isomorphism was first discovered by Eichler [Eic57] and there are many different ways to state it. We choose a version described in [Ant92], which is close to Shimura's formulation in [Shi59]. Let $\Gamma = \Gamma_0(N)$ and $k \geq 2$ be an even integer. To $f \in \mathcal{S}_k(\Gamma)$ and $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ we associate the polynomial

$$\sigma_{f,\gamma}(X) = \int_{\gamma^{-1}\infty}^{\infty} f(\tau)(\tau - X)^{k-2} d\tau. \tag{1.0.5}$$

Here the paths of integration are contained in the upper half plane (except for the endpoints). Let $\mathbb{R}[X]_{k-2}$ and $\mathbb{C}[X]_{k-2}$ be the space of polynomials of degree $\leq k-2$ with real and complex coefficients respectively. The group Γ acts on each of these spaces via the slash action $|_{2-k}$ and it is easy to show that

$$\sigma_f :\ \gamma \mapsto \sigma_{f,\gamma}(X)$$

is a cocycle with values in $\mathbb{C}[X]_{k-2}$, i.e., it satisfies

$$\sigma_{f,\gamma\delta}(X) = \sigma_{f,\gamma}(X)|_{2-k}\delta + \sigma_{f,\delta}(X),\ \forall \gamma, \delta \in \Gamma.$$

It is in fact a parabolic cocycle and the map $f \mapsto \sigma_f$ induces a linear map from $\mathcal{S}_k(\Gamma)$ to the parabolic cohomology group $\tilde{H}^1(\Gamma, \mathbb{C}[X]_{k-2}) \subseteq H^1(\Gamma, \mathbb{C}[X]_{k-2})$ (for definitions see §4.1.1). Denoting by $\mathrm{Re}(\sigma_{f,\gamma}(X))$ the polynomial that has as coefficients the real parts of the coefficients of $\sigma_{f,\gamma}(X)$, we can state the Eichler–Shimura isomorphism as follows.

Theorem 1.0.8 (Eichler–Shimura isomorphism)**.** *For all $k \geq 2$ we have an isomorphism*

$$\mathcal{S}_k(\Gamma) \overset{\cong}{\to} \tilde{H}^1(\Gamma, \mathbb{R}[X]_{k-2}).$$

given by

$$f \mapsto [\mathrm{Re}(\sigma_f)],$$

where $\mathrm{Re}(\sigma_f)$ is the cocycle that maps γ to $\mathrm{Re}(\sigma_{f,\gamma}(X))$ and $[\mathrm{Re}(\sigma_f)]$ is its associated cohomology class.

Theorem 1.0.8 has many applications in the theory of modular forms and the study of critical values of their L-functions, e.g., in algebraicity results like Manin's period theorem [Man73]. As mentioned before it is also an essential ingredient in the theory of modular symbols. Indeed the maps ξ_f that we use in §2.2, are closely connected to σ_f by the relation

$$\xi_f([(X-Y)^{k-2},g]) = \int_0^\infty f|_k g(\tau)(\tau-1)^{k-2}d\tau = \sigma_{f|_k g,\sigma}(1),$$

where $g \in \mathrm{SL}_2(\mathbb{Z})$ and $\sigma = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$.

The subject of Chapter 4 is an analogue of Theorem 1.0.8 in the case of arbitrary real weight. Knopp first formulated it in 1974 [Kno74]. Let $f \in \mathcal{S}_k(\Gamma, v)$, where v is a multiplier system of weight k for Γ. The first problem one encounters when allowing arbitrary real weights $k \in \mathbb{R}$, is that the factor $(\tau - X)^{k-2}$ in the integrand of (1.0.5) is no longer a polynomial in X. Viewing it as a function in $X = z$ it is not even well-defined for z in the upper half plane. Knopp solved this problem by conjugating z and conjugating the whole integral in (1.0.5) again, so that

$$\phi_f^\infty : \gamma \mapsto \phi_{f,\gamma}^\infty = \left[\int_{\gamma^{-1}\infty}^\infty f(\tau)(\tau-\overline{z})^{k-2}d\tau\right]^-$$

is, once we choose a branch for the exponentiation by $k-2$, a well-defined holomorphic function on the upper half plane. In fact $\phi_{f,\gamma}^\infty$ is an element of $\mathcal{P}$, a space of holomorphic functions with polynomial growth conditions. Viewing $\mathcal{P}$ as a Γ-module under the $|_{2-k,\overline{v}}$ action, ϕ_f^∞ is a cocycle of Γ with values in $\mathcal{P}$. With this Γ-action on $\mathcal{P}$ we denote the first cohomology group with coefficients in $\mathcal{P}$ by $H^1_{2-k,\overline{v}}(\Gamma, \mathcal{P})$. With the larger coefficient module $\mathcal{P}$ all cocycles are parabolic, i.e.,

$$\tilde{H}^1_{2-k,\overline{v}}(\Gamma, \mathcal{P}) = H^1_{2-k,\overline{v}}(\Gamma, \mathcal{P}).$$

This is the content of Theorem 4.1.3.

Knopp conjectured that the map $f \mapsto \phi_f^\infty$ is an isomorphism from $\mathcal{S}_k(\Gamma, v)$ to $\tilde{H}^1_{r,v}(\Gamma, \mathcal{P})$ but was only able to prove this for the cases $k \geq 2$ and $k \leq 0$. In the case $k > 2$ he relied heavily on previous work by Niebur [Nie74] on automorphic integrals. Later, in 2000, a partial result on the missing cases in Knopp's conjecture was obtained by Wang [Wan00] and it was resolved in 2010 by Knopp and Mawi [KM10], using Petersson's principal part theorem and generalised Poincaré series.

Theorem 1.0.9. *For all $k \in \mathbb{R}$ we have an isomorphism*

$$\mathcal{S}_k(\Gamma, v) \overset{\cong}{\to} H^1_{2-k,\overline{v}}(\Gamma, \mathcal{P})$$

given by

$$f \mapsto [\phi_f^\infty].$$

A recent preprint [BCD14] by Bruggeman, Choie and Diamantis gives a similar isomorphism for a much wider class of automorphic forms. They also provide several motivations to study cocycles of real weight. One of them is a formula of Goldfeld [Gol95] that suggests a connection between special values of derivatives of L-functions and cocycles. To be precise, let $f = \sum_{n\geq 1} a_n q^n$ be a Hecke cusp form of weight 2 for the group $\Gamma_0(N)$, and assume that f is invariant under the Fricke involution $W_N = \left(\begin{smallmatrix} 0 & -1 \\ N & 0 \end{smallmatrix}\right)$. The L-function of

f, $L(f,s)$, is defined as the analytic continuation to $\mathbb{C}$ of the Dirichlet series $\sum a_n n^{-s}$. In [BCD14, §9.4] it is shown that Goldfeld's formula leads to the following expression:

$$-\pi i r L'(f,1) + \mathcal{O}_{r\to 0}(r^2) = \phi_{f_r}(\sigma)(0),$$

where $f_r(z) = f(z)(\eta(z)\eta(Nz))^r$ is a cusp form of weight $2+r$.

In Chapter 4 we present a new proof of Theorem 1.0.9 for positive weights $k \neq 1$ that views the isomorphism in Knopp and Mawi's theorem as a duality. The results in that chapter have been accepted for publication in the Ramanujan Journal [Neu16]. The key construction is a pairing between $\mathcal{S}_k(\Gamma, v)$ and $H^1_{2-k,\overline{v}}(\Gamma, \mathcal{P})$ which we introduce in Section 4.3 when $k > 0$. In Section 4.4 we show that this pairing is perfect if $k \neq 1$, which implies Theorem 4.2.1 for the weights we consider. The proof also implies Theorem 4.2.1 for the weights $k \leq 0$, and hence for all real weights except $k = 1$.

One of the advantages of the new proof is that once all the constructions are in place the problem can be solved with standard techniques from the spectral theory of automorphic forms. With the new pairing some previously difficult facts become remarkably easy to derive. For example one can see immediately that $f \mapsto [\phi_f^\infty]$ is injective from the fact that $(f, [\phi_f^\infty]) = (f, f)$, where the first pairing is the one we construct and the latter is the Petersson inner product. Another advantage is, that the proof can easily be generalised to the case of vector-valued cusp forms. We sketch this generalisation in the last section of this chapter.

1.1 Preliminaries

1.1.1 Modular forms

Let $\mathcal{H} = \{x+iy|y>0\}$ be the upper half plane and $\overline{\mathcal{H}} = \mathcal{H} \cup \mathbb{R}^* \cup i\infty$ be its closure in $\mathbb{P}^1(\mathbb{C})$. The group $\mathrm{GL}_2^+(\mathbb{R})$ of real 2×2 matrices with positive determinant acts on $\overline{\mathcal{H}}$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}.$$

Since scalar matrices act trivially, this action induces an action of $\mathrm{PSL}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})/\{\pm \mathrm{I}\}$, where $\mathrm{SL}_2(\mathbb{R}) \leq GL_2^+(\mathbb{R})$ is the subgroup of matrices with determinant 1.

Let $k \in \mathbb{Z}$ be an integer. $\mathrm{GL}_2^+(\mathbb{R})$ also acts on functions on the upper half plane $\mathcal{H}$ by the weight k slash action $|_k$

$$f|_k\gamma(z) = \frac{(\det\gamma)^{k/2}}{j(\gamma,z)^k} f(\gamma z),$$

where $j(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), z) = cz+d$.

We denote the modular group $\mathrm{SL}_2(\mathbb{Z})$ by $\Gamma(1)$ and note that it is generated by the translation $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $S = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. We introduce the following important subgroups of $\Gamma(1)$:

$$\Gamma(N) = \{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} | \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod N\} \tag{1.1.1}$$

$$\Gamma_1(N) = \{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} | \gamma \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod N\} \tag{1.1.2}$$

$$\Gamma_0(N) = \{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} | \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod N\} \tag{1.1.3}$$

The group $\Gamma(N)$ is normal in $\Gamma(1)$ and called the *principal congruence group* of level N. A *congruence (sub-)group* of level N is any subgroup of $\Gamma(1)$ that contains $\Gamma(N)$, e.g., $\Gamma_0(N)$ and $\Gamma_1(N)$. Let $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$ be a congruence subgroup.

Definition 1.1.1. Let k be an integer. A holomorphic function $f : \mathcal{H} \to \mathbb{C}$ is called *weakly modular* of weight k with respect to Γ if

$$f|_k\gamma = f, \quad \forall \gamma \in \Gamma.$$

Since there exists an N such that $\left(\begin{smallmatrix} 1 & N \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma$, a weakly modular function f with respect to Γ must be invariant under translation by N, i.e., $f(z+N) = f(z)$ for all $z \in \mathcal{H}$. This means that for $\mathrm{Im}\, z \gg 0$ we have a Fourier expansion of the form

$$f(z) = \sum_{n \geq n_0} a_n q_N^n, \quad \text{where } q_N = e^{\frac{2\pi i}{N} z}. \tag{1.1.4}$$

We say that f is *holomorphic at* $i\infty$ if in (1.1.4) $a_n = 0$ for $n < 0$. This is equivalent to the existence of the limit $\lim_{\mathrm{Im}\, z \to \infty} f(z)$. If f is weakly modular of weight k with respect to a congruence subgroup Γ, then $f|_k\alpha$ is weakly modular with respect to $\alpha^{-1}\Gamma\alpha$ for any $\alpha \in \mathrm{SL}_2(\mathbb{Z})$. By exercise 1.2.5 in [DS10] $\alpha^{-1}\Gamma\alpha$ is again a congruence subgroup so $f|_k\alpha$ has a Fourier-expansion of the form (1.1.4) (for a possibly different choice of N) and the following definition is justified.

Definition 1.1.2. A function $f : \mathcal{H} \to \mathbb{C}$ is a *modular form* of weight k with respect to Γ if

1. f is weakly modular of weight k with respect to Γ.
2. $f|_k\alpha$ is holomorphic at $i\infty$ for all $\alpha \in \mathrm{SL}_2(\mathbb{Z})$.

If in addition $a_0 = 0$ in the Fourier expansion at $i\infty$ of $f|_k\alpha$ for all $\alpha \in \mathrm{SL}_2(\mathbb{Z})$, then f is a *cusp form.* We denote the space of modular forms of weight k with respect to Γ by $\mathcal{M}_k(\Gamma)$. The space of cusp forms is denoted by $\mathcal{S}_k(\Gamma)$. If f is a weight k modular form then we often write $f|\gamma$ instead of $f|_k\gamma$ for any $\gamma \in \mathrm{GL}_2^+(\mathbb{R})$.

One of the key facts about modular forms is that for any congruence group Γ the space $\mathcal{M}_k(\Gamma)$ is finite dimensional. This implies that in order to determine a modular form of a given weight and congruence group one only needs to know a finite number of its Fourier coefficients.

Let χ be a Dirichlet character modulo N. It can be extended to a character of $\Gamma_0(N)$ by defining $\chi((\begin{smallmatrix} a & b \\ c & d \end{smallmatrix})) = \chi(d)$. We write $\mathcal{M}_k(N, \chi)$ (or $\mathcal{M}_k(N)$, if χ is principal) for the space of weight k modular forms for $\Gamma_1(N)$ that satisfy the transformation law

$$f|_k\gamma = \chi(\gamma)f, \ \forall \gamma \in \Gamma_0(N)$$

and $\mathcal{S}_k(N, \chi)$ for the subspace of cusp forms in $\mathcal{M}_k(N, \chi)$. A modular form in $\mathcal{M}_k(N, \chi)$ is said to have *nebentypus* χ. Then we have

$$\mathcal{M}_k(\Gamma_1(N)) = \oplus_\chi \mathcal{M}_k(N, \chi),$$

where the sum is over all Dirichlet characters modulo N.

We write $\mathbf{1}_N$ for the principal character modulo N, which satisfies $\mathbf{1}_N(n) = 1$ for $(n, N) = 1$, and $\mathbf{1}_N(n) = 0$ otherwise. The trivial character is denoted by $\mathbf{1}$; it satisfies $\mathbf{1}(n) = 1$ for all n. Any character χ modulo $N = \prod_{p \text{ prime}} p^{v_p(N)}$ splits into a product of characters modulo the prime powers dividing N:

$$\chi = \prod_{\substack{p|N \\ p \text{ prime}}} \chi_p,$$

where χ_p is a character modulo $p^{v_p(N)}$ for each p. If S is a set of prime divisors of N, then we write $\chi_S = \prod_{p \in S} \chi_p$ for the S-part of χ.

1.1.2 Petersson inner product

Definition 1.1.3. A *fundamental domain* $\mathcal{F}$ for a subgroup of $\mathrm{SL}_2(\mathbb{R})$ is a connected open subset of $\mathcal{H}$ that satisfies the following properties:

1. For every $z \in \mathcal{H}$ there exists $\gamma \in \Gamma$ such that $\gamma z \in \overline{\mathcal{F}}$, where $\overline{\mathcal{F}}$ denotes the topological closure of $\mathcal{F}$.

2. Distinct points of $\mathcal{F}$ are not in the same Γ-orbit.

Definition 1.1.4. Let Γ be a congruence subgroup and $\mathcal{F}$ a fundamental domain for Γ. For $f, g \in \mathcal{M}_k(\Gamma)$ such that either f or g is a cusp form we define the *Petersson inner product*

$$\langle f, g\rangle = \int_{\mathcal{F}} f(z)\overline{g(z)} y^k \frac{dxdy}{y^2}. \tag{1.1.5}$$

Since the hyperbolic measure $\mu = \frac{dxdy}{y^2}$ and the function $f(z)\overline{g(z)}y^k$ are both Γ-invariant, the integral in (1.1.5) does not depend on a choice of a fundamental domain $\mathcal{F}$.

1.1.3 Hecke operators and Atkin–Lehner theory

In this section we introduce Hecke operators on $\mathcal{M}_k(\Gamma_1(N))$ and recall some facts from Atkin–Lehner theory. For more details and proofs we refer the reader to the original article by Atkin and Lehner [AL70] or [DS10].

Definition 1.1.5. Let $f \in \mathcal{M}_k(\Gamma_1(N))$ and p be a prime. We define the *Hecke operators* T_p and U_q for primes p, q with $p \nmid N$ and $q \mid N$ on $\mathcal{M}_k(\Gamma_1(N))$ by

$$f|_k U_p = \sum_{j=0}^{p-1} f|_k \begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix} \qquad p|N, \tag{1.1.6}$$

$$f|_k T_p = \sum_{j=0}^{p-1} f|_k \begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix} + f|_k \begin{pmatrix} mp & n \\ Np & p \end{pmatrix} \qquad p \nmid N \text{ where } mp - nN = 1. \tag{1.1.7}$$

Let χ be a Dirichlet character modulo N. Again, we will often omit the weight k in the notation if the weight of f is clear. The action of the Hecke operators on a modular form $f = \sum a_n q^n$ in $\mathcal{M}_k(N, \chi)$ is given by

$$f|T_p = \sum_{n\geq 0} (a_{np} + \chi(p)p^{k-1}a_{n/p})q^n, \tag{1.1.8}$$

$$f|U_q = \sum_{n\geq 0} a_{np}q^n, \tag{1.1.9}$$

where we set $a_{n/p} = 0$ if $n/p \notin \mathbb{Z}$.

For $r \geq 1$ we define the Hecke operator T_{p^r} inductively by setting T_1 to be the identity operator and

$$T_{p^r} = T_p T_{p^{r-1}} - p^{k-1}\chi(p)T_{p^{r-2}}$$

and extend the U_q multiplicatively. Then we can define a Hecke operator T_n for any $n = \prod_{p\nmid N} p^{v_p(n)} \prod_{q|N} q^{v_q(N)}$ by

$$T_n = \prod_{p\nmid N} T_{p^{v_p(n)}} \prod_{q|N} U_{q^{v_q(n)}}.$$

Hecke operators map cusp forms to cusp forms and the operators T_p for $p \nmid N$ commute. Their adjoints on $\mathcal{S}_k(N,\chi)$ with respect to the Petersson inner product are given by

$$T_p^* = \chi(p)^{-1} T_p$$

and hence they are normal, i.e., they commute with their adjoints. Thus $\mathcal{M}(N,\chi)$ has an orthonormal basis of eigenvectors of all T_n where $(n,N)=1$.

The Hecke operators U_q for $q \mid N$ behave very differently. They are not normal operators in general so one cannot find always find an orthonormal basis of eigenvectors of all Hecke operators T_n.

A solution to this problem was given by Atkin–Lehner [AL70] with what is now known as Atkin–Lehner theory. They introduced the *old subspace* of $\mathcal{S}_k(\Gamma_1(N))$ defined by

$$\mathcal{S}_k(\Gamma_1(N))^{\mathrm{old}} = \bigcup_{M,d\colon Md \mid N} \mathcal{S}_k(\Gamma_1(M)) | B_d,$$

where B_d is the operator

$$f|_k B_d(z) = f|_k \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix}(z) = d^{k/2} f(dz).$$

The *new subspace* $\mathcal{S}_k(\Gamma_1(N))^{\mathrm{new}}$ is defined as the orthogonal complement of $\mathcal{S}_k(\Gamma_1(N))^{\mathrm{old}}$, and $\mathcal{S}_k^{\mathrm{old}}(N,\chi)$ and $\mathcal{S}_k^{\mathrm{new}}(N,\chi)$ are the intersections of the old and new subspace with $\mathcal{S}_k(N,\chi)$. The Hecke operators act on the old and new subspaces and one of the main results of Atkin–Lehner was that on $\mathcal{S}_k^{\mathrm{new}}(N,\chi)$ *all* Hecke operators are normal and commute with each other. There is therefore an orthogonal basis of common eigenfunctions of all Hecke operators on $\mathcal{S}_k^{\mathrm{new}}(N,\chi)$. One can show that if $f = \sum a_n q^n$ is such an eigenfunction, then $a_1 \neq 0$ and hence we can normalise the basis by setting $a_1 = 1$ for all eigenfunctions. Such a modular form is called a *newform* and they play an important role in the theory of modular forms. Newforms satisfy the property

$$f|T_n = a_n f, \ \forall n \in \mathbb{N},$$

where a_n is the n-th Fourier coefficient. By using the recursive definition of the Hecke operators we see that one can obtain all Fourier coefficients of a newform from the Fourier coefficients at primes.

Theorem 1.1.1. *Let $\mathcal{N}_k^{new}(N,\chi)$ be the set of newforms of $\mathcal{S}_k^{new}(N,\chi)$. Then the set*

$$\bigcup_{M: cond(\chi) | M | N} \ \bigcup_{d\colon Md|N} \mathcal{N}_k^{new}(N,\chi) | B_d$$

is a basis of $\mathcal{S}_k(N,\chi)$.

For a set of prime divisors S of N and a divisor M of N, we write M_S for the S-part of M, i.e. $\prod_{p\in S} p^{v_p(M)}$. By $\overline{S}$ we denote the complement of S in the set of prime divisors of N.

Definition 1.1.6. For a set of prime divisors S of N we define the *Atkin–Lehner operator*

$$W_S^N = \begin{pmatrix} N_S x & y \\ Nz & N_S w \end{pmatrix} \in \mathrm{M}_2(\mathbb{Z}),$$

where $y \equiv 1 \pmod{N_S}$, $x \equiv 1 \pmod{N_{\overline{S}}}$ and $\det W_S^N = N_S$.

In the case when S is the set of all primes dividing N we simply write W_N for $W_S^N = \left(\begin{smallmatrix} 0 & -1 \\ N & 0 \end{smallmatrix}\right)$. This Atkin–Lehner operator is often called the *Fricke-involution* and it acts on functions on the upper half plane by

$$f|_k W_N(z) = (\sqrt{N}z)^{-k} f\left(-\frac{1}{Nz}\right).$$

The following properties of W_S^N are well-known (see for example [AL78]):

Proposition 1.1.2. *(i) Let S be a set of prime divisors of N. If*

$$M = \begin{pmatrix} N_S x' & y' \\ Nz' & N_S w' \end{pmatrix}$$

is any matrix with $x', y', z', w' \in \mathbb{Z}$ of determinant N_S then

$$f|M = \overline{\chi}_S(y')\overline{\chi_{\overline{S}}}(x') f|W_S^N. \tag{1.1.10}$$

In particular, W_S^N does not depend on the choice of x, y, z, w.

(ii) Let $f \in \mathcal{M}_k(N, \chi)$. Then

$$f|W_S^N \in \mathcal{M}_k(N, \overline{\chi}_S \chi_{\overline{S}}),$$

and cusp forms are preserved. Furthermore

$$f|W_S^N|W_S^N = \chi_S(-1)\overline{\chi_{\overline{S}}}(N_S) f. \tag{1.1.11}$$

(iii) The adjoint of W_S^N on $\mathcal{M}_k(N, \chi)$ with respect to the Petersson inner product is given by

$$W_S^{N,*} = \chi_S(-1)\chi_{\overline{S}}(N_S) W_S^N.$$

(iv) Let p be a prime divisor of N such that $(p, S) = 1$. Then

$$f|U_p|W_S^N = \chi_S(p) f|W_S^N|U_p.$$

If $f \in \mathcal{M}_k(N)$ is a newform, then it is automatically an eigenfunction of all Atkin–Lehner operators. We denote the W_S^N-eigenvalue of f by $\lambda_S(f)$. If f is a newform in $\mathcal{M}_k(N, \chi)$, then W_S^N does not necessarily act on $\mathcal{M}(N, \chi)$. However $f|W_S^N$ will be a scalar multiple of a newform $g \in \mathcal{M}_k(N, \overline{\chi}_S \chi_{\overline{S}})$. The W_S^N *pseudo-eigenvalue* of f is defined to be the constant $\lambda_S(f)$ satisfying

$$f|_k W_S^N = \lambda_S(f) g.$$

Let q be a prime divisor of N. On the new subspace there is a close connection between the Hecke operator U_q and the Atkin–Lehner operator W_q^N. The following proposition is a combination of results from [AL78]:

Proposition 1.1.3. *Let χ be a Dirichlet character modulo N and suppose χ_q is principal. Let f be a newform of $\mathcal{S}_k(N,\chi)$ with q-th Fourier coefficient a_q and W_q^N-eigenvalue $\lambda_q(f)$.*

- *If $q^2 \mid N$ then $a_q = 0$.*
- *If $q^2 \nmid N$ then $\lambda_q(f) = -q^{1-\frac{k}{2}} a_q$ and hence we have the equality of operators*
$$W_q^N = -q^{-\frac{k}{2}+1} U_q.$$
on $\mathcal{S}_k^{new}(N,\chi)$.

1.1.4 Twisting

The third class of operators that play a major role for us are various twisting operators. Let $f \in \mathcal{S}_k(N,\chi)$ with Fourier expansion $f(z) = \sum_{n\geq 1} a_n e(nz)$, let α be a Dirichlet character of modulo M, and define

$$f_\alpha(z) = \sum_{n\geq 1} a_n \alpha(n) e(nz).$$

With α, f as above, define also

$$S_\alpha(f) = \sum_{a \bmod M} \overline{\alpha(a)} f|_k \begin{pmatrix} 1 & a/M \\ 0 & 1 \end{pmatrix}.$$

Note that if α is primitive modulo M we have

$$S_\alpha(f) = G(\overline{\alpha}) f_\alpha, \tag{1.1.12}$$

where

$$G(\overline{\alpha}) = \sum_{n \bmod M} \overline{\alpha}(n) e^{2\pi i \frac{n}{M}}$$

is the Gauss sum of $\overline{\alpha}$.

For any $z \in \mathcal{H}$ we can view the function $F : n' \mapsto \left(f|_k \left(\begin{smallmatrix} 1 & n'/N' \\ 0 & 1 \end{smallmatrix}\right)\right)(z)$ as a function from $(\mathbb{Z}/N'\mathbb{Z})^\times$ to $\mathbb{C}^\times$. The Fourier coefficient at a given multiplicative character α modulo N' is

$$\widehat{F}(\alpha) = \sum_{n' \in (\mathbb{Z}/N'\mathbb{Z})^\times} \overline{\alpha(a)} F(n') = \sum_{n' \bmod N'} \overline{\alpha(a)} F(n') = S_\alpha(f)(z),$$

so by Fourier inversion

$$f|_k \begin{pmatrix} 1 & n'/N' \\ 0 & 1 \end{pmatrix} = \sum_{\alpha \bmod N'} \frac{\alpha(n')}{\varphi(N')} S_\alpha(f), \tag{1.1.13}$$

the sum being over all Dirichlet characters modulo N'.

Finally we state some standard facts about the commutation relations for the operators we have defined. These can be proved by direct computation (see also [AL78] §3).

Proposition 1.1.4. *Let $N \in \mathbb{Z}_{\geq 1}$, let $f \in \mathcal{M}_k(N, \chi)$, let α be a Dirichlet character modulo $N' \mid N$. Then*

$$S_\alpha(f) \in \mathcal{M}_k(NN', \chi\alpha^2).$$

Let q be any divisor of N that is coprime to N', then

$$S_\alpha(f)|U_q = \alpha(q) S_\alpha(f|U_q).$$

Similarly, if S is a set of prime divisors of N such that N_S and N' are coprime, then

$$S_\alpha(f)|W_S^{NN'} = \overline{\alpha}(S) S_\alpha(f|W_S^N).$$

1.1.5 Eisenstein series

The orthogonal complement of $\mathcal{S}_k(\Gamma)$, the *Eisenstein subspace* $\mathcal{E}_k(\Gamma)$, is well understood and we give a brief overview of the theory for $\Gamma = \Gamma_1(N)$; a detailed discussion can be found in [Miy06] or [CS15].

Let ϕ and ψ be two Dirichlet characters modulo N_1 and N_2 such that $N_1 N_2 = N$ and let ψ_0 be the primitive character that induces ψ. Define the Eisenstein series

$$E_k^{\phi,\psi}(z,s) = \frac{(k-1)! N_1^k}{(-2\pi i)^k G(\overline{\psi_0})} \sum_{(c,d) \in \mathbb{Z}^2 \setminus \{(0,0)\}} \frac{\phi(c)\overline{\psi}(d)}{(N_1 cz + d)^k |N_1 cz + d|^{2s}},$$

which converges uniformly and absolutely for $k + 2\,\mathrm{Re}(s) \geq 2 + \epsilon$, for any $\epsilon > 0$. In the region of absolute convergence it satisfies the transformation law

$$E_k^{\phi,\psi}(\delta z, s) = \phi(\delta)\psi(\delta) j(\delta, z)^k \left|j(\delta, z)\right|^{2s} E_k^{\phi,\psi}(z, s) \tag{1.1.14}$$

for $\delta \in \Gamma_0(N)$. Now set $E_k^{\phi,\psi}(z) = E_k^{\phi,\psi}(z, 0)$. This is possible because the $E_k^{\phi,\psi}(z,s)$ can be analytically continued in the s-variable. Moreover, unless $k = 2$ and ϕ and ψ are principal, the value at $s = 0$ is a holomorphic function of z, so (1.1.14) along with some growth estimates shows that in fact $E_k^{\phi,\psi} \in \mathcal{M}_k(N, \phi\psi)$.

