Aspects of Galileons and Generalised Scalar-Tensor Theories

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For my Mum and Dad

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Abstract

This thesis is devoted to the study of modified gravity theories, especially, the scalar-tensor theories. A theorem due to Weinberg which states, that the equivalence principle is a necessary consequence of Lorentz invariance in a gravitational theory described by spin-2 massless particles is presented in Chapter 2. In view of this theorem modified gravity models either attempt to make *graviton* massive or add other spin degrees of freedom. Scalar tensor theories are a simple and natural choice. An overview of some important scalar-tensor theories such as, Brans-Dicke model, DGP theory (although not a scalar-tensor theory it reduces to one in the so called *decoupling* limit as we would see in chapter 2), Galileon model, Horndeski theory is also given in Chapter 2. The Hamiltonian analysis of the Galileon model is presented in Chapter 3. Chapter 4 presents the boundary terms and junction conditions of the Horndeski theory in the presence of codimension-1 branes. A generalised multiple-scalar-tensor theory analogous to Horndeski theory is developed in Chapter 5. We conclude with the proof of the most general multiple scalar field theory in arbitrary dimensions and flat-space time in Chapter 6. Chapters 3,4,5,6 are original work where the first 3 are based on the following journal articles.

i) Hamiltonian of galileon field theory, Phys. Rev. D **85** (2012) 084018 [arXiv:1111.3558 [hep-th]].

ii) Boundary Terms and Junction Conditions for Generalized Scalar-Tensor Theories, JHEP **1208** (2012) 122 [arXiv:1206.1258 [gr-qc]].

iii) Covariant multi-galileons and their generalisation, JHEP **1304** (2013) 032 [arXiv:1210.4026 [gr-qc]].

Chapter 6 is based on an article to be published soon.

Notations

We use the signature $(- + + \cdots +)$ for the metric throughout this thesis. The Following notations have been used for brevity,

∇_a	Covariant derivative with respect to the coordinate x^a
∂_a	Derivative with respect to the coordinate x^a, $\frac{\partial}{\partial x^a}$
∂_a^b	$\partial_a \partial^b$
∂_{ab}	$\partial_a \partial_b$
$\phi(x)_{a\ldots b}^{c\ldots d}$	$\partial_b \ldots \partial_a \partial^d \ldots \partial^c \phi(x)$ Here $\phi(x)$ is a scalar field

We have introduced special notations relevant to some Chapters and defined them there.

Chapter 1

Introduction

The discovery of the general theory of relativity (GR) is considered an important event in the history of physics. Einstein's deep and penetrative insight has forever changed our conception of space and time. Newton conceived of space and time as absolute entities independent of the observers. At the turn of the 20th century these concepts that had been the bedrock of physics for centuries, were beginning to show some cracks. Experiments showed that this old view is incompatible with how nature works. The speed of light was observed to be constant independent of how the observer moves. Observations like these led Einstein to formulate general relativity. Space and time are no longer thought of as rigid entities but according to GR they are unified into a single construct called space-time that can be stretched and warped by the presence of matter. In turn the structure of space-time dictates how material bodies move in it.

GR has been spectacularly successful. The predictions of the theory such as the bending of light by massive objects, and the precession of mercury's orbit etc., has been borne out by experiments to a high degree of precision. General relativity is considered to be insufficient to describe phenomena at very small distances where quantum mechanical effects become important and an as yet unknown quantum theory of gravity is needed. However, despite the expected breakdown of GR at very small scales, it was generally believed that it would be an excellent approximation for large distances such as our solar system or even the universe. This notion has come under scrutiny in the recent years with the startling discovery of the accelerated expansion of the universe which was recognised with the award of the Nobel Prize in physics in 2011. Thus we are forced to accept a tiny non-zero cosmological constant, that is not forced upon us *a priori* by classical GR. This fine-tuning is known as the *cosmological constant problem.* Let us discuss this in greater detail.

Einstein postulated his field equations to be,

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{1.1}$$

Where $G_{\mu\nu}$ describes the curvature of space-time and $T_{\mu\nu}$ is the stress tensor describing the *energy* of matter fields. The action that leads to this field equation is

$$\int d^4x\sqrt{-g}\frac{1}{16\pi G}R + L_{matter} \tag{1.2}$$

However, one can arrive at this from more general mathematical considerations. It has been shown [5] that the most general tensor $A^{\mu\nu}$ that satisfies the following properties,

i) Symmetric in its indices $A^{\mu\nu} = A^{\nu\mu}$

ii) Depends on the metric and it first two derivatives $A^{\mu\nu} = A^{\mu\nu}\left(g_{\alpha\beta}, \partial_\gamma g_{\alpha\beta}, \partial_\gamma\partial_\delta g_{\alpha\beta}\right)$

iii) Divergence free $\nabla_\mu A^{\mu\nu} = 0$

is given by

$$A^{\mu\nu} = aG^{\mu\nu} + bg^{\mu\nu} \tag{1.3}$$

where a, b are arbitrary constants. This means, classically the vacuum field equations of GR should naturally contain an arbitrary constant b, known as the cosmological constant, unless there is a fundamental reason that forces us to set it to zero. The action that corresponds to this generality is then,

$$S = \int d^4x\sqrt{-g}\left\{\frac{1}{16\pi G}\left[R - 2\Lambda\right] + L_{matter}\right\} \tag{1.4}$$

and the field equations now include the cosmological constant term Λ,

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{1.5}$$

Here $T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}\int d^4x\sqrt{-g}L_{matter}$. The cosmological constant contribution to the field equations has the same structure as that of the vacuum energy of a field. To see this consider a simple scalar field theory

$$S = \int d^4x\sqrt{-g}\left[-\partial_\mu\phi\partial^\mu\phi - V(\phi)\right] \tag{1.6}$$

The stress tensor is given by

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \partial^\sigma\phi\partial_\sigma\phi g_{\mu\nu} - V(\phi)g_{\mu\nu} \tag{1.7}$$

For the classical vacuum state ϕ_0 the gradiant terms vanish yielding

$$T^{vac}_{\mu\nu} = -V(\phi_0)g_{\mu\nu} \tag{1.8}$$

Here we can see that the vacuum energy yields an identical contribution to that of the cosmological constant Λ. For this reason the terms *cosmological constant* and *vacuum energy* are used interchangeably. It is not necessary to have scalar fields in vacuum state as above to generate vacuum energy, one can set a bare cosmological constant in the definition of Einstein action. More importantly, however, there is a contribution to vacuum energy from quantum fluctuations. A quantum field can be thought of as a collection of harmonic oscillators each labelled by the momentum in the momentum space. The sum of zero point energies associated with each of them would be infinite. However, we expect quantum field theory to breakdown at some energy-scale, thus cutting off higher momentum modes above $k > k_{max}$ one could estimate the energy density of the vacuum. On dimensional grounds this is given by

$$\rho_{quantum} \approx |k_{max}|^4 \tag{1.9}$$

Generally we might expect our effective theory to be valid up to the Planck scale, where gravity is expected to kick in. Therefore the theoretical estimate for vacuum energy from quantum fluctuations is of the order $(10^{18} GeV)^4$. But the observed acceleration of the universe implies the vacuum energy density of around $(10^{-12} GeV)^4$. This then is the origin of the cosmological constant problem *i.e*, the observed value is 120 orders of magnitude smaller than what is expected theoretically.

An attempt towards a resolution of the cosmological constant problem is to assume that the effective description of GR breaks down at distances much larger than our solar system and try to modify it consistently, *i.e*, both modifying gravity appropriately in the long distance while replicating the successful prediction of GR in the solar system. Modified theories of gravity are generally based on two main ideas.

- Employ a mechanism such that the large cosmological constant is *degravitated*, that is to say, the cosmological constant does not affect the geometry of space-time or only negligibly so.

- Assume that an as yet unknown symmetry completely cancels the vacuum energy, and try to effect the cosmological acceleration by modifying gravity.

Let us discuss these ideas in a little more detail. A phenomenological argument in order to *degravitate* the cosmological constant was given in [6]. The fundamental idea employed in this work is that the *gravitational constant* is promoted to a *high-pass filter*, such that the sources that are uniform over a large-region in space-time effectively feel a small *gravitational constant*. To be precise one modifies the Einstein field equation to be [1],

$$M_p^2(1 + F(L^2\nabla^2))G_{\mu\nu} = T_{\mu\nu} \tag{1.10}$$

[1]Note that this is only a phenomenological equation and has not been derived from an action principle, it is easy to see that the Bianchi identity is not satisfied here.

where,

$$F(x) \to 0 \quad \text{for} \quad x >> 1$$
$$F(x) >> 1 \quad \text{for} \quad x << 1 \tag{1.11}$$

Here $F(L^2\nabla^2)$ plays the role of *high-pass filter* and L is a length scale above which gravity is modified. More precisely if one expands $G_{\mu\nu}$ in a complete set of eigen-modes of the laplacian ∇^2, for each mode, we could replace $F(L^2\nabla^2)$ with $F(L^2 l^{-2})$, where l is a length scale and l^{-2} the eigenvalue. Thus any source characterised by wave-length $l << L$ or frequency $\frac{1}{l} >> \frac{1}{L}$, does not feel the effect of the modification, in other words, they gravitate normally. But large wave-length sources, such as the cosmological constant feel a tiny effective Newton's constant and are screened. Let us see how this idea plays out by considering the case where stress-tensor is pure vacuum energy, $T_{\mu\nu} = \lambda g_{\mu\nu}$. For maximally symmetric spaces, $G_{\mu\nu} = -\frac{1}{4} g_{\mu\nu} R$, thus we get from 1.10,

$$M_p^2(1 + F(0))G_{\mu\nu} = \lambda g_{\mu\nu} \tag{1.12}$$

tracing this equation we get,

$$R = -\frac{4\lambda}{M_p^2 + F(0)M_p^2} \tag{1.13}$$

Thus we see the effective suppression of the cosmological constant for sufficiently large $F(0)$, without much need for fine tuning.

Self-tuning is another idea similar to this, where one modifies gravity by adding new scalar degrees of freedom, and tries to degravitate the cosmological constant. Here a dynamical scalar field is added to the theory that generates a vacuum energy equal and opposite to the vacuum energy generated by matter fields including whatever cosmological constant terms that may originally be present. This mechanism was recently resurrected with a concrete proposal by Padilla *et al* [119], who started with the most general scalar-tensor theory in 4-dimensions and identified the subset of this theory that satisfies the following constraints,

i) The theory has a Minkowski vacuum solution for any value of net cosmological constant.

ii) Phase transitions where the vacuum energy has an almost instantaneous jump does not have any effect on the geometry of space-time

iii) The theory allows for non-trivial cosmology, especially, FLRW cosmology to fit with the standard picture.

The resulting subset of the most general scalar-tensor theory was found to be given by,

$$S = \int \sqrt{-g}\left[\mathcal{L}_{john} + \mathcal{L}_{paul} + \mathcal{L}_{ringo} + \mathcal{L}_{george}\right] \tag{1.14}$$

where,

$$\begin{aligned}
\mathcal{L}_{john} &= V_{john}(\phi) G_{\mu\nu} \nabla^{\mu}\phi \nabla^{\nu}\phi \\
\mathcal{L}_{paul} &= V_{paul}(\phi) P^{\mu\nu\alpha\beta} \nabla_{\mu}\phi \nabla_{\alpha}\phi \nabla_{\nu}\nabla_{\beta}\phi \\
\mathcal{L}_{george} &= V_{george}(\phi) R \\
\mathcal{L}_{ringo} &= V_{ringo}(\phi) \hat{\mathcal{G}}
\end{aligned} \tag{1.15}$$

Here $P^{\mu\nu\alpha\beta} = -\frac{1}{4}\epsilon^{\mu\nu\lambda\sigma}\epsilon^{\alpha\beta\gamma\delta}R_{\lambda\sigma\gamma\delta}$ is the double dual of the Riemann tensor, when interpreted as a two form taking value in two form with $\epsilon^{\mu\nu\alpha\beta}$ the standard Levi-Civita tensor and $\hat{\mathcal{G}} = R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - 4R_{\mu\nu}R^{\mu\nu} + R^2$ is the Gauss-Bonnet term. The attempts at modifying gravity that we briefly discussed above, such that the effective cosmological constant is reduced to zero are known to be the proposals towards the resolution of the *old cosmological constant problem.* This basically points to the era before the discovery of the accelerated expansion of the universe, when it was generally believed that the cosmological constant is rendered zero by an unknown dynamical mechanism, symmetry, anthropic principle etc. Now we have an additional problem to deal with, that the cosmological constant is in fact non-zero but tiny. There have been various proposals over the past decade, that assume the de-gravitation of vacuum energy, yet attempt to explain the cosmic acceleration by some other dynamical mechanism. We would discuss theories that yield acceleration without the need for vacuum energy in chapter 2, such as DGP [27] and Galileon [22] field theory. Here, let us briefly address a popular mechanism called *quintessence* that is proposed to yield a natural explanation for the late time cosmic acceleration among others [15][16][17]. The main idea here is to introduce a scalar field that *mimics* the properties of the cosmological constant. With the *ansatz* of a homogenous and isotropic back-ground space-time, one postulates a quintessence field that is minimally coupled to gravity, with the equation of motion given by,

$$\ddot{Q} + 3H\dot{Q} + V'(Q) = 0 \tag{1.16}$$

Here over dots denote time derivatives, H is the Hubble scale and $V'(Q) \equiv \frac{dV}{dQ}$. The field gives right to the pressure p_Q and energy density ρ_Q given by,

$$\begin{aligned} p_Q &= \frac{1}{2}\dot{Q}^2 - V \\ \rho_Q &= \frac{1}{2}\dot{Q}^2 + V \end{aligned} \tag{1.17}$$

yielding and equation of state parameter,

$$\omega = \frac{p_Q}{\rho_Q} = \frac{\frac{1}{2}\dot{Q}^2 - V}{\frac{1}{2}\dot{Q}^2 + V} \tag{1.18}$$

We see here that for slowly varying fields $\omega \approx -1$, mimicking that of the cosmological constant. Introducing the quintessence field would be a vain attempt that merely shifts the fine tuning of the cosmological constant to that of fine tuning initial data for quintessence field, if not for the existence of *tracker solutions.* Tracker solutions are achieved by having some special form of the potential $V(Q)$. With these potentials one can make the energy density ρ_Q to be close to the radiation density up until the matter-radiation equality after which quintessence starts to dominate having characteristics similar to cosmological constant. These tracker solutions turn out to be quite insensitive to the initial data, in other words they have robust behaviour, emerging from a large class of initial data, thereby ameliorating the fine tuning problem. Despite this, fine tuning of the parameters that define the tracker potentials are still needed. The following are two such potentials inducing tracker behaviour,

$$\begin{aligned} V(Q) &= M^4 \left(\exp(M_p/Q) - 1\right) \\ V(Q) &= \frac{M^{4+\alpha}}{Q^\alpha} \end{aligned} \tag{1.19}$$

Here M_p is the Planck scale and M,α are free parameters. For a review of cosmological constant problem see [18][19] and for a comprehensive review of the models of modified gravity to date see [30].

The particular focus of this thesis is scalar-tensor theories that modify gravity in the infrared, where additional light scalar degrees of freedom modify the standard GR dynamics. In chapter 2 we give an overview of some important scalar-tensor theories, starting from Brans-Dicke model which is historically the first such attempt. We explain how some models such as DGP and Galileon manifest *Vainshtein screening* where the additional scalar modes are screened near massive objects, making them consistent with GR. We also investigate the ghost modes about self-accelerating backgrounds in DGP and how these are cured by having the freedom to choose appropriate parameters in Galileon model. We conclude chapter 2 with Ostrogradski theorem which states that theories with equation of motion of derivative order more than two suffer from linear instabilities, and present the most general scalar-tensor theory discovered first by Horndeski, that evades Ostrogradski ghosts. In chapter 3 we investigate the instabilities in the Galileon model at a non-linear level. This is done by first computing the Hamiltonian of Galileon field theory, and calculating the energies associated with the two branches of solution in cubic theory sourced by a point mass. We find they are of equal magnitude and opposite sign. In Chapter 4 we build up the necessary mathematical machinery for Horndeski's most general theory, by calculating the boundary terms that make the action well posed and the junction conditions across a thin brane. This opens up the possibility to apply Euclidean path integral methods, study tunnelling in generalised scalar-tensor theories and analyse brane world dynamics. Chapter 5 extends the results in [108] to multiple fields and illustrates how the covariantization procedure that yields equations of motion of derivative order at most two. We conclude with chapter 6 which is a proof of the most general multiple scalar field theory with field equations of derivative order upto two.

Chapter 2

An Outline of Scalar-Tensor Theories of Gravity

2.1 Introduction

Einstein's General theory of relativity(GR) is the unique interacting massless theory with two spin degrees of freedom in the low energy limit (see section **??**). Thus any attempt to modify gravity in the infrared amounts to introducing new degrees of freedom in the action or making the *graviton* massive. The simplest way to achieve this is to introduce scalar fields in the action. Such theories are commonly known as *scalar-tensor theories*, because of the existence of both the metric tensor and scalar field(s) introducing spin-2 and scalar degrees of freedom. Scalar-tensor theories are ubiquitous in the current drive towards modifying gravity in the infrared, as an attempt to resolve the *cosmological constant problem*. Scalar fields naturally satisfy the isotropic requirement in cosmology since they do not pick out a direction. Furthermore most modified gravity theories reduce to scalar-tensor models at appropriate limits, where one focuses on the dynamics induced by extra degrees of freedom - as the departure from standard GR. We would discuss such a limit known as the *decoupling* limit in more detail as it arises in the DGP model, in subsequent sections of this chapter. Scalar fields typically arise in theories with extra dimensions and characterize, although not exclusively, the position of the brane in unitary gauge, moduli-fields characterising the size of compactified dimensions in string theory etc.. This chapter begins with the discussion on the uniqueness of GR as a low-energy theory. We will also give an overview of some important scalar-tensor theories in the context of modifying gravity and discuss their theoretical and phenomenological properties. All of these models are constrained by Ostrogradski theorem [110], which states that theories with equations of motion of derivative order greater than two suffer from pathological instabilities. We discuss this theorem in the final section and state the most general single scalar-tensor theory in four dimensions that satisfies this constraint.

2.2 Uniqueness of Einstein field equations

This section focuses on the uniqueness of GR as a low-energy theory of interacting massless particles with spin-2. We discuss how gauge redundancy and equivalence principle are necessary consequences of Lorentz invariant theories, which uniquely determines the structure of GR action [2] [3].

2.2.1 Gauge Redundancy in massless field operators

Massless particles are labelled by their momenta and helicity which is defined as the component of the angular momentum in the direction of motion [1]. In the standard reference frame where the spin-1 particle's 4-momentum is given by $k^\mu = \{k, 0, 0, k\}$ the associated *polarisation vector* is,

$$e^\mu_\pm \equiv \{0, 1, \pm i, 0\} \tag{2.1}$$

Where $\pm$ stands for equal and opposite helicities. For a generic 3-momentum $\mathbf{p}$ we define the *polarisation vector* as follows,

$$e^{\mu}_{\pm}(\mathbf{p}) = R^{\mu}{}_{\nu} e^{\nu}_{\pm} \tag{2.2}$$

where $R^{\mu}{}_{\nu}$ is the rotation matrix that takes the z-axis to the direction of motion. The following properties are obvious,

$$e^{0}_{\pm}(\mathbf{p}) = 0 \tag{2.3}$$

$$p_{\mu} e^{\mu}_{\pm}(\mathbf{p}) = 0 \tag{2.4}$$

We can take products of this *polarisation vectors* to define polarisation tensors associated with higher integer spin particles. In particular for *gravitons* we have,

$$e^{\mu\nu}_{\pm}(\mathbf{p}) = e^{\mu}_{\pm}(\mathbf{p}) e^{\nu}_{\pm}(\mathbf{p}) \tag{2.5}$$

Now let us try to build a field operator for *photons*. Naturally we define it as,

$$A^{\mu}(x) = \frac{1}{(2\pi)^{3/2}} \sum_{h=\pm 1} \int \frac{\mathbf{d^3p}}{\sqrt{2p^0}} \left[e^{\mu}_{h}(\mathbf{p}) a_h(\mathbf{p}) e^{ipx} + e^{*\mu}_{h}(\mathbf{p}) a^{\dagger}_{h}(\mathbf{p}) e^{-ipx} \right] \tag{2.6}$$

This looks like a 4-vector, however to see how this operator transforms under lorentz group, we use the very important relation of the *polarisation vector* (see [2] for the proof),

$$\Lambda^{\mu}{}_{\nu} e^{\nu}_{h}(\mathbf{p}) = e^{\mu}_{h}(\Lambda\mathbf{p}) e^{i\theta(\Lambda,\mathbf{p})} - p^{\mu}\Omega(\Lambda,\mathbf{p}) \tag{2.7}$$

Here Λ is a Lorentz matrix. Further more the creation and annhilation operators for massless fields transform in the same way as the one-particle states giving,

$$U(\Lambda) a^{\dagger}_{h}(\mathbf{p}) U(\Lambda)^{-1} = e^{ih\theta} a^{\dagger}_{h}(\mathbf{p}) \tag{2.8}$$

$$U(\Lambda) a_{h}(\mathbf{p}) U(\Lambda)^{-1} = e^{-ih\theta} a_{h}(\mathbf{p}) \tag{2.9}$$

Using 2.7, 2.8, 2.9, we readily find the transformation of the field operator A^{μ} to be,

$$U(\Lambda) A^{\mu}(x) U(\Lambda)^{-1} = \Lambda^{-1\,\mu}{}_{\nu} A^{\nu}(\Lambda x) + \partial^{\mu}\phi_{\Lambda}(\Lambda, x) \tag{2.10}$$

Here $\Phi_\Lambda(x)$ is a Λ dependent linear combination of the creation and annhilation operators, the exact form of which does not concern us. We see here vividly that the field operator does not transform like a 4-vector under the Lorentz group. We could do a similar analysis for the *graviton* field operator $h_{\mu\nu}$. Just as before we construct

$$h^{\mu\nu} = \sum_{h=\pm 2} \int \frac{\mathbf{d^3p}}{\sqrt{2p^0}} \left[e_h^{\mu\nu}(\mathbf{p}) a_h^\dagger(\mathbf{p}) e^{ipx} + \text{h.c} \right] \tag{2.11}$$

Here h.c stands for hermitian conjugate and $e_h^{\mu\nu}(\mathbf{p})$ is defined as in 2.5. Again using 2.8 ,2.9,2.7 we find,

$$U(\Lambda) h^{\mu\nu} U(\Lambda)^{-1} = \Lambda_\rho^{-1\mu} \Lambda_\sigma^{-1\nu} h^{\rho\sigma}(\Lambda x) + \partial^\mu \zeta_\Lambda^\nu(x) + \partial^\nu \zeta_\Lambda^\mu(x) \tag{2.12}$$

It is now plain to see that in order to construct a Lorentz invariant Lagrangian in terms of massless field operators we need to demand that it is also invariant under the formal replacement

$$\begin{aligned} A_\mu(x) &\rightarrow A_\mu + \partial_\mu \phi(x) \\ h_{\mu\nu}(x) &\rightarrow h_{\mu\nu}(x) + \partial_\mu \zeta_\nu(x) + \partial_\nu \zeta_\mu(x) \end{aligned} \tag{2.13}$$

where $\phi(x), \zeta_\mu(x)$ are arbitrary functions. This is nothing but the gauge redundancy of electro-dynamics and GR. We see here that gauge-invariance arises naturally from the symmetry considerations in QFT.

2.2.2 Lorentz invariance of the S-matrix and the equivalence principle

Having discussed gauge redundancy as a consequence of Lorentz invariance let us see how charge conservation and equivalence principle emerges from the Lorentz invariance of the S-matrix. We give a sketch of this proof by Weinberg [2]. Consider a scattering process where a bunch of particles of generic type take part. The amplitude of this reaction would depend on the external momenta (p_i^μ) and other

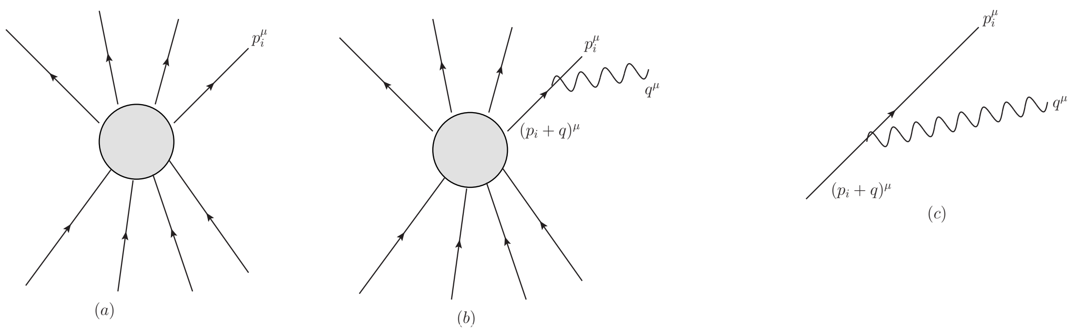

Figure 2.1: An arbitrary scattering process

,

discrete variables (σ_i) of the incoming and outgoing particles. We denote this amplitude by,

$$M_{orig} \equiv M_{orig}(p_i^\mu, \sigma_i) \tag{2.14}$$

Now let us modify this scattering process slightly where a *soft* massless particle of momentum q^μ is also emitted. By *soft* we mean the limit where $q^\mu \to 0$,that is to say, we keep only the leading order terms in the expansion in powers of q. First we take this massless particle to be a *photon* and focus on the part of the diagram where the photon line is attached to an outgoing external line with momenta (p_i^μ) (see fig-(2.1-c)). Notice that this can only be part of a larger diagram as in fig-(2.1-b) because of momentum conservation and the requirement to be on-shell. For simplicity we assume that the emitting particle is a boson, it can be generalised to particles with arbitrary spin or helicity (see [2]). Now the amplitude associated with this vertex (fig-(2.1-c)) can in general be written as,

$$M_{vertex} = M_\mu(p_i)e_h^\mu(q) \tag{2.15}$$

Here $M_\mu(p_i)$ is a 4-vector depending on the emitting particle. Since $p_i^\mu p_{i\mu} = M_i^2$, we can express M_μ as

$$M_\mu = p_{i\,\mu} f(M_i^2) \tag{2.16}$$

Here $f(M_i^2)$ is a constant that is characteristic of the emitting particle species, we define this to be the *charge* associated with this particle, $f(M_i^2) \equiv e_i$. Thus we get,

$$M_{vertex} = e_i p_{i\,\mu} e_h^{\mu}(q) \tag{2.17}$$

To compute the amplitude where a *photon* is emitted we need to sum over the diagrams where the *photon* line is attached to both the internal and the external lines of the all the diagrams that constitute the original amplitude M_{orig}. Let us consider the case where the *photon* line is attached to one of the external legs as in fig-(2.1-c). In addition to the vertex factor 2.17 now we have an internal line with momentum $(p_i^{\mu} + q^{\mu})$, this leads to the propagator,

$$\frac{1}{-(p_i + q)^2 + M_i^2} \tag{2.18}$$

since p_i is on shell we get,

$$\frac{1}{-(p_i + q)^2 + M_i^2} = -\frac{1}{2p.q} \tag{2.19}$$

The appearance of this divergent pole in the limit $q^{\mu} \to 0$, is the key point in the argument. It is first important to notice that in the $q^{\mu} \to 0$ limit, the original amplitude *before* the photon is emitted is unaltered. This is because the line that connects the blob to the vertex carries the momentum $(p_i + q)^{\mu}$ which in our limit approaches the momentum p_i as in the original process. Since none of the kinematic variables have changed in the original amplitude M_{orig}, we would be able to extract this factor in all the instances the *photon* line is connected to an external leg. Now the appearance of the pole would give a significantly large contribution to the total amplitude. This clearly would not be the case when the photon is emitted from an internal line, since the internal lines are off-shell and would not yield the cancellation necessary to give a divergent denominator. Therefore, in this *soft* limit, only the diagrams of the type shown in fig-(2.1-b) would give the leading order contribution and we have to sum over all such diagrams. Extracting out the original amplitude, we get the amplitude associated with the modified process as,

$$M_{q\to 0} = M_{orig}(p_i) \sum_{i=1}^{N} \left[s_i \frac{e_i p_i^{\mu} e_{h\,\mu}(q)}{2p_i.q} \right] \tag{2.20}$$

Here $s_i = -1(+1)$ for outgoing(incoming) particles. Now we make use of the Lorentz invariance. As we saw above Lorentz invariance implies gauge redundancy,*i.e*, under the formal replacement 2.13 states should be invariant. This means that our S-matrix should be invariant under the following shift in the *polarisation vector* induced by the gauge redundancy.

$$e_h^\mu(q) \to e_h^\mu(q) + \alpha q^\mu \tag{2.21}$$

where α is an arbitrary complex number. In other words, replacing e_h^μ with q^μ in the amplitude should give zero. Making this replacement in 2.20 we get,

$$\sum_{i(\text{incoming})} e_i = \sum_{j(\text{outgoing})} e_j \tag{2.22}$$

We have derived charge conservation. Having seen how charge conservation emerges naturally from Lorentz invariance, let us apply the same argument for *soft graviton* emission. The argument remains the same except we now have a *polarisation tensor* $e_h^{\mu\nu}(q) = e_h^\mu(q) e_h^\nu(q)$. The amplitude for the modified process, where, now a gravition is emitted, is given by,

$$M_{q\to 0}^{\text{graviton}} = M_{orig} \sum_{i=1}^{N} \left[s_i \frac{\kappa_i p_i^\mu p_i^\nu e_{h\,\mu\nu}(q)}{2p_i.q} \right] \tag{2.23}$$

Just as before here κ_i's are the coupling constants of the *graviton* and the individual particle species, which in principle can be different. Now gauge redundancy implies vanishing amplitude under the formal replacement of the *polarisation tensor* $e_h^{\mu\nu}$ by q^μ. This yields the constraint,

$$M_{orig} \sum_i s_i\, \kappa_i\, p_i^\mu = 0 \tag{2.24}$$

On account of the 4-momentum conservation this additional constraint can only be satisfied trivially, that is, the coupling constants κ_i have to be equal to each other.

$$\kappa_i = \kappa \qquad \forall i \tag{2.25}$$

We have established that the conservation of charge and the universality of gravity is a dynamical consequence of Lorentz invariance. This peculiarity of *photons* and *gravitons* give a powerful obstruction to any arbitrary modification of these theories using only massless spin-1 or spin-2 particles. It is possible to reconstruct the entire non-linear GR from this starting point (see [3]). We however will take a less demanding route and show how linear GR can be constructed from gauge redundancy and the universality of gravitation.

2.2.3 Linearised gravitational field equations

Here we attempt to reconstruct the linearised gravitational field equations using gauge redundancy and the universality of gravitation or equivalence principle. Let us begin by writing down all possible combinations of the firs order derivatives of the *field tensor* $h_{\mu\nu}$. There are 5 possibilities listed below.

$$\partial_\sigma h_{\mu\nu}\, \partial^\sigma h^{\mu\nu} \tag{2.26}$$

$$\partial_\sigma h_{\mu\nu}\, \partial^\nu h^{\mu\sigma} \tag{2.27}$$

$$\partial_\nu h^{\mu\nu}\, \partial_\sigma h^\sigma{}_\mu \tag{2.28}$$

$$\partial_\nu h^{\mu\nu}\, \partial_\mu h^\sigma{}_\sigma \tag{2.29}$$

$$\partial_\mu h_\nu{}^\nu\, \partial^\mu h_\sigma{}^\sigma \tag{2.30}$$

Note that 2.27 is equivalent to 2.28 via integration by parts. So we only have 4 independent combinations. Before writing down the general form of the action we need to add an interaction term where $h_{\mu\nu}$ is coupled to the stress tensor $T_{\mu\nu}$. In principle the coupling constant can be different for different particle species but we have already shown that Lorentz invariance forces them to be equal. So we can combine the contribution from all the particles species to have the total stress tensor $T_{\mu\nu}$ and write the interaction term as,

$$-\kappa h_{\mu\nu} T^{\mu\nu} \tag{2.31}$$

This term would not be Lorentz invariant in general since it transforms according to 2.27. The only way to make it Lorentz invariant is to enforce the following conservation equation,

$$\partial_\mu T^{\mu\nu} = 0 \tag{2.32}$$

Writing down the general action and including the interaction term we get,

$$\begin{aligned} S \;=\; & \int d^4x \Big[a\,\partial_\sigma h_{\mu\nu}\,\partial^\sigma h^{\mu\nu} + b\,\partial_\nu h^{\mu\nu}\,\partial_\sigma h^\sigma{}_\mu + c\,\partial_\nu h^{\mu\nu}\,\partial_\mu h^\sigma{}_\sigma \\ & + d\,\partial_\mu h_\nu{}^\nu\,\partial^\mu h_\sigma{}^\sigma - \kappa h_{\mu\nu} T^{\mu\nu} \Big] \end{aligned} \tag{2.33}$$

To resolve the arbitrary constants in the action, we find the equation of motion to be

$$2a\,\Box h_{\alpha\beta} + b\,(\partial_\beta\partial^\sigma h_{\alpha\sigma} + \partial_\alpha\partial^\sigma h_{\beta\sigma}) + c\,(\partial_\alpha\partial_\beta h^\sigma_\sigma + \eta_{\alpha,\beta}\partial^\mu\partial^\nu h_{\mu\nu}) + 2d\,\eta_{\alpha\beta}\Box h^\sigma_\sigma = -\kappa T_{\alpha\beta} \tag{2.34}$$

Taking the derivative ∂^β we get,

$$(2a+b)\,\partial_\beta\Box h^{\alpha\beta} + (b+c)\,\partial_\sigma\partial_\beta\partial^\alpha h^{\beta\sigma} + (c+2d)\,\partial_\beta\partial^\alpha\partial^\beta h^\sigma_\sigma = \kappa\partial_\beta T^{\alpha\beta} = 0 \tag{2.35}$$

We can choose a normalisation and set $a = \frac{1}{2}$ obtaining

$$b = -1 \quad c = 1 \quad d = -\frac{1}{2} \tag{2.36}$$

We have recovered the correct action for linearised gravity.

2.3 Jordan-Fierz-Brans-Dicke theory

Historically, a significant attempt to modify GR is the Jordan -Fierz -Brans -Dicke theory(Brans-Dicke theory for short). Brans and Dicke based their theory [4] primarily as an attempt to incorporate Mach's principle in a modified theory of GR. However, more recently, this model has been extensively investigated in the context of cosmic acceleration, inflation etc. Mach's principle asserts that the only meaningful motion is the relative motion with respect to the matter distribution in the universe, thus an inertial reaction experienced by a massive body is entirely determined by the relative matter distribution. Accordingly inertial forces appearing in an accelerated laboratory have their origin in the distant matter of the universe, since the distant matter can, in equal rights, be taken to be accelerating with respect to the laboratory. The standard GR implements this idea only partially in the form of equivalence principle. Equivalence principle is the statement that *at every space-time point there exists a free-falling frame where gravitational effects are cancelled by inertial reaction and physical laws hold according to special relativity.* Although the gravitational field of the entire matter distribution of the universe determines which frames are free-falling, all such frames are identical patches of Minkowskian space-time contrary to Mach's principle. Thus the principle of equivalence lies in between Mach's and Newton's idea of space. Brans and Dicke based their model to quantify Mach's principle, atleast to first approximation, on the following discussion by D.W.Sciama[45]. The acceleration caused by a massive body of mass m, at distance r is given according to Newtons formula $a = \frac{Gm}{r^2}$. Acceleration that can be determined by a dimensional argument compatible with Mach's principle is roughly, $a \approx \frac{mRc^2}{Mr^2}$. Here M is the total mass of the causally connected patch of the universe, and R is the radius of this region. Combining these results yields,

$$\frac{GM}{Rc^2} \approx 1 \tag{2.37}$$

This approximate relation suggests that either the ratio of M/R should fixed by the theory, or G should vary depending on the mass distribution about the point in question. Second alternative is in direct violation of strong equivalence principle that states all physical laws together with the numerical constants in a free-falling laboratory are independent of space-time position. The proposed Brans-Dicke model with a dynamical gravitaional *constant* is given by the action

$$S = \int d^4x\sqrt{g}\left\{\frac{1}{16\pi}\left[\phi R - \frac{\omega\partial_a\phi\partial^a\phi}{\phi}\right] + L_m[\Psi, g_{ab}]\right\} \tag{2.38}$$

Here ω is a constant that measures the deviation of this theory from GR and $\phi(x)$ is a scalar field that plays the role G^{-1} as the coefficient of the Ricci scalar R. GR is the limiting case when $\omega \to \infty$. The field equations for the metric and the scalar field $\phi(x)$ are given by,

$$\phi G^{ab} + g^{ab}\Box\phi - \nabla^a\nabla^b\phi - \frac{\omega}{2\phi}\left(-g^{ab}\nabla_c\phi\nabla^c\phi + \nabla^a\phi\nabla^b\phi\right) = 16\pi T^{ab} \quad (2.39)$$

$$R + 2\omega\left(\frac{\Box\phi}{\phi} - \frac{\nabla_a\phi\nabla^a\phi}{2\phi^2}\right) = 0 \quad (2.40)$$

Spherically symmetric solutions of these field equations have been extensively explored. We have no occasion here to be that detailed, however the present day solar system constraints imply $\omega > 40,000$. This makes Brans-Dicke theory practically indistinguishable from GR, and the general consensus is that this model lacks any interesting deviations in view of these observational constraints.

