Higher Dimensional Adeles, Mean-Periodicity and Zeta Functions of Arithmetic Surfaces

Thomas Oliver

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Abstract

This thesis is concerned with the analytic properties of arithmetic zeta functions, which remain largely conjectural at the time of writing. We will focus primarily on the most basic amongst them - meromorphic continuation and functional equation. Our weapon of choice is the so-called "mean-periodicity correspondence", which provides a passage between nicely behaved arithmetic schemes and mean-periodic functions in certain functional spaces. In what follows, there are two major themes.

1. The comparison of the mean-periodicity properties of zeta functions with the much better known, but nonetheless conjectural, automorphicity properties of Hasse–Weil L functions. The latter of the two is a widely believed aspect of the Langlands program. In somewhat vague language, the two notions are dual to each other. One route to this result is broadly comparable to the Rankin-Selberg method, in which Fesenko's "boundary function" plays the role of an Eisenstein series.

2. The use of a form of "lifted" harmonic analysis on the non-locally compact adele groups of arithmetic surfaces to develop integral representations of zeta functions. We also provide a more general discussion of a prospective theory of $\mathrm{GL}_1(\mathbb{A}(\mathcal{S}))$ zeta-integrals, where $\mathcal{S}$ is an arithmetic surface. When combined with adelic duality, we see that mean-periodicity may be accessible through further developments in higher dimensional adelic analysis.

The results of the first flavour have some bearing on questions asked first by Langlands, and those of the second kind are an extension of the ideas of Tate for Hecke L-functions. The theorems proved here directly extend those of Fesenko and Suzuki on two-dimensional adelic analysis and the interplay between mean-periodicity and automorphicity.

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In loving memory of Mark Knight.

R.I.P

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CHAPTER 1

Introduction

This chapter has three main components. We begin with an overview of the basic conjectures addressed in this thesis, followed by a rough explanation of how we proceed, and we conclude with a statement of the resulting theorems. References and formal definitions will be postponed until Chapter 2.

1.1 Motivation

If one desires to understand the arithmetic of an object, a good place to begin is with its zeta function. The simplest and best known example of such a function is the Riemann's zeta ζ, even the most basic analytic properties of which encode fundamental information about the ring of integers $\mathbb{Z}$. For example, the elementary observation that ζ has a pole at 1 is a manifestation of the infinitude of the set of prime numbers, as a consequence of the Euler product formula. A more pertinent example for this thesis is the functional equation of ζ, which can be viewed as a symptom of Poisson summation and adelic self-duality[1]. This is a harmono-analytic analogue of the Riemann-Roch theorem for curves over finite fields. Even in the classical case of ζ, not everything is known. In this direction, one must, of course, mention the conjectural Riemann hypothesis stating that the nontrivial complex zeros of the meromorphically continued Riemann zeta function all have real part $\frac{1}{2}$. Needless to say, ζ is merely the tip of the iceberg - we have many generalizations, exhibiting all manner of arithmetic phenom-

[1] There are several other proofs of meromorphic continuation and functional equation, as, for example, found in [Tit87].

ena, at least conjecturally.

There is something of a blurry distinction between the functions denoted by ζ and those denoted by L. In some sense, both are generating functions of arithmetic quantities, a priori only defined in some half plane. In this work, "zeta functions" will be associated to algebra/scheme of finite type over $\mathbb{Z}$. The "L-Functions" will arise in the context of representations of many different groups on vector spaces over both the complex numbers and p-adic fields. In dimension 1, meaning, number fields and function fields in one variable, one has the vague mantra that "L-functions are twisted zeta functions" - this is too simplistic for our purposes, although it maintains a kernel of truth. The source of the vast majority of connections between the zeta functions and L-functions appearing here is the fact that there are various ways in which an arithmetic scheme can give rise to a Galois, or even, Weil group, representation. These representations are, in turn, supposed to be connected to automorphic representations through the famous Langlands program.

By way of example, consider an arithmetic surface $\mathcal{S}$ over $\mathrm{Spec}(\mathcal{O}_k)$, where k is an algebraic number field, such that the generic fibre of $\mathcal{S}$ is a smooth, projective, geometrically connected curve C over k. The Jacobian $J(C)$ of C is an abelian variety, which has the structure of a finitely generated abelian group. For a prime l, the l-adic Tate module of $J(C)$ is a Galois representation defined by the inverse limit of l^n-torsion of $J(C)$, equipped with Galois action. At a (good) prime ideal $\mathfrak{p} \in \mathrm{Spec}(\mathcal{O}_k)$, one can understand the $\mathfrak{p}$-Euler factor of "the" L-function of C as the characteristic polynomial of the Frobenius at $\mathfrak{p}$ acting on an l-adic Tate module, where l differs from the residual characteristic at $\mathfrak{p}$. The Tate module is dual to the first étale cohomology group, and, in general, for any smooth, projective algebraic variety V over k one can define non-archimedean Euler factors of an L-function by considering Galois action on the inertia invariants of an l-adic étale cohomology group, which is a finite dimensional $\mathbb{Q}_l$-vector space. In a certain derived category, one can let Frobenius act on a chain complex of étale cohomology groups - the characteristic polynomial then gives a zeta function, which is more easily described by a very simple Euler product that we will see later. If $\mathcal{S} \to \mathrm{Spec}(\mathcal{O}_k)$ is an arithmetic scheme whose generic fibre is V, one can deduce the "Hasse–Weil" factorization of the zeta function of $\mathcal{S}$. For example, if $V = E$ is an elliptic curve over k for which there exists a global proper, regular, minimal model

$\mathcal{E}$:

$$\zeta(\mathcal{E}, s) = \frac{\zeta(k, s)\zeta(k, s-1)}{L(E, s)}.$$

In general, a global minimal model does not exist and there is an additional factor $n(\mathcal{S}, s)$ depending on the choice of model which must be taken into consideration. The Hasse–Weil decomposition of the zeta function corresponds to a motivic decomposition of a smooth projective variety. It is these motivic Hasse–Weil L-functions that are expected to be automorphic, at least in the traditional sense, not zeta itself, which is at best a quotient of automorphic L-functions.

Another important factorization is the Artin factorization, which arises when the zeta function of a scheme $\mathcal{S}$ is understood as the L-function of the character of the regular representation of a finite group G with a well-behaved[2] action on $\mathcal{S}$. The decomposition of this character then gives rise to the factorization in question:

$$\zeta(\mathcal{S}, s) = \prod_{\chi \in \mathrm{Irr}(G)} L(\mathcal{S}, \chi, s)^{\chi(1)},$$

where $\mathrm{Irr}(G)$ denotes the set of irreducible characters of G.

There are certain properties we expect from the zeta functions of (regular) arithmetic schemes and their motivic factors. A priori, our zeta functions will be defined as an Euler product which converges only in some (right) half plane - as such, we are interested knowing to what extent its domain of definition can be extended. One conjectures that the zeta functions admit meromorphic continuation to the complex plane. This extended meromorphic function should, in turn, be of order one and satisfy a functional equation with respect to reflection about some vertical line in $\mathbb{C}$. Moreover, one expects an analogue of the Riemann hypothesis. This is far from an exhaustive list, for instance, one has the conjecture of Birch and Swinnerton-Dyer, and many other conjectures on special values.

Having mentioned above the Euler product formulation and the functional equation expected of L- and zeta functions, it is natural to mention the archimedean "gamma factors" appearing in their "completed" formulations. At an archimedean place of a number field, the analogue of Galois structure of étale cohomology, which gives rise to the Euler factor non-archimedean places, is the Hodge structure of Betti cohomology. The real and complex Hodge decompositions give rise to the Euler factor appearing

[2]This means, roughly, that the quotient $\mathcal{S}/G$ exists as a scheme.

at each infinite place, which is a product of shifted gamma functions. Also appearing in the functional equation is the "conductor" which accounts for ramification and is deeply connected to the reduction type of the variety in question.

Harmonic analysis has long been known to be an effective tool for the study of zeta and L-functions. For example, some of the simplest computations of values of the Riemann zeta function at positive even integers are in terms of Fourier series, and its functional equation may be verified by Poisson summation. This is the basis for the adelic approach taking by Tate in his famous thesis, recovering the results of Riemann and Hecke. The significance of this work is not these well-established results in their own right, rather, it is the approach taken, which utilised the uniformity of adeles and can be understood as a prototype of the type of approach needed in verifying the analytic properties of automorphic L-functions. Indeed, the Langlands program originated in non-abelian harmonic analysis in locally compact adelic algebraic groups.

In this thesis we will pursue two further implementations of harmonic analysis in the theory of zeta functions. Firstly, there is the theory of mean-periodic functions, which extends that of periodic functions. Given a nice[3] function space on which a locally compact group acts continuously, a function in the space is mean-periodic if its orbit under the group is not dense in the function space. By letting $\mathbb{R}$ act on the space $\mathscr{C}^{\infty}(\mathbb{R})$ of smooth real functions by $y \cdot f(x) = f(x-y)$, we recover the set of smooth periodic functions. We will see that mean-periodicity is very relevant to meromorphic continuation and functional equations of Mellin transforms.

Secondly, we will pursue abelian, but non-locally compact, harmonic analysis on two-dimensional local fields and adele rings. We are interested in the interplay between these two areas and with more traditional approaches to conjectures concerning L-functions such as Langlands-type automorphicity results. The following diagram, and the discussion below, show the basic connections:

$$\begin{tikzcd}
\text{Automorphicity} \arrow[r] & \text{Mean-Periodicity} \arrow[r] & \text{Analytic Properties} \arrow[ll, bend right] \\
& \text{Higher-dimensional adelic duality} \arrow[u] &
\end{tikzcd}$$

[3]Though they will share basic common properties, the exact properties of our "nice" spaces will vary, as, correspondingly, will the definition of mean-periodicity.

The arrows are "morally", although not always mathematically, implications. Both arrows from left to right are certainly implications, at least for certain specifications of "analytic properties" such as meromorphic continuation - we will discuss this at length in chapters 2 and 3. The long arrow from right to left, which one might understand as a converse theorem, is valid only when we have sufficiently refined analytic properties, which are not implied by mean-periodicity alone. One motivation throughout is the interaction of mean-periodicty and converse theorems, though we will not make any truly satisfactory statements - for a discussion, see the remarks in chapter 6. The vertical arrow is somewhat speculative, and forms the basis of chapters 4 and 5.

1.2 Conjectures

In this section we will collect together the main problems to be considered in this thesis. Some of the conjectures written below serve only illustrative purposes and will not be studied in detail, whilst others will be motivated and explained fully in the main body of the text. In the latter case, references to the relevant sections are provided.

Given a smooth, projective algebraic variety over a number field, one has a family of L-functions, which are expected to exhibit nice analytic behaviour. These "Hasse–Weil" L-functions are constructed from the Galois action on étale cohomology. A priori, $L(H^m(X), s)$ are defined only on the half plane $\Re(s) > m+1$, where $\Re(s)$ denotes the real part of s (definition 2.5). The most basic problem is to verify that these functions can be extended meromorphically to the complex plane $\mathbb{C}$. Moreover, one expects these functions to have a certain symmetry best expressed as a simple functional equation of their completion (definition 2.12).

Conjecture 1. *Let X be a smooth, projective algebraic variety over a number field k, and let $m \geq 0$ be an integer. $\Lambda(H^m(X), s)$ has meromorphic continuation to $\mathbb{C}$ and satisfies the following functional equation:*

$$\Lambda(H^m(X), s) = \pm\Lambda(H^m(X), m+1-s),$$

where

$$\Lambda(H^m(X), s) = A(H^m(X))^{s/2}\Gamma(H^m(X), s)L(H^m(X), s)$$

is the completed L-function defined in the main body.

We will generally avoid the ambiguity in the sign of the functional equation.

A smooth, projective, geometrically connected algebraic variety X over a number field k can be viewed as the generic fibre of a regular arithmetic scheme $\mathcal{X} \to \mathrm{Spec}(\mathcal{O}_k)$[4]. The L-functions of X are then motivic factors of the zeta function of $\mathcal{X}$, which is defined by an Euler product over closed points for $\Re(s) > \dim(\mathcal{X})$ (definition 2.1).

If we believe in the conjecture above, we must also believe that the zeta function $\zeta(\mathcal{X}, s)$ admits meromorphic continuation to the complex plane with a simple functional equation. For an arithmetic scheme $\mathcal{X}$, the (completed) zeta function $\xi(\mathcal{X}, s)$ is defined in definition 2.17.

Conjecture 2. *Let $\mathcal{S}$ be a regular scheme of finite type over $\mathbb{Z}$ of pure dimension n. The completed zeta function $\xi(\mathcal{S}, s)$ admits meromorphic continuation to $\mathbb{C}$ and satisfies the following functional equation:*

$$\xi(\mathcal{S}, s) = \pm\xi(\mathcal{S}, n-s).$$

In the case that C is a smooth, projective, geometrically connected curve over a number field k and $\mathcal{S}$ is any proper, regular model of C, the conjectures for $\xi(\mathcal{S}, s)$ and $\Lambda(H^1(C), s)$ are equivalent (corollary 2.22). This is to be expected, but not entirely trivial - to begin, one must modify the conductor in passing to the relative curve setting as will be explained in the next chapter. We will use this equivalence to infer facts about the Hasse–Weil L-functions of curves from zeta functions of arithmetic surfaces. In this case, both conjectures 1 and 2 are equivalent to conjecture 4.26, which concerns zeta integrals, and are a consequence of conjecture 2.33, which involves mean-periodicity.

Situations will naturally arise in which we would like to consider a finite group acting on a scheme $\mathcal{S}$ of finite type over $\mathbb{Z}$. If $\mathcal{S}/G$ is a union of affine open sets stable by G, then, for each character χ of G, one defines a twisted analogue $L(\mathcal{S}, \chi, s)$ of $\zeta(\mathcal{S}, s)$. $\zeta(\mathcal{S}, s)$ is recovered when χ is the character of the regular representation. Again, we expect meromorphic continuation and a functional equation, though the latter is more complicated.

Conjecture 3. *Let $\mathcal{S}$ be a regular two-dimensional scheme of finite type of $\mathbb{Z}$ such that a finite group G acts on $\mathcal{S}$ with the quotient $\mathcal{S}/G$ admitting a cover by open affine subschemes stable by G. For each character χ of G, the completed L-function*

$$\Lambda(\mathcal{S}, \chi, s) = A(\mathcal{S}, \chi)^{s/2}\Gamma(\mathcal{S}, \chi, s)L(\mathcal{S}, \chi, s),$$

[4]We will call a scheme "arithmetic" if it is proper, flat and of finite type over $\mathbb{Z}$.

admits meromorphic continuation to $\mathbb{C}$ *and satisfies a following functional equation of the form:*

$$\Lambda(\mathcal{S}, \chi, s) = \varepsilon(\chi, \mathcal{S})\Lambda(\mathcal{S}, \overline{\chi}, 2-s),$$

where $\varepsilon(\mathcal{S}, \chi)$ *is a complex number of absolute value* 1.

We will discuss the completion $\Lambda(\mathcal{S}, \chi, s)$ and the constant $\varepsilon(\mathcal{S}, \chi)$ in chapter 2. It is a relatively complicated object, taking into account the ramification of $\mathcal{S}$ and χ.

One aspect of the Langlands program is the identification of motivic L-functions with their automorphic counterparts. This would certainly imply the analytic properties of the Hasse–Weil L-functions, and, subsequently, zeta functions of arithmetic schemes. One goal of this thesis is to provide a comparison between higher dimensional adelic analysis, the mean-periodicity correspondence and specific conjectures falling within the Langlands philosophy, which are best described in the language of representations. For a summary of the basic passage from automorphic representations to Galois representations, the reader is referred to [Ram94, Introduction] and the references therein. We will be more interested in the reverse direction of associating an automorphic representation to the Galois representation on the étale cohomology groups of an algebraic variety. A famous example is provided by elliptic curves over $\mathbb{Q}$, which have been proved to be modular, implying Fermat's last theorem [Wil95], [TW95], [BCDT01]. On the level of L-functions, this result shows that the L-function of such an elliptic curve is that of a modular form. Recently, there have been some developments of this result. Now it is known that elliptic curves over real quadratic fields are modular [FHS13] and elliptic curves over an arbitrary totally real field are "potentially" modular - meaning there is a totally real extension of the base field such that the elliptic curve is modular over this field. Such results were first obtained in [Tay02], for further references see the surveys [Buz10] and [Sno09]. The general case of an elliptic curve over a number field remains elusive, and the situation for smooth projective curves of higher genus is no better.

The most abstract conjectures in this direction are formulated as relations between the motivic Galois group and the automorphic Langlands group [Lan79, Section 2], [Art02]. In terms of L-functions, we have the following conjecture, which will be simplified further after a brief discussion.

Conjecture 4. *Let E and F be number fields and let M be a pure motive over F with coefficients in E. There exists a connected reductive algebraic group G over F, an irreducible*

automorphic representation $\pi = \otimes_v \pi_v$ of $G(\mathbb{A}_F)$ and a representation r of the L-group ${}^L G$ such that $L(M,s) = L(s,\pi,r)$.

The triple (G,π,r) is not unique, but one can try to obtain the "minimal" such with respect to the principle of functoriality [Yos11, Main Conjecture]. For example, the L-function of an elliptic curve over $\mathbb{Q}$ could be identified with that of a GL_2-automorphic representation, however, if the elliptic curve has CM, then the minimal representation is actually of $\mathrm{GL}_1 \times \mathrm{GL}_1$[5]. Though not "optimal" in this sense, we will concern ourselves mostly with automorphic representations of GL_{2g} associated to smooth projective curves of genus g. In this case, one does not need to mention the L-group, which is simply $\mathrm{GL}_{2g}(\mathbb{C})$, and one can make a much more concrete conjecture, which we will use later. Roughly, if n is the rank of the pure motive M, then we expect π to exist on GL_n, see [Clo90, Question 4.16].

Conjecture 5. *Let E and F be number fields and let M be a degree n, absolutely irreducible motive over F with coefficients in E and weights w. There exists an algebraic, cuspidal automorphic $GL_n(\mathbb{A}_F)$-representation defined over E with weights w such that for all finite places v of F*

$$L_v(\pi, s - \frac{w}{2}) = L_v(M,s).$$

In Chapter 3, this will be compared with the mean-periodicity hypothesis, which is conjecture 2.33.

It is known that such standard automorphic L-functions have the expected analytic properties, so that the identifications above imply the conjectural analytic properties of the L-function of C. On the other hand, one can ask what sort of analytic properties are required for a representation of GL_n to be automorphic. This is the world of converse theorems. For illustrative purposes, we will begin with a conjecture taken from [CPS94, conjecture], which proves weaker versions in which higher dimensional twists are required. Improvements of this result can be found in [CPS99].

Conjecture 6. *Let $\pi = \otimes \pi_v$ be an irreducible admissible representation of $GL_2(\mathbb{A})$ whose central character ω_π is invariant under $k^\times$ and whose L-function $L(\pi,s)$ is absolutely convergent in some half-plane. Assume that $L(\pi \otimes \omega, s)$ is "nice" for all characters ω of $k^\times \backslash \mathbb{A}^\times$. Then there exists an automorphic representation π' of $GL_n(\mathbb{A})$ which is quasi-isomorphic to π and such that $L(\pi \otimes \omega, s) = L(\pi' \otimes \omega, s)$ and $\varepsilon(\pi \otimes \omega, s) = \varepsilon(\pi' \otimes \omega, s)$.*

[5] That is, the L-function is a product of Hecke L-functions.

By "nice" it is meant that the L-function has holomorphic continuation, satisfies the expected functional equation and has certain boundedness properties in vertical strips. The above conjecture has been proved for $n = 2, 3$. The case $n = 2$ is [JL70, Theorem 12.2], and the case $n = 3$ is proved in [JPSS79]. In these cases $\pi = \pi'$, but this need not be true in general. In terms of Dirichlet series, it is often helpful to understand converse theorems as the statement that sufficiently refined analytic behaviour of an L-function implies that it is an automorphic L-function. For example, one might consult [Bum98, Theorem 1.5.1].

Of course, we expect other aspects of the analytic behaviour of the zeta functions of a scheme to encode something of its arithmetic. One manifestation of this is in terms of its zeros and poles. There are (at least) two very famous conjectures in this direction: the generalized Riemann hypothesis (GRH) and the conjecture of Birch and Swinnerton-Dyer (BSD). We will state the former here, which will be discussed informally in chapter 6. Observe that it would not be possible to formulate either without meromorphic continuation of zeta.

The zeta function of an arithmetic scheme $\mathcal{S}$ has no zeros or poles for $\Re(s) > \dim(\mathcal{S})$. By the expected functional equation relating values at s to those at $\dim(\mathcal{S}) - s$, one should expect some "trivial" zeros (resp. poles) of zeta in the half-plane $\Re(s) < 0$ corresponding to the poles (resp. zeros) of the gamma factor. One conjectures the following:

Conjecture 7. *Let $\mathcal{V}$ be a proper, regular scheme of finite type over $\mathbb{Z}$, whose zeta function admits meromorphic continuation to $\mathbb{C}$, and let V denote the irreducible generic fibre of $\mathcal{V}$. For all $m \in \mathbb{Z}$ such that $0 \leq m \leq 2\dim(V)$, the only zeros of $L(H^m(V), s)$ in the strip $\{0 < \Re(s) < m + 1\}$ lie on the line $\Re(s) = \frac{m+1}{2}$.*

Consequently, one expects the zeros (resp. poles) of $\zeta(\mathcal{V}, s)$ in the strip $\{0 < \Re(s) < \dim(\mathcal{V})\}$ to lie on one of the lines $\Re(s) = n$ (resp. $\Re(s) = \frac{2n+1}{2}$), for some $n \in \mathbb{Z}$ such that $0 \leq n \leq \dim(V)$. We will not consider this again until the final chapter.

1.3 Theorems

For convenience, and to give a flavour of the forthcoming material, the main theorems of this thesis are collected here. The statements are numbered as they appear in the

main body of the text.

Our primary tool in the consideration of the above open problems will be the mean-periodicity correspondence. A relatively detailed introduction to mean-periodicity will be provided in section 2.2. The idea that mean-periodicity could be implemented in the verification of analytic properties of zeta functions were published first in [Fes08] as an answer to a question of Langlands [Lan97] concerning manifestations of automorphicity on the level of zeta functions. This work subsequently inspired the developments in [FRS12]. Explicit connections were made to modular forms in [Suz12]. We begin by refining the mean-periodicity correspondence of [FRS12, Theorem 5.18] in the manner most suitable for our purposes.

Theorem (2.32). *On the locally compact multiplicative group $\mathbb{R}_+^\times$ of positive real numbers, let $\mathfrak{X}$ denote the strong Schwartz space or space of smooth functions of polynomial growth. Let C/k be a smooth projective curve over a number field k.*

1. *Assume that $L(C,s)$ admits meromorphic continuation to $\mathbb{C}$ with the expected functional equation $\Lambda(C,s) = \varepsilon\Lambda(C,2-s)$, the logarithmic derivative of $L(C,s)$ is an absolutely convergent Dirichlet series in the right half plane $\Re(s) > 2$ and there exists a polynomial $P(s)$ such that $P(s)L(C,s)$ is an entire function on $\mathbb{C}$ of order 1. Then, for each choice of proper, regular model $\mathcal{S} \to Spec(\mathcal{O}_k)$, there exists $m_{\mathcal{S}} \in \mathbb{N}$ such that the following function is $\mathfrak{X}$-mean-periodic for every $m \geq m_{\mathcal{S}}$:*

$$h_{\mathcal{S},\{k_i\}}(x) := f_{\mathcal{S},\{k_i\}}(x) - \varepsilon x^{-1} f_{\mathcal{S},\{k_i\}}(x^{-1}),$$

   where, for $i = 1, \ldots, m$, k_i is a finite extension of k, and

$$f_{\mathcal{S},\{k_i\}}(s) = \frac{1}{2\pi i}\int_{(c)} \xi(\mathcal{S},2s)(\prod_{i=1}^{m}\xi(k_i,s))x^{-s}ds.$$

2. *Conversely, suppose that there exists $m_{\mathcal{S}} \in \mathbb{N}$ such that, for some set $\{k_i\}$ of $m_{\mathcal{S}}$ finite extensions of k, the function $h_{\mathcal{S},\{k_i\}}$ is $\mathfrak{X}$-mean-periodic and that*

$$\Gamma(\mathcal{S},2s)\prod\xi(k_i,s) \ll |t|^{-1-\delta},$$

   where $t = \Im(s)$, then $\xi(\mathcal{S},s)$ admits meromorphic continuation and satisfies the functional equation

$$\xi(\mathcal{S},s) = \xi(\mathcal{S},\dim(\mathcal{S})-s).$$

This is a consequence of the more abstract theorem 2.31. We will give a twisted analogue as theorem 2.35, which requires a small modification of the boundary function. The statement is made slightly unwieldy by the inclusion of field extensions, this is needed primarily in the adelic interpretation in which the Dedekind zeta factors will arise from the closures of points on the generic fibre, which need not be rational.

Mean-periodicity is an easy consequence of automorphicity, [FRS12, Remark 5.20], however it is desirable to understand the relationship of the two conjectures in more detail. We will make the connection more explicit by solving certain convolution equations in terms of matrix coefficients of automorphic representations.

An elementary abelian case is explored at length in section 3.1. Given a CM elliptic curve E, one has an associated Hecke character χ from which we construct a complex vector space $\mathcal{W}_\chi$ as the span of certain Schwartz functions on $\mathbb{R}_+^\times$. This construction is inspired by an iterated integral decomposition of the zeta integral representation of $L(s,\chi)$. On the other hand, we let $\mathcal{E}$ denote a model of E and let $\mathcal{T}(h_\mathcal{E})^\perp$ denote the orthogonal complement of the "translates" of the boundary function $h_\mathcal{E}$ with respect to convolution. The relationship can be expressed concisely as an inclusion as follows.

Theorem (3.1). *With respect to the convolution on the strong Schwartz space $\mathbf{S}(\mathbb{R}_+^\times)$, every function in $\mathcal{W}_\chi$ is orthogonal to every translate of $h_\mathcal{E}$ with respect to the action of $\mathbb{R}_+^\times$, ie.*

$$\mathcal{W}_\chi \subset \mathcal{T}(h_\mathcal{E})^\perp.$$

This result is proved using techniques dating back to Tate's thesis to directly compute the convolution.

Under the assumption of conjecture 5, we will extend this to the non-abelian case in section 3.2. The Hasse–Weil L-functions are known to be automorphic when $C = E$ is an elliptic curve which is either defined over $\mathbb{Q}$ or has CM. Therefore we recover the above theorem and [Suz12, Theorem 3.1].

Theorem (3.4). *Let C be a smooth projective curve over a number field k, let $\mathcal{S} \to Spec(\mathcal{O}_k)$ be a proper regular model of C and let π be as in conjecture 5. If $h_\mathcal{S}$ denotes the boundary term, then*

$$\mathcal{W}_\pi \subset \mathcal{T}(h_\mathcal{S})^\perp,$$

where $\mathcal{T}(h_\mathcal{S})^\perp$ denotes the set of convolutors $\{g \in \mathbf{S}(\mathbb{R}_+^\times) : g * \tau = 0, \forall \tau \in \mathcal{T}(h_\mathcal{S})\}$.

The method of proof is comparable to the Rankin-Selberg method - we explicitly integrate a matrix coefficient against the boundary function, which plays the role of an Eisenstein series.

In the GL_n-case, we will also explore an adelic analogue and dual inclusions to the above result. By functoriality, one should expect similar result for any automorphic representation giving rise to the Hasse–Weil L-function $L(C,s)$. The above theorems depend on integral representations of automorphic L-functions. In section 3.3, we also provide generalized Fourier expansions for the boundary function.

Whilst explicit, the above theorems do not tell us much beyond what we already know - they merely relate conjectures, without simplifying them. Next we stop thinking about automorphic representations altogether, and instead develop an approach to mean-periodicity in terms of higher dimensional adelic duality.

Our first result in this direction is a representation of the zeta function as an integral over two-dimensional analytic adeles. The factors of Dedekind zeta functions of number fields k_i correspond to horizontal curves - theorem 2.32 says that we need only take finitely many. These horizontal curves correspond to the closures of k_i-rational points on the curve C. The case $g = 1$ can be found it [Fes08, Fes10].

Theorem (4.22). *Let $\mathbb{A}(\mathcal{S},\{y_i\})$ denote the analytic adelic space associated to the set containing every fibre on $\mathcal{S}$ and finitely many horizontal curves y_i. Let $\zeta(f,h,s)$ denote the renormalized zeta integral associated to the adelic space $(\mathbb{A}(\mathcal{S},\{y_i\}) \times \mathbb{A}(\mathcal{S},\{y_i\}))^\times$ then*

$$\zeta(f,h,s) = Q(s)^2\Gamma(\mathcal{S},s)^2 A(\mathcal{S})^{1-s}\zeta(\mathcal{S},s)^2 \prod_{i=1}^{n} \xi(k(y_i),s/2)^2,$$

where f is an explicit function

$$f : \mathbb{A}(\mathcal{S},\{y_i\}) \times \mathbb{A}(\mathcal{S},\{y_i\}) \to \mathbb{C},$$

h is an explicit function

$$h : \mathbb{A}(\mathbb{P}^1(\mathcal{O}_k)) \times \mathbb{A}(\mathbb{P}^1(\mathcal{O}_k)) \to \mathbb{C},$$

and $Q(s)$ is an explicit rational function in $\mathbb{C}(s)$ such that

$$Q(2-s) = \pm Q(s).$$

The meromorphic continuation and functional equation of the zeta integrals $\zeta(f,h,s)$ is thus equivalent to that of the completed zeta function $\xi(\mathcal{S},s)$, up to sign. These results

are proved through a renormalized analogue Fesenko's theory of integration on two-dimensional analytic adeles, the zeta integrals of which diverge for curves of genus $g \neq 1$. There is a slightly intricate argument at the archimedean places, utilising the zeta function of the projective line over $\mathcal{O}_k$:

$$\zeta(\mathbb{P}^1(\mathcal{O}_k), s) = \zeta(k, s)\zeta(k, s-1).$$

We mention also theorem 6.5, which provides a similar integral representation for twisted zeta functions, provided the character is sufficiently simple.

In order to approach mean-periodicity through these integral representations, one expects to incorporate adelic duality. The additive duality of higher dimensional adeles is related to the cohomology of quasi-coherent sheaves, whereas their multiplicative and K-theoretic duality appears in the class field theory of arithmetic schemes. The two-dimensional theta formula that follows depends on the compatibility between additive and multiplicative structures. We introduce measures on various local-global subspaces of the adeles, which do not factorize as the product of the equivalent measures over the fibre factors. For this reason, it is not sufficient to simply rescale at each factor, instead we have to perform adelic analysis on the non-connected arithmetic scheme

$$\mathcal{X} = \mathcal{S} \coprod_{i=1}^{g-1} \mathcal{P},$$

where $\coprod$ denotes the disjoint union and $\mathcal{P}$ is the relative projective line $\mathbb{P}^1(\mathcal{O}_k)$.

