Expression Polygons

Colin Foster

A problem-solving task can generate extensive opportunities for developing mathematical fluency in solving linear equations.

Solving linear equations is an important mathematical technique at the core of the Algebra 1 curriculum which forms an essential foundation for more advanced work. Too often, developing fluency with linear equations entails ploughing through pages of repetitive exercises. How can students master this topic while employing their mathematical sense-making faculties?

In addition to generating essential practice with the technique of solving linear equations, this lesson engages students in authentic mathematical thinking (Foster, 2013a, 2014). In particular, as advocated by the Common Core State Standards (2010), this activity asks students to:

- Make sense of problems and persevere in solving them.
- Reason abstractly and quantitatively.

This is a rich task which students of all ages from Grade 6 upwards can access.

Solving an expression polygon

The diagram in Figure 1 – an *expression polygon* – shows four algebraic expressions with lines connecting each pair of expressions. Each line forms an equation from the two expressions that it joins, and the students' initial task is to solve the six equations, writing the solution to each equation next to each line. For example, the line at the top joining $x + 5$ to $2x + 2$ corresponds to the equation $x + 5 = 2x + 2$, and the solution is $x = 3$, so students write 3 next to this line. In addition to recording their solutions on the sheet, students could write out full solutions on separate paper.

Solving the equations in the expression polygon also serves as formative assessment:

- Do students make errors where there are negative signs?
- Do they sometimes do *opposite* operations to both sides of the equation (e.g., add 3 to one side and subtract 3 from the other side) rather than the *same* operation?
- Do they sometimes try to change, for example, $3x$ into x by *subtracting* 3 instead of *dividing by* 3?
- Do they sometimes stop when they have found the value of a *multiple* of x, such as $3x$, and fail to complete the solution by dividing by the multiplier, 3?

Useful questions to ask are: "What operation are you going to do to both sides of your equation?" and "Why did you choose that operation?"

If students select operations that are unhelpful, let them explore what happens when they carry out those operations. For example, with $x – 2 = 5$, if the student suggests subtracting 2 from both sides, allow them to do this. Obtaining $x – 4 = 3$, they will realize that they are no closer to the solution, and that *adding* 2 would have been a better move. Although poor strategic choices, such steps preserve equality, which is something to recognise and build on (Foster, 2012). If students take other approaches to solving equations, such as “moving” letters and numbers from one side to the other, explore with them why these rules work and what their limitations might be. For example, to understand rules such as “change the side; change the sign,” consider what *identical* operation is being done simultaneously to *both* sides.

