Complex Analysis Using Nevanlinna Theory

by

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Abstract

In this thesis, we mainly worked in the following areas: value distributions of meromorphic functions, normal families, Bank-Laine functions and complex oscillation theory. In the first chapter we will give an introduction to those areas and some related topics that are needed. In Chapter 2 we will prove that for a meromorphic function f and a positive integer k, the function $af(f^{(k)})^n - 1,\ n \geq 2$, has infinitely many zeros and then we will prove that it is still true when we replace $f^{(k)}$ by a differential polynomial. In Chapter 3 we will prove that for a meromorphic function f and a positive integer k, the function $aff^{(k)} - 1$ with $N_1(r, \frac{1}{f^{(k)}}) = S(r, f)$ has infinitely many zeros and then we will prove that it is still true when we replace $f^{(k)}$ by a differential polynomial. In Chapter 4 we will apply Bloch's Principle to prove that a family of functions meromorphic on the unit disc $B(0,1)$, such that $f(f')^m \neq 1,\ m \geq 2$, is normal. Also we will prove that a family of functions meromorphic on $B(0,1)$, such that each $f \neq 0$ and $f(f^{(k)})^m,\ k, m \in \mathbb{N}$ omits the value 1, is normal. In the fifth chapter we will generalise Theorem 5.1.1 for a sequence of distinct complex numbers instead of a sequence of real numbers. Also, we will get very nice new results on Bank-Laine functions and Bank-Laine sequences. In the last chapter we will work on the relationship between the order of growth of A and the exponent of convergence of the solutions of $y^{(k)} + Ay = 0$, where A is a transcendental entire function with $\rho(A) < \frac{1}{2}$.

Published Papers

We published Chapter 2, Chapter 3 and Chapter 4 as listed below. Every one of them is in a single paper. Furthermore, we submitted Chapter 5 and Chapter 6 to be published.

1. Abdullah Alotaibi, On the zeros of $af(f^{(k)})^n - 1$, $n \geq 2$, Computational Methods and Function Theory, 4(1):227-235, 2004.

2. Abdullah Alotaibi, On the zeros of $aff^{(k)} - 1$, Complex Variables: Theory and Application, 49(1):977-989, 2004.

3. Abdullah Alotaibi, On normal families, Arab Journal of Mathematical Sciences, 10(1):33-42, 2004.

4. Abdullah Alotaibi, On Bank-Laine functions, submitted to Journal of Mathematical Analysis and Applications.

5. Abdullah Alotaibi, On complex oscillation theory, submitted to Results in Mathematics.

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Dedication

To

My Father and Mother,

My Brothers and Sisters,

My Relatives and Friends.

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Chapter 1

Preliminaries

In this chapter, we will give an introduction to each area that we worked on besides some needed topics.

1.1 Analytic and meromorphic functions

Definition 1.1.1 *[21] Let U be an open set in $\mathbb{C}$, and let $z_0 \in U$. We say that $f : U \to \mathbb{C}$ is complex differentiable at $z_0 \in U$ if the following limit exists and is finite:*

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}.$$

Moreover, we say that f is complex differentiable on U if it is complex differentiable at every point in U.

Definition 1.1.2 *[4] We say that f is analytic at $a \in \mathbb{C}$ if f is complex differentiable on some open disc centred at a. Moreover, if f is analytic at every point in $\mathbb{C}$, we say that f is entire.*

Example 1.1.1 $z + 1, e^z, \sin z$ *are all entire functions.*

Definition 1.1.3 *[30] We say that f is meromorphic at $a \in \mathbb{C}$ if f is analytic at a, or a is a pole of f.*

Example 1.1.2 $e^z, \frac{1}{z+1}, \frac{1}{\sin \pi z}$ *are all meromorphic functions.*

Throughout this thesis meromorphic means meromorphic in the complex plane $\mathbb{C}$ unless otherwise stated.

Definition 1.1.4 *[27] Let f be a complex-valued function, and let g be a real function, both of which are defined on $[a, \infty)$, $a \in \mathbb{R}$. We say that $f(r) = O(g(r))$ as $r \to \infty$ if there exist constants K_1, K_2 such that*

$$|f(r)| \leq K_1 g(r) \quad \forall\, r \geq K_2.$$

Also we say that $f(r) = o(g(r))$ as $r \to \infty$ if

$$\frac{f(r)}{g(r)} \to 0 \quad as\ r \to \infty.$$

Example 1.1.3 $r^3 + r + 2 = O(r^3)$ *as* $r \to \infty$.

Example 1.1.4 *Suppose that P is a polynomial in r; then $P(r) = O(r^n)$ as $r \to \infty$, where n is the degree of P.*

Example 1.1.5 $\log r = o(r)$ *as* $r \to \infty$ *since* $\frac{\log r}{r} \to 0$ *as* $r \to \infty$.

Theorem 1.1.1 *(Liouville's theorem)[32]*
Suppose that f is a bounded entire function. Then f is constant.

1.2 Nevanlinna Theory

We use the standard terminology of Nevanlinna theory as defined in [14], [20]. All theorems in this section are standard results from Nevanlinna theory. Let f be a meromorphic function; then we have the following definitions.

Definition 1.2.1 *For all $x \in (0, \infty)$, we set*

$$\log^+ x = \begin{cases} \log x & \text{if } x \geq 1 \\ 0 & \text{if } 0 < x < 1. \end{cases}$$

Definition 1.2.2 *(Proximity function)*

$$m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| f(re^{i\theta}) \right| d\theta,$$

and for $f \not\equiv a \in \mathbb{C}$, we have

$$m(r, \frac{1}{f-a}) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| \frac{1}{f(re^{i\theta}) - a} \right| d\theta.$$

Definition 1.2.3 *(Integrated counting function)*

$$N(r, f) = \int_0^r [n(t, f) - n(0, f)] \frac{dt}{t} + n(0, f) \log r$$

where $n(t, f)$ is the number of poles of $f(z)$ in $|z| \leq t$, counting multiplicity. Also for any complex number a, we have

$$N(r, \frac{1}{f-a}) = \int_0^r \left[n(t, \frac{1}{f-a}) - n(0, \frac{1}{f-a}) \right] \frac{dt}{t} + n(0, \frac{1}{f-a}) \log r.$$

where $n(t, \frac{1}{f-a})$ is the number of zeros of $f(z) - a$ in $|z| \leq t$.

Definition 1.2.4

$$\bar{N}(r, f) = \int_0^r [\bar{n}(t, f) - \bar{n}(0, f)] \frac{dt}{t} + \bar{n}(0, f) \log r$$

where $\bar{n}(t, f)$ is the number of poles of $f(z)$ in $|z| \leq t$ counting just once. Also for any complex number a, we have

$$\bar{N}(r, \frac{1}{f-a}) = \int_0^r \left[\bar{n}(t, \frac{1}{f-a}) - \bar{n}(0, \frac{1}{f-a}) \right] \frac{dt}{t} + \bar{n}(0, \frac{1}{f-a}) \log r.$$

where $\bar{n}(t, \frac{1}{f-a})$ is the number of zeros of $f(z) - a$ in $|z| \leq t$ counting just once.

$f(z)$	$n(r,f)$	$\bar{n}(r,f)$	$n(r,\frac{1}{f})$	$\bar{n}(r,\frac{1}{f})$
$\frac{(z-1)^2(z-2)^6}{(z-3)^5}$	5	1	8	2
e^{3z}	0	0	0	0
e^z-1	0	0	$\frac{r}{\pi}+O(1)$	$\frac{r}{\pi}+O(1)$
$\sin z$	0	0	$\frac{2r}{\pi}+O(1)$	$\frac{2r}{\pi}+O(1)$

Table 1.1: Counting the number of zeros and poles.

Example 1.2.1 *We calculate $n(r,f)$, $\bar{n}(r,f)$, $n(r,\frac{1}{f})$ and $\bar{n}(r,\frac{1}{f})$ as $r\to\infty$ for certain standard meromorphic functions. The results are displayed in Table 1.1.*

Definition 1.2.5 *(Characteristic function)*

$$T(r,f)=m(r,f)+N(r,f).$$

1.2.1 Theorems and Propositions in Nevanlinna Theory

Proposition 1.2.1 *Assume that the f_k are meromorphic functions. Then we have the following properties:*

1. $m(r,\sum_{k=1}^n f_k(z))\le\sum_{k=1}^n m(r,f_k(z))+O(1)$.
2. $m(r,\prod_{k=1}^n f_k(z))\le\sum_{k=1}^n m(r,f_k(z))$.
3. $N(r,\sum_{k=1}^n f_k(z))\le\sum_{k=1}^n N(r,f_k(z))$.
4. $N(r,\prod_{k=1}^n f_k(z))\le\sum_{k=1}^n N(r,f_k(z))$.
5. $T(r,\sum_{k=1}^n f_k(z))\le\sum_{k=1}^n T(r,f_k(z))+O(1)$.
6. $T(r,\prod_{k=1}^n f_k(z))\le\sum_{k=1}^n T(r,f_k(z))$.

Theorem 1.2.1 *(The first fundamental theorem)*

Suppose that f is a non-constant meromorphic function, and suppose that $a \in \mathbb{C}$. Then

$$T(r, \frac{1}{f-a}) = T(r, f) + O(1) \quad \text{as } r \to \infty.$$

Theorem 1.2.2 *(The second fundamental theorem)*

Suppose that f is a non-constant meromorphic function, and suppose that $q \geq 2$. Suppose that $a_1, \cdots, a_q$ are distinct complex numbers. Then

$$m(r, f) + \sum_{n=1}^{q} m(r, \frac{1}{f - a_n}) \leq 2T(r, f) + S(r, f)$$

where $S(r, f)$ means any quantity such that

$$S(r, f) = o(T(r, f)) \quad \text{as } r \to \infty$$

possibly outside a set of finite measure.

If we take $q = 2$, $a_1 = 0$, $a_2 = 1$ then Nevanlinna second fundamental theorem (Theorem 1.2.2) can also be expressed as follows.

Theorem 1.2.3 *(The second fundamental theorem)*

Suppose that f is a non-constant meromorphic function. Then

$$T(r, f) \leq \bar{N}(r, f) + \bar{N}(r, \frac{1}{f}) + \bar{N}(r, \frac{1}{f-1}) + S(r, f).$$

Theorem 1.2.4 *Let f be an entire function and let $0 < r < R < \infty$. Let the maximum modulus be $M(r, f) = \max_{|z|=r} |f(z)|$. Then*

$$T(r, f) \leq \log^+ M(r, f) \leq \frac{R+r}{R-r} T(R, f).$$

A fundamental result in Nevanlinna theory, which is the key to the proof of the second fundamental theorem, is the following.

Proposition 1.2.2 *Suppose that f is a transcendental meromorphic function, and suppose that k is a positive integer. Then*

$$m(r, \frac{f^{(k)}}{f}) = S(r, f).$$

If f is of finite order of growth, we have

$$m(r, \frac{f^{(k)}}{f}) = O(\log r).$$

Proposition 1.2.2 is called the lemma of the logarithmic derivative.

Proposition 1.2.3 *Let f be a meromorphic function, and let k be a positive integer. Then we have the following.*

1. $T(r, f^{(k)}) \leq (k+1)T(r, f) + S(r, f)$.

2. *The function $\bar{N}(r, f)$ counts the points at which f has poles and satisfies*

$$\bar{N}(r, f) = N_1(r, f) + \bar{N}_2(r, f)$$

where $N_1(r, f)$ counts the simple poles and $\bar{N}_2(r, f)$ counts the multiple poles just once.

3. $N(r, \frac{f'}{f}) = \bar{N}(r, f) + \bar{N}(r, \frac{1}{f})$.

4. $N(r, f') = N(r, f) + \bar{N}(r, f)$.

Proposition 1.2.4 *Suppose that f is a non-constant meromorphic function, and suppose that $a, b, c, d \in \mathbb{C}$ with $ad - bc \neq 0$. Then*

$$T(r, \frac{af + b}{cf + d}) = T(r, f) + O(1).$$

Proposition 1.2.5 *Suppose that f is a rational function. Then*

$$T(r, f) = O(\log r) \quad \textit{as } r \to \infty.$$

Proposition 1.2.6 *Suppose that f is a transcendental meromorphic function, i.e. f is meromorphic but not a rational function. Then*

$$\frac{T(r,f)}{\log r} \to \infty \quad \textit{as } r \to \infty.$$

Theorem 1.2.5 *(The Argument Principle)*
Suppose that f is a meromorphic function. Then, provided f has no poles or zeros on $|z| = r$,

$$\frac{1}{2\pi i}\int_{|z|=r} \frac{f'(z)}{f(z)}\, dz = n(r, \frac{1}{f}) - n(r, f).$$

1.2.2 The order of growth

Let us denote the order of growth of f by $\rho(f)$ which is defined as follows.

Definition 1.2.6 *Suppose that f is an entire function and that*

$$M(r,f) = \max_{|z|=r} |f(z)|.$$

Then

$$\rho(f) = \limsup_{r\to\infty} \frac{\log^+ \log^+ M(r,f)}{\log r}.$$

Definition 1.2.7 *Let f be a meromorphic function. Then*

$$\rho(f) = \limsup_{r\to\infty} \frac{\log^+ T(r,f)}{\log r}.$$

If f is entire then Theorem 1.2.4 shows that Definition 1.2.6 and Definition 1.2.7 give the same value of $\rho(f)$.

Proposition 1.2.7 *Let f and g be meromorphic functions. Then*

1. $\rho(f+g) \le \max\{\rho(f), \rho(g)\}$.

2. $\rho(fg) \le \max\{\rho(f), \rho(g)\}$.

3. *If* $g(z) = f(az+b)$, *then* $\rho(g) = \rho(f)$ *where* $a, b \in \mathbb{C}$, $a \neq 0$.

4. *If* $g(z) = f(z^k)$, *then* $\rho(g) = k\rho(f)$ *where* k *is a positive integer.*

5. *If* f *is a polynomial, then* $\rho(f) = 0$.

6. *Let* P *be a polynomial of degree* n. *Then* $\rho(e^{P(z)}) = n$.

7. *Suppose that* f *is a transcendental entire function. Then* $\rho(e^{f(z)}) = \infty$.

Example 1.2.2 *Here are some examples concerning the order of growth.*

1. $\rho(z^2 + 3) = 0$.

2. $\rho(\sin z) = \rho(\cos z) = 1$.

3. $\rho(e^z) = 1$.

4. $\rho(e^{z^k}) = k, \quad k \in \mathbb{N}$.

5. $\rho(e^{\sin z}) = \rho(e^{e^z}) = \infty$.

1.2.3 The exponent of convergence

Definition 1.2.8 *Let* $f \not\equiv 0$ *be a meromorphic function. The exponent of convergence* $\lambda(f) = \lambda(f, 0)$ *of the zeros of* f *is defined by*

$$\lambda(f) = \limsup_{r\to\infty} \frac{\log^+ N(r, \frac{1}{f})}{\log r} \tag{1.1}$$

Example 1.2.3 *Here are some examples concerning the exponent of convergence.*

1. $\lambda(e^z) = \lambda(P) = 0$, *where* P *is a polynomial.*

2. $\lambda(e^z + 1) = \lambda(\sin z) = \lambda(\cos z) = 1$.

3. *If* (a_n) *is a sequence tending to infinity then we may define its exponent of convergence by (1.1) using* $N(r)$*, where* $n(r)$ *is the number of* a_n *lying in* $|z| \leq r$*. For example, suppose that* p *is a positive integer and* (a_n) *is a sequence such that* $a_n = n^p$, $n \in \mathbb{N}$*. Then* $\lambda((a_n)) = \frac{1}{p}$.

The following two results are standard [20].

Proposition 1.2.8 *For all entire functions, we have* $\lambda(f) \leq \rho(f)$.

Proposition 1.2.9 *Suppose that* f *is an entire function. Then* $\lambda(f) < \rho(f) < \infty$ *implies that* $\rho(f)$ *is a positive integer.*

1.3 Normal families

Definition 1.3.1 *[11] Let* $\mathcal{F}$ *be a family of functions meromorphic on* $B(0,1)$*. We say that* $\mathcal{F}$ *is normal if every sequence in* $\mathcal{F}$ *has a subsequence, uniformly convergent in every compact subset of* $B(0,1)$*, with respect to the spherical metric.*

The easiest way to know whether a family of functions is normal or not is to apply Marty's theorem (Theorem 1.3.1).

Theorem 1.3.1 *(Marty's theorem)[11]*
Let $\mathcal{F}$ *be a family of functions. The family* $\mathcal{F}$ *is normal on* $B(z_0, r)$ *if and only if for every compact subset* $K \subset B(z_0, r)$ *there exists* $M = M_k > 0$ *such that*

$$\frac{|f'(z)|}{1 + |f(z)|^2} \leq M \quad \forall\, z \in K,\ f \in \mathcal{F}.$$

Example 1.3.1 $\mathcal{F} = \{\frac{z}{n} : n \in \mathbb{N}\}$ *is normal in* $\mathbb{C}$. *We can see this by applying Marty's theorem (Theorem 1.3.1), since we get*

$$\frac{|f'(z)|}{1+|f(z)|^2} = \frac{\frac{1}{n}}{1+|\frac{z}{n}|^2}$$
$$\leq \frac{1}{n}$$
$$\leq 1.$$

Example 1.3.2 $\mathcal{F} = \{nz : n \in \mathbb{N}\}$ *is not normal in* $\mathbb{C}$. *Again this follows from applying Marty's theorem (Theorem 1.3.1), since we get*

$$\frac{|f'(z)|}{1+|f(z)|^2} = \frac{n}{1+n^2|z|^2}$$
$$= n \quad \text{when } z = 0$$
$$\to \infty.$$

Theorem 1.3.2 *(Hurwitz' theorem)[11]*

Let G be a domain and suppose that $\{f_n\}$ is a sequence of analytic functions in G converging to the analytic function f. If $f \not\equiv 0$, $\bar{B}(a,R) \subset G$ and $f(z) \neq 0$ for $|z-a| = R$ then there is an integer N such that, for $n \geq N$, f and f_n have the same number of zeros in $B(a,R)$. Here $\bar{B}(a,R)$ denotes the closure of $B(a,R)$.

Theorem 1.3.3 *(Montel's theorem)[11]*

Suppose that $\mathcal{F}$ is a family of functions analytic on the open set G. Then $\mathcal{F}$ is normal if $\mathcal{F}$ is locally bounded, i.e. given $z_0 \in G$ there exist $\delta > 0$ and $M > 0$ such that $|f(z)| \leq M$ for all $z \in B(z_0,\delta)$ and all $f \in \mathcal{F}$.

Bloch's Principle: [37] *A family of meromorphic functions which have a property P in common in a domain D is usually a normal family in D if P cannot be possessed by non-constant meromorphic functions in the finite plane.*

Example 1.3.3 *Let M be a positive real number, and let f have the property P if $|f(z)| \leq M$ for $z \in D$. Applying Liouville's theorem (Theorem 1.1.1), we see that any entire function with property P is constant. Using Montel's theorem (Theorem 1.3.3), we see that a family of analytic functions which have the property P on a domain D is normal.*

1.4 Wronskian determinant

The properties defined in this section are standard and may be found in [20].

Definition 1.4.1 *Suppose that $f_1, \cdots, f_n$ are meromorphic functions in the plane. The Wronskian determinant $W(f_1, \cdots, f_n)$ is given by*

$$W(f_1, \cdots, f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & \vdots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}.$$

Moreover, for $k = 0, \cdots, n-1$, the $W_k(f_1, \cdots, f_n)$ means the determinant which comes from $W(f_1, \cdots, f_n)$ by replacing the row $f_1^{(k)}, \cdots, f_n^{(k)}$ by $f_1^{(n)}, \cdots, f_n^{(n)}$.

Proposition 1.4.1 *Suppose that $f_1, \cdots, f_n$ are meromorphic functions in the plane. Then $f_1, \cdots, f_n$ are linearly dependent over $\mathbb{C}$ if and only if the Wronskian $W(f_1, \cdots, f_n) \equiv 0$.*

Proposition 1.4.2 *Suppose that $f_1, \cdots, f_n, g$ are meromorphic functions and $c_1, \cdots, c_n$ are complex numbers. Then*

1. *$W(c_1 f_1, \cdots, c_n f_n) = c_1 \cdots c_n W(f_1, \cdots, f_n)$.*

2. *$W(1, z, \cdots, \frac{z^{n-1}}{(n-1)!}, g) = g^{(n)}$.*

3. $W(f_1, \cdots, f_n, 1) = (-1)^n W(f_1', \cdots, f_n')$.

4. $W(gf_1, \cdots, gf_n) = g^n W(f_1, \cdots, f_n)$.

5. $W(f_1, \cdots, f_n) = f_1^n W((\frac{f_2}{f_1})', \cdots, (\frac{f_n}{f_1})')$.

Proposition 1.4.3 *Suppose that $f_1, \cdots, f_n$ are meromorphic functions in the plane. Then*

$$\frac{d}{dz} W(f_1, \cdots, f_n) = W_{n-1}(f_1, \cdots, f_n).$$

The following proposition comes at once from Definition 1.4.1.