Theorem (5.6). *Let $\mathcal{X}$ denote the non-connected, two-dimensional arithmetic scheme above, let $T_0(\mathcal{X})$ denote the local-global subspace of $(\mathbb{A}(\mathcal{X}) \times \mathbb{A}(\mathcal{X}))^\times$ defined in chapter 5, and let $\partial T_0(\mathcal{X})$ be its boundary with respect to the inductive limit of lifted weak topologies, then, for all $\alpha \in T(\mathcal{X}) := (\mathbb{A}(\mathcal{X}) \times \mathbb{A}(\mathcal{X}))^\times$ we have the following equality of convergent integrals*

$$\int_{T_0(\mathcal{X})} (f_\alpha(\beta) - \mathcal{F}(f_\alpha)(\beta))d\mu(\beta) = \int_{\partial T_0(\mathcal{X})} (\mathcal{F}(f_\alpha)(\beta) - f_\alpha(\beta))d\mu(\beta),$$

where

$$f : (\mathbb{A}(\mathcal{X}) \times \mathbb{A}(\mathcal{X}))^\times \to \mathbb{C}$$

is an integrable function on $T_0(\mathcal{X}) \subset (\mathbb{A}(\mathcal{X}) \times \mathbb{A}(\mathcal{X}))^\times$, $\mathcal{F}$ denotes the Fourier transform on $T(\mathcal{X})$, and, for $\alpha \in T(\mathcal{X})$, $f_\alpha(\beta) := f(\alpha\beta)$.[6]

[6]The statement here differs slightly to that given in the main body, where one finds an explicit evaluation of the Fourier transform $\mathcal{F}(f_\alpha)$.

We will apply this to $f \coprod h$, ie. the function on $T(\mathcal{X})$ associated to the functions f and h of the preceding theorem. With the two-dimensional theta formula, we obtain various integral representations of the boundary functions $h_{\mathcal{S},\{k_i\}}$ as integrals over two-dimensional adelic subspaces.

We conclude in chapter 7 with an outlook of possible future directions.

CHAPTER 2

Background

The main result of this chapter is the mean-periodicity correspondence, as stated in theorem 2.32. This can be viewed as an answer to the long standing question of which conditions are "essentially" equivalent to meromorphic continuation and functional equation of higher dimensional zeta functions in analogy the way that automorphic properties are closely related to such properties of L-functions of Weil group representations. This idea has its origins in [Fes08].
The purpose of section 2.1 is the recollection of the definitions and basic properties of various L- and zeta functions to be used throughout the text. In section 2.2, the reader will find an introduction to mean-periodic functions and their connection to zeta functions.

2.1 Arithmetic Zeta Functions

A scheme is a locally ringed, locally affine topological space. Every scheme X is, in a unique way, a $\mathbb{Z}$-scheme, as every ring has one and only one $\mathbb{Z}$-algebra structure. A $\mathbb{Z}$-scheme X is said to be of finite type over $\mathbb{Z}$ if its structural morphism is, ie. if the morphism $f : X \to \mathbb{Z}$ is quasi-compact, and, for every affine open subset V of $\mathrm{Spec}(\mathbb{Z})$ and every affine open subset U of $f^{-1}(V)$, the canonical homomorphism $\mathcal{O}_{\mathrm{Spec}(\mathbb{Z})}(V) \to \mathcal{O}_X(U)$ makes $\mathcal{O}_X(U)$ into a finitely generated $\mathcal{O}_{\mathrm{Spec}(\mathbb{Z})}(V)$-algebra. Given such a scheme X, for any $p \in \mathrm{Spec}(\mathbb{Z})$, the fibre $X_p = X \times_{\mathbb{Z}} \mathbb{F}_p$ is an algebraic variety over $\mathbb{F}_p$. X can be viewed as a family of algebraic varieties, parameterized by $p \in \mathrm{Spec}(\mathbb{Z})$. Algebraic varieties over finite fields are schemes of finite type over $\mathbb{Z}$, for

which it happens that there is only one non-empty fibre.

If X is a scheme, then let X_0 denote the set of closed points in X [1]. When X is of finite type over $\mathbb{Z}$, the residue field $k(x)$ of a $x \in X_0$ has finite cardinality, which we will denote by $N(x)$. Following [Ser63], we make the following definition.

Definition 2.1. Let X be a scheme of finite type over $\mathbb{Z}$, the zeta function $\zeta(X,s)$ of X is defined for $\Re(s) > \dim(X)$ as the Euler product

$$\zeta(X,s) = \prod_{x \in X_0} \frac{1}{1-(N(x))^{-s}}.$$

If k is an algebraic number field, and $X = \mathrm{Spec}(\mathcal{O}_k)$, we recover the Dedekind zeta function of k on the half plane $\Re(s) > 1$. In general, one can check that the Euler product is absolutely convergent for $\Re(s) > \dim(X)$, as explained in [Bru63, Part III, Chapter 6].

One conjectures that $\zeta(X,s)$ extends to $s \in \mathbb{C}$, although generally one only knows weaker statements. For example, it can be proved that $\zeta(X,s)$ can be continued analytically as a meromorphic function to the half plane $\Re(s) > \dim(X) - \frac{1}{2}$. Still, this permits us to talk about the poles in the strip $\dim(X) - \frac{1}{2} < \Re(s) \leq \dim(X)$. For example, it is known that the point $s = \dim(X)$ is always a pole of $\zeta(X,s)$ whose order is equal to the number of irreducible components of X with dimension equal to $\dim(X)$. We therefore deduce that the Dirichlet series defined by the Euler product in definition 2.1 has domain of convergence $\Re(s) > \dim(X)$.

We will now give a more complete classification of the poles in the half-plane $\Re(s) > \dim(X) - \frac{1}{2}$. Let X be irreducible and let E denote the residue field of the generic point of X, if $\mathrm{char}(E) = 0$ then $s = \dim(X)$ is the only pole in this half plane and if $\mathrm{char}(E) = p$ then the only poles are the points $s = \dim(X) + \frac{2\pi i n}{\log(q)}$, where q is the maximal power of p such that $\mathbb{F}_q \subset E$ and $n \in \mathbb{Z}$.

Let G be a finite group acting on a scheme X of finite type over $\mathbb{Z}$, such that the quotient X/G is a union of affine open sets which are stable by G. If χ denotes a character of G[2], then the Artin L-functions $L(X,\chi,s)$ are uniquely characterized by the following properties, [Ser63, §2].

1. $L(X,\chi,s)$ depends on X only through its closed points X_0.

[1] More generally, X_n denotes the set of points on X whose closure have dimension n. The analogous notation for codimension n points is X^n.

[2] By this we mean a linear combination of characters of representations of G.

2. $L(X, \chi + \chi', s) = L(X, \chi, s)L(X, \chi', s)$.

3. If X_0 is the disjoint union of schemes $(X_i)_0$ such that each X_i is stable by G, then $L(X, \chi, s)$ is equal to the product of $L(X_i, \chi, s)$ with absolute convergence for $\Re(s) > \dim(X)$.

4. Let $\pi : G \to G'$ be a group homomorphism and let $\pi_* X$ denote the scheme deduced from X by extension of the structural group. If χ' denotes a character of G' and $\pi^*\chi' = \chi' \circ \pi$ denotes its pull-back to G, then:

$$L(X, \pi^*\chi', s) = L(\pi_* X, \chi', s).$$

5. Let $\pi : G' \to G$ be a homomorphism, and let $\pi^* X$ denote the scheme X equipped with an action of G' through π. If χ' denotes a character of G' and $\pi_*\chi'$ is the character of G induced from $\pi(G')$, then

$$L(X, \pi_*\chi', s) = L(\pi^* X, \chi', s).$$

6. If $X = \mathrm{Spec}(\mathbb{F}_{q^n})$, $G = \mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)$ and $X/G = \mathrm{Spec}(\mathbb{F}_q)$, then

$$L(X, \chi, s) = \frac{1}{1 - \chi(\mathrm{Frob}_q)q^{-s}},$$

where Frob_q is the Frobenius element of the finite field $\mathbb{F}_q$.

We provide an example of the constructions involved in each of properties 4 and 5. The context is a scheme X defined over a number field k

Example 2.2. Let $K \supset k' \supset k$ be a chain of Galois extensions of number fields, then a character χ' of $\mathrm{Gal}(k'/k)$ determines a character $\pi^*\chi'$ of $\mathrm{Gal}(K/k)$ through the quotient map $\mathrm{Gal}(K/k) \to \mathrm{Gal}(k'/k)$. The scheme $\pi_* X$ is simply the scheme X with the action of $\mathrm{Gal}(k'/k)$.

Example 2.3. Property 5 is often known as "inductivity" of Artin L-functions as when $\pi : G' \to G$ is the inclusion map of a subgroup $G' \leq G$, $\pi_*\chi'$ is the induced character of χ. In particular, this applies when $K \supset k' \supset k$ is a tower of Galois extensions and $G = \mathrm{Gal}(K/k)$, $G' = \mathrm{Gal}(K/k')$.

Explicitly, we may define Artin L-functions as follows. The atomization[3] Y_0 of Y may be identified with X_0/G. Let $y \in Y_0$ denote the image of some $x \in X_0$, the decomposition

[3]That is, the set of closed points.

group of x is the subgroup of G defined by

$$D(x) = \{g \in G : g \cdot x = x\}.$$

The residue field $k(x)$ is an extension of $k(y)$ and there is a natural epimorphism

$$D(x) \to \mathrm{Gal}(k(x)/k(y)),$$

whose kernel is the inertia subgroup. Therefore, $D(x)/I(x)$ is cyclic with canonical generator denoted F_x, the so-called "Frobenius element" of x.
For each $y \in Y_0$, let $\chi(y^n)$ denote the mean value of χ on the set $\{F_x^n \in D(x)/I(x) : x \in X_0$ is a lifting of $y\}$. The Artin L-function of the pair (X, χ) can now be defined.

Definition 2.4. Let X be a scheme of finite type over $\mathbb{Z}$ on which a finite group G acts such that $Y = X/G$ is a union of affine open sets stable by G, and let χ denote a character of G. For $\Re(s) > \dim(X)$,

$$\log L(X, \chi, s) = \sum_{y \in Y_0} \sum_{n=1}^{\infty} \frac{\chi(y^n) N(y)^{-ns}}{n}.$$

In particular, when χ is the character of a linear (complex) representation $g \mapsto M(g)$,

$$L(X, \chi, s) = \prod_{y \in Y_0} \frac{1}{\det(1 - M(F_x)/N(y)^s)},$$

where $M(F_x)$ is the mean value of $M(g)$ on the set $\{g \in G : f \mapsto F_x\}$.
If $\chi = 1$, then $L(X, \chi) = \zeta(X/G)$. If χ is the character of the regular representation, then $L(X, \chi) = \zeta(X)$. Combining this with the additivity property 2 above, we obtain the following factorization of the zeta function:

$$\zeta(X) = \prod_{\chi \in \mathrm{Irr(G)}} L(X, \chi)^{\deg(\chi)},$$

where $\mathrm{Irr}(G)$ denotes the set of irreducible characters of G and $\deg(\chi) = \chi(1)$.
These L-functions are a simultaneous generalization of Artin L-functions of number fields and the zeta functions of arithmetic schemes. To see this, let K/k be a finite extension of number fields and $X \to \mathrm{Spec}(\mathcal{O}_k)$ be a scheme of finite type. If $G =$ Gal(K/k) and $X_K = X \times_{\mathcal{O}_k} \mathcal{O}_K$, then

$$\zeta(X, s) = L(X_K, 1, s),$$

where 1 denotes the trivial character, and

$$\zeta(X_K, s) = L(X_K, r, s),$$

where r is the character of the regular representation. If $X = \mathrm{Spec}(\mathcal{O}_k)$, then, for any character χ of G

$$L(X_K, \chi, s) = L(K/k, \chi, s),$$

the Artin L-function of the extension K/k.

In this thesis, the finite group G will always be the Galois group of a finite Galois extension of number fields.

Of course, we are also interested in absolute (profinite) Galois groups such as $G_{\mathbb{Q}} := \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. For topological reasons, any continuous complex representation $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_n(\mathbb{C})$ factors through a finite Galois group $\mathrm{Gal}(K/\mathbb{Q})$. The l-adic representations we now consider provide examples of interesting Galois representations for which this is no longer true.

2.1.1 Hasse–Weil L-Functions

We enter the world of Hasse–Weil L-functions, which we will construct through the action of absolute Galois groups on the étale cohomology groups[4] of algebraic varieties over number fields. For further details, the reader is referred to [Ser69].

Let X be a smooth projective variety defined over a number field k and let $m \geq 0$ be an integer. The absolute Galois group $G_k = \mathrm{Gal}(\overline{\mathbb{Q}}/k)$ acts on X. If $\overline{\mathfrak{p}}$ is a prime of $\overline{\mathbb{Q}}$ lying above a prime ideal $\mathfrak{p} \in \mathrm{Spec}(\mathcal{O}_k)$, we therefore have an action of the decomposition group $G_{\overline{\mathfrak{p}}} \subset G_k$ on X. If $I_{\overline{\mathfrak{p}}} \subset G_{\overline{\mathfrak{p}}}$ is the inertia subgroup, then the quotient $G_{\mathfrak{p}}/I_{\mathfrak{p}} \cong \widehat{\mathbb{Z}}$ is generated by the geometric Frobenius $f_{\mathfrak{p}}$. Let l be a prime number different to the residual characteristic at $\mathfrak{p}$, by transport of structure, $G_{\overline{\mathfrak{p}}}/I_{\overline{\mathfrak{p}}}$ acts on the inertia invariants of étale cohomology $H^m_{\text{ét}}(X_{\overline{k}}, \mathbb{Q}_l)^{I_{\overline{\mathfrak{p}}}}$. The Euler factor at $\mathfrak{p}$, which does not depend on the choice of l[5], is defined as a characteristic polynomial.

Definition 2.5. Let X be a smooth projective variety over a number field k, and let $\mathfrak{p}$ denote a non-zero prime ideal of k. For each $m \in \mathbb{Z}$, the Euler factor at $\mathfrak{p}$ is the following

[4]For the most part, we will not use the technical details of étale cohomology.

[5]This is true, at least, for curves and abelian varieties. In general, this sort of statement is a consequence of the so-called standard conjectures, which are not proved in general.

characteristic polynomial

$$L_{\mathfrak{p}}(H^m(X), s) := \det(1 - f_{\mathfrak{p}} \cdot N\mathfrak{p}^{-s} | H^m_{\text{ét}}(X_{\bar{k}}, \mathbb{Q}_l)^{I_{\bar{\mathfrak{p}}}}),$$

and the mth Hasse–Weil L-function of X is then the Euler product

$$L(H^m(X), s) = \prod_{\mathfrak{p}} L_{\mathfrak{p}}(H^m(X), s).$$

For each m, this product converges absolutely for $\Re(s) > 1 + \frac{m}{2}$.

Example 2.6. Let $X = C$ be a smooth, projective curve over a number field k, then the cohomology groups are nontrivial for $m = 0, 1, 2$ and we may explicitly identify the Galois representations and recover:

$$L(H^0(C), s) = \zeta(k, s),$$

$$L(H^2(C), s) = \zeta(k, s - 1),$$

where $\zeta(k, s)$ is the Dedekind zeta-function. We will call the Hasse–Weil L-function for H^1 the L-function of the curve:

$$L(H^1(C), s) = L(C, s).$$

This terminology is due to the fact that the Galois action on first étale cohomology of a curve is equivalent to that of the Tate module of the Jacobian of C.

The Hasse–Weil L-functions provide an important factorization of the zeta function as we now explain. First we observe that the zeta function of an arithmetic scheme $\mathcal{X} \to \text{Spec}(\mathcal{O}_k)$ has a "fibral" factorization:

$$\zeta(\mathcal{X}, s) = \prod_{\mathfrak{p} \in \text{Spec}(\mathcal{O}_k)} \zeta(\mathcal{X}_{\mathfrak{p}}, s),$$

where $\mathcal{X}_{\mathfrak{p}}$ is the fibre of $\mathcal{X}$ over $\mathfrak{p}$. For a non-archimedean place v of k, let k_v denote the corresponding completion of k and let $\mathcal{O}_v$, $\mathfrak{p}_v$, $k(v)$ denote its ring of integers, its maximal ideal and residue field respectively. If $k(v)$ has residual characteristic p_v, then

$$Nv := p_v^{\deg(v)},$$

where

$$\deg(v) = [k(v) : \mathbb{F}_{p_v}].$$

Let X denote the generic fibre of the arithmetic scheme $\mathcal{X}$, which is a smooth, projective variety over the number field k. If X has good reduction at v, then there exists a smooth projective $\mathcal{O}_v$-scheme X_v such that

$$X_v \times_{\mathcal{O}_v} k_v = X \times_k k_v.$$

Choose such an X_v and let

$$X(v) = X_v \times_{\mathcal{O}_v} k(v).$$

$X(v)$ is a smooth projective variety over a finite field $k(v)$, which depends on the choice of X_v. The Weil conjectures, which are a theorem proved by Deligne, tell us everything we need to know about the zeta functions $\zeta(X(v), s)$. For example, after a change of variables it is a rational function:

$$\begin{aligned} Z(X(v), q^{-s}) &= \prod_{j=0}^{2n} \det(1 - f_{\mathfrak{p}} q^{-s} | H^j(X(p)))^{(-1)^{j+1}} \\ &= \prod_{j=0}^{2n} \det(1 - f_{\mathfrak{p}} q^{-s} | H^j(X))^{(-1)^{j+1}}, \end{aligned}$$

where the second equality is a consequence of smooth base change in étale cohomology and each polynomial is of degree B_m. At bad primes we take inertia invariants as in definition 2.5. Let

$$n_{\mathcal{X}}(s) = \frac{\zeta(\mathcal{X}, s)}{\prod_{i=0}^{2n} L(H^i(X), s)^{(-1)^i}}.$$

When there are only finitely many bad primes $\mathfrak{p}_i$, this is a rational function in q_i^s, where q_i is the maximal power of the residual characteristic p_i such that $\mathbb{F}_{q_i} \subset k(\mathfrak{p}_i)$. We deduce that the zeta function of $\mathcal{X}$ has the following factorization:

$$\zeta(\mathcal{X}, s) = n_{\mathcal{X}}(s) \prod_{i=0}^{2n} L(H^i(X), s)^{(-1)^i}.$$

We will now define the completed Hasse–Weil L-functions, thus explaining conjecture 1.

Let v be a non-archidean place of the number field k and choose a separable closure k_v^{sep} of k_v and write G for the Galois group of this extension. Let $l \neq p_v$ be a prime number and let V be a vector space over $\mathbb{Q}_l$ of finite dimension d. Assume we have a continuous homomorphism

$$\rho : G \to \text{Aut}(V) \cong \text{GL}_d(\mathbb{Q}_l).$$

Let I denote the inertia group, recall that I is the Galois group of the maximal unramified extension k_v^{ur}/k_v. The subspace of V invariant with respect to I is denoted V^I. Define

$$\varepsilon = \mathrm{codim} V^I = d - \dim V^I.$$

ε measures part of the ramification of ρ, in the sense that $\varepsilon = 0$ if and only if ρ is unramified, ie $\rho(1) = \{1\}$.

Since $k(v)$ is finite there exists an open subgroup $I' \leq I$ such that $\rho(g)$ is unipotent for all $g \in I'$, introduce the following notation:

$$V_n = \{x \in V : \forall g \in I',\ (\rho(g) - 1)^n x = 0\}.$$

The V_n form an increasing filtration of V, stable under I, and, for n big enough, we have $V_n = V$.

$$\mathrm{gr}(V) = \oplus_{n=0}^{\infty} V_{n+1}/V_n.$$

I acts on $\mathrm{gr}(V)$ through the finite group $\Phi = I/I'$. For $g \in I$, the trace of $\rho(g)$ depends only on the image of g in Φ. We have a function

$$\mathrm{Tr}_\rho : \Phi \to \mathbb{Q}_l,$$

the character of ρ on I. By the main theorem of Galois theory, the group Φ is the Galois group of a finite extension k_Φ/k_v^{ur}. Let v_Φ denote the normalized valuation of k_v and let t be a uniformizer of k_Φ. The Swan character of Φ is defined by the formulas

$$\mathrm{sw}_\Phi(g) = \begin{cases} 1 - v_\Phi(g(t) - t) & g \in \Phi - \{1\} \\ -\sum_{g \neq 1} \mathrm{sw}_\Phi(g) & g = 1. \end{cases}.$$

We define an invariant δ by the scalar product of the trace and the Swan character (the reader is referred to [Ser69, 2.2] or [Ser77, Chapter 19] for details):

$$\delta =< \mathrm{Tr}_\rho, \mathrm{sw}_\Phi >= \frac{1}{|\Phi|} \sum_{g \in \Phi} \mathrm{Tr}_\rho(g) \mathrm{sw}_\Phi(g).$$

δ is a non-negative integer which does not depend on the choice of I'. For example, $\delta = 0$ if and only if the action of I on $\mathrm{gr}(V)$ is tame, that is, if $\rho_v = 0$ or if the Sylow-p_v-subgroup of I acts trivially on $\mathrm{gr}(V)$.

Definition 2.7. The exponent of the conductor of ρ is $f = \varepsilon + \delta$. Let Σ_k denote the set of non-archimedean places of k, the conductor $\mathfrak{f}$ is the following divisor

$$\mathfrak{f} = \sum_{v \in \Sigma_k} f(v) \cdot v,$$

we also introduce

$$N(\mathfrak{f}) = \prod_{v \in \Sigma_k} Nv^{f(v)}.$$

The integers ε, δ and f do not change if we make an unramified field extension of the base - they only depend on the restriction of ρ to I. We will apply this definition when ρ is the representation of the absolute Galois group on étale cohomology. For curves over number fields, ε and δ, f are independent of the choice of l.

Example 2.8. Let E be an elliptic curve over a local field K of residue characteristic p, the Swan character is trivial if $p \geq 5$ or the reduction is good or multiplicative - see [Sil94, Theorem 10.2]. More generally, let A be an abelian variety with potential good reduction at all primes p, then the Swan character is trivial at all primes of good reduction and at all primes $p > 2\dim(A) + 1$, [ST68]. In particular, if C is a smooth projective curve of genus g, at all primes of good reduction or above $2g + 1$ the Swan conductor of its Jacobian is trivial. This fact will be of use to us in fixing the conventions of 4.2.

An archimedean place of k may be real or complex. Correspondingly, we write

$$\Gamma(\mathbb{R}, s) = \pi^{-s/2}\Gamma(s/2),$$

$$\Gamma(\mathbb{C}, s) = (2\pi)^{-s}\Gamma(s).$$

We have

$$2\Gamma(\mathbb{C}, s) = \Gamma(\mathbb{R}, s)\Gamma(\mathbb{R}, s+1).$$

Definition 2.9. Let V be a vector space over $\mathbb{C}$,

1. A $\mathbb{C}-$Hodge decomposition of V is an expression for V as a direct sum of vector spaces over $\mathbb{C}$ of the form:
$$V = \oplus_{(p,q) \in \mathbb{Z}^2} V^{p,q}.$$

2. A $\mathbb{R}-$Hodge decomposition of V is a $\mathbb{C}$-decomposition with an automorphism σ of V such that $\sigma^2 = 1$ and $\sigma(V^{p,q}) = V^{q,p}$, for all p, q.

For example, let $V = H^i(C(\mathbb{C}), \mathbb{C})$. V then has a well-known $\mathbb{C}$-Hodge decomposition $V = \oplus_{p,q} V^{p,q}$, for example [GH94, Chapter 0, Section 7]. Let σ denote complex conjugation, this induces an $\mathbb{R}$-Hodge decomposition on V. In short, V has a k_v-Hodge

decomposition at each archimedean place v of k.

If a complex vector space V has a $\mathbb{C}$-Hodge decomposition, let $h(p,q) = \dim(V^{p,q})$, then

$$\Gamma(V,s) := \prod_{p,q} \Gamma(\mathbb{C}, s - \min\{p,q\})^{h(p,q)}.$$

Let V have a $\mathbb{R}$-Hodge decomposition, and let n be an integer. Put

$$V^{n,n} = V^{n,+} \oplus V^{n,-},$$

where

$$V^{n,+} = \{x \in V^{n,n} | \sigma(x) = (-1)^n x\},$$

$$V^{n,-} = \{x \in V^{n,n} | \sigma(x) = (-1)^{n+1} x\}.$$

Write

$$h(n,+) = \dim(V^{n,+}),$$

$$h(n,-) = \dim(V^{n,-}),$$

then

$$\Gamma(V,s) = \prod_{n} \Gamma(\mathbb{R}, s-n)^{h(n,+)} \Gamma(\mathbb{R}, s-n+1)^{h(n,-)} \prod_{p<q} \Gamma(\mathbb{C}, s-p)^{h(p,q)}.$$

In particular, if v is a real or complex place, we have defined the gamma factor

$$\Gamma_v(H^i(C), s) := \Gamma(H^i(C(\mathbb{C}), \mathbb{C}), s).$$

Definition 2.10. The Gamma factor for $H^i(C), i = 0, 1, 2$ is

$$\Gamma(H^i(C), s) = \prod_{v|\infty} \Gamma_v(H^i(C), s).$$

We can compute the Gamma factors explicitly.

Lemma 2.11. *Let C be a smooth projective curve over a number field k, and let r_1 (resp. r_2) denote the number of real (resp. conjugate pairs of complex) embeddings of k, and $n = r_1 + 2r_2$. If g is the genus of C, then*

$$\Gamma(H^i(C), s) = \begin{cases} \Gamma(\mathbb{R}, s)^{r_1} \Gamma(\mathbb{C}, s)^{r_2}, & i = 0 \\ \Gamma(\mathbb{C}, s)^{gn}, & i = 1 \\ \Gamma(\mathbb{R}, s-1)^{r_1} \Gamma(\mathbb{C}, s-1)^{r_2}, & i = 2. \end{cases}$$

Proof.

$$H^0(C(\mathbb{C}),\mathbb{C}) = H^{0,0}(C(\mathbb{C}),\mathbb{C}),$$

so $h^{0,0} = 1$ and

$$\Gamma_v(H^0(C),s) = \begin{cases} \Gamma(\mathbb{R},s), \text{ v real} \\ \Gamma(\mathbb{C},s), \text{ v complex} \end{cases}$$

Next,

$$H^1(C(\mathbb{C}),\mathbb{C}) = H^{0,1}(C(\mathbb{C}),\mathbb{C}) \oplus H^{1,0}(C(\mathbb{C}),\mathbb{C}),$$

with $h^{0,1} = h^{1,0} = g$, so

$$\Gamma_v(H^1(C),s) = \begin{cases} \Gamma(\mathbb{C},s)^g, \text{ v real} \\ \Gamma(\mathbb{C},s)^{2g}, \text{ v complex} \end{cases}$$

Finally,

$$\begin{aligned} H^2(C(\mathbb{C}),\mathbb{C}) &= H^{0,2}(C(\mathbb{C}),\mathbb{C}) \oplus H^{1,1}(C(\mathbb{C}),\mathbb{C}) \oplus H^{2,0}(C(\mathbb{C}),\mathbb{C}) \\ &= H^{1,1}(C(\mathbb{C}),\mathbb{C}). \end{aligned}$$

So, $h^{0,2} = h^{2,0} = 0$, $h^{1,1} = 1$ and

$$\Gamma_v(H^2(C),s) = \begin{cases} \Gamma(\mathbb{R},s-1), \text{ v real} \\ \Gamma(\mathbb{C},s-1), \text{ v complex} \end{cases}$$

□

Bringing together all of the above ingredients, we can define the completed Hasse–Weil L-functions.

Definition 2.12. Let X be a smooth projective variety over a number field k, and let $D = d(k/\mathbb{Q})$ denote the discriminant of k. For all integers $m \geq 0$, denote by $A(H^m(X))$ the integer $N(\mathfrak{f}) \cdot D^{B_m}$. On $\Re(s) > 1 + m/2$,the completed mth Hasse–Weil L-function is given by

$$\Lambda(H^m(X),s) = A(H^m(X))^{s/2} L(H^m(X),s)\Gamma(H^m(X),s).$$

Conjecture 1 states that this function extends to $s \in \mathbb{C}$ and admits the functional equation.

$$\Lambda(H^m(X),s) = \pm\Lambda(H^m(X),m+1-s).$$

In particular, if C is a smooth projective curve over a number field, we expect that

$$\Lambda(C,s) = \pm\Lambda(C,2-s),$$

where $\Lambda(C,s) = \Lambda(H^1(C),s)$.

2.1.2 Hasse–Weil Zeta Functions

Let C be a smooth, projective, geometrically connected curve[6]. In this thesis, the aim is to infer properties of $L(C,s)$ from study of $\zeta(\mathcal{S},s)$ for suitable choices of proper, regular models $\mathcal{S}$ of C. In this section we show that the analytic properties of $\zeta(\mathcal{S},s)$ depend only on the generic fibre C, in that they do not depend on the choice of proper, regular model $\mathcal{S} \to \mathrm{Spec}(\mathcal{O}_k)$.

The conductor of a model of a curve is slightly modified from that of the L-function of the curve, this is explained in [Sai88, Section 1], [LS00, Section 2], or [Blo87, Section 1] . For a nonarchimedean prime $\mathfrak{p}$ of k, we have the formula:

$$A_{\mathfrak{p}}(\mathcal{S}) = \chi(C) - \chi(\mathcal{S}_{\mathfrak{p}}) + \mathrm{sw}_{\mathfrak{p}}(\chi(C)),$$

where χ denotes the Euler characteristic and $\mathrm{sw}_{\mathfrak{p}}(\chi(C))$ is the Swan character of the virtual representation associated to the alternating sum $\chi(C)$ of the cohomology groups of C.

Example 2.13. If $\mathcal{S}$ is proper over $\mathrm{Spec}(\mathcal{O}_k)$, with smooth generic fibre, and the special fiber $\mathcal{S}_{\mathfrak{p}}$ over $\mathfrak{p} \in \mathrm{Spec}(\mathcal{O}_k)$ is a reduced normal crossings divisor on $\mathcal{S}$, then

$$A_{\mathfrak{p}}(\mathcal{S}) = -|\{\text{singular points of } \mathcal{S}_{\mathfrak{p}}\}|,$$

see [Sai88, Section 1].

Piecing together the local components, the conductor of $\mathcal{S}$ is

$$A(\mathcal{S}) = \prod_{\mathfrak{p}} N\mathfrak{p}^{A_{\mathfrak{p}}(\mathcal{S})}.$$

Notice that the discriminant of k no longer appears in the conductor of $\mathcal{S}$, this is explained by [Ser00, VI, §2].

The gamma factor $\Gamma(\mathcal{S},s)$ of $\mathcal{S}$ is the alternating product of the gamma factors for each cohomology group, over all archimedean places.

[6]This means that the curve becomes connected upon extension of the base field to its algebraic closure.