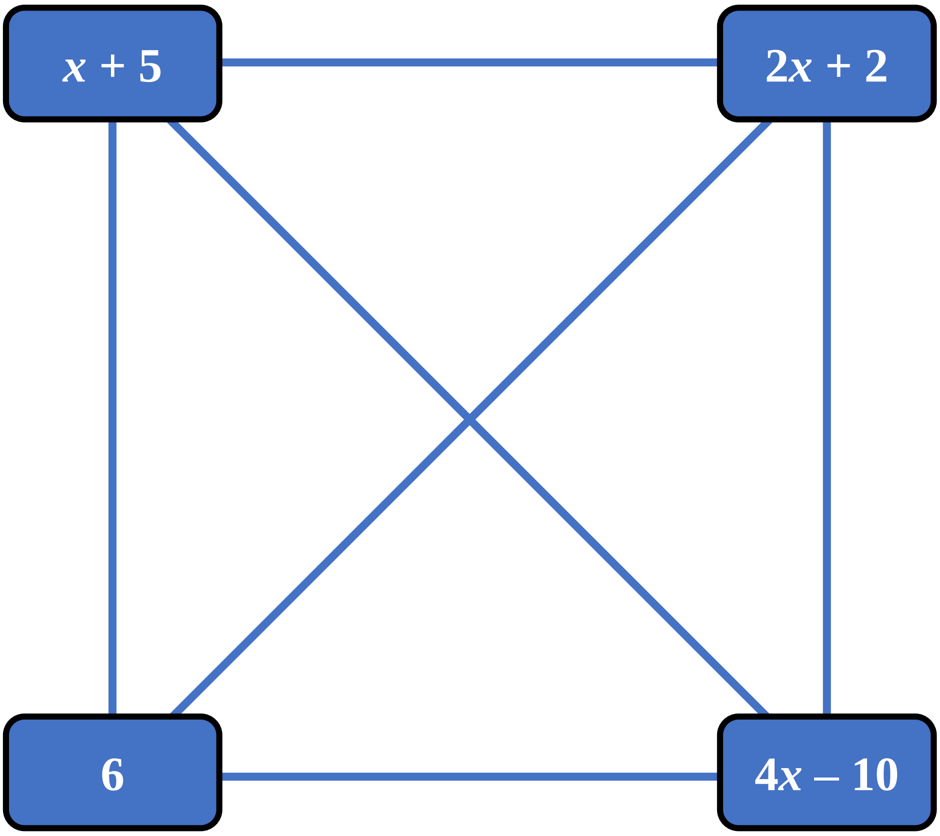

Figure 1. An expression polygon

Constructing expression polygons

When solving the given expression polygon, students find that the solutions are 1, 2, 3, 4, 5, and 6 (Figure 2). The pattern is provocative, and students normally comment on it (Foster, 2012). Uncovering an interesting pattern leads naturally to a challenge: *Can you make up an expression polygon of your own that has a nice, neat set of solutions?*