Proposition 1.4.4 *Suppose that f_1, f_2 are meromorphic functions in the plane. Then*

$$W(f_1, f_2) = f_1 f_2' - f_2 f_1'.$$

Proposition 1.4.5 *Suppose that $f_1, \cdots, f_n$ are linearly independent meromorphic solutions of*

$$y^{(n)} + \sum_{k=0}^{n-1} a_k(z) y^{(k)} = 0.$$

Then

$$a_k = -\frac{W_k(f_1, \cdots, f_n)}{W(f_1, \cdots, f_n)}.$$

Proposition 1.4.6 *Suppose that $f_1, \cdots, f_n$ are linearly independent meromorphic solutions of*

$$y^{(n)} + \sum_{k=0}^{n-1} a_k(z) y^{(k)} = 0$$

such that a_k are meromorphic functions. Then the Wronskian $W(f_1, \cdots, f_n)$ satisfies the differential equation

$$W' + a_{n-1}(z) W = 0.$$

1.5 Bank-Laine functions

Suppose that A is a transcendental entire function and suppose that we have the following equation

$$y'' + A(z)y = 0. \tag{1.2}$$

Cauchy [16] proved that every solution of (1.2) is entire. Moreover, it is shown [20], [36] that every non-trivial solution f of (1.2) has $\rho(f) = \infty$.

Definition 1.5.1 *[25] A Bank-Laine function E is an entire function such that $E'(z_0) = \pm 1$ at every zero z_0 of E.*

Example 1.5.1 *$E(z) = e^z - 1$ is a Bank-Laine function since the zeros of E are $a_k = 2k\pi i,\ \ k = 0, \pm 1, \pm 2, \cdots$ and $E'(a_k) = 1$.*

The following theorem gives another way to define Bank-Laine functions [25].

Theorem 1.5.1 *An entire function E is a Bank-Laine function if and only if E is the product $f_1 f_2$ of linearly independent normalised solutions of (1.2) such that A is an entire function. Here normalised means that $W(f_1, f_2) = 1$.*

Definition 1.5.2 *[12] Let (a_n) be a sequence of distinct complex numbers. We say that (a_n) is a Bank-Laine sequence if it is precisely the zero sequence of a Bank-Laine function E of finite order.*

Example 1.5.2 *The sequence (a_n) of integer numbers is a Bank-Laine sequence since the zero sequence of the Bank-Laine function $E(z) = \frac{1}{\pi}\sin \pi z$ is (a_n).*

Example 1.5.3 *Suppose that (a_n) is a finite sequence of distinct complex numbers $a_j,\ \ 1 \le j \le n$. Then (a_n) is a Bank-Laine sequence since if we let*

$$E(z) = P(z)e^{Q(z)}$$

where Q is defined using Lagrange interpolation so that

$$P(z) = (z - a_1) \cdots (z - a_n),$$

$$Q(a_j) = -\log P'(a_j), \quad 1 \leq j \leq n,$$

then we get

$$\begin{aligned} E'(a_j) &= P'(a_j)e^{-\log P'(a_j)} \\ &= P'(a_j)e^{\log \frac{1}{P'(a_j)}} \\ &= 1. \end{aligned}$$

So (a_n) is the zero sequence of the Bank-Laine function E and therefore (a_n) is a Bank-Laine sequence.

In 1985, Shen [33] proved the following.

Theorem 1.5.2 *Suppose that $a_1, a_2, \cdots$ are distinct complex numbers such that (a_n) tends to infinity. Then there is a Bank-Laine function E such that the zero sequence of E is precisely (a_n).*

Bank and Laine proved the following theorems [5].

Theorem 1.5.3 *Suppose that A is a transcendental entire function with $\rho(A) < \frac{1}{2}$, and suppose that $E = f_1 f_2$ is the product of normalised linearly independent solutions of (1.2). Then $\lambda(E) = \infty$.*

Theorem 1.5.4 *Suppose that A is a transcendental entire function of finite order ρ, and suppose that (1.2) has normalised linearly independent solutions f_1, f_2 such that $\lambda(f_1 f_2) < \rho$. Then ρ is a positive integer.*

Regarding the non-trivial solutions of (1.2), there is a very nice result about their order of growth and the exponent of convergence of their zeros when A is a polynomial in (1.2), given by the following theorem [5].

Theorem 1.5.5 *Let A be a polynomial of positive degree n, and let f be a non-trivial solution of (1.2). Then*

1. $\rho(f) = \frac{n+2}{2}$.

2. *If n is odd, then* $\lambda(f) = \frac{n+2}{2}$.

3. *If n is even and f_1, f_2 are linearly independent solution of (1.2), then*

$$\max\{\lambda(f_1), \lambda(f_2)\} = \frac{n+2}{2}.$$

When A is transcendental in (1.2), we have the following.

Theorem 1.5.6 *Suppose that $A(z)$ is a transcendental entire function with $\rho(A) \leq \frac{1}{2}$ and that f_1, f_2 are linearly independent solutions of (1.2). Then*

$$\max\{\lambda(f_1), \lambda(f_2)\} = +\infty.$$

As noted above Theorem 1.5.6 was proved by Bank and Laine [5] for $\rho(A) < \frac{1}{2}$, and by Rossi [31] and Shen [34] when $\rho(A) = \frac{1}{2}$.

1.6 The density of sets

We will be concerned only with logarithmic density and we will use it in Chapter 6. Let E be a measurable subset of $[0, +\infty)$. The lower logarithmic density of E is defined by

$$\underline{\log \operatorname{dens}}(E) = \liminf_{r\to\infty} \frac{\int_1^r \frac{\chi(t)dt}{t}}{\log r}$$

where $\chi(t)$ is the characteristic function of E which is defined as

$$\chi(t) = \begin{cases} 1 & \text{if } t \in E \\ 0 & \text{if } t \notin E. \end{cases}$$

The upper logarithmic density of E is defined by

$$\overline{\log\text{dens}}(E) = \limsup_{r\to\infty} \frac{\int_1^r \frac{\chi(t)dt}{t}}{\log r}.$$

The logarithmic density gives us an idea how big the set E is. The following fact is obvious.

Proposition 1.6.1

$$0 \le \underline{\log\text{dens}}(E) \le \overline{\log\text{dens}}(E) \le 1.$$

Example 1.6.1 *Suppose that E has finite measure. Then* $\overline{\log\text{dens}}(E) = 0$. *This is because* $\int_1^r \frac{\chi(t)dt}{t} \le \int_1^r \chi(t)dt = O(1)$.

We will prove our main results in Chapter 6 by using the following theorems [8], [9].

Theorem 1.6.1 *($\cos\pi\rho$ theorem)*
Suppose that f is a non-constant entire function with $\rho(f) < \frac{1}{2}$. Let

$$A(r) = \inf_{|z|=r} \log|f(z)|$$

$$B(r) = \sup_{|z|=r} \log|f(z)| = \log M(r,f).$$

If $\rho < \alpha < 1$, then

$$\underline{\log\text{dens}}\{r :\ A(r) > (\cos\pi\alpha)B(r)\} \ge 1 - \frac{\rho}{\alpha}.$$

Note that $\cos\pi\alpha > 0$ if $0 \le \alpha < \frac{1}{2}$.

Theorem 1.6.2 *(Modified $\cos \pi \rho$ theorem)*
Suppose that f is an entire function with $\rho(f) = \rho < \frac{1}{2}$ and suppose that $A(r)$ is defined as in Theorem 1.6.1. If $\sigma < \rho$, then the set $\{r : A(r) > r^{\sigma}\}$ has positive upper logarithmic density.

In the $\cos \pi \rho$ theorem and the modified $\cos \pi \rho$ theorem we cannot let $\rho = \frac{1}{2}$. For example, let $f(z) = \cos(z^{\frac{1}{2}})$. Here $\rho = \frac{1}{2}$ but $A(r) \leq 0$.

Going back to the relation between $T(r, f)$ and $T(r, f^{(k)})$, we see that Proposition 1.2.3 shows that $T(r, f^{(k)})$ is generally not much bigger than $T(r, f)$. A result in the opposite direction is given by the following.

Theorem 1.6.3 *(Hayman-Miles theorem)[15]*
Suppose that f is a transcendental (i.e non rational) meromorphic function, and suppose that $K > 1$. Then there exists a set $M(K)$ of upper logarithmic density at most

$$\delta(K) = \min\{(2e^{K-1} - 1)^{-1},\ (1 + e(K-1)) \exp(e(1-K))\}$$

such that for every positive integer q,

$$\limsup_{r \to \infty,\, r \notin M(K)} \frac{T(r, f)}{T(r, f^{(q)})} \leq 3eK. \tag{1.3}$$

If f is entire we can replace $3eK$ by $2eK$ in (1.3).

Chapter 2

On the Zeros of $af(f^{(k)})^n - 1,\ n \geq 2$

[1] In this chapter, we will consider the following. Let f be a transcendental meromorphic function and n, k be two positive integers. Then the function $af(f^{(k)})^n - 1$, $n \geq 2$, has infinitely many zeros, where $a(z) \not\equiv 0$ is a meromorphic function with $T(r, a) = S(r, f)$.

2.1 Introduction

In 1959, Hayman [13] proved the following theorem.

Theorem 2.1.1 *Suppose that f is a transcendental meromorphic function and n is a positive integer. Then $f'f^n$ assumes every finite non-zero value infinitely often when $n \geq 3$.*

Hayman conjectured in [13] that the same result remains true when $n \geq 1$. In 1995, Bergweiler and Eremenko [10] settled the remaining cases, i.e. when $n \geq 1$, by proving the following theorem.

[1] We published this chapter as a paper [3].

Theorem 2.1.2 *Suppose that f is a transcendental meromorphic function and $m > l$ are two positive integers. Then $(f^m)^{(l)}$ assumes every finite non-zero value infinitely often.*

In 1993, C.C. Yang, L. Yang and Y. Wang [35] conjectured the following.

Conjecture 2.1.1 *Suppose that f is a transcendental entire function and n, k are two positive integers. Then $f(f^{(k)})^n$ assumes every finite non-zero value infinitely often when $n \geq 2$.*

In 1998, Zhang and Song [38] stated the following.

Theorem 2.1.3 *Suppose that f is a transcendental meromorphic function and n, k are two positive integers. Then $f(f^{(k)})^n - A(z)$, $n \geq 2$, has infinitely many zeros, where $A(z) \not\equiv 0$ is a small function such that $T(r, A) = S(r, f)$.*

In fact, the proof of Theorem 2.1.3 is very complicated and there appear to be some gaps in it. We give a much simpler proof, with some generalisations, by proving our main result which is Theorem 2.1.4. Before stating this theorem, we make some assumptions which we need throughout this chapter. Suppose that f is a transcendental meromorphic function in the plane and $a(z) \not\equiv 0$ is a meromorphic function such that

$$T(r, a) = S(r, f). \tag{2.1}$$

Let

$$L(w) = w^{(k)} + b_{k-1}w^{(k-1)} + \cdots + b_0 w, \quad k \in \mathbb{N}, \tag{2.2}$$

where each $b_j(z)$, $j = 0, 1, \cdots, k-1$, is a meromorphic function such that

$$T(r, b_j) = S(r, f). \tag{2.3}$$

Let $n \in \mathbb{N}$, $n \geq 2$, and set

$$g = L(f) \tag{2.4}$$

$$\psi = afg^n - 1. \tag{2.5}$$

Theorem 2.1.4 *Suppose that f is a transcendental meromorphic function in the plane. Suppose that L, g, ψ and the b_j are given by (2.2), (2.3), (2.4), (2.5). Suppose that*

$$T(r, g) \neq S(r, f) \tag{2.6}$$

$$T(r, \phi) = S(r, f) \tag{2.7}$$

for every solution ϕ of $L(w) = 0$ which is meromorphic in the plane. Then

$$T(r, f) \leq \left(\frac{1}{1-\delta_k}\right)\left(\frac{n(k+1)}{n-1}\right)\bar{N}(r, \frac{1}{\psi}) + S(r, f),$$

where

$$\delta_k = \left(\frac{n(k+1)}{n-1}\right)\left(\frac{1}{1+n(k+1)}\right), \quad 0 < \delta_k < \frac{1}{n-1} \leq 1. \tag{2.8}$$

We will give examples in Section 2.4 to show that the hypotheses on g and ϕ in Theorem 2.1.4 are necessary.

2.2 A lemma needed for Theorem 2.1.4

Lemma 2.2.1 *Provided $g \not\equiv 0$, we have*

$$T(r, f) \leq T(r, \psi) + S(r, f).$$

In particular, $afL(f)^n$ is a non-constant function.

Proof: Using (2.5), we have

$$f = \frac{\psi + 1}{ag^n}.$$

So

$$\begin{aligned} N(r, \frac{1}{f}) &\leq N(r, \frac{1}{\psi+1}) + N(r, a) + n \sum_{j=0}^{k-1} N(r, b_j) \\ &= N(r, \frac{1}{\psi+1}) + S(r, f). \end{aligned}$$

Also, using (2.2), (2.4) and (2.5), we have

$$\frac{1}{f} = \frac{ag^n}{\psi+1}$$

$$\frac{1}{f^{n+1}} = \frac{a}{\psi+1}\left(\frac{f^{(k)}}{f} + b_{k-1}\frac{f^{(k-1)}}{f} + \cdots + b_0\right)^n.$$

Therefore

$$\begin{aligned} m(r, \frac{1}{f}) &\leq (n+1)m(r, \frac{1}{f}) \\ &= m(r, \frac{1}{f^{n+1}}) \\ &\leq m(r, \frac{1}{\psi+1}) + m(r, a) + S(r, f) \\ &= m(r, \frac{1}{\psi+1}) + S(r, f). \end{aligned}$$

Hence, using the first fundamental theorem of Nevanlinna theory (Theorem 1.2.1),

$$\begin{aligned} T(r, f) &= T(r, \frac{1}{f}) + O(1) \\ &= m(r, \frac{1}{f}) + N(r, \frac{1}{f}) + O(1) \\ &\leq m(r, \frac{1}{\psi+1}) + N(r, \frac{1}{\psi+1}) + S(r, f) \\ &= T(r, \frac{1}{\psi+1}) + S(r, f) \\ &= T(r, \psi) + S(r, f). \end{aligned}$$

This completes the proof of Lemma 2.2.1. $\square$

2.3 Proof of Theorem 2.1.4

By (2.5) a zero of g of multiplicity p with $a \neq 0, \infty$ and with $b_j \neq \infty$ is a zero of ψ' of multiplicity at least $np - 1 \geq (n-1)p$. Also, $\psi = -1 \neq 0$ at such a zero of g. Thus

$$\begin{aligned}
\bar{N}(r, \frac{1}{g}) &\leq \frac{1}{n-1} N(r, \frac{\psi}{\psi'}) + \bar{N}(r, a) + \bar{N}(r, \frac{1}{a}) + \sum_{j=0}^{k-1} \bar{N}(r, b_j) \\
&\leq \frac{1}{n-1} N(r, \frac{\psi'}{\psi}) + S(r, f) \\
&= \frac{1}{n-1} \left[\bar{N}(r, \psi) + \bar{N}(r, \frac{1}{\psi}) \right] + S(r, f) \\
&\leq \frac{1}{n-1} \left[\bar{N}(r, f) + \bar{N}(r, a) + \sum_{j=0}^{k-1} \bar{N}(r, b_j) + \bar{N}(r, \frac{1}{\psi}) \right] + S(r, f) \\
&= \frac{1}{n-1} \left[\bar{N}(r, f) + \bar{N}(r, \frac{1}{\psi}) \right] + S(r, f). \qquad (2.9)
\end{aligned}$$

Put $b_k = 1$ and $b_{-1} = b_{k+1} = 0$. Hence

$$g = \sum_{j=0}^{k} b_j f^{(j)} = \sum_{j=0}^{k+1} b_j f^{(j)}. \qquad (2.10)$$

Using the fact that $b'_{k+1} = 0$ and $b_{-1} = 0$, we have

$$\begin{aligned}
g' &= \sum_{j=0}^{k} (b'_j f^{(j)} + b_j f^{(j+1)}) \\
&= \sum_{j=0}^{k} b'_j f^{(j)} + \sum_{j=1}^{k+1} b_{j-1} f^{(j)} \\
&= \sum_{j=0}^{k+1} (b'_j + b_{j-1}) f^{(j)}.
\end{aligned}$$

Since

$$g = \sum_{j=0}^{k+1} b_j f^{(j)},$$

we find that $w = f$ solves

$$\sum_{j=0}^{k+1} c_j w^{(j)} = 0 \tag{2.11}$$

where

$$c_j = b_j' + b_{j-1} - \frac{g'}{g} b_j, \quad c_{k+1} = 1. \tag{2.12}$$

Let

$$w = uv, \quad v = \frac{1}{ag^n}. \tag{2.13}$$

Using Leibnitz' rule in (2.11), we get

$$\begin{aligned} 0 &= \sum_{j=0}^{k+1} c_j (uv)^{(j)} \\ &= \sum_{j=0}^{k+1} c_j \sum_{m=0}^{j} \binom{j}{m} u^{(m)} v^{(j-m)} \\ &= \sum_{j=0}^{k+1} c_j \sum_{m=0}^{k+1} \binom{j}{m} u^{(m)} v^{(j-m)} \end{aligned}$$

using the convention that $\binom{j}{m} = 0$ for $m > j$. Dividing through by v, we get

$$\begin{aligned} 0 &= \sum_{m=0}^{k+1} u^{(m)} \sum_{j=0}^{k+1} \binom{j}{m} c_j \frac{v^{(j-m)}}{v} \\ &= \sum_{m=0}^{k+1} u^{(m)} A_m \end{aligned} \tag{2.14}$$

where, again since $\binom{j}{m} = 0$ for $m > j$,

$$A_m = \sum_{j=m}^{k+1} \binom{j}{m} c_j \frac{v^{(j-m)}}{v}. \tag{2.15}$$

In particular, this gives using (2.12),

$$A_{k+1} = c_{k+1} = 1$$

$$\begin{aligned} A_0 &= \sum_{j=0}^{k+1} c_j \frac{v^{(j)}}{v} \\ &= \sum_{j=0}^{k+1} (b_j' + b_{j-1} - \frac{g'}{g} b_j) \frac{v^{(j)}}{v} \\ &= \frac{L(v)' - \frac{g'}{g} L(v)}{v}. \end{aligned} \tag{2.16}$$

Claim: $A_0 \not\equiv 0$.

To prove the claim, suppose that $A_0 \equiv 0$. Using (2.16), we get

$$L(v)' = \frac{g'}{g} L(v). \tag{2.17}$$

We consider two cases:

Case (1): $L(v) \not\equiv 0$.

Using (2.17), we have

$$\begin{aligned} \frac{L(v)'}{L(v)} &= \frac{g'}{g} \\ L(v) &= cg, \end{aligned}$$

where $0 \neq c \in \mathbb{C}$ since $L(v) \not\equiv 0$. Using (2.4), we have

$$L(v) = cL(f).$$

Solving this, we get

$$v = cf + h, \quad L(h) = 0$$

$$v = c(f + H), \quad L(H) = 0.$$

Let $F = f + H$. This gives $v = cF$ and

$$\begin{aligned} L(f) &= L(f+H) \\ &= L(F). \end{aligned}$$

Since $v = cF$ and $v = \frac{1}{ag^n}$ by (2.13), we have $\frac{1}{ag^n} = cF$ and

$$\begin{aligned} 1 &= acL(f)^n F \\ &= acL(F)^n F. \end{aligned}$$

Since $L(H) = 0$ and $H = \frac{v-cf}{c}$ is meromorphic in the plane, we get by (2.7) $T(r,H) = S(r,f)$. Hence

$$T(r,f) = T(r,F) + S(r,F)$$

$$T(r,a) + \sum_{j=0}^{k-1} T(r,b_j) = S(r,f) = S(r,F).$$

This contradicts Lemma 2.2.1, applied to $F = f + H$.

Case 2: $L(v) \equiv 0$.

Using (2.7), we get $T(r,v) = S(r,f)$ and using (2.13) we have $v = \frac{1}{ag^n}$. Thus, $T(r,g) = S(r,f)$ which is a contradiction to (2.6). This completes the proof of the claim.