Lemma 2.14. *Let $\mathcal{S}$ be a proper, regular model of a smooth projective curve C of genus g, then*

$$\Gamma(\mathcal{S}, s) = \frac{2^{r_2}\pi^{r_1+r_2}}{(s-1)^{r_1+r_2}} \cdot \frac{1}{\Gamma(\mathbb{C}, s)^{(g-1)(r_1+2r_2)}}.$$

Proof. Recall the following basic property of the Gamma function for $s \in \mathbb{C}$:

$$\Gamma(s) = (s-1)\Gamma(s-1).$$

Combining this with the definitions of $\Gamma(\mathbb{R})$ and $\Gamma(\mathbb{C})$, it follows that, for all $s \in \mathbb{C}$:

$$\Gamma(\mathbb{C}, s) = \pi^{-1}(s-1)\Gamma(\mathbb{C}, s-1),$$

$$\Gamma(\mathbb{R}, s) = (2\pi)^{-1}(s-1)\Gamma(\mathbb{R}, s-1).$$

Also, we recall that:

$$\Gamma(\mathbb{C}, s) = \Gamma(\mathbb{R}, s)\Gamma(\mathbb{R}, s+1).$$

Using Lemma 2.11, we have:

$$\begin{aligned}
\Gamma(\mathcal{S}, s) &= \frac{\Gamma(\mathbb{R}, s)^{r_1}\Gamma(\mathbb{C}, s)^{r_2}\Gamma(\mathbb{R}, s-1)^{r_1}\Gamma(\mathbb{C}, s-1)^{r_2}}{\Gamma(\mathbb{C}, s)^{gr_1}\Gamma(\mathbb{C}, s)^{2gr_2}} \\
&= \frac{\Gamma(\mathbb{R}, s)^{r_1}\Gamma(\mathbb{R}, s-1)^{r_1}}{\Gamma(\mathbb{C}, s)^{gr_1}} \cdot \frac{\Gamma(\mathbb{C}, s)^{r_2}\Gamma(\mathbb{C}, s-1)^{r_2}}{\Gamma(\mathbb{C}, s)^{2gr_2}} \\
&= \frac{\Gamma(\mathbb{C}, s-1)^{r_1}}{\Gamma(\mathbb{C}, s)^{gr_1}} \cdot \frac{\Gamma(\mathbb{C}, s)^{r_2}\Gamma(\mathbb{C}, s-1)^{r_2}}{\Gamma(\mathbb{C}, s)^{2gr_2}} \\
&= \frac{\pi^{r_1}}{(s-1)^{r_1}} \cdot \frac{\Gamma(\mathbb{C}, s)^{r_1}}{\Gamma(\mathbb{C}, s)^{gr_1}} \cdot \frac{(2\pi)^{r_2}}{(s-1)^{r_2}} \cdot \frac{\Gamma(\mathbb{C}, s)^{2r_2}}{\Gamma(\mathbb{C}, s)^{2gr_2}} \\
&= \frac{2^{r_2}\pi^{r_1+r_2}}{(s-1)^{r_1+r_2}} \cdot \frac{1}{\Gamma(\mathbb{C}, s)^{(g-1)(r_1+2r_2)}}.
\end{aligned}$$

□

Example 2.15. If $C = E$ is an elliptic curve over a number field, then $g - 1 = 0$, and so the gamma factor is a simple rational function (over $\mathbb{R}$):

$$R(s) = \frac{2^{r_2}\pi^{r_1+r_2}}{(s-1)^{r_1+r_2}}.$$

Recall that, in passing from a curve to its model, the discriminant of the base field was accounted for in the definition of conductor of an arithmetic surface (more precisely, in the Swan character). A comparable thing happens for the gamma factor - in the case of elliptic curves, its transcendental part for H^1 agrees with those for H^0 and H^2, which concern only the base field. For higher genus curves, the dimension of H^1 is bigger, meaning the gamma factor still has a transcendental part, even after passing to a model.

Remark 2.16. More generally, the gamma factor of a model of an abelian variety A is a rational function. Indeed, the Euler characteristic of A is zero and so the following expression has the same number of Γ-functions on the numerator as it does on the denominator:

$$\frac{\prod_{i=0}^{d}\Gamma(H^{2i}(A),s)}{\prod_{i=0}^{d-1}\Gamma(H^{2i+1}(A),s)},$$

where d denotes the dimension of A. In fact, it is not hard to compute that the denominator of the gamma factor of A is an integral power of

$$(s-1)\dots(s-d).$$

Definition 2.17. The completed zeta function of an arithmetic surface $\mathcal{S}$ is

$$\xi(\mathcal{S},s) = A(\mathcal{S})^{s/2}\zeta(\mathcal{S},s)\Gamma(\mathcal{S},s).$$

At this juncture, we recall conjecture 2, which states that $\xi(\mathcal{S},s)$ extends to a meromorphic on $\mathbb{C}$, and satisfies the following functional equation:

$$\xi(\mathcal{S},s) = \pm\xi(\mathcal{S},2-s).$$

It is possible to be more precise about the sign of the above functional equation in terms of that of $L(C,s)$ - again we stress that we will not have anything useful to say about the sign of the functional equations in this thesis. Note that since the rational function $R(s) = \pm R(2-s)$ (depending on the parity of r_1+r_2), it can be neglected from questions concerning meromorphicity and functional equation of the zeta-function.

It is well understood that the L-function of a curve is intimately related to the zeta-function of each of its models. It is crucial that we are clear on this point as the upshot is the equivalence of the two main conjectures (1 and 2) for Hasse–Weil L-functions of curves over number fields and for zeta-functions of their models. Once this is established we will focus entirely on conjecture 2.

Let C be a smooth, projective, geometrically connected curve over a number field k and let $\mathcal{S} \to \mathrm{Spec}(\mathcal{O}_k)$ be a proper regular model of C. By this we mean that $\mathcal{S}$ is a regular scheme with a proper, flat morphism $\mathcal{S} \to \mathrm{Spec}(\mathcal{O}_k)$ and an isomorphism $\mathcal{S}_\eta \cong C$ of the generic fiber with the curve C.

When $\Re(s) > 2$, the Hasse–Weil zeta-function of C is defined to be:

$$\zeta(C,s) = \prod_{m=0}^{2} L(H^m(C),s)^{(-1)^m} = \frac{\zeta(k,s)\zeta(k,s-1)}{L(C,s)}.$$

This function depends only on the generic fiber of $\mathcal{S}$, which is isomorphic to C. The completion $\xi(C,s)$ of the Hasse–Weil zeta-function of C is given by

$$\xi(C,s) = \zeta(C,s)\Gamma(\mathcal{S},s)A(C)^{s/2},$$

where $\mathcal{S}$ is any model of C over $\mathrm{Spec}(\mathcal{O}_k)$, [Blo87, Prop. 1.1]. The gamma factor can also be described in terms of a complex in Betti cohomology, without referring to a model of C. We will now show the equivalence of 1 and 2. We begin by recalling the function $n_{\mathcal{S}}(s)$:

$$n_{\mathcal{S}}(s) = \frac{\zeta(\mathcal{S},s)}{\zeta(C,s)}.$$

The function $n_{\mathcal{S}}(s)$ is meromorphic, as a finite product of rational functions in q_i^{-s} (taken over the bad primes). Thus we see instantly that the meromorphic continuation of $\zeta(\mathcal{S},s)$ is equivalent to that of $\zeta(C,s)$. We want to be able to make this statement for the functional equation.

Remark 2.18. If C has a global minimal model, then $\zeta(C)$ is in fact the Hasse zeta function of this model, and $n_{\mathcal{S}}$ is a product of finitely many zeta functions of affine lines induced from the blowing-up process from $\mathcal{S}$ to an arbitrary proper regular model. This would prove the equivalence of the functional equations. But, of course, C will not usually have a global minimal model. We will instead present a cohomological argument, which does not require such a model to exist.

Let $\mathfrak{p}$ be a prime of k and let $M_{\mathfrak{p}}$ be the free abelian group spanned by the irreducible components of the geometric fiber $\mathcal{S}_{\mathfrak{p}} \times \overline{k(\mathfrak{p})}$. This is a $\widehat{\mathbb{Z}}$-module.

Lemma 2.19. *There is an exact sequence of $\widehat{\mathbb{Z}}$-modules:*

$$0 \to \mathbb{Q}_l(-1) \to M_{\mathfrak{p}} \otimes \mathbb{Q}_l(-1) \to H^2_{et}(\mathcal{S}_{\mathfrak{p}} \times \overline{k(\mathfrak{p})}, \mathbb{Q}_l) \to H^2(C_{\overline{k}}, \mathbb{Q}_l) \to 0.$$

Proof. [Blo87, Lemma 1.2]. See also [Del74]. □

Clearly, in studying $n_{\mathcal{S}}(s)$ our attention is focussed on the bad primes. In fact, the lemma implies that

$$n_{\mathcal{S}}(s) = \prod_{\mathfrak{p}\text{ bad}} \det(1 - f_{\mathfrak{p}} N\mathfrak{p}^{-s} | M_{\mathfrak{p}}(-1)/\mathbb{Q}_l(-1))^{-1}$$

A factor of the right hand side is denoted $n_{\mathcal{S},\mathfrak{p}}(s)$, and the local and global completions are

$$\mathfrak{n}_{\mathcal{S},\mathfrak{p}}(s) = N\mathfrak{p}^{s\cdot(A_{\mathfrak{p}}(\mathcal{S})-A_{\mathfrak{p}}(C))/2} \cdot n_{\mathcal{S},\mathfrak{p}}(s).$$

$$\mathfrak{n}_{\mathcal{S}}(s) = \prod_{\mathfrak{p}\text{ bad}} \mathfrak{n}_{\mathcal{S},\mathfrak{p}}(s).$$

We have a functional equation:

Proposition 2.20. *Let $\mathcal{S}$ be a proper regular model of the smooth projective curve C. If $\mathfrak{n}_{\mathcal{S}}$ is as above, then*

$$\mathfrak{n}_{\mathcal{S}}(s) = \pm\mathfrak{n}_{\mathcal{S}}(2-s).$$

Proof. We begin with the following observation:

$$1 - \dim M_{\mathfrak{p}} = A_{\mathfrak{p}}(\mathcal{S}) - A_{\mathfrak{p}}(C).$$

We break the $\widehat{\mathbb{Z}}$-action on $M_{\mathfrak{p}}$ into orbits and apply the fact that if $\widehat{\mathbb{Z}}$ acts on $M \cong \mathbb{Z}^{\oplus n}$ by cyclic permutation of a basis, then

$$\det(1 - f_{\mathfrak{p}} N\mathfrak{p}^{-s}|M(-1)) = 1 - N\mathfrak{p}^{n(1-s)}.$$

The result then follows from:

$$(N\mathfrak{p}^{-ns/2}) \cdot (1 - N\mathfrak{p}^{n(1-s)})^{-1} = -(N\mathfrak{p}^{-n(2-s)/2}) \cdot (1 - N\mathfrak{p}^{n(s-1)})^{-1}.$$

□

Remark 2.21. When $\mathcal{S} = \mathcal{E}$ is a proper, regular model of an elliptic curve, Fesenko uses the function $n_{\mathcal{E}}(s)$ in his original work on two-dimensional adelic analysis. The comparison to our definition comes from the equality $1 - \dim M_{\mathfrak{p}} = A_{\mathfrak{p}}(\mathcal{S}) - A_{\mathfrak{p}}(C)$ used in the proof above.

Corollary 2.22. *Let C be smooth, projective, geometrically connected curve over a number field k and let $\mathcal{S} \to Spec(\mathcal{O}_k)$ be proper regular model of C, then conjecture 1 is equivalent to conjecture 2. In particular,*

$$\Lambda(C,s) = \pm\Lambda(C,2-s) \Leftrightarrow \xi(\mathcal{S},s) = \pm\xi(\mathcal{S},2-s).$$

Proof. Observe that

$$\xi(\mathcal{S},s) = \mathfrak{n}_{\mathcal{S}}(s)\frac{\xi(k,s)\xi(k,s-1)}{\Lambda(C,s)}.$$

By the previous proposition, $\mathfrak{n}_{\mathcal{S}}(s)$ is invariant with respect to $s \mapsto 2-s$. The theorem then follows from the functional equation for Dedekind zeta functions. □

Although it will not be crucial in what follows, we observe that, if $\mathcal{X}$ is a proper, regular scheme of finite type over $\mathbb{Z}$ with smooth, projective, geometrically connected generic fibre X, we may also write the L-functions $L(\mathcal{X}, \chi, s)$ as a product over the fibres of $\mathcal{X}$, with the fibre factors being described as ratios characteristic polynomials in [Gro66], using the theory of étale cohomology with compact supports. We will not need their explicit form here, however, analogously to zeta functions, we may write

$$L(\mathcal{X}, \chi, s) = \prod_{\mathfrak{p} \in \mathrm{Spec}(\mathcal{O}_k)} L(\mathcal{X}_\mathfrak{p}, \chi, s),$$

$$L_\mathfrak{p}(\mathcal{X}_\mathfrak{p}, \chi, s) = \prod_{m=0}^{2\dim X} L_\mathfrak{p}(H^m(\mathcal{X}_\mathfrak{p}), \chi, s)^{(-1)^{m-1}}.$$

We define

$$L(H^i(X), \chi, s) = \prod_{\mathfrak{p} \in \mathrm{Spec}(\mathcal{O}_k)} L_\mathfrak{p}(H^i(\mathcal{X}_\mathfrak{p}), \chi, s).$$

We will use the following shorthand

$$L(X, \chi, s) = L(H^1(X), \chi, s).$$

Let $\mathcal{S} \to \mathrm{Spec}(\mathcal{O}_k)$ be a proper, regular model of a smooth, projective geometrically irreducible curve C over a number field k, and let G be a finite group acting on $\mathcal{S}$ such that $\mathcal{S}/G$ is a union of affine open sets which are stable by G. Let χ be a character of a representation $\rho : \mathrm{Gal}(K/k) \to \mathrm{GL}(V)$, where V is an m-dimensional vector space over $\mathbb{C}$. Let $c \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ denote the automorphism induced by complex conjugation, and let $V^\pm$ denote the eigenspaces corresponding to the eigen vectors ± 1 of $\tilde{\rho}(c)$, where $\tilde{\rho}$ is the induced representation from $\mathrm{Gal}(K/k)$ to $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. The Gamma factor for $L(\mathcal{S}, \chi, s)$ is then

$$\Gamma(\mathcal{S}, \chi, s) = \frac{1}{(\Gamma(\mathbb{R}, s)^{m^+}\Gamma(\mathbb{R}, s+1)^{m^-})^{(g-1)}}.$$

One also defines a conductor

$$A(\mathcal{S}, \chi) = \prod_{\mathfrak{p} \in \mathrm{Spec}(\mathcal{O}_k)} N\mathfrak{p}^{A_\mathfrak{p}(\mathcal{S}, \chi)},$$

which satisfies $A_\mathfrak{p}(\mathcal{S}, \chi) = 1$ for almost all $\mathfrak{p}$, such that the completed L-function

$$\Lambda(\mathcal{S}, \chi, s) = A(\mathcal{S}, \chi)^{s/2} L(\mathcal{S}, \chi, s)\Gamma(\mathcal{S}, \chi, s)$$

satisfies the functional equation

$$\Lambda(\mathcal{S}, \chi, s) = \varepsilon(\mathcal{S}, \chi)\Lambda(\mathcal{S}, \overline{\chi}, 2-s),$$

for some $\varepsilon(\mathcal{S}, \chi) \in \mathbb{C}$ of absolute value 1.

2.2 Mean-Periodicity

Mean-periodicity is an important part of modern harmonic analysis ([CB84], [BT80]), first introduced in 1935 [Del35]. The general theory can be found in [Kah59].
Harmonic analysis has long been established as an effective tool for the study of number theory, with recent research placing mean-periodicity at the heart of some of the basic open questions concerning the analytic properties of zeta functions of arithmetic schemes [Fes08], [Fes10], [FRS12], [Suz12].
In order to explain this, we will give some indication of what it means for a function to be mean-periodic. Mean-periodicity is a property of a function in a given topological space $\mathfrak{X}$, and its precise definition depends on the properties of $\mathfrak{X}$. Let a locally compact topological group G act continuously on $\mathfrak{X}$, it is possible to define the set of "translates" of a function $f \in \mathfrak{X}$ as

$$T(f) := \{g \cdot f : g \in G\}.$$

We say that f is mean-periodic if $T(f)$ is not dense in $\mathfrak{X}$. For example, let $\mathfrak{X}$ be the set of smooth functions on $\mathbb{R}$ and let G be the group $\mathbb{R}$, which acts on $\mathfrak{X}$ by $g \cdot f(x) = f(x-g)$. In this way we recover what it means to be periodic. When $\mathfrak{X}$ satisfies the Hahn-Banach theorem, the definition of mean-periodicity just given is equivalent to the existence of a nontrivial homogeneous convolution equation $f * f^* = 0$ for some $f^* \in \mathfrak{X}^*$. For example, if a smooth function $f : \mathbb{R} \to \mathbb{C}$ has period $a \in \mathbb{R}$, then it satisfies the convolution equation $f * (\delta_a - \delta_0) = 0$, where δ_x denotes the Dirac measure at x. When $\mathfrak{X}$ has the correct spectral properties, there is a notion of generalized Fourier series for f.
If V is an algebraic variety over a number field k, then, for each m, $L(H^m(V), s)$ is expected to be automorphic, that is, the L-function of a certain automorphic representation. A seemingly suitable replacement of this notion for the zeta function $\zeta(\mathcal{V}, s)$ of a model $\mathcal{V} \to \mathrm{Spec}(\mathcal{O}_k)$ is "mean-periodicity of the boundary function". In this thesis we will address two natural questions:

1. To what extent can mean-periodicity of zeta functions of arithmetic schemes be regarded as a manifestation of automorphicity of the L-functions of their generic fibres?

2. Can one use the two-dimensional geometry of $\mathcal{S}$ to address the mean-periodicity condition, independently of any automorphicity results?

The answer to either of these questions has implications for the other.
One pragmatic motivation for introducing mean-periodicity is that proving modularity results is hard work, and often requires restrictions to the algebraic variety and the base number field. If we can understand mean-periodicity as an aspect of the duality of two-dimensional adeles, which are functorial with respect to the base field, we will address some of these problems. Moreover, writing in 1997, Langlands expressed some slightly unsatisfactory aspects of relying of the identification with automorphic L-functions to verify meromorphic continuation and functional equation [Lan97, Section 6]. His key point concerned special values of motivic L-functions: an automorphic representation has to have special properties in order to be motivic in nature, whereas the continuation of automorphic L-functions is completely uniform and does not consider such properties. With mean-periodicity, one can obtain the most simple analytic properties of Hasse–Weil L-functions without first solving the far more difficult problem of automorphicity, and, with some more work, one could hope to use mean-periodicity as a stepping stone to the deep automorphicity results, after the application of an appropriate converse theorem.

2.2.1 Definitions

We stress again that when dealing with mean-periodicity it is very important to fix the space in which a particular function is considered. Different function spaces will provide different flavours of results. When considering convolution equations, which we will do in 3.1 and 3.2, we choose the strong Schwartz space, following the inspiring work of Suzuki [Suz12]. Moreover, this space appears elsewhere in the spectral theory of zeta functions [Mey05a], and (weak) tempered distributions have long been connected with the fundamental properties of Hecke L-Functions [Wei56], [Mey05b]. On the other hand, spectral synthesis holds in the space of smooth functions of exponential growth [Pla11], so that in this space we have a theory of generalized Fourier expansions. Such expansions for the boundary term introduced below provide some comparisons to the theory of modular forms, see 3.3. We briefly mention that the spaces $\mathscr{C}(\mathbb{R})$ of continuous functions, or $\mathscr{C}^{\infty}(\mathbb{R})$ of smooth functions, is not appropriate for the theory of zeta functions [FRS12, Remark 5.12]. The same statement is true for their multiplicative analogues.

So, we begin by defining certain function spaces and the class of mean-periodic functions there. Recall that the exponential and logarithm maps provide topological isomorphisms between the locally compact abelian groups $\mathbb{R}$ and $\mathbb{R}_+^\times$, this will be used frequently to avoid repetition of definitions.

Strong Schwartz Functions

For all pairs of positive integers (m, n) define a seminorm on smooth functions $f : \mathbb{R} \to \mathbb{C}$ as follows

$$|f|_{m,n} = \sup_{x \in \mathbb{R}} |x^m f^{(n)}(x)|.$$

The Schwartz space $S(\mathbb{R})$ on the additive group $\mathbb{R}$ is the set of all smooth complex valued functions f such that $|f|_{m,n} < \infty$, for all positive integers m and n. The set $\{| \,|_{m,n} : m, n \in \mathbb{N}\}$ is a family of seminorms on $S(\mathbb{R})$ which induce a topology for which $S(\mathbb{R})$ is a Fréchet space over $\mathbb{C}$.

The Schwartz space on $\mathbb{R}_+^\times$, along with its topology can be defined via the homeomorphism

$$S(\mathbb{R}) \to S(\mathbb{R}_+^\times)$$

$$f(t) \mapsto f(-\log(x)),$$

where $x = e^{-t}$.

Definition 2.23. Strong Schwartz space on $\mathbb{R}_+^\times$ is the following set of functions:

$$\mathbf{S}(\mathbb{R}_+^\times) := \bigcap_{\beta \in \mathbb{R}} \{f : \mathbb{R}_+^\times \to \mathbb{C} : [x \mapsto x^{-\beta} f(x)] \in S(\mathbb{R}_+^\times)\},$$

with the topology given by the family of seminorms

$$||f||_{m,n} = \sup_{x \in \mathbb{R}_+^\times} |x^m f^{(n)}(x)|.$$

$\mathbf{S}(\mathbb{R})$ is defined by the homeomorphism

$$f(x) \to f(e^{-t}).$$

$\mathbb{R}_+^\times$ acts on $\mathbf{S}(\mathbb{R}_+^\times)$ by

$$\forall x \in \mathbb{R}_+^\times, \tau_x^\times f(y) := f(y/x).$$

Dual to the strong Schwartz space on $\mathbb{R}_+^\times$, we have the space $\mathbf{S}(\mathbb{R}_+^\times)^*$. When we endow the dual space with the weak $*$-topology, this is the space of weak-tempered distributions.

Let ϕ be a weak-tempered distribution, and let f be a strong Schwartz function. We have a pairing

$$< \ , \ >: \mathbf{S}(\mathbb{R}_+^\times) \times \mathbf{S}(\mathbb{R}_+^\times)^* \to \mathbb{C}$$

$$< f, \phi >= \phi(f).$$

The convolution $f * \phi$ is defined by

$$\forall x \in \mathbb{R}_+^\times, (f * \phi)(x) =< \tau_x^\times \check{f}, \phi >,$$

where

$$\check{f}(x) = f(x^{-1}).$$

Explicitly, the convolution is given by the following formula:

$$(f * \phi)(x) = \int_0^\infty f(x/y)\phi(y)\frac{dy}{y}.$$

We now give an abstract definition of mean-periodicity, which is valid in $\mathbf{S}(\mathbb{R}_+^\times)$

Definition 2.24. Let X be a locally convex, separated topological $\mathbb{C}$-vector space, with topological dual X^*. Assume that the Hahn-Banach theorem holds in X, and X is equipped with an involution:

$$\check{} : X \to X$$

$$f \mapsto \check{f}.$$

Let G be a locally compact topological abelian group with a continuous representation:

$$\tau : G \to \mathrm{End}(X)$$

$$g \mapsto \tau_g.$$

For $f \in X$, consider the following topological space:

$$\mathcal{T}(f) = \overline{\mathrm{Span}_{\mathbb{C}}\{\tau_g(f) : g \in G\}},$$

where the overline denotes topological closure. $f \in X$ is X-mean-periodic if either of the following two equivalent conditions holds:

1. $\mathcal{T}(f) \neq X$,

2. $\exists \psi \in X^* \backslash \{0\}$ such that $f * \psi = 0$,

where, for $\psi \in X^*$, the convolution $f * \psi : G \to \mathbb{C}$ is given by

$$(f * \psi)(g) := < \tau_g \check{f}, \psi >,$$

where $< \ , \ >$ denotes the natural pairing between a topological space and its dual.

In light of these definitions, we will introduce the following spaces

$$\mathcal{T}(f) = \overline{\mathrm{Span}_{\mathbb{C}}(\{\tau_a^{\times} f : a \in \mathbb{R}_+^{\times}\})},$$

$$\mathcal{T}(h)^{\perp} = \{\tau \in \mathbf{S}(\mathbb{R}_+^{\times})^* : g * \tau = 0, \forall g \in \mathcal{T}(h)\}.$$

Smooth Functions of Exponential/Polynomial Growth

We will also work with functions satisfying certain growth conditions:

Definition 2.25. The space $\mathscr{C}^{\infty}_{\mathrm{exp}}(\mathbb{R})$ of smooth functions on $\mathbb{R}$ with at most exponential growth at $\pm\infty$ is the topological space of smooth functions on $\mathbb{R}$ such that

$$\forall n \in \mathbb{Z}_+, \exists m \in \mathbb{Z}_+, f^{(n)}(x) = O(\exp(m|x|)) \text{ as } x \to \pm\infty.$$

This space is an inductive limit of Fréchet spaces $(F_m)_{m \geq 1}$, where F_m is the space of smooth functions on $\mathbb{R}$ such that for each positive integer n, $f^{(n)}(x) = O(\exp(m|x|))$ as $x \to \pm\infty$. There is a topology on each F_m which is induced by the family of seminorms

$$\|f\|_{m,n} = \sup_{x \in \mathbb{R}} |f^{(n)}(x) \exp(-m|x|)|.$$

The space $\mathscr{C}^{\infty}_{\mathrm{poly}}(\mathbb{R}_+^{\times})$ of smooth functions from $\mathbb{R}_+^{\times}$ to $\mathbb{C}$ which have at most polynomial growth at 0^+ and $+\infty$ is defined as those smooth functions $\mathbb{R}_+^{\times}$ such that

$$\forall n \in \mathbb{Z}_+, \exists m \in \mathbb{Z}, f^{(n)}(t) = O(t^m) \text{ as } x \to 0^+, +\infty,$$

and its topology is such that the bijection

$$\mathscr{C}^{\infty}_{\mathrm{exp}}(\mathbb{R}) \to \mathscr{C}^{\infty}_{\mathrm{poly}}(\mathbb{R}_+^{\times})$$

$$f(t) \mapsto f(-\log(x)),$$

becomes a homeomorphism.

Dual to these spaces, with respect to the weak $*$-topology, we have the spaces of distributions of over eponetial/polynomial decay.

As proved in [Pla11], the advantage of these spaces is they admit spectral synthesis in the sense of definition 2.26 below.

Let $G = \mathbb{R}$ (resp. $\mathbb{R}_+^\times$) and let X be a complex vector space of functions on G. We will assume that there exists an open $\Omega \subset \mathbb{C}$ such that for all $P(T) \in \mathbb{C}[T]$ and any $\lambda \in \Omega$ the exponential polynomials:

$$\begin{cases} P(t)e^{\lambda t}, \ G = \mathbb{R} \\ x^\lambda P(\log(x)), \ G = \mathbb{R}_+^\times, \end{cases}$$

belongs to X.

Definition 2.26. With the conditions above, we will say that spectral synthesis holds in X if, for all $f \in X$ such that $\mathcal{T}(f) \neq X$,

$$\mathcal{T}(f) = \begin{cases} \overline{\mathrm{Span}\{P(t)e^{\lambda t} \in \mathcal{T}(f), \lambda \in \Omega\}}, & G = \mathbb{R}, \\ \overline{\mathrm{Span}\{x^\lambda P(\log(x)) \in \mathcal{T}(f), \lambda \in \Omega\}}, & G = \mathbb{R}_+^\times. \end{cases}$$

The following result is standard in the theory of mean-periodic functions:

Lemma 2.27. *When X has spectral synthesis, $f \in X$ is X-mean-periodic if f is a limit of a sum of exponential polynomials in $\mathcal{T}(f)$ with respect to the topology of X.*

This is clearly an analogue of Fourier series of periodic functions.

It is known that there are continuous injections

$$\mathscr{C}^\infty_{\mathrm{poly}}(\mathbb{R}_+^\times) \hookrightarrow \mathbf{S}(\mathbb{R}_+^\times)^*,$$

$$\mathbf{S}(\mathbb{R}_+^\times) \hookrightarrow \mathscr{C}^\infty_{\mathrm{poly}}(\mathbb{R}_+^\times)^*.$$

2.2.2 Connection to Zeta Functions

We now work towards stating the mean-periodicity correspondence, which is a major theme throughout this thesis, providing a bridge between zeta functions of arithmetic schemes and mean-periodic functions in appropriate functional spaces. Throughout we will fix $\mathfrak{X}$ to be either the strong Schwartz space or the space of continuous functions of polynomial growth on $\mathbb{R}_+^\times$.

Motivation

Often, special analytic properties of Mellin transforms can be inferred from the mean-periodicity of related functions. For example, consider a real-valued function

$$f : \mathbb{R}_+^\times \to \mathbb{R},$$

such that for all $A > 0$,

$$f(x) = O(x^{-A}), \text{ as } x \to +\infty,$$

and for some $A > 0$,

$$f(x) = O(x^{A}), \text{ as } x \to 0^+,$$

then its Mellin transform defines a holomorphic function in some half plane $\mathrm{Re}(s) \gg 0$:

$$M(f)(s) := \int_0^{+\infty} f(x) x^s \frac{dx}{x}.$$

One can ask when $M(f)$ admits a meromorphic continuation and a functional equation of the following form:

$$M(f)(s) = \varepsilon M(f)(1-s),$$

where $\varepsilon = \pm 1$. Define

$$h_{f,\varepsilon}(x) = f(x) - \varepsilon x^{-1} f(x^{-1}),$$

then we see that

$$M(f)(s) = \int_1^{+\infty} f(x) x^s \frac{dx}{x} + \varepsilon \int_1^{+\infty} f(x) x^{1-s} \frac{dx}{x} + \int_0^1 h_{f,\varepsilon}(x) x^s \frac{dx}{x}.$$

To avoid cumbersome notation, introduce

$$\varphi_{f,\varepsilon}(s) := \int_1^{+\infty} f(x) x^s \frac{dx}{x} + \varepsilon \int_1^{+\infty} f(x) x^{1-s} \frac{dx}{x},$$

$$\omega_{f,\varepsilon}(s) := \int_0^1 h_{f,\varepsilon}(x) x^s \frac{dx}{x}.$$

Due to the first assumption on f, $\varphi_{f,\varepsilon}$ is an entire function satisfying

$$\varphi_{f,\varepsilon} = \varepsilon \varphi_{f,\varepsilon}(1-s),$$

so, the meromorphic continuation and functional equation of $M(f)$ is equivalent to that of $\omega_{f,\varepsilon}$. We proceed by noting that $\omega_{f,\varepsilon}$ is in fact the Laplace transform of $h_{f,\varepsilon}(e^{-t})$:

$$\omega_{f,\varepsilon}(s) = \int_0^{\infty} h_{f,\varepsilon}(e^{-t}) e^{-st} dt.$$

General theory of Laplace transforms says that if $h_{f,\varepsilon}$ is $\mathfrak{X}$-mean-periodic, then $\omega_{f,\varepsilon}$ admits meromorphic continuation to $\mathbb{C}$, given by the Mellin-Carleman transform of $h_{f,\varepsilon}$. Moreover, in this case we have the functional equation

$$MC(h_{f,\varepsilon})(s) = \varepsilon MC(h_{f,\varepsilon})(-s),$$

which is equivalent to

$$\omega_{f,\varepsilon}(s) = \varepsilon\omega_{f,\varepsilon}(1-s).$$

Loosely speaking, the above outline will be applied to the inverse Mellin transform of

$$Z(\mathcal{S},\{k_i\},s)^m = \xi(\mathcal{S},s)^m \prod_{i=1}^{n} \xi(k_i,s/2)^m,$$

where m,n are positive integers, $\mathcal{S}$ is a proper, regular model of a smooth projective curve C over a number field k, and, for each i, k_i is a finite extension of k. The meromorphic continuation and functional equation (up to sign) of $Z(\mathcal{S},\{k_i\},s)^m$ is equivalent to that of $\zeta(\mathcal{S},s)$. If k_i is the function field $k(y_i)$ of a horizontal curve y_i on $\mathcal{S}$, we will also use the notation $Z(\mathcal{S},\{y_i\},s)^m$. The case where $m=2$ is of special interest in later chapters, in which we will provide a higher dimensional adelic interpretation.

Abstract Mean-Periodicity Correspondence

We begin by abstracting the expected analytic properties of zeta functions - this is broadly comparable to the notion of the Selberg class of L-functions. Let $\mathfrak{Z}$ be a complex valued function defined in some half plane $\Re(s) > \sigma_1$, which has a decomposition

$$\mathfrak{Z}(s) = \frac{\mathscr{L}_1(s)}{\mathscr{L}_2(s)}.$$

Definition 2.28. We will say that $\mathfrak{Z}$ is of "expected analytic shape" if, for $i = 1,2$, the following hold:

1. $\mathscr{L}_i(s)$ is an absolutely convergent Dirichlet series in $\Re(s) > \sigma_1$ and has meromorphic continuation to $\mathbb{C}$.