2.4 Dvali-Gabadadze-Porrati Model

Models of extra dimensions have been used to modify gravity. Most of these models try to modify gravity in the ultraviolet. Kaluza Klein theories and Randal Sundrum models are famous examples. The DGP model developed by Dvali, Gabadadze and Porrati [27] is an extra-dimensional model distinct from most others since it tries to modify gravity in the infra-red. In DGP matter fields are confined to a 3-brane embedded in 5-dimensional bulk space-time. Only gravity is allowed to leak into the bulk, as it is the dynamics of space-time itself. This model attracted a lot of attention because of the ingenious mechanism to dilute gravity at large distances. The theory behaves like 5-D gravity at large distances, yet it achieves the standard Newtonian potential $V(r) \propto 1/r$ below the characteristic cross-over scale r_c, by the use of an induced curvature term on the brane action [14]. DGP model realizes two vacuum solutions on the brane, termed, *normal branch* and *self accelerating branch* [7]. *Normal branch* is the standard Minkowskian space-time on the brane, where as *self-accelerating branch* is the de-Sitter space-time. This remarkable property of DGP model to realize cosmic acceleration without the need for dark energy or cosmological constant on the brane lends itself as an alternative to dark-energy. However further investigations on the perturbative stability of self-accelerating solutions have shown the existence of ghost mode [12] [13]. We would discuss this in more detail in the following sections.

2.4.1 The Action and the vacuum solutions

In DGP model a time-like 3-brane is embedded in the 5-D bulk space-time, that is allowed to be infinite. Matter fields are confined to the 3-brane. The action for the DGP model is given by,

$$S = M_5^3 \int_{\mathcal{M}} d^5x\sqrt{-\gamma}\mathcal{R} + \int_{\partial\mathcal{M}} d^4x\sqrt{-g}\left[-2M_5^3 K + \frac{M_4^2}{2}R - \sigma + \mathcal{L}_{matter}\right] \quad (2.41)$$

Here we have two mass scales in the theory the 5-D Planck mass M_5 and the 4-D Planck mass M_4 which is taken to be of the standard scale, $M_4 \approx 10^{18}GeV$. The bulk integral contains $\mathcal{R}$, the 5-D Ricci scalar. The 3-brane by virtue of it's embedding in the bulk inherits the extrinsic curvature, $K = \frac{1}{2}\mathcal{L}_n g_{ab}$,where n_a denotes the unit-normal vector pointing inwards. The brane action consists of two terms, the trace of the extrinsic curvature, K, is the gibbons-hawking term needed for a well-posed variational principle and the intrinsic curvature term, R, built out of the induced metric, $g_{\mu\nu} = \frac{\partial X(\xi)^a}{\partial\xi_\mu}\frac{\partial X(\xi)^b}{\partial\xi^\nu}\gamma_{ab}$. The intrinsic curvature R might be generated due to the quantum corrections of the stress-tensor or the finite width corrections of the brane [8]. Here, $X^a(\xi)$ are the embedding functions of the brane, and ξ^μ are brane coordinates. It is the induced curvature term R that generates all of the interesting dynamics of the theory. The constant σ is the *tension* of the brane. The field equations for the bulk are the standard Einstein's equations for the vacuum in the absence of the cosmological constant,

$$\mathcal{G}_{\mu\nu} = 0 \tag{2.42}$$

Imposing the junction condition for a Z_2 symmetric brane yields,

$$\left[2M_5^2\left(K_{ab} - Kg_{ab}\right) = M_4^2 G_{ab} + \sigma g_{ab} - T_{ab}\right]\Big|_{brane} \tag{2.43}$$

Here $T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}\left(\int_{\partial M} d^4x\sqrt{-g}\mathcal{L}_{matter}\right)$ is the stress-tensor of the brane matter. In order to investigate the vacuum solutions we set $T_{\mu\nu} = 0$ and $\sigma = 0$. The maximally symmetric vacuum solutions of this model were found in [7] [9] and are given by (in conformal coordinates),

$$ds^2 = \gamma_{ab}dx^a dx^b = e^{2\epsilon Hy}\left(dy^2 + g_{\mu\nu}dx^\mu dx^\nu\right) \tag{2.44}$$

where,

$$H = \frac{M_5^3}{M_4^2}\left[\epsilon + \sqrt{1 + \frac{\sigma M_4^2}{6M_5^3}}\right], \qquad \epsilon = \pm 1 \tag{2.45}$$

and the metric seen by the brane observer, situated at $y = 0$ is de Sitter, given in Poincare coordinates as,

$$g_{\mu\nu} = -dt^2 + e^{2\epsilon Ht}d\mathbf{x}^2 \tag{2.46}$$

The two solutions corresponding to the different signs of ϵ are termed, *normal branch* ($\epsilon = -1$) and the *self-accelerating branch* ($\epsilon = +1$). It is clear that the *self-accelerating branch* retains the de-Sitter metric with curvature $H = \frac{2M_5^3}{M_4^2} = \frac{1}{r_c}$, even in the absence of a cosmological constant term. Here the length scale r_c also turns out to be the cross-over scale between the 5D and 4D ramifications of the model as we would see later. Despite being phenomenologically favourable, the self-accelerating branch suffers from disastrous physical instabilities/ghosts [12] [13]. We will explore this in more detail in subsequent sections.

2.4.2 The cross-over from 4D to 5D

Now we will see how DGP model, despite being a 5-D theory, realizes 4-D Newtonian potential on the brane below the cross-over scale r_c. Newtonian potential results in the weak-field and non-relativistic limit of the theory. Weak field corresponds to the metric perturbations around flat background, $\gamma_{ab} = \eta_{ab} + h_{ab}$. At linear order the general coordinate transformations (GCT) or gauge redundancies reduce to,

$$x^a \to x^a + \zeta^a \tag{2.47}$$

this induces the following transformation of the metric,

$$\gamma_{ab} \to \gamma_{ab} + \nabla_{(a}\zeta_{b)} \tag{2.48}$$

As both of these metrics represent the same physical space-time, we can use this relationship to choose a convenient gauge. By solving the equation

$$\partial^b \partial_a \zeta_a = -\partial^b \gamma_{ab} + \frac{1}{2}\partial_a \gamma \tag{2.49}$$

for ζ^a, we can choose the harmonic gauge that satisfies,

$$\partial^b \gamma_{ab} - \frac{1}{2}\partial_a \gamma = 0 \tag{2.50}$$

In this gauge the bulk equations of motion simplify to

$$\left[\partial_y^2 + \partial_a \partial^a\right] h_{ab} = 0 \tag{2.51}$$

This is easily solved by taking Fourier transforms with respect to the brane coordinates, $\tilde{h}_{ab}(p^\mu, y) := \frac{1}{(2\pi)^2}\int d^4x e^{ip_\mu x^\mu} h_{ab}(x^\mu, y)$, giving

$$\tilde{h}_{ab}(p^\mu, y) = e^{-py}\tilde{h}_{ab}(p^\mu, 0) \tag{2.52}$$

We have a further gauge redundancy $x^a \to x^a + \zeta'^a$, where $(\partial_y^2 + \partial^2)\zeta'^a = 0$. This can be used to place the position of the brane at $y = 0$ and set $h_{\mu y} = 0$. Reading off the relevant components of the equation 2.51 we get

$$\begin{aligned} h_{yy} &= \eta^{\mu\nu} h_{\mu\nu} = h \\ \partial^\mu h_{\mu\nu} &= \partial_\nu h \end{aligned} \tag{2.53}$$

Now perturbing the junction conditions 2.43 to linear order,

$$2M_5^3\delta(K_{\mu\nu} - Kg_{\mu\nu}) = M_4^2\delta G_{\mu\nu} - T_{\mu\nu} \tag{2.54}$$

yields,

$$\tilde{h}_{\mu\nu}(p^\mu, 0) = \frac{2}{M_4^2p^2 + 2M_5^3p}\left[\tilde{T}_{\mu\nu} - \frac{1}{3}\tilde{T}\eta_{\mu\nu}\right] \tag{2.55}$$

Let us study the Newtonian potential induced by localized sources in the non-relativistic limit. At classical level and linear order the interaction term in the Hamiltonian of DGP model is given by,

$$H_{int} = \int d^3x\,[T_{\mu\nu}(t,\mathbf{x})h^{\mu\nu}(t,\mathbf{x})] \tag{2.56}$$

We take a localized non-relativistic source with two *lumps* at $\mathbf{x_1}, \mathbf{x_2}$.

$$\begin{aligned} T_{\mu\nu} &= T^1_{\mu\nu} + T^2_{\mu\nu} \qquad \text{where,} \\ T^1_{\mu\nu} &= m_1\delta^3(\mathbf{x} - \mathbf{x_1})\delta^0_\mu\delta^0_\nu \qquad T^2_{\mu\nu} = m_2\delta^3(\mathbf{x} - \mathbf{x_2})\delta^0_\mu\delta^0_\nu \end{aligned} \tag{2.57}$$

Plugging this expression in 2.56, and keeping only the interaction terms between $T^1_{\mu\nu}$ and $T^2_{\mu\nu}$ we get,

$$H_{int} \supset V(\mathbf{x^1} - \mathbf{x^2}) = m_1m_2\int d^3\mathbf{p}\left[\frac{e^{ip.(\mathbf{x_1}-\mathbf{x_2})}}{M_4^2\mathbf{p^2} + M_5^3\mathbf{p}}\right] \tag{2.58}$$

It can be shown that this potential scales like $V(r) \propto 1/r$ for $r << r_c = \frac{M_4^2}{2M_5^3}$ and $V(r) \propto 1/r^2$ for $r >> r_c$. Here $r = |\mathbf{x_1} - \mathbf{x_2}|$. Thus it is clear that the model exhibits a transitionary behaviour from 4D gravity to 5D gravity at the cross over scale r_c. However this is true only in the Newtonian approximation. To see if the short distance limit of DGP model is consistent with GR, we should also consider the amplitude for interaction between two relativistic sources $T_{\mu\nu}, T'_{\mu\nu}$ given by $\mathcal{A} = h_{\mu\nu}T^{\mu\nu}$. The Fourier image of this amplitude for GR and DGP at short distance is given by,

$$\begin{aligned} \tilde{\mathcal{A}}_{\mathcal{GR}} &= \frac{2}{M_p^2p^2}\left[\tilde{T}_{\mu\nu}\tilde{T}'^{\mu\nu} - \frac{1}{2}\tilde{T}(p)\tilde{T}'(-p)\right] \\ \tilde{\mathcal{A}}_{\mathcal{DGP}} &= \frac{2}{M_p^2p^2}\left[\tilde{T}_{\mu\nu}\tilde{T}'^{\mu\nu} - \frac{1}{3}\tilde{T}(p)\tilde{T}'(-p)\right] \end{aligned} \tag{2.59}$$

For non-relativistic sources $T'_{\mu\nu}T^{\mu\nu} = TT'$ and to agree with GR we demand, $M_4^2 = \frac{4}{3}M_p^2$. If however, we consider photons for example, the trace of the stress-tensor vanishes identically and the two amplitudes differ by a factor of $\frac{3}{4}$. This implies that light bending around a massive body differs by a factor $\frac{3}{4}$ from GR. A similar discontinuity arises in massive gravity [10, 11], where in, the limit of the vanishing mass for *graviton* does not lead to GR. This effect is known as the vDVZ discontinuity. An argument proposed by Vainshtein[20] suggests that the linearised analysis breaks down at the so called *Vainshtein* radius which can be larger than the Schwarzchild radius and the discontinuity appearing in the linear analysis may be salvaged. We will discuss a version of this mechanism applicable to DGP and Galileon gravity in coming sections.

2.4.3 The *decoupling* limit and the π-Lagrangian

The boundary effective action for the DGP model was computed in [43]. This was done by integrating out the bulk degrees of freedom and imposing a boundary condition at the brane. Namely

$$e^{i\Gamma[\Phi]} = \int_{\Phi|_{\partial\mathcal{M}}} d[\Phi] e^{iS_{\mathcal{M}} + iS_{\partial\mathcal{M}}} \tag{2.60}$$

Here, Φ collectively denotes the dynamical fields living in the bulk space-time. We present here only the main results of the calculation. By fixing the brane at y=0, and expressing the bulk integrand in terms of a 4+1 ADM split we get,

$$S_{bulk} = 2M_5^3 \int d^4x \int_0^\infty dy \sqrt{-g} N \left[R(g) - K^{\mu\nu}K_{\mu\nu} + K^2\right]] \tag{2.61}$$

Here $N = \frac{1}{\gamma^{yy}}, N_\mu = \gamma_{y\mu}, g_{\mu\nu} = \gamma_{\mu\nu}$ are lapse, shift, and the induced metric on the brane resp. The extrinsic curvature tensor is given by,

$$K_{\mu\nu} = \frac{1}{2N}\left(\partial_y g_{\mu\nu} - \nabla_\mu N_\nu - \nabla_\nu N_\mu\right) \tag{2.62}$$

By expanding the metric γ_{ab} and other geometric quantities around the flat background, $\gamma_{ab} = \eta_{ab} + h_{ab}$ and integrating out the bulk fields, one obtains the final result.

$$\Gamma\left[\hat{h}_{\mu\nu},\hat{N}_\mu,\hat{\pi}\right] = \int d^4x\frac{1}{2}\hat{h}^{\mu\nu}\left(\partial^2-\frac{\sqrt{-\partial^2}}{r_c}\right)\hat{h}_{\mu\nu}-\frac{1}{4}\hat{h}\left(\partial^2-\frac{\sqrt{-\partial^2}}{r_c}\right)\hat{h} \quad (2.63)$$
$$-\frac{1}{2}\hat{N}^\mu\left(\sqrt{-\partial^2}+\frac{1}{r_c}\right)\hat{N}_\mu+\frac{1}{2}\hat{\pi}\left(\partial^2-\frac{\sqrt{-\partial^2}}{r_c}\right)\hat{\pi}$$
$$+\frac{1}{M_4}\hat{h}_{\mu\nu}T^{\mu\nu}+\frac{1}{\sqrt{6}M_4}\hat{\pi}T+\Gamma_{int}\left[\hat{h}_{\mu\nu},\hat{N}_\mu,\hat{\pi}\right]$$

In arriving at this result the following field redefinitions were performed in order to extract the scalar mode and to diagonalize the kinetic term,

$$h_{\mu\nu}=\tilde{h}_{\mu\nu}+\pi\eta_{\mu\nu}, \qquad N_\mu=\tilde{N}_\mu+r_c\partial_\mu\pi, \qquad h_{yy}=-2r_c\left(\sqrt{-\partial^2}+\frac{1}{r_c}\right)\pi \quad (2.64)$$

Further, the fields were canonically normalized using,

$$\hat{h}_{\mu\nu}=\frac{M_4}{2}\tilde{h}_{\mu\nu}, \qquad \hat{N}_\mu=\frac{M_4}{\sqrt{2r_c}}\tilde{N}_\mu, \qquad \hat{\pi}=\sqrt{\frac{3}{2}}M_4\pi \quad (2.65)$$

Higher order terms in the bulk will induce boundary interaction terms that take the following schematic form,

$$\int d^4xM_5^3\sqrt{-\partial^2}(h_{\mu\nu})^a(N_\mu)^b(h_{yy})^c \quad (2.66)$$

where $a+b+c\geq 3$. The leading order term giving the highest amplitude in low energy scattering processes is given by,

$$\Gamma_{int}=-\frac{1}{3\sqrt{6}\Lambda^3}\int d^4x(\partial^\mu\hat{\pi}\partial_\mu\hat{\pi})\partial^2\hat{\pi}+\text{sub-leading terms} \quad (2.67)$$

The coefficient of this term suggests that the scalar sector of the theory becomes strongly interacting, in the quantum sense, well before other modes kick in. The scale of this interaction is determined by $\Lambda\equiv\left(\frac{M_4}{r_c^2})^{1/3}\right)$, which, for $M_4\approx M_{pl}$ and $r_c\approx 1/H$ is roughly,

$$\Lambda\approx 10^{-13}eV\approx 1/1000km \quad (2.68)$$

Given this strong coupling scale in DGP, it is instructive to isolate it's dynamics taking appropriate limits such that other degrees of freedom decouple. A formal mechanism called the *decoupling* limit to do just this was suggested in [21]. The *decoupling* limit is defined as follows,

$$M_4, r_c, T_{\mu\nu} \to \infty \qquad \text{keeping} \qquad \Lambda, \frac{T_{\mu\nu}}{M_4} = constant \tag{2.69}$$

The limiting action is given by

$$\Gamma \approx \int d^4x \left[\mathcal{L}_{GR} + \mathcal{L}_\pi\right] \tag{2.70}$$

and valid in the region $\frac{1}{M_4} << r << r_c$ where

$$\mathcal{L}_{GR} = \text{Linearised Einstein action} \tag{2.71}$$
$$\mathcal{L}_\pi = \frac{1}{16\pi G}\left\{\frac{1}{2}\left[3\pi\partial^2\pi - r_c^2(\partial\pi)^2\partial^2\pi\right]\right\} + \frac{1}{2}\pi T \tag{2.72}$$

The leading modification or departure from GR in the DGP model is captured succinctly by the *decoupling* limit. However this limit is only valid when the interactions involving metric perturbations can be neglected *i.e,* $\tilde{h}_{\mu\nu} << 1$. We would now show how decoupling limit facilitates the study of some important phenomenology of the DGP model, such as stability and Vainshtein screening.

2.4.4 Ghosts on the self-accelerating branch

In order to investigate the perturbative stability of the self-accelerating solution, we first write the de-Sitter metric as a *small* deviation about flat space-time. The suitable gauge to do this is the Newtonian gauge where the line element for de-Sitter space with curvature scale H^2 seen by an observer at origin and in the region where $|x^\mu| << H^{-1}$ becomes,

$$ds^2 = \left[1 - \frac{1}{2}H^2 x_\mu x^\mu\right]\left(-dt^2 + d\mathbf{x}^2\right) \tag{2.73}$$

In the decoupling limit the tensor mode satisfies linearised Einstein equations which in the absence of sources has the solution, $\tilde{h}_{\mu\nu} = 0$. However the physical metric according to eq 2.64 is given by,

$$h_{\mu\nu} = \tilde{h}_{\mu\nu} + \pi\eta_{\mu\nu} \tag{2.74}$$

Thus according to (2.73) we would have a self-accelerating solution if,

$$\pi = -\frac{1}{2}H^2 x_\mu x^\mu \tag{2.75}$$

Now, the action for the fluctuations of the scalar mode in the *decoupling* limit around a classical solution π_{cl} can be shown to be,

$$\mathcal{L}_\psi = \frac{M_4^2}{4}\left[-Z^{\mu\nu}(x)\partial_\mu\psi\partial_\nu\psi - r_c^2(\partial\psi)^2\partial^2\psi\right] \tag{2.76}$$

Here,

$$Z^{\mu\nu}(x) = 3\eta^{\mu\nu} - 2r_c^2(\partial^\mu\partial^\nu - \eta^{\mu\nu})\pi_{cl} \tag{2.77}$$

Plugging in the de-Sitter solution (2.75) in (2.77) we get,

$$Z^{\mu\nu} = -6r_c^2 H^2 \eta\mu\nu \tag{2.78}$$

This immediately suggests that the kinetic term in (2.76) has the wrong sign signalling ghost mode.

2.4.5 The Vainshtein Screening

We mentioned in section 2.4.2 that the DGP model has an apparent discontinuity termed vDVZ discontinuity which might be redeemed due to the breakdown of linear analysis below certain length scale. Here we would study such a mechanism called *Vainshtein screening* in which the scalar mode is screened at short distances making the DGP model consistent with GR. For simplicity we take a massive non-relativistic source with spherical symmetry represented by the stress-tensor,

$$T_{\mu\nu} = Diag[\rho(r), 0, 0, 0] \tag{2.79}$$

The classical solution in the decoupling limit consists of both the tensor mode $h^{cl}_{\mu\nu}$ and the scalar mode π^{cl}. If we are to make contact with GR, the contribution from the scalar mode that is responsible for any modification should be negligible compared to the tensor mode governed by linearised Einstein field equations.

Thus,

$$|h^{cl}_{\mu\nu}| >> |\pi^{cl}| \tag{2.80}$$

For the spherically symmetric source we have the standard Newtonian potential for the tensor mode thus,

$$|h^{cl}_{\mu\nu}| \approx r_s/r \tag{2.81}$$

where $r_s = 2G_4M$ is the Schwarzchild radius, G_4 is the Newtons constant and M is the total mass of the spherical body. The field equations for the scalar becomes,

$$\frac{d}{dr}\pi^{cl} + \left(\frac{2r_c^2}{3r}\right)\frac{d^2}{dr^2}\pi^{cl} - \frac{r_s}{3r^2} = 0 \tag{2.82}$$

neglecting the non-linear term at large distances the linear solution is given by,

$$\pi^{cl}_{linear} = -\frac{r_s}{3r} \tag{2.83}$$

Comparing this to (2.81) we see, there is an O(1) modification to GR from the scalar mode. It follows that the only possibility for the scalar to be screened away is when the above linear approximation breaks down. There is no *apriori* reason to suspect that the non-linear terms would rectify this problem, but remarkably this is precisely what happens in the non-linear or short-distance regime. The Vainshetein radius(r_v) is defined to be the scale at which non-linear term is comparable to linear term thus,

$$\left|\frac{d}{dr}\pi^{cl}\right| \approx \left|\frac{r_c^2}{r_v^2}\left(\frac{d}{dr}\pi^{cl}\right)^2\right| \tag{2.84}$$

giving

$$r_v \approx \left(r_s r_c^2\right)^{\frac{1}{3}} \tag{2.85}$$

For $r << r_v$ we could neglect the linear term and the solution becomes,

$$\pi^{cl}_{non-linear} = \frac{\sqrt{2r_s r}}{r_c} \tag{2.86}$$

Taking the ratio between the tensor and the scalar amplitude,

$$\frac{|\pi^{cl}_{non-linear}|}{|\tilde{h}_{\mu\nu}|} \approx \left(\frac{r}{r_v}\right)^{\frac{3}{2}} \tag{2.87}$$

we see that the scalar in fact gets screened for $r_v \to \infty$.

2.5 Galileon field theory

The Galileon field theory was proposed by Nicolis *et al* [22] as a scalar-tensor theory with derviative self-interactions. It was originally motivated by focusing on the effective field theory on the brane of the DGP model or more precisely the *decoupling* limit. As we have seen in section 2.4, most of the DGP phenomenology at distances shorter than the curvature scale and in the weak field limit is captured by the $\pi - lagrangian$ where a scalar field is coupled to 4D linearised GR. The $\pi - lagrangian$ 2.72, has the following important properties

i) It is invariant under the transformation $\pi \to \pi + a_\mu x^\mu + b$ up to a total derivative.

ii) Even though the action contains second order derivative the field equations are only up to derivative order two.

iii) The scalar field is universally coupled to matter via πT or equivalently in the Jordan frame it is kinetically mixed with the metric.

One may take these constraints as the logical starting point of the Galileon field theory, and define the Galileon Lagrangian to include all possible terms that satisfy these properties. Naturally this would be a generalisation of the DGP 4D effective theory.

2.5.1 The Set-Up

Let us again recall the decoupling limt of DGP efective boudndary action, where the deviations from GR is due to the scalar degree of freedom π. A generic modification of this type can be achieved by replacing the curvature term in the Einstein action $R \to (1 - 2\pi)\, R$ at quadratic order. We can formally remove this mixing of the metric and the scalar field by performing a Weyl transformation, such that,

$$\tilde{g}_{\mu\nu} = (1 - 2\pi) g_{\mu\nu} \tag{2.88}$$

which at linear order becomes,

$$\tilde{h}_{\mu\nu} = h_{\mu\nu} - 2\pi\eta_{\mu\nu} \tag{2.89}$$

where $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, $\tilde{g}_{\mu\nu} = \eta_{\mu\nu} + \tilde{h}_{\mu\nu}$. The resulting action after the Weyl transformation is known as the Einstein frame action, where in, the kinetic term for the tensor mode $\tilde{h}_{\mu\nu}$ is given by the standard Einstein-Hilbert term.

$$S = \int d^4x \left[\frac{1}{16\pi G} \frac{1}{4} \left[\partial^2 \left(\tilde{h_{\mu\nu}} - \frac{1}{2} \tilde{h} \eta_{\mu\nu} \right) + \dots \right] + \frac{1}{2} \tilde{h}_{\mu\nu} T^{\mu\nu} + L_{gal} + \frac{1}{2} \pi T \right] \tag{2.90}$$

Notice that removing the kinetic mixing via Weyl rescaling has introduced a universal coupling of the π field to matter. Here L_{gal} denotes the so far undetermined Lagrangian of the scalar field.

2.5.2 The Structure of L_{gal}

Now we would like to resolve the structure of L_{gal}, that satisfies conditions i, ii. The most general scalar field theory that yields equations of motion of derivative order up to two in D space-time dimensions was found in [108] and given by,

$$S = \int d^d x \sum_{m=0}^{D} L_m \tag{2.91}$$

where

$$L_m = f_m(\pi, X) \partial_{a_1} \partial^{[a_1} \pi \dots \partial_{a_m} \partial^{a_m]} \pi \tag{2.92}$$

Here indices a_i denote space-time coordinates, $X = \frac{1}{2} \partial_a \pi \partial^a \pi$, and anti-symmetrisation does not include the usual factor of $1/m!$. **Note that this convention for anti-symmetrisation would be used throughout this thesis**. Further, $f_m(\pi, X)$ is an arbitrary function. In order for this action to describe Galileon fields each term in the summation should also be invariant under the symmetry $\pi \to \pi + A_a x^a + b$. For infinitesimal A_a, b this induces the following transformation in L_m.

$$L_m \to L_m + \frac{\partial f_m}{\partial \pi}\partial_a J^a + (2A.\pi)\frac{\partial f_m}{\partial X}\partial_b J'^b \tag{2.93}$$

Where

$$\begin{aligned} J^a &= (b + A.x)\,\partial^{[a}\pi\partial^{b_1}\partial_{b_1}\pi\ldots\partial^{b_{m-1}]}\partial_{b_{m-1}}\pi + A^{[a}\partial^{b_1}\partial_{b_1}\pi\ldots\partial^{b_{m-1}]}\partial_{b_{m-1}} \\ J'^a &= \partial^{[a}\pi\partial^{b_1}\partial_{b_1}\pi\ldots\partial^{b_{m-1}]}\partial_{b_{m-1}}\pi \end{aligned} \tag{2.94}$$

We see that the induced variation in L_m can be a total derivative if and only,

$$\begin{aligned} \frac{\partial f_m}{\partial \pi} &= \text{constant} \\ \frac{\partial f_m}{\partial X} &= 0 \end{aligned} \tag{2.95}$$

Thus the Galileon Lagrangian respecting conditions $(i), (ii)$ is given by,

$$L_{Gal} = \sum_{m=0}^{D} L_m^{Gal} \tag{2.96}$$

where,

$$L_m^{Gal} = c_m \pi \partial_{a_1}\partial^{[a_1}\pi\ldots\partial_{a_m}\partial^{a_m]}\pi \tag{2.97}$$

Here c_m's are constant parameters of the theory. Now the equation of motion for π becomes

$$\sum_{m=0}^{m=D} c_m \left[\partial_{a_1}\partial^{[a_1}\pi\ldots\partial_{a_m}\partial^{a_m]}\pi\right] + \frac{1}{2}T = 0 \tag{2.98}$$

2.5.3 Galileon Cosmology in the weak-field limit

Having defined the Galileon field theory we seek to study the weak field cosmology supported by the model. A FLRW space-time can be approximated *locally* as a perturbation around Minkowski space. Here *local* denotes the region around the origin ($\mathbf{x} = 0, t = 0$) such that $|\mathbf{x}| << H^{-1}, t << H$. H is the Hubble scale. In this regime the line element becomes,

$$ds^2 \approx \left[1 - \frac{1}{2}H^2|\mathbf{x}^2| + \frac{1}{2}(2\dot{H} + H^2)t^2\right](-dt^2 + dx^2), \tag{2.99}$$

Any perturbation around Minkowski space-time can be written in terms of the Newtonian potentials ϕ, ψ

$$ds^2 = -(1 + 2\psi)dt^2 + (1 - 2\phi)\mathbf{dx}^2 \tag{2.100}$$

We read off the corresponding potentials from this expression

$$\psi = -\frac{1}{4}H^2|\mathbf{x}^2| + \frac{1}{4}(2\dot{H} + H^2)t^2, \quad \phi = -\psi \tag{2.101}$$

In order to study the Galileon scalar field solution that gives rise to the FLRW space-time 2.99 we recall,

i) The tensor mode $\tilde{h}_{\mu\nu}$ has been decoupled from the scalar mode by Weyl rescaling 2.88. The tensor mode is governed by the linearised Einstein equations and the scalar mode by it's own separate dynamics. This is an approximation valid when the back-reaction of the scalar on to the geometry is negligible. For the weak-field cosmology that we are considering this is a reasonable assumption.

ii) The true physical metric $h_{\mu\nu}$ is given by undoing Weyl rescaling which at linear order gives, $h_{\mu\nu} = \tilde{h}_{\mu\nu} + 2\pi\eta_{\mu\nu}$.

Thus for a given stress-tensor $T_{\mu\nu}$, $\tilde{h}_{\mu\nu}$ satisfies the linearised GR equations with the corresponding Newtonian potentials $\phi_{GR},\ \psi_{GR}$. The scalar field is also sourced by matter via the coupling πT and satisfies Galileon equation of motion. We have for FLRW space-time,

$$\tilde{h}_{\mu\nu} = 2\psi_{GR}\eta_{\mu\nu}, \tag{2.102}$$

and the physical solution is,

$$h_{\mu\nu} = 2\psi\eta_{\mu\nu} \tag{2.103}$$

Using $h_{\mu\nu} = \tilde{h}_{\mu\nu} + 2\pi\eta_{\mu\nu}$ we get,

$$\pi = \psi - \psi_{GR} \tag{2.104}$$

We see here vividly how the scalar mode causes the deviation from GR.

2.5.4 The Self-accelerating solution

We saw in section 2.4.1 that DGP model has a phenomenologically appealing, self-accelerating (deSitter) solution even in the absence of sources and cosmological constant. Indeed one expects the same in Galileon field theory as it is a generalisation of DGP effective theory. However the freedom to choose the coupling-constants and the existence of terms greater than cubic order enlarges its parameter space. As we will explore in more detail, this enables one to find a ghost-free self-accelerating solution with appropriate choice of the parameters. In the absence of sources, the linearised GR has only the Minkowski space-time as the solution, which corresponds to $\tilde{h}_{\mu\nu} = 0$ and $\psi_{GR} = \phi_{GR} = 0$. Furthermore, the de-sitter metric has $\psi = \frac{1}{4}H^2 x_\mu x^\mu$. Plugging this into 2.104 we have

$$\pi_{dS} = -\frac{1}{4}H^2 x_\mu x^\mu \tag{2.105}$$

We still need to check if π_{dS} is a solution of the Galileon equations of motion. We switch off the tadpole term in 2.98 ($c_0 = 0$) in order to have Minkowski space-time as a vacuum solution ($\pi = 0, \tilde{h}_{\mu\nu} = 0$). Plugging in π_{dS} in 2.98 we get

$$-2c_1 H^2 + 3c_2 H^4 - 3c_3 H^6 + \frac{3}{2}c_4 H^8 = 0 \tag{2.106}$$

Existence of non-trivial solutions is guaranteed since for a given $H \neq 0$ there exist suitable parameters c_i satisfying 2.106.

Stability of self-accelerating solution

Now we proceed to investigate the perturbative stability of fluctuations about the deSitter solution $\pi_{dS} \to \pi_{dS} + \xi$. The effective action of the fluctuation is constrained by the Galileon symmetry, which is of the same structure as the underlying theory [22].

$$L_\xi = \sum_{m=0}^{m=4} d_m L_m(\xi, \partial\xi, \partial\partial\xi) \tag{2.107}$$

Here d_m's depend on the background solution and the underlying parameters c_i given by,

$$d_m = M_{mn} c_n \tag{2.108}$$

where

$$M_{mn} = \begin{bmatrix} 1 & -3H^2 & \frac{9}{2}H^4 & -3H^6 \\ 0 & 1 & -3H^2 & 3H^4 \\ 0 & 0 & 1 & -2H^2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.109}$$

It can be shown that one can choose the parameters such that the coefficient of kinetic term $d_1 > 0$ thus avoiding ghosts. We can easily see from 2.78 that the corresponding constant for the DGP decoupling limit theory is given by $-6r_c^2 H^2$ where r_c is the cross-over scale in DGP.

Vainshtein screening

Having shown that Galileons pass the zero'th order test of having ghost free self-accelerating solution, we ask if it realizes self-screening in the solar-system scales yielding the predictions of GR. This works precisely in the same manner as in DGP model. We now study the spherically symmetric solution of Galileon with de-Sitter asymptotics. We can forget about the underlying action with parameters c_m and just focus on the effective action for the perturbations around de-sitter solution given by 2.107. The analysis proceeds in the same line as for DGP. For the spherically symmetric configuration $\xi = \xi(r)$, and the equations of motion simplifies to

$$d_1 \left(\frac{\xi'}{r}\right) + 2d_2 \left(\frac{\xi'}{r}\right)^2 + 2d_3 \left(\frac{\xi'}{r}\right)^3 = \frac{M}{4\pi r^3} \tag{2.110}$$

For large distances the linear term dominates with the solution given by

$$\xi_{lin}(r) = -\frac{M}{4\pi d_1 r} \tag{2.111}$$

Thus giving an O(1) modification to GR. However at shorter distances non-linear terms dominate this happens when either one of the non-linear terms become comparable to the linear solution i.e,

$$\left| d_1 \left(\frac{\xi'_{lin}}{r_{v_1}} \right) \right| \approx \left| 2d_2 \left(\frac{\xi'}{r_{v_1}} \right)^2 \right| \implies r_{v_1} = \left(\frac{d_3 M}{d_2^2} \right)^{\frac{1}{3}} \tag{2.112}$$

or

$$\left| d_1 \left(\frac{\xi'_{lin}}{r_{v_2}} \right) \right| \approx \left| 2d_3 \left(\frac{\xi'}{r_{v_2}} \right)^3 \right| \implies r_{v_2} = \left(\frac{d_3 M^2}{d_1^3} \right)^{\frac{1}{6}} \tag{2.113}$$

Thus the Vainshetein radius is given by,

$$r_v = max\,(r_{v_1}, r_{v_2}) \tag{2.114}$$

The profile of the Galileon field scales according to which term dominates,

$$\xi_{non-lin}(r) \approx \left(\begin{array}{c} \left(\frac{M}{d_2}\right)^{\frac{1}{2}} \sqrt{r} \quad \text{for } r << r_v = r_{v_1} \\ \left(\frac{M}{d_3}\right)^{\frac{1}{3}} r \quad \text{for } r << r_v = r_{v_2} \end{array} \right) \tag{2.115}$$

It is clearly seen that the correction to the GR solution due to the scalar field is screened for $r_v \to \infty$. As anticipated the Galileon theory is more robust than the DGP model, for it realises the phenomenologically interesting features of the DGP and also over-comes some of its inconsistencies. The perturbatively stable self-accelerating solution with *Vainshtein screening* is one such feature. Despite the many interesting features of the Galileon field theory, it is still far from being conclusive for the following reasons.

i) The quantum fluctuations are scale dependent so is the classical background solution. This means there exists a critical radius below which quantum corrections invalidate the classical theory.

ii) Radial fluctuations at large distance from the source can propagate superluminally . However it has been recently shown that even though Galileon possess superluminal excitations closed time-like curves do not form within the acceptable energy scale of the theory [24].

iii) As an effective field theory Galileons cannot be the low-energy limit of any fundamental *local* quantum field theory. Thus the existence of Galileon field theory implies the existence of a fundamental field theory that violates locality [25].

iv) Although Vainshetein mechanism has been investigated for simple systems, it has not been proven in complete generality. Particularly, problems arise when one considers extended objects where the concept of Vainshtein radius is ambiguous [82]. One could raise the question whether the Vainshetein radius of the object is the same as that of a point object with the same mass situated at its center of mass. This is known as the *elephant problem.*

Multi-Galileons

The generalisation of the Galileon field theory to multiple fields [121][122] and arbitrary p-forms [47] has been developed. The action for multi-Galileons with N scalar fields respecting the symmetry $\pi_i \to \pi_i + A_b x^b + c$ is given by

$$S = \int d^D x \sum_{m=1}^{D+1} L_m^{multi-gal} \tag{2.116}$$

where,

$$L_m = \alpha^{i_1 \dots i_m} \pi_{i_1} \partial_{a_1} \partial^{[a_1} \pi_{i_2} \dots \partial_{a_{m-1}} \partial^{a_{m-1}]} \pi_{i_m} \tag{2.117}$$

Here the internal indices i_r run from $1 - N$ and the space-time indices a_r run from $0 - (D-1)$. Each term in the lagrangian is invariant under the exchange of down-stair indices i_r, therefore the coefficient $\alpha^{i_1 \dots i_m}$ can be taken to be completely symmetric without loss of generality. The number of free parameters in this theory is given by,

$$\sum_{m=1}^{D+1} \binom{N+m-1}{m} \tag{2.118}$$

Presence of large number of free parameters substantially enlarges the parameter space of the model and also the possibility to find regions in the parameter space where inconsistencies such as superluminality, quantum strong coupling etc.. can be overcome. This atleast was the motivation for the bi-Galileon model where

N=2 (see [30] [121]). However very recently it has been shown that superluminal modes are present regardless of the number of fields present in the theory [44].

