CONSTRUCTIONS OF SPECTRAL TRIPLES ON C*-ALGEBRAS

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I hereby declare that the work in this thesis is my own, except as indicated, and that an electronic version shall be available from the e-theses archive.

Abstract

We present some techniques in the construction of spectral triples for C^*-algebras, in particular those which determine a compatible metric on the state space, which provides a noncommutative analogue of geodesic distance between points on a manifold.

The main body of the thesis comprises three sections. In the first, we provide a further analysis on the existence of spectral triples on crossed products by discrete groups and their interplay with classical metric dynamics. Dynamical systems arising from non-unital C^*-algebras and certain semidirect products of groups are considered. The second section is a construction of spectral triples for certain unital extensions by stable ideals, using the language of unbounded Kasparov theory as presented by Mesland, Kaad and others. These ideas can be implemented for both the equatorial Podleś spheres and quantum SU_2 group. Finally, we investigate the potential of the construction of twisted spectral triples, as outlined by Connes and Moscovici. We achieve a construction of twisted spectral triples on all simple Cuntz-Krieger algebras, whose unique KMS state is obtained from the asymptotics of the Dirac.

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1: Introduction

1.1 Background and history.

The development of noncommutative geometry as a new field of mathematical research is based on a fundamental object known as a *spectral triple*. Spectral triples can be viewed as analytical tools that describe the geometric aspects of manifolds, such as the dimension, differential forms and geodesics. A prototype is given by a compact Riemannian manifold $\mathcal{M}$, equipped with a spin$^{\mathbb{C}}$ (or spin) structure $\mathcal{S}$ and the formal square root of the Laplace operator, or *Dirac* operator $\mathcal{D}$, acting on the smooth sections, $L^2(\mathcal{M}, \mathcal{S})$, of $\mathcal{M}$.

Connes introduced spectral triples as a potential means of describing the homology and index theoretic aspects in the more general language of (locally) compact topological spaces [30], as well as to develop a theory of cyclic cohomology mimicking the de-Rham cohomology theory of manifolds [31], [35], [34]. Moreover, the Gelfand-Naimark-Segal theorem makes the definition of a spectral triple, as a "Dirac-type" operator on a Hilbert space, applicable to general C^*-algebras, the latter viewed naturally as a noncommutative counterpart of a locally compact topological space in Gelfand duality theory.

Although everyone agrees that an object in noncommutative geometry is given by spectral triple, on a C^*-algebra, with "good" properties, there is considerable uncertainty into what these properties should be, since Connes' formal definition of a spectral triple, or *unbounded Fredholm module*, offers little by way of immediate information of the C^*-algebra besides its fundamental class in K-homology. In this respect the research seems be focused into two viewpoints: the first is centred on Connes' reconstruction programme [32], [79], in which it is considered under what axioms a spectral triple can provide a complete invariant of a commutative C^*-algebra as a manifold. Several reconstruction theorems have been suggested in what has become a very prominent area of research. Besides the *noncommutative tori*, there do not seem to be many examples of noncommutative C^*-algebras at present for which this sort of analysis is well understood.

Another way of looking at spectral triples is as noncommutative (quantum) metric spaces, beginning with Connes' observation that the Dirac triple on a Riemannian spin$^{\mathbb{C}}$ manifold $\mathcal{M}$ recovers the geodesics between two points on the manifold [31]. To use more recent language, a spectral triple on a C^*-algebra determines a *Lipschitz seminorm* on the self-adjoint part of the smooth subalgebra, an analogue of the classical notion of Lipschitz continuous functions. Rieffel studies seminorms of this kind extensively [100], [101], [102], and suggests that a question of particular interest might be when the metric on the state space it induces is compatible with the weak*-topology. Rieffel provided an answer for unital C^*-algebras and Latrémolière later extended much of this work to non-unital C^*-algebras [76], [77].

On one hand, examples of spectral triples for general C^*-algebras which are understood to satisfy such a metric condition are relatively few. Much of the focus so far has been centred on C^*-algebras arising from discrete groups, when a prototype for such a metric is given by a spectral triple coming from a length function on the group. On the other hand, analysis of this kind has proved successful on fractal structures, such as Cantor sets, which do not possess most of the attributes of a manifold (see [28], [92], [68] for examples).

The construction of such spectral triples gives a natural context to Rieffel's theory of Gromov-Hausdorff convergence for quantum metric spaces [103], [78], so that we can ask whether deformations of manifolds "converge", in a noncommutative metric sense, to their classical counterparts. Rieffel's Gromov-Hausdorff convergence analysis has already been considered in the case of the Moyal plane [12] and the Connes-Landi spheres [33], among other examples.

The construction of spectral triples is of interest to quantum group theorists, for which a natural question is to study the noncommutative geometry of q-deformations of classical Lie groups, such as Woronowicz's SU_q2 group [123], by writing down equivariant Dirac operators for the group co-action. By now there are many constructions, notably that of Neshveyev and Tuset [83], but difficulties remain in finding a unified approach which incorporates the different points of view.

Spectral triples define Baaj-Julg cycles [4], the unbounded counterpart to Kasparov's KK-theory. In this point of view, there is plenty of interest concerning the construction of these unbounded cycles, which can be seen to play the role of correspondences between spectral triples. The most interesting aspect is to present an unbounded version of the external and internal Kasparov product, which has received attention from various authors [80], [61].

In this thesis, we shall consider the unbounded Kasparov product as a tool in the explicit construction of spectral triples, especially in situations in which the relative K-homology is well understood. We will collate some of the theory of compact quantum metric spaces for C^*-algebras, which we can use to construct spectral triples with good metric properties.

1.2 Outline of the thesis and major developments.

Our focus is very much the **construction of spectral triples** for C^*-algebras, in a way which offers the maximum possible insight into further developments into the subject. The first two non-introductory chapters address some techniques and existing results which will be used later on, whilst in Chapters 4 and 5 these methods are applied to particular C^*-algebraic constructions, respectively **crossed products by discrete groups** and **extensions**. In Chapter 6, we offer an analysis of geometric aspects of the much studied **Cuntz-Krieger algebras**, for which it is typically not possible to write down finitely summable spectral triples satisfying all of Connes' axioms.

1.2.1 Spectral triples.

The definition of a spectral triple (**Definition 2.2.1**) should be considered an abstract list of axioms for an object in noncommutative geometry, whose initial objects are C^*-algebras. The properties which are of most interest to us are documented in **section 2.2**. We wish to regard spectral triples as *first order differential operators*. In Chapter 2 we therefore study the structure of algebras such as $C^1(\mathcal{M}) := \{f \in C_0(\mathcal{M}) : \ df \in C_0(\mathcal{M})\}$. With this machinery we will introduce spectral triples, which we can think of as providing an abstract notion of first order differentiation. The first example is the spectral triple on the circle: the algebra $C(\mathbb{T})$ acts faithfully over the Hilbert space $L^2(\mathbb{T})$, from which we can study the triple,

$$(C^1(\mathbb{T}), L^2(\mathbb{T}), \mathcal{D} := \frac{1}{i}\frac{\partial}{\partial t}). \tag{1.2.1}$$

The Product rule then implies that $\|\mathcal{D}f - f\mathcal{D}\| = \|f'\|$ for each $f \in C^1(\mathbb{T})$. We shall call $(C^1(\mathbb{T}), L^2(\mathbb{T}), \mathcal{D} := \frac{1}{i}\frac{\partial}{\partial t})$ the usual spectral triple on the circle algebra. It is the first example of a spectral triple and the simplest to study in nearly all respects.

1.2.2 The Kasparov Product.

We have found the idea of constructing new spectral triples from old ones to be implicit in much of classical differential geometry. In **Chapter 2** we study a version of the unbounded Kasparov product developed by Kaad and Lesch [61]. Their result is an associative pairing,

$$\mathfrak{K}\mathfrak{K}^p(A, B) \otimes_B \mathfrak{K}\mathfrak{K}^q(B, C) \mapsto \mathfrak{K}\mathfrak{K}^{p+q}(A, C), \tag{1.2.2}$$

where $\mathfrak{K}\mathfrak{K}(\cdot, \cdot)$ is the semigroup of Baaj-Julg cycles [4]. The construction requires the existence of a *correspondence* (**Definition 2.6.8**). The definition is based on an abstraction of the algebra of "first order derivatives", rather than the whole Sobolev chain which is the approach taken elsewhere (such as in [80]).

A spectral triple on A determines a class in $\mathfrak{K}\mathfrak{K}(A, \mathbb{C})$. Given another C^*-algebra B and a representative of $\mathfrak{K}\mathfrak{K}(B, A)$, the pairing offers the possibility of constructing spectral triples on B, which we have managed to exploit in a few instances in this work.

1.2.3 Quantum metric spaces.

The duality between metric spaces and Lipschitz seminorms was observed with the works of Kantorovich and Rubenstein, who show that a compact metric space (X, d) can be recovered from the algebra $C_{\mathrm{Lip}}(X)$ of globally d-Lipschitz functions in $C(X)$ by the formula

$$d(x, y) := \sup\{|f(x) - f(y)| : \ f \in C_{\mathrm{Lip}}(X), \|f\|_{\mathrm{Lip}} \leqslant 1\}. \tag{1.2.3}$$

and a similar metric can be defined, not only on X, but also the set of Borel probability measures $S(C(X))$ of X. It is possible to formalise this to the context of both unital and non-unital C^*-algebras, much of the analysis of which has been developed by Rieffel and, more recently, Latrémolière. The definition we shall work with is that of a *Lipschitz pair* $(\mathcal{A}, L)$ (**Definition 3.1.3**), which determines a metric $d_{\mathcal{A},L}$ on $S(A)$. A well-known result of Rieffel gives necessary conditions on $(\mathcal{A}, L)$ for $d_{\mathcal{A},L}$ to metrise the weak*-topology (**Definition**

3.1.11), which Latrémolière extended to the non-unital case (**Proposition 3.1.13**).

Because we are usually interested in Lipschitz pairs coming from spectral triples, there is the problem to address of whether the metric $d_{\mathcal{A},L}$ might depend on $\mathcal{A}$. Our first major result confirms that this is indeed the case:

Result 1 (Proposition 3.1.13.) *Let* A *be a unital* C^**-algebra and* $(\mathcal{A}, L)$ *be a closed Lipschitz pair on* A *which gives* $(S(A), d_{\mathcal{A},L})$ *finite diameter. Let* d_1, d_2 *be any metrics on* $S(A)$ *with* $d_1, d_2 \leqslant d_{\mathcal{A},L}$ *and with the property that the spaces* $\mathcal{A}_{d_1}$ *and* $\mathcal{A}_{d_2}$ *defined by*

$$\mathcal{A}_{d_i} := \{a \in \mathcal{A} : \exists L_a > 0 \text{ such that } \|\omega_1(a) - \omega_2(a)\| \leqslant L_a d_i(\omega_1, \omega_2),\ \forall \omega_1, \omega_2 \in S(A)\}$$

are dense in $\mathcal{A}$. *Then* $d_{\mathcal{A}_{d_1},L} = d_{\mathcal{A}_{d_2},L}$ *if and only if* d_1 *and* d_2 *are Lipschitz equivalent metrics.*

1.2.4 Equicontinuous actions.

In [5] and [57] it was shown that the ability to write down spectral triples on the reduced crossed product $A \rtimes_\alpha G$ of C^*-algebras by discrete groups, starting from a spectral triple on the "co-efficient" algebra A and via a natural description coming from the external Kasparov product formula [60], was a characteristic of a particular compatibility condition between the group action and the spectral triple on the coefficient algebra, which was labelled *equicontinuity* in [5] and does indeed turn out to be related to the classical equicontinuity condition of group actions on compact metric spaces.

In **Chapter 4** we study the interplay between quantum metric spaces and C^*-dynamical systems; we call the resulting structures *metric* C^**-dynamical systems* (**Definition 4.1.2**). We established some necessary C^*-algebraic conditions for a metric C^*-dynamical system to be equicontinuous (**Definition 4.2.8**). A necessary condition turns out to be that the action $\alpha : G \mapsto \mathrm{Aut}(A)$ is almost periodic; in other words, the G-orbit of every $a \in A$ is norm-relatively compact. (**Corollary 4.2.11**).

The almost periodicity condition is relatively obscure, so we decided to relate it to a C^*-dynamical invariant of natural interest, at least for the case of single automorphisms, *Voiculescu-Brown entropy* [116] [10], which is related to the classical definition of topological entropy of a compact topological space using open covers. Positive topological entropy of a dynamical system is generally interpreted as a chaos criterium and conversely we reach the expected conclusion that almost periodic actions are non-chaotic:

Result 2 (Proposition 4.2.15.) *Let A be a unital exact C^*-algebra and $\alpha \in \mathrm{Aut}(A)$ be an almost periodic automorphism. Then α has zero Voiculescu-Brown entropy, i.e $ht(\alpha) = 0$.*

At the end of the chapter we apply these ideas to group actions on AF-algebras.

1.2.5 Spectral triples, semidirect products and crossed products.

To write down examples of crossed product C^*-algebras with good metric properties coming from the crossed product constructions of [5] and [57], it is necessary to understand for which group C^*-algebras it is possible to write down spectral triples with good metric properties. For those groups arising as semidirect products, a special instance of the crossed product construction applies and we were able to use this to prove the next result:

Result 3 (Theorem 4.4.4.) *Let N be a finitely generated discrete group, L_N a length function on N and $\alpha : N \mapsto N$ be a group automorphism such that $L_N(\alpha(n)) = L_N(n)$ for each $n \in N$. Let $(\mathbb{C}[N], \ell_2(N), M_{L_N})$ be the associated spectral triple on $C^*_r(N)$ and let us suppose further that the Lipschitz seminorm induced by this spectral triple satisfies Rieffel's criteria (**Proposition 3.1.10**). Let $\Gamma := N \rtimes_\alpha \mathbb{Z}$. Then, with $\mathcal{D}$ and M_{L_N} defined respectively on $\ell_2(\Gamma)$ as multiplication operators with respect to length functions on $\mathbb{Z}$ and N,*

1. *$(\mathbb{C}[\Gamma], \ell_2(\Gamma) \oplus \ell_2(\Gamma), \begin{bmatrix} 0 & \mathcal{D} - iM_{L_N} \\ \mathcal{D} + iM_{L_N} & 0 \end{bmatrix})$ is a spectral triple on $C^*_r(\Gamma)$.*
2. *The Lipschitz seminorm induced by this spectral triple satisfies Rieffel's criteria also.*

The proof follows the same ideas as in the crossed product construction; integral to the proof is the existence of so-called *Fourier coefficients*, which can be controlled using the Ozawa-Rieffel cut-down procedure (**Lemma 4.3.3**). We remark that, unlike the length function triples, the above construction has a $\mathbb{Z}/2\mathbb{Z}$- grading, which seems crucial, although does not immediately offer much insight to groups which are not semidirect products. Finally, we state a generalised statement of the existence of spectral triples on crossed products, for which the C^*-algebra A is permitted to be non-unital (**Result 4: Theorem 4.5.3, Result 5: Theorem 4.5.5**).

Finally, we show that our crossed product spectral triple by $\mathbb{Z}$ arises as an internal Kasparov product of the spectral triple on A and an unbounded representative of the Toeplitz element (see [95], [55] for further details):

Result 6 (Lemma 4.6.2.) *Let A be a separable C^*-algebra and $\alpha \in \mathrm{Aut}(A)$ be an automorphism. Let $(\mathbb{C}[\mathbb{Z}], \ell_2(\mathbb{Z}), \mathcal{D}_{\mathbb{Z}})$ be the usual triple on $C^*_r(\mathbb{Z})$. Suppose that $(\mathcal{A}, \mathcal{H}_A, \mathcal{D}_A)$, $\mathcal{A} = C^1(A)$, is an ungraded spectral triple on A such that the action of α is equicontinuous (and leaves $\mathcal{A}$ invariant). Then the graded spectral triple on $A \rtimes_\alpha \mathbb{Z}$ defined in* ***Theorem 4.5.3*** *is an unbounded Kasparov product of $(C^1(\mathbb{Z}, A), \ell_2(\mathbb{Z}) \bar{\otimes} A), \mathcal{D}_{\mathbb{Z}} \bar{\otimes} 1) \in \mathfrak{KK}^1(A \rtimes_\alpha \mathbb{Z}, A)$ and $(\mathcal{A}, \mathcal{H}_A, \mathcal{D}_A) \in \mathfrak{KK}^1(A, \mathbb{C})$.*

1.2.6 Extensions.

In **Chapter 5** we present a construction of spectral triples and compact quantum metric spaces via certain extensions by stable ideals, that is, we study short exact sequences of the form

$$0 \longrightarrow \mathcal{K} \otimes B \xrightarrow{\iota} E \xrightarrow{\sigma} A \longrightarrow 0\,, \tag{1.2.4}$$

where A, E and B are unital C^*-algebras, the inclusion of $\mathcal{K} \otimes B$ in E is essential and the quotient map $\sigma : E \mapsto A$ admits a completely positive splitting. Our method is to study a cross section of the corresponding six-term sequence in K-homology,

$$K^*(A) \xrightarrow{\sigma^*} K^*(E) \xrightarrow{\iota^*} K^*(B)\,, \tag{1.2.5}$$

where ι^* admits a right inverse, represented by an element $\tau^* \in KK(E, B)$ (**Proposition 5.3.1**).

To construct spectral triples, we do not consider all extensions, but only *smooth* extensions of *Toeplitz type* (**Definition 5.4.1**, **Definition 5.4.2**) and in particular those for whom the representation theory has a "nice" form. The first two conditions were needed for the construction proposed by Christensen and Ivan, who primarily looked at extensions by compact operators [27]. We make a further commentary of the smoothness criteria, namely that it can be interpreted as those extensions for which the respective first order differential structures arising in the short exact sequence are compatible, so the extension has a natural "C^1- pullback structure" (**Corollary 5.4.8**). This gives us a natural operator algebra, $E_1 \subset E$, to consider. Under these conditions, the construction is given as follows:

Result 7 (Theorem 5.5.4.) *Let* A *and* B *be unital* C^**-algebras, endowed with spectral triples* $(\mathcal{A}, H_A, \mathcal{D}_A)$ *and* $(\mathcal{B}, H_B, \mathcal{D}_B)$ *respectively. Let* E *be a smooth extension of the form (5.3.1), where* $P \in B(H_A)$ *is an orthogonal projection of Toeplitz type. Then, for each dense* **-subalgebra* $\mathcal{E} \subset E_1$, $(\mathcal{E}, H \oplus H, \mathcal{D}_1 \oplus \mathcal{D}_2)$*, represented via* $\pi_1 \oplus \pi_2$*, defines a spectral triple on* E*. Moreover, the spectral dimension of this spectral triple is given by the identity*

$$s_0(\mathcal{E}, H \oplus H, \mathcal{D}_1 \oplus \mathcal{D}_2) = s_0(\mathcal{A}, H_A, \mathcal{D}_A) + s_0(\mathcal{B}, H_B, \mathcal{D}_B). \tag{1.2.6}$$

Furthermore, the spectral triple represents the Fredholm module $\sigma^*(\mathcal{A}, H_A, \mathcal{D}_A) \oplus \tau^*(\mathcal{B}, H_B, \mathcal{D}_B)$ *in K-homology.* (See **Section 5.5** for details and terminology).

When the spectral triples on A and B satisfy Rieffel's metric condition, so does the spectral triple on the extension:

Result 8 (Theorem 5.6.4.) $(\mathcal{E}_{sa}, L_{\mathcal{D}})$ *is a compact quantum metric space.*

We show that this construction leads to a spectral triple, with good metric properties, for the case of the equatorial Podleś spheres and the quantum SU_2 groups introduced in [96] and [123] respectively as deformations of the classical spheres (**Section 5.7**). For both these quantum spheres, we do not see an obvious connection between our analysis and any other approaches in the literature.

1.2.7 Twisted spectral triples.

It is well known that the nonexistence of a unital trace prevents the possibility of constructing finitely summable spectral triples [30]. For C^*-algebras of this kind, such as the Cuntz algebra [40], a different approach is required. We pursue a recent idea by Connes and Moscovici [36], where the definition of a spectral triple is modified so that instead of requiring $[\mathcal{D}, a] \in B(\mathcal{H})$ we have

$$[\mathcal{D}, a]_\sigma := \mathcal{D}a - \sigma(a)\mathcal{D} \in B(\mathcal{H}); \;\; a \in \mathcal{A} \tag{1.2.7}$$

(**Definition 6.1.8**), where σ is a regular automorphism. Typically σ will be implented by the action of a one-parameter group and in this context the motivation is to seek spectral triples for which the Dixmier functional satisfies a KMS-condition for this action. It turns out that the behaviour of such "twisted" spectral triples is, for the most part, similar to that of ordinary spectral triples (**Proposition 6.1.10**). Connes and Moscovici point out that, contrary to expectations, twisted spectral triples can be used to build untwisted Chern characters (so that the resultant cycles in cohomology are also untwisted).

Techniques in the constructions of twisted spectral triples are not widely known at present. We seek spectral triples for all simple Cuntz-Krieger algebras ([41], **Definition 6.2.4**), probably among the easiest families of C^*-algebra to describe which fit into the purely infinite category. We achieve this by viewing these algebras as Exel crossed products [50], encoding the expanding metric geometry of the one-sided subshift.

Result 9 (Theorem 6.4.2). Let $\sigma = \sigma_{-i\log\lambda}$ be the regular automorphism on $\mathcal{O}_A \cong C(\Sigma_A^+)\rtimes_{\alpha_T,\mathcal{L}_T}\mathbb{N}$ defined by $\sigma(f) = f\ \forall f \in C(\Sigma_A^+)$ and $\sigma(s) = \lambda s$. Let $\mathcal{B} \subset C^{1,\sigma}(\mathcal{O}_{\mathcal{A}})$ be any dense *-subalgebra of the Cuntz-Krieger algebra $\mathcal{O}_A$ containing the algebra span of s, s^* and the natural AF-filtration $\cup_{k\geqslant 0}A_k$ of $C(\Sigma_A^+)$, where A is irreducible. Then $(\mathcal{B}, L^2(C(\Sigma_A^+),\tau),\mathcal{D},\sigma)$, with $\mathcal{D}$ as in **section 6.3**, defines a twisted spectral triple on $\mathcal{O}_A$.

The spectral triples arising this way do not seem to be so interesting from the metric point of view and, perhaps disappointingly, the question of finding interesting Lipschitz seminorms to study on algebras such as the Cuntz algebra remain unanswered. On the other hand, twisted spectral triples allow one to recover KMS_β-states on the C^*-algebras from the twisted version of the Dixmier trace. We conclude with the observation that, from the spectral triple on $\mathcal{O}_A$ constructed above, we can recover the unique KMS state on $\mathcal{O}_A$ whose "departure from a trace" coincides with the topological entropy of the Markov shift representing $\mathcal{O}_A$. I am not sure how our ideas relate to the modular spectral triples constructed by Carey, Phillips and Rennie [15].

Contents

2: Preliminaries

2.1 Operator *-algebras and operator *-modules.

2.1.1 Operator *-algebras.

In this section, a space $(X, \|\cdot\|)$ is taken to mean a Banach space over the field of complex numbers.

In the context of noncommutative geometry, there are many operator algebra structures besides C^*-algebras which are of natural interest. As a motivating example, let us consider the algebra of differentiable functions on a locally compact manifold, $\mathcal{M}$, given by

$$C_0^1(\mathcal{M}) := \{f \in C_0(\mathcal{M}) : f \text{ has uniformly bounded derivative }\},$$

Observe that, when equipped with the norm $\|f\|_1 := \|f\| + \sup_{x\in\mathcal{M}} |df(x)|$, then $C_0^1(\mathcal{M})$ becomes a Banach algebra in its own right. Furthermore, given a Borel probability measure μ with full support on $\mathcal{M}$, we have a faithful representation of $(C_0(\mathcal{M}), \|\cdot\|)$ on $L^2(\mathcal{M}, \mu)$ and a faithful representation of $(C_0^1(\mathcal{M}), \|\cdot\|_1)$ on $L^2(\mathcal{M}, \mu) \oplus L^2(\mathcal{M}, \mu)$ given by

$$f \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} := \begin{pmatrix} f\psi_1 \\ df\psi_1 + f\psi_2 \end{pmatrix}; \quad f \in C_0(\mathcal{M}),\ \psi_1, \psi_2 \in L^2(\mathcal{M}, \mu). \tag{2.1.1}$$

In fact, Kaad and Lesch [61] introduced the terminology *operator *-algebra* to study algebras of this kind and we shall base our analysis closely on theirs. Some standard references for theory of operator spaces and operator algebras can be found for example in [91] and [106].

Definition 2.1.1. A Banach space X is an *operator space* if is isometrically isomorphic to a subspace of the C^*-algebra $B(\mathcal{H})$ of bounded operators on a Hilbert space.

It is more usual to give an abstract version of the definition of an order unit space. In this version, it turns out that a necessary and sufficient condition for this is requiring the matricial amplifications $M_n(X) := \{x_{i,j} \in X;\ 1 \leqslant i, j \leqslant n\}$ themselves to be Banach spaces in a way compatible with both the order structure $M_n(X) \subset M_{n+1}(X)$ and the Banach algebra structure of $M_n(\mathbb{C})$. If X is an operator space, $M_n(X)$ is an amplification of X and there is a multiplication map $m : X \times X \mapsto X$ on X then the induced maps $m_n : M_n(X) \times M_n(X) \mapsto M_n(X)$ are the multiplication maps given by

$$((m_n(x, y))_{i,j})_{1 \leqslant i,j \leqslant n} := ((\sum_{k=1}^{n} m(x_{i,k}, y_{k,j}))_{1 \leqslant i,j \leqslant n};\ \ x, y \in M_n(X).$$

If $j : X \mapsto X$ is an involution, the induced maps $j_n : M_n(X) \mapsto M_n(X)$ are given by

$$((j_n(x))_{i,j})_{1 \leqslant i,j \leqslant n} = (j(x_{j,i}))_{1 \leqslant i,j \leqslant n};\ \ x \in M_n(X).$$

Definition 2.1.2. An operator algebra X is an operator space X with a (linear) completely bounded multiplication map $m : X \times X \mapsto X$, i.e there exists a $K > 0$ such that

$$\|m_n(x, y)\| \leqslant K\|x\|\|y\|,\ \ \forall x, y \in M_n(X),\ n \in \mathbb{N}. \tag{2.1.2}$$

It is well known that operator algebras are precisely the closed subalgebras of bounded operators on a Hilbert space, a proof of which can be found in Blecher's article [7].

Definition 2.1.3. An operator *-algebra X is an operator algebra with a completely bounded involution $j : X \mapsto X$, i.e there exists an $L > 0$ such that

$$\|j_n(x)\| \leqslant L\|x\|,\ \ \forall x \in M_n(X),\ n \in \mathbb{N}. \tag{2.1.3}$$

An operator *-algebra X differs from a C^*-algebra in that the involution j on X is in general different from the involution coming from an algebra embedding $X \subset B(\mathcal{H})$, even if $K = L = 1$ can be chosen in the above definitions.

The most important varieties of operator *-algebras are those defined by *derivations* on C^*-algebras, many examples of which will be constructed later on.

Definition 2.1.4. A derivation $\delta : A \mapsto B$, where A and B are C^*-algebras, and $\pi : A \mapsto B$ is an injective *-homomorphism, is a densely defined linear map which satisfies the product rule $\delta(ab) = \pi(a)\delta(b) + \delta(a)\pi(b)$ whenever $a, b \in A$. We suppose additionally that $\delta(a^*) = -\delta(a)^*$ for each $a \in A$.

Proposition 2.1.5. *[62] Let A, B be C^*-algebras, equipped with an injective *-homomorphism $\pi : A \mapsto B$ and a derivation $\delta : A \mapsto B$. Then the completion of the algebra $A_1 := \mathrm{dom}\delta$ with respect with the norm $\|a\|_1 := \|\pi(a)\|_B + \|\delta(a)\|_B$ and equipped with the involution from A , is an operator *-algebra. There is an algebra embedding $A \subset M_2(B)$ defined by*

$$\rho(a) := \begin{pmatrix} \pi(a) & 0 \\ \delta(a) & \pi(a) \end{pmatrix}. \tag{2.1.4}$$

Example 2.1.6. Going back to the first example, we could require instead that the algebra $C^1_0(\mathcal{M})$ consists of *continuously differentiable functions vanishing at infinity* (whose derivatives also vanish at infinity), rather than just functions with uniformly bounded derivative. It turns out that this algebra too can be given the structure of an operator *-algebra ([61]). This raises the question of just how many examples there are and we expect this subject to dominate the literature in the near future.

2.1.2 *Operator *-modules.*

Let us consider operator *-algebras of the variety considered in Proposition 2.1.5, where $B = B(\mathcal{H})$ for some separable Hilbert space $\mathcal{H}$ and write $A_1 := \mathrm{dom}(\delta)$. We can form the exterior tensor product $\mathcal{H} \otimes A$, regarded as a right A-module. The completion E_1 of $\mathcal{H} \otimes A_1$ with respect to the inner product norm $\langle \cdot, \cdot \rangle_1$, where

$$\langle a_1 \otimes \xi_1, a_2 \otimes \xi_2 \rangle_1 := \langle a_1 \otimes \xi_1, a_2 \otimes \xi_2 \rangle + \langle \delta(a_1) \otimes \xi_1, \delta(a_2) \otimes \xi_2 \rangle; \quad a_1, a_2 \in A_1, \tag{2.1.5}$$

is itself an operator space, with a natural contractive right action of A_1.

The operator *-module is another idea discussed and motivated in [61]. Essentially, an operator *-module is to an operator *-algebra what a countably generated Hilbert C^*-module is to a C^*-algebra. When X is an operator space, A is an operator algebra and $r : X \times A \mapsto X$ is a right action, the induced maps $r_n : M_n(X) \times M_n(A) \mapsto M_n(X)$ are given by

$$((r_n(e, a))_{i,j})_{1 \leqslant i,j \leqslant n} := (\sum_{k=1}^{n} r(e_{i,k}, a_{k,j}))_{1 \leqslant i,j \leqslant n}.$$

Definition 2.1.7. Let A be an operator algebra with completely bounded multiplication map m and let X be an operator space. We call X a *right A-module* if there exists a completely bounded linear map $r : X \times A \mapsto X$ such that $r_n(r_n(e, a), b) = r_n(e, m_n(a, b))$ for each $a, b \in M_n(A)$, $e \in M_n(X)$ and $n \in \mathbb{Z}^+$.

Definition 2.1.8. Let X be a right A-module, where X is an operator space and A is an operator *-algebra with a completely bounded multiplication map m and a completely bounded involution j. We say that X is *hermitian* if there exists a completely bounded sesquilinear form on X which is right A-linear. In other words, there exists a map $(\cdot|\cdot) : X \otimes X \mapsto A$ which is linear in the right co-ordinate such that

1. $(e|r(f, a)) = r((e|f), a)\ \forall e, f \in X,\ a \in A$ and
2. $j((e|f)) = (f|e)\ \forall e, f \in X$.

In particular the inner product extends to $(\cdot|\cdot)_n : M_n(X) \otimes M_n(X) \mapsto M_n(A)$ with the same properties, defined by

$$((e|f)_n)_{i,j} = \sum_{k=1}^{\infty} (e_{k,i}|f_{k,j})$$

and there exists a $M > 0$ such that $\|(e|f)_n\|_A \leqslant M\|e\|_X\|f\|_X,\ \ \forall e, f \in M_n(X),\ n \in \mathbb{N}$.

Alternatively we can define a operator space X to be a (hermitian) left A-module in precisely the same way, except that linearity and A-linearity is now defined in the left argument.

Definition 2.1.9. Let A be an operator *-algebra. By the *standard operator *-module* A_∞ over A we mean the completion of the space $c_0(A)$ of finite sequences in A, viewed as a diagonal subspace of $M(A) := \cup_{n\in\mathbb{N}} M_n(A)$ with respect to the standard embeddings $M_n(A) \subset M_{n+1}(A)$, whose norm is the norm induced by the inner product $((a_n)|(b_n)) = \sum_n j_n(a_n)b_n$ and the obvious right module action of A.

Definition 2.1.10. A hermitian operator right A-module $\mathcal{E}$ is called a *operator *-module* if $\mathcal{E} \cong PA_\infty$, i.e $\mathcal{E}$ is an orthogonal direct summand in A_∞.

Example 2.1.11. Every countably generated Hilbert module which is a right A-module for some C^*-algebra A is automatically an operator *-module when one views A as an operator *-algebra, an obvious consequence of Kasparov's stabilisation theorem.

Example 2.1.12. In [61] it was shown that an operator space X is a *-module over $C_0^1(\mathcal{M})$ (the algebra of continuous functions with continuous derivatives) if and only if X is a direct summand in the operator space $C_0^1(\mathcal{M}, \mathcal{H})$ (i.e the space of Hilbert-valued continuous functions with continuous derivatives) for some Hilbert space $\mathcal{H}$.

Example 2.1.13. More generally, the operator space E_1 above is an operator module over A_1, with $E_1 = A_{1\infty}$, as the completion of the finite sequences in $c_0(A_1) \subset \mathcal{H} \otimes A_1$.

2.2 Spectral triples: background and definition.

The basic objects in noncommutative geometry are *spectral triples*. Everyone agrees that on the most basic level a spectral triple, defined on a general C^*-algebra, should provide firstly a generalised index theory, defined by a pairing with K-theory, and secondly an abstract notion of a first order differential operator. Because a "Dirac-type" operator on a compact manifold also recovers the dimension of the manifold from the Weyl asymptotics of the Dirac, it is usual to associate a dimensionality invariant to a spectral triple also. In the language of C^*-algebras, the first condition can be interpreted as saying that a spectral triple should define a representative of K-homology. Consequently, spectral triples are also called *unbounded Fredholm modules*.

2.2.1 Definition of a spectral triple.

Definition 2.2.1. Let A be a C^*-algebra. A spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is given by a *-representation $\pi : A \mapsto B(\mathcal{H})$, a dense *-subalgebra $\mathcal{A} \subset A$ and a linear densely defined unbounded self-adjoint operator $\mathcal{D}$ on $\mathcal{H}$ such that

1. $\pi(\mathcal{A})\mathrm{dom}\mathcal{D} \subset \mathrm{dom}\mathcal{D}$ and $[\mathcal{D}, \pi(a)] : \mathrm{dom}\mathcal{D} \mapsto \mathcal{H}$ extends to a bounded operator for each $a \in \mathcal{A}$ and
2. $\pi(a)(1 + \mathcal{D}^2)^{-1}$ is a compact operator for each $a \in A$.

2.2.2 *Relation to operator *-algebras and Sobolev spaces.*

In the definition of a spectral triple, the algebra $\mathcal{A}$ is formally viewed as an algebra of differentiable elements and, for $a \in \mathcal{A}$, the operator $[\mathcal{D}, a] : \mathrm{dom}\mathcal{D} \mapsto \mathcal{H}$ corresponds to a first order derivative. To each spectral triple, we can study the "maximal" set of all such elements. We introduce the set,

$$C^1(A) := \{a \in A : \ a(\mathrm{dom}\mathcal{D}) \subset \mathrm{dom}\mathcal{D} \text{ and } [\mathcal{D}, \pi(a)] \text{ extends to a bounded operator in } B(\mathcal{H})\}.$$

Our first assertion is that $C^1(A)$ is a *-algebra. Given $a, b \in C^1(A)$ then also ab leaves the domain of $\mathcal{D}$ invariant, so that $[\mathcal{D}, ab] : \mathrm{dom}\mathcal{D} \mapsto B(\mathcal{H})$ makes sense and $[\mathcal{D}, ab] = [\mathcal{D}, a]b + a[\mathcal{D}, b]$, so that $[\mathcal{D}, ab]$ extends to a bounded operator on $B(\mathcal{H})$. That $a \in C^1(A)$ implies $a^* \in C^1(A)$ is less clear, however. In a recent article [25], Christensen provides a more thorough analysis of operators of the form $[\mathcal{D}, a]$, such as arising from spectral triples. The following definition will be useful:

Definition 2.2.2. [25] Given a spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ on A, an element $a \in A$ will be called *weakly $\mathcal{D}$-differentiable* if and only if the sesquilinear form $S([\mathcal{D}, a]) : \mathcal{H}^2 \mapsto \mathbb{C}$ defined by

$$S([\mathcal{D}, a])(\xi, \eta) := \langle a\xi, \mathcal{D}\eta\rangle - \langle a\mathcal{D}\xi, \eta\rangle, \ \ \xi, \eta \in \mathrm{dom}\mathcal{D}, \tag{2.2.1}$$

is bounded. On one hand, given $a \in C^1(A)$ then the sesquilinear form given by $S([\mathcal{D}, a])$ is evidently bounded. The converse is true by Theorem 2.11 of [25]. We can use this fact to show that the set of weakly $\mathcal{D}$-differentiable elements is closed under involution, for indeed for each $\xi, \eta \in \mathrm{dom}\mathcal{D}$,

$$S([\mathcal{D}, a^*])(\xi, \eta) := \langle a^*\xi, \mathcal{D}\eta\rangle - \langle a^*\mathcal{D}\xi, \eta\rangle = -(\langle a\eta, \mathcal{D}\xi\rangle - \langle a\mathcal{D}\eta, \xi\rangle)^* =: -S([\mathcal{D}, a](\eta, \xi))^*.$$

Proposition 2.2.3. *The vector space $\mathcal{H}_1 := \mathrm{dom}\mathcal{D}$ becomes a Hilbert space with respect to the inner product $\langle \eta_1, \eta_2\rangle_1 := \langle \eta_1, \eta_2\rangle + \langle \mathcal{D}\eta_1, \mathcal{D}\eta_2\rangle$. The algebra $C^1(A)$ is an operator *-algebra with respect to the involution on A and the norm $\|a\|_1 := \|\pi(a)\| + \|[\mathcal{D}, \pi(a)]\|$. Moreover, the map*

$$\rho : C^1(A) \mapsto B(\mathcal{H}_1); \ \ \rho(a)(\eta) := \pi(a)\eta, \tag{2.2.2}$$

is an injective norm-decreasing algebra homomorphism.