2. There exist $r_i \geq 1$ such that, for $1 \leq j \leq r_i$, there are $\lambda_{i,j} > 0$ with $\mu_{i,j} \in \mathbb{C}$ such that $\Re(\mu_{i,j}) > \sigma_1\lambda_{i,j}$ and there are $q_i > 0$ such that the function

$$\widehat{\mathscr{L}_i}(s) := \gamma_i(s)\mathscr{L}_i(s),$$

where

$$\gamma_i(s) = q_i^{s/2} \prod_{j=1}^{r_1} \Gamma(\lambda_{i,j}s + \mu_{i,j}),$$

satisfies

$$\widehat{\mathscr{L}_i} = \epsilon_i \widehat{\mathscr{L}_i}(1-s),$$

for some $\varepsilon_i \in \mathbb{C}$ such that $|\varepsilon_i| = 1$.

3. There exists a polynomial $P(s)$ such that $P(s)\widehat{\mathscr{L}_i}(s)$ is an entire function of order one.

4. The logarithmic derivative of $\mathscr{L}_2(s)$ is an absolutely convergent Dirichlet series in some right half plane $\Re(s) > \sigma_2 \geq \sigma_1$.

For example, $\zeta(s)$ is of expected analytic shape if each of its Hasse–Weil L-factors satisfy the conjectures in [Ser69] - for details see [FRS12, Remark 5.20].
When $\mathfrak{Z}$ is of expected analytic shape, one can define its completion as

$$\widehat{\mathfrak{Z}}(s) = \frac{\widehat{\mathscr{L}_1}(s)}{\widehat{\mathscr{L}_2}(s)},$$

and the inverse Mellin transform

$$f(\mathfrak{Z}, \{k_i\}, x) := \frac{1}{2\pi i} \int_{(c)} \widehat{\mathfrak{Z}}(s) \prod_{i=1}^{m} \xi(k_i, s) x^{-s} dx,$$

for $c > 1/2 + \omega$, $m \in \mathbb{N}$ and finite extensions k_i/k.

Definition 2.29. Let $\varepsilon = \frac{\varepsilon_1}{\varepsilon_2}$. The boundary function associated to $(\mathfrak{Z}, \{k_i\})$ is

$$h(\mathfrak{Z}, \{k_i\}) : \mathbb{R}_+^\times \to \mathbb{C}$$

$$h(\mathfrak{Z}, \{k_i\}, x) := f(\mathfrak{Z}, \{k_i\}, x) - \varepsilon x^{-1} f(\mathfrak{Z}, \{k_i\}, x^{-1}),$$

In additive language, we have

$$H(\mathfrak{Z}, \{k_i\}) : \mathbb{R} \to \mathbb{C}$$

$$H(\mathfrak{Z}, \{k_i\}, t) = h(\mathfrak{Z}, \{k_i\}, e^{-t}).$$

The terminology "boundary function" pertains to the idea that these functions should be integrals over topological boundaries of subsets of the related adelic space - we will not address this in detail until chapter 5.

Let $\gamma(s) = \frac{\gamma_1(s)}{\gamma_2(s)}$, where the numerator and denominator are quantities from the definition of "expected analytic shape". For analytic reasons, it will be useful to control the behaviour of $\gamma(s)$.

Definition 2.30. Let $\mathcal{S}$ be a scheme of dimension d and let $\{k_i\}$ be a finite set of number fields, and denote by $\xi(k_i, s)$ the completed Dedekind zeta function of k_i. We will say that $\gamma(s)\prod_{i=1}^{n}\xi(k_i, s)$ has "expected uniform bound" if for all $a, b \in \mathbb{R}$ such that $a \leq b$ and all $\Re(s) \in [a, b]$,

$$\gamma(s)\prod_{i=1}^{n}\xi(k_i, s) \ll |\Im(s)|^{-1-\delta}.$$

for some $\delta > 0$ and for all $|t| \geq t_0$.

The mean-periodicity correspondence is encapsulated in the following theorem:

Theorem 2.31. *Let $\mathfrak{Z}(s)$ be as above.*

1. *If $\mathfrak{Z}(s)$ is of expected analytic shape then there exists an $n_{\mathfrak{Z}} \in \mathbb{Z}$ such that, if $\{k_i\}_{1\leq i\leq n}$ is a finite set of (not necessarily distinct) number fields for any $n \geq n_{\mathfrak{Z}}$, then $h(\mathfrak{Z}, \{k_i\}, x)$ is $\mathscr{C}^{\infty}_{poly}(\mathbb{R}^{\times}_{+})$- and $\mathbf{S}(\mathbb{R}^{\times}_{+})$-mean-periodic. Similarly for $H(\mathfrak{Z}, \{k_i\}, t)$.*

2. *If, for some $n \in \mathbb{Z}$, we have number fields k_i such that $\gamma(s)\prod_{i=0}^{n}\xi(k_i, s)$ has expected uniform bound, and $h(\mathfrak{Z}, \{k_i\}, x)$ is $\mathscr{C}^{\infty}_{poly}(\mathbb{R}^{\times}_{+})$– or $\mathbf{S}(\mathbb{R}^{\times}_{+})$-mean-periodic, then $\mathfrak{Z}(s)$ extends to a meromorphic function on $\mathbb{C}$ and*

$$\mathfrak{Z}(s) = \pm\mathfrak{Z}(1-s).$$

Similarly for $H(\mathfrak{Z}, \{k_i\}, t)$.

Proof. This is an immediate consequence of [FRS12, Theorem 5.18]. □

We will not spend any time trying to optimise this statement in terms of k_i and $n_{\mathfrak{Z}}$.
By rescaling the argument of the zeta functions of arithmetic schemes, we can apply the above result to the problem of their meromorphic continuation and functional equation. We will spell this out in below. Let $\mathscr{S}(\mathcal{O}_k)$ denote the set of arithmetic schemes over the ring of integers $\mathcal{O}_k$ in a number field k whose zeta functions have the expected analytic properties, and the $\mathrm{MP}(\mathfrak{X})$ denote the mean-periodic functions on $\mathbb{R}^{\times}_{+}$ in an appropriate function space. We have a family of maps

$$\mathscr{H}(\{k_i\}_{i=1}^{n\gg 0}) : \mathscr{S}(\mathcal{O}_k) \to \mathrm{MP}(\mathfrak{X})$$

$$\mathcal{S} \mapsto h(\mathcal{S}, \{k_i\}_{i=1}^{n \geq n_S}) : \mathbb{R}_+^\times \to \mathbb{C}.$$

An open problem is to find the image of the maps $\mathscr{H}$.

Mean-Periodicity for Arithmetic Surfaces

In practice, we will only work with proper, regular models $\mathcal{S}$ of smooth, projective geometrically connected algebraic curves C over a number fields k. We aim to apply the above abstract mean-periodicity correspondence, and so begin by factorising the Hasse–Weil zeta function $\zeta(C, s)$ as an (alternating) product of L-functions:

$$\zeta(C, s) = \frac{\zeta(k, s)\zeta(k, s-1)}{L(C, s)}.$$

In fact, we will be interested in functions of the form

$$Z(C, \{k_i\}, s) = \zeta(C, 2s) \prod_{i=1}^{n} \zeta(k_i, s).$$

A basic property of the geometry of arithmetic surfaces is that irreducible curves $y \subset \mathcal{S}$ are either "vertical" or "horizontal" [Liu02, Definition 8.3.5]. This means that y is either an irreducible component of a fibre $\mathcal{S}_\mathfrak{p}$ for $\mathfrak{p} \in \mathrm{Spec}(\mathcal{O}_k)$ or the closure of a closed point on the generic fibre. Geometrically, $\zeta(\mathcal{S}, s)$ is the vertical (or "fibral") part of $Z(\mathcal{S}, \{k_i\}, s)$ and the finite product $\prod_{i=1}^{n} \zeta(k_i, s/2)$ corresponds to the function fields k_i of finitely many horizontal curves. Note that the points need not be rational. Indeed, in general we may have a shortage of such points by Falting's theorem.

We will rescale $\zeta(\mathcal{S}, s)$ so that the expected functional equation of conjecture 2 is with respect to $s \mapsto 1 - s$. The resulting function of s will also be denoted $\zeta(\mathcal{S}, s)$ and converges on some half plane $\Re(s) > \sigma_1$.

Theorem 2.31 reduces to the following.

Theorem 2.32. *On the locally compact multiplicative group $\mathbb{R}_+^\times$ of positive real numbers, let $\mathfrak{X}$ denote the strong Schwartz space or space of smooth functions of polynomial growth. Let C/k be a smooth, projective, geometrically connected curve over a number field k.*

1. *Assume that $L(C, s)$ admits meromorphic continuation to $\mathbb{C}$ with the expected functional equation $\Lambda(C, s) = \varepsilon\Lambda(C, 1-s)$, the logarithmic derivative of $L(C, s)$ is an absolutely convergent Dirichlet series in the right half plane $\Re(s) > 1$ and there exists a polynomial $P(s)$ such that $P(s)L(C, s)$ is an entire function on $\mathbb{C}$ of order 1. Then, for each choice*

of proper, regular model $\mathcal{S} \to Spec(\mathcal{O}_k)$, there exists $m_{\mathcal{S}} \in \mathbb{N}$ such that the following function is $\mathfrak{X}$-mean-periodic for every $m \geq m_{\mathcal{S}}$:

$$h_{\mathcal{S},m}(x) := f_{\mathcal{S},\{k_i\}}(x) - \varepsilon x^{-1} f_{\mathcal{S},\{k_i\}}(x^{-1}),$$

where, for $i = 1, \ldots, m$, k_i is a finite extension of k, and

$$f_{\mathcal{S},\{k_i\}}(x) = \frac{1}{2\pi i} \int_{(c)} \xi(\mathcal{S}, 2s) x^{-s} (\prod_{i=1}^{m} \xi(k_i, s)) ds.$$

2. *Conversely, suppose that there exists $m_{\mathcal{S}} \in \mathbb{N}$ such that, for some set $\{k_i\}$ of $m_{\mathcal{S}}$ finite extensions of k, the function $h_{\mathcal{S},\{k_i\}}$ is $\mathfrak{X}$-mean-periodic and that*

$$\Gamma(\mathcal{S}, 2s) \prod \xi(k_i, s) \ll |t|^{-1-\delta},$$

then $\xi(\mathcal{S}, s)$ admits meromorphic continuation and satisfies the functional equation

$$\xi(\mathcal{S}, s) = \xi(\mathcal{S}, dim(\mathcal{S}) - s).$$

We are lead to make the following conjecture:

Conjecture 2.33. *Let $\mathcal{S}$ be a proper regular model of a smooth, projective, geometrically connected curve. There exists $n_{\mathcal{S}}$ such that, for any finite set of $n \geq n_{\mathcal{S}}$ number fields, the associated (additive) boundary function is mean-periodic in $\mathscr{C}^{\infty}_{exp}(\mathbb{R})$ or $\mathbf{S}(\mathbb{R})$. Similarly for the (multiplicative) boundary function on $\mathbb{R}^{\times}_{+}$.*

Unlike automorphicity conjectures, this "mean-periodicity hypothesis" is expected to be essentially commutative in its nature. One can hope that adelic analysis will provide a proof of the mean-periodicity of the boundary function without assuming any automorphicity conjectures. The first steps are the subject of chapters 4 and 5.

Remark 2.34. There is a completely analogous theory for curves over function fields. In this case it is known that mean-periodicity would follow from rationality of the boundary term. For simplicity, we will only address characteristic zero in the main body of this thesis.

The abstract mean-periodicity correspondence also implies a result for the twisted zeta functions $L(\mathcal{S}, \chi, s)$. Let f and g be two functions whose Mellin transforms exist and let ε be a constant complex number, then, we may write

$$M(f)(s) = \int_0^{\infty} f(x) x^s \frac{dx}{x}$$

$$= \int_1^\infty f(x)x^s \frac{dx}{x} + \varepsilon \int_1^\infty g(x)x^{1-s}\frac{dx}{x} + \int_0^1 [f(x) - \varepsilon x^{-1} g(x^{-1})]x^s \frac{dx}{x}.$$

We will apply this to the inverse Mellin transforms

$$f_\chi(x) = M^{-1}(L(\mathcal{S}, \chi, s))(x),$$

$$f_{\overline{\chi}}(x) = M^{-1}(L(\mathcal{S}, \chi, s))(x),$$

where χ is the character of a finite group acting on $\mathcal{S}$. In this case

$$M(f_\chi)(s) = \int_0^\infty f_\chi(x)x^s \frac{dx}{x}$$

$$= \int_1^\infty f_\chi(x)x^s \frac{dx}{x} + \varepsilon(\chi) \int_1^\infty f_{\overline{\chi}}(x)x^{1-s}\frac{dx}{x} + \int_0^1 [f_\chi(x) - \varepsilon(\chi)x^{-1} f_{\overline{\chi}}(x^{-1})]x^s \frac{dx}{x}.$$

We will denote the sum of the first two terms by $\phi_{f,\varepsilon}(s,\chi)$ and the third term by $\omega_{f,\varepsilon}(s,\chi)$.

$\phi_{f,\varepsilon}$ defines a holomorphic function, and we have that

$$\phi_{f,\varepsilon}(1-s, \overline{\chi}) = \int_1^\infty f_{\overline{\chi}}(x)x^{1-s}\frac{dx}{x} + \varepsilon(\overline{\chi}) \int_1^\infty f_\chi(x)x^s \frac{dx}{x}.$$

Therefore, since $|\varepsilon(\chi)| = 1$, we have the functional equation

$$\phi_{f,\varepsilon}(s,\chi) = \varepsilon(\chi)\phi_{f,\varepsilon}(1-s, \overline{\chi}).$$

So, the functional equation of $L(\mathcal{S}, \chi, s)$ is equivalent to the functional equation

$$\omega_{f,\varepsilon}(s,\chi) = \varepsilon(\chi)\omega(1-s, \overline{\chi}).$$

Introduce the notation

$$h_{f,\varepsilon}(x,\chi) = f_\chi(x) - \varepsilon(\chi)x^{-1} f_{\overline{\chi}}(x^{-1})$$

By the change of variables $x = e^{-t}$, we see that $\omega_{f,\varepsilon}$ is the Laplace transform of $H_{f,\varepsilon}(t) = h_{f,\varepsilon}(e^{-t})$, ie.

$$\omega_{f,\varepsilon}(s) = \int_0^\infty H_{f,\varepsilon}(t)e^{-st}dt.$$

As has been discussed before, the meromorphic continuation and functional equations of such Laplace transforms is related to mean-periodicity.

Observe that $h_{f,\varepsilon}(x, \overline{\chi})$ satisfies the functional equation:

$$h_{f,\varepsilon}(x^{-1}, \overline{\chi}) = \varepsilon(\overline{\chi})x h_{f,\varepsilon}(x,\chi).$$

With this observation, we can state a twisted consequence of [FRS12, Theorem 5.18]. From now on, let K/k be a finite Galois extension of number fields. $\mathcal{S} \to \mathrm{Spec}(\mathcal{O}_K)$ will be a proper, regular model of a smooth, projective, geometrically connected curve C over K, for some finite extension of number fields K/k. χ will be a character of $\mathrm{Gal}(K/k)$.

Theorem 2.35. *On the locally compact multiplicative group $\mathbb{R}_+^\times$ of positive real numbers, let $\mathfrak{X}$ denote the strong Schwartz space or space of smooth functions of polynomial growth. Let C/k be a smooth projective curve over a number field k, and assume that a group G acts on C and let χ denote a character of G.*

1. *Assume that $L(\mathcal{S},\chi,s)$ admits meromorphic continuation to $\mathbb{C}$ with the expected functional equation $\Lambda(\mathcal{S},\chi,s) = \varepsilon(\mathcal{S},\chi)\Lambda(\mathcal{S},\overline{\chi},2-s)$, the logarithmic derivative of the twisted L-function $L(\mathcal{S},\chi,s)$ is an absolutely convergent Dirichlet series in the right half plane $\Re(s) > 2$ and there exists a polynomial $P(s)$ such that $P(s)L(\mathcal{S},\chi,s)$ is an entire function on $\mathbb{C}$ of order 1. Then, for each choice of proper, regular model $\mathcal{S} \to \mathrm{Spec}(\mathcal{O}_k)$, there exists $m_{\mathcal{S}} \in \mathbb{N}$ such that the following function is $\mathfrak{X}$-mean-periodic for every $m \geq m_{\mathcal{S}}$:*
$$h_{\mathcal{S},m}(x,\chi) := f_{\mathcal{S},\{k_i\}}(x,\chi) - \varepsilon(\chi)x^{-1}f_{\mathcal{S},\{k_i\}}(x^{-1},\overline{\chi}),$$
where and $\{k_i/k\}$ is a set of m finite extensions of number fields, and
$$f_{\mathcal{S},\{k_i\}}(x,\chi) = \frac{1}{2\pi i}\int_{(c)} \Lambda(\mathcal{S},\chi,2s)x^{-s}(\prod_{i=1}^{m}\Lambda(k_i,\chi,s))ds.$$

2. *Conversely, suppose that there exists $m_{\mathcal{S}} \in \mathbb{N}$ such that, for some set $\{k_i\}$ of $m_{\mathcal{S}}$ finite extensions of k, the function $h_{\mathcal{S},\{k_i\}}(\chi,s)$ is $\mathfrak{X}$-mean-periodic and that*
$$\Gamma(\mathcal{S},\chi,2s)\prod\Lambda(k_i,\chi,s) \ll |t|^{-1-\delta},$$
then $\Lambda(\mathcal{S},\chi,s)$ admits meromorphic continuation and satisfies the functional equation
$$\Lambda(\mathcal{S},\chi,s) = \Lambda(\mathcal{S},\chi,2-s).$$

Proof. In the notation of [FRS12], write $Z_1(s) = L(\mathcal{S},\chi,s)$ and $Z_2(s) = \varepsilon(\chi)L(\mathcal{S},\overline{\chi},s)$. We then obtain $h_{12}(x) = h_{f,\varepsilon}(x,\chi)$ and we can apply [FRS12, Theorems 4.2, 4.7, 5.18]. □

CHAPTER 3

Automorphicity and Mean-Periodicty

In this chapter we begin exploring the interplay between mean-periodicity properties of zeta functions of arithmetic schemes and automorphicity properties of L-functions of their generic fibre. We will directly show the orthogonality of certain invariants of GL_{2g}-automorphic representations to the spaces $\mathcal{T}(h(\mathcal{S},\{k_i\}))$ of the last chapter, using a technique emulating the Rankin-Selberg method in which the boundary function plays the role of an Eisenstein series. We make use of the spectral interpretation of zeros of automorphic L-fucntions [Con99], [Sou01], [Dei01]. In section 3.3 we will substantiate the analogy to the Rankin-Selberg method by using Whittaker expansions of Eisenstein series to compute generalized Fourier expansions of the boundary functions. Before we begin we remark that, whilst comparisons with the Langlands program are interesting, perhaps the greater significance of mean-periodicity lies in the fact that it is open to proof through two-dimensional adelic analysis as explored in the following chapters. One might hope that this could be combined with converse theorems to deduce the full automorphicity results, although this thesis will not get that far.

3.1 Hecke Characters

In this section we explore a rather concrete manifestation of the general connection between mean-periodicity of arithmetic schemes and automorphic properties of their generic fibre using techniques dating back to Tate's thesis [Tat50]. A more technical

investigation of this connection will be undertaken in the next section.

In some sense, the CM case to which this section applies, being completely abelian, is far from remarkable - the analytic and automorphic properties having being deduced decades ago [Deu57], [Shi71], [Ser98]. However, the technique used will be essentially the same in the non-commutative case, and it is beneficial to see it stripped away of the technical difficulties. Moreover, the underlying result for Hecke characters (which we do not explicitly state) can be viewed as the first example of the twisted mean-periodicity considered in chapter 7.

Let F be a number field. Recall that $F^\times$ diagonally embeds into $\mathbb{A}_F^\times$. By a Hecke character, we mean a multiplicative character of $\mathbb{A}_F^\times$, trivial on $F^\times$.

Let E be an elliptic curve over F. There is always an inclusion of the ring $\mathbb{Z}$ of integers into the endomorphism ring $\mathrm{End}(E)$ defined by

$$n \mapsto [n] : P \mapsto nP = \overbrace{P + \cdots + P}^{n \text{ times}}.$$

In most cases, this inclusion is actually an equality. In general, it is easy to deduce that $\mathrm{End}(E)$ is isomorphic to $\mathbb{Z}$ or an order $\mathcal{O}$ in a quadratic imaginary field K [Sil94, Chapter 2]. When $\mathrm{End}(E) = \mathcal{O}$, we say that E has complex multiplication ("CM") by $\mathcal{O}$ and we have an isomorphism of $\mathbb{Q}$-algebras $K \cong \mathrm{End}(E) \otimes_{\mathbb{Z}} \mathbb{Q}$, motivating the terminology "CM by K".

Let E be an elliptic curve over a number field F with CM by K[1], which we assume, for now, to be a subfield of F. By the first main theorem of CM, [Sil94, Chapter 2, Theorem 8.2], for all $x \in \mathbb{A}_F^\times$ there is a unique $\alpha_x \in K^\times$ such that $\alpha_x \mathcal{O}_K$ is the principal fractional ideal generated by $s = N_K^F x \in \mathbb{A}_K^\times$ and, for all fractional ideals $\mathfrak{a} \subset K$ and analytic isomorphisms

$$f : \mathbb{C}/\mathfrak{a} \to E(\mathbb{C}),$$

the following diagram commutes

$$\begin{tikzcd} K/\mathfrak{a} \arrow[r, "\alpha_x s^{-1}"] \arrow[d, "f"'] & K/\mathfrak{a} \arrow[d, "f"] \\ E(F^{\mathrm{ab}}) \arrow[r, "r_F(x)"] & E(F^{\mathrm{ab}}) \end{tikzcd},$$

[1]This is an example of a motive defined over one field, with coefficients in another, as included in the statement of conjecture 4.

where $r_F : \mathbb{A}_F^\times \to \mathrm{Gal}(F^{\mathrm{ab}}/F)$ is the reciprocity map. Thus we have a homomorphism

$$\alpha_{E/F} : \mathbb{A}_F^\times \to \mathbb{C}^\times$$

$$\alpha_{E/F}(x) = \alpha_x$$

The Hecke character associated to the CM elliptic curve E/F is then defined as

$$\psi_{E/F} : \mathbb{A}_F^\times \to \mathbb{C}^\times$$

$$\psi_{E/F}(x) = \alpha_{E/F}(x) N_K^F(x^{-1})_\infty,$$

where, for all $\beta \in \mathbb{A}_K^\times$,

$$N_K^F(\beta)_\infty = \prod_{\tau : F \hookrightarrow \mathbb{C}} \beta^\tau,$$

and the product is taken over embeddings τ such that $\tau|_K = 1$.

The L-function of an elliptic curve E over a number field F with CM by the quadratic imaginary field K is given by

$$L(E,s) = \begin{cases} L(s, \psi_{E/F}) L(s, \overline{\psi_{E/F}}), & \text{if } K \subseteq F, \\ L(s, \psi_{E/F'}), & \text{otherwise,} \end{cases}$$

where $F' = FK$ in the second case.

On the locally compact group $\mathbb{A}_F$ we have a Haar measure inducing surjective "module" map on the idele group $\mathbb{A}_F^\times$:

$$|\,| : \mathbb{A}_F^\times \to \mathbb{R}_+^\times,$$

which may be chosen so that $|x| = 1$, for all $x \in F^\times$.

For a positive real number $x \in \mathbb{R}_+^\times$, we introduce the following notation:

$$(\mathbb{A}_F^\times)_x := \{\alpha \in \mathbb{A}_F^\times : |\alpha| = x\}.$$

Let $S(\mathbb{A}_F)$ be the adelic Schwartz-Bruhat space, that is, factorizable functions $f : \mathbb{A}_F \to \mathbb{C}$ whose archimedean components are Schwartz and whose non-archimedean components are locally constant, compactly supported and almost always the characteristic function of the maximal compact subring.

For a Hecke character $\chi : \mathbb{A}_F^\times \to \mathbb{C}^\times$, let $S(\chi)$ denote the set of triples $\{(\chi, f, s) : f \in S(\mathbb{A}_F), s \in \mathbb{C}\}$. We can define a map:

$$Z : S(\chi) \to \mathbf{S}(\mathbb{R}_+^\times)$$

$$Z(\chi, f, s)(x) = \int_{(\mathbb{A}_F^\times)_x} f(\alpha)\chi(\alpha)|\alpha|^s d\alpha.$$

This integral converges absolutely and its image is contained in $\mathbf{S}(\mathbb{R}_+^\times)$ because for any integer N there exists a positive constant C such that, for all $x \in \mathbb{R}_+^\times$,

$$|Z(\chi, f, s)(x)| \leq Cx^{-N}.$$

We will denote the image of $Z(\chi, f, s)$ by $\mathcal{V}_\chi$.

There is a well known relationship between $Z(\chi, f, s)(x)$ and the Hecke L-function $L(s, \chi)$, namely, there exists a nonzero holomorphic function $\Psi_{f,\chi}(s)$ depending on f and χ such that

$$\int_0^\infty Z(\chi, f, s)(x)\frac{dx}{x} = \int_{\mathbb{A}_F^\times} f(\alpha)\chi(\alpha)|\alpha|^s \frac{d\alpha}{|\alpha|} = \Psi_{f,\chi}(s)L(s, \chi).$$

Let E be an elliptic curve defined over F. If $\mathcal{E} \to \mathrm{Spec}(\mathcal{O}_F)$ is a proper, regular model of E, then we expect that the square of its completed zeta function has meromorphic continuation to $\mathbb{C}$ and the complex variable s can be rescaled to give the functional equation

$$\xi(\mathcal{E}, s)^2 = \xi(\mathcal{E}, 1-s)^2.$$

As explained in the previous chapter, we can construct a family of "boundary functions" $h_{\mathcal{E},\{k_i\}} \in \mathbf{S}(\mathbb{R}_+^\times)$ associated to $\xi(\mathcal{E}, s)^2$. Amongst them, we will take

$$h_{\mathcal{E}} : \mathbb{R}_+^\times \to \mathbb{C}$$

$$h_{\mathcal{E}}(x) = f_{\mathcal{E}}(x) - x^{-1}f_{\mathcal{E}}(x^{-1}),$$

where

$$f_{\mathcal{E}}(x) = \frac{1}{2\pi i}\int_{(c)} \xi(F, \frac{s}{2} + \frac{1}{4})^2 A(\mathcal{E})^{-s-\frac{1}{2}}\zeta(\mathcal{E}, s + \frac{1}{2})^2 x^{-s} ds,$$

Notice the absence of the gamma factor for $\mathcal{E}$, which, in the case where $g = 1$, is a rational function invariant with respect to $s \mapsto 1 - s$.

Let $\mathcal{T}(h_{\mathcal{E}})$ and $\mathcal{T}(h_{\mathcal{E}})^\perp$ be as in 2.2.1. Clearly, if $\mathcal{T}(h_{\mathcal{E}})^\perp$ is nonempty, then $h_{\mathcal{E}}$ is mean-periodic. In this chapter, given a CM curve E over F, we will construct a nonempty subset of $\mathcal{T}(h_{\mathcal{E}})^\perp$ from $\mathcal{V}_\chi$, where $\chi = \psi_{E/F}$ is the associated Hecke character.

Let $A(\mathcal{E})$ and $A(E)$ denote the conductors of $\mathcal{E}$ and E respectively. Recall that the quotient $\mathfrak{n}_{\mathcal{E}}(s) = \xi(\mathcal{E}, s)/\xi(E, s)$ is meromorphic on $\mathbb{C}$ and is invariant with respect to $s \mapsto 2 - s$. The gamma factor $\Gamma(E, s)$ of the smooth projective curve E is given explicitly by

$$\Gamma(E, s) = \Gamma(\mathbb{C}, s)^n,$$

where $n = [F : \mathbb{Q}]$. With these notations, we define

$$w_0(x) = \frac{1}{2\pi i}\int_{(c)} \Gamma(E, \frac{s}{4})^2 \frac{(A(\mathcal{E})/A(E))^s}{n_{\mathcal{E}}(s)^2} s^4(s-2)^4(s-1)^2 x^{-s} ds,$$

If $\psi_{E/F}$ denotes the Hecke character associated to the CM elliptic curve E over F and $F' = FK$, then define

$$\mathcal{W}_{\psi_{E/F}} = \begin{cases} w_0 * \mathcal{V}_{\psi_{E/F}} * \mathcal{V}_{\psi_{E/F}} * \mathcal{V}_{\overline{\psi}_{E/F}} * \mathcal{V}_{\overline{\psi}_{E/F}}, & K \subset F \\ w_0 * \mathcal{V}_{\psi_{E/F'}} * \mathcal{V}_{\psi_{E/F'}}, & \text{otherwise.} \end{cases}$$

where we use the following shorthand:

$$\mathcal{U} * \mathcal{V} := \overline{\mathrm{Span}_{\mathbb{C}}\{u * v : u \in \mathcal{U}, v \in \mathcal{V}\}}.$$

We can now state and prove the main theorem of this section. The fact that the boundary term is mean-periodic could already have been established from the analytic properties it implies, however the explicit connection to Hecke characters is new.

Theorem 3.1. *Let E/F be an elliptic curve with CM by K, then each proper, regular model $\mathcal{E} \to Spec(\mathcal{O}_F)$ of E satisfies the mean-periodicity hypothesis. Indeed,*

$$\mathcal{W}_{\psi_{E/F}} \subset \mathcal{T}(h_{\mathcal{E}})^{\perp},$$

where $\psi_{E/F}$ is the Hecke character associated to E and $\mathcal{W}_{\psi_{E/F}}$ is as above.

The proof uses two lemmas to be found below.

