# PERTURBATION AND ASYMPTOTIC METHODS IN MECHANICS AND WAVES

by

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## ABSTRACT

In this thesis perturbation and asymptotic methods for the solution of three non-linear problems are considered.

In Part I approximate methods for the analysis of linear and non-linear waves are developed. Waves in a rod of varying cross-section are examined as are waves propagating through an inhomogeneous region in which the wave speed is continuous but has discontinuous derivative. Iterative procedures are used in both these problems and an estimate is obtained for the region in which these methods converge.

In Part II non-linear Rayleigh waves, elastic surface waves of permanent form, are analysed. A straight-forward perturbation about linear sinusoidal waveforms is shown to fail. Retaining the full solution of the linear equations it is found that the surface elevation profiles of non-distorting waveforms must satisfy a certain non-linear functional equation which reduces in the small strain limit to a quadratic functional equation. In Chapter Four, periodic, but non-sinusoidal surface waves on a compressible material with non-linear constitutive law are obtained. Non-periodic waveforms are also considered. Periodic Rayleigh waves on an incompressible material are obtained in Chapter Five.

A fibre-reinforced belt stretched round a system of pulleys is analysed in Part III. The general theory, developed in Chapter Six, is applied in Chapter Seven to the case of a belf round two pulleys. A mathematical consequence of using the ideal theory in which the constraints of incompressibility and inextensibility are imposed, is the occurrence of singular sheets of fibres which carry infinite stress, but finite force. The ideal theory also gives an undetermined contribution to the tension carried by the fibres. This is determined by considering the case when the fibres are slightly extensible. The boundary layers are examined and the tension throughout the belt obtained.

# CHAPTER 1

## INTRODUCTION

Exact analytical solutions often cannot be found for physical problems involving non-linearities, variable coefficients, non-linear boundary conditions or unknown boundaries. When an exact solution can be found it may be of such complexity that it is not useful for mathematical or physical interpretation or numerical evaluation. Perturbation and asymptotic methods are frequently useful in providing an approximate solution from which the nature of the actual solution may be deduced. The expansions may be in terms of a small or large parameter or coordinate. In this thesis we consider three topics involving asymptotic or perturbation techniques.

First in Chapter Two we consider approximate methods for linear and non-linear waves in inhomogeneous regions. Whitham (62), (63) and Luke (27) consider the slow variation in time and space of a non-linear wavetrain. In (37) Parker analyses the situation where non-linearity dominates dispersion and when the disturbances are 'modulated simple waves'. We examine waves in a rod of varying cross-section for which we obtain a system of hyperbolic equations. We also consider a linear problem for wave propagation through an inhomogeneous region, in the example analysed the wave speed is continuous, but has discontinuous derivative. In both these examples we set up iterative procedures and derive an estimate for the region in which these procedures converge. In particular we obtain an estimate for the region in which an iteration based on 'modulated simple waves' converges.

In the first case we show how iterative solution based on the usual procedure of integration along $X = 0$ and $\alpha = 0$ may fail, where X is the independent space variable and the curves $\alpha$ = constant denote the forward characteristics. We therefore amend the procedure by integrating along the backward characteristics. The example under consideration is that of waves propagating along a rod with non-uniform, but slowly varying cross-section which allows us to use a perturbation method of solution. We consider 'modulated simple waves', set up an iterative procedure and derive an estimate for the region in which this procedure converges.

In the second case, for wave propagation through an inhomogeneous region, we solve a particular example exactly, where the wave speed depends on a small parameter. We then consider the region in which the wave speed is varying and obtain asymptotic solutions in this region by series solutions and Laplace transforms. In this region we also transform the equation to characteristic coordinates and consider the solution using an iterative procedure based on Riemann's method. In Section 2.6 we examine a further method, which has the advantage that since it is applied directly to the wave equation, it is not restricted to the particular inhomogeneity considered in the earlier sections.