Proof. If $(\eta_n)_{n\geqslant 1} \subset \mathcal{H}_1$ is a Cauchy sequence then $(\eta_n)_{n\geqslant 1}$ is certainly Cauchy with respect to $\|\cdot\|_{\mathcal{H}}$. Therefore it converges to some vector $\eta \in \mathcal{H}$. By the closed graph theorem $\lim_{m\to\infty}(\mathcal{D}\eta_m - \mathcal{D}\eta_n)$ exists for each $n \in \mathbb{N}$ so that $\mathcal{D}\eta = \mathcal{D}\eta_n + \lim_{m\to\infty}(\mathcal{D}\eta_m - \mathcal{D}\eta_n) \in \mathcal{H}$ and $\eta \in \mathrm{Dom}(\mathcal{D})$. So $\mathcal{H}_1$ is complete.

In view of Proposition 2.1.5, to show that $C^1(A)$ is an operator algebra it is only necessary to verify that $C^1(A)$ is already complete in the norm $\|a\|_1 := \|\pi(a)\| + \|[\mathcal{D}, \pi(a)]\|$. A proof of this is provided in Theorem 2.16 of [25].

Finally, ρ is an injective algebra homomorphism, which is norm decreasing because of the estimate

$$\begin{aligned}
\|\rho(a)\|^2_{B(\mathcal{H}_1)} &= \sup_{\eta\in\mathcal{H}_1, \|\eta\|_1\leqslant 1} \langle \pi(a)(\eta), \pi(a)(\eta)\rangle_1 \\
&= \sup_{\eta\in\mathcal{H}_1, \|\eta\|_1\leqslant 1} (\langle \pi(a)\eta, \pi(a)\eta\rangle + \langle \mathcal{D}\rho(a)\eta, \mathcal{D}\pi(a)\eta\rangle) \\
&= \sup_{\eta\in\mathcal{H}_1, \|\eta\|_1\leqslant 1} (\langle \pi(a)\eta, \pi(a)\eta\rangle + \langle \pi(a)\mathcal{D}\eta, \pi(a)\mathcal{D}\eta\rangle + \langle [\mathcal{D}, \pi(a)]\eta, [\mathcal{D}, \pi(a)]\eta\rangle) \\
&\leqslant \sup_{\eta\in\mathcal{H}_1, \|\eta\|_1\leqslant 1} (\|\pi(a)\|^2\|\eta\|_1^2 + \|[\mathcal{D}, \pi(a)]\|^2\|\eta\|^2) \\
&\leqslant (\|\pi(a)\| + \|[\mathcal{D}, \pi(a)]\|)^2.
\end{aligned}$$

□

2.2.3 *Dimension and differential forms.*

Definition 2.2.4. Let A be a C^*-algebra and $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ a spectral triple on A. It will be useful to introduce the *algebra of differential 1-forms*, given by

$$\Omega_{\mathcal{D}}(A) := \{\sum_{i\in I} \pi(a_i^{(0)})[\mathcal{D}, \pi(a_i^{(1)})] \ldots [\mathcal{D}, \pi(a_i^{(r)})],\ a_i^{(r)} \in \mathcal{A},\ r \in \mathbb{Z}^+\}. \tag{2.2.3}$$

This is can be viewed as an $\mathcal{A}$-bimodule. Moreover $\Omega_{\mathcal{D}}(A)$ is an algebra since, given $a, b \in \mathcal{A}$, then also $ab \in \mathcal{A}$ and $[\mathcal{D}, \pi(a)]\pi(b) = [\mathcal{D}, \pi(ab)] - \pi(a)[\mathcal{D}, \pi(b)]$. When the spectral triple is unital, then the spectral triple is called p-*summable* for each $p \in (0, \infty)$ such that $(1 + \mathcal{D}^2)^{-p/2} \in B(\mathcal{H})$ is trace-class (note that $(1 + \mathcal{D}^2)^{-p/2} \in \mathcal{K}(\mathcal{H})$ for $p \geqslant 1$).

The *spectral dimension* of $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is then defined by

$$s_0(\mathcal{A}, \mathcal{H}, \mathcal{D}) = \inf\{p \in (0, \infty) : \ \mathrm{Tr}(1 + \mathcal{D}^2)^{-p/2} < \infty\} \tag{2.2.4}$$

When the dimension is integer valued then the spectral triple is applicable to Connes' reconstruction programme [32].

2.2.4 Relation to K-homology.

The standard identification between spectral triples and Fredholm modules is made in the following way: when $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a spectral triple on A with the additional *regularity* condition that also $[(1 + \mathcal{D}^2)^{1/2}, \pi(a)]$ is bounded for each $a \in A$ then $(A, \mathcal{H}, F_{\mathcal{D}} := (1 + \mathcal{D}^2)^{-1/2}\mathcal{D})$ is a Fredholm module on A. The main thing to show is that $[F_{\mathcal{D}}, \pi(a)] \in \mathcal{K}(\mathcal{H})$, which follows from

$$[F_{\mathcal{D}}, \pi(a)] = (1 + \mathcal{D}^2)^{-1/2}([\mathcal{D}, \pi(a)] + [(1 + \mathcal{D}^2)^{1/2}, \pi(a)]F_{\mathcal{D}}), \tag{2.2.5}$$

Note that the index of $F_{\mathcal{D}}$ is then precisely the index of $\mathcal{D} : \mathcal{H}_1 \mapsto \mathcal{H}$. In this way the grading structure on $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ automatically passes to $(A, \mathcal{H}, F_{\mathcal{D}})$. Because K-homology admits a formal Bott periodicity, it is enough for many purposes to decide whether the Fredholm $(A, \mathcal{H}, F_{\mathcal{D}})$ is a representative of $K^0(A)$ or $K^1(A)$. Consequently spectral triples are often distinguished into *odd* and *even* varieties:

Definition 2.2.5. A spectral triple on A is called *graded* or even if there exists an operator $\gamma \in B(\mathcal{H})$ such that $\gamma^2 = \mathrm{id}$, $\gamma\pi(a) = \pi(a)\gamma$ for each $a \in A$ and $\gamma\mathfrak{D} = -\mathfrak{D}\gamma$. Otherwise it will be called *ungraded* or odd. An even triple is formally represented via a direct sum representation $\pi^+ \oplus \pi^-$ over an orthogonal direct sum $\mathcal{H}^+ \oplus \mathcal{H}^-$ with a skew-diagonal self-adjoint operator $\mathcal{D}$ of the form $\begin{pmatrix} 0 & \mathcal{D}^- \\ \mathcal{D}^+ & 0 \end{pmatrix}$.

2.2.5 The Dixmier trace and connection to cyclic cohomology.

The summability criteria is more significant in the development of the Dixmier trace in noncommutative integration theory: when $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a spectral triple of dimension $s_0 > 0$, we can ask whether additionally the spectral triple is $\mathcal{L}^{(s_0,\infty)}$- summable, that is, the operator $(1+\mathcal{D}^2)^{-s_0/2}$ belongs to the Dixmier ideal $\mathcal{L}^{(1,\infty)}$.In this case, a well known procedure as highlighted in [31] leads to the construction of a trace on A, for each suitably chosen generalised limit Lim_ω, via the map

$$\tau_\omega : a \mapsto \mathrm{Tr}_\omega(a(1+\mathcal{D}^2)^{-s_0/2}), \tag{2.2.6}$$

(see **Appendix A** for details). Connes shows that the Dixmier trace provides a geometric realisation of the Chern character, which represents each finitely summable spectral triple in the corresponding Hochschild class $HC^n(\mathcal{A})$ of cyclic cohomology, for $n \in \mathbb{Z}^+$ sufficiently large. For each $(n+1)$-tuple in $\mathcal{A}$, the formula (in the odd case) is

$$\psi_\omega(a_0, \dots, a_n) := \mathrm{Tr}_\omega(a_0[\mathcal{D}, a_1]\dots[\mathcal{D}, a_n](1+\mathcal{D}^2)^{-n/2}), \tag{2.2.7}$$

Connes and Moscovici improved on this work in their paper [35], achieving a noncommutative statement of the Atiyah-Singer index theorem for spectral triples in relative generality.

2.2.6 Connes' distance formula on the state space.

One of the most interesting properties which comes out of a spectral triple in differential geometry is the ability to recover the geodesics of the manifold. For a given spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ on A, Connes' *extended metric*(1). $d : S(A) \times S(A) \mapsto [0, \infty]$ is defined by

$$d_C(\omega_1, \omega_2) := \sup\{|\omega_1(a) - \omega_2(a)| : \; a = a^* \in \mathcal{A}, \; \|[\mathcal{D}, \pi(a)]\| \leqslant 1\} \tag{2.2.8}$$

The motivating example is prescribed by a the Dirac triple on a connected spin^c manifold $\mathcal{M}$ for which $\|[\mathcal{D}, f]\| = \|\mathrm{grad}(f)\|$. The restriction of Connes' metric to the point evaluation measures $d_C(p_x, p_y)$ then coincides with the path metric $d_\gamma(x, y)$ along $\mathcal{M}$ [31]. If we require instead that the metric d_C is merely *equivalent* to the geodesic metric along $\mathcal{M}$, then for each spectral triple on $\mathcal{M}$ we require only that there exists a constant $K > 0$ such that

$$\frac{1}{K}\|\mathrm{grad}(f)\| \leqslant \|[\mathcal{D}, f]\| \leqslant K\|\mathrm{grad}(f)\|, \; f \in C^1(\mathcal{M}),$$

(1) This is usually called a pseudometric in the literature but this confuses the common definition of a pseudometric, which takes only finite values but where two non identical points can have distance zero.

and this can be fulfilled with a wide range of spectral triples, meaning that the spin^c condition is not really necessary. *A priori*, Connes' metric d_C depends on the algebra $\mathcal{A}$.

In general, a spectral triple on a unital C^*-algebra A determines, via Connes' formula, an extended metric d on $S(A)$. The topology that d generates is stronger than the weak*-topology of $S(A)$, so that, as Rieffel points out in [100], the natural question to ask is when the topologies coincide. We shall discuss this in more detail in Chapter 3.

2.2.7 Riemannian metrics.

In general, the operator *-algebra $C^1(A)$ acts over $\mathcal{H}_1$, viewed as an operator space, as a left module as in the proof of Proposition 2.2.3. It seems reasonable, for spectral triples for which the Dixmier trace $\tau_\omega(a)$ is well defined for each $a \in C^1(A)$ (that is, a is *measurable*, c.f **Appendix A**), to provide an abstract definition of a Riemannian metric.

Definition 2.2.6. A *Riemannian metric* on a finitely summable spectral triple $(C^1(A), \mathcal{H}, \mathcal{D}) \in \mathcal{L}^{p,\infty}$, such that each $a \in C^1(A)$ is a measurable operator, is a left A-linear hermitian form,

$$\mathfrak{g}(\cdot|\cdot) : \mathcal{H}_1 \times \mathcal{H}_1 \mapsto C^1(A), \tag{2.2.9}$$

such that $\mathcal{H}_1$ is a countably generated operator *-module over $C^1(A)$ and furthermore there exists $n \in \mathbb{Z}^+$ such that,

$$\langle \xi, \eta \rangle_{\mathcal{H}} = \tau(\mathfrak{g}(\eta|\xi)), \quad \eta, \xi \in \mathcal{H}_1, \tag{2.2.10}$$

where $\tau(a) = \mathrm{Tr}_\omega(a|\mathcal{D}|^{-n})$ for any suitably chosen generalised limit Lim_ω.

2.3 Examples of spectral triples.

The study of spectral triples in differential geometry has a vast literature and we highly recommend the survey articles [115], [98] for well-known examples and theory. As we are more focused on the functional analytic aspects of these constructions, we shall not again visit many of these ideas, except where there operator algebraic language also applies. The first examples of a spectral triple for C^*-algebras are those arising from discrete groups.

Example 2.3.1. Connes ([30]) used this correspondence between $C(\mathbb{T})$ and $C^*(\mathbb{Z})$, coming from Pontryagin duality theory, to construct spectral triples on all **reduced group C^*-algebras** coming from discrete finitely generated groups Γ, equipped with a proper length function, that is, a map $\ell : \Gamma \mapsto \mathbb{Z}^+$ such that

$$\ell(g) = 0 \iff g = 1_\Gamma,\ \ell(gh) \leqslant \ell(g) + \ell(h),\ \ell(g) = \ell(g^{-1})\ \forall g, h \in \Gamma,$$

$$\ell^{-1}(\{0, \dots, n\}) \text{ is finite } \forall n \in \mathbb{Z}^+.$$

It is seen that the triple $(C^1(\Gamma), \ell_2(\Gamma), M_\ell)$, where $M_\ell e_g = \ell(g) e_g$ is a multiplication operator and $C^1(\Gamma) := \{x \in C^*_r(\Gamma) :\ [\mathcal{D}, x] \in B(\ell_2(\Gamma))\}$ contains the group ring $\mathbb{C}[\Gamma]$, satisfies the axioms of a spectral triple.

Example 2.3.2. A straightforward example of a spectral triple on a finite dimensional C^*-algebra is the algebra $M_n(\mathbb{C})$, represented over itself and with the Dirac operator given by transposition $M \mapsto M^t$. Finite dimensional C^*-algebras are the only instances in which a spectral triple can have dimension zero.

Example 2.3.3. Spectral triples have been relatively successful in describing **fractal geometry**. Some of the well studied examples include Cantor sets, Sierpinski gaskets [29] and aperiodic tiling spaces [68]. Although such spaces are not manifolds, the study of the metric aspects is quite well developed. For example, the authors [92] showed that starting from any ultrametric d on the Cantor set X, there is a spectral triple on $C(X)$ whose metric recovers d, at least up to Lipschitz equivalence.

Example 2.3.4. Constructions of spectral triples for approximately finite dimensional (AF) C^*-algebras were proposed by Christensen and Ivan [26]. The idea is to use each AF filtration $(A_n)_{n\geqslant 1}$ of A to construct an odd spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ on A such that $\cup_{n\geqslant 1} A_n \subset \mathcal{A}$. In this way $(A_n)_{n\geqslant 1}$ can be identified naturally with the filtration of A coming from the spectral projections $\{Q_n : n \geqslant 1\}$ of $\mathcal{D}$, namely

$$A_n := \{a \in \mathcal{A} :\ [Q_i, \pi(a)] = 0\ \forall i \geqslant n\}. \tag{2.3.1}$$

We shall outline this construction a little later on. The idea of using a filtration of a C^*-algebra, not necessarily AF, has been pursued in a few places (e.g [110], where general constructions of spectral triples were proposed for all quasidiagonal C^*-algebras) but, without the extra structure present in Christensen-Ivan's construction, it generally becomes difficult to say anything about the summability and metric properties.

2.4 Unbounded KK-theory.

Following the seminal works of Baaj and Julg [4], the study of unbounded Kasparov theory is all over the literature these days. Unbounded KK-theory provides a similar refinement to ordinary KK-theory that spectral triples provide to refine K-homology. Unbounded KK-theory arises naturally in the geometry of fibrations and also provides a useful description of the spectral flow formula, among many other things. We will tend to view unbounded KK-theory as correspondences between spectral triples, which is broadly how they were introduced in Mesland's dissertation [80].

Definition 2.4.1. [4] Let A and B be separable C^*-algebras. An unbounded (Kasparov) A-B cycle is given by a triple $(\mathcal{A}, E_B, \mathfrak{D})$, where E_B is a countably generated right Hilbert B-module, together with a *-homomorphism $\pi : A \mapsto \mathcal{L}(E_B)$, a norm-dense *-subalgebra $\mathcal{A} \subset A$ and a densely defined self-adjoint linear operator $\mathfrak{D}$ on E_B such that

1. $\mathfrak{D}$ is *regular*, i.e the operator $1 + \mathfrak{D}^*\mathfrak{D}$ is also densely defined and self-adjoint,
2. $\pi(\mathcal{A})\mathrm{dom}\mathfrak{D} \subset \mathrm{dom}\mathfrak{D}$ and $[\mathfrak{D}, \pi(a)] : \mathrm{dom}\mathcal{D} \mapsto E_B$ extends to an adjointable bounded operator in $\mathcal{L}(E_B)$ for each $a \in \mathcal{A}$ and
3. $\pi(a)(1 + \mathfrak{D}^*\mathfrak{D})^{-1/2}$ extends to an element of $\mathcal{K}(E_B)$ for each $a \in A$.

The cycle $(\mathcal{A}, E_B, \mathfrak{D})$ is said to be *graded* or *even* if there exists a self-adjoint operator $\gamma \in \mathcal{L}(E_B)$ such that $\gamma^2 = \mathrm{id}$, $\gamma\pi(a) = \pi(a)\gamma$ for each $a \in A$ and $\gamma\mathfrak{D} = -\mathfrak{D}\gamma$. Otherwise $(\mathcal{A}, E_B, \mathfrak{D})$ is called *ungraded* or *odd*.

A distinction is usually made between odd and even cycles, as with ordinary K-homology. When $B = \mathbb{C}$ in the definition, so that the cycle is expressed using a Dirac operator over a Hilbert space with compact resolvent, then the definition, up to faithfulness of the map $\pi : A \mapsto \mathcal{L}(E_B)$ is that of a spectral triple.

Remark 2.4.2. In the literature, the algebra $\mathcal{A}$ is not usually specified. We will often find it convenient to assume $\mathcal{A} = C^1(A, B)$, defined as the algebra,

$$C^1(A, B) := \{a \in A : a(\mathrm{dom}\mathfrak{D}) \subset \mathrm{dom}\mathfrak{D} \text{ and } [\mathfrak{D}, a] \text{ extends to a adjointable operator in } \mathcal{L}(E_B)\},$$

which becomes an operator algebra over E_B with respect to the norm $\|a\|_1 := \|a\|_A + \|[\mathfrak{D}, \pi(a)]\|$. In some instances, we may want to stress the $*$-homomorphism $\pi : A \mapsto \mathcal{L}(E_B)$ as well and write each unbounded cycle as a triple $(\pi, E_B, \mathfrak{D})$.

Proposition 2.4.3. *Let $(E_B, \mathfrak{D})$ be an unbounded A-B cycle, where A and B are separable C^*-algebras. Then $E_1 := \mathrm{dom}(\mathfrak{D})$ becomes a right Hilbert B-bimodule with respect to the inner product $\langle e_1, e_2\rangle_1 := \langle e_1, e_2\rangle + \langle \mathfrak{D}e_1, \mathfrak{D}e_2\rangle$. The natural left action of A on E_B restricts to a left action of $C^1(A, B)$ on E_1 and the natural map $(C^1(A, B), \|\cdot\|_1) \mapsto (\mathcal{L}(E_1), \|\cdot\|_{\mathcal{L}(E_1)})$ is a norm-decreasing algebra homomorphism.*

Proof. Essentially the same as in Proposition 2.2.3. □

There is, to my knowledge, no general understanding of the equivalences between unbounded cycles which captures the homotopy invariance of KK-theory. Orthogonal direct sum trivially gives the sets $\mathfrak{K}\mathfrak{K}^1(A, B)$ and $\mathfrak{K}\mathfrak{K}^0(A, B)$ of all unbounded odd (respectively even) A-B cycles of the form $(E_B, \mathfrak{D})$ an abelian semigroup structure. The correspondence between bounded and unbounded KK-cycles is then provided by the following result:

Theorem 2.4.4. *[4] Let A and B be separable C^*-algebras, let $p \in \mathbb{Z}/2\mathbb{Z}$ and suppose $(E_B, \mathfrak{D}) \in \mathfrak{K}\mathfrak{K}^p(A, B)$ is an unbounded A-B cycle. Then $(E_B, \mathfrak{D}(1 + \mathfrak{D}^*\mathfrak{D})^{-1/2}) \in KK^p(A, B)$. The map $\mathfrak{b} : \mathfrak{K}\mathfrak{K}^p(A, B) \mapsto KK^p(A, B)$ defined this way is a surjective group homomorphism.*

Example 2.4.5. If A, B, C are C^*-algebras, $(\pi, E_C, \mathfrak{D}) \in \mathcal{KK}^1(B, C)$ and $\sigma : A \mapsto B$ is a $*$-homomorphism such that $\sigma^{-1}(C^1(B, C))$ is dense in A then $(\pi \circ \sigma, E_C, \mathfrak{D}) \in \mathcal{KK}^1(A, C)$. We shall label this cycle $\sigma^*(\pi, E_C, \mathfrak{D})$. This is clearly compatible with the contravariant functor σ^* in ordinary KK-theory: $\mathfrak{b}(\pi, E_C, \mathfrak{D}) \in KK^1(B, C) \mapsto \sigma^*(\mathfrak{b}(\pi, E_C, \mathfrak{D})) \in KK^1(A, C)$.

2.5 The external Kasparov product.

One of the most useful properties in unbounded Kasparov theory is the ability to write down representatives of the external Kasparov product using explicit cycles. The external Kasparov product in ordinary KK-theory is an associative map

$$KK^p(A_1, B_1) \times KK^q(A_2, B_2) \mapsto KK^{p+q}(A_1 \otimes_{min} A_2, B_1 \otimes_{min} B_2), \tag{2.5.1}$$

but it is hard to write down explicitly. Conversely, Baaj and Julg show that this becomes relatively easy when the cycles are unbounded.

Given two cycles $(E_{B_1}, \mathfrak{D}_1) \in \mathcal{KK}(A_1, B_1)$, $(E_{B_2}, \mathfrak{D}_2) \in \mathcal{KK}(A_2, B_2)$, there is a natural representation of $A_1 \otimes_{min} A_2$ on the external tensor product $E_{B_1} \bar{\otimes} E_{B_2}$. The derivations $\delta_i(a) := [\mathfrak{D}_i, a]$ determine a derivation δ on the algebraic tensor product $C^1(A_1, B_1) \odot C^1(A_2, B_2)$ via

$$\delta(a_1 \otimes a_2) := \delta_1(a_1) \otimes a_2 + a_1 \otimes \delta_2(a_2). \tag{2.5.2}$$

Let $C^1(A_1, B_1) \otimes_1 C^1(A_2, B_2)$ be the completion of this algebra with respect to the norm $\|x\|_1 := \|x\|_{A_1 \otimes_{min} A_2} + \|\delta(x)\|_{A_1 \otimes_{min} A_2}$, which is then a Banach *-algebra. The statement of the external tensor product, in the ungraded case, is the following:

Theorem 2.5.1. *[4] Let A_1, A_2, B_1 and B_2 be separable C^*-algebras and let $(E_{B_i}, \mathfrak{D}_i)$ be unbounded odd A_i-B_i cycles for each $i \in \{1, 2\}$. Write $A := A_1 \otimes_{min} A_2$ and $B := B_1 \otimes_{min} B_2$. As usual let $\bar{\otimes}$ denote the external tensor product. Then the triple*

$$\left((E_{B_1} \bar{\otimes} E_{B_2}) \oplus (E_{B_1} \bar{\otimes} E_{B_2}), \begin{bmatrix} 0 & \mathfrak{D}_1 \bar{\otimes} 1 - i \bar{\otimes} \mathfrak{D}_2 \\ \mathfrak{D}_1 \bar{\otimes} 1 + i \bar{\otimes} \mathfrak{D}_2 & 0 \end{bmatrix} \right), \tag{2.5.3}$$

defines an unbounded even A-B cycle. It defines a pairing

$$\times_e : \mathfrak{KK}^1(A_1, B_1) \times \mathfrak{KK}^1(A_2, B_2) \mapsto \mathfrak{KK}^0(A, B). \tag{2.5.4}$$

Moreover there is a commuting diagram;

$$\begin{tikzcd} \mathfrak{KK}^1(A_1, B_1) \times \mathfrak{KK}^1(A_2, B_2) \arrow[r, "\times_e"] \arrow[d, "\mathfrak{b} \times \mathfrak{b}"] & \mathfrak{KK}^0(A, B) \arrow[d, "\mathfrak{b}"] \\ KK^1(A_1, B_1) \times KK^1(A_2, B_2) \arrow[r, "\times_e"] & KK^0(A, B). \end{tikzcd}$$

Remark 2.5.2. Similar results may be obtained, via standard changes to the grading, in the situation when either or both of the given cycles are even. In the most general form, the external Kasparov product in unbounded KK-theory is the existence of a commuting diagram

$$\begin{tikzcd} \mathfrak{KK}^p(A_1,B_1)\times\mathfrak{KK}^q(A_2,B_2) \arrow[r, "\times_e"] \arrow[d, "\mathfrak{b}\times\mathfrak{b}"] & \mathfrak{KK}^{p+q}(A,B) \arrow[d, "\mathfrak{b}"] \\ KK^p(A_1,B_1)\times KK^q(A_2,B_2) \arrow[r, "\times_e"] & KK^{p+q}(A,B). \end{tikzcd}$$

where both p, q and $p+q$ are understood mod 2. Explicit expressions can be found elsewhere (see the exercises in [60]).

2.6 The internal Kasparov product.

The internal Kasparov product in ordinary KK-theory is a pairing,

$$KK^p(A,B)\otimes_B KK^q(B,C)\mapsto KK^{p+q}(A,C), \tag{2.6.1}$$

with $p, q, p+q \in \mathbb{Z}/2\mathbb{Z}$. To show that this product can be represented by unbounded cycles is rather more of a challenge, but has attracted plenty of interest elsewhere.

Definition 2.6.1. Recall that, given two Hilbert modules E and F, such that E is an A-B bimodule and F is a B-C bimodule, the inner tensor product Hilbert module $E\otimes_B F$ is the A-C bimodule given by completion of the vector tensor product $E\odot F$ with respect to the inner product rule

$$\langle e_1\odot f_1, e_2\odot f_2\rangle := \langle f_1, \langle e_1, e_2\rangle f_2\rangle. \tag{2.6.2}$$

The difficulties associated with this product entail lifting unbounded Fredholm operators on each of E and F to operators on $E\otimes_B F$. This can be addressed by the use of connections, as developed by Cuntz and Quillen [42]. Kucerovsky established, under natural assumptions, a set of sufficient criteria for a given unbounded cycle to represent the internal Kasparov product of two other given cycles:

Theorem 2.6.2. *[74] Let A, B, C be separable C^*-algebras and let (E,S), (F,T) be cycles in $\mathfrak{KK}^1(A,B)$ and $\mathfrak{KK}^1(B,C)$ respectively, Then $(E\otimes_B F,\mathfrak{D})\in\mathfrak{KK}^0(A,C)$, equipped with the usual $\mathbb{Z}_2$ grading and the natural representation of A on $E\otimes_B F$, represents the Kasparov product of (E,S) and (F,T) provided:*

1. $\left[\left(\begin{pmatrix}\mathfrak{D} & 0\\ 0 & \hat{T}\end{pmatrix}, \begin{pmatrix}0 & T_e\\ T_e^* & 0\end{pmatrix}\right]$ *extends to a bounded operator on* $(E\otimes_B F)^2\oplus F^2$ *for* e *contained in a dense subset of* $A\cdot E$,
2. $\mathrm{Dom}(\mathfrak{D})\subset \mathrm{Dom}(\hat{S})$ *and*
3. *there exists a constant* $C>0$ *such that* $\langle \mathfrak{D}x, \hat{S}x\rangle + \langle \hat{S}x, \mathfrak{D}x\rangle \leqslant C\langle x, x\rangle$, *where* $x\in \mathrm{Dom}(\mathfrak{D})$.

Here, T_e *is the completely bounded adjointable operator* $(f_1, f_2)\mapsto (e\otimes f_1, e\otimes f_2)$, $\hat{T} := \begin{pmatrix}0 & -iT\\ iT & 0\end{pmatrix}$ *and* $\hat{S} := \begin{pmatrix}0 & S\otimes 1\\ S\otimes 1 & 0\end{pmatrix}$.

Example 2.6.3. Let $(\mathcal{M}, \mathfrak{g})$ be a Riemannian manifold and F a Clifford bundle over $\mathcal{M}$. A spectral triple over $C(\mathcal{M})$ can be defined as usual by defining a Dirac operator on $L^2(\mathcal{M}, F)$ as the composition of a Hermitian connection $\nabla^F : L^2(\mathcal{M}, F)\mapsto \mathcal{A}^1(\mathcal{M})\otimes L^2(\mathcal{M}, F)$ and the Clifford map $c : \mathcal{A}^1(\mathcal{M})\otimes L^2(\mathcal{M}, F)\mapsto L^2(\mathcal{M}, F)$, where $\mathcal{A}^1(\mathcal{M})$ is the (graded) algebra of differential 1-forms. This defines an unbounded cycle,

$$(C^1(\mathcal{M}), L^2(\mathcal{M}, F), c\circ\nabla^F)\in \mathfrak{KK}^p(C(\mathcal{M}), \mathbb{C}), \tag{2.6.3}$$

where p is determined by the grading of F. Moreover, each Hermitian Clifford bundle E over $\mathcal{M}$, equipped with a Hermitian pairing $(\cdot|\cdot) : E\times E\mapsto C(\mathcal{M})$ defines, for some q determined by the grading of E, an unbounded cycle

$$(C(\mathcal{M}), E, \mathrm{Id}_E)\in \mathfrak{KK}^q(C(\mathcal{M}), C(\mathcal{M})). \tag{2.6.4}$$

Now let $\nabla^E : E\mapsto E\otimes_{C^1(\mathcal{M})}\mathcal{A}^1(\mathcal{M})$ be any connection on E which is compatible with $\langle\cdot,\cdot\rangle$. In particular ∇^E is a well defined linear map with the properties,

$$(1)\ \nabla^E(fe) = f\nabla^E(e) + df\otimes e;\ \forall e\in E,\ f\in C^1(\mathcal{M}), \tag{2.6.5}$$

$$(2)(\nabla^E(e_1)|e_2) - (e_1|\nabla^E(e_2))^* = d(\langle e_1, e_2\rangle);\ \forall e_1, e_2\in E, \tag{2.6.6}$$

where $(\cdot|\cdot) : E\times(E\otimes_{C^1(\mathcal{M})}\mathcal{A}^1(\mathcal{M}))\mapsto \mathcal{A}^1(\mathcal{M})$ is the usual Hermitian pairing defined by $(e_1|e_2\otimes df) := \langle e_1, e_2\rangle df$. In this way, the map $\nabla^{E,F} : E\otimes_{C^1(\mathcal{M})} L^2(\mathcal{M}, F)\mapsto E\otimes_{C^1(\mathcal{M})}$

$\mathcal{A}^1(\mathcal{M}) \otimes_{C^1(\mathcal{M})} L^2(\mathcal{M}, F)$ given by:

$$\nabla^{E,F}(e \otimes_{C^1(\mathcal{M})} f) = \nabla^E(e) \otimes_{C(\mathcal{M})} f + e \otimes_{C^1(\mathcal{M})} \nabla^F(f) \tag{2.6.7}$$

is well defined for each $e \in E$ and $f \in C^1(\mathcal{M})$. Composition with the Clifford map, along with the natural identification $E \otimes_{C^\infty(\mathcal{M})} L^2(\mathcal{M}, F)) \cong L^2(\mathcal{M}, E \otimes_{C^\infty(\mathcal{M})} F)$, defines a self-adjoint operator $c \circ \nabla^{E,F}$ over $L^2(\mathcal{M}, E \otimes_{C^\infty(\mathcal{M})} F)$ and a spectral triple

$$(C^1(\mathcal{M}), L^2(\mathcal{M}, E \otimes_{C^1(\mathcal{M})} F), c \circ \nabla^{E,F}) \in \mathfrak{KK}^{p+q}(C(\mathcal{M}), \mathbb{C}). \tag{2.6.8}$$

We can view this construction as a pairing

$$\mathfrak{KK}^q(C(\mathcal{M}), C(\mathcal{M})) \otimes_{C(\mathcal{M})} \mathfrak{KK}^p(C(\mathcal{M}), \mathbb{C}) \mapsto \mathfrak{KK}^{p+q}(C(\mathcal{M}), \mathbb{C}). \tag{2.6.9}$$

In order for a construction of this kind to be accessible to operator algebras, there are various things to be addressed. Firstly, an algebraic treatment of Hermitian connections on more general operator *-algebras is needed. Then there is the challenge of adapting the treatment to operator *-modules defined over *-algebras, i.e to keep track of the first order differential structure. Most challenging of all is to establish conditions in which the resultant triple has compact resolvent. The presentations of [80], [79] and [61] address each of these concerns, but it is the latter approach which is most relevant to our purposes. The final part of this chapter reviews section 4-7 of that paper.

The motivation behind the construction and terminology therein is that given two unbounded modules $(E, S) \in \mathfrak{KK}(A, B)$ and $(F, T) \in \mathfrak{KK}(B, C)$, the cycles should be composible if and only if the smooth structures are aligned in the right way. Kaad and Lesch achieve this by constructing correspondences, which can be viewed as those connections which themselves satisfy first order differential criteria.

Definition 2.6.4. Let $\mathcal{B}$ be an operator *-algebra and δ a derivation on $\mathcal{B}$. As usual, the algebra of *1-forms* on $\mathcal{B}$ is given by $\Omega^1_\delta(\mathcal{B}) := \{\sum a_i\delta(b_i) : a_i, b_i \in \mathcal{B}\}$, which we will typically view as a right operator *-module over itself and a left operator *-module over $\mathcal{B}$ in the obvious way.

δ will be called *essential* if $\mathcal{B}\Omega^1_\delta(\mathcal{B})$ is dense in $\Omega^1_\delta(\mathcal{B})$.

Given (E,S) in $\mathfrak{KK}^1(A,B)$ and (F,T) in $\mathfrak{KK}^1(B,C)$, suppressing notation, we can consider the operator module $\Omega^1_T(B_1)$, where B_1 is a dense *-subalgebra of $C^1(B,C)$ and $\delta_T = [T,\cdot]$. We can view this as a B-$\mathcal{L}(F)$ bimodule. After completing in the norm $\|b\|_1 := \|b\|_B + \delta_T(b)$, we can view B_1 as an operator *-algebra.

Definition 2.6.5. Let (E,S) in $\mathfrak{KK}^1(A,B)$ and (F,T) in $\mathfrak{KK}^1(B,C)$ as above and suppose that $E_1 \subset E$ is an operator A-B submodule which is also a countably generated right operator *-module over B_1. In other words $E_1 \cong P(B_1)_\infty$ is an orthogonal summand in the standard right module $(B_1)_\infty$. Suppose that δ_T is an essential derivation on B_1. The Graßmannian connection is the completely bounded linear map $\nabla_T : E_1 \mapsto E_1 \otimes_{B_1} \Omega^1(B_1)$ defined by:

$$\nabla_T((b_n)_{n\in\mathbb{N}}) := (P\otimes 1)(\delta_T(b_n)_{n\in\mathbb{N}}). \tag{2.6.10}$$

Lemma 2.6.6. *[80] [61] $\nabla_T : E_1 \mapsto E_1 \otimes_{B_1} \Omega^1(B_1)$ is a Hermitian δ_T connection. It satisfies*

$$\nabla_T(eb) = \nabla_T(e)b + e\otimes\delta_T(b), \quad e\in E_1,\ b\in B_1, \text{ and} \tag{2.6.11}$$

$$(e_1|\nabla_T(e_2)) - (\nabla_T(e_1)|e_2)^* = \delta_T(\langle e_1,e_2\rangle_B),\ e_1,e_2\in E_1. \tag{2.6.12}$$

where $(\cdot|\cdot) : E_1 \times (E_1 \otimes_{B_1} \mathcal{L}(F)) \mapsto \mathcal{L}(F)$ is given by $(e_1|e_2\otimes_{B_1} x) := \langle e_1,e_2\rangle_{B_1} x$.

Corollary 2.6.7. *[80] [61] There is a densely defined linear operator $1\otimes_{\nabla_T} T :\in \mathcal{L}(E\otimes_B F)$ defined by*

$$(1\otimes_{\nabla_T} T)(e\otimes_B f) := e\otimes_B Tf + c\circ(\nabla_T(e)\otimes_{\Omega^1(B_1)} f);\ e\in E,\ f\in F, \tag{2.6.13}$$

where $c : E\otimes_{B_1}\Omega^1(B_1)\otimes_{\Omega^1(B_1)} F \mapsto E\otimes_B F$ is the natural map.

Definition 2.6.8. (compare with [61].) Let (E,S) in $\mathfrak{KK}^1(A,B)$ and (F,T) in $\mathfrak{KK}^1(B,C)$, where (F,T) is essential (that is, the left action of B on $\mathcal{L}(F)$ is essential and the derivation δ_T is essential). A *correspondence* from (E,S) to (F,T) comprises a pair (B_1,E_1), where E_1 is a right operator *-module over an operator *-algebra B_1, equipped with the Graßmannian δ_T- connection $\nabla_T : E_1 \mapsto E_1\otimes_{B_1}\Omega^1(B_1)$ such that the following hold:

1. $B_1 \subset C^1(B, C))$ as a dense *-subalgebra and $E_1 \subset E$ as a dense pre-Hilbert submodule.

2. There exists a dense *-subalgebra $A_1 \subset C^1(A, B)$ such that the commutators $[1 \otimes_{\nabla_T} T, a]$ are densely defined and extend to a bounded operator in $\mathcal{L}(E \otimes_B F)$ for each $a \in A_1$.
3. The operators $s := S \otimes 1, t := 1 \otimes_{\nabla_T} T : \mathrm{Dom}(s) \cap \mathrm{Dom}(t) \mapsto E \otimes_B F$ have a dense common domain of definition and furthermore for each $\mu \in \mathbb{R} \setminus \{0\}$ the operator $[s, t](s - i\mu)^{-1} : \mathrm{Dom}(t) \mapsto E \otimes_B F$ extends to a bounded operator in $\mathcal{L}(E \otimes_B F)$.