Proof. We use spectral interpretation of zeros of Hecke L-functions, as found in [Con99]. First assume that $K \subset F$, we will put $\psi := \psi_{E/F}$. Let w be any function in $\mathcal{W}_{\psi}$ - we will show that its convolution with the boundary function is 0.
If λ is a pole of $\xi(F, \frac{s}{2} + \frac{1}{4})^2 A(\mathcal{E})^{-s-\frac{1}{2}} \zeta(\mathcal{E}, s + \frac{1}{2})$ of order m_λ, for $0 \leq k \leq m_\lambda - 1$, let $f_{\lambda,k}$ be

$$f_{\lambda,k}(x) = x^{-\lambda}(\log(x)^k).$$

By the following lemma, we have

$$h_{\mathcal{E}}(x) = \lim_{T\to\infty} \sum_{\mathrm{Im}(\lambda)\leq T} \sum_{m=0}^{m_\lambda - 1} C_{m+1}(\lambda) \frac{(-1)^m}{m!} f_{\lambda,m}(x).$$

So, we have, for $w \in \mathcal{W}_{\psi}$,

$$w * h_{\mathcal{E}}(x) = \lim_{T\to\infty} \sum_{\mathrm{Im}(\lambda)\leq T} \sum_{m=0}^{m_\lambda - 1} C_{m+1}(\lambda) \frac{(-1)^m}{m!} f_{\lambda,m}(x) * h_{\mathcal{E}}(x).$$

$$w * f_{\lambda,k}(x) = \int_0^\infty w(y) f_{\lambda,k}(x/y) \frac{dx}{x}.$$

It is elementary to deduce that

$$w * f_{\lambda,k}(x) = \sum_{i=1}^{k} (-1)^j \binom{k}{j} x^{-\lambda} (\log(x))^{k-j} \int_0^\infty w(y) y^\lambda (\log(y))^j \frac{dy}{y}.$$

Next we observe that

$$\int_0^\infty w(y) y^\lambda (\log(y))^j \frac{dy}{y} = \frac{d^j}{d\lambda^j} \int_0^\infty w(y) y^\lambda \frac{dy}{y}.$$

For $i = 1, \ldots, 4$, there are $f_i \in S(\mathbb{A}_F)$ such that

$$w = w_0 * Z_{\psi,f_1} * Z_{\psi,f_2} * Z_{\overline{\psi},f_3} * Z_{\overline{\psi},f_4},$$

and so

$$\int_0^\infty w(y) y^\lambda \frac{dy}{y} = \int_0^\infty w_0(y) y^\lambda \frac{dy}{y} \prod_{i=1}^{2} \int_0^\infty Z_{\psi,f_i}(y) \frac{dy}{y} \prod_{i=3}^{4} \int_0^\infty Z_{\overline{\psi},f_i}(y) \frac{dy}{y}.$$

Recall that, for each $i = 1, \ldots, 4$ there exists a holomorphic function h_i such that

$$\int_0^\infty Z_{\chi,f_i} \frac{dy}{y} = h_i(s,\chi) L(s,\chi).$$

Finally, observe that

$$\int_0^\infty w_0(y) y^\lambda \frac{dy}{y} = \Gamma(\mathcal{E}, \frac{\lambda}{2})^2 \lambda^4 (\lambda-2)^4 (\lambda-1)^2 (A(\mathcal{E})/A(E))^\lambda n_{\mathcal{E}}(\lambda)^{-2}.$$

Altogether, the integral evaluates as

$$\Gamma(\mathcal{E}, \frac{\lambda}{2})^2 \lambda^4 (\lambda-2)^4 (\lambda-1)^2 (A(\mathcal{E})/A(E))^\lambda n_{\mathcal{E}}(\lambda)^{-2} L(E,\lambda)^2 \prod_{i=1}^{4} h_{f_i}(s,\psi).$$

The result then follows from the classification of poles λ given in the lemma below. The case $K \not\subset F$ is completely similar. □

Lemma 3.2.

$$h_{\mathcal{E}}(x) = \lim_{T\to\infty} \sum_{Im(\lambda)\le T} \sum_{m=0}^{m_\lambda - 1} C_{m+1}(\lambda) \frac{(-1)^m}{m!} x^{-\lambda} (\log(x)^m),$$

where λ runs over the poles of $\xi(F, \frac{s}{2}+\frac{1}{4})^2 A(\mathcal{E})^{-s-\frac{1}{2}} \zeta(\mathcal{E}, s+\frac{1}{2})^2$, m_λ denotes the multiplicity of λ and $C_m(\lambda)$ is the coefficient of $(s-\lambda)^{-m}$ in the principal part of

$$\xi(F, \frac{s}{2}+\frac{1}{4})^2 A(\mathcal{E})^{-s-\frac{1}{2}} \zeta(\mathcal{E}, s+\frac{1}{2})^2$$

at $s = \lambda$.

Proof. [FRS12, Section 5]. □

We also require a classification of the poles, which is easily verified.

Lemma 3.3. *The poles $\tau = \lambda + \frac{1}{2}$ that appear in the expansion above are classified as follows:*

1. *0 or 2, in which case, $m_\lambda = 4$,*
2. *Zero of $\Lambda(E,s)$, different from 1, with $n_{\mathcal{E}}(\lambda)^{-1} \neq 0$. In this case m_λ is non-negative and at most the multiplicity M of the zero of $\Lambda(E,s)^2$ at $s = \lambda$,*
3. *A zero of $\Lambda(E,s)$, different from 1, with $n_{\mathcal{E}}(\lambda)^{-1} = 0$. In this case $-2 \leq m_\lambda - 2 \leq M$.*
4. *A zero of $n_{\mathcal{E}}(\lambda)^{-1}$, different from 1, with $\Lambda(E,\lambda) \neq 0$. In this case, $m_\lambda = 2$,*
5. *1, in which case, $-2-2J \leq m_\lambda - 2 - 2J \leq M$.*

In the more general framework of the next section we will develop some inclusions in the opposite direction. We will also study an equivalent adelic construction.

3.2 $\mathbf{GL}_n$

Let C be a smooth projective curve of genus g over a number field k. The étale cohomology groups $H^1_{\text{ét}}(C, \mathbb{Q}_l)$ are the l-adic realisations of a rank $n = 2g$ motive. The motivic L-function is what we mean by "the" L-function of C:

$$L(C,s) = L(H^1(C), s).$$

Let π be a (cuspidal) automorphic representation of $\mathrm{GL}_n(\mathbb{A}_k)$. Following [GJ72], or via Satake parameters, one may define an L-function $L(s,\pi)$ and $\varepsilon(s,\pi,\psi)$, which depends on a choice of additive character ψ, such that $L(s,\pi)$ has analytic continuation to $\mathbb{C}$ and satisfies the functional equation

$$L(s,\pi) = \varepsilon(s,\pi,\psi)L(s,\widetilde{\pi}),$$

where $\widetilde{\pi}$ denotes the contragredient representation to π.

In this section, we will assume conjecture 5, that there exists an algebraic cuspidal automorphic representation π of $\mathrm{GL}_{2g}(\mathbb{A}_k)$ such that

$$L(C,s) = L(\pi, s - \frac{1}{2}).$$

Let G denote GL_{2g} and let

$$G(\mathcal{O}) := \prod_{\mathfrak{p}\in\mathrm{Spec}(\mathcal{O}_k)} G(\mathcal{O}_{\mathfrak{p}}) \subset G(\mathbb{A}_k),$$

so that the maximal compact subgroup of $G(\mathbb{A})$ is

$$\mathcal{K} = G(\mathcal{O}) \times \overbrace{\mathrm{SO}(2)\times\cdots\times\mathrm{SO}(2)}^{r_1} \times \overbrace{\mathrm{SU}(2)\times\cdots\times\mathrm{SU}(2)}^{r_2},$$

where r_1 (resp, r_2) denotes the number of real (resp. complex) places of k.

$G(\mathbb{A}_k)$ operates by right translation on $L^2(G(k)\backslash G(\mathbb{A}_k), 1)$, the space of all functions ϕ on $G(k)\backslash G(\mathbb{A}_k)$ such that, for $a \in \mathcal{K}$ and $g \in G(\mathbb{A}_k)$:

$$\phi(ag) = \phi(g),$$

$$\int_{G(k)\mathcal{K}\backslash G(\mathbb{A}_k)} |\phi(g)|^2 dg < \infty.$$

In the context of the twisted mean-periodicity correspondence of theorem 2.36, we could equally work with an arbitrary character ω of $\mathrm{GL}_1(\mathbb{A}_k)$, and the space

$$L^2(G(k)\backslash G(\mathbb{A}_k), \omega).$$

$L_0^2(G(k)\backslash G(\mathbb{A}_k))$ denotes the subspace of cuspidal elements, ie. those functions in $L^2(G(k)\backslash G(\mathbb{A}_k))$ such that for all proper k-parabolic subgroups H of G, the integral

$$\int_{U(k)\backslash U(\mathbb{A}_k)} \phi(ug)du = 0,$$

where U is the unipotent radical of H.

An admissible matrix coefficient

$$f_\pi : G \to \mathbb{C}$$

of the representation π of $G(\mathbb{A}_k)$ on $L_0^2(G(k)\backslash G(\mathbb{A}_k))$ has the form:

$$f_\pi(g) = \int_{G(\mathcal{O})G(k)\backslash G(\mathbb{A}_k)} \phi(hg)\widetilde{\phi}(h)dh,$$

for cuspidal automorphic forms $\phi, \tilde{\phi}$, the sense of [GJ72, Chapter II, §10]. The module $|\,| : G(\mathbb{A}_k) \to \mathbb{R}_+^\times$ is defined by:

$$|g| = |\det(g)|_{\mathbb{A}_k},$$

where the module on the right hand side is the usual adelic module.

Let Φ be a Schwartz-Bruhat function on $M_n(\mathbb{A}_k)$. According to [GJ72, § 13], we have an integral representation for the automorphic L-function

$$\int_{G(\mathbb{A}_F)} \Phi(g) f_\pi(g)|g|^s dg = L(\pi, s - \frac{1}{2}),$$

which can be analytically continued to $s \in \mathbb{C}$.

For $x \in \mathbb{R}_+^\times$, define G_x to be the set $\{g \in G(\mathbb{A}_k) : |g| = x\}$. For $\Phi \in S(M_n(\mathbb{A}_k))$ and an admissible matrix coefficient f_π, the following integral converges absolutely

$$\mathfrak{Z}(\Phi, f_\pi) : \mathbb{R}_+^\times \to \mathbb{C}$$

$$x \mapsto \int_{G_x} \Phi(g) f_\pi(g) dg$$

Moreover, for any positive integer N, there exists a positive constant C such that, for all $x \in \mathbb{R}_+^\times$, $|\mathfrak{Z}(\phi, f_\pi)(x)| \leq Cx^{-N}$ and, if $\widehat{\Phi}$ denotes the Fourier transform of Φ and $\check{f}_\pi(g) = f_\pi(g^{-1})$, then we have the functional equation

$$\mathfrak{Z}(\Phi, f_\pi)(x) = x^{-2}\mathfrak{Z}(\widehat{\Phi}, \check{f}_\pi)(x^{-1}).$$

Let $S(\pi)$ denote the set of pairs (Φ, f_π), where $\Phi \in S(M_n(\mathbb{A}))$ is a Schwartz-Bruhat function and f_π is an admissible coefficient of π. The properties above verify that we have defined a map

$$\mathfrak{Z} : S(\pi) \to \mathbf{S}(\mathbb{R}_+^\times)$$

$$(\Phi, f_\pi) \mapsto \mathfrak{Z}(\Phi, f_\pi).$$

We will denote the image of $\mathfrak{Z}$ by $\mathcal{V}_\pi \subset \mathbf{S}(\mathbb{R}_+^\times)$.

Let $\mathcal{S} \to \mathrm{Spec}(\mathcal{O}_k)$ be a proper regular model of C. In 2.1.2, we defined a conductor $\mathcal{A}_\mathcal{S}$ and a function $n_\mathcal{S}(s) = \zeta(\mathcal{S}, s)/\zeta(C, s)$. We will let $A(C) = A(H^1(C))$ denote the conductor of $L(C, s)$. Let

$$w_0(x) = \frac{1}{2\pi i}\int_{(c)} \Gamma(C, s/4)^2 \frac{(A(\mathcal{S})/A(C))^s}{(n_\mathcal{S}(s))^2} s^4(s-2)^4(s-1)^2 ds,$$

The following space gives a family of convolutors of the boundary term $h(\mathcal{S})$ associated to $\zeta(\mathcal{S}, s)^2\zeta(k, s/2)^2$:

$$\mathcal{W}_\pi = w_0 * \mathcal{V}_\pi * \mathcal{V}_\pi := \overline{\mathrm{Span}_\mathbb{C}\{w_0 * v_1 * v_2 : v_i \in \mathcal{V}_\pi\}}.$$

More precisely,

Theorem 3.4. *Let $C, \mathcal{S}$ and π be as above and let $h_\mathcal{S}$ denote the boundary term associated to $Z(\mathcal{S}, \{k, k\})$, then*

$$\mathcal{W}_\pi \subset \mathcal{T}(h_\mathcal{S})^\perp,$$

where $\mathcal{T}(h_\mathcal{S})^\perp$ denotes the set of convolutors $\{g \in \mathbf{S}(\mathbb{R}_+^\times) : g * \tau = 0, \forall \tau \in \mathcal{T}(h_\mathcal{S})\}$.

Proof. With the formulation as above, this is no different to the previous section. □

Remark 3.5. In order to recover the result for Hecke characters one must induce the associated representation of $GL_1(\mathbb{A}_k) \times GL_1(\mathbb{A}_k)$ to GL_2.

Dually, one can also prove that the set of translates to the boundary function are orthogonal the the modified image of the zeta integrals.

Theorem 3.6. *Let $\mathcal{S}$ denote a proper regular model of a smooth projective curve C over a number field k, and let π be as above, then*

$$\mathcal{T}(h_{\mathcal{S}}) \subset \mathcal{W}_{\pi}^{\perp}.$$

It is not much more work to show analogously that the twisted mean-periodicity correspondence in theorem 2.36 can be viewed a constraint on admissible matrix coefficients of twisted automorphic representations on the space $L^2(G(k)\backslash G(\mathbb{A}_k), \omega)$.
We will now use an adelic construction to prove a similar result to theorem 3.4. To do so, we introduce the following subset of the Schwartz-Bruhat space $S(M_n(\mathbb{A}_k))$:

$$S(M_n(\mathbb{A}_k))_0 = \{\Phi \in S(M_n(\mathbb{A}_k)) : \Phi(x) = \widehat{\Phi}(x) = 0, \text{ for } x \in M(\mathbb{A}_k) - G(\mathbb{A}_k)\}.$$

For $\Phi \in S(M_n(\mathbb{A}_k))_0$, following Deitmar [Dei01], we define the functions

$$E(\Phi), \widehat{E}(\Phi) : G(\mathbb{A}_k) \to \mathbb{C}$$

$$E(\Phi)(g) = |g|^{n/2} \sum_{\gamma \in M_n(\mathbb{A}_k)} \Phi(\gamma g)$$

$$\widehat{E}(\Phi)(g) = |g|^{n/2} \sum_{\gamma \in M_n(\mathbb{A}_k)} \Phi(g\gamma).$$

By [Dei01, theorem 3.1, lemma 3.5], when $\phi_\pi = \otimes_v \phi_{\pi,v}$ is such that $\phi_{\pi,v}$ is a normalized class one vector for almost all places, we have that there is an entire function F_{Φ,ϕ_π} of $s \in \mathbb{C}$ such that

$$\int_{G(k)\backslash G(\mathbb{A}_k)} E(\Phi)(g)\phi_\pi(g)|g|^{s-n/2} dg = L(\pi, s - \frac{1-n}{2}) F_{\Phi,\phi_\pi}(s)$$

We will denote the analogue of $\mathfrak{Z}$ above by $\mathfrak{X}$, which we construct following [Suz12, 3.2]. For $\Phi \in S(M_n(\mathbb{A}_k))_0$, define

$$\mathfrak{X}(\Phi), \widehat{\mathfrak{X}}(\Phi) \in \mathbf{S}(G(k)\backslash G(\mathbb{A}_k)) = \bigcap_{\beta \in \mathbb{R}} |\,|^{\beta} S(G(k)\backslash G(\mathbb{A}_k))$$

$$\mathfrak{X}(\Phi)(g) = \frac{E(\Phi)(g)}{|g|^{n/2}} = \sum_{\gamma \in M_n(\mathbb{A}_k)} \Phi(\gamma g)$$

$$\widehat{\mathfrak{X}}(\Phi)(g) = \frac{\widehat{E}(\Phi)(g)}{|g|^{n/2}} \sum_{\gamma \in M_n(\mathbb{A}_k)} \Phi(g\gamma).$$

The fact that these functions are in the strong Schwartz space follows from [Dei01, Proposition 3.1]. We have a split exact sequence

$$1 \to G(\mathbb{A}_k)_1 \to G(\mathbb{A}_k) \to \mathbb{R}_+^\times \to 1,$$

where $G(\mathbb{A}_k)_1$ denotes the kernel of the module map $g \mapsto |g|$, and the arrows are the natural ones. We may fix a splitting κ such that we have as isomorphism

$$(\iota, \kappa) : G(\mathbb{A}_k)_1 \times \mathbb{R}_+^\times \to G(\mathbb{A}_k).$$

Define $\mathrm{R} = \mathrm{image}(\kappa) \subset G(\mathbb{A}_k)$. With ϕ_π as above, define

$$\mathcal{W}_\pi = \mathrm{span}_{\mathbb{C}}\{(w_0 \circ \kappa) * (\mathfrak{X}(\phi_1) \cdot \phi_\pi) * (\mathfrak{X}(\phi_2) \cdot \phi_\pi) : \phi_i \in S(M_n(\mathbb{A}_k))\},$$

where $\mathfrak{X}(\phi) \cdot \phi_\pi(x) = \mathfrak{X}(\phi)(x)\phi_\pi(x)$ and $*$ is the convolution via the right regular representation. The orthogonal complement of $\mathcal{W}_\pi$ with respect to convolution is denoted $\mathcal{W}_\pi^\perp$:

$$\mathcal{W}_\pi^\perp := \{\eta \in \mathbf{S}(G(k)\backslash G(\mathbb{A}_k))^* : w * \eta = 0, \forall w \in \mathcal{W}_\pi\}.$$

For all $\eta \in \mathbf{S}(G(k)\backslash G(\mathbb{A}_k))^*$, define

$$\mathcal{T}(\eta) = \overline{\mathrm{span}_{\mathbb{C}}\{R^*(g)\eta : g \in G(\mathbb{A}_k)\}},$$

where R is the right regular representation of $G(\mathbb{A}_k)$ on $S(G(k)\backslash G(\mathbb{A}_k))$ and R^* is its transpose with respect to the pairing of $S(G(k)\backslash G(\mathbb{A}_k))$ and $S(G(k)\backslash G(\mathbb{A}_k))^*$.

Theorem 3.7. *Let C and $\mathcal{S}$ be as above and let $h_{\mathcal{S}}$ denote the boundary term associated to $Z(\mathcal{S}, \{k, k\})$, then*

$$\mathcal{T}(h_{\mathcal{S}} \circ \kappa) \subset \mathcal{W}_\pi^\perp.$$

Proof. The proof is no different to [Suz12, Theorem 3.2]. □

3.3 Whittaker Expansions

We claim that the mean-periodic properties of boundary functions are closely related to automorphicity properties of certain representations. One might hope that the generalized Fourier series of $\mathscr{C}^{\infty}_{\text{poly}}(\mathbb{R}^{\times}_{+})$-mean-periodic functions manifest this fact, by giving some discrete spectrum. We will entertain this thought before moving onto the adelic interpretation of mean-periodicity.

Let C be a smooth projective curve over a number field k and consider the following product of completed zeta functions:

$$Z_C(s) = \xi(k,s)^2 \xi(C,2s)^2,$$

where $\xi(C,s)$ denotes the (completed) Hasse–Weil zeta function of C. In this section we will assume that $L(C,s)$ admits meromorphic continuation and satisfies the expected functional equation. Consequently

$$Z_C(s) = \varepsilon Z_C(1-s),$$

for some $\varepsilon = \pm 1$. The boundary function $h_C : \mathbb{R}^{\times}_{+} \to \mathbb{C}$ associated to Z_C is

$$h_C(x) = f_C(x) - \varepsilon x^{-1} f_C(x^{-1}),$$

where f_C is the inverse Mellin transform

$$f_C(s) = \frac{1}{2\pi i} \int_{(c)} Z_C(s) x^{-s} ds.$$

We can express $\xi(C,s)$ as a ratio of completed L-functions

$$\xi(C,s) = \frac{\xi(k,s)\xi(k,s-1)}{\Lambda(C,s)},$$

so that, according to [FRS12, Theorem 4.7], $h_C(s)$ satisfies the convolution equation

$$v * h_C = 0,$$

where v is the inverse Mellin transform of the denominator

$$v(x) = \frac{1}{2\pi i} \int_{(c)} \Lambda(C,2s) P(s) x^{-s} dx,$$

where $P(x)$ is the following polynomial

$$P(s) = s(s-1)$$

The boundary function $h_C(x)$ has a generalized Fourier expansion and is $\mathscr{C}^{\infty}_{\text{poly}}$-mean-periodic [FRS12, Theorem 4.2]. In fact, if we write

$$Z_C(s) = \gamma(s)D(s),$$

where $D(s)$ is the Dirichlet series representing $\zeta(C, 2s)$:

$$D(s) = \sum_{m\geq 1} \frac{c_m}{m^s},$$

then

$$h_C(x) = \sum c_m[\kappa(mx) - \varepsilon x^{-1}\kappa(mx^{-1})],$$

where $\kappa(x)$ is the inverse Mellin transform of $\gamma(s)$. $\gamma(s)$ is a product involving the square of the completed Dedekind zeta function $\xi(k, s)$, and so its inverse Mellin transform is related to Eisenstein series for $\mathrm{GL}_2(\mathbb{A}_k)$, and hence has a Whittaker expansion which we will compute below. This connection to Eisenstein series goes a small way to substantiating the comparison to the Rankin-Selberg method of the convolutions studied in the previous sections.

For simplicity, let $C = E$ be an elliptic curve over $\mathbb{Q}$, then the gamma factor of $\zeta(C, s)$ is trivial and

$$\kappa(x) = \frac{1}{2\pi i}\int_{(c)} \xi(k, s)^2 x^{-s} dx.$$

We will understand this inverse Mellin transform as an Eisenstein series on $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ and state the associated Whittaker expansion.

For $w \in \mathbb{C}$ such that $\Re(w) > 1$, define

$$E(z, w) = \frac{1}{2}\sum_{(c,d)=1} \frac{y^w}{|cz + d|^{2w}},$$

where $z = x + iy$, $y > 0$ and the sum is taken over coprime integers c and d.

According to [Gol06, Theorem 3.1.8], we have the Fourier-Whittaker expansion:

$$E(z, w) = y^w + \phi(w)y^{1-w} \tag{3.3.1}$$

$$+ \frac{2^{1/2}\pi^{w-1/2}}{\Gamma(\mathbb{Q}, w)\zeta(\mathbb{Q}, 2w)} \sum_{n\neq 0} \sigma_{1-2w}(n)|n|^{w-1}\sqrt{2\pi|n|y}K_{w-1/2}(2\pi|n|y)e^{2\pi nx}, \tag{3.3.2}$$

where ϕ is the following function

$$\phi(s) = \sqrt{\pi}\frac{\Gamma(s - \frac{1}{2})}{\Gamma(s)}\frac{\zeta(\mathbb{Q}, 2s - 1)}{\zeta(\mathbb{Q}, 2s)}.$$

Note that the modified Bessel functions appearing in this expansion are closely related to Whittaker functions. To be precise

$$K_v(z) = (\frac{\pi}{2z})^{1/2} W_{0,v}(2z).$$

The L-function of $E(z,w)$ is defined as follows, for $\Re(s) > \min\{1/2 - w, w - 1/2\}$

$$L(E,s) = \sum_{n=1}^{\infty} \sigma_{1-2w}(n) n^{w-s-1/2}.$$

As in [Gol06, §3.14], one can deduce that this is simply a product of Riemann zeta functions

$$L(E,s) = \zeta(\mathbb{Q}, s + w - 1/2)\zeta(\mathbb{Q}, s - w - 1/2).$$

We therefore have meromorphic continuation and a simple functional equation for $L(E,s)$. We recover the following expansion for $h_C(x)$:

$$h_C(x) = \sum_{n \geq 1} (\sum_{d|n} c_d \sigma_0(n/d))[K_0(2\pi n e^{-t}) - e^t K_0(2\pi n e^t)].$$

This is slightly more complicated for curves of higher genus because of the non-trivial gamma factor.

CHAPTER 4

Two-Dimensional Zeta Integrals

Now that we have seen some of the connections between automorphicity and mean-periodicity, we turn our attention towards the development of a framework in which one can expect to prove mean-periodicity without resorting to automorphic properties. Again, we work in the context of a smooth, projective, geometrically connected curve C over a global field k, with L-function $L(C,s)$. As demonstrated in the chapter 2, standard conjectural properties of $L(C,s)$ correspond to analogous properties of zeta functions $\zeta(\mathcal{S},s)$ of proper regular models $\mathcal{S}$ of C. By taking into account the additional contribution of finitely many horizontal curves on $\mathcal{S}$, we are lead to study a modified zeta function

$$Z(\mathcal{S},\{k_i\},s) = \zeta(\mathcal{S},s)\prod_{i=1}^{n}\zeta(k_i,s/2),$$

where the number fields k_i/k are determined by the horizontal curves. Up to sign, the functional equation of $L(C,s)$ is equivalent to

$$\mathcal{Z}(\mathcal{S},\{k_i\},s)^2 = \mathcal{Z}(\mathcal{S},\{k_i\},2-s)^2,$$

where $\mathcal{Z}$ is not quite the product of the completions:

$$\mathcal{Z}(\mathcal{S},\{k_i\},s) = A(\mathcal{S})^{(1-s)/2}\Gamma(\mathcal{S},s)\zeta(\mathcal{S},s)\prod_{i=1}^{n}\xi(k_i,s/2).$$

When C has simple reduction properties (which can always be obtained after base change), we show that $\mathcal{Z}(\mathcal{S},\{k_i\},s)^2$ is an integral over certain two dimensional "analytic adeles", up to the square of a rational function $Q(s)$ of the following form

$$Q(s) = \frac{C}{(1-s)^m},$$

for constants $C \in \mathbb{C}$ and $m \in \mathbb{N}$, each of which depend on the base field k. Therefore $Q(s)^2$ is invariant with respect to $s \mapsto 2-s$. The required reduction properties for the integral expressions are explained in 4.2, and are broadly comparable to semistablity. When C possesses these reduction properties, the conductor $A(\mathcal{S})$ arises from counting singularities on bad fibres.

In dimension 1, it is well understood that the analytic properties of zeta functions can be obtained through harmonic analysis on a commutative adelic group - this is reviewed in section 4.1. In this chapter, our goal is to develop this idea on certain two-dimensional adelic groups. We will review the theory of two dimensional local fields in section 4.3, and two-dimensional analytic adeles in 4.4. The most fundamental issue is that these groups are not locally compact, and so what we mean by "harmonic analysis" has to be somewhat modified. We remark that the techniques of Tate's thesis have long since been extended to various non-commutative algebraic groups, as utilised in earlier chapters.

Following a sketch given in [Fes10, Section 57], section 4.5 introduces zeta integrals extending those of Fesenko in the case where $C = E$ is an elliptic curve, the primary difference being a renormalising factor whose arithmetic interpretation is a power of $\zeta(\mathbb{P}^1(\mathcal{O}_k))$. Fesenko's original zeta integrals for elliptic curves diverge for higher genus curves, due to a certain incompatibility of the additive and multiplicative measures, which is rectified by the renormalizing factor. Moreover, there is a very neat connection between this factor and the gamma factor of the completed zeta function, as will be explained.

The fact that it is the square of the modified completed zeta function appearing in the integral expressions is due to the fact we integrate over two-copies of the multiplicative group of the analytic adeles. Of course, one might expect that integrating over a single copy would give rise to the completed zeta function itself - this is not true at finitely many factors. There are two further reasons for considering the square of the zeta function. Firstly, in this way we avoid issues with the sign of the functional equation. The second reason concerns a compatibility with two-dimensional class field theory, which will be considered again in the final chapter where we consider the foundations of a $\mathrm{GL}_1(\mathbb{A}(\mathcal{S}))$-theory.

4.1 Tate's Thesis

We will begin by summarizing the content of Tate's thesis for Dedekind zeta functions [Tat50]. Independently of Tate, the method described below was discovered by Iwasawa, although he did not incorporate the local analogue [Iwa92]. For this reason, we refer to this circle of ideas as Iwasawa–Tate theory.

Let k be a number field with ring of integers $\mathcal{O}_k$. The Dedekind zeta function of $\mathcal{O}_k$ is then the zeta function of the arithmetic scheme $\mathcal{S} = \mathrm{Spec}(\mathcal{O}_k)$:

$$\zeta(\mathcal{S}, s) = \zeta(k, s).$$

Up to a constant factor, the locally compact topological group of ideles $\mathbb{A}_k^\times$ of k has a unique Haar measure μ. Each Haar measure defines module map

$$|\,| : \mathbb{A}_k^\times \to \mathbb{R}_+^\times,$$

which we will normalize so that, for $\alpha \in k^\times \hookrightarrow \mathbb{A}_k^\times$ (diagonal embedding):

$$|\alpha| = 1.$$

Let $f \in S(\mathbb{A}_k)$ be defined as follows

$$f = \otimes_v f_v$$

$$f_v(\alpha) = \begin{cases} \mathrm{char}(\mathcal{O}_v)(\alpha), & v \nmid \infty \\ \exp(-\pi\alpha^2), & v \text{ real.} \\ \exp(-2\pi|\alpha|^2), & v \text{ complex.} \end{cases}$$

For all $s > 1$, the following integral absolutely converges

$$\zeta(f, s) := \int_{\mathbb{A}_k^\times} f(x)|x|^s d\mu(x) = \xi(k, s).$$

We will call integrals of this form "one-dimensional zeta integrals." Such integrals were known and studied by Artin, Weil, Iwasawa, Tate and many other mathematicians.

In order to proceed, one applies basic techniques of integration (the Fubini property) and harmonic analysis on the locally compact multiplicative group of ideles. Adelic duality, whose incarnation is the theta formula and Riemann-Roch theorem, then implies the analytic continuation and functional equation of the zeta-function. More precisely, one shows that there exists an entire function $\eta_f(s)$ such that

$$\xi(k, s) = \eta_f(s) + \eta_{\hat{f}}(1 - s) + \omega_f(s),$$

where $\widehat{f}$ is the Fourier transform of f, and ω_f is the Laplace transform of a rational function:

$$\omega_f(s) = \int_0^1 h_f(x) x^s \frac{dx}{x},$$
$$h_f(x) = -\mu(\mathbb{A}_k^1/k^\times)(f(0) - x^{-1}\widehat{f}(0)).$$

The explicit form of $h_f(x)$ clearly implies the meromorphic continuation and functional equation of $\xi(k,s)$. The function $h_f(x)$ is closely related an integral over the weak topological boundary of a global subspace of the adeles. More precisely, if $\mathbb{A}_k^1$ denotes the set of ideles of norm 1, then

$$h_f(x) := -\int_{\gamma\in\mathbb{A}_k^1/k^\times}\int_{\beta\in\partial k^\times}(f(x\gamma\beta) - x^{-1}\widehat{f}(x^{-1}\gamma\beta))d\mu(\beta)d\mu(\gamma).$$

The boundary $\partial k^\times$ is with respect to the weak (or "initial" topology - see [Bou66, I, 2.3]) on $\mathbb{A}_k$, which is simply $k\backslash k^\times = \{0\}$.

More generally, one can consider one-dimensional zeta integrals where $|\ |^s$ is replaced by an arbitrary quasi-character χ of the multiplicative group of ideles. In this setting one deduces the basic analytic properties of Hecke L-functions. Further details are explained in example 5.4.

Remark 4.1. Let $\mathcal{E}$ be a proper, regular model of an elliptic curve E over a number field k, then [Fes10, Section 3] shows that there is an entire function $\eta_{\mathcal{E}}$ such that

$$A(\mathcal{E})^{1-s}\zeta(\mathcal{E},s)^2 = \eta_{\mathcal{E}}(s) + \eta_{\mathcal{E}}(2-s) + \omega_{\mathcal{E}}(s),$$

where $\omega_{\mathcal{E}}(s)$ is defined for $\Re(s) > 2$. Generalizing the embedding $k \hookrightarrow \mathbb{A}_k$, there is a semi-global ring of adeles $\mathbb{B}(\mathcal{E}) \hookrightarrow \mathbb{A}(\mathcal{E})$ such that, with respect to an inductive limit of weak topologies for a given family of characters, $\omega_{\mathcal{E}}(s)$ is closely related to an integral over the boundary of $\mathbb{B}(\mathcal{E})$. We will generalize this in chapter 5

4.2 Choice of Model for Adelic Analysis

We now specify a model $\mathcal{S}$ of C simplistic enough for application of two dimensional adelic analysis in its current form. A further development of the theory of lifted harmonic analysis should allow for application to a more general class of arithmetic surfaces. If one is willing to base-change, no restrictions are required. The final chapter considers peripherally the problem of descent to the ground field, invoking integral

representations of certain twisted zeta functions.