Covariant Galileon

Let us now discuss how we can extend the Galileon theory beyond the weak-field limit *i.e* take account of arbitrary backgrounds. We saw how Galileon theory was set up as a scalar-tensor theory such that the scalar field was coupled to the curvature term R where standard GR is reproduced in the limit $\pi \to 0$. This was done by taking $R \to (1-2\pi)R$ in the action where R was truncated at quadratic order. For a fully covariant theory we can make the following generalisations

i) Use the full non-linear curvature term R in the action.

ii) Choose a non-linear function ($f(\pi)$) instead of $(1-2\pi)$ such that $f(\pi) \to 1-2\pi$ when $\pi \to 0$. There is no unique choice, however, a natural choice would be $e^{2\pi}$.

iii) Promote derivatives to covariant derivatives $\partial_a \to \nabla_a$

Let us start with the following action,

$$S = \int d^D x \sqrt{-g} \left[\frac{1}{16\pi G} e^{-2\pi} R + L_{gal} + L_{matter} \right] \tag{2.119}$$

Performing Weyl rescaling $g \to \hat{g} = e^{-2\pi} g$ we get

$$S' = \int d^D x \sqrt{-\hat{g}} \left[\hat{R} + L'_{gal} + L'_{matter} \right] \tag{2.120}$$

Now to make contact with Galileon theory in the limit $\hat{g}_{ab} \to \eta_{ab}$, the minimal procedure would be to promote the derivatives in the flat-space Galileon Lagrangian to covariant derivatives. However, this yields equation of motion with derivative order more than two. It has been shown that to remove these higher order terms some non-minimal coupling to π has to be introduced [112]. We would study this procedure for more general multi-Galileons and generalised multi-scalar theories in greater detail in chapter 5. For now we just present the results obtained in [112].

$$L'_{gal} = \sum_m c_m L_m^{cov} \tag{2.121}$$

where,

$$L_2^{cov} = -\frac{1}{2}(\nabla\pi)^2 \tag{2.122}$$

$$L_3^{cov} = -\frac{1}{2}\Box\pi(\nabla\pi)^2 \tag{2.123}$$

$$L_4^{cov} = -\frac{1}{2}\left[(\Box\pi)^2 - \nabla_\mu\nabla_\nu\pi\nabla^\mu\nabla^\nu\pi - \frac{1}{4}R(\nabla\pi)^2\right](\nabla\pi)^2 \tag{2.124}$$

$$L_5^{cov} = -\frac{1}{2}\left[(\Box\pi)^3 - 3(\Box\pi)(\nabla_\mu\nabla_\nu\pi\nabla^\mu\nabla^\nu\pi) + 2\nabla_\alpha\nabla^\beta\pi\nabla_\beta\nabla^\mu\pi\nabla_\mu\nabla^\alpha\pi\right. \tag{2.125}$$

$$\left. -6G_{\mu\nu}\nabla^\mu\pi\nabla^\nu\nabla_\alpha\pi\nabla^\alpha\pi\right](\nabla\pi)^2 \tag{2.126}$$

2.6 Ostrogradski's theorem

A recurrent feature of all the scalar-tensor theories we discussed so far is that their equation of motion is of derivative order at most two. This is not a mere coincidence but tantamount to a deep fact about physically viable theories known as Ostrogradsky theorem [110]. Ostrogradsky theorem states that any physical system with equations of motion having derivative order greater than two suffer from the linear instability, provided that the system is *non-degenerate.* Here non-degeneracy means that the highest derivative term cannot be written as a function of canonical coordinates or does not have additional constraints that effectively reduce the dimension of phase space. Higher derivative theories are drastically different from lower derivative theories, this is so even when higher derivatives enter as small corrections. There is a discontinuity when the small coefficient is taken to zero *i.e,* the limiting theory is not the original lower derivative one. To demonstrate this let us study a simple system where a harmonic oscillator is modified by a small acceleration term. We take the action

$$S = \int dt \left[\frac{1}{2}\left(1+\epsilon^2\omega^2\right)\dot{q}^2 - \frac{1}{2}\omega^2 q^2 - \frac{1}{2}\epsilon^2\ddot{q}^2\right] \tag{2.127}$$

The equation of motion is

$$\epsilon^2 q^{(4)} + \left(1+\epsilon^2\omega^2\right) q^{(2)} + \omega^2 q = 0 \tag{2.128}$$

Here $q^{(n)} = \frac{d^n q(t)}{dt^n}$. As the equation of motion is fourth order in time derivative, a unique solution requires four pieces of initial data thus the dimension of the phase space is four. We define the canonical coordinates and their corresponding momenta (Q_i, Π_i) as

$$Q_1 = q, \quad \Pi_1 := \frac{\partial L}{\partial \dot{q}} - \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{q}}\right) = (1+\epsilon^2\omega^2)\dot{q} + \epsilon^2 \ddot{q} \tag{2.129}$$

$$Q_2 = \dot{q}, \quad \Pi_2 := \frac{\partial L}{\partial \ddot{q}} = -\epsilon^2 \dddot{q} \tag{2.130}$$

The Hamiltonian is given by the Legendre transformation

$$H = \sum_{i=1}^{2} \Pi_i \dot{Q}_i - L \tag{2.131}$$

giving,

$$H = \frac{1}{2}\left[2\Pi_1 Q_1 - \epsilon^{-2} P_2^2 - \left(1+\epsilon^2\omega^2\right) Q_2^2 + \omega^2 Q_1^2\right] \tag{2.132}$$

Furthermore it can be checked easily that this choice of the Hamiltonian generates the correct time evolution consistent with the equation of motion via,

$$\dot{Q}_i = [H, Q_i]_{P.B} \tag{2.133}$$

$$\dot{P}_i = [H, P_i]_{P.B}$$

Here $[]_{P.B}$ stands for Poisson bracket. Of course $\dot{H} = [H, H]_{P.B} = 0$. Thus we can conclude that our choices of canonical coordinates and momenta are consistent and the Hamiltonian evaluated on any solution of the system can rightly be interpreted as the energy. Notice that two features are immediately evident from the inspection of this Hamiltonian.

i) The limit $\epsilon \to 0$ does not exist which means that however small the higher-derivative correction is, the resulting theory is distinctly different. This is owing to the fact that the dimension of phase space of this theory is double that of the simple harmonic oscillator.

ii) The linear term $\Pi_1 Q_2$ makes the Hamiltonian unbounded from below. By entropic arguments, Π_1 would get arbitrarily negative,that is to say, there is a huge volume of phase space the degrees of freedom can populate to make Π_1 and consequently the Hamiltonian arbitrarily negative.

We can see this explicitly for our example by calculating the energy for a general solution $q_{cl}(t)$ given by

$$q_{cl}(t) = A_+ \cos(\omega t + \theta_+) + A_- \cos(\epsilon^{-1} t + \theta_-) \tag{2.134}$$

Substituting this in 2.132 we get

$$E = \frac{1}{2}\left(1 - \epsilon^2\omega^2\right)\left(\omega^2 A_+^2 - \epsilon^{-2} A_-^2\right) \tag{2.135}$$

which is manifestly unbounded. This indefiniteness of the Hamiltonian is not particularly a problem although it means there is no ground state with lowest energy when quantized. However when this system is interacting with other degrees of freedom, the negative energy modes would get arbitrarily negative while, by conservation of energy, pushing the interacting degrees of freedom to arbitrarily high energies. This would cause a catastrophic instability in the classical system - termed *linear instability.* In the quantum realm such a system would lead to the pair production of ghost and non-ghost particles at a divergent rate (see[40]). It is easy to generalize this result for a generic theory of arbitrary derivative order, which we shall now proceed to do. Let us consider a one-dimensional system defined by the Lagrangian that contains up to N derivatives in the dynamical variable $q(t)$

$$L_N = L_N\left(q, \dot{q}, \dots q^{(N)}\right) \tag{2.136}$$

The Euler-Lagrange equation of motion is given by

$$\sum_{i=0}^{N}\left(-\frac{d}{dt}\right)^i \frac{\partial L_N}{\partial q^{(i)}} = 0 \tag{2.137}$$

if L_N dependes non-linearly on $q^{(N)}$, the equation of motion is of derivative order 2N. Therefore to uniquely specify the trajectory of motion one needs to specify

2N initial data at $t = 0$ namely, $\{q(0), \dot{q(0)} \ldots q(0)^{(N-1)}\}$. This means that the dimension of the phase space is $2N$. Hence we define the generalised coordinates and their momenta according to Ostrogradski's prescription giving,

$$Q_j := q^{(j-1)}, \qquad \Pi_j := \sum_{k=j}^{N-1} -\left(\frac{d}{dt}\right)^{k-j} \frac{\partial L_N}{\partial q^{(k)}} \tag{2.138}$$

Non-linearity of $q^{(N)}$ in L_N implies that it is also non-degenerate meaning $q^{(N)}$ can be written in terms of a function depending on Q_i's and Π_N, $q^{(N)} = F(Q_i, \Pi_N)$. Given this condition the Hamiltonian for arbitrary N is the Legendre transform giving,

$$\begin{aligned} H &:= \sum_{i=1}^{N} \Pi_i q^{(i)} - L_N \\ &= \Pi_1 Q_2 + \Pi_2 Q_3 + \cdots + \Pi_N F(Q_i, \Pi_N) - L(Q_1 \ldots Q_N, F) \end{aligned} \tag{2.139}$$

It is easy to check that this choice of the Hamiltonian leads to time evolution and it is the conserved Noether charge with respect to time translation invariance (if it exists), thus we can identify this with the energy of the system. Again we notice just like in the example that the first N-1 canonical momenta enter the Hamiltonian linearly giving rise to *linear instability* in the system.

The most general scalar-tensor theory in 4 dimensions

The most general scalar-tensor theory in 4D, that is free of Ostrogradsky instability was written down by Horndeski [107] in 1971, and was rediscovered by Deffayet *et al* in 2011 [108]. It is remarkable that the most general theory in arbitrary background proven by Horndeski in 4D coincides with the covariantization of the most general scalar field theory in Minkowski space. Deffayet *et al* showed computed the covariantization such that the resulting theory does not suffer from Ostrogradski instability i.e the equation of motion is of derivative order at most two in both the metric and the scalar field. In 4D this takes the following form,

$$S_{Horn} = \int d^4x\sqrt{-g}\Big[K(\phi,X) - G_3(\phi,X)\Box\phi + G_4(\phi,X)R + G_4(\phi,X)_X\nabla^{[\mu}\nabla_\mu\phi\nabla^{\nu]}\nabla_\nu\phi$$
$$+G_5(\phi,X)G_{\mu\nu}\nabla^\mu\nabla^\nu\phi - \frac{1}{6}G(\phi,X)_X\nabla^{[\mu}\nabla_\mu\phi\nabla^\nu\nabla_\nu\phi\nabla^{\sigma]}\nabla_\sigma\phi\Big] \quad (2.140)$$

Horndeski's action was written in some what different but equivalent representation given below,

$$\mathcal{L}_H = \delta^{\alpha\beta\gamma}_{\mu\nu\sigma}\Big[\kappa_1\nabla^\mu\nabla_\alpha\phi R_{\beta\gamma}{}^{\nu\sigma} + \frac{2}{3}\kappa_{1X}\nabla^\mu\nabla_\alpha\phi\nabla^\nu\nabla_\beta\phi\nabla^\sigma\nabla_\gamma\phi + \kappa_3\nabla_\alpha\phi\nabla^\mu\phi R_{\beta\gamma}{}^{\nu\sigma}$$
$$+2\kappa_{3X}\nabla_\alpha\phi\nabla^\mu\phi\nabla^\nu\nabla_\beta\phi\nabla^\sigma\nabla_\gamma\phi\Big] + \delta^{\alpha\beta}_{\mu\nu}\left[(F+2W)R_{\alpha\beta}{}^{\mu\nu} + 2F_X\nabla^\mu\nabla_\alpha\phi\nabla^\nu\nabla_\beta\phi\right.$$
$$\left.+2\kappa_8\nabla_\alpha\phi\nabla^\mu\phi\nabla^\nu\nabla_\beta\phi\right] - 6\left(F_\phi + 2W_\phi - X\kappa_8\right)\Box\phi + \kappa_9, \quad (2.141)$$

where $\delta^{\alpha_1\alpha_2...\alpha_n}_{\mu_1\mu_2...\mu_n} = \delta^{[\alpha_1}_{\mu_1}\delta^{\alpha_2}_{\mu_2}...\delta^{\alpha_n]}_{\mu_n}$, and $\kappa_1,\ \kappa_3,\ \kappa_8,\ \kappa_9$ and $F = F(\phi,X)$ are arbitrary functions of ϕ and X. $W = W(\phi)$ depends only on ϕ. Further $F_X = 2(\kappa_3 + 2X\kappa_{3X} - \kappa_{1\phi})$. As we can absorb W into F via the redefinition of F, the Lagrangian effectively contains only four arbitrary functions of ϕ, X as expected. The equivalence of the two forms of this theory can be shown by the mapping

$$K = \kappa_9 + 4X\int^X dX'\left(\kappa_{8\phi} - 2\kappa_{3\phi\phi}\right), \quad (2.142)$$

$$G_3 = 6F_\phi - 2X\kappa_8 - 8X\kappa_{3\phi} + 2\int^X dX'(\kappa_8 - 2\kappa_{3\phi}), \quad (2.143)$$

$$G_4 = 2F - 4X\kappa_3, \quad (2.144)$$

$$G_5 = -4\kappa_1, \quad (2.145)$$

where F was redefined to be $F + 2W \to F$. Recently there has been a flurry of interest in this most general scalar-tensor theory in the context of cosmology and modified gravity, ranging from applications to inflation [109, 113], a discussion of the Vainshtein mechanism [114], and the derivation of boundary terms and junction conditions [115]. Utilising this generalised framework Kobayashi *et al* [109] parametrised the most general single field inflation model, termed G-inflation. As we discussed in chapter 1, the fab-4 is also a subset of this most general frame work obeying the self-tuning constraints [119]. Furthermore Horndeski's theory provides a general framework to most of the higher derivative theories such as Galileons. These theories have a number of important applications ranging from consistent violations of the null energy condition [86], to soliton stabilisation [123]. When coupled to gravity, such theories can exhibit self acceleration [111, 121], self tuning [121], and Vainshtein screening [126, 111, 121, 82, 90]. These properties are inherited, of course, by Horndeski's generalisation, but Horndeski's theory can offer even more. It also includes chameleons [88], quintessence and k-essence [116], as well as accomodating Higgs inflation [113].

2.7 Summary

In this chapter we studied some important scalar-tensor theories starting with the first attempt to modify gravity - Brans-Dicke theory. Any attempt to modify gravity has the necessity to comply with GR in solar system scales. This we studied in the section on DGP and Galileon field theories where a mechanism known as *Vainshtein screening* screens out the extra scalar mode thus enabling these models to approach GR. We note however that this is not the only case of such screening mechanism - Chameleon mechanism is another interesting scheme where the mass of the scalar-fields become massive in dense environments making them effective non-dynamical. We investigated the interesting phenomenology of self-acceleration in DGP and Gaileon models, however there were fundamental concerns coming from the peculiar derivative interaction terms - strong coupling, super-luminal propagation, ghost modes, and the inability for these models to arise as the effective field theory of a local UV complete theory are few such pathologies. One realizes that in order to achieve the necessary phenomenological characteristics as an alternative to dark energy, the theoretical models have to push the boundary of conventional quantum field theories. These non-linear field theories are interesting to study in their own right, whether or not they give a successful and consistent implementation of the cosmological constant problem. In closing this section we mentioned the most general scalar-tensor theory evading Ostrogradski ghosts. We would study in greater detail a generalisation of this result to multiple-scalar fields in chapter 5. In the next chapter we investigate the Hamiltonian of Galileon field theory and show how the energy of the two branches of solution sourced by a point mass differ in sign. This is reminiscent of the perturbative ghost discussed in section 2.5.4, at the non-linear level.

Chapter 3

Hamiltonian of Galileon field theory

3.1 Introduction

We saw in section 2.5 that Galileon field theory has novel field theoretic properties both at the classical and quantum mechanical level [50][37]. In particular it is possible to choose suitable parameters to avoid ghost instabilities in the self-accelerating branch as opposed to the DGP model where there is no freedom to choose these parameters appropriately [22]. In this chapter we seek to investigate the stability of the Galileon field theory at the level of the Hamiltonian, this non-linear analysis might help uncover any stability issues that are not captured in the pertubative analysis. But we find that our results are consistent with the latter. To this end, we derive the Hamiltonian for a single galileon field living in Minkowski background space-time with an arbitrary time-like boundary at spatial infinity. This has previously been done for multi-galileons without taking into account the boundary contribution [28]. Here we keep careful track of all the boundary terms and investigate the energy of the static spherically symmetric galileon field at cubic order sourced by a point-mass at the origin. We find that the energies for the non-trivial and normal (Minkowski) branch have equal magnitude but opposite signs depending on the sign of the coefficient of the quadratic term α_2 (see (3.1)). Setting $\alpha_2 > 0$ gives positive (negative) energy for the normal (non-trivial) branch and vice versa, indicating ghost like behaviour in the branch with negative energy as we discuss later. This is a non-linear manifestation of the perturbative ghost instability we studied in section 2.5.4. Moreover the energy is regularized in the short distance (ultra-violet) regime by the dominant cubic term even though the source is divergent at the origin.

Section 3.2 illustrates the framework used in computing the Hamiltonian. We use the normalization of energy with respect to a reference solution as was done in [33][42]. In section 3.3 we present the ADM 3+1 splitting for the bulk Lagrangian density. In section 3.4 we do a subsequent decomposition of the boundary terms. We present the general expression for the Hamiltonian in section 3.5 and in the final section we use this result for a static spherically symmetric single galileon field and explore the implications of this result.

3.2 Infrared regularization of the Hamiltonian

Our aim is to calculate the Hamiltonian for single galileon field theory living in Minkowski space-time with closed boundary(see fig1). The boundary is made up of constant-time hypersurfaces at far-past and far-future, $\Sigma_{-\infty}$, $\Sigma_{+\infty}$ and bounded by an arbitrary time-like hypersurface, B, at spatial infinity, with no inner boundaries. Usually it is fairly straightforward to calculate the Hamiltonian from the action of a field theory, where the Hamiltonian is the Legendre transformation of the Lagrangian, but it is slightly non-trivial when the action has boundary terms as in GR (Gibbons-Hawking-York boundary term). We follow a method that is conceptually similar to that followed by [33] in defining a physically meaningful notion of Hamiltonian for unbounded space-times. This is done by regularizing the action with respect to a reference field as explained below.

The most general action for a single galileon field, $\pi(x)$, in 4-D is given by[22][35],

$$S_{galileon} = S_{bulk} + S_{boundary} \tag{3.1}$$

where,

$$S_{bulk} = \sum_{n=2}^{n=5} \int_M L^n \tag{3.2}$$

$$L^n = \left\{ -\alpha_n \, \pi_{a_2} \pi^{[a_2} \pi^{a_3}_{a_3} \dots \pi^{a_n]}_{a_n} \right\} \tag{3.3}$$

$$S_{boundary} = \sum_{n=3}^{n=5} \int_{\partial M} \left\{ \alpha_n \, (n-2) \pi_\perp \; \pi_{\tilde{a}_3} \pi^{[\tilde{a}_3} \pi^{\tilde{a}_4}_{\tilde{a}_4} \dots \pi^{\tilde{a}_n]}_{\tilde{a}_n} \right\} \tag{3.4}$$

Here $\pi_a = \partial_a \pi$ and $\pi_\perp$, $\pi_{\tilde{a}_n}$ are orthogonal and tangential derivatives with respect to the boundary. We use the convention that antisymmetrization over the a indices do not involve the prefactor $\frac{1}{n!}$. Note that in this chapter index, a, runs over 0..3 and i runs over 1..3. This is a consistent action for Dirichlet boundary condition, ie when the fields and their tangential derivatives are held fixed at the boundary [35]. $S_{boundary}$ is the analog of the Gibbons-Hawking-York boundary term in GR. Note that $\alpha_2 > 0$ as we have defined in (3) yields a stable Minkowski branch free of ghost-like behaviour due to the positivity of the kinetic term, however this would make some other branches unstable. A concrete example of this is infact what we discuss in the final section. The action defined above is finite for compact geometries but diverges for non-compact space-times. To renormalize this action for non-compact space-times we choose a reference background π_0 that asymptotes to the value of π and also a solution of the theory. Then we demand the physical action to be given by,

$$S_{physical} = S_{galileon}[\pi] - S_{galileon}[\pi_0] \tag{3.5}$$

consequently the physical Hamiltonian is,

$$H_{physical} = H_{galileon}[\pi] - H_{galileon}[\pi_0] \tag{3.6}$$

In order to derive the Hamiltonian for Galileon field theory, one must do an ADM decomposition of the action. We postpone the final result until we have presented the decomposition of the bulk-space-time and the decomposition of boundary terms in terms of relevant derivatives and geometrical quantities.

3.3 Bulk Decomposition

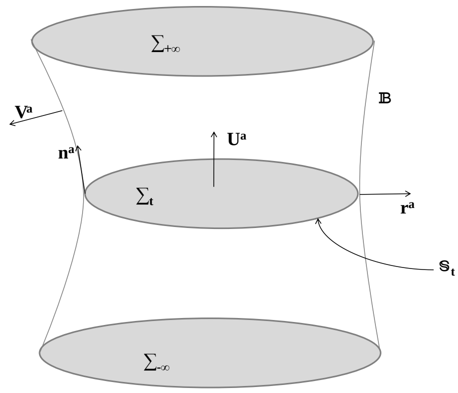

Figure 3.1: Space-time with boundary

In fig(3.1) V^a, U^a *are vectors orthogonal to hyper-surfaces* B, Σ_t *resp.* r^a, n^a *are vectors lying on* Σ_t, B *respectively, and orthogonal to* S_t*.*

We start with decomposing the action for single galileon field in terms of time and spatial derivatives. This is similar to the ADM formalism in GR except there is a prefered time direction since we are working in Minkowski space-time. We consider the galileon field in Minkowski space-time bounded by a time-like boundary at spatial infinity B (see FIG. 3.1). We foliate the bulk space-time in constant-time space-like hypersurfaces Σ_t. Thus the natural embedding is as follows,

$$\Sigma_t : [x^i] \to [t, x^i] \tag{3.7}$$

where t, x^i define the standard cartesian coordinates giving the line element as,

$$ds^2 = -dt^2 + \delta_{ij} dx^i dx^j \tag{3.8}$$

Ignoring the boundary terms (4), the most general Lagrangian for galileon fields in 4-dimensions can be expressed as follows,

$$L_{galileon} = \sum_{n=2}^{5} L^n \tag{3.9}$$

We consider a general term L^n of order n in π and seek to do a 3+1 split in terms of time and space. In the spirit of integrating by parts, we rewrite the Lagrangian as a piece that contains no 2^{nd} order time derivatives, L^n_{bulk}, and a total derivative term, $L^n_{left-over}$, (see Appendix (A) for details). Thus,

$$L^n = L^n_{bulk} + L^n_{left-over} \tag{3.10}$$

where,

$$L^{(n)}_{bulk} = \alpha_n \left\{ {}^nC_2 \ \dot{\pi}^2 \pi^{[i_3}_{i_3} \pi^{i_4}_{i_4} .... \pi^{i_n]}_{i_n} - \pi_{i_2} \pi^{[i_2} \pi^{i_3}_{i_3} ... \pi^{i_n]}_{i_n} \right\}$$

here, ${}^nC_2 = \frac{(n)(n-1)}{2}$.

$$L^n_{left-over} = \alpha_n \Bigg\{ - \frac{(n-2)(n+1)}{2} \partial_{i_3} \left[\dot{\pi}^2 \pi^{[i_3} \pi^{i_4}_{i_4} ... \pi^{i_n]}_{i_n} \right] - (n-2) \partial^a \left[\pi_a \pi_{i_3} \pi^{[i_3} \pi^{i_4}_{i_4} \dots \pi^{i_n]}_{i_n} \right] \tag{3.11}$$

$$+ (n-2) \partial^i \left[\pi_i \pi_{i_3} \pi^{[i_3} \pi^{i_4}_{i_4} ... \pi^{i_n}_{i_n} \right] + (n-2)(n-3) \partial_{i_3} \left[\dot{\pi} \pi_{i_4} \pi^{[i_3} \pi^{i_4}_t \pi^{i_5}_{i_5} ... \pi^{i_n]}_{i_n} \right] \Bigg\}$$

Inserting the boundary term (4) back into the action and using Stoke's Theorem to convert bulk-integrals to boundary-integrals and including terms of all-order in 4D we recast the total action as follows,

$$S_{total} = S_{bulk} + S_{total-boundary} \tag{3.12}$$

where

$$S_{bulk} = \sum_{n=2}^{5} \int dt \int_{\Sigma_t} \alpha_n \left\{ {}^nC_2 \; \dot{\pi}^2 \pi^{[i_3}_{i_3} \pi^{i_4}_{i_4} ....\pi^{i_n]}_{i_n} - \pi_{i_2} \pi^{[i_2} \pi^{i_3}_{i_3} ...\pi^{i_n]}_{i_n} \right\} \tag{3.13}$$

$$S_{total-boundary} = \sum_{n=3}^{5} S^n_{total-boundary} \tag{3.14}$$

with,

$$\begin{aligned} S^n_{total-boundary} = \alpha_n \int dt \int_{S_t} \Big\{ & - \frac{(n-2)(n+1)}{2} r_{i_3} \left[\dot{\pi}^2 \pi^{[i_3} \pi_{i_4} \pi^{i_4} \dots \pi^{i_n]}_{i_n} \right] \\ & + (n-2)\pi_r \left[\pi_{i_3} \pi^{[i_3} \pi^{i_4}_{i_4} \dots \pi^{i_n]}_{i_n} \right] \\ & + (n-2)(n-3) \dot{\pi} \pi_{i_4} r_{i_3} \pi^{[i_3} \dot{\pi}^{i_4} \pi^{i_5}_{i_5} \dots \pi^{i_n]}_{i_n} \Big\} \\ & + \alpha_n \int_{\partial M} \left\{ (n-2) \pi_V \left[\pi_{\bar{a}_3} \pi^{[\bar{a}_3} \pi^{\bar{a}_4}_{\bar{a}_4} \dots \pi^{\bar{a}_n]}_{\bar{a}_n} - \pi_{i_3} \pi^{[i_3} \pi^{i_4}_{i_4} \dots \pi^{i_n]}_{i_n} \right] \right\} \end{aligned} \tag{3.15}$$

Note that $S^2_{total-boundary} = 0$. Here $\pi_V = V^a \partial_a \pi$, $\pi_r = r^a \partial_a \pi$ denote the derivatives along the normal vectors V^a, r^a (see FIG. 3.1) respectively. $\pi_{\bar{a}}$ denotes the covariant derivative with respect to the boundary, B (see next section). Also, $\int_{\partial M} = \int dt N \int_{S_t}$ where $N = (1+\theta^2)^{-1/2}, \theta = n_a r^a$ is the lapse function. We have completed the decomposition of the bulk-terms in the action. In the next section we decompose the boundary term $S_{total-boundary}$ with respect to the relevant derivatives to be defined below.

3.4 Boundary decomposition

We seek to decompose the boundary terms in terms of derivatives with respect to the closed 2-surface $S_t = B \cap \Sigma_t$ and derivatives along r^a, U^a. We work in full-space time coordinates and begin by presenting the definitions of various derivatives and projection operators,

$$\gamma_{ab} = g_{ab} + U_a U_b := \text{Projection operator for } \Sigma_t \tag{3.16}$$

$$H_{ab} = g_{ab} - V_a V_b := \text{Projection operator for } B \tag{3.17}$$

$$q_{ab} = H_{ab} + n_a n_b = \gamma_{ab} - r_a r_b := \text{Projection operator for } S_t$$

We use $D_a, \bar{D}_a, \hat{D}_a$ to denote covariant-derivatives with respect to Σ_t, B, S_t. For brevity this convention is used on the indices in long expressions. $\pi_n, \pi_V, \pi_r, \dot{\pi}$ are derivatives along the corresponding vector fields defined as $\pi_n := D_n\pi := n^a \bar{D}_a \pi$ etc. Also, $\pi_{n\hat{a}} := \hat{D}_a D_n \pi$, $\pi_{r\hat{a}} := \hat{D}_a D_r \pi$, $\pi_{n^2} := D_n^2 \pi := D_n D_n \pi$, $\pi_{r^2} := D_r^2 \pi := D_r D_r \pi$. The action of a covariant derivative $\tilde{D}_a$ on a hypersurface (with an associated projection tensor h_{ab}) on a given tensor lying on the surface is given by [48],

$$\tilde{D}_a T^{b_1 \ldots b_i}_{c_1 \ldots c_j} = h^b_a h^{b_1}_{d_1} \ldots h^{b_i}_{d_i} h^{e_1}_{c_1} \ldots h^{e_j}_{c_j} \nabla_b T^{d_1 \ldots d_i}_{e_1 \ldots e_j} \tag{3.18}$$

Boundary terms contain derivatives $D_a, \bar{D}_a, D_a D_b, \bar{D}_a \bar{D}_b$ which can be decomposed as follows (see Appendix (A)).

$$\begin{aligned} D_a \pi &= \hat{D}_a \pi + r_a D_r \pi \\ \bar{D}_a \pi &= \hat{D}_a \pi - n_a D_n \pi \\ D_a D_b \pi &= \hat{D}_a \hat{D}_b \pi + K^1_{ab} D_r \pi + 2 r_{(a} \hat{D}_{b)} D_r \pi - 2 r_{(a} K^1_{b)c} \hat{D}^c \pi + r_a r_b D_r^2 \pi \\ \bar{D}_a \bar{D}_b \pi &= \hat{D}_a \hat{D}_b \pi - K^2_{ab} D_n \pi - 2 n_{(a} \hat{D}_{b)} D_n \pi + 2 n_{(a} K^2_{b)c} \hat{D}^c \pi + n_a n_b D_n^2 \pi \end{aligned} \tag{3.19}$$

Here $K^1_{ab}, K^2_{ab} = \theta K^1_{ab}$ are extrinsic curvatures of the 2-surface S_t with respect to the hypersurfaces Σ_t, B respectively. $T_{(ab)} = \frac{1}{2}(T_{ab} + T_{ba})$ denotes symmetrization of the indices. We can now express the boundary terms given in (16) by substituting the decomposition given above. Thus the boundary terms are (here we omit the result for 5^{th} order for brevity, see Appendix (A)),

$$S^3_{total-boundary} = \alpha_3 \int dt \int_{S_t} \Big\{ -3(1+\theta^2)\dot{\pi}^2\pi_r - \theta^2\pi_r^3 - 3\theta(1+\theta^2)^{\frac{1}{2}}\dot{\pi}\pi_r^2 - \theta(1+\theta^2)^{1/2}\dot{\pi}^3 + (\hat{D}\pi)^2\pi_r \Big\} \tag{3.20}$$

$$\begin{aligned}
S^4_{total-boundary} = \alpha_4 \int dt \int_{S_t} \Big\{ & -2\pi_{\hat{a}}\pi^{[\hat{a}}K_b^{1b]}\left[\theta^2\dot{\pi}^2 + 2\theta(1+\theta^2)^{\frac{1}{2}}\dot{\pi}\pi_r + (1+\theta^2){\pi_r}^2\right] \\
& -2(1+\theta^2)^{-\frac{1}{2}}(\hat{D}\pi)^2\Big[\theta(1+\theta^2)\dot{\pi}\ddot{\pi} + 2\theta^2(1+\theta^2)^{\frac{1}{2}}\dot{\pi}\dot{\pi}_r + \theta(1+\theta^2)\dot{\pi}\pi_{r^2} \\
& + (1+\theta^2)^{\frac{3}{2}}\pi_r\ddot{\pi} + 2\theta(1+\theta^2)\dot{\pi}_r\pi_r + (1+\theta^2)^{\frac{3}{2}}\pi_r\pi_{r^2} - K^B_{nn}\left(\theta^2\dot{\pi}^2 + (1+\theta^2)\pi_r^2 + 2\theta(1+\theta^2)^{\frac{1}{2}}\dot{\pi}\pi_r\right)\Big] \\
& + 4\Big[\theta(1+\theta^2)^{\frac{1}{2}}\dot{\pi}^2\pi_{\hat{a}}\dot{\pi}^{\hat{a}} + (1+\theta^2){\pi_r}^2\pi_{\hat{a}}\pi_r^{\hat{a}} + 2\theta(1+\theta^2)^{\frac{1}{2}}\dot{\pi}\pi_r\pi_{\hat{a}}\pi_r^{\hat{a}} + (1+2\theta^2)\dot{\pi}\pi_r\pi_{\hat{a}}\dot{\pi}^{\hat{a}} \\
& + \theta^2\dot{\pi}^2\pi_{\hat{a}}\pi_r^{\hat{a}} + \theta(1+\theta^2)^{\frac{1}{2}}{\pi_r}^2\pi_{\hat{a}}\dot{\pi}^{\hat{a}}\Big] - 4\left[\pi_{\hat{a}}\pi_{\hat{b}}K^{1ab}\left[\theta^2\dot{\pi}^2 + (1+\theta^2)\pi_r^2 + 2\theta(1+\theta^2)^{\frac{1}{2}}\dot{\pi}\pi_r\right]\right] \\
& - 2(\hat{D}^2\pi)\left[\theta(1+\theta^2)^{\frac{1}{2}}\dot{\pi}^3 + (1+\theta^2){\pi_r}^3 + (1+3\theta^2)\dot{\pi}^2\pi_r + 3\theta(1+\theta^2)^{\frac{1}{2}}\dot{\pi}{\pi_r}^2\right] \\
& + 2(1+\theta^2)^{-\frac{1}{2}}(\theta\dot{\pi} + (1+\theta^2)^{\frac{1}{2}}\pi_r)K^1\left[\theta(1+\theta^2)^{\frac{3}{2}}\dot{\pi}^3 + 3\theta^2(1+\theta^2)\dot{\pi}^2\pi_r + 3\theta^3(1+\theta^2)^{\frac{1}{2}}{\pi_r}^2\dot{\pi} + (\theta^4-1){\pi_r}^3\right] \\
& - 5\dot{\pi}^2{\pi_r}^2K^1 - 6\dot{\pi}^2(\hat{D}^2\pi)\pi_r - 5\dot{\pi}^2K^1_{ab}\pi^{\hat{a}}\pi^{\hat{b}} + 4\dot{\pi}^2\pi_{r\hat{a}}\pi^{\hat{a}} - 4{\pi_r}^2\pi_{r\hat{a}}\pi^{\hat{a}} + 4{\pi_r}^2K^1_{ab}\pi^{\hat{a}}\pi^{\hat{b}} \\
& + 2{\pi_r}^3(\hat{D}^2\pi) + 2{\pi_r}^4K^1 + 2\pi_r\pi_{\hat{a}}\pi^{[\hat{a}}\pi_{\hat{b}}^{\hat{b}]} + 2{\pi_r}^2\pi_{\hat{a}}\pi^{[\hat{a}}K_{\hat{b}}^{1\hat{b}]} + 2\pi_r\pi_{r^2}(\hat{D}\pi)^2 - 2\dot{\pi}\dot{\pi}_r(\hat{D}\pi)^2\Big\}
\end{aligned} \tag{3.21}$$

3.5 Derivation of the Hamiltonian

Having recast the galileon action interms of ADM decomposition we can now write down the Hamiltonian directly. The Hamiltonian density of the galileon theory described by the Lagrangian $L(\pi, \partial\pi, \partial\partial\pi)$ is given by the Legendre transform,

$$H = p\dot{\pi} - L \tag{3.22}$$

where the canonical momenta p is given by,

$$p = \frac{\partial L}{\partial \dot{\pi}} = \sum_{n=2}^{5} 2\alpha_n \left\{ {}^nC_2 \ \dot{\pi}\pi_{i_3}^{[i_3}\pi_{i_4}^{i_4}....\pi_{i_n}^{i_n]} \right\} \tag{3.23}$$

Thus the Hamiltonian of single galileon field theory is,

$$H_{galileon} = \left\{ \sum_{n=2}^{5} \alpha_n \int_{\Sigma_t} \left[{}^nC_2 \; \dot{\pi}^2 \pi^{[i_3}_{i_3} \pi^{i_4}_{i_4} \dots \pi^{i_n]}_{i_n} + \pi_{i_2} \pi^{[i_2} \pi^{i_3}_{i_3} \dots \pi^{i_n]}_{i_n} \right] \right\} \tag{3.24}$$

$$- S^3_{total-boundary} - S^4_{total-boundary} - S^5_{total-boundary}$$

Where the last 3 boundary-terms are given by (3.20),(3.21) and (A.22).

