Critical Points of Discrete Potentials in the Plane and in Space

Robert V. Trickey, MMath.

Thesis submitted to The University of Nottingham
for the degree of Doctor of Philosophy

December 2007

To Joyce Trickey and Patricia Phillips.

Abstract

This thesis looks at various problems relating to the value distribution of certain discrete potentials.

Chapter 1 - Background material is introduced, the motivation behind this work is explained, and existing results in the area are presented.

Chapter 2 - By using a method based on a result of Cartan, the existence of zeros is shown for potentials in both the complex plane and real space.

Chapter 3 - Using an argument of Hayman, we expand on an established result concerning these potentials in the complex plane. We also look at the consequences of a spacing of the poles.

Chapter 4 - We extend the potentials in the complex plane to a generalised form, and establish some value distribution results.

Chapter 5 - We examine the derivative of the basic potentials, and explore the assumption that it takes the value zero only finitely often.

Chapter 6 - We look at a new potential in real space which has advantages over the previously examined ones. These advantages are explained.

Appendix - The results of computer simulations relating to these problems are presented here, along with the programs used.

Acknowledgements

My first thanks go to my supervisor Jim Langley, for his tireless support, encouragement and guidance, and for not laughing when I ask really stupid questions, or present two line proofs of ancient unsolved problems.

Thanks to my family for supporting me through this, and providing a safe haven to fall back on at any point, even though I haven't been visiting nearly as often as I should.

Thanks to all the other mathematical PhD students who have lightened my days with all kinds of mathematical and non-mathematical distractions, in particular thanks to Eleanor, Lawrence, Rich, Tim, Andy and Matt, who have all helped in different ways, whether being travel companions, part of musical ensembles, giving me a place to live, or providing an unending supply of fascinating trivia and online comics.

Thanks to Rachael for helping me get here, and all my friends who I haven't seen or phoned nearly often enough. Thanks to Antony Clare for helping me rediscover something I really care about.

Finally, thanks to Ileana, as without you this couldn't have ended nearly so smoothly.

Contents

CHAPTER 1

Introduction

This chapter consists of the background mathematics which is used throughout this thesis, and also introduces the reader to the problem which is the main focus of this work. Work by other authors in the direction of this problem is also highlighted. Since the majority of results in this chapter are well established, proofs are omitted, but can be found in the cited sources.

1.1 Nevanlinna Theory

Nevanlinna Theory is primarily concerned with describing the growth of meromorphic functions. The results here follow [11, 17], where the proofs can be found. The growth of meromorphic functions is more difficult to describe than that of entire functions, as the Maximum Principle (see [24] or any elementary text on complex analysis) does not apply.

If a function f is entire, we can consider the two real valued functions of the positive variable r.

$$\begin{aligned} m_o(r,f) &= \min_{|z|=r} |f(z)| \\ M(r,f) &= \max_{|z|=r} |f(z)|. \end{aligned}$$

By the Maximum principle, $M(r,f)$ gives the maximum value the function takes on the disc $\{|z| \leq r\}$, not just the boundary, and is hence non-decreasing.

This leads us to the following definition.

Definition 1.1.1. ***Order of an Entire Function*** - *The order ρ of an entire function is defined as follows:*

$$\rho(f) = \limsup_{r\to\infty} \frac{\log\log M(r,f)}{\log r}$$

Example 1.1.2. *Consider the function $f_k(z) = e^{z^k}$, where $k > 0$ is an integer. Simple calculation gives $M(r, f_k) = e^{r^k}$, and hence $\rho(f_k) = k$.*

Notice that if we multiply f_k by any polynomial, it does not change the order.

In [5], Conway states the following lemma regarding the order of an entire function.

Lemma 1.1.3 ([5])**.** *Assume f is entire, and has finite order α. We write $f(z) = \sum_{n=1}^{\infty} c_n z^n$ as a power series. Then*

$$\frac{1}{\alpha} = \liminf_{n\to\infty} \frac{-\log|c_n|}{n\log n}.$$

From this formula we can see that adding a polynomial will only change finitely many terms in the sequence c_n, and hence leave the limit unchanged.

Turning our attention now to meromorphic functions, we immediately observe that $M(r, f)$ is no longer of use, as these functions may have poles, so the Maximum Principle does not apply, and hence $M(r, f)$ is not non-decreasing. Nevanlinna Theory, amongst other things, provides an alternative functional to $M(r, f)$ which can be used to describe the growth of a meromorphic function, and also gives rise to a definition of order which extends to meromorphic functions, but is equivalent to (1.1.1) in the case of entire functions. The material here mainly follows [17], but see also [11] for a comprehensive approach.

In order to monitor the behaviour of meromorphic functions, we need to observe how often the poles occur. To this end, we describe the following function.

Definition 1.1.4. ***The Pole Counting Function*** $n(r, f)$ - *We define the value of the function $n(r, f)$ as being equal to the number of poles (counting multiplicities) of the function f inside a closed disc of radius r, centred on the origin.*

Clearly $n(r, f)$ is non-decreasing, and if f has finitely many poles, then it will be constant valued for large r. We now expand on this function and introduce the first of the two Nevanlinna Functionals.

Definition 1.1.5. ***The (integrated) Counting Function (Anzahlfunktion)*** *$N(r, f)$ - We define the function $N(r, f)$ as follows:*

$$N(r, f) = \int_0^r \frac{n(t, f) - n(0, f)}{t} dt + n(0, f) \log r.$$

Clearly $N(r, f)$ is non-decreasing, and if f has only finitely many poles then $N(r, f) \sim M \log r$ for large r (where M is the number of poles). In this thesis we shall use $A \sim B$ to mean $A = B(1 + o(1))$. $N(r, f)$ grows according to the number of poles, but does not take into account how fast the function is growing apart from at the poles. To this end we introduce the following, which is the second Nevanlinna Functional.

Definition 1.1.6. ***The Proximity Function (Schmiegungsfunktion)*** *$m(r, f)$ - We define the function $m(r, f)$ as follows:*

$$m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| f(re^{i\theta}) \right| d\theta,$$

where $\log^+ x = \max\{\log x, 0\}$.

The Proximity Function gives us an idea of how large the function f is in an 'averaged' sense.

The particular definitions of these two Nevanlinna Functionals in fact arise out of manipulation of the Poisson-Jensen Formula. See [11, 17] for a full explanation.

Now we define the Nevanlinna Characteristic, which is essential to understanding the growth and value distribution of meromorphic functions in the complex plane.

Definition 1.1.7. ***The Nevanlinna Characteristic*** *$T(r, f)$ - We define $T(r, f)$ as follows:*

$$T(r, f) = N(r, f) + m(r, f),$$

with $N(r, f)$ and $m(r, f)$ as defined above.

Nevanlinna's characteristic function is a remarkably powerful tool. The following lemma describes some elementary, yet highly useful results concerning it. We will use "meromorphic" to mean meromorphic in the plane, unless specifically stated otherwise.

Lemma 1.1.8. *The Nevanlinna Characteristic satisfies the following, for meromorphic f_1 and f_2:*

$$\begin{aligned} T(r, f_1 f_2) &\leq T(r, f_1) + T(r, f_2) \\ T(r, f_1 + f_2) &\leq T(r, f_1) + T(r, f_2) + O(1) \end{aligned}$$

Furthermore,

$$T(r, f) = O(\log r)$$

*is satisfied for a meromorphic f **if and only if** f is a rational function.*

These results lead us to the first of two key results in Nevanlinna Theory which are critical to understanding the behaviour of meromorphic functions.

Theorem 1.1.9 (Nevanlinna's First Fundamental Theorem). *Let f be meromorphic, and $a \in \mathbb{C}$. Then*

$$T\left(r, \frac{1}{f-a}\right) = T(r, f) + O(1).$$

This is an equidistribution theorem. It tells us that f does not have a special affinity with any complex number. We can observe that since $T(r, f)$ grows big as r grows large (indeed for transcendental f it must be growing at more than a logarithmic rate), so this correspondingly tells us that $T\left(r, \frac{1}{f-a}\right)$ must also be big for large r. Splitting this into its two composite functionals, either $N\left(r, \frac{1}{f-a}\right)$ must be big, which implies that f is taking the value a lots of times, or $m\left(r, \frac{1}{f-a}\right)$ must be big, which means that f is very close to the value a on at least part of the circle $|z| = r$.

Example 1.1.10. *Let us look at the function $f(z) = e^z$. Here f is entire and zero free, and simple calculation gives*

$$\begin{aligned} N\left(r, e^z\right) &= N\left(r, \frac{1}{e^z}\right) = 0 \\ m\left(r, e^z\right) &= m\left(r, \frac{1}{e^z}\right) = \frac{r}{\pi}, \end{aligned}$$

and hence

$$T\left(r, e^z\right) = T\left(r, \frac{1}{e^z}\right) = \frac{r}{\pi}.$$

Furthermore, $m\left(r, \frac{1}{f-1}\right)$ is small, but $f(z) = 1$ often (at points $z = \{2\pi i k | k \in \mathbb{Z}\}$).

An important property of $M(r, f)$ which leads to the definition of order for entire functions, is that $M(r, f)$ is non-decreasing. Here we present a result from which we can draw the same conclusion regarding $T(r, f)$.

Theorem 1.1.11 (Cartan [11]). *Let f be a function, non-constant and meromorphic on $|z| < R$, and let $r \in (0, R)$. Then*

$$T(r, f) = \log^+ |f(0)| + \frac{1}{2\pi} \int_0^{2\pi} N\left(r, \frac{1}{f - e^{is}}\right) ds,$$

with minor modifications if $f(0) = 0$.

This result clearly demonstrates that $T(r, f)$ is non-decreasing. Now we can use this to define the order of a meromorphic function.

Definition 1.1.12. ***Order of a Meromorphic Function*** - *The order ρ of a meromorphic function is defined as follows:*

$$\rho(f) = \limsup_{r \to \infty} \frac{\log^+ T(r, f)}{\log r}$$

Sometimes we will refer to this as the **Upper Order**. If we replace the lim sup with lim inf in the definition above, we also get a definition for the **Lower Order** of a meromorphic function, which we refer to as $\lambda(f)$.

If a function has finite order, we can describe the growth even more precisely, by looking at its type.

Definition 1.1.13. ***Order Type*** - *Let f be a meromorphic function of finite order ρ, and let*

$$s = \limsup_{r \to \infty} \frac{T(r, f)}{r^\rho}.$$

Then we say f has minimal type, mean type, *or* maximal type *if $s = 0$, s is finite non-zero, or $s = \infty$, respectively.*

Looking at Example 1.1.10 above, we can clearly see that

$$\rho(e^z) = 1$$

which agrees with the order obtained when using the definition of order for entire functions (see Definition 1.1.1). Furthermore this function clearly has mean type.

Comparing the example above with Example 1.1.2, the two different definitions of order give the same value for an entire function. In fact this is always the case. This fact is immediate from the next Theorem.

Theorem 1.1.14. *Let f be meromorphic and non-constant in $|z| \leq R$. If f has no poles in $|z| \leq R$ and $0 < r < R$, then*

$$T(r, f) \leq \log^+ M(r, f) \leq \left(\frac{R + r}{R - r}\right) T(R, f).$$

The next result concerns the Proximity Function of the logarithmic derivative $\left(\frac{f'}{f}\right)$, and is used many times in this thesis.

Lemma 1.1.15 (Lemma of the Logarithmic Derivative)**.** *Let f be non-constant and meromorphic in the plane. Then there are positive constants C_1 and C_2, independent of f, such that we have*

$$m\left(r, \frac{f'}{f}\right) \leq C_1 \log r + C_2 \log T(r, f),$$

as r tends to ∞ outside a set of finite measure.

Next comes the second of Nevanlinna's two key results, which is more powerful than the first. Before we can state it, we require the following notation.

$\overline{n}(r, f)$ counts the poles of f in $|z| \leq r$, but ignores multiplicities, and

$$\begin{aligned} \overline{N}(r, f) &= \int_0^r \frac{\overline{n}(t, f) - \overline{n}(0, f)}{t} dt + \overline{n}(0, f) \log r \\ N_1(r, f) &= N(r, f) - \overline{N}(r, f) + N\left(r, \frac{1}{f'}\right), \end{aligned}$$

and we use $S(r, f)$ to signify any term which can be described as $O(\log^+(rT(r, f)))$ outside a set of finite measure, and can hence be considered small compared to $T(r, f)$.

Using this notation, we can state the second key result as follows.

Theorem 1.1.16 (Nevanlinna's Second Fundamental Theorem)**.** *Let f be meromorphic in the plane, and let b_j, $j = 1, ..., s$ be finitely many distinct values in $\mathbb{C}^*$ $(= \mathbb{C} \cup \infty)$. Then*

$$\sum_{j=1}^{s} m\left(r, \frac{1}{f - b_j}\right) \leq 2T(r, f) - N_1(r, f) + S(r, f).$$

By adding in $N\left(r, \dfrac{1}{f-b_j}\right)$ *terms to both sides, and then applying The First Fundamental Theorem, we can reformulate this as*

$$(s-2)T(r,f) \leq \sum_{j=1}^{s} N\left(r, \frac{1}{f-b_j}\right) - N_1(r,f) + S(r,f).$$

The usefulness of this result in Value Distribution theory is easily demonstrated with the following definition.

Definition 1.1.17. ***Nevanlinna Deficiency*** - *Given a meromorphic* f*, and* $a \in \mathbb{C}$*, we can define the Nevanlinna Deficiency of* f *at* a*,* $\delta(a,f)$*, to be*

$$\delta(a,f) = \liminf_{r\to\infty} \frac{m\left(r, \frac{1}{f-a}\right)}{T(r,f)} = 1 - \limsup_{r\to\infty} \frac{N\left(r, \frac{1}{f-a}\right)}{T(r,f)}.$$

The deficiency takes values in $[0,1]$ and can be viewed as a measure of how rarely the function f takes the value a. If a transcendental meromorphic f does not take the value a, or only takes it finitely often, then $\delta(a,f) = 1$. However, the converse of this is not true.

Example 1.1.18. *The function* e^z *does not take the value* 0 *or* ∞ *anywhere in the plane, and hence*

$$\delta(0, e^z) = \delta(\infty, e^z) = 1.$$

However it is possible for a function to take a value a *infinitely often, and still have* $\delta(a,f) = 1$*, consider*

$$f(z) = e^{z^2} \tan z.$$

A routine calculation shows that

$$\delta(0,f) = \delta(\infty,f) = 1,$$

but f *clearly takes both* 0 *and* ∞ *infinitely often.*

This example is constructed by having a higher order function with deficient values at 0 and ∞ (in this case e^{z^2}) multiplied by a lower order function without deficient values (in this case $\tan z$). The zeros and poles provided by $\tan z$ cannot occur regularly enough to be significant next to the higher order e^{z^2} and hence 0 and ∞ remain deficient.

Using Definition 1.1.17, we can deduce an immediate corollary of Nevanlinna's Second Fundamental Theorem, which can be considered a much more powerful version of classical distribution results such as Picard's Theorem.

Corollary 1.1.19 (The Defect Relation)**.** *Let f be any non constant transcendental meromorphic function. Then*

$$\sum_{a\in\mathbb{C}^*} \delta(a, f) \leq 2.$$

1.2 Harmonic and Subharmonic Functions

The study of harmonic functions has long been integral to understanding the behaviour of analytic functions. In the study of meromorphic functions, it is frequently beneficial to extend this study to *Subharmonic Functions.* This class of functions, related to harmonic functions, is described below. The results here follow [11, 13, 17], where the proofs can be found.

Before we can define subharmonic functions, we need to describe some of their properties.

Definition 1.2.1. ***Upper Semi-Continuous*** - *Let X be a metric space. We call a function $u : X \to [-\infty, \infty)$* upper semi-continuous *if for every real t the set $\{x \in X : u(x) < t\}$ is open.*

Definition 1.2.2. ***Sub-Mean-Value-Property*** - *Let D be a domain. An upper semi-continuous function $u : D \to [-\infty, \infty)$ is said to have the* sub-mean-value-property *if for each point $z_0 \in D$, there is an $r_0 > 0$ such that*

$$u(z_0) \leq \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{it})dt \quad (0 < r \leq r_0).$$

This is clearly a generalisation of the mean-value-property which harmonic functions satisfy.

With these two properties defined, we can define subharmonic functions quite simply.

Definition 1.2.3. ***Subharmonic Function*** - *An function u is called* subharmonic *on a domain D if $u : D \to [-\infty, \infty)$, and*
1) u is upper semi-continuous in D,
and
2) u has the sub-mean-value-property in D.

It is worth noting that all harmonic functions are also subharmonic. Also there is a related class of *Superharmonic Functions*, which can be defined as follows: u is superharmonic, if $-u$ is subharmonic.

Example 1.2.4. *For an example of a subharmonic function, if we take any function f, analytic in a domain D, and then look at*

$$u(z) = \log |f(z)|.$$

Then u is subharmonic in D.

Now we look at some key properties of functions of this type.

Definition 1.2.5. $A(r,u)$**,** $B(r,u)$ - *Let u be a function subharmonic on $\mathbb{C}$. Then we define $A(r,u)$ and $B(r,u)$ as follows.*

$$\begin{aligned} A(r,u) &= \inf\{u(re^{i\theta}) : \theta \in [0,2\pi)\} \\ B(r,u) &= \sup\{u(re^{i\theta}) : \theta \in [0,2\pi)\}. \end{aligned}$$

Remark - For Harmonic Functions we can replace inf and sup above with min and max, as they are continuous.

$B(r,u)$ can be considered as a replacement for $T(r,f)$ for subharmonic functions, and we use it accordingly to define the order of a subharmonic function.

Definition 1.2.6. ***Order of a Subharmonic Function*** - *Given a subharmonic function u, we define its order as*

$$\lambda(u) = \limsup_{r\to\infty} \frac{\log B(r,u)}{\log r}.$$

The next two results are very important, and can be viewed as equivalents of the Maximum Principle, and Liouville's Theorem.

Theorem 1.2.7. *Let D be a domain in $\mathbb{C}$ and define $\partial_\infty D$ to be the set of all boundary points of D in $\mathbb{C}^*$. Now let u be subharmonic in D with*

$$\limsup_{z\to\zeta, z\in D} u(z) \leq M \in [-\infty, \infty)$$

for every $\zeta \in \partial_\infty D$; then either $u(z) \equiv M$ on D, or $u(z) < M$ for all $z \in D$.

Theorem 1.2.8. *Let u be subharmonic in the plane. Suppose*

$$B(r, u) < M < \infty,$$

for all $r \in [0, \infty)$. Then u is constant.

The final theorem in this section is a very powerful result concerning subharmonic functions of small order, and gives a strict relationship between their lower growth and upper growth functionals. This can also be applied to entire functions.

Theorem 1.2.9 (The $\cos \pi\rho$ Theorem for Subharmonic Functions). *Suppose that $0 \leq \lambda < \alpha < 1$. Then if $u(z)$ is subharmonic of order λ and not constant in the plane, we have*

$$A(r, u) > \cos(\pi\alpha) B(r, u),$$

for a set E of r such that

$$\underline{\Lambda}(E) \geq \left(1 - \frac{\lambda}{\alpha}\right).$$

Here $\underline{\Lambda}(E)$ is the lower logarithmic density of E, defined as

$$\underline{\Lambda}(E) = \liminf_{r\to\infty} \frac{\int_1^r \chi_E(t)\frac{dt}{t}}{\log r},$$

where $\chi_E(t)$ is 1 if t is in E, and 0 otherwise.

1.3 Discrete Potentials in the Complex Plane

The majority of this thesis looks at functions corresponding to the fields generated by arrangements of point masses or evenly charged wires in two or more dimensional space.

Motivation

Following [6], in $\mathbb{R}^m$, if at each x_k we place a point mass, magnitude $a_k > 0$, the gravitational force at a point x is given by

$$P(x) = \sum_{k=1}^{\infty} \frac{a_k(x_k - x)}{|x - x_k|^m}. \tag{1.3.1}$$

In particular for $m = 3$ this corresponds to the inverse square law of gravitation. This series converges absolutely when

$$\sum_{k=1}^{\infty} \frac{a_k}{|x_k|^{m-1}} < \infty.$$

Remark - Throughout this thesis, where series are referred to as converging, the reader may take this to mean converging absolutely.