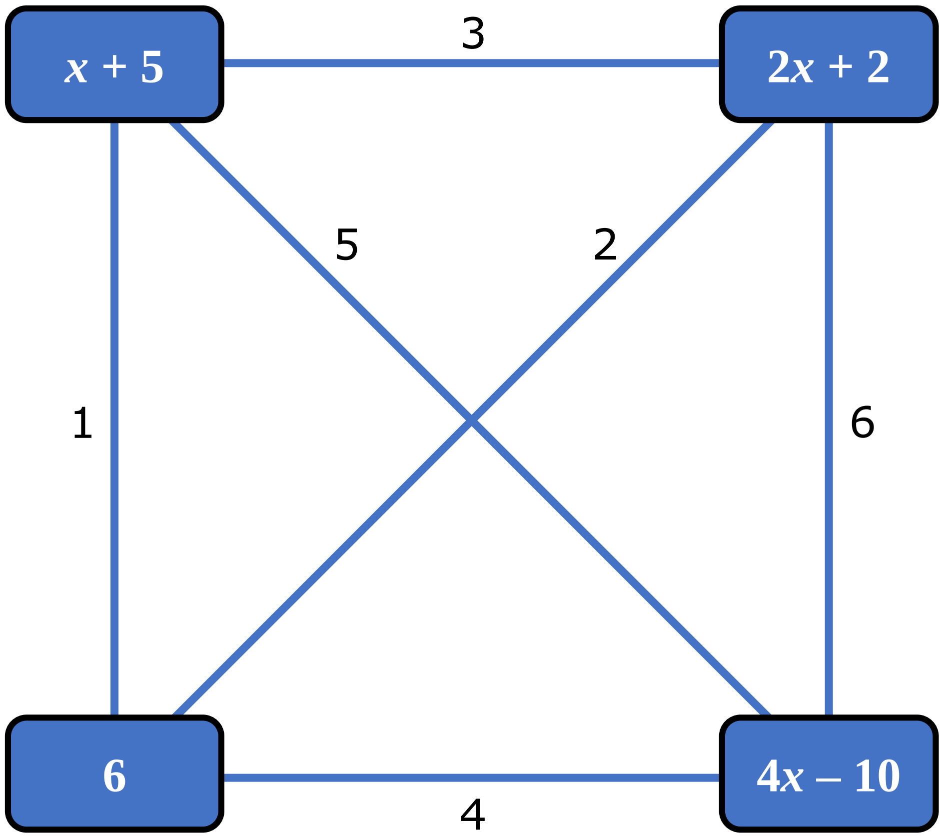

Figure 2. The completed expression polygon

Of course, students decide what counts as “nice” and “neat.” They might choose to aim for the first six even numbers, first six prime numbers, first six squares, or some other significant set of six numbers. Regardless of the specific targets students choose, the trial and improvement involved in producing their expression polygon will generate useful practice at solving linear equations. Because students are aiming for particular solutions, they must deconstruct the equation-solving process. As students gain confidence solving equations, they can focus their attention on their *strategic* decisions about which expressions to choose. In this way, the task naturally differentiates for students.

Producing a nice set of solutions is more difficult than it first appears – try it! Initially, aiming for *integer* solutions may be enough of a challenge. Help students who are struggling by encouraging them to think about how they could simplify the task to get started. For example, they could begin with an expression *triangle*, with *three* expressions, rather than an expression square. (Figure 3 shows an expression triangle created by a Grade 8 student.) Another possibility is to remove the two diagonal lines from the expression square, so that only four equations are required. After solving these simpler versions of the problem, students will be better prepared to attempt the original task. Although at different levels of complexity, all versions of the task require strategic choices and problem-solving skills while also developing fluency with linear equations. Two additional approaches have helped students struggling to make progress with the main activity.

- **Begin with the original expression polygon (Figure 1).** "Try starting with the original expression square and changing one or two of the numbers in one or two of the expressions. What happens when you solve the new equations that you get? Why?"
- **Model an experimentation process.** "Choose two small positive integers." If the student offers, for example, 3 and 5, write down for them $3x + 5$ in one of the empty expression boxes. Then ask: "Can you put a number into one of the other expression boxes to create an equation with an integer answer?" If the student is still unsure, encourage them to begin with *any* number, solve the resulting equation, and then adjust the choice if the solution is not an integer. By trial and improvement they will begin to discover what is needed to obtain integer solutions.