Let $\mathcal{B}$ be a Dedekind scheme of dimension 1, and let $\pi : \mathcal{S} \to \mathcal{B}$ be a regular, integral, projective, flat $\mathcal{B}$-scheme. We will call such an $\mathcal{S}$ an arithmetic surface. Closed, irreducible curves on $\mathcal{S}$ are either horizontal or vertical. More precisely, such curves are either an irreducible component of a special fiber or the closure of a closed point of the generic fiber, the latter being finite and surjective onto the base $\mathcal{B}$.

Since $\mathcal{S}$ is regular, the special fibre $\mathcal{S}_b$ over a closed point $b \in \mathcal{B}$ is the Cartier divisor $\pi^* b$. If a given special fiber $\mathcal{S}_b$ contains r irreducible components $\mathcal{S}_{b,i}$, with multiplicity d_i, then, as Weil divisors

$$\mathcal{S}_b = \sum_{1 \leq i \leq r} d_i \mathcal{S}_{b,i}.$$

An effective divisor D on a regular Noetherian scheme X is said to have normal crossings if, at each point $x \in X$, there exist a system of parameters $f_1, \ldots, f_n$ of X at x such that, for some positive integer $m \leq n$, there are integers $r_1, \ldots, r_m$ such that $\mathcal{O}_X(-D)_x$ is generated by $f_1^{r_1} \ldots f_m^{r_m}$. If $D = \mathcal{S}_b$ is the fibre over $b \in \mathcal{B}$, below we will ask for this property to be true over the residue field $k(b)$, in short, we will be asking for split singularities.

The zeta function depends only on the atomization of $\mathcal{S}$, in particular the zeta function agrees with that of the reduced part $\mathcal{S}_{\text{red}}$. With that in mind, for the purposes of adelic analysis we will only work with the reduced part of each fiber, tacitly using the same notation:

$$\mathcal{S}_b := \sum_{1 \leq i \leq r} \mathcal{S}_{b,i}.$$

On finitely many reduced fibres $\mathcal{S}_b$, there may well be non-smooth points. In this chapter we will only work with ordinary double points. Moreover, we need S_b to be a normal crossing divisor over $k(b)$, so we will assume the ordinary double points are split. To summarize, we will assume that the reduced part of each fiber on $\mathcal{S}$ has only split ordinary double points.

From now on, $\mathcal{B}$ will be $\mathrm{Spec}(\mathcal{O}_k)$, where k is a number field. Let C be a smooth, projective geometrically irreducible curve of genus g over k, such that C has good reduction in all residual characteristics less than $2g + 1$. This ensures that the Swan character is trivial and the conductor of $\mathcal{S}$ can be computed by counting singularities as in example 2.13.

Remark 4.2. Some authors describe an arithmetic surface $\mathcal{S} \to \mathrm{Spec}(\mathcal{O}_k)$ as semistable

if $\mathcal{S}$ is a regular $\mathrm{Spec}(\mathcal{O}_k)$-curve with smooth generic fiber and all closed fibers are reduced normal crossing divisors.

A fully developed theory of two-dimensional adelic analysis would have implications for a wide range of problems. As such, the foundational theory needs to be well understood. The development of harmonic analysis on more general topological groups is of the utmost importance, as mentioned as far back as Weil [Wei74, Foreword]. Fesenko's two-dimensional theta formula is potentially a very powerful tool, and it would be beneficial to understand it in a more general setting. In particular, it is desirable to be able to apply adelic analysis to a more general class of arithmetic surfaces, including models of non-semistable curves.

One could work with a more general class of curves by extending the base field. More precisely, let C be a smooth, projective geometrically connected curve of genus ≥ 2 over the function field K of a one-dimensional Dedekind scheme $\mathcal{B}$. By the Deligne–Mumford theorem ([DM69], [Liu02, Theorem 10.4.3]), there exists a Dedekind scheme $\mathcal{B}'$ with function field K' such that the extension $C_{K'}$ has a unique stable model over S'. One can take the extension K'/K to be separable. The extension of base field required by the Deligne–Mumford theorem is not intractable. Let $G := \mathrm{Gal}(K'/K)$, which has a natural action on $\mathcal{S}$, lifting its natural action on $\mathrm{Spec}(\mathcal{O}_{K'})$. The stable reduction, along with its natural G-action determines the local factors of the L-function (for example, see [BW12, Theorem 1.1]).

Throughout, the function field of $\mathcal{S}$ is denoted by K. Closed points of $\mathcal{S}$ are denoted x, and y will denote an irreducible fibre or horizontal curve. When y is an irreducible component of a fibre, its genus is denoted g_y and function field $k(y)$. The maximal finite subfield of $k(y)$ has cardinality denoted $q(y)$. The set of components of a fibre $\mathcal{S}_\mathfrak{p}$ is denoted by $\mathrm{comp}(\mathcal{S}_\mathfrak{p})$. If x is a singular point on a fibre $\mathcal{S}_\mathfrak{p}$, then

$$\mathcal{S}_\mathfrak{p}(x) = \cup_{y \in \mathrm{comp}(\mathcal{S}_\mathfrak{p})} y(x),$$

where $y(x)$ denotes the set of local branches of y at x.

4.3 Two-Dimensional Local Fields

Let $\mathcal{S}$ be a two dimensional, irreducible, Noetherian scheme and let $x \in y \subset \mathcal{S}$ be a complete flag of irreducible closed subschemes. If $\mathfrak{m}$ is a local equation for x and $\mathfrak{p}$ is a

local equation for y, then let $\mathcal{O} = \widehat{\mathcal{O}}_{\mathcal{S},x}$ and

$$K_{x,y} = \mathrm{Frac}(\widehat{(\mathcal{O})_{\mathfrak{p}\mathcal{O}}}),$$

see, for example, [Bei80], [Par83], [FK00], [Mor12b, Sections 6, 7] and [Fes12, Part 1].
If x is a smooth point of y, then $K_{x,y}$ is an example of a two-dimensional local field; it is a complete discrete valuation field whose residue field is a one-dimensional local field. If x is a singular point on y, the same construction yields a direct product of two-dimensional local fields. Recall that $y(x)$ is the set of local branches of y at x, then

$$K_{x,y} = \prod_{z \in y(x)} K_{x,z},$$

where $K_{x,z}$ is the two-dimensional local field associated to x and the minimal prime z. The residue field of $K_{x,z}$ will always be denoted $E_{x,z}$. A lift of a local parameter from $E_{x,z}$ to $K_{x,z}$ will be denoted $t_{1,x,z}$, and the cardinality of the residue field of $E_{x,z}$ (which is the second residue field of $K_{x,z}$) is denoted $q(x,z)$.
Let F be a two-dimensional local field. As a complete discrete valuation field, it has the discrete valuation

$$v_2 : F \twoheadrightarrow \mathbb{Z}.$$

For this valuation we fix a local parameter t_2. We denote the ring of integers with respect to this valuation $\mathcal{O}_F$.
On the residue field $\overline{F}$ we have the discrete valuation

$$v_1 : \overline{F} \twoheadrightarrow \mathbb{Z}.$$

Together, v_1 and v_2 induce a "rank 2" valuation on F, which depends on t_2:

$$\underline{v} : F \to \mathbb{Z}^2$$

$$\alpha \mapsto (v_1(\alpha t_2^{-v_2(\alpha)}), v_2(\alpha)),$$

where $\mathbb{Z}^2$ is given the lexicographic ordering. Let O_F denote the ring of integers with respect to $\underline{v}$, we have:

$$O_F = \{x \in \mathcal{O}_F : \overline{x} \in \mathcal{O}_F\}.$$

Unlike the classical situation, there are infinitely many different rank 2 discrete valuations on F, however, the ring of integers and maximal ideal do not depend on this choice.

For further details on higher local fields the reader is referred to [FK00, Part I, Section 1] and [Mor12b].

When $F = K_{x,y}$ we use the notations:

$$\mathcal{O}_F = \mathcal{O}_{x,y},$$

$$O_F = O_{x,y},$$

and when $y = \mathcal{S}_{\mathfrak{p}}$ is the fibre of $\mathcal{S}$ over $\mathfrak{p} \in \mathrm{Spec}(\mathcal{O}_k)$ we will write

$$\mathcal{O}_{x,\mathfrak{p}} := \mathcal{O}_{x,\mathcal{S}_{\mathfrak{p}}},$$

$$O_{x,\mathfrak{p}} := O_{x,\mathcal{S}_{\mathfrak{p}}}$$

It is well known that complete discrete valuation fields have a nontrivial $\mathbb{R}$-valued Haar measure only when their residue field is finite. In particular, there is no $\mathbb{R}$-valued Haar measure on higher dimensional local fields. A lifted $\mathbb{R}((X))$-valued Haar measure and integration theory appeared in [Fes06, Fes10]. In these papers Fesenko develops two approaches to the theory of higher Haar measure on higher local fields, taking values in formal power series over $\mathbb{R}$. A third, lifting approach, suggested in [Fes10] was further developed by Morrow in [Mor08]. All these approaches give essentially the same translation invariant measure on a class of measurable subsets of F. There is also a model-theoretic approach of Hrushovski-Kazdhan [HK06].

Example 4.3. Let F be a two-dimensional local field, with a fixed local parameter t_2 and residue field K. On the locally compact field K we have a Haar measure μ_K, normalized so that $\mu_K(\mathcal{O}_K) = 1$. Let $\mathcal{A}$ be the minimal ring of sets generated by $\alpha + t_2^j p^{-1}(S)$, where S is μ_K-measurable, the "measure" of a generator of $\mathcal{A}$ is $X^i \mu_K(S) \in \mathbb{R}((X))$. For example $\mu_F(O_F) = 1$, where O_F is the rank two ring of integers. This measure extends to a well-defined additive function on $\mathcal{A}$, which is moreover, countably additive in a certain refined sense, [Fes03, Part 6], [Fes06], [Fes10].

Remark 4.4. We observe the following:

1. Essential role was played by a choice of a local parameter t_2, i.e. the splitting of the residue map. An analogous statement will be true in the adelic counterpart of example 4.6.

2. In the mixed characteristic case there are nonlinear changes of variables for which the Fubini property of the measure does not hold [Mor07]. This could be considered as an example of the non-commutativity inherent in studying L-functions of curves over global fields. In this chapter, such considerations will not cause a problem.

4.4 Analytic Adeles

Let X be a Noetherian scheme, let M be a quasi-coherent sheaf on X, and let T be a set of reduced chains on X. To such a triple (X, M, T), one can associate an abelian group $\mathbf{A}(X, M, T)$ of adeles. We will call these groups "geometric adeles" and recommend the following references for details [Par76], [Par83], [Bei80], [Hub91], [Mor12b, Section 8] and [Fes12]. The adelic group $\mathbf{A}(X, M, T)$ can be interpreted as a restricted product over T of local factors, which are obtained by localising and completing along each flag. Often, one takes T to be the set of all reduced chains on X, and we denote the resulting group by $\mathbf{A}(X, M)$. $\mathbf{A}(X, M)$ has more structure than that of an abelian group - it admits a semi-cosimplicial structure whose cohomology is that of M.

Let y be an irreducible curve on $\mathcal{S}$. If T is the set of all reduced chains formed by closed points on y, then we will denote $\mathbf{A}(\mathcal{S}, \mathcal{O}_{\mathcal{S}}, T)$ by $\mathbf{A}(y)$. Later (remark 4.7) we will see that this space is "too big" for integration, which motivates us to introduce the smaller spaces of "analytic" adeles $\mathbb{A}(y)$, following the constructions of [Fes10, Section 1][1].

As mentioned in 4.2, there are two types of irreducible curves on $\mathcal{S}$ - irreducible components of fibres and horizontal curves. Whilst there is no real difference in the construction of $\mathbb{A}(y)$, we will treat the two cases separately so as to emphasize some important aspects in each setting. In particular, the fibres may well be singular, and the horizontal curves contain archimedean information.

[1] The geometric adeles $\mathbf{A}(\mathcal{S})$ exist for arbitrary Noetherian schemes $\mathcal{S}$. One can consider what the general definition of the analytic space $\mathbb{A}(\mathcal{S})$ is and what role it plays in algebraic geometry.

4.4.1 Fibres

Let y be an irreducible component of the fibre $\mathcal{S}_{\mathfrak{p}}$ over $\mathfrak{p} \in \mathrm{Spec}(\mathcal{O}_k)$. For any $n \geq 0$ and any point $x \in y$, one can define local lifting maps

$$l^n_{x,y} : E^{\oplus n}_{x,y} \to \begin{cases} \mathcal{O}_{x,y}, & \text{if } K_{x,y} \text{ is of equal characteristic.} \\ \mathcal{O}_{x,y}/t^n\mathcal{O}_{x,y}, & \text{otherwise,} \end{cases}$$

and, subsequently, adelic lifting maps

$$L^n_y : \mathbb{A}(k(y))^{\oplus n} \to \begin{cases} (K_{x,y})_{x\in y} \\ (\mathcal{O}_{x,y}/t^n\mathcal{O}_{x,y})_{x\in y}. \end{cases}$$

For details of these constructions, the reader is referred to [Fes10, Section 1.1]. The y-component of the analytic adeles is the following ring:

$$\mathbb{A}(y) = \{(a_{x,y})_{x\in y} : a_{x,y} \in K_{x,y}, \forall n \geq 0,\ (a_{x,y}) + t^n_y\mathcal{O}_y \in \mathrm{im}(L^n_y)\}.$$

Recall that, when x is a singular point on y, $K_{x,y}$ is in fact $\prod_{z\in y(x)} K_{x,z}$
For $a_{x,y} \in \mathcal{O}_{x,y} = \prod_{z\in y(x)} \mathcal{O}_{x,z}$, let $\overline{a}_{x,y} = (\overline{a}_{x,z})_{z\in y(x)}$ denote the image of $a_{x,y}$ under the residue map to $\prod E_{x,z}$. We thus have

$$p_y : \mathbb{A}(y) \to \mathbb{A}(k(y))$$

$$(a_{x,y}) \mapsto (\overline{a}_{x,y}).$$

Definition 4.5. Let $\mathcal{S}_{\mathfrak{p}}$ denote the fibre of $\mathcal{S}$ over $\mathfrak{p}$. The $\mathcal{S}_{\mathfrak{p}}$-component $\mathbb{A}(\mathcal{S}_{\mathfrak{p}})$ of the analytic adeles is

$$\mathbb{A}(\mathcal{S}_{\mathfrak{p}}) = \prod_{y\in \mathrm{comp}(\mathcal{S}_{\mathfrak{p}})} \mathbb{A}(y).$$

We have a residue map

$$p_{\mathfrak{p}} = (p_y) : \mathbb{A}(\mathcal{S}_{\mathfrak{p}}) \to \prod_{y\in \mathrm{comp}(\mathcal{S}_{\mathfrak{p}})} \mathbb{A}(k(y)).$$

The following example gives a concrete interpretation of analytic adelic spaces and the subsequent remark explains why their measure theory cannot be extended to geometric adeles.

Example 4.6. Let S be a surface over a finite field and let y be a nonsingular irreducible curve on S, with function field $k(y)$. Associated to y we have the complete discrete valuation field

$$K_y = \mathrm{Frac}(\widehat{\mathcal{O}_y}).$$

We fix a local parameter and denote it by t_y, it can be taken as a second local parameter for all two-dimensional local fields associated to closed points x on y. We will refer to it as a local parameter of y. The ring $\mathbb{A}(k(y))$ is locally compact and has a Haar measure $\mu_{\mathbb{A}(k(y))}$. We have a non-canonical isomorphism

$$\mathbb{A}(y) \cong \mathbb{A}(k(y))((t_y)).$$

Let p be the map to $\mathbb{A}(k(y))$ sending a power series to its free coefficient. As in example 4.3, one can construct an $\mathbb{R}((X))-$measure μ_y on $\mathbb{A}(y)$. Let S be a measurable subset of $\mathbb{A}(k(y))$, then

$$\mu_y(t_y^i p^{-1}(S)) = X^i \mu_{\mathbb{A}(k(y))}(S).$$

Remark 4.7. As in the above situation, let y be an irreducible curve on $\mathcal{S}$. If $\mathbf{A}(y)$ is the group of geometric adeles associated to y on $\mathcal{S}$, $M = \mathcal{O}_{\mathcal{S}}$ and T is the set of all reduced chains of the form $x \in y \subset S$, then

$$\mathbf{A}(y) = \cup_{r \in \mathbb{Z}} t_y^r \mathbb{A}(y).$$

$\mathbf{A}(y)$ can be understood as a restricted direct product of $\mathbf{A}(y)$ in which almost all components lie in $\mathbb{A}(y)$. Since the measure of $\mathbb{A}(k(y))$ is infinite, the measure of $\mathbb{A}(y)$ in the previous example is infinite. The geometric adeles are therefore a restricted product with respect to a set of infinite measure, and so we cannot extend the measure to $\mathbf{A}$.

4.4.2 Horizontal Curves

Horizontal curves on $\mathcal{S}$ will play a crucial role in this chapter. We will begin by explaining their archimedean content - roughly, each horizontal curve intersects the archimedean fibers of the surface, as we now explain in more detail (see also [Mor12a, Section 5]).

By an archimedean fibre, we mean the fiber product

$$\mathcal{S}_\sigma = \mathcal{S} \times_{\mathrm{Spec}(\mathcal{O}_k)} k_\sigma.$$

where σ is an archimedean place of the base field k, with corresponding completion k_σ. There is a natural morphism from an archimedean fiber to the generic fiber

$$\mathcal{S}_\sigma \to \mathcal{S} \times_{\mathcal{O}_k} k = \mathcal{S}_\eta \cong C.$$

The fibre over any closed point on the generic fibre $C \cong \mathcal{S}_\eta$ is a finite reduced scheme. A horizontal curve y on $\mathcal{S}$ is the closure $\overline{\{z\}}$ of a unique closed point $z \in C$, which has residue field $k(z)$. There are only finitely many points on $\mathcal{S}_\sigma$ which map to z, and they are the primes of $k_\sigma \otimes_k k(z)$, which correspond to the infinite places of $k(z)$ extending σ on k.

At a closed point ω on $\mathcal{S}_\sigma$ we have a two-dimensional local field

$$K_{\omega,\sigma} = \mathrm{Frac}(\widehat{\mathcal{O}_{\mathcal{S}_\sigma,\omega}}).$$

The residue field of $K_{\omega,\sigma}$ is denoted $k_\sigma(\omega)$ and is either $\mathbb{R}$ or $\mathbb{C}$. We have, respectively

$$K_{\omega,\sigma} \cong \begin{cases} \mathbb{R}((t)) \\ \mathbb{C}((t)) \end{cases}$$

Let y be a horizontal curve on $\mathcal{S}$, and let σ be an archimedean place of k. By the correspondence just described, we have an archimedean place ω of $k(y)$ and a two-dimensional local field

$$K_{\omega,y} = k(y)_\omega((t_y)),$$

where $k(y)_\omega$ is the completion of $k(y)$ at ω.

Repeating the construction from 4.4.1, we obtain a lifting map

$$l^n_{\omega,y} : k(y)_\omega^{\oplus n} \to \mathcal{O}_{\omega,y} \cong \begin{cases} \mathbb{R}[[t]], \\ \mathbb{C}[[t]]. \end{cases}$$

Also, at a closed point $x \in y$, we have a local lifting

$$l^n_{x,y} : E_{x,y} \to \mathcal{O}_{x,y}.$$

Altogether, we have an adelic map:

$$L^r_y : \oplus_{x\in y} l^n_{x,y} \oplus_\omega l^n_{\omega,y} : \mathbb{A}(k(y))^{\oplus n} \to \prod_{x\in y} K_{x,y} \prod_\omega K_{\omega,y}.$$

Definition 4.8. Let y be a horizontal curve on $\mathcal{S}$. The y-component of the analytic adelic space is:

$$\mathbb{A}(y) = \{((a_{x,y})_{x\in y}, (a_{\omega,y})_\omega) \in \prod_{x\in y} K_{x,y} \prod_\omega K_{\omega,y} : \forall n \geq 1,\ ((a_{x,y})_{x\in y}, (a_{\omega,y})_\omega) \in \mathrm{im}(L_y^n)\}.$$

The residue maps $\mathcal{O}_{x,z} \to E_{x,z}$ and $\mathcal{O}_{\omega,y} \to k(y)_\omega$ induce

$$p_y : \mathbb{A}(y) \to \mathbb{A}(k(y)).$$

4.4.3 Additive Normalization

We want to extend example 4.6 to $\mathbb{A}(\mathcal{S}_\mathfrak{p})$ and $\mathbb{A}(y)$, where $\mathfrak{p} \in \mathrm{Spec}(\mathcal{O}_k)$ and y is a horizontal curve on $\mathcal{S}$.

If y is a fiber or horizontal curve, the additive group of the ring $\mathbb{A}(k(y))$ is a locally compact abelian group. It thus has a Haar measure, which is unique up to scalar multiplication. The $\mathbb{R}((X))$-measure on $\mathbb{A}(y)$ will depend on a choice of normalization of the Haar measure on $\mathbb{A}(k(y))$.

Let F be a two-dimensional local field. If F is nonarchimedean and $\psi_F : F \to \mathbb{C}^\times$ is a character, then we will refer to the orthogonal complement of O_F as the conductor of ψ_F. If F is an archimedean two-dimensional local field, the conductor is the orthogonal complement of $\mathcal{O}_F$.

When $F = K_{x,z}$ (resp. $K_{\omega,y}$), we denote ψ_F by $\psi_{x,z}$ (resp. $\psi_{\omega,y}$). The aim is to define the normalization of the measure on $\mathbb{A}(k(y))$ through the characters $\psi_{x,z}$.

Lemma 4.9. *For any closed point x on the fibre $\mathcal{S}_\mathfrak{p}$ of $\mathcal{S}$, let z be a branch of an irreducible component of $\mathcal{S}_\mathfrak{p}$ at x. There are characters $\psi_{x,z}$ of the two-dimensional local fields $K_{x,z}$ such that if*

$$\psi_{x,\mathfrak{p}} = \otimes_{z\in\mathcal{S}_\mathfrak{p}(x)} \psi_{x,z},$$

then the following is defined on $\mathbb{A}(\mathcal{S}_\mathfrak{p})$:

$$\psi_\mathfrak{p} = \otimes_{x\in\mathcal{S}_\mathfrak{p}} \psi_{x,\mathfrak{p}}$$

Moreover, the conductor $A_{x,\mathfrak{p}}$ is commensurable with $O_{x,\mathfrak{p}}$, with equality at almost all $x \in \mathcal{S}_\mathfrak{p}$, including the singular points. There is a nontrivial

$$\varphi_\mathfrak{p} : \mathbb{A}(k(\mathcal{S}_\mathfrak{p})) \to \mathbb{C}^\times$$

such that

$$\psi_{\mathfrak{p}} = \varphi_{\mathfrak{p}} p_{\mathfrak{p}}.$$

Similarly, if y is a horizontal curve, for all points $x \in y$ and archimedean places ω of $k(y)$, there are local characters $\psi_{x,y}$ and $\psi_{\omega,y}$ such that

$$\psi_y = \otimes_{x \in y} \psi_{x,y} \otimes_\omega \psi_{\omega,y}$$

is defined on $\mathbb{A}(y)$ with the same properties.

Proof. [Fes10, Proposition 27]. □

From now on we fix such $\psi_{x,\mathfrak{p}}$ (resp. $\psi_{x,y}$) for all closed points on fibres $\mathcal{S}_{\mathfrak{p}}$ (resp. horizontal curves y).

Definition 4.10. Let y be an irreducible component of a fibre. If x is a nonsingular point on y, define $d(x,y)$ by

$$A_{x,y} = t_{1,x,y}^{d(x,y)} O_{x,y},$$

where $A_{x,y}$ is the conductor of $\psi_{x,y}$ and $t_{1,x,y}$ denotes the local parameter of $K_{x,y}$. If x is a split ordinary double point on y, and z, z' are local branches of y at x, we will write $d(x,z) = d(x,z') = -1$.

Lemma 4.11. *Let $\mathcal{S}_{\mathfrak{p}}$ be a smooth fibre on $\mathcal{S}$, then*

$$\prod_{x \in \mathcal{S}_{\mathfrak{p}}} q_{x,\mathfrak{p}}^{d(x,\mathfrak{p})} = 1$$

Proof. Let $k(\mathcal{S}_{\mathfrak{p}})$ denote the function field of $\mathcal{S}_{\mathfrak{p}}$ and let v be a place of this global field. The residue field at v has cardinality q_v. The lemma then follows from the representation of the canonical divisor $\mathfrak{C}$ on z as

$$\mathfrak{C} = \sum_v m_v v,$$

where $P_v^{m_v}$ is the v-component of the conductor of the standard character on $\mathbb{A}(k(z))$. We know:

$$N(P_v^{m_v}) = q_v^{m_v} = q^{\deg(v) m_v}$$

So

$$\prod_v N(P_v^{m_v}) = q^{-\deg(\mathfrak{C})},$$

and formula follows from the fact that $\deg(\mathfrak{C}) = 2g - 2$. □

Let $\mu_{\mathbb{A}(k(y))}$ be the Haar measure on $\mathbb{A}(k(y))$ which is self dual with respect to the character φ_y from lemma 4.9. If y is an irreducible curve on $\mathcal{S}$, let L denote a measurable subset of $\mathbb{A}(k(y))$ with respect to the above Haar measure. Consider the lifted measure $M_{\mathbb{A}(y)}$ on $\mathbb{A}(y)$ such that

$$M_{\mathbb{A}(y)}(t_y^i p_y^{-1}(L)) = X^i \mu_{\mathbb{A}(k(y))}(L).$$

For example, let $\mathcal{S}_\mathfrak{p}$ be a smooth fibre of $\mathcal{S}$ over $\mathfrak{p}$. Consider the subset

$$O\mathbb{A}(\mathcal{S}_\mathfrak{p}) = \prod_{x \in \mathcal{S}_\mathfrak{p}} O_{x,\mathfrak{p}},$$

then

$$\begin{aligned} M_{\mathbb{A}(\mathcal{S}_\mathfrak{p})}(O\mathbb{A}(\mathcal{S}_\mathfrak{p})) &= \mu_{\mathbb{A}(k(\mathcal{S}_\mathfrak{p}))}(\prod_{x \in \mathcal{S}_\mathfrak{p}} O_x), \\ &= \prod_{x \in \mathcal{S}_\mathfrak{p}} \mu_{k(\mathcal{S}_\mathfrak{p})_x}(O_x) \\ &= \prod_{x \in \mathcal{S}_\mathfrak{p}} q_{x,\mathfrak{p}}^{d(x,y)/2} \\ &= q_\mathfrak{p}^{1-g(\mathcal{S}_\mathfrak{p})}, \end{aligned}$$

where $g(\mathcal{S}_\mathfrak{p}) = g$ denotes the genus of the special fibre $\mathcal{S}_\mathfrak{p}$.

Definition 4.12. Let g be the genus of C and $\mathcal{S}_\mathfrak{p}$ be a smooth fibre, define:

$$\mu_{\mathbb{A}(\mathcal{S}_\mathfrak{p})} = q(\mathcal{S}_\mathfrak{p})^{g-1} M_{\mathbb{A}(y)}.$$

If $\mathcal{S}_\mathfrak{p}$ is a singular fiber, then

$$\mu_{\mathbb{A}(\mathcal{S}_\mathfrak{p})} = M_{\mathbb{A}(\mathcal{S}_\mathfrak{p})}.$$

If y is a horizontal curve, let $\mu_{\mathbb{A}(y)} = M_{\mathbb{A}(y)}$.

So, for all smooth fibres and horizontal curves $\mu_{\mathbb{A}(y)}(O\mathbb{A}(y)) = 1$.
In this chapter we will only integrate linear combinations of characteristic functions of measurable sets. For a more general theory, see [Fes10, 1.3].

4.5 Zeta Integrals on $\mathcal{S}$

Two-dimensional zeta integrals were first studied by Fesenko for proper regular models of elliptic curves [Fes08, Fes10]. We extend his results to a model $\mathcal{S}$ as in 4.2, following the sketch in [Fes08, Part 57].

4.5.1 Multiplicative Normalization

First, we recall the relationship between the measure on the additive and multiplicative group of one-dimensional local fields and adeles.

Example 4.13. Let k be a number field. At each non-archimedean prime $\mathfrak{p}$ we have a normalized Haar measure $d\mu$ on the locally compact additive abelian group $k_{\mathfrak{p}}$, which has finite residue field of cardinality $q(\mathfrak{p})$. One then integrates on the multiplicative group $k_{\mathfrak{p}}^{\times}$ with the measure

$$(1-q(\mathfrak{p})^{-1})\frac{d\mu}{|\ |_{k_{\mathfrak{p}}}}.$$

In turn, one integrates over the idele group $\mathbb{A}_k^{\times}$ with the tensor product of these measures.

Similarly, on an arithmetic surface, we need a measure compatible with the multiplicative structure of a two-dimensional local field.

Let F be a two-dimensional local field with local parameter t_2 and let t_1 be a lift of the local parameter of the residue field of F. Let U denote the group of principal units. Let q denote the cardinality of the final (finite) residue field. One can decompose the multiplicative group $F^{\times}$ as follows:

$$F^{\times} =< t_1 >^{\times} < t_2 >^{\times} U.$$

Using this decomposition we define the $\mathbb{R}((X))-$valued module $|\ |_F$ by

$$|t_2^i t_1^j u|_F = q^{-j} X^i.$$

When $F = K_{x,y}$, we use the notation $|\ |_F = |\ |_{x,y}$.