The main points of interest are the zeros of these functions, which correspond to the equilibrium points in the gravitational field. These functions can also be used to model point charge electrostatic distributions, but in electrostatic models, you may require some of the a_k to be negative.

The majority of the work in this thesis focuses on the 2-dimensional case. Look at the following meromorphic function,

$$f(z) = \sum_{k=1}^{\infty} \frac{a_k}{z - z_k}.$$

We can write

$$\begin{aligned} -\overline{f(z)} &= -\overline{\sum_{k=1}^{\infty} \frac{a_k}{z - z_k}} \\ &= -\sum_{k=1}^{\infty} \frac{\overline{a_k}}{\overline{(z - z_k)}} \\ &= \sum_{k=1}^{\infty} \frac{\overline{a_k}(z_k - z)}{\overline{(z - z_k)}(z - z_k)} \\ &= \sum_{k=1}^{\infty} \frac{\overline{a_k}(z_k - z)}{|z - z_k|^2}. \end{aligned}$$

which is in direct analogy with (1.3.1), i.e. the meromorphic function f has the same zeros as the corresponding discrete potential P.

There is a further application of these functions in two dimensional electrostatics: If an infinite wire carrying a charge density a_k is placed perpendicular to the complex plane at each z_k, then the resulting electrostatic field is given by the vector $2\overline{f(z)} = 2(\mathrm{Re}(f(z)), -\mathrm{Im}(f(z)))$. This is calculated by working out the force on a point z from any point on the wire, then integrating this expression over the entire wire (see [16]). The zeros of the function f will correspond to equilibrium points in this electrostatic field.

An important observation, which is valid in all dimensions, is that if the a_k are all positive, then the zeros will lie in their convex hull. This is obvious when we consider the problem as a charge distribution.

This class of meromorphic functions can be extended further, by allowing the a_k to take any values in $\mathbb{C}$. Although these functions no longer have an obvious application, because a complex-valued charge is at best ill-defined, they are interesting as a problem in value distribution function theory nevertheless. In this thesis, we will refer to this larger class as *Keldysh Functions*, as he was one of the primary exponents of their study (see [8]). We shall define them as follows.

Definition 1.3.1. ***Keldysh Functions*** - *A Keldysh function is one which can be written in the following way, with z_k distinct:*

$$f(z) = \sum_{k=1}^{\infty} \frac{a_k}{z - z_k}, \quad z_k, a_k \in \mathbb{C} \setminus \{0\}, \quad z_k \to \infty, \quad \sum_{k=1}^{\infty} \left| \frac{a_k}{z_k} \right| < \infty. \tag{1.3.2}$$

Since we are interested in the zeros of these functions, it is reasonable to ask if there are any reasons why we should think they have any zeros at all. The evidence for the existence of zeros comes from when we look at the case of replacing the infinite sum in (1.3.2) above with a finite sum. This clearly eliminates the need for any convergence criteria or limiting behaviour of the z_k.

Example 1.3.2. *Let $z_1, ..., z_N \in \mathbb{C}$, distinct,*

$$g(z) = \sum_{k=1}^{N} \frac{a_k}{z - z_k}, \quad a_k \in \mathbb{C} \setminus 0, \quad \sum_{k=1}^{N} a_k \neq 0.$$

Then g has, counting multiplicities, $N-1$ zeros.

Explanation

We can observe that g is in fact a rational function, as we can write it as

$$g(z) = \frac{\displaystyle\sum_{j=1}^{N} a_j \prod_{k \neq j} (z - z_k)}{\displaystyle\prod_{k=1}^{N} (z - z_k)}.$$

Clearly the denominator of the fraction is a polynomial of degree N. The numerator is a polynomial of degree $N-1$, because the multiplicative products consist of $N-1$ linear terms, so each of these products gives rise to a polynomial of degree $N-1$, and when we sum all N of these, we get a polynomial of degree $N-1$ with leading coefficient $\sum a_k$, non-zero by assumption. Using The Fundamental Theorem of Algebra, we know that this function will have (counting multiplicities) $N-1$ zeros, as the numerator and the denominator will have no common factors. This tells us that as the number of z_k increase, so too does the number of zeros.

The obvious next question to ask is, does a Keldysh Function always have zeros? The answer to this is no, there do in fact exist some which do not take the value zero anywhere, as the following example, which is from [3], demonstrates.

Example 1.3.3. *Let*

$$f(z) = \frac{2}{\sqrt{z} \sin 2\sqrt{z}}.$$

Then f can be written in the following way, i.e. as a Keldysh function with one of the z_k at zero:

$$f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} \frac{(-1)^k 2}{z - \frac{k^2 \pi^2}{4}}.$$

To write f in this way, we merely need make the following observation. We have

$$\begin{aligned} f(z) &= \frac{F'(z)}{F(z)} \\ \text{where} \quad F(z) &= \tan^2 \sqrt{z} \end{aligned}$$

Routine calculation shows that $F(z)$ is meromorphic and $\rho(F) = \frac{1}{2} < 1$. Now we use The Weierstrass Factorisation Theorem (see [5, p.170]) to write it in the following manner. Here u_k are the zeros of F, and v_k are the poles, which are all of order two, with the exception of the zero at the origin which has order one.

$$F(z) = z \prod_{k=1}^{\infty} (z-u_k)^2 \prod_{k=1}^{\infty} (z-v_k)^{-2}$$

If we then calculate the logarithmic derivative of the function in this form, we immediately obtain

$$f(z) = \frac{1}{z} + 2\sum_{k=1}^{\infty} \frac{1}{(z-u_k)} - 2\sum_{k=1}^{\infty} \frac{1}{(z-v_k)},$$

which is in the stated form.

Notation

Throughout this thesis, the following notation shall be used:

$$\begin{aligned} n(r) &= \sum_{|z_k|\le r} |a_k|; \\ \overline{n}(r) &= \sum_{|z_k|\le r} 1, \end{aligned} \tag{1.3.3}$$

Now we prove several basic lemmas concerning $n(r)$, all under the assumption that $|z_k| \ge 2\delta > 0$ for all k, and the generalised convergence criteria $\sum_{k=1}^{\infty} \frac{|a_k|}{|z_k|^m} < \infty$, where $m \in \mathbb{N}$.

Lemma 1.3.4. *We assert that the following hold:*

$$\lim_{r\to\infty} \int_r^{\infty} \frac{n(t)}{t^{m+1}} dt = 0;$$
$$\lim_{r\to\infty} \frac{n(r)}{r^m} = 0.$$

Proof of Lemma 1.3.4

Using Riemann-Stieltjes integration by parts (see [2]) and our initial assumptions we observe the following:

$$\begin{aligned}
M = \sum_{k=1}^{\infty} \frac{|a_k|}{|z_k|^m} &= \lim_{R\to\infty} \sum_{|z_k|\le R} \frac{|a_k|}{|z_k|^m} \\
&= \lim_{R\to\infty} \int_{\delta}^{R} \frac{1}{t^m} dn(t) \\
&= \lim_{R\to\infty} \left(\left[\frac{n(t)}{t^m} \right]_{\delta}^{R} + \int_{\delta}^{R} \frac{mn(t)}{t^{m+1}} dt \right) \\
&= \lim_{R\to\infty} \frac{n(R)}{R^m} + \int_{\delta}^{\infty} \frac{mn(t)}{t^{m+1}} dt.
\end{aligned}$$

Both terms above, being non-negative, must be bounded above by M. In particular, this tells us that

$$\int_{\delta}^{\infty} \frac{n(t)}{t^{m+1}} dt < \frac{M}{m} < \infty,$$

and hence

$$\lim_{r\to\infty} \int_{r}^{\infty} \frac{n(t)}{t^{m+1}} dt = 0,$$

as required.

For the second part, we observe the following:

$$\begin{aligned}
\frac{n(R)}{mR^m} &= n(R) \int_{R}^{\infty} \frac{1}{t^{m+1}} dt \\
&\le \int_{R}^{\infty} \frac{n(t)}{t^{m+1}} dt,
\end{aligned}$$

and the result follows by the first part.

Next we show the following property about integrals with respect to $n(t)$.

Lemma 1.3.5. *We assert the following holds for all integers $k \ge m$, where m is from the convergence criterion:*

$$\int_{\delta}^{\infty} \frac{1}{t^k} dn(t) = \int_{\delta}^{\infty} \frac{kn(t)}{t^{k+1}} dt.$$

The proof of this is immediate from the proof and result of Lemma 1.3.4.

1.4 The Logarithmic Potential

Definition 1.4.1. ***The Logarithmic Potential*** *- If we have a Keldysh Function f defined as in (1.3.2), we can define* The Logarithmic Potential *corresponding to f as follows,*

$$u(z) = \sum_{k=1}^{\infty} a_k \log \left| 1 - \frac{z}{z_k} \right| .$$

Remark - It is worth noting that $\overline{f(z)} = \nabla u$. Furthermore, if $a_k > 0$ for all k, then $u(z)$ is subharmonic. This is because each term in the sum is subharmonic (see Example 1.2.4), provided the corresponding a_k is positive (if a_k is negative, the kth term is superharmonic), and a convergent sum of subharmonic functions is still subharmonic.

Lemma 1.4.2. *Let $u(z)$ be the logarithmic potential relating to a Keldysh Function f as given above. Let $a_k > 0$ and $|z_k| \geq 2\delta$ for all k.*

Then $B(r, u) = o(r)$, as $r \to \infty$.

Proof of Lemma 1.4.2

We observe the following.

$$\begin{aligned} u(z) &= \sum_{k=1}^{\infty} a_k \log \left| 1 - \frac{z}{z_k} \right| \\ &\leq \sum_{k=1}^{\infty} a_k \log \left(1 + \left| \frac{z}{z_k} \right| \right) \\ &= \int_{\delta}^{\infty} \log \left(1 + \frac{|z|}{t} \right) dn(t), \end{aligned}$$

where $n(t) = \sum_{|z_k| < t} a_k$ as in (1.3.3). This is using Riemann-Stieljes integration, for an explanation see a standard text such as [2]. Next we utilise integration by parts for Riemann-Stieljes integrals:

$$\begin{aligned} u(z) &\leq \int_{\delta}^{\infty} \log \left(1 + \frac{|z|}{t} \right) dn(t) \\ &= \lim_{R \to \infty} \left(\left[n(t) \log \left(1 + \frac{|z|}{t} \right) \right]_{\delta}^{R} + \int_{\delta}^{R} \frac{n(t)|z|}{t(t+|z|)} dt \right) . \end{aligned}$$

By Lemma 1.3.4, we have that $\lim_{r\to\infty} \dfrac{n(r)}{r} \to 0$, which implies that the first term above tends to zero. This leaves us with

$$\begin{aligned} u(z) &\leq |z| \int_{\delta}^{\infty} \frac{n(t)}{t(t+|z|)} dt \\ &\leq |z| \int_{\delta}^{|z|} \frac{n(t)}{t(t+|z|)} dt + |z| \int_{|z|}^{\infty} \frac{n(t)}{t(t+|z|)} dt \\ &\leq \int_{\delta}^{|z|} \frac{n(t)}{t} dt + |z| \int_{|z|}^{\infty} \frac{n(t)}{t^2} dt, \end{aligned}$$

Next we use the fact that $\lim_{r\to\infty} \int_r^{\infty} \dfrac{n(t)}{t^2} = o(1)$ from Lemma 1.3.4 to leave us with,

$$\begin{aligned} u(z) &\leq \int_{\delta}^{|z|} \frac{n(t)}{t} dt + |z| \int_{|z|}^{\infty} \frac{n(t)}{t^2} dt \\ &\leq \int_{\delta}^{|z|} o(1) dt + |z| o(1) = o(|z|), \end{aligned}$$

as required. To see that $\int_{\delta}^{|z|} o(1)dt = o(|z|)$, we observe the following. Given $\epsilon > 0$, we have $o(1) < \epsilon$ for $|z| > T$, and $o(1) < M$ for $0 < |z| < T$. Now

$$\begin{aligned} \int_{\delta}^{|z|} o(1) dt &\leq \int_{\delta}^{T} M dt + \int_{T}^{|z|} o(1) dt \\ &\leq MT + \epsilon|z| \\ &\leq \epsilon|z| + \epsilon|z|, \end{aligned}$$

if $|z| > \frac{MT}{\epsilon}$. Since ϵ can be chosen to be arbitrarily small, the result follows.

1.5 Discrete Potentials in Several Dimensional Real Space

Definition 1.5.1. ***Potentials in*** $\mathbb{R}^m$ - *For $m \geq 3$ the following functions are used to look for equilibrium points of electrostatic and gravitational fields in $\mathbb{R}^m$.*

$$u(x) = \sum_{k=1}^{\infty} \frac{a_k}{|x - x_k|^{m-2}}, x_k \in \mathbb{R}^m, a_k > 0, |x_k| \to \infty, \sum_{k=1}^{\infty} \left| \frac{a_k}{{x_k}^{m-2}} \right| < \infty. \quad (1.5.1)$$

It is important to notice that these are not the same as the functions which describe mass distributions in $\mathbb{R}^m$ given in (1.3.1). However, the gradient (vector derivative) of a function of the form (1.5.1) is equal (up to a constant) to the function given in (1.3.1). We utilise these functions because they are easier to deal with, as they have a single real variable output (always positive), and their critical points correspond to the zeros of functions (1.3.1), and hence the equilibrium points in the electrostatic or gravitational field.

An important feature to notice is that in switching from looking at functions of the form (1.3.1) to (1.5.1), we now require a stricter convergence criterion, which means that some cases of the original problem cannot be covered by these functions. Chapter 6 describes a modification to overcome this problem.

When dealing with these potentials in real space, the notation given in (1.3.3) is modified as follows:

$$\begin{aligned} n(r) &= \sum_{|x_k| \leq r} a_k \\ \overline{n}(r) &= \sum_{|x_k| \leq r} 1. \end{aligned} \tag{1.5.2}$$

Currently the only method we have available to find critical points in $\mathbb{R}^m$ utilises the following Lemma, which traces its origins back to James C. Maxwell, in his famous book, A Treatise on Electricity and Magnetism, see [23, Section 113]. The following notation will be used, which is an analogue of the notation used for entire functions. We define

$$\begin{aligned} m_0(r, u) &= \min_{|x|=r} |u(x)| \\ M(r, u) &= \max_{|x|=r} |u(x)|. \end{aligned}$$

Lemma 1.5.2. *Let $u : \mathbb{R}^m \to (-\infty, +\infty]$, be superharmonic, and C^1 apart from at isolated poles x_k, $|x_k| \to \infty$, at which $u(x_k) = +\infty$. Assume that there exists $r_n \to \infty$ such that*

$$M(r_n, u) \to \mu, \tag{1.5.3}$$

where

$$\mu = \inf\{u(x) : x \in \mathbb{R}^m\}.$$

Then u has infinitely many critical points.

A Sketch of the Proof of Lemma 1.5.2

For a more detailed proof see [4, 19].
Since u is superharmonic, $m(r) = m_0(r, u)$ is strictly decreasing.

In fact there does not exist $x \in \mathbb{R}^m$ with $u(x) = \mu$ by the maximum principle applied to $-u$, which is subharmonic. Hence $m_0(r, u) > \mu$.

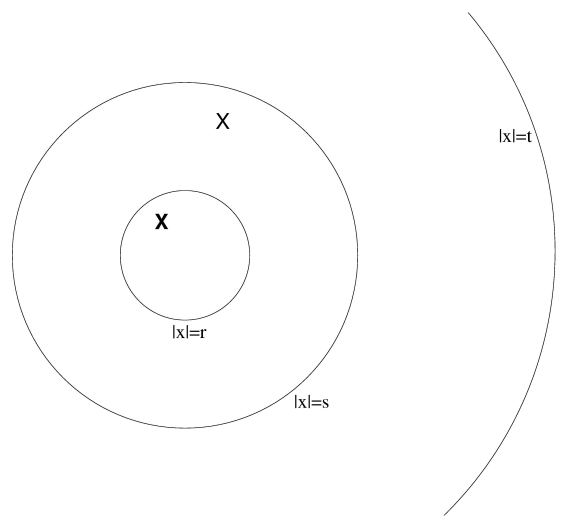
Choose r, s, and t with

$$r < s < t,$$

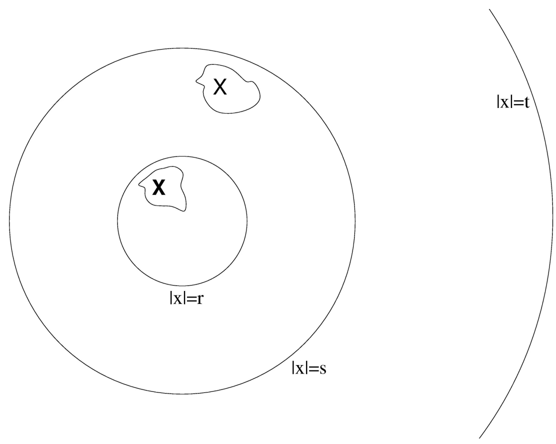
and

$$M(r, u) \geq m_0(r, u) > m_0(s, u) > M(t, u) = M(t).$$

Choose them furthermore, so that there is at least one pole inside $|x| = r$, and at least one inside the annular region $r < |x| < s$, as shown below. This is possible because of (1.5.3). The two large crosses mark the location of the poles, x_1, and x_2. Although these diagrams are two dimensional, the argument works exactly the same in higher dimensional real space.

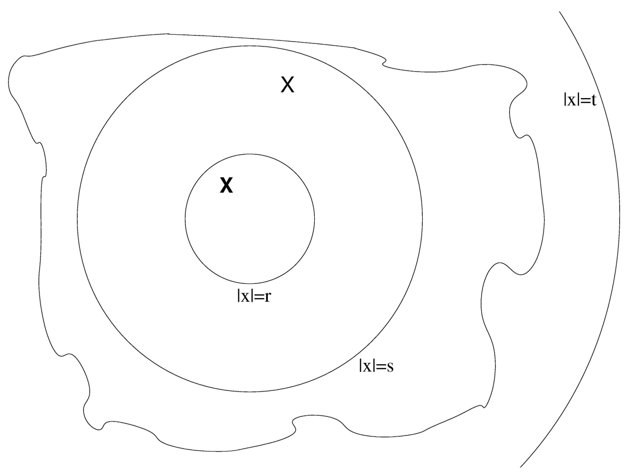


Now we look at the region where $u(x) > T$, for some real T. Clearly, if we choose $T > M(r)$, then we get the following.



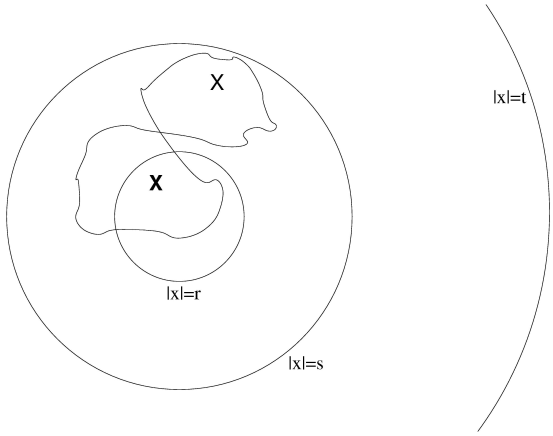
The curved lines around each pole represent part of the region in which $u(x) > T$. These clearly cannot touch the sphere $|x| = r$, from the way in which we have chosen r and s.

Next we look at the region $u(x) > T$ for a T satisfying $M(t) < T < m(s)$.



Since $u(x) \geq m(s)$ for $|x| \leq s$, the whole closed ball $B(0, s)$ is contained in the set where $u(x) > T$.