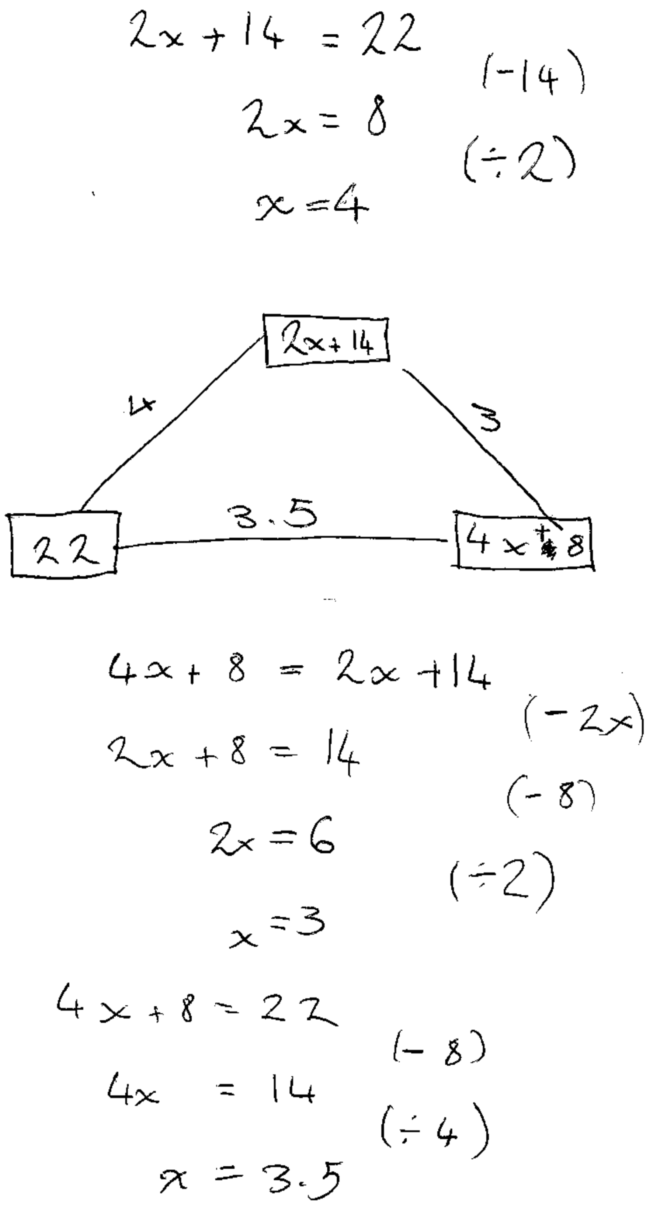

Figure 3. An expression triangle (Grade 8)

Adapting the original expression square (Figure 1) can be a productive way to generate new sets of solutions. For example, one Grade 7 student carefully doubled each expression, with the intention of obtaining the solutions 2, 4, 6, 8, 10, and 12, and was very surprised when he found that he obtained exactly the same solutions as for the original expression polygon!

Through discussion with other students, the student realized that this happened because multiplying both sides of an equation by the same factor leaves the solution unchanged. Older students realize sooner that solutions are preserved when scaling or translating all of the expressions by the same constant. Adopting a more sophisticated approach, another student replaced every "*x*" with "½*x*" (so 4*x* – 20 becomes 2*x* – 20, etc.) and then doubled all of the expressions (to clear the fractions), obtaining the expression polygon shown in Figure 4, whose solutions are the first six even numbers. Another "neat" set of solutions students sometimes consider involves making all of the solutions the same (Figure 5).

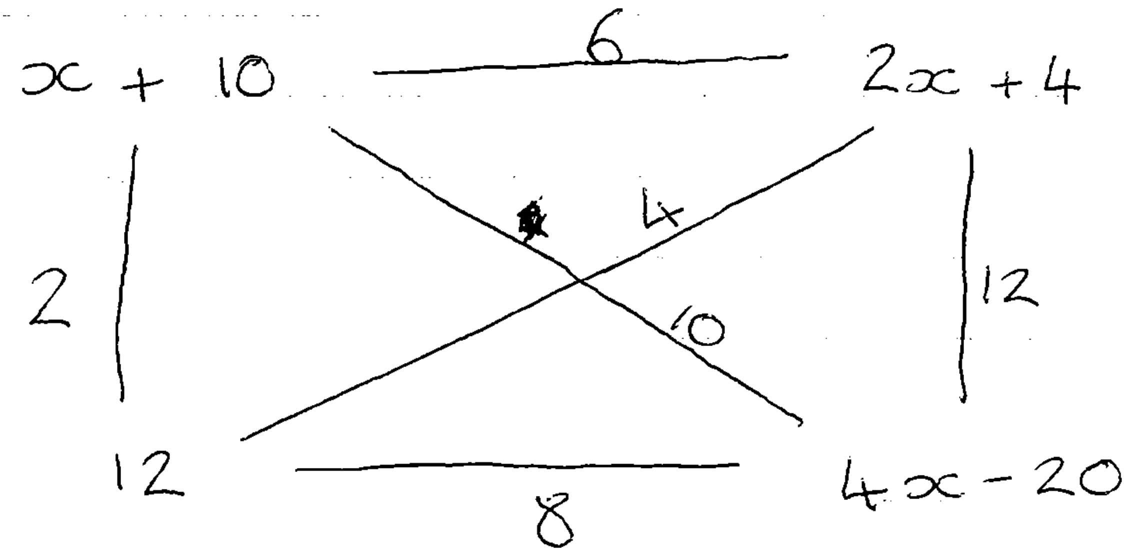

Figure 4. A completed expression polygon with the first six even numbers as solutions