3.6 Energy of static galileon fields coupled to a point-source

Let us use our Hamiltonian to compute the energy of a single galileon field at cubic order in a static configuration with $SO(3)$ symmetry, coupled to a point mass, m, at the origin. We take S_t to be a 2-sphere with fixed radius R. Here the theory contains two vacua: a normal branch (X^{ref}_+) and a non-trivial branch (X^{ref}_-) (see (3.28)). The stability of these branches depends on the sign of α_2, where $\alpha_2 > 0$ leads to a stable normal branch but an unstable non-trivial branch and vice versa for $\alpha_2 < 0$. Here we demonstrate that this perturbative instability is consistent with our non-linear calculation using the full Hamiltonian where it manifests as negative energy for non-trival (normal) branch when α_2 is positive (negative). The natural coordinates to work with are the spherical coordinates (r, θ, ϕ). The Hamiltonian function for this set-up becomes,

$$H = 4\pi \int dr r^2 \left\{ \alpha_2 \pi'^2 + 2\alpha_3 \frac{\pi'^3}{r} + \frac{\rho}{M_p} \pi \right\} \tag{3.25}$$

Here M_p is a dimension-full coupling constant with mass dimension, usually this is of order planck mass for gravitational theories. Also, $(') = \frac{d}{dr}$ and $\rho = m\delta^{(3)}(r)$. Note that $S^3_{total-boundary}$ vanishes for this set up, since for static $SO(3)$ symmetric galileon field, $\dot{\pi} = \hat{D}_a \pi = 0$, and time invariance of the 3-boundary, B, implies $\theta := n.r = 0$. The equation of motion is given by [22],(The π appearing in the expressions from (26) to (33) is the number π not to be confused with the field.)

$$\alpha_2 X + 3\alpha_3 X^2 = \frac{m}{8M_p\pi r^3} \tag{3.26}$$

where $X = \frac{\pi'}{r}$. The normal and non-trivial branch solutions of (3.26) are given implicitly by,

$$\begin{aligned} X_+ &:= \frac{\pi'_+}{r} = \frac{-\alpha_2 + \sqrt{\alpha_2^2 + \frac{3m\alpha_3}{2M_p\pi r^3}}}{6\alpha_3} \\ X_- &:= \frac{\pi'_-}{r} = \frac{-\alpha_2 - \sqrt{\alpha_2^2 + \frac{3m\alpha_3}{2M_p\pi r^3}}}{6\alpha_3} \end{aligned} \tag{3.27}$$

The corresponding reference solutions which we choose to be the normal and non-trivial vacuum solutions are given by setting $m = 0$ in (3.27).

$$\begin{aligned} X_+^{ref} &= 0 \\ X_-^{ref} &= -\frac{\alpha_2}{3\alpha_3} \end{aligned} \tag{3.28}$$

It is convenient to rewrite the integrand in (3.25) using the equation of motion to eliminate the π dependence. Thus,

$$H = -4\pi \int_0^R drr^4 \left\{\alpha X^2 + 4\alpha_3 X^3\right\} + \frac{m}{M_p}\int_0^R dr\,\{rX\} \tag{3.29}$$

The energy for positive and negative branches is now given by,

$$E_\pm = H[X_\pm] - H[X_\pm^{ref}]|_{m=0} \tag{3.30}$$

Substituting (3.27), (3.28) above we get,

$$\begin{aligned} E_+ = -E_- = \frac{2\pi\alpha_2^3 R^5}{135\alpha_3^2} + \frac{\alpha_2 m}{18M_p\alpha_3}\int_0^R drr\left(1 + \frac{3\alpha_3 m}{2M_p\pi\alpha_2^2}r^{-3}\right)^{1/2} \\ - \frac{2\alpha_2^3\pi}{27\alpha_3^2}\int_0^R drr^4\left(1 + \frac{3\alpha_3 m}{2M_p\pi\alpha_2^2}r^{-3}\right)^{1/2} \end{aligned} \tag{3.31}$$

After some change of variables the integrals can be recognized as a linear combination of hypergeometric functions given by,

$$\begin{aligned} E_+ =& \frac{2\pi\alpha_2^3 R^5}{135\alpha_3^2} + \frac{sign(\alpha_2)(\frac{m}{M_p})^{3/2}\sqrt{R}}{3\sqrt{6\pi\alpha_3}} {}_2F_1\left[-1/2, 1/6, 7/6, -\frac{2M_p\alpha_2^2\pi R^3}{3\alpha_3 m}\right] \\ &- sign(\alpha_2)\frac{2\alpha_2^2}{63}\sqrt{\frac{2\pi m}{3M_p\alpha_3^3}} R^{\frac{7}{2}} {}_2F_1\left[-1/2, 7/6, 13/6, -\frac{2M_p\alpha_2^2\pi R^3}{3\alpha_3 m}\right] \end{aligned} \tag{3.32}$$

Here the hypergeometric functions are real and positive and defined for the range $\alpha_3 > -\frac{2M_p\alpha_2^2\pi R^3}{3m}$. However for real values of E_+, E_-, α_3 is forced to be positive.We now take the limit $R \to \infty$ and the energy becomes,

$$E_+^\infty = -E_-^\infty = -\left(\frac{2}{3}\right)^{\frac{7}{3}} \Gamma\left(-\frac{8}{3}\right)\Gamma\left(\frac{7}{6}\right)\frac{(\alpha_2\alpha_3)^{-\frac{1}{3}}(\frac{m}{M_p})^{\frac{5}{3}}}{\pi^{\frac{7}{6}}} > 0 \tag{3.33}$$

We get a finite expression for energy with equal magnitude and opposite sign. The infra-red divergence is regularized by substracting the vacuum energy contribution. As a non-trivial check for our calculation we take the limit $m \to 0$ in (3.32) and obtain,

$$\lim_{m\to 0} E_\pm = 0 \tag{3.34}$$

as expected.

3.7 Summary

We conclude with a few remarks on our analysis of the energy of Galileon field theory. Having presented the expression for Hamiltonian in ADM formalism carefully keeping track of all the boundary terms, we calculated the energy of static spherically symmetric configuration. In particular, the results of our calculation shows,

i) The two branches of the cubic theory coupled to a point source have energies of equal magnitude and opposite sign.

ii) The expression for energy flips sign when the sign of α_2 is changed.

iii) Even though we couple Galileon field to a divergent source at the origin, energy is still finite where non-linear cubic contribution dominates the divergent quadratic term and regularizes it.

We argue that the negative energy of the non-trivial(normal) branch when $\alpha_2 > 0(< 0)$ with a coupling to a point mass indicates a ghost like instability. Our calculations have been entirely classical and as was argued in [40] the appearance of negative energy can be traced back to the wrong sign in the propagator, at quantum level. If one evades negative probabilities by shifting the poles in the denominator of the propagator it leads to negative energy. Scattering processes involving ghost like particles and ordinary matter particles can generate ghost particles with unbounded negative energy and matter particles with unbounded positive energy. We believe the sign flip of the energy when changing the sign of α_2 further reinforces this argument, for it is the correct sign of α_2 in ordinary field theories that ensures the positivity of the kinetic term in the Lagrangian. It is interesting to note that a similar calculation was done for Gauss-Bonnet gravity in [42] and the authors found that the energies for the 2-branches match both in magnitude and sign. Further more it was shown that one of the vacua of Gauss-Bonnet gravity was unstable despite the fact that ghost like modes were not excited by the spherically symmetric black-hole[29]. In contrast here we find that point source which can be taken to be a spherically symmetric source in the limiting case, does seem to excite ghost-like modes giving negative energy. This investigation can be taken further, it would be interesting to do this calculation for covariant Galileon model [112] and Multi-galileon theories [122]. In the next chapter we investigate the most general scalar-tensor theory in 4-D in arbitrary back-grounds and compute the necessary boundary terms that makes the action well-posed and the junction conditions associated with a co-dimension 1 brane.

Chapter 4

Boundary Terms and Junction Conditions for Generalized Scalar-Tensor Theories

4.1 Introduction

In preceding chapters we studied various scalar tensor theories. As we mentioned in section 2.6 in chapter-2 all such theories fall into a general framework discovered by Horndeski in 1974 [107]. This generality is constrained by the requirement that the field equations are of derivative order up to two, in order to avoid Ostrogradski ghosts. What is surprising is that this theory was forgotten about until very recently, where it was resurrected in [119], and discovered independently in [108]. Horndeski's theory is the "most general" scalar-tensor theory up to the requirement of second order field equations in four dimensions. Higher order field equations can be interpreted as propagating extra fields, and in any event, they typically suffer from the Ostrogradski instability [110]. Here we will work with the formulation of Hornedeski's theory presented in [108] by Deffayet, Gao, Steer and Zahariade [DGSZ], as it is more aesthetic and is valid in any number of dimensions[1]. This is given by

$$S[g_{ab},\phi] = \int_{\mathcal{M}} k(\phi,X) - G_3(\phi,X)\Box\phi + G_4(\phi,X)R \\ + G_{4X}\nabla^a_{[a}\phi\nabla^b_{b]}\phi + G_5(\phi,X)G_{ab}\nabla^a\nabla^b\phi - \frac{G_{5X}}{6}\nabla^a_{[a}\phi\nabla^b_b\phi\nabla^c_{c]}\phi \quad (4.1)$$

where $X = -\frac{1}{2}(\nabla\phi)^2$, and **the antisymmetrisation does not include the usual factor of** $1/n!$. The covariant measure on the manifold is omitted for brevity. First, let us discuss some generalities regarding well-posed actions and boundary terms.

4.2 Well-posedness of the action principle

The action plays an important role in physics, especially in field theory. A naive point of view is to take the variational principle as a formal device to arrive at the equations of motion. Let us take a generic action that depends on the scalar field ϕ and it's derivatives,

$$S = \int d^dx\mathcal{L}([\phi]) \quad (4.2)$$

Here $[\phi]$ denotes the field and it's derivatives to arbitrary order. To find the equation of motion one varies the action and integrates by parts to find the stationary point giving,

$$\frac{\partial\mathcal{L}}{\partial\phi} + \sum_{m=1}^{N}(-1)^m\partial_{a_1}\dots\partial_{a_m}\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_{a_1}\dots\partial_{a_m}\phi\right)}\right) = 0 \quad (4.3)$$

Notice that we have neglected the boundary terms arising from integrating by parts or assumed that the field and its derivatives vanish at infinity. This is a valid approach if we are only interested in the local dynamics of the field. The role of

[1] In four dimensions, the Horndeski and DGSZ actions were shown to be equivalent [109], and given Horndeski's proof, we know this to be the most general scalar-tensor theory admitting second order field equations. In higher dimensions the DGSZ action is known to yield second order field equations, but it is not proven to be the most general theory.

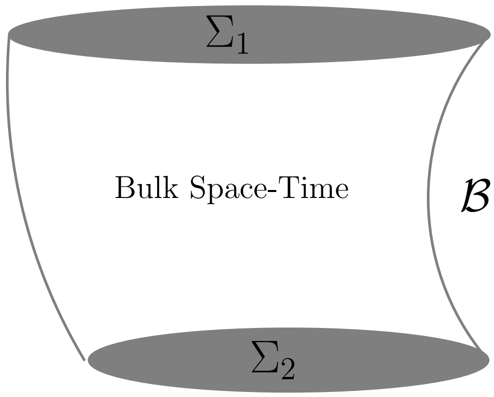

Figure 4.1: Field theory living in space-time with boundary

the action here is manifestly secondary since every local quantity can be calculated from the Lagrangian alone. However, it is clearly understood that the action and the variational principle have their origin in quantum mechanics, where the action functional appears naturally in the path integral formulation and consequently it is crucial for the euclidean path integral methods. On account of the fact that the action is a function of the boundary data specified to yield a unique extremal solution, one can no longer neglect the boundary terms and it is necessary to have the *correct* form of the action. Let us elucidate what we mean by the *correct* action. Consider the field theory living in a space-time with boundary as shown in fig(4.1). The *correct* action for this theory can be defined as the one that gives a unique extremal solution for a given boundary data. That is, for a unique classical solution $\phi_c(x)$ satisfying boundary data defined on sub-manifolds $\Sigma_1, \Sigma_2, \mathcal{B}$

$$\frac{\delta S}{\delta \phi} = 0 \Longleftrightarrow \phi(x) = \phi_c(x) \tag{4.4}$$

This implies the following,

i) The Euler-Lagrange equations must be satisfied and it should have a unique solution for a given boundary data.

ii) The boundary terms arising from the variation should vanish without having to impose any more constraints than what is already implied by the fixed boundary data.

Now, a well-posed problem is defined to be such that given appropriate set of boundary data there is a unique solution satisfying the partial differential equation, this notion of well-posedness can thus be extended to the corresponding action principle satisfying conditions (i),(ii). The well-posed action is a functional of all the competing paths of the field satisfying the boundary data, and it picks out a unique extremising path. Therefore it is this well-posed action that should enter any quantum mechanical calculations, for it does not forbid any path that satisfies the boundary data. We will now discuss a toy example that illustrates these ideas.

4.2.1 An example

In this simple one-dimensional example we will see how an action that is not well-posed in the above sense can be made well posed by adding a suitable boundary term. Consider the action,

$$S = \int_{t=t_1}^{t_2} dt \left[-\frac{1}{2} q(t)\ddot{q}(t) - V(q) \right] \tag{4.5}$$

varying this we get,

$$\delta S = \int_{t=t_1}^{t_2} dt \left[-\ddot{q}(t) - \frac{\partial V}{\partial q} \right] \delta q + \left[-\frac{1}{2} q(t)\delta\dot{q}(t) + \frac{1}{2}\dot{q}(t)\delta q \right] \Bigg|_{t_1}^{t_2} \tag{4.6}$$

The resulting Euler-Lagrange equation has a unique solution for given arbitrary values of $q(t)$ at the the end-points/boundary ($t = t_1, t_2$). However we see from 4.6 that the 3'rd term would vanish only if we demand $\delta\dot{q} = 0$. This however is an additional constraint that is not implied by the fixed boundary data $q(t_1), q(t_2)$, thus it is clear that this action is not well-posed in the sense described above. In order to rectify this problem we add the following boundary term,

$$B_1 \equiv -\frac{1}{2} \left[q(t)\dot{q}(t) \right] \Bigg|_{t_1}^{t_2} \tag{4.7}$$

varying this we get,

$$\delta B_1 = -\frac{1}{2} \left[\delta q(t)\dot{q}(t) + q(t)\delta\dot{q}(t) \right] \Bigg|_{t_1}^{t_2} \tag{4.8}$$

exactly cancelling the problematic boundary term. Incidentally the well-posed action with the boundary term (B_1) is the canonical action of this system.

$$S + B_1 = \int_{t=t_1}^{t_2} dt \left[\frac{1}{2}\dot{q}(t)^2 - V(q) \right] \tag{4.9}$$

As they stand, neither the original Horndeski action [107], nor the recent reformulation [108] given above admit a well defined variational principle on a manifold with a boundary. This is problematic if one wishes to apply (Euclidean) path-integral methods to Horndeski's theory, or if one wishes to consider the dynamics of domain walls configurations. The same is true, of course, of the Einstein-Hilbert action, where the Gibbons-Hawking boundary term [73, 74] is added such that the full theory can be extremised with Dirichilet boundary conditions on the spacetime metric. In this chapter, we derive the analogue of the Gibbons-Hawking boundary term for Horndeski's theory.

Armed with a well defined action, we can derive the junction conditions across a co-dimension one brane, or domain wall, embedded within the manifold. This leads to the analogue of the Israel junction conditions [75] in Horndeski's theory, and opens up the possibility of studying plenty of new physics from bubble nucleation to braneworld dynamics. Our derivation makes use of the standard technique of treating the brane as the common boundary of the bulk geometry on either side of the brane. The methods used for deriving the boundary terms and junction conditions will be described in more detail in section 4.3, where we will explicitly present the relevant calculation for the first two terms in (4.1). The boundary terms and junction conditions for the full theory will be presented in section 4.4. In section 4.5 we will discuss some special cases such as Brans-Dicke gravity [103], flat space galileon [111] and covariant galileon [112] theory. We will conclude in section 4.6.

4.3 Methodology

Let us briefly outline the methodology we used in deriving the results that will be presented in the next section. Consider the incomplete action (4.1) defined on a manifold $\mathcal{M}$ with boundary $\partial\mathcal{M}$. The boundary may be spacelike ($s = -1$) or timelike ($s = +1$). We begin by computing the variation of (4.1) keeping track of all surface terms. The result is

$$\delta S[g_{ab}, \phi] = \int_{\mathcal{M}} \varepsilon^{ab}\delta g_{ab} + \varepsilon^{\phi}\delta\phi + \int_{\partial\mathcal{M}} X^{ij}\delta h_{ij} + X^{\phi}\delta\phi + Y^{ij}\delta(h_{ij,n}) + Y^{\phi}\delta\phi_n \quad (4.10)$$

where ε^{ab} and ε^{ϕ} are the equations of motion. We are using bulk coordinates x^a, and boundary coordinates ξ^i, and we may think of the boundary as an embedding $x^a = X^a(\xi)$. This defines tangent vectors $\partial_i X^a$, each of which is orthogonal to the unit outward pointed normal n^a. The induced metric on the boundary is defined as

$$h_{ij} = \partial_i X^a \partial_j X^b g_{ab}|_{\partial\mathcal{M}} \quad (4.11)$$

This can also be identified with the projector on to the boundary, which we denote $h_{ab} = g_{ab} - s n_a n_b$, where $s = g_{ab} n^a n^b$.

Dirichilet boundary conditions require that $\delta\phi$ and δh_{ij} vanish on $\partial\mathcal{M}$, so the boundary terms $X^{ij}\delta h_{ij}$ and $X^{\phi}\delta\phi$ are not considered problematic. The same cannot be said of the remaining boundary terms $Y^{ij}\delta(h_{ij,n})$ and $Y^{\phi}\delta\phi_n$. $\phi_n = n^a\partial_a\phi|_{\partial\mathcal{M}}$ is the normal derivative to the scalar on the boundary, and its variation is not necessarily vanishing. Similarly, $h_{ij,n} = \partial_i X^a \partial_j X^b n^c \partial_c g_{ab}|_{\partial\mathcal{M}}$, which is the normal derivative to the metric on the boundary. These troublesome boundary terms are present because the DGSZ action (4.1) contains terms with second order derivatives.

To fix this problem, we must add a boundary term $B[h_{ij}, \phi, h_{ij,n}, \phi_n]$ whose variation cancels off the troublesome contributions described above. In other words, we must choose B such that

$$\delta B[h_{ij}, \phi, h_{ij,n}, \phi_n] = \int_{\partial\mathcal{M}} Z^{ij}\delta h_{ij} + Z^{\phi}\delta\phi - Y^{ij}\delta(h_{ij,n}) - Y^{\phi}\delta\phi_n \tag{4.12}$$

It then follows that the total action $S_{total} = S+B$ admits a well defined variational principle, since

$$\delta S_{total} = \int_{\mathcal{M}} \varepsilon^{ab}\delta g_{ab} + \varepsilon^{\phi}\delta\phi + \int_{\partial\mathcal{M}} J^{ij}\delta h_{ij} + J^{\phi}\delta\phi \tag{4.13}$$

where $J^{ij} = X^{ij} + Z^{ij}$ and $J^{\phi} = X^{\phi} + Z^{\phi}$. Now, it is immediately clear that the choice of B is not unique: if B is a good boundary term, then so is $B + \eta[h_{ij}, \phi]$, since the variation of η acts only to renormalise Z^{ij} and Z^{ϕ}. The same is of course true for the Gibbons-Hawking term in General Relativity. To eliminate this ambiguity, we impose a minimal construction, requiring that $B \to 0$ as both $h_{ij,n} \to 0$ and $\phi_n \to 0$.

The junction conditions across a domain wall, $\Sigma \in \mathcal{M}$, can now be derived in one of two ways. The first is to treat the wall as a delta-function source in the field equations. A completely equivalent approach, and the one we will adopt here, is to note that the wall splits the manifold $\mathcal{M}$ into two manifolds, $\mathcal{M}_+$ and $\mathcal{M}_-$, and is treated as the common boundary to each. Of course, this statement neglects the contribution of boundary components far away from the wall, since they play no role here. The action describing the system is given by

$$S_{DW} = S^+_{total} + S^-_{total} + S_{\Sigma} \tag{4.14}$$

where $S^{\pm}_{total}$ is the total action defined on $\mathcal{M}_{\pm}$ with boundary $\partial\mathcal{M}_{\pm}$. Variation of the full action yields

$$\delta S_{DW} = \int_{\mathcal{M}_+} \varepsilon^{ab}\delta g_{ab} + \varepsilon^{\phi}\delta\phi + \int_{\partial\mathcal{M}_+} J^{ij}\delta h_{ij} + J^{\phi}\delta\phi \\ + \int_{\mathcal{M}_-} \varepsilon^{ab}\delta g_{ab} + \varepsilon^{\phi}\delta\phi + \int_{\partial\mathcal{M}_-} J^{ij}\delta h_{ij} + J^{\phi}\delta\phi + \int_{\Sigma} \frac{1}{\sqrt{-h}}\frac{\delta S_{\Sigma}}{\delta h_{ij}}\delta h_{ij} + \frac{1}{\sqrt{-h}}\frac{\delta S_{\Sigma}}{\delta\phi}\delta\phi \tag{4.15}$$

Now because of the orientation, it is clear that $\int_{\partial\mathcal{M}_+} = -\int_{\partial\mathcal{M}_-} = \int_\Sigma$. It follows that

$$\delta S_{DW} = \int_{\mathcal{M}_+\cup\mathcal{M}_-} \varepsilon^{ab}\delta g_{ab} + \varepsilon^\phi \delta\phi + \int_\Sigma \left(\Delta J^{ij} + \frac{1}{\sqrt{-h}}\frac{\delta S_\Sigma}{\delta h_{ij}}\right)\delta h_{ij} + \left(\Delta J^\phi + \frac{1}{\sqrt{-h}}\frac{\delta S_\Sigma}{\delta\phi}\right)\delta\phi \tag{4.16}$$

where $\Delta Q = Q_{\partial\mathcal{M}_+} - Q_{\partial\mathcal{M}_-}$. The resulting junction conditions are given by the continuity relations $\Delta h_{ij} = \Delta\phi = 0$ and the analogue of the Israel equations,

$$\Delta J^{ij} = -\frac{1}{\sqrt{-h}}\frac{\delta S_\Sigma}{\delta h_{ij}}, \qquad \Delta J^\phi = -\frac{1}{\sqrt{-h}}\frac{\delta S_\Sigma}{\delta\phi} \tag{4.17}$$

Note that the continuity relations ensure that equations (4.17) are invariant under $B \to B + \eta[h_{ij}, \phi]$.

We shall now demonstrate explicitly how this methodology was applied to the first two terms in (4.1). We begin with the k-essence term [116], $S_k = \int_\mathcal{M} k(\phi, X)$. Variation yields

$$\delta S_k = \int_\mathcal{M} \frac{1}{2}\left[k_X \nabla^a\phi\nabla^b\phi + k g^{ab}\right]\delta g_{ab} + \left[k_\phi + \nabla_a(k_X\nabla^a\phi)\right]\delta\phi + \int_{\partial\mathcal{M}} -k_X\phi_n\delta\phi \tag{4.18}$$

Because there were no second derivatives in S_k this piece of the action is already well defined, and there is no need to add a boundary term. The contribution to the equations of motion and junction conditions can be immediately read off:

$$\varepsilon_k^{ab} = \frac{1}{2}\left[k_X\nabla^a\phi\nabla^b\phi + kg^{ab}\right] \qquad \varepsilon_k^\phi = k_\phi + \nabla_a(k_X\nabla^a\phi) \tag{4.19}$$

$$J_k^{ij} = 0 \qquad J_k^\phi = \Delta\left[-k_X\phi_n\right] \tag{4.20}$$

Next we consider the second term in the DGSZ action (4.1), $S_3 = -\int_\mathcal{M} G_3(\phi, X)\Box\phi$. We shall perform the variation with respect to ϕ and g_{ab} separately. Starting with the ϕ variation, we find,

$$\begin{aligned}\delta_\phi S_3 &= \int_\mathcal{M}\Big\{ -G_{3\phi}\Box\phi - (G_{3X}\phi^a)_{;a}\Box\phi - G_{3X}\phi_b\nabla^b\nabla^c\nabla_c\phi - \Box G_3\Big\}\delta\phi \\ &\quad + \int_{\partial\mathcal{M}} \left[G_{3X}\phi_n\Box\phi + G_{3n}\right]\delta\phi - G_3\delta\phi_n \end{aligned} \tag{4.21}$$

where $G_{3n} \equiv n^a \nabla_a G_3$. The boundary terms contain the problematic contribution from $\delta\phi_n$. To cancel this off, we add the following:

$$B_3 = \int_{\partial\mathcal{M}} F_3(\phi, Y, \phi_n) \tag{4.22}$$

where $Y = -\frac{1}{2}h^{ij}\partial_i\phi\partial_j\phi$ is the boundary analogue of X, and

$$F_3(\phi, Y, \phi_n) = \int_0^{\phi_n} dx \; G_3\left(\phi, Y - \frac{1}{2}sx^2\right) \tag{4.23}$$

To see that this works, we note that

$$\delta_\phi B_3 = \int_B G_3\delta\phi_n + \left[F_{3\phi} - (F_{3Y}\phi^i)_{;i}\right]\delta\phi \tag{4.24}$$

The ϕ variation of the completed action is well behaved, and yields

$$\delta_\phi(S_3 + B_3) = \int_{\mathcal{M}} \varepsilon_3^\phi \delta\phi + \int_{\partial\mathcal{M}} J_3^\phi \delta\phi \tag{4.25}$$

where

$$\varepsilon_3^\phi = -G_{3\phi}\Box\phi - \left(G_{3X}\phi^b\right)_{;b}\Box\phi + G_{3X}R_{ab}\phi^a\phi^b + (G_{3X}\phi^a)^{;b}\phi_{ab} - (G_{3\phi}\phi_a)^{;a} \tag{4.26}$$

$$J_3^\phi = G_{3X}C\phi_n + G_{3\phi}\phi_n + G_{3X}K_{ij}\phi^i\phi^j - F_{3YY}\phi^i\phi^j\phi_{ij} + F_{3Y}\bar{\Box}\phi + F_{3\phi} + F_{3Y\phi}\phi_i\phi^i$$

A few comments are in order here. In arriving at the expression for ε_3^ϕ we have eliminated the apparent third derivative terms using the Riemann identity, giving

$$-G_{3X}\phi^b\nabla_b\nabla^c\nabla_c\phi - \Box G_3 = G_{3X}R_{ab}\phi^a\phi^b + (G_{3X}\phi^b)^{;a}\phi_{ab} - (G_{3\phi}\phi_a)^{;a} \tag{4.27}$$

This serves as a good check of our calculation as we know that the equations of motion are second order. Note that we sometimes denote covariant derivatives using superscripts and subscripts, ie $\phi_a = \nabla_a\phi$, $\phi^a = \nabla^a\phi$ etc. Covariant derivatives along the normal direction attain the super/subscript n, ie $\phi_n = n^a\phi_a, \phi_{nn} = n^a n^b\phi_{ab}$.

Similarly, the final expression for J_3^ϕ has made use of the following identity

$$G_{3n} \equiv n^a\nabla_a G_3 = G_{3\phi}\phi_n - sG_{3X}\phi_{nn} - G_{3X}\phi_{ni}\phi^i + G_{3X}K_{ij}\phi^{ij} \tag{4.28}$$

where K_{ij} is the extrinsic curvature of the boundary, defined as the Lie derivative of the induced metric with respect to the normal

$$K_{ij} = \frac{1}{2}\mathcal{L}_n h_{ij} \tag{4.29}$$

We also introduce the covariant derivative on the boundary, $\bar{D}_i$, which we will sometimes denote using superscripts and subscripts, as with the bulk covariant derivative, ie $\phi_i = \bar{D}_i\phi$, $\phi^i = \bar{D}^i\phi$. The covariant d'Alembertian on the boundary is written as $\bar{\Box} = \bar{D}_i\bar{D}^i$, while the boundary scalar C is defined as the trace $C = h^{ij}C_{ij}$, where

$$C_{ij} = \bar{D}_i\bar{D}_j\phi + s\phi_n K_{ij} \tag{4.30}$$

In other words $C = \bar{\Box}\phi + s\phi_n K$ where $K = h^{ij}K_{ij}$. Further details of the useful formulae used in our derivations can be found in appendix B.1. Once again we note that J_3^ϕ contains no more than second derivatives along the boundary, and first derivatives along the normal. This is to be expected for a second order system in the bulk.

We now consider the variation of S_3 with respect to the metric g_{ab}. This gives,

$$\begin{aligned}\delta_g S_3 &= \int_{\mathcal{M}} -\frac{1}{2}\left[G_3\Box\phi g^{ab} + G_{3X}\Box\phi\phi^a\phi^b + G_3^{;a}\phi^b + G_3^{;b}\phi^a - (G_3\phi^c)_{;c}\, g^{ab}\right]\delta g_{ab} \\ &\quad + \int_{\partial\mathcal{M}} -\frac{1}{2}G_3\phi_n h^{ij}\delta h_{ij} \end{aligned} \tag{4.31}$$

Although the metric variation does not lead to any troublesome boundary terms, we must account for any additional contributions coming from B_3. The metric variation of B_3 yields

$$\delta_g B_3 = \int_{\partial\mathcal{M}} \frac{1}{2}\left[F_3 h^{ij} + F_{3Y}\phi^i\phi^j\right]\delta h_{ij} \tag{4.32}$$

It follows that

$$\delta_g(S_3 + B_3) = \int_{\mathcal{M}} \varepsilon_3^{ab}\delta g_{ab} + \int_{\partial\mathcal{M}} J_3^{ij}\delta h_{ij} \tag{4.33}$$

where

$$\varepsilon_3^{ab} = -\frac{1}{2}\left[G_3\Box\phi g^{ab} + G_{3X}\Box\phi\phi^a\phi^b + G_3^{;a}\phi^b + G_3^{;b}\phi^a - (G_3\phi^c)_{;c}\, g^{ab}\right] \tag{4.34}$$

$$J_3^{ij} = \frac{1}{2}\left[F_3 h^{ij} + F_{3Y}\phi^i\phi^j - G_3\phi_n h^{ij}\right] \tag{4.35}$$

Again we see that the metric equations of motion are second order in the bulk, and the junctions conditions contain no more than second derivatives along the boundary, and first derivatives along the normal.

Analogous calculations were applied to the remaining terms in the DGSZ action, which we denote

$$S_4 = \int_{\mathcal{M}} G_4(\phi, X)R + G_{4X}\nabla^a_{[a}\phi\nabla^b_{b]}\phi \tag{4.36}$$

$$S_5 = \int_{\mathcal{M}} G_5(\phi, X)G_{ab}\nabla^a\nabla^b\phi - \frac{G_{5X}}{6}\nabla^a_{[a}\phi\nabla^b_b\phi\nabla^c_{c]}\phi \tag{4.37}$$

However, the algebra is extremely long so we shall not present it here, being content to present the results in the next section.

4.4 Boundary terms and junction conditions for Horndeski theory

In this section we shall simply quote the results of lengthy calculations, as described in the previous section. Our starting point is the DGSZ action for a general scalar-tensor theory [108], which is equivalent to Horndeski's original theory [107] in four dimensions [109]. Let us repeat the form of this action in order to make this section self-contained:

$$S[g_{ab}, \phi] = \int_{\mathcal{M}} \Big[k(\phi, X) - G_3(\phi, X)\Box\phi + G_4(\phi, X)R + G_{4X}\nabla^a_{[a}\phi\nabla^b_{b]}\phi + G_5(\phi, X)G_{ab}\nabla^a\nabla^b\phi - \frac{G_{5X}}{6}\nabla^a_{[a}\phi\nabla^b_b\phi\nabla^c_{c]}\phi\Big]$$

where $X = -\frac{1}{2}(\nabla\phi)^2$. Recall that the antisymmetrisation does not include the usual factor of $1/n!$, and that the covariant measure on the manifold is omitted for brevity. In order to admit a well defined variational principle under Dirichilet boundary conditions, this action must be supplemented by the following boundary term

$$B[h_{ij}, \phi, h_{ij,n}, \phi_n] = \sum_{\alpha=3}^{5} B_\alpha[h_{ij}, \phi, h_{ij,n}, \phi_n] \tag{4.38}$$

where

$$B_3 = \int_{\partial\mathcal{M}} F_3 \tag{4.39}$$
$$B_4 = \int_{\partial\mathcal{M}} 2(G_4 K - F_{4Y}\phi^i_i)$$
$$B_5 = \int_{\partial\mathcal{M}} -\frac{1}{2}sG_5 K^{[i}_i K^{j]}_j \phi_n - G_5 \phi^{[i}_i K^{j]}_j + \frac{1}{2}\bar{R}F_5 + \frac{1}{2}F_{5Y}\phi^{[i}_i \phi^{j]}_j$$

Here we define

$$F_\alpha(\phi, Y, \phi_n) = \int_0^{\phi_n} dx\ G_\alpha\left(\phi, Y - \frac{1}{2}sx^2\right), \qquad Y = -\frac{1}{2}\phi_i\phi^i \tag{4.40}$$

from which it follows that $\frac{\partial F_\alpha}{\partial \phi_n} = G_\alpha$. Note that any curvature terms with an "overbar" correspond to *boundary* curvatures, eg $\bar{R}_{ijkl}$ is the boundary Riemann tensor, $\bar{G}_{ij}$ is the boundary Einstein tensor, $\bar{R}$ is the boundary Ricci scalar, etc etc. Of course, if the "overbar" is absent, it corresponds to a bulk curvature.