In Part II, we consider non-linear Rayleigh waves, that is elastic surface waves of permanent form (46). Such waves may be expected to appear when a disturbance travels for large distances near the traction-free surface of a homogeneous elastic half-space. We show that a straight-forward perturbation about linear sinusoidal waveforms fails, since after the leading approximation, solutions can only be found which contain terms growing exponentially with depth. The assumption that the surface elevation is close to a sinusoidal wavetrain must be relaxed by allowing the leading approximation to the disturbance to be an arbitrary non-distorting solution of the linear elastic equations. The full solution of the linear equations is retained and it is found that the surface elevation profiles of non-distorting waveforms must satisfy a certain non-linear functional equation. In the small strain limit, this reduces to a quadratic functional equation. This mathematical feature will arise whenever the linear waves have no dispersion. The general theory is developed in Chapter Three following the analysis of Parker in (41) and in Chapter Four this is applied to an example of a surface wave on a compressible material with non-linear constitutive law. In Chapter Five, Rayleigh waves on an incompressible material are examined. Methods are presented for the analysis of both periodic, but non-sinusoidal, waveforms and non-periodic waveforms.

In Part III, we consider a third topic, the analysis of a fibre-reinforced belt stretched round a system of pulleys as has been considered by Everatt (11). The general theory is developed in Chapter Six and is applied in Chapter Seven to the case of a belt round two pulleys. The properties of a fibre-reinforced material have been considered by Spencer (52) and may be specified for the composite as a whole, although they derive ultimately from the properties and geometrical arrangement of its constituents. The ideal theory, in which the constraints of incompressibility and inextensibility are imposed, is considered. A mathematical consequence of this is the occurrence of singular sheets of fibres which carry infinite stress but finite force, as has been shown by Pipkin and Rogers (43). Also, use of the ideal theory leads to an undetermined contribution to the tension carried by the fibres, which Everatt does not calculate. To determine this function we consider the situation when the fibres are slightly extensible. We perform a perturbation analysis using the small parameter introduced here. The constraint of periodicity round the belt gives an equation for this contribution to the tension. The boundary layers are also examined in more detail and the tension throughout the belt obtained. It is found that the large stresses are confined to narrow layers near the surfaces for a range of geometrical parameters. The method of determination of this tension function is analogous to the elimination of secular terms familiar in perturbation processes describing periodic oscillations.

# PART I

# INTRODUCTION

There are two main classes of linear waves: dispersive and non-dispersive. The dispersion relation between the frequency $\omega$ and the wavenumber k may be used to distinguish between the two classes of waves. Waves are called non-dispersive if the phase speed $\omega/k$ is independent of k, otherwise they are called dispersive.

For dispersive waves, solutions representing infinitely long, periodic wavetrains may be readily obtained. Water waves and plasma waves are examples of these. For the usual linear examples these waves are sinusoidal and a general solution may then be constructed by the superposition of these wavetrains in a Fourier integral. Since the different uniform wavetrains generally have different velocities of propagation, a local disturbance expressed in this way tends to disperse into its various component waves. The saddle point or stationary phase approximation (10) shows for typical examples that a nearly uniform wavetrain eventually develops in any locality.

The equations for a linear dispersive system are often obtained by linearization of governing equations, which are originally non-linear. In the examination of these non-linear equations the superposition principle cannot be applied and an approximation becomes necessary. Nayfeh and Mook (33) consider linear longitudinal waves along a uniform elastic bar as an example of non-dispersive waves and linear transverse waves along a uniform elastic bar as an example of dispersive waves. They extend this to consider the behaviour of non-linear waves in a bar, whose properties vary slowly along its length. For non-linear waveforms the speed is a function of the wavelength and the amplitude. In (62) Whitham introduces an averaging technique to determine the slow variation in time and space of a non-linear wavetrain. These slowly varying wavetrains occur in two main problems, the first of which has already been mentioned, that of an initial disturbance in the linear theory dispersing into a slowly varying wavetrain. The second one occurs when wavetrains enter a slowly varying medium, examples of this are water waves over a sloping beach and plasma waves propagating through a slowly varying magnetic field. In (62) Whitham considers the first problem and in (63) he applies an averaging procedure to the Lagrangian of the original system to obtain the results in a simpler way and also to extend the theory to include more space dimensions and the propagation in a non-uniform medium. In (27) Luke shows how the same results may be obtained from the differential equation as the first approximation in a formal perturbation expansion. Whitham (64) justifies formally the results obtained by considering a perturbation expansion, but working directly on the variational principle rather than on the Euler equations as Luke does.