If ϕ and ψ are primitive, the Fourier expansion of $E_k^{\phi,\psi}$ can be deduced from Theorems 7.13, 7.2.12, and 7.2.13 of [Miy06]:

$$E_k^{\phi,\psi}(z) = a_k^{\phi,\psi} + 2\sum_{n \geq 1} \sigma_{k-1,\phi,\psi}(n) q^n \in \mathcal{M}_k(M, \phi\psi)$$

where $\sigma_{k-1,\phi,\psi}(n) = \sum_{d|n} \phi(n/d)\psi(d) d^{k-1}$ and

$$a_k^{\phi,\psi} = \begin{cases} L(\psi, 1-k) & N_1 = 1, \\ L(\phi, 0) & N_2 = 1 \text{ and } k = 1, \\ 0 & \text{else.} \end{cases}$$

The special case $\phi = \mathbf{1}$ is particularly important in this section. In this case we define the normalised Eisenstein series

$$\begin{aligned} E_k^{\psi,*}(z,s) &= \frac{2(-2\pi i)^k L(\overline{\psi}, k+2s) G(\overline{\psi_0})}{(k-1)! N^l} E_k^{\mathbf{1},\psi}(z,s) \\ &= \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} \frac{\overline{\psi(\gamma)}}{j(\gamma,z)^k \left|j(\gamma,z)\right|^{2s}}. \end{aligned} \tag{1.1.15}$$

Theorem 1.1.5. *Let $A_{N,k}$ be the set of $(\{\psi,\phi\},t)$ such that ϕ and ψ are primitive Dirichlet characters modulo N_1 and N_2 such that $(\phi\psi)(-1) = (-1)^k$ and t is a positive integer such that $tN_1N_2|N$. If $k = 1$ we require furthermore that ϕ is odd. If $k \neq 2$ the set*

$$\{E_1^{\phi,\psi,t}; (\{\psi,\phi\},t) \in A_{N,k}\}$$

is a basis of $\mathcal{E}_1(\Gamma_1(N))$. If $k = 2$ the series $E_2^{\mathbf{1},\mathbf{1}}$ is no longer holomorphic. To replace it we introduce $E_{2,t} = E_2^{\mathbf{1},\mathbf{1},1} - tE_2^{\mathbf{1},\mathbf{1},t}$ which is a holomorphic Eisenstein series of level t. Let $B_{N,2}$ be the set of triples (ϕ,ψ,t) such that ϕ and ψ are primitive Dirichlet characters modulo N_1 and N_2 with $(\phi\psi)(-1) = 1$, and t is a positive integer such that $1 < tN_1N_2|N$. Then

$$\{E_2^{\psi,\phi,t}; (\psi,\phi,t) \in B_{N,2}\} \cup \{E_2^{\mathbf{1},\mathbf{1},1} - tE_2^{\mathbf{1},\mathbf{1},t}; t|N\}$$

forms a basis of $\mathcal{E}_2(\Gamma_1(N))$.

In [Wei77] the action of all Atkin–Lehner operators on $E_k^{\phi,\psi}$ is derived:

Theorem 1.1.6 (Proposition 14 in [Wei77])**.** *Let ϕ and ψ be primitive Dirichlet characters (not both trivial if $k = 2$) of conductors N_1, N_2 with $\phi(-1)\psi(-1) = (-1)^k$, and S a set of prime divisors of $N = N_1N_2$.*

$$E_k^{\phi,\psi}|W_S^N = \left(\frac{N_2}{N_1}\right)_S^{\frac{k-1}{2}} \tau(\phi_S)\tau(\psi_S) E_k^{\phi_{\overline{S}}\overline{\psi}_S, \overline{\phi}_S\psi_{\overline{S}}},$$

where for a character χ modulo M

$$\tau(\chi) := \frac{G(\chi)}{\sqrt{M}} = \frac{1}{\sqrt{M}} \sum_{n \bmod M} \chi(n) e^{2\pi i \frac{n}{M}}$$

is the normalised Gauss sum of χ.

1.1.6 L-functions

For a holomorphic function g we denote the *Mellin transform* of g by

$$\mathcal{M}g(s) := \int_0^\infty g(it) t^s \frac{dt}{t}.$$

For a modular form $f = \sum a_n q^n \in \mathcal{M}_k(\Gamma_1(N))$ the *L-series*

$$L(f,s) = \sum_{n \geq 1} \frac{a_n}{n^s}$$

converges absolutely when $\operatorname{Re} s > \frac{k+1}{2}$.

Definition 1.1.7. The *completed L-function* of f is defined as:

$$\Lambda(f,s) := \Gamma(s)\left(\frac{\sqrt{N}}{2\pi}\right)^s L(f,s) = N^{s/2}\mathcal{M}(f-a_0)(s). \tag{1.1.16}$$

Let a_0 be the constant term of f and b_0 the constant term of $g = f|W_N$. By [Iwa97, Theorem 7.3], the function

$$\Lambda(f,s) + \frac{a_0}{s} + \frac{i^k b_0}{k-s}$$

can be continued to an entire function on $\mathbb{C}$. Furthermore we have the functional equation

$$\Lambda(f,s) = i^k \Lambda(g, k-s).$$

If $f \in \mathcal{M}_k(N,\chi)$ we also have functional equations for each twist of f by a character of modulus coprime to N. If ψ is a Dirichlet character of level M, then f_ψ has level M^2N. Accordingly we define the completed L-function of f_ψ as the meromorphic continuation of

$$\Lambda(f_\psi, s) = \frac{\Gamma(s)(M^2N)^{s/2}}{(2\pi)^s} L(f_\psi, s).$$

We then have the functional equation

$$\Lambda(f_\psi, s) = \overline{\chi(m)}\psi(-N)\frac{\tau(\psi)}{\tau(\overline{\psi})}\Lambda((f|W_N)_{\overline{\psi}}, s). \tag{1.1.17}$$

1.1.7 Modular symbols

We give a brief introduction to the theory of modular symbols for the group $\Gamma_1(N)$. For details we refer the reader to [Mer94] or [Ste07, §8]. Let k be an integer ≥ 2, and let $\mathbb{C}[X,Y]_{k-2}$ be the vector space of homogeneous polynomials of degree $k-2$. We define a left $\mathrm{SL}_2(\mathbb{Z})$-action on this space by

$$(gP)(X,Y) = P(dX - bY, -cX + aY), \text{ if } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Let $\mathbb{M}$ be the torsion-free abelian group generated by the symbols $\{\alpha,\beta\}$, where $\alpha,\beta \in \mathbb{P}^1(\mathbb{Q}) = \mathbb{Q}\cup\infty$, with the relations

$$\{\alpha,\beta\} + \{\beta,\gamma\} + \{\gamma,\alpha\} = 0, \ \forall \alpha,\beta,\gamma \in \mathbb{P}^1(\mathbb{Q}).$$

Set

$$\mathbb{M}_k = \mathbb{C}[X,Y]_{k-2} \otimes \mathbb{M},$$

so $\mathbb{M}_k$ is a vector space over $\mathbb{C}$, generated by elements of the form $P \otimes \{\alpha,\beta\}$, where $P \in \mathbb{C}[X,Y]_{k-2}$ and $\{\alpha,\beta\} \in \mathcal{M}$. This space has an $\mathrm{SL}_2(\mathbb{Z})$-action defined by

$$g(P \otimes \{\alpha,\beta\}) = gP \otimes \{g\alpha, g\beta\}, \quad \text{for } g \in \mathrm{SL}_2(\mathbb{Z}),$$

where the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{P}^1(\mathbb{Q})$ comes from the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\overline{\mathcal{H}}$.

We define the space of *modular symbols of weight k for $\Gamma_1(N)$*, $\mathbb{M}_k(\Gamma_1(N))$, as the quotient vector space obtained from $\mathbb{M}_k$ by imposing $gx = x$ for all $g \in \Gamma_1(N)$ and $x \in \mathbb{M}_k$.

The space $\mathbb{M}_k(\Gamma_1(N))$ is generated by the *Manin symbols* $[P, g] = gP \otimes \{g0, g\infty\}$, where $P \in \mathbb{C}[X, Y]_{k-2}$ and $g \in \mathrm{SL}_2(\mathbb{Z})$. The action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{M}_k(\Gamma_1(N))$ translates to

$$[P, g]h = [h^{-1}P, gh]$$

and the Manin symbols satisfy the following defining relations: the symbol $[P, g]$ depends only on P and the coset $\Gamma_1(N)g$, and

$$[P, g] + [P, g]\sigma = 0, \qquad (1.1.18)$$
$$[P, g] + [P, g]\tau + [P, g]\tau^2 = 0, \qquad (1.1.19)$$
$$[P, g] - [P, g]J = 0, \qquad (1.1.20)$$

where

$$\sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \ \tau = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \ \text{ and } J = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let $\mathbb{B}$ be the torsion free abelian group generated by the elements of $\mathbb{P}^1(\mathbb{Q})$. We define

$$\mathbb{B}_k = \mathbb{C}[X, Y]_{k-2} \otimes \mathbb{B},$$

and an $\mathrm{SL}_2(\mathbb{Z})$-action by $g(P \otimes \alpha) = gP \otimes g\alpha$. As before we define the space of *boundary symbols of weight k for $\Gamma_1(N)$*, $\mathbb{B}_k(\Gamma_1(N))$ as the quotient vector space obtained from $\mathbb{B}_k$ by imposing the relations $gx = x$ for all $g \in \Gamma_1(N)$ and $x \in \mathbb{B}_k$. There is a natural boundary map from $\mathbb{M}_k(\Gamma_1(N))$ to $\mathbb{B}_k(\Gamma_1(N))$ defined by

$$b(P \otimes \{\alpha, \beta\}) = P \otimes \{\alpha\} - P \otimes \{\beta\}$$

and a modular symbol in the kernel of b is called *cuspidal*. The space of cuspidal modular symbols is denoted by $\mathbb{S}_k(\Gamma_1(N))$. We can now state one of the main theorems in the theory of modular symbols.

Theorem 1.1.7 (Theorem 3 in [Mer94]). *Define a pairing*

$$(\mathcal{S}_k(\Gamma_1(N)) \oplus \overline{\mathcal{S}_k(\Gamma_1(N))}) \times \mathbb{M}_k(\Gamma_1(N)) \to \mathbb{C}$$

by

$$\langle (f_1, f_2), P\{\alpha, \beta\} \rangle = \int_\alpha^\beta f_1(z)P(z, 1)dz + \int_\alpha^\beta f_2(z)P(\overline{z}, 1)dz.$$

Then $\langle \cdot, \cdot \rangle$ is non-degenerate when restricted to

$$(\mathcal{S}_k(\Gamma_1(N)) \oplus \overline{\mathcal{S}_k(\Gamma_1(N))}) \times \mathbb{S}_k(\Gamma_1(N)).$$

Let $\eta = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. There is an involution on $\mathbb{M}_k(\Gamma_1(N))$ given on Manin symbols by

$$\iota^*([P, g]) = -[\tilde{P}, \eta g \eta^{-1}],$$

where $\tilde{P}(X, Y) = P(-X, Y)$. Denoting by $\mathbb{S}_k(\Gamma_1(N))^+$ and $\mathbb{S}_k(\Gamma_1(N))^-$ the $+1$ and -1 eigenspaces of $\mathbb{S}_k(\Gamma_1(N))$ under ι^* we have

Proposition 1.1.8 (Proposition 8 in [Mer94]). *The pairing $\langle \cdot , \cdot \rangle$ is non-degenerate when restricted to*

$$\mathcal{S}_k(\Gamma_1(N)) \times \mathbb{S}_k(\Gamma_1(N))^+, \ \text{or } \mathcal{S}_k(\Gamma_1(N)) \times \mathbb{S}_k(\Gamma_1(N))^-.$$

By mapping a matrix g to its bottom row modulo N, the cosets of $\Gamma_1(N)\backslash \mathrm{SL}_2(\mathbb{Z})$ are in bijection with the set

$$E_N = \{(u,v) \in (\mathbb{Z}/N\mathbb{Z})^2;\ (u,v) \text{ has additive order } N\}.$$

We therefore write $[P,(u,v)] = [P,g]$ for any $g \in \mathrm{SL}_2(\mathbb{Z})$ with bottom row congruent to (u,v) modulo N. Define

$$\xi_f(j;u,v) := \langle f, [X^jY^{k-2-j},(u,v)]\rangle \ j \in \{0,\dots,k-2\} \text{ and } (u,v) \in E_N.$$

A consequence of Theorem 1.1.7 is that if the map ξ_f is identically zero, then f vanishes. Proposition 1.1.8 allows one to say more. We define

$$\begin{aligned}\xi_f^{\pm}(j;u,v) &= \frac{\langle f, [X^jY^{k-2-j},(u,v)] \pm \iota^*[X^jY^{k-2-j},(u,v)]\rangle}{2} \\ &= \frac{\xi_f(j;u,v) \pm (-1)^{j+1}\xi_f(j;-u,v)}{2}.\end{aligned}$$

It is a consequence of Proposition 1.1.8 that f is determined by the map ξ_f^+ or ξ_f^-. In particular, if one of them vanishes, then so does f. This is the crucial fact about modular symbols that we use in the proof of Theorem 2.2.2.

The pairing of Theorem 1.1.7, and hence the map ξ_f, is related to values of L-functions associated to f. Indeed, taking $g = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ with $(c,v) \equiv (u,v) \bmod N$ we have

$$\xi_f(j;u,v) = \frac{j!}{(-2\pi i)^{j+1}} L(f|g, j+1). \tag{1.1.21}$$

Chapter 2

Spaces generated by products of two Eisenstein series

2.1 Outline

The aim of this chapter is to prove Theorem 1.0.2 and Theorem 1.0.3. Here we will give a brief sketch of the proof of the first of these theorems, the other requires only minor modifications. By an inductive argument it suffices to show that, for $N = N'p^n$ as in the statement of Theorem 1.0.2 and $k \geq 4$, we have

$$\mathcal{S}_k^{\mathrm{new}}(N) = \overline{\mathcal{Q}_k(N)},$$

where $\overline{\mathcal{Q}_k(N)}$ is the projection of $\mathcal{Q}_k(N)$ to the new space. In §2.5 we show that this projection is equal to the projection of $P_k(N)$, the space generated by the products

$$(E_l^{\mathbf{1},\alpha} E_{k-l}^{\mathbf{1},\overline{\alpha_N}})|W_S^{NM}; \tag{2.1.1}$$

where α is primitive of level $M \mid N$, α_N its extension to a character modulo N, and the W_S^{NM} vary over all the partial Atkin–Lehner operators. So the proof reduces to showing that

$$\mathcal{S}_k^{\mathrm{new}}(N) = \overline{P_k(N)}. \tag{2.1.2}$$

Let $g \in \mathcal{S}_k^{\mathrm{new}}$ be orthogonal to $\overline{P_k(N)}$. We need to show that this implies $g = 0$. If g is a newform, a standard calculation using the Rankin–Selberg method shows that for any α as in the definition of $P_k(N)$ all the critical L-values $L(g_\alpha|W_S^{NM}, j)$ must vanish (except for some cases when $\alpha = \mathbf{1}$ and $j = 2, k-2$, when technical difficulties coming from weight two Eisenstein series enter). At this point one can use a calculation in modular symbols to show that such a g must be zero. However g will in general not be a newform but a sum of newforms. Since $P_k(N)$ is closed under the action of the partial Atkin–Lehner operators W_p^N, we can at least assume that g is an eigenfunction of all these operators. With a little more care in the modular symbols calculation, this assumption is enough to prove a satisfactory criterion for the vanishing of g. The proof of the vanishing criterion, Theorem 2.2.2, will be given in the next section.

The reason the assumption $N = p^n N'$ enters is because we want to be in a situation where, if g is a newform (or a sum of newforms with the same W_p^N-eigenvalue for all $p \mid N$) and α is a primitive character modulo $M \mid N$, then the W_p^{NM} (pseudo-)eigenvalues of g_α for each $p \mid (N/M)$ are determined by those of g. With our methods, this condition arises naturally in the proof of Theorem 1.0.2, and our argument would extend immediately to any situation where it holds. When N is squarefree or twice squarefree, this condition is automatic by a Theorem of Atkin and Li in [AL78]. When N is not squarefree this is a much more difficult question, and it seems unlikely that a purely local argument will work. Indeed our extension to level $N = p^n N'$ stems from a rather different argument involving the (global) functional equation.

In §2.6 we explain how similar arguments can be used to prove the analogue of Theorems 1.0.2 and 1.0.3 when $N = p$ is prime and χ is primitive modulo p.

In the last section we give a few more selected examples of the main theorems.

2.2 A vanishing condition

The main goal of this section is to prove Theorem 2.2.2, which states that if a cusp form f has sufficiently many special values of certain twisted L-functions equal to zero, then f must be zero. The result is in the spirit of Corollaire 2 of [Mer09], although we require some modifications since we do not assume that f is a newform, or even an eigenfunction of almost all Hecke operators. First we recall an identity from the proof of Proposition 6 in [Mer09]:

Lemma 2.2.1. *Let $N \in \mathbb{Z}_{\geq 1}$, let $(u,v) \in E_N$, let S denote the set of prime divisors of N which divide u, let $\overline{S}$ denote the remaining prime divisors of N, and let N' be the order of uv in $\mathbb{Z}/N\mathbb{Z}$. Let $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ be such that $(c,d) \equiv (u,v) \bmod N$. Then*

$$\Gamma_1(N)g = \Gamma_1(N)\begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}\begin{pmatrix} 1 & \frac{n}{N} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} NN'_S & 0 \\ 0 & N_S \end{pmatrix}^{-1},$$

where n is chosen so that $n \equiv uv \bmod N_{\overline{S}}$ and $n \equiv -uv \bmod N_S$, and $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathbb{Z}^{2\times 2}$ has $AD - BC = N_S N'_S$, $A \equiv uN'_S \bmod N_{\overline{S}}$, $B \equiv v/N_{\overline{S}} \bmod N_S$, and $N_S N'_S \mid A$, $N_S N'_S \mid D$, $NN' \mid C$, $N_{\overline{S}} N'_{\overline{S}} \mid B$.

Proof. The existence of n and A, B, C, D satisfying the conditions of the lemma follows from the Chinese Remainder Theorem. So it suffices to verify that, under these conditions, the claimed identity holds. Note that the condition on the determinant is necessary, since the matrix on the right hand side must have determinant one. Computing the matrix on the right hand side we get

$$\begin{pmatrix} -\frac{C}{NN'_S} & -\frac{D}{N_S} \\ \frac{A}{N'_S} + \frac{nC}{NN'_S} & N_{\overline{S}}B + \frac{nD}{N_S} \end{pmatrix}.$$

To prove the claim, it suffices to show that the top row is integral and that the bottom row is congruent to (u,v) modulo N. Now our conditions imply that we can write $C = NN'C'$ for some integer C', and $D = N_S N'_S D'$ for some integer D', so the top row is indeed integral. Note that the divisibility of A by N'_S is also necessary for the bottom row to be integral; we use the full strength of our assumption and write $A = N_S N'_S A'$. With this notation the matrix we are considering is

$$\begin{pmatrix} -N'_{\overline{S}}C' & -N'_S D' \\ N_S A' + nN'_{\overline{S}}C' & N_{\overline{S}}B + nN'_S D' \end{pmatrix}.$$

To show that the bottom row is congruent to (u,v) modulo N, we check this modulo $N_{\overline{S}}$ and modulo N_S separately. For the former,

$$\left(N_S A' + nN'_{\overline{S}}C', N_{\overline{S}}B + nN'_S D'\right) \equiv (N_S A', uvN'_S D') \bmod N_{\overline{S}},$$

since $uvN'_{\overline{S}} \equiv 0 \bmod N_{\overline{S}}$ by definition of N'. Since $A = N_S N'_S A' \equiv uN'_S \bmod N_{\overline{S}}$ and N'_S is invertible modulo $N_{\overline{S}}$, we see $N_S A' \equiv u \bmod N_{\overline{S}}$. For the second component, consider the equation $AD - BC = (N_S N'_S)^2 A'D' - NN'BC' = N_S N'_S$, so $N_S N'_S A'D' - N_{\overline{S}} N'_{\overline{S}} BC' = 1$. This gives $N_S N'_S A'D' \equiv 1 \bmod N_{\overline{S}}$, so using $A = N_S N'_S A' \equiv uN'_S \bmod N_{\overline{S}}$ again we get $uN'_S D' \equiv 1 \bmod N_{\overline{S}}$, hence $uvN'_S D' \equiv v \bmod N_{\overline{S}}$ as required.

Now consider the bottom row modulo N_S:

$$\left(N_S A' + nN'_{\overline{S}} C', N_{\overline{S}} B + nN'_S D'\right) \equiv (-uvN'_{\overline{S}} C', N_{\overline{S}} B) \bmod N_S,$$

again using the definition of N'. Since $B \equiv v/N_{\overline{S}} \bmod N_S$, the second component is congruent to v modulo N_S. For the first component we again argue from the determinant condition. We have $N_S N'_S A'D' - N_{\overline{S}} N'_{\overline{S}} BC' = 1$. This gives $-vN'_{\overline{S}} C' \equiv 1 \bmod N_S$, so $-uvN'_{\overline{S}} C' \equiv u \bmod N_S$ as required. □

Theorem 2.2.2. *Let N be a positive integer, $k \geq 2$, and let $f \in \mathcal{S}_k^{new}(N)$ be an eigenfunction of all partial Atkin–Lehner operators W_S^N. Assume that $L(f_\alpha | W_S^{NM}, j+1) = 0$ for all characters α primitive modulo $M \mid N$ and all sets of primes S such that $\prod_{p \in S} p \cdot M \mid N$, and all $j = 0, 1, \ldots, k-2$ such that $\alpha(-1) = (-1)^{j+1}$ (resp. $\alpha(-1) = (-1)^j$). Then $f = 0$.*

Proof. We will present the argument for the case $\alpha(-1) = (-1)^{j+1}$, which uses the function ξ_f^+. The other case, using ξ_f^-, is almost identical, the only difference being which characters cancel in (2.2.4). We will show that the conditions in the theorem imply $\xi^+_{f|W_N}(j; u, v) = 0$ for all $j = 0, 1, ..., k-2$ and $(u, v) \in E_N$, which in turn implies $f = 0$ by the last remarks in §1.1.7. Let us therefore fix $(u, v) \in E_N$ and consider

$$\xi^+_{f|W_N}(j; u, v) = \frac{\xi_{f|W_N}(j; u, v) + (-1)^{j+1} \xi_{f|W_N}(j; -u, v)}{2}.$$

As in the statement of Lemma 2.2.1, let S be the set of those prime divisors of N that divide u. Write N' for the order of uv in $\mathbb{Z}/N\mathbb{Z}$. Choose $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ such that $(c, d) \equiv (u, v) \bmod N$. By Lemma 2.2.1 we have

$$\Gamma_1(N) g = \Gamma_1(N) \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix} \begin{pmatrix} 1 & \frac{n}{N} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} NN'_S & 0 \\ 0 & N_S \end{pmatrix}^{-1}, \tag{2.2.1}$$

with A, B, C, D and n satisfying the conditions of Lemma 2.2.1. Since $f|W_N|W_N$ equals f, we have

$$f|W_N|g = f| \begin{pmatrix} 1 & \frac{n}{N} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} NN'_S & 0 \\ 0 & N_S \end{pmatrix}^{-1}.$$

Now $n \equiv uv \bmod N_{\overline{S}}$ and $n \equiv -uv \bmod N_S$, so n also has order N' modulo N. Hence $nN' = n'N$ for some n' which is coprime to N'. Writing this as $n/N = n'/N'$ and using (1.1.13) we get

$$f|W_N|g = \sum_{\alpha \bmod N'} \frac{\alpha(n')}{\phi(N')} S_\alpha(f)| \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} NN'_S & 0 \\ 0 & N_S \end{pmatrix}^{-1},$$

where α varies over all Dirichlet characters modulo N'.

By Proposition 1.1.4 we have $S_\alpha(f) \in \mathcal{S}_2(NN', \alpha^2)$. Now the conditions of Lemma 2.2.1 and Proposition 1.1.2 give

$$S_\alpha(f)| \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \overline{\alpha^2_S}(B) \overline{\alpha^2_{\overline{S}}}\left(\frac{A}{N_S N'_S}\right) S_\alpha(f)|W_S^{NN'}.$$

Hence, using (1.1.21), we see that $\xi_{f|W_N}(j;u,v)$ equals

$$\frac{j!(N_{\overline{S}}/N'_S)^{\frac{k}{2}-j-1}}{(-2\pi i)^{j+1}\phi(N')}\sum_{\alpha \bmod N'}\alpha(n')\overline{\alpha_S^2}(B)\overline{\alpha_{\overline{S}}^2}\left(\frac{A}{N_S N'_S}\right)L\left(S_\alpha(f)|W_S^{NN'},j+1\right), \qquad (2.2.2)$$

where the sum is over all characters modulo N'.

To compute $\xi_{f|W_N}(j;-u,v)$ we proceed analogously with $\tilde{g}=\left(\begin{smallmatrix} a & -b \\ -c & d\end{smallmatrix}\right)$, since this has bottom row $(-c,d)\equiv(-u,v) \bmod N$. With A,B,C,D,n as in (2.2.1) we see that

$$\Gamma_1(N)\tilde{g}=\Gamma_1(N)\begin{pmatrix}0 & -1\\ N & 0\end{pmatrix}\begin{pmatrix}1 & -\frac{n}{N}\\ 0 & 1\end{pmatrix}\begin{pmatrix}-A & B\\ C & -D\end{pmatrix}\begin{pmatrix}NN'_S & 0\\ 0 & N_S\end{pmatrix}^{-1}. \qquad (2.2.3)$$

The argument is as above, with n' replaced by $-n'$, and each individual summand in the final expression for $\xi_{f|W_N}(j;u,v)$ changes by a factor of $\alpha(-1)\overline{\alpha_{\overline{S}}^2}(-1)=\alpha(-1)$. From the definition of $\xi^+_{f|W_N}$ we then see $\xi^+_{f|W_N}(j;u,v)$ equals

$$\frac{j!(N_{\overline{S}}/N'_S)^{\frac{k}{2}-j-1}}{(-2\pi i)^{j+1}\phi(N')}\sum_{\alpha}\alpha(n')\overline{\alpha_S^2}(B)\overline{\alpha_{\overline{S}}^2}\left(\frac{A}{N_S N'_S}\right)L\left(S_\alpha(f)|W_S^{NN'},j+1\right), \qquad (2.2.4)$$

where the sum is over all characters α modulo N' with $\alpha(-1)=(-1)^{j+1}$.