Let z denote a local branch of an irreducible curve on $\mathcal{S}$ at a point x. Motivated by example 4.13, we will use the following measure on $K_{x,z}^{\times}$:

$$M_{K_{x,z}^{\times}} = \frac{M_{K_{x,z}}}{(1-q(x,z)^{-1})|\ |_{x,z}}.$$

Example 4.14. Let $\mathcal{S}_{\mathfrak{p}}$ be a smooth fibre of $\mathcal{S}$ and consider the following measurable function for each closed $x \in y$

$$f_{x,\mathcal{S}_{\mathfrak{p}}} = |\ |_{x,\mathcal{S}_{\mathfrak{p}}}^{s} \mathrm{char}(O_{x,\mathfrak{p}}),$$

and define $f_{\mathcal{S}_{\mathfrak{p}}} = \otimes_{x\in\mathcal{S}_{\mathfrak{p}}} f_{x,\mathcal{S}_{\mathfrak{p}}}$. Then, $f_{\mathcal{S}_{\mathfrak{p}}}$ is integrable and

$$\int_{\mathbb{A}(\mathcal{S}_{\mathfrak{p}}^{\times})} f_{\mathcal{S}_{\mathfrak{p}}} d\mu_{\mathbb{A}(\mathcal{S}_{\mathfrak{p}})}^{\times} = \zeta(\mathcal{S}_{\mathfrak{p}}, s) \prod_{x\in\mathcal{S}_{\mathfrak{p}}} q(x,\mathcal{S}_{\mathfrak{p}})^{(d(x,\mathcal{S}_{\mathfrak{p}}))(1-s)} = \zeta(\mathcal{S}_{\mathfrak{p}}, s) \prod_{x\in\mathcal{S}_{\mathfrak{p}}} q(x,\mathcal{S}_{\mathfrak{p}})^{(1-g)(1-s)}.$$

In our case the special fibres have at worst split ordinary double singularities and we will use an ad hoc variant of the function in example 4.14 to recover the corresponding factor of the zeta-function - for a more complete approach see [Fes10, 36, Remark 1, 37]. When x is a singular point of $\mathcal{S}_{\mathfrak{p}}$, define

$$M_{K^{\times}_{x,\mathcal{S}_{\mathfrak{p}}}} = \otimes_{z\in\mathcal{S}_{\mathfrak{p}}(x)} M_{K^{\times}_{x,z}}.$$

4.5.2 Zeta Integrals on the Projective Line

We would like to take the product over all the fibres in order to obtain the non-archimedean part of the zeta function of $\mathcal{S}$, including the conductor. Unfortunately, the product diverges due to the additional factors appearing in examples 4.14, and 4.17.
To resolve this, we begin by observing something complementary that happens when we apply the adelic analysis on the scheme $\mathcal{P} := \mathbb{P}^1(\mathcal{O}_k)$. At a non-archimedean place $\mathfrak{p}$ of the base field k, the fibre $\mathcal{P}_{\mathfrak{p}} = \mathbb{P}^1(k(\mathfrak{p}))$. At a closed point $x \in \mathcal{P}_{\mathfrak{p}}$, define

$$g_{x,\mathfrak{p}} = \mathrm{char}(O_{x,\mathcal{P}_{\mathfrak{p}}}),$$

and subsequently,

$$g_{\mathfrak{p}} = \otimes_{x\in\mathcal{P}_{\mathfrak{p}}} g_{x,\mathfrak{p}}.$$

Then

$$\int_{\mathbb{A}(\mathcal{P}_{\mathfrak{p}})^{\times}} g_{\mathfrak{p}} |\,|^{s}_{\mathcal{P}_{\mathfrak{p}}} d\mu_{\mathbb{A}(\mathcal{P}_{\mathfrak{p}})^{\times}} = \zeta(\mathcal{P}_{\mathfrak{p}}, s) \prod_{x\in\mathcal{P}_{\mathfrak{p}}} q_x^{(1-s)}.$$

Combining this computation with examples 4.14 and 4.17, we see that:

$$\prod_{\mathfrak{p}\in\mathrm{Spec}(\mathcal{O}_k)} \int_{\mathbb{A}(\mathcal{S}_{\mathfrak{p}})^{\times}} f_{\mathfrak{p}} |\,|^{s}_{\mathcal{S}_{\mathfrak{p}}} d\mu_{\mathbb{A}(\mathcal{S}_{\mathfrak{p}})^{\times}} \cdot \left(\int_{\mathbb{A}(\mathcal{P}_{\mathfrak{p}})^{\times}} g_{\mathfrak{p}} |\,|^{s}_{\mathcal{P}_{\mathfrak{p}}} d\mu_{\mathbb{A}(\mathfrak{p})^{\times}} \right)^{g-1}.$$

$$= \zeta(\mathbb{P}^1(\mathcal{O}_k), s)^{1-g} \zeta(\mathcal{S}, s) A(\mathcal{S})^{(1-s)/2}.$$

This is essentially the non-archimedean part of the zeta integral in 4.5.5 below.
The $(1-g)$th power of the zeta integral on $\mathbb{P}^1(\mathcal{O}_k)$ conveniently cancels the divergent part of the zeta integral over $\mathcal{S}$. But that is not all, as by a completion process for the zeta function of $\mathbb{P}^1(\mathcal{O}_k)^{1-g}$, we can recover the gamma factor of $\mathcal{S}$ up to an $[s \mapsto 2-s]$—invariant rational function. We will now make this idea precise.

4.5.3 The Gamma Factor

The renormalizing factor in fact induces the gamma factor in a very natural way. The zeta function of $\mathbb{P}^1(\mathcal{O}_k)$ is very simple:

$$\zeta(\mathbb{P}^1(\mathcal{O}_k), s) = \zeta(k, s)\zeta(k, s-1),$$

and so its gamma factor is

$$\Gamma(\mathbb{P}^1(\mathcal{O}_k), s) = \Gamma(\mathbb{C}, s)^{r_2}\Gamma(\mathbb{R}, s)^{r_1}\Gamma(\mathbb{C}, s-1)^{r_2}\Gamma(\mathbb{R}, s-1)^{r_1}.$$

Applying well-known identities of the gamma function,

$$\begin{aligned}
\Gamma(\mathbb{P}^1(\mathcal{O}_k), s)^{1-g} &= \frac{1}{(\Gamma(\mathbb{C}, s)^{r_2}\Gamma(\mathbb{C}, s-1)^{r_2}\Gamma(\mathbb{R}, s)^{r_1}\Gamma(\mathbb{R}, s-1)^{r_1})^{g-1}} \\
&= \frac{1}{(\Gamma(\mathbb{C}, s)^{r_2}\Gamma(\mathbb{C}, s-1)^{r_2}\Gamma(\mathbb{C}, s-1)^{r_1})^{g-1}} \\
&= \frac{\pi^{-r_2(g-1)}(s-1)^{r_2(g-1)}}{(\Gamma(\mathbb{C}, s)^{2r_2}\Gamma(\mathbb{C}, s-1)^{r_1})^{g-1}} \\
&= \frac{\pi^{-(r_1+r_2)(g-1)}(s-1)^{(r_1+r_2)(g-1)}}{(\Gamma(\mathbb{C}, s)^{r_1+2r_2})^{g-1}} \\
&= \frac{\pi^{-(r_1+r_2)(g-1)}(s-1)^{(r_1+r_2)(g-1)}}{R(s)}\Gamma(\mathcal{S}, s) \\
&= Q(s)\Gamma(\mathcal{S}, s),
\end{aligned}$$

where

$$Q(s) = \frac{\pi^{-(r_1+r_2)(g-1)}(s-1)^{(r_1+r_2)(g-1)}}{R(s)},$$

so

$$Q(2-s) = \pm Q(s).$$

We thus see that completing the normalizing factor gives us the transcendental part of the gamma factor of $\mathcal{S}$.

4.5.4 Integration on Horizontal Curves

We will remind ourselves of the Haar measure on $\mathbb{R}$ and $\mathbb{C}$.

$$\mu_{k_\sigma(\omega)} = \begin{cases} \text{Lebesgue measure, } dx & k_\sigma(\omega) = \mathbb{R}, \\ \text{twice Lebesgue measure, } 2dz & k_\sigma(\omega) = \mathbb{C}. \end{cases}$$

One then integrates on the multiplicative group $\mathbb{R}^\times$ (resp. $\mathbb{C}^\times$) with the measure $\frac{dx}{|x|}$ (resp. $\frac{2dx}{|z|^2}$).

Example 4.15. We have the well-known identities:

$$\int_{\mathbb{R}^\times} e^{-\pi x^2}|x|^s \frac{dx}{x} = \Gamma(\mathbb{R}, s),$$

$$\int_{\mathbb{C}^\times} e^{-2\pi|z|^2}|z|^s \frac{2dz}{|z|^2} = 2\pi\Gamma(\mathbb{C}, s).$$

These are precisely the Gamma factors required for the Dedekind-zeta function of a number field at a real (respectively complex) place.

We will integrate on $K_{\omega,y}$ with the lifted measure from $k(y)_\omega$. At all closed points x of y we have the two-dimensional local field $K_{x,y}$ and the natural higher Haar measure. Altogether, we have a measure on $\mathbb{A}(y) = \prod_x' K_{x,y} \prod_\omega \mathbb{A}(\omega, y)$ for a horizontal curve y.

On a horizontal curve y, we redefine[2] $|\ |_y$ to be $\prod_{x\in y} |\ |_y^{1/2}$. On a set S of fibers and finitely many horizontal curves:

$$|\ |_S = \prod_{y\in S} |\ |_y,$$

where

$$|\ |_y = \prod_{x\in y} |\ |_{x,y}.$$

On a horizontal curve y, we redefine $\mathbb{A}(y)^\times$ to be a maximal subgroup such that the image of $|\ |_y$ is equal to that of $|\ |_y^2$.

Example 4.16. At a nonsingular $x \in y$, let $f_{x,y} = \mathrm{char}(O_x)$. At an archimedean place ω of y, let $f_{\omega,y}(\alpha) = \mathrm{char}(\mathcal{O}_{\omega,y})(\alpha)\exp(\mathrm{Tr}_{k(y)_\omega/\mathbb{R}}(1)|\mathrm{res}_{t_\omega^0}(\alpha)|)$. Then

$$\int_{\mathbb{A}(y)^\times} f|\ |_y^s d\mu_{\mathbb{A}(y)^\times} = \zeta(k, \tilde{f}, |\ |_k^{s/2}),$$

where $\zeta(k, g, \chi)$ is a classical Iwasawa–Tate zeta integral and

$$\tilde{f} = \otimes_v \tilde{f}_v$$

$$\tilde{f}_v(\alpha_v) = \begin{cases} \mathrm{char}(\mathcal{O}_v)(\alpha_v), & v \text{ archimedean} \\ e^{-\pi\alpha_v^2}, & v \text{ real} \\ e^{-2\pi|\alpha_v|^2}, & v \text{ complex} \end{cases}$$

by the well-known theory of Iwasawa–Tate this integral defines a meromorphic function on $\mathbb{C}$ and satisfies a functional equation with respect to $s \mapsto 2 - s$.

[2]This potentially confusing notation will be used throughout without much further comment

4.5.5 Zeta Integrals

We are missing the factor at a bad prime $\mathfrak{p}$. At a split ordinary double singularity on the fibre $\mathcal{S}_{\mathfrak{p}}$ we have two local branches, so that integrating over multiplicative group of the two-dimensional analytic adelic space for $\mathcal{S}_{\mathfrak{p}}$ gives us an additional factor that is not present in the zeta function. One way of treating singular and smooth fibres y in a regular way is by integrating over $\mathbb{A}(y)^\times \times \mathbb{A}(y)^\times$, which we give the product measure.

Example 4.17. Let $y = \mathcal{S}_{\mathfrak{p}}$ be a fibre over $\mathfrak{p}$ with singular point x. Let z be a branch of y at x, then define $f_{x,y}$ on $O_{x,y} \times O_{x,y}$ as follows[3]:

$$f_{x,y} = q_x^{-1}\mathrm{char}(O_{x,z}, t_{1,x,z}^{-1}O_{x,z}).$$

For nonsingular points $x \in y$ put

$$f_{x,y} = \mathrm{char}(O_{x,y}, O_{x,y}).$$

Combining, put $f_y = \otimes_{x\in y} f_{x,y}$, then

$$\int f_y d\mu_{\mathbb{A}(y)^\times \times \mathbb{A}(y)^\times} = A_{\mathfrak{p}}(\mathcal{S})^{(1-s)}\zeta(y,s)^2 \prod_{z\in\mathcal{S}_{\mathfrak{p}}} q_z^{2(1-g_z)(1-s)}.$$

All that remains is to put everything together as an integral over the whole adelic space $\mathbb{A}(S)^\times$, where S is contains all fibres of $\mathcal{S}$ and finitely many horizontal curves.

Definition 4.18. Combining the previous examples, let

$$f = \otimes_{y\in S} f_y,$$

where y runs over all curves in S. f_y is defined as follows:

1. Let y be a nonsingular fiber and $x \in y$ be a closed point, put

$$f_{x,y} = \mathrm{char}((O_{x,y}, O_{x,y})),$$

$$f_y = \otimes f_{x,y}$$

2. Let y be a fiber with singular point x. Choose branches $z, z' \in y(x)$ and put

$$f_{x,y} = q_x^{-1}\mathrm{char}(O_{x,z}, t_{1,x,z}O_{x,z}).$$

[3]This is the image of $\mathrm{Char}(O_{x,y})$ under Fesenko's "diamond operator".

3. Let y be a (nonsingular) horizontal curve, for nonarchimedean places of k define $f_{x,y}$ as in point (1). At archimedean places ω take

$$f_{\omega,y}(\alpha) = \mathrm{char}(\mathcal{O}_{\omega,y})(\alpha)\exp(\mathrm{Tr}_{k(y)_\omega/\mathbb{R}}(1)|\mathrm{res}_{t_\omega^0}(\alpha)|).$$

We introduce the following abbreviated notation:

Definition 4.19. Let k be a number field and $\mathcal{P} = \mathbb{P}^1(\mathcal{O}_k)$. Let S denote a set of curves on $\mathcal{S}$, consisting of all fibres and a finite set S_h of horizontal curves on $\mathcal{S}$. If f is an integrable function on $\mathbb{A}(\mathcal{S})^\times \times \mathbb{A}(\mathcal{S})^\times$ and h is an integrable function on $\mathbb{A}(\mathcal{P})^\times \times \mathbb{A}(\mathcal{P})^\times$, then the zeta integral $\zeta_S^{(2)}(f,h,s)$ is defined to be the following product:

$$\prod_{\mathfrak{p}\in\mathrm{Spec}\left(\mathcal{O}_k\right)} \left(\int_{\mathbb{A}(\mathcal{P}_\mathfrak{p})^\times\times\mathbb{A}(\mathcal{P}_\mathfrak{p})^\times} h_\mathfrak{p}|\,|_{\mathcal{P}_\mathfrak{p}}^s d\mu_{\mathbb{A}(\mathcal{P}_\mathfrak{p})^\times}\right)^{g-1} \int_{\mathbb{A}(\mathcal{S}_\mathfrak{p})^\times\times\mathbb{A}(\mathcal{S}_\mathfrak{p})^\times} f_{\mathcal{S}_\mathfrak{p}}|\,|_{\mathcal{S}_\mathfrak{p}}^s d\mu_{\mathbb{A}(\mathcal{S}_\mathcal{P})^\times}$$

$$\times \prod_{y\in S_h} \int_{\mathbb{A}(y)^\times} f_y|\,|_y^s d\mu_{\mathbb{A}(y)^\times} \times \left(\xi(\mathbb{P}^1(\mathcal{O}_k),s)\right)^{1-g}.$$

Remark 4.20. This can be viewed as a "renormalized" integral over the adelic spaces $\mathbb{A}(\mathcal{S},S)^\times$ and $\mathbb{A}(\mathcal{P},S)^\times$ [Fes10, Part 57]. In the next chapter we will consider this as an integral over the analytic adeles of the non-connected arithmetic scheme

$$\mathcal{S}\coprod_{i=1}^{g-1}\mathcal{P}$$

Remark 4.21. In this chapter we have only specified one integrable function, for a more complete theory see [Fes10, Section 1.3]. In general, integrable functions will only differ at finitely many components.

At a closed point $x \in \mathcal{P}_\mathfrak{p}$, define $h_{x,\mathcal{P}_\mathfrak{p}} = \mathrm{char}(O_{x,\mathcal{P}_\mathfrak{p}}, O_{x,\mathcal{P}_\mathfrak{p}})$, and let $h_{\mathcal{P}_\mathfrak{p}} = \otimes_{x\in\mathcal{P}_\mathfrak{p}} h_{x,\mathcal{P}_\mathfrak{p}}$. Convergence of the preliminary zeta integral in some specified half plane will be a corollary (4.23) of the following computation.

Theorem 4.22. *Let S be a set of curves consisting of all fibers and finitely many horizontal curves $\{y_i\}$, each of function field $k(y_i)$. If f is as in definition 4.18 and h is as above, then*

$$\zeta^{(2)}(f,h,s) = Q(s)^2\Gamma(\mathcal{S},s)^2 A(\mathcal{S})^{(1-s)}\zeta(\mathcal{S},s)^2\prod_i \xi(k(y_i),s/2)^2,$$

where $Q(s)$ is a rational function such that

$$Q(s) = \pm Q(2-s),$$

and $\xi(k(y_i), s)$ is the completed Dedekind zeta function of the finite extension $k(y_i)/k$ which satisfies the functional equation:

$$\xi(k(y_i))(\frac{s}{2}) = \xi(k(y_i))(\frac{2-s}{2}).$$

Proof. This follows from combining examples 4.14, 4.17, 4.15 and 4.16. □

Corollary 4.23. *Assume that the integral in the above definition is defined at f, h, then it converges for $s \in \{\Re(s) > 2\}$.*

Proof. In the case of f, h as in the theorem, the convergence of the zeta integral follows from the well known properties of $\zeta(\mathcal{S}, s)$ which are described in [Ser63]. For arbitrary f, g such that the zeta integral is defined, the zeta integral will differ by only finitely many factors. □

We will introduce the notation

$$\mathcal{Z}(\mathcal{S}, \{y_i\}, s) = \zeta(\mathcal{S}, s) A(\mathcal{S})^{(1-s)/2} \Gamma(\mathcal{S}, s) Q(s) \prod_i \xi(k(y_i), s/2).$$

Clearly, the zeta function verifies conjecture 2 if and only if $\mathcal{Z}(\mathcal{S}, \{y_i\}, s)$ does.
Let $\mathcal{T}$ be a two-dimensional arithmetic scheme over $\mathrm{Spec}(\mathcal{O}_k)$. For $\mathfrak{p} \in \mathrm{Spec}(\mathcal{O}_k)$, define $|\ |_{\mathcal{T}_\mathfrak{p}}^{(n)}$ on $(\mathbb{A}(\mathcal{T}_\mathfrak{p})^\times)^{\times n}$ by $|(a_1, \ldots, a_n)|^{(n)} = |a_1| \ldots |a_n|$. We will use the product measure on $(\mathbb{A}(\mathcal{T}_\mathfrak{p})^\times)^{\times n}$. Let $f^{(n)} = (f, \ldots, f)$ and $g^{(n)} = (g, \ldots, g)$, and define $\zeta^{(n)}(\mathcal{S}, f, h, s)$ as the following product:

$$\prod_{\mathfrak{p} \in \mathrm{Spec}(\mathcal{O}_k)} \Big(\int_{(\mathbb{A}(\mathcal{P}_\mathfrak{p})^\times)^{\times n}} g_\mathfrak{p}^{(n)} (|\ |_{\mathcal{P}_\mathfrak{p}}^{(n)})^s d\mu_{(\mathbb{A}(\mathcal{P}_\mathfrak{p})^\times)^{\times n}} \Big)^{g-1} \int_{(\mathbb{A}(\mathcal{S}_\mathfrak{p})^\times)^{\times n}} f_{\mathcal{S}_\mathfrak{p}}^{(n)} (|\ |_{\mathcal{S}_\mathfrak{p}}^{(n)})^s d\mu_{(\mathbb{A}(\mathcal{S}_\mathcal{P})^\times)^{\times n}}$$

$$\times \prod_{y \in S_h} \int_{(\mathbb{A}(y)^\times)^{\times n}} f_y^{(n)} (|\ |^{(n)})_y^s d\mu_{(\mathbb{A}(y)^\times)^{\times n}} \xi(\mathbb{P}^1(\mathcal{O}_k), s)^{n(1-g)/2}.$$

Corollary 4.24. *For each positive integer m, we have*

$$\zeta^{(2m)}(\mathcal{S}, f, h, s) = \mathcal{Z}(\mathcal{S}, \{y_i\}, s)^{2m}.$$

Proof. This follows from Theorem 4.22 and the definitions of measures above. □

When the genus of C is 1 and $2m = 2$, we recover Fesenko's zeta integrals for elliptic curves and the formula in corollary 4.24 agrees with his first calculation. This motivates the following definition:

Definition 4.25. Let S be a set of curves on $\mathcal{S}$, for integrable functions on

$$f : (\mathbb{A}(\mathcal{S}, S)^\times)^{\times 2} \to \mathbb{C}$$

and h on $(\mathbb{A}(\mathbb{P}^1(\mathcal{O}_k))^\times)^{\times 2}$, the "two-dimensional unramified zeta integral" is

$$\zeta(\mathcal{S}, S, f, h, |\,|^s) := \zeta^{(2)}(S, f, h, s).$$

We make the following conjecture, extending that of [Fes10, Section 4]:

Conjecture 4.26. *Provided the set S of curves on $\mathcal{S}$ contains finitely many horizontal curves, the zeta integral $\zeta(\mathcal{S}, S, f, h, |\,|^s)$ meromorphically extends to the complex plane and satisfies the following functional equation*

$$\zeta(\mathcal{S}, S, f, h, |\,|^s) = \zeta(\mathcal{S}, S, f, h, |\,|^{2-s}).$$

Remark 4.27. Let S contain all fibres of $\mathcal{S}$ and finitely many horizontal curves, then

$$\begin{aligned}\zeta(\mathcal{S}, S, f, h, |\,|^s) = \zeta(\mathcal{S}, S, f, h, |\,|^{2-s}) &\iff \xi(\mathcal{S}, s)^2 = \xi(\mathcal{S}, 2-s)^2 \\ &\iff \xi(\mathcal{S}, s) = \pm\xi(\mathcal{S}, s) \\ &\iff \Lambda(C, s) = \pm\Lambda(C, s).\end{aligned}$$

The final implication follows from corollary 2.22.

We have integrated over two copies of the multiplicative group of the ring of analytic adeles so as to get the correct factor of the zeta function at split ordinary double points. This is not the only motivation for doing so. In fact, there is a certain compatibility with two-dimensional class field theory that allows us to define "twisted" zeta integrals whose evaluation is an analogue of Hecke L-functions for arithmetic surfaces. This will be the subject of a later chapter. Before then, we will formulate the mean periodicity correspondence in terms of this "two dimensional adelic analysis" on $\mathcal{S}$.

CHAPTER 5

Adelic Duality and Filtrations

In dimension two, there are three "levels" to the adeles. On the purely local level, one has the products of fields associated to closed points on irreducible curves. The other extreme is the global object, ie. the function field of the surface. In between one has the local-global complete discrete valuation fields associated to irreducible curves, or closed points on the surface. In combining these spaces, one obtains a filtration on the adeles, and, moreover, semi-cosimplicial complexes which compute the cohomology of quasi-coherent sheaves. The additive duality of the adeles and associated quotients can then be used to deduce the Riemann–Roch theorem.

Results in class field theory can be viewed in terms of the duality of multiplicative, and K-theoretic, adelic structures. In studying zeta integrals, we use these ideas implicitly. In this chapter we derive a harmono-analytic expression of a mixture of additive and multiplicative duality and apply this to the zeta integrals of the previous chapter. This will allow us to understand an important component of the boundary function as an integral over the boundary of a local-global adelic space. More precisely, we will write the boundary function as a sum of two adelic integrals, one of which is an integral over the boundary, in such a way that the meromorphic continuation, functional equation, and even poles, of the zeta integrals are all reduced to the analogous properties of the boundary integral.

The formula we deduce is the so-called "two-dimensional theta formula." This terminology is by analogy to the classical theta formula, expressing the functional equation of the theta function. This classical result can be verified through Poisson summation on the adeles and is used in the Iwasawa–Tate method for the functional equation of Hecke L-functions. The two-dimensional theta formula for elliptic surfaces was first

proved by Fesenko in [Fes10, §3.6].

First we must construct integrals on the local-global adelis spaces. The measures required do not factorize as a product of local factors, so the ad hoc method of renormalizing in the previous chapter is not sufficient. Instead, we will consider convergent integrals on certain non-connected arithmetic schemes.

5.1 A Second Calculation of the Zeta Integral

As always, let $\mathcal{S}$ be a proper, regular model of a smooth, projective, geometrically connected curve C over a number field k, and let $\mathcal{P}$ denote the relative projective line $\mathbb{P}^1(\mathcal{O}_k)$. Due to the renormalizing factors of the previous chapter, we are interested in the zeta function of the disjoint union

$$\mathcal{X} = \mathcal{S} \coprod_{i=1}^{g-1} \mathcal{P}.$$

Given any disjoint union $X = \bigcup X_i$ of schemes of finite type over $\mathbb{Z}$, one has

$$\zeta(X, s) = \prod \zeta(X_i, s),$$

so that

$$\zeta(\mathcal{X}, s) = \zeta(\mathcal{S}, s)\zeta(\mathcal{P}, s)^{g-1}.$$

Let $S(\mathcal{S})$ denote a set of curves on $\mathcal{S}$, and $S(\mathcal{P})$ denote a set of curves on $\mathcal{P}$. We will assume throughout that $S(\mathcal{S})$ contains at least one horizontal curve and each set contains all fibres. Let $S(\mathcal{X})$ denote the union:

$$S(\mathcal{X}) = S(\mathcal{S}) \cup S(\mathcal{P}).$$

We will define an analytic adelic space on $\mathcal{X}$ as the following product

$$\mathbb{A}(\mathcal{X}, S(\mathcal{X})) = \mathbb{A}(\mathcal{S}, S(\mathcal{S})) \times \overbrace{\mathbb{A}(\mathcal{P}, S(\mathcal{P})) \times \cdots \times \mathbb{A}(\mathcal{P}, S(\mathcal{P}))}^{g-1 \text{ copies}}.$$

To avoid cumbersome notation, for an arithmetic surface $\mathcal{A}$ and a set S of curves on $\mathcal{A}$ we will use the notation

$$T(\mathcal{A}, S) = (\mathbb{A}(\mathcal{A}, S) \times \mathbb{A}(\mathcal{A}, S))^{\times}.$$

Note that if S contains only finitely many horizontal curves on $\mathcal{S}$ then the following integral converges for $s > 1$:

$$\int_{T(\mathcal{X})} (f \coprod h)(\alpha)|\alpha|^s d\mu(\alpha),$$

where the measure on $T(\mathcal{X})$ is simply the product measure on the multiplicative adelic groups. Indeed, in the notation of the previous chapter, this is equal to the zeta integral

$$\zeta(\mathcal{S}, S, f, h, |\,|^s).$$

Often, we will omit the set of curves from the notation, and simply use $T(\mathcal{A}, S)$ and

$$\zeta(f, h, s) := \zeta(\mathcal{S}, S, f, h, |\,|^s).$$

Due to the presence of a horizontal curve in $S(\mathcal{S})$, we have a surjective module on $T(\mathcal{X})$,

$$|\,| : T(\mathcal{X}) \to \mathbb{R}_+^\times,$$

given as the product of modules on $\mathcal{S}$ and $\mathcal{P}$, which are modified at horizontal curves as in the previous chapter. $T_1(\mathcal{X})$ denotes the kernel of this module, namely

$$T_1(\mathcal{X}) = \{x \in T(\mathcal{X}) : |x| = 1\}.$$

We may choose a splitting

$$T(\mathcal{X}) \cong \mathbb{R}_+^\times \times T_1(\mathcal{X}).$$

The aim is to integrate over $T_1(\mathcal{X})$. In order to do so, we must first consider a finite subset $S^0 \subset S(\mathcal{X})$ containing at least one horizontal curve. S^0 can be decomposed into a union

$$S^0(\mathcal{S}) \cup S^0(\mathcal{P}),$$

where the first set contains only curves on $\mathcal{S}$ and the second only curves on $\mathcal{P}$. For such an S^0, its complement will be denoted $S_0 = S(\mathcal{X}) - S^0$. We define

$$T_{S^0}(\mathcal{X}) = \prod_{y \in S^0(\mathcal{S})} (\mathbb{A}(\mathcal{S}, y) \times \mathbb{A}(\mathcal{S}, y))^\times \prod_{y \in S^0(\mathcal{P})} \prod_{i=1}^{g-1} (\mathbb{A}(\mathcal{P}, y) \times \mathbb{A}(\mathcal{P}, y))^\times.$$

Again, we have a surjective map

$$|\,|_{S^0} : T_{S^0}(\mathcal{X}) \to \mathbb{R}_+^\times,$$

defined in the obvious manner. Its kernel is denoted

$$T_{S^0,1}(\mathcal{X}),$$

and we have a splitting

$$T_{S^0}(\mathcal{X}) \cong \mathbb{R}_+^\times \times T_{S^0,1}(\mathcal{X}).$$

Let $p(S^0)$ denote the product of projections to one-dimensional adelic spaces as introduced in the previous chapter. We will fix a Haar measure on $p(T_{S^0,1}(\mathcal{X}))$ such that the Haar measure on $p(T_{S^0}(\mathcal{X}))$ is the product of this Haar measure and that on $\mathbb{R}_+^\times$, and let $\mu(T_{S^0,1})$ denote the lift of this Haar measure. For an integrable $\mathscr{F}$ on $T(\mathcal{X})$, for example $\mathscr{F} = f \coprod h$ defined by the functions in the previous chapter, let

$$\int_{T_1(\mathcal{X})} \mathscr{F} = \int_{T_{S_0}(\mathcal{X})} \int_{T_{S^0,1}(\mathcal{X})} \mathscr{F}(\alpha^0\gamma) d\mu(T_{S^0,1}) d\mu(T_{S_0}),$$

where $\alpha^0 \in T_{S^0}$ is such that

$$|\alpha^0|_{S^0} = |\alpha|_{S_0}^{-1}.$$

The integral does not depend on the choice of S^0, and we have the following lemma as a consequence of [Fes10, Lemma 43].

Lemma 5.1. *For an integrable function $\mathscr{F}$ on $T(\mathcal{X})$, we have the following*

$$\int_{T(\mathcal{X})} \mathscr{F} = \int_{\mathbb{R}_+^\times} \int_{T_1(\mathcal{X})} \mathscr{F}(x\alpha) d\mu(\alpha) \frac{dx}{x}.$$

In particular, we can decompose the zeta integrals of the previous section as

$$\zeta(f,h,s) = \int_{\mathbb{R}_+^\times} \zeta_x(f \coprod h, s) \frac{dx}{x} \cdot \xi(\mathcal{P},s)^{1-g},$$

where

$$\zeta_x(f \coprod h, s) = \int_{T_1(\mathcal{X})} (f \coprod h)(m_x\alpha)|m_x\alpha|^s d\mu(\alpha).$$

This decomposition is the key to our second calculation of the zeta integral.

Proposition 5.2. *Let f, h be as in the previous chapter, then we may decompose the zeta function as a sum of the form*

$$\zeta(f,h,s) = \eta(s) + \eta(2-s) + \omega(s),$$

where $\eta(s)$ absolutely converges for all s, and so extends to an entire function on $\mathbb{C}$.

Proof. We decompose the multiplicative group $M = \mathbb{R}_+^\times$ of positive real numbers as $M = M^+ \cup M^-$, where

$$M^\pm = \{m \in M : \pm(|m| - 1) \geq 0\}.$$

We give these spaces the measure

$$\mu_{M^\pm} = \begin{cases} \mu_M & \text{on } M - M \cap T_1 \\ \frac{1}{2}\mu_M & \text{on } M \cap T_1. \end{cases}$$

The result then follows directly from

$$\zeta(f,h,s) = \int_{M^+} \zeta_m(f,h,s) d\mu_{M^+}(m) + \int_{M^-} \zeta_m(f,h,s) d\mu_{M^-}(m),$$

and

$$\omega_m(s) = \zeta_m(f,h,s) - |m|^{-2}\zeta_{m^{-1}}(f,h,s),$$

by writing

$$\eta(s) = \int_{M^+} \zeta_m(f,s) d\mu_{M^+}(m),$$
$$\omega(s) = \int_{M^-} \omega_m(s) d\mu_{M^-}(m).$$

□

Let $\{y_i\}$ denote the complete set of horizontal curves in $S(\mathcal{S})$. We will define the adelic boundary function $\mathfrak{h}(\mathcal{S},\{y_i\}) : \mathbb{R}_+^\times \to \mathbb{C}$ as follows

$$\mathfrak{h}(\mathcal{S},\{y_i\})(x) = \int_{T_1(\mathcal{X})} (x^2 f(m_x\gamma) - f(m_x^{-1}\gamma)) d\mu(\gamma),$$

where $m_x \in M \subset T(\mathcal{X})$ is a choice of representative of $x \in \mathbb{R}_+^\times$.
From the above proposition we deduce the following:

Corollary 5.3. *Let $\mathfrak{f}(\mathcal{S},\{y_i\})$ be the inverse Mellin transform of $\mathcal{Z}(\mathcal{S},\{y_i\})$, then*

$$\mathfrak{h}(\mathcal{S},\{y_i\})(x) = \mathfrak{f}(\mathcal{S},\{y_i\})(x)x^2 - \mathfrak{f}(\mathcal{S},\{y_i\})(x^{-1}).$$

In particular

$$\mathfrak{h}(\mathcal{S},\{y_i\})(x) = x^{-1/2}h(\mathcal{S},\{y_i\})(x).$$

In this way, we understand the boundary function h as an adelic integral. The next step is to understand the role of local-global adelic boundaries.