Returning to the proof of Theorem 2.1.4, we have using (2.5) and (2.13),

$$\begin{aligned} u &= \frac{w}{v} \\ &= afg^n \\ &= \psi + 1. \end{aligned}$$

So $\psi + 1$ solves (2.14). This gives

$$\begin{aligned}(\psi+1)^{(k+1)} + A_k(\psi+1)^{(k)} + \cdots + A_1(\psi+1)' + A_0(\psi+1) &= 0\\ \psi^{(k+1)} + A_k\psi^{(k)} + \cdots + A_1\psi' + A_0(\psi+1) &= 0\\ \frac{\psi^{(k+1)}}{A_0} + \frac{A_k}{A_0}\psi^{(k)} + \cdots + \frac{A_1}{A_0}\psi' + \psi + 1 &= 0\\ \frac{1}{A_0}\left[\frac{\psi^{(k+1)}}{\psi} + A_k\frac{\psi^{(k)}}{\psi} + \cdots + A_1\frac{\psi'}{\psi}\right] + 1 + \frac{1}{\psi} &= 0.\end{aligned}$$

Therefore

$$\frac{1}{\psi} = \frac{-1}{A_0}\left[\frac{\psi^{(k+1)}}{\psi} + A_k\frac{\psi^{(k)}}{\psi} + \cdots + A_1\frac{\psi'}{\psi}\right] - 1. \tag{2.18}$$

Using (2.15), we note that

$$A_m = \begin{pmatrix} k+1 \\ m \end{pmatrix} \frac{v^{(k+1-m)}}{v} + \sum_{j=m}^{k} \begin{pmatrix} j \\ m \end{pmatrix} c_j \frac{v^{(j-m)}}{v}.$$

Hence the contribution to $n(r, A_m)$ from the terms $\frac{v^{(j-m)}}{v}$, $\frac{g'}{g}$ is at most $k+1-m$, and the contribution to $n(r, A_0)$ and $n(r, A_m\frac{\psi^{(m)}}{\psi})$ from these terms is at most $k+1$. Furthermore, using (2.13) and (2.16) we see that any pole of A_0 can only occur at poles or zeros of g, poles or zeros of $a(z)$, or poles of $b_j(z)$. So

$$\begin{aligned}N(r, A_0) &\leq (k+1)\left[\bar{N}(r,g) + \bar{N}(r,\frac{1}{g}) + \bar{N}(r,a) + \bar{N}(r,\frac{1}{a})\right] + 2\sum_{j=0}^{k-1} N(r,b_j)\\ &\leq (k+1)\left[\bar{N}(r,f) + \bar{N}(r,\frac{1}{g})\right] + S(r,f).\end{aligned}$$

Using (2.9), we have

$$\begin{aligned}N(r, A_0) &\leq (k+1)\left[\bar{N}(r,f) + \left(\frac{1}{n-1}\right)\bar{N}(r,f) + \left(\frac{1}{n-1}\right)\bar{N}(r,\frac{1}{\psi})\right] + S(r,f)\\ &= (k+1)\left(1 + \frac{1}{n-1}\right)\bar{N}(r,f) + \left(\frac{k+1}{n-1}\right)\bar{N}(r,\frac{1}{\psi}) + S(r,f)\\ &= \left(\frac{n(k+1)}{n-1}\right)\bar{N}(r,f) + \left(\frac{k+1}{n-1}\right)\bar{N}(r,\frac{1}{\psi}) + S(r,f).\end{aligned} \tag{2.19}$$

Using (2.2), (2.4) and (2.5), we have

$$\begin{aligned}\frac{\psi+1}{a} &= fg^n \\ &= f\left(f^{(k)} + b_{k-1}f^{(k-1)} + \cdots + b_0 f\right)^n \\ &= f(f^{(k)})^n \left(1 + b_{k-1}\frac{f^{(k-1)}}{f^{(k)}} + \cdots + b_0\frac{f}{f^{(k)}}\right)^n .\end{aligned}$$

So a pole of f of multiplicity p with $b_j \neq \infty$ is a pole of $\frac{\psi+1}{a}$ of multiplicity $p + (p+k)n \geq 1 + (1+k)n$. Thus

$$\begin{aligned}\bar{N}(r,f) &\leq \left(\frac{1}{1+n(1+k)}\right) N(r, \frac{\psi+1}{a}) + \sum_{j=0}^{k-1} \bar{N}(r, b_j) \\ &\leq \left(\frac{1}{1+n(1+k)}\right) N(r, \psi+1) + S(r,f) \\ &\leq \left(\frac{1}{1+n(1+k)}\right) T(r,\psi) + S(r,f). \end{aligned} \tag{2.20}$$

Using (2.8), (2.19) and (2.20), we get

$$\begin{aligned}N(r, A_0) &\leq \left(\frac{n(k+1)}{n-1}\right)\left(\frac{1}{1+n(1+k)}\right) T(r,\psi) + \left(\frac{k+1}{n-1}\right) \bar{N}(r, \frac{1}{\psi}) + S(r,f) \\ &= \delta_k T(r,\psi) + \left(\frac{k+1}{n-1}\right) \bar{N}(r,\frac{1}{\psi}) + S(r,f). \end{aligned} \tag{2.21}$$

Using (2.15) and (2.18), we have

$$N(r, \frac{1}{\psi}) \leq N(r, \frac{1}{A_0}) + (k+1)\bar{N}(r, \frac{1}{\psi}) + S(r,f) \tag{2.22}$$

$$m(r, \frac{1}{\psi}) \leq m(r, \frac{1}{A_0}) + S(r,f). \tag{2.23}$$

Using (2.16), (2.21), (2.22) and (2.23), we get

$$\begin{aligned}
T(r,\psi) &= T(r,\frac{1}{\psi}) + O(1) \\
&= m(r,\frac{1}{\psi}) + N(r,\frac{1}{\psi}) + O(1) \\
&\leq m(r,\frac{1}{A_0}) + N(r,\frac{1}{A_0}) + (k+1)\bar{N}(r,\frac{1}{\psi}) + S(r,f) \\
&= T(r,\frac{1}{A_0}) + (k+1)\bar{N}(r,\frac{1}{\psi}) + S(r,f) \\
&= T(r,A_0) + (k+1)\bar{N}(r,\frac{1}{\psi}) + S(r,f) \\
&= N(r,A_0) + (k+1)\bar{N}(r,\frac{1}{\psi}) + S(r,f) \\
&\leq \delta_k T(r,\psi) + \left(\frac{k+1}{n-1}\right)\bar{N}(r,\frac{1}{\psi}) + (k+1)\bar{N}(r,\frac{1}{\psi}) + S(r,f) \\
&= \delta_k T(r,\psi) + \left(\frac{n(k+1)}{n-1}\right)\bar{N}(r,\frac{1}{\psi}) + S(r,f).
\end{aligned}$$

Thus

$$(1-\delta_k)T(r,\psi) \leq \left(\frac{n(k+1)}{n-1}\right)\bar{N}(r,\frac{1}{\psi}) + S(r,f)$$

$$T(r,\psi) \leq \left(\frac{1}{1-\delta_k}\right)\left(\frac{n(k+1)}{n-1}\right)\bar{N}(r,\frac{1}{\psi}) + S(r,f).$$

Using Lemma 2.2.1, we get

$$T(r,f) \leq \left(\frac{1}{1-\delta_k}\right)\left(\frac{n(k+1)}{n-1}\right)\bar{N}(r,\frac{1}{\psi}) + S(r,f).$$

Hence, Theorem 2.1.4 is proved.

2.4 Examples and Corollaries

Now we will give two examples to show that without the hypotheses which are in Theorem 2.1.4 we can find $\psi = afg^n - 1$ with no zeros. In the first example, we do not have the first hypothesis which is the equation (2.6), and in the second example we do not have the second hypothesis which is the equation (2.7).

Example 2.4.1 *Suppose that* $f(z) = e^z + z$ *and* $L(w) = w'' - 2w' + w$*. This gives* $g = L(f) = z - 2$, $T(r, g) = S(r, f)$ *and* $fg^2 = (e^z + z)(z-2)^2$*. Let* $a = \frac{1}{z(z-2)^2}$ *which gives* $T(r, a) = S(r, f)$*. From all of these, we see that*

$$\begin{aligned} \psi &= afg^2 - 1 \\ &= \frac{1}{z(z-2)^2}(e^z + z)(z-2)^2 - 1 \\ &= \frac{e^z + z}{z} - 1 = \frac{e^z}{z} + 1 - 1 \\ &= \frac{e^z}{z} \neq 0. \end{aligned}$$

Example 2.4.2 *Suppose that* $L(w) = w' - w$ *and* $\psi = fg^2 - 1$*, where* $f = e^z - \frac{2}{3}e^{\frac{-z}{2}}$*. Hence*

$$\begin{aligned} g &= f' - f \\ &= e^z + \frac{1}{3}e^{\frac{-z}{2}} - e^z + \frac{2}{3}e^{\frac{-z}{2}} \\ &= e^{\frac{-z}{2}}. \end{aligned}$$

Thus

$$\begin{aligned} \psi &= fg^2 - 1 \\ &= (e^z - \frac{2}{3}e^{\frac{-z}{2}})(e^{\frac{-z}{2}})^2 - 1 \\ &= (e^z - \frac{2}{3}e^{\frac{-z}{2}})e^{-z} - 1 \\ &= 1 - \frac{2}{3}e^{\frac{-3z}{2}} - 1 \\ &= -\frac{2}{3}e^{\frac{-3z}{2}} \\ &\neq 0. \end{aligned}$$

Here, $\phi = e^z$ *solves* $L(w) = 0$*, but* $T(r, \phi) \neq S(r, f)$*.*

Corollary 2.4.1 *Suppose that f is a transcendental meromorphic function in the plane, and suppose that a, g, ψ are given by (2.1), (2.4) and (2.5) such that $T(r,g) \neq S(r,f)$ and $T(r,\phi) = S(r,f)$ for every solution ϕ of $L(w) = 0$ which is meromorphic in the plane. Then $\psi = afg^n - 1$ has infinitely many zeros and the function afg^n assumes every non-zero value infinitely often.*

Corollary 2.4.2 *Suppose that f is a transcendental meromorphic function in the plane, and suppose that $a \not\equiv 0$ is a meromorphic function with $T(r,a) = S(r,f)$. Let $\psi = af(f^{(k)})^n - 1, \quad n \geq 2, \quad n,k \in \mathbb{N}$. Then*

$$T(r,f) \leq \left(\frac{1}{1-\delta_k}\right)\left(\frac{n(k+1)}{n-1}\right)\bar{N}(r,\frac{1}{\psi}) + S(r,f).$$

Proof: We have here $g = f^{(k)}$. Using the Hayman-Miles theorem (Theorem 1.6.3), we have $T(r,g) \neq S(r,f)$. Also, we have here $L(w) = w^{(k)}$. Thus, every ϕ which is a solution of $L(w) = 0$ would be a polynomial. So $T(r,\phi) = S(r,f)$. Applying Theorem 2.1.4, this completes the proof of Corollary 2.4.2. □

Theorem 2.1.3 follows at once from Corollary 2.4.2, using $A = \frac{1}{a}$.

Corollary 2.4.3 *Suppose that f is a transcendental meromorphic function in the plane. Let a, ψ be as in Corollary 2.4.2. Then $\psi = af(f^{(k)})^n - 1$ has infinitely many zeros and the function $af(f^{(k)})^n$ assumes every non-zero value infinitely often.*

Chapter 3

On the Zeros of $aff^{(k)} - 1$

[1] In this chapter, we will consider the following. Let k be a positive integer, and let f be a transcendental meromorphic function with $N_1(r, \frac{1}{f^{(k)}}) = S(r, f)$. Then the function $aff^{(k)} - 1$ has infinitely many zeros, where $a(z) \not\equiv 0$ is a meromorphic function such that $T(r, a) = S(r, f)$. Here, $N_1(r, \frac{1}{f^{(k)}})$ denotes the integrated counting function of simple zeros of $f^{(k)}$.

3.1 Introduction

Let us recall the Conjecture 2.1.1, which we restate as follows.

Conjecture 3.1.1 *Let n, k be two positive integers, and let f be a transcendental entire function. Then $f(f^{(k)})^n - 1$ has infinitely many zeros.*

In Chapter 2, we proved the following (See Corollary 2.4.3).

Theorem 3.1.1 *Let n, k be two positive integers, and let f be a transcendental meromorphic function in the plane. Let $a(z) \not\equiv 0$ be a meromorphic function with*

[1]We published this chapter as a paper [2].

$T(r,a) = S(r,f)$. *Then* $af(f^{(k)})^n - 1$ *has infinitely many zeros when* $n \geq 2$.

Thus the case $n \geq 2$ is completely proved in Chapter 2 and the remaining case is when $n = 1$. In 1995, Bergweiler and Eremenko [10] proved the following.

Theorem 3.1.2 *Suppose that* f *is a transcendental meromorphic function . Then* $ff' - 1$ *has infinitely many zeros.*

The next result is by Langley [26] from 2003.

Theorem 3.1.3 *Suppose that* f *is a transcendental entire function. Then* $ff''-1$ *has infinitely many zeros.*

In 1998, Zhang and Song [38] proved the following.

Theorem 3.1.4 *Let* k *be a positive integer, and let* f *be a transcendental entire function with* $N_1(r, \frac{1}{f^{(k)}}) = S(r,f)$. *Then* $ff^{(k)} - 1$ *has infinitely many zeros, where* $N_1(r, \frac{1}{f^{(k)}})$ *denotes the integrated counting function of simple zeros of* $f^{(k)}$.

In fact, few details of the proof of Theorem 3.1.4 are given in [38]. We give a simple proof of a more general result. Before stating our main results, Theorem 3.1.5 and Theorem 3.1.6, we make some assumptions which we need throughout this chapter and are similar to those made in Chapter 2. Suppose that f is a transcendental meromorphic function in the plane and $a(z) \not\equiv 0$ is a meromorphic function such that

$$T(r,a) = S(r,f). \tag{3.1}$$

Let

$$L(w) = w^{(k)} + b_{k-1}w^{(k-1)} + \cdots + b_0 w, \quad k \in \mathbb{N}, \tag{3.2}$$

where each $b_j(z)$, $j = 0, 1, \cdots, k-1$, is a meromorphic function such that

$$T(r,b_j) = S(r,f). \tag{3.3}$$

Let

$$g = L(f) \tag{3.4}$$

$$\psi = afg - 1. \tag{3.5}$$

Theorem 3.1.5 *Suppose that f is a transcendental meromorphic function in the plane. Suppose that L, g, ψ and the b_j are given by (3.2), (3.3), (3.4) and (3.5). Suppose that*

$$T(r, g) \neq S(r, f) \tag{3.6}$$

$$T(r, \phi) = S(r, f) \tag{3.7}$$

for every solution ϕ of $L(w) = 0$ which is meromorphic in the plane. Then

$$T(r, f) \leq (k+1)(k+2)\left[\bar{N}(r, \frac{1}{\psi}) + \bar{N}(r, \frac{1}{g})\right] + S(r, f).$$

Theorem 3.1.6 *Suppose that f is a transcendental meromorphic function in the plane. Suppose that L, g, ψ and the b_j are given by (3.2), (3.3), (3.4), (3.5), and that (3.6) holds, as does (3.7) for every solution ϕ of $L(w) = 0$ which is meromorphic in the plane. Then*

$$T(r, f) \leq 2(k+1)\bar{N}(r, f) + 2(k+1)\bar{N}(r, \frac{1}{\psi}) + (k+1)N_1(r, \frac{1}{g}) + S(r, f).$$

3.2 A lemma needed for Theorem 3.1.5 and Theorem 3.1.6

This lemma is the analogue of Lemma 2.2.1. We include the proof for completeness.

Lemma 3.2.1 *Provided $g \not\equiv 0$, we have*

$$T(r, f) \leq T(r, \psi) + S(r, f).$$

In particular, $afL(f)$ is a non-constant function.

Proof: Using (3.5), we have

$$f = \frac{\psi + 1}{ag}.$$

So

$$\begin{aligned} N(r, \frac{1}{f}) &\leq N(r, \frac{1}{\psi+1}) + N(r, a) + \sum_{j=0}^{k-1} N(r, b_j) \\ &= N(r, \frac{1}{\psi+1}) + S(r, f). \end{aligned}$$

Also, using (3.2), (3.4) and (3.5) we have

$$\frac{1}{f} = \frac{ag}{\psi+1}$$

$$\frac{1}{f^2} = \frac{a}{\psi+1}\left(\frac{f^{(k)}}{f} + b_{k-1}\frac{f^{(k-1)}}{f} + \cdots + b_0\right).$$

Therefore

$$\begin{aligned} m(r, \frac{1}{f}) &\leq 2m(r, \frac{1}{f}) \\ &= m(r, \frac{1}{f^2}) \\ &\leq m(r, \frac{1}{\psi+1}) + m(r, a) + S(r, f) \\ &= m(r, \frac{1}{\psi+1}) + S(r, f). \end{aligned}$$

Hence, using the first fundamental theorem of Nevanlinna theory (Theorem 1.2.1),

$$\begin{aligned} T(r, f) &= T(r, \frac{1}{f}) + O(1) \\ &= m(r, \frac{1}{f}) + N(r, \frac{1}{f}) + O(1) \\ &\leq m(r, \frac{1}{\psi+1}) + N(r, \frac{1}{\psi+1}) + S(r, f) \\ &= T(r, \frac{1}{\psi+1}) + S(r, f) \\ &= T(r, \psi) + S(r, f). \end{aligned}$$

This completes the proof of Lemma 3.2.1. □

3.3 Proof of Theorem 3.1.5 and Theorem 3.1.6

Some steps in this proof are similar to those in Chapter 2 but we include all details for completeness. Put $b_k = 1$ and $b_{-1} = b_{k+1} = 0$. So

$$g = \sum_{j=0}^{k} b_j f^{(j)} = \sum_{j=0}^{k+1} b_j f^{(j)}. \tag{3.8}$$

Using the fact that $b'_{k+1} = 0$ and $b_{-1} = 0$, we have

$$\begin{aligned} g' &= \sum_{j=0}^{k} (b'_j f^{(j)} + b_j f^{(j+1)}) \\ &= \sum_{j=0}^{k} b'_j f^{(j)} + \sum_{j=1}^{k+1} b_{j-1} f^{(j)} \\ &= \sum_{j=0}^{k+1} (b'_j + b_{j-1}) f^{(j)}. \end{aligned}$$

Since

$$g = \sum_{j=0}^{k+1} b_j f^{(j)},$$

we find that $w = f$ solves

$$\sum_{j=0}^{k+1} c_j w^{(j)} = 0 \tag{3.9}$$

where

$$c_j = b'_j + b_{j-1} - \frac{g'}{g} b_j, \quad c_{k+1} = 1. \tag{3.10}$$

Let

$$w = uv, \quad v = \frac{1}{ag}. \tag{3.11}$$

Using Leibnitz' rule in (3.9), we get

$$\begin{aligned} 0 &= \sum_{j=0}^{k+1} c_j (uv)^{(j)} \\ &= \sum_{j=0}^{k+1} c_j \sum_{m=0}^{j} \binom{j}{m} u^{(m)} v^{(j-m)} \\ &= \sum_{j=0}^{k+1} c_j \sum_{m=0}^{k+1} \binom{j}{m} u^{(m)} v^{(j-m)} \end{aligned}$$

using the convention that $\binom{j}{m} = 0$ for $m > j$. Dividing through by v, we get

$$\begin{aligned} 0 &= \sum_{m=0}^{k+1} u^{(m)} \sum_{j=0}^{k+1} \binom{j}{m} c_j \frac{v^{(j-m)}}{v} \\ &= \sum_{m=0}^{k+1} u^{(m)} A_m \end{aligned} \tag{3.12}$$

where, again since $\binom{j}{m} = 0$ for $m > j$,

$$A_m = \sum_{j=m}^{k+1} \binom{j}{m} c_j \frac{v^{(j-m)}}{v}. \tag{3.13}$$

In particular, this gives using (3.10),

$$A_{k+1} = c_{k+1} = 1$$

$$\begin{aligned} A_0 &= \sum_{j=0}^{k+1} c_j \frac{v^{(j)}}{v} \\ &= \sum_{j=0}^{k+1} (b_j' + b_{j-1} - \frac{g'}{g} b_j) \frac{v^{(j)}}{v} \\ &= \frac{L(v)' - \frac{g'}{g} L(v)}{v}. \end{aligned} \tag{3.14}$$

Claim: $A_0 \not\equiv 0$.

To prove the claim, suppose that $A_0 \equiv 0$. Using (3.14), we get

$$L(v)' = \frac{g'}{g} L(v). \tag{3.15}$$

We consider two cases:

Case (1): $L(v) \not\equiv 0$.

Using (3.15), we have

$$\frac{L(v)'}{L(v)} = \frac{g'}{g}$$
$$L(v) = cg,$$

where $0 \neq c \in \mathbb{C}$ since $L(v) \not\equiv 0$. Using (3.4), we have

$$L(v) = cL(f).$$

Solving this, we get

$$v = cf + h, \quad L(h) = 0$$

$$v = c(f + H), \quad L(H) = 0.$$

Let $F = f + H$. This gives $v = cF$ and

$$\begin{aligned} L(f) &= L(f + H) \\ &= L(F). \end{aligned}$$

Since $v = cF$ and $v = \frac{1}{ag}$, we have $\frac{1}{ag} = cF$ and

$$\begin{aligned} 1 &= acL(f)F \\ &= acL(F)F. \end{aligned}$$

Since $L(H) = 0$ and $H = \frac{v-cf}{c}$ is meromorphic in the plane, we get by (3.7) $T(r, H) = S(r, f)$. Hence

$$T(r, f) = T(r, F) + S(r, F)$$

$$T(r, a) + \sum_{j=0}^{k-1} T(r, b_j) = S(r, f) = S(r, F).$$

This contradicts Lemma 3.2.1, applied to $F = f + H$.

Case 2: $L(v) \equiv 0$.

Using (3.7), we get $T(r, v) = S(r, f)$ and using (3.11) we have $v = \frac{1}{ag}$. Thus $T(r, g) = S(r, f)$ which is a contradiction to the first hypothesis in the equation (3.6). So the claim is proved.