The next step is to relate $S_\alpha(f)$ to the twist by the primitive character underlying α. The key to this is the following lemma, the proof of which will be given after the completion of the current argument:

Lemma 2.2.3. *Let N and k be positive integers, let χ be a Dirichlet character modulo N, and let $f\in\mathcal{S}_k(N,\chi)$. Let $N'\in\mathbb{Z}_{\geq 1}$, let α be a character modulo N' with conductor M. Assume that $M<N'$, let p be any prime dividing N'/M, and let β be the character modulo N'/p inducing α. Then*

$$S_\alpha(f)=p^{1-k/2}S_\beta(f|T_p)|\begin{pmatrix}p & 0\\ 0 & 1\end{pmatrix}-\overline{\beta}(p)S_\beta(f).$$

In our case $f\in\mathcal{S}_k^{\mathrm{new}}(N)$ is an eigenfunction of each W_p^N, so it is also an eigenfunction of each U_p for $p\mid N$ by Proposition 1.1.3. Write a_p for the eigenvalue, which may be zero. Then Lemma 2.2.3 gives

$$S_\alpha(f)=p^{1-k/2}a_pS_\beta(f)|\begin{pmatrix}p & 0\\ 0 & 1\end{pmatrix}-\overline{\beta}(p)S_\beta(f),$$

and so

$$L(S_\alpha(f)|W_S^{NN'},j+1)=(p^{-j}a_p-\overline{\beta}(p))L(S_\beta(f)|W_S^{NN'},j+1).$$

Applying this repeatedly we see that $L(S_\alpha(f)|W_S^{NN'},j+1)$ is a multiple of $L(S_{\alpha_0}(f)|W_S^{NN'},j+1)$, where α_0 is the the primitive character modulo $M\mid N'$ inducing α modulo N'. Finally we note that $S_{\alpha_0}(f)=G(\overline{\alpha_0})f_{\alpha_0}\in\mathcal{S}_k(NM,\alpha_0^2)$. We then use $S_{\alpha_0}(f)|W_S^{NN'}=S_{\alpha_0}(f)|W_S^{NM}|B_d$, where $d=(\frac{N}{M})_S$ (c.f. (2.5.2) below). Thus $L(S_\alpha(f)|W_S^{NN'},j+1)$ is a multiple of $L(f_{\alpha_0}|W_S^{NM},j+1)$, and using (2.2.4) we see that $\xi^+_{f|W_N}$ is a linear combination of L-values which we have assumed to be equal to zero, as required. □

Proof of Lemma 2.2.3. With the notation of the lemma, note that

$$\begin{aligned} p^{1-k/2} S_\beta(f|U_p)| \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} &= \sum_{a=0}^{N'/p-1} \sum_{u=0}^{p-1} \overline{\beta}(a) f| \begin{pmatrix} 1 & u \\ 0 & p \end{pmatrix} \begin{pmatrix} 1 & \frac{a}{N'/p} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \\ &= \sum_{a=0}^{N'/p-1} \overline{\beta}(a) \sum_{u=0}^{p-1} f| \begin{pmatrix} p & \frac{a}{N'/p} + u \\ 0 & p \end{pmatrix} \\ &= \sum_{a=0}^{N'/p-1} \overline{\beta}(a) \sum_{u=0}^{p-1} f| \begin{pmatrix} 1 & \frac{a+u\frac{N'}{p}}{N'} \\ 0 & 1 \end{pmatrix} \\ &= \sum_{a=0}^{N'} \overline{\beta}(a) f| \begin{pmatrix} 1 & a/N' \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Now if $(u,p) = 1$ then $\alpha(u) = \beta(u)$, and if $(u,p) > 1$ then $\alpha(u) = 0$ but $\beta(u)$ may be non-zero:

$$\begin{aligned} p^{1-k/2} S_\beta(f|U_p)| \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} &= \sum_{a=0}^{N'} \overline{\alpha}(a) f| \begin{pmatrix} 1 & a/N' \\ 0 & 1 \end{pmatrix} + \sum_{a=0}^{N'/p} \overline{\beta}(pa) f| \begin{pmatrix} 1 & ap/N' \\ 0 & 1 \end{pmatrix} \\ &= S_\alpha(f) + \overline{\beta}(p) S_\beta(f). \end{aligned}$$

Re-arranging this proves the lemma. □

A technical difficulty arises in our application of Theorem 2.2.2 when $k \geq 4$ due to the fact that the weight two Eisenstein series $E_2^{\mathbf{1},\mathbf{1}}$ is not holomorphic. To this end we prove a result which states that the problematic cases are in fact already a consequence of the other assumptions:

Proposition 2.2.4. *Let $N \in \mathbb{Z}_{\geq 1}$, $k \geq 4$ be even and $f \in \mathcal{S}_k^{new}(N)$ be an eigenform of the Atkin–Lehner operators W_S^N. Assume that $L(f_\alpha|W_S^{NM}, j+1) = 0$ for all primitive characters α modulo $M \mid N$ where $M > 1$, all sets of primes S such that $\prod_{p \in S} p \cdot M \mid N$, and all $j = 0, \ldots, k-2$ such that $\alpha(-1) = (-1)^{j+1}$. Assume moreover that $L(f|W_S^N, j+1) = 0$ for all sets S of prime divisors of N and all $j \neq 1, k-3$. Then $L(f,2) = 0$ and $L(f, k-2) = 0$ must hold as well.*

Proof. From the second relation (1.1.19) for Manin symbols with $P(X,Y) = Y^{k-2}$ and $g = -\sigma$ we have

$$[Y^{k-2}, -\sigma] + \sum_{j=0}^{k-2} (-1)^{k-2-j} \binom{k-2}{j} [X^j Y^{k-2-j}, -\sigma\tau] + [X^{k-2}, -\sigma\tau^2] = 0.$$

If we denote this modular symbol by M then $\langle f|W_N, M + \iota^* M \rangle$ equals

$$\xi^+_{f|W_N}(0; -1, 0) + \sum_{j=0}^{k-2} (-1)^{k-2-j} \binom{k-2}{j} \xi^+_{f|W_N}(j; 0, 1) + \xi^+_{f|W_N}(k-2; 1, -1) = 0. \quad (2.2.5)$$

We already know that $\xi^+_{f|W_N}(j;u,v) = 0$ for $(u,v) = (-1,0),(0,1),(1,-1)$, unless $j=1$ or $j = k-3$. To see this we argue as in the proof of Theorem 2.2.2: By (2.2.4) $\xi^+_{f|W_N}(j;u,v)$ is a linear combination of $L(S_\alpha(f)|W_S^{NN'}, j+1)$, and we can reduce this to a linear combination of $L(f_{\alpha_0}|W_S^{NM}, j+1) = 0$ with α_0 the underlying primitive character as in the proof of Theorem 2.2.2. When $j \neq 1, k-3$ these L-values are zero by assumption, so $\xi^+_{f|W_N}(j;u,v) = 0$ for all $(u,v) \in E_N$ and $j \neq 1, k-3$. Thus (2.2.5) reduces to

$$-(k-2)\left(\xi^+_{f|W_N}(1;0,1) + \xi^+_{f|W_N}(k-3;0,1)\right) = 0.$$

Since $k \geq 4$ this is equivalent to

$$\xi^+_{f|W_N}(1;0,1) + \xi^+_{f|W_N}(k-3;0,1) = 0.$$

Now applying (1.1.21) we get

$$\frac{1}{(-2\pi i)^2}L(f|W_N,2) + \frac{(k-3)!}{(-2\pi i)^{k-2}}L(f|W_N,k-2) = 0;$$

since f is an eigenfunction of W_N by assumption this is equivalent to

$$\frac{1}{(-2\pi i)^2}L(f,2) + \frac{(k-3)!}{(-2\pi i)^{k-2}}L(f,k-2) = 0.$$

Writing this in terms of the completed L-functions,

$$\frac{1}{N}\Lambda(f,2) + \frac{i^k}{N^{\frac{k-2}{2}}}\Lambda(f,k-2) = 0.$$

Applying the functional equation,

$$\left(\frac{1}{N} + \frac{\epsilon}{N^{\frac{k}{2}-1}}\right)\Lambda(f,2) = 0,$$

where ϵ is the eigenvalue of f under W_N. This implies that $\Lambda(f,2) = 0$, unless $k=4$ and $\epsilon = -1$. However, when $k=4$ and $\epsilon=-1$, $s=2$ is the central value of $L(f,s)$ so $L(f,2) = 0$ since the sign in the functional equation is negative. □

2.3 The Rankin–Selberg method

Let $k \in \mathbb{Z}_{\geq 1}$, χ be a Dirichlet character modulo N with $\chi(-1) = (-1)^k$, and let $f \in \mathcal{S}_k(N,\chi)$. Given any $g \in \mathcal{M}_l(N, \overline{\psi}\chi)$, we consider the inner product

$$\langle gE^{\psi,*}_{k-l}(\cdot,s), f\rangle = \int_{\mathcal{F}} g(z)E^{\psi,*}_{k-l}(z,s)\overline{f(z)}y^{s+k}d\mu(z),$$

where $\mathcal{F}$ is a fundamental domain for $\Gamma_0(N)$ and $d\mu(z) = \frac{dxdy}{y^2}$ is the hyperbolic measure on $\mathcal{H}$. Note that integrand is $\Gamma_0(N)$-invariant so the integral over this quotient makes sense, at least when it converges. This is certainly the case if s has sufficiently large real part, which we assume during these next manipulations.

Let $f = \sum a_n q^n \in \mathcal{S}_k(N, \chi)$ and $g = \sum b_n q^n \in \mathcal{M}_l(N, \phi)$. The Rankin–Selberg method (see [Shi76]) was originally applied to study the L-function

$$L(f \times g, s) = \sum_{n \geq 1} \frac{a_n b_n}{n^s}$$

and derive its meromorphic continuation to $\mathbb{C}$ and functional equation. We will use it to find an expression for the Petersson inner product between a cusp form and a product of Eisenstein series. Let $\psi = \chi\phi^{-1}$. By the definition of $E_{k-l}^{\psi,*}(z, s)$ we get

$$\begin{aligned}
&\langle g E_{k-l}^{\psi,*}(z, s), f \rangle \\
&\quad = \int_{\mathcal{F}} g(z) \left(\sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} \frac{\overline{\psi}(\gamma)}{j(\gamma, z)^{k-l} \left| j(\gamma, z) \right|^{2s}} \right) \overline{f(z)} y^{s+k} d\mu(z) \\
&\quad = \int_{\mathcal{F}} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} \frac{g(\gamma z) \overline{f(\gamma z)}}{\left| j(\gamma, z) \right|^{2(s+k)}} y^{s+k} d\mu(z) \\
&\quad = \int_{\Gamma_0(N) \backslash \mathcal{H}} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} g(\gamma z) \overline{f(\gamma z)} \, \mathrm{Im}(\gamma z)^{s+k} d\mu(z) \\
&\quad = \int_{\Gamma_\infty \backslash \mathcal{H}} g(z) \overline{f(z)} y^{s+k-2} dx dy.
\end{aligned}$$

Now substitute in the Fourier expansion $f(z) = \sum_{n \geq 1} a_n e(nz)$ and $g(z) = \sum_{m \geq 0} b_m e(mz)$; using orthogonality of the characters $x \mapsto e(nx)$ of $\mathbb{R}/\mathbb{Z}$ we obtain

$$\begin{aligned}
&\langle g E_{k-l}^{\psi,*}(z, s), f \rangle \\
&\quad = \int_{y=0}^{\infty} \int_{x=0}^{1} \left(\sum_{n \geq 1} \overline{a_n} e^{-2\pi i n x - 2\pi n y} \right) \left(\sum_{m \geq 0} b_m e^{2\pi i m x - 2\pi m y} \right) y^{s+k-2} dx dy \\
&\quad = \int_{y=0}^{\infty} \sum_{n \geq 1} \overline{a_n} b_n e^{-4\pi n y} y^{s+k-2} dy.
\end{aligned}$$

For any value of s, the exponential decay in y means that the integrand is rapidly decaying, so we can swap the order of summation and integration. Thus

$$\begin{aligned}
\langle g E_{k-l}^{\psi,*}(z, s), f \rangle &= \sum_{n \geq 1} \frac{\overline{a_n} b_n}{n^{s+k-1}} \int_{y=0}^{\infty} e^{-4\pi y} y^{s+k-2} dy \\
&= \frac{\Gamma(s + k - 1)}{(4\pi)^{s+k-1}} \sum_{n \geq 1} \frac{\overline{a_n} b_n}{n^{s+k-1}}.
\end{aligned} \tag{2.3.1}$$

Write f^c for the function defined by $f^c(z) = \overline{f(-\overline{z})}$. It has Fourier expansion

$$f^c(z) = \sum_{n \geq 1} \overline{a_n} e(nz).$$

By results of Shimura we have $f^c \in \mathcal{S}_k(N, \overline{\chi})$ and this construction preserves newforms. Alternatively, if $f \in \mathcal{S}_k(N, \chi)$ is a newform, then one easily sees that f^c is the newform associated to $f_{\overline{\chi}}$.

Proposition 2.3.1. *Let $N, k, l \in \mathbb{Z}_{\geq 1}$, χ be a Dirichlet character modulo N, and f be a newform in $\mathcal{S}_k(N, \chi)$, let ϕ, ψ be Dirichlet characters such that $\phi\psi = \chi$ and $\phi(-1) = (-1)^l$. Let ϕ_0 be the primitive character modulo $M = cond(\phi)$ associated to ϕ and exclude the two cases $\phi_0 = \mathbf{1}$ and $l = 2$, and $\phi = \chi$ and $l = k-2$. Then*

$$\langle E_l^{\mathbf{1},\phi_0} E_{k-l}^{\psi,*}(\cdot, s), f\rangle = \frac{2\Gamma(s+k-1)}{(4\pi)^{s+k-1}} \cdot \frac{L(f^c, s+k-1)L((f^c)_{\phi_0}, s+k-l)}{L(\overline{\chi}\phi_0, 2s+k-l)}. \tag{2.3.2}$$

Proof. Recall that the Fourier coefficients of $E_l^{\mathbf{1},\phi_0}$ are given by $b_n = 2\sigma_{l-1,\mathbf{1},\phi_0}(n)$ for $n \geq 1$. Substituting this into (2.3.1) and using a standard computation (see e.g. [Rau14] Proposition 4.1[1]) gives

$$\sum_{n\geq 1} \frac{\overline{a_n}\sigma_{l-1,\mathbf{1},\phi_0}(n)}{n^{s+k-1}} = \frac{L(f^c, s+k-1)L((f^c)_{\phi_0}, s+k-l)}{L(\overline{\chi}\phi_0, 2s+k-l)}.$$

and the result follows. □

Note that both sides of (2.3.2) have analytic continuation to $s = 0$ and by the uniqueness of analytic continuation the equality remains true there. Using the fact that

$$E_{k-l}^{\psi,*}(z, 0) = \frac{2(-2\pi i)^{k-l}L(\overline{\psi}, k-l)G(\overline{\psi_0})}{(k-l-1)!N^{k-l}} E_{k-l}^{\mathbf{1},\psi}$$

we obtain

Corollary 2.3.2.

$$\langle E_l^{\mathbf{1},\phi_0} E_{k-l}^{\mathbf{1},\psi}, f\rangle = \frac{(k-l-1)!(k-2)!N^{k-l}}{(-2\pi i)^{k-l}(4\pi)^{k-1}L(\overline{\psi}, k-l)G(\overline{\psi_0})} \frac{L(f^c, k-1)L((f^c)_{\phi_0}, k-l)}{L(\overline{\chi}\phi_0, k-l)}.$$

2.4 Generating spaces of cusp forms by products of Eisenstein series

Let N be any positive integer, and define $P_k(N) \subset \mathcal{M}_k(N)$ to be the space generated by the products

$$(E_l^{\mathbf{1},\alpha} E_{k-l}^{\mathbf{1},\overline{\alpha_N}})|W_S^N$$

where $1 \leq l \leq k-1$, S is a set of prime divisors of N, α is a primitive character modulo M with $\alpha(-1) = (-1)^l$ and α_N is its extension to a character modulo N. The cases when $\alpha = \mathbf{1}$ and l equals 2 or $k-2$ are excluded.

[1]Our divisor function is $\sigma_{l-1,\phi,\mathbf{1}}$ in Raum's notation.

In this section we will prove Theorem 2.4.2, which describes the projection of $P_k(N)$ on to the new subspace. The proof requires us to deduce vanishing of the L-values $L(G_\alpha|W_S^{NM}, j{+}1)$ of the Atkin–Lehner images of a form G_α from vanishing of $L(G_\alpha, j{+}1)$, for which we prove the following technical lemma:

Lemma 2.4.1. *Let $N = N'p^n$ where N' is squarefree or twice a squarefree number, p is a prime and $p \nmid N'$. Let $G \in \mathcal{S}_k^{new}(N)$ be an eigenfunction of all W_q for $q \mid N$ and fix $M \mid N$. Suppose $L(G_\alpha, j+1) = 0$ for all primitive characters α modulo M and all $j \in \{0, \ldots, k-2\}$ such that $\alpha(-1) = (-1)^{j+1}$. Then, for all such α, j, and all sets of primes S such that $\prod_{p\in S} p \cdot M \mid N$, we have $L(G_\alpha|W_S^{NM}, j+1) = 0$.*

Proof. If $p \notin S$ then M and N_S are coprime, so by Proposition 1.1.4 we have

$$L(G_\alpha|W_S^{NM}, j+1) = \lambda_S(G)\overline{\alpha}(S)L(G_\alpha, j+1) = 0.$$

If $p \in S$, by the functional equation we have

$$L(G_\alpha|W_S^{NM}, j+1) = cL(G_\alpha|W_{\overline{S}}^{NM}, k-j-1) \tag{2.4.1}$$

for a non-zero constant c. Note that $p \notin \overline{S}$. Let $\alpha = \alpha_{M'}\alpha_p$, where α_p is the p-primary part of α. Then $G_\alpha = (G_{\alpha_{M'}})_{\alpha_p}$ and by Proposition 1.1.4 we have

$$G_\alpha|W_{\overline{S}}^{NM} = (G_{\alpha_{M'}}|W_{\overline{S}}^{NM'})_{\alpha_p}.$$

Since G is a W_q-eigenform for all $q \mid N$ it is a linear combination of newforms $f_1, \ldots, f_r$ which all have the same W_q-eigenvalues. Since N' is squarefree or twice a squarefree number, we know that $\alpha_{M'}$ is maximally ramified at primes where it is non-trivial[2], so we can apply Theorem 4.1 of [AL78] to see that $(f_i)_{\alpha_{M'}}$ is again a newform for all i and the corresponding W_q-eigenvalues are independent of i. Hence $G_{\alpha_{M'}}$ is a pseudo-eigenfunction of $W_{\overline{S}}^{NM'}$, say with pseudo-eigenvalue $\lambda_{\overline{S}}^{NM'}(G_{\alpha_{M'}})$, which means

$$G_{\alpha_{M'}}|W_{\overline{S}}^{NM} = \lambda_{\overline{S}}^{NM'}(G_{\alpha_{M'}})G_{\overline{\alpha_{M'}}}.$$

In summary

$$\begin{aligned} L(G_\alpha|W_S^{NM}, j+1) &= cL((G_{\alpha_{M'}}|W_{\overline{S}}^{NM})_{\alpha_p}, k-j-1) \\ &= c\lambda_{\overline{S}}^{NM'}(G_{\alpha_{M'}})L(G_{\overline{\alpha_{M'}}\alpha_p}, k-j-1), \end{aligned}$$

which equals 0 by our assumptions. □

Theorem 2.4.2. *Let $N \in \mathbb{Z}_{\geq 1}$ be such that Lemma 2.4.1 holds. Then for $k \geq 4$ even*

$$\overline{P_k(N)} = \mathcal{S}_k^{new}(N).$$

In the case $k = 2$ we define $\overline{\mathcal{S}_2^{rk=0}(N)} \subset \mathcal{S}_2^{new}(N)$ to be the subspace generated by newforms f with non-zero central L-value, i.e. $L(f,1) \neq 0$ [3]; note that $\overline{\mathcal{S}_2^{rk=0}(N)} \subset \mathcal{S}_2^{new}(N)^-$. Then

$$\overline{P_2(N)} = \overline{\mathcal{S}_2^{rk=0}(N)}.$$

[2] I.e. if $q \neq p$ is prime such that α_q is non-trivial, then $\mathrm{ord}_q(M) = \mathrm{ord}_q(N)$.

[3] The subspace $\mathcal{S}_2^{\mathrm{rk}=0}(N)$ whose projection to the new space is $\overline{\mathcal{S}_2^{\mathrm{rk}=0}(N)}$ is defined in the next section.

Proof. Let f be a weight k, level N newform, and write $\lambda_S(f)$ for the W_S^N-eigenvalue of f. By Proposition 1.1.2 the operators W_S^N are self-adjoint, so

$$\begin{aligned}\langle (E_l^\alpha E_{k-l}^{\overline{\alpha_N}})|W_S^N, f\rangle &= \langle E_l^\alpha E_{k-l}^{\overline{\alpha_N}}, f|W_S^N\rangle \\ &= \lambda_S(f)\langle E_l^\alpha E_{k-l}^{\overline{\alpha_N}}, f\rangle\end{aligned}$$

Using Corollary 2.3.2 (note $f = f^c$ since f has trivial character) we get that $\langle (E_l^\alpha E_{k-l}^{\overline{\alpha_N}})|W_S^N, f\rangle$ equals

$$\lambda_S(f)\frac{(k-l-1)!(k-2)!N^{k-l}}{(-2\pi i)^{k-l}(4\pi)^{k-1}L(\alpha_N, k-l)L(\alpha, k-l)G(\alpha)} \cdot L(f, k-1)L(f_\alpha, k-l). \quad (2.4.2)$$

First assume $k > 2$. Suppose that the containment $\overline{P_k(N)} \subset \mathcal{S}_k^{\text{new}}(N)$ is proper. Since $P_k(N)$ is closed under the action of the Atkin–Lehner operators W_q^N for $q \mid N$, so is the orthogonal complement of $\overline{P_k(N)}$ in $\mathcal{S}_k^{\text{new}}(N)$. Therefore there exists a non-zero form $g \in \mathcal{S}_k^{\text{new}}(N)$ that is orthogonal to $P_k(N)$ and an eigenform of the W_q^N. We can write

$$g = \sum_{i=1}^{r} \beta_i f_i,$$

where $f_1, \ldots, f_r$ are the newforms in $\mathcal{S}_k^{\text{new}}(N)$ with the same W_q^N-eigenvalues as g. Using (2.4.2) we see that orthogonality of g to $P_k(N)$ is equivalent to

$$\sum_{i=1}^{r} \lambda_S(f_i)\beta_i L(f_i, k-1)L((f_i)_\alpha, k-l) = 0.$$

for all l, α, S as specified in the definition of $P_k(N)$. However, by definition of g, $\lambda_S(f_i) = \lambda_S(f_j)$ for each i, j, so the orthogonality of g to $P_k(N)$ is equivalent to

$$\sum_{i=1}^{r} \beta_i L(f_i, k-1)L((f_i)_\alpha, k-l) = 0. \quad (2.4.3)$$

Following an idea from the proof of Theorem 1 in [KM08], we define another form in $G \in \mathcal{S}_k^{\text{new}}(N)$ by

$$G = \sum_{i=1}^{r} \beta_i L(f_i, k-1)f_i.$$

Since the f_i all have the same W_q^N-eigenvalues as g, so does G. Then (2.4.3) translates to

$$L(G_\alpha, k-l) = 0 \quad (2.4.4)$$

for all primitive characters α modulo $M \mid N$ with $\alpha(-1) = (-1)^{k-l}$, excluding the cases $\alpha = \mathbf{1}$ and $l = 2$ or $l = k-2$.

Using Lemma 2.4.1, we get

$$L(G_\alpha|W_S^{NM}, k-l) = 0$$

for all primitive characters α modulo $M \mid N$ with $\alpha(-1) = (-1)^{k-l}$, and all sets of primes S such that $\prod_{p \in S} p \cdot M \mid N$, excluding the cases $\alpha = \mathbf{1}$ and $l = 2$ or $l = k-2$. Now applying Proposition 2.2.4 we see that $L(G, 2) = 0$ and $L(G, k-2) = 0$. We now have

$$L(G_\alpha | W_S^{NM}, k-l) = 0$$

for all α primitive modulo M, $S \mid\mid N$ such that $\mathrm{rad}(S)M \mid N$, and $l = 1, ..., k-1$. By Theorem 2.2.2 we can conclude that $G = 0$. Since $k \geq 4$, $L(f_i, k-1) \neq 0$, so we must have that all β_i are zero, and we arrive at the contradiction $g = 0$.

In the case where $k = 2$ the proof is similar. The containment $\overline{P_2(N)} \subset \overline{\mathcal{S}_2^{\mathrm{rk}=0}(N)}$ comes from (2.4.2), since $P_k(N)$ is orthogonal to every newform f with $L(f, 1) = 0$. The rest of the argument works as above. □

2.5 The new part of $P_k(N)$

In this section we will analyse the new parts of the generators of $P_k(N)$ for any N. We use this to construct another space $Q_k(N)$ with the same projection to the new space as $P_k(N)$ whose generators do not involve partial Atkin–Lehner operators. While $P_k(N)$ was more useful for the proof of Theorem 2.4.2, $Q_k(N)$ is more explicit and easy to implement on a computer.

First we find the new part of $E_{k-l}^{\mathbf{1}, \overline{\alpha_N}}$:

Lemma 2.5.1. *Let α be a primitive character modulo M with $\alpha(-1) = (-1)^k$. Writing $N = \prod_i p_i^{e_i}$, let $N_M = \prod_{p_i | M} p_i^{e_i}$ be the M-part of N, so that $M \mid N_M$ and* $\gcd(M, N/N_M) =$ *1. Then*

$$E_{k-l}^{\mathbf{1}, \overline{\alpha_N}} = \left(\frac{N}{M}\right)^{\frac{k}{2}-l} \sum_{e | N/N_M} \mu(e)\alpha(e)e^{-\frac{k}{2}+l} E_{k-l}^{\mathbf{1}, \overline{\alpha}} | B_{N/Me}$$

Proof. For $\mathrm{Re}(s) \gg 0$ we have

$$E_{k-l,N}^{\overline{\mathbf{1}, \alpha_N}}(z, s) = \frac{(k-l-1)! N^{k-l}}{(-2\pi i)^{k-l} G(\overline{\alpha})} \sum_{(c,d) \neq (0,0)} \frac{\alpha_N(d)}{(cNz+d)^{k-l} |cNz+d|^{2s}}.$$

Using the formula

$$\sum_{d | n} \mu(d) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{else,} \end{cases}$$

for the Möbius function μ, we get

$$\begin{aligned}
&\sum_{(c,d)\neq(0,0)} \frac{\alpha_N(d)}{(cNz+d)^{k-l}|cNz+d|^{2s}} \\
&\qquad = \sum_{(c,d)\neq(0,0)} \sum_{e|\gcd(d,N/N_M)} \mu(e) \frac{\alpha(d)}{(cNz+d)^{k-l}|cNz+d|^{2s}} \\
&\qquad = \sum_{e|N/N_M} \mu(e)\alpha(e)e^{-k+l-2s} \sum_{(c,d)\neq(0,0)} \frac{\alpha(d)}{(cM(\frac{N}{Me})z+d)^{k-l}|cM(\frac{N}{Me})z+d|^{2s}} \\
&\qquad = \frac{(-2\pi i)^{k-l}G(\overline{\alpha})}{(k-l-1)!M^{k-l}} \sum_{e|N/N_M} \mu(e)\alpha(e)e^{-k+l-2s} E^{\mathbf{1},\overline{\alpha}}_{k-l}((N/Me)z,s).
\end{aligned}$$

We obtain an equality of holomorphic functions

$$E^{\mathbf{1},\overline{\alpha_N}}_{k-l}(z,s) = \left(\frac{N}{M}\right)^{k-l} \sum_{e|N/N_M} \mu(e)\alpha_0(e)e^{-k+l-2s} E^{\mathbf{1},\overline{\alpha}}_{k-l}((N/Me)z,s),$$

which must also be true at $s=0$. □

Thus the product $E^{\mathbf{1},\alpha}_l E^{\mathbf{1},\overline{\alpha_N}}_{k-l}$ is a linear combination of products of the form

$$E^{\mathbf{1},\alpha}_l \cdot \left(E^{\mathbf{1},\overline{\alpha}}_{k-l}|B_{N/Me}\right)$$

for $e \mid N/N_M$. If $e \neq 1$ these products are clearly old forms. Hence the projection of $P_k(N)$ to the new space, $\overline{P_k(N)}$, is generated by the projections of the products

$$\left(E^{\mathbf{1},\alpha}_l|W^N_S\right) \cdot \left(E^{\mathbf{1},\overline{\alpha}}_{k-l}|B_{N/M}|W^N_S\right). \tag{2.5.1}$$

where S is a set of prime divisors of N. Let us focus on the first factor for now. Let $x,y,z,w \in \mathbb{Z}$ as in the definition of W^N_S. We have

$$W^N_S = \begin{pmatrix} N_S x & y \\ Nz & N_S w \end{pmatrix} = \begin{pmatrix} M_S x & y \\ N_{\overline{S}} M_S z & N_S w \end{pmatrix} \begin{pmatrix} (\frac{N}{M})_S & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.5.2}$$

The first matrix on the right has determinant M_S and satisfies all other conditions in the definition of $W^M_{S_M}$, where S_M is the set of primes in S that divide M. So, as operators on $\mathcal{M}_l(M,\alpha)$, we have the equality $W^N_S = W^M_{S_M}|B_{(\frac{N}{M})_S}$.

As mentioned in the preliminaries the action of the partial Atkin–Lehner operators on Eisenstein series was studied in [Wei77], and using Theorem 1.1.6 we see that the first factor in (2.5.1) is a multiple of

$$E^{\overline{\alpha}_{S_M},\alpha_{\overline{S_M}}}_l|B_{(\frac{N}{M})_S},$$

where $\overline{S_M} = \{p \mid M\} \setminus S_M$. To study the second factor in (2.5.1) we use an extension of Proposition 1.5 of [AL78] that allows us to swap the order of the lifting operator and the Atkin–Lehner operator above:

Proposition 2.5.2. *Let $F \in \mathcal{M}_k(M, \chi)$, $d \in \mathbb{Z}_{\geq 1}$, and S be a set of primes dividing dM. Let $\overline{S}$ be the complement of S in the set of prime divisors of dM, S_M the elements of S that divide M, and define $d_S = \prod_{p \in S} p^{v_p(d)}$ and $d_{\overline{S}}$ as usual. Then*

$$F|B_d|W_S^{Md} = \overline{\chi}_S(d_{\overline{S}})\overline{\chi}_{\overline{S}}(d_S)F|W_{S_M}^M|B_{d_{\overline{S}}}$$

Proof. Choose $x, y, z, w \in \mathbb{Z}$ as in the definition of W_S^{Md}, i.e. satisfying $y \equiv 1 \pmod{d_S M_S}$, $x \equiv 1 \pmod{d_{\overline{S}} M_{\overline{S}}}$ and $(M_S d_S)^2 xw - Mdzy = M_S d_S$. As operators on $\mathcal{M}_k(N, \chi)$, we have

$$B_d W_S^{Md} = \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} d_S M_S x & y \\ Mdz & d_S M_S w \end{pmatrix} = \begin{pmatrix} M_S d_S x & d_{\overline{S}} y \\ Mz & M_S w \end{pmatrix} \begin{pmatrix} d & 0 \\ 0 & d_S \end{pmatrix}.$$

The determinant of $\left(\begin{smallmatrix} M_S d_S x & d_{\overline{S}} y \\ Mz & M_S w \end{smallmatrix}\right)$ is M_S and so by Proposition 1.1.2 and the fact that $y \equiv 1 \pmod{M_S}$ and $x \equiv 1 \pmod{M_{\overline{S}}}$ it equals $\overline{\chi}_S(d_{\overline{S}})\overline{\chi}_{\overline{S}}(d_S)W_{S_M}^M$.