Since for $T > M(r)$ we get two distinct regions, and for $M(t) < T < m(s)$ we only get one region, clearly somewhere in between the two values $m(s)$ and $M(r)$, there must be a T_1 for which these two regions first coalesce, as shown in this diagram.



More precisely, let T_1, satisfying $m(s) < T_1 \leq M(r)$ be the infimum of T for which there is a continuous path in $u(x) \geq T$ from x_1 to x_2. Then there exists an x_0, with $r < |x_0| < s$ and $u(x_0) = T_1$, such that x_0 is a critical point of u.

Note that nothing in this argument prevents the regions from joining at more than just a point, so it is possible that we would get a continuum of critical points. This argument does however guarantee that there will be at least one.

This argument is then repeatedly applied at successively larger spheres to obtain an infinite number of critical points.

The last result in this section is a classic result regarding mass distributions in $\mathbb{R}^m$, which is used extensively in this thesis, in particular in Chapter 2. This is from [13, p.366].

Lemma 1.5.3 (Cartan's Lemma). *If μ is a mass distribution in $\mathbb{R}^m$ with finite total mass M, and*

$$V(x) = \int \log |x - \zeta| d\mu(\zeta),$$

then if $h > 0$ and $A > 2e$, we have

$$V(x) > M \log h.$$

Furthermore provided that x lies outside a set E of balls, the sum of whose radii is less than Ah, then

$$\mu(C(x,t)) < \frac{Mt}{eh},$$

for $0 < t < \infty$, where $C(x,t)$ is a ball of radius t around x.

1.6 Known Results and Conjectures concerning Keldysh Functions

The main motivation for this thesis, is the following conjecture which is due to Alexandre Eremenko, and is given in [4].

Conjecture 1.6.1. *Let* $z_k \in \mathbb{C}$*, with* $\lim_{k\to\infty} |z_k| = \infty$*, and let*

$$f(z) = \sum_{k=1}^{\infty} \frac{a_k}{z - z_k}, \quad a_k > 0, \quad \sum_{k=1}^{\infty} \frac{a_k}{|z_k|} < \infty.$$

Then f *has infinitely many zeros.*

A highly important result in the study of these functions is the following, which is due to Keldysh.

Theorem 1.6.2. [8] *Let* f *be a Keldysh Function. Then* $m(r, f) = o(1)$.

From this we can immediately notice the following corollary.

Corollary 1.6.3. *For a Keldysh Function* f*, the Nevanlinna Characteristic is determined by the occurrence of the poles,*

$$T(r, f) = \int_0^r \frac{\overline{n}(t)}{t} dt + o(1),$$

with $\overline{n}(t)$ *as defined in (1.3.3).*

This follows since all the poles of Keldysh Functions are simple.

Next follow the most significant results so far concerning the existence of zeros of these Keldysh Functions. These are included here without proofs, so that the reader can obtain a better idea of what is already known.

The first two results here are also attributed to Keldysh.

Theorem 1.6.4. [8] *Let* f *be a Keldysh Function of finite lower order. Then* $\delta(a, f) = 0$ *for all* $a \neq 0$.

Theorem 1.6.5. [8] *Let f be a Keldysh Function of finite lower order, with the following additional criteria:*

$$\begin{aligned}\sum_{k=1}^{\infty}|a_k| &< \infty;\\ \sum_{k=1}^{\infty}a_k &\neq 0.\end{aligned}$$

Then $\delta(0,f)=0$.

The next result and its corollary are from [7] and represent the best steps currently made towards the proof of Conjecture 1.6.1.

Theorem 1.6.6. [20] *Let f be a Keldysh Function with $a_k > 0$ for all k, and order at most one (mean type). Then f has infinitely many zeros.*

Corollary 1.6.7. [7] *Suppose that f is a Keldysh Function with $a_k \geq a > 0$. Then f has infinitely many zeros.*

These results prove special cases of Conjecture 1.6.1, under the additional conditions that the a_k are either small (Theorem 1.6.5) or large (Corollary 1.6.7).

Chapter 2

An Application of Cartan's Lemma

This chapter builds upon an existing result for potentials in real space which requires a condition on upper growth, and shows an alternative result requiring a condition on lower growth instead. The original result is then extended to encompass functions over $\mathbb{C}$. The results of this chapter have been published as a paper [25] in the journal CMFT.

2.1 Introduction

In [19], J.K. Langley and J. Rossi prove the following result, for potentials in $\mathbb{R}^m$ $(m \geq 3)$, see (1.5.1).

Theorem 2.1.1. [19] *Let $x_k \in \mathbb{R}^m, m \geq 3$, with $\lim_{k\to\infty} |x_k| = \infty$, and let*

$$u(x) = \sum_{k=1}^{\infty} \frac{a_k}{|x - x_k|^{m-2}}, \quad a_k > 0, \quad \sum_{k=1}^{\infty} \frac{a_k}{|x_k|} < \infty.$$

Suppose further that $\phi(t)$ is continuous, positive and non-decreasing on $[1, \infty)$ with

$$\lim_{t\to\infty} \phi(t) = \infty, \quad \int_1^{\infty} \frac{1}{t\phi(t)} dt = \infty,$$

and that the number $\overline{n}(r)$ of x_k lying in $|x| \leq r$ satisfies

$$\limsup_{r\to\infty} \frac{\log \overline{n}(r)}{\phi(r)} < \infty \quad (m = 3),$$
$$\limsup_{r\to\infty} \left(\frac{\overline{n}(r)}{r}\right)^{m-3} \frac{1}{\phi(r)} < \infty \quad (m \geq 4).$$

Then u has infinitely many critical points.

An example of the type of function ϕ mentioned in this theorem is $\phi(t) = \log t$.

Using a method similar to the proof of Theorem 2.1.1, we can prove the following result, which allows us to establish several value distribution results for functions either of the form (1.3.2) on $\mathbb{C}$ or (1.5.1) on $\mathbb{R}^m$.

Proposition 2.1.2. *Let m, q be integers with $m \geq 2$ and $q \geq 1$. Let $y_k \to \infty$ in $\mathbb{R}^m$, with $y_k \neq 0$, and let $b_k > 0$, and assume that*

$$\sum_{k=1}^{\infty} \frac{b_k}{|y_k|} < \infty. \tag{2.1.1}$$

For $0 < r < \infty$, let

$$n(r) = \sum_{|y_k| \leq r} b_k, \quad \overline{n}(r) = \sum_{|y_k| \leq r} 1. \tag{2.1.2}$$

For $x \in \mathbb{R}^m$, let

$$u(x) = \sum_{k=1}^{\infty} \frac{b_k}{|x - y_k|^q}. \tag{2.1.3}$$

Then the following conclusions hold.

a) If $q = 1$ and

$$\limsup_{r\to\infty} \frac{\log \overline{n}(r)}{\phi(r)} < \infty \tag{2.1.4}$$

for some continuous, positive, non-decreasing function $\phi(t)$ on $[1, \infty)$ which satisfies

$$\int_1^{\infty} \frac{1}{t\phi(t)} dt = \infty, \tag{2.1.5}$$

then

$$\liminf_{r\to\infty} M(r, u) = 0. \tag{2.1.6}$$

b) If $q = 1$ and

$$\liminf_{r\to\infty} \frac{\log \overline{n}(r)}{\log r} < \infty \tag{2.1.7}$$

then (2.1.6) holds.

c) If $q \geq 2$ and

$$\liminf_{r\to\infty} \frac{\overline{n}(r)}{r} < \infty \tag{2.1.8}$$

then (2.1.6) holds.

In this Proposition the condition $\phi(t) \to \infty$ is a consequence of 2.1.4, since $\overline{n}(r) \to \infty$.

Proposition 2.1.2 will be used to prove the following four theorems.

Theorem 2.1.3. *Let $z_k \in \mathbb{C}$ with $\lim_{k\to\infty} |z_k| = \infty$, and let*

$$f(z) = \sum_{k=1}^{\infty} \frac{a_k}{z - z_k}, \quad a_k \in \mathbb{C} \setminus \{0\}, \quad \sum_{k=1}^{\infty} |a_k| < \infty, \quad \lambda = \sum_{k=1}^{\infty} a_k \neq 0. \tag{2.1.9}$$

Suppose further that $\phi(t)$ is continuous, positive and non-decreasing on $[1, \infty)$ with

$$\lim_{t\to\infty} \phi(t) = \infty, \qquad \int_1^{\infty} \frac{1}{t\phi(t)} dt = \infty,$$

and that the number $\overline{n}(r)$ of z_k lying in $|z_k| \leq r$ satisfies,

$$\limsup_{r\to\infty} \frac{\log \overline{n}(r)}{\phi(r)} < \infty. \tag{2.1.10}$$

Then $\delta(0, f) = 0$.

Here $\delta(0, f)$ denotes the Nevanlinna deficiency of the zeros of f, see Definition 1.1.17.

Remark - In the case $\phi(t) = \log t$, condition (2.1.10) is equivalent to f having finite upper order. When ϕ is larger,

$$\begin{aligned} \text{i.e.} \quad \phi(t) &= \log t (\log \log t) \\ \text{or} \quad \phi(t) &= \log t (\log \log t)(\log \log \log t), \quad \text{etc.} \end{aligned}$$

it is equivalent to f having "small" infinite order. See Chapter 1, Corollary 1.6.3 for an explanation of this.

Theorem 2.1.4. *Let $z_k \in \mathbb{C}$ with $\lim_{k\to\infty} |z_k| = \infty$, and let $f(z)$ be given by and satisfy (2.1.9).*

Suppose further that the number $\overline{n}(r)$ of z_k lying within $|z| \le r$ satisfies

$$\liminf_{r\to\infty} \frac{\log \overline{n}(r)}{\log r} < \infty. \tag{2.1.11}$$

Then $\delta(0, f) = 0$.

Remark - In view of Corollary 1.6.3, condition 2.1.11 is equivalent to f having finite lower order. This theorem is therefore equivalent to Theorem 1.6.5, although the proof uses a different method to that of Keldysh.

Theorem 2.1.5. *Let $x_k \in \mathbb{R}^3$ with $\lim_{k\to\infty} |x_k| = \infty$, and let*

$$u(x) = \sum_{k=1}^{\infty} \frac{a_k}{|x - x_k|}, \qquad a_k > 0, \qquad \sum_{k=1}^{\infty} \frac{a_k}{|x_k|} < \infty.$$

Suppose further that the number $\overline{n}(r)$ of x_k lying within $|x| \le r$ satisfies

$$\liminf_{r\to\infty} \frac{\log \overline{n}(r)}{\log r} < \infty.$$

Then u has infinitely many critical points.

Theorem 2.1.6. *Let $x_k \in \mathbb{R}^m$ $(m \ge 4)$, with $\lim_{k\to\infty} |x_k| = \infty$, and let*

$$u(x) = \sum_{k=1}^{\infty} \frac{a_k}{|x - x_k|^{m-2}}, \qquad a_k > 0, \qquad \sum_{k=1}^{\infty} \frac{a_k}{|x_k|} < \infty.$$

Suppose further that the number $\overline{n}(r)$ of x_k lying within $|x| \le r$ satisfies

$$\liminf_{r\to\infty} \frac{\overline{n}(r)}{r} < \infty.$$

Then u has infinitely many critical points.

Theorems 2.1.5 and 2.1.6 can be viewed as improved versions of Theorem 2.1.1, as the upper growth condition has been replaced with a lower growth condition. However the lower growth condition is not an exact analogue of the replaced upper growth condition, so they do not render Theorem 2.1.1 obsolete.

2.2 Proof of Proposition 2.1.2

This proof covers all three cases (a), (b), and (c), however in case (b) for $\phi(t)$ read $\log t$, and in case (c), for $\phi(t)$ read 1. We now need the following lemma.

Lemma 2.2.1. *With the hypotheses of Proposition 2.1.2, there exists a sequence $r_n \to \infty$, with*

$$n(r_n) = o\left(\frac{r_n}{\phi(r_n)}\right). \tag{2.2.1}$$

Moreover, in cases (a) and (b),

$$\log \overline{n}(r_n) = O(\phi(r_n)), \tag{2.2.2}$$

while

$$\overline{n}(r_n) = O(r_n) \tag{2.2.3}$$

in case (c).

Proof of Lemma 2.2.1

First note that (2.1.1) and (2.1.2) give

$$\begin{aligned} \infty > \sum_{k=1}^{\infty} \frac{b_k}{|y_k|} &= \int_0^\infty \frac{1}{t} dn(t) \\ &\geq \int_0^\infty \frac{n(t)}{t^2} dt \end{aligned} \tag{2.2.4}$$

using the fact that $n(0) = 0$ (see Lemma 1.3.5 for an explanation of this).

Consider first case (a). If no sequence (r_n) satisfying (2.2.1) exists, then there exist δ, $r_1 > 0$ with

$$n(r) > \frac{\delta r}{\phi(r)} \qquad \text{for } r \geq r_1.$$

This can be combined with (2.1.5) to give

$$\int_{r_1}^\infty \frac{n(t)}{t^2} dt \geq \int_{r_1}^\infty \frac{\delta}{t\phi(t)} dt = \infty,$$

which contradicts (2.2.4).

Hence in this case a sequence (r_n) as in (2.2.1) must exist. Note that (2.2.2) follows at once from (2.1.4).

Now consider case (b). From (2.1.7) we know there exists a sequence $s_n \to \infty$ with

$$\log \overline{n}(s_n) = O(\log s_n) = O(\phi(s_n)). \tag{2.2.5}$$

We assert that a sequence (r_n) may be chosen with $\sqrt{s_n} \leq r_n \leq s_n$ such that (2.2.1) holds. Assume that this is not the case. Then there exists a $\delta > 0$ such that, for arbitrarily large n, we have

$$n(t) > \frac{\delta t}{\log t} \quad \text{for all t in } [\sqrt{s_n}, s_n]. \tag{2.2.6}$$

Now, (2.2.6) gives, for these n,

$$\begin{aligned} \int_{\sqrt{s_n}}^{s_n} \frac{n(t)}{t^2} dt &> \delta \int_{\sqrt{s_n}}^{s_n} \frac{dt}{t \log t} \\ &= \delta \log 2, \end{aligned}$$

which contradicts (2.2.4) since

$$\int_{\sqrt{s_n}}^{s_n} \frac{n(t)}{t^2} \leq \int_{\sqrt{s_n}}^{\infty} \frac{n(t)}{t^2} \to 0 \quad \text{as } n \to \infty.$$

Thus a sequence (r_n) satisfying (2.2.1) may be chosen with $\sqrt{s_n} \leq r_n \leq s_n$. Moreover (2.2.5) gives

$$\begin{aligned} \log \overline{n}(r_n) &\leq \log \overline{n}(s_n) \\ &= O(\log s_n) \\ &= O(2 \log r_n) \\ &= O(\phi(r_n)) \end{aligned}$$

so that (2.2.2) also holds.

Finally consider case (c). Then by (2.1.8) we may choose $r_n \to \infty$ such that (2.2.3) holds. Now set $M = \sum_{k=1}^{\infty} \frac{b_k}{|y_k|}$, which is finite by (2.1.1). Then

$$\sum_{|y_k| \geq s} \frac{b_k}{|y_k|} = o(1) \quad \text{as} \quad s \to \infty.$$

Now

$$
\begin{aligned}
\frac{n(r)}{r} &= \frac{1}{r}\sum_{|y_k|\le r} b_k \\
&= \frac{1}{r}\sum_{|y_k|\le\sqrt{r}} b_k + \frac{1}{r}\sum_{\sqrt{r}<|y_k|\le r} b_k \\
&\le \frac{1}{r}\left(\sqrt{r}\sum_{|y_k|\le\sqrt{r}}\frac{b_k}{|y_k|}\right) + \sum_{\sqrt{r}<|y_k|\le r}\frac{b_k}{|y_k|} \\
&\le \frac{1}{\sqrt{r}}M + \sum_{|y_k|>\sqrt{r}}\frac{b_k}{|y_k|} \\
&= o(1).
\end{aligned}
$$

Hence (2.2.1) is satisfied, since $\phi(t) = 1$ in this case. This proves Lemma 2.2.1.

In order to complete the proof of Proposition 2.1.2, let the sequence (r_n) be as in Lemma 2.2.1. Let $r = r_n$ be large, and set $N = \overline{n}(r)$.

For $x \in \mathbb{R}^m$ and $t > 0$, set

$$\mu(x, r, t) = \sum_{|y_k|\le r, |x-y_k|\le t} b_k.$$

We now apply the version of Cartan's Lemma from [13, p.366] (see Lemma 1.5.3) to the mass distribution $\mu = \sum_{|y_k|\le r} b_k\delta_{y_k}$ with

$$M = n(r), \qquad h = \frac{r}{192}, \qquad A = 6.$$

This gives a union of balls E, with sum of radii at most

$$Ah = \frac{r}{32},$$

such that for all x lying outside of E, then

$$\mu(x, r, t) \le \frac{Mt}{eh} = \frac{192n(r)t}{er}, \quad 0 < t < \infty.$$

Also, let F be the union of balls given by

$$F = \bigcup_{|y_k|\le r} D\left(y_k, \frac{r}{32N}\right), \quad \text{where} \quad N = \overline{n}(r).$$

Then the sum of the diameters of all balls in the combined set $E \cup F$ is at most $\frac{r}{8}$, so we can choose $r' \in [\frac{r}{2}, \frac{3r}{4}]$ such that the sphere $|x| = r'$ does not meet $E \cup F$.

Consider now the function u. For $|x| = r'$, since $r' < r$, we use (2.1.1) to get,

$$\begin{aligned} \sum_{|y_k|>r} \frac{b_k}{|x-y_k|^q} &\le \sum_{|y_k|>r} \frac{b_k}{(|y_k|-|x|)^q} \\ &\le \sum_{|y_k|>r} \frac{b_k}{(|y_k|-\frac{3|y_k|}{4})^q} \qquad (2.2.7) \\ &\le \sum_{|y_k|>r} \frac{4^q b_k}{|y_k|^q} = o(1) \quad \text{as} \quad r \to \infty. \end{aligned}$$

Also for $|x| = r'$ we have, since $x \notin F$ and consequently $\mu\left(x, r, \frac{r}{64N}\right) = 0$,

$$\begin{aligned} \sum_{|y_k|\le r} \frac{b_k}{|x-y_k|^q} &= \int_{\frac{r}{64N}}^{2r} \frac{1}{t^q} d\mu(x,r,t) \\ &= \frac{1}{(2r)^q}\mu(x,r,2r) + q\int_{\frac{r}{64N}}^{2r} \frac{\mu(x,r,t)}{t^{q+1}}dt - \left(\frac{64N}{r}\right)^q \mu\left(x,r,\frac{r}{64N}\right) \\ &\le \frac{n(r)}{(2r)^q} + q\frac{192n(r)}{er}\int_{\frac{r}{64N}}^{2r} \frac{1}{t^q}dt. \qquad (2.2.8) \end{aligned}$$

$$(2.2.9)$$

At this point two separate cases arise. In the case of (a) and (b), when $q = 1$, we get the following:

$$\begin{aligned} \frac{n(r)}{2r} + \frac{192n(r)}{er}\int_{\frac{r}{64N}}^{2r} \frac{1}{t}dt &= \frac{n(r)}{2r} + \frac{192n(r)}{er}\log(128N) \\ &= o(1) \end{aligned}$$

by (2.2.1) and (2.2.2).

In case (c), where $q > 1$, we get

$$\begin{aligned} \frac{n(r)}{(2r)^q} + q\frac{192n(r)}{er}\int_{\frac{r}{64N}}^{2r} \frac{1}{t^q}dt &\le c_1\frac{n(r)}{r^q} + c_2\frac{n(r)}{r}\left(\frac{N}{r}\right)^{q-1} \\ &= o(1) \end{aligned}$$

using (2.2.1) and (2.2.3), with $\phi(t) = 1$, and the c_j positive constants. In both cases, substitution into (2.2.8) and combining (2.2.7) with (2.2.8) gives $u(x) = o(1)$. This proves Proposition 2.1.2.