For non-linear waves, not only does no superposition principle apply, so there is no guarantee that disturbances do resolve themselves into slowly modulating waves, but even if a single wave does emerge it may suffer continual profile distortion leading to shock formation. In some disturbances this is prevented by dispersion, in non-dispersive systems it arises for arbitrarily small non-linearity, provided that propagation distances are sufficiently large. In the work of Whitham and Luke, already discussed, the non-linear distortion is everywhere held in check by dispersion. In (37) Parker discusses the situation where non-linearity dominates dispersion, and where the disturbances are 'modulated simple waves'. Each of these leads to asymptotic methods based on two scales of both length and time. In (38) Parker describes the underlying assumptions common to these techniques and also discusses the distinguishing features of the two procedures.

When the equations are linear and the coefficients independent of $(\underset{\sim}{x},t)$, where the $x_i$ are space coordinates and t is time, there exist plane wave solutions $\underset{\sim}{u} = \mathrm{Re}(\underset{\sim}{U}_0 e^{i(\underset{\sim}{k}.\underset{\sim}{x}-\omega t)})$ corresponding to wave vector $\underset{\sim}{k}$ and frequency $\omega$. For non-constant coefficients there exist solutions having the limiting form

$$\underset{\sim}{u} \sim \mathrm{Re}(\underset{\sim}{U}(\underset{\sim}{x},t)e^{i(\underset{\sim}{k}.\underset{\sim}{x}-\omega t)})$$

and valid for large $|\underset{\sim}{k}|$ and $|\omega|$. These disturbances travel along the rays of geometric acoustics.

For non-linear systems of equations, a similar description is applicable if the sinusoidal waveforms are replaced by non-distorting wave profiles or simple waves. In the non-distorting solutions, variables depend only on the single phase variable $\alpha \equiv \underset{\sim}{k}.\underset{\sim}{x} - \omega t$, where the phase planes $\alpha$ = constant are perpendicular to the vector $\underset{\sim}{k}$ and each propagates with the same speed $\omega/|\underset{\sim}{k}|$. In conservative dispersive systems the functions $U_i(\alpha)$ are periodic and the disturbances $u_i = U_i(\alpha)$ are known as periodic wavetrains. Important examples are cnoidal waves, involving the elliptic function $\mathrm{cn}\alpha$, arising in water wave theory. For simple waves the variables $u_i = U_i(\alpha)$ are again functions of a single variable $\alpha$ and the surfaces of constant $\alpha$ are plane. However, the 'wavelets' $\alpha$ = constant are given implicitly by some equation

$$\underset{\sim}{k}(\alpha).\underset{\sim}{x} - \omega(\alpha)t \quad = \quad \phi(\alpha)$$

and need not be parallel. Even when they are parallel, their propagation speed $\omega/|\underset{\sim}{k}|$ usually depends on $\alpha$, so that the wavelets tend either to coalesce or to spread out and the profile distorts.

Both periodic wavetrains and simple waves may be used as approximations to the local behaviour in more general disturbances. Since the local structure of each type of wave is determined by the solution of certain ordinary differential equations, the functions $U_i(\alpha)$ contain some arbitrary constants of integration, and in the case of simple waves, also certain arbitrary functions related to the wave profile at a reference time. In (38) Parker describes procedures for disturbances in which these 'constants' and 'profile functions' slowly vary. For example, a periodic wavetrain may encounter a region having slowly varying physical properties. The consequent disturbance will approximate to a wavetrain in which the amplitude, wavelength and propagation direction slowly vary. Similarly an acoustic or elastic wave of arbitrary initial profile may be refracted and modulated by inhomogeneities or interactions with other 'slowly varying' disturbances. 'Modulated simple wave' theory then determines how the profile of the wave changes as the disturbance propagates.