□

Applying Proposition 2.5.2 with $d = N/M$ to $E_{k-l}^{\mathbf{1},\overline{\alpha}}|B_{N/M}|W_S^N$ and using Proposition 14 of [Wei77], we see that the second factor in (2.5.1) is a multiple of

$$E_{k-l}^{\alpha_{S_M},\overline{\alpha}_{\overline{S_M}}}|B_{(\frac{N}{M})_{\overline{S}}},$$

so the product in (2.5.1) a multiple of

$$\left(E_l^{\overline{\alpha}_{S_M},\alpha_{\overline{S_M}}}|B_{(\frac{N}{M})_S}\right) \cdot \left(E_{k-l}^{\alpha_{S_M},\overline{\alpha}_{\overline{S_M}}}|B_{(\frac{N}{M})_{\overline{S}}}\right).$$

Set

$$\begin{aligned} M_1 &= M_S, \\ M_2 &= M_{\overline{S}}, \\ d_1 &= (N/M)_S = N_S/M_1, \\ d_2 &= (N/M)_{\overline{S}} = N_{\overline{S}}/M_2. \end{aligned}$$

With these definitions, α_{S_M} and $\overline{\alpha}_{\overline{S_M}}$ are primitive characters modulo M_1, M_2 respectively; we rename them to ϕ and ψ. We then define the space $Q_k(N)$ to be space generated by the products

$$E_l^{\phi,\psi}|B_{d_1} \cdot E_{k-l}^{\overline{\phi},\overline{\psi}}|B_{d_2} \tag{2.5.3}$$

for any set S of prime divisors of N and two primitive characters ϕ of modulus $M_1|N_S$ and ψ of modulus $M_2|N_{\overline{S}}$. In (2.5.3) $\overline{S}$ denotes the complement of S among the set of prime divisors of N. The above calculation shows that $Q_k(N)$ and $P_k(N)$ have the same projection on to the new subspace $\mathcal{S}_k^{\text{new}}(N)$. Using the spaces $Q_k(N)$ and their lifts we can extend Theorem 2.4.2 to the full space $\mathcal{S}_k(N)$:

Theorem 2.5.3. *Let N be as in Theorem 2.4.2 and $\mathcal{Q}_k(N) = \bigcup\limits_{Md|N} Q_k(M)|B_d$ be the subspace of $\mathcal{M}_k(N)$ generated by the products*

$$E_l^{\phi,\psi}|B_{d_1 d} \cdot E_{k-l}^{\overline{\phi},\overline{\psi}}|B_{d_2 d}$$

for $1 \leq l \leq k-1$ *and all pairs of primitive characters* ϕ, ψ *of modulus* M_1, M_2 *and* $d_1, d_2, d \in \mathbb{Z}_{\geq 1}$ *such that* $\gcd(d_1M_1, d_2M_2) = 1$ *and* $d_1M_1d_2M_2d \mid N$. *As usual we exclude the case* $\phi = \psi = \mathbf{1}$ *and* $l = 2$ *or* $l = k-2$. *Then for* $k \geq 4$

$$\mathcal{M}_k(N) = \mathcal{Q}_k(N) + \mathcal{E}_k(N).$$

Proof. Follows from Theorem 2.4.2, the previous calculations, and the fact that

$$\mathcal{S}_k(N) = \bigcup_{M|N} \bigcup_{d|N/M} \mathcal{S}_k^{new}(M)|B_d.$$

by induction. □

To treat the case $k = 2$ we need one more result.

Proposition 2.5.4. *Let* $f \in \mathcal{S}_2^{new}(N')$ *be a newform of level* $N' \mid N$ *with* $L(f,1) = 0$, *and let* d *be such that* $dN' \mid N$. *Then* $f|B_d$ *is orthogonal to* $P_2(N)$.

Proof. It suffices to show that $f|B_d$ is orthogonal to each of the generators of $P_2(N)$, so we fix a product

$$(E_1^{\mathbf{1},\alpha} E_1^{\overline{\alpha_N},*})|W_S^N$$

where α is a primitive odd character modulo M and S is a set of prime divisors of N. Since W_S^N is self-adjoint,

$$\langle (E_1^{\mathbf{1},\alpha} E_1^{\overline{\alpha_N},*})|W_S^N, f|B_d \rangle = \langle E_1^{\mathbf{1},\alpha} E_1^{\overline{\alpha_N},*}, f|B_d|W_S^N \rangle.$$

Using Lemma 2.5.2 and the fact that f is an eigenfunction of all $W_{S'}^M$ for sets S' of prime divisors of M, we see that $f|B_d|W_S^N$ is a multiple of $f|B_{d'}$ for some $d'|d$. By the Rankin-Selberg method (see (2.3.1)) we get that for, Re(s) $\gg 0$,

$$\langle E_1^{\mathbf{1},\alpha} E_1^{\overline{\alpha_N},*}, f|B_{d'} \rangle = \frac{\Gamma(s+1)}{d'^{s+1}(4\pi)^{s+1}} \sum_{n \geq 1} \frac{a_n \sigma_{1,\mathbf{1},\alpha}(d'n)}{n^{s+1}},$$

where a_n are the Fourier coefficients of f. Let $d' = \prod p^{e_p}$. Then

$$\sum_{n \geq 1} \frac{a_n \sigma_{1,\mathbf{1},\alpha}(d'n)}{n^{s+1}} = \sum_{\gcd(n,d')=1} \frac{a_n \sigma_{1,\mathbf{1},\alpha}(n)}{n^{s+1}} \prod_{p|d'} \left(\sum_{a=0}^{\infty} \frac{a_{p^a} \sigma_{1,\mathbf{1},\alpha}(p^{a+e_p})}{(p^a)^{s+1}} \right) \tag{2.5.4}$$

The first sum over n coprime to d' is, up to the Euler factors corresponding to the prime divisors of d', given in the proof of Proposition 2.3.1. It has analytic continuation to $s = 0$ and vanishes there, since $L(f,1) = 0$. It remains to show that the sums

$$f_p(s) = \left(\sum_{a=0}^{\infty} \frac{a_{p^a} \sigma_{1,\mathbf{1},\alpha}(p^{a+e_p})}{(p^a)^{s+1}} \right)$$

can be analytically continued to $s = 0$. If $\alpha(p) = 1$, $f_p(s)$ equals

$$\sum_{a=0}^{\infty} \frac{a_{p^a}(a+e_p)}{(p^a)^{s+1}} = -\log(p)^{-1} L_p'(f, s+1) + e_p L_p(f, s+1)$$

where $L_p(f,\cdot)$ is the Euler factor of $L(f,\cdot)$ at p. So f_p can indeed be analytically continued to $s=0$, since local Euler factors are entire. If $\alpha(p)\neq 1$ then

$$(1-\alpha(p))f_p(s)=\sum_{a=0}^{\infty}\frac{a_{p^a}(1-\alpha(p^{a+e_p+1}))}{(p^a)^{s+1}}=L_p(f,s+1)+\alpha(p^{e_p+1})L_p(f_\alpha,s+1),$$

which is again entire. □

Using Proposition 2.5.4 we can also show that any lift of an old form, of the form $f|B_d$, with $L(f,1)=0$ is orthogonal to $Q_2(N)$. Define the subspace

$$\mathcal{S}_2^{\mathrm{rk}=0}(N)=\bigcup_{M|N}\bigcup_{d|N/M}\overline{\mathcal{S}_2^{\mathrm{rk}=0}(M)}|B_d.$$

Then as for Theorem 2.5.3 we can use induction to prove

Theorem 2.5.5. *Let N be as in Theorem 2.4.2 and $\mathcal{Q}_2(N)$ be the subspace of $\mathcal{M}_2(N)$ generated by the products*

$$E_1^{\phi,\psi}|B_{d_1d}\cdot E_1^{\overline{\phi},\overline{\psi}}|B_{d_2d}$$

for all pairs of primitive characters ϕ,ψ of modulus M_1,M_2 and $d_1,d_2,d\in\mathbb{N}$ such that $\gcd(d_1M_1,d_2M_2)=1$ *and $d_1M_1d_2M_2d\mid N$. Then*

$$\mathcal{S}_2^{rk=0}(N)+\mathcal{E}_2(N)=\mathcal{Q}_2(N)+\mathcal{E}_2(N).$$

2.6 Non-trivial nebentypus

Most of the methods we have developed in the previous sections also work for the spaces $\mathcal{M}_k(N,\chi)$, where χ is a non-trivial character modulo N. However some significant complications arise, in particular because the Atkin–Lehner operators W_S^N are not endomorphisms of $\mathcal{M}_k(N,\chi)$ anymore. To avoid these complications we restrict our treatment to the case of prime level and weight 2.

Theorem 2.6.1. *Let p be a prime and χ a character modulo p. Let $P_2(p,\chi)$ be the space generated by $E_1^{\mathbf{1},\overline{\alpha}}E_1^{\mathbf{1},\overline{\chi}\alpha}$, for odd characters α modulo p and $\overline{P_2(p,\chi)}$ be its projection to $\mathcal{S}_2(p,\chi)$. Then*

$$\overline{P_2(p,\chi)}=\mathcal{S}_2^{rk=0}(p,\chi)$$

Proof. Note that $\mathcal{S}_2^{\mathrm{new}}(p,\chi)=\mathcal{S}_2(p,\chi)$ since $\mathcal{S}_2(1)=\{0\}$. Proposition 2.3.1 shows that the products $E_1^{\mathbf{1},\overline{\alpha}}E_1\mathbf{1},\overline{\chi}\alpha$ are orthogonal to any newform f with $L(f,1)=0$ and hence $\overline{P_2(p,\chi)}\subseteq\mathcal{S}_2^{\mathrm{rk}=0}(p,\chi)$. Suppose for a contradiction that the reverse inclusion does not hold. Then there exists a non-zero form $g\in\mathcal{S}_2^{\mathrm{rk}=0}(p,\chi)$ that is orthogonal to $\overline{P_2(p,\chi)}$. Let $f_1,\ldots,f_r$ be a basis of newforms of $\mathcal{S}_2^{\mathrm{rk}=0}(p,\chi)$ and

$$g=\sum_{i=1}^{r}\beta_i f_i.$$

Orthogonality to $\overline{P_2(p,\chi)}$ translates to

$$\sum_{i=1}^{r} \beta_i L(f_i^c, 1) L((f^c)_\alpha, 1) = 0, \tag{2.6.1}$$

for every odd character α modulo p. Again we introduce

$$G = \sum_{i=1}^{r} \beta_i L(f_i^c, 1) f_i^c \in \mathcal{S}_2(p, \overline{\chi})$$

and note that (2.6.1) is equivalent to

$$L(G_\alpha, 1) = 0$$

for every odd character α. We will show that this implies $\xi^+_{G|W_p}(0; u, v) = 0$ for all $(u, v) \in E_p$ and hence $G = 0$. If $p \mid u$ or $p \mid v$, then automatically $\xi^+_{G|W_p}(0; u, v) = 0$, so we can assume that p does not divide u or v. Repeating the calculations in the proof of Theorem 2.2.2 we obtain

$$\xi^+_{G|W_p}(0; u, v) = \frac{1}{(-2\pi i)} \sum_{\alpha} \frac{\overline{\chi\alpha^2(u)}}{p-1} L(G_\alpha, 1)$$

where the sum is over all odd characters modulo p. Since $L(G_\alpha, 1) = 0$ for all such characters this shows $G = 0$. Since conjugation acts continuously on $\mathbb{C}$ we have $L(f_i^c, 1) = \overline{L(f_i, 1)} \neq 0$, so we see that $\beta_i = 0$ for all $i = 1, \ldots, r$ and hence we reach the contradiction $g = 0$. □

2.7 Examples

Since the Fourier expansions of the generators of $\mathcal{Q}_k(N)$ are all given explicitly in terms of twisted divisor sums, it is straightforward to implement an algorithm that takes a newform f of weight k as input and calculates its representation as a linear combination of generators of $\mathcal{Q}_k(N)$ and $\mathcal{E}_k(N)$. According to Theorems 2.5.3 and 2.5.5 this is always possible when $k > 2$ and in the case $k = 2$ only possible when $f \in \mathcal{S}_2^{rk=0}(N)$. We implemented this algorithm in the Sage Mathematics Software [Sage] and present a few selected examples here. The level and weight were always chosen so that $\mathcal{M}_k(N)$ contains only one newform, that we denote by $f_{N,k}$. We use the notation

$$E^{\phi,\psi,t}(z) := t^{-k/2} E^{\phi,\psi}|_k B_t(z) = E^{\phi,\psi}(tz)$$

that we will also be useful in the next chapter. To make the examples more readable we denote Dirichlet characters by bold numbers, ordered as in Sage, i.e., the character **i** is the one obtained by the Sage command DirichletGroup(N)[i-1].

$N = 14, k = 2$:

$$f_{14,2} = \frac{1}{4} E_1^{\mathbf{2},\mathbf{1}} E_1^{\overline{\mathbf{2}},\mathbf{1}} + \frac{1}{2} E_1^{\mathbf{2},\mathbf{1},2} E_1^{\overline{\mathbf{2}},\mathbf{1},2} + (\frac{3}{4}\zeta_6 - \frac{3}{4}) E_1^{\mathbf{2},\mathbf{1}} E_1^{\overline{\mathbf{2}},\mathbf{1},2} - \frac{3}{4}\zeta_6 \cdot E_1^{\mathbf{2},\mathbf{1},2} E_1^{\overline{\mathbf{2}},\mathbf{1}}.$$

$N = 15, k = 2$:

$$f_{15,2} = -\frac{3}{8}E_1^{\mathbf{2},\mathbf{1}}E_1^{\bar{\mathbf{2}},\mathbf{1}} - \frac{15}{8}\cdot E_1^{\mathbf{2},\mathbf{1},5}E_1^{\bar{\mathbf{2}},\mathbf{1},2} + \frac{9}{4}E_1^{\mathbf{2},\mathbf{1}}E_1^{\bar{\mathbf{2}},\mathbf{1},5} + \frac{1}{8}E_1^{\mathbf{2},\mathbf{5}}E_1^{\bar{\mathbf{2}},\bar{\mathbf{5}}}.$$

$N = 19, k = 2$:

$$\begin{aligned} f_{19,2} =& (\frac{1}{3}\zeta_{18}^5 - \frac{5}{12}\zeta_{18}^4 + \frac{1}{12}\zeta_{18}^2 + \frac{1}{12}\zeta_{18} + \frac{1}{4})E_1^{\mathbf{2},\mathbf{1}}E_1^{\bar{\mathbf{2}},\mathbf{1}} \\ &+ (-\frac{1}{12}\zeta_{18}^5 + \frac{1}{6}\zeta_{18}^4 - \frac{1}{12}\zeta_{18}^2 - \frac{1}{12}\zeta_{18} - \frac{1}{4})E_1^{\mathbf{4},\mathbf{1}}E_1^{\bar{\mathbf{4}},\mathbf{1}}. \end{aligned}$$

$N = 20, k = 2$:

$$f_{20,2} = -\frac{1}{4}E_1^{\mathbf{2},\mathbf{1}}E_1^{\bar{\mathbf{2}},\mathbf{1}} - \frac{5}{4}E_1^{\mathbf{2},\mathbf{1},5}E_1^{\bar{\mathbf{2}},\mathbf{1},5} + \frac{3}{2}E_1^{\mathbf{2},\mathbf{1}}E_1^{\bar{\mathbf{2}},\mathbf{1},5}.$$

$N = 27, k = 2$:

$$\begin{aligned} f_{27,2} =& (\frac{1}{12}\zeta_{18}^5 - \frac{1}{12}\zeta_{18}^4 - \frac{1}{6}\zeta_{18}^3 + \frac{1}{12}\zeta_{18}^2 - \frac{1}{12}\zeta_{18} + \frac{1}{12})\cdot E_1^{\mathbf{2},\mathbf{1}}E_1^{\bar{\mathbf{2}},\mathbf{1}} \\ &+ (\frac{1}{4}\zeta_{18}^4 + \frac{1}{2}\zeta_{18}^3 + \frac{1}{4}\zeta_{18} - \frac{1}{4})\cdot E_1^{\mathbf{4},\mathbf{1}}E_1^{\bar{\mathbf{4}},\mathbf{1},3}. \end{aligned}$$

$N = 5, k = 4$:

$$f_{5,4} = (-\frac{7}{16}\zeta_4 + \frac{1}{16})E_1^{\mathbf{2},\mathbf{1}}E_3^{\bar{\mathbf{2}},\mathbf{1}} + (\frac{7}{16}\zeta_4 + \frac{1}{16})E_1^{\mathbf{4},\mathbf{1}}E_3^{\bar{\mathbf{4}},\mathbf{1}}.$$

$N = 6, k = 4$:

$$f_{6,4} = -\frac{1}{2}E_1^{\mathbf{2},\mathbf{1}}E_3^{\bar{\mathbf{2}},\mathbf{1}} + 3\cdot E_1^{\mathbf{2},\mathbf{1},2}E_3^{\bar{\mathbf{2}},\mathbf{1},2} - \frac{5}{2}E_1^{\mathbf{2},\mathbf{1}}E_3^{\bar{\mathbf{2}},\mathbf{1},2}.$$

$N = 7, k = 4$:

$$f_{7,4} = (-\frac{2}{7}\zeta_6 + \frac{4}{21})E_1^{\mathbf{2},\mathbf{1}}E_3^{\bar{\mathbf{2}},\mathbf{1}} + (\frac{4}{21}\zeta_6 - \frac{1}{14})E_1^{\mathbf{4},\mathbf{1}}E_3^{\bar{\mathbf{4}},\mathbf{1}}.$$

$N = 8, k = 4$:

$$f_{8,4} = E_1^{\mathbf{2},\mathbf{1},2}E_3^{\bar{\mathbf{2}},\mathbf{1},2} - E_1^{\mathbf{2},\mathbf{1}}E_3^{\bar{\mathbf{2}},\mathbf{1},2}.$$

$N = 9, k = 4$:

$$f_{9,4} = -\frac{1}{8}\zeta_6 E_1^{\mathbf{2},\mathbf{1}}E_3^{\bar{\mathbf{2}},\mathbf{1}} + (\frac{27}{4}\zeta_6 + \frac{9}{4})E_1^{\mathbf{4},\mathbf{1},3}E_3^{\bar{\mathbf{4}},\mathbf{1},3}.$$

$N = 6, k = 6$:

$$\begin{aligned} f_{6,6} =& \frac{5}{52}E_1^{\mathbf{2},\mathbf{1}}E_5^{\bar{\mathbf{2}},\mathbf{1}} - \frac{10}{13}E_1^{\mathbf{2},\mathbf{1},2}E_5^{\bar{\mathbf{2}},\mathbf{1},2} + \frac{7}{13}E_1^{\mathbf{2},\mathbf{1}}E_5^{\bar{\mathbf{2}},\mathbf{1},2} + \frac{7}{52}E_1^{\mathbf{2},\mathbf{1},2}E_5^{\bar{\mathbf{2}},\mathbf{1}} \\ &+ \frac{45}{26}E_3^{\mathbf{1},\mathbf{2}}E_3^{\mathbf{1},\bar{\mathbf{2}}} - \frac{180}{13}E_3^{\mathbf{1},\mathbf{2},2}E_3^{\mathbf{1},\bar{\mathbf{2}},2} - \frac{315}{26}E_3^{\mathbf{1},\mathbf{2}}E_3^{\mathbf{1},\bar{\mathbf{2}},2}. \end{aligned}$$

$N = 8, k = 6$:

$$f_{8,6} = \frac{1}{4}E_1^{\mathbf{2},\mathbf{1}}E_5^{\bar{\mathbf{2}},\mathbf{1},2} - \frac{1}{4}E_1^{\mathbf{2},\mathbf{1},2}E_5^{\bar{\mathbf{2}},\mathbf{1}}.$$

$N = 3, k = 8$:

$$f_{3,8} = -\frac{1}{18}E_1^{\mathbf{2},\mathbf{1}}E_7^{\bar{\mathbf{2}},\mathbf{1}} + \frac{7}{12}E_3^{\mathbf{2},\mathbf{1}}E_5^{\bar{\mathbf{2}},\mathbf{1}}.$$

Chapter 3

A correspondence of modular forms

Recall the definition of a completed L-function from §1.1.6.

$$\Lambda(f,s) := \Gamma(s)\left(\frac{\sqrt{N}}{2\pi}\right)^{s} L(f,s) = N^{s/2}\mathcal{M}(f-a_0)(s) \tag{3.0.1}$$

and of the Fricke involution

$$f|_k W_N(z) = (\sqrt{N}z)^{-k} f\left(-\frac{1}{Nz}\right).$$

Recall also that when f is a modular form of weight k, we write $f|W_N$ for $f|_k W_N$. In this chapter we will often look at L-functions of modular forms that aren't cusp forms. For this reason we introduce the regularised Λ-values as in [Bru]:

Definition 3.0.1. Let $f \in \mathcal{M}_k(\Gamma_1(N))$. Then the *regularised values* of $\Lambda(f,s)$ at $s=0$ and $s=k$ are defined by

$$\begin{aligned}\Lambda^*(f,0) &= \lim_{s\to 0} \Lambda(f,s) + \frac{a_0}{s}\\ \Lambda^*(f,k) &= \lim_{s\to k} \Lambda(f,s) + \frac{i^k b_0}{k-s}.\end{aligned}$$

The regularised values of $\Lambda(f,s)$ still satisfy the functional equations

$$\Lambda^*(f,0) = i^k\Lambda^*(f|W_N,k), \text{ and } \Lambda^*(f,k) = i^k\Lambda^*(f|W_N,0).$$

The following Lemma will be useful later.

Lemma 3.0.1 (Lemma 8 in [Bru]). *Let $e \in \mathcal{M}_l(\Gamma_1(N))$ and $f \in \mathcal{M}_k(\Gamma_1(N))$ with constant terms e_0 and f_0. Let $e^* = e - e_0$ and $f^* = f - f_0$. Then*

$$\begin{aligned}&N^{s/2}\mathcal{M}(e^*\cdot(f^*|_0 W_N))(s)\\ &\quad = i^{-k}\Lambda(e\cdot f|W_N, s+k) - e_0 i^{-k}\Lambda(f|W_N, s+k) - f_0\Lambda(e,s)\end{aligned} \tag{3.0.2}$$

for all $s\in\mathbb{C}$. Note that for $s=-k$, $s=0$ and $s=l$ the poles on the right hand side cancel and using the regularised values of $\Lambda^*(f,s)$ we can specialise to $s=l$:

$$\begin{aligned}&N^{l/2}\mathcal{M}(e^*\cdot(f^*|_0 W_N))(l)\\ &\quad = i^k\Lambda^*(e\cdot f|W_N, l+k) - i^k e_0\Lambda(f|W_N, l+k) - f_0\Lambda^*(e,l)\end{aligned}$$

Proof. First note that since e^* and f^* have exponential decay at $i\infty$ the Mellin transform in (3.0.2) converges. Let g_0 be the constant term of $g = f|W_N$ and let $g^* = g - g_0$. For $\operatorname{Re} s \gg 0$

$$\begin{aligned}N^{s/2}\mathcal{M}(e^*\cdot f^*|_0 W_N)(s) &= i^k N^{\frac{s+k}{2}}\mathcal{M}(e^*\cdot f|_k W_N)(s+k) - a_0 N^{s/2}\mathcal{M}(e^*)(s)\\ &= i^k N^{\frac{s+k}{2}}\mathcal{M}(e^* g)(s+k) - a_0\Lambda(e,s)\\ &= i^k\Lambda(eg)(s+k) - i^k e_0\Lambda(g)(s+k) - a_0\Lambda(e,s).\end{aligned}$$

By the uniqueness of meromorphic continuation this equation is true for all $s\in\mathbb{C}$. □

3.1 A correspondence of modular forms

Let $f_1(z) = \sum_{m_1=0}^{\infty} \alpha(m_1)e^{2\pi i m_1 z}$ and $f_2(z) = \sum_{m_2=0}^{\infty} \beta(m_2)e^{2\pi i m_2 z}$ be functions on $\mathcal{H}$. By applying the Möbius inversion formula we can rewrite f_1 and f_2 as double sums:

$$f_1(z) = \sum_{m_1,n_1 \geq 0}^{\infty} a(m_1)e^{2\pi i m_1 n_1 z} \text{ and } f_2(z) = \sum_{m_2,n_2 \geq 0}^{\infty} b(m_2)e^{2\pi i m_2 n_2 z},$$

where $a(n) := \sum_{r|n} \alpha(r)\mu(n/r)$ and $b(n) := \sum_{r|n} \beta(r)\mu(n/r)$. We then define a new function $\Phi_t(f_1, f_2)$ on $\mathcal{H}$ by the Fourier expansion

$$\Phi_t(f_1, f_2)(z) = \sum_{m=1}^{\infty} \sum_{d|m} a(m/d)b(d)d^{t-1}e^{2\pi m i z} = \sum_{m,n \geq 1} a(m)b(n)n^{t-1}e^{2\pi i m n z}.$$

This construction leads to a correspondence on spaces of Eisenstein series. Assume that ψ is odd and that the positive integer l satisfies $\phi(-1) = (-1)^l$. If $f_1 = E_1^{\mathbf{1},\psi}$, $f_2 = E_l^{\mathbf{1},\phi}$ and k is such that $k - l$ is even then the function $\Phi_k(f_1, f_2) = E_{k+l-1}^{\psi,\phi}$ is an Eisenstein series of weight $k + l - 1$. This fact will be used in the proof of Theorem 1.0.7.

An analogous construction can be carried out when f_1 and f_2 are cusp forms of weight 1, level N and Dirichlet characters v_1 and v_2, respectively. Although we do not expect $\Phi_t(f_1, f_2)$ to be a modular form, Proposition 3.1.1 shows that if t is even then all its twisted L-series satisfy the functional equations of a weight t cusp form of level N^2 and character $v_1 v_2$.

Specifically, for each prime $r \nmid N$, consider a primitive character ψ of conductor r such that $\psi(-1) = (-1)^u$ ($u = 0$ or 1). For convenience ψ can also stand for the trivial character $\mathbf{1}$ (mod 1). For $\mathrm{Re}(s) \gg 0$ consider

$$\begin{aligned} L(\Phi_t(f_1, f_2), \psi; s) &= \sum_{m \geq 1} \frac{\psi(m)}{m^s} \left(\sum_{d|m} a\left(\frac{m}{d}\right) d^{t-1} b(d) \right) \\ &= \left(\sum_{m \geq 1} \frac{b(m)\psi(m)}{m^{s-t+1}} \right) \left(\sum_{l \geq 1} \frac{a(l)\psi(l)}{l^s} \right). \end{aligned}$$

The second factor is connected to the L-series of $(f_1)_\psi$ by

$$L(f_1, \psi; s) = \sum_{m \geq 1} \frac{\psi(n)}{n^s} \left(\sum_{m|n} a(m) \right) = \left(\sum_{m \geq 1} \frac{a(m)\psi(m)}{m^s} \right) \cdot L(\psi, s)$$

and similarly the first factor is connected to the L-series of $(f_2)_\psi$. From the definition of $L(\Phi_t(f_1, f_2), \psi; s)$ we immediately deduce that

$$L(\Phi_t(f_1, f_2), \psi; s) = \frac{L((f_1)_\psi, s)}{L(\psi, s)} \frac{L((f_2)_\psi, s - t + 1)}{L(\psi, s - t + 1)}. \tag{3.1.1}$$

If f_1 and f_2 are Hecke eigenforms, this equality implies that $L(\Phi_t(f_1, f_2), \psi; s)$ has an Euler product representation. Defining the completion of $L(\Phi_t(f_1, f_2), \psi; s)$ as

$$\Lambda(\Phi_t(f_1, f_2), \psi; s) := \frac{\Gamma(s)(Nr)^s}{(2\pi)^s} L(\Phi_t(f_1, f_2), \psi; s)$$

we have

Proposition 3.1.1. *Let f_1 and f_2 be cusp forms of weight 1, level N and Dirichlet characters υ_1 and υ_2, respectively. Let ψ be a primitive character ψ of prime conductor $r \nmid N$. Then for even $t > 1$ the completed L-series $\Lambda(\Phi_t(f_1, f_2), \psi; s)$ has meromorphic continuation to $s \in \mathbb{C}$ and satisfies the functional equation*

$$\Lambda(\Phi_t(f_1, f_2), \psi; s) = (-1)^{t/2} \upsilon_1(r) \upsilon_2(r) \psi(N^2) \tau(\psi) \Lambda(\Phi_t(f_2|_1 W_N, f_1|_1 W_N), \overline{\psi}; t - s),$$

where we recall that

$$\tau(\psi) := \frac{G(\psi)}{\sqrt{r}} = \frac{1}{\sqrt{r}} \sum_{n \bmod r} \psi(n) e^{2\pi i \frac{n}{r}}$$

is the normalised Gauss sum of ψ.