Using Proposition 2.1.2, we can now prove Theorems 2.1.3 through 2.1.6, where Theorem 2.1.4 may be viewed as an alternative proof of a result of Keldysh given in [8, p.327], which is Theorem 1.6.5 in Chapter 1.

2.3 Proof of Theorem 2.1.3

Following [8, p.333] construct a new function $g(z) = zf(z)$. Note that

$$\begin{aligned} zf(z) = g(z) &= \sum_{k=1}^{\infty} \frac{za_k}{z - z_k} \\ &= \sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} \frac{a_k z_k}{z - z_k} \\ &= \lambda + \sum_{z_k \neq 0} \frac{a_k z_k}{z - z_k} \\ &= \lambda + h(z). \end{aligned} \tag{2.3.1}$$

Now applying Proposition 2.1.2(a), with $b_k = |a_k||z_k|$, and $y_k = z_k$, gives us a sequence of circles $|z| = R_j \to \infty$ on which $|h(z)| = o(1)$.

On the circles $|z| = R_j$, we have $h(z) = o(1)$. This means that for all $j \geq j_1$, say, we have $|h(z)| < \frac{|\lambda|}{2}$ on $|z| = R_j$, which gives $m\left(R_j, \frac{1}{h+\lambda}\right) = O(1)$. Since the R_j tend to infinity, $T(R_j, h) \to \infty$ as $j \to \infty$ and

$$\delta(0, f) = \delta(-\lambda, h) \leq \liminf_{j \to \infty} \frac{m\left(R_j, \frac{1}{h+\lambda}\right)}{T(R_j, h)} = 0.$$

This completes the proof of Theorem 2.1.3.

This idea comes from Keldysh's proof that $\delta(a, f) = 0$ for all $a \in \mathbb{C} \setminus 0$ and for Keldysh Functions f, which is Theorem 1.6.4 in Chapter 1, and can be found in [9].

2.4 Proof of Theorems 2.1.4, 2.1.5 and 2.1.6

Proof of Theorem 2.1.4

The proof of Theorem 2.1.4 starts by constructing new functions $g(z)$ and $h(z)$, as in (2.3.1).
Now we can apply Proposition 2.1.2(b), with $b_k = |a_k||z_k|$, and $y_k = z_k$, to give us a sequence of circles on which $|h(z)| = o(1)$.

Now the same reasoning as in the proof of Theorem 2.1.3 completes the result.

Proof of Theorem 2.1.5

Applying Proposition 2.1.2(b) to u, with $b_k = a_k$, and $y_k = x_k$, gives us a sequence of spheres $|x| = r_n \to \infty$ on which $|u(x)| = o(1)$. Now using the argument of Lemma 1.5.2 (see also [4, 19]), it follows that u has infinitely many critical points.

Proof of Theorem 2.1.6

Following the same reasoning as in the proof of Theorem 2.1.5, we apply Proposition 2.1.2(c) with $b_k = a_k$, $y_k = x_k$, and $q = m - 2$. Combining this with Lemma 1.5.2 completes the result.

Chapter 3

On the Frequency of the Zeros

This chapter primarily contains the proof of a result regarding the frequency, not just existence of zeros of some Keldysh Functions. Following this is a result concerning the effect that the spacing of the z_k has on the zeros of these functions. These results, along with those from Chapter 4, have been published as a paper [26] in the CMFT journal.

3.1 Introduction

The first result in this chapter is an expansion of the following result, which was first proved by J.K. Langley and J. Rossi. The notation used is from [11].

Theorem 3.1.1. [20] *Let f be given as in (1.3.2), with $a_k > 0$. Assume that f has finite order, and that*

$$\sum_{k=1}^{\infty} a_k = \infty, \quad \liminf_{r\to\infty} \frac{T(r,f)}{r} < \infty.$$

Let $S(z)$ be a rational function. Then $f(z) - S(z)$ has infinitely many zeros.

Although this result ensures the occurrence of the zeros, it tells us nothing of their frequency. With the aim of describing their frequency, the following result is proved.

Theorem 3.1.2. *Let $f(z)$ be as in (1.3.2), with $a_k > 0$, and let $\overline{n}(r) = O(r^d)$, as $r \to \infty$, for some real $d > 0$. Furthermore, let f satisfy*

$$\liminf_{r\to\infty} \frac{T(r,f)}{r} < \infty. \tag{3.1.1}$$

Then f must satisfy

$$\liminf_{r\to\infty}\left(T(r,f)-2H\log r-\frac{r^{\frac{1}{2}}}{2}\int_r^\infty t^{-\frac{3}{2}}N\left(t,\frac{1}{f}\right)dt\right)<\infty, \tag{3.1.2}$$

where H is any integer with $H \geq (5+d)/2$.

Remark - Under the hypotheses of Theorem 3.1.2, $N(t,\frac{1}{f})$ cannot grow too slowly because otherwise the term in parentheses in (3.1.2) would tend to $+\infty$.

Notice that this result does not require the condition that $\sum_{k=1}^{\infty} a_k = \infty$, as in Theorem 3.1.1. However, for positive a_k with $\sum_{k=1}^{\infty} a_k < \infty$, Theorem 1.6.5, which is originally from [8, p.327], may be applied to give $\delta(0,f)=0$.

The other result in this chapter uses the residue theorem combined with a result of Anderson and Clunie [1], to show what effect a separation condition on the z_k has on these functions.

Theorem 3.1.3. *Let f be given by (1.3.2). Further, suppose that there exists an $\eta > 0$ such that*

$$|z_k - z_m| > \eta |z_m|, \tag{3.1.3}$$

for all distinct k and m.

Then as $k \to \infty$ we have

$$|a_k| < e^{-[\delta(0,f)-o(1)]T(|z_k|,f)}, \tag{3.1.4}$$

and furthermore, if we assume $\delta(0,f)>0$, we get

$$\sum_{k=1}^{\infty} |a_k| \quad < \quad \infty$$

and

$$\sum_{k=1}^{\infty} a_k \quad = \quad 0.$$

3.2 Proof of Theorem 3.1.2

The proof of this result is based on a method used by Hinchliffe in [15, Theorem 2.1.3] and Langley and Shea in [21, Theorem 2]. It utilises several established results, the first of which was proved by Langley and Shea in [21], and is stated below. An alternative proof is given in [19].

Lemma 3.2.1. [21] *Let h be transcendental and meromorphic in the plane, and suppose that $d \geq 1$ is such that $T(r,h) = O(r^d)$ for all large r. Denote by $L(r,R,h)$ the total length of the level curves $|h(z)| = R$ lying in the region $|z| < r$. Then there exist arbitrarily large positive R such that $h(z)$ has no critical values on the circle $S(0,R) = \{w : |w| = R\}$ and*

$$L(r,R,h) = O(r^{d_1})$$

as $r \to \infty$, where $d_1 = (3+d)/2$.

To prove Theorem 3.1.2, assume that f, d and H are as in the hypotheses, but that (3.1.2) fails. In particular, the integral in (3.1.2) must then converge, and this tells us that

$$N\left(t, \frac{1}{f}\right) + n\left(t, \frac{1}{f}\right) = o\left(t^{\frac{1}{2}}\right) \text{ as } t \to \infty. \tag{3.2.1}$$

This uses the fact that

$$\begin{aligned} n\left(r, \frac{1}{f}\right)\log 2 &= n\left(r, \frac{1}{f}\right)\int_r^{2r} \frac{dt}{t} \\ &\leq \int_r^{2r} \frac{n\left(t, \frac{1}{f}\right)}{t} dt \\ &\leq N\left(2r, \frac{1}{f}\right), \end{aligned}$$

while the fact that

$$\int_r^{\infty} t^{-\frac{3}{2}} N\left(t, \frac{1}{f}\right) dt = o(1),$$

leads to

$$\frac{r^{\frac{1}{2}}}{2}\int_r^{\infty} t^{-\frac{3}{2}} N\left(t, \frac{1}{f}\right) dt = o(r^{\frac{1}{2}}),$$

and then to

$$\begin{aligned} \frac{r^{\frac{1}{2}}}{2}\int_r^\infty t^{-\frac{3}{2}} N\left(t, \frac{1}{f}\right) dt &\geq N\left(r, \frac{1}{f}\right)\frac{r^{\frac{1}{2}}}{2}\int_r^\infty t^{-\frac{3}{2}} dt \\ &= N\left(r, \frac{1}{f}\right). \end{aligned}$$

Thus (3.2.1) holds. Let

$$\begin{aligned} f_1(z) &= z^H f(z), \\ h(z) &= \frac{1}{f_1(z)}, \\ u(z) &= \sum_{k=1}^{\infty} a_k \log\left|1 - \frac{z}{z_k}\right|. \end{aligned} \tag{3.2.2}$$

Notice that u is the Logarithmic Potential relating to f (see Definition 1.4.1) and hence $\overline{f(z)} = \nabla u$. Also, using Keldysh's result from [8], (Theorem 1.6.2 in this thesis) we get that

$$\begin{aligned} N(r,f) + o(1) = T(r,f) &= T\left(r, hz^H\right) + O(1) \\ &\leq T(r,h) + T\left(r, z^H\right) + O(1) \\ &= T(r,h) + H\log r + O(1). \end{aligned}$$

Estimating $T(r,h)$ similarly we get

$$|T(r,h) - T(r,f)| \leq H\log r + O(1). \tag{3.2.3}$$

Since $T(r,f) = O(r^d)$, it follows that $T(r,h) = O(r^d)$ as $r \to \infty$. Thus we may apply Lemma 3.2.1 to h.

Let $R > 0$ be such that the level curves of $|h(z)| = R$ in $|z| < r$ have total length $\leq O(r^{d_1})$ as $r \to \infty$, where d_1 is as in Lemma 3.2.1, and also such that $h(z)$ has no critical points with $|h(z)| = R$. This ensures that each level curve $|h(z)| = R$ is either a simple closed curve, or a simple curve going to infinity in both directions, since any points at which the curves intersected would be critical points of $h(z)$. A choice of such R exist by Lemma 3.2.1.

Assume that $h(z)$ satisfies Hayman's condition from [12];

$$\lim_{r\to\infty}\left(m(r,h) - r^{\frac{1}{2}}\int_r^\infty \frac{n(t,h)}{t^{\frac{3}{2}}} dt\right) = \infty. \tag{3.2.4}$$

Next we need the following result, which is a consequence of a theorem of Hayman [12, Theorem 8]. Langley and Shea highlight this consequence explicitly in [21].

Theorem 3.2.2. [12, 21] *Suppose h is transcendental and meromorphic in the plane, such that (3.2.4) is satisfied. Suppose further that there exists a path Γ tending to infinity, on which $|h(z)| \leq M < \infty$.*

Then there exists a function v_h which is non-constant, non-negative and subharmonic in the plane, such that

$$v_h(z) \leq \log^+ |h(z)| + O(1) \tag{3.2.5}$$

for all z.

Furthermore, property (3.2.4) suffices to ensure that ∞ is an asymptotic value of h.

Theorem 3.2.2 ensures that ∞ is an asymptotic value of h. Therefore the set $\{z : |h(z)| > R\}$ must have at least one unbounded component U, where R is as in Lemma 3.2.1. Now we need the following lemma.

Lemma 3.2.3. *Let V be an unbounded component of the set $\{z : |h(z)| > S\}$, where S satisfies the conclusions of Lemma 3.2.1. Then $u(z)$ is bounded on V.*

Proof of Lemma 3.2.3

Partition the boundary ∂V into its intersections with the disc $D(0,1)$ and the annuli $A_m = \{z : 2^{m-1} \leq |z| < 2^m\} \quad m = 1, 2, ...$

On ∂V, we have $|h(z)| = S$, so on $A_m \cap \partial V$, we have $|h(z)|^{-1} = \frac{1}{S}$, and $\frac{2^{-mH}}{S} \leq |f(z)| \leq \frac{2^{-(m-1)H}}{S}$ by (3.2.2)

Now, using c_n to denote real positive constants, but not necessarily the same constant each time they are used,

$$\begin{aligned}
\int_{A_m \cap \partial V} |f(z)||dz| &\leq c_1 L(2^m, S, h) \frac{1}{S 2^{(m-1)H}} \\
&\leq c_1 (2^m)^{d_1} 2^{-(m-1)H} \\
&\leq c_1 2^{d_1 m - (m-1)H} \\
&= c_1 2^{m(d_1 - H) + H}
\end{aligned}$$

We sum this over all values of m, to get

$$\begin{aligned}\int_{\partial V}|f(z)||dz| &\leq c_1+c_2\sum_{m=1}^{\infty}2^{m(d_1-H)+H}\\ &= c_1+c_2 2^H\sum_{m=1}^{\infty}2^{m(d_1-H)}\\ &\leq c_1+c_2 2^H\sum_{m=1}^{\infty}2^{-m}<\infty, \qquad (3.2.6)\end{aligned}$$

since $d_1-H\leq -1$. Fix a point z_0 in V. Any point ζ in the closure $\overline{V}$ of V can be joined to z_0 by a path μ_ζ in $\overline{V}$ consisting of part of the circle $|z|=|z_0|$, part of the ray $\arg z=\arg\zeta$, and part of ∂V. Split μ_ζ into its intersections μ_1 with the ray, μ_2 with the circle, and μ_3 with ∂V. Now,

$$\int_{\mu_1}|f(z)||dz| \leq c_1\int_{|z_0|}^{\infty}\frac{s^{-H}}{S}ds\leq c_1 \qquad (3.2.7)$$

$$\int_{\mu_2}|f(z)||dz| \leq 2\pi\frac{|z_0|^{1-H}}{S}\leq c_1 \qquad (3.2.8)$$

$$\int_{\mu_3}|f(z)||dz| \leq c_1, \qquad (3.2.9)$$

by (3.2.6). Now if we split f into its real and imaginary parts, using $\overline{f(z)}=\nabla u$, and $dz=dx+idy$, we get

$$\begin{aligned}\int_\gamma f(z)dz &= \int_\gamma(u_x-iu_y)(dx+idy)\\ &= \int_\gamma(u_xdx+u_ydy)+i\int_\gamma(u_xdy-u_ydx)\\ &= \int_\gamma\nabla u\cdot d\underline{r}+\text{imaginary part.}\end{aligned}$$

Where $d\underline{r}=(dx,dy)$. We then use these facts to tell us that:

$$|u(\zeta)-u(z_0)|\leq\left|\int_\gamma\nabla u\cdot d\underline{r}\right|=\left|Re\left(\int_\gamma f(z)dz\right)\right|\leq\left|\int_\gamma f(z)dz\right|\leq\int_\gamma|f(z)||dz|$$

We combine this with (3.2.7), (3.2.8), and (3.2.9), to tell us that $|u(z)|$ is bounded on V. This proves the lemma.

Returning to the proof of Theorem 3.1.2, we apply Lemma 3.2.3 to U, which gives us that $u(z)$ is bounded on U, say $u(z)\leq m_1$.

From this we deduce that the boundary ∂U of U, which consists of countably many pairwise disjoint level curves $|h(z)| = R$, cannot be such that every component of ∂U is bounded.

To see this, let C_1 be a bounded component of ∂U. Then C_1 is a simple closed curve (by Lemma 3.2.1) and the maximum principle gives $u(z) \leq m_1$ inside C_1. If all components C_i of ∂U are bounded, then $u(z) \leq m_1$ in all of $\mathbb{C}$. We could then apply Liouville's theorem for subharmonic functions giving us that u is constant. This cannot be the case, since $u(z_k) = -\infty$.

This tells us that there has to be a path γ_U which tends to infinity and is contained in ∂U. Note that on this path $|h(z)| \leq R$, which means we can now apply (3.2.4) and Hayman's result (Theorem 3.2.2), to give us a non-negative, non-constant, subharmonic function v_h, satisfying (3.2.5) throughout $\mathbb{C}$.

Note that v_h is subharmonic, hence unbounded, and further that if $v_h(z)$ is large, then $h(z)$ is also large. Since v_h is bounded on γ_U, it follows from the $\cos \pi\rho$ theorem for subharmonic functions (see Theorem 1.2.9) that,

$$\frac{B(r, v_h)}{\log r} \to \infty \quad \text{as r } \to \infty, \tag{3.2.10}$$

where $B(r, v_h) = \sup\{v_h(re^{i\theta}) : \theta \in [0, 2\pi)\}$ (see Definition 1.2.5).

Now we combine (3.2.10) and the following result of Lewis, Rossi and Weitsman from [22].

Theorem 3.2.4 ([22])**.** *Let u be subharmonic in the plane, and assume that*

$$\lim_{r\to\infty} \frac{B(r, u)}{\log r} = \infty.$$

Then there exists a path Γ tending to infinity with

$$\int_\Gamma e^{-\lambda u} |dz| < \infty,$$

for each $\lambda > 0$, and

$$\frac{u(z)}{\log |z|} \to \infty, \tag{3.2.11}$$

as $z \to \infty$, $z \in \Gamma$. Note that Γ is independent of λ.

We apply this to v_h to give a path Γ with all the conditions specified above.

Now, we can combine (3.2.11) with (3.2.5) to give us

$$\frac{\log|h(z)|}{\log|z|} \to +\infty \quad \text{on } \Gamma.$$

This tells us that there must be an unbounded component U_1 of $\{z : |h(z)| > T_1\}$ such that T_1 satisfies the conclusions of Lemma 3.2.1 and $\Gamma \setminus U_1$ is bounded.

Applying Lemma 3.2.3 to U_1, we get that u is bounded above on U_1, by M say, but we also get that v_h is unbounded on U_1, as $\Gamma \cap U_1$ is unbounded. However

$$v_h \leq \log T_1 + O(1) \leq M_1$$

say, on ∂U_1.

On U_1 we have $u(z) \leq M$, and now we take an unbounded component U_2 of the set on which u is also unbounded. $\{z : u(z) > M + 1\}$. Clearly U_1 and U_2 must be disjoint.

Now we need the following from [21, Lemma 5].

Lemma 3.2.5 ([21])**.** *Let v_1, v_2 be non-constant subharmonic functions in the plane, let s_1, s_2 be real constants, and let U_1, U_2 be disjoint, unbounded domains such that $v_j \leq s_j$ on ∂U_j and $v_j(z_j) > s_j$ for at least one point z_j in U_j. Then*

$$\log B(r, v_1) + \log B(r, v_2) \geq 2\log r - O(1),$$

for all large r.