5.2 The Adelic Boundary Term

The second goal of the chapter is to understand the boundary function as a boundary integral (thus motivating the terminology used throughout). This is the first step towards a verification of the mean-periodicity conjecture 2.33 through two-dimensional adelic duality.

We require the notion of "weak" or "initial" topology, and "final" topology, see, for example [Bou66, I, 2.3, 2.4]. If G is a topological group, then the weak topology is the weakest topology with respect to which every character of G is continuous.

Example 5.4. Let k be a number field, to verify the meromorphic continuation and functional equation of $\zeta(k,s)$, Tate utilizes the following decomposition of his zeta integrals:

$$\zeta(f,\chi) = \int_0^\infty \int_{\mathbb{A}_k^\times/k^\times} f(t\alpha)\chi(t\alpha)d\alpha\frac{dt}{t},$$

see [Tat50, Main Theorem 4.4.1]. Let $\mathbb{A}_k$ have the weak topology, then the boundary $\partial k^\times$ of $k^\times$ in $\mathbb{A}_k$ is $\{0\}$. After applying the theta formula, the functional equation of the zeta integrals is equivalent to that of

$$\int_0^1 h_f(x)x^s\frac{dx}{x},$$

where the boundary function h_f has the following integral representation

$$h_f(x) := -\int_{\gamma\in\mathbb{A}_k^1/k^\times}\int_{\beta\in\partial k^\times}(f(x\gamma\beta) - x^{-1}\widehat{f}(x^{-1}\gamma\beta))d\mu(\beta)d\mu(\gamma).$$

Explicitly, this is the following rational function:

$$-\mu(\mathbb{A}_k^1/k^\times)(f(0) - x^{-1}\widehat{f}(0)).$$

We will need to use a two-dimensional analogue of the inclusion $k^\times \hookrightarrow \mathbb{A}_k^\times$.

Let $\mathcal{A}$ be an arithmetic surface with function field K. If y is a curve on $\mathcal{A}$, the field $K_y = \mathrm{Frac}(\widehat{\mathcal{O}_y})$ is a complete discrete valuation field whose residue field is the global field $k(y)$. It is therefore neither truly local, nor truly global in nature. For all closed points $x \in y$, we have an embedding

$$K_y \hookrightarrow K_{x,y},$$

which together induce a diagonal embedding

$$K_y \hookrightarrow \prod_{x\in y} K_{x,y}.$$

For a curve y on $\mathcal{A}$, let $\mathbb{B}(\mathcal{A}, y)$ denote the intersection of the image of this embedding with $\mathbb{A}(\mathcal{A}, y)$. The counting measure on $k(y)$ lifts to an $\mathbb{R}((X))$-valued measure on $\mathbb{B}(\mathcal{A}, y)$.

Let S^0 be a finite set of curves on $\mathcal{A}$, and define

$$\mathbb{B}(\mathcal{A}, S^0) = (\prod_{y \in S^0} \mathbb{B}(\mathcal{A}, y)) \cap \mathbb{A}(\mathcal{A}, S^0).$$

We integrate on $\mathbb{B}(\mathcal{A}, S^0)$ with the measure induced from the product measure on each $\mathbb{B}(\mathcal{A}, y)$, for $y \in S^0$. We then take the product measure on $\mathbb{B}(\mathcal{A}, S^0) \times \mathbb{B}(\mathcal{A}, S^0)$.

Let F be a two-dimensional local field, and let ψ be a choice of character such that all continuous characters of F are of the form

$$\psi_a : \alpha \mapsto \psi(a\alpha),$$

for $a \in F$. For an integrable function f on F, the Fourier transform $\mathcal{F}(f)$ with respect to ψ is defined by

$$\mathcal{F}(f)(\beta) = \int_F f(\alpha)\psi(\alpha\beta)d\alpha.$$

In particular, this applies to fields of the form $K_{x,y}$ and we denote the Fourier transform on these fields by $\mathcal{F}_{x,y}$. For any integrable function f_y on $\mathbb{A}(\mathcal{A}, y)$, we may define

$$\mathcal{F}_y(f_y) = \otimes_{x \in y} \mathcal{F}_{x,y}(f_{x,y}).$$

By [Fes10, Proposition 32], we have a "summation formula"[1]:

Proposition 5.5. *Let S^0 be a finite set of curves on an arithmetic surface $\mathcal{A}$, and f be an integrable function on $\mathbb{B}(\mathcal{A}, S^0)$ then*

$$\int f(\alpha\beta)d\mu_{\mathbb{B}\times\mathbb{B}}(\beta) = \frac{1}{|\alpha|}\int \mathcal{F}(f)(\alpha^{-1}\beta)d\mu_{\mathbb{B}\times\mathbb{B}}(\beta).$$

Let y be a curve on $\mathcal{A}$, we introduce the notation

$$T_0(\mathcal{A}, y) = \mathbb{B}(\mathcal{A}, y)^\times \times \mathbb{B}(\mathcal{A}, y)^\times \subset T(\mathcal{A}, y).$$

We may take the product measure on $\mathbb{B}(\mathcal{A}, y) \times \mathbb{B}(\mathcal{A}, y)$, which induces the subspace measure M_y on $T_0(\mathcal{A}, y)$. For a finite subset S^0 of curves on $\mathcal{A}$, let

$$T_0(\mathcal{A}, S^0) = \prod_{y \in S^0} T_0(\mathcal{A}, y) \subset T(\mathcal{A}, S^0).$$

[1] The semi-global adelic object $\mathbb{B}$ is discrete in $\mathbb{A}$.

On $T_0(\mathcal{A}, S^0)$, we introduce the measure

$$\mu(T_{0,S^0}) = \prod_{y \in S^0} (q_y - 1)^{-2} M_y.$$

Finally, let S be a set containing all fibres and finitely many horizontal curves, we define

$$T_0(\mathcal{A}, S) = \prod_{y \in S} T_0(\mathcal{A}, y).$$

On this space, we integrate using the following rule

$$\int_{T_0(\mathcal{A},S)} \mathscr{F} = \lim_{S_0 \subset S} \int_{T_0(\mathcal{A},S^0)} \mathscr{F}.$$

We have a filtration

$$T_0(\mathcal{A}, S) \subset T_1(\mathcal{A}, S) \subset T(\mathcal{A}, S).$$

In [Fes10, Section 3.5], Fesenko introduces a measure on the quotient such that, for an integrable function g on $T(\mathcal{A}, S)$:

$$\int_{T_1(\mathcal{A},S)} g = \int_{T_1(\mathcal{A},S)/T_0(\mathcal{A},S)} \int_{T_0(\mathcal{A},S)} g(\gamma\beta) d\mu(\beta) d\mu(\gamma)$$

Let $S^0 \subset S$ be a finite subset. We endow $\mathbb{A}(\mathcal{A}, S^0) \times \mathbb{A}(\mathcal{A}, S^0)$ with the weakest topology such that each character lifted by p is continuous. With respect to this topology, we will call the boundary $\partial T_0(\mathcal{A}, S^0)$ of $T_0(\mathcal{A}, S^0) \subset \mathbb{A}(\mathcal{A}, S^0) \times \mathbb{A}(\mathcal{A}, S^0)$ the "weak boundary". We are interested in an inductive limit of these weak boundaries:

$$\partial T_0(\mathcal{A}, S) = \bigcup_{S_0 \subset S} T_0(\mathcal{A}, S^0),$$

where the union runs over finite subsets $S^0 \subset S$. If g is an integrable function on $\mathbb{A}(\mathcal{A}, S) \times \mathbb{A}(\mathcal{A}, S)$, then one defines

$$\int_{\partial T_0(\mathcal{A},S)} g = \lim_{S_0 \subset S} \int_{\partial T_0(\mathcal{A},S^0)} g,$$

where

$$\int_{\partial T_0(\mathcal{A},S^0)} g = d(S^0) \int_{\partial(\mathbb{B}(\mathcal{A},S^0) \times \mathbb{B}(\mathcal{A},S^0))^\times} g d\mu_{\mathbb{B}(\mathcal{A},S^0) \times \mathbb{B}(\mathcal{A},S^0)},$$

$$d(S^0) = \prod_{y \in S^0} (q_y - 1)^{-2}.$$

The limit expressing the integral is only finite in exceptional circumstances. One such case is expressed by the so-called "two-dimensional theta formula".

Theorem 5.6. *Let $\mathcal{A}$ be an arithmetic surface, S denote a set of curves on $\mathcal{A}$, and f be an integrable function on $\mathbb{A}(\mathcal{A}, S) \times \mathbb{A}(\mathcal{A}, S)$, then*

$$\int_{T_0(\mathcal{A})} (f(\alpha\beta) - \mathcal{F}(f_\alpha)(\beta))d\mu(\beta) = \int_{\partial T_0(\mathcal{A})} (\mathcal{F}(f_\alpha(\beta)) - f(\alpha\beta))d\mu(\beta).$$

One applies this result to $\mathcal{A} = \mathcal{X}$ and $g = f \coprod h$. Consequently, one obtains the boundary integral contribution to the boundary term. It transpires that this boundary integral knows much about the analytic properties of zeta - this will be discussed in chapter 7.

Corollary 5.7. *Let $\{y_i\}$ be the set of horizontal curves in S. We may decompose the boundary integral as follows*

$$\mathfrak{h}(\mathcal{S}, \{y_i\})(x) = \mathfrak{h}_1(\mathcal{S}, \{y_i\})(x) + \mathfrak{h}_2(\mathcal{S}, \{y_i\})(x),$$

where

$$\mathfrak{h}_1(x) = \int_{T_1(\mathcal{X},S)} (|\alpha^{-1}| - 1) f(m_x^{-1}\alpha^{-1}) d\mu(\alpha)$$

$$\mathfrak{h}_2(x) = x^2 \int_{[T_1/T_0](\mathcal{X},S)} \int_{\partial T_0(\mathcal{X},S)} (|m_x\gamma|^{-1} f(m_x^{-1}\nu^{-1}\gamma^{-1}\beta) - f(m_x\gamma\beta)) d\mu(\beta) d\mu(\gamma),$$

and m_x are lifts of $x \in \mathbb{R}_+^\times$ to $T(\mathcal{X})$, and ν is as in [Fes10, Section 3.4].

For elliptic curves, this result was first deduced by Fesenko. An analogous decomposition, not involving adelic integrals, appeared in [FRS12, Remark 5.11].

CHAPTER 6

Outlook

We now provide a rough overview of the new perspectives opened by this thesis and the inspiring related works. The material found here is somewhat speculative.

Zeta Functions, Distributions and the GRH

We have not yet given much consideration to the poles and zeros of zeta functions of arithmetic surfaces. As well as the conditions of previous chapters, for this discussion we will assume moreover that the generic fibre is geometrically irreducible.

According to an old idea of Hilbert and Pólya, one can hope to interpret the nontrivial zeros of the Riemann zeta function as the spectrum of a self-adjoint operator on a Hilbert space. A result along these lines was first achieved by Connes in [Con99], building on the work of Weil re-interpreting the results Tate's thesis as statements in terms of tempered distributions [Wei56]. The results of Connes were extended independently by Soulé and Deitmar to certain automorphic L-functions [Sou01], [Dei01]. When it comes to these generalized Riemann hypotheses, there is thus much interest in associating zeta functions to distributions. For the zeta functions of arithmetic schemes, one could hope to interpret the zeta integrals introduced by Fesenko and in chapter 4 in this way.

A key idea in the prospective proof of generalized Riemann hypotheses is that of "positivity" of the Pólya-Hilbert operators. This notion also features strongly in an approach we now state, first appearing in Fesenko's pioneering paper [Fes10]. The statement can be formulated independently of adelic analysis, as in [FRS12, Proposition 4.10]. The main theorem is:

Theorem 6.1. *Assume that $L(C,s)$ is continued meromorphically to $\mathbb{C}$ and satisfies the conjectural functional equation. In addition assume that there exists $x_0 > 0$ such that the function*

$$(-x\frac{d}{dx})^4 h_{\mathcal{S}}(x)$$

does not change its sign in $(0, x_0)$ and $L(C,s)$ has no real zeros in $(1,2)$. Then, all poles of $Z(\mathcal{S}, \{k_i\}, s)$ in the critical strip lie on the line $\Re(s) = 1$.

This is supported by [Suz11] - in which it is shown that positivity is also a necessary condition under reasonable hypotheses. It would be interesting to understand the relationship between positivity of the boundary term and positivity of the operators introduced by Connes, Soulé and Deitmar.

Langlands Program

The analytic properties of L-functions are intimately connected to their automorphic properties. Mean-periodicity therefore surely plays a role in the Langlands program. For example, one may be able to view mean-periodicity as an extension of the Rankin-Selberg method, and use it to deduce the continuation of more exotic L-functions. When generalized Fourier series are available, one can hope to compare to the discrete spectra of automorphic representations. One natural question is, to what does the polynomial contribution correspond? More ambitiously, one could hope to use the convolution equations provided by mean-periodicity as a constraint on the matrix coefficients of the automorphic representations associated to algebraic varieties. The difficulty here is proving an appropriate converse theorem for mean-periodicity, which only grants meromorphic continuation. On the other hand, meromorphic continuation is suitable for the "potential" modularity results briefly mentioned in the introduction, and maybe mean-periodicity has a suitable role to play there.

Of course, in order to apply converse theorems, one expects to include twisted L-functions, at the very least by characters. Another motivation for considering such L-functions is descent - in earlier chapters we have had to extend the base field in order to have suitably simple singularities to perform the adelic integration. Rather than developing a more intricate integration theory, it may be possible to deduce the analytic properties of the zeta function over the original base field by invoking twisted zeta functions. A third, more novel, motivation is the desire to find the image of the

mean-periodicity correspondence, which we will see in section 6.1 also includes functions constructed from twisted zeta functions.

We will now discuss some notions from a prospective "higher dimensional Langlands program", over two-dimensional local and global fields. For a notion of Satake isomorphism for two-dimensional local fields, the reader is referred to [KL04]. Below we will discuss the rudiments of integral representations of $\mathrm{GL}_1(\mathbb{A}(\mathcal{S}))$-automorphic representations. Much remains to be done even in this most basic case. What we consider are "two-dimensional" twisted zeta functions, which are analogues of Artin L-functions for characters of two-dimensional global fields. When the character factors through a Galois group on the residue level, we recover the usual Artin L-functions. This factorization occurs when the character has suitable ramification properties. For sufficiently unramified, abelian characters, we will interpret these new twisted zeta functions as integrals over two-dimensional adelic spaces.

The function field K of an arithmetic surface $\mathcal{S}$ is a two-dimensional global field, and abelian aspects of its arithmetic can be studied through two-dimensional class field theory [KS83b], [KS83a], [Ker11], [KS09] and [Fes10, Theorem 34]. The first pair of papers use higher K-groups, whereas the second pair do not. The final reference describes explicit adelic class field theory.

Let F be a two-dimensional local field. According to [FK00, Section 10], there is an injective map with dense image:

$$K_2^{\mathrm{top}}(F) \to \mathrm{Gal}(F^{\mathrm{ab}}/F),$$

where the topological (Milnor) K_2 group is defined in [FK00, Section 6]. $K_2^{\mathrm{top}}(F)$ was introduced because the reciprocity map $\psi_F : K_2(F) \to \mathrm{Gal}(F^{\mathrm{ab}}/F)$ is not injective. In fact, the topological K_2 group is equal to the quotient $K_2(F)/\ker(\psi_F)$. On the other hand, $\ker(\psi_F)$ is equal to the intersection of all neighbourhoods of 0 in $K_2(F)$ with respect to a certain topology.

The boundary map ∂ from K-theory induces a map, also denoted by ∂:

$$K_2^{\mathrm{top}}(F) \to K_1(\overline{F}) = \overline{F}^{\times},$$

where $\overline{F}$ is the residue field of F. Let $\mathbb{F}_q$ be the second residue field of F. Upon applying ∂ again, we obtain a map

$$v_F : K_2^{\mathrm{top}}(F) \to K_0(\mathbb{F}_q) = \mathbb{Z}.$$

We will refer to an element mapped to 1 under v as a prime of $K_2^{\mathrm{top}}(F)$. v induces a module

$$|\ |_{2,F} : K_2^{\mathrm{top}}(F) \to \mathbb{R}_+^\times$$
$$\alpha \mapsto q^{-v(\alpha)}.$$

This map is continuous in the topologies of [Fes03]. The module $|\ |_{2,F}$ has a certain compatibility with $|\ |_F$, see equation 6.0.3.

Let z be a branch at a point x of an irreducible curve y on $\mathcal{S}$, and let $F = K_{x,z}$. We will use the notations:

$$v_F = v_{x,z},$$
$$|\ |_{2,x,z} = |\ |_{2,F},$$
$$|\ |_{2,x,y} = \prod_{z \in y(x)} |\ |_{2,x,y}.$$

An adelic analogue of $K^{\mathrm{top}}(F)$ was constructed in [Fes10, Section 2], by restricting with respect to the rank 2 structure:

$$J_{\mathcal{S}} = \prod_y {}' \prod_{x \in y} {}' K_2^{\mathrm{top}}(K_{x,y}) \times \prod_\sigma \prod_{\omega \in \mathcal{S}_\sigma} {}' K_2^{\mathrm{top}}(K_{\omega,\sigma}),$$

where y runs over all irreducible curves on $\mathcal{S}$ and σ over all archimedean places of k. The restricted product over closed points ω on the archimedean fibre $\mathcal{S}_\sigma$ means that all but finitely many components lie in $K_2^{\mathrm{top}}(\mathcal{O}_{\omega,\sigma})$. The first pair of restricted products mean that for almost all (x, y), the (x, y)-component of an element in $J_{\mathcal{S}}$ lies in $K_2^{\mathrm{top}}(O_{x,y})$.

Let x be a closed point on an irreducible curve y and let σ be a closed point on the archimedean fibre S_ω, where ω is an archimedean place of k. We have field embeddings

$$K_x \hookrightarrow K_{x,y},$$
$$K_y \hookrightarrow K_{x,y},$$
$$K_\sigma \hookrightarrow K_{\omega,\sigma}.$$

These embeddings induce diagonal maps

$$\Delta : \begin{cases} K_2(K_y) \to K_2^{\mathrm{top}}(K_{x,y}) \\ K_2(K_x) \to K_2^{\mathrm{top}}(K_{x,y}) \\ K_2(K_\sigma) \to K_2^{\mathrm{top}}(K_{\omega,\sigma}) \end{cases}$$

The product of all Δ gives a map to $J_{\mathcal{S}}$, which we use to define the semi-global adelic objects:

$$P_{\mathcal{S}} = \Delta \prod_{y}{}' K_2(K_y) + \Delta \prod_{x}{}' K_2(K_x) + \Delta \prod_{\sigma} K_2(K_\sigma), \tag{6.0.1}$$

where the restricted direct product is short hand for intersection with $\Delta^{-1}(J_{\mathcal{S}})$.
The product of all $|\ |_{2,x,y}$ induces a module on $J_{\mathcal{S}}$. Let $J^1_{\mathcal{S}}$ denote those elements of module 1. Note that

$$P_{\mathcal{S}} \subset J^1_{\mathcal{S}}.$$

We will use the topology on $J_{\mathcal{S}}$ as in [Fes10, theorem 34]. Then, by two-dimensional class field theory, finite order characters of the Galois group $\mathrm{Gal}(K^{\mathrm{ab}}/K)$ correspond to characters of $J_{\mathcal{S}}$ which are trivial on $P_{\mathcal{S}}$.
If x is a closed point on the local branch z of an irreducible fibre y, and $\chi : J_{\mathcal{S}} \to \mathbb{C}^\times$ is a character, let $\chi_{x,z}$ denote the composite

$$K_2^{\mathrm{top}}(K_{x,z}) \hookrightarrow J_{\mathcal{S}} \to \mathbb{C}^\times.$$

If ω is an archimedean point on a horizontal curve y, let $\chi_{\omega,y}$ denote the composite

$$K_2^{\mathrm{top}}(K_{\omega,y}) \to K_2^{\mathrm{top}}(K_{\omega,\sigma}) \hookrightarrow J_{\mathcal{S}} \to \mathbb{C}^\times,$$

where the first map uses the isomorphism $K_{\omega,y} \cong K_{\omega,\sigma}$, for an archimedean fibre $\mathcal{S}_\sigma$.
We are interested in finite order characters χ of the group $J_{\mathcal{S}}/P_{\mathcal{S}}$. We will assume the following:

Assumption 6.2. there is a finite order character χ_0, lifting a character of $J^1_{\mathcal{S}}/P_{\mathcal{S}}$ of the same order, such that

$$\chi(\alpha) = \chi_0(\alpha)|\alpha|_2^s,$$

where χ_0 is of finite order, lifting a character of $J^1_{\mathcal{S}}/P_{\mathcal{S}}$ of the same order.

In this case, both χ_0 and s are both uniquely determined.
Bearing in mind that, for a two-dimensional local field F, $K_2^{\mathrm{top}}(F)$ is the Galois group of the maximal abelian extension F^{ab}/F, we will say that $\chi_{x,z}$ is unramified if it is trivial on the subgroup of $K_2^{\mathrm{top}}(K_{x,z})$ generated by $K_1(O_{x,z})$.
Let x be a (possibly singular) point on a fibre $\mathcal{S}_{\mathfrak{p}}$. On each branch $\mathcal{S}_{\mathfrak{p}}(x)$, let $\pi_{x,z}$ be a prime with respect to $v_{x,z}$. When χ is unramified, let $t = q_x^{-s}$, where $\chi = \chi_0|\ |_2^s$. We have polynomials

$$f_{x,z}(t) = 1 - \chi_{x,z}(\pi_{x,z}) \in \mathbb{C}[t].$$

We define

$$L(\mathcal{S},\chi_0,s) = L(\mathcal{S},\chi) = \prod_{\mathfrak{p}\in \mathrm{Spec}(\mathcal{O}_k)} \prod_{x\in\mathcal{S}_{\mathfrak{p}}}^{\chi_{x,\mathcal{S}_{\mathfrak{p}}}\text{ unramified}} (\gcd\{f_{x,z}\}_{z\in\mathcal{S}_{\mathfrak{p}}(x)})^{-1}.$$

In particular, if $x \in \mathcal{S}_{\mathfrak{p}} =: y$ is nonsingular, then the (x,y)-factor is simple

$$(1-\chi_{x,y}(t))^{-1},$$

where t is a prime of $K_{x,y}$.

This is a two dimensional analogue the Hecke L-function of a character of the idele class group of a number field, and first appeared in [Fes10, Section 3.2]. We will discuss its convergence after our main theorem.

Let F be a two-dimensional local field. An element $\alpha \in \mathcal{O}_F^\times$ can be written in the form $t_1^i u$, where $u \in O_F^\times$, for a choice of local parameter t_1. We have a map

$$\mathfrak{t} : \mathcal{O}_F^\times \times \mathcal{O}_F^\times \to K_2^{\mathrm{top}}(F) \tag{6.0.2}$$

$$(t_1^j u_1, t_2^j u_2) = (j+l)\{t_1,t_2\} + \{t_1,u_1\} + \{u_2,t_2\}.$$

The compatibility of the modules $|\ |$ and $|\ |_2$ is described through this map:

$$|\mathfrak{t}(\alpha_1,\alpha_2)|_2 = |\alpha_1||\alpha_2|. \tag{6.0.3}$$

For a character χ of $K_2^{\mathrm{top}}(F)$, put

$$\chi_{\mathfrak{t}} : F^\times \times F^\times \to \mathbb{C}^\times$$

$$\chi_{\mathfrak{t}}(\alpha) = \begin{cases} \chi\circ\mathfrak{t}(\alpha), & \alpha \in \mathcal{O}_F^\times \times \mathcal{O}_F^\times, \\ 0, & \text{otherwise.} \end{cases}$$

When $F = K_{x,z}$, we will use the notation $\chi_{\mathfrak{t},x,z}$. We will make a second assumption about the character $\chi_{x,z}$.

Assumption 6.3. $\forall x \in z$, $\chi_{0,\mathfrak{t},x,z}$ factors through the residue map:

$$T \to \overline{F}^\times \times \overline{F}^\times.$$

This assumption concerns the ramification of $\chi_{0,x,z}$, as illustrated in the following examples.

Example 6.4. Let F be a two-dimensional local field, and let L be a finite abelian extension of F. A character of $\mathrm{Gal}(L/F)$ corresponds to a character of $K_2^{\mathrm{top}}(F)$ which vanishes on $N_{L/F}K_2^{\mathrm{top}}(F)$ by two-dimensional class field theory. Consider two possibilities:

- L/F is an unramified extension of complete discrete valuation fields. So, $\overline{L}/\overline{F}$ is separable and $|\overline{L} : \overline{F}| = |L : F|$.
- L/F is a totally unramified extension in the second parameter. In this case $\overline{L} = \overline{F}$, $p \nmid |L : F|$ and $t_2 \in N_{L/F}L^\times$.

In either case, the induced character $\chi_{\mathfrak{t}}$ of T factors through the residue map according to [Mor08, Examples 7.11, 7.12].

Under assumption 6.3, we can write

$$\chi_{\mathfrak{t}}(a,b) = \omega_1(\overline{a})\omega_2(\overline{b}),$$

where ω_1 and ω_2 are quasi-characters of the local field $\overline{F}$, the residue field of F. When $F = K_{x,z}$, we will write $\omega_{1,x,z}$ and $\omega_{2,x,z}$.
Integration against an unramified character see [Fes03, Section 1], [Fes10, Section 1] and [Mor12b, Section 7].

Theorem 6.5. *Let χ be a character of $J_{\mathcal{S}}/P_{\mathcal{S}}$, satisfying assumptions 6.2 and 6.3. Then, for all f, h such that the integrals are defined, there is an entire meromorphic function $\Phi_{\chi,f,h}(s)$ such that*

$$\begin{aligned}\zeta_{na}^{(2)}(\mathcal{S}, f, h, \chi) =& \Phi_{\chi,f,h}(s) A(\mathcal{S},\chi)^{1-s} L(\mathcal{S}, s, \chi_0)^2 Q(s)^2 \\ &\cdot \prod_{y \in S_h} \Lambda(k(y), \omega_{1,y}, s)\Lambda(k(y), \omega_{2,y}, s),\end{aligned}$$

where f and g are the functions from chapter 4, $\Lambda(k(y), \omega, s)$ is a completed Hecke L-function for the Hecke character ω on the number field $k(y)$, S_h is a finite set of horizontal curves, and $\zeta_{na}^{(2)}(\mathcal{S}, f, g, \chi)$ is the following product

$$\prod_{\mathfrak{p} \in Spec(\mathcal{O}_k)} \int_{(\mathbb{A}(\mathcal{S}_\mathfrak{p})^\times)^{\times 2}} f_\mathfrak{p}(\alpha)((\chi_0|\,|_{2,\mathfrak{p}}^s)_{\mathfrak{t}})(\alpha) d\mu(\alpha) (\int_{(\mathbb{A}(\mathcal{P}_\mathfrak{p})^\times)^{\times 2}} g_\mathfrak{p}(\alpha)|\alpha|^s d\mu(\alpha))^{g-1}$$

$$\cdot \prod_{y \in S_h} \int_{(\mathbb{A}(y)^\times)^{\times 2}} f_y(\alpha)\chi_0(\alpha)|\alpha|_y^s d\mu(\alpha) \cdot .$$

Proof. We have

$$L(\mathcal{S},\chi) = \prod_{\mathfrak{p}} L_{\mathfrak{p}}(\mathcal{S},\chi),$$

where

$$L_{\mathfrak{p}}(\mathcal{S},\chi) = \prod_{x\in\mathcal{S}_{\mathfrak{p}}}^{\chi_{x,\mathcal{S}_{\mathfrak{p}}}\text{ unramified}} (\gcd\{f_{x,z}\}_{z\in\mathcal{S}_{\mathfrak{p}}(x)})^{-1}$$

Due to assumption 6.3, if $\mathcal{S}_{\mathfrak{p}}$ is a smooth fibre, then

$$L_{\mathfrak{p}}(\mathcal{S},\chi) = L(k(\mathcal{S}_{\mathfrak{p}}),\omega_{1,\mathfrak{p}},s)L(k(\mathcal{S}_{\mathfrak{p}}),\omega_{2,\mathfrak{p}},s),$$

where, for $i = 1,2$, $\omega_{i,\mathfrak{p}} = \otimes_{x\in\mathcal{S}_{\mathfrak{p}}}\omega_{1,x,\mathcal{S}_{\mathfrak{p}}}$, for some characters on the local field $k(\mathcal{S}_{\mathfrak{p}})_x$. By [Mor08, Proposition 7.3], $L_{\mathfrak{p}}(\mathcal{S},\chi)$ is equal to

$$\int_{(\mathbb{A}(\mathcal{S}_{\mathfrak{p}})^{\times})^{\times 2}} f_{\mathfrak{p}}(\alpha)((\chi_0|\,|_{2,\mathfrak{p}}^{s})_{\mathfrak{t}})(\alpha)d\mu(\alpha)(\int_{(\mathbb{A}(\mathcal{P}_{\mathfrak{p}})^{\times})^{\times 2}} h_{\mathfrak{p}}(\alpha)|\alpha|^{s}d\mu(\alpha))^{g-1}.$$

Finally, let

$$\Phi_{\chi,f,h}(s) = \varepsilon(\mathcal{S},\chi)A(\mathcal{S},\chi)^{1-s}\prod_{\mathfrak{p}\text{ bad}}\frac{1}{\zeta(\mathcal{S},f_{\mathfrak{p}},h_{\mathfrak{p}},\chi)}.$$

The result follows from taking the product over all primes $\mathfrak{p}$ of the integrals. □

The above computation does not incorporate an archimedean contribution - indeed, it is not clear how to do this in general. However, if K/k is a finite Galois extension of number fields and χ is a character of Gal(K/k), we can apply the above theorem to the character ψ lifted to $\mathbb{A}(\mathcal{S},S)$ and $\mathbb{A}(\mathcal{P},S)$. Observe that

$$L(\mathcal{P}_K,\chi,s) = L(K/k,\chi,s)L(K/k,\chi,s),$$

and

$$\Gamma(\mathcal{P},\chi,s)^{1-g} = Q(\chi,s)\Gamma(\mathcal{S},\chi,s),$$

where $Q(\chi,s) \in \mathbb{C}(s)$ is invariant with respect to $s \mapsto 2-s$. Using this, and applying the previous theorem, one obtains

Corollary 6.6. *With the conventions above*

$$\begin{aligned}\xi(\mathbb{P}^1(\mathcal{O}_k),\psi,s)^{1-g}\zeta_{na}^{(2)}(\mathcal{S},f,h,\chi) =& \Phi_{\chi,f,h}(s)A(\mathcal{S},\psi)^{1-s}L(\mathcal{S},s,\psi_0)^2Q(s)^2\Gamma(\mathcal{S},\psi,s)\\ &\cdot\prod_{y\in S_h}\Lambda(k(y),\omega_{1,y},s)\Lambda(k(y),\omega_{2,y},s).\end{aligned}$$

Unfortunately, this integral description is nowhere near detailed enough to investigate the functional equation, however we deduce for such a restricted class of characters that $L(\mathcal{S}, \chi, s)$ is a meromorphic function on the half plane $\Re(s) > 2$ due to the convergence property of the zeta integrals, and $L(\mathcal{S}, \chi, s)$ has meromorphic continuation if and only if the zeta integral does.

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