Proof. We first express $\Lambda(\Phi_t(f_1, f_2), \psi; s)$ in terms of the completed L-series,

$$\Lambda(f, \psi; s) = \frac{\Gamma(s)(r\sqrt{N})^s}{(2\pi)^s} L(f, \psi; s) \text{ and } \Lambda(\psi, s) := \left(\frac{r}{\pi}\right)^{s/2} \Gamma\left(\frac{s+u}{2}\right) L(\psi, s),$$

where $u = 0$ or 1 is determined by $\psi(-1) = (-1)^u$. We then have

$$\begin{aligned} \Lambda(\Phi_t(f_1, f_2), \psi; s) = \left(\frac{Nr}{\pi}\right)^{\frac{t-1}{2}} & 2^{s-t+1} \frac{\Gamma((s-t+1+u)/2)\Gamma((s+u)/2)}{\Gamma(s-t+1)} \\ & \times \frac{\Lambda(f_1, \psi; s)}{\Lambda(\psi; s)} \frac{\Lambda(f_2, \psi; s-t+1)}{\Lambda(\psi; s-t+1)}. \end{aligned} \tag{3.1.2}$$

We recall the functional equations for the L-functions which appear in the expression above:

$$\begin{aligned} \Lambda(f_j, \psi; s) &= i \upsilon_j(r) \psi(N) \tau(\psi)^2 \Lambda(f_j|_k W_N, \overline{\psi}; 1-s), \text{ for } j = 1, 2, \text{ and} \\ \Lambda(\psi; s) &= i^{-u} \tau(\psi) \Lambda(\overline{\psi}; 1-s). \end{aligned}$$

By using these functional equations we can rewrite the right-hand side of (3.1.2) to obtain

$$\Lambda(\Phi_t(f_1, f_2), \psi; s) = \epsilon \cdot \Lambda(\Phi_t(f_2|_1 W_N, f_1|_1 W_N), \overline{\psi}; t - s),$$

where

$$\begin{aligned} \epsilon = & 2^{2s-t} \frac{\Gamma((s-t+1+u)/2)\Gamma((s+u)/2)\Gamma(1-s)}{\Gamma((-s+1+u)/2)\Gamma((-s+t+u)/2)\Gamma(s-t+1)} \\ & \times (-1)^{u+1} \upsilon_1(r) \upsilon_2(r) \psi(N^2) \tau(\psi)^2. \end{aligned}$$

The final version of the functional equation now follows from the identity

$$\frac{\Gamma((s-t+1+u)/2)\Gamma((s+u)/2)\Gamma(1-s)}{\Gamma((-s+1+u)/2)\Gamma((-s+t+u)/2)\Gamma(s-t+1)} = 2^{t-2s}(-1)^{t/2+u+1},$$

which is valid for even t and can be shown using standard properties of the Gamma function, including the reflection and duplication formulas. □

Remark. It follows immediately from (3.1.1) that $L(\Phi_t(f_1, f_2), s)$ has infinitely many poles (assuming the Grand Simplicity Hypothesis [RS94]) and therefore $\Phi_t(f_1, f_2)$ can not be a modular form. However, the extension of the converse theorem of [Dau14] to general levels implies that $\Phi_t(f_1, f_2)$ is a modular *integral.*

3.2 A reinterpretation of the method of Rogers–Zudilin

The method of [Zud13] relies crucially on a simple change of variables in an integral of the product of two series which leads to a product of two different functions. This part of the method can be expressed as the following simple "duality relation" involving the functions rather than their Fourier expansions. For a function h on $\mathcal{H}$ and $x \in \mathbb{Z}$ it will be convenient to use the notation $h^{(x)}$ for the function $h|_0 B_x(z) = h(xz)$.

Lemma 3.2.1. *Let $f, g : \mathcal{H} \to \mathbb{C}$ be holomorphic functions with exponential decay at infinity and at most polynomial growth at 0. For each m, $n \in \mathbb{N}$ and $s \in \mathbb{C}$ we have*

$$\mathcal{M}(f^{(m)} \cdot (g^{(n)}|_0 W_N))(s) = (n/m)^s \mathcal{M}(f^{(n)} \cdot (g^{(m)}|_0 W_N))(s).$$

Proof. From the growth conditions at infinity and 0 it follows that the product $f \cdot g|_0 W_N$ has exponential decay at both infinity and 0 and thus the Mellin transforms on both sides are well defined. By the change of variables $t \to (n/m)t$ we see that $\mathcal{M}(f^{(m)} \cdot g^{(n)}|_0 W_N)(s)$ equals

$$\int_0^\infty f(mit) g\left(\frac{ni}{Nt}\right) t^s \frac{dt}{t} = (n/m)^s \int_0^\infty f(nit) g\left(\frac{mi}{Nt}\right) t^s \frac{dt}{t}.$$

□

With the above lemma we obtain the following

Theorem 3.2.2. *Let $F_1, F_2 : \mathcal{H} \to \mathbb{C}$ be given by the Fourier expansions*

$$F_1(z) = \sum_{m_1, n_1 \geq 1} a_1(m_1) b_1(n_1) e^{2\pi i m_1 n_1 z},$$
$$F_2(z) = \sum_{m_2, n_2 \geq 1} a_2(m_2) b_2(n_2) e^{2\pi i m_2 n_2 z},$$

where we assume, additionally, that the coefficients $a_j(n)$ and $b_j(n)$ grow at most polynomially in n. If, for $j = 1, 2$, we define the functions

$$f_j(z) = \sum_{m_j, n_j \geq 1} b_j(n_j) e^{2\pi i m_j n_j z} \quad \textit{and} \quad g_j(z) = \sum_{m_j, n_j \geq 1} a_j(m_j) e^{2\pi i m_j n_j z}$$

then we have the following relation between Mellin transforms

$$\mathcal{M}(F_1 \cdot F_2|_0 W_N)(s) = \mathcal{M}(\Phi_{s+1}(f_1, f_2) \cdot (\Phi_{-s+1}(g_2, g_1)|_0 W_N))(s) \quad \textit{for all } s \in \mathbb{C}.$$

Proof. Set $h_j(z) := \sum_{n_j \geq 1} b_j(n_j)e^{2\pi i n_j z}$ for $j = 1, 2$. The growth conditions on $b_j(n)$ imply that h_1, h_2 have exponential decay at infinity and at most polynomial growth at 0. Hence Lemma 3.2.1 implies

$$\begin{aligned}\mathcal{M}(h_1^{(m_1)} \cdot h_2^{(m_2)}|_0 W_N)(s) &= \left(\frac{m_2}{m_1}\right)^s \int_0^\infty h_1(m_2 it) \cdot h_2\left(\frac{im_1}{Nt}\right) t^s \frac{dt}{t} \\ &= \left(\frac{m_2}{m_1}\right)^s \int_0^\infty \sum_{n_1, n_2 \geq 1} b_1(n_1)b_2(n_2)e^{-\frac{2\pi m_1 n_2}{Nt}} e^{-2\pi n_1 m_2 t} t^s \frac{dt}{t}.\end{aligned}$$

The growth condition of b_j justifies the interchange of integration and summation, so, upon the further change of variables $t \to (n_2/m_2)t$ we deduce that

$$\begin{aligned}\mathcal{M}(h_1^{(m_1)} \cdot h_2^{(m_2)}|_0 W_N)(s) &= m_1^{-s} \int_0^\infty \sum_{n_1, n_2 \geq 1} b_1(n_1)b_2(n_2)n_2^s e^{-\frac{2\pi m_1 m_2}{Nt}} e^{-2\pi n_1 n_2 t} t^s \frac{dt}{t} \\ &= m_1^{-s} \int_0^\infty \Phi_{s+1}(f_1, f_2)(it) e^{\frac{-2\pi m_1 m_2}{Nt}} t^s \frac{dt}{t}.\end{aligned}$$

The desired conclusion now follows from the fact that

$$F_1 \cdot F_2|_0 W_N(z) = \sum_{m_1, m_2 \geq 1} a_1(m_1)a_2(m_2)h_1^{(m_1)}(z) \cdot (h_2^{(m_2)}|_0 W_N)(z).$$

□

3.3 An application to products of Eisenstein series

We recall the weight k Eisenstein series $E_k^{\psi_1, \phi_2}$ assigned to primitive Dirichlet characters ψ_1 mod N_1 and ψ_2 mod N_2 which satisfy $\psi_1(-1)\psi_2(-1) = (-1)^k$. Its Fourier expansion at infinity is given by

$$E_k^{\psi_1, \psi_2}(z) = a_k^{\psi_1, \psi_2} + 2 \sum_{m, n \geq 1} \psi_1(m)\psi_2(n)n^{k-1}e^{2\pi i n m z}.$$

To ease notation we will write $E_k^{\psi_1, \psi_2, t}(z)$ for the function $t^{-k/2}E_k^{\psi_1, \psi_2}|B_t(z) = E_k^{\psi_1, \psi_2}(tz)$ for any $t \in \mathbb{N}$. In the sequel we will often use the following identity

$$E_k^{\psi_1, \psi_2}|_k W_{tN_1N_2} = (-1)^k \tau(\psi_1)\tau(\psi_2) \left(\frac{N_2}{N_1}\right)^{\frac{k-1}{2}} t^{k/2} E_k^{\bar{\phi}, \bar{\psi}, t}, \tag{3.3.1}$$

which is valid for any $t > 0$ and follows from Theorem 1.1.6.

We can now use Theorem 3.2.2 to prove a relation between L-values of $E_l^{\chi_1, \chi_2} \cdot E_k^{\bar{\psi}_2, \bar{\psi}_1, M}$ and L-values of $E_j^{\chi_1, \psi_2} \cdot E_{k+l-j}^{\bar{\chi}_2, \bar{\psi}_1, M_1 N_2}$. Let ψ_i and χ_i $(i = 1, 2)$ are primitive characters modulo N_i and M_i such that $(\chi_1 \cdot \chi_2)(-1) = (-1)^l$ and $(\psi_1 \cdot \psi_2)(-1) = (-1)^k$. We will regard both Eisenstein series $E_l^{\chi_1, \chi_2}$ and $E_k^{\psi_1, \psi_2}$ as modular forms of level MN where

$M = M_1M_2$ and $N = N_1N_2$. It follows immediately from (3.3.1) that

$$(-1)^k\tau(\psi_1)\tau(\psi_2)\left(\frac{N_2}{N_1}\right)^{\frac{k-1}{2}} M^{k/2}\Lambda(E_l^{\chi_1,\chi_2}\cdot E_k^{\bar{\psi}_2,\bar{\psi}_1,M}, j) = \Lambda(E_l^{\chi_1,\chi_2}\cdot(E_k^{\psi_1,\psi_2}|_kW_{MN}), j),$$

and by Lemma 3.0.1 this equals

$$\begin{aligned} &i^{-k}(MN)^{\frac{j-k}{2}}\mathcal{M}((E_l^{\chi_1,\chi_2} - a_l^{\chi_1,\chi_2})\cdot(E_k^{\psi_1,\psi_2} - a_k^{\psi_1,\psi_2})|_0W_{MN})(j-k) \\ &\quad + a_l^{\chi_1,\chi_2}\Lambda(E_k^{\psi_1,\psi_2}|_kW_{MN}, j) + i^{-k}a_k^{\psi_1,\psi_2}\Lambda(E_l^{\chi_1,\chi_2}, j-k)\end{aligned} \tag{3.3.2}$$

We are now in a position to apply Theorem 3.2.2 to the Mellin transform (3.3.2) for $j \in \{1,\dots,k+l-1\}$ with $\chi_1(-1)\psi_2(-1) = (-1)^j$. In the notation of the theorem set

$$\begin{aligned} a_1(m_1) &= \chi_2(m_1)m_1^{l-1},\ b_1(n_1) = \chi_1(n_1) \\ a_2(m_2) &= \psi_1(m_2),\ b_2(n_2) = \psi_2(n_2)n_2^{k-1}\end{aligned}\qquad s = j-k$$

Then

$$\begin{aligned} &\mathcal{M}((E_l^{\chi_1,\chi_2} - a_l^{\chi_1,\chi_2})\cdot(E_k^{\psi_1,\psi_2} - a_k^{\psi_1,\psi_2})|_0W_{MN})(j-k) \\ &\quad = 4\mathcal{M}(\Phi_{j-k+1}(f_1,f_2)\cdot(\Phi_{k-j+1}(g_2,g_1)|_0W_{MN}))(j-k) \\ &\quad = \mathcal{M}((E_j^{\chi_1,\psi_2} - a_j^{\chi_1,\psi_2})\cdot(E_{k+l-j}^{\psi_1,\chi_2} - a_{k+l-j}^{\psi_1,\chi_2})|_0W_{MN})(j-k).\end{aligned} \tag{3.3.3}$$

Another application of Lemma 3.0.1 shows that this equals

$$\begin{aligned} &(MN)^{\frac{k-j}{2}}i^{k+l-j}\Lambda(E_j^{\chi_1,\psi_2}\cdot(E_{k+l-j}^{\psi_1,\chi_2}|_{k+l-j}W_{MN}), l) \\ &\quad - (MN)^{\frac{k-j}{2}}i^{k+l-j}a_j^{\chi_1,\psi_2}\Lambda(E_{k+l-j}^{\psi_1,\chi_2}|W_{MN}, l) - (MN)^{\frac{k-j}{2}}a_{k+l-j}^{\psi_1,\chi_2}\Lambda(E_j^{\chi_1,\psi_2}, j-k)\end{aligned}$$

Collecting everything together

$$\begin{aligned} \Lambda(E_l^{\chi_1,\chi_2}\cdot(E_k^{\psi_1,\psi_2}|_kW_{MN}), j) &= a_l^{\chi_1,\chi_2}\Lambda(E_k^{\psi_1,\psi_2}|_kW_{MN}, j) &(3.3.4) \\ &\quad + i^{-k}a_k^{\psi_1,\psi_2}\Lambda(E_l^{\chi_1,\chi_2}, j-k) \\ &\quad + i^{l-j}\Lambda(E_j^{\chi_1,\psi_2}\cdot(E_{k+l-j}^{\psi_1,\chi_2}|_{k+l-j}W_{MN}), l) \\ &\quad - i^{l-j}a_j^{\chi_1,\psi_2}\Lambda(E_{k+l-j}^{\psi_1,\chi_2}|W_{MN}, l) &(3.3.5) \\ &\quad - a_{k+l-j}^{\psi_1,\chi_2}\Lambda(E_j^{\chi_1,\psi_2}, j-k).\end{aligned}$$

Applying the functional equation to (3.3.4) and (3.3.5) we arrive at the following theorem.

Theorem 3.3.1. *Let ψ_i and χ_i $(i = 1,2)$ are primitive characters modulo N_i and M_i such that $(\chi_1\cdot\chi_2)(-1) = (-1)^l$ and $(\psi_1\cdot\psi_2)(-1) = (-1)^k$. Let $j \in \{1,\dots,k+l-1\}$ with $\chi_1(-1)\psi_2(-1) = (-1)^j$. Then we have the following relation of L-values*

$$\begin{aligned} \Lambda(E_l^{\chi_1,\chi_2}\cdot(E_k^{\psi_1,\psi_2}|_kW_{MN}), j) &= i^{l-j}\Lambda(E_j^{\chi_1,\psi_2}\cdot(E_{k+l-j}^{\psi_1,\chi_2}|_{k+l-j}W_{MN}), l) \\ &\quad + i^{-k}a_l^{\chi_1,\chi_2}\Lambda(E_k^{\psi_1,\psi_2}, k-j) \\ &\quad + i^{-k}a_k^{\psi_1,\psi_2}\Lambda(E_l^{\chi_1,\chi_2}, j-k) \\ &\quad - i^{-k}a_j^{\chi_1,\psi_2}\Lambda(E_{k+l-j}^{\psi_1,\chi_2}, k-j) \\ &\quad - a_{k+l-j}^{\psi_1,\chi_2}\Lambda(E_j^{\chi_1,\psi_2}, j-k).\end{aligned}$$

In full generality the theorem looks complicated but note that the L-functions of Eisenstein series come from Dirichlet L-functions,

$$L(E_k^{\phi,\psi}, s) = 2L(\psi, s)L(\phi, s-k+1),$$

and moreover only in special cases the constant terms of the Eisenstein series are non-zero. In particular if all characters are non-trivial we can apply (3.3.1) to obtain:

Corollary 3.3.2. *In the conditions of Theorem 3.3.1 assume furthermore that all characters ψ_i and χ_i are non-trivial. Then*

$$\Lambda(E_l^{\chi_1,\chi_2} \cdot E_k^{\bar\psi_2,\bar\psi_1,M}, j) = C \cdot \Lambda(E_j^{\chi_1,\psi_2} \cdot E_{k+l-j}^{\bar\chi_2,\bar\psi_1,M_1N_2}, l), \tag{3.3.6}$$

where

$$C = (-i)^{l-j}\tau(\chi_2)\tau(\psi_2)^{-1}M_1^{\frac{l-j}{2}}M_2^{\frac{l-j-1}{2}}N_1^{-\frac{l-j}{2}}N_2^{\frac{l-j+1}{2}}.$$

3.4 Application to derivatives of L-functions

Let ψ and ϕ be odd, primitive Dirichlet characters modulo N_1 and N_2 respectively. Using the notation of the last section we set $N = N_1N_2$ and

$$E_1^{\psi} := E_1^{\psi,\mathbf{1}}, a_\psi := a_1^{\psi,\mathbf{1}}, \text{ and } f_r^{\psi,\phi} := \frac{\sqrt{N}}{4}\left(E_1^{\psi} - a_\psi\right) \cdot \left(\left(E_1^{\phi,r} - a_\phi\right)|_1 W_N\right).$$

The goal of this section is to evaluate a particular linear combination of the special values $\mathcal{M}(f_r^{\psi,\phi})(2)$ in two different ways thereby obtaining a relation between values and derivatives of certain L-functions. We first observe that for a fixed positive integer r we can write

$$\mathcal{M}(f_r^{\psi,\phi})(2) = \frac{1}{4i}\mathcal{M}\left(\left(E_1^{\psi} - a_\psi\right) \cdot \left(\left(E_1^{\phi,r} - a_\phi\right)|_0 W_N\right)\right)(1).$$

Since we now have a weight 0 action in the right-hand side we can use Theorem 3.2.2 with

$$s = 1, a_1(n) = 1, b_1(n) = \psi(n), a_2(n) = \delta_r(n), b_2(n) = \phi(n),$$

where $\delta_r(n) = 1$ if $r|n$ and 0 otherwise. This implies that $\mathcal{M}(f_r^{\psi,\phi})(2)$ equals

$$\frac{1}{i}\mathcal{M}\left(\Phi_2(f_1, f_2) \cdot \Phi_0(g_2, g_1)|_0 W_N\right)(1) = \frac{1}{2i}\mathcal{M}\left(E_1^{\psi,\phi}(it) \cdot \sum_{n_1,n_2\geq 1}\frac{1}{n_1}e^{\frac{-2\pi r n_1 n_2}{Nt}}\right)(1).$$

From the following well-known expression for the logarithm of the Dedekind eta function

$$\sum_{m,n\geq 1}\frac{1}{n}e^{\frac{-2\pi rmn}{Nu}} = -\sum_{m\geq 1}\log(1 - e^{\frac{-2\pi rm}{Nu}}) = -\log\left(\eta(ri/(Nu))e^{\frac{2\pi r}{24\cdot Nu}}\right),$$

we deduce that

$$\mathcal{M}(f_r^{\psi,\phi})(2) = \frac{i}{2}\int_0^\infty E_2^{\psi,\phi}(iu)\log\left(\eta(ri/(Nu))e^{\frac{2\pi r}{24\cdot Nu}}\right)du. \tag{3.4.1}$$

The integral above is well-defined since $E_2^{\psi,\phi}$ decays exponentially at both ∞ and 0. The decay at infinity is immediate since ψ is not trivial and the decay at 0 follows from (3.3.1). By using (3.3.1) to rewrite $f_r^{\bar{\phi},\bar{\psi}}$ it follows from (3.4.1) that

$$\begin{aligned}\mathcal{M}(F_r^{\psi,\phi})(2) &= \frac{i}{2}\int_0^\infty (E_2^{\psi,\phi}|_2(1+W_N))(iu)\log\left(\eta(ri/(Nu))e^{\frac{2\pi r}{24\cdot Nu}}\right)du \\ &= -\frac{i}{2}\int_0^\infty (E_2^{\psi,\phi}|_2(1+W_N))(iu)\log\left(\eta(riu))e^{\frac{2\pi ru}{24}}\right)du \quad (3.4.2)\end{aligned}$$

where

$$F_r^{\psi,\phi} := f_r^{\psi,\phi} + \sqrt{\frac{N_2}{N_1}}\tau(\psi)\tau(\phi)f_r^{\bar{\phi},\bar{\psi}}.$$

It is clear from (3.4.2) that we can find a linear combination of $F_r^{\psi,\phi}$'s such that the exponentials inside the logarithm on the right-hand side are eliminated:

$$\begin{aligned}\mathcal{M}((N_1+N_2)(F_1^{\psi,\phi}+F_N^{\psi,\phi}) - (1+N)(F_{N_1}^{\psi,\phi}+F_{N_2}^{\psi,\phi}))(2) = \\ -\frac{i}{2}\int_0^\infty (E_2^{\psi,\phi}|_2(1+W_N))(iu)\log(V(iu))du, \quad (3.4.3)\end{aligned}$$

where

$$V(z) := \frac{(\eta(z)\eta(Nz))^{N_1+N_2}}{(\eta(N_1z)\eta(N_2z))^{1+N}}.$$

We will now proceed to evaluate the two sides of (3.4.3) separately.

3.4.1 The right-hand side of (3.4.3)

We first recall the principle behind Goldfeld's expression for derivatives of L-functions:

Proposition 3.4.1. *Let f and g be holomorphic functions on $\mathcal{H}$ such that for some $N \in \mathbb{N}$:*
(i) $f|_2W_N = f$
(ii) $g|_kW_N = \pm g$, for some non-zero constant $k \in \mathbb{R}$. Then

$$\int_0^\infty f(z)dz = 0 \ \text{ and } \ 2\int_0^\infty f(iy)\log(g(iy))dy = k\int_0^\infty f(iy)\log(y)dy.$$

Proof. Condition *(i)* is equivalent to $f(W_Nz)d(W_Nz) = f(z)dz$. Therefore

$$\int_0^\infty f(z)dz = \int_{W_N\infty}^{W_N0} f(z)dz = \int_\infty^0 f(z)dz$$

and hence $\int_0^\infty f(z)dz = 0$. Similarly, we see that

$$\begin{aligned}\int_0^\infty f(z)\log(g(z))dz &= \int_{W_N\infty}^{W_N0} f(z)\log(g(z))dz = \int_\infty^0 f(z)\log(g(W_Nz))dz \\ &= \int_\infty^0 f(z)\log(g(z))dz + ik\int_\infty^0 f(iy)\log(y)dy \\ &\quad + c'\int_0^\infty f(z)dz\end{aligned}$$

for some $c' \in \mathbb{C}$. This equality, together with $\int_0^\infty f(z)dz = 0$, implies the conclusion. $\square$

Since Proposition 3.4.1 holds for $f = E_2^{\psi,\phi}|_2(1+W_N)$ and $g = V$ with $k = N_1+N_2-1-N$, we deduce that

$$\int_0^\infty f(iu)\log(V(iu))du = \frac{k}{2}\int_0^\infty f(iu)\log(u)du = \frac{k}{2}(\mathcal{M}f)'(1). \tag{3.4.4}$$

By using Proposition 3.4.1 together with (3.0.1) we can express the the right-hand side of (3.4.4) as

$$\frac{(N_1-1)(1-N_2)}{2\sqrt{N}}\Lambda'(E_2^{\psi,\phi}|_2(1+W_N),1).$$

If h is a modular form of weight 2 and level N it is easy to see from the functional equation of $\Lambda(h,s)$ that $\Lambda'(h|_2(1+W_N),1) = 2\Lambda'(h,1)$. It follows that the right-hand side of (3.4.3) equals

$$\frac{i(N_1-1)(N_2-1)}{2\sqrt{N}}\Lambda'(E_2^{\psi,\phi},1). \tag{3.4.5}$$

3.4.2 The left-hand side of (3.4.3)

To compute the left-hand side of (3.4.3) we first express $\mathcal{M}(f_r^{\psi,\phi})(s)$ in a form where we can apply Lemma 3.0.1:

$$\mathcal{M}(f_r^{\psi,\phi})(s) = \frac{i}{4}\mathcal{M}\left((E_1^{\psi}-a_\psi)\cdot(E_1^{\phi,r}-a_\phi)|_0W_N\right)(s-1) \tag{3.4.6}$$

Applying Lemma 3.0.1 and the functional equation for the completed L-functions we deduce that $4N^{(s-1)/2}\mathcal{M}(f_r^{\psi,\phi})(s)$ equals

$$\begin{aligned}
&\Lambda(E_1^{\psi}\cdot E_1^{\phi,r}|_1W_N,s) + a_\phi i\Lambda(E_1^{\psi},s-1) + a_\psi\Lambda(E_1^{\phi,r}|_1W_N,s)\\
&= \Lambda((E_1^{\psi}|_1W_N)\cdot E_1^{\phi,r},2-s) + a_\phi i\Lambda(E_1^{\psi},s-1) + a_\psi i\Lambda(E_1^{\phi,r},1-s)\\
&= -\tau(\psi)\sqrt{N_2}\Lambda(E_1^{\bar\psi,N_2}E_1^{\phi,r},2-s) + a_\phi i\Lambda(E_1^{\psi},s-1) + a_\psi i\Lambda(E_1^{\phi,r},1-s).
\end{aligned} \tag{3.4.7}$$

For the last equality we again used (3.3.1) and we have an analogous expression for $4N^{(s-1)/2}\mathcal{M}(f_r^{\bar\phi,\bar\psi})(s)$. Note that (3.4.7) is valid for all $s\in\mathbb{C}$ if we use regularised L-values whenever one of the L-functions in (3.4.7) has a pole.

We will now compute the value of the linear combination

$$\mathcal{M}((N_1+N_2)(F_1^{\psi,\phi}+F_N^{\psi,\phi}) - (1+N)(F_{N_1}^{\psi,\phi}+F_{N_2}^{\psi,\phi}))(s) \tag{3.4.8}$$

at $s=2$ by considering each of the three summands of (3.4.7) and the analogue for $f_r^{\bar\phi,\bar\psi}$.

First we treat the contributions from L-functions associated to products of Eisenstein series. In $\mathcal{M}(F_1^{\psi,\phi}+F_N^{\psi,\phi})(s)$ they are

$$\frac{N^{(1-s)/2}N_2^{1/2}}{4}\tau(\psi)\Lambda(-E_1^{\bar\psi,N_2}E_1^{\phi,1} + E_1^{\phi,N_1}E_1^{\bar\psi,1} - E_1^{\bar\psi,N_2}E_1^{\phi,N} + E_1^{\phi,N_1}E_1^{\bar\psi,N},2-s). \tag{3.4.9}$$

By using the trivial fact

$$\Lambda(f,s) = a^s\Lambda(f^{(a)},s), \tag{3.4.10}$$

combined with $(E_1^{\phi,N_1}E_1^{\bar\psi,1})^{N_2} = E_1^{\phi,N}E_1^{\bar\psi,N_2}$ and $(E_1^{\phi,1}E_1^{\bar\psi,N_2})^{N_1} = E_1^{\phi,N_1}E_1^{\bar\psi,N}$, (3.4.9) becomes

$$\frac{N^{(1-s)/2}N_2^{1/2}}{4}\tau(\psi) \cdot \left[(N_1^{s-2}-1)\Lambda(E_1^{\bar\psi,N_2}E_1^{\phi,1},2-s)+(1-N_2^{s-2})\Lambda(E_1^{\phi,N_1}E_1^{\bar\psi,1},2-s)\right]. \quad (3.4.11)$$

Both $\Lambda(E_1^{\bar\psi,N_2}E_1^{\phi,1},2-s)$ and $\Lambda(E_1^{\phi,N_1}E_1^{\bar\psi,1},2-s)$ have a simple pole at $s=2$ with residue $-a_{\bar\psi}a_\phi$. Therefore (3.4.11) is equal to $\tau(\psi)a_{\bar\psi}a_\phi\log(N_1/N_2)/\sqrt{N_1}$ at $s=2$. It is easy to verify that the contribution of products of Eisenstein series in $\mathcal{M}(F_{N_1}^{\psi,\phi}+F_{N_2}^{\psi,\phi})(2)$ is exactly the same as that in $\mathcal{M}(F_1^{\psi,\phi}+F_N^{\psi,\phi})(2)$ and hence the products of Eisenstein series contribute

$$\frac{\tau(\psi)a_{\bar\psi}a_\phi(N_1+N_2-1-N)}{4\sqrt{N_1}}\log\left(\frac{N_1}{N_2}\right)$$

to $\mathcal{M}((N_1+N_2)(F_1^{\psi,\phi}+F_N^{\psi,\phi})-(1+N)(F_{N_1}^{\psi,\phi}+F_{N_2}^{\psi,\phi}))(s)$.