We use this lemma to give

$$\log B(r, v_h) + \log B(r, u) \geq 2\log r - O(1). \tag{3.2.12}$$

Note that $B(r, u) = o(r)$, by Lemma 1.4.2, so substituting into (3.2.12) and exponentiating gives

$$cr^2 \leq B(r, v_h)o(r)$$

for some $c > 0$, so this tells us that

$$\liminf_{r\to\infty} \frac{B(r, v_h)}{r} = \infty. \tag{3.2.13}$$

Using Poisson's formula and (3.2.5), we get that

$$\begin{aligned} B(r, v_h) &\leq \frac{3}{2\pi}\int_0^{2\pi} v_h(2re^{i\phi})d\phi \\ &\leq 3m(2r,h) + O(1) \\ &\leq 3T(2r,h) + O(1). \end{aligned}$$

We can combine this with (3.2.13) to give

$$\liminf_{r\to\infty} \frac{T(r,h)}{r} = \infty,$$

and using (3.2.3) we get

$$\liminf_{r\to\infty} \frac{T(r,f)}{r} = \infty.$$

This clearly contradicts our initial assumption (3.1.1), thus telling us that our assumption (3.2.4) cannot hold. Therefore

$$\liminf_{r\to\infty} \left(m(r,h) - r^{\frac{1}{2}} \int_r^\infty \frac{n(t,h)}{t^{\frac{3}{2}}} dt \right) < \infty. \tag{3.2.14}$$

Now integration by parts using (3.2.1) and (3.2.2) gives us

$$\begin{aligned} m(r,h) - r^{\frac{1}{2}} \int_r^\infty \frac{n(t,h)}{t^{\frac{3}{2}}} dt &= m(r,h) - r^{\frac{1}{2}} \left[t^{-\frac{1}{2}} N(t,h) \right]_r^\infty - \frac{r^{\frac{1}{2}}}{2} \int_r^\infty t^{-\frac{3}{2}} N(t,h) dt \\ &= m(r,h) + N(r,h) - \frac{r^{\frac{1}{2}}}{2} \int_r^\infty t^{-\frac{3}{2}} N(t,h) dt \\ &\geq T(r,h) - \frac{r^{\frac{1}{2}}}{2} \int_r^\infty t^{-\frac{3}{2}} (N\left(t, \frac{1}{f}\right) + H \log t) dt) \\ &= T(r,h) - \frac{r^{\frac{1}{2}}}{2} \int_r^\infty t^{-\frac{3}{2}} N\left(t, \frac{1}{f}\right) dt - \frac{r^{\frac{1}{2}}}{2} \left(\frac{4H}{r^{\frac{1}{2}}} + \frac{2H \log r}{r^{\frac{1}{2}}} \right) \\ &= T(r,h) - H\log r - 2H - \frac{r^{\frac{1}{2}}}{2} \int_r^\infty t^{-\frac{3}{2}} N\left(t, \frac{1}{f}\right) dt. \end{aligned}$$

Combining (3.2.14) with (3.2.3) and taking the limit infinum gives

$$\liminf_{r\to\infty} \left(T(r,f) - H\log r - O(1) - H\log r - 2H - \frac{r^{\frac{1}{2}}}{2} \int_r^\infty t^{-\frac{3}{2}} N\left(t, \frac{1}{f}\right) dt \right) < \infty$$

and so

$$\liminf_{r\to\infty} \left(T(r,f) - 2H\log r - \frac{r^{\frac{1}{2}}}{2} \int_r^\infty t^{-\frac{3}{2}} N\left(t, \frac{1}{f}\right) dt \right) < \infty$$

as required.

3.3 Proof of Theorem 3.1.3

Before proving this theorem, we examine further what implication condition (3.1.3) has. With this aim, we prove the following lemma.

Lemma 3.3.1. *Let f be as in the hypotheses of Theorem 3.1.3. Then $T(r,f) = O(\log r)^2$, and in particular this gives $\rho(f) = 0$.*

Proof of Lemma 3.3.1

We shall show that there exists an upper bound, M, independent of r, for the number of distinct z_k in the annulus $2r < |z| \leq 4r$. It then follows that in the annulus $r < |z| \leq 2r$ there is the same upper bound for the number of z_k as in the annulus $2r < |z| \leq 4r$. So if we let $c_1 = \overline{n}(r)$, then we deduce that,

$$\overline{n}(2^n r) \leq nM + c_1.$$

To calculate such an upper bound M, let r_1 be large and fixed and notice that the annulus $\frac{r_1}{2} < |z| \leq 3r_1$ has area $\pi(9{r_1}^2 - \frac{{r_1}^2}{4}) = \frac{35}{4}\pi {r_1}^2$. Surround each z_k in $r_1 < |z| \leq 2r_1$ by a disc of radius $\frac{r_1\eta}{2}$. Then these discs are disjoint, by (3.1.3), and each lies wholly within the annulus $\frac{r_1}{2} < |z| \leq 3r_1$, since we may assume that η is less than 1. Since each disc has area $\frac{\pi {r_1}^2\eta^2}{4}$, this gives us that

$$\begin{aligned} M &\leq \frac{\frac{35}{4}\pi {r_1}^2}{\frac{1}{4}\pi {r_1}^2\eta^2} \\ &= \frac{35}{\eta^2}. \end{aligned}$$

This then gives us,

$$\overline{n}(2^n r_1) \leq \frac{35n}{\eta^2} + c_1.$$

If $r \in (2^n r_1, 2^{n+1} r_1]$, then

$$\overline{n}(r) \leq \frac{35(n+1)}{\eta^2} + c_1.$$

Since $r \geq 2^n r_1$, we get that $n \leq \frac{\log r - \log r_1}{\log 2}$. This tells us that

$$\overline{n}(r) \leq \frac{35 \log r}{\eta^2 \log 2} + c_2 = O(\log r),$$

as $r \to \infty$. From this we can estimate $N(r,f)$ in the usual way, see Corollary 1.6.3, which gives us $N(r,f) \leq O((\log r)^2)$. Combining this with Theorem 1.6.2, given in Chapter 1, gives us that $T(r, f) = O((\log r)^2)$, which completes the proof of the lemma.

Returning to Theorem 3.1.3, we need the following definition from Hayman [10].

Definition 3.3.2. ***ε-set*** - *We call any countable set of discs not containing the origin and subtending angles at the origin whose sum s is finite an ε*-set.

This proof will utilise the following result of Anderson and Clunie, which says, roughly speaking, that a deficient value of a meromorphic function of slow growth is an asymptotic value in a very strong sense.

Theorem 3.3.3. [1] *Let $f(z)$ be meromorphic, and satisfy $T(r, f) = O(\log r)^2$. Suppose that $\delta(a, f) > 0$ for some $a \in \mathbb{C}$. Then*

$$\log|f(z) - a| < -[\delta(a, f) - o(1)]T(|z|, f), \tag{3.3.1}$$

as $z \to \infty$ outside of an ε-set of discs whose centres tend to infinity.

In order to prove Theorem 3.1.3, we let f be as in the hypotheses, and for the time being we assume $\delta(0, f) > 0$, and denote by

$$E = \bigcup_{n=1}^{\infty} B(A_n, r_n), \quad |A_n| \to \infty, \quad \sum_{n=1}^{\infty} \frac{r_n}{|A_n|} < \infty,$$

the ε-set described in Theorem 3.3.3. If we take r large, then the discs of E which meet the annulus $\frac{r}{2} < |z| < 2r$ have sum of radii $o(r)$. This is shown in [10].

We choose a small, positive constant ϵ less than $\frac{\eta}{2}$. Then for large m there exists an $s = s_m$ with

$$\epsilon|z_m| < s < 2\epsilon|z_m|$$

such that the circle $|z - z_m| = s$ does not meet the discs of E, and such that z_m is the only member of the set $\{z_k\}$ lying inside the circle.

Now consider

$$\int_{|z-z_m|=s} |f(z)||dz|.$$

From (3.3.1) we get that

$$|f(z)| < e^{-[\delta(0,f)-o(1)]T(|z|,f)}$$

for $|z - z_m| = s$.

Now using the following integration estimate

$$\int_{|z-z_m|=s} |f(z)||dz| \leq \max\{|f(z)| : |z-z_m| = s\} \cdot (\text{length of the curve}),$$

and the fact that $T(r,f)$ is strictly increasing, we get that

$$\begin{aligned}\int_{|z-z_m|=s} |f(z)||dz| &\leq 2\pi s e^{-(\delta(0,f)-o(1))T(|z_m|-s,f)} \\ &\leq 4\pi\epsilon|z_m|e^{-(\delta(0,f)-o(1))T((1-2\epsilon)|z_m|,f)}\end{aligned}$$

By the residue theorem, and the fact that for functions of the type f, the residue at each pole z_k is equal to a_k, we get that

$$2\pi|a_m| = \left|\int_{|z-z_m|=s} f(z)dz\right| \leq \int_{|z-z_m|=s} |f(z)||dz|.$$

When we combine these results we get that

$$|a_m| \leq 2\epsilon|z_m|e^{-(\delta(0,f)-o(1))T((1-2\epsilon)|z_m|,f)}.$$

Note that for $0 < \kappa < 1$, we have

$$T(r,f) - T(\kappa r, f) = O(\log r). \tag{3.3.2}$$

To see this note that, by [11, p.9],

$$\begin{aligned}\phi(r) &= r\frac{d}{dr}T(r,f) \\ &= \frac{1}{2\pi}\int_0^{2\pi} n(r, e^{i\theta}, f)d\theta\end{aligned}$$

is non-decreasing. Hence

$$\begin{aligned}\phi(r)\log r &= \int_r^{r^2} \phi(r)\frac{dt}{t} \\ &\leq \int_r^{r^2} \phi(t)\frac{dt}{t} \\ &= T(r^2,f) - T(r,f) \\ &= O((\log r)^2)\end{aligned}$$

and so $\phi(r) = O(\log r)$, from which (3.3.2) follows by integration.

This tells us that

$$\begin{aligned} |a_m| &\leq 2\epsilon|z_m|e^{-[\delta(0,f)-o(1)](T(|z_m|,f)-O(\log|z_m|))} \\ &= 2\epsilon|z_m|e^{-[\delta(0,f)-o(1)](T(|z_m|,f))} \\ &= e^{-[\delta(0,f)-o(1)](T(|z_m|,f))} \end{aligned}$$

as required.

We continue for now to assume that $\delta(0,f) > 0$. For large k we have

$$|a_k| < e^{-\delta' T(|z_k|,f)}, \tag{3.3.3}$$

for some $\delta' > 0$.

Now we use the fact that f is transcendental, and hence

$$\lim_{k\to\infty} \frac{T(|z_k|,f)}{\log|z_k|} = \infty.$$

By combining this with (3.3.3), we obtain the following inequality.

$$|a_k| < |z_k|^{-L}, \tag{3.3.4}$$

for any $L > 0$. However, standard results (see [11, p.26]) give

$$\sum_{k=1}^{\infty} |z_k|^{-1} < \infty, \tag{3.3.5}$$

for any function of order 0, with poles z_k.

We now combine (3.3.4) and (3.3.5) to give,

$$\sum_{k=1}^{\infty} |a_k| < \infty.$$

Now we can apply Theorem 1.6.5 or Theorem 2.1.4 to give us that

$$\sum_{k=1}^{\infty} a_k = 0,$$

as required.

Remark - This method only works when $\delta(0,f) > 0$, however the result (3.1.4) still holds even when $\delta(0,f) = 0$, since the convergence criteria (1.3.2) gives

$$a_k = o(|z_k|) \leq e^{o(T(|z_k|,f))}.$$

3.4 An Interesting Observation

In an attempt to find an improved version of Theorem 1.6.6 without the condition $a_k > 0$, the author has made an observation, which arises instantly from an application of the following result.

Lemma 3.4.1 ([5]). *Let f be a non-constant entire function of order λ with $f(0) \neq 0$, and let $\{z_1, z_2, ...\}$ be the zeros of f counted according to multiplicity and arranged so that $|z_1| \leq |z_2| \leq ....$ If an integer p satisfies $p > \lambda - 1$ then*

$$\frac{d^p}{dz^p}\left[\frac{f'(z)}{f(z)}\right] = -p! \sum_{n=1}^{\infty} \frac{1}{(z_n - z)^{p+1}}$$

for $z \neq z_1, z_2, ...$

If we let f be a Keldysh Function, and assume that $\rho(f) < 1$ and that f has no zeros, we can then apply Lemma 3.4.1 to $\frac{1}{f}$ to give

$$-\frac{f'(z)}{f(z)} = -\sum_{n=1}^{\infty} \frac{1}{(z_n - z)}$$

for $z \neq z_k$.

The interesting fact about this is that it means that the derivative of f and the Keldysh Function with the same z_k but with all a_k set to 1 must have the exact same zero sets. However, f' is dependent on a_k, whereas the other function is not. Since we can apply Corollary 1.6.7 to the right hand side, these zero sets must contain infinitely many points. Furthermore, all the zeros of the right hand side must lie in the convex hull of the z_k, so the same must also be true of f'. Also f' must have infinitely many zeros to match those on the right hand side. If the derivative does not satisfy these properties, then we can conclude that f must have at least one zero. Furthermore, if $\rho(f) < 1$, f has no zeros, and we change exactly one a_k, then the new function must have at least one zero, as its derivative will no longer have the same zero set.

The author has so far been unable to use this observation to prove any new results, however it is included here, as it may be of interest to anyone studying this area.

Chapter 4

A Generalisation to Mittag-Leffler Sums

This chapter looks at a generalisation of Keldysh Functions to Mittag-Leffler type functions. By examining two distinct cases, the existence of zeros of these new functions is shown. These results, along with those from Chapter 3, have been published as a paper in the CMFT journal ([26]).

4.1 Introduction

The two results in this chapter concentrate on a problem which generalises the standard functions to Mittag-Leffler-type functions as follows. Let

$$F(z) = \sum_{n=1}^{\infty} \frac{z^{k_n} a_n}{{z_n}^{k_n}(z - z_n)}, \quad a_n \neq 0, \quad 0 \leq k_n \in \mathbb{Z}, \quad \sum_{n=1}^{\infty} \left| \frac{a_n}{{z_n}^{k_n+1}} \right| < \infty. \quad (4.1.1)$$

This extension was suggested by Rod Halburd of Loughborough University, and allows for a weakened convergence criterion. Clearly the original functions f can be viewed as functions of this type with all the k_n set equal to zero.

These functions come from viewing the original functions f as linear combinations of logarithmic derivatives of simple Weierstrass primary factors. These are defined as follows in [11, p.21],

$$\begin{aligned} E(z,q) &= (1-z)e^{z+\frac{z^2}{2}+\ldots+(\frac{z^q}{q})} \qquad (q \geq 1) \\ E(z,0) &= 1-z. \end{aligned}$$

The logarithmic derivatives are as follows: if $p_n(z) = E\left(\frac{z}{z_n}, k_n\right)$, then

$$\frac{p_n'(z)}{p_n(z)} = \frac{z^{k_n}}{{z_n}^{k_n}(z - z_n)}.$$

In this way, the extended functions $F(z)$ are linear combinations of logarithmic derivatives of higher order primary factors.

The first result describes the case where there is an upper bound for the k_n, and adapts the method used in Chapter 2 to these extended functions.

Theorem 4.1.1. *Let $F(z)$ be as in (4.1.1), with $z_n \in \mathbb{C}\backslash\{0\}$ and $\lim_{n\to\infty} |z_n| = \infty$. Suppose $k_n \leq M < \infty$ for all n, and*

$$\sum_{n=1}^{\infty} \left| \frac{a_n}{{z_n}^{k_n}} \right| < \infty, \qquad \sum_{n:k_n=M} \frac{a_n}{{z_n}^{k_n}} \neq 0. \tag{4.1.2}$$

Suppose further that either
a) $\phi(t)$ is continuous, positive and non-decreasing on $[1,\infty)$ with

$$\int_1^{\infty} \frac{1}{t\phi(t)}\, dt = \infty$$

and

$$\limsup_{r\to\infty} \frac{\log \overline{n}(r)}{\phi(r)} < \infty,$$

or b)

$$\liminf_{r\to\infty} \frac{\log \overline{n}(r)}{\log r} < \infty.$$

Then $\delta(0, F) = 0$.

Next we prove another result for such Mittag-Leffler-type functions, but in the case where the k_n are allowed to be unbounded. The proof uses the $\cos \pi\rho$ theorem combined with the residue theorem.

Theorem 4.1.2. *Let $F(z)$ be as in (4.1.1), with $z_n \in \mathbb{C} \setminus \{0\}$ and $\lim\limits_{n\to\infty} |z_n| = \infty$. Suppose further that $\rho(F) = \rho < \frac{1}{2}$, and that*

$$\liminf_{r\to\infty} \left| \sum_{|z_n|<r} a_n \right| > 0. \tag{4.1.3}$$

Then $\delta(0, F) \leq 1 - \cos \pi\rho$.

We observe that (4.1.3) is satisfied if $Re(a_n) > 0$ for all n. We remark that a sufficient condition for $\rho(F) < \frac{1}{2}$ is given in §4.4.

4.2 Proof of Theorem 4.1.1

Let F be as in the hypotheses of Theorem 4.1.1. We adapt a method from [7, p.333], by writing

$$\begin{aligned} zF(z) &= \sum_{n=1}^{\infty} \frac{a_n}{{z_n}^{k_n}} z^{k_n} + \sum_{n=1}^{\infty} \frac{a_n}{{z_n}^{k_n-1}(z-z_n)} z^{k_n}. \\ &= H(z) + G(z). \end{aligned} \tag{4.2.1}$$

Both series converge, by (4.1.2). Here $H(z)$ is entire and non-constant, and in fact is a polynomial of degree M, by (4.1.2). This tells us that $|H(z)| \approx c|z|^M$ as $z \to \infty$ for some $c > 0$.

If $zF(z)$ is to be small then $|H(z)| \approx |G(z)|$, and so we seek an upper bound for $|G(z)|$.

We construct a new function defined as follows: Let

$$J(z) = \sum_{n=1}^{\infty} \left| \frac{a_n}{z_n^{k_n-1}(z-z_n)} \right| \tag{4.2.2}$$

We apply Proposition 2.1.2 from Chapter 2 to J, noting that in this case we have $b_n = \left| \frac{a_n}{{z_n}^{k_n-1}} \right|$, $y_n = z_n$.

This gives us circles $|z| = R_j \to \infty$ on which $|J(z)| \leq o(1)$.

However, for $|z| = R_j$ (4.2.1) gives

$$\begin{aligned} |G(z)| &\leq \sum_{n=1}^{\infty} \left| \frac{a_n}{z_n^{k_n-1}(z-z_n)} \right| |z|^{k_n} \\ &\leq \sum_{n=1}^{\infty} \left| \frac{a_n}{z_n^{k_n-1}(z-z_n)} \right| |z|^{M} \\ &\leq J(z)|z|^M \\ &= o(1)|z|^M \\ &= o(|H(z)|). \end{aligned}$$

Hence $zF(z)$ cannot be small on $|z| = R_j$, and so for large j, and $|z| = R_j$

$$|H + G| \geq |H| - |G| \geq \frac{c}{2}|z|^M.$$

Hence,

$$\log^+\left(\frac{1}{|H+G|}\right) \leq \log^+\left(\frac{2}{cR_j^M}\right) = O(1),$$

since $R_j \to \infty$. Thus

$$\begin{aligned}
\delta(0, zF) &= \delta(0, H+G) \\
&= \liminf_{r\to\infty} \frac{m\left(r, \frac{1}{(H+G)}\right)}{T(r, H+G)} \\
&\leq \liminf_{j\to\infty} \frac{m\left(R_j, \frac{1}{(H+G)}\right)}{T(R_j, H+G)} \\
&= 0,
\end{aligned}$$

since $m\left(R_j, \dfrac{1}{(H+G)}\right) = O(1)$, and $T(R_j, H+G) \to \infty$ as $R_j \to \infty$. This gives $\delta(0, F) = 0$.

A Remark on Theorem 4.1.1

Note that for functions F as in (4.1.1), where $k_n \leq M$ for all n, we can make the following observation. We have

$$F = F_0 + F_1 + \ldots + F_M,$$

and

$$\begin{aligned}
|F_j(z)| &= \left| \sum_{k_n=j} \frac{a_n z^j}{{z_n}^j (z - z_n)} \right| \\
&= |z|^j \left| \sum_{k_n=j} \frac{a_n}{{z_n}^j (z - z_n)} \right|.
\end{aligned}$$

Using this, we can estimate $m(r, F_j)$ for large r as follows. We have

$$\begin{aligned}
m(r, F_j) &= \frac{1}{2\pi} \int_0^{2\pi} \log^+ |F_j(re^{i\theta})| d\theta \\
&\leq \frac{1}{2\pi} \int_0^{2\pi} \log^+ |re^{i\theta}|^j d\theta + \frac{1}{2\pi} \int_0^{2\pi} \log^+ |h(re^{i\theta})| d\theta \\
&\leq \frac{1}{2\pi} \int_0^{2\pi} j \log r \ d\theta + m(r, h)
\end{aligned}$$

where

$$h(z) = \sum_{k_n=j} \frac{a_n}{{z_n}^j (z - z_n)}.$$

Now we combine this with Theorem 1.6.2 to obtain,

$$m(r, F_j) \leq j \log r + o(1),$$

and hence,

$$\begin{aligned} m(r, F) &\leq \sum_{j=0}^{M} j \log r + o(1) + \log M \\ &= O(\log r). \end{aligned}$$

Therefore the criteria a) and b) in Theorem 4.1.1 determine the order of the function F, and are equivalent to F having finite lower order, or finite or "small infinite" upper order.