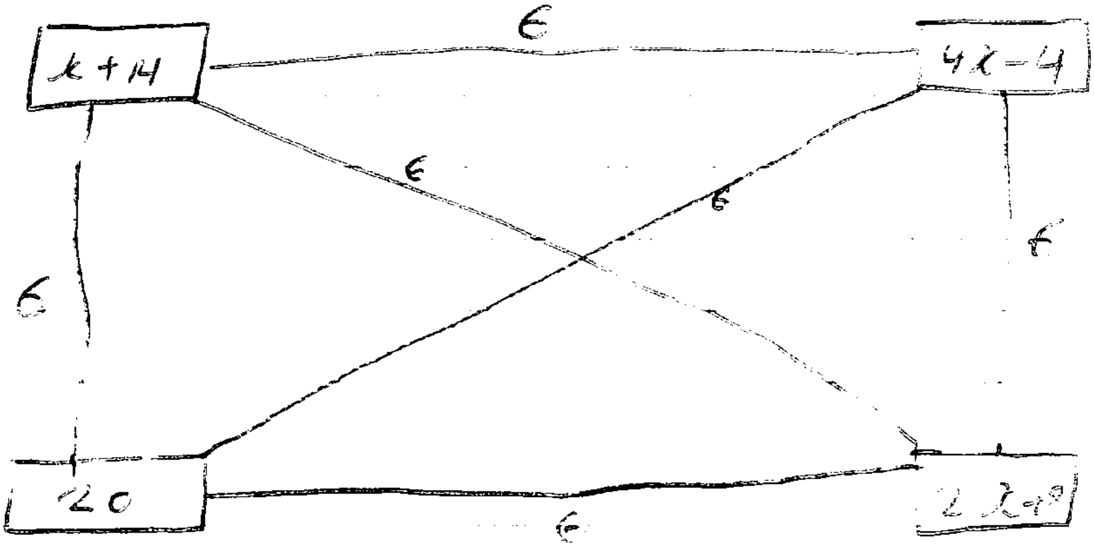

Figure 5. A completed expression polygon with all solutions equal to 6

Students sometimes observe patterns among the polygons themselves. For example, expression triangles have *three* solutions but expression squares have *six* solutions, rather than four. In general, an expression *n*-gon will have ½*n*(*n* – 1) solutions, one per diagonal. Since the number of required solutions is quadratic in *n*, completing an expression *n*-gon becomes rapidly more difficult as *n* increases. A Grade 9 student produced the expression *pentagon* shown in Figure 6, in which all the solutions are integers, albeit not distinct. Students may consider creating an expression *polyhedron*, in three dimensions, and realize that, for example, an expression square is equivalent to an expression tetrahedron, and an expression cube is equivalent to an expression octagon. For further challenge, students could include quadratic expressions (Foster, 2013b) or ask whether a given six-tuple can be the solutions to an expression polygon. For example, is it possible to design an expression square with the solutions 1, 1, 1, 1, 1, and 2?