Theorem 2.6.9. *[61] Let $(E, S) \in \mathfrak{KK}^1(A, B)$ and $(F, T) \in \mathfrak{KK}^1(B, C)$, where (F, T) is essential, and let (B_1, E_1) be any correspondence from (E, S) to (F, T). Then*

$$\left((E \otimes_B F) \oplus (E \otimes_B F), \begin{bmatrix} 0 & s - it \\ s + it & 0 \end{bmatrix}\right), \tag{2.6.14}$$

defines an element of $\mathfrak{KK}^0(A, C)$. Moreover, the pairing $\otimes_{(B_1, E_1)} : \mathfrak{KK}^1(A, B) \times \mathfrak{KK}^1(B, C) \mapsto \mathfrak{KK}^0(A, C)$ defined by

$$(E, S) \otimes_{(B_1, E_1)} (F, T) := (E \otimes_B F \otimes \mathbb{C}^2, \begin{bmatrix} 0 & s - it \\ s + it & 0 \end{bmatrix}), \tag{2.6.15}$$

represents the Kasparov product $\otimes_B : KK^1(A, B) \times KK^1(B, C) \mapsto KK^0(A, C)$. That is, we have a commuting diagram:

$$\begin{tikzcd} \mathfrak{KK}^1(A, B) \times \mathfrak{KK}^1(B, C) \arrow[r, "\otimes_{(B_1, E_1)}"] \arrow[d, "\mathfrak{b}"] & \mathfrak{KK}^0(A, C) \arrow[d, "\mathfrak{b}"] \\ KK^1(A, B) \times KK^1(B, C) \arrow[r, "\otimes_B"] & KK^0(A, C). \end{tikzcd}$$

By making suitable minor adjustments analogous to the external tensor product construction, it is possible to define correspondences between even cycles, or between cycles one of which is even and the other odd.

3: Quantum metric spaces

3.1 Compact quantum metric spaces.

The purpose of this section is in part to give a review of the development of the *quantum metric space* and to include some relevant observations and examples for later chapters. The *compact quantum metric space* was a definition coined by Rieffel in a series of papers in the late nineties [100], [101], [102] to describe those C*-algebras admitting a so-called *Lipschitz seminorm*, with which the state space can be endowed with a metric structure. The terminology owes itself to the setting of ordinary Lipschitz functions, since a compact metric space (X, d) defines a densely defined seminorm $L_{\mathtt{Lip}}$ on the algebra ring $C(X, \mathbb{R})$ of real-valued functions via

$$L_{\mathtt{Lip}}(f) := \sup\left\{\frac{|f(x) - f(y)|}{d(x, y)} : \ x, y \in X, \ x \neq y\right\}. \tag{3.1.1}$$

and this can in some sense be thought to encode *all* the metric information for X, since the metric is recovered via $d(x, y) := \sup\{|f(x) - f(y)| : \ L_{\mathtt{Lip}}(f) \leqslant 1\}$. More significantly, Monge and Kantorovich [64] show that the metric extends to the state space $\mathcal{S}(C(X))$ of Borel probability measures on X, where the formula becomes

$$d(\mu, \nu) := \sup\{|\mu(f) - \nu(f)| : \ f \in C(X, \mathbb{R}), \ L_{\mathtt{Lip}}(f) \leqslant 1\}. \tag{3.1.2}$$

In the greatest generality, this metric can be used in the study of order-unit spaces, which is the context of Rieffel's programme. As a rule we are more interested in C*-algebras than order-unit spaces, so the change in approach which we present can be motivated by "keeping track of the topology".

In general, when $L : C(X, \mathbb{R}) \mapsto [0, \infty]$ is a densely defined seminorm, the function defined by $d(\mu, \nu) := \sup\{|\mu(f) - \nu(f)| : \ f \in C(X, \mathbb{R}), \ L(f) \leqslant 1\}$ is an *extended metric*: it is a metric up to the possibility that it may take infinite values on certain states. The first task is, then, to find necessary and sufficient conditions on L such that d is a metric.

Proposition 3.1.1. *Let* X *be a compact Hausdorff topological space,* $d : X \times X \mapsto \mathbb{R}$ *be an extended metric and* $L_{\mathrm{Lip}} : C(X, \mathbb{R}) \mapsto [0, \infty]$ *be as in (3.1.1). Let* $\mathcal{O} := \{(x, y) \in X \times X : \ d(x, y) < \infty\}$*. Then (1)* $\mathcal{O}$ *is dense in* $X \times X$ *if and only if (2)* $\{f \in C(X, \mathbb{R}) : \ L(f) = 0\} = \mathbb{R}I$.

Proof. If (2) fails then there exists an $f \in C(X, \mathbb{R})$ and a pair $(x_0, y_0) \in X \times X$ such that $L(f) = 0$ and $f(x_0) \neq f(y_0)$. By continuity of f this holds if and only if there is an open set $\mathcal{O}$ containing (x_0, y_0) such that $f(x) \neq f(y)$ for each pair $(x, y) \in \mathcal{O}$. From (3.1.2), d takes infinite values on $\mathcal{O}$ and hence (1) fails. Conversely if (1) fails then there is an open subset $\mathcal{O}$ of $X \times X$ on which d takes infinite values, so $L(f) = 0$ for any non-constant $f \in C(X, \mathbb{R})$ supported in $\mathcal{O}$. □

The above observation supports the idea that a good seminorm for metric purposes should satisfy a nondegeneracy condition.

Definition 3.1.2. A seminorm L on a unital algebra $\mathcal{A}$ is called *nondegenerate* if $\{a \in \mathcal{A} : L(a) = 0\} = \mathbb{R}I$.

Definition 3.1.3. Let A be a unital separable C^*-algebra A and $A_{sa} := \{a = a^* \in A\}$. A *Lipschitz pair* $(\mathcal{A}, L)$ on A comprises a dense unital subalgebra $\mathcal{A} \subset A_{sa}$ and a nondegenerate seminorm $L : \mathcal{A} \mapsto [0, \infty)$.

A Lipschitz pair $(\mathcal{A}, L)$ on A is seen to determine an extended metric $d_{\mathcal{A},L}$ over $S(A)$ (occasionally written $d_{\mathcal{A}}$, or d_L, if the Lipschitz pair is understood) in a way which provides a noncommutative analogue of the Monge-Kantorovich distance when A is commutative; the metric is just

$$d_{\mathcal{A},L}(\omega_1, \omega_2) := \sup\{|\omega_1(a) - \omega_2(a)| : \ a \in \mathcal{A}, \ L(a) \leqslant 1\}. \tag{3.1.3}$$

Conversely a metric d on $S(A)$ defines a nondegenerate seminorm L_d on A via

$$L_d(a) := \sup\left\{\frac{|\mu(a) - \nu(a)|}{d(\mu, \nu)} : \ \mu, \nu \in S(A), \ \mu \neq \nu\right\}. \tag{3.1.4}$$

3.1.1 Recovering a metric from a Lipschitz pair, and vice-versa.

Proposition 3.1.4. *Let A be a unital C^*-algebra and let d be a metric on $S(A)$. Consider the pair $(\mathcal{A}, L_d)$, defined for some algebra $\mathcal{A} \subset A_{sa} \cap \mathrm{dom} L_d$, with L_d defined as in equation (3.1.4), which may or may not be a Lipschitz pair. Then $d_{\mathcal{A},L_d}$ defined by (3.1.3) is a pseudometric and $d_{\mathcal{A},L_d} \leqslant d$. Moreover if $d_{\mathcal{A},L_d}$ is a metric and the topology d generates is compact, then the two metrics are topologically equivalent.*

Proof. The first part is immediate from the definitions. With the extra assumptions in place, it follows that the identity map $\iota : (S(A), d) \mapsto (S(A), d_{\mathcal{A},L_d})$ is a continuous map from a compact to a Hausdorff topological space, whence a homeomorphism. □

The opposite problem, namely when a Lipschitz seminorm may be recovered from the metric that it induces, was considered in Chapter 4 of [101]. Using (3.1.3) and (3.1.4), it is clear that if $(\mathcal{A}, L)$ is a Lipschitz pair on A then so is $(\mathcal{A}, L_{d_{\mathcal{A},L}})$. The most important feature of the seminorm $L_{d_{\mathcal{A},L}}$ is that it is lower semicontinuous, whether or not L itself is.

Definition 3.1.5. A Lipschitz pair $(\mathcal{A}, L)$ is called *lower semicontinuous* if whenever $(a_n) \subset \mathcal{A}$ is a sequence converging to $a \in \mathcal{A}$ such that $L(a_n) \leqslant R$ for each $n \in \mathbb{N}$ then also $L(a) \leqslant R$.

Proposition 3.1.6. *[101] If A is a unital C^*-algebra and $(\mathcal{A}, L)$ is any Lipschitz pair on A such that $d_{\mathcal{A},L}$ is a metric, then $L_{d_{\mathcal{A},L}}$ is the largest lower semicontinuous Lipschitz seminorm smaller than L. Consequently, $L_{d_{\mathcal{A},L}}(a) = L(a)$ for each $a \in \mathcal{A}$ if and only if $(\mathcal{A}, L)$ is a lower semicontinuous Lipschitz pair.*

Definition 3.1.7. A Lipschitz pair $(\mathcal{A}, L)$ is called *closed* if it is lower semicontinuous and $\mathcal{A} = \mathrm{Dom}(L_{d_{\mathcal{A},L}})$.

By replacing $(\mathcal{A}, L)$ by $(\mathrm{Dom}(L_{d_{\mathcal{A},L}}), L_{d_{\mathcal{A},L}})$, we can sometimes reduce problems about Lipschitz pairs to the closed case. To ensure that $\mathrm{Dom}(L_{d_{\mathcal{A},L}})$ is an algebra, it is important to introduce the *Leibniz rule*:

$$L(ab) \leqslant L(a)\|b\| + \|a\|L(b) \text{ for each } a, b \in \mathcal{A}. \tag{3.1.5}$$

Lemma 3.1.8. *[102] Let A be a unital C^*-algebra and $(\mathcal{A}, L)$ be a closed Lipschitz pair on A. Then $(\mathcal{A}, L)$ becomes a Banach space when equipped with the norm $\|a\|_1 := \|a\| + L(a)$, which is a Banach algebra provided $(\mathcal{A}, L)$ satisfies the Leibniz rule.*

The Leibniz rule can be thought of as a heuristic for Lipschitz seminorms coming from unbounded derivations, such as those coming from spectral triples. For a spectral triple, the induced metric coincides with Connes' extended metric. The existence of a spectral triple is a somewhat stronger than necessary hypothesis, however, since the Dirac operator involved in the definition is assumed to have compact resolvent, which is a condition more relevant to summability and Fredholm properties. Nevertheless, Rennie and Varilly came up with a very interesting statement, which suggests that the extra conditions of a spectral triple may be of metric significance to the resulting Lipschitz seminorms:

Proposition 3.1.9. *[99] Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a spectral triple over a C^*-algebra A coming from a faithful representation $\pi : \mathcal{A} \mapsto B(\mathcal{H})$ with the property $[\mathcal{D}, \pi(a)] = 0 \iff a \in \mathbb{C}I_A$. Then $(\mathcal{A}_{sa}, L_{\mathcal{D}})$, where $\mathcal{A}_{sa} := \{a \in \mathcal{A} : a^* = a\}$ and $L_{\mathcal{D}}(a) := \|[\mathcal{D}, \pi(a)]\|$, is a lower semicontinuous Lipschitz pair satisfying the Leibniz rule. If the representation π is nondegenerate and the spectral triple comes with a cyclic vector ξ for (A, π) such that* $\ker \mathcal{D} = \mathbb{C}\xi$*, then Connes' extended metric is a metric. Further, $(\mathcal{A}_{sa}, L_{\mathcal{D}})$ is closed if and only if $\mathcal{A} = C^1(A)$.*

Finally, Rieffel addresses the question of whether a metric induced by a Lipschitz seminorm has finite diameter and whether it induces the weak*-topology of $S(A)$, which in particular turns $(S(A), d_{\mathcal{A},L})$ into a compact topological space.

For each Lipschitz pair $(\mathcal{A}, L)$, define $B_L(\mathcal{A}) := \{a \in \mathcal{A};\ L(a) \leqslant 1\}$, $\tilde{B_L}(\mathcal{A}) := \{\tilde{a} \in \mathcal{A}/\mathbb{R}I;\ L(\tilde{a}) \leqslant 1\}$ and $B_{1,L}(\mathcal{A}) := \{a \in B_L(\mathcal{A});\ \|a\| \leqslant 1\}$. Formulated differently, the statements are:

Proposition 3.1.10. *[100], [101]*

1. *Given a unital C^*-algebra A equipped with a Lipschitz pair $(\mathcal{A}, L)$, equation (3.1.3) determines a metric $d_{L,\mathcal{A}}$ of finite diameter if and only if $\tilde{B_L}(\mathcal{A}) \subset A/\mathbb{R}I$ is norm- bounded, and further diam$\tilde{B_L}(\mathcal{A}) \leqslant r$ if and only if diam$(S(A), d_L) \leqslant 2r$, for each $r > 0$.*
2. *$d_{\mathcal{A},L}$ metrises the weak*-topology of $S(A)$ if and only if (1) $d_{\mathcal{A},L}$ has finite diameter and (2) $B_{1,L}(\mathcal{A}) \subset A$ is norm- totally bounded.*

Definition 3.1.11. [102] A C^*-algebra A, equipped with a Lipschitz pair $(\mathcal{A}, L)$ with the property that $d_{\mathcal{A},L}$ metrises the weak*-topology of $S(A)$, is called a *compact quantum metric space*.

3.1.2 *Dependence on the choice of algebra.*

A priori, every property of a Lipschitz pair that we have discussed so far depends on the algebra $\mathcal{A}$ defining it. A question which was raised in [57] was the following: given two Lipschitz pairs $(\mathcal{A}_1, L)$ and $(\mathcal{A}_2, L)$ defined on the same C^*-algebra, what is the relationship between $d_{L,\mathcal{A}_1}$ and $d_{L,\mathcal{A}_2}$? Are there instances, for example, in which $d_{L,\mathcal{A}_1}$ is a metric on $S(A)$ whilst $d_{L,\mathcal{A}_2}$ is not? We begin with another of Rieffel's observations:

Lemma 3.1.12. *[101] Let* A *be a separable unital* C^**-algebra and* $(\mathcal{A}_1, L)$ *and* $(\mathcal{A}_2, L)$ *be two Lipschitz pairs on* A*. Then* $d_{\mathcal{A}_1,L} = d_{\mathcal{A}_2,L}$ *if and only if* $\overline{B_L(\mathcal{A}_1)} = \overline{B_L(\mathcal{A}_2)}$.

The lemma tells us that the metrics $d_{\mathcal{A}_1,L}$ and $d_{\mathcal{A}_2,L}$ agree whenever $\mathcal{A}_1$ and $\mathcal{A}_2$ have the same closure in the $\|\cdot\|_1$-norm.

In general, if A is any unital C^*-algebra, $(\mathcal{A}, L)$ any closed Lipschitz pair on A and d is any metric on $S(A)$ which is smaller than $d_{\mathcal{A},L}$, then we can contemplate the pair $(\mathcal{A}_d, L)$, where $\mathcal{A}_d$ is the set of all functions which are Lipschitz for d in the sense of (3.1.4). Since $d \leqslant d_{\mathcal{A},L}$ and L is closed, then necessarily $L_{d_{\mathcal{A},L}} \leqslant L_d$. So long as the inclusion $\mathcal{A}_d \subset \mathcal{A}$ is dense and the former is an algebra, then $(\mathcal{A}_d, L)$ is also a Lipschitz pair. Of course, when $A = C(X)$ is unital, $\mathcal{A}_d$ is certainly an algebra and density of $\mathcal{A}_d$ follows from the Stone-Weierstrass theorem. The question is, what does the metric $d_{\mathcal{A}_d,L}$ give us? We were able to come up with the following proposition:

Proposition 3.1.13. *Let* A *be a unital* C^**-algebra and* $(\mathcal{A}, L)$ *be a closed Lipschitz pair on* A *which gives* $(S(A), d_{\mathcal{A},L})$ *finite diameter. Let* d_1, d_2 *be any metrics on* $S(A)$ *with* $d_1, d_2 \leqslant d_{\mathcal{A},L}$ *and with the property that the spaces* $\mathcal{A}_{d_1}$ *and* $\mathcal{A}_{d_2}$ *defined by*

$$\mathcal{A}_{d_i} := \{a \in \mathcal{A} : \exists L_a > 0 \text{ such that } |\omega_1(a) - \omega_2(a)| \leqslant L_a d_i(\omega_1, \omega_2),\ \forall \omega_1, \omega_2 \in S(A)\}$$

are dense in $\mathcal{A}$*. Then* $d_{\mathcal{A}_{d_1},L} = d_{\mathcal{A}_{d_2},L}$ *if and only if* d_1 *and* d_2 *are Lipschitz equivalent metrics.*

Proof. One one hand, if d_1 and d_2 are Lipschitz equivalent then by definition there exists a $C \geqslant 1$ such that $\frac{1}{C} d_2(\omega_1, \omega_2) \leqslant d_1(\omega_1, \omega_2) \leqslant C d_2(\omega_1, \omega_2)$ for each $\omega_1, \omega_2 \in S(A)$. It follows that $\mathcal{A}_{d_1} = \mathcal{A}_{d_2}$ and so immediately $d_{\mathcal{A}_{d_1},L} = d_{\mathcal{A}_{d_2},L}$.

Conversely if $d_{\mathcal{A}_{d_1},L} = d_{\mathcal{A}_{d_2},L}$ then, from Lemma 3.1.12, $\overline{B_L(\mathcal{A}_{d_1})} = \overline{B_L(\mathcal{A}_{d_2})}$. By our standing hypotheses, $\mathcal{A}_{d_1}$ and $\mathcal{A}_{d_2}$ become Banach spaces when equipped with the norms

$\|a\|_i := \|a\| + L_{d_i}(a)$ for each $i \in \{1,2\}$. Since $(\mathcal{A}, L)$ is closed, it follows that $\overline{B_L(\mathcal{A}_{d_1})}$ and $\overline{B_L(\mathcal{A}_{d_2})}$ are closed Banach subspaces of $\mathcal{A}_{d_1}$ and $\mathcal{A}_{d_2}$ in the $\|\cdot\|_1$ and $\|\cdot\|_2$ norms respectively. For indeed, if $(a_n)_n$ is a sequence in $\overline{B_L(\mathcal{A}_{d_1})}$ converging to $a \in \mathcal{A}_{d_1}$ then we can replace $(a_n)_n$ with a sequence $(b_n)_n \in B_L(\mathcal{A}_{d_1})$ with the same limit. Since $L(b_n) \leqslant 1$ for each n, also $L(a) \leqslant 1$ and hence $a \in \overline{B_L(\mathcal{A}_{d_1})}$. The same argument goes for $\overline{B_L(\mathcal{A}_{d_2})}$. All this means that

$$\iota : (\overline{B_L(\mathcal{A}_{d_1})}, \|\cdot\|_1) \mapsto (\overline{B_L(\mathcal{A}_{d_2})}, \|\cdot\|_2)$$

is an everywhere defined linear bijection between Banach spaces. As a consequence of the uniform boundedness principle, the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent. In particular there exists an $R_1 > 0$ such that

$$L_{d_2}(a) \leqslant R_1(\|a\| + L_{d_1}(a)), \ \forall a \in \overline{B_L(\mathcal{A}_{d_1})}. \tag{3.1.6}$$

Since L_{d_1} and L_{d_2} are zero on multiples of the identity, this inequality passes to the quotient algebra and becomes

$$L_{d_2}(\tilde{a}) \leqslant R_1(\|\tilde{a}\| + L_{d_1}(\tilde{a})), \ \forall \tilde{a} \in \overline{B_L(\mathcal{A}_{d_1})}/\mathbb{R}I. \tag{3.1.7}$$

Since d_1 gives $S(A)$ finite diameter, the first part of Proposition (3.1.10) applies and we can write

$$L_{d_2}(\tilde{a}) \leqslant R_1\left(1 + \frac{\mathrm{diam}_{d_1}(S(A))}{2}\right)L_{d_1}(\tilde{a}), \ \forall \tilde{a} \in \overline{B_L(\mathcal{A}_{d_1})}/\mathbb{R}I. \tag{3.1.8}$$

By symmetry, there also exists an $R_2 > 0$ such that

$$L_{d_1}(\tilde{a}) \leqslant R_2\left(1 + \frac{\mathrm{diam}_{d_2}(S(A))}{2}\right)L_{d_2}(\tilde{a}), \ \forall \tilde{a} \in \overline{B_L(\mathcal{A}_{d_2})}/\mathbb{R}I. \tag{3.1.9}$$

Consequently L_{d_1} and L_{d_2} are equivalent Lipschitz seminorms on $\overline{B_L(\mathcal{A}_{d_1})}/\mathbb{R}I = \overline{B_L(\mathcal{A}_{d_2})}/\mathbb{R}I$ and so the metrics d_1 and d_2 are Lipschitz equivalent. $\square$

3.1.3 Continuity and Lipschitz continuity for quantum metric spaces.

In this section we make a brief commentary on the morphisms between compact quantum metric spaces. The terminology in the next definition is motivated by the relationship between Lipschitz functions over general compact metric spaces and differentiable functions on manifolds.

Definition 3.1.14. [70] [124] Let A, B be C^*-algebras, equipped with lower semicontinuous Lipschitz pairs $(\mathcal{A}, L_A)$ and $(\mathcal{B}, L_B)$. We say a *-homomorphism $\alpha : A \to B$ is *smooth* if $\alpha(\mathcal{A}) \subset \mathcal{B}$ and *Lipschitz* if there is a $\lambda > 0$ with $L_B(\alpha(a)) \leqslant \lambda L_A(a)$ for all $a \in \mathcal{A}$.

It is easy to see that a map $\alpha : A \to B$ of bounded quantum metric spaces is Lipschitz if and only if the induced map $T_\alpha : (S(B), d_{L_B}) \to (S(A), d_{L_A})$ given as

$$T_\alpha(\phi)(a) := \phi(\alpha(a)), \tag{3.1.10}$$

is Lipschitz (with the same constant λ), for indeed if $\phi, \psi \in S(B)$ then:

$$\begin{aligned} d_{L_A}(T_\alpha(\phi), T_\alpha(\psi)) &:= \sup\{|T_\alpha(\phi)(a) - T_\alpha(\psi)(a)| : \ a \in \mathcal{A}, \ L_A(a) \leqslant 1\} \\ &= \sup\{|\phi(\alpha(a)) - \psi(\alpha(a))| : \ a \in \mathcal{A}, \ L_A(a) \leqslant 1\} \\ &\leqslant \lambda \sup\{|\phi(\alpha(a)) - \psi(\alpha(a))| : \ a \in \mathcal{A}, \ L_B(\alpha(a)) \leqslant 1\} \\ &\leqslant \lambda d_{L_B}(\phi, \psi). \end{aligned}$$

Conversely, if the identity $d_{L_A}(T_\alpha(\phi), T_\alpha(\psi)) \leqslant \lambda d_{L_B}(\phi, \psi)$ applies then the expression for $L_{d_{L_B}}$ becomes

$$L_{d_{L_B}}(\alpha(a)) := \sup\left\{\frac{|\phi(\alpha(a)) - \psi(\alpha(a))|}{d_{L_B}(\phi, \psi)} : \ \phi, \psi \in S(B), \ \phi \neq \psi\right\}. \tag{3.1.11}$$

Therefore, by the lower semicontinuity of L_B, $L_B(\alpha(a)) = L_{d_{L_B}}(\alpha(a)) \leqslant \lambda L_A(a)$ for each $a \in \mathcal{A}$.

Lemma 3.1.15. *Let A, B be C^*-algebras equipped with closed Lipschitz pairs $(\mathcal{A}, L_A)$ and $(\mathcal{B}, L_B)$, Then $\alpha : A \to B$ is smooth if and only if it is Lipschitz.*

Proof. See [124]. Alternatively, we could argue that $\alpha : (\mathcal{A}, \| \cdot \|_1) \mapsto (\mathcal{B}, \| \cdot \|_1)$ is an everywhere defined Banach algebra map and argue as in the proof of Proposition 3.1.13. $\square$

3.2 Locally compact quantum metric spaces.

A full treatment of Rieffel's quantum metric picture for nonunital C^*-algebras has only very recently become available, thanks to the works of Latrémolière in [76] and later [77]. The difficulties can be traced to the classical *Wasserstein* metric [120], which is defined in the same way as the Kantorovich metric for probability measures on locally compact topological spaces. Starting from each such space X, the Wasserstein metric may take infinite values and thus cannot be expected to recover the weak*-topology on the set of probability measures of X, at least if X has infinite diameter. More specifically, the Wasserstein metric recovers the weak*-topology only on the so-called *tight* subsets of $S(C(X))$. A collection $\mathcal{P} \subset S(C(X))$, where (X, ρ) is a metric space, is called *Dobrushin-tight* if, for any $x_0 \in X$,

$$\lim_{r \to \infty} \sup \left\{ \int_{\{x \in X : \rho(x, x_0) \geqslant r\}} \rho(x, x_0)\, d\mu(x) : \ \mu \in \mathcal{P} \right\} = 0. \tag{3.2.1}$$

This suggests a dichotomy between the study of *bounded* quantum metric spaces, namely when the induced metric on the (no longer compact) state space has finite diameter, and *unbounded* quantum metric spaces. The former is generally easier to deal with.

Most of the complexities arise when trying to establish a counterpart of Proposition 3.1.10. There is a technical difficulty, however, in trying to write down a precise definition of a Lipschitz pair which accounts for both unital and non-unital C^*-algebras. For example, given a locally compact metric space (X, d) which is not compact, the Lipschitz seminorm $L_{\mathtt{Lip}_d}$ on $C_0(X)$ has trivial kernel, which is never the case with a unital C^*-algebra. One solution would be to replace A, or rather its subalgebra $\mathcal{A}$, with its unitisation, which is again the approach in [77], for a seminorm $L : \mathcal{A} \mapsto [0, \infty)$ certainly extends to a Lipschitz seminorm $\hat{L} : \hat{\mathcal{A}} \mapsto [0, \infty)$ simply by setting $\hat{L}(a, \lambda) = L(a)$. We are ready to bring together some of the ideas of [77]:

Definition 3.2.1. Let A be a separable nonunital C^*-algebra. A (lower semicontinuous, closed) Lipschitz pair $(\mathcal{A}, L)$ on A is a seminorm $L : \mathcal{A} \mapsto [0, \infty)$ defined on a dense *-subalgebra $\mathcal{A}$ of A, such that $(\hat{\mathcal{A}}, \hat{L})$ is a (lower semicontinuous, closed) Lipschitz pair on $\hat{A}$.

The pair $(\mathcal{A}, L)$ will be called *bounded* if and only if $(\hat{\mathcal{A}}, \hat{L})$ is bounded. Letting $B_L(\mathcal{A}) :=$

$\{a \in \mathcal{A};\ L(a) \leqslant 1\}$, a calculation then shows (see [77]),

$$(\mathcal{A}, L) \text{ is bounded } \iff B_L(\mathcal{A}) \text{ is norm-bounded } \iff (S(A), d_{\mathcal{A},L}) \text{ is bounded.}$$

There still remains the question of whether $d_{\mathcal{A},L}$ recovers the weak*-topology. For *bounded* Lipschitz pairs, a necessary and sufficient criterium was given:

Theorem 3.2.2. *[77] Let A be a separable nonunital C^*-algebra and $(\mathcal{A}, L)$ a bounded Lipschitz pair on A. Then the induced metric $d_{\mathcal{A},L}$ on the state space recovers the weak*-topology if and only if there exists a strictly positive element $h \in A^+$ such that the set*

$$hB_L(\mathcal{A})h := \{hah;\ a \in B_L(\mathcal{A})\} \subset A \tag{3.2.2}$$

is norm- totally bounded.

If A admits such a Lipschitz pair then we will call A with $(\mathcal{A}, L)$ a *bounded quantum metric space*$^{(1)}$. Notice that the unitisation of a bounded quantum metric space is a compact quantum metric space and conversely every compact quantum metric space is a bounded quantum metric space. Latrémolière's earlier paper [76] essentially reduces all information about Lipschitz pairs to the bounded case, but this was only achieved by replacing the Wasserstein-type metric with so-called *bounded Lipschitz distances*, for which the above theorem applies, but conversely may not be sufficient to recover the complete metric information. An illustration of this point of view is seen in the setting of the usual Lipschitz seminorm on the continuous functions on the real numbers, since the usual metric

$$d(s,t) := \sup\{|f(s) - f(t)| :\ f \in C_0(\mathbb{R}, \mathbb{R}),\ L_{Lip}(f) \leqslant 1\}$$

is unbounded and consequently the set of Borel measures on $\mathbb{R}$ cannot be Dobrushin-tight. A family of bounded Lipschitz distances on $\mathbb{R}$ can at least be defined via

$$d_\lambda(s,t) := \sup\{|f(s) - f(t)| :\ \|f\| \leqslant \lambda,\ L_{Lip}(f) \leqslant 1\}.$$

The main features of d_λ are that it defines the weak*- topology on the state space of $C_0(\mathbb{R})$

(1)in view of Theorem 3.2.2 and the terminology in [77], a more accurate title would be "bounded locally compact quantum metric space" and that $d_\lambda = \min\{d, 2\lambda\}$ on the restriction of the metric to $\mathbb{R}$.

Broadly speaking, bounded Lipschitz distances tell us the precise behaviour of a metric on each bounded set, though not globally. In many respects the Wasserstein picture, rather than the bounded-Lipschitz picture, is the more natural one, provided we can establish suitable analogue of tight subsets. The proposal in [102] comprises developing a noncommutative notion of topography, which can capture the sense of points "escaping at infinity", which underpins the idea of Dobrushin tight sets. In the noncommutative world, Latrémolière translates this to replacing a Lipschitz pair $(\mathcal{A}, L)$ with *topographic space* $(\mathcal{A}, \mathcal{M}, L)$, where $\mathcal{M}$ is a suitable abelian subalgebra of $\mathcal{A}$ containing an approximate unit of $\mathcal{A}$. In [77], it is shown that this point of view is general enough to capture both the bounded Lipschitz distances on a quantum metric with finite diameter and the extended Monge-Kantorovich distance on a locally compact spaces as special cases, by making a natural choice of topography. Even with this in place, much of the theory that we have considered so far does not behave so well when dealing with unbounded quantum metric spaces. Thus, we will say no more about this approach and focus our analysis on bounded quantum metric spaces.

3.3 Examples.

3.3.1 Ergodic actions of compact groups.

An early example of a compact quantum metric space was observed by Rieffel in [100]. Let A be a unital C^*-algebra with a continuous weakly ergodic action α of a compact group G. Recall that an action $\alpha : G \mapsto \mathrm{Aut}(A)$ is called *weakly ergodic* if and only if the fixed point algebra contains only multiples of the identity. A continuous map $\ell : G \mapsto [0, \infty)$ is called a *length function* on G if it has the properties: (1) $\ell(g) = 0 \iff g = e$, (2) $\ell(g^{-1}) = \ell(g)\ \forall g \in G$ and (3) $|\ell(gh) - \ell(h)| \leqslant \ell(g)\ \forall g, h \in G$. It is known in this case that

$$L(a) := \sup\{\|\alpha_g(a) - a\|/\ell(g) :\ g \neq e\}, \tag{3.3.1}$$

turns A into a compact quantum metric space.

3.3.2 Algebra of compact operators.

We consider the algebra of compact operators $\mathcal{K} \cong \mathcal{K}(\mathcal{H})$, where $\mathcal{H}$ is a separable Hilbert space. With the next result we provide a general recipe for writing down quantum metric spaces on the compacts, which gives a metric on $S(\mathcal{K})$ with finite diameter:

Proposition 3.3.1. *Let $\mathcal{D} : \mathcal{H} \mapsto \mathcal{H}$ be a densely defined linear self-adjoint operator such that $\mathcal{D}^{-1} \in \mathcal{K}(\mathcal{H})$. Then the Lipschitz pair $(\mathcal{C}(\mathcal{H}), L)$, where*

$$\mathcal{C} := \{x \in \mathcal{K} : x = x^*,\ \mathcal{D}x,\ x\mathcal{D}\ \in \mathcal{K}\} \tag{3.3.2}$$

and $L(x) := \sup\{\|\mathcal{D}x\|, \|x\mathcal{D}\|\}$, turns $\mathcal{K}(\mathcal{H})$ into a bounded quantum metric space.

Proof. Since $\mathcal{D}^{-1}$ is compact and self-adjoint, density of $\mathcal{C}$ in the self-adjoint part of the compacts follows from the spectral theorem. Since $\mathcal{K}(\mathcal{H})$ is non-unital, it suffices to show that the set $\{x \in \mathcal{C};\ L(x) \leqslant 1\}$ is norm- bounded and the set $\{x \in \mathcal{C};\ \|x\| \leqslant 1,\ L(x) \leqslant 1\}$ is norm- totally bounded.

To this end, let $\{P_k\}_{k\in\mathbb{N}}$ be the spectral projections of $Y := \mathcal{D}^{-1}$ and write $Q_n := \sum_{k=1}^{n} P_k$. It follows by assumption that for each $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $\|Y - YQ_n\| \leqslant \frac{\epsilon}{2}$ and $\|Y - Q_n Y\| \leqslant \frac{\epsilon}{2}$ whenever $n > N$. For $x \in \mathcal{C}$ with $L(x) < 1$ and $n > N$, we obtain

$$\|Q_n x Q_n\| \leqslant \|Q_n x\mathcal{D}YQ_n\| \leqslant \|x\mathcal{D}\|\|Y\| \leqslant \|Y\|,$$

Hence $\|x\| \leqslant \|Y\|$. Moreover,

$$\|x - xQ_n\| \leqslant \|x\mathcal{D}Y - x\mathcal{D}YQ_n\| \leqslant \|x\mathcal{D}\|\|Y - YQ_n\| \leqslant \frac{\epsilon}{2},$$

and $\|x - Q_n x\| \leqslant \frac{\epsilon}{2}$ by symmetry, so that $\|x - Q_n x Q_n\| \leqslant \|x - xQ_n\| + \|xQ_n - Q_n x Q_n\| \leqslant \epsilon$. Moreover, the set $Q_n\{x \in \mathcal{C};\ \|x\| \leqslant 1,\ L(x) \leqslant 1\}Q_n$ is bounded and the vector space it generates is finite dimensional, so it is totally bounded. The result follows. □

3.3.3 Further questions: tensor products and continuous fields.

There are two fundamental building block constructions in C^*-algebra theory for which, to my knowledge, the interplay with quantum metrics is not fully known. The first is the (minimal) tensor product construction of two C^*-algebras. If $(\mathcal{A}, L_A)$ and $(\mathcal{B}, L_B)$ are compact quantum metrics on A and B, how does one write down a compact quantum metric structure on $A \otimes_{min} B$? The other examples of interest are C^*-algebras arising as continuous fields.

Let X be a locally compact Hausdorff topological space. Recall that C^*-algebra A is called a $C(X)$-algebra if there is a *-homomorphism $\psi : C(X) \mapsto \mathcal{Z}(A)$. When this is so then A has a natural fibration over X consisting of the family $\{A_x : x \in X\}$ of quotient spaces, where

$$A_x := A/\psi(\{f \in C(X) : f(x) = 0\})A. \tag{3.3.3}$$

The quotient maps will be written $\pi_x : A \mapsto A_x$.

Definition 3.3.2. A $C(X)$-algebra is called a *continuous field of C^*-algebras* over X if the map $x \mapsto \|\pi_x(a)\|$ is continuous and $\|a\| = \sup_{x\in X} \|\pi_x(a)\|$ for each $a \in A$ (i.e the representation $\prod_{x\in X} \pi_x$ is faithful).

(Some authors prefer to weaken the continuity condition to requiring the maps $x \mapsto \|\pi_x(a)\|$ to be upper semicontinuous).

The first example of a continuous field C^*-algebra is the algebra $C_0(X, A)$, which we remark can also be regarded as $A \otimes_{min} C_0(X)$. Following the discussion of this chapter, the first object should be to realise $C_0(X, A)$ as a compact quantum metric space, starting from a metric space (X, d) and a compact quantum metric structure on A. This is the context of the next result:

Proposition 3.3.3. *Let A be a unital C^*-algebra and (X, d) be a compact metric space. Let $(\mathrm{Dom}(L_A), L_A)$ be a compact quantum metric space on A. Then the seminorm L defined by*

$$L_1(f) := \sup_{x\in X}\{L_A(f(x))\}, \quad L_2(f) := \sup_{x\neq y}\left\{\frac{\|f(x)-f(y)\|_A}{d(x,y)}\right\}, \quad L(f) := L_1(f) \vee L_2(f),$$

is densely defined on $C_0(X, A_{sa})$ and $(\mathrm{dom}(L), L)$ is compact quantum metric space.

Proof. Most of the properties to be verified are elementary, so we shall focus on Rieffel's criteria (Proposition 3.1.10).

For fixed $\epsilon \in (0,1)$, we write down a finite open covering $\{B_{\epsilon/2}(x_i) : i \in I\}$ of X by $\epsilon/2$-balls centred at $\{x_i\}_{i\in I}$ with respect to the metric d on X.