In (30) Miura and Kruskal consider a non-linear generalization of the usual WKB method, which is similar to Whitham's averaging method and apply it to the Korteweg-de Vries equation.

In (58) Varley and Cumberbatch derive equations which govern the mode of propagation and change in strengths of wavefronts whose behaviour are controlled by quite general systems of quasi-linear hyperbolic equations. A simple example of the type of problem under consideration is provided by the one-dimensional flow of a compressible gas which is generated by the motion of a piston in a cylindrical tube. If the piston is accelerated from rest, the wavefront, which separates the disturbed region from the undisturbed region, propagates with sound speed down the tube. The variation in particle acceleration behind the wavefront can be determined independently of the remaining flow up to the time that the acceleration becomes infinite when a shock begins to form. The general theory is applied to discuss conditions at the head of a sound wave of arbitrary initial form and strength and also at the head of gravity waves on sloping beaches. Conditions under which shocks and bores occur are derived. If a shock does not form, the decay rate is always more rapid than that predicted by linear theory, which becomes progressively worse with increasing time. Asymptotically for 'large' time, conditions at an expansion front 'forget' the details of conditions at any finite time. This aspect of non-linearity is in direct contradiction with linear theory and has been noted for classes of exact solutions. In the next section we show how, for waves in a rod, integration along $X = 0$ and $\alpha = 0$, where $\alpha$ is a characteristic variable may fail if the cross-sectional area does not have suitable continuity properties. To overcome this we set up an iterative procedure and show that this converges for some region.

A generalization of the technique of Varley and Cumberbatch is used by Varley (57) to discuss the mode of propagation and decay in strength of an acceleration front of arbitrary initial form and strength, as it advances into a finitely strained viscoelastic material, which until the time of arrival of the front is undergoing any admissible deformation. Such a front is produced, say, when the boundary of a material is suddenly accelerated from rest.

Parker and Seymour (42) have applied the theory of 'modulated simple waves' to the problem of pulses in an inhomogeneous granular material. This theory yields exact equations for signal growth which are generalizations of those in acceleration wave theory.

In Section 2.2 we consider waves in a rod of varying cross-section, which leads to a system of linear hyperbolic equations with non-uniform characteristic speed. We set up an iterative solution for these equations and show that for some region this procedure converges. In Section 2.3 we determine the solution to the wave equation for an example where the wave speed is continuous but has discontinuous derivative.

An asymptotic solution to this problem is considered in Section 2.4. In Section 2.5 an alternative method of solution is developed using Riemann's method and we show that an iterative solution to this equation converges for some region of the $(x,t)$ plane. In Section 2.6 another method of solution is considered, which does not depend on any special choice of inhomogeneity.

# CHAPTER 2

# WAVES IN INHOMOGENEOUS REGIONS

## 2.1 CONVERGENCE OF AN ITERATIVE SCHEME FOR WAVES IN A NON-UNIFORM REGION

We consider a rod of reference cross-sectional area $\tilde{A}(\tilde{X})$ and density $\rho$ in its unstressed reference configuration, where $\tilde{X}$ is a material coordinate. We let $\tilde{x} = \tilde{x}(\tilde{X},t)$ be a current coordinate and define the stretch $\lambda \equiv \frac{\partial \tilde{x}}{\partial \tilde{X}}$ and velocity $\tilde{u} \equiv \frac{\partial \tilde{x}}{\partial \tilde{t}}$. The compatibility condition

$$\frac{\partial}{\partial \tilde{t}}\left(\frac{\partial \tilde{x}}{\partial \tilde{X}}\right) = \frac{\partial}{\partial \tilde{X}}\left(\frac{\partial \tilde{x}}{\partial \tilde{t}}\right)$$

then implies that

$$\frac{\partial \tilde{u}}{\partial \tilde{X}} = \frac{\partial \lambda}{\partial \tilde{t}} \quad . \qquad (2.1.1)$$

We assume that the stress $\tilde{T}$ is a function of $\lambda$, so that the load is $\tilde{A}\ \tilde{T}(\lambda)$. The forces acting on a small section length $\delta\tilde{X}$ are shown in figure 2.1.