Returning to the proof of Theorem 3.1.5 and Theorem 3.1.6, we have using (3.5) and (3.11),

$$\begin{aligned} u &= \frac{w}{v} \\ &= afg \\ &= \psi + 1. \end{aligned}$$

So $\psi + 1$ solves (3.12). This gives

$$\begin{aligned} (\psi + 1)^{(k+1)} + A_k(\psi + 1)^{(k)} + \cdots + A_1(\psi + 1)' + A_0(\psi + 1) = 0 \\ \psi^{(k+1)} + A_k\psi^{(k)} + \cdots + A_1\psi' + A_0(\psi + 1) = 0 \\ \frac{\psi^{(k+1)}}{A_0} + \frac{A_k}{A_0}\psi^{(k)} + \cdots + \frac{A_1}{A_0}\psi' + \psi + 1 = 0 \\ \frac{1}{A_0}\left[\frac{\psi^{(k+1)}}{\psi} + A_k\frac{\psi^{(k)}}{\psi} + \cdots + A_1\frac{\psi'}{\psi}\right] + 1 + \frac{1}{\psi} = 0. \end{aligned}$$

Thus

$$\frac{1}{\psi} = \frac{-1}{A_0}\left[\frac{\psi^{(k+1)}}{\psi} + A_k\frac{\psi^{(k)}}{\psi} + \cdots + A_1\frac{\psi'}{\psi}\right] - 1. \tag{3.16}$$

Using (3.13), we note that

$$A_m = \binom{k+1}{m}\frac{v^{(k+1-m)}}{v} + \sum_{j=m}^{k}\binom{j}{m}c_j\frac{v^{(j-m)}}{v}.$$

Hence the contribution to $n(r, A_m)$ from the terms $\frac{v^{(j-m)}}{v}, \frac{g'}{g}$ is at most $k+1-m$, and the contribution to $n(r, A_0)$ and $n(r, A_m\frac{\psi^{(m)}}{\psi})$ from these terms is at most $k+1$. Furthermore, using (3.14) we see that any pole of A_0 can only occur at poles or zeros of g, poles or zeros of $a(z)$, or poles of $b_j(z)$. So

$$\begin{aligned} N(r, A_0) &\leq (k+1)\left[\bar{N}(r,g) + \bar{N}(r,\frac{1}{g}) + \bar{N}(r,a) + \bar{N}(r,\frac{1}{a})\right] + 2\sum_{j=0}^{k-1}N(r,b_j) \\ &\leq (k+1)\left[\bar{N}(r,f) + \bar{N}(r,\frac{1}{g})\right] + S(r,f). \end{aligned} \tag{3.17}$$

Using (3.16), we have

$$m(r,\frac{1}{\psi}) \leq m(r,\frac{1}{A_0}) + S(r,f) \tag{3.18}$$

$$N(r,\frac{1}{\psi}) \leq N(r,\frac{1}{A_0}) + (k+1)\bar{N}(r,\frac{1}{\psi}) + S(r,f). \tag{3.19}$$

3.4 The completion of the proof of Theorem 3.1.5

Using (3.4) and (3.5), we have

$$\begin{aligned} \frac{\psi+1}{a} &= fg \\ &= f(f^{(k)} + b_{k-1}f^{(k-1)} + \cdots + b_0 f) \\ &= ff^{(k)}\left(1 + b_{k-1}\frac{f^{(k-1)}}{f^{(k)}} + \cdots + b_0\frac{f}{f^{(k)}}\right). \end{aligned}$$

So a pole of f of multiplicity p with $b_j \neq \infty$ is a pole of $\frac{\psi+1}{a}$ of multiplicity $p + (p+k) \geq 1 + (1+k) = k+2$. Thus

$$\begin{aligned}
\bar{N}(r,f) &\leq \left(\frac{1}{k+2}\right) N(r, \frac{\psi+1}{a}) + \sum_{j=0}^{k-1} \bar{N}(r, b_j) \\
&\leq \left(\frac{1}{k+2}\right) N(r, \psi+1) + S(r,f) \\
&\leq \left(\frac{1}{k+2}\right) T(r,\psi) + S(r,f). \qquad (3.20)
\end{aligned}$$

Using (3.17) and (3.20), we have

$$N(r, A_0) \leq \left(\frac{k+1}{k+2}\right) T(r,\psi) + (k+1)\bar{N}(r, \frac{1}{g}) + S(r,f). \qquad (3.21)$$

Using (3.14), (3.18), (3.19) and (3.21), we get

$$\begin{aligned}
T(r,\psi) &= T(r, \frac{1}{\psi}) + O(1) \\
&= m(r, \frac{1}{\psi}) + N(r, \frac{1}{\psi}) + O(1) \\
&\leq m(r, \frac{1}{A_0}) + N(r, \frac{1}{A_0}) + (k+1)\bar{N}(r, \frac{1}{\psi}) + S(r,f) \\
&= T(r, \frac{1}{A_0}) + (k+1)\bar{N}(r, \frac{1}{\psi}) + S(r,f) \\
&= T(r, A_0) + (k+1)\bar{N}(r, \frac{1}{\psi}) + S(r,f) \\
&= N(r, A_0) + (k+1)\bar{N}(r, \frac{1}{\psi}) + S(r,f) \\
&\leq \left(\frac{k+1}{k+2}\right) T(r,\psi) + (k+1)\bar{N}(r, \frac{1}{g}) + (k+1)\bar{N}(r, \frac{1}{\psi}) + S(r,f).
\end{aligned}$$

So

$$\left(1 - \frac{k+1}{k+2}\right) T(r,\psi) \leq (k+1)\left[\bar{N}(r, \frac{1}{\psi}) + \bar{N}(r, \frac{1}{g})\right] + S(r,f)$$

$$\left(\frac{1}{k+2}\right) T(r,\psi) \leq (k+1)\left[\bar{N}(r, \frac{1}{\psi}) + \bar{N}(r, \frac{1}{g})\right] + S(r,f)$$

$$T(r,\psi) \leq (k+1)(k+2)\left[\bar{N}(r,\frac{1}{\psi}) + \bar{N}(r,\frac{1}{g})\right] + S(r,f).$$

Using Lemma 3.2.1, we get

$$T(r,f) \leq (k+1)(k+2)\left[\bar{N}(r,\frac{1}{\psi}) + \bar{N}(r,\frac{1}{g})\right] + S(r,f).$$

This completes the proof of Theorem 3.1.5.

3.5 The completion of the proof of Theorem 3.1.6

By (3.5) a multiple zero of g of multiplicity p with $a \neq 0, \infty$ and with $b_j \neq \infty$ is a zero of ψ' of multiplicity $p-1 \geq 1$, but is not a zero of ψ. So, with $\bar{N_2}$ as defined in Proposition 1.2.3,

$$\begin{aligned}
\bar{N_2}(r,\frac{1}{g}) &\leq N(r,\frac{\psi}{\psi'}) + \bar{N}(r,a) + \bar{N}(r,\frac{1}{a}) + \sum_{j=0}^{k-1}\bar{N}(r,b_j) \\
&\leq N(r,\frac{\psi'}{\psi}) + S(r,f) \\
&= \bar{N}(r,\psi) + \bar{N}(r,\frac{1}{\psi}) + S(r,f) \\
&\leq \bar{N}(r,f) + \bar{N}(r,\frac{1}{\psi}) + S(r,f). \qquad (3.22)
\end{aligned}$$

Using (3.17), (3.18), (3.19) and (3.22), we get

$$\begin{aligned}
T(r,\psi) &= T(r,\frac{1}{\psi}) + O(1) = m(r,\frac{1}{\psi}) + N(r,\frac{1}{\psi}) + O(1) \\
&\leq m(r,\frac{1}{A_0}) + N(r,\frac{1}{A_0}) + (k+1)\bar{N}(r,\frac{1}{\psi}) + S(r,f) \\
&= T(r,\frac{1}{A_0}) + (k+1)\bar{N}(r,\frac{1}{\psi}) + S(r,f) \\
&= T(r,A_0) + (k+1)\bar{N}(r,\frac{1}{\psi}) + S(r,f) \\
&= N(r,A_0) + (k+1)\bar{N}(r,\frac{1}{\psi}) + S(r,f) \\
&\leq (k+1)\bar{N}(r,f) + (k+1)\bar{N}(r,\frac{1}{g}) + (k+1)\bar{N}(r,\frac{1}{\psi}) + S(r,f) \\
&= (k+1)\bar{N}(r,f) + (k+1)N_1(r,\frac{1}{g}) + (k+1)\bar{N}_2(r,\frac{1}{g}) + (k+1)\bar{N}(r,\frac{1}{\psi}) \\
&+ S(r,f) \\
&\leq (k+1)\bar{N}(r,f) + (k+1)N_1(r,\frac{1}{g}) + (k+1)\left[\bar{N}(r,f) + \bar{N}(r,\frac{1}{\psi})\right] \\
&+ (k+1)\bar{N}(r,\frac{1}{\psi}) + S(r,f) \\
&= 2(k+1)\bar{N}(r,f) + 2(k+1)\bar{N}(r,\frac{1}{\psi}) + (k+1)N_1(r,\frac{1}{g}) + S(r,f).
\end{aligned}$$

Using Lemma 3.2.1, we get

$$T(r,f) \leq 2(k+1)\bar{N}(r,f) + 2(k+1)\bar{N}(r,\frac{1}{\psi}) + (k+1)N_1(r,\frac{1}{g}) + S(r,f).$$

This complete the proof of Theorem 3.1.6.

3.6 Examples and Corollaries

In this section, we will give two examples to show that without the hypotheses in Theorem 3.1.5 and Theorem 3.1.6 we can find $\psi = afg - 1$ with no zeros. In the first example, we do not have the first hypothesis which is the equation (3.6), and in the second example we do not have the second hypothesis which is the equation (3.7).

Example 3.6.1 *Suppose that $f(z) = e^z + z$ and $L(w) = w'' - 2w' + w$. This gives $g = L(f) = z-2$, $T(r,g) = S(r,f)$ and $fg = (e^z+z)(z-2)$. Let $a = \frac{1}{z(z-2)}$ which gives $T(r,a) = S(r,f)$. From all of these, we see that*

$$\begin{aligned}
\psi &= afg - 1 \\
&= \frac{1}{z(z-2)}(e^z + z)(z-2) - 1 \\
&= \frac{e^z + z}{z} - 1 \\
&= \frac{e^z}{z} + 1 - 1 \\
&= \frac{e^z}{z} \neq 0.
\end{aligned}$$

Example 3.6.2 *Suppose that $L(w) = w' - w$ and $\psi = fg - 1$, where $f = e^z - \frac{1}{2}e^{-z}$. Hence*

$$\begin{aligned}
g &= f' - f \\
&= e^z + \frac{1}{2}e^{-z} - e^z + \frac{1}{2}e^{-z} \\
&= e^{-z}.
\end{aligned}$$

Thus

$$\begin{aligned}
\psi &= fg - 1 \\
&= (e^z - \frac{1}{2}e^{-z})e^{-z} - 1 \\
&= 1 - \frac{1}{2}e^{-2z} - 1 \\
&= -\frac{1}{2}e^{-2z} \\
&\neq 0.
\end{aligned}$$

Here, $\phi = e^z$ solves $L(w) = 0$, but $T(r,\phi) \neq S(r,f)$.

Corollary 3.6.1 *Let k be a positive integer, and let f be a transcendental meromorphic function. Let $a(z) \not\equiv 0$ be a meromorphic function with $T(r,a) = S(r,f)$. Suppose that $\psi = aff^{(k)} - 1$. Then*

$$T(r,f) \leq (k+1)(k+2)\left[\bar{N}(r,\frac{1}{\psi}) + \bar{N}(r,\frac{1}{f^{(k)}})\right] + S(r,f).$$

Corollary 3.6.2 *Let k be a positive integer, and let f be a transcendental meromorphic function. Let $a(z) \not\equiv 0$ be a meromorphic function with $T(r,a) = S(r,f)$. Suppose that $\psi = aff^{(k)} - 1$. Then*

$$T(r,f) \leq 2(k+1)\bar{N}(r,f) + 2(k+1)\bar{N}(r,\frac{1}{\psi}) + (k+1)N_1(r,\frac{1}{f^{(k)}}) + S(r,f).$$

Corollary 3.6.3 *Let k be a positive integer, and let f be a transcendental entire function. Let $a(z) \not\equiv 0$ be a meromorphic function with $T(r,a) = S(r,f)$. Suppose that $\psi = aff^{(k)} - 1$. Then*

$$T(r,f) \leq 2(k+1)\bar{N}(r,\frac{1}{\psi}) + (k+1)N_1(r,\frac{1}{f^{(k)}}) + S(r,f).$$

Theorem 3.1.4 follows at once from Corollary 3.6.3. However, we still need to prove the following to complete the proof of Conjecture 3.1.1.

Conjecture 3.6.1 *Let k be a positive integer, and let f be a transcendental entire function. Then $ff^{(k)} - 1$ has infinitely many zeros.*

Chapter 4

On Normal Families

[1] In this chapter, we will prove that a family of functions meromorphic on the unit disc $B(0,1)$, such that $f(f')^m \neq 1$, $m \geq 2$, is normal. Also, we will prove that a family of functions meromorphic on $B(0,1)$, such that $f \neq 0$, $f(f^{(k)})^m \neq 1$, $k, m \in \mathbb{N}$, is normal. Moreover, we will generalise both of these results.

4.1 Introduction

Theorem 4.1.1 *Let n be a positive integer, and let $\mathcal{F}$ be a family of functions meromorphic such that $f'f^n \neq 1$ on $B(0,1)$, for each $f \in \mathcal{F}$. Then $\mathcal{F}$ is normal on $B(0,1)$.*

The proof of Theorem 4.1.1 is due to Yang and Chang [28] for $n \geq 5$, Ku [19] for $n = 3, 4$, Pang [29] for $n = 2$, and Bergweiler and Eremenko [10] for $n = 1$. In 1979, Ku [18] proved the following theorem.

Theorem 4.1.2 *Let k be a positive integer, and let $\mathcal{F}$ be a family of meromorphic*

[1] We published this chapter as a paper [1].

functions such that $f \neq 0$ and $f^{(k)} \neq 1$ for each $f \in \mathcal{F}$ on $B(0,1)$. Then $\mathcal{F}$ is normal on $B(0,1)$.

For proving the normality, we will mainly use the following lemma.

Lemma 4.1.1 *(Zalcman Lemma)[37]*
Let $\mathcal{F}$ be a family of meromorphic functions on the unit disc $B(0,1)$ such that all zeros of functions in $\mathcal{F}$ have multiplicities greater than or equal to l and all poles of functions in $\mathcal{F}$ have multiplicities greater than or equal to j. Let α be a real number satisfying $-l < \alpha < j$. Then $\mathcal{F}$ is not normal in any neighbourhood of $z_0 \in B(0,1)$ if and only if there exist

1. *points $z_n \in B(0,1)$, $z_n \to z_0$;*
2. *positive numbers ρ_n, $\rho_n \to 0$;*
3. *functions $f_n \in \mathcal{F}$;*
4. *a non-constant meromorphic function g;*

such that $g_n(z) = \rho_n^{\alpha} f_n(z_n + \rho_n z) \to g(z)$ locally uniformly on $\mathbb{C}$ with respect to the spherical metric.

Lemma 4.1.1 holds without any restrictions on the zeros and the poles of all $f \in \mathcal{F}$ for $-1 < \alpha < 1$. However, we can take $-1 < \alpha < \infty$ for a family of analytic functions and we can also take $-\infty < \alpha < 1$ for a family of meromorphic functions which do not vanish [37], i.e. have no zeros.

4.2 Some theorems required for the subsequent results

In Chapter 2, we have proved the following theorem (See Corollary 2.4.3).

Theorem 4.2.1 *Let m, k be two positive integers with $m \geq 2$, and let f be a transcendental meromorphic function. Then $f(f^{(k)})^m - 1$ has infinitely many zeros in $\mathbb{C}$.*

J. Hinchliffe [17] proved the following theorem.

Theorem 4.2.2 *Let f be a transcendental meromorphic function and let $a_k \not\equiv 0$ be a meromorphic function with $T(r, a_k) = S(r, f)$. Let $P[f]$ be a non-constant differential polynomial in f defined by*

$$P[f](z) = \sum_{k=1}^{n} a_k(z) \prod_{j=0}^{p} (f^{(j)}(z))^{S_{k,j}},$$

where the minimum degree of $P[f])$ is given by

$$\underline{d}(P[f]) = \min_{1 \leq k \leq n} \left\{ \sum_{j=0}^{p} S_{k,j} \right\} \geq 2.$$

Let

$$Q = \max_{1 \leq k \leq n} \left\{ \sum_{j=1}^{p} j S_{k,j} \right\}.$$

Then

$$T(r, f) \leq \frac{Q+1}{\underline{d}(P[f]) - 1} \bar{N}(r, \frac{1}{f}) + \frac{1}{\underline{d}(P[f]) - 1} \bar{N}(r, \frac{1}{P[f] - 1}) + S(r, f).$$

Corollary 4.2.1 *Let f be a transcendental meromorphic function with no zeros, and let $P[f]$ be a non-constant differential polynomial in f with $\underline{d}(P[f]) \geq 2$. Then $P[f] - 1$ has infinitely many zeros in $\mathbb{C}$.*

Example 4.2.1 *In this example, we will evaluate $\underline{d}(P[f])$ and Q for given differential polynomials $P[f](z)$, using Theorem 4.2.2. The results are given in Table 4.1.*

We will require the following result on rational functions.

$P[f](z)$	$\underline{d}(P[f])$	Q
$ff'' - f'^3$	2	3
$f''' + f'^2 f''$	1	4

Table 4.1: Evaluating the minimum degree of $P[f]$) and Q.

Theorem 4.2.3 *Let k, m be two positive integers and let g be a non-constant rational function. Then either $g^{(k)} \equiv 0$ or $g(z)(g^{(k)}(z))^m = 1$ has at least one solution $z \in \mathbb{C}$.*

Proof: Assume that $g^{(k)} \not\equiv 0$. Then $g \not\equiv 0$ and so $R(z) = g(z)(g^{(k)}(z))^m$ is not identically zero. We show first that

$$R(\infty) = \alpha, \quad \alpha \in \mathbb{C}\backslash\{0\} \tag{4.1}$$

is impossible. Assuming that (4.1) holds, we may suppose that $\alpha = 1$. Thus $g(z)(g^{(k)}(z))^m \to 1$ as $z \to \infty$. Hence $g(\infty) = \infty$ since if not we would have $g(\infty) \in \mathbb{C}$ and so $g'(\infty) = 0$ and then $(g^{(k)})^m(\infty) = 0$. This gives $g(g^{(k)})^m \to 0$ as $z \to \infty$ which is a contradiction. Since $g(\infty) = \infty$, we get $(g^{(k)})^m(\infty) = 0$. Using the Laurent expansion, we get

$$g(z) = c_n z^n + \cdots + c_0 + \frac{d_1}{z} + \cdots \quad \text{as } z \to \infty,$$

$$g^{(k)}(z) = n(n-1)\cdots(n-k+1)c_n z^{n-k} + \cdots + (-1)^k \frac{k! d_1}{z^{k+1}} + \cdots .$$

But $(g^{(k)})^m(\infty) = 0$, so $g^{(k)}(\infty) = 0$ and this gives

$$(g^{(k)})^m(z) = (-1)^{mk} \frac{(k! d_1)^m}{z^{m(k+1)}} + \cdots \quad \text{as } z \to \infty.$$

Since $g(z)(g^{(k)})^m(z) \to 1$ as $z \to \infty$, we have

$$g(z) = (-1)^{mk} \frac{z^{m(k+1)}}{(k! d_1)^m} + \cdots \quad \text{as } z \to \infty,$$

and this gives as $z \to \infty$, since $m(k+1) > k$,

$$g^{(k)}(z) = \frac{(-1)^{mk}}{(k!d_1)^m}(mk+m)(mk+m-1)\cdots(mk+m-k+1)z^{m(k+1)-k} + \cdots,$$

$$(g^{(k)})^m(z) = \left(\frac{(-1)^{mk}}{(k!d_1)^m}(mk+m)(mk+m-1)\cdots(mk+m-k+1)\right)^m z^{m^2(k+1)-mk} + \cdots.$$

Hence, $(g^{(k)})^m(\infty) = \infty$ which is a contradiction.