Let us first suppose $f \in C_0(X, A_{sa})$, where $L(\tilde{f}) \leqslant 1$: We can instead write $\tilde{f} = f + \mathbb{R}I$, where $L(f) > 0$. By assumption, $L_A(f(x_i)) \leqslant 1$ for each $i \in I$ and the Lipschitz seminorm on A gives the metric on the state space finite diameter, so that there is a constant $K_i > 0$ such that $\|\tilde{f}(x_i)\| \leqslant K_i$. Since also $L_2(\tilde{f}) < 1$, it follows that

$$\|\tilde{f}(x)\| \leqslant \|\tilde{f}(x_i)\| + \|\tilde{f}(x) - \tilde{f}(x_i)\| \leqslant K_i + d(x, x_i) \tag{3.3.4}$$

and hence $\|\tilde{f}\| = \sup_{x\in X} \|\tilde{f}(x)\| \leqslant \max_{i\in I} K_i + 1/2$ and the set $\{\tilde{f} \in C_0(X, A)/\mathbb{R}I : L(f) \leqslant 1\}$ is norm-bounded.

By assumption, the collection $\{a \in A : L_A(a) \leqslant 1\}$ is norm-totally bounded, so there is a finite family $F_\epsilon \subset A$ such that $\mathrm{dist}(f(x_i), F_\epsilon) < \epsilon/2$ for each $i \in I$. Let $f_i \in F_\epsilon$ be the (not necessarily unique) closest element to $f(x_i)$ and let $\{\theta_i\}_{i\in I} \subset C(X)$ be a finite partition of unity for $\{x_i\}_{i\in I}$. A calculation then shows that for each $x \in X$:

$$\begin{aligned} \|f(x) - \sum_{i\in I} f_i\theta_i(x)\| &\leqslant \| \sum_{d(x,x_i)<\epsilon/2} d(x, x_i)\theta_i(x)\| + \| \sum_{d(x,x_i)<\epsilon/2} (f(x_i) - f_i)\theta_i(x)\| \\ &< \epsilon/2 + \epsilon/2 = \epsilon, \end{aligned}$$

showing that the set $\{f \in C_0(X, A) : L(f) \leqslant 1\}$ is totally bounded. □

Remark 3.3.4. We were not successful in trying to generalise the above to the setting of tensor products or continuous fields of C^*-algebras. In the tensor product case, we could find a suitable analogue of the Lipschitz seminorm defined in the previous theorem. For continuous fields, an analogous choice of seminorm seems reasonable, but we could not find any non-trivial examples (including the C^*-algebra of the discrete Heisenberg group) for which the proof above still applies.

4: Metric C^*-dynamical systems and constructions of spectral triples on crossed products

4.1 Metric C^*- dynamical systems.

The motivation for some of the ideas in this chapter is to explore further the relationship between the construction of spectral triples on crossed products, as studied by the authors [5] [57], and ordinary metric and topological dynamics, as well as to develop a repertoire for constructing spectral triples over crossed products of non-unital C^*-algebras. The question of what "noncommutative metric dynamics" should entail has not really been much considered and what follows is merely a proposal based on observations made in recent papers. Some of our findings were somewhat surprising but we record most of these for completeness.

As usual, C^*-algebras will be assumed separable and groups second countable and locally compact throughout. As a convention, every Lipschitz pair $(\mathcal{A}, L)$ is assumed to be lower semicontinuous and to satisfy the Leibniz rule.

Definition 4.1.1. A *metric dynamical system* is a triple (X, d, G) comprising a locally compact metric space (X, d) and a group G with a continuous action of (X, d).

As usual, a continuous action of G on (X, d) induces a family $(\alpha_\gamma)_{\gamma \in G}$ of automorphisms of the algebra ring $C_0(X)$ given by $\alpha_g(f)(x) = f(g^{-1} \cdot x)$, which in turn defines a continuous group action of the space of probability measures on X given by $\tilde{g}(\mu)(f) := \mu(\alpha_{g^{-1}}(f))$. A metric dynamical system is certainly a topological dynamical system when one forgets the extra metric structure. For continuous group actions of C^*-algebras, we will as usual view the groups as topological groups, induced by the point norm topology on $\mathrm{Aut}(A)$. Recall this is the topology defined by the neighbourhood base of sets of the form

$$\mathcal{U}(\alpha; a, \epsilon) := \{\beta \in \mathrm{Aut}(\mathcal{A}, L) : \|\beta(a) - \alpha(a)\| < \epsilon\} : \ a \in \mathcal{A}, \ \epsilon > 0. \tag{4.1.1}$$

Definition 4.1.2. A metric C^*-dynamical system comprises a C^*-dynamical system (A, G, α), where A is a C^*-algebra, equipped with a compact quantum metric structure $(\mathcal{A}, L)$ on A, such that the action of G on $(\mathcal{A}, L)$ is *smooth*, that is, $\alpha_g(\mathcal{A}) = \mathcal{A}$ for each $g \in G$.

Remark 4.1.3. One might ask whether we should impose an extra assumption on the relationship between the group action and the seminorm L. It seems clear from our analysis that requiring $g \mapsto L(\alpha_g(a))$ is actually continuous (with respect to the point norm topology on G) is too strong an assumption to make. We point out that this is not implied from continuity of the induced action of G on the metric space $(S(A), d_{\mathcal{A},L})$.

So how many metric C^*- dynamical systems are there floating around? The next lemma shows that under additional assumptions, namely when the Lipschitz pair $(\mathcal{A}, L)$ is closed and gives the metric on $S(A)$ finite diameter, then we can provide a very natural characteristic of the group action:

Lemma 4.1.4. *(see also [124], [5], [57]) Let $(\mathcal{A}, L, G)$ be a metric C^*- dynamical system where $(\mathcal{A}, L)$ is closed. Then the following equivalent conditions apply:*

1. *The action of G on $(\mathcal{A}, L)$ is bi-Lipschitz, i.e there exists constants $K_g \geqslant 1, g \in G$, such that whenever $a \in \mathcal{A}$ then $K_g^{-1} L(a) \leqslant L(\alpha_g(a)) \leqslant K_g L(a)$,*
2. *The induced action of G over the compact metric space $(S(A), d_{\mathcal{A},L})$ defined by $\tilde{g}(\omega)(a) = \omega(\alpha_g^{-1}(a))$ is bi-Lipschitz with respect to the same constants: if $\omega_1, \ \omega_2 \in S(A)$ then $K_g^{-1} d_{\mathcal{A},L}(\omega_1, \omega_2) \leqslant d_{\mathcal{A},L}(\tilde{g}(\omega_1), \tilde{g}(\omega_2)) \leqslant K_g d_{\mathcal{A},L}(\omega_1, \omega_2)$.*

If in addition $A = C_0(X)$, where X is locally compact and Hausdorff, then equivalently α is spatially implemented by a continuous action of G on X which is bi-Lipschitz in the classical sense, i.e for $x, y \in X$ we have, for the same constants, $K_g^{-1} d_{\mathcal{A},L}(x, y) \leqslant d_{\mathcal{A},L}(gx, gy) \leqslant K_g d_{\mathcal{A},L}(x, y)$.

Proof. This follows immediately from the comments in section 3.4 and the evident fact that a metric dynamical system (X, d, G), such that $\mathrm{diam}_d(X) < \infty$, is bi-Lipschitz if and only if the extension to $(S(C_0(X)), d, G)$ is bi-Lipschitz with the same constant, with respect to the Wasserstein metric. □

4.2 Equicontinuity, almost periodicity and the noncommutative Arzela-Ascoli theorem.

4.2.1 *Equicontinuity for metric dynamical systems.*

Equicontinuity is a very important notion in the study of metric dynamics; indeed it has been studied by a number of people [47] [75]. One can think of equicontinuity as a topological stability criterium; points in close proximity do not wander too far apart in time.

Definition 4.2.1. A metric dynamical system (X, d, G) is called *equicontinuous* if $\forall \epsilon > 0$ there exists a $\delta > 0$ such that whenever $x, y \in X$, $g \in G$ and $d(x, y) < \delta$ then $d(gx, gy) < \epsilon$.

A closely related concept when applied to compact metric spaces is almost periodicity, which was originally introduced by Bohr and examined further by Ellis. It turns out to be the characteristic of those group actions which admit continuous extensions to actions of compact groups, which has many useful applications in ergodic theory.

Definition 4.2.2. Let α be a homeomorphism of a compact metric space (X, d), which defines an automorphism of the algebra ring $C(X)$ in the usual way, denoted still by α. A collection of functions $\mathcal{F} \subset C(X)$ is called *α-almost periodic* if the collection $\{\alpha^n(f) \in C(X) : n \in \mathbb{Z}\}$ is relatively compact in the sup-norm whenever $f \in \mathcal{F}$. We call α itself *almost periodic* if every $f \in C(X)$ is α-almost periodic.

Theorem 4.2.3. *[47] Let α be a homeomorphism of a compact metric space. Then (X, d, α) is equicontinuous if and only if α is almost periodic.*

The classical Arzela-Ascoli theorem characterises norm relatively compact families of functions over a compact metric space. There are various generalisations, as well as some statements appearing for actions of C^*-algebras. The following will do for our purposes.

Definition 4.2.4. If A is a unital C^*-algebra with a metrisable state space and d is any metric on $S(A)$ which induces the weak*-topology, then a family $\mathcal{F} \subset A$ will be called *bounded* if $\sup\{\|a\| : a \in \mathcal{F}\}$ is bounded and *equicontinuous* if $\forall \epsilon > 0$ there exists a $\delta > 0$ such that whenever $\omega_1, \omega_2 \in S(A)$, $a \in \mathcal{F}$ and $d(\omega_1, \omega_2) < \delta$ then $|\omega_1(a) - \omega_2(a)| < \epsilon$.

Theorem 4.2.5. *[3] Let A be a unital C^*-algebra. A family $\mathcal{F} \subset A$ is norm-relatively compact if and only if it is bounded and equicontinuous.*

The proof is made by identifying elements in $\mathcal{F}$ with their image in the Kadison bidual $\mathrm{Aff}(S(A))$ of affine maps in $C(S(A))$.

Definition 4.2.6. Let (A, G, α) be a C^*-dynamical system on a unital C^*-algebra A. We say that α is *almost periodic* if the family $\{\alpha_g(a) : \ g \in G\}$ is norm-relatively compact for each $a \in A$, that is, the image of G in $\mathrm{Aut}(A)$ is relatively compact in the point-norm topology.

Corollary 4.2.7. *Let (A, G, α) be a C^*-dynamical system on a unital C^*-algebra A. Let d be any metric on $S(A)$ which metrises the weak*- topology and suppose that the induced action of G on $(S(A), d)$ is equicontinuous. Then α is almost periodic.*

4.2.2 Equicontinuity for metric C^*-dynamical systems.

With the next definition we present a concept of equicontinuity for groups acting on compact quantum metric spaces which is slightly more general than [57] and [5] and motivated from the point of view of constructing spectral triples.

Definition 4.2.8. Let $(\mathcal{A}, L, G)$ be a metric C^*- dynamical system on a C^*-algebra A. An element $a \in \mathcal{A}$ is said to be *equicontinuous* if

$$\sup_{g \in G} L(\alpha_g(a)) < \infty \tag{4.2.1}$$

Let $\mathcal{A}_G$ be the algebra of equicontinuous elements of $(\mathcal{A}, L, G)$. When $(\mathcal{A}, L)$ is closed, $\mathcal{A}_G$ then becomes a Banach algebra under the norm $\|a\|_{1,G} := \|a\| + \sup_g L(\alpha_g(a))$.

Definition 4.2.9. We say that the metric C^*- dynamical system $(\mathcal{A}, L, G)$ is *equicontinuous* when $\mathcal{A}_G \subset \mathcal{A}$ is dense.

For commutative C^*-algebras, a spatial interpretation of equicontinuous metric C^*- dynamical systems in terms of the dynamics of metric spaces is given in the following result, which is somewhat surprising, although it seems to be related to a functoriality problem relating the maximum of two metrics with the corresponding Lipschitz seminorm and vice-versa.

Proposition 4.2.10. *Let (X, d) be a compact metric space, $\alpha : G \mapsto \mathrm{Homeo}(X, d)$ a continuous group action and $(C_{\mathrm{Lip}_d}(X), L_d)$ the usual compact quantum metric structure on $C(X)$. Suppose that the action of G on $C(X)$ leaves the dense subalgebra $C_{\mathrm{Lip}_d}(X)$ invariant, so that*

$(C_{\mathrm{Lip}_d}(X), L_d, G)$ *is a metric C*-dynamical system. Then* $(C_{\mathrm{Lip}_d}(X), L_d, G)$ *is equicontinuous if and only if*

1. *The system* (X, d, G) *is equicontinuous in the classical sense and*
2. *there exists an equivalent metric* $d' \leqslant d$ *such that* (X, d', G) *is isometric: we have* $d'(gx, gy) = d'(x, y)$ *for all* $x, y \in X$ *and* $g \in G$.

Proof. Supposing $(C_{\mathrm{Lip}_d}(X), L_d, G)$ is equicontinuous, we can write down a pseudometric d on X of the form

$$d'(x, y) := \sup\{|f(x) - f(y)| : \ f \in C^G_{\mathrm{Lip}_d}(X, \mathbb{R}), \ \sup_g L(\alpha_g(f)) \leqslant 1\}, \tag{4.2.2}$$

where $C^G_{\mathrm{Lip}_d}(X) \subset C_{\mathrm{Lip}_d}(X)$ is the subalgebra of equicontinuous functions. Since this is dense, d' is a metric, which is clearly isometric and smaller than d and must also generate the same topology as d, since $\iota : (X, d) \mapsto (X, d')$ is continuous. This proves the forward implication. On the other hand, if the system (X, d, G) is equicontinuous and $d' \leqslant d$ is as above then we may extend d' to the Monge-Kantorovich metric on the space $S(C(X))$ of probability measures of X, so that $(S(C(X)), d', G)$ is isometric. It follows $L_{d'}$ is a Lipschitz seminorm on $C(X)$ and clearly $(C_{\mathrm{Lip}_{d'}}(X), L_{d'}, G)$ is equicontinuous. Moreover $C_{\mathrm{Lip}_{d'}}(X) \subset C_{\mathrm{Lip}_d}(X)$ since $d' \leqslant d$, whilst the former is dense by the Stone-Weierstrass theorem. Hence $(C_{\mathrm{Lip}_d}(X), L_d, G)$ is equicontinuous. □

Corollary 4.2.11. *Let* $(\mathcal{A}, L, G)$, $\alpha : G \mapsto \mathrm{Aut}(A)$, *be an equicontinuous metric C*- dynamical system on a C*-algebra* A. *Then* α *is almost periodic.*

Proof. The same proof as in the previous theorem shows that the system $(S(A), d, G)$ is equicontinuous, so the result follows from Corollary 4.2.7. □

When additionally the quantum metric space $(\mathcal{A}, L)$ is closed and *every* $a \in \mathcal{A}$ is equicontinuous, then we proved the following in [57], Proposition 2.6. In fact, analysis of the proof reveals that it is not necessary to assume $(\mathcal{A}, L)$ is a compact quantum metric space; only that it gives the metric on $S(A)$ finite diameter. The statement becomes:

Proposition 4.2.12. *Let* $(\mathcal{A}, L, G)$ *be a metric C*- dynamical system on a C*-algebra* A, *where* $(\mathcal{A}, L)$ *is a closed Lipschitz pair. Then (1)* $(\mathcal{A}, L, G)$ *is equicontinuous and furthermore* $\mathcal{A}_G = \mathcal{A}$ *if* *and only if (2) there exists a metric* d *on* $S(A)$ *equivalent to* $d_{\mathcal{A},L}$ *(i.e* $\frac{1}{K}d \leqslant d_{\mathcal{A},L} \leqslant Kd$ *for some* $K > 1$*) such that* $(S(A), d, G)$ *is isometric.*

Proof. Same as in [57]. □

4.2.3 *Relation to the compact group actions and invariant states.*

Following the last section, we can study almost periodic automorphisms of C^*-algebras via their compact extensions. When a C^*-algebra A admits an action of a compact group G, then there is a normalised Haar measure μ on G faithful conditional expectation $E : A \mapsto A^G$ into the fixed point subalgebra A^G given by

$$E(a) := \int_G \alpha_g(a) d\mu.$$

In this way we are able to identify the states on $S(A^G)$ with G-invariant states on A. When the group action is weakly ergodic, so that $A^G = \mathbb{C}$, then the conditional expectation E defines a G-invariant trace on A. A natural consequence is that almost periodic actions of C^*-algebras ensure the existence of G- invariant states on the algebra. This leads us to another observation which will be needed later on.

Lemma 4.2.13. *Let* A *be a* C^**-algebra which contains a strictly positive element and suppose that* $\alpha : G \mapsto \mathrm{Aut}(A)$ *is an almost periodic action. Then* A *contains a* G*-invariant strictly positive element.*

Proof. When A is unital then the unit is clearly the desired element, so we may assume A to be non-unital. Let $h \in A^+$ be any strictly positive element, so that $\omega(h) > 0$ for each $\omega \in S(A)$. We may extend α to a continuous action $\bar{\alpha}$ of the point-norm completion $\bar{G}$ of G. Let $E : A \mapsto A^{\bar{G}}$ be a faithful conditional expectation. Then $E(h)$ is G-invariant. Moreover $\omega(E(h)) = \int_{\bar{G}} \omega(\alpha_g(h)) d\mu = \int_{\bar{G}} \tilde{g}(\omega)(h) d\mu > 0$ for each $\omega \in S(A)$, so that $E(h)$ is strictly positive. □

4.2.4 *Relation to Voiculescu-Brown entropy and induced entropy on the state space.*

Voiculescu presented a definition of entropy for automorphisms of C^*-algebras, which is closely based on the classical topological entropy of a homeomorphism coming from a resolving sequence of open covers [116]. His definition was instead based on a nuclear decomposition of the C*-algebra (that is, a net of completely positive maps factoring through finite dimensional approximations). Brown [10] later extended the definition to exact C*-algebras.

The usual definition of Voiculescu-Brown entropy for unital C*-algebras is as follows: Given an exact C*-algebra A, an an automorphism α of A and a faithful representation $\pi : A \mapsto B(\mathcal{H})$, we define the set $CPA(\Omega, \delta)$, with $\Omega \subset A$ finite and $\delta > 0$, to be the set of all triples (ϕ, F, ψ) such that F is a finite dimensional C*-algebra, $\psi : A \mapsto F$ and $\phi : F \mapsto B(\mathcal{H})$ are unital completely positive maps and $\|\phi \circ \psi(a) - \pi(a)\| < \delta$ whenever $a \in \Omega$. By definition of exactness, $CPA(\Omega, \delta)$ is non-empty. Define

$$\mathrm{rcp}(\Omega, \delta) := \inf\{\mathrm{rank}(F) : \ (\phi, F, \psi) \in CPA(\Omega, \delta)\}. \tag{4.2.3}$$

This does not depend on π ([10]).

Definition 4.2.14. The *Voiculescu-Brown entropy* $ht(\alpha)$ of an automorphism α of an exact C*-algebra A is defined by

$$ht(\alpha) = \sup_{\Omega \subset\subset A} \left(\sup_{\delta > 0} \left(\limsup_{n \to \infty} \frac{1}{n} \log \ \mathrm{rcp}(\Omega \cup \alpha(\Omega) \cup \cdots \cup \alpha^{n-1}(\Omega), \delta) \right) \right). \tag{4.2.4}$$

Proposition 4.2.15. *Let A be a unital exact C*-algebra and $\alpha \in \mathrm{Aut}(A)$ be an almost periodic automorphism. Then $ht(\alpha) = 0$.*

Proof. Fix a faithful representation $\pi : A \mapsto B(\mathcal{H})$. By assumption, for each fixed $a \in A$ the family $\{\alpha^n(a) : \ n \in \mathbb{Z}\}$ is norm-totally bounded. Therefore, given a finite set $\Omega \subset A$ and a $\delta > 0$, the set $\{\bigcup_{n \in \mathbb{Z}} \alpha^n(\Omega)\}$ may be $\frac{\delta}{3}$-norm-totally bounded by some finite set Ω'. If (ϕ, F, ψ) is a unital completely positive approximation in $CPA(\Omega', \frac{\delta}{3})$ then we can see that $(\phi, F, \psi) \in CPA(\Omega \cup \alpha(\Omega) \cup \cdots \cup \alpha^{n-1}(\Omega)a, \delta)$ for each $n \geqslant 0$. To see this, fix $a \in \Omega$, fix $n \geqslant 0$ and fix $b \in \Omega'$ with the property that $\|\alpha^n(a) - b\| \leqslant \frac{\delta}{3}$. Then

$$\|\pi(\alpha^n(a)) - \phi \circ \psi(\alpha^n(a))\| \leqslant \frac{2\delta}{3} + \|\pi(b) - \phi \circ \psi(b)\| < \delta.$$

Therefore, $\mathrm{rcp}(\Omega \cup \alpha(\Omega) \cup \cdots \cup \alpha^{n-1}(\Omega), \delta) \leqslant \mathrm{rcp}(\Omega', \frac{\delta}{3})$. Since this holds for each $n \geqslant 0$, it quickly follows that $ht(\alpha) = 0$. □

It would be remiss not to mention a result by Kerr in [69], in which it was shown that when (A, α) is any C^*-dynamical system with zero Voiculescu-Brown entropy, then the induced topological entropy of α on the state space $S(A)$ is also zero. It would be interesting to examine the "noncommutative Kolmogorov" entropies for automorphisms of compact quantum metric spaces $(\mathcal{A}, L)$ introduced by the same author in [70] as well.

4.2.5 *Examples*

Example 4.2.16. Diffeomorphisms on manifolds. Let $(\mathcal{M}, \mathfrak{g})$ be a compact Riemannian manifold, $\mathcal{S}$ be a spin^c-structure on $(\mathcal{M}, \mathfrak{g})$ and $(C^\infty(\mathcal{M}, L^2(\mathcal{M}, \mathcal{S}), \mathcal{D})$ be the usual "Dirac-type" triple on $\mathcal{M}$. A Lipschitz pair on $C(\mathcal{M})$ can be given by $(C^\infty(\mathcal{M}), \|[\mathcal{D}, \cdot]\|)$, where $\|[\mathcal{D}, f]\| = \|\mathrm{grad}(f)\|$. Given a group Γ of diffeomorphisms of $\mathcal{M}$, the triple $(C^\infty(\mathcal{M}), \|[\mathcal{D}, \cdot]\|, \Gamma)$ is a C^*-metric dynamical structure on $C(\mathcal{M})$. It is equicontinuous provided

$$\sup\{\|\mathrm{grad}(\alpha_\gamma(f))\| : \gamma \in \Gamma\} < \infty, \ \forall f \in C^\infty(\mathcal{M}). \tag{4.2.5}$$

When further $\mathcal{M} \subset \mathbb{R}^n$ and $\gamma : \mathcal{M} \mapsto \mathcal{M}$ is represented by local coordinates in $\mathbb{R}^n$ for each $\gamma \in \Gamma$ then one can write $\mathrm{grad}(f \circ \gamma)(x) = (J\gamma(x))^T \mathrm{grad}(f)(\gamma^{-1}(x))$, where $J\gamma(x)$ is the Jacobi matrix of γ at x, so that the group action is equicontinuous provided the partial derivatives $\{\frac{\partial \gamma_i}{\partial x_j} : 1 \leqslant i, j \leqslant n, \ x \in \mathcal{M}, \ \gamma \in \Gamma\}$ are uniformly bounded.

Example 4.2.17. Cantor sets. Cantor sets admit a particularly important class of group actions known as *subodometers*, generalising the odometer action of $\mathbb{Z}$. These were studied in [73] and [39]. Starting with a countable discrete group Γ and a strictly decreasing sequence $(\Gamma_k)_{k \in \mathbb{N}}$ of finite index subgroups of Γ, the Cantor space X is identified with the projective limit of the discrete spaces $(X_k)_{k \in \mathbb{N}}$, where Γ acts on $X_k := \Gamma/\Gamma_k$ by left multiplication. By equipping X with the compatible metric $d((x_k), (y_k)) := \inf\{\frac{1}{n} : x_k = y_k, \ \forall 1 \leqslant k \leqslant n\}$, it is easy to see that (X, d, Γ) is an isometric, and hence equicontinuous metric dynamical system. d is an ultrametric, so that the construction of [92] leads to spectral triples with good Lipschitz properties on the Cantor set.

Subodometers were studied from a more C^*-algebraic perspective in [85], [18] under additional hypotheses (i.e the group Γ is amenable, residually finite and $\bigcap_{k \in \mathbb{N}} \Gamma_n = \{e\}$, so that X is equipped with a group structure as a profinite completion of the Γ_k). In [57] it was showed that every equicontinuous action of a discrete group on a Cantor set was conjugate to a subodometer, along the lines of Ellis' original result [75].

Example 4.2.18. AF-algebras. Lipschitz seminorms for approximately finite dimensional (AF)-algebras satisfying the regularity conditions of Rieffel (Theorem 3.1.10) were studied by Christensen and Ivan [27]. The content of their works was to show that for each AF-filtration $(A_n)_{n\in\mathbb{N}}$ of a unital AF-algebra A, there exists a spectral triple on A such that $\cup_{n\in\mathbb{N}}A_n \subset \mathcal{A}$ in such a way that the induced metric recovers the weak*-topology on $S(A)$, defining a compact quantum metric space structure on A.

Recall that the construction [27] comprises the Hilbert space $\mathcal{H} = L^2(A,\phi)$, where ϕ is a faithful state on ϕ. Writing $\mathcal{H} = A\xi$, where ξ is a cyclic vector for the nondegenerate representation of A on $\mathcal{H}$, the Dirac operator is given by an increasing sequence $(\lambda_n)_{n\in\mathbb{Z}^+} \subset [0,\infty)$ which diverges to infinity and $\mathcal{D} := \sum_{n\geqslant 1}\lambda_n Q_n$, where Q_n are finite rank orthogonal projections into the spaces $A_n\xi \ominus A_{n-1}\xi$ for each $n \geqslant 1$ and Q_0 is the projection into $\mathbb{C}\xi$. This defines a spectral triple $(\mathcal{A},\mathcal{H},\mathcal{D})$ (e.g when $\mathcal{A} = \cup_{n\in\mathbb{Z}^+}A_n$). Rieffel's metric condition is achieved provided the λ_n grow sufficiently fast and this typically depends on both the choice of state ϕ and the growth rate of the filtration $(A_n)_{n\in\mathbb{N}}$.

The automorphisms of AF-algebras are numerous and varied, and have been studied by a number of authors. Perhaps the most famous is Bratteli's *bilateral shift* on the two-sided CAR-algebra [9]: writing A as the UHF algebra $\overline{\otimes_{k=-\infty}^{\infty}M_2}$, such that $A_n \cong \otimes_{k=-n}^{n}M_2$ embeds unitally into $A_{n+1} \cong \otimes_{k=-n}^{n}M_2$ via the map $x \mapsto 1\otimes x\otimes 1$, the algebra A carries a unique faithful trace τ which is invariant for the shift action α on A given by

$$\alpha(\dots a_{-k}\otimes\dots a_{-1}\otimes a_0.\otimes a_1\cdots\otimes a_k\dots) := \dots a_{-(k+1)}\otimes\dots a_{-2}\otimes a_{-1}.\otimes a_0\cdots\otimes a_{(k-1)}\dots$$

The bilateral shift has positive Voiculescu-Brown entropy [9], so that it is not almost periodic from Proposition 4.2.15. Thus there does not exist a compact quantum metric structure $(\mathcal{A}, L_A)$ on A such that $(\mathcal{A}, L_A, \alpha)$ is equicontinuous.

Compact actions of AF-algebras were studied in [56]. The authors provided a complete invariant in terms of the representation rings for the class of so-called *locally representable* compact group actions. Essentially these are the actions which fix a particular AF-filtration

$(A_n)_{n\in\mathbb{N}}$ and for which the restriction of $\alpha : G \mapsto \mathrm{Aut}(A)$ to A_n is given by $\mathrm{Ad}(\gamma_n)$, where the $\gamma_n : G \mapsto U(A_n)$ are unitary representations.

A natural question to consider is, given a unital AF-algebra A and compact group action $\alpha : G \mapsto \mathrm{Aut}(A)$, is it always possible to write down a compact quantum metric structure on A such that the action of G is isometric? One situation in which an answer can be given is when there is an increasing AF-filtration $(A_n)_{n\in\mathbb{N}}$ of A with the property that $\alpha_g(A_n) = A_n$ for each $g \in G$ and $n \in \mathbb{N}$. Such actions were studied in [57] and it was shown that in this case the answer is positive and such a construction is achieved by writing down a Christensen-Ivan triple on A affliated to $(A_n)_{n\in\mathbb{N}}$ and a G-invariant faithful state on A. We do not know if all compact actions of unital AF-algebras have this property, however.

4.3 Group C^*-algebras and length functions.

Let us recall from Example 2.4.2 the fundamental construction of a spectral triple on the reduced C^*-algebra of a discrete group. It is the triple $(\mathbb{C}[\Gamma], \ell_2(\Gamma), M_L)$ defined by the usual left-regular representation $\lambda : C^*_r(\Gamma) \mapsto B(\ell_2(\Gamma))$ and where M_L is the multiplication operator determined by a proper length function $L : \Gamma \mapsto \mathbb{Z}^+$([30]). We should recall that for each $g \in \Gamma$,

$$\|[\mathcal{D}, \lambda(g)]\| = L(g), \tag{4.3.1}$$

which follows since $\|[\mathcal{D}, \lambda(g)]e_h\| = \|(L(gh) - L(h))e_{gh}\| \leqslant L(g)$, with equality when h is the identity.

Proposition 4.3.1. *[30] Provided there exists constants $c, r > 0$ such that $|B_n| \leqslant c(1+n)^r$, where $B_n := \{g \in \Gamma;\ L(g) \leqslant n\}$, the spectral triple $(\mathbb{C}[\Gamma], \ell_2(\Gamma), M_L)$ is p summable whenever $p > r+1$.*

A question which has been investigated by a number of people now is when does the Lipschitz seminorm on this triple satisfy Rieffel's condition, thus giving $C^*_r(\Gamma)$ the structure of a compact quantum metric space? From Proposition 3.1.10, the question can be formulated in the following way:

Question 4.3.2. For which discrete groups Γ and for which length functions $L : \Gamma \mapsto \mathbb{Z}^+$

are the sets

$$\tilde{B}_L(C^*_r(\Gamma)) := \{\tilde{x} \in C^*_r(\Gamma)/\mathbb{C}I;\ \|\lambda(x)\| \leqslant 1,\ \|[M_L, \lambda(x)]\| \leqslant 1\} \subset C^*_r(\Gamma)/\mathbb{C}I, \quad (4.3.2)$$

$$B_{1,L}(C^*_r(\Gamma)) := \{x \in C^*_r(\Gamma);\ \|\lambda(x)\| \leqslant 1,\ \|[M_L, \lambda(x)]\| \leqslant 1\} \subset C^*_r(\Gamma) \quad (4.3.3)$$

respectively norm-bounded and norm- totally bounded?

A positive answer can be given for specific examples:

1. L is the word-length function and additionally Γ satisfies a "Haagerup-type" condition [87]. In particular this condition is satisfied when Γ is a hyperbolic group, which is to say that there exists a $\delta > 0$ such that whenever $w, x, y, z \in \Gamma$ then

$$L(x^{-1}y) + L(z^{-1}w) \leqslant \max\{L(x^{-1}z) + L(y^{-1}w), L(x^{-1}w) + L(y^{-1}z)\} + \delta. \quad (4.3.4)$$

   (the standard examples of such are the free groups on finitely many generators.)
2. $\Gamma = \mathbb{Z}^d$ and L is an arbitrary proper length function ([100]),

(see also [3]). The approaches of [100] and [87] are entirely different. Significantly, the methods of Ozawa and Rieffel are not sufficient to give a proof in the case $\Gamma = \mathbb{Z}^d$ whenever $d \geqslant 2$. To resolve this, it is important to see how the "near diagonalisation" trick in Chapter 2 of that article can be applied to groups on multiple generators.

4.3.1 The Ozawa-Rieffel near-diagonal cut-down procedure.

Lemma 4.3.3. *[87] Let $\mathcal{D}$ be an unbounded self-adjoint diagonal operator on $\ell_2(\mathbb{Z})$ with spectrum contained in the integers. Then for each $\epsilon > 0$, there exists an $N_0 > 0$ such that whenever $x \in B(\mathcal{H})$ has the property that $[\mathcal{D}, x] := \mathcal{D}x - x\mathcal{D}$ extends to a bounded operator in $B(\mathcal{H})$ with $\|[\mathcal{D}, x]\| \leqslant 1$ and $N \geqslant N_0$ then*

$$\|\sum_{|m-n|>N} P_m x P_n\| \leqslant \epsilon, \quad (4.3.5)$$

where P_n is the orthogonal projection onto $\langle e_n \rangle$.

Proof. Write $\mathcal{D} = \sum_{n\in E} nP_n$, where $E \subset \mathbb{Z}$ and for each $z \in \mathbb{T}$ define $U_z := z^{\mathcal{D}} = \sum_{n\in E} z^n P_n$. This defines a strongly continuous gauge action of $B(\mathcal{H})$ defined by $\alpha_z(x) :=$

$U_z x U_z^*$. This can be extended to all Borel probability measures μ, where the formula becomes

$$\alpha_\mu(x) := \int_{\mathbb{T}} U_z x U_z^* d\mu(z). \tag{4.3.6}$$

In particular, if $x \in B(\mathcal{H})$ is such that $[\mathcal{D}, x] \in B(\mathcal{H})$ and additionally $\mu \in L^2(\mathbb{T})$ then a calculation shows that for $m, n \in E$,

$$P_m \alpha_\mu([\mathcal{D}, x]) P_n := \hat{\mu}(n-m) P_m [\mathcal{D}, x] P_n = (m-n)\hat{\mu}(n-m) P_m x P_n. \tag{4.3.7}$$

where $\hat{\mu} \in \ell_2(\mathbb{Z})$ is the Fourier transform of $\mu \in L^2(\mathbb{T})$. For each $N > 0$, we can introduce the function $\hat{\mu}_N \in \ell_2(\mathbb{Z})$ which is defined by $\hat{\mu}_N(n) = -\frac{1}{n}$ for $|n| > N$ and $\hat{\mu}_N(n) = 0$ otherwise. Letting $\mu_N \in L^1(\mathbb{T})$ be the inverse Fourier transform of $\hat{\mu}_N$, the Cauchy-Schwartz inequality implies $\|\mu_N\|_1 \leqslant \sqrt{2\pi}\|\hat{\mu}_N\|_2 \to 0$ as $N \to \infty$.

To complete the argument, fix $\epsilon > 0$ and write $x^N := \sum_{|m-n|>N} P_m x P_n$. Let $N \in \mathbb{N}$ be large enough such that $\|\mu_N\| \leqslant \epsilon$. Then for each $x \in B(\mathcal{H})$ such that $[\mathcal{D}, x]$ is a bounded operator we have

$$x^N = \sum_{|m-n|>N} (m-n)\hat{\mu_N}(n-m) P_m x P_n = \sum_{|m-n|>N} P_m \alpha_{\mu_N}([\mathcal{D}, x]) P_n, \tag{4.3.8}$$

so that $\|x^N\| \leqslant \|\mu_N\|_1 \|[\mathcal{D}, x]\| \leqslant \epsilon$. □

4.4 Semidirect product group C^*-algebras.

Recall that a semidirect product $N \rtimes_\alpha H$ is a group Γ with a unique decomposition $\Gamma = NH$ such that H is a subgroup, N is a normal subgroup and the product rule is given by $(n_1, h_1)_\alpha \star (n_2, h_2)_\alpha = (n_1 \alpha_{h_1}(n_2), h_1 h_2)_\alpha$, where $\alpha : H \mapsto \mathrm{Aut}(N)$ is a group action.

When N is a discrete group, $\alpha : N \mapsto N$ is a group automorphism and L_N is a proper length function on N, there is a natural candidate for a length function on $N \rtimes_\alpha \mathbb{Z}$. It is defined by

$$L((n, k)_\alpha) := L_N(n) + |k|, \; n \in N, \; k \in \mathbb{Z}. \tag{4.4.1}$$

Proposition 4.4.1. L *defines a proper length function on* $N \rtimes_\alpha \mathbb{Z}$*, provided that also* $L_N(\alpha(n)) = L_N(n)$ *for each* $n \in N$.

Proof. It is clear that L defines a map from Γ to $\mathbb{Z}^+$ and that $L(n,k)_\alpha = 0$ if and only if $n = 1_N$ and $k = 0$. Moreover, $L(\alpha^{-k}(n), -k)_\alpha = L_N(n) + |k| = L((n,k)_\alpha)$, proving that $L(g^{-1}) = L(g)$ for each $g \in \Gamma$. Finally for $n_1, n_2 \in N$ and $k_1, k_2 \in \mathbb{Z}$, $L((n_1,h_1)_\alpha \star (n_2,h_2)_\alpha) = L_N(n_1\alpha_{h_1}(n_2)) + |k_1 + k_2| \leqslant L_N(n_1) + |k_1| + L_N(n_2) + |k_2| = L((n_1,h_1)_\alpha) + L((n_2,h_2)_\alpha)$. That L is proper is obvious. □

Example 4.4.2. We can view $\mathbb{Z}^{d+1}$ as the untwisted semidirect product $\mathbb{Z}^d \rtimes_\alpha \mathbb{Z}$. When the length function on $\mathbb{Z}^d$ is the usual one then L defined above is the usual length function on $\mathbb{Z}^{d+1}$.

Example 4.4.3. The "Klein bottle group" $\mathbb{Z} \rtimes_\alpha \mathbb{Z}$ is defined by the conjugation $\alpha(n) = -n$, which necessarily preserves any length function on $\mathbb{Z}$. Hence we get a proper length function on $\mathbb{Z} \rtimes_\alpha \mathbb{Z}$.