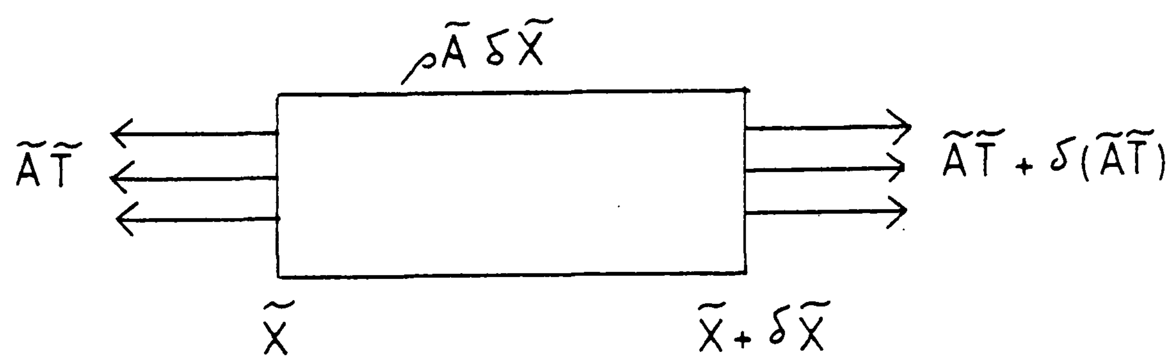



Figure 2.1 Forces acting on a small section

The momentum equation gives

$$\frac{\partial}{\partial \tilde{X}}(\tilde{A}\ \tilde{T}) = \frac{\partial}{\partial \tilde{t}}(\rho\tilde{A}\ \tilde{u}) = \rho\tilde{A}\ \frac{\partial \tilde{u}}{\partial \tilde{t}}$$

which may be rewritten

$$\tilde{T}'(\lambda)\ \frac{\partial\lambda}{\partial\tilde{X}} + \frac{d\tilde{A}}{d\tilde{X}}\ \frac{\tilde{T}}{\tilde{A}} \;=\; \rho\ \frac{\partial\tilde{u}}{\partial\tilde{t}} \quad . \tag{2.1.2}$$

We now non-dimensionalize by writing $\tilde{T}(\lambda) = ET(\lambda)$, where E is a typical Young's modulus, and defining

$$c_o \;=\; \left(\frac{E}{\rho}\right)^{\frac{1}{2}} \;,\quad \tilde{u} \;=\; c_o u,\quad \tilde{A} \;=\; A_o A \;,\quad \tilde{t} \;=\; t_o t \;,\quad \tilde{X} \;=\; c_o t_o X$$

then the compatibility condition (2.1.1) becomes

$$\frac{\partial u}{\partial X} \;=\; \frac{\partial\lambda}{\partial t} \tag{2.1.3}$$

and the momentum equation (2.1.2) becomes

$$T'(\lambda)\ \frac{\partial\lambda}{\partial X} + \frac{\partial A}{\partial X}\ \frac{T}{A} \;=\; \frac{\partial u}{\partial t} \quad . \tag{2.1.4}$$

Equations (2.1.3) and (2.1.4) form a hyperbolic system with characteristics $\frac{dX}{dt} = \pm c(\lambda) = \pm\sqrt{T'(\lambda)}$. Along the characteristic $\frac{dX}{dt} = +c(\lambda)$, we introduce a label $\alpha$, where $\frac{\partial\alpha}{\partial t} + c\ \frac{\partial\alpha}{\partial X} = 0$.