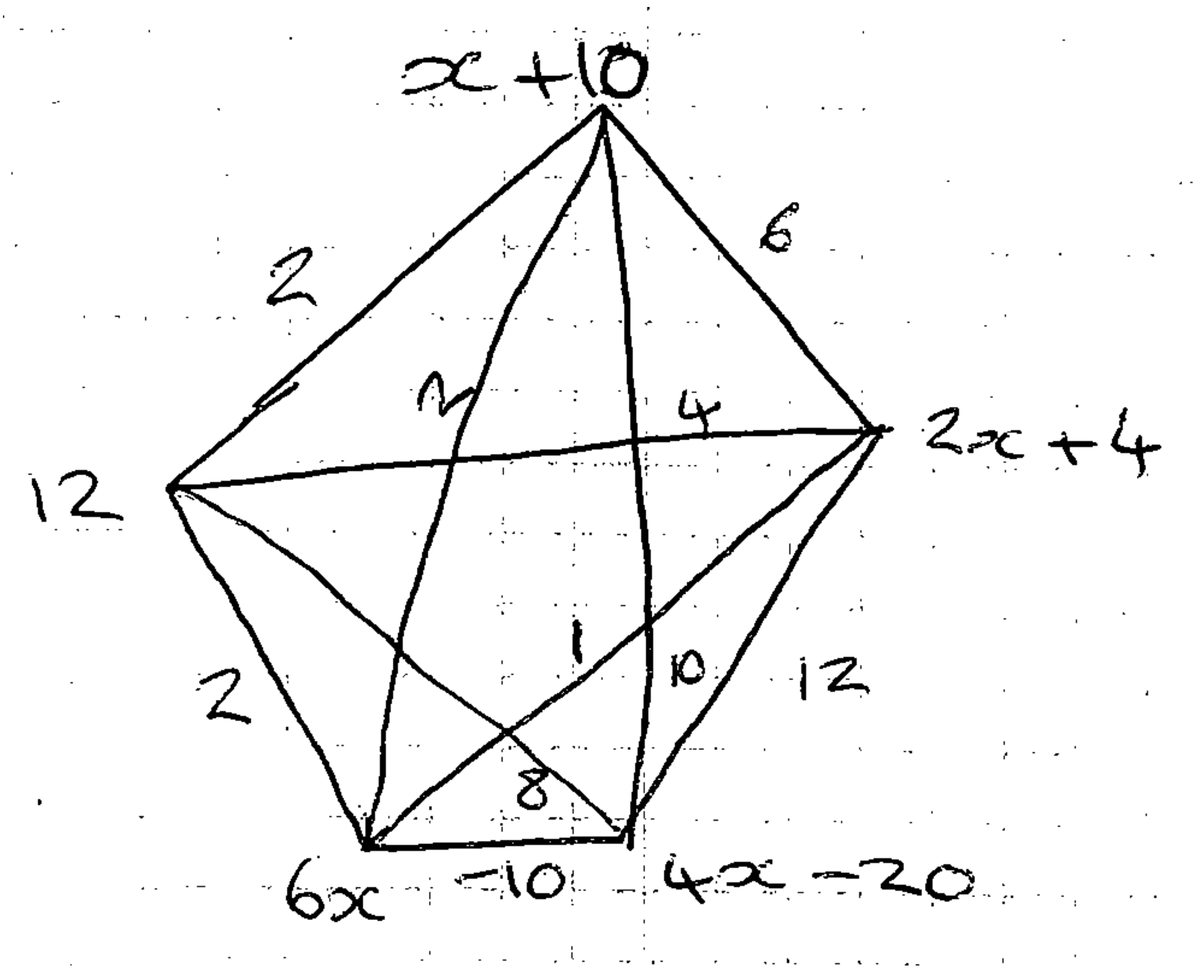

Figure 6. A completed expression pentagon

Discussion

Towards the end of the lesson, ask students to share the expression polygons that they have created and to talk about how they made them. Some helpful prompts include:

- "How did you choose your expressions?"
- "What changes did you make to your expressions and why?"
- "Which was the hardest part of making your expression polygon and why?"

Often students succeed with three expressions, but have difficulty adding a fourth. They may need to change one of the three expressions that "work" in order to find a way forward.

Describing their process and how they worked through their difficulties can reinforce the value of persistence.

Students may have noticed that the set of solutions is preserved when scaling or translating all of the expressions by the same constant, and this could be valuable to draw out. They may have noticed that it is relatively easy to generate an expression polygon in which all the solutions are the same, and it could be useful to ask them to explain a method for doing this. Students may comment on the fact that an expression n-gon has $\frac{1}{2}n(n-1)$ solutions, but if not then it may be worth asking them about why an expression triangle has three solutions but an expression square has six solutions, rather than four.

The expression polygon task provides extensive practice solving linear equations while simultaneously developing students' creativity and problem solving. Beyond fluency with equation-solving techniques, the activity moves students' attention to a higher strategic level, thereby deepening their understanding of algebraic operations and equality.