Thus (4.1) is impossible. In particular, $R(z)$ cannot be a non-zero constant. Suppose now that $R(z)$ is non-constant, but never takes the value 1 in $\mathbb{C}$. Then $\frac{1}{R-1} \neq \infty$ on $\mathbb{C}$. So

$$\begin{aligned} \frac{1}{R-1} &= P, \quad \text{where } P \text{ is a polynomial} \\ R-1 &= \frac{1}{P} \\ R &= 1+\frac{1}{P}. \end{aligned}$$

But this gives $R(\infty) = 1$, and we have already excluded this case. This completes the proof. □

4.3 The normality when $f(f'+af)^m \neq 1,\ m \geq 2$

Theorem 4.3.1 *Let m be a positive integer with $m \geq 2$, and let $a(z)$ be an analytic function on $B(0,1)$. Suppose that $\mathcal{F}$ is a family of functions meromorphic on $B(0,1)$ such that, for each $f \in \mathcal{F}$, $f(f'+af)^m \neq 1$ on $B(0,1)$. Then $\mathcal{F}$ is normal on $B(0,1)$.*

Proof: Suppose that $\mathcal{F}$ is not normal on $B(0,1)$. So $\mathcal{F}$ is not normal at at least one point in $B(0,1)$, say z_0. Using the Zalcman lemma (Lemma 4.1.1), with $\alpha = \frac{-m}{m+1} \in (-1,1)$, there exist

1. $z_n \in B(0,1)$, $z_n \to z_0$;

2. positive numbers ρ_n, $\rho_n \to 0$;

3. functions $f_n \in \mathcal{F}$;

4. a non-constant meromorphic function g;

such that

$$g_n(z) = \rho_n^{\alpha} f_n(z_n + \rho_n z) \to g(z) \tag{4.2}$$

locally uniformly on $\mathbb{C}$ with respect to the spherical metric. Let $P = g^{-1}(\{\infty\})$ be the set of all poles of g. So, $g_n' \to g'$ locally uniformly on $\mathbb{C}\backslash P$. Using (4.2), we have $g_n'(z) = \rho_n^{\alpha+1} f_n'(z_n + \rho_n z)$. Thus, on $\mathbb{C}\backslash P$,

$$\begin{aligned}
&f_n(z_n + \rho_n z)[f_n'(z_n + \rho_n z) + a(z_n + \rho_n z) f_n(z_n + \rho_n z)]^m \\
&= \rho_n^{-\alpha} g_n(z)[\rho_n^{-\alpha-1} g_n'(z) + a(z_n + \rho_n z)\rho_n^{-\alpha} g_n(z)]^m \\
&= \rho_n^{-\alpha} g_n(z)\rho_n^{-m(\alpha+1)}[g_n'(z) + \rho_n a(z_n + \rho_n z) g_n(z)]^m \\
&= \rho_n^{-\alpha(m+1)-m} g_n(z)[g_n'(z) + \rho_n a(z_n + \rho_n z) g_n(z)]^m \\
&= g_n(z)[g_n'(z) + \rho_n a(z_n + \rho_n z) g_n(z)]^m \\
&\to g(z)[g'(z) + 0.a(z_0) g(z)]^m \\
&= g(z)[g'(z)]^m.
\end{aligned}$$

Using Theorem 4.2.1 and Theorem 4.2.3, we have at least one $\zeta_0 \in \mathbb{C}$ with $g(\zeta_0)(g'(\zeta_0))^m = 1$, and $\zeta_0 \notin P$. Applying the Hurwitz' theorem (Theorem 1.3.2), there exist points $\zeta_n \to \zeta_0$ with

$$g_n(\zeta_n)[g_n'(\zeta_n) + \rho_n a(z_n + \rho_n \zeta_n) g_n(\zeta_n)]^m = 1.$$

Thus

$$f_n(z_n + \rho_n \zeta_n)[f_n'(z_n + \rho_n \zeta_n) + a(z_n + \rho_n \zeta_n) f_n(z_n + \rho_n \zeta_n)]^m = 1.$$

But $z_n + \rho_n\zeta_n \in B(0,1)$ for large n since $z_n \to z_0 \in B(0,1)$ and $\rho_n \to 0$. So we obtain points in $B(0,1)$ at which $f_n(f_n' + af_n)^m = 1$, $f_n \in \mathcal{F}$, which give us a contradiction. Hence, $\mathcal{F}$ is normal. This completes the proof of Theorem 4.3.1. $\square$

Corollary 4.3.1 *Let m be a positive integer with $m \geq 2$. Suppose that $\mathcal{F}$ is a family of functions meromorphic on $B(0,1)$ such that for each $f \in \mathcal{F}$, $f(f')^m \neq 1$ on $B(0,1)$. Then $\mathcal{F}$ is normal on $B(0,1)$.*

We cannot let $a(z)$ be a meromorphic function in Theorem 4.3.1. The counter example is the following.

Example 4.3.1 *Let n be a positive integer, and let $f(z) = \frac{1}{nz}$. This gives $f'(z) = \frac{-1}{nz^2}$. Let $a(z) = \frac{1}{z}$, which is a meromorphic function in $B(0,1)$. From all of this, we get $f' + a(z)f = \frac{-1}{nz^2} + \frac{1}{z}\frac{1}{nz} = \frac{-1}{nz^2} + \frac{1}{nz^2} = 0$. Hence*

$$f(f' + af)^2 = 0 \neq 1 \quad \text{on } B(0,1).$$

However, $\mathcal{F} = \{\frac{1}{nz} : \; n \in \mathbb{N}\}$ is not normal on $B(0,1)$ since the family $\{nz : n \in \mathbb{N}\}$ is not normal (Example 1.3.2).

4.4 The normality when $P[f] \neq 1$, $f \neq 0$

Lemma 4.4.1 *Let g be a non constant rational function such that $g(z)$ is never 0 in the plane. Let*

$$Q(\zeta) = \prod_{j=0}^{p} g^{(j)}(\zeta)^{S_{\delta,j}}, \quad S_{\delta,j} \geq 0, \quad \sum_{j=0}^{p} S_{\delta,j} \geq 1.$$

Let $\alpha \in \mathbb{C}\backslash\{0\}$. Then Q takes the value α at least once in $\mathbb{C}$.

Proof: We have given that g is never 0. This gives $g(\infty) = 0$ and so $g^{(j)}(\infty) = 0$. Thus $Q(\infty) = 0$. So provided Q is not constant, Q takes the value $\alpha \in \mathbb{C}\backslash\{0\}$. If Q is constant, $Q \equiv 0$. Hence, $g^{(j)} \equiv 0$ for some j. This gives either g is constant (for $j = 1$) or g takes the value 0. This is a contradiction and hence Lemma 4.4.1 is proved. □

Theorem 4.4.1 *Let n, p be two positive integers and let $T = \{1, 2, \cdots, n\}$. For each $k \in T$ and each $j \in \{0, 1, \cdots, p\}$, let $S_{k,j}$ be a non-negative integer. For each $k \in T$ let α_k be the solution of the following equation:*

$$\sum_{j=0}^{p} S_{k,j}(-\alpha_k - j) = 0. \tag{4.3}$$

Assume that there is a unique $\delta \in T$ such that $\alpha_\delta < \alpha_k$ for all $k \in T\backslash\{\delta\}$, and assume that

$$\begin{aligned} \sum_{j=0}^{p} S_{\delta,j} &\geq 2, \\ \sum_{j=0}^{p} S_{k,j} &\geq 1 \quad \forall\, k \in T\backslash\{\delta\}. \end{aligned} \tag{4.4}$$

For each $k \in T$ let $a_k(z)$ be an analytic function on $B(0,1)$, and assume further that $a_\delta(z)$ has no zeros in $B(0,1)$. Let $\mathcal{F}$ be a family of functions meromorphic on $B(0,1)$ such that, for each $f \in \mathcal{F}$, f has no zeros in $B(0,1)$ and the function $P[f]$ defined by

$$P[f](z) = \sum_{k=1}^{n} a_k(z) \prod_{j=0}^{p} (f^{(j)}(z))^{S_{k,j}}$$

does not take the value 1 in $B(0,1)$. Then $\mathcal{F}$ is normal on $B(0,1)$.

Example 4.4.1 *In this example, we evaluate α_k for a given differential polynomial $P[f]$, using (4.3). The results are given in Table 4.2.*

$P[f]$	α_1	α_2	α_3	α_δ
$ff''+f'^2$	-1	-1	0	-1
$f^2f'+f'''^2-f'^3f''^4$	$\frac{-1}{3}$	-3	$\frac{-11}{7}$	-3

Table 4.2: Examples on α_k.

Proof: Suppose that $\mathcal{F}$ is not normal on $B(0,1)$. So $\mathcal{F}$ is not normal at at least one point on $B(0,1)$, say z_0. We note that $\alpha_\delta \leq 0$ by (4.3). Applying the Zalcman lemma (Lemma 4.1.1), with $\alpha = \alpha_\delta$, we find that there exist

1. $z_n \in B(0,1)$, $z_n \to z_0$;
2. positive numbers ρ_n, $\rho_n \to 0$;
3. functions $f_n \in \mathcal{F}$;
4. a non-constant function g meromorphic in $\mathbb{C}$;

such that

$$g_n(z) = \rho_n^\alpha f_n(z_n + \rho_n z) \to g(z) \tag{4.5}$$

locally uniformly on $\mathbb{C}$ with respect to the spherical metric. Since g is non-constant and each g_n omits 0 on $B(0,1)$, it follows from the Hurwitz' theorem (Theorem 1.3.2) that g omits the value 0 on $\mathbb{C}$. Let $R = g^{-1}(\{\infty\})$ be the set of all poles of g. Thus, $g_n^{(j)} \to g^{(j)}$ locally uniformly on $\mathbb{C}\backslash R$, for $j = 1, 2, \cdots, p$. Using (4.5), we have

$$g_n^{(j)}(z) = \rho_n^{\alpha+j} f_n^{(j)}(z_n + \rho_n z), \quad j = 1, 2, \cdots, p.$$

Hence, locally uniformly on $\mathbb{C}\backslash R$, using (4.3) with $k = \delta$,

$$\sum_{k=1}^{n} a_k(z_n + \rho_n z) \prod_{j=0}^{p} (f_n^{(j)}(z_n + \rho_n z))^{S_{k,j}}$$

$$= \sum_{k=1}^{n} a_k(z_n + \rho_n z) \prod_{j=0}^{p} \rho_n^{S_{k,j}(-\alpha-j)} (g_n^{(j)}(z))^{S_{k,j}}$$

$$= \sum_{k=1}^{n} a_k(z_n + \rho_n z) \rho_n^{\sum_{j=0}^{p} S_{k,j}(-\alpha-j)} \prod_{j=0}^{p} (g_n^{(j)}(z))^{S_{k,j}}$$

$$= a_\delta(z_n + \rho_n z) \prod_{j=0}^{p} (g_n^{(j)}(z))^{S_{\delta,j}} + \sum_{k\in T,\, k\neq\delta} a_k(z_n + \rho_n z) \rho_n^{\sum_{j=0}^{p} S_{k,j}(-\alpha_\delta-j)} \prod_{j=0}^{p} \left(g_n^{(j)}(z)\right)^{S_{k,j}}$$

$$\to a_\delta(z_0) \prod_{j=0}^{p} \left(g^{(j)}(z)\right)^{S_{\delta,j}} = G(z)$$

since $\rho_n \to 0$, $\quad z_n \to z_0 \in B(0,1)$, each a_k is analytic on $B(0,1)$ and for $k \neq \delta$

$$\sum_{j=0}^{p} S_{k,j}(-\alpha_\delta - j) = \sum_{j=0}^{p} S_{k,j}(\alpha_k - \alpha_\delta) > 0,$$

using (4.3) and (4.4).

We note that if g is transcendental then G cannot be constant. For if G is constant then clearly $G \neq 0$ since g is transcendental. Assuming that G is a non-zero constant we see that g has no poles. Let m be the largest integer such that $S_{\delta,m} \neq 0$. Then we can write

$$(g^{(m)})^{\sum_{j=0}^{p} S_{\delta,j}} = \frac{G}{a_\delta(z_0)} \prod_{0\leq j\leq m} \left(\frac{g^{(m)}}{g^{(j)}}\right)^{S_{\delta,j}}$$

which gives

$$T(r, g^{(m)}) = m(r, g^{(m)}) = S(r, f)$$

by the lemma of the logarithmic derivative. This contradicts the Hayman-Miles theorem (Theorem 1.6.3).

Using Corollary 4.2.1 and Lemma 4.4.1, we have at least one $\zeta_0 \in \mathbb{C}$ with

$$a_\delta(z_0)\prod_{j=0}^{p}\left(g^{(j)}(\zeta_0)\right)^{S_{\delta,j}} = 1,$$

and $\zeta_0 \notin R$. Applying the Hurwitz' theorem, we get points $\zeta_n \to \zeta_0$ with

$$a_\delta(z_n+\rho_n\zeta_n)\prod_{j=0}^{p}\left(g_n^{(j)}(\zeta_n)\right)^{S_{\delta,j}} + \sum_{k\in T,\ k\neq\delta} a_k(z_n+\rho_n\zeta_n)\rho_n^{\sum_{j=0}^{p} S_{k,j}(-\alpha_\delta - j)}\prod_{j=0}^{p}\left(g_n^{(j)}(\zeta_n)\right)^{S_{k,j}} = 1.$$

Thus

$$\sum_{k=1}^{n} a_k(z_n+\rho_n\zeta_n)\prod_{j=0}^{p}\left(f_n^{(j)}(z_n+\rho_n\zeta_n)\right)^{S_{k,j}} = 1.$$

But $z_n + \rho_n\zeta_n \in B(0,1)$ since $z_n \to z_0 \in B(0,1)$ and $\rho_n \to 0$. So, we have functions $f_n \in \mathcal{F}$ such that $P[f_n] = \sum_{k=1}^{n} a_k \prod_{j=0}^{p}\left(f_n^{(j)}\right)^{S_{kj}}$ takes the value 1 on $B(0,1)$. This gives a contradiction and so $\mathcal{F}$ is normal. This completes the proof of Theorem 4.4.1 □

Corollary 4.4.1 *Let k, m be positive integers, and f be a meromorphic function. Let $\mathcal{F}$ be a family of functions meromorphic on $B(0,1)$ such that for each $f \in \mathcal{F}$, we have $f \neq 0$, $f(f^{(k)})^m \neq 1$ on $B(0,1)$. Then $\mathcal{F}$ is normal on $B(0,1)$.*

We cannot omit the condition that $a_\delta(z)$ have no zeros in Theorem 4.4.1. The counter example is the following.

Example 4.4.2 *Let n denote a positive integer, and let $P[f](z) = a(z)f(z)f'(z)$, where $f(z) = \frac{1}{nz} \neq 0$ on $B(0,1)$. This gives $f'(z) = \frac{-1}{nz^2}$. Let $a(z) = \frac{z^3}{z^4+10}$. From* *all of these, we get*

$$\begin{aligned} P[f](z) &= a(z)f(z)f'(z) \\ &= \frac{z^3}{z^4+10}\frac{1}{nz}\frac{-1}{nz^2} \\ &= \frac{-1}{n^2(z^4+10)} \\ &\neq 1 \quad on \ \ B(0,1). \end{aligned}$$

However, $\mathcal{F} = \{\frac{1}{nz} : \quad n \in \mathbb{N}\}$ *is not normal on* $B(0,1)$.

Furthermore, we cannot allow $a(z)$ to have poles in $B(0,1)$ in Theorem 4.4.1. The counter example is the following.

Example 4.4.3 *Let* n *be a positive integer and let* $f \in \mathcal{F} = \{nz : \quad n \geq 2\}$. *Suppose that* $a(z) = \frac{1}{z}$. *This means that* 0 *is a pole of* $a(z)$. *So*

$$\begin{aligned} P[f](z) &= a(z)f(z)f'(z) \\ &= \frac{1}{z}.nz.n \\ &= n^2 \\ &\neq 1 \quad on \ \ B(0,1). \end{aligned}$$

However, $\mathcal{F} = \{nz : \quad n \geq 2\}$ *is not normal on* $B(0,1)$.

Also, we cannot ignore the condition that δ is unique in Theorem 4.4.1. The counter example is the following.

Example 4.4.4 *Let* n *be a positive integer, and let* $P[f](z) = f(z)f''(z) - 2(f'(z))^2$, *where* $f(z) = \frac{1}{nz} \neq 0$. *This gives* $f'(z) = \frac{-1}{nz^2}$, *and* $f''(z) = \frac{2}{nz^3}$.

From all of these, we get

$$\begin{aligned} P[f](z) &= f(z)f''(z) - 2(f'(z))^2 \\ &= \frac{2}{n^2z^4} - \frac{2}{n^2z^4} \\ &= 0 \\ &\neq 1 \quad \text{on } B(0,1). \end{aligned}$$

However, $\mathcal{F} = \{\frac{1}{nz} : \quad n \in \mathbb{N}\}$ *is not normal on* $B(0,1)$, *and* $\alpha_1 = \alpha_2 = -1$. *Hence there is not a unique* δ.

Chapter 5

On Bank-Laine Functions

[1] In this chapter, we will consider the following. Suppose that (a_n) is a sequence of distinct complex numbers such that the imaginary part of a_n is very close to zero and such that for $\epsilon > 0$, we have $|a_m - a_n| > \epsilon|a_n| > 0 \quad \forall\, n \neq m$. Then (a_n) is not a Bank-Laine sequence. Also, we will prove a new result concerning the Bank-Laine functions.

5.1 Introduction

Let A be an entire function, and let f_1, f_2 be two linearly independent solutions of

$$y'' + A(z)y = 0, \tag{5.1}$$

normalised so that the Wronskian, as defined in Proposition 1.4.4,

$$W = W(f_1, f_2) = f_1 f_2' - f_1' f_2 \tag{5.2}$$

[1]We submitted this chapter to be published as a paper in Journal of Mathematical Analysis and Applications.

satisfies $W = 1$. Then $E = f_1 f_2$ is a Bank-Laine function as in Definition 1.5.1 and satisfies the following Bank-Laine product formula [5]

$$4A = \left(\frac{E'}{E}\right)^2 - 2\frac{E''}{E} - \frac{1}{E}. \tag{5.3}$$

Conversely, if E is a Bank-Laine function then [6] the function A which is defined by (5.3) is entire and the function E is the product of linearly independent normalised solutions of (5.1).

Example 5.1.1 *$E(z) = \sin z$ is a Bank-Laine function and we may write*

$$\sin z = 2 \sin \frac{z}{2} \cos \frac{z}{2}.$$

It is very easy to see that $f_1 = \sqrt{2} \cos \frac{z}{2}$ and $f_2 = \sqrt{2} \sin \frac{z}{2}$ are normalised solutions of

$$y'' + \frac{1}{4} y = 0.$$

We recall from Theorem 1.5.2 that any sequence (a_n) tending to infinity without repetition is the zero sequence of a Bank-Laine function and we refer the reader to the definition of a Bank-Laine sequence (see Definition 1.5.2). In 1999, Elzaidi [12] proved the following theorem.

Theorem 5.1.1 *Suppose that λ is a positive real constant with $\lambda > 1$, and suppose that (a_n) is a sequence of distinct real numbers such that for large n we have $a_{n+1} > \lambda a_n > 0$. Then (a_n) is not a Bank-Laine sequence.*

In 2001, Jim Langley [25] gave an example of a Bank-Laine sequence (a_n) such that $|a_{n+1}| > \lambda |a_n|$ where λ is a positive real constant with $\lambda > 1$. In 1999, Jim Langley [24] proved the following theorem.

Theorem 5.1.2 *Let K and M be two positive real constants with $K > 1$. Let A be a transcendental entire function of finite order, and let $E = f_1 f_2$ be the*

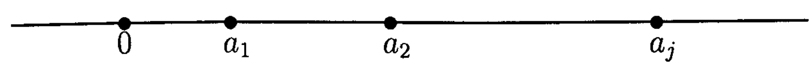

Figure 5.1: An example of a sequence (a_n) in Theorem 5.1.1.

product of two linearly independent solutions of the equation (5.1) normalised so that the Wronskian satisfies $W = 1$. Suppose that there exists a positive sequence r_m tending to infinity, such that for each large positive integer m, the number of zeros of E in the annulus

$$\Omega(r_m, K) = \{z \in \mathbb{C} : \frac{r_m}{K} < |z| < Kr_m\} \tag{5.4}$$

is at most r_m^M. Suppose that

$$\limsup_{m\to\infty} \frac{\log r_{m+1}}{\log r_m} < \infty.$$

Then E has finite order.

In 1999, Elzaidi [12] proved the following theorem.

Theorem 5.1.3 *Let (a_n) be a Bank-Laine sequence. Then there is a positive constant M such that for all large n we have*

$$|a_n - a_m| \geq \exp(-|a_n|^M) \quad \forall\, m \neq n.$$

5.2 Generalising Theorem 5.1.1

Theorem 5.2.1 *Let λ be a positive real number with $\lambda > 1$, and let $\phi : [0, \infty) \to [0, \infty)$ be a function which satisfies*

$$\lim_{x\to\infty} x^n \phi(x) = 0 \quad \forall\, n \in \mathbb{N}. \tag{5.5}$$

Suppose that (a_n) is a sequence of distinct complex numbers with, for all large n,

$$|a_{n+1}| > \lambda |a_n| > 0 \tag{5.6}$$

$$|Im(a_n)| < \phi(|a_n|). \tag{5.7}$$

Then (a_n) is not a Bank-Laine sequence.