First we consider the case of uniform cross-section so that $A'(X) = 0$. Multiplying equation (2.1.3) by $\mu$ and adding it to equation (2.1.4) gives

$$c^2\lambda_X - u_t + \mu u_X - \mu\lambda_t \;=\; 0$$

which may be written as

$$\left(c^2\ \frac{\partial}{\partial X} - \mu\ \frac{\partial}{\partial t}\right)\lambda + \left(\mu\ \frac{\partial}{\partial X} - \frac{\partial}{\partial t}\right)u \;=\; 0\ .$$

Choosing $\mu = \pm c$ and introducing $\sigma(\lambda) = \int c\,d\lambda$ we obtain

$$\left(c\ \frac{\partial}{\partial X} \pm \frac{\partial}{\partial t}\right)\ (\sigma \pm u) \;=\; 0\ . \tag{2.1.5}$$

Therefore $\sigma$-u is the Riemann invariant taking a value $r(\alpha)$ which is constant along $\frac{dX}{dt} = c(\lambda)$, and similarly $\sigma$+u is constant along each curve $\frac{dX}{dt} = -c(\lambda)$, on which $\beta$ is constant.

In the remaining analysis we consider waves propagating along a rod with non-uniform, but slowly varying, cross-section. When variations in $\sigma$ and u across the $\alpha$ curves are very much greater than variations along the $\alpha$ curves, this allows a 'modulated simple wave' formulation. We transform the (X,t) coordinates to (X,$\alpha$) coordinates, where as above

$$\frac{\partial\alpha}{\partial t} + c(\lambda)\,\frac{\partial\alpha}{\partial X} = 0 .$$

Now

$$\left.\frac{\partial}{\partial X}\right|_t = \left.\frac{\partial}{\partial X}\right|_\alpha + \left.\frac{\partial}{\partial\alpha}\right|_X \left.\frac{\partial\alpha}{\partial X}\right|_t$$

and

$$\left.\frac{\partial}{\partial t}\right|_X = \left.\frac{\partial}{\partial\alpha}\right|_X \left.\frac{\partial\alpha}{\partial t}\right|_X$$

hence writing $\frac{\partial\alpha}{\partial X} = \kappa$ gives $\frac{\partial\alpha}{\partial t} = -c\kappa$, so derivatives are transformed as

$$\left.\frac{\partial}{\partial X}\right|_t = \left.\frac{\partial}{\partial X}\right|_\alpha + \kappa\left.\frac{\partial}{\partial\alpha}\right|_X \quad ,$$

$$\left.\frac{\partial}{\partial t}\right|_X = -\,c\kappa\left.\frac{\partial}{\partial\alpha}\right|_X \quad .$$

Equations (2.1.3) and (2.1.4) then give

$$\frac{\partial u}{\partial X} + \kappa\,\frac{\partial u}{\partial\alpha} + c\kappa\,\frac{\partial\lambda}{\partial\alpha} = 0 \tag{2.1.6}$$

$$-c\kappa\,\frac{\partial u}{\partial\alpha} - c^2\,\frac{\partial\lambda}{\partial X} - c^2\kappa\,\frac{\partial\lambda}{\partial\alpha} = \frac{A'(X)}{A(X)}\,T . \tag{2.1.7}$$

Multiplying equation (2.1.6) by c and adding it to equation (2.1.7) then gives

$$c \frac{\partial u}{\partial X} - c^2 \frac{\partial \lambda}{\partial X} = \frac{A'}{A} T . \tag{2.1.8}$$

Equation (2.1.8) must then be solved with either equation (2.1.6) or (2.1.7). We rewrite equation (2.1.6) in the form

$$\frac{\partial u}{\partial \alpha} + c \frac{\partial \lambda}{\partial \alpha} = - \frac{1}{\kappa} \frac{\partial u}{\partial X} \tag{2.1.9}$$

and consider equations (2.1.8) and (2.1.9)