Secondly, to compute the contribution of the terms coming from $E_1^{\phi,r}$ and $E_1^{\bar\psi,r}$ we apply (3.4.10) to $\Lambda(E_1^{\phi,r},1-s)$ and $\Lambda(E_1^{\bar\psi,r},1-s)$. Thus their contribution to $\mathcal{M}(F_r^{\phi,\psi})(s)$ is

$$\frac{N^{(1-s)/2}r^{s-1}}{4}\left(a_\psi i\Lambda(E_1^{\phi},1-s)+a_{\bar\phi}i\sqrt{\frac{N_2}{N_1}}\tau(\psi)\tau(\phi)\Lambda(E_1^{\bar\psi},1-s)\right),$$

which implies that the contribution of these terms to (3.4.8) at $s=2$ is 0. We are now left with

$$\mathcal{M}((N_1+N_2)(F_1^{\psi,\phi}+F_N^{\psi,\phi})-(1+N)(F_{N_1}^{\psi,\phi}+F_{N_2}^{\psi,\phi}))(2) = \frac{(N_1+N_2-1-N)}{4\sqrt{N_1}}\left[\tau(\psi)a_{\bar\psi}a_\phi\log\left(\frac{N_1}{N_2}\right)+\frac{i}{\sqrt{N_2}}\Lambda(\mathcal{E}^{\psi,\phi},1)\right] \quad (3.4.12)$$

where $\mathcal{E}^{\psi,\phi}$ is given by

$$\mathcal{E}^{\psi,\phi} := L(\phi,0)E_1^{\psi}+\sqrt{\frac{N_2}{N_1}}\tau(\psi)\tau(\phi)L(\bar\psi,0)E_1^{\bar\phi}. \quad (3.4.13)$$

We note that the last term of (3.4.12) is well-defined because the residues of $\Lambda(E_1^{\bar\phi},s)$ and $\Lambda(E_1^{\psi},s)$ at 1 cancel when we take the linear combination giving $\mathcal{E}^{\psi,\phi}$. Equations (3.4.5) and (3.4.12) together finally give give

Theorem 3.4.2. *Let ψ and ϕ be odd, primitive Dirichlet characters modulo N_1 and N_2 respectively and $\mathcal{E}^{\psi,\phi}\in\mathcal{E}_1(\Gamma_1(N))$ be defined as in* (3.4.13)*. Then*

$$i\sqrt{N_2}\tau(\psi)a_{\bar\psi}a_\phi\log\left(\frac{N_1}{N_2}\right)-\Lambda(\mathcal{E}^{\psi,\phi},1)=2\Lambda'(E_2^{\psi,\phi},1). \quad (3.4.14)$$

Chapter 4

Eichler-cohomology for arbitrary real weights

4.1 Preliminaries

In this chapter we will work with modular forms with respect to a Fuchsian group of the first kind. We sketch the definition of such groups here and refer the reader to [Shi71, §1] for a more thorough introduction. The groups we have worked with in the previous chapters, congruence groups, are special cases of Fuchsian groups of the first kind. Let Γ be a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$ or of $\mathrm{SL}_2(\mathbb{R})/\{\pm I\}$. A *cusp* of Γ is any element of $\mathbb{R} \cup \{\infty\}$ that is fixed by a parabolic element of Γ, i.e., an element of Γ that has only one fixed point in $\mathbb{R} \cup \{\infty\}$. Let $\mathcal{H}^*$ be the union of $\mathcal{H}$ with the cusps of Γ. The quotient space $\Gamma\backslash\mathcal{H}^*$ can be given the structure of a Riemann surface such that the natural projection

$$\pi : \mathcal{H} \to \Gamma\backslash\mathcal{H}^*$$

is an open map. The group Γ is called a *Fuchsian group of the first kind*, if $\Gamma\backslash\mathcal{H}^*$ is compact. For the rest of this chapter we assume that $\Gamma \subseteq \mathrm{SL}_2(\mathbb{R})$ is a Fuchsian group of the first kind that contains a translation. This condition is not very restrictive since any Fuchsian group of the first kind that has cusps is conjugate to a Fuchsian group of the first kind that contains translations. The only Fuchsian groups of the first kind that are excluded by this requirement are cocompact groups, i.e., groups for which $\Gamma\backslash\mathcal{H}$ is compact. For convenience we will also assume that Γ contains $-I$.

In contrast to the previous chapters, the weight of modular forms in this chapter will not necessarily be integral. We refer the reader to [Iwa97] for a good introduction to modular forms of real weight. In order to define the slash operator $|_r$ of $\mathrm{SL}_2(\mathbb{R})$, we have to fix a branch of the logarithm on $\mathbb{C}^\times$. We choose the principal branch, i.e.,

$$\log(z) = \log|z| + i\arg(z), \text{ where } \arg(z) \in (-\pi, \pi].$$

Then we set $j(\gamma, z)^r = \exp(r \cdot \log(j(\gamma, z))$ and, for a function f on $\mathcal{H}$,

$$f|_r\gamma(z) = j(\gamma, z)^{-r} f(\gamma z).$$

While we have the formula

$$j(\gamma\delta, z)^r = j(\gamma, \delta z)^r j(\delta, z)^r$$

for all $r \in \mathbb{Z}$, this is no longer true if $r \in \mathbb{R}$ and so $|_r$ is not necessarily a group action of $\mathrm{SL}_2(\mathbb{R})$ any more. To get a useful notion of modular forms we will introduce multiplier systems.

Two important functions when dealing with real weights, introduced by Petersson in [Pet38], are

$$\omega(\gamma, \delta) = \frac{1}{2\pi}\left[-\arg(j(\gamma\delta, z)) + \arg(j(\gamma, \delta z)) + \arg(j(\delta, z))\right]$$

and

$$\sigma_r(\gamma, \delta) = e^{2\pi i r\omega(\gamma,\delta)}.$$

The value of $\omega(\gamma, \delta)$ is independent of z and in $\{-1, 0, 1\}$. From the definition it follows that

$$\sigma_r(\gamma, \delta) j(\gamma\delta, z)^r = j(\gamma, \delta z)^r j(\delta, z)^r, \quad \gamma, \delta \in \Gamma. \tag{4.1.1}$$

A *multiplier system* of weight r for Γ is a function $v : \Gamma \to \mathbb{C}$ which satisfies the consistency condition

$$v(\gamma\delta)j(\gamma\delta, z)^r = v(\gamma)v(\delta)j(\gamma, \delta z)^r j(\delta, z)^r, \ \forall \gamma, \delta \in \Gamma,$$

or equivalently

$$v(\gamma\delta) = \sigma_r(\gamma, \delta)v(\gamma)v(\delta).$$

Note that v is also a multiplier system of any weight $r' \in \mathbb{R}$ with $r' \equiv r \bmod 2$ and $\overline{v}$ is a multiplier system of weight $-r$. A multiplier system is called *unitary* if $|v(\gamma)| = 1$ for all $\gamma \in \Gamma$. For the rest of this chapter we fix a unitary multiplier system v of weight r.

For a function f on the upper half plane $\mathcal{H}$ and $\gamma \in \mathrm{SL}_2(\mathbb{R})$ we define a new slash operator $|_{r,v}$ by

$$f|_{r,v}\gamma(z) = \overline{v}(\gamma)j(\gamma, z)^{-r} f(\gamma z)$$

The consistency condition for v implies that

$$f|_{r,v}\gamma\delta(z) = (f|_{r,v}\gamma)|_{r,v}\delta(z), \quad \forall \gamma, \delta \in \Gamma,$$

and hence $|_{r,v}$ is a group operation, in contrast to $|_r$.

Let $q_0 = \infty$ and $q_1, \dots, q_m$ be a set of representatives of the cusps of Γ. For every cusp q the stabiliser subgroup Γ_q is generated by $-I$ and one generator $\sigma_q \in \Gamma$. For $q = \infty$ we choose $\sigma_\infty = \left(\begin{smallmatrix} 1 & \lambda \\ 0 & 1 \end{smallmatrix}\right)$, the minimal translation matrix in Γ with $\lambda > 0$. Let f be holomorphic on $\mathcal{H}$ and invariant under $|_{r,v}$. The equation $f(z + \lambda) = v(\sigma_\infty)f(z)$ implies that f has a Fourier expansion at ∞ of the form

$$f(z) = \sum_{n=-\infty}^{\infty} a_{n,0} \exp\left(2\pi i(n + \kappa_0)z/\lambda\right), \tag{4.1.2}$$

where $\kappa_i \in [0, 1)$ is defined for any cusp by $v(\sigma_{q_i}) = e^{2\pi i \kappa_i}$. To find the expansion at the other cusps, choose σ_{q_i} so that if

$$A_i \sigma_{q_i} A_i^{-1} = \begin{pmatrix} 1 & \lambda_i \\ 0 & 1 \end{pmatrix},$$

where $A_i = \left(\begin{smallmatrix} 0 & -1 \\ 1 & -q_i \end{smallmatrix}\right)$, we have $\lambda_i > 0$. The Fourier expansion of f at q_i is then given by

$$f|_r A_i^{-1}(z) = \sum_{n=-\infty}^{\infty} a_{n,i} \exp\left(2\pi i(n + \kappa_i)z/\lambda_i\right). \tag{4.1.3}$$

Definition 4.1.1. Let f be holomorphic in $\mathcal{H}$ and invariant under $|_{r,v}$. Then f is called a *modular form*[1] of weight r and multiplier system v with respect to Γ, if in the Fourier expansions in (4.1.2) and (4.1.3) all $a_{n,i}$ with $n + \kappa_i < 0$ are zero. If in addition all $a_{n,i}$ with $n + \kappa_i = 0$ vanish, then f is called a cusp form. The set of modular forms is denoted by $M_r(\Gamma, v)$, the set of cusp forms by $S_r(\Gamma, v)$.

Remark. Just like in the case of integral weight the space $\mathcal{M}_k(\Gamma, v)$ is always finite-dimensional.

Remark. By the main theorem of [Kno67] the only modular form of negative weight is the zero function. By [**?**] the only non-zero modular forms of weight 0 are constant functions.

[1] Another common term for modular forms that is used e.g., in [Kno74], is *entire automorphic forms*.

4.1.1 Cohomology

Definition 4.1.2. Let M be an abelian group with a right group action by Γ that we denote by $m \cdot \gamma$ for $m \in M$ and $\gamma \in \Gamma$. The group M is called a *(right)* Γ*-module* if the Γ-action is compatible with the group structure on M, i.e.,

$$(m_1 + m_2) \cdot \gamma = m_1 \cdot \gamma + m_2 \cdot \gamma,\ \forall m_1, m_2 \in M,\ \gamma \in \Gamma.$$

Let M be a Γ-module. A *cocycle* of Γ with values in M is a function $\phi : \Gamma \to M$ that satisfies

$$\phi(\gamma\delta) = \phi(\gamma) \cdot \delta + \phi(\delta),\ \forall \gamma, \delta \in \Gamma.$$

We denote the space of cocycles by $Z^1(\Gamma, M)$. There is a natural map d from M to $Z^1(\Gamma, M)$ that associates to $m \in M$ the cocycle

$$dm :\ \gamma \mapsto m \cdot \gamma - m.$$

A cocycle of the form dm for $m \in M$ is called a *coboundary* and the space of coboundaries is denoted by $B^1(\Gamma, M)$. The (first) *Eichler cohomology group* $H^1(\Gamma, M)$ is the quotient space $Z^1(\Gamma, M)/B^1(\Gamma, M)$.

A cocycle ϕ is called *parabolic* if for all cusps q_i there exists an element $m_{q_i} \in m$ such that

$$\phi(\sigma_{q_i}) = m_{q_i} \cdot \sigma_{q_i} - m_{q_i}.$$

We denote the space of parabolic cocycles by $\tilde{Z}^1(\Gamma, M)$. Since coboundaries are clearly parabolic we can form the parabolic cohomology group $\tilde{H}^1(\Gamma, M) = \tilde{Z}^1(\Gamma, M)/B^1(\Gamma, M)$.

The classical Eichler–Shimura isomorphism (see 1.0.8) for even weights $k = 2-r \geq 2$ is an isomorphism between $\mathcal{S}_{2-r}(\Gamma)$ and $\tilde{H}^1(\Gamma, \mathbb{R}[X]_r)$, where $\mathbb{R}[X]_r$ is the space of polynomials of degree $\leq k-2$ with coefficients in $\mathbb{R}$.

If we allow arbitrary real weights we have to work with the much larger coefficient module $\mathcal{P}$.

Definition 4.1.3. Let $\mathcal{P}$ be the space of holomorphic functions on $\mathcal{H}$ such that there exist positive constants K, A and B with

$$|f(z)| < K(|z|^A + y^{-B}),\ \forall z = x + iy \in \mathcal{H},$$

We can view $\mathcal{P}$ as a Γ-module with the $|_{r,v}$ action for any weight r and multiplier system v. To emphasise the dependence of the action on r and v we denote the cocycles, coboundaries, cohomology group and parabolic cohomology group associated to $\mathcal{P}$ with the $|_{r,v}$ action by $Z^1_{r,v}(\Gamma, \mathcal{P})$, $B^1_{r,v}(\Gamma, \mathcal{P})$, $H^1_{r,v}(\Gamma, \mathcal{P})$, and $\tilde{H}^1_{r,v}(\Gamma, \mathcal{P})$.

We will also call elements of $Z^1_{r,v}(\Gamma, \mathcal{P})$ cocycles of weight r and multiplier system v.

It turns out that all cocycles in $Z^1_{r,v}(\Gamma, \mathcal{P})$ are parabolic. This follows from a result that Knopp attributes to B.A. Taylor in [Kno74].

Proposition 4.1.1. *Let $\epsilon \in \mathbb{C}$ with $|\epsilon| = 1$ and $g \in \mathcal{P}$. Then there exists an $f \in \mathcal{P}$ with*

$$\bar{\epsilon} f(z+1) - f(z) = g(z), \quad \forall z \in \mathcal{H}. \tag{4.1.4}$$

Proof. This is Proposition 9 in [Kno74] and a full proof is given there. We will only present the main idea here. A formal solution of (4.1.4) is given by the one-sided average

$$f(z) = -\sum_{n=0}^{\infty} \overline{\epsilon}^n g(z+n).$$

However this sum does not always converge. Knopp uses the fact that $\mathcal{P}$ is closed under integration and differentiation to replace g with a function $\tilde{g} = g_1 + g_2$ such that the one-sided averages $f_1(z) = -\sum_{n=0}^{\infty} \overline{\epsilon}^n g_1(z+n)$ and $f_2(z) = -\sum_{n=0}^{\infty} \overline{\epsilon}^n g_2(z+n)$ converge and are in $\mathcal{P}$. □

Corollary 4.1.2. *Let $\epsilon \in \mathbb{C}$ with $|\epsilon| = 1$, $s \in \mathbb{R} \setminus \{0\}$ and $g \in \mathcal{P}$. Then there exists an $f \in \mathcal{P}$ with*

$$\overline{\epsilon} f(z+s) - f(z) = g(z), \quad \forall z \in \mathcal{H}. \tag{4.1.5}$$

Proof. First assume $s > 0$ and set $\hat{g}(z) = g(sz)$. By Proposition 4.1.1 there exists $\hat{f} \in \mathcal{P}$ that satisfies

$$\overline{\epsilon} \hat{f}(z+1) - \hat{f}(z) = \hat{g}(z), \quad \forall z \in \mathcal{H}.$$

Then $f(z) = \hat{f}(z/s)$ solves (4.1.5).

Now we treat the case $s < 0$. By the first part of this proof there exists an $\hat{f} \in \mathcal{P}$ that satisfies

$$\epsilon \hat{f}(z-s) - \hat{f}(z) = g(z), \quad \forall z \in \mathcal{H}.$$

The function $f(z) = -\epsilon \hat{f}(z-s)$ solves (4.1.5). □

Theorem 4.1.3 ([Kno74], p.627)**.** *Every cocycle in $Z^1_{r,v}(\Gamma, \mathcal{P})$ is parabolic, i.e.,*

$$Z^1_{r,v}(\Gamma, \mathcal{P}) = \tilde{Z}^1_{r,v}(\Gamma, \mathcal{P}).$$

Proof. Let $\phi \in Z^1_{r,v}(\Gamma, \mathcal{P})$. We will show that for every parabolic $\gamma \in \Gamma$ there exists $f \in \mathcal{P}$ such that

$$\phi(\gamma) = f|_{r,v}\gamma - f. \tag{4.1.6}$$

First suppose $\gamma = \left(\begin{smallmatrix} 1 & s \\ 0 & 1 \end{smallmatrix}\right)$ is a translation by $s \in \mathbb{R} \setminus \{0\}$. Then by Corollary 4.1.2 a function $f \in \mathcal{P}$ with the desired property exists.

For the general case let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ and fix a cusp q. Then there exists an $s \in \mathbb{R} \setminus \{0\}$ such that

$$A\gamma A^{-1} = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} = U, \text{where } A = \begin{pmatrix} 0 & -1 \\ 1 & -q \end{pmatrix}.$$

Replacing z by $A^{-1}z$ in equation (4.1.6) we see that it is sufficient to show the existence of $f \in \mathcal{P}$ with

$$\overline{v(\gamma)} j(A^{-1}UA, A^{-1}z)^{-r} f(\gamma A^{-1}z) - f(A^{-1}z) = \phi(\gamma)(A^{-1}z). \tag{4.1.7}$$

Setting $\hat{f}(z) = f(A^{-1}z)$ this is equivalent to

$$\overline{v(\gamma)} j(A^{-1}UA, A^{-1}z)^{-r} \hat{f}(z+s) - \hat{f}(z) = \phi(\gamma)(A^{-1}z). \tag{4.1.8}$$

Equation (4.1.1) implies the two relations

$$1 = j(AA^{-1}U, z)^{-r} = \sigma_r(A, A^{-1}U) j(A, A^{-1}Uz)^{-r} j(A^{-1}U, z)^{-r}, \tag{4.1.9}$$
$$j(A^{-1}UA, A^{-1}z)^{-r} = \sigma_r(A^{-1}U, A) j(A^{-1}U, z)^{-r} j(A, A^{-1}z)^{-r}. \tag{4.1.10}$$

After multiplying equation (4.1.8) by $j(A, A^{-1}z)^r$ and using the two relations (4.1.9) and (4.1.10) we get

$$\overline{\epsilon} F(z+s) - F(z) = j(A, A^{-1}z)^r \phi(\gamma)(A^{-1}z), \tag{4.1.11}$$

where we set $F(z) = j(A, A^{-1}z)^r \hat{f}(z)$ and $\epsilon = v(\gamma)\overline{\sigma_r(A^{-1}U, A)}\sigma_r(A, A^{-1}U)$. Note that $|\epsilon| = 1$ and $j(A, A^{-1}z)^r \phi(\gamma)(A^{-1}z) \in \mathcal{P}$. The existence of such an $F \in \mathcal{P}$ again follows from Corollary 4.1.2. □

4.2 Outline

The aim of this chapter is to give a new proof of the following theorem for $r \neq 1$.

Theorem 4.2.1 (Knopp–Mawi (2010)). *For all $r \in \mathbb{R}$ the map $f \mapsto [\phi_f^\infty]$ is an isomorphism*

$$\mathcal{S}_{2-r}(\Gamma, \overline{v}) \overset{\cong}{\to} H^1_{r,v}(\Gamma, \mathcal{P}).$$

This theorem is equivalent to Theorem 1.0.9 in the introduction, except that we replaced k with $2-r$ and v with $\overline{v}$. This choice of notation will be more convenient in the following sections.

We now give a brief outline of the proof of Theorem 4.2.1 in the case $0 < 2-r \neq 1$. This is the harder case of the theorem, for the proof in the case $2-r \leq 0$ we can skip §4.3.

In that section we construct a pairing $(\cdot\,,\cdot)$ between $\mathcal{S}_{2-r}(\Gamma, \overline{v})$ and $H^1_{r,v}(\Gamma, \mathcal{P})$. From the construction it follows immediately (see Corollary 4.3.4) that the map $f \mapsto [\phi_f^\infty]$ is injective. In order to prove Theorem 4.2.1 for $2-r > 0$ it remains to show that this pairing is perfect.

In §4.4 we first show, in Theorem 4.4.2 and Corollary 4.4.5, that every cocycle ϕ in $Z^1_{r,v}(\Gamma, \mathcal{P})$ is a coboundary in $Z^1_{r,v}(\Gamma, \mathcal{Q})$, where $\mathcal{Q}$ is a larger space of functions than $\mathcal{P}$.

Suppose $\phi \in Z^1_{r,v}(\Gamma, \mathcal{P})$ is orthogonal to $\mathcal{S}_{2-r}(\Gamma, \overline{v})$. Using the description of ϕ as a coboundary in $Z^1_{2-k,\overline{v}}(\Gamma, \mathcal{Q})$, we apply classic results from the spectral theory of automorphic forms to show that $y^{\frac{4-k}{2}}\overline{\frac{\partial g}{\partial \overline{z}}(z)}$ is in the image of the Maass weight-raising operator K_{k-2} (see Proposition 4.4.11). This then implies that ϕ is a coboundary in $Z^1_{2-k,\overline{v}}(\Gamma, \mathcal{P})$ and hence that the pairing $(\cdot\,,\cdot)$ is perfect.

In the case $k = 1$ only the last step of the proof fails, since some technical complications arise in the proof of Proposition 4.4.11.

In the last section we sketch our proof of a vector-valued version of Theorem 4.2.1.

4.3 Petersson inner product

An essential ingredient in the proof of 4.2.1 for $r \neq 1$ is the pairing that we define in this section. We make use of the auxiliary integral of a cusp form of positive real weight. For weights greater than 2 it was introduced in [Nie74] and for any positive weight it first appeared in [Pri05], where also the transformation formula (4.3.1) is mentioned. Corollary 4.3.4 can also be deduced from results in these papers and [Pri99] but the proof presented here is new.

Definition 4.3.1. Let $r \in \mathbb{R}$ with $2 - r > 0$ and g be a cusp form for the group Γ of weight $2 - r$ and unitary multiplier system $\overline{v}$. The *auxiliary integral* of g is defined as

$$G(z) = \left[-\int_z^\infty g(\tau)(\tau - \overline{z})^{-r} d\tau \right]^-,$$

where $[\cdot]^-$ indicates complex conjugation. The path of integration is the vertical line $p(t) = z + it$ where t ranges from 0 to ∞.

Since g decays exponentially at $i\infty$ the integral converges and G is a smooth function from $\mathcal{H}$ to $\mathbb{C}$. We can define a cocycle by

$$\phi_g^\infty : \gamma \mapsto \phi_{g,\gamma}^\infty(z) = G|_{r,v}\gamma(z) - G(z).$$

Proposition 4.3.1. *The cocycle ϕ_g^∞ is in $Z^1_{r,v}(\Gamma, \mathcal{P})$ and*

$$\phi_{g,\gamma}^\infty(z) = \left[\int_{\gamma^{-1}\infty}^\infty g(\tau)(\tau - \overline{z})^{-r} d\tau \right]^-, \tag{4.3.1}$$

for all $\gamma \in \Gamma$.

Proof. Let $\gamma \in \Gamma$:

$$\begin{aligned} \overline{G(\gamma z)} &= \int_\infty^{\gamma z} g(\tau)(\tau - \gamma\overline{z})^{-r} d\tau \\ &= \int_{\gamma^{-1}\infty}^{z} g(\gamma\tau)(\gamma\tau - \gamma\overline{z})^{-r} d(\gamma\tau) \\ &= j(\gamma, \overline{z})^r \int_{\gamma^{-1}\infty}^{z} g(\gamma\tau) j(\gamma,\tau)^{-2+r} (\tau - \overline{z})^{-r} d\tau. \end{aligned}$$

In the last equality we used

$$(\gamma\tau - \gamma\overline{z})^{-r} = \left(\frac{\tau - \overline{z}}{j(\gamma,\tau) j(\gamma,\overline{z})} \right)^{-r} = \frac{(\tau - \overline{z})^{-r}}{j(\gamma,\tau)^{-r} j(\gamma,\overline{z})^{-r}}.$$

To prove this let

$$\alpha = \arg(\gamma\tau - \gamma\overline{z}) \text{ and } \beta = \arg(\tau - \overline{z}) - \arg(j(\gamma,\tau)) - \arg(j(\gamma,\overline{z})).$$

We know that $\alpha \equiv \beta \mod 2\pi$ and want to show $\alpha = \beta$. Both $(\gamma\tau - \gamma\overline{z})$ and $\tau - \overline{z}$ are in $\mathcal{H}$, so their arguments are in $(0, \pi)$. Furthermore exactly one of $j(\gamma,\tau)$ and $j(\gamma,\overline{z})$ will be in $\mathcal{H}$ and one in $\overline{\mathcal{H}}$, so $-\pi < \beta < 2\pi$ and $0 < \alpha < \pi$. Together with $\beta \equiv \alpha \mod 2\pi$ this implies $\alpha = \beta$. Now we use the modularity of g to obtain

$$G(\gamma z) = j(\gamma, z)^r v(\gamma) \left[\int_{\gamma^{-1}\infty}^{z} g(\tau)(\tau - \overline{z})^{-r} d\tau \right]^{-}, \tag{4.3.2}$$

or $G|_{r,v}\gamma(z) = \left[\int_{\gamma^{-1}\infty}^{z} g(\tau)(\tau - \overline{z})^{-r} d\tau \right]^{-}$. An application of Cauchy's theorem now gives us

$$\begin{aligned} \phi_{g,\gamma}^{\infty}(z) &= G|_{r,v}\gamma(z) - G(z) \\ &= \left[\left(\int_{\gamma^{-1}\infty}^{z} - \int_{\infty}^{z} \right) g(\tau)(\tau - \overline{z})^{-r} d\tau \right]^{-} \\ &= \left[\int_{\gamma^{-1}\infty}^{\infty} g(\tau)(\tau - \overline{z})^{-r} d\tau \right]^{-}. \end{aligned}$$

To see that $\phi_{g,\gamma}^{\infty}$ is in $\mathcal{P}$ first note that $(\tau - \overline{z})^{-r}$ is antiholomorphic in $\mathcal{H}$ as a function of z (actually even in the slit plane $\mathbb{C} \setminus \{\mathbb{R}_{\geq 0} + \overline{\tau}\}$) and the integrals in the definition of G and ϕ_g^{∞} converge absolutely because g is a cusp form. Therefore $\phi_{g,\gamma}^{\infty}(z)$ is holomorphic in $\mathcal{H}$. To prove that $\phi_{g,\gamma}^{\infty}$ is in $\mathcal{P}$ one can use simple bounds for $|\tau - \overline{z}|^{-r}$. We sketch the procedure for the case $r \leq 0$ and $\mathrm{Im}(z) > 1$. In this case

$$|\tau - \overline{z}|^{-r} \leq |\tau - \overline{z}|^{\lceil -r \rceil} \leq \sum_{j=0}^{\lceil -r \rceil} \binom{\lceil -r \rceil}{j} |\tau|^{\lceil -r \rceil - j} |z|^j.$$

One can use this to bound $\phi_{g,\gamma}^{\infty}(z)$ by a polynomial in $|z|$. If $r > 0$ then for any z we can use the bound $|\tau - \overline{z}|^{-r} < |\overline{z}|^{-r} < |\mathrm{Im}(z)|$, so in this case we can bound $\phi_{g,\gamma}^{\infty}(z)$ by a negative power of $\mathrm{Im}(z)$. The missing case $r \leq 0$ and $\mathrm{Im}(z) \leq 1$ is dealt with similarly. □

Let f be another modular form of the weight $2 - r$ and multiplier system $\overline{v}$. Then, since f is holomorphic

$$\frac{\partial Gf}{\partial \overline{z}}(z) = f(z)\frac{\partial G}{\partial \overline{z}}(z) = \overline{g(z)}(\overline{z} - z)^{-r} f(z) = (-2i)^{-r} f(z)\overline{g(z)} y^{-r}.$$

This is just a scalar times the integrand occurring in the Petersson inner product of g and f, which was defined in 1.1.5 as

$$(f, g) = \int_{\mathcal{F}} f(z)\overline{g(z)} y^{-r} dx dy,$$

where $\mathcal{F}$ is a fundamental domain of Γ (Definition 1.1.3). Then by Stokes' theorem we have

$$(f, g) = -\frac{i}{2} \int_{\mathcal{F}} f(z)\overline{g(z)} y^{-r} d\overline{z} \wedge dz = C_{2-r} \int_{\partial \mathcal{F}} f(z)G(z)dz,$$

for $C_{2-r} = -\frac{i}{2}(-2i)^r$. Now we choose a fundamental domain according to the following Proposition 4.2 in [Coh13].

Proposition 4.3.2. *The fundamental domain $\mathcal{F}$ can be chosen such that $\partial\mathcal{F} = \overline{\mathcal{F}} \setminus \mathcal{F}^\circ$ consists of an even number of geodesic segments $[A_i, A_{i+1}[$[2] for $i = 1, \dots, 2n$ (the indices are taken modulo $2n$) and $\alpha_i \in \Gamma$ for $i = 1, \dots, 2n$ such that there exists an involution of $\{1, \dots, 2n\}$, denoted by π, such that*

1. *π does not have any fixed points,*
2. *$\alpha_i A_i = A_{\pi(i)+1}, \ \alpha_i A_{i+1} = A_{\pi(i)}$,*
3. *$\alpha_{\pi(i)} = \alpha_i^{-1}$,*
4. *α_i maps $[A_i, A_{i+1}[$ to $[A_{\pi(i)+1}, A_{\pi(i)}[$.*

Example 4.3.1. For $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ we choose the classic fundamental domain with $A_1 = \infty, A_2 = e^{2\pi i/3}, A_3 = i, A_4 = A_2 + 1$. Then $\alpha_1 = T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ maps $[A_1, A_2[$ to $[A_1, A_4[$ and $\alpha_2 = \sigma = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ maps $[A_2, A_3[$ to $[A_4, A_3[$. So π is the permutation that swaps 1 with 4 and 2 with 3.

Remark. For general Fuchsian groups Γ of the first kind an example of such a fundamental domain is the Ford fundamental domain (see [For25])

$$\mathcal{F} = \{z \in \mathcal{H} | |z| \leq \lambda/2 \text{ and } |j(\gamma, z)| > 1 \ \forall \gamma \in \Gamma \setminus \Gamma_\infty\}, \tag{4.3.3}$$

where λ, the width of the cusp ∞, was defined in the last section. For the rest of this chapter, we will fix this fundamental domain for Γ.