With the next result we aim to show that the set of groups for which a positive answer to Question 4.3.2 can be given is closed under semidirect products. We have not managed to answer the question exactly, rather we are motivated by the ordinary spectral triple on the algebra $C^*(\mathbb{Z}^d) \cong C(\mathbb{T}^d)$. The idea is to modify the spectral triple from an ungraded triple to a graded one and this grading structure is what enables us to "cut down" the coefficients of the group algebra which lie in the Lipschitz ball.

Set up. Write $\Gamma := N \rtimes_\alpha \mathbb{Z}$, where N is a discrete group with a proper length function L_N and α is a group automorphism such that $L_N(\alpha(n)) = L_N(n)$ for each $n \in N$. With some abuse of notation, let M_{L_N} denote either the usual multiplication operator on $\ell_2(N)$ or the multiplication operator on $\ell_2(\Gamma)$ given by $M_{L_N} e_{(n,k)_\alpha} := L_N(n) e_{(n,k)_\alpha}$. We can also define $\mathcal{D}$ on $\ell_2(\Gamma)$ by $\mathcal{D} e_{(n,k)_\alpha} := |k| e_{(n,k)_\alpha}$, so that $\mathrm{Dom}(\mathcal{D}), \mathrm{Dom}(M_{L_N}) \supset \mathrm{Dom}(M_L)$ and

$$M_L = M_{L_N} + \mathcal{D} : \ \mathrm{Dom}(M_L) \mapsto \ell_2(N \rtimes_\alpha \mathbb{Z}). \tag{4.4.2}$$

Theorem 4.4.4. *Let* N *be a finitely generated discrete group,* L_N *a length function on* N *and* $\alpha : N \mapsto N$ *be a group automorphism such that* $L_N(\alpha(n)) = L_N(n)$ *for each* $n \in N$*. Let* $(\mathbb{C}[N], \ell_2(N), M_{L_N})$ *be the associated spectral triple on* $C^*_r(N)$ *and let us suppose further that the*

Lipschitz seminorm induced by this spectral triple satisfies Rieffel's criteria, i.e the sets

$$\tilde{S}_N := \{\tilde{x} \in \mathbb{C}[N]/\mathbb{C}I;\ \|[M_{L_N}, \tilde{x}]\| \leqslant 1\} \subset C^*(N)/\mathbb{C}I \tag{4.4.3}$$

$$S_N := \{x \in \mathbb{C}[N];\ \|x\| \leqslant 1,\ \|[M_{L_N}, x]\| \leqslant 1\} \subset C^*(N) \tag{4.4.4}$$

are respectively norm- bounded and norm- totally bounded. Then,

1. $(\mathbb{C}[\Gamma], \ell_2(\Gamma) \oplus \ell_2(\Gamma), \begin{bmatrix} 0 & \mathcal{D} - iM_{L_N} \\ \mathcal{D} + iM_{L_N} & 0 \end{bmatrix})$ *is a spectral triple on* $C^*_r(\Gamma)$.
2. *The Lipschitz seminorm induced by this spectral triple satisfies Rieffel's criteria also.*

Proof. First we observe the inequality

$$\max\{\|[M_L, x]\|, \|[\mathcal{D}, x]\|\} \leqslant 2\max\{\|[\mathcal{D} - iM_{L_N}, x]\|, \|[\mathcal{D} + iM_{L_N}, x]\|\}, \tag{4.4.5}$$

for $x \in \mathbb{C}[\Gamma]$ or $x \in \mathbb{C}[\Gamma]/\mathbb{C}I$. To prove the theorem, it suffices to show that the sets

$$\tilde{S} := \{\tilde{x} \in \mathbb{C}[\Gamma]/\mathbb{C}I;\ \|[\mathcal{D}, \tilde{x}]\| \leqslant 1,\ \|[M_{L_N}, \tilde{x}]\| \leqslant 1\} \subset C^*(\Gamma)/\mathbb{C}I, \tag{4.4.6}$$

$$S := \{x \in \mathbb{C}[\Gamma];\ \|x\| \leqslant 1,\ \|[\mathcal{D}, x]\| \leqslant 1,\ \|[M_{L_N}, x]\| \leqslant 1\} \subset C^*(\Gamma) \tag{4.4.7}$$

are respectively norm bounded and norm totally bounded. To this end, let U be the unitary generator of the element $(i_N, 1)_\alpha \in \Gamma$, where i_N is the identity of N. By construction, an element of $\mathbb{C}[\Gamma]$ has a unique decomposition as a finite sum of the form $\sum x_k U^k$ where $k \in \mathbb{Z}$ and $x_k \in \mathbb{C}[N]$.

It is known how to construct a conditional expectation $E : C^*_r(\Gamma) \mapsto C^*_r(N)$: there is a circle action defined by $\gamma_z(U^k) = z^k U^k$ and $\gamma_z(x) = x$ for $x \in \mathbb{C}[N]$ and the conditional expectation is the map $E(x) := \int_{\mathbb{T}} \gamma_z(x) dz$. Notice that $E(\mathbb{C}[\Gamma]) = \mathbb{C}[N]$.

The resulting *Fourier coefficients* of each $x \in C^*_r(\Gamma)$ for γ are the elements $x_k := E(xU^{-k})$ for $k \in \mathbb{Z}$. Note that for each $x \in \mathbb{C}[\Gamma]$ and $k \in \mathbb{Z}$,

$$\|x_k\| = \|x_k U^k\| = \|\int_{\mathbb{T}} z^{-k} \gamma_z(xU^{-k}) dz U^k\| \leqslant \|x\| \tag{4.4.8}$$

Moreover, $[M_{L_N}, U^k] = 0$ for each $k \in \mathbb{Z}$, as revealed by the calculation

$$[M_{L_N}, U^k]e_{(n,l)_\alpha} = (L_N(\alpha^k(n)) - L_N(n))e_{(\alpha^k(n),l+k)_\alpha}, \; n \in N, \; k \in \mathbb{Z}. \tag{4.4.9}$$

Therefore, for each $x \in \mathbb{C}[\Gamma]$ and $k \in \mathbb{Z}$,

$$\|[M_{L_N}, x_k]\| = \|[M_{L_N}, x_k U^k]\| = \|[M_{L_N}, \int_{\mathbb{T}} z^{-k}\gamma_z(xU^{-k})dzU^k]\| \leqslant \|[M_{L_N}, x]\| \tag{4.4.10}$$

and for each $\tilde{x} \in \mathbb{C}[\Gamma]/\mathbb{C}I$, then $E(\tilde{x}) = \tilde{x_0}$ and

$$\|[M_{L_N}, \tilde{x_0}]\| = \|[M_{L_N}, \int_{\mathbb{T}} z^{-k}\gamma_z(\tilde{x_0})dzU^k]\| \leqslant \|[M_{L_N}, \tilde{x}]\|. \tag{4.4.11}$$

Consequently, $x \in S$ implies $x_k \in S_N$ for each Fourier coefficient k and $\tilde{x} \in \tilde{S}$ implies $\tilde{x_0} \in \tilde{S_N}$.

Now let $\{P_k\}_{k\geqslant 0}$ be the spectral projections of $\mathcal{D}$, which are rank two for $k \geqslant 1$. These projections are related to the Fourier coefficients by the formula

$$E(x) := \sum_{m\geqslant 0} P_m x P_m. \tag{4.4.12}$$

For each $x \in S$, the proof of Lemma 4.3.3 shows that for each $\epsilon > 0$ there exists an $K \in \mathbb{N}$ such that

$$\|x - \sum_{k=0}^{K} x_k U^k\| \leqslant \epsilon. \tag{4.4.13}$$

where $x_k \in S_K$ for $k \neq 0$. Hence also

$$\|\tilde{x} - (\tilde{x_0} + \sum_{k=1}^{K} x_k U^k)\| \leqslant \epsilon, \tag{4.4.14}$$

where $\tilde{x_0} \in \tilde{S}_N$ and $x_k \in S_N$ for $k \neq 0$. By hypothesis, $\tilde{S}_N$ and S_N are respectively bounded and totally bounded, so that $\tilde{S}$ and S are respectively bounded and totally bounded, proving the lemma. □

Corollary 4.4.5. *Let Γ be a finitely generated iterated discrete semidirect product group of the form*

$(((\mathbb{Z} \rtimes_{\alpha_1} \mathbb{Z}) \rtimes_{\alpha_2} \mathbb{Z}) \ldots \rtimes_{\alpha_k} \mathbb{Z})$. *Let the length function on* Γ *be that induced by equation (4.2.1), starting from the usual word-length function on* $\mathbb{Z}$.

Suppose that each $\alpha_i \in \mathrm{Aut}((\mathbb{Z} \rtimes_{\alpha_1} \mathbb{Z}) \rtimes_{\alpha_2} \mathbb{Z}) \ldots \rtimes_{\alpha_{i-1}} \mathbb{Z}$ *is determined by a permutation matrix on the generators with length 1. Then the Lipschitz seminorm induced by the spectral triple on* $C^*_r(\Gamma)$, *obtained from applying the previous lemma inductively, satisfies Rieffel's condition: we obtain a compact quantum metric structure on* $C^*_r(\Gamma)$.

Remark 4.4.6. It is worth making a brief observation how we would expect the above ideas to work for "bad cases", such as the discrete Heisenberg group,

$$H_3 := \langle g, h, n \mid gh = hg,\ hn = nh,\ ng = ghn \rangle, \tag{4.4.15}$$

which can be identified as a semidirect product of $\mathbb{Z}^2$ by $\mathbb{Z}$, with conjugation defined by the group automorphism $\phi^n(g, h) = (g + nh, h)$. Since the orbit of ϕ is infinite, there does not exist a proper length function L on $\mathbb{Z}^2$ such that $L(\phi^n(g, h)) = L(g, h)$ for each $(g, h) \in \mathbb{Z}^2$.

On $C^*_r(H_3) \cong C^*(U, V, W \mid UV = VU,\ VW = WV,\ WU = UVW)$ there are two circle actions, related to the description of the algebra as a continuous field of rotation algebras. They are given by

$$\sigma : U \mapsto zU,\ V \mapsto V,\ W \mapsto W;\ \ \tau : U \mapsto U,\ V \mapsto wV,\ W \mapsto W, \tag{4.4.16}$$

where $w, z \in \mathbb{T}$ and U,V and W are the unitaries representing the generators n, g and h respectively. Just as in the proof of Lemma 4.4.4, we can use these actions to construct "Fourier coefficients" in U and V. There are natural well-defined operators on $\ell_2(H_3)$ given on the generators by

$$\delta_1 e_{n^\alpha g^\beta h^\gamma} = |\alpha| e_{n^\alpha g^\beta h^\gamma},\ \delta_2 e_{n^\alpha g^\beta h^\gamma} = |\beta| e_{n^\alpha g^\beta h^\gamma}, \tag{4.4.17}$$

where $\alpha, \beta, \gamma \in \mathbb{Z}$. By construction, $0 \leqslant \delta_1, \delta_2 \leqslant \delta_1 + \delta_2 \leqslant M_L$, when L is the word-length on H_3. Consequently, if we wanted to write down a spectral triple on $C^*(H_3)$ with good metric properties, we might try something like

$$(\mathbb{C}[H_3], \ell_2(H_3) \oplus \ell_2(H_3), \begin{bmatrix} (M_L - (\delta_1 + \delta_2)) & \delta_1 - i\delta_2 \\ \delta_1 + i\delta_2 & -(M_L - (\delta_1 + \delta_2)) \end{bmatrix}), \tag{4.4.18}$$

where the representation is given by two copies of the left-regular representation of $\mathbb{C}[H_3]$ on $\ell_2(H_3)$. We would be very interested in exploring the metric properties of this triple in future work.

4.5 Crossed products of C^*-algebras by discrete groups.

4.5.1 Review

Both [5] and [57] addressed the following question: given a unital C^*-algebra A equipped with a spectral triple $(\mathcal{A}, \mathcal{H}_A, \mathcal{D}_A)$ and a discrete group Γ, under what assumptions is it possible to write down a spectral triple on the reduced crossed product $A \rtimes_{r,\alpha} \Gamma$? Moreover when $(\mathcal{A}, \mathcal{H}_A, \mathcal{D}_A)$ determines a compact quantum metric structure for A, under what circumstances does the resultant crossed product triple determine a compact quantum metric structure on $A \rtimes_{r,\alpha} \Gamma$?

As well as provide a brief summary of these findings, we would like to consider the effect of allowing the initial algebra A to be non-unital and prescribed with a quantum metric structure which determines a metric on $S(A)$ with finite diameter. Moreover, we would like our findings to be applicable for as large a class of discrete groups as possible. The groups which we are able to consider are those groups in Corollary 4.4.5 which arise as iterated semidirect products by $\mathbb{Z}$.

Lemma 4.5.1. *Let N and H be discrete and finitely generated groups and $\alpha : H \mapsto \mathrm{Aut}(N)$ act as conjugations. Then there is a natural isomorphism $C^*_r(N \rtimes_\alpha H) \cong C^*_r(N) \rtimes_{\tilde{\alpha}} H$, where $\tilde{\alpha_h}(U_n) = U_{\alpha_h(n)}$ for each $n \in \mathbb{N}$.*

If $(\mathbb{C}[N], L)$ is the usual Lipschitz pair on $C^*_r(N)$ coming from a proper length function l on N then the action of $\tilde{\alpha}$ on $(\mathbb{C}[N], L)$ is equicontinuous for every $x \in \mathbb{C}[U_N]$ if and only if there exists a proper length function l_α on N fixed by α.

Proof. On one hand if α fixes the word-length of N then clearly the action of $\tilde{\alpha}$ on $\mathbb{C}[N]$

is smooth and $\sup_{h\in H} L(\tilde{\alpha_h}(U_n)) = \sup_{h\in H} L(U_{\alpha_h(n)}) = l(n) < \infty$ for each $n \in N$, so that $\tilde{\alpha}$ is equicontinuous. On the other hand if the action of $\tilde{\alpha}$ on $(\mathbb{C}[N], L_N)$ is equicontinuous and l is any proper length function (such as word-length) on N then $l_\alpha(n) := \sup_{h\in H} l(\alpha_h(n)) < \infty$ is clearly the desired length function on N. □

4.5.2 Set up.

We suppose that we have a *spectral* metric C^*-dynamical system, i.e

1. A C^*-dynamical system (A, Γ, α), where A is a separable C^*-algebra and $\alpha : \Gamma \mapsto \mathrm{Aut}(A)$ is an action of a discrete and finitely generated group,
2. a spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ on A such that
3. $(\mathcal{A}_{sa}, L_{\mathcal{D}}(a) := \|[\mathcal{D}, \pi(a)]\|)$ determines a quantum metric structure on A which gives the metric $d_{\mathcal{A},L_{\mathcal{D}}}$ on $S(A)$ finite diameter and
4. $\alpha_g(\mathcal{A}) = \mathcal{A}$ for each $g \in \Gamma$, i.e $(\mathcal{A}, L_{\mathcal{D}}, \Gamma)$ is a metric C^*-dynamical system.

Recall that α is equicontinuous if the property

$$L_\Gamma(a) := \sup_{g\in\Gamma} L(\alpha_g(a)) < \infty, \quad \forall a \in \mathcal{A}_\Gamma, \tag{4.5.1}$$

holds for some norm dense *-subalgebra $\mathcal{A}_\Gamma \subset \mathcal{A}$.

Definition 4.5.2. Recall that if A is a C^*-algebra, $\alpha : \Gamma \mapsto \mathrm{Aut}(A)$ is a continuous action and $\pi_A : A \mapsto B(H_A)$ is a faithful representation then the *left-regular representation* is the faithful representation of $A \rtimes_\alpha \mathbb{Z}$ on $B(H_A \otimes \ell_2(\Gamma))$ defined on the vector space generators $C_c(\Gamma, A) := \{\sum_g a_g u_g : \ a_g \in A, \ u_g \in U(\Gamma)\}$ by the relations

$$\begin{aligned} \pi(a)(\xi \otimes e_h) &= \pi_A(\alpha_{h^{-1}}(a))\xi \otimes e_h && (4.5.2)\\ \pi(u_g)(\xi \otimes e_h) &= \xi \otimes e_{gh} \quad a \in A, \ \xi \in H_A, \ g, h \in \Gamma, && (4.5.3)\end{aligned}$$

for each $a \in A, \ \xi \in H_A, \ g, h \in \Gamma$. π is also non-degenerate whenever π_A is nondegenerate. When $\xi \in H_A$ is a cyclic and separating vector for π_A then $\xi \otimes e_{1_\Gamma}$ is a separating and cyclic vector for π.

The existence of spectral triples on crossed products is given in the next result:

Theorem 4.5.3. *Let A be a separable C^*-algebra and Γ be a discrete group. Let $(\mathbb{C}[\Gamma], \ell_2(\Gamma), \mathcal{D}_\Gamma)$ be the spectral triple on $C_r^*(\Gamma)$ arising from either a proper word-length function on Γ or the usual spectral triple on $C_r^*(\mathbb{Z})$. Suppose $(\mathcal{A}, H_A, \mathcal{D}_A)$ is an odd spectral triple on A which satisfies (1), (3) and (4) above. Then there exists a norm dense *-subalgebra $C_c(\Gamma, \mathcal{A}) \subset \mathcal{B} \subset A \rtimes_{\alpha,r} \Gamma$ such that*

$$(\mathcal{B}, H_A \otimes \ell_2(\Gamma) \otimes \mathbb{C}^2, \mathcal{D} := \begin{bmatrix} 0 & \mathcal{D}_A \otimes 1 - i \otimes \mathcal{D}_\Gamma \\ \mathcal{D}_A \otimes 1 + i \otimes \mathcal{D}_\Gamma & 0 \end{bmatrix}), \tag{4.5.4}$$

represented via $\pi \oplus \pi$, is an even spectral triple for the reduced crossed product $A \rtimes_{\alpha,r} \Gamma$.

Proof. It remains to show that the crossed product triple satisfies the resolvent condition whenever the C^*-algebra A is non-unital and the spectral triple $(\mathcal{A}, H_A, \mathcal{D}_A)$ itself satisfies the resolvent condition. We set out to show, therefore,

$$(\pi(x) \oplus \pi(x))(1 + \mathcal{D}^2)^{-1/2} \in \mathcal{K}(H_A \otimes \ell_2(\Gamma) \otimes \mathbb{C}^2), \; x \in C_c(\Gamma, \mathcal{A}).$$

By writing $x = \sum a_g u_g = \sum u_g \alpha_{g^{-1}}(a_g)$ for each $x \in C_c(\Gamma, \mathcal{A})$, we see that it suffices to show this for $x = a \in \mathcal{A}$. By expanding $(\pi(x) \oplus \pi(x))(1 + \mathcal{D}^2)^{-1/2}$ carefully, it suffices to show

$$\pi(a)(1 + \mathcal{D}_A^2 \otimes 1 + 1 \otimes \mathcal{D}_\Gamma^2)^{-1/2} \in \mathcal{K}(H_A \otimes \ell_2(\Gamma)); \;\; a \in \mathcal{A}.$$

Letting P_g denote the orthogonal projection into $H_A \otimes \mathbb{C}e_g$, we can write

$$\pi(a)((1 + \mathcal{D}_A^2) \otimes 1 + 1 \otimes \mathcal{D}_\Gamma^2)^{-1/2} P_g = (\pi_A(\alpha_{g^{-1}}(a))(1 + \mathcal{D}_A^2) \otimes 1 + \lambda_g^2)^{-1/2} P_g$$

Since $(\mathcal{A}, H_A, \mathcal{D}_A)$ is a spectral triple on A, it satisfies the resolvent condition, so that $\pi_A(\alpha_{g^{-1}}(a))(1 + \mathcal{D}_A^2)^{-1/2} \in \mathcal{K}(H_A)$ for each $g \in \Gamma$ and for each $a \in \mathcal{A}$. Thus

$$\pi(a)(\mathcal{D}_A^2 \otimes 1 + 1 \otimes (1 + \mathcal{D}_\Gamma^2))^{-1/2} P_g \in \mathcal{K}(H_A \otimes \ell_2(\Gamma)).$$

Now the operator $(1 + \mathcal{D}_\Gamma^2)^{-1/2}$ is also compact and by assumption this means that the eigenvalues of $\mathcal{D}_\Gamma^2$ are discrete and increase to infinity. Therefore, for each $\epsilon > 0$ there exists a finite subset $F \subset \Gamma$ such that

$$\|(1+\mathcal{D}_\Gamma^2)^{-1/2})(1-p_F)\| < \epsilon \tag{4.5.5}$$

where $p_F = \sum_{g\in F} P_g$. Therefore, since the projection $1 \otimes p_F$ commutes with $\pi(a)$ and $(\mathcal{D}_A^2 \otimes 1)$, we have

$$\|\pi(a)(\mathcal{D}_A^2 \otimes 1 + 1 \otimes (1+\mathcal{D}_\Gamma^2))^{-1/2}(1 \otimes (1-p_F))\| < \epsilon. \tag{4.5.6}$$

from which it quickly follows that $\pi(a)(\mathcal{D}_A^2 \otimes 1 + 1 \otimes (1+\mathcal{D}_\Gamma^2))^{-1/2}$ is compact. □

Remark 4.5.4. When the spectral triple on A is even, so that the representation is of the form $\pi_A := \pi_A^+ \oplus \pi_A^-$ and $\mathcal{D}_A = \begin{bmatrix} 0 & \mathcal{D}_A^+ \\ \mathcal{D}_A^- & 0 \end{bmatrix}$, the argument is the same, except that the crossed product triple is now given by an odd cycle,

$$(\mathcal{B}, H_A \otimes \ell_2(\Gamma) \otimes \mathbb{C}^2, \mathcal{D} := \begin{bmatrix} 1 \otimes \mathcal{D}_\Gamma & \mathcal{D}_A^+ \otimes 1 \\ \mathcal{D}_A^- \otimes 1 & -1 \otimes \mathcal{D}_\Gamma \end{bmatrix}). \tag{4.5.7}$$

Our next objective is to show that the spectral triple in Theorem 4.5.3 defines a bounded quantum metric structure on $A \rtimes_{r,\alpha} \Gamma$, whenever the given triple $(\mathcal{A}, H_A, \mathcal{D}_A)$ defines a bounded quantum metric structure on A. We can provide a direct proof of this when $\Gamma = \mathbb{Z}$:

Theorem 4.5.5. *Let A be a C^*-algebra and $\alpha \in \mathrm{Aut}(A)$. Let $(C_c(\mathbb{Z}), \ell_2(\mathbb{Z}), \mathcal{D}_\mathbb{Z})$ be the usual spectral triple on $C^*(\mathbb{Z})$. Suppose $(\mathcal{A}, H_A, \mathcal{D}_A)$, with $\mathcal{A} = C^1(A)$, is an odd spectral triple on A which turns $(\mathcal{A}, L_{\mathcal{D}_A})$ into a bounded quantum metric space, that $\alpha_g(\mathcal{A}) = \mathcal{A}$ for each $g \in \mathbb{Z}$ and that the metric C^*- dynamical system $(\mathcal{A}, L_{\mathcal{D}_A}, \mathbb{Z})$ is equicontinuous. Let $(\mathcal{B}, H_A \otimes \ell_2(\mathbb{Z}), \hat{\mathcal{D}})$ be the spectral triple in Theorem 4.5.3, where $\mathcal{B} = C^1(A \rtimes_\alpha \mathbb{Z})$. Then $(\mathcal{B}, L_\mathcal{D})$ is a bounded quantum metric space.*

Proof. We shall assume that A is non-unital and invoke Theorem 3.2.2. It is very straightforward to extend the proof to the unital case, essentially by replacing $\mathcal{A}$ by $\mathcal{A}/\mathbb{R}I$ where necessary. To this end, let $h \in A^+$ be an α-invariant strictly positive element and u be the unitary implementing α, so that $[h, u] = 0$. Since α is equicontinuous, it is almost periodic and, from Lemma 4.2.13, we can certainly find such a h.

Write $\mathcal{L}^1(A) = \{a = a^* \in \mathcal{A} : \ \|[\mathcal{D}_A, \pi_A(a)]\| \leqslant 1\}$ and $\mathcal{L}^1(A \rtimes_{\alpha,r} \mathbb{Z}) = \{x = x^* \in \mathcal{B} : \|[\hat{\mathcal{D}}, x \oplus x]\| \leqslant 1\}$. Our standing assumptions are

1. $\{a \in \mathcal{A} : \ [\mathcal{D}_A, \pi_A(a)] = 0\} = 0$,
2. $\mathcal{L}^1(A) \subset A$ is norm-bounded and
3. $h\mathcal{L}^1(A)h \subset A$ is norm-totally bounded,

and we are required to show

1. $\{x \in \mathcal{B} : \ [\hat{\mathcal{D}}, x \oplus x] = 0\} = 0$,
2. $\mathcal{L}^1(A \rtimes_{\alpha,r} \mathbb{Z})$ is norm-bounded and
3. $h\mathcal{L}^1(A \rtimes_{\alpha,r} \mathbb{Z})h \subset A \rtimes_\alpha \mathbb{Z}$ is norm-totally bounded.

The content of the proof is much the same as in Lemma 4.4.4 and we will write down the steps in the same way. As in the beginning of that proof, we can deduce that, for all $x \in \mathcal{L}^1(A \rtimes_{\alpha,r} \mathbb{Z})$, we have

$$\max\{\|[\mathcal{D}_A \otimes 1, x]\|, \|[1 \otimes \mathcal{D}_{\mathbb{Z}}, x]\|\} \leqslant 2. \tag{4.5.8}$$

Another important feature of our proof is the faithful conditional expectation $E : A \rtimes_\alpha \mathbb{Z} \mapsto A$, defined by

$$E(x) := \int_{\mathbb{T}} \gamma_z(x) dz, \quad \text{where } \gamma_z(au^n) := z^n a u^n. \tag{4.5.9}$$

If we write by $(P_n)_{n \in \mathbb{Z}} \in B(H_A \otimes \ell_2(\mathbb{Z}))$ the usual family of pairwise orthogonal projections into $H_A \otimes \mathbb{C}e_n$ then the expectation E is spatially implemented by the formula $E(x) = \sum_{m \in \mathbb{Z}} P_m x P_m$. For $x \in A \rtimes_\alpha \mathbb{Z}$ the n^{th} "Fourier coefficient" is given by $x_n := E(u^{-n}x)$.

Observe that, for each $n \in \mathbb{Z}$, $u^n x_n = \sum_{m \in \mathbb{Z}} P_{m+n} x P_m$. Moreover,

$$\|[\mathcal{D}_A \otimes 1, u^n x_n]\| \leqslant \|[\mathcal{D}_A \otimes 1, x]\| \text{ and } \|[1 \otimes \mathcal{D}_{\mathbb{Z}}, u^n x_n]\| \leqslant \|[1 \otimes \mathcal{D}_{\mathbb{Z}}, x]\|. \tag{4.5.10}$$

A calculation reveals that, for each $n \in \mathbb{Z}$, $\|[\mathcal{D}_A \otimes 1, u^n x_n]\| = \|[\mathcal{D}_A \otimes 1, \int_{\mathbb{T}} z^{-n} \gamma_z(x) dz]\| = \|\int_{\mathbb{T}} z^{-n} [\mathcal{D}_A \otimes 1, \gamma_z(x)] dz\|$. The above identities hold because the gauge action $x \mapsto \gamma_z(x)$

is isometric for $\|[\mathcal{D}_A \otimes 1, \cdot]\|$ and a similar argument holds for $\|[1 \otimes \mathcal{D}_{\mathbb{Z}}, \cdot]\|$. By explicitly evaluating the operators $[\mathcal{D}_A \otimes 1, u^n x_n]$ and $[1 \otimes \mathcal{D}_{\mathbb{Z}}, u^n x_n]$ on the basis vectors, we find

$$\|[\mathcal{D}_A \otimes 1, u^n x_n]\| = \sup_{m \in \mathbb{Z}} \|[\mathcal{D}_A, \pi_A(\alpha^m(x_n))]\|, \tag{4.5.11}$$

$$\|[1 \otimes \mathcal{D}_{\mathbb{Z}}, u^n x_n]\| = |n| \|\pi_A(x_n)\|. \tag{4.5.12}$$

It follows that whenever $x \in \mathcal{B}$ is self-adjoint and $[\hat{\mathcal{D}}, x \oplus x] = 0$ then the Fourier coefficients vanish and so $x = 0$, proving the first part of the theorem. To prove the second and third parts, we can apply the Ozawa-Rieffel cut-down procedure ([87], Chapter 2) with the projections $\{P_n : n \in \mathbb{Z}\}$ to show that for each $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $\|x - \sum_{|n-m| \leqslant N} P_m x P_n\| \leqslant \epsilon/2$ whenever $x \in \mathcal{B}_{sa}$ and $\|[\hat{\mathcal{D}}, x \oplus x]\| \leqslant 1$. In other words,

$$\|x - \sum_{n=-N}^{N} u^n x_n\| \leqslant \epsilon/2, \tag{4.5.13}$$

It suffices to show that the sets $\mathcal{S}_N$ defined for each $N \in \mathbb{N}$ by

$$\mathcal{S}_N := \{ \sum_{n=-N}^{N} u^n x_n : \ \|[\mathcal{D}_A, \pi_A(x_n)]\| \leqslant 1, \ \|\pi_A(x_n)\| \leqslant \frac{1}{n} \} \tag{4.5.14}$$

are norm-bounded, whilst $h\mathcal{S}_N h$ is also norm-totally bounded. Norm-boundedness of $\mathcal{S}_N$ follows immediately from our hypothesis, whilst $h\mathcal{S}_N h$ can be written:

$$h\mathcal{S}_N h := \{ \sum_{n=-N}^{N} u^n h x_n h : \ \|[\mathcal{D}_A, \pi_A(x_n)]\| \leqslant 1, \ \|\pi_A(x_n)\| \leqslant \frac{1}{n} \} \tag{4.5.15}$$

and norm-total boundedness of $h\mathcal{S}_N h$ then follows from our hypothesis as well. $\square$

Let us investigate Connes' metric for the state space of $A \rtimes_\alpha \mathbb{Z}$. We shall study the states of the form $\{\omega \circ E : \ \omega \in S(A)\}$, which is a closed subset of $S(A)$. Using the proof of the previous theorem, we have for each $\omega_1, \omega_2 \in S(A)$:

$$\begin{aligned} d(\omega_1 \circ E, \omega_2 \circ E) &= \sup\{|(\omega_1 \circ E)(x) - (\omega_2 \circ E)(x)| : x \in \mathcal{B}, \ \|[\hat{\mathcal{D}}, x \oplus x]\| \leqslant 1\} \\ &= \sup\{|\omega_1(E(x)) - \omega_2(E(x))| : x \in \mathcal{B}, \ \|[\mathcal{D}_A \otimes 1, x]\| \leqslant 1\} \\ &= \sup\{|\omega_1(E(x)) - \omega_2(E(x))| : x \in \mathcal{B}, \ \sup_{m \in \mathbb{Z}} \|[\mathcal{D}_A, \alpha_m(E(x))]\| \leqslant 1\} \\ &= \sup\{|\omega_1(a) - \omega_2(a)| : a \in \mathcal{A}, \ \sup_{m \in \mathbb{Z}} \|[\mathcal{D}_A, \alpha_m(a)]\| \leqslant 1\} \\ &=: d_{\mathbb{Z}}(\omega_1, \omega_2) \end{aligned}$$

where $d_{\mathbb{Z}}$ is a compatible metric on $S(A)$ that is isometric for α.

Remark 4.5.6. The iterated procedure highlighted in [57] to handle the existence of compact quantum metric spaces on crossed products by $\mathbb{Z}^d$ can be generalised to iterated crossed products by $\mathbb{Z}$ (Corollary 4.4.5). This is immediate from Lemma 4.5.1. This extends the class of crossed product C*-algebras for which we can construct spectral triples with good metric properties, e.g the Generalised Bunce-Deddens algebras corresponding to the crossed product of a Cantor set by any such group (see [85] and [18] for details).

4.6 A description of the crossed product spectral triple as an unbounded Kasparov product.

In this section we would like to discuss the interplay between some of the spectral triples in this section and their representatives in K-homology, which leads us nicely towards the next chapter. The starting point will be the famed six-term exact sequence of Pimsner and Voiculescu.

Theorem 4.6.1. *[95] Let A be a σ-unital C^*-algebra and $\alpha \in \mathrm{Aut}(A)$. Then there is a six-term exact sequence in K-homology given by:*

$$\begin{tikzcd} K^0(A \rtimes_\alpha \mathbb{Z}) \arrow[r, "\iota^*"] & K^0(A) \arrow[r, "1-\alpha^*"] & K^0(A) \arrow[d, "\delta_0"] \\ K^1(A) \arrow[u, "\delta_1"] & K^1(A) \arrow[l, "1-\alpha^*"] & K^1(A \rtimes_\alpha \mathbb{Z}) \arrow[l, "\iota^*"] \end{tikzcd}$$

where $\iota^* : K^i(A \rtimes_\alpha \mathbb{Z}) \mapsto K^i(A)$ *and* $1 - \alpha^* : K^i(A) \mapsto K^i(A)$ *are the induced group homomorphisms in K-homology and* δ^0, δ^1 *are the boundary maps defined by left multiplication with the generalised Toeplitz element* $[x] \in KK^1(A \rtimes_\alpha \mathbb{Z}, A)$.

The Toeplitz element can be expressed as the relative Kasparov bimodule of an unbounded $A \rtimes_\alpha \mathbb{Z}$-$A$ cycle. The algebra $A \rtimes_\alpha \mathbb{Z}$ is canonically represented over the Hilbert module $\ell_2(\mathbb{Z}) \bar{\otimes} A$ via

$$a(e_n \bar{\otimes} b) = e_n \bar{\otimes} \alpha^{-n}(a)b, \quad u(e_n \bar{\otimes} b)) = e_{n+1} \bar{\otimes} b, \tag{4.6.1}$$

and there is a densely defined self-adjoint regular operator $\mathcal{D}_\mathbb{Z} \bar{\otimes} 1$ on $\ell_2(\mathbb{Z}) \bar{\otimes} A$ defined by $\mathcal{D}_\mathbb{Z} \bar{\otimes} 1(e_n \bar{\otimes} b) = n(e_n \bar{\otimes} b)$. It is known that $(C^1(\mathbb{Z}, A), \ell_2(\mathbb{Z}) \bar{\otimes} A, \mathcal{D}_\mathbb{Z} \bar{\otimes} 1)$ is the desired cycle (see [55] for details). The next result shows that the spectral triples obtained in the crossed product construction arise by composition with $[x]$:

Proposition 4.6.2. *Let* A *be a separable* C^**-algebra and* $\alpha \in \mathrm{Aut}(A)$ *be an automorphism. Let* $(\mathbb{C}[\mathbb{Z}], \ell_2(\mathbb{Z}), \mathcal{D}_\mathbb{Z})$ *be the usual triple on* $C^*_r(\mathbb{Z})$*. Suppose that* $(\mathcal{A}, \mathcal{H}_A, \mathcal{D}_A)$*,* $\mathcal{A} = C^1(A)$*, is an ungraded spectral triple on* A *such that the action of* α *is equicontinuous (and leaves* $\mathcal{A}$ *invariant). Then the graded spectral triple on* $A \rtimes_\alpha \mathbb{Z}$ *defined in Theorem 4.5.3 is an unbounded Kasparov product of* $(C^1(\mathbb{Z}, A), \ell_2(\mathbb{Z}) \bar{\otimes} A), \mathcal{D}_\mathbb{Z} \bar{\otimes} 1) \in \mathfrak{K}\mathfrak{K}^1(A \rtimes_\alpha \mathbb{Z}, A)$ *and* $(\mathcal{A}, \mathcal{H}_A, \mathcal{D}_A) \in \mathfrak{K}\mathfrak{K}^1(A, \mathbb{C})$.

Proof. Rather than verify that Kucerovsky's criteria are satisfied, we shall use Theorem 2.6.9. The module of differential 1-forms on A coming from $(\mathcal{A}, \mathcal{H}_A, \mathcal{D}_A)$ is given by

$$\Omega^1(A) := \{\sum_{i \in I} a_i \delta_{\mathcal{D}_A}(b_i), \quad a_i, b_i \in \mathcal{A}\}.$$

where $\delta_{\mathcal{D}_A}(b) := [\mathcal{D}_A, \pi_A(b)]$. Since α is equicontinuous and the Lipschitz pair defined on the spectral triple $(\mathcal{A}, \mathcal{H}_A, \mathcal{D}_A)$ is closed, the algebra A_1 of equicontinuous entries with the norm defined by $\|a\|_1 := \|a\| + \sup_{n \in \mathbb{Z}} \|\delta_{\mathcal{D}_A}(\alpha^n(a))\|$ is an operator *-algebra, and $E_1 := c_0(\mathbb{Z}, A_1)$ becomes a countably generated right operator *-module.

We are required to show that (E_1, A_1) is a correspondence in the sense of Definition 2.6.8, at which point it is easy to see that the triple is precisely the form described in Theorem 2.6.9.