Variation of the full action, $S_{total} = S + B$ now yields(see Appendix (B) for the identities used in the computation),

$$\delta S_{total} = \int_{\mathcal{M}} \varepsilon^{ab}\delta g_{ab} + \varepsilon^\phi \delta\phi + \int_{\partial\mathcal{M}} J^{ij}\delta h_{ij} + J^\phi \delta\phi \tag{4.41}$$

where the bulk equations of motion are given by

$$\varepsilon^{ab} = \frac{1}{2}(\mathcal{E}^{ab} + \mathcal{E}^{ba}), \qquad \mathcal{E}^{ab} = \mathcal{E}^{ab}_k + \sum_{\alpha=3}^{5}\mathcal{E}^{ab}_\alpha; \qquad \varepsilon^\phi = \varepsilon^\phi_k + \sum_{\alpha=3}^{5}\varepsilon^\phi_\alpha \tag{4.42}$$

with

$$
\begin{aligned}
\mathcal{E}_k^{ab} &= \frac{1}{2}(k_X\phi^a\phi^b + kg^{ab}) && (4.43)\\
\mathcal{E}_3^{ab} &= -\frac{1}{2}\left[G_3\Box\phi g^{ab} + G_{3X}\Box\phi\phi^a\phi^b + 2G_3^{;a}\phi^b - (G_3\phi^c)_{;c}\, g^{ab}\right] && (4.44)\\
\mathcal{E}_4^{ab} &= \frac{1}{2}\left(g^{ab}G_{4X}\phi_f^{[f}\phi_g^{g]} + G_{4X}R\phi^a\phi^b - 2G_4G^{ab} + G_{4XX}\phi_f^{[f}\phi_g^{g]}\phi^a\phi^b\right) - (G_{4\phi}\phi_c)^{;[c}g^{a]b} \\
&\quad +(G_{4X}\phi_d)^{;[c}g^{a]b}\phi_c^d + G_{4X}\phi^d g^{a[b}R^{c]}_{dce}\phi^e + 2G_{4X}^{;[a}\phi_c^{c]}\phi^b - 2G_{4X}R_c^a\phi^b\phi^c && (4.45)\\
&\quad -(G_{4X}\phi^d)_{;d}g^{a[b}\phi_c^{c]}\\
\mathcal{E}_5^{ab} &= \frac{1}{2}\Big[G_5(R^{ab}\Box\phi - R\phi^{ab}) - 4G_5G_c^a\phi^{cb} + 2(G_5\phi^a)_{;d}G^{bd} - (G_5\phi^c)_{;c}G^{ab} && (4.46)\\
&\quad -G_{5X}^{;[a}\phi_c^c\phi_d^{d]}\phi^b + \frac{1}{2}G_{5X;d}\phi^d g^{a[b}\phi_c^c\phi_e^{e]} + \frac{1}{2}G_{5X}(\Box\phi)g^{a[b}\phi_c^c\phi_d^{d]} + G_{5X}G_{cd}\phi^{cd}\phi^a\phi^b\\
&\quad -\frac{1}{6}G_{5XX}\phi_f^{[f}\phi_g^g\phi_h^{h]}\phi^a\phi^b g^{a[b}\phi_c^c\nabla^{d]}(G_{5\phi}\phi_d) - g^{a[b}\phi_c^c\nabla^{d]}(G_{5X}\phi_e)\phi_d^e\\
&\quad -G_{5X}g^{a[b}\phi_c^c R^{d]}{}_{edf}\phi^e\phi^f - G_{5X}\phi^a\phi_c^{[c}R^{bd]}{}_{de}\phi^e + 2G_5^{;a}R_c^b\phi^c - 2G_{5;c}R^{cabd}\phi_d\\
&\quad -2G_{5;c}R^{cd}\phi_d g^{ab} + 2G_5R_d^a\phi^{bd} - \frac{1}{2}G_5R\Box\phi g^{ab} - \frac{G_{5X}}{6}g^{ab}\phi_f^{[f}\phi_g^g\phi_h^{h]}\Big]
\end{aligned}
$$

and

$$
\begin{aligned}
\varepsilon_k^\phi &= k_\phi + (k_X\phi_a)^{;a} && (4.47)\\
\varepsilon_3^\phi &= -G_{3\phi}\Box\phi - \left(G_{3X}\phi^b\right)_{;b}\Box\phi + G_{3X}R_{ab}\phi^a\phi^b + (G_{3X}\phi^a)^{;b}\phi_{ab} - (G_{3\phi}\phi_a)^{;a} && (4.48)\\
\varepsilon_4^\phi &= G_{4\phi}R + (G_{4X}\phi_a)^{;a}R + G_{4X\phi}\phi_f^{[f}\phi_g^{g]} + (G_{4XX}\phi_a)^{;a}\phi_f^{[f}\phi_g^{g]} && (4.49)\\
&\quad -2G_{4XX}\phi_b^{[b}R^{a]}_{cad}\phi^c\phi^d + 2(G_{4X\phi}\phi_a)^{[;a}\phi_b^{b]} - 2(G_{4XX}\phi_c)^{[;a}\phi_b^{b]}\phi_a^c\\
&\quad -4R_{ab}G_{4X}^{;a}\phi^b - 2G_{4X}R_{ab}\phi^{ab}\\
\varepsilon_5^\phi &= G_{5\phi}G_{ab}\phi^{ab} + (G_{5X}\phi_c)^{;c}G_{ab}\phi^{ab} - \frac{1}{6}G_{5X\phi}\phi_f^{[f}\phi_g^g\phi_h^{h]} - \frac{1}{6}(G_{5XX}\phi_c)^{;c}\phi_f^{[f}\phi_g^g\phi_h^{h]}\\
&\quad +(G_{5\phi}\phi^a)^{;b}G_{ab} - (G_{5X}\phi_c)^{;b}\phi^{ac}G_{ab} + G_{5X}R_{abcd}G^{ad}\phi^b\phi^c - \frac{1}{2}(G_{5X\phi}\phi_a)^{;[a}\phi_b^b\phi_c^{c]}\\
&\quad +\frac{1}{2}(G_{5XX}\phi^d)^{;[a}\phi_b^b\phi_c^{c]}\phi_{ad} + \frac{1}{2}G_{5XX}\phi^d\phi^e\phi_a^{[a}\phi_b^b R^{c]}{}_{dce} - G_{5X;a}\phi_b^{[b}R^{ac]}{}_{cd}\phi^d\\
&\quad -\frac{1}{2}G_{5X}R^{[ab}{}_{bd}R_a{}^{c]}{}_{ce}\phi^d\phi^e - G_{5X}\phi_d^{[a}\phi_b^b R_a{}^{c]}{}_c{}^d && (4.50)
\end{aligned}
$$

Note that we have written the bulk equations of motion in a form that is explicitly second order, something that did not appear previously in the literature, as far as we are aware.

As explained in the previous section, the junction conditions (4.17) can be obtained from the boundary equations of motion, which are given by

$$J^{ij} = \frac{1}{2}(\mathcal{J}^{ij} + \mathcal{J}^{ji}), \qquad \mathcal{J}^{ij} = \sum_{\alpha=3}^{5} \mathcal{J}^{ij}_{\alpha}; \qquad J^{\phi}_{\alpha} = J^{\phi}_{k} + \sum_{\alpha=3}^{5} J^{\phi}_{\alpha} \tag{4.51}$$

with

$$\mathcal{J}^{ij}_3 = \frac{1}{2}\left[F_3 h^{ij} + F_{3Y}\phi^i\phi^j - G_3\phi_n h^{ij}\right] \tag{4.52}$$

$$\begin{aligned}\mathcal{J}^{ij}_4 = {} & -G_4(K^{ij} - Kh^{ij}) + G_{4\phi}\phi_n h^{ij} - G_{4X}\phi^k\phi_{nk}h^{ij} + G_{4X}\phi^k\phi^l K_{kl}h^{ij} + 2sG_{4X}B^i\phi^j \\ & +G_{4X}\phi_n h^{i[j}C^{k]}_k + G_{4X}K\phi^i\phi^j - F_{4YY}\phi^i\phi^j\bar{\Box}\phi - 2F^{;i}_{4Y}\phi^j + F_{4Y;k}\phi^k h^{ij}\end{aligned} \tag{4.53}$$

$$\mathcal{J}^{ij}_5 = \frac{1}{2}\Bigg[-\frac{1}{2}sG_{5X}K^{[k}_k K^{l]}_l\phi^i\phi^j\phi_n - G_{5X}\phi^{[k}_k K^{l]}_l\phi^i\phi^j - 2G^{;[i}_5 K^{k]}_k\phi^j \tag{4.54}$$

$$\begin{aligned} & +G_{5;k}\phi^k h^{i[j}K^{l]}_l - F_5\bar{G}^{ij} + \frac{1}{2}F_{5Y}\bar{R}\phi^i\phi^j + \frac{1}{2}F_{5YY}\phi^{[k}_k\phi^{l]}_l\phi^i\phi^j + 2F^{[;i}_{5Y}\phi^{k]}_k\phi^j \\ & +2sG_{5;k}B^k h^{ij} - 2sG^{;i}_5 B^j - \phi_{nk}G^{;[k}_5 h^{i]j} + (F_{5Y}\phi_k)^{[;l}h^{i]j}\phi^k_l - (F_{5Y}\phi^k)_{;k}h^{i[j}\phi^{l]}_l \\ & +F_{5Y}\phi^k\bar{R}_{klm}{}^{[l}h^{i]j}\phi^m + G_5\phi_n\bar{G}^{ij} - G_{5\phi}\phi_n h^{i[j}C^{k]}_k \\ & +G_{5X}\phi_{nk}\phi^k h^{i[j}C^{l]}_l - G_{5X}K_{kl}\phi^k\phi^l h^{i[j}C^{m]}_m - 2sG_{5X}\phi^i B^{[j}C^{k]}_k + sG_{5X}\phi_n h^{i[j}B^{k]}B_k \\ & -\frac{1}{2}G_{5X}\phi_n h^{i[j}C^k_k C^{l]}_l + \frac{1}{2}F_{5Y}\phi^{[k}_k\phi^{l]}_l h^{ij} - 2F_{5Y}\bar{R}^i_k\phi^j\phi^k \\ & +h^{ij}\left(-\frac{1}{2}sG_5 K^{[k}_k K^{l]}_l\phi_n - G_5\phi^{[k}_k K^{l]}_l + \frac{1}{2}\bar{R}F_5 + \frac{1}{2}F_{5Y}\phi^{[k}_k\phi^{l]}_l\right)\Bigg]\end{aligned} \tag{4.55}$$

and

$$J_k^\phi = -k_X\phi_n \quad (4.56)$$

$$J_3^\phi = G_{3X}C\phi_n + G_{3\phi}\phi_n + G_{3X}K_{ij}\phi^i\phi^j - F_{3YY}\phi^i\phi^j\phi_{ij} + F_{3Y}\bar{\Box}\phi + F_{3\phi} + F_{3Y\phi}\phi_i\phi^i \quad (4.57)$$

$$\begin{aligned} J_4^\phi &= -G_{4X}\phi_n(\bar{R} - sK_i^{[i}K_j^{j]}) - G_{4XX}\phi_n[-2sB^iB_i + C_i^{[i}C_j^{j]}] + 4sG_{4X;i}B^i \\ &\quad -2G_{4X\phi}\phi_nC + 2G_{4XX}C\phi_{ni}\phi^i - 2G_{4XX}CK_{ij}\phi^i\phi^j - 2G_{4X}K_{ij}\phi^{ij} \\ &\quad +2G_{4\phi}K + 2(G_{4X}\phi_i)^{;i}K - 2(F_{4YY}\phi_i)^{;i}\bar{\Box}\phi + 2F_{4YY}\bar{R}_{ij}\phi^i\phi^j \\ &\quad -2G_{4X;i}\phi_n^i + 2(F_{4YY}\phi^i)_{;j}\phi_i^j - 2(F_{4Y\phi}\phi_i)^{;i} - 2F_{4Y\phi}\bar{\Box}\phi \end{aligned} \quad (4.58)$$

$$\begin{aligned} J_5^\phi &= -sG_{5X;i}B^{[i}C_j^{j]} - G_{5X}C^{ij}\phi_n\left(\bar{G}_{ij} - s\left[KK_{ij} - 2K_{ik}K_j^k - \frac{1}{2}h_{ij}(K^2 + K_{kl}K^{kl})\right]\right) \\ &\quad -G_{5\phi}\phi^i(K_{ij}{}^{;j} - K_{;i}) + \frac{1}{2}(G_{5\phi}\phi_n - sG_{5X}\phi^iB_i)(\bar{R} - sK_k^{[k}K_l^{l]}) + G_{5X}C^{ij}\phi_j(K_{ik}{}^{;k} \\ &\quad -K_{;i}) + \frac{1}{6}G_{5XX}C_i^{[i}C_j^jC_k^{k]}\phi_n + \frac{1}{2}G_{5\phi X}\phi_nC_i^{[i}C_j^{j]} - \frac{1}{2}sG_{5XX}\phi_iB^iC_k^{[k}C_l^{l]} \\ &\quad -sG_{5\phi X}\phi_iB^{[i}C_i^{i]} + sG_{5XX}\phi_iC_j^iB^{[j}C_k^{k]} - G_{5X}C\phi^i(K_{ij}{}^{;j} - K_{;i}) + sG_{5X}C\phi_nK_{ij}K^{ij} \\ &\quad +sG_{5X}\phi^iB^j(\bar{R}_{ij} - sKK_{ij} + 2sK_{ik}K_j^k) + G_{5X}C^{ij}\phi^kK_{i[k;j]} - G_{5X}B^i\phi^jK_{ik}K_j^k \\ &\quad +sG_{5X}\phi_nK_{ik}K^{kj}C_j^i - \frac{1}{2}sG_{5\phi}K_i^{[i}K_j^{j]}\phi_n - \frac{1}{2}s(G_{5X}K_i^{[i}K_j^{j]}\phi_n\phi_k)^{;k} - G_{5\phi}\phi_i^{[i}K_j^{j]} \\ &\quad -(G_{5X}\phi_i)^{;i}\phi_j^{[j}K_k^{k]} - G_{5X}\phi_i^{[i}K^{j]}{}_{j;k}\phi^k + \frac{1}{2}(F_{5Y}\phi_i)^{;i}\bar{R} - (G_{5\phi}\phi_i)^{[;i}K^{j]}{}_j \\ &\quad +(G_{5X}\phi^i)^{[;j}K^{k]}{}_k\phi_{ij} + s(G_{5X}\phi_n)^{[;i}K_j^{j]}\phi_{ni} - 2G_5^{[;i}K^{j]}{}_{j;i} + \frac{1}{2}(F_{5YY}\phi_i)^{;[i}\phi_j^j\phi_k^{k]} \\ &\quad +G_{5X;i}\phi_n{}^{;[i}\phi_j^{j]} + G_{5X}(\phi^iK_{ij})^{[;j}C_k^{k]} - G_{5X}B^{[i}K^{j]}{}_{j;i}\phi_n - G_{5X}K^{[i}{}_i\bar{R}_{jk}{}^{k]l}\phi^j\phi_l \\ &\quad +F_{5YY}\phi_i^{[i}\bar{R}_{jk}{}^{k]l}\phi^j\phi_l - F_{5Y}\bar{R}_{ij}\phi^{ij} - 2F_{5Y;i}\phi_j\bar{R}^{ij} + sG_{5X}\bar{R}_{ij}B^i\phi^j \\ &\quad -G_{5;i}(K^{ij}{}_{;j} - K^{;i}) + (F_{5Y\phi}\phi_i)^{[;i}\phi_j^{j]} + \frac{1}{2}F_{5\phi}\bar{R} + \frac{1}{2}F_{5Y\phi}\phi_i^{[i}\phi_j^{j]} - G_{5X}\phi_{ni}B^{[i}K_j^{j]} \end{aligned} \quad (4.59)$$

Here we recall that

$$C_{ij} = \bar{D}_i\bar{D}_j\phi + s\phi_nK_{ij}, \qquad C = h^{ij}C_{ij} = \bar{\Box}\phi + s\phi_nK \quad (4.60)$$

and we introduce the boundary vector

$$B_i = s\bar{D}_i\phi_n - sK_{ij}\bar{D}^j\phi \quad (4.61)$$

Note also that $\phi_{ni} = \bar{D}_i\phi_n$. The formulae for J^{ij} and J^ϕ have been written so that they are explictly second order in boundary derivatives, and first order in normal derivatives.

4.5 Examples

We shall now present the boundary terms and junction conditions for certain important subclasses of Horndeski's theory, specifically: General Relativity (as a check), Brans-Dicke gravity [103], covariant galileon theory [112], and the original flat space galileon theory [111]. Of course, one can use the results of the previous section to infer the boundary terms and junction conditions for many other theories such as *the Fab Four* [119], DBI theories [79], conformal galileon [111], KGB theories [117] and so on.

4.5.1 General Relativity

General Relativity is perhaps the most "special" special case of Horndeski's theory, corresponding to the choice

$$G_4 = \frac{1}{16\pi G}, \qquad k = G_3 = G_5 = 0$$

so that the bulk action is given by the standard Einstein-Hilbert action

$$S = \frac{1}{16\pi G}\int_{\mathcal{M}} R \tag{4.62}$$

and, as expected, the boundary term is given by the Gibbons-Hawking term [73, 74]

$$B = \frac{1}{8\pi G}\int_{\partial\mathcal{M}} K \tag{4.63}$$

The bulk equations of motion are simply the Einstein tensor

$$\varepsilon^{ab} = -\frac{1}{16\pi G}G^{ab} \tag{4.64}$$

while the boundary equations of motion take the form expected from the Israel junction conditions [75]

$$J^{ij} = -\frac{1}{16\pi G}\left(K^{ij} - Kh^{ij}\right) \tag{4.65}$$

4.5.2 Brans-Dicke theory

Brans-Dicke theory [103] is the most well studied scalar-tensor theory, and corresponds to the choice

$$k = \frac{\omega}{8\pi\phi}X \qquad G_4 = \frac{\phi}{16\pi}, \qquad G_3 = G_5 = 0 \tag{4.66}$$

so that the bulk action is given by

$$S = \frac{1}{16\pi}\int_{\mathcal{M}} \phi R - w\frac{(\nabla\phi)^2}{\phi} \tag{4.67}$$

and the boundary term by

$$B = \frac{1}{8\pi}\int_{\partial\mathcal{M}} \phi K \tag{4.68}$$

The bulk equations of motion are the usual Brans-Dicke field equations

$$\begin{aligned} \varepsilon^{ab} &= -\frac{1}{16\pi}\left[\phi G^{ab} + g^{ab}\Box\phi - \phi^{ab} - \frac{\omega}{\phi}\left(Xg^{ab} + \phi^a\phi^b\right)\right] && (4.69)\\ \varepsilon^{\phi} &= \frac{1}{16\pi}\left[R + 2w\left(\frac{\Box\phi}{\phi} + \frac{X}{\phi^2}\right)\right] && (4.70)\end{aligned}$$

while the boundary equations of motion are

$$\begin{aligned} J^{ij} &= \frac{1}{16\pi}\left[-\phi\left(K^{ij} - Kh^{ij}\right) + \phi_n h^{ij}\right] && (4.71)\\ J^{\phi} &= \frac{1}{8\pi}\left[K - \frac{\omega}{\phi}\phi_n\right] && (4.72)\end{aligned}$$

It is easy to check that these are consistent with the junction conditions presented in [81].

4.5.3 Covariant galileon

Covariant galileon theory [112] was developed in order to couple the original galileon theory [111] to gravity without introducing any new higher derivatives.

We will ignore the historical timeline and begin by discussing the covariant model because the flat space galileon is easily obtained by decoupling the graviton. Static spherically symmetric thin shells for the covariant galileon, up to cubic order, were studied in [82] in order to explore aspects of the Vainshtein mechanism [126]. This suggests that the following formulae will ultimately lend themselves to undertstanding screening mechanisms in modified gravity.

The covariant galileon theory corresponds to the choice,

$$k = c_2 X, \qquad G_3 = -c_3 X, \qquad G_4 = \frac{1}{2} c_4 X^2, \qquad G_5 = -3c_5 X^2 \tag{4.73}$$

where c_i are constant. This gives the bulk action,

$$S = \int_{\mathcal{M}} c_2 X + c_3 X \Box\phi + c_4 X \left(\phi_f^{[f} \phi_g^{g]} + \frac{1}{2} X R \right) + c_5 X \left(\phi_f^{[f} \phi_g^{g} \phi_h^{h]} - 3X G_{ab} \phi^{ab} \right) \tag{4.74}$$

and the boundary term

$$B = \int_{\partial\mathcal{M}} \left(c_3 + 2c_4 \bar{\Box}\phi + 3c_5 \phi_f^{[f} \phi_g^{g]} \right) \phi_n \left(\frac{1}{6} s \phi_n^2 - Y \right) - \frac{3}{2} c_5 \bar{R} \phi_n \left[\left(\frac{1}{6} s\phi_n^2 - Y \right)^2 + \frac{\phi_n^4}{45} \right] + c_4 X^2 K + \frac{3}{2} c_5 X^2 \left(s K_i^{[i} K_j^{j]} \phi_n + 2\phi_i^{[i} K_j^{j]} \right) \tag{4.75}$$

where we recall that $Y = -\frac{1}{2}\phi_i \phi^i$ is the boundary analogue of X. The bulk equations of motion now give $\varepsilon^{ab} = \frac{1}{2}(\mathcal{E}^{ab} + \mathcal{E}^{ba})$ with

$$
\begin{aligned}
\mathcal{E}^{ab} \;=\; & \frac{1}{2}c_2(\phi^a\phi^b + Xg^{ab}) \\
& + c_3\left[\frac{1}{2}\Box\phi\left[\phi^a\phi^b + Xg^{ab}\right] + X^{(;a}\phi^{b)} - \frac{1}{2}\left(X\phi^c\right)_{;c}g^{ab}\right] \\
& + c_4\Big[\frac{1}{2}\big(g^{ab}X\phi_f^{[f}\phi_g^{g]} + XR\phi^a\phi^b - X^2G^{ab} + \phi_f^{[f}\phi_g^{g]}\phi^a\phi^b\big) \\
& \quad + (X\phi_d)^{;[c}g^{a]b}\phi_c^d + X\phi^d g^{a[b}R^{c]}_{dce}\phi^e + 2X^{;[a}\phi_c^{c]}\phi^b - 2XR_c^a\phi^b\phi^c - (X\phi^d)_{;d}g^{a[b}\phi_c^{c]}\Big] \\
& + \frac{1}{2}c_5\Big[-3X^2(R^{ab}\Box\phi - R\phi^{ab}) + 12X^2G_c^a\phi^{cb} - 6(X^2\phi^a)_{;d}G^{bd} \\
& \quad + 3(X^2\phi^c)_{;c}G^{ab} + 6X^{;[a}\phi_c^c\phi_d^{d]}\phi^b \\
& \quad - 3X_{;d}\phi^d g^{a[b}\phi_c^c\phi_e^{e]} - 3X\Box\phi g^{a[b}\phi_c^c\phi_d^{d]} - 6XG_{cd}\phi^{cd}\phi^a\phi^b + \phi_f^{[f}\phi_g^g\phi_h^{h]}\phi^a\phi^b \\
& \quad + 6g^{a[b}\phi_c^c\nabla^{d]}(X\phi_e)\phi_d^e + 6Xg^{a[b}\phi_c^cR^{d]}{}_{edf}\phi^e\phi^f + 6X\phi^a\phi_c^{[c}R^{bd]}{}_{de}\phi^e - 6(X^2)^{;a}R_c^b\phi^c \\
& \quad + 6(X^2)_{;c}R^{cabd}\phi_d + 6(X^2)_{;c}R^{cd}\phi_dg^{ab} - 6X^2R_d^a\phi^{bd} + \frac{3}{2}X^2R\Box\phi g^{ab} \\
& \quad + Xg^{ab}\phi_f^{[f}\phi_g^g\phi_h^{h]}\Big]
\end{aligned}
\tag{4.76}
$$

and

$$
\begin{aligned}
\varepsilon^{\phi} \;=\; & c_2\Box\phi \\
& + c_3\left[(\Box\phi)^2 - R_{ab}\phi^a\phi^b - \phi^{ab}\phi_{ab}\right] \\
& + c_4\left[(X\phi_a)^{;a}R + \Box\phi\phi_f^{[f}\phi_g^{g]} - 2\phi_b^{[b}R^{a]}_{cad}\phi^c\phi^d - 2\phi_c^{[a}\phi_b^{b]}\phi_a^c - 4R_{ab}X^{;a}\phi^b - 2c_4XR_{ab}\phi^{ab}\right] \\
& + c_5\Big[-6(X\phi_c)^{;c}G_{ab}\phi^{ab} + \Box\phi\phi_f^{[f}\phi_g^g\phi_h^{h]} + 6(X\phi_c)^{;b}\phi^{ac}G_{ab} - 6XR_{abcd}G^{ad}\phi^b\phi^c \\
& \quad - 3\phi^{d;[a}\phi_b^b\phi_c^{c]}\phi_{ad} - 3\phi^d\phi^e\phi_a^{[a}\phi_b^bR^{c]}{}_{dce} + 6X_{;a}\phi_b^{[b}R^{ac]}{}_{cd}\phi^d \\
& \quad + 3XR^{[ab}{}_{bd}R_a{}^{c]}{}_{ce}\phi^d\phi^e + 6X\phi_d^{[a}\phi_b^bR_a{}^{c]}{}_c{}^d\Big]
\end{aligned}
\tag{4.77}
$$

We have checked the consistency of these equations with the corresponding formulae presented in [112][2]. The boundary equations of motion for variation of the

[2]The formula for ε^{ab} matches exactly, while the formulae for ε^{ϕ} differ by a term proportional to $XR^{abc}{}_dR_{abce}\phi^d\phi^e$. We believe that [112] contains a typo and that the formula (4.77) presented here is correct.

metric are given by $J^{ij} = \frac{1}{2}\left(\mathcal{J}^{ij} + \mathcal{J}^{ji}\right)$ with,

$$
\begin{aligned}
\mathcal{J}^{ij} \;=\; & \frac{1}{2}c_3\left[Z\phi_n h^{ij} - \phi_n\phi^i\phi^j + X\phi_n h^{ij}\right] \\
& + c_4\Big[-\frac{1}{2}X^2(K^{ij} - Kh^{ij}) - X\phi^k\phi_{nk}h^{ij} + X\phi^k\phi^l K_{kl}h^{ij} + 2sXB^i\phi^j \\
& \quad + X\phi_n h^{i[j}C_k^{k]} + XK\phi^i\phi^j - \phi_n\phi^i\phi^j\bar{\Box}\phi + 2Z\phi_n^{;i}\phi^j - (Z\phi_n)_{;k}\phi^k h^{ij}\Big] \\
& + \frac{1}{2}c_5\Big[3sXK_k^{[k}K_l^{l]}\phi^i\phi^j\phi_n + 6X\phi_k^{[k}K_l^{l]}\phi^i\phi^j + 6(X^2)^{;[i}K_k^{k]}\phi^j \\
& \quad - 3(X^2)_{;k}\phi^k h^{i[j}K_l^{l]} + 3\phi_n\left(Z^2 + \frac{\phi_n^4}{45}\right)\bar{G}^{ij} + 3Z\phi_n\bar{R}\phi^i\phi^j - 3\phi_n\phi_k^{[k}\phi_l^{l]}\phi^i\phi^j \\
& \quad + 12(Z\phi_n)^{[;i}\phi_k^{k]}\phi^j - 6s(X^2)_{;k}B^k h^{ij} + 6s(X^2)^{;i}B^j + 3\phi_{nk}(X^2)^{;[k}h^{i]j} \\
& \quad + 6(Z\phi_n\phi_k)^{[;l}h^{i]j}\phi_l^k - 6(Z\phi_n\phi^k)_{;k}h^{i[j}\phi_l^{l]} \qquad + 6Z\phi_n\phi^k\bar{R}_{klm}{}^{[l}h^{i]j}\phi^m \\
& \quad - 3X^2\phi_n\bar{G}^{ij} - 6X\phi_{nk}\phi^k h^{i[j}C_l^{l]} + 6XK_{kl}\phi^k\phi^l h^{i[j}C_m^{m]} \\
& \quad + 12sX\phi^i B^{[j}C_k^{k]} - 6sX\phi_n h^{i[j}B^{k]}B_k + 3X\phi_n h^{i[j}C_k^k C_l^{l]} + 3Z\phi_n\phi_k^{[k}\phi_l^{l]}h^{ij} \\
& \quad - 12Z\phi_n\bar{R}_k^i\phi^j\phi^k + 3h^{ij}\Big(\frac{1}{2}sX^2K_k^{[k}K_l^{l]}\phi_n \\
& \quad + X^2\phi_k^{[k}K_l^{l]} - \frac{1}{2}\bar{R}\phi_n\left(Z^2 + \frac{\phi_n^4}{45}\right) + Z\phi_n\phi_k^{[k}\phi_l^{l]}\Big)\Big]
\end{aligned}
\tag{4.78}
$$

where $X = -\frac{1}{2}s\phi_n^2 + Y$, $Z = \frac{1}{6}s\phi_n^2 - Y$. The boundary equations of motion for variation of the scalar, meanwhile, are given by

$$
\begin{aligned}
J_\phi &= -c_2\phi_n + c_3\left[-C\phi_n - K_{ij}\phi^i\phi^j - \phi_n\bar{\Box}\phi\right] \\
&\quad +c_4\Big[-X\phi_n(\bar{R} - sK_i^{[i}K_j^{j]}) - \phi_n[-2sB^iB_i + C_i^{[i}C_j^{j]}] + 4sX_{;i}B^i + 2C\phi_{ni}\phi^i \\
&\qquad -2CK_{ij}\phi^i\phi^j - 2XK_{ij}\phi^{ij} + 2(X\phi_i)^{;i}K - 2(\phi_n\phi_i)^{;i}\bar{\Box}\phi \\
&\qquad +2\phi_n\bar{R}_{ij}\phi^i\phi^j - 2X_{;i}\phi_n^i + 2(\phi_n\phi^i)_{;j}\phi_i^j\Big] \\
&\quad +c_5\Bigg[6sX_{;i}B^{[i}C_j^{j]} + 6XC^{ij}\phi_n\left(\bar{G}_{ij} - s\left[KK_{ij} - 2K_{ik}K_j^k - \frac{1}{2}h_{ij}(K^2 + K_{kl}K^{kl})\right]\right) \\
&\qquad +3sX\phi^iB_i(\bar{R} - sK_k^{[k}K_l^{l]}) - 6XC^{ij}\phi_j(K_{ik}{}^{;k} - K_{;i}) - C_i^{[i}C_j^jC_k^{k]}\phi_n \\
&\qquad +3s\phi_iB^iC_k^{[k}C_l^{l]} - 6s\phi_iC_j^iB^{[j}C_k^{k]} + 6XC\phi^i(K_{ij}{}^{;j} - K_{;i}) \\
&\qquad -6sXC\phi_nK_{ij}K^{ij} - 6sX\phi^iB^j(\bar{R}_{ij} - sKK_{ij} + sK_{ik}K_j^k) - 6XC^{ij}\phi^kK_{i[k;j]} \\
&\qquad -6sX\phi_nK_{ik}K^{kj}C_j^i + 3s(XK_i^{[i}K_j^{j]}\phi_n\phi_k)^{;k} + 6(X\phi_i)^{;i}\phi_j^{[j}K_k^{k]} \\
&\qquad +6X\phi_i^{[i}K^{j]}{}_{j;k}\phi^k + 3(Z\phi_n\phi_i)^{;i}\bar{R} - 6(X\phi^i)^{[;j}K^{k]}{}_k\phi_{ij} - 6s(X\phi_n)^{[;i}K_j^{j]}\phi_{ni} \\
&\qquad +6(X^2)^{[;i}K^{j]}{}_{j;i} - 3(\phi_n\phi_i)^{;[i}\phi_j^j\phi_k^{k]} - 6X_{;i}\phi_n{}^{;[i}\phi_j^{j]} - 6X(\phi^iK_{ij})^{[;j}C_k^{k]} + 6XB^{[i}K^{j]}{}_{j;i}\phi_n \\
&\qquad +6XK^{[i}{}_i\bar{R}_{jk}{}^{k]l}\phi^j\phi_l - 6\phi_n\phi_i^{[i}\bar{R}_{jk}{}^{k]l}\phi^j\phi_l - 6Z\phi_n\bar{R}_{ij}\phi^{ij} - 12(Z\phi_n)_{;i}\phi_j\bar{R}^{ij} \\
&\qquad -6sX\bar{R}_{ij}B^i\phi^j + 3(X^2)_{;i}(K^{ij}{}_{;j} - K^{;i}) + 6X\phi_{ni}B^{[i}K_j^{j]}\Bigg]
\end{aligned}
\tag{4.79}
$$

4.5.4 Galileon in flat space

The original galileon theory [111] corresponds to a single scalar field propagating in Minkowksi space, satisfying the "galileon" symmetry $\phi \to \phi + b_\mu x^\mu + c$, where b_μ and c are constants. We can obtain the equations of motion and boundary terms for this theory by taking the limit $g_{\mu\nu} \to \eta_{\mu\nu}$ of the covariant galileon theory. It follows that we recover the (by now) well known galileon action in the bulk [111],

$$
S = \int_{\mathcal{M}} c_2X + c_3X\Box\phi + c_4X\phi_f^{[f}\phi_g^{g]} + c_5X\phi_f^{[f}\phi_g^g\phi_h^{h]} \tag{4.80}
$$

The boundary terms do not simplify as much, and are given by

$$B = \int_{\partial\mathcal{M}} \left(c_3 + 2c_4\bar{\Box}\phi + 3c_5\phi^{[f}_f\phi^{g]}_g\right)\phi_n\left(\frac{1}{6}s\phi_n^2 - Y\right) - \frac{3}{2}c_5\bar{R}\phi_n\left[\left(\frac{1}{6}s\phi_n^2 - Y\right)^2 + \frac{\phi_n^4}{45}\right]$$
$$+ c_4X^2K + \frac{3}{2}c_5X^2\left(\bar{R}\phi_n + 2\phi^{[i}_iK^{j]}_j\right) \quad (4.81)$$

One might be puzzled by the presence of curvature terms in this expression. However, even though the bulk geometry is flat, the same need not be true of the boundary if it corresponds to a non-trivial embedding. That is not to say that there is no simplification whatsoever. Because the bulk is flat, the Gauss-Codazzi relations lead to the following identities,

$$\bar{R}_{ijkl} = sK_{k[i}K_{j]l}$$
$$0 = \bar{D}^jK_{ij} - \bar{D}_iK$$

We have already used the first of these in expressing (4.81).

The bulk equations of motion are the usual galileon equations [111],

$$\varepsilon^\phi = c_2\Box\phi + c_3\phi^{[f}_f\phi^{g]}_g + c_4\phi^{[f}_f\phi^g_g\phi^{h]}_h + c_5\phi^{[f}_f\phi^g_g\phi^h_h\phi^{l]}_l \quad (4.82)$$

while the boundary equations of motion are given by

$$
\begin{aligned}
J_\phi \;=\; & -c_2\phi_n + c_3\left[-C\phi_n - K_{ij}\phi^i\phi^j - \phi_n\bar{\Box}\phi\right] \\
& +c_4\Big[-\phi_n[-2sB^iB_i + C_i^{[i}C_j^{j]}] + 4sX_{;i}B^i + 2C\phi_{ni}\phi^i \\
& \quad -2CK_{ij}\phi^i\phi^j - 2XK_{ij}\phi^{ij} + 2(X\phi_i)^{;i}K - 2(\phi_n\phi_i)^{;i}\bar{\Box}\phi \\
& \quad +2\phi_n\bar{R}_{ij}\phi^i\phi^j - 2X_{;i}\phi_n^i + 2(\phi_n\phi^i)_{;j}\phi_i^j\Big] \\
& +c_5\Bigg[6sX_{;i}B^{[i}C_j^{j]} + 6XC^{ij}\phi_n\left(\bar{G}_{ij} - s\left[KK_{ij} - 2K_{ik}K_j^k - \frac{1}{2}h_{ij}(K^2 + K_{kl}K^{kl})\right]\right) \\
& \quad -C_i^{[i}C_j^jC_k^{k]}\phi_n + 3s\phi_iB^iC_k^{[k}C_l^{l]} - 6s\phi_iC_j^iB^{[j}C_k^{k]} - 6sXC\phi_nK_{ij}K^{ij} - 6XC^{ij}\phi^kK_{i[k;j]} \\
& \quad -6sX\phi_nK_{ik}K^{kj}C_j^i + 3s(XK_i^{[i}K_j^{j]}\phi_n\phi_k)^{;k} + 6(X\phi_i)^{;i}\phi_j^{[j}K_k^{k]} \\
& \quad +6X\phi_i^{[i}K^{j]}{}_{j;k}\phi^k + 3(Z\phi_n\phi_i)^{;i}\bar{R} - 6(X\phi^i)^{[;j}K^{k]}{}_k\phi_{ij} - 6s(X\phi_n)^{[;i}K_j^{j]}\phi_{ni} \\
& \quad +6(X^2)^{[;i}K^{j]}{}_{j;i} - 3(\phi_n\phi_i)^{;[i}\phi_j^j\phi_k^{k]} - 6X_{;i}\phi_n{}^{;[i}\phi_j^{j]} - 6X(\phi^iK_{ij})^{[;j}C_k^{k]} + 6XB^{[i}K^{j]}{}_{j;i}\phi_n \\
& \quad +6XK^{[i}{}_i\bar{R}_{jk}{}^{k]l}\phi^j\phi_l - 6\phi_n\phi_i^{[i}\bar{R}_{jk}{}^{k]l}\phi^j\phi_l - 6Z\phi_n\bar{R}_{ij}\phi^{ij} - 12(Z\phi_n)_{;i}\phi_j\bar{R}^{ij} \\
& \quad -6sX\bar{R}_{ij}B^i\phi^j + 6X\phi_{ni}B^{[i}K_j^{j]}\Bigg]
\end{aligned}
\tag{4.83}
$$

where we recall that C_{ij} and B_i are given by equations (4.60) and (4.61) respectively. Note that all of these formulae agree with those presented in [84], save for the boundary curvature terms. It seems that the possibility of a non-trivial embedding and the resulting boundary curvature was not considered in [84]. It might be interesting to see what effect these additional terms have on the value of the on-shell Hamiltonian calculated in [85] and discussed in chapter 2.