We can restate Proposition 4.3.2 as

$$\partial\mathcal{F} = \bigsqcup_{m=1}^{n} \left([A_{i_m}, A_{i_m+1}[\sqcup \alpha_{i_m}]A_{i_m}, A_{i_m+1}] \right).$$

Thus the Petersson inner product of f and g becomes

$$C_{2-r} \sum_{m=1}^{n} \left(\int_{A_{i_m}}^{A_{i_m+1}} - \int_{\alpha_{i_m} A_{i_m}}^{\alpha_{i_m} A_{i_m+1}} \right) f(z) G(z) dz.$$

Using the modularity of f, the second integral in the sum becomes

$$\begin{aligned} \int_{\alpha_{i_m} A_{i_m}}^{\alpha_{i_m} A_{i_m+1}} f(z) G(z) dz &= \int_{A_{i_m}}^{A_{i_m+1}} f(\alpha_{i_m} z) G(\alpha_{i_m} z) d(\alpha_{i_m} z) \\ &= \int_{A_{i_m}}^{A_{i_m+1}} f(z) G|_{r,v} \alpha_{i_m}(z) dz. \end{aligned}$$

Finally we arrive at

$$\begin{aligned} (f, g) &= C_{2-r} \sum_{m=1}^{n} \int_{A_{i_m}}^{A_{i_m+1}} f(z) \left(G(z) - G|_{r,v} \alpha_{i_m}(z) \right) dz \\ &= -C_{2-r} \sum_{m=1}^{n} \int_{A_{i_m}}^{A_{i_m+1}} f(z) \phi^{\infty}_{g, \alpha_{i_m}}(z) dz. \end{aligned}$$

[2] $[A_i, A_{i+1}[$ denotes the geodesic in $\mathcal{H}$ that connects A_i and A_{i+1} and includes A_i but not A_{i+1}.

Motivated by the previous calculations we define a pairing between cusp forms and cocycles:

Definition 4.3.2. Let $2-r>0$, $f \in \mathcal{S}_{2-r}(\Gamma, \overline{v})$ and $\phi \in Z^1_{r,v}(\Gamma, \mathcal{P})$. Define the pairing

$$(f,\phi) = -C_{2-r}\sum_{m=1}^{n}\int_{A_{i_m}}^{A_{i_m+1}} f(z)\phi(\alpha_{i_m})(z)dz.$$

The integrals in the sum converge because $\phi(\alpha_{i_m})$ is in $\mathcal{P}$ and therefore can increase only polynomially towards the cusps, while f decreases exponentially.

Lemma 4.3.3. *Let $f \in \mathcal{S}_{2-r}(\Gamma, \overline{v})$ and $[\phi] \in H^1_{r,v}(\Gamma, \mathcal{P})$ be represented by $\phi \in Z^1_{r,v}(\Gamma, \mathcal{P})$. The value (f,ϕ) does not depend on a choice of representative of $[\phi]$, i.e., the pairing*

$$(f,[\phi]) = (f,\phi),$$

between $\mathcal{S}_{2-r}(\Gamma, \overline{v})$ and $H^1_{r,v}(\Gamma, \mathcal{P})$, is well-defined.

Proof. It suffices to show that if ϕ is a coboundary, then $(f,\phi)=0$. If ϕ is a coboundary there exists a function $h \in \mathcal{P}$ with $\phi(\gamma) = h|_{r,v}\gamma - h$. We have

$$\begin{aligned}
\int_{A_{i_m}}^{A_{i_m+1}} f(z)h|_{r,v}\alpha_{i_m}(z)dz &= \int_{A_{i_m}}^{A_{i_m+1}} f(z)j(\alpha_{i_m},z)^{2-r}\overline{v(\alpha_{i_m})}h(\alpha_{i_m}z)d(\alpha_{i_m}z) \\
&= \int_{A_{i_m}}^{A_{i_m+1}} f(\alpha_{i_m}z)h(\alpha_{i_m}z)d(\alpha_{i_m}z) \\
&= \int_{\alpha_{i_m}A_i}^{\alpha_{i_m}A_{i_m+1}} f(z)h(z)dz.
\end{aligned} \tag{4.3.4}$$

So

$$\begin{aligned}
(f,\phi) &= -C_{2-r}\sum_{m=1}^{n}\int_{A_{i_m}}^{A_{i_m+1}} f(z)\phi(\alpha_{i_m})(z)dz \\
&= -C_{2-r}\sum_{m=1}^{n}\int_{A_{i_m}}^{A_{i_m+1}} f(z)(h|_{r,v}\alpha_{i_m}(z)-h(z))dz \\
&= -C_{2-r}\sum_{m=1}^{n}\left(\int_{\alpha_{i_m}A_i}^{\alpha_{i_m}A_{i_m+1}} - \int_{A_{i_m}}^{A_{i_m+1}}\right) f(z)h(z)dz \\
&= C_{2-r}\int_{\partial\mathcal{F}} f(z)h(z)dz.
\end{aligned} \tag{4.3.5}$$

The integral over the boundary is 0 because, since $f(z)h(z)$ decreases exponentially at the cusps, we can approach $\int_{\partial\mathcal{F}} f(z)h(z)dz$ by integrals over closed paths contained in $\mathcal{H}$, which are all equal to zero, since $f(z)h(z)$ is holomorphic. □

Corollary 4.3.4. *The map $f \mapsto [\phi_f^\infty]$ from $\mathcal{S}_{2-r}(\Gamma, \overline{v})$ to $H^1_{r,v}(\Gamma, \mathcal{P})$ is injective.*

Proof. If $[\phi_f^\infty]$ is represented by a coboundary in $Z^1_{r,v}(\Gamma, \mathcal{P})$ then, by the above calculations $0 = (f,[\phi_f^\infty]) = (f,f)$ and hence $f=0$. □

4.4 The Duality theorem

In this section we prove that the pairing we defined in Lemma 4.3.3, between $\mathcal{S}_{2-r}(\Gamma, \overline{v})$ and $H^1_{r,v}(\Gamma, \mathcal{P})$, is perfect for $0 < 2 - r \neq 1$. For such weights r this implies Theorem 4.2.1.

We already know that for every non-zero f in $\mathcal{S}_{2-r}(\Gamma, \overline{v})$ there exists a cocycle ϕ such that $(f, [\phi]) \neq 0$, since $(f, [\phi_f^\infty]) = (f, f) \neq 0$. To show that the pairing is perfect, we therefore need to prove the following theorem.

Theorem 4.4.1. *Let $1 \neq r < 2$ and $[\phi] \in H^1_{r,v}(\Gamma, \mathcal{P})$. If $(f, [\phi]) = 0$ for all $f \in \mathcal{S}_{2-r}(\Gamma, \overline{v})$, then $[\phi] = 0$. Together with Corollary 4.3.4 this implies that $\mathcal{S}_{2-r}(\Gamma, \overline{v})$ and $H^1_{r,v}(\Gamma, \mathcal{P})$ are dual to each other.*

The proof of Theorem 4.4.1 will be given at the end of this section. Most constructions that follow will be valid for any real r and so, if not explicitly stated otherwise, we work in this generality. In particular we will also show Theorem 4.2.1 for $r \geq 2$.

Let $\overline{\mathcal{H}} = \mathcal{H} \cup \mathbb{R} \cup \{\infty\}$ be the closure of $\mathcal{H}$ in $\mathbb{P}^1(\mathbb{C})$. A basis of neighbourhoods of ∞ in $\overline{\mathcal{H}}$ is given by the sets

$$H_Y(\infty) = \{z \in \mathcal{H} | \operatorname{Im}(z) > Y\} \cup \{\infty\}.$$

Let q be a cusp with $\tau_q \infty = q$ for $\tau_q \in \mathrm{SL}_2(\mathbb{R})$ such that $\tau_q^{-1}\Gamma_q\tau_q$ is generated by $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Then the open sets $H_Y(q) = \tau_q H_Y(\infty)$ for $Y > 0$ form a basis of neighbourhoods of q.

We define a variation of the space $\mathcal{P}$ that will be useful in our proof. Let $\tilde{\mathcal{Q}}$ be the space of C^∞-functions f on $\mathcal{H}$ such that, for every cusp q of Γ, there exists a neighbourhood $U_q \subseteq \mathcal{H}$ and $K_q, A_q, B_q > 0$ such that f is holomorphic in U_q and

$$|f(z)| < K_q(|z|^{A_q} + y^{-B_q}), \ \forall z \in U_q.$$

For the purpose of proving Theorem 4.4.1 we will actually be interested in a subspace $\mathcal{Q} \subseteq \tilde{\mathcal{Q}}$, that we introduce in Definition 4.4.1.

Theorem 4.4.2. *Every element of $Z^1_{r,v}(\Gamma, \mathcal{P})$ is a coboundary in $Z^1_{r,v}(\Gamma, \tilde{\mathcal{Q}})$.*

Proof. Let $\phi \in Z^1_{r,v}(\Gamma, \mathcal{P})$. We need to show that there exists a function $G \in \tilde{\mathcal{Q}}$ with $\phi(\gamma) = G|_{r,v}\gamma - G$ for all γ in Γ. Choose Y large enough, so that all the $H_Y(q)$ are disjoint and contain no elliptic fixed points. Define $U = \bigcup_{q \text{ cusp of } \Gamma} H_Y(q)$ and $V = \bigcup_{q \text{ cusp of } \Gamma} H_{2Y}(q)$. Then U and V are Γ-invariant. Recall that the projections $\pi(U)$ and $\pi(V)$ are open in $\Gamma\backslash\mathcal{H}^*$. By the smooth Urysohn lemma (see for example [Con01, Corollary 3.5.5]), there exists a smooth function $\hat{\eta}$ on $\Gamma\backslash\mathcal{H}^*$ such that $\hat{\eta}(\pi(z)) = 1$ for all $\pi(z) \in \pi(V)$ and $\hat{\eta}(\pi(z)) = 0$ for all $\pi(z)$ outside $\pi(U)$. Define $\eta(z) = \hat{\eta}(\pi(z))$ to be the pullback of $\hat{\eta}$. It is a Γ-invariant C^∞-function on $\mathcal{H}$ that satisfies $\eta(z) = 1$ on V and $\eta(z) = 0$ outside U.

We will first construct a function that has $\eta\phi$ as a coboundary. By Theorem 4.1.3, ϕ is a parabolic cocycle, so for every cusp q there exists a function $g_q \in \mathcal{P}$ such that $\phi(\sigma_q) = g_q|_{r,v}\sigma_q - g_q$, where σ_q is the generator of $\Gamma_q/\{\pm I\}$. We define G on U as follows:

if $z \in H_Y(q_i)$ for some i then $G(z) = g_{q_i}(z)$. If $z = \delta w$ for $\delta \in \Gamma$ and $w \in H_Y(q_i)$ we define

$$G(z) = v(\delta) j(\delta, w)^r (\phi(\delta)(w) + g_{q_i}(w)).$$

Note that this is equivalent to defining $G|_{r,v}\delta(w) = \phi(\delta)(w) + G(w)$, so once we show that the definition of $G(z)$ does not depend on the choice of δ, the coboundary of ηG will be $\eta\phi$. Suppose $z = \delta w = \delta' w'$, for $\delta, \delta' \in \Gamma$ and $w, w' \in H_Y(q_i)$. We need to check that

$$v(\delta) j(\delta, w)^r (\phi(\delta)(w) + g_{q_i}(w)) = v(\delta') j(\delta', w')^r (\phi(\delta')(w') + g_{q_i}(w')).$$

Multiplying both sides by $v(\delta)^{-1} j(\delta, w)^{-r}$ and using the consistency condition of the multiplier system v, we see that this is equivalent to

$$\phi(\delta)(w) + g_{q_i}(w) = [\phi(\delta') + g_{q_i}]\,|_{r,v}(\delta'^{-1}\delta)(w).$$

This follows from the cocycle condition on ϕ and the choice of g_{q_i}. Indeed, since $w' \in \delta'^{-1}\delta H_Y(q_i) \cap H_Y(q_i) \neq \emptyset$ and since we assumed that all the $H_Y(q)$ are disjoint, $\delta'^{-1}\delta$ must fix q_i. Hence $\delta'^{-1}\delta = \pm\sigma_{q_i}^n$ for some $n \geq 1$. This implies

$$g_{q_i}|_{r,v}(\delta'^{-1}\delta)(w) = \phi(\delta'^{-1}\delta)(w) + g_{q_i}(w),$$

and so

$$\begin{aligned}[\phi(\delta') + g_{q_i}]\,|_{r,v}(\delta'^{-1}\delta)(w) &= \phi(\delta)(w) - \phi(\delta'^{-1}\delta)(w) + g_{q_i}|_{r,v}(\delta'^{-1}\delta)(w) \\ &= \phi(\delta)(w) + g_{q_i}(w).\end{aligned}$$

So ηG is a well-defined function in $\tilde{\mathcal{Q}}$. We have thus shown that $\eta\phi$ is a coboundary in $Z^1_{r,v}(\Gamma, \tilde{\mathcal{Q}})$.

It remains to show that $(1-\eta)\phi$ is a coboundary. We first construct a partition of unity on $\mathcal{H}$ that is Γ-invariant. The construction we describe here is due to Gunning [Gun59]. Since Γ acts discontinuously on $\mathcal{H}$, every $z \in \mathcal{H}$ has a neighbourhood O_z such that $\gamma O_z = O_z$ if $\gamma \in \Gamma_z$ (the stabiliser of z), and $\gamma O_z \cap O_z = \emptyset$ if $\gamma \in \Gamma \setminus \Gamma_z$. Let V be as in the construction of η, a Γ-invariant open set that contains all cusps of Γ with $\eta|_V = 1$. Since $\Gamma\backslash\mathcal{H}^*$ is compact, there exist $z_1, \ldots, z_n \in \mathcal{H}$ such that the sets $\pi(O_{z_i})$ together with $\pi(V)$ cover $\Gamma\backslash\mathcal{H}^*$. Let $\hat{\epsilon}_1, \ldots, \hat{\epsilon}_n, \hat{\epsilon}_V$ be a partition of unity corresponding to this cover, i.e., smooth functions supported in $\pi(O_{z_1}), \ldots, \pi(O_{z_n})$ and $\pi(V)$ respectively, satisfying

$$\sum_{i=1}^n \hat{\epsilon}_i(\pi(z)) + \hat{\epsilon}_V(\pi(z)) = 1, \quad \forall z \in \mathcal{H}.$$

We define functions $H_1, \ldots, H_n$ on $\mathcal{H}$ as follows. If there exists $g_i(z) \in \Gamma$ such that $g_i(z)z \in O_{z_i}$ we set

$$H_i(z) = -(1-\eta(z))\frac{\epsilon_i(z)}{|\Gamma_i|}\sum_{g \in \Gamma_i} \phi(g \cdot g_i(z))(z),$$

where Γ_i is the stabiliser of z_i and $|\Gamma_i|$ is its order. This does not depend on the choice of $g_i(z)$: if $\gamma z \in O_{z_i}$ with $\gamma \in \Gamma$, then we must have $\gamma^{-1} g_i(z) \in \Gamma_i$. Thus the set $\Gamma_i g_i(z)$

is equal to $\Gamma_i\gamma$ and we see that a different choice of $g_i(z)$ just permutes the summands in the definition of $H_i(z)$. If no such $g_i(z) \in \Gamma$ exists we set $H_i(z) = 0$.

Clearly H_i is a function in $\tilde{\mathcal{Q}}$ and defining $H = \sum_{i=1}^n H_i$, we will see that $H|_{r,v}\gamma(z) - H(z) = (1-\eta(z))\phi(\gamma)(z)$ for all $\gamma \in \Gamma$ and $z \in \mathcal{H}$. First note that if z is in V, then $H(z)$ and $H(\gamma z)$ vanish and so does $(1-\eta(z))\phi(\gamma)(z)$. If z is not in V we have

$$H|_{r,v}\gamma(z) = -(1-\eta(z))\sum_i \frac{\epsilon_i(\gamma z)}{|\Gamma_i|}\sum_{g\in\Gamma_i}\phi(g\cdot g_i(\gamma z))|_{r,v}\gamma(z),$$

where the first sum is over all i such that there exists a $g_i(\gamma z) \in \Gamma$ with $g_i(\gamma z)\gamma z \in O_{z_i}$. Now we choose $g_i(\gamma z) = g_i(z)\gamma^{-1}$, to get that $H|_{r,v}\gamma(z)$ equals

$$\begin{aligned} &= -(1-\eta(z))\sum_i \frac{\epsilon_i(z)}{|\Gamma_i|}\sum_{g\in\Gamma_i}[\phi(g\cdot g_i(z))(z) - \phi(\gamma)(z)] \\ &= (1-\eta(z))(\phi(\gamma)(z) + H(z)). \end{aligned}$$

□

In the definition of $\tilde{\mathcal{Q}}$, the constants K_q, A_q, B_q may vary from cusp to cusp, in the following definition we impose stricter growth conditions, requiring the constants to be fixed.

Definition 4.4.1. Let $\mathcal{Q}$ be the space of functions F in $\tilde{\mathcal{Q}}$ such that there exist positive constants K, A, B with

$$|F(z)| < K(|z|^A + y^{-B}), \quad \forall z \in \mathcal{H}.$$

Note that the functions in $\mathcal{P}$ are the holomorphic functions in $\mathcal{Q}$.

Proposition 4.4.3. *Let F be in $\tilde{\mathcal{Q}}$. If $\gamma \mapsto F|_{r,v}\gamma - F = \psi(\gamma)$ is in $Z^1_{r,v}(\Gamma,\mathcal{P})$ then F is in $\mathcal{Q}$.*

Proof. This proof is similar to the proof of the main theorem of [Kno85]. Let M be the set of matrices γ in Γ with $\lambda/2 \le \mathrm{Re}(\gamma i) < \lambda/2$. M is a complete set of coset representatives of $\Gamma_\infty \setminus \Gamma$. We need a technical lemma from [Kno74]:

Lemma 4.4.4. *(Lemma 8 in [Kno74]) There exist positive constants K_1, A_1, B_1 such that for all $\tau \in \overline{\mathcal{F}} \cap \mathcal{H}$ and all $\gamma \in M$*

$$|\psi(\gamma)(\tau)| < K_1(Im(\gamma\tau)^{A_1} + Im(\gamma\tau)^{-B_1}).$$

Since only finitely many cusps are in $\overline{\mathcal{F}}$ and since the real part of $z \in \mathcal{F}$ is bounded, we can also find positive K_2, A_2, B_2 with

$$|F(\tau)| < K_2(\mathrm{Im}(\tau)^{A_2} + \mathrm{Im}(\tau)^{-B_2}), \quad \forall \tau \in \overline{\mathcal{F}} \cap \mathcal{H}. \tag{4.4.1}$$

As in the proof of Theorem 4.4.2, we use the fact that ψ is parabolic and hence there exists a function $g_\infty \in \mathcal{P}$ such that $\psi(\sigma_\infty) = g_\infty|_{r,v}\sigma_\infty - g_\infty$. The equation $F|_{r,v}\sigma_\infty - F = \psi(\sigma_\infty)$ implies

$$(F - g_\infty)|_{r,v}\sigma_\infty - (F - g_\infty) = 0.$$

F is in $\mathcal{Q}$ if and only if $F - g_\infty$ is in $\mathcal{P}$, so we can assume without loss of generality that $F(z+\lambda) = v(\sigma_\infty)F(z)$. Let $z \in \mathcal{H}$. There exists $\tau \in \overline{\mathcal{F}}$ and $\gamma \in \Gamma$ such that $z = \gamma\tau$. Since M is a complete set of representatives of $\Gamma_\infty \setminus \Gamma$, there is an integer m and $\delta \in M$ such that $z = \sigma_\infty^m \delta\tau$. If $\delta = I$ then we can deduce

$$|F(z)| < K_2(\mathrm{Im}(\tau)^{A_2} + \mathrm{Im}(z)^{-B_2}),$$

from equation (4.4.1) and the fact that $|F|$ is Γ_∞-invariant. Suppose $\delta = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is not the identity. Then $c \neq 0$, because the only member of M that fixes ∞ is I. We have

$$\begin{aligned} |F(z)| &= |F(\sigma_\infty^m \delta\tau)| = |F(\delta\tau)| && (4.4.2) \\ &\leq |j(\delta,\tau)|^r \left(|F(\tau)| + |\psi(\delta)(\tau)|\right) && (4.4.3) \\ &< |j(\delta,\tau)|^r [K_2(\mathrm{Im}(\tau)^{A_2} + \mathrm{Im}(\tau)^{-B_2}) && (4.4.4) \\ &\qquad + K_1(\mathrm{Im}(\delta\tau)^{A_1} + \mathrm{Im}(\delta\tau)^{-B_1})]. \end{aligned}$$

By our choice of fundamental domain we have $|j(\delta,\tau)| \geq 1$, since $\delta \notin \Gamma_\infty$. So $y = \mathrm{Im}(z) = \frac{\mathrm{Im}(\tau)}{|j(\delta,\tau)|^2} \leq \mathrm{Im}(\tau)$. On the other hand, using $\tau = \delta^{-1}\sigma_\infty^{-m} z$ we have $\mathrm{Im}(\tau) = \frac{y}{|j(\delta^{-1}\sigma_\infty^{-m}, z)|^2}$ and

$$|j(\delta^{-1}\sigma_\infty^{-m}, z)|^2 = |-cz + cm\lambda + a|^2 = c^2y^2 + (cm\lambda + a - cx)^2 \geq cy^2 > c_0 y^2,$$

where $c_0 > 0$ depends only on Γ. Such a c_0 exists because Γ is discrete. Therefore $y \leq \mathrm{Im}(\tau) < c_0^{-1} y^{-1}$, $\mathrm{Im}(\tau)^{A_2} < c_0^{-A_2} y^{-A_2}$ and $\mathrm{Im}(\tau)^{-B_2} \leq y^{B_2}$. Also $|j(\delta,\tau)|^r = (\frac{y}{\mathrm{Im}(\tau)})^{-r/2}$ is either ≤ 1 (if $r \leq 0$), or $\leq c_0^{-r/2} y^{-r}$ (if $r \geq 0$). These inequalities inserted into (4.4.4) lead to the desired inequality of the form

$$|F(z)| < K(|z|^A + y^{-B}),$$

for positive constants K, A, B and all $z \in \mathcal{H}$. □

Corollary 4.4.5. *Every cocycle in $Z^1_{r,v}(\Gamma, \mathcal{P})$ is a coboundary in $Z^1_{r,v}(\Gamma, \mathcal{Q})$.*

Let $\phi \in Z^1_{r,v}(\Gamma, \mathcal{P})$. By Corollary 4.4.5 there exists a function $g \in \mathcal{Q}$ such that $g|_{r,v}\gamma - g = \phi(\gamma)$ for all $\gamma \in \Gamma$. By the same calculation as in equation (4.3.5), for any $f \in \mathcal{S}_{2-r}(\Gamma, \overline{v})$, we have

$$\begin{aligned} (f, \phi) &= -C_{2-r} \sum_{m=1}^{n} \int_{A_{i_m}}^{A_{i_m+1}} f(z)(g|_{r,v}\alpha_{i_m}(z) - g(z))dz \\ &= C_{2-r} \int_{\partial\mathcal{F}} f(z)g(z)dz \\ &= C_{2-r} \int_{\mathcal{F}} \frac{\partial g}{\partial \overline{z}} d\overline{z} \wedge f(z)dz. \end{aligned}$$

Here we note again that the integrals above exist because g can only increase polynomially towards the cusps of Γ, while f decreases exponentially.

4.4.1 Spectral theory of automorphic forms

To carry out the proof of Theorem 4.4.1, we will apply spectral theory. We only give a very brief introduction here; for more details and proofs, see the exposition [Roe66] by Roelcke. In these articles Roelcke uses a variation of the slash operator which we denote by $|_{r,v}^R$

$$f|_{r,v}^R\gamma(z) = \left(\frac{j(\gamma,\overline{z})}{j(\gamma,z)}\right)^{r/2}\overline{v}(\gamma)f(\gamma z).$$

The connection to our slash operator is given by the following lemma:

Lemma 4.4.6. *Let $f:\mathcal{H}\to\mathbb{C}$, $F(z)=y^{\frac{r}{2}}f(z)$ and $\gamma\in\Gamma$. Then*

$$y^{\frac{r}{2}}\left(f|_{r,v}\gamma(z)\right) = F|_{r,v}^R\gamma(z).$$

So a function f is invariant under $|_{r,v}$ if and only if $F(z)=y^{\frac{r}{2}}f(z)$ is invariant under $|_{r,v}^R$.

Definition 4.4.2. Let $H_{r,v} = H_r(\Gamma,v)$ be the Hilbert space of functions f that are invariant under $|_{r,v}^R$ and have finite norm with respect to the scalar product

$$(f_1,f_2)^R = \int_{\mathcal{F}} f_1(z)\overline{f_2(z)}\frac{dxdy}{y^2}.$$

The weight r hyperbolic Laplacian and the Maass weight-raising and weight-lowering operators are defined as

$$\begin{aligned}
\Delta_r &= -(z-\overline{z})^2\frac{\partial^2}{\partial z\partial\overline{z}} - \frac{r}{2}(z-\overline{z})\left(\frac{\partial}{\partial z}+\frac{\partial}{\partial\overline{z}}\right),\\
K_r &= (z-\overline{z})\frac{\partial}{\partial z}+\frac{r}{2},\\
\Lambda_r &= (z-\overline{z})\frac{\partial}{\partial\overline{z}}+\frac{r}{2}.
\end{aligned}$$

Before we sum up the main properties of these operators in Proposition 4.4.7, we recall some definitions from operator theory.

Definition 4.4.3. Let H and H' be Hilbert spaces and let T be a linear operator from a subspace D of H to H'. T is called *closed* if, for every sequence x_n in D that converges to $x\in H$ such that Tx_n converges to $y\in H'$, we have $x\in D$ and $Tx=y$.

Definition 4.4.4. If D is dense in H then for any operator T from D to H, we can define its *adjoint* T^* on the domain

$$\{y\in H : x\mapsto\langle Tx,y\rangle \text{ is continuous on } D\}.$$

Any y in this set defines a linear functional on D by $\phi_y : x\mapsto\langle Tx,y\rangle$. This functional can be extended to H and by the Riesz representation theorem there exists $z\in H$ such that $\phi_y(x)=\langle x,z\rangle$ for all x in H. We define $T^*y=z$.

An operator is called *self-adjoint* if it is equal to its adjoint. An operator is called *essentially self-adjoint* if $T\subseteq T^*=(T^*)^*$, where $T\subseteq T^*$ means that T^* extends T.

Let
$$\mathcal{D}_r^2 = \{f \in H_{r,v} | f \text{ twice differentiable and } -\Delta_r f \in H_{r,v}\}.$$

Proposition 4.4.7.

(i) $\Delta_r : \mathcal{D}_r^2 \to H_{r,v}$ *is essentially self-adjoint. It has a self-adjoint extension to a dense subset of* $H_{r,v}$ *that we denote by* $\tilde{\mathcal{D}}_r$.

(ii) The eigenfunctions of Δ_r *are smooth (in fact they are real analytic).*

(iii) $K_r : \mathcal{D}_r^2 \to H_{r+2,v}$ *and* $\Lambda_r : \mathcal{D}_r^2 \to H_{r-2,v}$ *can be extended to closed operators defined on* $\tilde{\mathcal{D}}_r$. *For* $f \in \tilde{\mathcal{D}}_r$ *and* $g \in \tilde{\mathcal{D}}_{2+r}$ *we have*
$$(K_r f, g)^R = (f, \Lambda_{2+r} g)^R.$$

(iv)
$$-\Delta_r = \Lambda_{r+2} K_r - \frac{r}{2}(1 + \frac{r}{2}) = K_{r-2}\Lambda_r + \frac{r}{2}(1 - \frac{r}{2}).$$

Proof. For proofs of the statements (i), (iii) and (iv) see [Roe66]. (i) is Satz 3.2, (iii) follows from the discussion after the proof of Lemma 6.2 on page 332 and (iv) is equation (3.4) on page 305. Statement (ii) follows from the fact that Δ_r is an elliptic operator and elliptic regularity applies. For an introduction to the theory of elliptic operators, see [GT01]. The result needed here is Corollary 8.11 in [GT01]. □

Definition 4.4.5. A *cuspidal Maass wave form* in $H_{r,v}$ with eigenvalue λ is an eigenfunction of $-\Delta_r$ with eigenvalue λ that decays exponentially at the cusps of Γ.

Remark. By [Roe66, Satz 5.2] all eigenfunctions in $H_{r,v}$ of $-\Delta_r$ of eigenvalue $\frac{r}{2}(1-\frac{r}{2})$ are of the form $y^{\frac{r}{2}} f$, where f is a modular form in $\mathcal{M}_r(\Gamma, v)$ that has finite Petersson norm, i.e., $(f, f) < \infty$. If f is a cusp form, then $y^{\frac{r}{2}} f$ is a cuspidal Maass wave form.