Proof: Suppose that (a_n) is a Bank-Laine sequence. Hence there is an entire function E of finite order with zero sequence (a_n), which satisfies the Bank-Laine property. We can relabel those a_n which satisfy (5.6) as $a_1, a_2, \cdots$ and then there exist polynomials $Q(z)$ and $P(z)$, such that $P(z)$ has simple zeros, and such that

$$E(z) = P(z)g(z)e^{Q(z)}. \tag{5.8}$$

$$g(z) = \prod_{j=1}^{\infty}\left(1 - \frac{z}{a_j}\right), \tag{5.9}$$

where each a_n in (5.9) satisfies (5.6). Using (5.8), we get

$$E'(z) = P'(z)g(z)e^{Q(z)} + P(z)g'(z)e^{Q(z)} + P(z)g(z)Q'(z)e^{Q(z)}.$$

At each a_m we have, using (5.8) and (5.9),

$$E'(a_m) = P(a_m)g'(a_m)e^{Q(a_m)} = \pm 1. \tag{5.10}$$

Now let us apply some analysis to $g'(a_m)$. Using (5.9), we have

$$\begin{aligned} g(z) &= \prod_{j=1}^{\infty}\left(1 - \frac{z}{a_j}\right) \\ &= \left(1 - \frac{z}{a_m}\right)\prod_{j\neq m}\left(1 - \frac{z}{a_j}\right). \end{aligned}$$

Therefore

$$g'(a_m) = \frac{-1}{a_m}\prod_{j\neq m}\left(1 - \frac{a_m}{a_j}\right). \tag{5.11}$$

Let

$$B_0 = \prod_{j=1}^{\infty}\left(1 - \frac{1}{\lambda^j}\right) \tag{5.12}$$

$$B_1 = \prod_{j=1}^{\infty}\left(1 + \frac{1}{\lambda^j}\right). \tag{5.13}$$

Both of these products, which are in (5.12) and (5.13), converge and

$$0 < B_0 < 1 < B_1. \tag{5.14}$$

Now let us rewrite the following

$$\begin{aligned}\left|\prod_{j\neq m}\left(1 - \frac{a_m}{a_j}\right)\right| &= \left|\prod_{j<m}\left(1 - \frac{a_m}{a_j}\right)\right|\left|\prod_{j>m}\left(1 - \frac{a_m}{a_j}\right)\right| \\ &= \left|\prod_{j<m}\frac{a_m}{a_j}\left(\frac{a_j}{a_m} - 1\right)\right|\left|\prod_{j>m}\left(1 - \frac{a_m}{a_j}\right)\right| \\ &= \prod_{j<m}\frac{|a_m|}{|a_j|}\prod_{j<m}\left|1 - \frac{a_j}{a_m}\right|\prod_{j>m}\left|1 - \frac{a_m}{a_j}\right|. \end{aligned} \tag{5.15}$$

For $j > m$, we have

$$1 - \frac{|a_m|}{|a_j|} \leq \left|1 - \frac{a_m}{a_j}\right| \leq 1 + \frac{|a_m|}{|a_j|}$$

and this gives, using (5.6),

$$1 - \frac{1}{\lambda^{j-m}} \leq \left|1 - \frac{a_m}{a_j}\right| \leq 1 + \frac{1}{\lambda^{j-m}}.$$

So

$$\prod_{j>m}\left(1 - \frac{1}{\lambda^{j-m}}\right) \leq \prod_{j>m}\left|1 - \frac{a_m}{a_j}\right| \leq \prod_{j>m}\left(1 + \frac{1}{\lambda^{j-m}}\right)$$

$$\prod_{j=m+1}^{\infty}\left(1 - \frac{1}{\lambda^{j-m}}\right) \leq \prod_{j>m}\left|1 - \frac{a_m}{a_j}\right| \leq \prod_{j=m+1}^{\infty}\left(1 + \frac{1}{\lambda^{j-m}}\right)$$

$$\prod_{j=1}^{\infty}\left(1 - \frac{1}{\lambda^j}\right) \leq \prod_{j>m}\left|1 - \frac{a_m}{a_j}\right| \leq \prod_{j=1}^{\infty}\left(1 + \frac{1}{\lambda^j}\right).$$

Hence, using (5.12) and (5.13),

$$B_0 \leq \prod_{j>m} \left| 1 - \frac{a_m}{a_j} \right| \leq B_1. \tag{5.16}$$

Using (5.13), we get

$$\begin{aligned} \prod_{j<m} \left| 1 - \frac{a_j}{a_m} \right| &\leq \prod_{j<m} \left(1 + \frac{|a_j|}{|a_m|} \right) \\ &\leq \prod_{j=1}^{m-1} \left(1 + \frac{1}{\lambda^j} \right) \\ &\leq B_1. \end{aligned} \tag{5.17}$$

Using (5.12), we get

$$\begin{aligned} \prod_{j<m} \left| 1 - \frac{a_j}{a_m} \right| &\geq \prod_{j<m} \left(1 - \frac{|a_j|}{|a_m|} \right) \\ &\geq \prod_{j=1}^{m-1} \left(1 - \frac{1}{\lambda^j} \right) \\ &\geq B_0. \end{aligned} \tag{5.18}$$

Using (5.11), (5.15), (5.16), (5.17) and (5.18), we have

$$|g'(a_m)| \leq \frac{1}{|a_m|} \prod_{j<m} \frac{|a_m|}{|a_j|} B_1^2,$$

$$|g'(a_m)| \geq \frac{1}{|a_m|} \prod_{j<m} \frac{|a_m|}{|a_j|} B_0^2.$$

Hence, using the last two inequalities,

$$\begin{aligned} \log |g'(a_m)| &= -\log |a_m| + \sum_{j<m} \log \frac{|a_m|}{|a_j|} + O(1) \\ &= -\log |a_m| + N(|a_m|, \frac{1}{g}) + O(1) \\ &= N(|a_m|, \frac{1}{g}) + O(\log |a_m|). \end{aligned} \tag{5.19}$$

Now let us estimate $N(r, \frac{1}{g})$.

1. The lower bound

$$\begin{aligned} N(r, \frac{1}{g}) &\geq \int_{r^{\frac{1}{2}}}^{r} n(t, \frac{1}{g}) \frac{dt}{t} \\ &\geq n(r^{\frac{1}{2}}, \frac{1}{g}) \int_{r^{\frac{1}{2}}}^{r} \frac{dt}{t} \\ &= \frac{1}{2} n(r^{\frac{1}{2}}, \frac{1}{g}) \log r. \end{aligned}$$

Therefore

$$\frac{N(r, \frac{1}{g})}{\log r} \to \infty \quad \text{as}\, r \to \infty.$$

This means that

$$\log r = o\left(N(r, \frac{1}{g})\right) \quad \text{as}\, r \to \infty. \tag{5.20}$$

2. The upper bound

For large m and

$$|a_m| \leq r < |a_{m+1}|, \tag{5.21}$$

we have $n(r, \frac{1}{g}) = m$. Using (5.6), we have

$$|a_m| > \lambda^{m-1} |a_1|$$

which gives, for r as in (5.21),

$$(m-1) \log \lambda + \log |a_1| < \log |a_m| \leq \log r,$$

and so

$$m < \frac{\log r - \log |a_1|}{\log \lambda} + 1.$$

Therefore

$$n(r, \frac{1}{g}) \leq O(\log r) \quad \text{as}\, r \to \infty.$$

Using the definition of $N(r, \frac{1}{g})$, we have

$$N(r, \frac{1}{g}) \leq \int_{|a_1|}^{r} n(r, \frac{1}{g}) \frac{dt}{t}$$
$$\leq n(r, \frac{1}{g}) \log \frac{r}{|a_1|}.$$

Hence

$$N(r, \frac{1}{g}) \leq O(\log r)^2 \quad \text{as}\, r \to \infty. \tag{5.22}$$

Taking the absolute value and logarithms in (5.10), we get

$$\log |P(a_m)| + \log |g'(a_m)| + Re(Q(a_m)) = 0. \tag{5.23}$$

Using (5.19), (5.20) and the fact that $\log |P(a_m)| = O(\log |a_m|)$, we get

$$-Re(Q(a_m)) = (1 + o(1)) N(|a_m|, \frac{1}{g}). \tag{5.24}$$

Returning to the polynomial $Q(z)$ which is in (5.8), we can write $Q(z)$ as

$$Q(z) = \sum_{k=0}^{q} c_k z^k, \quad c_k \in \mathbb{C}.$$

Letting $a_m = x_m + iy_m$, we get

$$Q(a_m) = \sum_{k=0}^{q} c_k (x_m + iy_m)^k$$
$$= \sum_{k=0}^{q} c_k \sum_{n=0}^{k} \binom{k}{n} x_m^{k-n} (iy_m)^n. \tag{5.25}$$

So every term in $Q(a_m)$ includes y_m except when $n = 0$. Using (5.5) and (5.7), we have $y_m \to 0$ as $m \to \infty$ and, for $1 \leq n \leq k$,

$$|x_m^{k-n} (iy_m)^n| \leq |a_m|^{k-n} |y_m|^n$$
$$\leq |a_m|^{k-n} \phi(|a_m|)^n$$
$$\to 0. \tag{5.26}$$

Using (5.25) and (5.26), we have

$$Q(a_m) = \sum_{k=0}^{q} c_k \left[x_m^k + \sum_{n=1}^{k} \binom{k}{n} x_m^{k-n} (iy_m)^n \right]$$
$$= \sum_{k=0}^{q} c_k x_m^k + o(1).$$

Thus, as $m \to \infty$ we have,

$$-Re(Q(a_m)) = \sum_{k=0}^{q} d_k x_m^k + o(1), \quad d_k = -Re(c_k). \tag{5.27}$$

Using (5.20), (5.24) and (5.27) we see that $\sum_{k=0}^{q} d_k x_m^k$ must be a non-constant polynomial in x_m. So it grows like a power of x_m and this contradicts the upper bound for $N(|a_m|, \frac{1}{g})$, which is in (5.22). This completes the proof of Theorem 5.2.1. □

It is very easy to see that Theorem 5.1.1 comes at once from Theorem 5.2.1. The sequence (a_n) could be as in Figure 5.2.

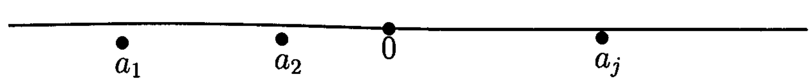

Figure 5.2: An example of a sequence (a_n) in Theorem 5.2.1.

Corollary 5.2.1 *Suppose that λ is a positive real constant with $\lambda > 1$, and suppose that (a_n) is a sequence of distinct non-zero real numbers such that for large n we have*

$$|a_{n+1}| > \lambda |a_n| > 0.$$

Then (a_n) is not a Bank-Laine sequence.

5.3 A refinement of Theorem 5.2.1

Theorem 5.3.1 *Let $\epsilon > 0$ and let $\phi : [0,\infty) \to [0,\infty)$ be a function which satisfies*

$$\lim_{x\to\infty} x^n \phi(x) = 0 \quad \forall n \in \mathbb{N}. \tag{5.28}$$

Suppose that (a_n) is a sequence of distinct complex numbers tending to infinity with, for all large n,

$$|a_m - a_n| > \epsilon |a_n| > 0 \quad \forall\, m \neq n \tag{5.29}$$

$$|Im(a_n)| < \phi(|a_n|). \tag{5.30}$$

Then (a_n) is not a Bank-Laine sequence.

Proof: We may assume that ϵ is small. Suppose that (a_n) is a Bank-Laine sequence. Hence there is an entire function E of finite order with zero sequence (a_n), which satisfies the Bank-Laine property. We can relabel those a_n which satisfy (5.29) and (5.30) as $a_1, a_2, \cdots$ and then there exist polynomials $P(z)$ and $Q(z)$, such that $P(z)$ has simple zeros, and such that

$$E(z) = P(z)g(z)e^{Q(z)} \tag{5.31}$$

where

$$g(z) = \prod_{j=1}^{\infty} \left(1 - \frac{z}{a_j}\right). \tag{5.32}$$

Using (5.31), we have

$$E'(z) = P'(z)g(z)e^{Q(z)} + P(z)g'(z)e^{Q(z)} + P(z)g(z)Q'(z)e^{Q(z)}.$$

Hence at each a_m we have, using (5.32),

$$E'(a_m) = P(a_m)g'(a_m)e^{Q(a_m)} = \pm 1. \tag{5.33}$$

Now let us apply some analysis to $g'(a_m)$. Using (5.32), we have

$$\begin{aligned} g(z) &= \prod_{j=1}^{\infty}\left(1-\frac{z}{a_j}\right) \\ &= \left(1-\frac{z}{a_m}\right)\prod_{j\neq m}\left(1-\frac{z}{a_j}\right). \end{aligned}$$

Therefore

$$g'(a_m) = \frac{-1}{a_m}\prod_{j\neq m}\left(1-\frac{a_m}{a_j}\right). \tag{5.34}$$

Let

$$\lambda = 1+\frac{\epsilon}{4} \tag{5.35}$$

$$B_0 = \prod_{j=1}^{\infty}\left(1-\frac{1}{\lambda^j}\right) \tag{5.36}$$

$$B_1 = \prod_{j=1}^{\infty}\left(1+\frac{1}{\lambda^j}\right). \tag{5.37}$$

Both of these products, which are in (5.36) and (5.37), converge and

$$0 < B_0 < 1 < B_1. \tag{5.38}$$

Now let us rewrite the following

$$\begin{aligned} \left|\prod_{j\neq m}\left(1-\frac{a_m}{a_j}\right)\right| &= \left|\prod_{j<m}\left(1-\frac{a_m}{a_j}\right)\right|\left|\prod_{j>m}\left(1-\frac{a_m}{a_j}\right)\right| \\ &= \left|\prod_{j<m}\frac{a_m}{a_j}\left(\frac{a_j}{a_m}-1\right)\right|\left|\prod_{j>m}\left(1-\frac{a_m}{a_j}\right)\right| \\ &= \prod_{j<m}\frac{|a_m|}{|a_j|}\prod_{j<m}\left|1-\frac{a_j}{a_m}\right|\prod_{j>m}\left|1-\frac{a_m}{a_j}\right|. \end{aligned} \tag{5.39}$$

Lemma 5.3.1 *Suppose that $a_1, a_2, \cdots$ satisfy (5.29) and (5.30). Then for large m there are at most two a_j such that*

$$\left(1-\frac{\epsilon}{4}\right)|a_m| \leq |a_j| \leq \left(1+\frac{\epsilon}{4}\right)|a_m|. \tag{5.40}$$

Proof: Suppose that there are three a_j satisfying (5.40), say a_α, a_β and a_γ. Then, using (5.29) and (5.30), either at least two lie near $\mathbb{R}^+$ or at least two lie near $\mathbb{R}^-$. Suppose that a_α and a_β lie near $\mathbb{R}^+$. Then we have, using (5.30) and (5.40),

$$\begin{aligned} |a_\alpha - a_\beta| &\leq \frac{\epsilon}{2}|a_m| + o(1) \\ &< \epsilon|a_\alpha|. \end{aligned}$$

This contradicts (5.29) and therefore Lemma 5.3.1 is proved. □

Relabelling (a_m) if necessary, we may assume that

$$|a_m| \leq |a_{m+1}| \leq |a_{m+2}| \leq \cdots$$

Using (5.35) and Lemma 5.3.1, we get for large m

$$\begin{aligned} &|a_{m+1}| \geq |a_m| \\ &|a_{m+2}| \geq \lambda|a_m| \\ &|a_{m+3}| \geq |a_{m+2}| \geq \lambda|a_m| \\ &|a_{m+4}| \geq \lambda|a_{m+2}| \geq \lambda^2|a_m|. \end{aligned}$$

Hence, if m is large and $j \in \mathbb{N}$,

$$|a_{m+j}| \geq \lambda^{[\frac{j}{2}]}|a_m|. \tag{5.41}$$

We may assume that (5.41) holds for all $m, j \in \mathbb{N}$, by otherwise incorporating finitely many terms from g into P and relabelling.

For $j > m$, we have

$$1 - \frac{|a_m|}{|a_j|} \leq \left|1 - \frac{a_m}{a_j}\right| \leq 1 + \frac{|a_m|}{|a_j|}$$

and this gives, using (5.41),

$$1-\frac{1}{\lambda^{[\frac{j-m}{2}]}} \leq \left|1-\frac{a_m}{a_j}\right| \leq 1+\frac{1}{\lambda^{[\frac{j-m}{2}]}}. \tag{5.42}$$

Now let us rewrite the following

$$\begin{aligned}\prod_{j>m}\left|1-\frac{a_m}{a_j}\right| &= \prod_{j=m+1}^{\infty}\left|1-\frac{a_m}{a_j}\right| \\ &= \left|1-\frac{a_m}{a_{m+1}}\right| \prod_{j=m+2}^{\infty}\left|1-\frac{a_m}{a_j}\right|. \end{aligned} \tag{5.43}$$

Using (5.29) and (5.41), we have

$$1+\frac{|a_m|}{|a_{m+1}|} \geq \left|1-\frac{a_m}{a_{m+1}}\right| = \frac{|a_{m+1}-a_m|}{|a_{m+1}|}$$

$$2 \geq \left|1-\frac{a_m}{a_{m+1}}\right| > \frac{\epsilon|a_{m+1}|}{|a_m|}$$

$$2 \geq \left|1-\frac{a_m}{a_{m+1}}\right| > \epsilon. \tag{5.44}$$

Using (5.36), (5.37) and (5.42), we have

$$\prod_{j=m+2}^{\infty}\left(1-\frac{1}{\lambda^{[\frac{j-m}{2}]}}\right) \leq \prod_{j=m+2}^{\infty}\left|1-\frac{a_m}{a_j}\right| \leq \prod_{j=m+2}^{\infty}\left(1+\frac{1}{\lambda^{[\frac{j-m}{2}]}}\right)$$

$$\prod_{j=1}^{\infty}\left(1-\frac{1}{\lambda^j}\right)^2 \leq \prod_{j=m+2}^{\infty}\left|1-\frac{a_m}{a_j}\right| \leq \prod_{j=1}^{\infty}\left(1+\frac{1}{\lambda^j}\right)^2$$

$$B_0^2 \leq \prod_{j=m+2}^{\infty}\left|1-\frac{a_m}{a_j}\right| \leq B_1^2. \tag{5.45}$$

Using (5.43), (5.44) and (5.45), we now have

$$\epsilon B_0^2 \leq \prod_{j>m}\left|1-\frac{a_m}{a_j}\right| \leq 2B_1^2. \tag{5.46}$$

For $j < m$, we have

$$\prod_{j<m}\left|1-\frac{a_j}{a_m}\right| = \prod_{k=1}^{m-1}\left|1-\frac{a_{m-k}}{a_m}\right|$$
$$= \left|1-\frac{a_{m-1}}{a_m}\right|\prod_{k=2}^{m-1}\left|1-\frac{a_{m-k}}{a_m}\right|. \tag{5.47}$$

Using (5.29) as in (5.44), we have

$$1+\frac{|a_{m-1}|}{|a_m|} \geq \left|1-\frac{a_{m-1}}{a_m}\right| = \frac{|a_m - a_{m-1}|}{|a_m|}$$

$$2 \geq \left|1-\frac{a_{m-1}}{a_m}\right| > \epsilon. \tag{5.48}$$

Using (5.36), (5.37) and (5.41), we have

$$\prod_{k=2}^{m-1}\left(1-\frac{|a_{m-k}|}{|a_m|}\right) \leq \prod_{k=2}^{m-1}\left|1-\frac{a_{m-k}}{a_m}\right| \leq \prod_{k=2}^{m-1}\left(1+\frac{|a_{m-k}|}{|a_m|}\right)$$

$$\prod_{k=2}^{m-1}\left(1-\frac{1}{\lambda^{[\frac{k}{2}]}}\right) \leq \prod_{k=2}^{m-1}\left|1-\frac{a_{m-k}}{a_m}\right| \leq \prod_{k=2}^{m-1}\left(1+\frac{1}{\lambda^{[\frac{k}{2}]}}\right)$$

$$\prod_{j=1}^{\infty}\left(1-\frac{1}{\lambda^j}\right)^2 \leq \prod_{k=2}^{m-1}\left|1-\frac{a_{m-k}}{a_m}\right| \leq \prod_{j=1}^{\infty}\left(1+\frac{1}{\lambda^j}\right)^2$$

$$B_0^2 \leq \prod_{k=2}^{m-1}\left|1-\frac{a_{m-k}}{a_m}\right| \leq B_1^2. \tag{5.49}$$

Using (5.47), (5.48) and (5.49), we thus have

$$\epsilon B_0^2 \leq \prod_{j<m}\left|1-\frac{a_j}{a_m}\right| \leq 2B_1^2. \tag{5.50}$$

Using (5.34), (5.39), (5.46) and (5.50), we now obtain

$$|g'(a_m)| \leq \frac{4}{|a_m|}B_1^4\prod_{j<m}\frac{|a_m|}{|a_j|},$$

$$|g'(a_m)| \geq \frac{\epsilon^2}{|a_m|} B_0^4 \prod_{j<m} \frac{|a_m|}{|a_j|}.$$

Using the last two inequalities, we get

$$\begin{aligned}\log|g'(a_m)| &= -\log|a_m| + \sum_{j<m} \log\frac{|a_m|}{|a_j|} + O(1) \\ &= -\log|a_m| + N(|a_m|, \frac{1}{g}) + O(1) \\ &= N(|a_m|, \frac{1}{g}) + O(\log|a_m|). \end{aligned} \tag{5.51}$$

Now let us estimate $N(r, \frac{1}{g})$.