By definition, the Graβmannian $\delta_{\mathcal{D}_A}$- connection $\nabla : E_1 \mapsto E_1 \otimes_{A_1} \Omega^1(A)$ is given by

$$\nabla(e_n \bar{\otimes} b) := e_n \bar{\otimes} \delta_{\mathcal{D}_A}(b), \; b \in A_1, \; n \in \mathbb{Z}. \tag{4.6.2}$$

Since the operators $\mathcal{D}_{\mathbb{Z}} \otimes 1$ and $1 \otimes \mathcal{D}_A$ clearly commute and their domain of intersection is dense, the third criterium of Definition 2.6.8 is satisfied. Further, a calculation shows that for each $au^m \in \mathbb{C}[\mathbb{Z}, A_1] \subset A \rtimes_\alpha \mathbb{Z}$ then

$$[\nabla, \pi(au^m)](e_n \bar{\otimes} b) = e_{m+n} \bar{\otimes} \delta_{\mathcal{D}_A}(\alpha^{-(m+n)}(a))b,$$

so that $\mathbb{C}[\mathbb{Z}, A_1] \subset \{x \in A \rtimes_\alpha \mathbb{Z} : \; [\mathcal{D}, \pi(x) \oplus \pi(x)] \in B(\mathcal{H})\}$ and the inclusion is dense. The result follows. $\square$

When the spectral triple on A is even to begin with, we can formulate a similar result, the only difference being that taking the Kasparov product with the Toeplitz element now corresponds to the boundary map $\delta_1 : K^0(A) \mapsto K^1(A \rtimes_\alpha \mathbb{Z})$ rather than $\delta_0 : K^1(A) \mapsto K^0(A \rtimes_\alpha \mathbb{Z})$. We leave the reader to fill in the details.

5: Constructions of spectral triples arising from extensions

5.1 Review of the theory of C^*-extensions.

5.1.1 *Extensions and Brown-Douglas-Filmore theory*

Recall that a C^*-algebra E is called an *extension* of A by B if B is contained in E as a closed two-sided ideal and $A \cong E/B$. An natural question to ask is under what assumptions the geometric aspects of E can then be read off those of A and B, although a question in this kind of generality is hard to answer. In this chapter we shall we restrict our attention to extensions of a certain kind, namely short exact sequences of the form

$$0 \longrightarrow \mathcal{K} \otimes B \xrightarrow{\iota} E \xrightarrow{\sigma} A \longrightarrow 0\,, \tag{5.1.1}$$

Such extensions fit into the framework of Brown-Douglas-Filmore theory and its various generalisations. Brown-Douglas-Filmore theory is motivated by the study of the properties of *essentially normal Fredholm operators*. These are operators $T \in B(\mathcal{H})$ for which $T^*T - TT^*$ is a compact operator. To each compact set $X \subset \mathbb{C}$ is associated a group $\mathrm{Ext}(X)$ comprising all essentially unitary equivalence classes of essentially normal operators with essential spectrum X. An important observation that was made was that $\mathrm{Ext}(X)$ can then be identified with the unitary equivalence classes of extensions of the form

$$0 \longrightarrow \mathcal{K} \longrightarrow E \longrightarrow C(X) \longrightarrow 0\,,, \tag{5.1.2}$$

(see [60]). The statement of the Brown-Douglas-Filmore theorem can then be phrased as saying that $\mathrm{Ext}(X)$ is an abelian group and that the natural map, $\mathrm{Index} : \mathrm{Ext}(X) \mapsto \mathrm{Hom}(K^{-1}(X), \mathbb{Z})$, is an isomorphism (which is well-defined). Here, $K^{-1}(X)$ is the ordinary K-group of X.

Kasparov later formulated the same map as an index pairing between K-theory and the classifying space $\mathrm{Ext}(A)$ of invertible extensions of A by compact operators [67], [66]. Following the well-known identification between the analytic and topological Fredholm index in Atiyah-Singer index theory, the homology group $\mathrm{Ext}(A)$ can be defined using Fredholm operators and, as a consequence of Voiculescu's theorem, the correspondence is given between homotopy equivalence classes of Fredholm modules and invertible compact extensions of A. In the more general language of KK-theory, the statement can be formulated:

$$\mathrm{Ext}^{-1}(A, B) \cong KK^1(A, B) \tag{5.1.3}$$

whenever A and B are σ-unital C^*-algebras and $\mathrm{Ext}^{-1}(A, B)$ is the homotopy invariant classifying space of extensions of A by the stabilisation $\mathcal{K} \otimes B$ of B. From a practical point of view, analysis of this correspondence tells us how to write down the boundary maps in the six-term exact sequence for K-theory (or K-homology) arising from a given extension of C^*-algebras.

5.1.2 *Objectives and literature review.*

In the light of the above remarks, we seek to address the following aspects of the geometry of extensions:

1. To construct spectral triples on E, based on given spectral triples on each of A and B which are related via the equation (5.1.1).
2. To show that in situations in which the spectral triples on each of A and B turn these algebras into compact metric spaces, then we can obtain, via Connes' metric, a natural compact quantum metric space on E.

There are three articles of particular interest to us: we expect the ability to construct spectral triples should be linked to the extension (5.1.1) admitting a "first order differential pullback structure" with respect to the noncommutative geometry of the ideal and quotient parts. This matter was considered in the context of extensions of C^*-algebras by Fréchet operator ideals [119]. The smoothness assumption which we formulate later in this chapter is based in part on their definition.

An important development in this area, focusing solely on the compact quantum metric aspects, was made by Chakraborty and Pal [19]. Their construction was not based on a spectral triple approach and it seems unlikely to me that their methods can be recovered precisely this way, though we think that the Lipschitz seminorms which come out of our analysis are similar (though certainly not identical) to theirs.

Most significantly, Christensen and Ivan study extensions similar to ours in the special case $B = \mathbb{C}$ ([27]), that is, the extension is by the algebra of compact operators, and the assumptions which we will need to make our analysis work essentially specialise to theirs. We have not provided an analysis of the Gromov-Hausdorff convergence aspects ([103]) of our construction as they do with theirs, which is based on associating two real parameters to the spectral triple on the ideal and quotient parts of the extension, but it seems reasonably evident we could introduce the same parameters they do at no loss of generality.

It has very recently been pointed out to me that the authors Carey *et al.* have recently studied the geometric aspects of extensions, focusing specifically on the commutative and noncommutative disc [13]. The language which they use is rather different from ours and somewhat more explicit, but it highlights an important principal in that we would like to compare examples arising in our approach from quantum deformations of manifolds to their classical counterparts.

5.1.3 Varieties of extensions.

A *topological extension* of X by Y, where X and Y are respectively compact and locally compact Hausdorff topological spaces, is a compact Hausdorff space $Z = X' \sqcup Y'$, together with continuous bijections $\iota_X : X \mapsto X'$, $\iota_Y : Y \mapsto Y'$. Note that $X' \subset Z$ is closed and ι_X is a homeomorphism. This determines a short exact sequence of C^*-algebras

$$0 \longrightarrow C_0(Y) \longrightarrow C(Z) \longrightarrow C(X) \longrightarrow 0\ . \tag{5.1.4}$$

Example 5.1.1. Recall that *Toeplitz algebra* $\mathcal{T}$ generated by the unilateral shift has a description as the short exact sequence of C^*-algebras,

$$0 \longrightarrow \mathcal{K}(H^+) \longrightarrow \mathcal{T} \longrightarrow C(\mathbb{T}) \longrightarrow 0\ , \tag{5.1.5}$$

Here, H^+ is the Hardy space of those functions in $H = L^2(\mathbb{T})$ with an analytic extension to the disc. The Toeplitz algebra is the algebra generated by operators $\{T_f : f \in C(\mathbb{T})\}$, $T_f := PM_fP$, where P is the projection into H^+ and M_f is the multiplication operator on $L^2(\mathbb{T})$. It is related to the commutative extension

$$0 \longrightarrow C_0(\text{open disc}) \longrightarrow C(\text{closed disc}) \longrightarrow C(\mathbb{T}) \longrightarrow 0\,, \tag{5.1.6}$$

Example 5.1.2. The quantum group SU_q2 was introduced by Woronowicz in [123]. as a 1-parameter deformation of the ordinary SU_2 group. When one considers the isomorphism $SU(2) \cong S^3$ of topological Lie groups, its C^*-algebra $C(S^3_q)$, viewed as the algebra of functions on the quantum 3-sphere, can be defined for each $q \in [0,1]$ as the universal C^*-algebra for generators α and β subject to the relations

$$\alpha^*\alpha + \beta^*\beta = I, \quad \alpha\alpha^* + q^2\beta\beta^* = I, \tag{5.1.7}$$

$$\alpha\beta = q\beta\alpha, \quad \alpha\beta^* = q\beta^*\alpha, \quad \beta^*\beta = \beta\beta^*. \tag{5.1.8}$$

Woronowicz shows that the C^*-algebras $C(S^3_q)$ are all isomorphic for $0 \leqslant q < 1$. For $q \in (0,1)$, there is an alternative description of $C(S^3_q)$ as a symplectic foliation ([112], [58], [8], [21]). We will use the description presented in [19]: write $\mathcal{H} := \ell_2(\mathbb{N}_0) \otimes \ell_2(\mathbb{Z})$ and let $S \in B(\ell_2(\mathbb{N}_0))$ and $T \in B(\ell_2(\mathbb{Z}))$ be the usual shift operators (i.e $Se_k := e_{k+1}\ \forall k \geqslant 0$, $Te_k := e_{k+1} \forall k \in \mathbb{Z}$). Let $N_q \in \mathcal{K}(\ell_2(\mathbb{N}_0))$ be defined by $N_q e_k := q^k e_k$. There exists a representation of $C(S^3_q)$ over $\mathcal{H}$ defined by:

$$\pi(\alpha) := S^*\sqrt{1-N_q^2} \otimes I, \quad \pi(\beta) \quad := \quad N_q \otimes T^*. \tag{5.1.9}$$

The representation π is faithful: if $x \in C(S^3_q)$ is such that $\pi(x) = 0$ then x must be invariant for the action of $\mathbb{T}^2$ given on the generators by $(\gamma_{z,w} : \alpha \mapsto z\alpha, \beta \mapsto w\beta)$. Thus x is in the closed linear span of polynomials in $\alpha^*\alpha, \alpha\alpha^*$ and $\beta^*\beta$. The relations (5.1.7), (5.1.8) soon imply $x = 0$.

By considering the map $\sigma : C(S^3_q) \to C(\mathbb{T})$ sending β to 0 and α to the generator T of

$C(\mathbb{T})$, we soon obtain a short exact sequence,

$$0 \to \mathcal{K} \otimes C(\mathbb{T}) \to C(S^3_q) \to C(\mathbb{T}) \to 0. \tag{5.1.10}$$

Writing $\pi(\alpha) = -PT^*P(1-\sqrt{1-N_q^2}) \otimes I + PT^*P \otimes I$, we see that we have an algebra embedding of $C(S^3_q)$ in the algebra $PC(\mathbb{T})P \otimes \mathbb{C}I + \mathcal{K}(\ell_2(\mathbb{N}_0)) \otimes C(\mathbb{T})$, where $P \in B(\ell_2(\mathbb{Z}))$ is the usual Toeplitz projection which has the property that $[P, x]$ is a compact operator for each $x \in \mathbb{C}(\mathbb{T})$ and $PxP \otimes 1 \in C(S^3_q)$ for each $x \in C(\mathbb{T})$. This embedding is an isomorphism.

Example 5.1.3. The Podleś spheres [96] were introduced as a family of *quantum homogeneous spaces* for the action of the quantum SU_2 group, which parallels the $SU(2)$ action of the ordinary 2-sphere. Probably the most widely studied algebraically non-trivial examples are the so-called *equatorial Podleś spheres*. They can be defined for each $q \in (0,1)$ as the universal C^*-algebra for generators α and β, subject to the relations $\beta^* = \beta$ and

$$\beta\alpha = q\alpha\beta, \quad \alpha^*\alpha + \beta^2 = I, \quad q^4\alpha\alpha^* + \beta^2 = q^4. \tag{5.1.11}$$

It is known in this case how to construct a faithful representation of $C(S^2_q)$ on the Hilbert space $\mathcal{H} := \ell_2(\mathbb{N}) \otimes \mathbb{C}^2$, namely,

$$\pi(\alpha) := T\sqrt{1-N_q^4} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \pi(\beta) \quad := \quad N_q^2 \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5.1.12}$$

This soon leads to the semisplit short exact sequence

$$0 \to \mathcal{K} \otimes \mathbb{C}^2 \to C(S^2_q) \to C(\mathbb{T}) \to 0. \tag{5.1.13}$$

The equatorial Podleś sphere can be viewed as a symplectiv q-deformation of the 2-sphere arising as gluing of two open discs about the equator, viewed as the short exact sequence,

$$0 \to C_0(\mathbb{R}^2) \otimes \mathbb{C}^2 \to C(S^2) \to C(\mathbb{T}) \to 0. \tag{5.1.14}$$

Many other varieties of noncommutative spheres, including those in higher dimensions and those defined using extra parameters, can be found in the literature. Some of the more prominent examples include the isospectral noncommutative 4-spheres of Connes and Landi [33] and those arising as symplectic foliations ([112], [58], [8]). The noncommutative geometry of quantum spheres has been studied in numerous articles ([108], [88], [43], [44], [45], [86], [118], [20] - [24] and more besides).

5.1.4 Set up.

Convention. We suppose that we are given

1. separable and unital C^*-algebras A and B,
2. a short exact sequence (5.1.1), where E is unital and
3. $\mathcal{K} \otimes B \subset E$ as an *essential ideal*; if $I \subset E$ is another ideal then $I \cap \mathcal{K} \otimes B \neq \{0\}$.

Extensions of this form are quite easy to describe by invariants, for each essential inclusion $\mathcal{K} \otimes B \subset E$ determines a faithful unital *-homomorphism π of E into the multiplier algebra $M(\mathcal{K} \otimes B)$ of $\mathcal{K} \otimes B$, further identified with the C^*algebra $\mathcal{L}(\mathcal{H}_B)$ of bounded adjointable operators on the standard right Hilbert module $\mathcal{H}_B$ of B. The *Busby invariant* of (5.1.1) is the faithful unital *-homomorphism,

$$\psi : A \cong E/\mathcal{K}(\mathcal{H}_B) \mapsto \mathcal{L}(\mathcal{H}_B)/\mathcal{K}(\mathcal{H}_B) =: Q(\mathcal{H}_B). \tag{5.1.15}$$

In short, we are studying unital extensions for which the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{K} \otimes B \arrow[r, hook, "\iota"] \arrow[d, hook] & E \arrow[r, "\sigma"] \arrow[d, "\pi"] & A \arrow[r] \arrow[d, "\psi"] & 0 \\
0 \arrow[r] & \mathcal{K}(\mathcal{H}_B) \arrow[r, hook] & L(\mathcal{H}_B) \arrow[r, "q"] & Q(\mathcal{H}_B) \arrow[r] & 0
\end{tikzcd}$$

commutes and each vertical map is injective.

Definition 5.1.4. An extension of the above type is called *semisplit* if and only if ψ admits a completely positive lift, i.e there exists a unital completely positive map $s : A \mapsto \mathcal{L}(\mathcal{H}_B)$ such that $q \circ s = \psi$, where $q : \mathcal{L}(\mathcal{H}_B) \mapsto Q(\mathcal{H}_B)$ is the quotient map.

Theorem 5.1.5. *[6] [60] Let A, B be unital C^*-algebras and let ψ, as in equation (5.1.15), admit a completely positive splitting. Then there exists a faithful unital representation $\rho : A \mapsto$*

$M_2(\mathcal{L}(\mathcal{H}_B)) \cong \mathcal{L}(\mathcal{H}_B \oplus \mathcal{H}_B)$ *and an orthogonal projection* $P \in M_2(\mathcal{L}(\mathcal{H}_B))$*:*

$$\rho = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}; \quad P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{5.1.16}$$

such that $\rho_{11}(a) = s(a)$ *for each* $a \in A$*. Moreover,* $[P, \rho(a)] \in \mathcal{K}(\mathcal{H}_B \oplus \mathcal{H}_B)$ *for each* $a \in A$*.*

Proof. The first statement is a well known formulation of Stinespring's theorem for Hilbert modules (see [6], Exercise 13.7.2). The second statement can be argued as in [60], Lemma 3.1.6: Notice that $s := P\rho(\cdot)P$ is a *-homomorphism up to $\mathcal{K}(\mathcal{H}_B)$. By assumption, ρ is a *-homomorphism so that in particular $\rho(aa^*) = \rho(a)\rho(a^*)$ for each a. Comparing matrix entries and using the first part of the result gives us

$$s(aa^*) = s(a)s(a^*) + \rho_{12}(a)\rho_{12}(a^*).$$

We deduce that $\rho_{12}(a)\rho_{12}(a^*) \in \mathcal{K}(\mathcal{H}_B)$ for each $a \in A$. This means that $q(\rho_{12}(a)\rho_{12}(a^*)) = 0$ where q is quotient map and, from the C*-algebra identity on $\mathcal{L}(\mathcal{H}_B)/\mathcal{K}(\mathcal{H}_B)$, we have that $q(\rho_{12}(a)) = 0$ and therefore $\rho_{12}(a) \in \mathcal{K}(\mathcal{H}_B)$. $\rho_{21} \in \mathcal{K}(\mathcal{H}_B)$ by a similar argument. □

As per convention, the pair (ρ, P) shall be called a *Stinespring dilation*. The next result is the essence of Kasparov's isomorphism between invertible (i.e semisplit) extensions and KK-cycles [66]:

Corollary 5.1.6. *Given a semisplit extension (5.1.1) and a Stinespring dilation* (ρ, P)*, the pair* $(\mathcal{H}_B \oplus \mathcal{H}_B, 2P - 1)$*, defined by the representation* $\rho : A \mapsto M_2(\mathcal{H}_B)$ *above, defines an ungraded Kasparov* A*-*B *module. We shall let* $[\psi]$ *denote its equivalence class in* $KK^1(A, B)$*.*

Theorem 5.1.7. *[66] Every invertible extension of the form equation (5.1.1), where* A *and* B *are* σ*-unital, defines a six-term exact sequence in* K*-homology,*

$$\begin{tikzcd} K^0(A) \arrow[r, "\sigma^*"] & K^0(E) \arrow[r, "\iota^*"] & K^0(B) \arrow[d, "\delta_0^*"] \\ K^1(B) \arrow[u, "\delta_1^*"] & K^1(E) \arrow[l, "\iota^*"] & K^1(A), \arrow[l, "\sigma^*"] \end{tikzcd} \tag{5.1.17}$$

where $\sigma^* : K^0(A) \mapsto K^0(E)$, $\iota^* : K^0(E) \mapsto K^0(B)$ *are the induced group homomorphisms and* $\delta_0^*[x] \in K^1(A)$ *is the internal Kasparov product of* $[\psi] \in KK^1(A,B)$ *and* $[x] \in K^0(B)$ *and* $\delta_1^*[x] \in K^0(A)$ *is the internal Kasparov product of* $[\psi] \in KK^1(A,B)$ *and* $[x] \in K^1(B)$.

5.2 Extensions and KK-theory.

The diagram (5.1.17) helps to describe the K-homology of the extension E in terms of the K-homology of A and B. When working with K-homology as a dual object to K-theory, one must distinguish between graded and ungraded varieties. From our point of view, however, all that we need is a way of associating K-homology to spectral triples and for this purpose we have found that a separate handling of graded and ungraded K-homology makes our analysis rather more difficult. Instead, we shall write $K^*(\cdot)$ to refer to possibly odd or even classes of K-homology. We are aiming to construct a well defined pairing,

$$\Phi : K^*(A) \times K^*(B) \mapsto K^*(E). \tag{5.2.1}$$

Given a *-homomorphism $\sigma : A_1 \mapsto A_2$ of C^*-algebras, as usual we shall let σ^* denote the induced group homomorphism from $K^*(A_2)$ to $K^*(A_1)$, which is clearly well defined. Starting from the sequence (5.1.1), we obtain another sequence,

$$K^*(A) \xrightarrow{\sigma^*} K^*(E) \xrightarrow{\iota^*} K^*(B)\ . \tag{5.2.2}$$

Both σ^* and ι^* are defined in the usual way by right composition with the left module action, and their representatives in ordinary KK-theory, respectively $KK(E,A)$ and $KK(B,E)$, can be expressed as $\sigma^* = [z_1]$, $\iota^* = [z_2]$, where

$$z_1 := \left(\sigma \oplus 0, \quad \mathcal{H}_A \oplus \mathcal{H}_A, \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right); \ z_2 := \left(\iota \oplus 0, \quad \mathcal{H}_B \oplus \mathcal{H}_B, \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right),$$

and $\sigma : E \mapsto A \subset \mathcal{L}(\mathcal{H}_A)$, $\iota : B \mapsto E \subset \mathcal{L}(\mathcal{H}_B)$ are the natural maps. When the sequence (5.1.1) splits, it is known how to construct a Kasparov E-B module. For semisplit extensions, this process can be mimicked using Theorem 5.1.5:

Proposition 5.2.1. *Given a semisplit extension (5.1.1) and a Stinespring dilation* (ρ, P)*, the triple*

$$z_3 := \left(\mathrm{id} \oplus (\rho \circ \sigma), \quad \mathcal{H}_B \oplus \mathcal{H}_B \oplus \mathcal{H}_B, \quad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \right),$$

where $\mathrm{id} : E \subset \mathcal{L}(\mathcal{H}_B)$ *and* $\rho \circ \sigma : E \mapsto \mathcal{L}(\mathcal{H}_B \oplus \mathcal{H}_B)$*, determines a Kasparov module* $\tau^* \in KK(E, B)$.

Remark 5.2.2. It is not immediately clear to me whether $\tau^* \in KK^0(E, B)$ or $\tau^* \in KK^1(E, B)$.

Proof. The non-trivial part of the proof showing that each of the commutators of the Fredholm operator defining z_3 are in $\mathcal{K}(\mathcal{H}_B \oplus \mathcal{H}_B \oplus \mathcal{H}_B)$ for every element in E. To this end fix $e \in E$ and let $a = \sigma(e)$. Then,

$$\left[\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} e & 0 & 0 \\ 0 & \rho_{11}(a) & \rho_{12}(a) \\ 0 & \rho_{21}(a) & \rho_{22}(a) \end{pmatrix} \right] = \begin{pmatrix} 0 & s(a) - e & \rho_{12}(a) \\ e - s(a) & 0 & \rho_{12}(a) \\ -\rho_{21}(a) & -\rho_{21}(a) & 0 \end{pmatrix}.$$

where the whole term is in $\mathcal{K}(\mathcal{H}_B \oplus \mathcal{H}_B \oplus \mathcal{H}_B)$, since each entry is in $\mathcal{K}(\mathcal{H}_B)$. □

Remark 5.2.3. We have chosen τ^* in such a way that it can be viewed as a cross section for the sequence 5.2.2 in the sense that $\iota^* \circ \tau^*$ acts as an identity on $K^*(B)$. To see this, note that $\iota^* \circ \tau^* \in KK^*(B, B)$ can be represented by a homotopy equivalence of Fredholm modules

$$z_4^t := \left(\mathrm{id} \oplus 0, \quad \mathcal{H}_B \oplus \mathcal{H}_B \oplus \mathcal{H}_B, \quad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -t \end{bmatrix} \right),$$

and z_4^0 represents the direct sum of the module representing $I_B^* \in KK^*(B, B)$ with a degenerate Fredholm module.

In conclusion, for each pair of Fredholm modules $[x_A] \in K^*(A)$ and $[x_B] \in K^*(B)$ we can construct a pairing, which will form the basis for our analysis,

$$\Phi([x_A], [x_B]) := \sigma^*([x_A]) \oplus \tau^*([x_B]) \in K^*(E). \tag{5.2.3}$$

5.3 Representation theory and smooth structures.

5.3.1 An ansatz for the representation theory of semisplit extensions.

From now on, because of the difficulties involved in trying to describe the representation theory for C^*-algebras arising as extensions, we do not consider all extensions by stable ideals but only unital extensions of the form

$$E \cong \mathcal{K}(PH_A) \otimes \pi_B(B) + P\pi_A(A)P \otimes \mathbb{C}I_{H_B}, \tag{5.3.1}$$

where A and B are unital C^*-algebras, $\pi_A : A \mapsto B(H_A)$ and $\pi_B : A \mapsto B(H_B)$ are faithful nondegenerate representations over separable Hilbert spaces H_A, H_B and $P \in B(H_A)$ is an orthogonal projection with the properties

1. $[P, \pi_A(a)] \in \mathcal{K}(H_A)$ for each $a \in A$,
2. $\mathcal{K}(PH_A) \otimes \pi_B(B) \cap P\pi_A(A)P \otimes \mathbb{C}I_{H_B} = \{0\}$.

The Busby invariant $\psi : A \mapsto \mathcal{Q}(PH_A) \otimes \pi_B(B)$ is recovered via $\psi(a) := P\pi_A(a)P \otimes 1 + \mathcal{K}(PH_A) \otimes \pi_B(B)$, which is faithful and unital. As usual, the ideal and quotient maps will be denoted by $\iota : B \mapsto E$ and $\sigma : E \mapsto A$ respectively.

Extensions given by this description generalise those studied in [27], for which $B = \mathbb{C}$ and, as we have seen, the noncommutative spheres discussed in Examples 5.1.2 and 5.1.3 also belong into this class. Conversely such representations do not apply to *all* extensions by stable C^*-algebras. One such exception is the generalised Toeplitz extension arising from a crossed product of a C^*-algebra by an automorphism ([95]). To date, it is not clear to me if it is possible to provide a natural invariant of extensions which fit the above description.

5.3.2 Toeplitz type extensions and smoothness criteria.

From now on, we shall assume the existence of spectral triples $(\mathcal{A}, H_A, \mathcal{D}_A)$ and $(\mathcal{B}, H_B, \mathcal{D}_B)$ on A and B respectively. Prompted by a very similar approach in [27], we make a couple of definitions:

Definition 5.3.1. An extension of the form (5.3.1) is said to be of *Toeplitz type* if $[P, \mathcal{D}_A] = 0$, that is, P commutes with $\mathcal{D}_A$ on $\mathrm{Dom}(\mathcal{D}_A)$.

Definition/Proposition 5.3.2. Following [119], an extension of the form (5.3.1), together with spectral triples $(\mathcal{A}, H_A, \mathcal{D}_A)$ and $(\mathcal{B}, H_B, \mathcal{D}_B)$ on A and B respectively, is called *smooth* if additionally $[P, \pi_A(a)] \in \mathcal{C}(H_A)$ for each $a \in \mathcal{A}$, where

$$\mathcal{C}(H_A) := \{y \in \mathcal{K}(H_A) : \ \mathcal{D}_A y,\ y\mathcal{D}_A \in \mathcal{B}(H_A)\}, \tag{5.3.2}$$

is the dense subalgebra of "differentiable compacts", which is a Banach *-algebra (c.f Definition 2.1.3) with respect to the graph norm $\|y\|_1 := \|y\| + \max\{\|\mathcal{D}_A y\|, \|y\mathcal{D}_A\|\}$.

Proof. It has already been seen that $\mathcal{C}(H_A) \subset \mathcal{K}(H_A)$ is dense. On the other hand a direct computation shows that $\|xy\|_1 \leqslant \|x\|_1\|y\|_1$ for all $x, y \in \mathcal{C}(H_A)$ and that $\mathcal{C}(H_A)$ is a Banach algebra. □

Remark 5.3.3. The smoothness criterium implies that the operators given by

$$[P\mathcal{D}_A, \pi_A(a)] = P[\mathcal{D}_A, \pi_A(a)] + [P, \pi_A(a)]\mathcal{D}_A, \tag{5.3.3}$$
$$[(1-P)\mathcal{D}_A, \pi_A(a)] = (1-P)[\mathcal{D}_A, \pi_A(a)] - [P, \pi_A(a)]\mathcal{D}_A, \tag{5.3.4}$$

are bounded and densely defined for each $a \in \mathcal{A}$ and that the quadruple $((\mathcal{A}, H_A, \mathcal{D}_A), P)$ is of *Toeplitz type* (c.f [27], Definition 1.2), up to the condition that also the operator $\mathcal{D}_A P$ has trivial kernel. As explained in the same article, this means that we can write $\mathcal{D}_A = P\mathcal{D}_A \oplus (1-P)\mathcal{D}_A$ as an orthogonal direct sum and decompose the commutators $[\mathcal{D}_A, \pi_A(a)]$ into its four matrix parts corresponding to the decomposition $H_A = PH_A \oplus (1-P)H_A$ in such a way that the matrix entries of the closure is the closure of each of the corresponding matrix entries.

Remark 5.3.4. As suggested in [27], it is canonical to associate P with the orthogonal projection into the closed span of the eigenspaces corresponding to the positive part of the spectrum for $\mathcal{D}_A$, so that extensions of Toeplitz type can be interpreted as extensions for which the spectral triple on A satisfies not only $[\mathcal{D}_A, \pi_A(a)] \in B(H_A)$ but also $[|\mathcal{D}_A|, \pi_A(a)] = [(2P-1)\mathcal{D}_A, \pi_A(a)] \in B(H_A)$ for all $a \in A$.

The next objective is to identify the algebra structure which characterises the first order differential geometry of the extension, on which we can then build a spectral triple. We shall refer back to chapter two for discussion on operator *-algebras. We shall adopt the notation, $x_{pp} := PxP$, $x_{pq} := Px(1-P)$, $x_{qp} := (1-P)xP$, $x_{qq} := (1-P)x(1-P)$ for $x \in B(H_A)$.

Lemma 5.3.5. *The maps* $\delta_A : A \mapsto B(H_A \otimes H_B)$, $\delta_B : \mathcal{K}(PH_A) \otimes \pi_B(B) \mapsto B(H_A \otimes H_B \otimes \mathbb{C}^2)$, $\kappa : A \mapsto B(H_A \otimes H_B \otimes \mathbb{C}^2)$, *given by*

$$\delta_A(a) = [\mathcal{D}_A \otimes 1, \pi_A(a) \otimes 1],\ \delta_B(x) = \begin{pmatrix} [1 \otimes \mathcal{D}_B, x] & x(P\mathcal{D}_A \otimes 1) \\ -(P\mathcal{D}_A \otimes 1)x & 0 \end{pmatrix},$$

$$\kappa(a) = \begin{pmatrix} 0_{pp} & 0_{pq} & [\mathcal{D}_A, \pi_A(a)]_{pp} \otimes 1 & \mathcal{D}_A \pi_A(a)_{pq} \otimes 1 \\ 0_{qp} & 0_{qq} & 0_{qp} & 0_{qq} \\ [\mathcal{D}_A, \pi_A(a)]_{pp} \otimes 1 & 0_{pq} & 0_{pp} & -\pi_A(a)_{pq} \mathcal{D}_A \otimes 1 \\ -\pi_A(a)_{qp} \mathcal{D}_A \otimes 1 & 0_{qq} & \mathcal{D}_A \pi_A(a)_{qp} \otimes 1 & [\mathcal{D}_A, \pi_A(a)]_{qq} \otimes 1 \end{pmatrix},$$

are densely defined derivations in the sense that

$$\delta_A(ab) = \delta_B(a)\pi_1(b) + \pi_1(a)\delta_B(b),\ a, b \in \mathrm{Dom}(\delta_A)$$
$$\delta_B(xy) = \delta_B(x)\pi_2(y) + \pi_2(x)\delta_B(y),\ x, y \in \mathrm{Dom}(\delta_B)$$
$$\kappa(cd) = \kappa(c)\pi_3(d) + \pi_3(c)\kappa(d),\ c, d \in \mathrm{Dom}(\kappa)$$

where $\pi_1 := \pi_A \otimes 1 : A \mapsto B(H_A \otimes H_B), \pi_2 := \mathrm{id} \oplus 0 : \mathcal{K}(PH_A) \otimes B \mapsto B(H_A \otimes H_B \otimes \mathbb{C}^2), \pi_3 := (\pi_A \otimes 1) \oplus 0 : A \mapsto B(H_A \otimes H_B \otimes \mathbb{C}^2)$.

The spaces $\mathcal{A}$ *and* $\mathrm{Dom}(\delta_B)$ *become* **-algebras when equipped with the norms* $\|a\|_{1,A} := \|a\| + \max\{\|\delta_A(a)\|, \|\kappa(a)\|\}$, $\|x\|_{1,B} := \|x\| + \|\delta_B(x)\|$ *respectively. The completion of these spaces will be denoted* A_1 *and* $\mathcal{C}(PH_A) \otimes_1 B_1$ *respectively, where* B_1 *is the completion of* $\mathcal{B}$ *in the norm* $\|b\|_1 := \|b\|_B + \|\delta_B(b)\|_B$, *so that* A_1 *and* $\mathcal{C}(PH_A) \otimes_1 B_1$ *are operator* **-algebras.*

Proof. Necessarily $\mathcal{A} \subset \mathrm{Dom}(\delta_A)$ and elementary tensors in $\mathcal{C}(PH_A) \otimes \pi_B(\mathcal{B})$ are in $\mathrm{Dom}(\delta_B)$, so that δ_A, δ_B and κ are densely defined. It is easy to verify that these are all derivations and the conclusion now follows from Proposition 2.1.5. $\square$

Definition 5.3.6. For each smooth extension of the form (5.3.1),where P is of Toeplitz type, we define the *smooth part of the extension* to be the dense *-subalgebra,

$$E_1 := \mathcal{C}(P\mathcal{H}_A) \otimes_1 B_1 + P\pi_A(A_1)P \otimes_1 \mathbb{C}I_{H_B}, \tag{5.3.5}$$

where the last term is understood to be the inclusion of $P\pi_A(A_1)P$ in $B(H_A \otimes H_B)$.

The assumption of smoothness ensures E_1 is an algebra, which is evidently closed under involution. Notice that necessarily $\mathcal{C}(PH_A) \otimes_1 B_1 \cap P\pi_A(A_1)P \otimes_1 \mathbb{C}I_{H_B} = \{0\}$, so that every element in E_1 has a unique decomposition of the form $x + y$, where $x \in \mathcal{C}(PH_A) \otimes_1 B_1$ and $y \in P\pi_A(A_1)P \otimes_1 \mathbb{C}I_{H_B}$.

Lemma 5.3.7. E_1 *becomes an operator* **-algebra when equipped with the norm*

$$\|x + y\|_{1,E} := \|x + y\| + \max\{\|\delta_A(\sigma_1(y))\|, \|\delta_B(x) + \kappa(\sigma_1(y))\|\}, \tag{5.3.6}$$

where $x \in \mathcal{C}(PH_A) \otimes_1 B_1$, $y \in P\pi_A(A_1)P \otimes_1 \mathbb{C}I_{H_B}$ *and* $\sigma_1 : E_1 \mapsto A_1$ *is the restriction of* $\sigma : E \mapsto A$ *to* E_1. *Moreover the natural maps* $\iota_1 : (\mathcal{C}(PH_A) \otimes_1 B_1, \|\cdot\|_{1,B}) \mapsto (E_1, \|\cdot\|_{1,E})$ *and* $\sigma_1 : (E_1, \|\cdot\|_{1,E}) \mapsto (A_1, \|\cdot\|_{1,A})$ *are norm-decreasing* **-homomorphisms, such that* ι_1 *is injective,* σ_1 *is surjective and* $\mathrm{Im}(\iota_1) = \mathrm{Ker}(\sigma_1)$.

Proof. We will show E_1 is an operator *-algebra in the next section. The other assertions are obvious. □

Corollary 5.3.8. *A smooth extension of the form (5.3.1), where* P *is an orthogonal projection of Toeplitz type, defines a continuous pullback diagram of operator* **-algebras:*

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{C}(PH_A) \otimes_1 B_1 \arrow[r, hook, "\iota_1"] \arrow[d, hook] & E_1 \arrow[r, "\sigma_1"] \arrow[d, hook] & A_1 \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & \mathcal{K}(PH_A) \otimes B \arrow[r, hook, "\iota"] & E \arrow[r, "\sigma"] & A \arrow[r] & 0
\end{tikzcd}$$

5.4 Construction of the spectral triple.

We are ready to write down spectral triples based on the smooth part of the extension E_1. The methods automatically extend to the setting when E_1 is replaced by any dense

*-subalgebra $\mathcal{E}$, which might be easier or more natural in certain situations. The Hilbert space in this construction is $H := \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathbb{C}^2 = (P\mathcal{H}_A \otimes \mathcal{H}_B \oplus (1-P)\mathcal{H}_A \otimes \mathcal{H}_B) \otimes \mathbb{C}^2$, equipped with two representations $\pi_i : E \mapsto B(H)$ for $i \in \{1,2\}$,

$$\pi_1 := \begin{pmatrix} (\pi_A \circ \sigma) \otimes 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \pi_2 := \begin{pmatrix} \tilde{\pi} & 0 \\ 0 & (\pi_A \circ \sigma) \otimes 1 \end{pmatrix},$$

where $\tilde{\pi} : E \mapsto B(\mathcal{H}_A \otimes \mathcal{H}_B)$ is the natural amplification of $\pi : E \mapsto B(P\mathcal{H}_A \otimes \mathcal{H}_B)$ by 0 (that is, $\tilde{\pi}$ acts as zero on the $(1-P)\mathcal{H}_A \otimes \mathcal{H}_B$ part of the Hilbert space). Faithfulness of π ensures that $\pi_1 \oplus \pi_2$ is a faithful representation.

Lemma 5.4.1. *The operators $\mathcal{D}_i : \mathrm{dom}(\mathcal{D}_i) \mapsto H$, $i \in \{1,2\}$ defined by*

$$\mathcal{D}_1 := \begin{bmatrix} \mathcal{D}_A \otimes 1 & 1 \otimes \mathcal{D}_B \\ 1 \otimes \mathcal{D}_B & -\mathcal{D}_A \otimes 1 \end{bmatrix}, \tag{5.4.1}$$

$$\mathcal{D}_2 := \begin{bmatrix} P \otimes \mathcal{D}_B & 0 & P\mathcal{D}_A \otimes 1 & 0 \\ 0 & (1-P) \otimes \mathcal{D}_B & 0 & 0 \\ P\mathcal{D}_A \otimes 1 & 0 & -P \otimes \mathcal{D}_B & 0 \\ 0 & 0 & 0 & (1-P)\mathcal{D}_A \otimes 1) \end{bmatrix}. \tag{5.4.2}$$

are linear, densely defined self-adjoint operators with compact resolvent. The common domain of $\mathcal{D}_1$ and $\mathcal{D}_2$ is also dense. Moreover, provided there exists positive numbers $p, q > 0$ such that the spectral triples $(\mathcal{A}, H_A, \mathcal{D}_A)$ and $(\mathcal{B}, H_B, \mathcal{D}_B)$ are respectively p- summable and q- summable then both $\mathcal{D}_1$ and $\mathcal{D}_2$ are $p+q$- summable.