4.6 Summary

By computing the boundary terms and junction conditions for thin shells in Horndeski's theory, we have opened up the possibilty of further applications. The boundary terms, being the analogue of the Gibbons-Hawking term [73, 74] in GR, enable us to apply Euclidean path integral methods to the theory. Armed with the junction conditions one can in principle construct Coleman-De Luccia instantons

[92], and use the well defined action to compute tunnelling rates. Indeed, such analyses may capture salient features of tunnelling within the string landscape [93], at least if we can treat Horndeski's theory as a toy representation.

The junction conditions will also enable us to study collapse of a spherical shell in a large class of modified gravity theories, along the lines initiated for the cubic covariant galileon in [82], helping to develop our understanding of Vainshtein screening. Furthermore, given that our results do not depend on spacetime dimension, we are now in a position to study the dynamics of braneworlds in a Horndeski bulk (for reviews of braneworld gravity, see [105, 106]). In particular it might be interesting to see what effect consistent violation of null energy [86] in the bulk has on the dynamics of the brane, especially in view of [94].

Given the most general scalar-tensor theory that we explored in some detail in this chapter, it is natural to ask, what is the most general multiple-scalar-tensor theory. In the next chapter we do precisely this and take the generalisation scheme one step further, to build a generalised scalar-tensor theory with multiple scalar fields.

Chapter 5

Covariant multi-Galileons and their generalisation

5.1 Introduction

In this chapter we work towards a multi-scalar analogue of Horndeski's theory, describing the most general theory of multiple scalars and a single tensor, admitting second order field equations. We begin with the multi-galileon theory described in D dimensional Minkowski space[1] [120, 121, 122, 123]

$$S_{\text{multi-gal}} = \int_{\mathcal{M}} d^D x \sum_{m=1}^{D+1} \alpha^{i_1 \dots i_m} \pi_{i_1} \partial^{[a_2} \partial_{a_2} \pi_{i_2} \cdots \partial^{a_m]} \partial_{a_m} \pi_{i_m} \tag{5.1}$$

where $\alpha^{i_1 \dots i_m}$ is symmetric, and derive its covariant completion. This is achieved by first minimally coupling the scalars to gravity which generically introduces higher order field equations. To restore the system to second order we add curvature dependent counter terms and arrive at the following

$$S_{\text{cov-multi-gal}} = \int_{\mathcal{M}} d^D x \sqrt{-g} \sum_{m=1}^{D+1} \alpha^{i_1 \dots i_m} \pi_{i_1} \nabla_{a_2} \nabla^{[a_2} \pi_{i_2} \cdots \nabla_{a_m} \nabla^{a_m]} \pi_{i_m} + \sum_{m=3}^{D+1} \sum_{n=1}^{\lfloor \frac{m-1}{2} \rfloor} C_n^m \tag{5.2}$$

[1] Throughout this thesis, antisymmetrization omits the usual factor of $1/n!$

where

$$C_n^m = \left(-\frac{1}{4}\right)^n \frac{(m-1)!}{(m-2n-1)!(n!)^2}\alpha^{i_1\ldots i_m}\pi_{i_1}X_{i_2i_3}\cdots X_{i_{2n}i_{2n+1}}$$
$$\times\nabla_{a_{2n+2}}\nabla^{[a_{2n+2}}\pi_{i_{2n+2}}\cdots\nabla_{a_m}\nabla^{a_m}\pi_{i_m}R^{b_1c_1}{}_{b_1c_1}\cdots R^{b_nc_n]}{}_{b_nc_n} \qquad (5.3)$$

for $n > 0$. We have also defined $X_{ij} = \frac{1}{2}\nabla_a\pi_i\nabla^a\pi_j$, using i, j, k to label the scalar, and a, b, c to label the spacetime indices. For N scalars, i, j, k run from $1 \ldots N$, and in D dimensions a, b, c run from $0 \ldots D-1$.

The covariant mutli-galileon theory (5.2) describes a multiple scalar-tensor theory, with potentially interesting applications, ranging from multi-field galileon inflation [124] to covariant self-tuning scenarios (see section 5.4). In the case of a single scalar field, our theory does not quite reduce to the covariant galileon theory presented in [112] owing to the fact that the flat-space Lagrangians differ by a total derivative and this affects the subsequent covariant completion. Of course, both versions of the covariant single galileon still correspond to a subset of Horndeski's theory. The derivation of our covariant multi-galileon theory is presented in section 5.2, with some details postponed to appendix (C). The appendix also includes the resulting field equations.

In section 5.3 we begin to generalise this theory, with a view to deriving a multi-scalar version of Horndeski. Using methods similar to those presented in [108], we first introduce the following generalised multi-galileon theory,

$$S_{\text{multi-scalar}} = \int_{\mathcal{M}} d^Dx\; A(\bar{X}_{ij}, \pi_l) + \sum_{m=1}^{D-1} A^{k_1\ldots k_m}(\bar{X}_{ij}, \pi_l)\partial^{[a_1}\partial_{a_1}\pi_{k_2}\cdots\partial^{a_m]}\partial_{a_m}\pi_{k_m} \qquad (5.4)$$

where $\bar{X}_{ij} = \frac{1}{2}\partial_a\pi_i\partial^a\pi_j$. We prove in chapter 6, that it is the most general multi-scalar theory defined on Minkowski space, preserving second order field equations, provided $\frac{\partial A^{i_1\ldots i_m}}{\partial \bar{X}_{kl}}$ is symmetric in *all* of its indices $i_1, \ldots i_m, k, l$. We can covariantise this theory in the way described earlier, thereby arriving at the following

$$S_{\text{cov-multi-scalar}} = \int_{\mathcal{M}} d^D x\sqrt{-g}\; A(X_{ij},\pi_l) + A^k(X_{ij},\pi_l)\Box\pi_k$$
$$+\sum_{m=2}^{D-1}\frac{(-4)^{\bar{n}}\bar{n}!(m-2\bar{n})!}{m!}\left[\frac{\partial^{\bar{n}}}{\partial X_{k_1k_2}\cdots\partial X_{k_{2\bar{n}-1}k_{2\bar{n}}}}B_m^{k_1\dots k_m}(X_{ij},\pi_l)\right]\nabla_{a_1}\nabla^{[a_1}\pi_{k_1}\cdots\nabla_{a_m}\nabla^{a_m]}\pi_{k_m}$$
$$+\sum_{m=2}^{D-1}\sum_{n=1}^{\bar{n}}Q_n^m, \quad (5.5)$$

where $\bar{n} = \lfloor \frac{m}{2} \rfloor$ and

$$Q_n^m = \frac{(-4)^{\bar{n}-n}\bar{n}!(m-2\bar{n})!}{n!(m-2n)!}\left[\frac{\partial^{\bar{n}-n}}{\partial X_{k_1k_2}\cdots\partial X_{k_{2(\bar{n}-n)-1}k_{2(\bar{n}-n)}}}B_m^{k_{2(\bar{n}-n)+1}\dots k_{m-2n}}(X_{ij},\pi_l)\right]$$
$$\nabla_{a_1}\nabla^{[a_1}\pi_{k_1}\cdots\nabla_{a_{m-2n}}\nabla^{a_{m-2n}}\pi_{k_{m-2n}}R^{b_1c_1}{}_{b_1c_1}\cdots R^{b_nc_n]}{}_{b_nc_n} \quad (5.6)$$

Note that it is convenient to rewrite $A^{k_1\dots k_m} = \frac{(-4)^{\bar{n}}\bar{n}!(m-2\bar{n})!}{m!}\frac{\partial^{\bar{n}}}{\partial X_{k_1k_2}\cdots\partial X_{k_{2\bar{n}-1}k_{2\bar{n}}}}B_m^{k_1\dots k_m}$ for $m \geq 2$. This generalised theory of multiple scalars and a single tensor reduces to Horndeski's theory for the case of a single scalar, and we conjecture that it represents the multi-scalar generalisation. Again, the potential applications of this theory are likely to be considerable, from multi-field inflation to a possible multi-field extension of the *Fab-Four* [119]. These and other future directions are discussed in greater detail in section 5.4.

5.2 Multi-Galileons and covariantization

We begin with the action describing multiple galileon fields in Minkowski space [123],

$$S_{\text{multi-gal}} = \int_{\mathcal{M}} d^D x \sum_{m=1}^{D+1}\alpha^{i_1\dots i_m}\pi_{i_1}\partial^{[a_2}\partial_{a_2}\pi_{i_2}\cdots\partial^{a_m]}\partial_{a_m}\pi_{i_m} \quad (5.7)$$

where $\alpha^{i_1\dots i_m}$ is completely symmetric. Recall that anitsymmetrization omits the usual factor of $1/n!$ and that the indices i, j, k label the scalar field, while a, b, c are spacetime indices. The first step towards covariantizing this theory is to couple gravity minimally, promoted partial derivatives to covariant ones, $\partial_a \to \nabla_a$, such that

$$S_{\text{multi-gal}} \to \int_{\mathcal{M}} d^D x\sqrt{-g} \sum_{m=1}^{D+1} \alpha^{i_1 i_2 \ldots i_m} \pi_{i_1} \nabla^{[a_2}{}_{a_2} \pi_{i_2} \ldots \nabla^{a_m]}{}_{a_m} \pi_{i_m} \tag{5.8}$$

Here we use the notation $\nabla_a{}^b \equiv \nabla_a \nabla^b$ and repeated indices are summed over. Indeed, let us summarize the notation we will adopt for the remainder of this chapter in the following table. It is also convenient define the following scalars for

Notation	Description	Definition/Example
$i, j, k \ldots$	Internal indices of the field	π_i, π_j etc., $\quad i, j, k \in \{1 \ldots N\}$
$a, b, c \ldots$	Space-time indices	$\nabla^a \quad a, b, c \in \{0 \ldots D-1\}$
$\nabla_{ab}, \nabla^a{}_b$	Double covariant derivative	$\nabla_{ab} \equiv \nabla_a \nabla_b,\ \nabla_a{}^b \equiv \nabla_a \nabla^b$
I_{2p}, J_q	Collective unordered internal index	$I_{2p} \equiv \{r_1 \ldots r_{2p}\},\ J_q \equiv \{s_1 \ldots s_q\}$
		Ex: $A_{I_2 J_3} B^{I_2} C^{J_3} = A_{i_1 \ldots i_5} B^{i_1 i_2} C^{i_3 i_4 i_5}$
$\hat{a}, \hat{b}, \hat{c} \ldots$	Antisymmetrized space-time index	$X^{\hat{a}\hat{b}} \times Y^{\hat{c}\hat{d}\hat{e}} \times Z^{\hat{f}\hat{g}} \equiv X^{[ab} Y^{cde} Z^{fg]}$

Table 5.1: Notations

the sake of brevity,

$$\begin{aligned} E_{I_{2p}} &= (\nabla_{a_1} \pi_{r_1} \nabla^{a_1} \pi_{r_2}) \ldots \left(\nabla_{a_p} \pi_{r_{2p-1}} \nabla^{a_p} \pi_{r_{2p}}\right) \\ F_{J_q} &= \left(\nabla_{a_1}{}^{\hat{a}_1} \pi_{s_1}\right) \ldots \left(\nabla_{a_q}{}^{\hat{a}_q} \pi_{s_q}\right) \\ G_r &= R^{\hat{a}_1 \hat{b}_1}{}_{a_1 b_1} \ldots R^{\hat{a}_r \hat{b}_r}{}_{a_r b_r} \end{aligned} \tag{5.9}$$

Here we take $E_{I_0} = F_{J_0} = G_0 = 1$ and $E_{I_{2p}} = F_{J_q} = G_r = 0$ when p, q, r are negative. According to our notations we can write the mth order Lagrangian term as

$$C_0^m \equiv \alpha^{i_1 i_2 \ldots i_m} \pi_{i_1} \nabla^{[a_2}{}_{a_2} \pi_{i_2} \ldots \nabla^{a_m]}{}_{a_m} \pi_{i_m} = \alpha^{i_1 J_{m-1}} \pi_{i_1} F_{J_{m-1}} \tag{5.10}$$

Variation of this term induced by $\pi_k \to \pi_k + \delta\pi_k$ where k is an arbitrary integer between 1 and N is given by,

$$\delta C_0^m = \alpha^{k J_{m-1}} F_{J_{m-1}} \delta\pi_k + (m-1) \alpha^{i_1 k J_{m-2}} \pi_{i_1} \nabla_a{}^{\hat{a}} \delta\pi_k F_{J_{m-2}} \tag{5.11}$$

and after integrating by parts we get,

$$\begin{aligned}
\delta C_0^m &= \Big\{ \alpha^{kJ_{m-1}} F_{J_{m-1}} + (m-1)\alpha^{i_1 k J_{m-2}} \nabla_a{}^{\hat{a}} \pi_{i_1} F_{J_{m-2}} \\
&\quad +(m-1)(m-2)\alpha^{i_1 k i_3 J_{m-3}} \big(2\, \nabla_a \pi_{i_1} \nabla^{\hat{a}\hat{b}} \nabla_b \pi_{i_3} F_{J_{m-3}} + \pi_{i_1} \nabla_a \nabla^{\hat{a}\hat{b}} \nabla_b \pi_{i_3} F_{J_{m-3}} \Big\} \delta\pi_k \\
&= \Big\{ \alpha^{kJ_{m-1}} F_{J_{m-1}} + (m-1)\alpha^{i_1 k J_{m-2}} \nabla_a{}^{\hat{a}} \pi_{i_1} F_{J_{m-2}} \\
&\quad +(m-1)(m-2)\alpha^{i_1 k i_3 J_{m-3}} \big(\nabla_a \pi_{i_1} \nabla^c \pi_{i_3} R^{\hat{a}\hat{b}}{}_{bc} F_{J_{m-3}} \\
&\quad -\frac{1}{4}\pi_{i_1} \nabla^c \pi_{i_3} (\nabla_c R^{\hat{a}\hat{b}}{}_{ab}) F_{J_{m-3}} + \frac{1}{2}\pi_{i_1} R^{\hat{a}\hat{b}}{}_{bc} \nabla_a{}^c \pi_{i_3} F_{J_{m-3}} \big) \Big\} \delta\pi_k \qquad (5.12)
\end{aligned}$$

Here we have used the Riemann and Bianchi identities in the second step. To remove the term containing third derivatives in the metric we add the following counter term to the action.

$$C_1^m = -\frac{1}{8}\alpha^{i_1 I_2 J_{m-3}} \pi_{i_1} E_{I_2} F_{J_{m-3}} G_1 \qquad (5.13)$$

Although the variation of E_{I_2} would generate the correct term to cancel the higher derivative term in (5.12, a further higher order term would be generated through the variation of $F_{J_{m-3}}$. Thus it is clear that a finite number of counter terms should be added recursively at each order in π. We find that the counter term needed at the n^{th} step is given by (see Appendix (C) for details),

$$C_n^m = T_n^{i_1 I_{2n} J_{m-2n-1}} \pi_{i_1} E_{I_{2n}} F_{J_{m-2n-1}} G_n \qquad (5.14)$$

where,

$$T_n^{i_1 \ldots i_m} = \left(-\frac{1}{8}\right)^n \frac{(m-1)!}{(m-2n-1)!(n!)^2} \alpha^{i_1 \ldots i_m} \qquad m \geq 3 \qquad (5.15)$$

It turns out that these counter terms are also sufficient to remove higher derivative terms generated in the g_{ab} equation of motion (see Appendix (C)). Thus a covariant generalization of multi-galileon theory, preserving second order field equations, is given by

$$S_{\text{cov-mutli-gal}} = \int d^D x \sqrt{-g} \sum_{m=1}^{D+1} \sum_{n=0}^{\lfloor \frac{m-1}{2} \rfloor} C_n^m \qquad (5.16)$$

Of course, this was already expressed using more familiar notation in equation (5.2). The corresponding field equations are given by equations (C.9) and (C.15) in Appendix (C). For a single scalar field this theory does not quite reduce to the one presented in [112], although it does still correspond to a subset of Honrdeski's theory [107, 108]. The reason for the slight discrepancy is that our starting point in Minkowski space differs from that in [112] by a total derivative and this affects the details of the subsequent covariantisation.

5.3 Towards Multi-scalar Horndeski

Having derived the covariant multi-galileon theory, it is natural to ask if we can go a stage further and find a multi-scalar generalisation of Horndeski's panoptic theory [107]. Recall that Horndeski's theory was rediscovered by DGSZ [108] using the following method: find the most general theory of a scalar in Minkowski space, with second order field equations, and then covariantise the resulting theory. Here we will conjecture the form of the most general multi-scalar theory in Minkowski space, with second order equations of motion, and covariantise the result in order to give a generalised multi-scalar tensor theory of gravity. We do not attempt to prove the generality of our theory here, and leave this question as a future project.

To arrive at our proposed multi-scalar theory in Minkowski we begin by performing an integration by parts on the multi-galileon action (5.1), and some relabelling, to arrive at the following

$$S_{\text{multi-gal}} = \int_{\mathcal{M}} d^D x \, -\frac{1}{2}\alpha^i \pi_i - \alpha^{ij}\bar{X}_{ij} - \sum_{m=1}^{D-1} \frac{m+2}{2}\alpha^{ijk_1\ldots k_m}\bar{X}_{ij}\partial^{[a_1}\partial_{a_1}\pi_{k_1}\cdots\partial^{a_m]}\partial_{a_m}\pi_{k_m} \tag{5.17}$$

where we recall that $\bar{X}_{ij} = \frac{1}{2}\partial_a\pi_i\partial^a\pi_j$. An obvious generalisation of this, consistent with the one for a single scalar presented in [108], is given by

$$S_{\text{multi-scalar}} = \int_{\mathcal{M}} d^D x \; A(\bar{X}_{ij}, \pi_l) + \sum_{m=1}^{D-1} A^{k_1\ldots k_m}(\bar{X}_{ij}, \pi_l)\partial^{[a_1}\partial_{a_1}\pi_{k_2}\cdots\partial^{a_m]}\partial_{a_m}\pi_{k_m} \tag{5.18}$$

Taken at face value, this action will yield higher order equations of motion. However, this can be avoided by imposing the condition that $\frac{\partial A^{i_1 \dots i_m}}{\partial X_{kl}}$ is symmetric in *all* of its indices $i_1, \dots i_m, k, l$. We prove that this theory is the most general multi-scalar theory in Minkowski space, with second order equations of motion in chapter 6.

The next step is to covariantise the theory (5.18). As before we begin by minimally coupling to gravity, promoting partial derivatives to covariant ones, $\partial_a \to \nabla_a$ yielding

$$S_{\text{multi-scalar}} \to \int d^D x \sqrt{-g} \sum_{m=0}^{D-1} A(X_{ij}, \pi_l)^{i_1 \dots i_m} \nabla_{a_1}{}^{[a_1} \pi_{i_1} \dots \nabla_{a_m}{}^{a_m]} \pi_{i_m} \qquad (5.19)$$

with $X_{ij} := \frac{1}{2} \nabla_a \pi_i \nabla^a \pi_j$. Analogous to the covariantized multi-galileons, this action would yield equations of motion of derivative order greater than two. Thus we introduce the following counter terms at each order in π_i to cancel those higher derivatives.

$$Q_n^m = A_n(X_{ij}, \pi_l)^{J_{m-2n}} F_{J_{m-2n}} G_n \qquad (5.20)$$

Note that $Q_0^m = A(X_{ij}, \pi_l)^{i_1 \dots i_m} \nabla_{a_1}{}^{[a_1} \pi_{i_1} \dots \nabla_{a_m}{}^{a_m]} \pi_{i_m}$. In order to impose the constraint that the equations of motion are second order, we take the variation of Q_n^m induced by $\pi_k \to \pi_k + \delta\pi_k$. We focus only on those terms that contain higher derivatives, and using the following short-hand,

$$A_n^{J_q} = A_n(X_{ij}, \pi_l)^{J_q}, \qquad \partial^{ij} A_n^{J_q} = \frac{\partial A_n^{J_q}}{\partial X_{ij}} \qquad (5.21)$$

we obtain

$$\begin{aligned} \delta Q_n^m &= \delta A_n^{J_{m-2n}} F_{J_{m-2n}} G_n + A_n^{J_{m-2n}} \delta F_{J_{m-2n}} G_n \qquad (5.22) \\ &= \partial^{ik} A_n^{J_{m-2n}} \nabla^a \delta\pi_k \nabla_a \pi_i F_{J_{m-2n}} G_n + (m-2n) A_n^{k J_{m-2n-1}} \nabla_a{}^{\hat{a}} \pi_k F_{J_{m-2n-1}} G_n \end{aligned}$$

After performing an integration by parts, and using both the Riemann and BIanchi identities, we find that δQ_n^m contains the following terms that will contribute higher derivatives in the equations of motion,

$$\begin{aligned}\delta Q_n^m \supset & \Big\{ -(m-2n)\partial^{ik} A_n^{J_{m-2n-1}j} \nabla_a \pi_i \nabla^a \nabla^{\hat{b}}{}_b \pi_j F_{J_{m-2n-1}} G_n \\ & +(m-2n)\partial^{ij} A_n^{J_{m-2n-1}k} \nabla_a \pi_i \nabla_b{}^{\hat{b}} \nabla^a \pi_i F_{J_{m-2n-1}} G_n \\ & -n\,\partial^{ik} A_n^{J_{m-2n}} \nabla^a \pi_i (\nabla_a R^{\hat{b}\hat{c}}{}_{bc}) F_{J_{m-2n}} G_{n-1} \\ & -\frac{(m-2n)(m-2n-1)}{4} A_n^{kiJ_{m-2n-2}} \nabla^a \pi_i (\nabla_a R^{\hat{b}\hat{c}}{}_{bc}) F_{J_{m-2n-2}} G_n \Big\} \delta\pi_k \end{aligned} \tag{5.23}$$

It turns out that the first two terms cancel when we make use of the Riemann identity $[\nabla_a \nabla_b]\nabla_c \pi_i = R_{abcd}\nabla^d \pi_i$. The cancellation follows from the fact that we have the following constraint on the functions $A_n^{J_q}$,

$$\partial^{ij} A_n^{I_q} = \partial^{(ij} A_n^{I_q)} \tag{5.24}$$

where the bracket $(ij\dots)$ stands for the symmetrization of the indices. We should also note that subsequent derivatives preserve this property of the function ie., $\partial^{ij}\partial^{kl} A_n^{J_q} = \partial^{(ij}\partial^{kl} A_n^{J_q)}$.

The remaining higher order terms are now

$$\begin{aligned}\delta Q_n^m \supset & \Big\{ -n\,\partial^{ik} A_n^{J_{m-2n}} \nabla^a \pi_i (\nabla_a R^{\hat{b}\hat{c}}{}_{bc}) F_{J_{m-2n}} G_{n-1} \\ & -\frac{(m-2n)(m-2n-1)}{4} A_n^{ikJ_{m-2n-2}} \nabla^a \pi_i (\nabla_a R^{\hat{b}\hat{c}}{}_{bc}) F_{J_{m-2n-2}} G_n \Big\} \delta\pi_k \end{aligned} \tag{5.25}$$

These can be cancelled off by successive counter terms, Q_{n+1}^m, provided the following recursive relationship holds.

$$(s+1)\partial^{ij} A_{s+1}^{J_{m-2s-2}} = -\frac{(m-2s)(m-2s-1)}{4} A_s^{ijJ_{m-2s-2}} \tag{5.26}$$

Operating with $(\partial^{ij})^{s-n}$ on both sides and taking the following product, we find that

$$\prod_{s=n}^{\bar{n}-1} \left(\frac{(\partial^{kl})^{s-n} A_s^{ijJ_{m-2s-2}}}{(\partial^{kl})^{s-n} \partial^{ij} A_{s+1}^{J_{m-2s-2}}} \right) = \frac{A_n^{J_{m-2n}}}{(\partial^{ij})^{\bar{n}-n} A_{\bar{n}}^{J_{m-2\bar{n}}}} = \frac{(-4)^{\bar{n}-n} \bar{n}!(m-2\bar{n})!}{n!(m-2n)!} \tag{5.27}$$

Here $\bar{n} = \left\lfloor \frac{m}{2} \right\rfloor$ denotes the last counter term. We take $A_{\bar{n}}^{J_{m-2\bar{N}}} = B_m^{J_{m-2\bar{n}}}$ to define an arbitrary function for each m, giving,

$$A_n^{J_{m-2n}} = \frac{(-4)^{\bar{n}-n} \bar{n}!(m-2\bar{n})!}{n!(m-2n)!} (\partial^{ij})^{\bar{n}-n} G_m^{J_{m-2\bar{n}}} \quad 0 \le n \le \bar{n} \tag{5.28}$$

We conclude that the following generalized multi-scalar tensor theory has second order field equations from variation of the scalars

$$S_{\text{cov-multi-scalar}} = \int d^D x \sqrt{-g} \sum_{m=0}^{D-1} \sum_{n=0}^{\lfloor \frac{m}{2} \rfloor} Q_n^m \tag{5.29}$$

This action is written using more familiar notation in equation (5.5). It remains to show that it does not give rise to higher derivatives in the g_{ab} equations of motion. Variation of the metric gives

$$\begin{aligned}
\delta Q_n^m \supset\ & A_n^{J_{m-2n}} \delta F_{J_{m-2n}} G_n + A_n^{J_{m-2n}} F_{J_{m-2n}} \delta G_n \\
=\ & (m-2n) A_n^{iJ_{m-2n-1}} \left(\nabla^{\hat{a}b} \pi_i \delta g_{ab} - \nabla_a \pi_i \delta g_{ab}{}^{;\hat{a}} + \frac{1}{2} g^{\hat{a}b} \nabla^c \pi_i \delta g_{ab;c} \right) F_{J_{m-2n-1}} G_n \\
& + n\, A_n^{J_{m-2n}} F_{J_{m-2n}} \left(R_c{}^{b\hat{a}\hat{c}} \delta g_{ab} - 2 g^{\hat{a}b} \delta g_{ab;c}{}^{\hat{c}} \right) G_{n-1}
\end{aligned} \tag{5.30}$$

where we again focus on terms that yield higher derivatives. Integrating by parts and making use of the geometric identities, we find that

$$\begin{aligned}
\delta Q_n^m \supset\ & \Bigg\{ - \frac{(m-2n)(m-2n-1)}{2} A_n^{ijJ_{m-2n-2}} \nabla_c \pi_i \nabla^c \nabla^{\hat{d}}{}_d \pi_j F_{J_{m-2n-2}} G_n g^{\hat{a}b} \\
& - 2n\, \partial^{ij} A_n^{J_{m-2n}} \nabla_c \pi_i \nabla^c \nabla^{\hat{d}}{}_d \pi_j F_{J_{m-2n}} G_{n-1} g^{\hat{a}b} \Bigg\} \delta g_{ab}
\end{aligned} \tag{5.31}$$

It is clear that these higher derivative terms would cancel if the same recursive relationship (5.26) holds. We therefore conclude that our generalised theory (5.5) gives at most second order field equations under variation of all fields. As a consistency check of our work, it is reassuring to see that (5.5) does indeed reduce to Horndeski's theory [107, 108] in the case of a single scalar.

5.4 Discussion

In this chapter, we have shown how gravity may be coupled to multi-galileons [121, 122, 123] without introducing higher order field equations, and generalised our result, proposing a multi-scalar version of Horndeski's panoptic scalar-tensor theory [107, 108]. The actions for these theories are given by equations (5.2) and (5.5) respectively, and both may have interesting applications in multi-field inflation and quintessence scenarios.

For the case of two galileons, the covariant theory (5.2) may be particularly relevant in the context of the cosmological constant problem. In [121] it was shown that certain classes of bigalileon theories can give rise to self-tuning, with the physical spacetime screened from the cosmological constant b y the self-adjusting galileon fields. This is not in contradiction to Weinberg's famous no-go theorem [125] as Poincaré invariance is broken in the galileon sector. However, the proposal presented in [121] was not entirely satisfactory for the following reason. Self-tuning a large cosmological constant represents a considerable deviation from General Relativity. Whilst this is desirable on cosmological scales, one requires some mechanism through which deviations are screened in the solar system. In the bigalileon theory, this was achieved through the Vainshtein mechanism (see eg. [126]). However, if we want to self-tune a large cosmological constant and at the same time exploit Vainshtein screening, one finds that the decoupled bigalileon description breaks down. This begs the question, what happens in a covariant completion of these theories when one is no longer forced to work in the decoupling limit? The covariant multi-galileon theory will now enable us to address this issue and was one of the original motivations for this work.

Let us present the covariant completion of the self-tuning bigalileon theory given as an example in [121]. In the decoupling limit, this example is given by the action

$$S = \int d^4x \, \frac{M_{pl}^2}{2}\sqrt{-g}R + \mathcal{L}_{\pi,\xi} + S_m[e^{2\pi}g_{ab}; \Psi_n] \tag{5.32}$$

where the bigalileon Lagrangian is given by

$$\mathcal{L}_{\pi,\xi} = 3M_{pl}^2 \pi \Box \pi - \frac{M^4}{\mu^2}\pi \Box \xi + \frac{1}{3\mu^2}\pi \partial_a \partial^{[a} \pi \partial_b \partial^b \xi \partial_c \partial^{c]} \xi \tag{5.33}$$

and S_m describes matter minimally coupled to the metric $e^{2\pi}g_{ab}$. The covariant completion of this theory can now be immediately read off from (5.2) and is given by

$$S = \int d^4x\, \sqrt{-g}\Bigg[\frac{M_{pl}^2}{2}R + 3M_{pl}^2\pi\Box\pi - \frac{M^4}{\mu^2}\pi\Box\xi + \frac{1}{3\mu^2}\pi\nabla_a\nabla^{[a}\pi\nabla_b\nabla^b\xi\nabla_c\nabla^{c]}\xi$$
$$-\frac{1}{6\mu^2}\pi(\nabla_a\pi\nabla^a\xi)\nabla_b\nabla^{[b}\xi R^{cd]}{}_{cd} - \frac{1}{12\mu^2}\pi(\nabla_a\xi\nabla^a\xi)\nabla_b\nabla^{[b}\pi R^{cd]}{}_{cd}\Bigg] + S_m[e^{2\pi}g_{ab};\Psi_n]$$

It would be interesting to study the behaviour of this model in some detail, as well as the covariant completions of other self-tuning models presented in [121]. How does self-tuning manifest itself? How large a cosmological constant can one tolerate and still be compatible with solar system tests?

Staying on the subject of self-tuning, we note that our proposed multi-scalar version of Horndeski's theory (5.5), puts us in a good position to generalise the so-called *Fab Four* theory [119] to multiple fields. The *Fab Four* Lagrangians were obtained by asking which subset of Horndeski's theory can "solve" the cosmological constant problem in that they screen the curvature from the vacuum energy. Given the generality of Horndeski, this enables one to say that a self-tuning single scalar-tensor theory in four dimensions *must* correspond to a *Fab Four* theory. This is a rather powerful statement, but we are now in position to make it even more powerful by generalising to multiple fields. Mutliple fields will open up new possibilities as well, allowing for greater flexibility in deriving stable, phenomenologically consistent solutions.

We drawing attention to an interesting two-scalar tensor theory which can now be seen as a subset of the two-scalar version of our generalised theory (5.5). This so-called *Fab Five* theory is an extension of certain *Fab Four* Lagrangians[127] and is given by the following

$$S_{\text{Fab5}} = \int d^4x\, \sqrt{-g}\left[\frac{M_{pl}^2}{2}R - \frac{c_1}{2}(\nabla\pi)^2 + f\left(-\frac{c_2}{2}(\nabla\pi)^2 + \frac{c_G}{M^2}G^{ab}\nabla_a\pi\nabla_b\pi\right)\right] \tag{5.34}$$

When the function f is the identity, this corresponds to a theory built from John and George from the *Fab Four*, along with a canonical kinetic term for the scalar.

Generalising f introduces an additional scalar degree of freedom, and the theory can be written as [127]

$$S_{\text{Fab5}} = \int d^4x \sqrt{-g} \left[\frac{M_{pl}^2}{2} R - \frac{1}{2}(c_1 + c_2 f'(\xi))(\nabla\pi)^2 + \frac{c_G}{M^2} f'(\xi) G^{ab} \nabla_a \pi \nabla_b \pi + f(\xi) - \xi f'(\xi) \right] \tag{5.35}$$

This corresponds to a particular two scalar-tensor theory contained within (5.5).

Whilst we have alluded to a few, at this stage it is impossible to envisage all the potential applications of our generalised mutli-scalar tensor theory. Horndeski's theory is currently the focus of plenty of research, and to that we can now add its generalisation. See [128] for some interesting recent use of Horndeski's theory that may now be extended to include the theory presented in this chapter.

Chapter 6

Proof: The most general multiple-scalar field theory in Minkowski space-time

In this chapter we prove that the most general multiple scalar field theory in flat-space time satisfying the conditions

i) Lagrangian contains up to second order derivatives of the fields

ii) Field equations contain up to second order derivatives of the fields

is given by,

$$S_{\text{multi-scalar}} = \int_{\mathcal{M}} d^D x \; A(\bar{X}_{ij}, \pi_l) + \sum_{m=1}^{D-1} A^{k_1 \dots k_m}(\bar{X}_{ij}, \pi_l) \partial^{[a_1} \partial_{a_1} \pi_{k_2} \cdots \partial^{a_m]} \partial_{a_m} \pi_{k_m} \tag{6.1}$$

where $\bar{X}_{ij} = \frac{1}{2} \partial_a \pi_i \partial^a \pi_j$ and $\frac{\partial A^{i_1 \cdots i_m}}{\partial \bar{X}_{kl}}$ is symmetric in *all* of its indices $i_1, \dots i_m, k, l$.

6.1 Polynomiality of the second derivatives of π_i in the Lagrangian

We start with the general multi-field action in D dimensions of the form,

$$S = \int d^D x \mathcal{L}\left(\pi_i, \partial_a \pi_j, \partial_b \partial_c \pi_k\right) \tag{6.2}$$

Here i, j, k are used to label different fields ($i \in \{1 \ldots N\}$). The Euler-Lagrange equations of this action is

$$\frac{\partial \mathcal{L}}{\partial \pi_i} - \partial_a \left(\frac{\partial \mathcal{L}}{\partial \pi_{i\,a}} \right) + \partial_a \partial_b \left(\frac{\partial \mathcal{L}}{\partial \pi_{i\,ab}} \right) = 0 \tag{6.3}$$

Note that we occasionally resort to the notation $\pi_{i\,a\ldots b} \equiv \partial_b \ldots \partial_a \pi_i$ for brevity. In general only the third term on the LHS will have fourth derivatives, explicitly this is,

$$\frac{\partial \mathcal{L}}{\partial \pi_{i\,cd} \partial \pi_{i\,ab}} \pi_{i\,abcd} \tag{6.4}$$

imposing the constraint that this term vanishes we get,

$$\mathcal{L}^{ab|cd} \pi_{abcd} = 0 \tag{6.5}$$

where we used the notation, $\mathcal{L}^{ab|cd} \equiv \frac{\partial \mathcal{L}}{\partial \pi_{i\,cd} \partial \pi_{i\,ab}}$ and suppressed the internal indices for brevity. Due to the complete symmetry in the space time indices of π_{abcd}, 6.5 implies

$$\mathcal{L}^{(ab|cd)} = 0 \tag{6.6}$$

where $(\ldots)$ stands for symmetrisation. Furthermore, the following symmetries are trivial.

$$\mathcal{L}^{ab|cd} = \mathcal{L}^{cd|ab} = \mathcal{L}^{cd|ba} = \mathcal{L}^{dc|ba} \tag{6.7}$$

On account of these symmetries 6.6 is satisfied *if and only if* the following cyclic identity holds

$$\mathcal{L}^{ab|cd} + \mathcal{L}^{ac|db} + \mathcal{L}^{ad|bc} = 0 \tag{6.8}$$

We define the tensor,

$$\mathcal{L}^{a_1b_1|\dots|a_nb_n} \equiv \frac{\partial}{\partial \pi_{a_nb_n}} \cdots \frac{\partial}{\partial \pi_{a_1b_1}} \mathcal{L} \tag{6.9}$$

This tensor naturally inherits the cyclic identity 6.8,

$$\mathcal{L}^{\dots|ab|\dots|cd|\dots} + \mathcal{L}^{\dots|ac|\dots|db|\dots} + \mathcal{L}^{\dots|ad|\dots|bc|\dots} = 0 \tag{6.10}$$

Consider the (D+1)th derivative of the Lagrangian with respect to π_{ab} that is given by the tensor 6.9 of rank (0,2D+2),

$$\mathcal{L}^{a_1b_1|\dots|a_Db_D|a_{D+1}b_{D+1}} \tag{6.11}$$

In D dimensions it is easy to see that the components of this tensor will have atleast 3 identical space-time indices. Using the symmetries **??** any component of this tensor can be cast in the following two forms,

$$\begin{aligned} F_1 &= \mathcal{L}^{\dots|aa|ab|\dots} \\ F_2 &= \mathcal{L}^{\dots|ab|ac|ad|\dots} \end{aligned} \tag{6.12}$$

Now using the cyclic identity on the identical indices a of F_1 yields,

$$F_1 = 0 \tag{6.13}$$

and using the cyclic identity on the first three indices a, b, a of F_2 and subsequently on the similar indices gives,

$$F_2 = \mathcal{L}^{\dots|ab|ac|ad|\dots} = -\frac{1}{2}\mathcal{L}^{\dots|aa|ad|bc|\dots} = 0 \tag{6.14}$$

Thus we have established,

Theorem 6.1. *The most general action that yields equations of motion of derivative order up to two, has the Lagrangian that depends polynomially on the second derivatives of the field and the polynomial order is bounded above by D in D-dimensions ie, Order of the polynomial* $\leq D$.