1. The lower bound

$$\begin{aligned} N(r, \frac{1}{g}) &\geq \int_{r^{\frac{1}{2}}}^{r} n(t, \frac{1}{g}) \frac{dt}{t} \\ &\geq n(r^{\frac{1}{2}}, \frac{1}{g}) \int_{r^{\frac{1}{2}}}^{r} \frac{dt}{t} \\ &= \frac{1}{2} n(r^{\frac{1}{2}}, \frac{1}{g}) \log r. \end{aligned}$$

Therefore

$$\frac{N(r, \frac{1}{g})}{\log r} \to \infty \quad \text{as } r \to \infty.$$

This means that

$$\log r = o\left(N(r, \frac{1}{g})\right) \quad \text{as } r \to \infty. \tag{5.52}$$

2. The upper bound

For large m and

$$|a_m| \leq r \leq |a_{m+1}|, \tag{5.53}$$

we have $n(r, \frac{1}{g}) = m$. Using (5.41), we have

$$|a_m| \geq \lambda^{[\frac{m-1}{2}]} |a_1|$$

which gives, for r as in (5.53),

$$\left[\frac{m-1}{2}\right] \log \lambda + \log |a_1| \leq \log |a_m| \leq \log r,$$

and so

$$\left[\frac{m-1}{2}\right] \leq \frac{\log r - \log |a_1|}{\log \lambda}.$$

Therefore, since $m \leq 2(\frac{m-1}{2}) + 1 \leq 2[\frac{m-1}{2}] + 3$ we have

$$n(r, \frac{1}{g}) \leq O(\log r) \quad \text{as } r \to \infty.$$

Using the definition of $N(r, \frac{1}{g})$, we have

$$\begin{aligned} N(r, \frac{1}{g}) &\leq \int_{|a_1|}^{r} n(r, \frac{1}{g}) \frac{dt}{t} \\ &\leq n(r, \frac{1}{g}) \log \frac{r}{|a_1|}. \end{aligned}$$

Hence

$$N(r, \frac{1}{g}) \leq O(\log r)^2 \quad \text{as} \, r \to \infty. \tag{5.54}$$

Taking absolute value and logarithms in (5.33), we get

$$\log |P(a_m)| + \log |g'(a_m)| + Re(Q(a_m)) = 0. \tag{5.55}$$

Using (5.51), (5.52) and the fact that $\log |P(a_m)| = O(\log |a_m|)$, we get

$$-Re(Q(a_m)) = (1 + o(1)) N(|a_m|, \frac{1}{g}). \tag{5.56}$$

Returning to the polynomial $Q(z)$ which is in (5.31), we can write $Q(z)$ as

$$Q(z) = \sum_{k=0}^{q} c_k z^k, \quad c_k \in \mathbb{C}.$$

Letting $a_m = x_m + iy_m$, we get

$$Q(a_m) = \sum_{k=0}^{q} c_k(x_m + iy_m)^k = \sum_{k=0}^{q} c_k \sum_{n=0}^{k} \binom{k}{n} x_m^{k-n}(iy_m)^n. \tag{5.57}$$

So every term in $Q(a_m)$ includes y_m except when $n = 0$. Using (5.28) and (5.30), we have $y_m \to 0$ as $m \to \infty$ and, for $1 \leq n \leq k$,

$$|x_m^{k-n}(iy_m)^n| \leq |a_m|^{k-n}|y_m|^n \leq |a_m|^{k-n}\phi(|a_m|)^n \to 0. \tag{5.58}$$

Using (5.57) and (5.58), we have

$$Q(a_m) = \sum_{k=0}^{q} c_k \left[x_m^k + \sum_{n=1}^{k} \binom{k}{n} x_m^{k-n}(iy_m)^n\right] = \sum_{k=0}^{q} c_k x_m^k + o(1).$$

Thus, as $m \to \infty$ we have,

$$-Re(Q(a_m)) = \sum_{k=0}^{q} d_k x_m^k + o(1), \quad d_k = -Re(c_k). \tag{5.59}$$

Using (5.52), (5.56) and (5.59) we see that $\sum_{k=0}^{q} d_k x_m^k$ must be a non-constant polynomial in x_m. So it grows like a power of x_m and this contradicts the upper bound for $N(|a_m|, \frac{1}{g})$, which is in (5.54). This completes the proof of Theorem 5.3.1. □

5.4 A further new result

Theorem 5.4.1 *Suppose that A is a transcendental entire function of finite order. Let λ be a positive real number with $\lambda > 1$, and let $\psi : [0, \infty) \to [0, \infty)$ be a function satisfying*

$$\lim_{x\to\infty} e^{x^n}\psi(x) = 0 \quad \forall\, n \in \mathbb{N}. \tag{5.60}$$

Suppose that each b_n is a real number with

$$|b_{n+1}| > \lambda^2 |b_n| > 0 \quad n = 1, 2, \cdots, \tag{5.61}$$

and suppose that

$$\rho_n = \psi(|b_n|). \tag{5.62}$$

Suppose that $E = f_1 f_2$ is the product of linearly independent normalised solutions of (5.1), and that all but finitely many zeros of E lie in $U_{n=1}^{\infty} B(b_n, \rho_n)$. Then E has finite order and finitely many zeros, and there are polynomials P_1, P_2 such that $E = P_1 e^{P_2}$.

Proof: Suppose that A and E are as in the hypotheses. First we want to show that such a function E has finite order.

Claim: There exist positive real numbers r_m, $m = 1, 2, \cdots$, with

$$\lambda r_m \le r_{m+1} \le \lambda^3 r_m \tag{5.63}$$

and such that, for each m,

$$\{|b_j|\} \cap (\lambda^{-1} r_m, \lambda r_m) = \phi. \tag{5.64}$$

To prove the claim we set $r_1 = \lambda |b_1|$. Assume $r_1, \cdots, r_m$ have been chosen satisfying (5.63) and (5.64). If

$$\{|b_j|\} \cap [\lambda r_m, \lambda^2 r_m] = \phi,$$

we set

$$r_{m+1} = \lambda r_m. \tag{5.65}$$

In this case we have, using (5.65),

$$\begin{aligned} &\{|b_j|\} \cap (\lambda^{-1} r_{m+1}, \lambda r_{m+1}) \\ &= \{|b_j|\} \cap (r_m, \lambda^2 r_m) \\ &= \phi. \end{aligned}$$

On the other hand if there exists

$$|b_k| \in [\lambda r_m, \lambda^2 r_m]$$

then this b_k is unique by (5.61), and we set

$$r_{m+1} = \lambda |b_k| \leq \lambda^3 r_m. \tag{5.66}$$

Then, using (5.61) and (5.66),

$$\begin{aligned} &\{|b_j|\} \cap (\lambda^{-1} r_{m+1}, \lambda r_{m+1}) \\ &= \{|b_j|\} \cap (|b_k|, \lambda^2 |b_k|) \\ &= \phi. \end{aligned}$$

This proves the claim.

Using (5.63), we have for some positive constant C,

$$\log r_{m+1} \leq \log r_m + C.$$

Hence

$$\limsup_{m \to \infty} \frac{\log r_{m+1}}{\log r_m} \leq 1 < \infty.$$

Using (5.62) and (5.64), we see that the number of zeros of E in the annulus

$$\Omega(r_m, K) = \{z \in \mathbb{C} : \frac{r_m}{K} < |z| < Kr_m\}, \quad K = \sqrt{\lambda},$$

is zero for large m. Returning to how we defined r_m, we see that (r_m) tends to infinity. Applying Theorem 5.1.2, we get that E has finite order.

Assume that E has infinitely many zeros. Using Theorem 5.1.3, there is a positive constant M such that, for all zeros a, a' of E with $a \neq a'$,

$$|a - a'| \geq \exp(-|a|^M).$$

Hence by (5.60) and (5.62) for large n there is at most one zero of E in each disc $B(b_n, \rho_n)$. We may delete any b_n such that $B(b_n, \rho_n)$ does not contain a zero of E. This does not affect (5.61). Relabelling the b_n and a_n if necessary, we can write

$$E(z) = P(z)e^{Q(z)} \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right)$$

where $P(z)$ and $Q(z)$ are polynomials and $a_n \in B(b_n, \rho_n)$. Using (5.60) and (5.62), we have

$$\begin{aligned} |Im(a_n)| < \psi(|b_n|) &< \psi(|a_n| + o(1)) \\ &= o(|a_n|^{-L}) \end{aligned}$$

for every $L > 0$. Using (5.61), we have for large n

$$|a_{n+1}| > \lambda |a_n|.$$

Applying Theorem 5.2.1, we see that the zero sequence of E is not a Bank-Laine sequence and this gives a contradiction. This completes the proof of Theorem 5.4.1. $\square$

Chapter 6

On Complex Oscillation Theory

[1] In this chapter, we will consider the following. Suppose that A is a transcendental entire function with $\rho(A) < \frac{1}{2}$. Suppose that $k \geq 2$ and $y^{(k)} + Ay = 0$ has a solution f with $\lambda(f) < \rho(A)$, and suppose that $A_1 = A + h$ where $h \not\equiv 0$ is an entire function with $\rho(h) < \rho(A)$. Then $y^{(k)} + A_1 y = 0$ does not have a solution g with $\lambda(g) < \rho(A)$.

6.1 Introduction

Suppose that $k \in \mathbb{N}$ and A is an entire function. Suppose that $f_j,\ j = \{1, 2, \cdots, k\}$ are solutions of

$$y^{(k)} + Ay = 0. \tag{6.1}$$

Cauchy [16] proved that any solution of (6.1) is entire. In recent years, a lot of work [5], [7], [22], [23] has been done in the connection between the order of growth ρ of A and the exponent of convergence λ of $f_j,\ j = \{1, 2, \cdots, k\}$. We

[1] We submitted this chapter to be published as a paper in Results in Mathematics.

recall from Definition 1.2.7 and Definition 1.2.8 the following.

$$\rho(A) = \limsup_{r\to\infty} \frac{\log^+ T(r, A)}{\log r}$$

$$\lambda(f_j) = \limsup_{r\to\infty} \frac{\log^+ N(r, \frac{1}{f_j})}{\log r}.$$

Hayman-Miles [15] proved the following, which follows from Theorem 1.6.3.

Theorem 6.1.1 *Let f be a meromorphic function and let k be a positive integer. Then $\rho(f^{(k)}) = \rho(f)$.*

We need the following definition and lemma [20] for our main results, Theorem 6.2.1 and Theorem 6.5.1.

Definition 6.1.1 *Let $B(z_n, r_n)$ be open discs in the complex plane. We say that the countable union $\bigcup B(z_n, r_n)$ is an R-set if $z_n \to \infty$ and $\sum r_n$ is finite.*

Lemma 6.1.1 *Suppose that f is a meromorphic function of finite order. Then there exists a positive integer N such that*

$$\frac{f'(z)}{f(z)} = O\left(|z|^N\right)$$

holds for large z outside of an R-set.

For the proof of Theorem 6.5.1, we will use the following theorem and lemma, which give an asymptotic representation for solutions of (6.1) with few zeros. The first of these is a special case of a result from [23].

Theorem 6.1.2 *(Langley's theorem) [27]*

Let A be a transcendental entire function of finite order, and let E_1 be a subset of $[1, \infty)$ of infinite logarithmic measure and with the following property. For each $r \in E_1$ there exists an arc

$$a_r = \{re^{it} :\ 0 \le \alpha_r \le t \le \beta_r \le 2\pi\}$$

of the circle $S(0,r)$ such that

$$\lim_{r\to\infty, r\in E_1} \frac{\min\{\log|A(z)| : z\in a_r\}}{\log r} = +\infty.$$

Let $k \geq 2$ and let f be a solution of (6.1) with $\lambda(f) < \infty$. Then there exists a subset $E_2 \subset [1,\infty)$ of finite measure, such that for large $r \in E_0 = E_1 \backslash E_2$, we have

$$\frac{f'(z)}{f(z)} = c_r A(z)^{\frac{1}{k}} - \frac{k-1}{2k}\frac{A'(z)}{A(z)} + O(r^{-2}), \quad z\in a_r,\ c_r^k = -1.$$

Here c_r depends on r but not on z, and the branch of $A^{\frac{1}{k}}$ is analytic on a_r.

We note that E_2 has finite measure and so finite logarithmic measure. Therefore, E_0 has infinite logarithmic measure. Moreover, we exclude the case $k=1$ because for $k=1$ the general solution of (6.1) is

$$y = C\exp\left(-\int_0^z A(t)dt\right), \quad C\in\mathbb{C}.$$

Lemma 6.1.2 *(Hayman's lemma) [14]*
Let f be an analytic function, and let $F = \frac{f'}{f}$. Then for $k \in \mathbb{N}$

$$\frac{f^{(k)}}{f} = F^k + \frac{k(k-1)}{2}F^{k-2}F' + P_{k-2}(F),$$

where P_{k-2} is a differential polynomial with constant coefficients, which vanishes identically for $k \leq 2$ and has degree $k-2$ when $k > 2$.

Before stating the first result, let us note the following example.

Example 6.1.1 *It is possible to have a solution f of (6.1) with no zeros. Let $f = e^B$ where B is an entire function, and let $F = \frac{f'}{f}$. It is clear to see that $F = B'$. Using the Hayman's lemma (Lemma 6.1.2), we have*

$$\frac{f^{(k)}}{f} = (B')^k + \frac{k(k-1)}{2}(B')^{k-2} + B'' + P_{k-2}(B').$$

By letting $-A = \frac{f^{(k)}}{f}$, we see that $f = e^B$, which has no zeros, solves (6.1).

6.2 A new result

Theorem 6.2.1 *Suppose that A is a transcendental entire function with $\rho(A) < \frac{1}{2}$. Suppose that*

$$y'' + Ay = 0 \tag{6.2}$$

has a solution f with $\lambda(f) < \rho(A)$. Suppose that

$$A_1 = A + h \tag{6.3}$$

where $h \not\equiv 0$ is an entire function with $\rho(h) < \rho(A)$. Then

$$y'' + A_1 y = 0 \tag{6.4}$$

does not have a solution g with $\lambda(g) < \rho(A)$.

Proof: We note first that $\rho(A_1) = \rho(A)$, using (6.3). Suppose that (6.2) has a solution f with $\lambda(f) < \rho(A)$, and suppose that (6.4) has a solution g with $\lambda(g) < \rho(A)$. We can let

$$f = Pe^U \tag{6.5}$$

$$g = Qe^V \tag{6.6}$$

where U and V are entire functions and P, Q either are polynomials or satisfy

$$P = z^{m_1} \prod_{k=1}^{\infty} \left(1 - \frac{z}{a_k}\right), \quad a_k \in \mathbb{C}, \quad m_1 \in \mathbb{N} \cup \{0\},$$

$$Q = z^{m_2} \prod_{k=1}^{\infty} \left(1 - \frac{z}{b_k}\right), \quad b_k \in \mathbb{C}, \quad m_2 \in \mathbb{N} \cup \{0\},$$

with $\rho(P) = \lambda(f)$ and $\rho(Q) = \lambda(g)$. We note that these products converge because $\lambda(f), \lambda(g)$ are both less than 1. Using (6.2) and (6.5), we get

$$PU'^2 + PU'' + 2P'U' + P'' + AP = 0. \tag{6.7}$$

Using (6.4) and (6.6), we get analogously

$$QV'^2 + QV'' + 2Q'V' + Q'' + AQ + hQ = 0. \tag{6.8}$$

Dividing (6.7) by P, we find that

$$U'^2 + U'' + 2\frac{P'}{P}U' + \frac{P''}{P} + A = 0. \tag{6.9}$$

Dividing (6.8) by Q, we obtain analogously

$$V'^2 + V'' + 2\frac{Q'}{Q}V' + \frac{Q''}{Q} + A + h = 0. \tag{6.10}$$

Let

$$F = \frac{f'}{f}, \tag{6.11}$$

$$G = \frac{g'}{g}. \tag{6.12}$$

Using (6.5) and (6.11), we get

$$F = \frac{P'}{P} + U'. \tag{6.13}$$

Using (6.6) and (6.12), we obtain

$$G = \frac{Q'}{Q} + V'. \tag{6.14}$$

Using (6.9) and (6.13), we find that

$$F' + F^2 + A = 0, \tag{6.15}$$

whereas, using (6.10) and (6.14), we get

$$G' + G^2 + A + h = 0. \tag{6.16}$$

Subtracting (6.16) from (6.15) gives

$$F' - G' + F^2 - G^2 = h. \tag{6.17}$$

6.3 A lemma and corollary needed for Theorem 6.2.1

Lemma 6.3.1 *Suppose that A is an entire function with $\rho(A) < \frac{1}{2}$. Let f, g be solutions of (6.2) and (6.4) respectively with $\lambda(f) < \rho(A)$ and $\lambda(g) < \rho(A)$. Let U and V be as in (6.5) and (6.6) respectively. Then*

$$\rho(A) = \rho(U) = \rho(V).$$

Proof: To prove that $\rho(A) = \rho(U)$, we need to prove the following.

1. $\rho(A) \leq \rho(U)$.

   Using (6.9), we have

$$A = -U'^2 - U'' - 2\frac{P'}{P}U' - \frac{P''}{P}.$$

   Using the fact that $\rho(\frac{P'}{P}) \leq \rho(P) = \lambda(f) < \rho(A)$, which gives $\rho(\frac{P''}{P}) < \rho(A)$, and using Theorem 6.1.1, we get

$$\begin{aligned}\rho(A) &\leq \max\{\rho(U'^2), \rho(U''), \rho(U')\}\\ &= \rho(U).\end{aligned}$$

2. $\rho(U) \leq \rho(A)$.

   Using (6.9), we have

$$U'^2 = -U'' - 2\frac{P'}{P}U' - \frac{P''}{P} - A.$$

   Dividing through by U', we get

$$U' = -\frac{U''}{U'} - 2\frac{P'}{P} - \frac{1}{U'}\left(\frac{P''}{P} + A\right).$$

Suppose $|U'(z)| \geq 1$. If not, we have nothing to prove. So

$$\begin{aligned}|U'(z)| &\leq \left|\frac{U''}{U'}(z)\right| + 2\left|\frac{P'}{P}(z)\right| + \frac{1}{|U'(z)|}\left(\left|\frac{P''}{P}(z)\right| + |A(z)|\right) \\ &\leq \left|\frac{U''}{U'}(z)\right| + 2\left|\frac{P'}{P}(z)\right| + \left|\frac{P''}{P}(z)\right| + |A(z)|.\end{aligned}$$

So since U is transcendental by $\rho(A) \leq \rho(U)$, and since $m(r, \frac{P^{(j)}}{P}) = O(\log r)$ because P has finite order,

$$m(r, U') \leq m(r, A) + S(r, U').$$

Hence, outside a set of finite measure,

$$(1 + o(1))T(r, U') \leq T(r, A).$$

This gives

$$\rho(U') \leq \rho(A)$$

and so, using Theorem 6.1.1,

$$\rho(U) \leq \rho(A).$$

Similarly we can prove that $\rho(A) = \rho(V)$ using the fact that $\rho(A_1) = \rho(A)$ as noted above. This completes the proof of Lemma 6.3.1. □

Corollary 6.3.1 *Suppose that A is an entire function with $\rho(A) < \frac{1}{2}$. Let f, g be as in Lemma 6.3.1, and let F, G be defined by (6.11) and (6.12) respectively. Then*

$$\rho(A) = \rho(F) = \rho(G).$$

Proof: Using (6.13), we have

$$F = \frac{P'}{P} + U'.$$

Using the assumption and Lemma 6.3.1, we know that $\rho(\frac{P'}{P}) \leq \rho(P) = \lambda(f) < \rho(A) = \rho(U)$. Hence, using Theorem 6.1.1,

$$\rho(F) = \rho(U).$$

Using (6.14), we have

$$G = \frac{Q'}{Q} + V'.$$

Using the assumption and Lemma 6.3.1, we know that $\rho(\frac{Q'}{Q}) \leq \rho(Q) = \lambda(g) < \rho(A_1) = \rho(A) = \rho(V)$. Hence, using Theorem 6.1.1,

$$\rho(G) = \rho(V).$$

This completes the proof of Corollary 6.3.1, using Lemma 6.3.1. □

6.4 The completion of the proof of Theorem 6.2.1

Let

$$\Phi = F - G. \tag{6.18}$$

It is very easy to estimate the order of Φ as follows, using (6.18) and Corollary 6.3.1. We have

$$\begin{aligned}\rho(\Phi) &\leq \max\{\rho(F), \rho(G)\} \\ &= \rho(A) \\ &< \frac{1}{2}.\end{aligned}$$

Claim: $\Phi \not\equiv 0$.

To prove the claim, suppose that $\Phi \equiv 0$. This gives, using (6.18), $F \equiv G$ and so $F' \equiv G'$. Using (6.17), we get $h = 0$ and this contradicts the hypothesis $h \not\equiv 0$. This completes the proof of the claim.

Dividing (6.17) by Φ, we get

$$\frac{\Phi'}{\Phi} + F + G = \frac{h}{\Phi}.$$

Using (6.13) and (6.14), we find that

$$\frac{\Phi'}{\Phi} + \frac{P'}{P} + U' + \frac{Q'}{Q} + V' = \frac{h}{\Phi}. \tag{6.19}$$

Pick α such that

$$\max\{\lambda(f), \lambda(g), \rho(h)\} < \alpha < \rho(A). \tag{6.20}$$

Let us consider the two cases for the order of Φ, i.e when $\rho(\Phi) \leq \alpha$ and when $\rho(\Phi) > \alpha$.

Case 1: $\rho(\Phi) \leq \alpha$.

(This includes the case when Φ is a rational function)

Using (6.18), we have

$$F = G + \Phi$$

$$F' = G' + \Phi'.$$

Using (6.15) and (6.16), we get

$$\begin{aligned} -A &= F^2 + F' \\ &= G^2 + 2G\Phi + \Phi^2 + G' + \Phi' \\ &= -(A + h) + 2G\Phi + \Phi^2 + \Phi'. \end{aligned}$$

This gives

$$G = \frac{h - \Phi^2 - \Phi'}{2\Phi}.$$

Therefore, using Theorem 6.1.1 and (6.20),

$$\begin{aligned}\rho(G) &\leq \max\{\rho(h), \rho(\Phi)\} \\ &\leq \alpha \\ &< \rho(A)\end{aligned}$$

which contradicts the fact that $\rho(G) = \rho(A)$ as in Corollary 6.3.1.

Case 2: $\rho(\Phi) > \alpha$.

Using (6.13), (6.14) and (6.18), we have

$$\begin{aligned}\Phi &= F - G \\ &= \frac{P'}{P} - \frac{Q'}{Q} + U' - V'.\end{aligned} \tag{6.21}$$

Using (6.20), it is clear that $\rho(\frac{P'}{P}) \leq \rho(P) = \lambda(f) < \alpha$ and $\rho(\frac{Q'}{Q}) \leq \rho(Q) = \lambda(g) < \alpha$. So

$$\rho(U' - V') > \alpha$$

since $\rho(\Phi) > \alpha$. Hence

$$\alpha < \rho(U' - V') < \frac{1}{2}.$$

Pick σ such that

$$\alpha < \sigma < \rho(U' - V') < \frac{1}{2}.$$

The modified $\cos \pi\rho$ theorem (Theorem 1.6.2) gives that the set

$$\left\{ r : \inf_{|z|=r} \log |U'(z) - V'(z)| > r^\sigma \right\}$$

has positive upper logarithmic density.