4.3 Proof of Theorem 4.1.2

For functions F of the form (4.1.1), we obtain,

$$\int_{|z|=r} F(z)dz = 2\pi i \sum_{|z_n|<r} a_n, \tag{4.3.1}$$

by the residue theorem, provided no z_n lie on $|z| = r$.

Now we shall estimate the integral using the $\cos \pi\rho$ theorem. The following is an immediate consequence of the $\cos \pi\rho$ theorem for meromorphic functions [9] (for the version applying to entire or subharmonic functions see Theorem 1.2.9 in Chapter 1).

Theorem 4.3.1 ([9]). *Let f be a transcendental meromorphic function, such that $\rho = \rho(f) < \frac{1}{2}$. If $\delta(0, f) > 1 - \cos \pi\rho$, then there exist $r_j \to \infty$ for which*

$$\frac{\log M(r_j, f)}{\log r_j} \to -\infty.$$

In particular,

$$\int_{|z|=r_j} f(z)dz = o(1).$$

Now if we apply this theorem to F, we clearly get a contradiction with our value (4.3.1) for the integral calculated using the residue theorem. This proves Theorem 4.1.2.

4.4 Remarks on Theorem 4.1.2: The Condition $\rho(F) < \frac{1}{2}$.

In this section we consider a sufficient condition to ensure that $\rho(F) < \frac{1}{2}$ in (4.1.1). To this end, the following lemma will be proved.

Lemma 4.4.1. *Let F be given by (4.1.1). Assume that F satisfies the following, for some $\epsilon > 0$:*

$$\limsup_{j\to\infty} j^{(2+\epsilon)j} \sum_{k_n=j} \left| \frac{a_n}{{z_n}^{k_n}} \right| \leq 1, \quad \limsup_{r\to\infty} \frac{\log \overline{n}(r)}{\log r} < \frac{1}{2}. \tag{4.4.1}$$

Then $\rho(F) < \frac{1}{2}$.

Proof of Lemma 4.4.1

For $|z| = r$, notice that

$$\begin{aligned} |F(z)| &\leq \sum_{n=1}^{\infty} \left| \frac{a_n}{{z_n}^{k_n}} z^{k_n} \right| \cdot \sum_{n=1}^{\infty} \frac{1}{|z - z_n|} \\ &= \quad G(z) \quad \cdot \quad H(z). \end{aligned}$$

If we look at G and H separately, we can write

$$\begin{aligned} g(z) &= \sum_{n=1}^{\infty} \left| \frac{a_n}{{z_n}^{k_n}} \right| z^{k_n} \\ \text{where} \quad G(z) &= g(|z|). \end{aligned}$$

Using (4.4.1) and the formula for the order of an entire function [5, p.286] (see also Chapter 1, Lemma 1.1.3), we can deduce that $\rho(g) < \frac{1}{2}$, and hence $G(z) \leq \exp(|z|^{\frac{1}{2}-\epsilon})$ as $|z| \to \infty$.

Now we look at H. Let E be the union of discs $D(z_n, 1)$. Then the discs of E which meet $D(0, 4r)$ have sum of diameters

$$\begin{aligned} \leq \sum_{|z_n| \leq 4r+1} 2 &= 2\overline{n}(4r+1) \\ &= o(r), \end{aligned}$$

by (4.4.1). We now pick r' with $r \leq r' \leq 2r$ such that the circle $|z| = r'$ meets none of the discs of E. Then for $|z| = r'$, we have

$$\begin{aligned} \sum_{|z_n|>4r} \frac{1}{|z-z_n|} &\leq \sum_{|z_n|>4r} \frac{1}{||z|-|z_n||} \\ &< \sum_{|z_n|>4r} \frac{4}{3|z_n|} = o(1), \end{aligned}$$

as $r \to \infty$, using the second condition of (4.4.1), and

$$\begin{aligned} \sum_{|z_n|\leq 4r} \frac{1}{|z-z_n|} &\leq \sum_{|z_n|\leq 4r} 1 \\ &= \overline{n}(4r) \\ &= o(r), \end{aligned}$$

using (4.4.1) again.

Adding together our estimates for G and H, we get that

$$|F(z)| \leq e^{(r')^{\frac{1}{2}-\epsilon}} + o(r).$$

for $|z| = r'$. This gives

$$m(r', F) \leq (r')^{\frac{1}{2}-\epsilon} + O(\log r)$$

and we know that

$$N(r', F) \leq (r')^{\frac{1}{2}-\epsilon}$$

by (4.4.1). Since

$$\begin{aligned} T(r, F) \leq T(r', F) &\leq 3(r')^{\frac{1}{2}-\epsilon} \\ &= O(r^{\frac{1}{2}-\epsilon}), \end{aligned}$$

the result follows.

Chapter 5

Behaviour of the Derivative

This chapter examines the family of functions which correspond to the derivatives of Keldysh Functions. In particular we examine the constraints that follow from the assumption that these functions have only finitely many zeros.

5.1 Introduction

Let us look at the functions of the following type,

$$g(z) = \sum_{k=1}^{\infty} \frac{a_k}{(z-z_k)^2}, \quad a_k \in \mathbb{C} \setminus \{0\}, \quad |z_k| \to \infty. \tag{5.1.1}$$

Without loss of generality we assume that $z_k \neq 0$. These require the following convergence criterion:

$$\sum_{k=1}^{\infty} \frac{|a_k|}{|z_k|^2} < \infty. \tag{5.1.2}$$

Bearing this in mind, we can view these functions g as the derivatives of functions F, given as follows,

$$\begin{aligned} F(z) &= \sum_{k=1}^{\infty} \left(\frac{a_k}{z-z_k} + \frac{a_k}{z_k} \right) \\ &= z \sum_{k=1}^{\infty} \frac{a_k}{z_k(z-z_k)}, \end{aligned} \tag{5.1.3}$$

where a_k and z_k are as before, providing that (5.1.2) is satisfied.

If however the stronger condition $\sum_{k=1}^{\infty} \frac{|a_k|}{|z_k|} < \infty$ is satisfied, then we can view g as the derivative of a Keldysh Function f, i.e.

$$f(z) = \sum_{k=1}^{\infty} \frac{a_k}{z - z_k},$$

again with a_k and z_k as before.

This next theorem demonstrates the tight restrictions upon a_k and z_k required for a function of this type to have finitely many zeros.

Theorem 5.1.1. *Let g and F be given as above, so that $g = F'$. Assume that F has finite order ρ and that g has only finitely many zeros.*
Then we have the following asymptotic behaviour for F, and hence also for g.

1. The order ρ of F satisfies $2 \leq 2\rho \in \mathbb{N}$.

2. There are $N = 2\rho$ critical rays $\arg z = \theta_i$, going from the origin out to infinity, separated by angles of opening $\frac{2\pi}{N}$.

3. If k is large, then $\arg z_k = \theta_i + o(1)$ for some i.

4. Let $\epsilon > 0$. Then in a sectorial region,

$$S_p = \{z \in \mathbb{C} : |z| > R, \theta_p - \frac{2\pi}{N} + \epsilon < \arg z < \theta_p + \frac{2\pi}{N} - \epsilon\}, \tag{5.1.4}$$

F can be written in the form $F = \frac{u_1}{u_2}$ where the u_j are analytic in S_p and are linearly independent solutions of

$$u'' + \frac{S(z)}{2} u = 0, \tag{5.1.5}$$

where $S(z)$ is the Schwarzian derivative of F (see (5.2.1) below), and is a rational function.

5. For each p there exists a constant $c_p \neq 0$ such that if $z_k \in S_p$ is large, then

$$a_k \sim \frac{c_p}{{z_k}^{\rho(F)-1}}.$$

5.2 Proof of Theorem 5.1.1

Looking at the Schwarzian derivative of F, which is defined in [10] as follows,

$$S(z) = \frac{F'''}{F'} - \frac{3}{2}\left(\frac{F''}{F'}\right)^2, \tag{5.2.1}$$

we immediately notice that S has only finitely many poles, as it gets its poles from the zeros of F', finitely many by assumption, and the multiple poles of F, of which there are none.

In this proof we will require the following basic facts about the Schwarzian derivative, which can be found in texts such as [14].

Lemma 5.2.1. *Let f be a meromorphic function, let T be any Möbius transformation, and let $S(f)$ be the Schwarzian derivative of f. The Schwarzian derivative has the following properties.*

$$S(T(f)) = S(f),$$

and furthermore, if we have

$$S(f) \equiv 0,$$

then f is a Möbius Transformation.

In this case S has finitely many poles, and this implies that S is a rational function, as clearly $N(r, S) = O(\log r)$, and also

$$\begin{aligned} m(r, S) &= \frac{1}{2\pi}\int_0^{2\pi} \log^+ |S(re^{i\phi})| d\phi \\ &\leq m\left(r, \frac{F'''}{F'}\right) + 2m\left(r, \frac{F''}{F'}\right) + O(1) \\ &= O(\log r), \end{aligned}$$

by the lemma of the logarithmic derivative (see Lemma 1.1.15), and the fact that F has finite order (by assumption). Using this fact, we can write, as $z \to \infty$,

$$S(z) = cz^m(1 + o(1)), \tag{5.2.2}$$

where $c \in \mathbb{C}$ is a constant and $m \in \mathbb{Z}$.

Since S is the Schwarzian derivative of F, it follows from standard results [14, 17] that if Ω is any simply connected domain on which S is analytic, then we may write,

$$F = \frac{u_1}{u_2}$$

in Ω, where u_1, u_2 are linearly independent solutions of (5.1.5) on Ω. This proves part four of the theorem.

To prove the rest of the result, we will utilise Lemma 5.2.1 from above.

We start by assuming that either $c = 0$ or $m \leq -2$. Clearly if $c = 0$, then by Lemma 5.2.1 F is Möbius, a contradiction. If $m \leq -2$, following the method in [14, Chapter 7], we choose a sectorial region $S = \{z : |z| > R, \quad 0 < \arg(z) < 2\pi\}$ with R large, and we can write $F = \frac{u_1}{u_2}$, where the u_i are linearly independent and satisfy (5.1.5). Returning to $F' = g$, we can write this as $F' = \frac{d}{{u_2}^2}$, where d is a constant.

Looking at the proximity function of $\frac{1}{F'}$, because u_2 is analytic in S we have

$$m\left(r, \frac{1}{F'}\right) = \frac{1}{\pi}\int_0^{2\pi} \log^+ |u_2| d\theta + O(1). \tag{5.2.3}$$

We now utilise the following, which is Lemma 4 from [18].

Lemma 5.2.2 ([18]). *Suppose $k \geq 2$, and that $a_0, \ldots, a_{k-1}$ are analytic in $|z| \geq R$ such that for some $\lambda \geq 0$,*

$$a_j(z) = O\left(|z|^{(\lambda-1)(k-j)}\right) \qquad \textit{as } z \to \infty.$$

Let $f(z)$ be a solution of

$$L(y) = y^{(k)} + a_{k-1}y^{(k-1)} + \cdots + a_0 y = 0,$$

in a sectorial region

$$S = \{z : |z| > R, \alpha < \arg z < \alpha + 2\pi\},$$

where α is real. Then as $z \to \infty$ in S,

$$\log^+ |f(z)| = O\left(|z|^\lambda + \log|z|\right).$$

We take $k = 2$, and $\lambda = 0$. Since $S(z)$ is analytic in $|z| > R$, we set $a_0(z) = \frac{1}{2}S(z)$, and let $a_1(z) = 0$. Note that $a_0(z) = O\left(|z|^{-2}\right)$ so the conditions of Lemma 5.2.2 are satisfied.

This gives us

$$\log^+ |u_2(z)| = O(\log |z|),$$

from which we get, using (5.2.3),

$$m\left(r, \frac{1}{F'}\right) = O(\log r).$$

Combining this with the fact that F' only has finitely many zeros, and hence $N\left(r, \frac{1}{F'}\right) = O(\log r)$, we deduce

$$T\left(r, \frac{1}{F'}\right) = O(\log r),$$

i.e. $\dfrac{1}{F'}$ is a rational function (and hence so are F and g), a contradiction.

Next we need more information about the asymptotics when $m \geq -1$. Following [14], since we know that $c \neq 0$ and $m \geq -1$, the critical rays for equation (5.1.5) are defined to be the rays $\arg z = \theta_p$, for which

$$\arg c + (m+2)\theta_p = 0 \quad (mod\ 2\pi).$$

If $m = -1$ there is only one ray. Take a small positive $\epsilon > 0$, and R large and positive, and define the sectorial region S_p by (5.1.4), where $N = m+2$. Then on the region S_p, the equation (5.1.5) has principal solutions v_1, v_2 satisfying

$$v_j = T(z)^{-\frac{1}{4}} e^{(-1)^j iZ},$$

where

$$T(z) = \frac{S(z)}{2},$$

and

$$\begin{aligned} Z &= \int_{Re^{i\theta_p}}^{z} T(t)^{\frac{1}{2}} dt \\ &\simeq \frac{2\sqrt{c} z^{\frac{m+2}{2}}}{m+2}(1 + o(1)), \end{aligned}$$

as $z \to \infty$ in S_p. Moreover, F can be written as $F = \dfrac{u_1}{u_2}$, where the u_i are linearly independent solutions of (5.1.5).

With this notation we can write, in S_p,

$$\begin{aligned} u_1 &= A_1 v_1 - B_1 v_2 \\ u_2 &= A_2 v_1 - B_2 v_2, \end{aligned}$$

and

$$F(z) = \frac{A_1 v_1 - B_1 v_2}{A_2 v_1 - B_2 v_2},$$

where the A_i and B_i are constants.

Since

$$\begin{aligned} W(u_1, u_2) &= W(A_1 v_1 - B_1 v_2, A_2 v_1 - B_2 v_2) \\ &= (A_2 B_1 - A_1 B_2) W(v_1, v_2), \end{aligned}$$

and since $F = \frac{u_1}{u_2}$ we are free to multiply u_1 and u_2 by the same constant, and so may assume that

$$A_1 B_2 - A_2 B_1 = 1, \tag{5.2.4}$$

as $A_1 B_2 - A_2 B_1 \neq 0$ by the linear independence of the u_i.

We next show that $m \geq 0$, and so begin by assuming $m = -1$, and work for a contradiction. With only one critical ray, we can view the principal solutions v_1 and v_2 as functions on the Riemann surface of $\log z$, as they are defined on

$$|z| > R, \quad |\arg z - \theta_p| < 2\pi - \epsilon.$$

On one side of this domain, say the subsector given by,

$$\epsilon < \arg z - \theta_p < 2\pi - \epsilon,$$

one of the v_i is dominant, so either $\dfrac{v_1}{v_2} \to 0$ or $\dfrac{v_2}{v_1} \to 0$.

In the other subsector,

$$-2\pi + \epsilon < \arg z - \theta_p < -\epsilon,$$

the other of the v_i is dominant.

Now we use the fact that F is a meromorphic function, and hence single valued in all of $\mathbb{C}$ to reach a contradiction.

We consider the ray L_1, which is defined by

$$\arg z = \theta_p + \pi,$$

which is clearly the same ray as

$$\arg z = \theta_p - \pi.$$

In one subsector, we get that $\frac{v_1}{v_2} \to 0$, which implies that $F \sim \frac{B_1}{B_2}$.

Repeating the argument for the other subsector, we get $F \sim \frac{A_1}{A_2}$. Since both expressions are valid on the ray L_1, we conclude that

$$\frac{A_1}{A_2} \sim \frac{B_1}{B_2}.$$

Since this expression is valid for the entirity of the ray, we conclude that $A_1B_2 - A_2B_1 = 0$, which clearly contradicts the linear independence of u_1 and u_2. This combined with our previous argument gives us that $m > -1$, and proves our assertion.

To prove part five of Theorem 5.1.1, we assume that $m \geq 0$ and start by writing $S(z)$ as in (5.2.2).

Using part four of Theorem 5.1.1, assume that the sector S_p contains infinitely many poles of F. Then $A_2B_2 \neq 0$ and these poles z_k of F must be both zeros of u_2 and solutions of,

$$\frac{A_2}{B_2} = \frac{v_2}{v_1} \simeq e^{2iZ}.$$

Hence $\arg z_k \sim \theta_p$ for k large, and the number of roots in $S_p \cap B(0, r)$ is $c_1(1 + o(1))r^{\frac{(m+2)}{2}}$ as $r \to \infty$, where $c_1 \neq 0$, (see [17] for details). Applying this for each such S_p we see that $N(r, F)$ has order $\frac{(m+2)}{2}$ and so has $T(r, f)$, since $m(r, f) = O(\log r)$ by (5.1.3).

It remains only to prove the asymptotic formula in part 5 of Theorem 5.1.1 for a_k when $z_k \in S_p$ is large.

Logarithmic differentiation of v_j gives

$$\begin{aligned} \frac{v_j'(z)}{v_j(z)} &= -\frac{T'(z)}{4T(z)} + (-1)^j i \frac{dZ}{dz} \\ &\sim (-1)^j i T(z)^{\frac{1}{2}}, \end{aligned}$$

and so

$$v_j'(z) \sim (-1)^j i T(z)^{\frac{1}{4}} e^{(-1)^j i Z}.$$

In particular,

$$\frac{v_2'(z)}{v_1'(z)} \sim -\frac{v_2(z)}{v_1(z)}.$$

At the pole z_k, we have

$$u_2 = A_2 v_1 - B_2 v_2 = 0,$$

i.e.

$$\frac{v_2(z_k)}{v_1(z_k)} = \frac{A_2}{B_2},$$

and

$$\begin{aligned} a_k &= \frac{u_1(z_k)}{u_2'(z_k)} = \frac{A_1 v_1(z_k) - B_1 v_2(z_k)}{A_2 v_1'(z_k) - B_2 v_2'(z_k)} \\ &= \frac{v_1(z_k)}{v_1'(z_k)} \cdot \frac{A_1 - B_1 \frac{v_2(z_k)}{v_1(z_k)}}{A_2 - B_2 \frac{v_2'(z_k)}{v_1'(z_k)}} \\ &\sim \frac{i}{T(z_k)^{\frac{1}{2}}} \cdot \frac{A_1 - B_1 \frac{A_2}{B_2}}{A_2 + B_2 \frac{A_2}{B_2(1+o(1))}} \\ &= \frac{i}{T(z_k)^{\frac{1}{2}}} \cdot \frac{A_1 B_2 - B_1 A_2}{2A_2 B_2 + o(1)} \end{aligned}$$

as required.

5.3 Example of Functions Satisfying the Hypotheses of Theorem 5.1.1

Example 5.3.1. *Let*

$$G_n(z) = \frac{1}{e^{iz^n}+1} - \frac{1}{2}.$$

Then each G_n is an example of a function satisfying the conditions of the above theorem, with order n, and has a representation of the form (5.1.3).

Calculating the derivative of G_n, we obtain

$$G_n'(z) = \frac{-inz^{n-1}e^{iz^n}}{(e^{iz^n}+1)^2},$$

which has one zero of order $n-1$ at 0, and hence the conditions of Theorem 5.1.1 are satisfied.

As a check, we can calculate the Schwarzian derivative of G_n, giving

$$S_n(z) = \frac{n^2z^{2n} - n^2 + 1}{2z^2},$$

which is clearly a rational function, of leading order $2n-2$ as $z \to \infty$. This is easily calculated using the fact that $G_n(z) = T(e^{iz^n})$ where T is a Möbius transformation (see Lemma 5.2.1).

Now we show that we can write $G_n(z)$ in the form (5.1.3). First we must locate the poles.

The poles of G_n lie along the $2n$ critical rays, which have arguments θ_j, $j \in \{0, 1, \ldots, 2n-1\}$, where $\theta_j = \frac{\pi j}{n}$, and they appear on all rays at points with modulus $\sqrt[n]{(2k+1)\pi}$ $(k \in \mathbb{N})$.