Proof. We start by making the comment that the operator $\mathcal{D}_1$ is precisely of the form arising in the Kasparov external product between graded and ungraded spectral triples. To show that $\mathcal{D}_1$ and $\mathcal{D}_2$ are self-adjoint operators with compact resolvent, we start by remarking that $\mathcal{D}_1$ and $\mathcal{D}_2$ are, by assumption, essentially self-adjoint operators with compact resolvent. As such, they are defined on the closure of the (necessarily finite dimensional) eigenspaces corresponding to their pure-point spectra. The spectra of $\mathcal{D}_1$ and $\mathcal{D}_2$ are countable subsets of $\mathbb{R}$, so we shall write $\mathrm{spec}\mathcal{D}_1 = \{\lambda_m\}_{m\in\mathbb{N}}$ and $\mathrm{spec}\mathcal{D}_2 = \{\mu_n\}_{n\in\mathbb{N}}$, where each eigenvalue has finite multiplicity.

We have the identities,

$$\mathcal{D}_1^2 = \begin{bmatrix} \mathcal{D}_A^2 \otimes 1 + 1 \otimes \mathcal{D}_B^2 & 0 \\ 0 & \mathcal{D}_A^2 \otimes 1 + 1 \otimes \mathcal{D}_B^2 \end{bmatrix}, \tag{5.4.3}$$

$$\mathcal{D}_2^2 = \begin{bmatrix} P\mathcal{D}_A^2 \otimes 1 + 1 \otimes \mathcal{D}_B^2 & 0 \\ 0 & (1-P)\mathcal{D}_A^2 \otimes 1 + P \otimes \mathcal{D}_B^2 \end{bmatrix}. \tag{5.4.4}$$

Thus, given every eigenvalue pair (λ, μ) for $(\mathcal{D}_1, \mathcal{D}_2)$ with eigenvectors $(\xi, \eta) \in \mathcal{H}_A \times \mathcal{H}_B$, we can associate two eigenvectors,

$$v_{\lambda,\nu}^{(1)} := \begin{pmatrix} \xi \otimes \eta \\ 0 \end{pmatrix}, \quad v_{\lambda,\nu}^{(2)} := \begin{pmatrix} 0 \\ \xi \otimes \eta \end{pmatrix} \tag{5.4.5}$$

corresponding to the eigenvalue $\lambda^2 + \mu^2$. Every eigenvalue of $\mathcal{D}_1^2$ arises this way. An application of Nelson's theorem then ensures that $\mathcal{D}_1$ is self-adjoint and has compact resolvent. A similar argument applies for $\mathcal{D}_2$, since the essential spectra of $\mathcal{D}_1^2$ and $\mathcal{D}_2^2$ coincide (though the multiplicity of each eigenvector may be smaller). The other properties are now obvious. $\square$

Lemma 5.4.2. *The operators* $[\mathcal{D}_i, \pi_i(e)] : \mathrm{Dom}(\mathcal{D}_i) \mapsto H$ *are bounded for* $e \in E_1$.

Proof. Let $e = x + y \in \mathcal{E}$, where $y \in P\pi_A(A_1)P \otimes_1 \mathbb{C}I_B$ and $x \in \mathcal{C}(PH_A) \otimes_1 B_1$. Direct computation shows,

$$[\mathcal{D}_1, \pi_1(x+y)] = \mathrm{diag}(\delta_A(\sigma_1(y)), 0), \tag{5.4.6}$$

$$[\mathcal{D}_2, \pi_2(x+y)] = \delta_B(x) + \kappa(\sigma_1(y)), \tag{5.4.7}$$

and the result follows. This also proves E_1 is an operator algebra (Lemma 5.3.7), since by definition $\|e\|_{1,E} := \|e\| + \max\{\|[\mathcal{D}_1, \pi_1(e)]\|, \|[\mathcal{D}_2, \pi_2(e)]\|\}$. $\square$

Theorem 5.4.3. *Let* A *and* B *be unital* C^**-algebras, endowed with spectral triples* $(\mathcal{A}, H_A, \mathcal{D}_A)$ *and* $(\mathcal{B}, H_B, \mathcal{D}_B)$ *respectively. Let* E *be a smooth extension of the form (5.3.1), where* $P \in B(H_A)$ *is an orthogonal projection of Toeplitz type. Then, for each dense* **-subalgebra* $\mathcal{E} \subset E_1$, $(\mathcal{E}, H \oplus$

$H, \mathcal{D}_1 \oplus \mathcal{D}_2)$, *represented via* $\pi_1 \oplus \pi_2$, *defines a spectral triple on* E. *Moreover, the spectral dimension of this spectral triple is given by the identity*

$$s_0(\mathcal{E}, H \oplus H, \mathcal{D}_1 \oplus \mathcal{D}_2) = s_0(\mathcal{A}, H_A, \mathcal{D}_A) + s_0(\mathcal{B}, H_B, \mathcal{D}_B). \tag{5.4.8}$$

Furthermore, the spectral triple represents the Fredholm module $\sigma^*(\mathcal{A}, H_A, \mathcal{D}_A) \oplus \tau^*(\mathcal{B}, H_B, \mathcal{D}_B)$ *in K-homology.*

Proof. This follows from 5.4.1 and 5.4.2. The last statement is illustrated in **Appendix B**. □

5.5 Extensions of compact quantum metric spaces.

We now divert our attention to the existence of compact quantum metric spaces. Therefore, through this chapter, we assume that the hypotheses of Theorem 5.4.3 hold and additionally the spectral triples $(\mathcal{A}, H_A, \mathcal{D}_A)$, $(\mathcal{B}, H_B, \mathcal{D}_B)$, where the induced Lipschitz pairs $(\mathcal{A}, L_{\mathcal{D}_A})$ and $(\mathcal{B}, L_{\mathcal{D}_B})$ are compact quantum metric spaces for A and B respectively. Because of this, we shall introduce the notation,

$$\mathcal{X} := \mathcal{C}(PH_A) \otimes_1 B_1 \cap \mathcal{E}, \quad \mathcal{Y} := P\pi_A(A_1)P \otimes_1 \mathbb{C}I_B \cap \mathcal{E},$$

$$L_A(x+y) := \|\delta_A(\sigma_1(y))\|, \; L_B(x+y) := \|\delta_B(x)\|, \;\; x \in \mathcal{X}_{sa}, \; y \in \mathcal{Y}_{sa},$$

$$\mathcal{U}_{A,1} := \{y \in \mathcal{Y}_{sa} : \; \|\sigma_1(y)\| \leqslant 1, \; L_A(y) \leqslant 1\}; \quad \tilde{\mathcal{U}_A} := \{\tilde{y} \in \mathcal{Y}_{sa}/\mathbb{R}I_A : \; L_A(\tilde{y}) \leqslant 1\},$$

$$\mathcal{U}_B := \{x \in \mathcal{X}_{sa} : \; L_B(x) \leqslant 1\}.$$

In addition to each of the previously stated assumptions, we will adopt an extra convention which was proposed in [27], namely $P\mathcal{D}_A$ has trivial kernel. We remark that necessarily $\mathrm{Ker}P\mathcal{D}_A$ is finite rank, so that if this convention fails then we can merely replace P with $P - P\mathrm{Ker}(\mathcal{D}_A)$, a procedure which does not affect any other aspects of the extension.

The Lipschitz pair coming from the spectral triple on E is $(\mathcal{E}_{sa}, L)$, where $\mathcal{E}_{sa} = \mathcal{X}_{sa} + \mathcal{Y}_{sa}$ is the self-adjoint part of $\mathcal{E}$ and $L(e) := \max\{\|[\mathcal{D}_1, \pi_1(e)]\|, \|[\mathcal{D}_2, \pi_2(e)]\|\}$ is the usual seminorm coming from the spectral triple on $\mathcal{E}$ in Theorem 5.4.3. In this way the seminorm L is automatically lower semicontinuous and satisfies the Leibniz rule.

Proposition 5.5.1. *$(\mathcal{E}_{sa}, L)$ is nondegenerate, i.e $(\mathcal{E}_{sa}, L)$ is a Lipschitz pair.*

Proof. The proof comprises showing $\mathcal{E}_{sa} \cap L^{-1}(0) = \mathbb{R}I_E$, where $I_E = I_A|_{PH_A}$ is the identity of E. Notice that the proof of Lemma 5.4.2 implies immediately $L_A(e) = 0$ for each $e \in \mathcal{E}$, so that if $e \in \mathcal{E}_{sa}$ and $L(e) = 0$ then $\sigma_1(e) = \lambda I_A$ for some $\lambda \in \mathbb{R}$, by nondegeneracy of the spectral triple on A. It also means that we can write $e = x + \lambda I_E$ for some $x \in \mathcal{X}_{sa}$. The same Lemma now implies $\delta_B(x) = 0$, so that $[1 \otimes \mathcal{D}_B, x] = 0$ and $(P\mathcal{D}_A \otimes 1)x = 0$. Since $P\mathcal{D}_A$ is invertible, $x = 0$. So $e = \lambda I_E$ and L is nondegenerate. □

Lemma 5.5.2. *Let $L_{A,B}$ be the seminorm on $\mathcal{E}_{sa}$ given by*

$$L_{A,B}(e) := \max\{L_A(e), L_B(e)\}, \tag{5.5.1}$$

where $e \in \mathcal{E}_{sa}$. Then $L_{A,B}$ is a Lipschitz seminorm, $(\mathcal{E}_{sa}, L_{A,B})$ is another Lipschitz pair on E and $L_{A,B}(e) \leqslant 2L(e)$ for all $e \in \mathcal{E}_{sa}$.

Proof. The same argument as in the proof of Proposition 5.5.1 shows that $L_{A,B}$ is nondegenerate. Fix x and y as above and write $e = x + y$. Expanding terms shows that $L(e) \geqslant \|\delta_A(\sigma_1(y))\| = L_A(y)$ and $L(e) \geqslant \|\kappa(\sigma_1(y)) + \delta_B(x)\|$. By projecting into the subspace of $\mathcal{H}$ corresponding to the upper left 3x3 block matrix, we find

$$\begin{aligned} L(e) &\geqslant \left\| \begin{pmatrix} [1 \otimes \mathcal{D}_B, x] & [\mathcal{D}_A, \pi_A(\sigma_1(y))]_{pp} \otimes 1 + x(P\mathcal{D}_A \otimes 1) \\ [\mathcal{D}_A, \pi_A(\sigma_1(y))]_{pp} \otimes 1 - (P\mathcal{D}_A \otimes 1)x & 0 \end{pmatrix} \right\| \\ &= \left\| \delta_B(x) + \begin{pmatrix} 0 & [\mathcal{D}_A, \pi_A(\sigma_1(y))]_{pp} \otimes 1 \\ [\mathcal{D}_A, \pi_A(\sigma_1(y))]_{pp} \otimes 1 & 0 \end{pmatrix} \right\|. \end{aligned}$$

Therefore, $L_B(x) = \|\delta_B(x)\| \leqslant \|[\mathcal{D}_A, \pi_A(\sigma_1(y))]_{pp}\| + L(e) \leqslant 2L(e)$ and hence $L_{A,B}(e) \leqslant 2L(e)$ for all $e \in \mathcal{E}_{sa}$, as required. □

Theorem 5.5.3. *$(\mathcal{E}_{sa}, L)$ is a compact quantum metric space.*

Proof. By the previous lemma, it is sufficient to show that $(\mathcal{E}_{sa}, L_{A,B})$ is a compact quantum metric space and we may take $\mathcal{E} = E_1$. The proof consists of showing the sets

$$\tilde{\mathcal{U}_E} := \{\tilde{e} \in \mathcal{E}_{sa}/\mathbb{R}I_E : \ L_{A,B}(\tilde{e}) \leqslant 1\}; \quad \mathcal{U}_{E,1} := \{e \in \mathcal{E}_{sa} : \|e\| \leqslant 1, \ L_{A,B}(e) \leqslant 1\},$$

are respectively norm bounded and norm- totally bounded. By construction,

$$\tilde{\mathcal{U}_E} \subset \tilde{\mathcal{U}_A} + \mathcal{U}_B, \quad \mathcal{U}_{E,1} \subset \mathcal{U}_{A,1} + \mathcal{U}_B, \tag{5.5.2}$$

Since the spectral triple on A implements a compact quantum metric space, the spaces

$$\tilde{\mathcal{U}_A} = \mathcal{Y}_{sa} \cap \sigma_1^{-1}(\{\tilde{a} \in \mathcal{A}_{sa}/\mathbb{R}I_A : \|[\mathcal{D}_A, \pi_A(\tilde{a})]\| \leqslant 1\}) \tag{5.5.3}$$

$$\mathcal{U}_{A,1} = \mathcal{Y}_{sa} \cap \sigma_1^{-1}(\{a \in \mathcal{A}_{sa} : \|a\| \leqslant 1, \ \|[\mathcal{D}_A, \pi_A(a)]\| \leqslant 1\}), \tag{5.5.4}$$

are respectively bounded and totally bounded as normed vector subspaces of $\mathcal{E}_{sa}/\mathbb{R}I_E$ and $\mathcal{E}_{sa}$ respectively. It therefore suffices to show the set $\mathcal{U}_B := \{x = x^* \in \mathcal{K}(P\mathcal{H}_A) \otimes \pi_B(B) : \|\delta_B(x)\| \leqslant 1\}$ is totally bounded. By self-adjointness, we can replace the last inequality with the inequality: $\max\{\|[1 \otimes \mathcal{D}_B, x]\|, \ \|x(P\mathcal{D}_A \otimes 1)\|, \ \|x(P\mathcal{D}_A \otimes 1)\|\} \leqslant 1$.

Effectively, we have reduced the proof to showing that we obtain a (noncompact!) quantum metric structure on the stabilised ideal space $\mathcal{K} \otimes B$, which must give the metric on the state space $S(\mathcal{K} \otimes B)$ finite diameter.

Lemma 5.5.4. *Let $\{P_k\}_{k \in \mathbb{N}}$ be the spectral projections of $Y := (P\mathcal{D}_A)^{-1} \otimes 1 = (P\mathcal{D}_A \otimes 1)^{-1}$ and write $Q_n := \sum_{k=1}^n P_k$. Then for each $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that*

$$\|x - Q_N x Q_N\| \leqslant \epsilon; \ x \in \mathcal{U}_B. \tag{5.5.5}$$

Moreover for each $x \in \mathcal{U}_B$ and for each $n \in \mathbb{N}$, $\|x_n\| \leqslant \|Y\|$, where $x_n := Q_n x Q_n$.

Proof. Since $(P\mathcal{D}_A)^{-1}$ is compact by assumption, it quickly follows that for each $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $\|Y - YQ_n\| \leqslant \frac{\epsilon}{2}$ and $\|Y - Q_n Y\| \leqslant \frac{\epsilon}{2}$. For $x \in \mathcal{U}_B$, we obtain

$$\|Q_n x Q_n\| \leqslant \|Q_n x (P\mathcal{D}_A \otimes 1) Y Q_n\| \leqslant \|x(P\mathcal{D}_A \otimes 1)\| \|Y\| \leqslant \|Y\|,$$

which proves the second statement. Moreover, for $x \in \mathcal{U}_B$,

$$\|x - xQ_n\| \leqslant \|x(P\mathcal{D}_A \otimes 1)Y - x(P\mathcal{D}_A \otimes 1)YQ_n\| \leqslant \|x(P\mathcal{D}_A \otimes 1)\| \|Y - YQ_n\| \leqslant \frac{\epsilon}{2},$$

and $\|x - Q_n x Q\| \leqslant \frac{\epsilon}{2}$ by symmetry, so that $\|x - xQ_n x\| \leqslant \|x - xQ_n\| + \|xQ_n - Q_n x Q_n\| \leqslant$

ϵ, proving the first statement. □

Returning to the proof of the theorem, it will suffice for us to show that the sets $Q_n \mathcal{U}_B Q_n$ are totally bounded for each $n \in \mathbb{N}$. Any given element in this set can be expressed in the form

$$x_n = \sum_{i,j=1}^{m_n} \pi_B(b_{i,j})|e_j\rangle\langle e_i|; , \tag{5.5.6}$$

where $b_{i,j} \in \mathcal{B}$ and $\{e_i\}_{i=1}^{m_n}$ is an orthonormal basis for the finite dimensional Hilbert space $E_n H_A$. We shall write the corresponding projections in $B(E_n H_A \otimes H_B)$ by $\{p_i\}_{i=1}^{m_n}$. Since these commute with $1 \otimes \mathcal{D}_B$, we have that for $x \in \mathcal{U}_B$ and $n \in \mathbb{N}$,

$$\|\pi_B(b_{i,j})\| = \|p_j x_n p_i\| \leqslant \|x_n\| \leqslant \|Y\|, \tag{5.5.7}$$

$$\|[\mathcal{D}_B, \pi_B(b_{i,j})]\| = \|[1 \otimes \mathcal{D}_B, p_j x_n p_i]\| = \|p_j [1 \otimes \mathcal{D}_B, x] p_i]\| \leqslant 1. \tag{5.5.8}$$

These estimates tell us that the sets $Q_n \mathcal{U}_B Q_n$ are contained in the sets

$$S_n := \{\sum_{i,j=1}^{m_n} \pi_B(b_{i,j})|e_j\rangle\langle e_i|; \quad b_{i,j} \in \mathcal{B},\ \|b_{i,j}\| \leqslant \|Y\|,\ \|[\mathcal{D}_B, \pi_B(b_{i,j})]\| \leqslant 1\}. \tag{5.5.9}$$

Now we can at long last use the assumption that the spectral triple on B implements a compact quantum metric structure, so that $\{b \in \mathcal{B} :\ \|b\| \leqslant Y,\ \|[\mathcal{D}_B, \pi_B(b)]\| \leqslant 1\}$ is totally bounded and consequently the sets S_n are totally bounded as well. This concludes the proof of the theorem. □

5.6 Application to the Podleś spheres, quantum $SU_q 2$ group and other examples.

5.6.1 *The algebra* $C(S^2_q)$.

Let us begin our analysis with an exposition of the construction for the equatorial Podleś spheres $(C(S^2_q) = C^*(\alpha, \beta :\ \beta\alpha = q\alpha\beta,\ \ \alpha^*\alpha + \beta^2 = I,\ \ q^4\alpha\alpha^* + \beta^2 = q^4)$, for $0 < q < 1$.

It is described by the short exact sequence

$$0 \to \mathcal{K} \otimes \mathbb{C}^2 \to C(S_q^2) \to C(\mathbb{T}) \to 0. \tag{5.6.1}$$

The quotient map $\sigma : C(S_q^2) \to C(\mathbb{T})$ appearing in this sequence is defined by $\sigma(\alpha) = T^*$, $\sigma(\beta) = 0$, where $T \in C(\mathbb{T})$ is the unitary identified with the bilateral shift on $\ell_2(\mathbb{Z})$. Clearly $s : T \mapsto S \otimes I_{\mathbb{C}^2}$, where $S = PTP$ the unilateral shift on $\ell_2(\mathbb{N})$ and $P : \ell_2(\mathbb{Z}) \mapsto \ell_2(\mathbb{N})$ is the Hardy space projection, is a splitting map for σ and we obtain a representation (5.3.1) of $C(S_q^2)$ as the algebra,

$$C(S_q^2) = \mathcal{K}(\ell_2(\mathbb{N})) \otimes \mathbb{C}^2 + PC(\mathbb{T})P \otimes I_{\mathbb{C}^2}. \tag{5.6.2}$$

We introduce the spectral triples;

$$A = (C^1(\mathbb{T}), L^2(\mathbb{T}), \frac{1}{2\pi i}\partial), \quad B = (\mathbb{C}^2, \mathbb{C}^2, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}).$$

where $C^1(T) = \{f \in C(\mathbb{T}) : f' \in C(\mathbb{T})\}$. Because P is the projection into the positive spectrum of ∂, the projection P is of Toeplitz type and, as in Remark 5.4.4, smoothness is implied by the regularity condition $C^1(\mathbb{T}) \subset \mathrm{Dom}(\delta)$, where $\delta(f) = [|\partial|, f]$. Consequently the construction highlighted in Section 5.5 gives a spectral triple on $C(S_q^2)$. This spectral triple has dimension one, which distinguishes it from the spectral triples proposed by D'Andrea and Dabrowski [43].

It is well known that the triple on $C(\mathbb{T})$ satisfies Rieffel's regularity condition. The same is true of the spectral triple on $\mathbb{C}^2$: the associated Lipschitz pair coming from the latter construction is $(\mathbb{R}^2, L_B)$, where $L_B(\lambda, \mu) = |\lambda - \mu|$ recovers the usual discrete metric on the two-point space. This, following the discussion in this chapter, provides us with a construction of a compact quantum metric structure on $C(S_q^2)$ for each $q \in (0, 1)$.

Remark 5.6.1. Although the Podleś spheres are isomorphic as C^*-algebras, the smooth structures for the spectral triple on $C(S_q^2)$ will certainly depend on q. For this reason, as well as considerations in quantum group theory, we might wish to study the Hopf algebra,

$\mathcal{A}(C(S_q^2))$ of polynomials in α and β, instead. Since:

$$\alpha = T \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + T(1 - (\sqrt{1 - N_q^4})) \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \beta = N_q^2 \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{5.6.3}$$

and because the operators N_q^2 and $(1 - \sqrt{1 - N_q^4})$ belong to the space of "differentiable compacts", it is relatively easy to see that α and β belong to the domain of the smooth structure of the spectral triple we are considering.

5.6.2 *The algebra* $C(S_q^3)$ *and higher dimensions.*

The analysis of the noncommutative 3-spheres $C(S_q^3)$ is notably similar to the case of the equatorial Podleś spheres. We have a short exact sequence, and as stated earlier there is a representation theory of $C(S_q^3)$ as the algebra $PC(\mathbb{T})P \otimes \mathbb{C}I + \mathcal{K}(\ell_2(\mathbb{N}_0)) \otimes C(\mathbb{T})$. When $A = B = C(\mathbb{T})$ is prescribed with the usual spectral triple then as before we may verify that the extension satisfies the criterium of smoothness and that we obtain a spectral triple on $C(S_q^3)$. The methods of section 5.6 ensure that this triple implements a compact quantum metric structure on $C(S_q^3)$, which has spectral dimension two.

The dimensionality properties of our spectral triples for this algebra is in stark contrast from the analysis of the noncommutative geometry of the quantum SU_q2 group provided by Chakraborty and Pal, who show that an equivariant spectral triple (for the co-action on the quantum group structure on SU_q2) must have dimension at least three [20]. Much of the analysis of the geometric aspects of this algebra seems to be based on the latter point of view and I do not see a correspondence with the theory in this chapter.

The noncommutative n-spheres for higher dimensions can be defined inductively on n. The spheres of odd dimension arise as short exact sequences of the form

$$0 \to \mathcal{K} \otimes C(S^1) \to C(S_q^{2n+1}) \to C(S_q^{2n-1}) \to 0, \quad n \geqslant 1. \tag{5.6.4}$$

and the spheres of even dimension as short exact sequences of the form

$$0 \to \mathcal{K} \otimes \mathbb{C}^2 \to C(S_q^{2n}) \to C(S_q^{2n-1}) \to 0, \quad n \geqslant 1. \tag{5.6.5}$$

(see [8] for an overview and references therein).

5.6.3 Future questions: Rieffel-Gromov-Hausdorff convergence and other matters.

In [103], Rieffel proposed a notion of distance between compact quantum metric spaces, modelled on the Gromov-Hausdorff distance. It has since been used in number of questions relating to C^*-algebras endowed with seminorms. Some of the results are quite surprising: see [104], for example, in which Rieffel shows how the common observation in quantum physics that "matrices converge to the 2-sphere" can be illustrated quite well using Rieffel-Gromov-Hausdorff convergence.

There are various perspectives that we could take with respect to convergence for extensions in this chapter, especially for algebras arising as q-deformations. If we wish, we can try to mimic the convergence studied by Christensen and Ivan in their approach [27]. They construct a two-parameter family of spectral triples $(\mathcal{T}, H_A, \mathcal{D}_{\alpha,\beta})$ for extensions of the form

$$0 \to \mathcal{K} \to \mathcal{T} \to A \to 0, \tag{5.6.6}$$

and for $\alpha, \beta > 0$, for which the quantum metric spaces "converge" to those on A and $\mathcal{K}$ as $\alpha \to 0$ and $\beta \to 0$. This is not sufficient, however, to study the Gromov-Hausdorff convergence aspects of varying the parameter q, for example in the case of the Podleś spheres. We would be most interested in the answer to the following two questions, the first of which seems easy and the latter rather unclear:

Question 5.6.2. Let $(q_n)_{n\in\mathbb{N}} \subset (0,1)$ be a sequence converging to $q \in (0,1)$ and let $(\mathcal{A}(C(S^2_{q_n})), L)$ be the compact quantum metric on the Podleś sphere $C(S^2_{q_n})$ for $n \in \mathbb{N}$ highlighted in this chapter. Is it true that $(\mathcal{A}(C(S^2_{q_n})), L)$ converges to $(\mathcal{A}(C(S^2_q)), L)$ for Rieffel-Gromov-Hausdorff convergence?

Question 5.6.3. Suppose now that $(q_n)_{n\in\mathbb{N}} \subset (0,1)$ converges to 1. Let $(C^1(S^2), L_{\mathcal{D}})$ be the usual Lipschitz seminorm on the algebra $C(S^2) \cong C(S^2_1)$, namely for which the restriction of the metric to S^2 is the path metric. Is it true that $(\mathcal{A}(C(S^2_{q_n})), L)$ converges to $(C^1(S^2)), L)$, or any equivalent Lipschitz pair on the two-sphere, for Rieffel-Gromov-Hausdorff convergence?

6: Construction of twisted spectral triples

6.1 Twisted spectral triples.

6.1.1 *Motivation and preliminaries.*

Simply stated, the main reason that that researchers are becoming increasingly interested in the construction of twisted spectral triples is that they are applicable to a broader class of examples than are ordinary spectral triples. Connes and Moscovici [36] suggested that twisted spectral triples might be a characteristic of the type III von Neumann algebras, for which the nonexistence of a trace effectively rules out the existence of finitely summable spectral triples.

Definition 6.1.1. Let $\mathcal{A}$ be a *-algebra. By a *trace* on $\mathcal{A}$ we mean a positive linear map $\tau : \mathcal{A} \mapsto \mathbb{R}$ such that $\tau(ab) = \tau(ba)$ for all $a, b \in \mathcal{A}$. Such a trace will be called *faithful* if whenever $a \in \mathcal{A}^+$ and $\tau(a) = 0$ then $a = 0$.

Theorem 6.1.2. *[30] If* A *is a unital* C^**-algebra and* $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ *is a finitely summable spectral triple then there is a trace* τ *on* $\mathcal{A}$ *such that* $\tau(1) = 1$.

Example 6.1.3. Another example of a C^*-algebra which cannot permit such a trace is the Cuntz algebra $\mathcal{O}_2$, the simplest example of a simple purely infinite C^*-algebra. This makes a geometric analysis of $\mathcal{O}_2$ a challenging, but rewarding, task.

Example 6.1.4. Another set of motivating examples for writing down twisted spectral triples arises in examples relating to conformal actions of manifolds (see [81]). Starting from a compact Riemannian spin$^{\mathbb{C}}$-manifold $(\mathcal{M}, \mathfrak{g})$ and the group $\Gamma = SCO(\mathcal{M}, \mathfrak{g})$ of conformal diffeomorphisms of $\mathcal{M}$ which preserve the orientation and spin structure and where $\Gamma_0 \subset \Gamma$ is the connected component of the identity, the crossed product $C(\mathcal{M}) \rtimes \Gamma_0$ provides a natural way to encode the diffeomorphism invariant structure of $\mathcal{M}$. Except when Γ_0 preserves the Riemannian metric, there is in general no conformal structure on which to construct an ordinary spectral triple. This seems consistent with our analysis in Chapter 4, where the requirement that the action of Γ_0 on $(\mathcal{M}, \mathfrak{g})$ is equicontinuous is needed.

6.1.2 *Regular automorphisms and KMS states.*

Kubo-Martin-Schwinger (KMS) states, a concept with its origins in quantum field theory, have been met with considerable success in the study and classification of von-Neumann algebras. Their role in C^*-algebra theory is no less intriguing, however. For example, it has long been known that there is exactly one such state on each Cuntz algebra $\mathcal{O}_n$, $n \geqslant 2$ [84]. The KMS-condition captures the notion of a state whose "departure from a trace" is determined by an analytic function, in the absence of a trace itself.

Definition 6.1.5. Let $\mathcal{A}$ be a *-algebra. A *regular automorphism* $\sigma : \mathcal{A} \mapsto \mathcal{A}$ is an invertible linear algebra homomorphism such that

$$\sigma(a^*) := (\sigma^{-1}(a))^*, \ a \in A. \tag{6.1.1}$$

Definition 6.1.6. Let $\mathcal{A}$ be a *-algebra and let $\sigma : \mathcal{A} \mapsto \mathcal{A}$ be a regular automorphism. A σ^{-n}-*twisted trace* on $\mathcal{A}$ is a positive linear map $\psi : \mathcal{A} \mapsto \mathbb{R}$ with the property,

$$\psi(a\sigma^{-n}(b)) = \psi(ba), \ a, b \in \mathcal{A}. \tag{6.1.2}$$

The formal definition of a KMS state on a C^*-algebra A depends on two parameters, namely a fixed strongly continuous "gauge" action of $\mathbb{R}$ on A and an *inverse temperature* $\beta > 0$.

Definition 6.1.7. Let A be a C^*-algebra, $\sigma : \mathbb{R} \mapsto \mathrm{Aut}(A)$ be a strongly continuous action of $\mathbb{R}$ which admits an analytic extension to a strongly continuous action of $\mathbb{C}$ on some dense *-subalgebra, $\mathcal{A}$, of A and let $\beta > 0$. A KMS(σ, β) (or just KMS$_\beta$) state on A is a positive linear map $\psi : \mathcal{A} \mapsto \mathbb{R}$ with the property,

$$\psi(a\sigma_{-i\beta}(b)) = \psi(ba), \ a, b \in \mathcal{A}. \tag{6.1.3}$$

6.1.3 *Definition and properties of twisted spectral triples.*

I am aware of two different approaches in noncommutative geometry which have been suggested to include those examples, such as type III von Neumann algebras, for which the construction of finitely summable spectral triples is not possible. In [36], and later [81], Connes and Moscovici set about establishing a different set of axioms for a spectral triple for which the Dixmier trace becomes a twisted trace with respect to a regular automorphism σ on A. Further analysis of these twisted spectral triples is provided by Fathizadeh and Khalkhali in [54], where the named authors explore the possibilities of writing down a version of the Connes-Chern character formula. The above analysis lead to the proposal of a spectral triple in which most of the regularity and Fredholm properties are preserved, but that the commutator is only bounded up to a "twisting" by σ. It is this approach we follow.

The other approach is the construction of *modular spectral triples*, the definition of which was suggested in [15] in which the geometry of the Cuntz algebra was considered, although this has since been extended to several other examples, e.g the quantum SU_2 group (see [63] and [72]). The axioms of a modular spectral triple are based on a semifinite von Neumann algebra, rather than an unbounded Fredholm module, however. I am not aware if the relationship between these two points of view is fully understood.

Definition 6.1.8. Let A be a C^*-algebra. A twisted spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \sigma)$ is given by a *-representation $\pi : A \mapsto B(\mathcal{H})$, a dense *-subalgebra $\mathcal{A} \subset A$, a regular automorphism σ of $\mathcal{A}$ and a linear densely defined unbounded self-adjoint operator $\mathcal{D}$ on $\mathcal{H}$ such that

1. $\pi(\mathcal{A})\mathrm{dom}\mathcal{D} \subset \mathrm{dom}\mathcal{D}$ and $[\mathcal{D}, \pi(a)]_\sigma := \mathcal{D}\pi(a) - \pi(\sigma(a))\mathcal{D} : \mathrm{dom}\mathcal{D} \mapsto \mathcal{H}$ extends to a bounded operator for each $a \in \mathcal{A}$ and
2. $\pi(a)(1 + \mathcal{D}^2)^{-1}$ is a compact operator for each $a \in A$.

Remark 6.1.9. A twisted spectral triple can be either even or odd as in the usual case. p-summability and spectral dimension is defined in the same way as in the untwisted case and the spectral triple is called *regular* if it satisfies the condition that also $[(1+\mathcal{D}^2)^{1/2}, \pi(a)]_\sigma$ extends to a bounded operator for each $a \in \mathcal{A}$. The *algebra of twisted differential 1-forms* $\Omega^\sigma_\mathcal{D}(\mathcal{A})$, spanned by $\{a \in \mathcal{A}\}$ and $\{[\mathcal{D}, a]_\sigma;\ b \in \mathcal{A}\}$, becomes a bimodule as in the untwisted case, but the left action is now given by $a \cdot b := \sigma(a)b$ for all $a \in \mathcal{A}$ and for all $b \in \Omega_\mathcal{D}(\mathcal{A})$ (see [81] for details).

When a C^*-algebra A admits a twisted spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \sigma)$ and $\tilde{\sigma}$ is a maximal analytic extension of σ, there is a natural space to study given by

$$C^{1,\sigma}(A) := \{a \in A :\ a(\mathrm{dom}\mathcal{D}) \subset \mathrm{dom}\mathcal{D},\ a \in \cap_{n\in\mathbb{Z}}\mathrm{Dom}(\tilde{\sigma}^n),\ \|[\mathcal{D}, \pi(\sigma^n(a))]_{\tilde{\sigma}}\| < \infty, \forall n \in \mathbb{Z}\}.$$

Proposition 6.1.10. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \sigma)$ be a finite dimensional twisted spectral triple on A. Then:*

1. $C^{1,\sigma}(A)$ *becomes a Banach algebra when equipped with the norm*

$$\|a\|_{1,\sigma} := \|a\| + \|\sigma(a)\| + \|[\mathcal{D}, \pi(a)]_\sigma\|. \tag{6.1.4}$$

   Moreover as in Proposition 2.2.3 we obtain an injective norm-decreasing algebra homomorphism $\rho : C^{1,\sigma}(A) \mapsto B(\mathcal{H}_1)$, where $\mathcal{H}_1 := \mathrm{Dom}(\mathcal{D})$ with the inner product $\langle \eta_1, \eta_2 \rangle_1 := \langle \eta_1, \eta_2 \rangle + \langle \mathcal{D}\eta_1, \mathcal{D}\eta_2 \rangle$ and ρ is left-multiplication.

2. *When additionally $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \sigma)$ is regular, then $(A, \mathcal{H}, F_{\mathcal{D}})$, where $F_{\mathcal{D}} := (1 + \mathcal{D}^2)^{-1/2}\mathcal{D}$, defines a Fredholm module over A with the same grading.*

3. *When additionally $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \sigma) \in \mathcal{L}^{(p,\infty)}$ for some $p > 0$, the map $\mathrm{Tr}_\omega(\cdot(1 + \mathcal{D}^2)^{-p/2})$ is a σ^{-p}-twisted trace for each suitably chosen generalised limit (as highlighted in **Appendix A**):*

$$\mathrm{Tr}_\omega(ab(1 + \mathcal{D}^2)^{-p/2}) = \mathrm{Tr}_\omega(b\sigma^{-p}(a)(1 + \mathcal{D}^2)^{-p/2}) \tag{6.1.5}$$

   whenever $a, b \in \mathcal{A}$.

4. *$(\mathcal{A}_{sa}, L_{\mathcal{D}})$, where $L_{\mathcal{D}}(a) := \|[\mathcal{D}, \pi(a)]_\sigma\|$, is a lower semicontinuous Lipschitz pair satisfying the twisted Leibniz rule $L(ab) \leqslant L(a)\|b\| + \|\sigma(a)\|L(b)$ for each $a, b \in \mathcal{A}_{sa}$.*

Proof. The proof of (2) and (3) follows as in [36] and [81], except that we do not assume that $\mathcal{D}$ is invertible. Thus, the proof of (2) follows instead from the identity:

$$[F_{\mathcal{D}}, \pi(a)] = (1 + \mathcal{D}^2)^{-1/2}([\mathcal{D}, \pi(a)]_\sigma + [(1 + \mathcal{D}^2)^{1/2}, \pi(a)]_\sigma F_{\mathcal{D}}), \tag{6.1.6}$$

whilst (4) is obvious and the main ingredient of (1) is the calculation

$$\begin{aligned}
\|ab\|_{1,\sigma} &:= \|ab\| + \|\sigma(ab)\| + \|[\mathcal{D}, \pi(ab)]_\sigma\| \\
&\leqslant \|a\|\|b\| + \|\sigma(a)\|\|\sigma(b)\| + \|[\mathcal{D}, \pi(a)]_\sigma\|\|b\| + \|\sigma(a)\|\|[\mathcal{D}, \pi(b)]_\sigma\| \\
&\leqslant (\|a\| + \|\sigma(a)\| + \|[\mathcal{D}, \pi(a)]_\sigma\|)(\|b\| + \|\sigma(b)\| + \|[\mathcal{D}, \pi(b)]_\sigma\|) =: \|a\|_{1,\sigma}\|b\|_{1,\sigma},
\end{aligned}$$

for each $a, b \in \mathcal{A}$, and arguments analogous to the proof of Proposition 2.2.3. $\square$

Example 6.1.11. Suppose we have a C^*-algebra A, an automorphism $\alpha \in \mathrm{Aut}(A)$ and a faithful α-invariant state ϕ on A. Let us suppose we have a spectral triple of the form

$(\mathcal{A}, L^2(A,\phi), \mathcal{D})$ on A, complete with a separating and cyclic vector ξ_ϕ for ϕ. Since ϕ is α-invariant, the algebra $A \rtimes_\alpha \mathbb{Z}$ is represented equivariantly over $L^2(A,\phi)$ via,

$$a(b\xi_\phi) := ab\xi_\phi; \;\; U(b\xi_\phi) := \alpha(a)\xi_\phi; \;\; U^*(b\xi_\phi) := \alpha^{-1}(b)\xi_\phi, \;\; a, b \in A. \tag{6.1.7}$$

Provided further the action of α is smooth (i.e $\alpha(\mathcal{A}) = \mathcal{A}$), U and U^* leave the domain of $\mathcal{D}$ invariant and there exists a positive invertible $h \in \mathcal{Z}(\mathcal{A})$ such that $U^*\mathcal{D}U = h\mathcal{D}$, there is a natural way of writing down a spectral triple on $A \rtimes_\alpha \mathbb{Z}$: A regular automorphism is defined on the algebra $\mathcal{B}$ generated by U and $\mathcal{A}$ by

$$\sigma(aU^n) := aU^n h^n_\alpha, \tag{6.1.8}$$

where $h^n_\alpha := \alpha^{-(n-1)}(h) \dots \alpha^{-1}(h)h$. Evaluation shows $[\mathcal{D}, aU^n]_\sigma = [\mathcal{D}, a]U^n$ is a bounded operator and so $(\mathcal{B}, L^2(A,\psi), \mathcal{D}, \sigma)$ is a twisted spectral triple on $A \rtimes_\alpha \mathbb{Z}$.