6.2 The Structure of the Lagrangian $\mathcal{L}$

Having established the polynomiality of the second derivatives in $\mathcal{L}$, we write down a generic term in the Lagrangian that is constrained by this fact.

$$\begin{aligned}\mathcal{L}_{2p,q} &= A(\bar{X}_{ij},\pi_l)^{i_1\ldots i_p j_1\ldots j_q}\sum_{\sigma\in S_{p+q}}\sum_{p\in S_q} c(\sigma)d(p)\left[\left(\partial_{a_1}\pi_{i_1}\partial^{\sigma(a_1)}\pi_{i_2}\right)\ldots\left(\partial_{a_p}\pi_{i_{(2p-1)}}\partial^{\sigma(a_p)}\pi_{i_{2p}}\right)\right.\\ &\qquad \left.\times\partial^{\sigma(b_1)}_{p(b_1)}\pi_{j_1}\ldots\partial^{\sigma(b_q)}_{p(b_q)}\pi_{j_1}\right] \end{aligned} \tag{6.15}$$

A note on notations is in order. Here $i, j, k\ldots$ denote internal indices and $a, b, c\ldots$ are space-time indices. $\bar{X}_{ij} = \frac{1}{2}\partial_a\pi_i\partial^a\pi_j$ and $A(\bar{X}_{ij},\pi_l)^{i_1\cdots}$ can in general be a non-polynomial function. $\sigma \in S_{p+q}$ is an element of the symmetric permutation group acting on the positions of the space-time labels a_i, b_j upstairs, similarly $p \in S_q$ is an element of the symmetric permutation group acting on the positions of the space-time labels b_j downstairs. C_σ, d_p are coefficients that depend on σ, p resp (in general they can also be functions of $\bar{X}_{ij}, \pi_k$). We observe the following facts.

i) Any term generated by the action of σ **within** the positions of the indices b_i can also be generated via the action of p on b_i downstairs since the indices are summed over.

ii) Any term generated by the action of σ **within** the positions of the indices a_i can also be generated by shuffling the order of internal indices i_j.

On account of these facts we can avoid over-counting by reducing the symmetry group of σ to the quotient group defined by,

$$S_{p+q} \to \frac{S_{p+q}}{S_p\,S_q} \tag{6.16}$$

6.2.1 Symmetricity of $A(X_{ij},\pi_k)^{i_1,i_2\cdots}$

So far we have established that a generic term in the Lagrangian can be cast in the form,

$$\mathcal{L}_{2p,q} = A(\bar{X}_{ij},\pi_l)^{i_1\ldots i_p j_1\ldots j_q} \sum_{\sigma\in\frac{S_{p+q}}{S_p\,S_q}} \sum_{p\in S_q} c(\sigma)d(p)\left[\left(\partial_{a_1}\pi_{i_1}\partial^{\sigma(a_1)}\pi_{i_2}\right)\ldots\left(\partial_{a_p}\pi_{i_{2p-1})}\partial^{\sigma(a_p)}\pi_{i_{2p}}\right)\right.$$
$$\left.\times\partial^{\sigma(b_1)}_{p(b_1)}\pi_{j_1}\ldots\partial^{\sigma(b_q)}_{p(b_q)}\pi_{j_1}\right] \tag{6.17}$$

We note that any cancellation of 3'rd and 4'th derivatives coming from the variational principle should occur within each of these terms as they are structurally different from each other. We ignore the variation of $A^{i_1\cdots}$ for now, and focus on terms that are of derivatives order 3 (containing $(\pi_{i\,abc})$). Let us isolate a generic term that would give rise to 3'rd order derivatives in $\mathcal{L}_{(2p,q)}$.

$$\mathcal{L}_{2p,q} \supset B_{k,l} \equiv A^{\ldots k\ldots l\ldots}\sum_{\sigma,p} c_\sigma d_p\left(\partial^{\sigma(a_r)}\pi_k\,\partial^{\sigma(a_s)}_{p(b_t)}\pi_l\right)\ldots \tag{6.18}$$

where we have suppressed the dependence of the arbitrary function $A^{i\cdots}(\bar{X}_{ij},\pi_k)$ and other factors of the fields for brevity. For the $\delta\pi_k$ variation this will contain the term ,

$$\delta B_{k,l} \supset -A^{\ldots k\ldots l\ldots}\sum_{\sigma,p} c_\sigma d_p\left(\delta\pi_k\,\partial^{\sigma(a_r)}\partial^{\sigma(a_s)}_{p(b_t)}\pi_l\right)\ldots \tag{6.19}$$

Note however that under the interchange of indices k,l the corresponding variation yields a similar term *i.e,*

$$\delta B_{l,k} \supset +A^{\ldots l\ldots k\ldots}\sum_{\sigma,p} c_\sigma d_p\left(\delta\pi_k\,\partial^{\sigma(a_r)}\partial^{\sigma(a_s)}_{p(b_t)}\pi_l\right)\ldots \tag{6.20}$$

These would cancel if,

$$A^{\ldots k\ldots l\ldots} = A^{\ldots l\ldots k\ldots} \tag{6.21}$$

This implies that $A^{i_1\cdots}(\bar{X}_{ij},\pi_k)$ is completely symmetric in it's indices.

As we have seen the symmetry group of the σ would just amount to the interchange of space-time labels $a_i \leftrightarrow b_j$. On account of the symmetry in $A^{i_1\cdots}$, it does not matter which labels are interchanged *i.e,* all such operations form an *equivalence class* labelled by how many pairs are interchanged. Now consider the case where two interchanges are made, we write a generic term of this type, and omit the internal indices and the function $A^{i\cdots}$ as they don't play any role in this argument, due to the symmetricity of $A^{\imath\cdots}$,

$$C \equiv c_p c_\sigma \pi_{a_i} \pi_{a_j} \pi^{b_k} \pi^{b_l} \pi^{a_i}_{p(b_k)} \pi^{a_j}_{p(b_l)} \cdots \tag{6.22}$$

Here we have interchanged $a_i \leftrightarrow b_k, a_j \leftrightarrow b_l$. This term would give rise to 4'th derivatives in the eom given by,

$$\delta_{ij} C \supset 2 c_p c_\sigma \pi_{a_i} \pi_{a_j} \pi^{b_k} \pi^{b_l} \left(\pi^{a_i \ \ a_j}_{p(b_k) p(b_l)} \right) \delta\pi \ldots \tag{6.23}$$

Here δ_{ij} denotes the variation and the subsequent integration by parts restricted to the factors $\pi^{a_i}_{p(b_k)} \pi^{a_j}_{p(b_l)}$. Note that C is invariant under the interchange of either $a_i \leftrightarrow a_j$ or $p(b_k) \leftrightarrow p(b_l)$, this means that the 4'th derivative term in δC would vanish only if all terms proportional to C vanish. Thus we have established,

Theorem 6.2. *The action of σ is necessarily limited to just the interchange of at most 1 index $a_i \leftrightarrow b_j$, if the condition (ii) is to be satisfied. Thus the $\mathcal{L}_{2p,q}$ is restricted to*

$$\mathcal{L}_{0,q} = A^{i_1 \ldots i_q} \sum_{p \in S_q} c_p \partial^{b_1}_{p(b_1)} \pi_{(i_1)} \ldots \partial^{b_q}_{p(b_q)} \pi_{(i_q)} \tag{6.24}$$

$$\mathcal{L}_{2,q} = A^{i_1 \ldots i_{q+2}} \sum_{p \in S_q} c_p \partial_a \pi_{i_1} \partial^{b_1} \pi_{(i_2)} \partial^{a}_{p(b_1)} \pi_{(i_3)} \ldots \partial^{b_q}_{p(b_q)} \pi_{(i_{q+2})} \tag{6.25}$$

Having resolved the σ permutations we focus on the permutation p acting on the indices b_i downstairs.

Definition 6.3. T^{ij} is a transposition map that acts on individual instances of the permutations generated by p and interchanges $b_i \leftrightarrow b_j$,

$$T^{ij} : P^{ij}_k \to P^{ji}_k$$

Note that $T^{ij} \circ T^{ji} = id$. Here P^{ij}_k is an instance of the permutation where b_i appears before b_j and $k \in \{1, \ldots q!/2\}$.

Two permutations $P_k^{ij}, P_{k'}^{i'j'}$ are identical if they are related by the relabelling of the indices b_i. Relabelling of $b_k \leftrightarrow b_l$ would change their order both upstairs and downstairs, which can be put back into the allowed form by interchanging the fields that carry the relabelled indices upstairs. Let us illustrate this by an example. Consider the following term in the series, where we have again omitted the symmetric function $A^{i_1\cdots}$ and the remaining factors of the field,

$$\partial_{b_4}^{b_2}\pi_{i_r}\partial_{b_6}^{b_3}\pi_{i_{r+1}}\dots\partial_{b_2}^{b_7}\pi_{i_{r+5}}\partial_{b_3}^{b_8}\pi_{i_{r+6}} \tag{6.26}$$

relabelling $b_2 \leftrightarrow b_3$ yields the identical term,

$$\partial_{b_4}^{b_3}\pi_{i_r}\partial_{b_6}^{b_2}\pi_{i_{r+1}}\dots\partial_{b_3}^{b_7}\pi_{i_{r+5}}\partial_{b_2}^{b_8}\pi_{i_{r+6}} \tag{6.27}$$

But permutation group p would not alter the indices upstairs therefore we have to move the fields back such that the order upstairs is unchanged. We can interchange the positions of the terms $\partial_{b_4}^{b_3}\pi_{i_r}, \partial_{b_6}^{b_2}\pi_{i_{r+1}}$ invariantly since the function $A^{i_1\cdots}$ is symmetric. We get the identical term

$$\partial_{b_6}^{b_2}\pi_{i_{r+1}}\partial_{b_4}^{b_3}\pi_{i_r}\dots\partial_{b_3}^{b_7}\pi_{i_{r+5}}\partial_{b_2}^{b_8}\pi_{i_{r+6}} \tag{6.28}$$

comparing this to 6.26 we see that it differs by two transpositions acting on the original term interchanging $b_2 \leftrightarrow b_3$ and $b_4 \leftrightarrow b_6$, in other words it is operated by the map $T^{23} \circ T^{46}$. Thus we have established,

Theorem 6.4. *Two identical terms generated by the permutation group p* ***necessarily*** *differ by an even number of transpositions acting downstairs.*

Definition 6.5. $M^{[ik][jl]}$ is a map acting on the permutation P_r^{ij} and relabels $b_i \leftrightarrow b_k$ and $b_j \leftrightarrow b_l$ i.e,

$$M^{[ik][jl]} : P_r^{ij} \to P_r^{kl} \tag{6.29}$$

The two maps T^{ij} and $M^{[ik][jl]}$ can be expressed succinctly with the following commutative diagram. It is clear from this diagram that there is an induced map $M^{[ik][jl]} : T^{ij} \to T^{kl}$.

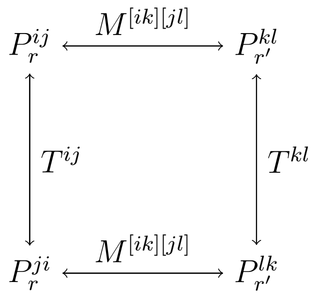

Figure 6.1: Commutative diagram of the maps $T^{ij}, M^{[ik][jl]}$

Consider a given term in the permutation series, where once again we drop the internal indices and focus only on the two relevant factors for brevity,

$$c_p P_k^{ij} = c_p \dots \partial_{b_i}^{b_r} \pi \dots \partial_{b_j}^{b_s} \pi \dots \tag{6.30}$$

Under the action of T^{ij} we get,

$$c_{p'} P_k^{ji} = c_{p'} \dots \partial_{b_j}^{b_r} \pi \dots \partial_{b_i}^{b_s} \pi \dots \tag{6.31}$$

Both of these terms are distinct on account of theorem 6.4. Varying the second derivatives and integrating by parts would yield for each of these terms

$$\delta_{ij} \left(c_p P_k^{ij} \right) \supset 2c_p \dots \delta\pi \dots \partial_{b_j}^{b_r} \partial_{b_i}^{b_s} \pi \dots \tag{6.32}$$

and

$$\delta_{ji} \left(c_{p'} P_k^{ji} \right) \supset 2c_{p'} \dots \delta\pi \dots \partial_{b_j}^{b_r} \partial_{b_i}^{b_s} \pi \dots \tag{6.33}$$

Note that there is also a natural induced map between the coefficients, $T_{\text{induced}}^{ij} : c_p \to c_{p'}$. Now let us observe the following facts,

i) Every 4'th derivative term occurring in the field equation originates from the pairs $(P_k^{ij}, P_k^{ji}), (P_{k'}^{kl}, P_{k'}^{lk})$ connected by transposition map and the map $M^{[ki][lj]}$ as shown in fig(6.1).

ii) We can set the coefficients c_p of terms in Lagrangian differing by relabelling of the indices(i.e same terms) to be equal to each other without loss of generality.

This implies that under the action of any transposition map T^{ij} we must have $T^{ij}_{\text{induced}} : c_p \to -c_p$, if the 4'th derivative terms are to cancel. At the level of Lagrangian this implies that, interchanging any indices b_i downstairs would flip the sign of the Lagrangian. Thus we have resolved the permutation coefficients c_p, that is to say, all the indices b_i downstairs are anti-symmetrised. Now we can express $\mathcal{L}_{2,q}, \mathcal{L}_q$ as,

$$\mathcal{L}_{2,q} = A^{i_1 i_2 \ldots i_{q+2}} \, \partial_a \pi_{i_1} \, \partial^{b_1} \pi_{i_2} \, \partial^a_{[b_1} \pi_{i_3} \, \partial^{b_2}_{b_2} \pi_{i_4} \ldots \partial^{b_q}_{b_q]} \pi_{i_{q+2}} \tag{6.34}$$

$$\mathcal{L}_q = A^{i_1 i_2 \ldots i_q} \, \partial^{b_1}_{[b_1} \pi_{i_1} \, \partial^{b_2}_{b_2} \pi_{i_2} \ldots \partial^{b_q}_{b_q]} \pi_{i_q} \tag{6.35}$$

It has been shown in [108] that $\mathcal{L}_{2,q}$ is in fact a linear combination of $\mathcal{L}_k$ with $k \in \{0 \ldots q\}$ and $\mathcal{L}_D$ is a linear combination of $\mathcal{L}_k$ with $k \in \{0 \ldots (D-1)\}$. Hence only one of these terms is independent. Now it is a simple matter to see that in order for the 3'rd derivative terms coming from the variation of the function $A(X_{ij}, \pi_k)$ to cancel, we must have the condition that, $\frac{\partial A^{i_1 \ldots i_m}}{\partial \bar{X}_{kl}}$ is symmetric in *all* of its indices $i_1, \ldots i_m, k, l$. Thus we have established our final result,

Theorem 6.6. *The most general multiple scalar field theory in flat space-time satisfying conditions (i), (ii) is given by,*

$$S_{\textit{multi-scalar}} = \int_{\mathcal{M}} d^D x \; A(\bar{X}_{ij}, \pi_l) + \sum_{m=1}^{D-1} A^{k_1 \ldots k_m}(\bar{X}_{ij}, \pi_l) \partial^{[a_1} \partial_{a_1} \pi_{k_2} \cdots \partial^{a_m]} \partial_{a_m} \pi_{k_m}$$

where $\bar{X}_{ij} = \frac{1}{2} \partial_a \pi_i \partial^a \pi_j$ *and* $\frac{\partial A^{i_1 \ldots i_m}}{\partial \bar{X}_{kl}}$ *is symmetric in all of its indices* $i_1, \ldots i_m, k, l$*.*

Chapter 7

Discussion and Outlook

7.1 Summary

This thesis has been focused towards some aspects and generalisations of scalar-tensor theories. In Chapter 1 we introduced and discussed the *cosmological constant problem*, we motivated the need to look for theories that modify gravity in the infrared and discussed *degravitation*, *self-tuning*, and *quintessence* in this context. In Section 2.2 we discussed how Lorentz invariance implies gauge redundancy and equivalence principle, placing GR as the unique low-energy theory of spin-2 interacting massless particles. Sections 2.3 - 2.5 were a brief survey of some historically important scalar-tensor theories discussing their characteristic features and inconsistencies. We discussed the DGP *decoupling limit* and the ghost mode around self accelerating solutions, and saw how this is overcome in the Galileon field theory that is originally set up as a generalisation of DGP decoupling limit. Furthermore we studied how *Vainshtein mechanism* plays a crucial role in making these theories consistent with the predictions of GR in solar system scales. This was shown to be achieved by the dominance of non-linear terms in dense environments. We concluded Chapter 2 with Ostrogradski theorem which states that any system with equations of motion of derivative order greater than two suffers from catastrophic instabilities.

In chapter 3 we computed the Hamiltonian of Galileon field theory living in a bounded space-time taking into account the boundary contributions arising from this computation. We saw how the ghost instability is manifested as negative energy for spherically symmetric profile sourced by a point source. Furthermore, we discussed how non-linear terms in the Lagrangian regularize the energy near the point source.

In Chapter 4 we analysed the most general scalar-tensor theory in 4-D, the Horndeski theory. We computed the boundary terms that make the action well posed and the junction conditions to be satisfied across a codimension-1 brane, this computation was done for arbitrary dimensions. We mentioned how the Gibbons-Hawking type boundary terms enables one to employ Euclidean path integral methods as well as study the brane world dynamics in Horndeski bulk.

In Chapter 5 we extended the generalisation scheme started in [108] to multiple scalar fields. Here we built up a generalised multiple scalar-tensor theory in arbitrary dimensions that is constrained by having equations of motion of derivative order up to two. This powerful framework provides a starting point for analysis on multi-field inflation, self-tuning theories, etc..

Chapter 6 provides a proof of the most general multiple scalar field theory in flat space-time with field equations of derivative order up to two.

7.2 Outlook

This thesis presents some mathematical analysis on Galileon field theory and Horndeski theory, further, a generalisation to multiple-scalar tensor theory was developed. These investigations of Galileon field theory and the generalisation scheme can be taken further. We discuss some of the possible directions.

Most of the field theories in physics have only solutions that satisfy the NEC condition, namely,

$$T^{\mu\nu} n_\mu n_\nu \geq 0 \tag{7.1}$$

with $n_\mu n^\mu = 0$. The usual field theories with quadratic kinetic terms are unstable and have superluminal modes around NEC violating solutions, if it exists. This is not so, for Galileon field theory due to its higher derivative structure. Furthermore satisfying NEC is a necessary condition for the general singularity theorems in GR. In particular for a FRLW metric NEC condition reduces to

$$p + \rho \geq 0 \tag{7.2}$$

Given that for spatially flat universe,

$$M_p^2 \dot{H} = -\frac{1}{2}(p + \rho) \tag{7.3}$$

any matter content that satisfies NEC forbids a non-singular bounce. Famously, flat-space Galileons have de-Sitter solutions that violate NEC and possess perturbative stability. Yet they suffer from superluminal propagation on rotationally symmetric backgrounds. Armed with the most general scalar-tensor theory with 2nd order field equations, one could investigate this further. Starting from Horndeski's scalar-tensor theory we can reverse engineer to identify the sectors within Horndeski's theory that have perturbatively stable cosmological solutions, violating NEC. At least in the flat space limit one of the sectors would be the Galileon theory itself but it is unclear whether any other theories exist obeying these constraints. Horndeski's theory being a superset which includes but is not limited to the existing modified gravity theories that pass the solar system tests, one expects to uncover novel theories with peculiar characteristics like Galileons. Once a catalogue of theories of this type has been established one can impose further constraints like subluminal propagation of the perturbations, and constrain the possible theories further. A recently such a theory was presented in [**?**]. However a systematic approach would uncover all such theories.

The weakest non-local energy condition is the averaged null energy condition(ANEC) that requires,

$$\int_{\Gamma} T_{ab}(x(\lambda))n^a n^b d\lambda \geq 0 \tag{7.4}$$

here Γ is any null curve with n^a being the tangent vector-field that generates it and λ is an affine parameter. Given that Galileons consist of NEC violating solutions as discussed above, one can ask whether these solutions violate ANEC as well. This straight forward analysis would lead to important conclusions. The existence of solutions that also violate ANEC clearly rules out many of the global results in GR including singularity theorems, positive energy theorem and topological censorship, since in order for these results to hold, atleast the ANEC has to be satisfied.

In a recent paper[24] Burrage et al argued that closed time-like curves (CTC) are forbidden in Galileon field theory coupled to gravity by a mechanism analogous to Hawking's[34] chronology protection conjecture. They assumed a compact Minkowski space-time with periodicity in one space-like direction, and showed that when a closed space-like curve is continuously deformed to form a CTC, quantum fluctuations become arbitrarily large and take the theory beyond its cut-off scale making the effective field description invalid. Given this scenario there appears to be a tension with regards to the quantum effects of field theories. Expectation value of the stress-energy tensor of all quantum fields violate NEC and naively, one expects that this quantum effect would aid the formation of pathologies like wormholes etc. However this is in contrast with the very same quantum back-reaction that underlies chronology protection. Using the formalism of semi-classical gravity defined by $G_{ab} = 8\pi G < T_{ab} >_{ren}$, quantum effects of non-linear field theories, in particular Galileon field theory, can be incorporated. This would be a generalization of the analysis in [48] for non-linear field theories. Where the authors studied linear, massless scalar field theories with arbitrary curvature coupling and showed that violation of ANEC is limited to the Planck scale. One can do a similar analysis for non-linear field theories with derivative interactions, the effects of which might be qualitatively different from free scalar field theories. The result of this investigation would be to resolve whether or not ANEC is violated on macroscopic scales in the presence of non-linear scalar field theories. If such a violation occurs it would be a strong counter argument against Hawking's chronology protection conjecture and would also invalidate singularity theorems.

Horndeski's proof of the most general scalar-tensor theory that yields field equations of derivative order up to two, starts from a very general Lagrangian of the form,

$$L = L\left(g_{ab}, g_{ab,c}; \dots g_{ab,c_1 \dots, c_p}; \phi; \phi_{,a}; \dots; \phi_{,a_1 \dots, a_q}\right) \tag{7.5}$$

we saw that the same action was rediscovered by Deffayet et al[108]. Where they found the most general scalar field theory in flat background with field equations of derivative order up to two in arbitrary dimensions. Remarkably, when this theory restricted to 4-dimensions, is covariantized as described in Chapter 5, it was shown that it yields the Horndeski action. It can be expected that this covariantizing prescription starting from the most general scalar/multiple-scalar field theories would also yield the most general scalar-tensor theories. However, a proof of this does not exist as yet. With the techniques developed in Chapter 6 one could try to prove that these generalised theories are in fact the most general theories.

Appendix A

A.1 Bulk decomposition in detail

Consider the general n^{th} order term appearing in the $\pi - Lagrangian$. The highest order possible is $(n+1)$ in space-time of n-dimensions.

$$L^{(n)} = -\alpha_n \left\{ \pi_{a_2} \pi^{[a_2} \pi^{a_3}_{a_3} \dots \pi^{a_n]}_{a_n} \right\} \tag{A.1}$$

First we make note of the following general identities which we would make use of repeatedly. Note that Einstein-summation is assumed for repeated indices.

$$T^{a_1 a_2 \dots a_n}_{a_1 a_2 \dots a_n} = T^{t i_2 \dots i_n}_{t i_2 \dots i_n} + \dots + T^{i_1 i_2 \dots t \dots i_n}_{i_1 i_2 \dots t \dots i_n} + \dots + T^{i_1 i_2 \dots i_{n-1} t}_{i_1 i_2 \dots i_{n-1} t} + T^{i_1 i_2 \dots i_n}_{i_1 i_2 \dots i_n} \tag{A.2}$$

$$\pi^{[t} \pi^{i_1}_{i_1} \pi^{i_2}_{i_2} \dots \pi^{i_n]}_{i_n} = \pi^t \pi^{[i_1}_{i_1} \pi^{i_2}_{i_2} \dots \pi^{i_n]}_{i_n} - n \pi^{i_1} \pi^{[t}_{i_1} \pi^{i_2}_{i_2} \dots \pi^{i_n]}_{i_n} \tag{A.3}$$

$$T^{[t, i_1, i_2 \dots, i_n]}_{i_1, i_2 \dots, i_n} = T^{t [i_1 i_2 \dots i_n]}_{i_1 i_2 \dots i_n} - \frac{1}{(n-1)!} T^{i_1 [t i_2 \dots i_n]}_{[i_1 i_2 \dots i_n]} \tag{A.4}$$

Using (A.2) L^n can be cast in the following form.

$$L^{(n)} = -\alpha_n \left\{ \pi_t \pi^{[t} \pi^{i_3}_{i_3} \dots \pi^{i_n]}_{i_n} + (n-2) \pi_{i_2} \pi^{[i_2} \pi^t_t \dots \pi^{i_n]}_{i_n} + \pi_{i_2} \pi^{[i_2} \pi^{i_3}_{i_3} \dots \pi^{i_n]}_{i_n} \right\} \tag{A.5}$$

Integrating by parts with respect to the upper time index in the second term before doing the anti-commutation operation we get,

$$L^{(n)} = -\alpha_n \Bigg\{ (n-1)\pi_t \pi^{[t} \pi^{i_3}_{i_3} \ldots \pi^{i_n]}_{i_n} + \pi_{i_2}\pi^{[i_2}\pi^{i_3}_{i_3} \ldots \pi^{i_n]}_{i_n} + (n-2)\partial^{[t|} \left[\pi_t \pi_{i_3} \pi^{|i_3} \pi^{i_4}_{i_4} \ldots \pi^{i_n]}_{i_n} \right] \Bigg\} \tag{A.6}$$

using (A.3) on the first term and (A.4) on the last term we get,

$$= -\alpha_n \Bigg\{ (n-1)\pi_t \pi^t \pi^{[i_3}_{i_3} \ldots \pi^{i_n]}_{i_n} - (n-1)(n-2)\pi_t \pi^{i_3} \pi^{[t}_{i_3} \pi^{i_4}_{i_4} \ldots \pi^{i_n]}_{i_n} + \pi_{i_2}\pi^{[i_2}\pi^{i_3}_{i_3} \ldots \pi^{i_n]}_{i_n}$$
$$+ (n-2)\partial^t \left[\pi_t \pi_{i_3} \pi^{[i_3} \pi^{i_4}_{i_4} \ldots \pi^{i_n]}_{i_n} \right] - \frac{(n-2)}{(n-3)!} \partial^{i_3} \left[\pi_t \pi^{[t} \pi_{[i_3} \pi^{i_4}_{i_4} \ldots \pi^{i_n]}_{i_n} \right] \Bigg\}$$

again using (A.3) for the last term we get,

$$= -\alpha_n \Bigg\{ (n-1)\pi_t \pi^t \pi^{[i_3}_{i_3} \ldots \pi^{i_n]}_{i_n} - (n-1)(n-2)\pi_t \pi^{i_3} \pi^{[t}_{i_3} \pi^{i_4}_{i_4} \ldots \pi^{i_n]}_{i_n}$$
$$+ (n-2)\partial^t \left[\pi_t \pi_{i_3} \pi^{[i_3} \pi^{i_4}_{i_4} \ldots \pi^{i_n]}_{i_n} \right] - \frac{(n-2)}{(n-3)!} \partial^{i_3} \left[\pi_t \pi^t \pi_{[i_3} \pi^{[i_4}_{i_4} \ldots \pi^{i_n]}_{i_n} \right]$$
$$+ \frac{(n-2)(n-3)}{(n-3)!} \partial^{i_3} \left[\pi_t \pi^{i_4} \pi_{[i_3} \pi^{[t}_{i_4} \pi^{i_5}_{i_5} \ldots \pi^{i_n]}_{i_n]} \right] + \pi_{i_2}\pi^{[i_2}\pi^{i_3}_{i_3} \ldots \pi^{i_n]}_{i_n} \Bigg\}$$

$$= \alpha_n \Bigg\{ {}^nC_2 \dot{\pi}^2 \pi^{[i_3}_{i_3} \ldots \pi^{i_n]}_{i_n} - \pi_{i_2}\pi^{[i_2}\pi^{i_3}_{i_3} \ldots \pi^{i_n]}_{i_n} - \frac{(n-2)(n+1)}{2} \partial_{i_3} \left[\dot{\pi}^2 \pi^{[i_3} \pi^{i_4}_{i_4} \ldots \pi^{i_n}_{i_n]} \right]$$
$$- (n-2)\partial^t \left[\dot{\pi} \pi_{i_3} \pi^{[i_3} \pi^{i_4}_{i_4} \ldots \pi^{i_n]}_{i_n} \right] + (n-2)(n-3)\partial_{i_3} \left[\dot{\pi} \pi_{i_4} \pi^{[i_3} \pi^{i_4}_t \pi^{i_5}_{i_5} \ldots \pi^{i_n]}_{i_n} \right] \Bigg\}$$

In the final step we have recast the following term as,

$$\pi_t \pi^{i_3} \pi^{[t}_{i_3} \pi^{i_4}_{i_4} \ldots \pi^{i_n]}_{i_n} = \frac{1}{2} \pi^{i_3} \left(\pi_t \pi^t \right)_{[,i_3|} \left[\pi^{i_4}_{|i_4} \ldots \pi^{i_n}_{i_n]} \right] \tag{A.7}$$

and integrated by parts with respect i_3 inside the commutator. Now we convert the term involving $\partial^t [..]$ into a total derivative in full space-time by adding and substracting a corresponding term involving a total derivative with respect to the spatial slices Σ_t. Thus we get,

$$L_n = \alpha_n \Big\{ {}^nC_2 \dot{\pi}^2 \pi^{[a_3}_{a_3} \ldots \pi^{a_n]}_{a_n} - \pi_{a_2} \pi^{[a_2} \pi^{a_3}_{a_3} \ldots \pi^{a_n]}_{a_n} - \frac{(n-1)(n+1)}{2} \partial_{a_3} \left[\dot{\pi}^2 \pi^{[a_3} \pi^{a_4}_{a_4} \ldots \pi^{a_n]}_{a_n} \right] \tag{A.8}$$

$$- (n-2) \partial^\mu \left[\pi_\mu \pi_{a_3} \pi^{[a_3} \pi^{a_4}_{a_4} \ldots \pi^{a_n]}_{a_n} \right] + (n-2) \partial^a \left[\pi_a \pi_{a_3} \pi^{[a_3} \pi^{a_4}_{a_4} \ldots \pi^{a_n]}_{a_n} \right]$$
$$+ (n-2)(n-3) \partial_{a_3} \left[\dot{\pi} \pi_{a_4} \pi^{[a_3} \pi^{a_4}_t \pi^{a_5}_{a_5} \ldots \pi^{a_n]}_{a_n} \right] \Big\}$$

as promised.

A.2 Decomposing the extrinsic curvature of B

Extrinsic curvature of the time-like surface B is given by,

$$K^B_{ab} = H^c_a \left[\partial_c V_b \right] \tag{A.9}$$

we wish to decompose this interms of the following basis of one forms,

$$E_V = V_a dx^a, \hat{E}_a = q_{ab} dx^b, E_n = n_a dx^a \tag{A.10}$$

we get,

$$K^B_{V\hat{a}} = V^b q^{ad} K^B_{bd} = 0 \tag{A.11}$$
$$K^B_{VV} = V^a V^b K^B_{ab} = 0$$
$$K^B_{\hat{a}\hat{b}} = q^c_a q^d_b K^B_{cd} = q^c_a q^d_b H^e_c \left[\partial_e V_d \right] = -q^e_a \left[\partial_e q^d_b \right] V_d = q^e_a \left[\partial_e [r_b r^d] \right] V_d = (V.r) K^1_{ab}$$
$$K^B_{\hat{a}n} = q^b_a n^c K^B_{bc} = q^b_a n^c H^d_b \left[\partial_d V_c \right] = q^d_a n^c \left[\partial_d V_c \right]$$
$$= q^b_a n^c H^d_c \left[\partial_d V_b \right] = q^b_a n^d \left[\partial_d V_b \right] = V_b n^d \left[\partial_d [r_a r^b] \right] = (V.r) n^d \left[\partial_d r_a \right]$$

Thus,

$$K^B_{ab} = K_{\hat{a}\hat{b}} - 2\, n_{(a|} K^B_{n|\hat{b})} + n_a n_b K^B_{nn} \tag{A.12}$$

as expected.

A.3 Decomposing the derivatives $D_aD_b\pi$, $\bar{D}_a\bar{D}_b\pi$

First we derive the following results to be used later,

$$K^1_{ab} = q^c_a D_c r_b = q^c_a \gamma^d_c \gamma^e_b \left[\partial_d r_e\right] = q^d_a \gamma^e_b \left[\partial_d r_e\right] = q^d_a \left[\partial_d [\gamma^e_b r_e]\right] = q^d_a \left[\partial_d r_b\right] \quad (A.13)$$

$$K^2_{ab} = q^c_a \left[\bar{D}_c n_b\right] = q^c_a H^d_c H^e_b \left[\partial_d n_e\right] = q^d_a H^e_b \left[\partial_d n_e\right] = q^d_a q^e_b \left[\partial_d n_e\right] = q^d_a n_e \left[\partial_d [r_b r_e]\right]$$

$$= (n.r) q^d_a \left[\partial_d r_b\right] = (n.r) K^1_{ab} \quad (A.14)$$

$$\hat{D}_a \hat{D}_b \pi = \hat{D}_a [q^c_b \ \partial_c \pi] = q^d_a q^e_b \partial_d [q^c_e \ \partial_c \pi] = q^d_a q^c_b \left[\partial_c \partial_d \pi\right] - q^d_a q^e_b \left[\partial_d r_e\right] D_r \pi \quad (A.15)$$

$$= q^d_a q^c_b \left[\partial_c \partial_d \pi\right] - K^1_{ab} D_r \pi$$

$$q^d_a r_b r^c \left[\partial_c \partial_d \pi\right] = q^d_a r_b \left[\partial_d (r^c \partial_c \pi)\right] - q^d_a r_b \left[\partial_d r^c\right] \partial_c \pi = r_b \left[\hat{D}_a D_r \pi\right] - r_b K^1_{ac} \hat{D}^c \pi \quad (A.16)$$

$$r_a r_b r^c r^d \partial_c \partial_d \pi = r_a r_b r^c \partial_c (r^d \partial_d \pi) - r_a r_b r^c (\partial_c r^d) \partial_d \pi$$

$$= r_a r_b D^2_r \pi - r_a r_b (r^c D_c r^d) \partial_d \pi = r_a r_b D^2_r \pi \quad (A.17)$$

we have used the result $r_a D^a r_b = 0$ in the last equality.

$$q^d_a n_b n^c \partial_c \partial_d \pi = n_b q^d_a \partial_d (n^c \partial_c \pi) - n_b q^d_a (\partial_d n^c)(\partial_c \pi) \quad (A.18)$$

$$= n_b \hat{D}_a D_n \pi - n_b q^d_a [H^c_e + V^c V_e](\partial_d n^e)(\partial_c \pi)$$

$$= n_b \hat{D}_a D_n \pi - n_b K^2_{ac} \hat{D}^c \pi - n_b q^d_a V_e \partial_d n^e D_V \pi$$

$$= n_b \hat{D}_a D_n \pi - n_b K^2_{ac} \hat{D}^c \pi + n_b K^B_{\hat{a}n} D_V \pi$$

$$n_a n_b n^c n^d \partial_c \partial_d \pi = n_a n_b n^c \partial_c (n^d \partial_d \pi) - n_a n_b (n^c \partial_c n^d) \partial_d \pi \quad (A.19)$$

$$= n_a n_b D^2_n \pi - n_a n_b [H^d_e + V^d V_e](n^c \partial_c n^e) \partial_d \pi$$

$$= n_a n_b D^2_n \pi - n_a n_b (n^c \bar{D}_c n^d) \partial_d \pi - n_a n_b n^c V_e \partial_c n^e D_V \pi$$

$$= n_a n_b D^2_n \pi + n_a n_b K^B_{nn} D_V \pi$$

we used $n_a \bar{D}^a n_b = 0$ in the last equality.

$$D_a D_b \pi = D_a [\gamma^c_b \partial_c \pi] = \gamma^d_a \gamma^e_b \partial [\gamma^c_e \partial_c \pi] = \gamma^d_a \gamma^c_b \partial_c \partial_d \pi \quad (A.20)$$

$$= q^d_a q^c_b \partial_c \partial_d \pi + 2 q^d_{(a|} r_{|b)} r^c \partial_c \partial_d \pi + r_a r_b r^c r^d \partial_c \partial_d \pi$$

$$= \hat{D}_a \hat{D}_b \pi + K^1_{ab} + 2 r_{(a|} \hat{D}_{|b)} D_r \pi - 2 r_{(a|} K^1_{|b)c} \hat{D}^c \pi + r_a r_b D^2_r \pi$$

where A.13, A.15, A.16, A.17 was used.

$$\bar{D}_a\bar{D}_b\pi = \bar{D}_a[H^c_b\partial_c\pi] = H^d_a H^e_b\partial_d[H^c_e\partial_c\pi] = H^d_a H^c_b\partial_c\partial_d\pi + H^d_a H^e_b(\partial_d H^c_e)(\partial_c\pi) \tag{A.21}$$

$$= q^d_a q^c_b\partial_c\partial_d\pi - 2q^d_{(a|}n_{|b)}n^c\partial_c\partial_d\pi + n_a n_b n^c n^d\partial_c\partial_d\pi - H^d_a H^e_b\partial_d[V_e V^c]\partial_c\pi$$

$$= q^d_a q^c_b\partial_c\partial_d\pi - 2q^d_{(a|}n_{|b)}n^c\partial_c\partial_d\pi + n_a n_b n^c n^d\partial_c\partial_d\pi - K^B_{ab}D_V\pi$$

$$= \hat{D}_a\hat{D}_b\pi - \theta K^1_{ab}D_n\pi - 2n_{(a|}\hat{D}_{|b)}D_n\pi + 2n_{(a|}K^2_{|b)c}\hat{D}^c\pi + n_a n_b D^2_n\pi$$

where A.11, A.12, A.13, A.14, A.15, A.18, A.19 was used.