We label these $z_{j,k}$, with $j \in \{0, 1, \ldots, 2n-1\}$ and $k \in \mathbb{N}$.
Next we calculate the residues of the poles. The residue at $z_{j,k}$ is equal to

$$a_{j,k} = \frac{i}{n{z_{k,j}}^{n-1}} \tag{5.3.1}$$

Now let us check the convergence of $\sum_{j,k} \left| \frac{a_{j,k}}{{z_{j,k}}^m} \right|$. We have

$$\begin{aligned}\sum_{j,k} \left| \frac{a_{j,k}}{{z_{j,k}}^m} \right| &= 2\sum_k \frac{1}{(\sqrt[n]{(2k+1)\pi})^{n+m-1}} \\ &= c\sum_k \frac{1}{(2k+1)\cdot(2k+1)^{\frac{m-1}{n}}},\end{aligned}$$

where c is a constant. This clearly converges for $m \geq 2$.

Proposition 5.3.2. *We have*

$$\begin{aligned}G_n(z) &= \frac{1}{e^{iz^n}+1} - \frac{1}{2} \\ &= \frac{i}{n}\sum_{j,k}\left(\frac{1}{{z_{j,k}}^{n-1}(z-z_{j,k})} - \frac{1}{{z_{j,k}}^n}\right) \\ &= \frac{iz}{n}\sum_{j,k}\frac{1}{{z_{j,k}}^n(z-z_{j,k})},\end{aligned}$$

with the $z_{j,k}$ defined as above.

Proof: We start by writing $G_n(z)$ in the following manner.

$$\begin{aligned}G_n(z) = \frac{1}{e^{iz^n}+1} - \frac{1}{2} &= \frac{e^{-iz^n}}{1+e^{-iz^n}} - \frac{1}{2} \\ &= \frac{inz^{n-1}}{inz^{n-1}}\frac{e^{-iz^n}}{1+e^{-iz^n}} - \frac{1}{2} \\ &= \frac{i}{nz^{n-1}}\frac{h_n'(z)}{h_n(z)} - \frac{1}{2} \\ &= R(z)\frac{h_n'(z)}{h_n(z)} - \frac{1}{2},\end{aligned}$$

where $h_n(z) = 1 + e^{-iz^n}$, which is entire of order n, mean type, and $R(z)$ is a rational function.

If we now calculate $m(r, G_n)$ we get

$$\begin{aligned}m(r,G_n) &= m\left(r, R\frac{h'}{h} - \frac{1}{2}\right) \\ &\leq m(r,R) + m\left(r,\frac{h'}{h}\right) + O(1) \\ &= O(\log r), \end{aligned} \tag{5.3.2}$$

by the lemma of the logarithmic derivative (Lemma 1.1.15). Since we know that $N(r, G_n) = O(r^n)$, we get $T(r, G_n) = O(r^n)$.

Now we look at the function $F_n(z)$, given as follows, using (5.3.1):

$$\begin{aligned} F_n(z) &= \frac{iz}{n}\sum_{j,k}\frac{1}{{z_{j,k}}^n(z-z_{j,k})} \\ &= z\sum_{j,k}\frac{a_{j,k}}{z_{j,k}(z-z_{j,k})}. \end{aligned}$$

Using Keldysh's upper bound for $m(r,f)$ when f is a Keldysh Function (Theorem 1.6.2), we can calculate $m(r,F_n)$ as follows. We write

$$m(r,F_n) \le m(r,z) + m\left(r,\frac{F_n}{z}\right) + O(1),$$

and since $\frac{F_n}{z}$ is a Keldysh Function (with a_k in the standard definition here replaced by $\frac{a_{j,k}}{{z_{j,k}}^n}$), we get that

$$m(r,F_n) \le \log r + o(1) + O(1) = O(\log r). \tag{5.3.3}$$

If we now compare F_n and G_n, we observe that they have the same poles with the same multiplicities and residues, so we can define an entire function H_n as follows. We set

$$H_n = F_n - G_n.$$

Since H_n is entire, $N(r,H_n) = 0$, and from our calculations (5.3.2) and (5.3.3), we get that $m(r,H_n) = O(\log r)$ and hence $T(r,H_n) = O(\log r)$. This tells us that H_n is a polynomial. Next we show that this polynomial is in fact a constant.

We look at the ray given by

$$\Gamma_n = \{z : \arg z = -\frac{\pi}{2n}\}$$

As we move away from the origin along Γ_n, we clearly have $|iz^n| \to \infty$, but we also have $\arg(iz^n) = 0$. This tells us that $e^{iz^n} = e^{|z|^n}$ on Γ_n. Hence on Γ_n, $G_n(z)$ tends to $-\frac{1}{2}$ rapidly.

Let z be large, and lying on Γ_n. Since Γ_n lies in between two of the critical rays of G_n, separated from them by an angle of $\frac{\pi}{2n}$, we can calculate an lower bound for $|z - z_{j,k}|$ as follows. We have

$$\begin{aligned} |z - z_{j,k}| &\ge |z_{j,k}| 2\sin\left(\frac{\pi}{2n}\right) \\ &= c_1|z_{j,k}|, \end{aligned} \tag{5.3.4}$$

where c_1 is a positive constant. The above inequality is obtained using basic trigonometry, as the minimal case arises when z has equal modulus to one of the $|z_{j,k}|$ and thus we can draw an isosceles triangle between z, $z_{j,k}$ and the origin.

For z as chosen before, we now consider the following decomposition of F_n. Write

$$F_n(z) = \sum_{k=1}^{N} \frac{z a_{j,k}}{z_{j,k}(z - z_{j,k})} + \sum_{k=N+1}^{\infty} \frac{z a_{j,k}}{z_{j,k}(z - z_{j,k})} \tag{5.3.5}$$

Take $\epsilon > 0$. Looking at the second term in (5.3.5), and using (5.3.4), we can choose N large enough so that

$$\begin{aligned} \left| \sum_{k=N+1}^{\infty} \frac{z a_{j,k}}{z_{j,k}(z - z_{j,k})} \right| &\leq \frac{|z|}{c_1} \sum_{k=N+1}^{\infty} \frac{|a_{j,k}|}{|z_{j,k}|^2} \\ &\leq \epsilon |z|. \end{aligned}$$

On the other hand, the first term in (5.3.5) is clearly a rational function, and the polynomials in the numerator and denominator have the same order, hence it is finite valued at infinity.

Choosing large N and also large z lying on Γ_n, we get

$$|H_n(z)| \leq |F(z)| + |G(z)| \leq 2\epsilon |z|.$$

However, we can choose ϵ to be arbitrarily small, which implies that H_n must in fact be constant.

To calculate the value of this constant, we need merely compare the values of F_n and G_n at any point which is not a pole. Looking at their values at 0, we get

$$F_n(0) = G_n(0) = 0,$$

and hence $H_n(z) \equiv 0$ and the functions are equal. These are therefore examples of functions of the form (5.1.3) which satisfy the conditions of Theorem 5.1.1.

Chapter 6

Further results in $\mathbb{R}^3$

This chapter returns to looking at potentials in real space, and by providing a correcting term, shows results concerning the existence of critical points of potentials with a weakened convergence criterion. The importance of this weakened criterion is explained.

6.1 Introduction

This chapter focuses on potentials in $\mathbb{R}^3$ of the following form. Let

$$u(x) = \sum_{k=1}^{\infty} a_k \left(\frac{1}{|x - x_k|} - \frac{1}{|x_k|} \right), \tag{6.1.1}$$

where $a_k > 0$, $|x_k| \to \infty$, $|x_k| \leq |x_{k+1}|$, and

$$\sum_{k=1}^{\infty} \frac{a_k}{|x_k|^2} < \infty. \tag{6.1.2}$$

Lemma 6.1.1. *The convergence criteria (6.1.2) is enough to ensure that (6.1.1) also converges absolutely (with the exception of the points where $x = x_k$).*

Proof of Lemma 6.1.1

Looking at $u(x)$ as defined in (6.1.1), we combine the fractions, and write it as

$$u(x) = \sum_{k=1}^{\infty} \frac{a_k(|x_k| - |x - x_k|)}{|x_k||x - x_k|}. \tag{6.1.3}$$

Since $|x_k| - |x - x_k| \leq |x|$,

$$\sum_{k=1}^{\infty} \left| a_k \left(\frac{1}{|x - x_k|} - \frac{1}{|x_k|} \right) \right| \leq \sum_{k=1}^{\infty} \frac{a_k |x|}{|x_k||x - x_k|}.$$

Now we split this sum as follows, with $R = |x|$,

$$\begin{aligned} \sum_{k=1}^{\infty} \left| a_k \left(\frac{1}{|x - x_k|} - \frac{1}{|x_k|} \right) \right| &\leq \sum_{|x_k|<2R} \frac{a_k |x|}{|x_k||x - x_k|} + \sum_{|x_k|\geq 2R} \frac{a_k |x|}{|x_k||x - x_k|} \\ &\leq \sum_{|x_k|<2R} \frac{a_k |x|}{|x_k||x - x_k|} + 2 \sum_{|x_k|\geq 2R} \frac{a_k |x|}{|x_k|^2}, \end{aligned}$$

and hence (6.1.2) is clearly enough to make (6.1.1) converge absolutely.

Notice that these potentials are not always positive, and have a weaker convergence criterion (6.1.2) than the ones described in Chapter 1 (Definition 1.5.1) and dealt with in [4, 19] and Chapter 2. We continue to use the same notation ($n(r)$, $\overline{n}(r)$) as defined in Chapter 1, (1.5.2).

With the aim of showing the existence of critical points, the following theorems have been proved. The first result considers the case where

$$\limsup_{k\to\infty} \frac{|x_{k+1}|}{|x_k|} > 1.$$

This represents a spacing of the x_k.

Theorem 6.1.2. *Let $u(x)$ be given by (6.1.1), with $|x_k| \geq 2\delta > 0$ and $\sum_{k=1}^{\infty} \frac{a_k}{|x_k|} = \infty$. Assume that there exist $\epsilon > 0$ and a sequence $r_n \to \infty$ such that*

a) No x_k lie in

$$r_n < |x| < (1 + \epsilon) r_n; \tag{6.1.4}$$

b)

$$\limsup_{n\to\infty} \frac{{r_n}^2}{n(r_n)} \left[\int_{r_n}^{\infty} \frac{1}{t^2} dn(t) \right] < \infty, \tag{6.1.5}$$

$$\text{and} \quad \lim_{n\to\infty} \frac{r_n}{n(r_n)} \left[\int_{\delta}^{r_n} \frac{1}{t} dn(t) \right] = \infty. \tag{6.1.6}$$

Then $u(x)$ has infinitely many critical points.

Remark - Using similar arguments to those in Chapter 1, Lemma 1.3.5 and in the proof of Lemma 1.3.4, we can use condition (6.1.2) to rewrite conditions (6.1.5) and (6.1.6) as follows:

$$\begin{aligned} & \limsup_{n\to\infty} \frac{{r_n}^2}{n(r_n)} \left[\int_{r_n}^{\infty} \frac{n(t)}{t^3} dt\right] &<& \infty, \\ \text{and} \quad & \lim_{n\to\infty} \frac{r_n}{n(r_n)} \left[\int_{\delta}^{r_n} \frac{n(t)}{t^2} dt\right] &=& \infty. \end{aligned}$$

Theorem 6.1.3. *Let $u(x)$ be given by (6.1.1), with $|x_1| = 2\delta > 0$, and $\sum_{k=1}^{\infty} \frac{a_k}{|x_k|} = \infty$. Suppose that there exists $r_n \to \infty$ for which*

$$\frac{n(2r_n)}{r_n} \log \overline{n}(2r_n) + r_n \int_{2r_n}^{\infty} \frac{1}{t^2} dn(t) \quad = \quad o\left(\int_{\delta}^{2r_n} \frac{1}{t} dn(t)\right). \tag{6.1.7}$$

Then $u(x)$ has infinitely many critical points.

Remark - Both Theorems 6.1.2 and 6.1.3 require the condition that $\sum_{k=1}^{\infty} \frac{a_k}{|x_k|} = \infty$. However if $\sum_{k=1}^{\infty} \frac{a_k}{|x_k|} < \infty$, we need not examine functions of the form (6.1.1), and can instead look at functions as described by Definition 1.5.1 in Chapter 1, and then apply results such as those in [4, 19] and Chapter 2.

6.2 Proof of Theorem 6.1.2

During this proof C will be used to designate a positive constant, but not necessarily the same constant each time it appears.

Proposition 6.2.1. *Let $u(x)$ satisfy the hypotheses of Theorem 6.1.2. Then*

$$\liminf_{r\to\infty} M(r, u) = -\infty.$$

Proof of Proposition 6.2.1

Start by splitting the function $u(x)$ into three parts as follows.

$$u(x) = \sum_{|x_k|>r_n} a_k \left(\frac{1}{|x-x_k|} - \frac{1}{|x_k|} \right) \tag{6.2.1}$$
$$+ \sum_{|x_k|\le r_n} \frac{a_k}{|x-x_k|} \tag{6.2.2}$$
$$- \sum_{|x_k|\le r_n} \frac{a_k}{|x_k|} \tag{6.2.3}$$

Define $r_n' = (\sqrt{1+\epsilon})r_n$. Now notice that for $|x| = r_n'$, and for all k,

$$|x - x_k| \ \ge \ C \max\{|x|, |x_k|\}. \tag{6.2.4}$$

This holds by (6.1.4).

Now we look at the three parts of the decomposition separately. First look at the term in (6.2.1).

Using the fact that $|a+b| \ge ||a| - |b||$, with $a = x_k$, and $b = x - x_k$, we get

$$\begin{aligned}
\sum_{|x_k|>r_n} a_k \left| \left(\frac{1}{|x-x_k|} - \frac{1}{|x_k|} \right) \right| &= \sum_{|x_k|>r_n} a_k \left| \left(\frac{|x_k| - |x-x_k|}{|x_k||x-x_k|} \right) \right| \\
&\le \sum_{|x_k|>r_n} a_k \frac{|x|}{|x-x_k||x_k|} \\
&\le C \sum_{|x_k|>r_n} a_k \frac{|x|}{|x_k|^2} \\
&= C r_n \sum_{|x_k|>r_n} \frac{a_k}{|x_k|^2} \\
&= C r_n \int_{r_n}^{\infty} \frac{1}{t^2} dn(t) \\
&\le C \frac{n(r_n)}{r_n}, \qquad (6.2.5)
\end{aligned}$$

using (6.1.5) and (6.2.4). Next we look at (6.2.2). Here

$$\begin{aligned}
\sum_{|x_k|\le r_n} \frac{a_k}{|x-x_k|} &\le \frac{C}{|x|} \sum_{|x_k|\le r_n} a_k \\
&= C \frac{n(r_n)}{r_n}, \qquad (6.2.6)
\end{aligned}$$

using (6.2.4). Finally we look at (6.2.3). This time we write

$$\sum_{|x_k|\le r_n} \frac{a_k}{|x_k|} = \int_\delta^{r_n} \frac{1}{t} dn(t). \tag{6.2.7}$$

We combine (6.2.5), (6.2.6), and (6.2.7), to get that

$$u(x) \le C\frac{n(r_n)}{r_n} - \int_\delta^{r_n} \frac{1}{t} dn(t)$$

Using (6.1.6), we obtain

$$M(r_n', u) \to -\infty,$$

which proves Proposition 6.2.1.

Returning to the proof of Theorem 6.1.2, we can now apply Lemma 1.5.2 from Chapter 1, to get that $u(x)$ has infinitely many critical points, since u is evidently harmonic apart from at the poles x_k.

6.3 Proof of Theorem 6.1.3

Let r_n be a sequence satisfying (6.1.7). As in the proof of Proposition 2.1.2 in Chapter 2, we apply the method of Cartan's Lemma [13, p.366] (see also Lemma 1.5.3 from Chapter 1) to the mass distribution $\sum a_k\delta_{x_k}$ on the spheres $|x| = 2r_n$, this time with

$$M = n(2r_n), \quad h = \frac{r_n}{96}, \quad \text{and} \quad A = 6.$$

This gives us a union E_1 of balls, having sum of radii at most $\frac{r_n}{16}$, such that for all x outside of E_1, we have

$$\begin{aligned} \mu(x, 2r_n, t) &= \sum_{|x_k|\le 2r_n, |x-x_k|\le t} a_k \\ &\le \frac{Mt}{eh} = \frac{96n(2r_n)t}{er_n}, \end{aligned}$$

for $0 < t < \infty$.

We also write $N = \overline{n}(2r_n)$, and take E_2 to be the union of balls

$$E_2 = \bigcup_{|x_k|\le 2r_n} B\left(x_k, \frac{r_n}{16N}\right).$$

Thus the balls of $E = E_1 \cup E_2$ have sum of diameters at most $\dfrac{r_n}{4}$ and we may choose an $s \in [r_n, \frac{3r_n}{2}]$ such that the sphere $|x| = s$ meets none of the balls of E. This then gives, for $|x| = s$,

$$\begin{aligned}
\sum_{|x_k| \le 2r_n} \frac{a_k}{|x - x_k|} &\le \int_{\frac{r_n}{64N}}^{4r_n} \frac{1}{t} d\mu(x, 2r_n, t) \\
&= \frac{1}{(4r_n)} \mu(x, 2r_n, 4r_n) + \int_{\frac{r_n}{64N}}^{4r_n} \frac{\mu(x, 2r_n, t)}{t^2} dt \\
&\quad - \left(\frac{64N}{r_n}\right) \mu\left(x, r_n, \frac{r_n}{64N}\right) \\
&\le C\left(\frac{n(2r_n)}{r_n} + \frac{n(2r_n)}{r_n} \int_{\frac{r_n}{64N}}^{4r_n} \frac{1}{t} dt\right) \\
&\le C\left(\frac{n(2r_n)}{r_n} + \frac{n(2r_n)}{r_n} \log \overline{n}(2r_n)\right) \\
&\le o\left(\int_\delta^{2r_n} \frac{1}{t} dn(t)\right) \qquad (6.3.1)
\end{aligned}$$

using (6.1.7). Now we write

$$\begin{aligned}
u(x) = &\sum_{|x_k| > 2r_n} a_k \left(\frac{1}{|x - x_k|} - \frac{1}{|x_k|}\right) \\
+ &\sum_{|x_k| \le 2r_n} \frac{a_k}{|x - x_k|} \\
- &\sum_{|x_k| \le 2r_n} \frac{a_k}{|x_k|},
\end{aligned}$$

and note that, for $|x| = s$,

$$\begin{aligned}
\sum_{|x_k| > 2r_n} a_k \left|\frac{1}{|x - x_k|} - \frac{1}{|x_k|}\right| &= \sum_{|x_k| > 2r_n} a_k \left|\frac{|x_k| - |x - x_k|}{|x_k||x - x_k|}\right| \\
&\le \sum_{|x_k| > 2r_n} a_k \frac{|x|}{|x - x_k||x_k|}.
\end{aligned}$$

But since $|x| = s \in [r_n, \frac{3r_n}{2}]$, we have

$$|x - x_k| \ge |x_k| - |x| \ge \frac{|x_k|}{4}$$

for $|x_k| > 2r_n$, which gives

$$\begin{aligned}
\sum_{|x_k|>2r_n} a_k \frac{|x|}{|x-x_k||x_k|} &< C \sum_{|x_k|>2r_n} a_k \frac{|x|}{|x_k|^2} \\
&\leq Cs \sum_{|x_k|>2r_n} \frac{a_k}{|x_k|^2} \\
&\leq Cs \int_{2r_n}^{\infty} \frac{1}{t^2} dn(t) \\
&\leq Cr_n \int_{2r_n}^{\infty} \frac{1}{t^2} dn(t) \\
&\leq o\left(\int_{\delta}^{2r_n} \frac{1}{t} dn(t) \right), \qquad (6.3.2)
\end{aligned}$$

using (6.1.7). On the other hand,

$$\sum_{|x_k|\leq 2r_n} \frac{a_k}{|x_k|} = \int_{\delta}^{2r_n} \frac{1}{t} dn(t). \qquad (6.3.3)$$

Now we combine (6.3.1), (6.3.2) and (6.3.3), to get for $|x| = s$,

$$\begin{aligned}
u(x) &\leq o\left(\int_{\delta}^{2r_n} \frac{1}{t} dn(t) \right) + o\left(\int_{\delta}^{2r_n} \frac{1}{t} dn(t) \right) - \int_{\delta}^{2r_n} \frac{1}{t} dn(t) \\
&= (-1 + o(1)) \int_{\delta}^{2r_n} \frac{1}{t} dn(t) \\
&= (-1 + o(1)) \sum_{|x_k|\leq 2r_n} \frac{a_k}{|x_k|}.
\end{aligned}$$

Since $\sum \frac{a_k}{|x_k|}$ is divergent the last term tends to $-\infty$ as $n \to \infty$, so that

$$\liminf_{r\to\infty} M(r, u) = -\infty.$$

Hence we can once again apply Lemma 1.5.2 from Chapter 1 to get that $u(x)$ has infinitely many critical points.