Remark 6.1.12. We point out that the above is a slight generalisation of the *scaling automorphisms* introduced by Moscovici in [81], in which $h = \lambda$ is a positive real number.

We come now to the matter that we would like to address in this chapter.

Problem 6.1.13. Following the work of Connes and Moscovici, establish existence results for finitely summable twisted spectral triples for Cuntz-Krieger algebras.

6.2 Crossed products by endomorphisms.

6.2.1 Exel crossed products.

When Cuntz introduced his eponymous algebra [40], it was shown amongst other things how the behaviour of the Cuntz algebra $\mathcal{O}_n$ closely resembles the crossed product of the fixed UHF algebra $\mathcal{F}_n$ under the circle action. There is a natural gauge-invariant endomorphism on $\mathcal{O}_n$ defined by $\alpha(x) = \sum_{1\leqslant j\leqslant n} S_j x S_j^*$, which restricts to a shift-type action on $\mathcal{F}_n$. Cuntz describes the same map as a natural automorphism $\tilde{\alpha}$ on the stabilisation of $\mathcal{F}_n$, whence the isomorphism

$$\mathcal{K} \otimes \mathcal{O}_n \cong (\mathcal{K} \otimes \mathcal{F}_n) \rtimes_{\tilde{\alpha}} \mathbb{Z}. \tag{6.2.1}$$

Many authors have have offered formal definitions of what a crossed product C^*-algebra by an endomorphism might be. The most important of these include the constructions of Paschke [89], Stacey [111] and Exel [50]. We choose to adopt the latter approach. Many comparisons have been made between Exel and Stacey crossed products, for example in [2], but in many cases the correspondence is not explicit.

The description of the Cuntz algebra and related examples using Exel crossed products has many nice features. For one, the Cuntz algebra can be expressed as a crossed product of a commutative C^*-algebra, which makes a geometric analysis easier. In fact, the algebra encodes the dynamics of the full subshift, whose C^*-algebra has totally disconnected spectrum. Moreover, it offers a relatively straightforward characterisation of KMS- states.

Given a dynamical system, comprising a unital C^*-algebra A and a *-endomorphism α of A, the philosophy behind Exel's crossed product construction is to construct a Toeplitz-Pimsner type algebra which contains a copy of an algebra A and an extra operator S which encodes the action of α. Because this algebra is "too big"- it gives a generalisation of the Toeplitz crossed product C^*-algebra of an endomorphism- the crossed product algebra itself is defined as a certain quotient, which mimics the passage from a Toeplitz-Pimsner to a Cuntz-Pimsner algebra.

Definition 6.2.1. [50] We suppose that we have a unital C^*-algebra A, a *-endomorphism $\alpha : A \mapsto A$ and a positive map $\mathcal{L} : A \mapsto A$, called the *transfer operator*, such that $\mathcal{L}(\alpha(a)b) = a\mathcal{L}(b)$ for each $a, b \in A$. The C^*-algebra $\mathcal{T}(A, \alpha, \mathcal{L})$ is defined as the universal C^*-algebra generated by a copy of A and an operator S such that:

$$Sa = \alpha(a)S, \;\; \mathcal{L}(a) := S^*aS \in A; \;\; a \in A. \tag{6.2.2}$$

Such a universal algebra exists in each case [50].

Warning 6.2.2. Khoshkam and Skandalis define a Toeplitz Pimsner-type algebra $\mathcal{T}_\alpha$ for each endomorphism α of a C^*-algebra A in [68], but these are in general quite different to the algebras constructed above.

Definition 6.2.3. [50] Given a unital C^*-algebra A, a *-endomorphism $\alpha : A \mapsto A$ and a positive map $\mathcal{L} : A \mapsto A$ such that $\mathcal{L}(\alpha(a)b) = a\mathcal{L}(b)$ for each $a, b \in A$, the Exel crossed product $A \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$ is defined as the quotient of $\mathcal{T}(A, \alpha, \mathcal{L})$ with respect to the ideal generated by $a - k$, where $a \in A$, $k \in \overline{ASS^*A}$ and $abS = kbS$ for all $b \in A$.

6.2.2 Cuntz-Krieger algebras.

Definition 6.2.4. The Cuntz-Krieger algebra $\mathcal{O}_A$ associated with a matrix $A \in M_n(\{0,1\})$ is the algebra generated by partial isometries $S_1, \dots, S_n$ satisfying the relations (see [41])

$$S_i^* S_i = \sum_{j=1}^{n} A_{i,j} S_j S_j^*, \quad S_i^* S_k = 0 \ (k \neq i). \tag{6.2.3}$$

The *graph* of $\mathcal{O}_A$ is the finite directed graph with vertices $\{v_1, \dots, v_n\}$ and a single edge from v_j to v_i if and only if $A_{i,j} = 1$.

The Cuntz-Krieger algebra is used to describe the dynamics of the underlying graph. Let $A \in M_n(\{0,1\})$ be a matrix with no zero rows or columns and let $\mathcal{O}_A$ be the Cuntz-Krieger algebra associated to A. The subshift space is defined by $\Sigma_A^+ := \{\xi = (\xi_i)_{i\in\mathbb{N}} \in \{1, \dots, n\}^{\mathbb{N}} : A_{\xi_i,\xi_{i+1}} = 1, \ \forall i \in \mathbb{N}\}$, which becomes a compact totally disconnected space with respect to the usual topology. The map $T : \Sigma_A^+ \mapsto \Sigma_A^+$, $T(\xi_1, \xi_2, \dots) = (\xi_2, \xi_3, \dots)$ is the usual *left-shift*, on Σ_A^+.

The dynamical system (Σ_A^+, T) is called the Markov subshift for A. It leads to an endomorphism σ_T on the algebra ting $C(\Sigma_A^+)$ in the usual way by $\sigma_T(f)(\xi) := f(T(\xi))$. Our standing assumption that A contains no trivial rows or columns ensures that T is surjective and a transfer operator for (Σ_A^+, T) can be defined by $\mathcal{L}_T(f)(\xi) := |T^{-1}(\{\xi\})|^{-1} \sum_{\eta\in T^{-1}(\{\xi\})} f(\eta)$. Thus the algebra $C(\Sigma_A^+) \rtimes_{\sigma_T,\mathcal{L}_T} \mathbb{N}$ can be formed.

Theorem 6.2.5. *[50] For every $A \in M_n(\{0,1\})$, there is an isomorphism $\mathcal{O}_A \cong C(\Sigma_A^+) \rtimes_{\sigma_T,\mathcal{L}_T} \mathbb{N}$.*

6.2.3 Cuntz-Krieger algebras as covering map C^*-algebras.

The abstract definition of an Exel crossed product is often hard to work with. Exel and Vershik [52] gave a much simplified definition of the algebra $A \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$ in the setting in which $A = C(X)$ is commutative, $\alpha = \alpha_T$ is spatially implemented by a continuous map $T : X \mapsto X$ and, provided T is a local homeomorphism, $\mathcal{L} = \mathcal{L}_T$ is spatially implemented as in the previous example.

Definition 6.2.6. A continuous surjection $T : X \mapsto X$, with X compact and Hausdorff, is called a *covering map* if it has the property that for each $x \in X$ there exists an open neighbourhood V_x such that $T^{-1}(V_x)$ is a finite collection of disjoint open sets $U_{x_1}, \ldots, U_{x_k}$ such that the restriction $T|_{U_{x_i}} : U_{x_i} \to V_x$ is a homeomorphism.

Every covering map $T : X \mapsto X$ defines an endomorphism α_T of $C(X)$ in the usual way and transfer operator by:

$$\mathcal{L}_T(f)(x) := \frac{1}{|T^{-1}(\{x\})|} \sum_{T(y)=x} f(y). \tag{6.2.4}$$

Notice that $\mathcal{L}_T \circ \alpha_T$ is the identity on $C(X)$.

Lemma 6.2.7. *[17] Let T be a covering map of a compact Hausdorff topological space X. Then there exists a minimal finite covering $\{U_i\}_{1 \leqslant i \leqslant l}$ of X such that the restriction of T to U_i is injective.*

Exel and Vershik introduce a positive family of function $\{v_i : 1 \leqslant i \leqslant l\}$ for each such covering given by $v_i := Qu_i$, where $Q(x) := |\{y \in X;\ T(x) = T(y)\}|$ and $\{v_i : 1 \leqslant i \leqslant l\}$ is a partition of unity for $\{U_i\}_{1 \leqslant i \leqslant l}$.

Definition 6.2.8. ([52], [17]) Let T be a surjective covering map of a compact Hausdorff topological space X. Then the C^*-algebra $\mathcal{C}(X, T)$ is defined as the universal C^*-algebra generated by a copy of $C(X)$ and an isometry s, subject to the relations:

1. $sf = \sigma_T(f)s$,
2. $s^*fs = \mathcal{L}_T(f)$ and
3. $1 = \sum_{i=1}^{l} v_i^{1/2} ss^* v_i^{1/2}$ $\ (f \in C(X))$.

Theorem 6.2.9. *[52] For every pair (X, T), comprising a compact Hausdorff space X and a surjective covering map $T : X \mapsto X$, there is an isomorphism $\mathcal{C}(X, T) \cong C(X) \rtimes_{\sigma_T, \mathcal{L}_T} \mathbb{N}$.*

Example 6.2.10. For the Markov subshift (Σ_A^+, σ) defined by a matrix $A \in M_n(\{0, 1\})$ with no zero rows or columns, the minimal covering is given by a clopen partition,

$$\Sigma_A^+ = \cup_{1 \leqslant i \leqslant n} \Sigma_{A,i}; \quad \Sigma_{A,i} := \{\xi \in \Sigma_A^+ : \ \xi_1 = i\}. \tag{6.2.5}$$

We shall define $v_i = QP_i$, where $Q(\xi) = \sum_{r=1}^{R} A_{r,\xi_2}$ and P_i is the projection into $C(\Sigma_{A,i})$. Let us show that the covering map C^*-algebra $\mathcal{C}(\Sigma_A^+, \sigma)$ satisfies the Cuntz-Krieger relation.

Define partial isometries $\{S_i : 1 \leqslant i \leqslant n\}$ by $S_i = v_i^{1/2}s$. A calculation then shows that for each $\xi \in \Sigma_A^+$ and $1 \leqslant i \leqslant n$,

$$S_i^* S_i(\xi) = \mathcal{L}_T(QP_i)(\xi) = \sum_{r=1}^{R} A_{r,\xi_1}\mathcal{L}_T(P_i)(\xi) = A_{i,\xi_1}. \tag{6.2.6}$$

On the other hand, clearly $S_i^* S_j = 0$ for $j \neq i$ and

$$\sum_{j=1}^{R} A_{i,j} S_j S_j^*(\xi) = \sum_{j=1}^{R} A_{i,\xi_1} v_j^{1/2} s s^* v_j^{1/2}(\xi) = A_{i,\xi_1}. \tag{6.2.7}$$

Hence we recover the usual Cuntz-Krieger relation (equation 6.2.3). Let us now document some important properties of covering map C^*-algebras for later use:

Lemma 6.2.11. *[52] [17] [49]*

1. *As a vector space, $\mathcal{C}(X,T)$ is a closure of the space $\mathcal{C}_F(X,T)$ of finite sums of the form*

$$\sum_{i,j \geqslant 0} f_{i,j} s^i s^{j^*} g_{i,j}, \quad f_{i,j}, g_{i,j} \in C(X). \tag{6.2.8}$$

2. *The canonical map from $C(X)$ to $\mathcal{C}(X,T)$ is injective.*
3. *$\mathcal{C}(X,T)$ is a simple C^*-algebra if and only if $T : X \mapsto X$ is irreducible, i.e there are no nontrivial open sets $U, V \subset X$ such that $T^n(U) \cap V = \emptyset$ for every $n \geqslant 0$.*
4. *There exists a gauge action $\sigma : S^1 \mapsto \mathrm{Aut}(\mathcal{C}(X,T))$ given by $\sigma_z(f) = f, \sigma_t(s) = zs,\ z \in \mathbb{T}$ and a faithful conditional expectation F on $\mathcal{C}(X,T)$ defined by $F(x) = \int_{\mathbb{T}} \sigma_z(x)dz$, whose image is the C^*-subalgebra of $\mathcal{C}(X,T)$ generated by vectors of the form*

$$\sum_{i \geqslant 0} f_i s^i s^{i^*} g_i, \quad f_i, g_i \in C(X). \tag{6.2.9}$$

5. *There exists another conditional expectation G on $\mathcal{C}(X,T)$ whose image is $C(X)$ and such that $G(F(x)) = G(x) \forall x \in \mathcal{C}(X,T)$. Furthermore*

$$G(fs^i s^{j^*} g) = \delta_{i,j} fg; \ f, g \in C(X),\ i, j \geqslant 0. \tag{6.2.10}$$

6. *The automorphism $\sigma_{-\beta i} : \mathcal{C}_F(X,T) \mapsto \mathcal{C}_F(X,T)$, where $\beta > 0$, is regular.*

Proof. Shown in [49] and [52]. □

As an immediate consequence of (6), the gauge action extends to a densely defined analytic action of $\mathbb{C}$ on $\mathcal{C}(X,T)$ defined by $\sigma_z(f) = f,\ \sigma_z(s) = e^{iz}s$. As such, Exel and Vershik establish necessary and sufficient conditions for the existence of KMS$_\beta$ states for σ.

Theorem 6.2.12. *[49] Let $\beta > 0$. Then a state ψ on $\mathcal{C}(X,T)$ is a KMS$_\beta$ state for σ if and only if $\psi = \tau \circ G$, where τ is a trace on $C(X)$ such that $\tau(\mathcal{L}_T(e^{-\beta}Qf)) = \tau(f)\ \forall f \in C(X)$.*

Remark 6.2.13. In [51], Exel describes the KMS states for Cuntz-Krieger algebras explicitly using this description. The operator $\mathcal{L}_\theta(f) := \mathcal{L}_T(e^\theta Qf)$ is also called the *Ruelle-Perron-Frobenius operator*, so the KMS states in this case correspond to Borel probability measures which leave the operator $f \mapsto \mathcal{L}_{-\beta}(f)$ invariant.

Definition/Proposition 6.2.14. A Markov shift (Σ_A^+, T) is called *irreducible* if equivalently T is irreducible, A is an irreducible matrix (that is, for $1 \leqslant i, j \leqslant n$ there exists a $k > 0$ such that $(A^k)_{i,j} = 1$), the graph of $\mathcal{O}_A$ is connected and the algebra $\mathcal{O}_A$ is simple.

The KMS-states of irreducible Cuntz-Krieger algebras have already been classified by An Huef *et al.* and the analysis above provides an alternative result to the same conclusion:

Corollary 6.2.15. *[1] [51] Let $A \in M_n(\{0,1\})$ be in irreducible matrix. There is a unique KMS$_\beta$ state on each Cuntz-Krieger algebra $\mathcal{O}_A$, and the inverse temperature at which this occurs is $\beta = \log\rho(A)$, where $\rho(A)$ is the spectral radius of A.*

6.3 The ordinary spectral triple on $C(\Sigma_A^+)$.

Let us now show that it is possible to write down at least one finitely summable twisted spectral triple on every simple Cuntz-Krieger algebra $\mathcal{O}_A$. Let us first note that (Σ_A^+, T) is canonically equipped with the structure of a compact metric space in the following way: For $\xi, \eta \in \Sigma_A^+$ define $N(\xi,\eta) = \inf\{k \geqslant 1 : \xi_k \neq \eta_k\}$. For $\lambda > 1$, the metric $d(\xi_1, \xi_2) := \lambda^{-N(\xi_1,\xi_2)}$ metrises the natural topology of Σ_A^+ and it is seen that $d(T(\xi), T(\eta)) = \lambda d(\xi,\eta)$, unless $\xi_1 = \eta_1$.

Starting from an invariant measure τ on (Σ_A^+, T) with full support, we establish the existence of a faithful trace τ on $C(\Sigma_A^+)$ such that $\tau(f) = \tau(\mathcal{L}_T(f)) = \tau(\alpha_T(f))$. In this way, as highlighted in [52], we can construct a covariant representation of $\mathcal{O}_A$ on $L^2(C(\Sigma_A^+),\tau)$. It is given by

$$\pi(s)(f\xi_\tau) := \alpha(f)\xi_\tau, \quad \pi(s^*)(f\xi_\tau) = \mathcal{L}(f)\xi_\tau, \tag{6.3.1}$$

where ξ_τ is a separating and cyclic vector for the representation of $C(\Sigma_A^+)$ on $L^2(C(\Sigma_A^+),\tau)$. A spectral triple of Christensen-Ivan type ([26]) shall now be chosen which reflects the metric d on Σ_A^+. Let P_0 be the orthogonal projection into $\mathcal{H}_0 := \mathbb{C}\xi$ and more generally $\{P_k\}_{k\geqslant 1}$ be the series of mutually orthogonal projections into $\mathcal{H}_k \ominus \mathcal{H}_{k-1}$, where

$$\mathcal{H}_k := \{\Phi \in L^2(C(\Sigma_A^+),\tau);\ \pi(s^{*k})(\Phi) \in \mathcal{H}_0\}. \tag{6.3.2}$$

The algebra $C(\Sigma_A^+)$ is viewed as an AF-algebra with its natural filtration coming from the representation of $C(\Sigma_A^+)$ on $B(\mathcal{H})$, so that A_k is the algebra generated by characteristic functions $\{\chi_{\xi_1,\dots,\xi_k};\ A_{\xi_i,\xi_{i+1}=1}\ \forall 1 \leqslant i < k\}$ and $A_k\xi = \mathcal{H}_k$ for $k \geqslant 0$. Evidently,

Proposition 6.3.1. *For $\cup_{k\geqslant 0}A_k \subset \mathcal{A} \subset C^1(A)$, the triple $(\mathcal{A}, L^2(C(\Sigma_A^+),\tau), \mathcal{D} := \sum_{k\geqslant 1}\lambda^k P_k)$ is a spectral triple on $C(\Sigma_A^+)$.*

The proof of all the details can be traced to the construction of Christensen and Ivan for unital AF-algebras [26].

6.4 The twisted spectral triple on $\mathcal{O}_A$.

Lemma 6.4.1. *The isometry s satisfies the relations $sP_0 = P_0 s = P_0$, $P_1 s = 0$ and $P_k s = sP_{k-1}$ for each $k \geqslant 2$. In particular both s and s^* map $\mathcal{H}_\infty := \cup_{k\geqslant 0}\mathcal{H}_k \subset \mathrm{Dom}\mathcal{D}$ onto itself and $s\mathcal{D}|_{\mathcal{H}_\infty} = \lambda^{-1}\mathcal{D}s|_{\mathcal{H}_\infty}$ and $s^*\mathcal{D}|_{\mathcal{H}_\infty} = \lambda\mathcal{D}s^*|_{\mathcal{H}_\infty}$.*

Proof. For convenience, let $\pi_k : C(\Sigma_A^+) \mapsto A_k$ be the projection defined by $\pi_k(f)\xi := P_k(f\xi)$. It suffices to prove the claims in the first sentence, since then (suppressing notation),

$$s\mathcal{D} = \sum_{k\geqslant 1}\lambda^k sP_k = \sum_{k\geqslant 1}\lambda^k P_{k+1}s = \sum_{k\geqslant 1}\lambda^{k-1}P_k s = \lambda^{-1}\mathcal{D}s. \tag{6.4.1}$$

Since both α and $\mathcal{L}$ fix multiples of the identity, the relations $sP_0 = P_0 s = P_0$ are clear. Also $P_1 s = s^*P_1 = 0$, since the only instance in which $\mathcal{L}(f)$ is a multiple of the identity is when also f is a multiple of the identity and $\mathcal{L}(f) = f$. To prove the other relation, it suffices to show that whenever $f \in C(\Sigma_A^+)$ and $k \geqslant 2$ then $\|(P_k s - sP_{k-1})f\xi\|_{\mathcal{H}}^2 = \|\pi_k(\alpha(f)) - \alpha(\pi_{k-1}(f))\xi\|_{\mathcal{H}}^2$ vanishes. However the latter is equal to $\tau(|g|^2)$, where $g = \pi_k(\alpha(f)) - \alpha(\pi_{k-1}(f))$. But for $k \geqslant 2$, $\alpha(A_{k-1}) = A_k$ so that $g = 0$, proving the claim. □

The consequence of Lemma 6.4.1 is that the ordinary spectral triple $(\mathcal{A}, L^2(C(\Sigma_A^+), \tau), \mathcal{D})$ on $C(\Sigma_A^+)$ can be extended to the whole of $\mathcal{O}_A$, provided that a twisting is introduced which leaves the subalgebra $C(\Sigma_A^+) \subset \mathcal{O}_A$ invariant. The regular automorphism is $\sigma = \sigma_{-i\log\lambda}$, which acts via,

$$\sigma(f) = f \;\forall f \in C(\Sigma_A^+), \quad \sigma(s) = \lambda s. \tag{6.4.2}$$

Necessarily $[\mathcal{D}, s]_\sigma = [\mathcal{D}, s^*]_\sigma = 0$, so that $\{x \in \mathcal{O}_A;\; [\mathcal{D}, x]_\sigma \in B(L^2(C(\Sigma_A^+), \tau))\}$ contains the span of $\{fss^*g;\; f, g \in \mathcal{A}\}$. We arrive at the main result of this chapter.

Theorem 6.4.2. *Let $\mathcal{B} \subset C^{1,\sigma}(\mathcal{O}_A)$ be any dense *-subalgebra of the Cuntz-Krieger algebra $\mathcal{O}_A$ containing the algebra span of s, s^* and the natural AF-filtration $\cup_{k\geqslant 0} A_k$ of $C(\Sigma_A^+)$, where A is irreducible. Then $(\mathcal{B}, L^2(C(\Sigma_A^+), \tau), \mathcal{D}, \sigma)$, with $\mathcal{D}$ as above, defines a twisted spectral triple on $\mathcal{O}_A$.*

6.5 The dimension and KMS-state for the triple on $\mathcal{O}_A$.

We shall now show that, at least for irreducible non-permutation matrices A, the summability properties for the twisted spectral triple on $\mathcal{O}_A$ have a natural interpretation in terms of the topological entropy of the Markov shift (Σ_A^+, T). The asymptotics of $\mathcal{D}$ for the twisted spectral triple on $\mathcal{O}_A$ depend only on the spectral triple on $C(\Sigma_A^+)$, so we shall focus on the latter. By construction, for each p such that $(\mathcal{A}, L^2(C(\Sigma_A^+), \tau), \mathcal{D})$ is p-summable,

$$\mathrm{Tr}((1+\mathcal{D}^2)^{-p/2}) = \sum_{k=0}^{\infty}(1+\lambda^{2kp})^{-1/2}\dim(\mathcal{H}_k) = 1 + \sum_{k=1}^{\infty}(1+\lambda^{2kp})^{-1/2}\|A^{k-1}\|, \tag{6.5.1}$$

where $\|A^k\|$ is the norm of the matrix A^k, which corresponds with the number of admissible words of length k. As is well known,

Lemma 6.5.1. *The topological entropy $h_{\mathrm{Top}}(T)$ of the Markov shift T on Σ_A^+, where $A \in M_n(\{0,1\})$*

is an irreducible non-permutation matrix, is given by

$$h_{Top}(T) = \rho(A) = \lim_{k\to\infty} \frac{1}{k} \log \|A^{k-1}\|.$$

where $\rho(A)$ *is the Perron-Frobenius eigenvalue of* A.

Proposition 6.5.2. *Let* $\mathcal{O}_A$ *be a Cuntz-Krieger algebra, where* A *is an irreducible non-permutation matrix, and* $(\mathcal{B}, \mathcal{H}, \mathcal{D}, \sigma)$ *be the twisted spectral triple on* $\mathcal{O}_A$ *constructed in Corollary 6.4.2. Then the spectral dimension is given by*

$$s_0(\mathcal{B}, \mathcal{H}, \mathcal{D}, \sigma) = \log_\lambda(e) h_{Top}(T). \tag{6.5.2}$$

Proof. This follows from the root test: letting $\kappa_n^{(p)} := \lambda^{-p}\|A^{n-1}\|^{1/n}$ then we see that

$$\begin{aligned} s_0(\mathcal{B}, \mathcal{H}, \mathcal{D}, \sigma) &= \inf\{p > 0 : \limsup_{n\to\infty} \log \kappa_n^{(p)} < 0\} \\ &= \inf\{p > 0 : h_{Top}(T) - p\log(\lambda) < 0\} \\ &= \log_\lambda(e) h_{Top}(T). \end{aligned}$$

□

We shall not give a detailed exposition of the resulting Dixmier functional, but it is relatively easy to make a few observations. It is easy to see from inspection of the eigenvalues of $\mathcal{D}$ (or $(1+\mathcal{D}^2)^{1/2}$) that the spectral triple on $C(\Sigma_A^+)$ is $\mathcal{L}^{(s_0,\infty)}$ summable with s_0 as above. This means that for each suitably chosen generalised limit, we can write a family of positive linear functionals on $\mathcal{O}_A$ given by

$$\tau_\omega(x) := \mathrm{Tr}_\omega(x(1+\mathcal{D}^2)^{-s_0/2}) = \nu_\omega \int_\omega (x), \tag{6.5.3}$$

where $\nu_\omega := \mathrm{Tr}_\omega((1+\mathcal{D}^2)^{-s_0/2})$ is interpreted as a volume constant and $\int_\omega : \mathcal{O}_A \mapsto \mathbb{C}$ is a state. Standard analysis using Riemannian zeta functions reveals that 1 is measurable and so the volume constant does not depend on the choice of Lim_ω. In fact,

Corollary 6.5.3. $\psi := \int_\omega : C^{1,\sigma}(\mathcal{O}_A) \mapsto \mathbb{C}$ *is the unique KMS*$_\beta$ *state for* $\mathcal{O}_A$ *at inverse temperature* $\beta = h_{Top}(T)$, *which does not therefore depend on* Lim_ω.

Proof. Recall that the Dixmier functional defined by the spectral triple $(\mathcal{B}, L^2(C(\Sigma_A^+), \tau), \mathcal{D}, \sigma)$ is a σ^{-s_0} twisted trace (Proposition 6.1.10). A computation shows that, for $a, b \in C^{1,\sigma}(\mathcal{O}_A)$,

$$\psi(ab) = \psi(b\sigma_{-i\log_e \lambda}^{-\log_\lambda e h_{\mathrm{Top}}(T)}(a)) = \psi(b\sigma_{-i(\log_e \lambda \log_\lambda e h_{\mathrm{Top}}(T))}(a)) = \psi(b\sigma_{-ih_{\mathrm{Top}(T)}}(a)),$$

completing the proof. $\square$

6.6 Further remarks and questions.

Remark 6.6.1. We do not see an obvious correspondence between our construction and the modular spectral triples of Carey *et. al* [15]. It would be interesting to see whether both approaches lead to the same local index formula, for example following the lead of Moscovici's work [81].

Remark 6.6.2. An interesting thesis of Whittaker [121] addresses the noncommutative aspects of an important subclass of hyperbolic dynamical systems known as *Smale spaces*. For these examples, it seems preferable to work with Ruelle's crossed product coming from a C^*-algebra of an étale groupoid of Smale space, rather than the usual crossed product. It can be observed that, in the finitely summable case, what Whittaker achieves is an implicit construction of a twisted spectral triple on the stable and unstable Ruelle algebras of each irreducible Smale space. It would be very interesting to explore the relationship between the two viewpoints further.

Remark 6.6.3. As well as Cuntz-Krieger algebras, the Exel crossed product construction captures the structure of many higher-rank K-graphs, as well as some examples arising from nonunital C^*-algebras, which gives natural possibilities for further development in this area.

A: The Dixmier trace for finitely summable spectral triples

We highlight a few facts about Dixmier ideals and Dixmier traces for general reference. Rennie's summary [98] is useful, but more detailed analyses can be found elsewhere (e.g [31]).

Definition A.0.4. The *Dixmier ideal* $\mathcal{L}^{(1,\infty)}(H)$ of $B(H)$, where H is a separable Hilbert space, is the two-sided ideal of operators

$$T \in \mathcal{K}(H); \quad \sigma_N(T) = O(\log N), \tag{A.0.1}$$

where $\sigma_N(T) = \sum_{k=1}^{N} \lambda_k$ is the sum of the first k eigenvalues of the positive operator $(1 + T^*T)^{1/2} \in \mathcal{K}(H)$ (including their multiplicities).

Definition A.0.5. A spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ for a unital C^*-algebra A is called $\mathcal{L}^{(p,\infty)}$-summable if $(1 + \mathcal{D}^2)^{-p/2} \in \mathcal{L}^{(1,\infty)}(H)$.

It is well known that the above is a slightly more refined version of p-summability for spectral triples: given a spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ on A which is $\mathcal{L}^{(q,\infty)}$-summable for $q \geqslant p$, but not $0 < q < p$, it is easy to see that $s_0(\mathcal{A}, \mathcal{H}, \mathcal{D}) = p$.

The idea is to then "define" a trace on A related to the asymptotics of the Dirac $\mathcal{D}$. It is natural to try

$$\mathrm{Tr}((1 + \mathcal{D}^2)^{-p/2}) := \lim_{N \to \infty} \frac{1}{\log N} \sigma_N((1 + \mathcal{D}^2)^{-p/2}), \tag{A.0.2}$$

but the right hand side need not converge in general. Dixmier addresses this by means of generalised limits. He does not consider all such limits, rather specifically those sequences $(\alpha_n)_{n \in \mathbb{N}} \in \ell_\infty(\mathbb{N})$ with the properties,

1. $\mathrm{Lim}_\omega(\alpha_n) \geqslant 0$ if and only if $\alpha_n \geqslant 0$, $\forall n \in \mathbb{N}$,

2. $\mathrm{Lim}_\omega(\alpha_n) = \lim(\alpha_n)$ whenever $(\alpha_n)_{n\in\mathbb{N}}$ converges and

3. $\mathrm{Lim}_\omega(\alpha_1, \alpha_1, \alpha_2, \alpha_2, \alpha_3, \dots) = \mathrm{Lim}_\omega(\alpha_n)$.

Definition A.0.6. For any generalised limit Lim_ω satisfying (1), (2) and (3) above, the *Dixmier trace* of an operator $T \in \mathcal{L}^{(1,\infty)}(H)$ with $\mathrm{spec}((1+T^*T)^{1/2}) = \{\lambda_1 \leqslant \lambda_2 \leqslant \dots\}$ is given by

$$\mathrm{Tr}_\omega(T) := \mathrm{Lim}_\omega \frac{1}{\log N} \sum_{k=1}^{N} \lambda_k. \tag{A.0.3}$$

The Dixmier trace satisfies most of the axioms of a trace: it defines a positive linear functional on $\mathcal{L}^{(1,\infty)}(H)$ for each generalised limit Lim_ω above and $\mathrm{Tr}_\omega(ST) = \mathrm{Tr}_\omega(TS)$ for each $T \in \mathcal{L}^{(1,\infty)}(H)$. The Dixmier trace vanishes on all trace-class operators.

Starting from a finitely summable spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ which is additionally $\mathcal{L}^{(p,\infty)}$-summable for some $p \geqslant 1$, the analysis above provides a family $\{\tau_\omega\}$ of traces on A (indeed, the whole of $B(\mathcal{H})$), namely

$$\tau_\omega(a) := \mathrm{Tr}_\omega(a(1+\mathcal{D}^2)^{-p/2}). \tag{A.0.4}$$

There is good reason to consider this process to be a generalisation of volume integration for functions on elliptic differential manifolds (see [31]). We are, however, faced with problems. Aside from the fact that it might be difficult to show whether indeed $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is $\mathcal{L}^{(p,\infty)}$-summable for a spectral triple with dimension p, there is also the following:

Problem A.0.7. For which $a \in A$ (or $B(H)$) is $\tau_\omega(a)$ independent of the generalised limit Lim_ω? (such functions are called *measurable*).

Problem A.0.8. Potentially it can happen that $\tau_\omega(1) = 0$, and thus we do not recover a trace on A with interesting properties.

The answers to both of the above are well-understood in certain contexts, but I do not know of any particularly easy ways to verify the above for abstract classes of spectral triples coming from C^*-algebras.

B: A Proof of Theorem 5.4.4. (sketch)

A full proof of the final assertion in Theorem 5.5.4 is beyond the scope of this report. However, we do outline the ideas involved, which are undoubtedly familiar to researchers in the field.

It will suffice to show that the spectral triples, labelled $(E_1, H, \mathcal{D}_1)$ and $(E_1, H, \mathcal{D}_2)$ in Theorem 5.5.4 represent the Kasparov modules $\sigma^*(\mathcal{A}, H_A, \mathcal{D}_A)$ and $\tau^*(\mathcal{B}, H_B, \mathcal{D}_B)$ respectively. Our analysis will focus on the situation in which the triples on A and B are trivially graded.

For the former, note that $(E_1, H, \mathcal{D}_1)$ represents the "pullback along σ" of the triple

$$\left(\begin{pmatrix} \pi_A \otimes 1 & 0 \\ 0 & 0 \end{pmatrix}, (H_A \oplus H_B) \otimes \mathbb{C}^2, \begin{pmatrix} \mathcal{D}_A \otimes 1 & 1 \otimes \mathcal{D}_B \\ 1 \otimes \mathcal{D}_B & -1 \otimes \mathcal{D}_A \end{pmatrix}\right) \in \mathcal{KK}^1(A, \mathbb{C}). \tag{B.0.1}$$

The operators $1 \otimes \mathcal{D}_B$ play no role in the K-homology, since these commute with operators of the form $\pi_A(a) \otimes 1$ for each $a \in \mathcal{A}$, so that this triple represents the direct sum of the triple $(\pi_A \otimes 1, H_A \otimes H_B, \mathcal{D}_A \otimes 1) \in \mathcal{KK}^1(A, \mathbb{C})$ and a degenerate module. The homological information coming from the former is just the spectral triple on A itself, which establishes the first part of the theorem.

We would like to sketch the fact that the triple $(E_2, H, \mathcal{D}_2)$ represents the internal Kasparov product of the spectral triple $(\mathcal{B}, H_B, \mathcal{D}_B) \in \mathcal{KK}^1(B, \mathbb{C})$ and the module

$$\left(\begin{pmatrix} \tilde{\pi} & 0 \\ 0 & (\pi_A \circ \sigma) \otimes 1 \end{pmatrix}, \quad (\mathcal{H}_A \oplus \mathcal{H}_B) \otimes \mathbb{C}^2, \quad \mathcal{D}\right) \in \mathcal{KK}(E, B),$$

where,

$$\mathcal{D}_0 = \begin{pmatrix} 0 & 0 & P\mathcal{D}_A \otimes 1 & 0 \\ 0 & 0 & 0 & 0 \\ P\mathcal{D}_A \otimes 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & (1-P)\mathcal{D}_A \otimes 1 \end{pmatrix},$$

the latter of which can be seen to represent the map $\tau^* \in KK(E, B)$ in Proposition 5.3.1 (this is perhaps most evident when P corresponds to the projection into the positive part of the spectrum of $\mathcal{D}_A$).

To see this, we remark that the Dirac operator for the spectral triple $(\pi_2, H, \mathcal{D}_2)$ can be replaced with

$$\mathcal{D} = \begin{pmatrix} P \otimes \mathcal{D}_B & 0 & P\mathcal{D}_A \otimes 1 & 0 \\ 0 & (1-P) \otimes \mathcal{D}_B & 0 & 0 \\ P\mathcal{D}_A \otimes 1 & 0 & -P \otimes \mathcal{D}_B & 0 \\ 0 & 0 & 0 & (1-P)\mathcal{D}_A \otimes 1 - (1-P) \otimes \mathcal{D}_B \end{pmatrix},$$

and the latter is, up to the grading of $\mathcal{D}_0$, the explicit formula for the external Kasparov product between graded and ungraded Fredholm modules. We expect, then, that a modified version of the procedure outlined in chapter 2 can lead us to the desired conclusion.

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