A.4 Boundary term at 5^{th} order

$$
\begin{aligned}
S^5_{total-boundary} &= \alpha_5 \int dt \int_{S_t} \Big\{ -9\dot{\pi}^2 \Big[\pi_{[r}\pi^{\hat{b}}_{\hat{b}}\pi^{\hat{c}}_{\hat{c}]} + (\pi_r)^3 K^{[1b}_b K^{1\hat{c}]}_{\hat{c}} + 2\pi_r K^{1[b}_b K^{1c]}_d \pi_c \pi^d \\
&+ (\pi_r)^2 \pi^{[\hat{b}}_{[\hat{b}} K^{1c]}_{c]} + 2\pi^{[\hat{b}}_{\hat{b}} K^{1c]}_d \pi_c \pi^d - 2\pi_r K^{[1b}_b \pi^{\hat{c}]} \pi_{r\hat{c}} \Big] \qquad \text{(A.22)} \\
&+ 3(1+\theta^2)^{-\frac{1}{2}} (\theta\dot{\pi} + (1+\theta^2)^{\frac{1}{2}}\pi_r) \Big[-(\pi_r \pi_{[r|} + \pi_n \pi_{[n|})\pi^{\hat{b}}_{|\hat{b}}\pi^{\hat{c}}_{\hat{c}]} - (\theta^2 \pi_n^4 + \pi_r^4) K^{1[\hat{b}}_{\hat{b}} K^{1\hat{c}]}_{\hat{c}} \\
&+ 2(\theta^2\pi_n^2 - \pi_r^2)\pi_{\hat{c}}\pi^{\hat{d}} K^{[1b}_b K^{1c]}_d + (\theta\pi_n^3 - \pi_r^3)\pi^{[\hat{b}}_{[\hat{b}} K^{1c]}_{c]} - 2(\theta\pi_n^2\pi_{n\hat{c}} - \pi_r^2\pi_{rc}) K^{[1\hat{b}}_{\hat{b}} \pi^{\hat{c}]} \\
&- 2(\theta\pi_n + \pi_r)\pi^{[\hat{b}}_{\hat{b}} K^{1c]}_d \pi_{\hat{c}}\pi^{\hat{d}} + (\theta^2\pi_n^2 - \pi_r^2)\pi_{\hat{a}}\pi^{[\hat{a}} K^{1b}_b K^{1c]}_c + 2\pi_{\hat{a}}\pi^{[\hat{a}}\pi^{\hat{b}]}_n \pi_{n\hat{b}} + 2\pi_{\hat{a}}\pi^{[\hat{a}}\pi^{b]}_r \pi_{r\hat{b}} \\
&- 2\theta\pi_{\hat{a}}\pi^{[\hat{a}} K^{1b]}_c \pi_{n\hat{b}}\pi^{\hat{c}} - 2\pi_{\hat{a}}\pi^{[\hat{a}} K^{1b]}_c \pi_{r\hat{b}}\pi^{\hat{c}} - 2\theta K^{1c}_b \pi_{\hat{a}}\pi^{[\hat{a}}\pi^{\hat{b}]}_n \pi_{\hat{c}} - 2K^{1b}_c \pi_{\hat{b}}\pi_{\hat{a}}\pi^{[\hat{c}}_r \pi^{\hat{a}]} \\
&+ 2\theta^2 \pi_{\hat{a}}\pi^{[\hat{a}} K^{1\hat{b}]}_c K^1_{bd} \pi^{\hat{c}}\pi^{\hat{d}} + 2\pi_{\hat{a}}\pi^{[\hat{a}} K^{1b]}_c K^1_{bd}\pi^{\hat{c}}\pi^{\hat{d}} - (\theta\pi_n + \pi_r)\pi_{\hat{a}}\pi^{[\hat{a}}\pi^{\hat{b}}_{[\hat{b}} K^{1c]}_{c]} \\
&- 2(\pi_{n^2} + \pi_{r^2})\pi^{[\hat{a}}\pi^{\hat{b}]}_{\hat{b}}\pi_{\hat{a}} + 2(\theta\pi_n\pi_{n^2} - \pi_r\pi_{r^2})\pi_{\hat{a}}\pi^{[\hat{a}} K^{1\hat{b}]}_{\hat{b}} \\
&- 2\theta\pi_n^2\pi_{\hat{a}}\pi^{[\hat{a}}_n K^{1b]}_b + 2\pi_r^2\pi_{\hat{a}}\pi^{[\hat{a}}_r K^{1b]}_b + 2(\theta^2\pi_n^2 - \pi_r^2)\pi_{\hat{a}} K^{1[a}_c K^{1b]}_b \pi^{\hat{c}} \\
&+ 2\pi_n\pi_{\hat{a}}\pi^{[\hat{a}}_n \pi^{\hat{b}]}_{\hat{b}} + 2\pi_r\pi_{\hat{a}}\pi^{[\hat{a}}_r\pi^{\hat{b}]}_{\hat{b}} - 2(\theta\pi_n + \pi_r)\pi^{[\hat{b}}_{\hat{b}} K^{1a]}_c \pi^{\hat{c}}\pi_{\hat{a}} \Big] \\
&+ 3\Big[\pi_r^2 \pi_{[r}\pi^{\hat{b}}_{\hat{b}}\pi^{\hat{c}}_{\hat{c}]} + \pi_r^5 K^{[1b}_b K^{1c]}_c + 2\pi_r^3 K^{[1b}_b K^{1c]}_d \pi_{\hat{c}}\pi^{\hat{d}} + \pi_r^4 \pi^{[\hat{b}}_{[\hat{b}} K^{1c]}_{c]} \\
&+ 2\pi_r^2 \pi^{[\hat{b}}_{\hat{b}} K^{1c]}_d \pi_{\hat{c}}\pi^{\hat{d}} - 2\pi_r^3 K^{1[b}_b \pi^{\hat{c}]}\pi_{r\hat{c}} + \pi_r\pi_{\hat{a}}\pi^{[\hat{a}}\pi^{\hat{b}}_{\hat{b}}\pi^{\hat{c}]}_{\hat{c}} \\
&+ \pi_r^3\pi_{\hat{a}}\pi^{[\hat{a}} K^{1b}_b K^{1c]}_c + 2\pi_r\pi_{r\hat{b}}\pi_{\hat{a}}\pi^{[\hat{a}}_r\pi^{\hat{b}]} - 2\pi_r\pi_{\hat{a}}\pi^{[\hat{a}} K^{1b]}_c K^1_{bd}\pi^{\hat{c}}\pi^{\hat{d}} + 2\pi_r\pi_{\hat{a}}\pi^{[\hat{a}} K^{1b]}_c \pi^{\hat{c}}\pi_{r\hat{b}} \\
&+ 2\pi_r K^{1b}_c \pi_{\hat{b}}\pi_{\hat{a}}\pi^{[\hat{c}}_r\pi^{\hat{a}]} + \pi_r^2\pi_{\hat{a}}\pi^{[\hat{a}}\pi^{\hat{b}}_{[\hat{b}} K^{1c]}_{c]} + 2\pi_{r^2}\pi_r\pi_{\hat{a}}\pi^{[\hat{a}}\pi^{\hat{b}]}_{\hat{b}} + 2\pi_r^2\pi_{r^2}\pi_{\hat{a}}\pi^{[\hat{a}} K^{1b]}_b \\
&- 2\pi_r^3\pi_{\hat{a}}\pi^{[\hat{a}}_r K^{1b]}_b + 2\pi_r^3\pi_{\hat{a}} K^{1[a}_c K^{1b]}_b \pi^{\hat{c}} - 2\pi_r^2\pi_{\hat{a}}\pi^{[\hat{a}}_r\pi^{\hat{b}]}_{\hat{b}} + 2\pi_r^2\pi^{[\hat{b}}_{\hat{b}} K^{1a]}_c \pi^{\hat{c}}\pi_{\hat{a}} \Big] \\
&+ 6\Big[\pi_r\dot{\pi}\dot{\pi}_{\hat{a}}\pi^{[\hat{a}}\pi^{\hat{c}]}_{\hat{c}} + \pi_r^2\dot{\pi}\dot{\pi}_{\hat{a}}\pi^{[\hat{a}} K^{1c]}_c - \dot{\pi}\dot{\pi}_r\pi_{\hat{b}}\pi^{[\hat{b}}\pi^{\hat{c}]}_{\hat{c}} - \dot{\pi}\dot{\pi}_r\pi_{\hat{b}}\pi^{[\hat{b}} K^{1c]}_c \pi_r \Big] \Big\}
\end{aligned}
$$

If one needs to restrict the above expression to the basis $E_u = U_a dx^a, E_r = r_a dx^a, \hat{E}_a = q_{ab}dx^b$, the following expressions can be used to convert the relevant terms (we omit this step for brevity),

$$
\begin{aligned}
\pi_n &= (1+\theta^2)^{\frac{1}{2}}\dot{\pi} + \theta\pi_r \qquad \text{(A.23)} \\
\pi_{n\hat{a}} &= (1+\theta^2)^{\frac{1}{2}}\dot{\pi}_{\hat{a}} + \theta\pi_{r\hat{a}} - K^B_{\hat{a}n}\Big[\theta\dot{\pi} + (1+\theta^2)^{\frac{1}{2}}\pi_r\Big] \\
\pi_{n^2} &= (1+\theta^2)\ddot{\pi} + 2\theta(1+\theta^2)^{\frac{1}{2}}\dot{\pi}_r + \theta^2\pi_{r^2} - K^B_{nn}\Big[(1+\theta^2)^{\frac{1}{2}}\pi_r + \theta\dot{\pi}\Big]
\end{aligned}
$$

Appendix B

B.1 Notations and Identities used in Chapter 4

This section contains identities that were useful in deriving many of the formulae presented in this paper. Recall that we are using bulk coordinates x^a, and boundary coordinates ξ^i, and we may think of the boundary as an embedding $x^a = X^a(\xi)$. This defines tangent vectors $\partial_i X^a$, each of which is orthogonal to the unit outward point normal n^a. **Note that** $s = -1(+1)$ **for space-like (time-like) boundary**. The induced metric on the boundary is defined as

$$h_{ij} = \partial_i X^a \partial_j X^b g_{ab}|_{\partial\mathcal{M}}$$

with the Lie derivative along the normal giving the extrinsic curvature

$$K_{ij} = \frac{1}{2}\mathcal{L}_n h_{ij}$$

We have repeatedly made use of the following expressions

$$\begin{aligned} B_i &= s\partial_i X^a n^b \nabla_a \nabla_b \phi = s\bar{D}_i \nabla_n \phi - sK_{ij}\bar{D}^j \phi \\ C_{ij} &= \partial_i X^a \partial_j X^b \nabla_a \nabla_b \phi = \bar{D}_i \bar{D}_j \phi + sK_{ij} \nabla_n \phi \end{aligned}$$

and the following identites

$$
\begin{aligned}
n^a n^b G_{ab} &= -\frac{s}{2}\left[\bar{R} - sK_i^{[i}K_j^{j]}\right] \\
n^a \partial_i X^b G_{ab} &= \bar{D}^j K_{ij} - \bar{D}_i K \\
\partial_i X^a \partial_i X^b G_{ab} &= \bar{G}_{ij} - s\left[KK_{ij} - 2K_{ik}K_j^k - \frac{1}{2}h_{ij}(K^2 + K_{kl}K^{kl}) + \mathcal{L}_n K_{ij} - h_{ij}\mathcal{L}_n K\right] \\
n^a n^b R_{ab} &= -\mathcal{L}_n K - K_{ij}K^{ij} \\
n^a n^c \partial_i X^b \partial_j X^d R_{abcd} &= -\mathcal{L}_n K_{ij} + K_{ik}K_j^k \\
\partial_i X^a \partial_i X^b R_{ab} &= \bar{R}_{ij} - sKK_{ij} + 2sK_{ik}K_j^k - s\mathcal{L}_n K_{ij} \\
R &= \bar{R} - sK^2 - sK_{ij}K^{ij} - 2s\mathcal{L}_n K
\end{aligned}
$$

We have assumed that the normal vector, n_a, to the boundary is extended along geodesics such that, $n_a \nabla^a n_b = 0$

Appendix C

C.1 Recursive cancellation of higher order terms via counter terms

Here we demonstrate the counter-terms defined for the covariant multi-galileon, C_n^m, give rise to recursive cancellation of higher order derivatives upon variation. The methods described here were also applied to the generalised Horndeski theory, although the details are slightly different. Note that in passing we will present the field equations for the covariant multi-galileon theory (5.2)

C.1.1 π_k equation of motion

Let us begin with the scalar equations of motion. The minimally coupled Lagrangian term at m^{th} order in π_r is given by,

$$C_0^m = \alpha^{i_1 \ldots i_m} \pi_{i_1} \nabla_{a_2}{}^{[a_2} \pi_{i_2} \ldots \nabla_{a_m}{}^{a_m]} \pi_{i_m} \tag{C.1}$$

More generally we define the corresponding counter term required at the n^{th} recursive step at teh same order to be ,

$$C_n^m = T_n^{i_1 I_{2n} J_{m-2n-1}} \pi_{i_1} E_{I_{2n}} F_{J_{m-2n-1}} G_n \tag{C.2}$$

where $T_n^{i_1 I_{2n} J_{m-2n-1}}$ is symmetric in the last $(m-1)$ indices and to be determined. Variation of C_n^m induced by the variation in π_k is,

$$\begin{aligned}\delta C_n^m &= T_n^{kI_{2n}J_{m-2n-1}} E_{I_{2n}} F_{J_{m-2n-1}} \delta\pi_k + 2n\, T_n^{i_1 k i_3 I_{2n-2} J_{m-2n-1}} \pi_{i_1} \nabla_a \pi_{i_3} E_{I_{2n-2}} F_{J_{m-2n-1}} G_n \nabla^a \delta\pi_k \\ &\quad +(m-2n-1) T_n^{i_1 k I_{2n} J_{m-2n-2}} \pi_{i_1} E_{I_{2n}} F_{J_{m-2n-2}} G_n \nabla_a{}^{\hat{a}} \delta\pi_k \end{aligned} \tag{C.3}$$

and subsequent integration by parts yields,

$$\begin{aligned}\delta C_n^m &= \Big\{ T_n^{kI_{2n}J_{m-2n-1}} E_{I_{2n}} F_{J_{m-2n-1}} G_n \\ &\quad -2n\, T_n^{i_1 k i_3 I_{2n-2} J_{m-2n-1}} \nabla_a (\pi_{i_1} \nabla^a \pi_{i_3} E_{I_{2n-2}}) F_{J_{m-2n-1}} G_n \\ &\quad +2n(m-2n-1)\, T_n^{i_1 k i_3 i_4 I_{2n-2} J_{m-2n-2}} \Big[-\pi_{i_1} \nabla_b \pi_{i_3} R^{b\hat{a}}{}_{ac} \nabla^c \pi_{i_4} E_{I_{2n-2}} F_{J_{m-2n-2}} G_n \\ &\quad +2\nabla_a \pi_{i_1} \nabla^{b\hat{a}} \pi_{i_3} \nabla_b \pi_{i_4} E_{I_{2n-2}} F_{J_{m-2n-2}} G_n + \pi_{i_1} (\nabla_{ab} \pi_{i_3} \nabla^{\hat{a}b} \pi_{i_4}) E_{I_{2n-2}} F_{J_{m-2n-2}} G_n \Big] \\ &\quad +T_n^{i_1 k I_{2n} J_{m-2n-2}} (m-2n-1) \nabla_a{}^{\hat{a}} \pi_{i_1} E_{I_{2n}} F_{J_{m-2n-2}} G_n \\ &\quad +2 \binom{n}{2} (m-2n-1) T_n^{i_1 k i_3 i_4 i_5 i_6 I_{2n-4} J_{m-2n-2}} \pi_{i_1} (\nabla_{ab} \pi_{i_3} \nabla^b \pi_{i_4} \nabla_c{}^{\hat{a}} \pi_{i_5} \nabla^c \pi_{i_6}) E_{I_{2n-4}} F_{J_{m-2n-2}} G_n \\ &\quad \frac{m-2n-1}{2} \binom{m-2n-2}{2} T_n^{i_1 k i_3 i_4 I_{2n} J_{m-2n-4}} \pi_{i_1} (R^{\hat{b}\hat{a}}{}_{ad} \nabla^d \pi_{i_3}) (R_{bc}{}^{p(c)e} \nabla_e \pi_{i_4}) E_{I_{2n}} F_{J_{m-2n-4}} G_n \\ &\quad -2\, T_n^{i_1 k i_3 I_{2n} J_{m-2n-3}} \pi_{i_1} \nabla_a^c \pi_{i_3} R^{\hat{a}\hat{b}}{}_{bc} E_{I_{2n}} F_{J_{m-2n-3}} G_n \\ &\quad -\frac{(m-2n-1)(m-2n-2)}{4} T_n^{i_1 k i_3 I_{2n} J_{m-2n-3}} (\pi_{i_1} \nabla^c \pi_{i_3} \nabla_c R^{\hat{a}\hat{b}}{}_{ab}) E_{I_{2n}} F_{J_{m-2n-3}} G_n \\ &\quad -2n^2\, T_n^{i_1 k i_3 I_{2n-2} J_{m-2n-1}} \pi_{i_1} \nabla_c \pi_{i_3} (\nabla^c R^{\hat{a}\hat{b}}{}_{ab}) E_{I_{2n-2}} F_{J_{m-2n-1}} G_{n-1} \Big\} \delta\pi_k \end{aligned} \tag{C.4}$$

Notice that the last two terms contain third derivative terms in the metric,

$$\begin{aligned}\delta C_n^m &\supset -\Big\{ \frac{(m-2n-1)(m-2n-2)}{4} T_n^{i_1 k i_3 I_{2n} J_{m-2n-3}} (\pi_{i_1} \nabla^c \pi_{i_3} \nabla_c R^{\hat{a}\hat{b}}{}_{ab}) E_{I_{2n}} F_{J_{m-2n-3}} G_n \\ &\quad -2n^2\, T_n^{i_1 k i_3 I_{2n-2} J_{m-2n-1}} \pi_{i_1} \nabla_c \pi_{i_3} (\nabla^c R^{\hat{a}\hat{b}}{}_{ab}) \pi_{i_3} E_{I_{2n-2}} F_{J_{m-2n-1}} G_{n-1} \Big\} \delta\pi_k \end{aligned} \tag{C.5}$$

It is clear that these terms can be absorbed into each other recursively if the following relationship holds,

$$\frac{T_{s+1}^{i_1 \ldots i_m}}{T_s^{i_1 \ldots i_m}} = -\frac{1}{8} \frac{(m-2s-1)(m-2s-2)}{(s+1)^2} \tag{C.6}$$

which implies that

$$\prod_{s=0}^{n-1} \left[\frac{T_{s+1}^{i_1 \ldots i_m}}{T_s^{i_1 \ldots i_m}} \right] = \frac{T_n^{i_1 \ldots i_m}}{\alpha^{i_1 \ldots i_m}} = \left(-\frac{1}{8}\right)^n \frac{(m-1)!}{(n!)^2\,(m-2n-1)!} \qquad m > 2 \tag{C.7}$$

This yields the result given by equation (5.15). Finally, to express the π_k equation of motion, we collect terms that at most second order in δC_n^m. These are given by

$$\begin{aligned}
\epsilon_n^{(k)\ m} &= T_n^{kI_{2n}J_{m-2n-1}} E_{I_{2n}} F_{J_{m-2n-1}} G_n \\
& -2n\, T_n^{i_1 k i_3 I_{2n-2} J_{m-2n-1}} \nabla_a (\pi_{i_1} \nabla^a \pi_{i_3} E_{I_{2n-2}}) F_{J_{m-2n-1}} G_n \\
& +2n(m-2n-1)\, T_n^{i_1 k i_3 i_4 I_{2n-2} J_{m-2n-2}} \Big[-\pi_{i_1} \nabla_b \pi_{i_3} R^{b\hat{a}}{}_{ac} \nabla^c \pi_{i_4} E_{I_{2n-2}} F_{J_{m-2n-2}} G_n \\
& +2 \nabla_a \pi_{i_1} \nabla^{b\hat{a}} \pi_{i_3} \nabla_b \pi_{i_4} E_{I_{2n-2}} F_{J_{m-2n-2}} G_n + \pi_{i_1} (\nabla_{ab} \pi_{i_3} \nabla^{\hat{a}b} \pi_{i_4}) E_{I_{2n-2}} F_{J_{m-2n-2}} G_n \Big] \\
& + T_n^{i_1 k I_{2n} J_{m-2n-2}} (m-2n-1) \nabla_a{}^{\hat{a}} \pi_{i_1} E_{I_{2n}} F_{J_{m-2n-2}} G_n \\
& +2 \binom{n}{2} (m-2n-1) T_n^{i_1 k i_3 i_4 i_5 i_6 I_{2n-4} J_{m-2n-2}} \pi_{i_1} (\nabla_{ab} \pi_{i_3} \nabla^b \pi_{i_4} \nabla^{\hat{a}}{}_c \pi_{i_5} \nabla^c \pi_{i_6}) E_{I_{2n-4}} F_{J_{m-2n-2}} G_n \\
& \frac{m-2n-1}{2} \binom{m-2n-2}{2} T_n^{i_1 k i_3 i_4 I_{2n} J_{m-2n-4}} \pi_{i_1} (R^{\hat{b}\hat{a}}{}_{ad} \nabla^d \pi_{i_3}) (R_{bc}{}^{p(c)e} \nabla_e \pi_{i_4}) E_{I_{2n}} F_{J_{m-2n-4}} G_n \\
& -2\, T_n^{i_1 k i_3 I_{2n} J_{m-2n-3}} \pi_{i_1} \nabla_a{}^c \pi_{i_3} R^{\hat{a}\hat{b}}{}_{bc} E_{I_{2n}} F_{J_{m-2n-3}} G_n
\end{aligned} \tag{C.8}$$

It follows that the π_k equation of motion is given by,

$$\sum_{m=1}^{D+1} \sum_{n=0}^{\lfloor \frac{m-1}{2} \rfloor} \epsilon_n^{(k)\ m} = 0 \tag{C.9}$$

which is, of course, at most second order in derivatives, as desired.

C.1.2 g_{ab} equation of motion

We now verify that our chosen counter-terms (5.14) also guarantee second order equations of motion from metric variation. To this end, we first note the following identities

$$\delta R^{\hat{a}\hat{b}}{}_{ab} X^{\hat{d}\hat{e}\cdots} = \left(\delta g^{\hat{b}c} R^{\hat{a}}{}_{cab} - 2 g^{\hat{b}c} \delta g_{bc;a}{}^{\hat{a}} \right) X^{\hat{d}\hat{e}\cdots} \tag{C.10}$$

$$\delta \nabla_a{}^{\hat{a}} \pi_s X^{\hat{d}\hat{e}\cdots} = \left(-\nabla^{b\hat{a}} \pi_s \delta g_{ab} - \nabla^a \pi_s \delta g_{ab}{}^{;\hat{a}} + \frac{1}{2} g^{b\hat{a}} \nabla^c \pi_s \delta g_{ab;c} \right) X^{\hat{d}\hat{e}\cdots} \tag{C.11}$$

The variation of the counter-term induced by the metric variation is,

$$\begin{aligned}
\delta C_n^m &= T_n^{i_1 I_{2n} J_{m-2n-1}} \left(\pi_{i_1}\delta E_{I_{2n}} F_{J_{m-2n-1}} G_n + \pi_{i_1} E_{I_{2n}} \delta F_{J_{m-2n-1}} G_n + \pi_{i_1} E_{I_{2n}} F_{J_{m-2n-1}} \delta G_n\right) \\
&= -T_n^{i_1 i_2 i_3 I_{2n-2} J_{m-2n-2}} \pi_{i_1} \nabla^a \pi_{i_2} \nabla^b \pi_{i_3} E_{I_{2n-2}} F_{J_{m-2n-1}} G_n \delta g_{ab} \\
&\quad + (m-2n-1)\, T_n^{i_1 i_2 I_{2n} J_{m-2n-2}} \pi_{i_1} E_{I_{2n}} F_{J_{m-2n-2}} G_n \\
&\quad \times \Big(-\nabla^{b\hat{a}} \pi_{i_2} \delta g_{ab} - \nabla^b \delta g_{ab}{}^{;\hat{a}} + \frac{1}{2} g^{\hat{a}b} \nabla^c \pi_{i_2} \delta g_{ab;c} \Big) \\
&\quad + n\, T_n^{i_1 I_{2n} J_{m-2n-1}} \pi_{i_1} E_{I_{2n}} F_{J_{m-2n-1}} G_{n-1} \left(R_c{}^{b\hat{a}\hat{c}} \delta g_{ab} - 2 g^{\hat{a}b} \delta g_{ab;c}{}^{\hat{c}} \right) \qquad \text{(C.12)}
\end{aligned}$$

so that after integration by parts we obtain,

$$\begin{aligned}
\delta C_n^m &= \Big\{ T_n^{i_1 i_2 i_3 I_{2n-2} J_{m-2n-1}} \Big[-n \pi_{i_1} \nabla^a \pi_{i_2} \nabla^b \pi_{i_3} E_{I_{2n-2}} F_{J_{m-2n-1}} G_n \\
&\quad -4n^2\, \pi_{i_1} \nabla_{cd} \pi_{i_2} \nabla^{\hat{c}d} \pi_{i_3} E_{I_{2n-2}} F_{J_{m-2n-1}} G_{n-1} g^{\hat{a}b} \\
&\quad -4n^2\, \pi_{i_1} \nabla^{\hat{c}} \nabla_{cd} \pi_{i_2} \nabla^d \pi_{i_3} E_{I_{2n-2}} F_{J_{m-2n-1}} G_{n-1} g^{\hat{a}b} \Big] \\
&\quad +(m-2n-1) T_n^{i_1 i_2 I_{2n} J_{m-2n-2}} \Big[-\pi_{i_1} \nabla^{b\hat{a}} \pi_{i_2} E_{I_{2n}} F_{J_{m-2n-2}} G_n + (\pi_{i_1} \nabla^b \pi_{i_2} E_{I_{2n}})^{;\hat{a}} F_{J_{m-2n-2}} G_n \\
&\quad -\frac{1}{2} (\pi_{i_1} \nabla^c \pi_{i_2} E_{I_{2n}})_{;c} F_{J_{m-2n-2}} G_n g^{\hat{a}b} - 2n\, R^{\hat{c}\hat{d}}{}_{de} \nabla^e \pi_{i_2} \nabla_c (E_{I_{2n}} \pi_{i_1}) F_{J_{m-2n-2}} G_{n-1} g^{\hat{a}b} \\
&\quad -n\, \pi_{i_1} \nabla_d{}^c \pi_{i_2} R^{\hat{d}\hat{e}}{}_{ec} E_{I_{2n}} F_{J_{m-2n-2}} G_{n-1} g^{\hat{a}b} \Big] \\
&\quad + T_n^{i_1 i_2 i_3 I_{2n} J_{m-2n-3}} \Bigg[\frac{(m-2n-1)(m-2n-2)}{2} \pi_{i_1} \nabla^b \pi_{i_2} R^{\hat{a}\hat{c}}{}_{cd} \nabla^d \pi_{i_3} E_{I_{2n}} F_{J_{m-2n-3}} G_n \\
&\quad - \frac{(m-2n-1)(m-2n-2)}{2} \pi_{i_1} \nabla^c \pi_{i_2} R_{cd}{}^{\hat{d}e} \nabla_e \pi_{i_3} E_{I_{2n}} F_{J_{m-2n-3}} G_n g^{\hat{a}b} \\
&\quad -n \binom{m-2n-1}{2} \pi_{i_1} R^{\hat{c}\hat{d}}{}_{de} \nabla^e \pi_{i_2} R_{cf}{}^{\hat{f}g} \nabla_g \pi_{i_3} E_{I_{2n}} F_{J_{m-2n-3}} G_{n-1} g^{\hat{a}b} \Bigg] \\
&\quad + T_n^{i_1 I_{2n} J_{m-2n-1}} \Big[n \pi_{i_1} E_{I_{2n}} F_{J_{m-2n-1}} G_{n-1} R_c{}^{b\hat{a}\hat{c}} - 2n \nabla_c^{\hat{c}} \pi_{i_1} E_{I_{2n}} F_{J_{m-2n-1}} G_{n-1} g^{\hat{a}b} \\
&\quad -4n\, \nabla_c \pi_{i_1} \nabla^c E_{I_{2n}} F_{J_{m-2n-1}} G_{n-1} g^{\hat{a}b} \Big] \\
&\quad - T_n^{i_1 i_2 i_3 I_{2n} J_{m-2n-3}} \frac{(m-2n-1)(m-2n-2)}{2} \pi_{i_1} \nabla^{\hat{c}} \nabla_{cd} \pi_{i_2} \nabla^d \pi_{i_3} E_{I_{2n}} F_{J_{m-2n-3}} G_n g^{\hat{a}b} \\
&\quad -4n \binom{n}{2} T_n^{i_1 i_2 i_3 i_4 i_5 I_{2n-4} J_{m-2n-1}} \pi_{i_1} \nabla^{\hat{c}} (\nabla_d \pi_{i_2} \nabla^d \pi_{i_3}) \nabla_c (\nabla_e \pi_{i_4} \nabla^e \pi_{i_5}) \\
&\qquad \times E_{I_{2n-4}} F_{J_{m-2n-1}} G_{n-1} g^{\hat{a}b} \Big\} \delta g_{ab} \qquad \text{(C.13)}
\end{aligned}$$

Again, focussing on the third derivative terms,

$$\begin{aligned}
\delta C_n^m \supset &\Big\{ -4n^2 T_n^{i_1 i_2 i_3 I_{2n-2} J_{m-2n-1}} \pi_{i_1} \nabla^{\hat{c}} \nabla_{cd} \pi_{i_2} \nabla^d \pi_{i_3} E_{I_{2n-2}} F_{J_{m-2n-1}} G_{n-1} g^{p(a)b} \qquad \text{(C.14)} \\
&- T_n^{i_1 i_2 i_3 I_{2n} J_{m-2n-3}} \frac{(m-2n-1)(m-2n-2)}{2} \pi_{i_1} \nabla^{\hat{c}} \nabla_{cd} \pi_{i_2} \nabla^d \pi_{i_3} E_{I_{2n}} F_{J_{m-2n-3}} G_n g^{\hat{a}b} \Big\} \delta g_{ab}
\end{aligned}$$

we see that they can are recursively cancelled if the same relationship (C.6) holds. As before, to express the metric equation of motion we collect terms up to second order in derivatives, remembering to include the term generated by the variation of the metric determinant $\sqrt{-g}$. We find that the g_{ab} equations of motion are given by,

$$\sum_{m=1}^{D+1}\sum_{n=0}^{\lfloor\frac{m-1}{2}\rfloor}\mathcal{E}_n^{(m)ab}=0 \tag{C.15}$$

where $\mathcal{E}_n^{(m)ab}=\frac{1}{2}\left(\epsilon_n^{(m)ab}+\epsilon_n^{(m)ba}\right)$ and

$$\begin{aligned}
\epsilon_n^{(m)ab} = & -n\,T_n^{i_1i_2i_3I_{2n-2}J_{m-2n-1}}\pi_{i_1}\nabla^a\pi_{i_2}\nabla^b\pi_{i_3}E_{I_{2n-2}}F_{J_{m-2n-1}}G_n \\
& -4n^2\,\pi_{i_1}\nabla_{cd}\pi_{i_2}\nabla^{\hat{c}\hat{d}}\pi_{i_3}E_{I_{2n-2}}F_{J_{m-2n-1}}G_{n-1}g^{\hat{a}b} \\
& +(m-2n-1)T_n^{i_1i_2I_{2n}J_{m-2n-2}}\Big[-\pi_{i_1}\nabla^{b\hat{a}}\pi_{i_2}E_{I_{2n}}F_{J_{m-2n-2}}G_n \\
& +(\pi_{i_1}\nabla^b\pi_{i_2}E_{I_{2n}})^{;\hat{a}}F_{J_{m-2n-2}}G_n \\
& -\frac{1}{2}(\pi_{i_1}\nabla^c\pi_{i_2}E_{I_{2n}})_{;c}F_{J_{m-2n-2}}G_ng^{\hat{a}b}-2n\,R^{\hat{c}\hat{d}}{}_{de}\nabla^e\pi_{i_2}\nabla_c(E_{I_{2n}}\pi_{i_1})F_{J_{m-2n-2}}G_{n-1}g^{\hat{a}b} \\
& -n\,\pi_{i_1}\nabla^c_d\pi_{i_2}R^{\hat{d}\hat{e}}{}_{ec}E_{I_{2n}}F_{J_{m-2n-2}}G_{n-1}g^{\hat{a}b}\Big] \\
& +T_n^{i_1i_2i_3I_{2n}J_{m-2n-3}}\Big[\frac{(m-2n-1)(m-2n-2)}{2}\pi_{i_1}\nabla^b\pi_{i_2}R^{\hat{a}\hat{c}}{}_{cd}\nabla^d\pi_{i_3}E_{I_{2n}}F_{J_{m-2n-3}}G_n \\
& -\frac{(m-2n-1)(m-2n-2)}{2}\pi_{i_1}\nabla^c\pi_{i_2}R_{cd}{}^{\hat{d}e}\nabla_e\pi_{i_3}E_{I_{2n}}F_{J_{m-2n-3}}G_ng^{\hat{a}b} \\
& -n\binom{m-2n-1}{2}\pi_{i_1}R^{\hat{c}\hat{d}}{}_{de}\nabla^e\pi_{i_2}R_{cf}{}^{\hat{f}g}\nabla_g\pi_{i_3}E_{I_{2n}}F_{J_{m-2n-3}}G_{n-1}g^{\hat{a}b}\Big] \\
& +T_n^{i_1I_{2n}J_{m-2n-1}}\Big[n\pi_{i_1}E_{I_{2n}}F_{J_{m-2n-1}}G_{n-1}R_c{}^{b\hat{a}\hat{c}}-2n\nabla^{\hat{c}}_c\pi_{i_1}E_{I_{2n}}F_{J_{m-2n-1}}G_{n-1}g^{\hat{a}b} \\
& -4n\,\nabla_c\pi_{i_1}\nabla^cE_{I_{2n}}F_{J_{m-2n-1}}G_{n-1}g^{\hat{a}b}\Big]+\frac{1}{2}g^{ab}T_n^{i_1I_{2n}J_{m-2n-1}}\pi_{i_1}E_{I_{2n}}F_{J_{m-2n-1}}G_n \\
& -4n\binom{n}{2}T_n^{i_1i_2i_3i_4i_5I_{2n-4}J_{m-2n-1}}\pi_{i_1}\nabla^{\hat{c}}(\nabla_d\pi_{i_2}\nabla^d\pi_{i_3})\nabla_c(\nabla_e\pi_{i_4}\nabla^e\pi_{i_5})E_{I_{2n-4}}F_{J_{m-2n-1}}G_{n-1}g^{\hat{a}b}
\end{aligned} \tag{C.16}$$

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