6.4 Examples Satisfying the Hypotheses of Theorems 6.1.2 and 6.1.3

Conditions (6.1.5) and (6.1.6) are satisfied by functions with the property

$$C_1 r(\log r)^d \leq n(r) \leq C_2 r(\log r)^d, \tag{6.4.1}$$

where d is a positive integer, and $0 < C_1 \leq C_2$.

To prove this we require the following two integrals.

$$\begin{aligned} \int_a^b \frac{(\log t)^d}{t^2} dt &= \left[-\frac{1}{t} \sum_{i=0}^{d} \frac{d!(\log t)^{d-i}}{(d-i)!} \right]_a^b \\ \int_a^b \frac{(\log t)^d}{t} dt &= \left[\frac{1}{d+1} (\log t)^{d+1} \right]_a^b \end{aligned}$$

Using these and the alternative forms of conditions (6.1.5) and (6.1.6) provided immediately after the statement of the theorem, we can see that for a function satisfying (6.4.1) we get

$$\begin{aligned} \frac{{r_n}^2}{n(r_n)} \left[\int_{r_n}^{\infty} \frac{n(t)}{t^3} dt \right] &\leq \frac{r_n}{C_1 (\log r_n)^d} \left[\int_{r_n}^{\infty} \frac{C_2 (\log t)^d}{t^2} dt \right] \\ &\leq \frac{C_2 r_n}{C_1 (\log r_n)^d} \frac{1}{r_n} \sum_{i=0}^{d} \frac{d!(\log r_n)^{d-i}}{(d-i)!} < \infty, \end{aligned}$$

and

$$\begin{aligned} \frac{r_n}{n(r_n)} \left[\int_{\delta}^{r_n} \frac{n(t)}{t^2} dt \right] &\geq \frac{1}{C_2 (\log r_n)^d} \left[\int_{\delta}^{r_n} \frac{C_1 (\log t)^d}{t} dt \right] \\ &\geq \frac{C_1}{C_2 (\log r_n)^d} \left(\frac{1}{d+1} (\log r_n)^{d+1} + O(1) \right) \\ &\sim C_3 \log r_n \to \infty, \end{aligned}$$

with $C_3 > 0$, and hence both criteria are satisfied.

Notice that if $n(r) = r^q$, with $q > 1$, condition (6.1.6) is not satisfied, although if $q \geq 2$ then the convergence criterion (6.1.2) will also not be satisfied.

We give an explicit construction which satisfies the hypotheses of Theorems 6.1.2 and 6.1.3. Let $\lambda > 1$, and $0 < C_1 < C_2$, and choose x_k, a_k with

$$\begin{aligned} C_1 \lambda^k \leq \ & |x_k| \leq C_2 \lambda^k \\ C_1 \lambda^k \leq \ & \ a_k \ \leq C_2 \lambda^k. \end{aligned}$$

for all $k \in \mathbb{N}$. Then it is evident that there are constants $C_4 > C_3 > 0$ such that

$$\begin{aligned} C_3 r \leq \quad n(r) \quad &\leq C_4 r \\ C_3 \log r \leq \quad \overline{n}(r) \quad &\leq C_4 \log r \end{aligned}$$

for all large r.

Moreover, with C_j representing positive constants, we have

$$\begin{aligned} \int_r^\infty \frac{1}{t^2} dn(t) &= \sum_{|x_k|>r} \frac{a_k}{|x_k|^2} \\ &\leq C_5 \sum_{|x_k|>r} \frac{1}{|x_k|} \\ &\leq \frac{C_6}{r} \end{aligned}$$

and, with δ small,

$$\begin{aligned} \int_\delta^r \frac{1}{t} dn(t) &= \sum_{|x_k|\leq r} \frac{a_k}{|x_k|} \\ &\geq \sum_{|x_k|\leq r} C_7 = C_7 \overline{n}(r). \end{aligned}$$

Then the integral conditions in Theorems 6.1.2 and 6.1.3 are clearly satisfied, and since $|x_k| \geq C_1 \lambda^k$ it is clear that (6.1.4) is satisfied for arbitrarily large r_n, provided ϵ is chosen small enough.

References

[1] J. M. Anderson and J. Clunie. Slowly growing meromorphic functions. *Comment. Math. Helv.*, 40:267–280, 1966.

[2] T. M. Apostol. *Mathematical Analysis 2nd Ed.* Addison-Wesley Publishing Co., Reading, MA, 1974.

[3] S. Bank and I. Laine. On the oscillation theory of $f'' + af = 0$ where a is entire. *Trans. Amer. Math. Soc.*, 273:351–363, 1986.

[4] J. Clunie, A. Eremenko, and J. Rossi. On equilibrium points of logarithmic and Newtonian potentials. *J. London Math. Soc.*, 47(2):309–320, 1993.

[5] J. B. Conway. *Functions of One Complex Variable I, 2nd edition.* Springer, New York, 1978.

[6] A. Eremenko. Equilibrium points of logarithmic and newtonian potentials. www at http://www.math.purdue.edu/∼eremenko/dvi/equil.pdf.

[7] A. Eremenko, J. Langley, and J. Rossi. On the zeros of meromorphic functions of the form $\sum_{k=1}^{\infty} a_k/(z - z_k)$. *J. Analyse Math.*, 62:271–286, 1990.

[8] A. A. Gol'dberg and I. V. Ostrovskii. *Distribution of Values of Meromorphic Functions.* Nauka, Moscow, 1970.

[9] A. A. Gol'dberg and O. P. Sokolovskaya. Some relations for meromorphic functions of order or lower order less than one. *Isz. Vyssh. Uchebn. Zaved. Mat.*, 31(6):26–31, 1987. Translation: Soviet Math., Isv. VUZ, 31, 1987, no.6, 29-35.

[10] W. K. Hayman. Slowly growing integral and subharmonic functions. *Comment. Math. Helv.*, 34:75–84, 1960.

[11] W. K. Hayman. *Meromorphic Functions.* Oxford at the Clarendon Press, Oxford, 1964.

[12] W. K. Hayman. On Iversen's theorem for meromorphic functions with few poles. *Acta Math.*, 141:115–145, 1978.

[13] W. K. Hayman. *Subharmonic Functions, Vol. 2.* Academic Press, London, 1989.

[14] E. Hille. *Ordinary differential equations in the complex domain.* Wiley, New York, 1976.

[15] J. D. Hinchliffe. *Some results on the value distribution of meromorphic functions.* PhD thesis, University of Nottingham, http://etheses.nottingham.ac.uk/archive/00000043/01/thesis.pdf, 2003.

[16] O. D. Kellogg. *Foundations of Potential Theory.* Springer, Berlin, 1967.

[17] J. K. Langley. Postgraduate notes in complex analysis. http://www.maths.nottingham.ac.uk/personal/jkl/pg1.pdf.

[18] J. K. Langley. An application of the Tsuji characteristic. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 38(2):299–318, 1991.

[19] J. K. Langley and J. Rossi. Critical points of certain discrete potentials. *Complex Variables*, 49:621–637, 2004. In special memory of Matts Essen.

[20] J. K. Langley and J. Rossi. Meromorphic functions of the form $f(z) = \sum_{n=1}^{\infty} a_n/(z-z_n)$. *Rev. Mat. Iberoamericana*, 20:285–314, 2004.

[21] J. K. Langley and D. Shea. On multiple points of meromorphic functions. *J. London Math. Soc.*, 57(2):371–384, 1998.

[22] J. Lewis, J. Rossi, and A. Weitsman. On the growth of subharmonic functions along paths. *Ark. Mat.*, 22(1):109–119, 1984.

[23] J. C. Maxwell. *A Treatise on Electricity and Magnetism, vol. 1, Republication of the 3rd revised edition.* Dover Publ. Inc., Mineola, NY, 1954.

[24] W. Rudin. *Real and Complex Analysis.* McGraw Hill, Inc., New York, 1987.

[25] R. V. Trickey. On critical points and zeros of certain discrete potentials,. *Comput. Methods Funct. Thy.*, 6(1):29–36, 2006.

[26] R. V. Trickey. On the frequency of zeros of certain meromorphic functions associated with subharmonic potentials. *Comput. Methods Funct. Thy.*, 7(1):227–291, 2007.

Appendix A

Appendix

In order to try and gain evidence as to whether Conjecture 1.6.1 is true or not, some computer simulations were run in MATLAB. This Appendix explains the way the simulations were run, contains the results of these simulations, and the programs used.

A.1 Problems with and Reasons for Computer Simulation

The main problem with running computer simulations of Keldysh Functions is that Keldysh Functions consist primarily of an infinite number of summed elements, and clearly a computer cannot handle a function given in such a form. However, a computer can cope with a finite number of these elements, so it was such functions that were simulated in MATLAB.

The other main problem with computer simulation is that it is highly unlikely that a computer will be able to find an actual zero of the function. All it can do is find places where the function is very small, and we have to just hope that this indicates that there is a zero nearby.

The Basic Idea

We already know that in the case where we have only N poles or point charges, the function will have $N-1$ zeros (see Example 1.3.2), but we do not know where these will lie. The idea behind running these simulations was to observe where the zeros lie as we increase the value of N.

If as the number of poles N increases, the zeros move further away from the origin, then it is possible that in the limiting case they may no longer exist. However, if instead they cluster close to the origin, then, although speculative, it seems likely that they will still be there in the infinite case.

To this end, the following program randomly generates a user specified number of poles (z_k), with their corresponding (positive) charges (a_k), and finds points where the function is very small. We will however, for brevity, refer to these as zeros. The program then draws a graph of where the poles are, and where the zeros lie. It then calculates as a percentage of the total number of zeros, the number of them which lie close to the origin, and also the number that lie far away. It also has a repeat option so that you can get it to run repeatedly, storing the results each time, and then perform basic statistical analysis. We note from Chapter 1 that all the actual zeros of such a function will lie in the convex hull of the z_k, and so we will limit our search to that region.

To this end, the program was run 1000 times with 100 z_k, and also 1000 times with 1000 z_k. The results of these runs are given below. It is worth noting that the second of these runs required 9 days computing time to complete.

Please note that this program is not entirely written by the author. A simpler version of this program was gratefully received from Lee Johnson from Virginia Tech University, which required the user to manually input the locations and charges of three points into the code, and the program then found and graphed the points where the function was very small. The modified program presented here represents the author's adaptation of Lee Johnson's program.

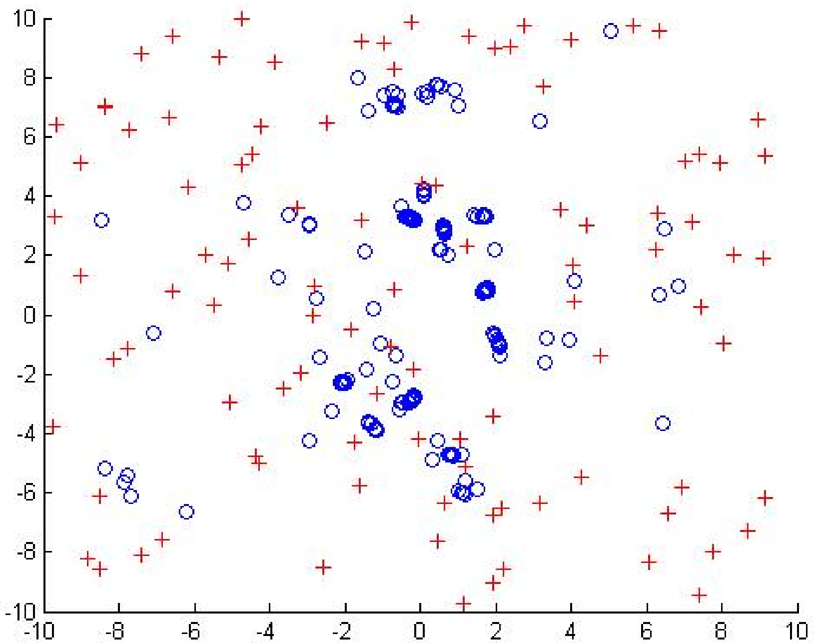
A.2 The Results

All the z_k were placed randomly inside the square with corners $\{(-10+10i),(10+10i),(10-10i),(-10-10i)\}$. To each z_k an a_k was also randomly generated taking values in the interval $[0, 10]$. When a zero is referred to as being 'close' to the origin, we actually mean within a disk of radius 2. When a zero is referred to as being 'far away' from the origin, we mean outside a disk of radius 7. Here is a table of the results of the simulations.

	100 charges	1000 charges
Averaged % zeros Far Away	3.64	1.19
Averaged % zeros Close	29.45	36.49
Standard Deviation of Far Away %s	4.55	0.78
Standard Deviation of Close %s	16.45	5.63

As evidenced by the results, as the number of charges increases, not only does the number of them packed close to the origin rise, but the number close to the extremes also drops significantly. On top of this, the standard deviation drops significantly, which implies that the models are getting more consistent, although this could be merely because with the larger number of generated points, the distribution is more uniform. Based upon the argument explained above, this provides evidence that the Conjecture 1.6.1 is probably true.

Here is an example of the output graph for a run with 100 z_k. The crosses represent the location of the z_k, and the circles are where zeros have been found.



A.3 The Program Used

Here is a copy of the program printed verbatim, with the line numbers included. The program was split into six parts, with *testcase.m* being the main body of the program, and the other five parts being additional functions.

testcase.m

```
1.  Fardistance=[];
2.  Closedistance=[];
3.  for k=1:1000
4.  %For multiple tests, change above to the number required,
5.  %also cancel lines 33 and 34 below, as well as lines 13 and 14
6.  %in f program.
7.  sizefactor=10;
8.  %Note to make it a random size up to maximum M, replace the 10
9.  %above with rand*M (choose M)
10. rmin=1.e-03;
11. ftol=1.e-08;
12. itmax=500;
13. maxnorm=2*sizefactor;
14. wxy=[0 0];
15. reals = [];
16. imags = [];
17. consts=[];
18. crosses =[];
19. stopgen=[0];
20. counter=[];
21. bigcounter=[];
22. for i=1:1000
23.     y=-(0.5*sizefactor)+i*(0.001*sizefactor);
24.     x=0.1;
25.     ws=[x;y];
26.     [wend,fnorm,ierr,stopgen, reals,imags, consts,crosses]=
qn(ws,rmin,ftol,itmax, stopgen, reals,imags, consts,crosses, sizefactor);
27.     if norm(wend) < maxnorm
28.         wxy=[wxy;wend'];
```

```
    end
end
npoints=length(wxy);
hold all
plot(wxy(2:npoints,1),wxy(2:npoints,2),'o')
plot(crosses(1:100),'r+')
for i=2:npoints
    if norm(wxy(i,:)) < (0.2*sizefactor)
        counter=[1,counter];
    end
end
small = length(counter);
Close = small/npoints;
for i=2:npoints
    if norm(wxy(i,:)) > (0.7*sizefactor)
        bigcounter=[1,bigcounter];
    end
end
big = length(bigcounter);
Far = big/npoints;
Fardistance=[Far,Fardistance];
Closedistance=[Close, Closedistance];
end
AverageFar=(sum(Fardistance)/length(Fardistance))
AverageClose=(sum(Closedistance)/length(Closedistance))
```

qn.m

```
function [wend,fnorm,ierr, stopgen, reals,imags, consts, crosses]=
qn(ws,rmin,ftol,itmax, stopgen, reals,imags, consts, crosses, sizefactor)
%
%  assume ws and fws are column vectors
%
sz=size(ws);
n=sz(1)*sz(2);
ierr=1;
itctr=0;
[fws, stopgen, reals,imags, consts, crosses]=
f(ws, stopgen, reals,imags, consts, crosses, sizefactor);
%
%  main loop
%
while ierr > 0
    if itctr > itmax
      ierr=-1;
      wend=ws;
      fnorm=(fws'*fws)/2;
      return
    else
      itctr=itctr+1;
      [a, stopgen, reals,imags, consts, crosses]=
jacob(ws,fws,n, stopgen, reals,imags, consts, crosses, sizefactor);
      [dir, stopgen, reals,imags, consts, crosses]=
direction(fws,a, stopgen, reals,imags, consts, crosses, sizefactor);
      [wnew,fnew,ierr, stopgen, reals,imags, consts, crosses]=
linesearch(ws,fws,dir,rmin, stopgen, reals,imags, consts, crosses, sizefactor);
      if ierr < 0
        wend=ws;
        fnorm=(fws'*fws)/2;
        return
      end
      ws=wnew;
      fws=fnew;
```

```
        fnorm=(fws'*fws)/2;
        if fnorm < ftol
          wend=ws;
          return
        end
    end
end
```

f.m

```
function [y, stopgen, reals,imags, consts, crosses]= f(x, stopgen,reals,imags, consts, crosses, sizefactor)
while length(stopgen) <= 1
    crosses = [];
    for j=1:1000
            real_part = 2*sizefactor*(rand-0.5);
            im_part = 2*sizefactor*(rand-0.5);
            const_part = sizefactor*rand;
            reals=[reals,real_part];
            imags=[imags, im_part];
            consts=[consts,const_part];
            compole = complex(real_part, im_part);
            v=[compole];
            constvals = sprintf('const_part = %4.2f, real_part = %4.2f, im_part = %4.2f', const_part, real_part, im_part);
            disp(constvals);
            crosses = [crosses;v];
            stopgen = [stopgen;1];
    end
end
functs=[];
for j=1:1000
    par1 = consts(j) * (x(1)-reals(j))/((x(1)-reals(j))^2 + (x(2)-imags(j))^2)^1.5;
    par2 = consts(j) * (x(2)-imags(j))/((x(1)-reals(j))^2 + (x(2)-imags(j))^2)^1.5;
    u=[par1, par2];
    functs = [functs;u];
end
summedfunct = sum(functs);
yx = summedfunct(1);
yy = summedfunct(2);
y = -[yx; yy];
```

linesearch.m

```
function [wnew,fnew,ierr, stopgen, reals,imags, consts, crosses]=
linesearch(ws,fws,dir,rmin, stopgen, reals,imags, consts, crosses, sizefactor)
ierr=1;
fnormold=fws'*fws;
r=2;
for i=1:20
  r=r/2;
  if r < rmin
    ierr=-2;
    return
  end
  wnew=ws+r*dir;
  [fnew, stopgen, reals,imags, consts, crosses]=
f(wnew, stopgen, reals,imags, consts, crosses, sizefactor);
  fnormnew=fnew'*fnew;
  if fnormnew < fnormold
    return
  end
end
```

jacob.m

```
function [a, stopgen, reals,imags, consts, crosses]= jacob(ws,fws,n, stopgen, reals,imags, consts, crosses, sizefactor)
for i=1:n
  wpert=ws;
  h=1e-01*(1+abs(ws(i)));
  wpert(i)=wpert(i)+h;
  [fpert, stopgen, reals,imags, consts, crosses]= f(wpert, stopgen, reals,imags, consts, crosses, sizefactor);
  a(:,i)=(fpert-fws)/h;
end
```

direction.m

```
function [dir, stopgen, reals,imags, consts, crosses]= direction(fws,a, stopgen, reals,imags, consts, crosses, sizefactor)
dir=-(a\fws);
```