Since $\rho(P) + \rho(Q) < \infty$, using Lemma 6.1.1, there exist $M > 0$ and a set of discs $B(z_n, \rho_n)$ with $\sum \rho_n < \infty$ such that, for large z,

$$\left|\frac{P'(z)}{P(z)}\right| + \left|\frac{Q'(z)}{Q(z)}\right| \leq |z|^M, \quad z \notin \bigcup B(z_n, \rho_n) = H. \tag{6.22}$$

The set H_1 of $r > 0$ such that the circle $|z| = r$ meets H has measure less than or equal to $2\sum \rho_n < \infty$. Hence

$$m(H_1) = \int_0^\infty \chi_{H_1}(t)dt < \infty.$$

So

$$\begin{aligned}\int_1^r \chi_{H_1}(t)\frac{dt}{t} &\leq \int_1^r \chi_{H_1}(t)dt \\ &\leq m(H_1),\end{aligned}$$

and so

$$\frac{\int_1^r \chi_{H_1}(t)\frac{dt}{t}}{\log r} \to 0.$$

Therefore, H_1 has zero upper logarithmic density. Hence we deduce that there exists a set H_2 of positive upper logarithmic density such that

$$\inf_{|z|=r} \log|U'(z) - V'(z)| > r^\sigma$$

and such that the inequality of (6.22) holds for $|z| = r$, which then gives

$$\begin{aligned}|\Phi(z)| &\geq |U'(z) - V'(z)| - O(r^M), \\ \log|\Phi(z)| &\geq \frac{r^\sigma}{2}.\end{aligned} \tag{6.23}$$

Hence, using (6.20),

$$\left|\frac{h(z)}{\Phi(z)}\right| \leq \frac{e^{r^{\rho(h)+o(1)}}}{e^{r^\sigma}} = o(1), \quad |z| = r \in H_2, r \to \infty. \tag{6.24}$$

Using (6.11), (6.12) and (6.18), we have

$$\Phi = F - G = \frac{f'}{f} - \frac{g'}{g}. \tag{6.25}$$

Using (6.25), we see that Φ has simple poles and these can only occur at the zeros of f or the zeros of g. Hence, using (6.5) and (6.6), we get

$$\begin{aligned} n(r, \Phi) &\leq \bar{n}(r, \frac{1}{f}) + \bar{n}(r, \frac{1}{g}) \\ &= \bar{n}(r, \frac{1}{P}) + \bar{n}(r, \frac{1}{Q}). \end{aligned} \tag{6.26}$$

Integrating (6.19) around $|z| = r_n$, $r_n \to \infty$, $r_n \in H_2$, we get using the Argument Principle (Theorem 1.2.5) and (6.24), and the fact that U' and V' are entire,

$$n(r_n, \frac{1}{\Phi}) - n(r_n, \Phi) + n(r_n, \frac{1}{P}) + n(r_n, \frac{1}{Q}) = o(1).$$

But the left hand side must be an integer. So

$$n(r_n, \frac{1}{\Phi}) - n(r_n, \Phi) + n(r_n, \frac{1}{P}) + n(r_n, \frac{1}{Q}) = 0.$$

Hence, using (6.26),

$$\begin{aligned} n(r_n, \frac{1}{\Phi}) &= n(r_n, \Phi) - n(r_n, \frac{1}{P}) - n(r_n, \frac{1}{Q}) \\ &\leq \bar{n}(r_n, \frac{1}{P}) - n(r_n, \frac{1}{P}) + \bar{n}(r_n, \frac{1}{Q}) - n(r_n, \frac{1}{Q}) \\ &\leq 0. \end{aligned}$$

Since the number of zeros of any function is a non-negative number, we get

$$n(r_n, \frac{1}{\Phi}) = 0$$

and so

$$N(r_n, \frac{1}{\Phi}) = 0.$$

Since Φ is very large on $|z| = r_n$ by (6.23), we have

$$m(r_n, \frac{1}{\Phi}) = 0.$$

Hence

$$T(r_n, \frac{1}{\Phi}) = 0$$

and so, using the first fundamental theorem of Nevanlinna Theory (Theorem 1.2.1),

$$T(r_n, \Phi) = O(1).$$

This contradicts the fact that Φ is a transcendental function, since $\rho(\Phi) > \alpha$, and completes the proof of Theorem 6.2.1. □

Corollary 6.4.1 *Suppose that A is a transcendental entire function with $\rho(A) < \frac{1}{2}$. Suppose that $y'' + Ay = 0$ has a solution with finitely many zeros. Suppose that $A_1 = A + h$, where $h \not\equiv 0$ is an entire function with $\rho(h) < \rho(A)$. Then $y'' + A_1 y = 0$ does not have a solution with finitely many zeros.*

Corollary 6.4.2 *Suppose that A is a transcendental entire function with $\rho(A) < \frac{1}{2}$. Suppose that $y''+Ay = 0$ has a solution with no zeros. Suppose that $A_1 = A+h$, where $h \not\equiv 0$ is an entire function with $\rho(h) < \rho(A)$. Then $y'' + A_1 y = 0$ does not have a solution with no zeros.*

It seems that we need another method to solve the following problem since we do not get a simple formula such as (6.17).

Problem 6.4.1 *Suppose that A is a transcendental entire function with $\rho(A) < \frac{1}{2}$. Suppose that $k \geq 2$ and (6.1) has a solution with no zeros. Suppose that $A_1 = A + h$, where $h \not\equiv 0$ is an entire function with $\rho(h) < \rho(A)$. Show that $y^{(k)} + A_1 y = 0$ does not have a solution with no zeros.*

6.5 The solution of Problem 6.4.1

In this section, we will prove a generalisation of Problem 6.4.1 and by doing this we give another proof of Theorem 6.2.1. Our main result in this section is the following theorem.

Theorem 6.5.1 *Suppose that A is a transcendental entire function with $\rho(A) < \frac{1}{2}$. Suppose that $k \geq 2$ and (6.1) has a solution f with $\lambda(f) < \rho(A)$, and suppose that (6.3) holds, where $h \not\equiv 0$ is entire with $\rho(h) < \rho(A)$. Then*

$$y^{(k)} + A_1 y = 0 \tag{6.27}$$

does not have a solution g with $\lambda(g) < \rho(A)$.

Proof: We note as before that $\rho(A_1) = \rho(A)$, using (6.3). Suppose that (6.1) has a solution f with $\lambda(f) < \rho(A)$, and suppose that (6.27) has a solution g with $\lambda(g) < \rho(A)$. We can let

$$f = Pe^U \tag{6.28}$$

$$g = Qe^V \tag{6.29}$$

where U and V are entire functions and P, Q either are polynomials or satisfy

$$P = z^{m_1} \prod_{k=1}^{\infty} \left(1 - \frac{z}{a_k}\right), \quad a_k \in \mathbb{C}, \quad m_1 \in \mathbb{N} \cup \{0\},$$

$$Q = z^{m_2} \prod_{k=1}^{\infty} \left(1 - \frac{z}{b_k}\right), \quad b_k \in \mathbb{C}, \quad m_2 \in \mathbb{N} \cup \{0\},$$

with $\rho(P) = \lambda(f)$ and $\rho(Q) = \lambda(g)$. We note that these products converge because $\lambda(f), \lambda(g)$ are both less than 1. Let

$$F = \frac{f'}{f}, \tag{6.30}$$

$$G = \frac{g'}{g}. \tag{6.31}$$

Using (6.28) and (6.30), we get

$$F = \frac{P'}{P} + U'. \tag{6.32}$$

Using (6.29) and (6.31), we obtain

$$G = \frac{Q'}{Q} + V'. \tag{6.33}$$

Applying the Hayman's lemma (Lemma 6.1.2), we obtain

$$\frac{f^{(k)}}{f} = F^k + \frac{k(k-1)}{2}F^{k-2}F' + P_{k-2}(F), \tag{6.34}$$

$$\frac{g^{(k)}}{g} = G^k + \frac{k(k-1)}{2}G^{k-2}G' + P_{k-2}(G), \tag{6.35}$$

where P_{k-2} is a differential polynomial with constant coefficients, which vanishes identically for $k \leq 2$ and has degree $k-2$ when $k > 2$.

Pick τ, σ such that

$$\max\{\lambda(f), \lambda(g), \rho(h)\} < \tau < \sigma < \rho(A) < \frac{1}{2}. \tag{6.36}$$

The modified *cosπρ* theorem (Theorem 1.6.2) gives that the set

$$E_1 = \{r : \inf_{|z|=r} \log|A(z)| > r^\sigma\} \tag{6.37}$$

has positive upper logarithmic density. Let $E_2 \subset [1, \infty)$ be a subset of finite measure so that, for some $M_1 \in \mathbb{N}$,

$$\left|\frac{A'(z)}{A(z)}\right| + \left|\frac{P'(z)}{P(z)}\right| + \left|\frac{Q'(z)}{Q(z)}\right| \leq r^{M_1}, \ |z| = r \geq 1, \ r \notin E_2. \tag{6.38}$$

Such E_2 and M_1 exist by [27]. For large $r \in E_1$ we also have, using (6.36) and (6.37),

$$\log |A_1(z)| > \frac{r^\sigma}{2}.$$

We can apply the Langley's theorem (Theorem 6.1.2) to (6.1) and (6.27), letting a_r be the whole circle $|z| = r, r \in E_1$. Hence for large $r \in E_0 = E_1 \backslash E_3$, where $E_2 \subset E_3$ and E_3 has finite measure, the following is true.

$$\frac{f'(z)}{f(z)} = c\,A(z)^{\frac{1}{k}} - \frac{k-1}{2k}\frac{A'(z)}{A(z)} + O(r^{-2}),\ |z| = r,\ c^k = -1, \tag{6.39}$$

$$\frac{g'(z)}{g(z)} = d\,A_1(z)^{\frac{1}{k}} - \frac{k-1}{2k}\frac{A_1'(z)}{A_1(z)} + O(r^{-2}),\ |z| = r,\ d^k = -1. \tag{6.40}$$

Here c, d may depend on r, but not on z. We remark that E_0 is infinite, because finite measure implies zero logarithmic density. It is clear from (6.36) and (6.37) that

$$\begin{aligned} \left|\frac{h(z)}{A(z)}\right| &\le \frac{e^{r^{\rho(h)+o(1)}}}{e^{r^\sigma}} = o(1),\ |z| = r,\ r \to \infty,\ r \in E_0, \\ \left|\frac{h'(z)}{A(z)}\right| &\le \frac{e^{r^{\rho(h')+o(1)}}}{e^{r^\sigma}} = \frac{e^{r^{\rho(h)+o(1)}}}{e^{r^\sigma}} = o(1),\ r \to \infty,\ |z| = r,\ r \in E_0, \end{aligned} \tag{6.41}$$

using $\rho(h') \le \rho(h)$. Now let us apply the binomial theorem to expand $A_1^{\frac{1}{k}}$, $\frac{A_1'}{A_1}$ in terms of $A^{\frac{1}{k}}$, $\frac{A'}{A}$. Using (6.3) and (6.41), we get

$$\begin{aligned} A_1^{\frac{1}{k}} &= (A+h)^{\frac{1}{k}} \\ &= A^{\frac{1}{k}}\left(1 + \frac{h}{A}\right)^{\frac{1}{k}} \\ &= A^{\frac{1}{k}}\left(1 + O\left(\frac{|h|}{|A|}\right)\right),\ |z| = r,\ r \in E_0. \end{aligned} \tag{6.42}$$

Also, using (6.3) and (6.41) again, we obtain

$$\begin{aligned}\frac{A_1'}{A_1} &= \frac{A'+h'}{A+h}\\ &= \frac{A'+h'}{A(1+\frac{h}{A})}\\ &= \frac{A'+h'}{A}\left(1-\frac{h}{A}+\frac{h^2}{A^2}-+\cdots\right)\\ &= \left(\frac{A'}{A}+\frac{h'}{A}\right)\left(1+O\left(\frac{|h|}{|A|}\right)\right)\\ &= \frac{A'}{A}\left(1+O\left(\frac{|h|}{|A|}\right)\right)+o\left(\frac{|h|}{|A|}\right),\ |z|=r,\ r\in E_0. \qquad (6.43)\end{aligned}$$

Using (6.38), (6.40), (6.42) and (6.43), we get for $|z| = r \in E_0$,

$$\frac{g'(z)}{g(z)} = d\,A(z)^{\frac{1}{k}} - \frac{k-1}{2k}\frac{A'(z)}{A(z)} + O(r^{-2}),\ d^k = -1. \qquad (6.44)$$

We recall that c and d may depend on r but, for a given r, do not depend on z.

6.6 A lemma needed for Theorem 6.5.1

Lemma 6.6.1 *Suppose that c, d are as in (6.39) and (6.44) respectively. Then $c = d$ for all large $r \in E_0$.*

Proof: We may write $d = \omega c$ where $\omega^k = 1$. Using (6.44), we obtain

$$\frac{g'(z)}{g(z)} = \omega c\,A(z)^{\frac{1}{k}} - \frac{k-1}{2k}\frac{A'(z)}{A(z)} + O(r^{-2}),\ \omega^k = 1. \qquad (6.45)$$

Multiplying (6.39) by ω and then subtracting (6.45) from it, we get

$$\omega\left(\frac{f'(z)}{f(z)} + \frac{k-1}{2k}\frac{A'(z)}{A(z)}\right) = \frac{g'(z)}{g(z)} + \frac{k-1}{2k}\frac{A'(z)}{A(z)} + O(r^{-2}).$$

Integrating around $|z| = r_n$, $r_n \to \infty$, $r_n \in E_0$, we find that

$$\omega\left[n(r_n,\frac{1}{f}) + \frac{k-1}{2k}n(r_n,\frac{1}{A})\right] = n(r_n,\frac{1}{g}) + \frac{k-1}{2k}n(r_n,\frac{1}{A}).$$

But the right hand side must be a positive integer since $n(r_n, \frac{1}{g}) \geq 0$, $n(r_n, \frac{1}{A}) > 0$. This is because if $n\left(r_n, \frac{1}{A}\right) = 0$ we get

$$N(r_n, \frac{1}{A}) = 0.$$

Since $\inf_{|z|=r} \log |A(z)|$ is very big for $r_n \to \infty$, $r_n \in E_0$, we get

$$m(r_n, \frac{1}{A}) = 0.$$

Hence,

$$T(r_n, \frac{1}{A}) = 0.$$

Using the first fundamental theorem of Nevanlinna Theory (Theorem 1.2.1), we obtain

$$T(r_n, A) = O(1).$$

This contradicts the fact that A is transcendental and proves the claim that $n(r_n, \frac{1}{A}) > 0$. For the same reason, $n(r_n, \frac{1}{f}) + n(r_n, \frac{1}{A})$ is a non-zero positive integer. Hence, ω is a positive rational number and since $|\omega| = 1$ we get $\omega = 1$ and so $c = d$. □

6.7 The completion of the proof of Theorem 6.5.1

We can now write on $|z| = r,\ r \in E_0$, using (6.44),

$$G(z) = \frac{g'(z)}{g(z)} = c\,A(z)^{\frac{1}{k}} - \frac{k-1}{2k}\frac{A'(z)}{A(z)} + O(r^{-2}),\ c^k = -1. \tag{6.46}$$

Subtracting (6.46) from (6.39), we get as $r \to \infty$, $r \in E_0$,

$$\frac{f'}{f} = \frac{g'}{g} + o(1),\ |z| = r.$$

Using (6.32) and (6.33), we get

$$\frac{P'}{P} + U' = \frac{Q'}{Q} + V' + o(1).$$

Using (6.38), we obtain

$$|U'(z) - V'(z)| \leq 2r^{M_1}, \ \ |z| = r, \ r \in E_0.$$

This gives

$$\liminf_{r\to\infty} \frac{T(r, U' - V')}{\log r} < \infty$$

and so $U' - V'$ is a polynomial. Hence, $U - V$ is a polynomial and so

$$U = P_0 + V$$

where P_0 is a polynomial. Thus, using (6.32),

$$F = \frac{P'}{P} + P_0' + V'.$$

Using (6.33), we find that

$$F = G + M \tag{6.47}$$

where

$$M = \frac{P'}{P} - \frac{Q'}{Q} + P_0'. \tag{6.48}$$

Using (6.1) and (6.34), we get

$$F^k + \frac{k(k-1)}{2} F^{k-2}F' + P_{k-2}(F) = -A, \tag{6.49}$$

where P_{k-2} is a differential polynomial with constant coefficients, which vanishes identically for $k \leq 2$ and has degree $k-2$ when $k > 2$. Using (6.27) and (6.35), we obtain

$$G^k + \frac{k(k-1)}{2} G^{k-2}G' + P_{k-2}(G) = -A - h. \tag{6.50}$$

Using (6.47) and (6.49), we get

$$(G+M)^k + \frac{k(k-1)}{2}(G+M)^{k-2}(G'+M') + P_{k-2}(G+M) = -A. \tag{6.51}$$

Expanding out $(G+M)^k$ and $(G+M)^{k-2}$ by the binomial theorem, we can write (6.51) as

$$G^k + kMG^{k-1} + \frac{k(k-1)}{2}G^{k-2}G' + R_{k-2}(M,G) = -A$$

and subtracting (6.50) from this we get

$$kMG^{k-1} + S_{k-2}(M,G) = h. \tag{6.52}$$

Here $R_{k-2}(M,G)$ and $S_{k-2}(M,G)$ are polynomials in M, G and their derivatives, and each have total degree at most $k-2$ in G and its derivatives.

Claim: $M \not\equiv 0$.

To prove the claim, We may assume that $M = 0$. Using (6.47), we get $F = G$. Using (6.49) and (6.50), we have $h = 0$. This contradicts the hypothesis $h \not\equiv 0$ and completes the proof of the claim.

Dividing (6.52) by MG^{k-2}, we get

$$kG + \frac{S_{k-2}(M,G)}{MG^{k-2}} = \frac{h}{MG^{k-2}}. \tag{6.53}$$

Suppose that $|G| > 1$. Now $\frac{S_{k-2}(M,G)}{MG^{k-2}}$ is a sum of terms

$$\frac{1}{MG^{k-2}}M^{j_0}(M')^{j_1}\cdots(M^{(k)})^{j_k}G^{q_0}(G')^{q_1}\cdots(G^{(k)})^{q_k}$$

where

$$q_0 + q_1 + \cdots + q_k \leq k-2$$

and hence such a term has modulus at most

$$|M|^{j_0+\cdots+j_k-1}\left|\frac{M'}{M}\right|^{j_1}\cdots\left|\frac{M^{(k)}}{M}\right|^{j_k}|G|^{q_0+\cdots+q_k-k+2}\left|\frac{G'}{G}\right|^{q_1}\cdots\left|\frac{G^{(k)}}{G}\right|^{q_k}$$
$$\leq |M|^{j_0+\cdots+j_k-1}\left|\frac{M'}{M}\right|^{j_1}\cdots\left|\frac{M^{(k)}}{M}\right|^{j_k}\left|\frac{G'}{G}\right|^{q_1}\cdots\left|\frac{G^{(k)}}{G}\right|^{q_k} \tag{6.54}$$

Using (6.53) and (6.54), we get

$$m(r,G) \leq C_1\, m(r,M) + m(r,\frac{1}{M}) + m(r,h) + S(r,G) + S(r,M)$$
$$\leq C_2T(r,M) + T(r,h) + S(r,G).$$

But, using (6.33),

$$N(r,G) \leq N(r,\frac{1}{Q}).$$

Hence

$$(1+o(1))T(r,G) \leq CT(r,M) + T(r,h) + N(r,\frac{1}{Q}).$$

Therefore, using (6.36) and (6.48),

$$T(r,G) = O(r^{\tau}),\ r \in E_0.$$

Using (6.50), we have

$$T(r,A) \leq C_3\, T(r,G) + T(r,h).$$

Hence

$$T(r,A) = O(r^{\tau}),\ r \in E_0.$$

Using (6.37), this contradicts the fact that $\log|A(z)| > r^{\sigma}$, $|z| = r$, $r \in E_0$, so that $T(r,A) = m(r,A) > r^{\sigma}$, $r \in E_0$. This completes the proof of Theorem 6.5.1. □

It is very clear to notice that Theorem 6.2.1 comes at once from Theorem 6.5.1. Also, it is very easy to see that Problem 6.4.1 comes directly from Theorem 6.5.1.

Corollary 6.7.1 *Suppose that A is a transcendental entire function with $\rho(A) < \frac{1}{2}$. Suppose that (6.1) has a solution with finitely many zeros. Suppose that A_1 is defined as $A_1 = A + h$, where $h \not\equiv 0$ is an entire function with $\rho(h) < \rho(A)$. Then $y^{(k)} + A_1 y = 0$ does not have a solution with finitely many zeros.*

Corollary 6.7.2 *Suppose that A is a transcendental entire function with $\rho(A) < \frac{1}{2}$. Suppose that (6.1) has a solution with no zeros. Suppose that A_1 is defined as $A_1 = A + h$, where $h \not\equiv 0$ is an entire function with $\rho(h) < \rho(A)$. Then $y^{(k)} + A_1 y = 0$ does not have a solution with no zeros.*

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