The main result in [Roe66] is a spectral decomposition of Δ_r. For this purpose we introduce the Eisenstein series. Let q be a cusp of Γ, σ_q the generator of $\Gamma_q/\{\pm I\}$ and $A_q \in \mathrm{SL}_2(\mathbb{R})$ chosen such that $q = A_q^{-1}\infty$. The cusp q is called *singular for the multiplier system* v, if $v(\sigma_q) = 1$ and *regular for* v otherwise. Let $q_1, \dots, q_{m^*}$ be a set of representatives of the cusps of Γ that are singular for v. For each of these cusps, we define the Eisenstein series
$$E^q_{r,v}(z,s) = \frac{1}{2} \sum_{M \in \Gamma_q \backslash \Gamma} \sigma_r(A_q, M)^{-1} \overline{v(M)} \left(\frac{j(A_q M, \overline{z})}{j(A_q M, z)}\right)^{r/2} (\mathrm{Im}\ A_q M z)^s.$$

The definition of $E^q_{r,v}$ depends on the choice of A_q, but a different choice of A_q will only multiply the Eisenstein series by a constant of absolute value 1. The series above converges absolutely and uniformly for (z, s) in sets of the form $K \times \{s | \mathrm{Re}\ s \geq 1 + \epsilon\}$, where K is a compact subset of $\mathcal{H}$ and $\epsilon > 0$. For a fixed s with real part $\geq 1 + \epsilon$, one can use the absolute and uniform convergence of the series to see that $E^q_{r,v}(\cdot, s)$ is invariant under $|^R_{r,v}$ and that
$$-\Delta_r E^q_{r,v}(\cdot, s) = s(1-s) E^q_{r,v}(\cdot, s).$$
These series can be meromorphically continued and play an important role in the spectral decomposition of Δ_r.

Theorem 4.4.8.

(i) For fixed $z \in \mathcal{H}$ the Eisenstein series $E^q_{r,v}(z,\cdot)$ can be meromorphically continued to the whole complex plane.

(ii) If, for one fixed z, $E^q_{r,v}(z,\cdot)$ has a pole of order n at s_0, then the function $f(z) := \lim_{s\to s_0}(s-s_0)^n E^q_{r,v}(z,s)$ is real analytic, invariant under $|^R_{r,v}$ and satisfies

$$-\Delta_r f = s_0(1-s_0)f.$$

If n is chosen so that $f(z)$ has no poles in $\mathcal{H}$, then f grows at most polynomially at each cusp of Γ, i.e., if q is a cusp of Γ and $\tau_q\infty = q$ for $\tau_q \in \mathrm{SL}_2(\mathbb{R})$, then there exists $A \in \mathbb{R}$ such that $f|_r\tau_q(z) = \mathcal{O}(y^A)$ as $y \to \infty$.

In particular, if $E^q_{r,v}(z,s)$ is holomorphic at $s = s_0$, then

$$-\Delta_r E^q_{r,v}(\cdot, s_0) = s_0(1-s_0)E^q_{r,v}(\cdot, s_0).$$

Furthermore we have the following equalities:

$$K_r E^q_{r,v}(\cdot, s_0) = \left(\frac{r}{2} + s_0\right) E^q_{r+2,v}(\cdot, s_0), \tag{4.4.5}$$

$$\Lambda_r E^q_{r,v}(\cdot, s_0) = \left(\frac{r}{2} - s_0\right) E^q_{r-2,v}(\cdot, s_0). \tag{4.4.6}$$

The poles of $E^q_{r,v}(z,\cdot)$ in the half plane defined by Re $s \geq \frac{1}{2}$ are all simple and in the interval $(\frac{1}{2}, 1]$. In particular there are no poles on the line Re $s = \frac{1}{2}$.

Theorem 4.4.9 (Spectral expansion)**.** *Let $f \in \tilde{\mathcal{D}}_r$ and e_n be a maximal orthonormal system of eigenfunctions*[3] *of Δ_r. Then f has a spectral expansion*

$$f = \sum_n (e_n, f)^R e_n + \sum_{i=1}^{m^*} \frac{1}{4\pi} \int_{-\infty}^{\infty} (E^{q_i}_{r,v}(\cdot, \frac{1}{2} + i\rho), f)^R E^{q_i}_{r,v}(z, \frac{1}{2} + i\rho) d\rho.$$

If f has compact support mod Γ, i.e., $\pi(supp(f))$ is compact in $\Gamma\backslash\mathcal{H}^*$, then both parts of the spectral expansion, $\sum (e_n, f)^R e_n$ and $\sum_{i=1}^{m^*} \frac{1}{4\pi} \int_{-\infty}^{\infty} (E^{q_i}_{r,v}(\cdot, \frac{1}{2} + i\rho), f)^R E^{q_i}_{r,v}(z, \frac{1}{2} + i\rho) d\rho$, converge absolutely and uniformly on compact subsets of $\mathcal{H}$.

The properties of Eisenstein series and the spectral expansion are proved in the second part of [Roe66] with the notable exception of the fact that Eisenstein series can be meromorphically continued to the whole complex plane. Roelcke attributes the meromorphic continuation to Selberg and a proof of it can be found in [Bru81, §11]. The version of the spectral expansion we state is a combination of Satz 7.2 and the second part of Satz 12.3 in [Roe66].

We turn back to the proof of Theorem 4.4.1: Let $[\phi] \in H^1_{r,v}(\Gamma, \mathcal{P})$ be represented by $\phi \in Z^1_{r,v}(\Gamma, \mathcal{P})$. By Corollary 4.4.5, there exists a function $g \in \mathcal{Q}$ such that

$$\phi(\gamma) = g|_{r,v}\gamma - g, \ \forall \gamma \in \Gamma. \tag{4.4.7}$$

[3] An *orthonormal system* of eigenfunctions of an operator T on a Hilbert space H is a set of eigenfunctions of T that are pairwise orthogonal and have norm 1.

By applying $\frac{\partial}{\partial \overline{z}}$ to (4.4.7), we see that

$$\frac{\partial g}{\partial \overline{z}}(z) = \overline{v(\gamma)} j(\gamma, z)^{-r} j(\gamma, \overline{z})^{-2} \frac{\partial g}{\partial \overline{z}}(\gamma z).$$

A short calculation shows that the function

$$G : z \mapsto y^{\frac{r+2}{2}} \overline{\frac{\partial g}{\partial \overline{z}}(z)} \tag{4.4.8}$$

is invariant under $|^R_{2-r,\overline{v}}$. Moreover G vanishes in a neighbourhood of every cusp since g is holomorphic there, so G has compact support mod Γ and is in $H_{2-r,\overline{v}}$.

To prove Theorem 4.4.1, we have to show that if ϕ is orthogonal to $\mathcal{S}_{2-r}(\Gamma, \overline{v})$, then $g \in \mathcal{Q}$ can be chosen to be holomorphic. This implies that ϕ is a coboundary in $Z^1_{r,v}(\Gamma, \mathcal{P})$.

Lemma 4.4.10. *Let $2 - r > 0$ and ϕ, g and G be as above. Then $(f, \phi) = 0$ for all $f \in \mathcal{S}_{2-r}(\Gamma, \overline{v})$ if and only if $(\tilde{f}, G)^R = 0$ for all cuspidal Maass wave forms $\tilde{f}$ with eigenvalue $\frac{r}{2}(1 - \frac{r}{2})$.*

Proof. We have the equality

$$\frac{i}{2C_{2-r}}(f, \phi) = \frac{i}{2} \int_{\mathcal{F}} \overline{\partial} g \wedge f(z) dz = \int_{\mathcal{F}} y^{\frac{2-r}{2}} f(z) \overline{G(z)} \frac{dxdy}{y^2} = (y^{\frac{2-r}{2}} f, G)^R,$$

so $(f, \phi) = 0$ for all $f \in \mathcal{S}_{2-r}(\Gamma, \overline{v})$ if and only if $(\tilde{f}, G)^R = 0$ for all functions $\tilde{f}$ of the form $y^{\frac{2-r}{2}} f$, $f \in \mathcal{S}_{2-r}(\Gamma, \overline{v})$. According to Remark 4.4.1, these functions are exactly the cuspidal Maass wave forms of eigenvalue $\frac{r}{2}(1 - \frac{r}{2})$. □

We can now use spectral theory to characterise functions which are orthogonal to cuspidal Maass wave forms of eigenvalue $\frac{r}{2}(1 - \frac{r}{2})$.

Proposition 4.4.11. *Let $2 - r \neq 1$ and H be a smooth function in $H_{2-r,\overline{v}}$ with compact support mod Γ. Then the following are equivalent:*

(i) *$(\tilde{f}, H)^R = 0$ for all cuspidal Maass wave forms $\tilde{f}$ with eigenvalue $\frac{r}{2}(1 - \frac{r}{2})$.*

(ii) *$H = K_{-r}F + K_{-r}E$, where F is a smooth function in $H_{-r,\overline{v}}$ and E is a linear combination of the functions $E^{q_i}_{-r,\overline{v}}(z, \frac{r}{2})$.*

If $2 - r > 1$ or $2 - r < 0$ this implies $E = 0$.

Remark. By [Kno67] and [?] we have $\mathcal{S}_{2-r}(\Gamma, \overline{v}) = \{0\}$, if $2 - r \leq 0$. Since, by [Roe66, Satz 5.2], all cuspidal Maass wave forms of eigenvalue $\frac{r}{2}(1 - \frac{r}{2})$ are of the form $y^{\frac{r}{2}} f$, where $f \in \mathcal{S}_{2-r}(\Gamma, \overline{v})$, the first condition is always satisfied in the case $2 - r \leq 0$.

Proof. (i)⇒(ii): By [Roe66, Satz 6.3] there is a maximal orthonormal system of eigenfunctions of Δ_{2-r} consisting of:

1. Images of eigenfunctions of Δ_{-r} under the Maass raising operator $K_{-r} = (z - \overline{z}) \frac{\partial}{\partial z} - \frac{r}{2}$. We denote these by $K_{-r} e_n$. By [Roe66, Satz 6.3] these eigenfunctions cannot have eigenvalue $\frac{r}{2}(1 - \frac{r}{2})$.

2. A (finite) orthonormal basis of the eigenfunctions of eigenvalue $\frac{r}{2}(1-\frac{r}{2})$. By Remark 4.4.1 this set is of the form $\{y^{\frac{2-r}{2}}f_1, \dots, y^{\frac{2-r}{2}}f_N\}$, where the f_i form an orthonormal basis of the subspace of $\mathcal{M}_{2-r}(\Gamma, \overline{v})$ of modular forms with finite Petersson norm. If $2-r \geq 1$ this subspace is equal to $\mathcal{S}_{2-r}(\Gamma, \overline{v})$, while for $2-r < 1$ every modular form in $\mathcal{M}_{2-r}(\Gamma, \overline{v})$ has finite Petersson norm.

Hence by Theorem 4.4.9 the spectral expansion of H is of the form

$$H = \underbrace{\sum_n (K_{-r}e_n, H)^R K_{-r}e_n}_{=K_{-r}F_1} + \underbrace{\sum_{i=1}^{N} (y^{\frac{2-r}{2}} f_i, H)^R y^{\frac{2-r}{2}} f_i}_{=y^{\frac{2-r}{2}}\tilde{E}} +$$

$$\underbrace{\sum_{i=1}^{m^*} \frac{1}{4\pi} \int_{-\infty}^{\infty} (E^{q_i}_{2-r,\overline{v}}(\cdot, \frac{1}{2}+i\rho), H)^R E^{q_i}_{2-r,\overline{v}}(z, \frac{1}{2}+i\rho) d\rho}_{=\tilde{F}_2}.$$

Here we used that $\sum_n (K_{-r}e_n, H)^R K_{-r}e_n$ converges absolutely and uniformly on compacta to swap differentiation and summation and write it as $K_{-r}F_1 = K_{-r}\left(\sum_n (K_{-r}e_n, H)^R e_n\right)$.

We now show that $\tilde{F}_2 = K_{-r}F_2$ for a smooth function $F_2 \in H_{-r,\overline{v}}$: Applying equation (4.4.5) twice and using Proposition 4.4.7, we see

$$\int_{-\infty}^{\infty} (E^{q_i}_{2-r,\overline{v}}(\cdot, \frac{1}{2}+i\rho), H)^R E^{q_i}_{2-r,\overline{v}}(z, \frac{1}{2}+i\rho) d\rho$$

$$= \int_{-\infty}^{\infty} \left(\frac{1-r}{2}+i\rho\right)^{-2} \underbrace{(K_{-r}E^{q_i}_{-r,\overline{v}}(\cdot, \frac{1}{2}+i\rho), H)^R}_{=(E^{q_i}_{-r,\overline{v}}, \Lambda_{2-r}H)^R} K_{-r}E^{q_i}_{-r,\overline{v}}(z, \frac{1}{2}+i\rho) d\rho.$$

If $r \neq 1$

$$F_2^i(z) = \int_{-\infty}^{\infty} \left(\frac{1-r}{2}+i\rho\right)^{-2} (E^{q_i}_{-r,\overline{v}}, \Lambda_{2-r}H)^R E^{q_i}_{-r,\overline{v}}(z, \frac{1}{2}+i\rho) d\rho, \tag{4.4.9}$$

converges absolutely and uniformly on compacta. To see this note the integrand can be bounded above by

$$|\frac{1-r}{2}|^{-2} \cdot |(E^{q_i}_{-r,\overline{v}}, \Lambda_{2-r}H)^R E^{q_i}_{-r,\overline{v}}(z, \frac{1}{2}+i\rho)|,$$

and

$$\int_{-\infty}^{\infty} (E^{q_i}_{-r,\overline{v}}, \Lambda_{2-r}H)^R E^{q_i}_{-r,\overline{v}}(z, \frac{1}{2}+i\rho) d\rho,$$

converges absolutely and uniformly on compacta as it occurs in the spectral expansion of $\Lambda_{2-r}H$. So when we apply K_{-r} to $F_2 = \sum_{i=1}^{m^*} \frac{1}{4\pi} F_2^i$ we can swap it with the integral and obtain

$$K_{-r}F_2 = \tilde{F}_2.$$

F_2 is clearly in $H_{-r,\overline{v}}$ by the bound we used for the F_2^i. We have thus shown that

$$H = K_{-r}F + y^{\frac{2-r}{2}}\tilde{E}, \text{ where } F = F_1 + F_2 \in H_{-r,\overline{v}}. \tag{4.4.10}$$

To see that F is smooth we apply Λ_{2-r} to (4.4.10) and obtain

$$\Lambda_{2-r}H = \Lambda_{2-r}K_{-r}F + \Lambda_{2-r}(y^{\frac{2-r}{2}}\tilde{E}) = -\Delta_{-r}F - \frac{r}{2}(1-\frac{r}{2})F + \Lambda_{2-r}(y^{\frac{2-r}{2}}\tilde{E}).$$

We see that F is a solution of an elliptic differential equation and so, by elliptic regularity, F is smooth.

It remains to show that $y^{\frac{2-r}{2}}\tilde{E}$ is in the image of K_{-r}. Since H is orthogonal to all cuspidal Maass wave forms with eigenvalue $\frac{r}{2}(1-\frac{r}{2})$, we see that in the expansion

$$\tilde{E} = \sum_{i=1}^{N}(y^{\frac{2-r}{2}}f_i, H)^R f_i$$

only the $f_i \in \mathcal{M}_{2-r}(\Gamma,\overline{v})$ that are orthogonal to $\mathcal{S}_{2-r}(\Gamma,\overline{v})$ can occur. Hence $\tilde{E}$ must be orthogonal to $\mathcal{S}_{2-r}(\Gamma,\overline{v})$ and has finite Petersson norm. If $2-r \geq 1$ this implies $\tilde{E} = 0$. If $2-r < 0$ we have $\mathcal{M}_{2-r}(\Gamma,\overline{v}) = \{0\}$ by [Kno67], so in this case we also have $\tilde{E} = 0$. We are left with the case $0 \leq 2-r < 1$. In this case all modular forms in $\mathcal{M}_{2-r}(\Gamma,\overline{v})$ have finite Petersson norm, so $\tilde{E}$ can be any form in the orthogonal complement of $\mathcal{S}_{2-r}(\Gamma,\overline{v})$. We can appeal to [Roe66, Satz 11.2], to see that $\tilde{E}$ is a linear combination of residues of Eisenstein series at $s = \frac{r}{2}$. Therefore there exist $a_i \in \mathbb{C}$ with

$$y^{\frac{2-r}{2}}\tilde{E}(z) = \sum_{i=1}^{m^*} a_i \mathrm{Res}_{s=\frac{r}{2}}(E^{q_i}_{2-r,\overline{v}}(z,s)).$$

Note that we can restrict the sum on the right hand side to include only Eisenstein series that have a pole at $s = \frac{r}{2}$. On the other hand Eisenstein series of weight $-r$ never have a pole at $s = \frac{r}{2}$ by [Roe66, Satz 13.2], since $-r < -1$. Equation (4.4.5) now implies

$$\mathrm{Res}_{s=\frac{r}{2}}(E^{q_i}_{2-r,\overline{v}}(z,s)) = \lim_{s\to\frac{r}{2}}(s-\frac{r}{2})E^{q_i}_{2-r,\overline{v}}(z,s) \tag{4.4.11}$$

$$= \lim_{s\to\frac{r}{2}} K_{-r}E^{q_i}_{-r,\overline{v}}(z,s) = K_{-r}E^{q_i}_{-r,\overline{v}}(z,\frac{r}{2}). \tag{4.4.12}$$

Setting $E = \sum_{i=1}^{m^*} a_i E^{q_i}_{-r,\overline{v}}(z,\frac{r}{2})$ we can confirm statement (ii).

(ii)$\Rightarrow$(i): Let $H = K_{-r}F + K_{-r}E$ as described in (ii) and let $\tilde{f}$ be a cuspidal Maass wave form with eigenvalue $\frac{r}{2}(1-\frac{r}{2})$. From the first part of the proof we know that $K_{-r}E$ has the form $y^{\frac{2-r}{2}}\tilde{E}$, where $\tilde{E} \in \mathcal{M}_{2-r}(\Gamma,\overline{v})$ is orthogonal to $\mathcal{S}_{2-r}(\Gamma,\overline{v})$. This implies that $y^{\frac{2-r}{2}}\tilde{E}$ is orthogonal to $\tilde{f}$ with respect to the scalar product of $H_{2-r,\overline{v}}$, so

$$(H,\tilde{f})^R = (K_{-r}F,\tilde{f})^R = (F,\Lambda_{2-r}\tilde{f})^R.$$

Since $f = y^{-\frac{2-r}{2}}\tilde{f}$ is in $\mathcal{S}_{2-r}(\Gamma,\overline{v})$ and hence holomorphic we have

$$\Lambda_{2-r}\tilde{f} = \Lambda_{2-r}(y^{\frac{2-r}{2}}f) = (z-\overline{z})\frac{\partial f}{\partial \overline{z}} = 0,$$

and therefore $(H,\tilde{f})^R = 0$.

$\square$

Theorem 4.4.1 now follows from Proposition 4.4.11.

Proof of Theorem 4.4.1 and of Theorem 4.2.1 for $2-r \neq 1$. Let $\phi \in Z^1_{r,v}(\Gamma,\mathcal{P})$ and g and G be constructed as in (4.4.7) and (4.4.8). In the case $2-r>0$ suppose additionally that $(f,\phi)=0$ for all $f \in \mathcal{S}_{2-r}(\Gamma,\overline{v})$. By Lemma 4.4.10 in the case $2-r>0$, and Remark 4.4.1 in the case $2-r \leq 0$, G satisfies condition (i) of Proposition 4.4.11. Hence there is a smooth $F \in H_{-r,\overline{v}}$ and a linear combination of Eisenstein series $E(z)=\sum_{i=1}^{m^*} a_i E^{q_i}_{-r,\overline{v}}(z,\frac{r}{2})$, with

$$G = K_{-r}F + K_{-r}E = K_{-r}(F+E).$$

As stated in Proposition 4.4.11, E is only non-zero if $0 \leq 2-r < 1$, and in this case the Eisenstein series $E^{q_i}_{-r,\overline{v}}(\cdot,\frac{r}{2})$ are smooth functions that grow at most polynomially at each cusp of Γ. Since F is smooth and in $H_{-r,\overline{v}}$, F also grows at most polynomially at each cusp and so the same is true for $D=E+F$. We have

$$G(z) = y^{\frac{r+2}{2}}\overline{\frac{\partial g}{\partial \overline{z}}(z)} = 2iy\frac{\partial D}{\partial z} - \frac{r}{2}D = 2iy^{\frac{r+2}{2}}\frac{\partial}{\partial z}(y^{-\frac{r}{2}}D).$$

Dividing by $y^{\frac{r+2}{2}}$ and taking the complex conjugate of both sides we arrive at

$$\frac{\partial g}{\partial \overline{z}}(z) = \frac{\partial}{\partial \overline{z}}(-2iy^{-\frac{r}{2}}\overline{D})(z). \tag{4.4.13}$$

Since D is invariant under $|^R_{-r,\overline{v}}$, $\overline{D}$ is invariant under $|^R_{r,v}$. By Lemma 4.4.6, the function $\tilde{D}(z) = -2iy^{-\frac{r}{2}}\overline{D}$ is invariant under $|_{r,v}$. This invariance implies that $\tilde{g} = g - \tilde{D}$ satisfies $\tilde{g}|_{r,v}\gamma - \tilde{g} = \phi(\gamma)$ for all $\gamma \in \Gamma$. Since $\tilde{D}$ grows at most polynomially at the cusps of Γ, $\tilde{g}$ satisfies the growth conditions for functions in $\tilde{Q}$. Proposition 4.4.3 now tells us that $\tilde{g} \in \mathcal{Q}$. Note also that equation (4.4.13) implies that $\tilde{g}$ is holomorphic, so $\tilde{g} \in \mathcal{P}$. We finally conclude that ϕ is indeed a coboundary in $Z^1_{r,v}(\Gamma,\mathcal{P})$.

The proof above shows in particular that for $2-r \leq 0$ every cocycle in $Z^1_{r,v}(\Gamma,\mathcal{P})$ is a coboundary and hence $H^1_{r,v}(\Gamma,v)=\{0\}$. This proves Theorem 4.2.1 for $2-r \leq 0$, since $\mathcal{S}_{2-r}(\Gamma,\overline{v})$ is also $\{0\}$ in this case. □

Remark. The proof fails if $2-r=1$, because Proposition 4.4.11 is not available in that case. The only point where we need the assumption $2-r \neq 1$ in the proof of that proposition, is when we show that $\tilde{F}_2$ is in the image of K_{-r}, in particular for the construction of the functions $F_2^i \in H_{-r,\overline{v}}$ in (4.4.9). The crucial consequence of Proposition 4.4.11 is that G is in the image of K_{-r}. In the case $2-r=1$ we only obtain

$$G = K_{-1}F + \sum_{i=1}^{m^*}\frac{1}{4\pi}\int_{-\infty}^{\infty}(E^{q_i}_{1,\overline{v}}(\cdot,\frac{1}{2}+i\rho),G)^R E^{q_i}_{1,\overline{v}}(z,\frac{1}{2}+i\rho)d\rho.$$

In the notation of the proof of Proposition 4.4.11 we have $F=F_1$ and $E=0$ since $r=1$. To prove Theorem 4.2.1 in this case, one would need to show that the second summand above is in the image of K_{-1}.

4.5 Vector-valued modular forms

In this section we generalise Theorem 4.2.1 to vector-valued cusp forms. Let $\rho : \Gamma \to U(n)$ be a unitary representation of Γ on $\mathbb{C}^n$ and v a unitary multiplier system of weight r. Let F be a function from $\mathcal{H}$ to $\mathbb{C}^n$. The slash operator $|_{\rho,v,r}$ is defined by

$$F|_{r,v,\rho}\gamma(z) = j(\gamma, z)^{-r}\overline{v(\gamma)}\rho(\gamma)^{-1}F(\gamma z).$$

Definition 4.5.1. A function $f : \mathcal{H} \to \mathbb{C}^n$ is a *modular form* for Γ of weight r, representation ρ, and multiplier system v if the following conditions are satisfied:

(i) f is holomorphic on $\mathcal{H}$.

(ii) $f(z) = f|_{r,v,\rho}\gamma(z)$ for all $\gamma \in \Gamma$ and $z \in \mathcal{H}$.

(iii) If q is a cusp of Γ and $A\infty = q$, then for any $\epsilon > 0$

$$j(A, z)^{-r}f(Az) \text{ is bounded for } y \geq \epsilon.$$

If f satisfies the additional condition

(iii') If q is a cusp of Γ and $A\infty = q$, then there exists an $\epsilon > 0$ such that

$$j(A, z)^{-r}f(Az) = \mathcal{O}_{y\to\infty}(e^{-\epsilon y}),$$

it is a *cusp form*. The set of modular forms or cusp forms of this kind is denoted by $M_r(\Gamma, v, \rho)$ and $S_r(\Gamma, v, \rho)$ respectively.

Let $\mathcal{P}^n$ be the set of vector-valued functions $f(z) = (f_1(z), \ldots, f_n(z))$ such that all f_i are in $\mathcal{P}$. The slash operator $|_{r,v,\rho}$ defines a Γ-action on $\mathcal{P}^n$ and so we can define the cohomology groups $H^1_{r,v,\rho}(\Gamma, \mathcal{P}^n)$ and $\tilde{H}^1_{r,v,\rho}(\Gamma, \mathcal{P}^n)$. Just as in the 1-dimensional case, they turn out to be the same. The proof of this fact relies on a generalisation of Corollary 4.1.2:

Proposition 4.5.1. *Let $U \in U(n)$, $s \in \mathbb{R} \setminus \{0\}$ and $g \in \mathcal{P}^n$. Then there exists an $f \in \mathcal{P}^n$ such that*

$$U^*f(z + s) - f(z) = g(z), \quad \forall z \in \mathcal{H}. \tag{4.5.1}$$

Proof. Since U is diagonalisable, there exists a $V \in \mathrm{U}(n)$ and a diagonal matrix $D \in \mathrm{U}(n)$ with

$$U = V^*DV.$$

Multiplying equation (4.5.1) by V, we get

$$D^*Vf(z + s) - Vf(z) = Vg(z). \tag{4.5.2}$$

Let $\epsilon_1, \ldots, \epsilon_n$ be the diagonal entries of D and $G = Vg = (G_1, \ldots, G_n) \in \mathcal{P}^n$. We can use Corollary 4.1.2 to find solutions $F_i \in \mathcal{P}$ for

$$\overline{\epsilon_i}F_i(z + s) - F_i(z) = G_i(z).$$

Then $f = V^{-1}(F_1, \ldots, F_n)$ is in $\mathcal{P}^n$ and satisfies (4.5.2). □

This can be used to show

Theorem 4.5.2. *Every cocycle in $Z^1_{v,\rho}(\Gamma, \mathcal{P}^n)$ is parabolic.*

4.5.1 Petersson inner product

Let $2-r>0$ and f,g be in $\mathcal{S}_{2-r}(\Gamma,\overline{v},\rho^{-1})$. The Petersson inner product of f and g is defined by

$$(f,g)=\int_{\mathcal{F}}\langle f(z),g(z)\rangle\, y^{-r}dxdy,$$

where $\langle(a_i),(b_i)\rangle=\sum_{i=1}^n a_i\overline{b_i}$ is the usual scalar product on $\mathbb{C}^n$. We will repeat the constructions of Section 4.3.

Lemma 4.5.3. *Let g be in $\mathcal{S}_{2-r}(\Gamma,\overline{v},\rho^{-1})$, then*

$$\phi_g^\infty(z):\gamma\mapsto\phi_{g,\gamma}^\infty(z)=\left[\int_{\gamma^{-1}\infty}^{\infty}g(\tau)(\tau-\overline{z})^{-r}d\tau\right]^{-},$$

is a cocycle in $Z^1_{v,\rho}(\Gamma,\mathcal{P}^n)$.

Again we can use Stokes' theorem to show

$$(f,g)=-C_{2-r}\sum_{m=1}^{n}\int_{A_{i_m}}^{A_{i_m+1}}\left\langle f(z),\overline{\phi_{g,\alpha_{i_m}}^\infty(z)}\right\rangle dz.$$

Using this we define a pairing between $\mathcal{S}_{2-r}(\Gamma,\overline{v},\rho^{-1})$ and $H^1_{r,v,\rho}(\Gamma,\mathcal{P}^n)$ as follows. Let $f\in\mathcal{S}_{2-r}(\Gamma,\rho^{-1},\overline{v})$ and $[\phi]\in H^1_{r,v}(\Gamma,\mathcal{P}^n)$ be represented by ϕ. Then

$$(f,[\phi])=(f,\phi)=-C_{2-r}\sum_{m=1}^{n}\int_{A_{i_m}}^{A_{i_m+1}}\left\langle f(z),\overline{\phi(\alpha_{i_m})(z)}\right\rangle dz,$$

is well-defined (independent of the representative ϕ), and furthermore we have the following theorem, analogous to Theorem 4.2.1.

Theorem 4.5.4. *Let v and ρ be as above and $0<2-r\neq 1$. The pairing defined above is perfect, so the map $f\mapsto\phi_f^\infty$ induces an isomorphism*

$$\mathcal{S}_{2-r}(\Gamma,\overline{v},\rho^{-1})\cong H^1_{r,v,\rho}(\Gamma,\mathcal{P}^n).$$

If $2-r\leq 0$ we have

$$\mathcal{S}_{2-r}(\Gamma,\overline{v},\rho^{-1})\cong H^1_{r,v,\rho}(\Gamma,\mathcal{P}^n)\cong\{0\}.$$

Proof. All the constructions of Section 4.4 work in the vector-valued case. In particular every statement we cited from [Roe66] is already formulated for vector-valued functions. The fact that every vector-valued modular form of negative weight is 0 is also stated in [Roe66] as a consequence of Satz 5.3; and this generalises the main theorem of [Kno67]. It is also shown that a vector-valued modular form of weight 0 is constant. □

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