REFERENCES

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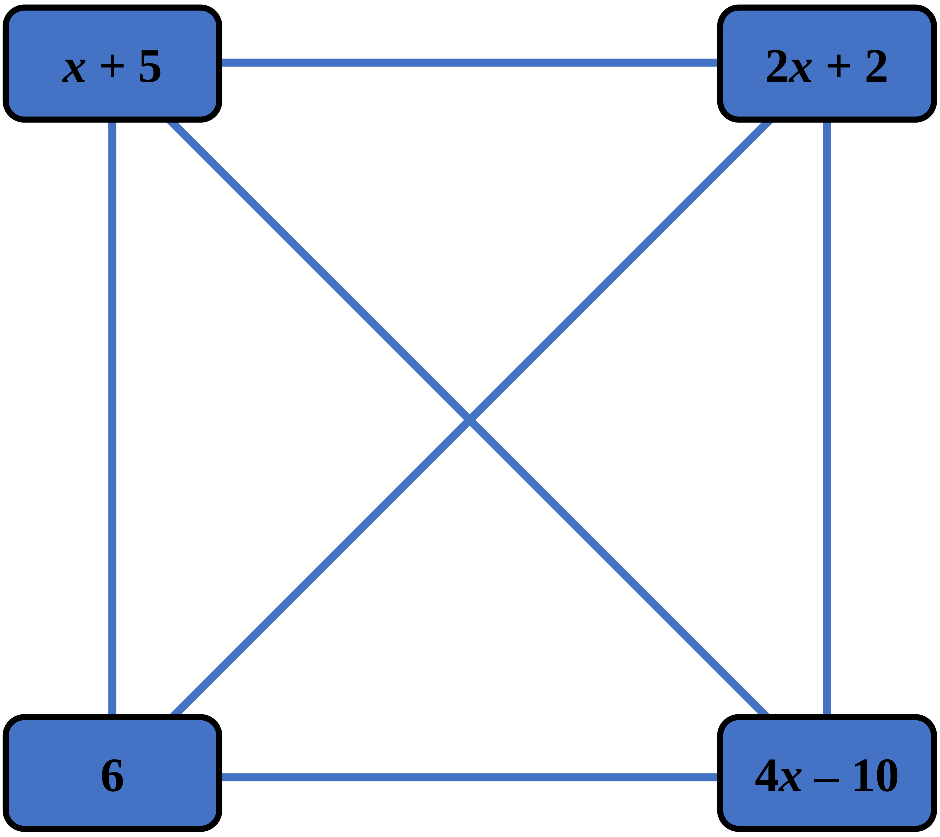
Expression Polygons

The figure below is an *expression polygon*. Each pair of algebraic expressions is joined by a line segment. Each line segment represents an equation equating the expressions that it connects. For example, the line segment at the top represents the equation $x + 5 = 2x + 2$.



Question 1: Write the six equations representing the six line segments in this expression square.

Question 2: Solve your six equations.

Question 3: Describe any patterns you notice among your six solutions.

Question 4: Construct another expression polygon containing different expressions. Can you make the solutions to your expression polygon a "nice" set of numbers?

Things to consider:

- **Number of expressions**. An expression *triangle* would be easier to begin with than an expression *square*. An expression *pentagon* would be challenging! An expression *cube*, in three dimensions, could be *very* challenging!
- **Type of solutions**. Here are some possible challenges:
    - Make all of the solutions *integers*.
    - Make all of the solutions *distinct* (i.e., different) *integers*.
    - Make the solutions a "nice" set of numbers, such as consecutive even numbers, or square numbers, or prime numbers.

  Is it always possible to design an expression polygon that will produce *any* given set of numbers? Why / why not?

- **Type of expressions**. It is best to start with linear expressions of the form $ax + b$, where a and b are constants. Including one or more *quadratic* expressions of the form $ax^2 + bx + c$, where a, b, and c are constants, could be a possibility for those with knowledge of how to solve quadratic equations. (With quadratic expressions, some of the equations may have *two* solutions, so there may have to be *two* numbers on some of the line segments.)

Question 5: How could you make an expression polygon in which all of the solutions were 7?

Question 6: How many solutions will an expression 10-gon have? What about an expression *n*-gon? Why?