From the compatibility condition

$$\frac{\partial}{\partial X} \left(\frac{\partial \alpha}{\partial t}\right) = \frac{\partial}{\partial t} \left(\frac{\partial \alpha}{\partial X}\right)$$

we have

$$\left.\frac{\partial \kappa}{\partial t}\right|_X = - \left.\frac{\partial}{\partial X} (c\kappa)\right|_t$$

which implies

$$-c\kappa \frac{\partial \kappa}{\partial \alpha} = - \frac{\partial}{\partial X} (c\kappa) - \kappa \frac{\partial}{\partial \alpha} (c\kappa)$$

this gives

$$\kappa \frac{\partial c}{\partial X} + c \frac{\partial \kappa}{\partial X} + \kappa^2 \frac{\partial c}{\partial \alpha} = 0$$

which may be rewritten as

$$\frac{\partial}{\partial X} (\kappa^{-1} c^{-1}) = - \frac{\partial}{\partial \alpha} (c^{-1}) . \tag{2.1.10}$$

We now define the slowness $s(\lambda)$ by

$$s(\lambda) = c^{-1} = [T'(\lambda)]^{-\frac{1}{2}}$$

and the incremental arrival time $\ell$ by

$$\ell = -(\kappa^{-1}c^{-1}) = \left.\frac{\partial t}{\partial \alpha}\right|_X$$

with boundary condition chosen as $\ell(\alpha,0) = 1$ to make $\alpha = t$ at $X = 0$. Equations (2.1.8), (2.1.9) and (2.1.10) then become

$$\frac{\partial \lambda}{\partial X} - s\frac{\partial u}{\partial X} = -s^2\frac{A'}{A}T \tag{2.1.11}$$

$$\frac{\partial \lambda}{\partial \alpha} + s\frac{\partial u}{\partial \alpha} = \ell\frac{\partial u}{\partial X} \tag{2.1.12}$$

$$\frac{\partial \ell}{\partial X} = \frac{\partial s}{\partial \alpha} \ . \tag{2.1.13}$$

These are three equations for $\lambda$, u and $\ell$ as functions of $\alpha$ and X. The time t is then found from

$$t = \int \ell(\alpha,X)d\alpha \tag{2.1.14}$$

or alternatively $t = \int s(\lambda)dX$ along the wavelets $\alpha$ = constant, because $s = \left.\frac{\partial t}{\partial X}\right|_\alpha$ .

We introduce the function $\sigma(\lambda) = \int c d\lambda$ as in the case of constant cross-section and we write $T(\lambda) = \Sigma(\sigma)$. Equations (2.1.11), (2.1.12) and (2.1.13) then become

$$\sigma_X - u_X = -\frac{A'}{A}s\Sigma \tag{2.1.15}$$

$$\sigma_\alpha + u_\alpha = \ell c u_X \tag{2.1.16}$$

$$\ell_X = s_\alpha \tag{2.1.17}$$

which we rewrite in the form

$$2\sigma_X + \frac{A'}{A}s\Sigma = u_X + \sigma_X \tag{2.1.18}$$

$$\sigma_\alpha + u_\alpha = \ell c u_X \tag{2.1.19}$$

$$\ell_X = s_\alpha \quad . \tag{2.1.20}$$

We may set up an iterative procedure:

$$u_\alpha^{(n)} + \sigma_\alpha^{(n)} = \ell^{(n-1)} c^{(n-1)} u_X^{(n-1)} \tag{2.1.21}$$

$$2\sigma_X^{(n)} + \frac{A'}{A} s^{(n)} \Sigma^{(n)} = \sigma_X^{(n)} + u_X^{(n)} \tag{2.1.22}$$

$$\ell_X^{(n)} = s_\alpha^{(n)} \quad . \tag{2.1.23}$$

If we suppose that $u^{(n-1)} \varepsilon \mathcal{L}^{p+1}$, where $\mathcal{L}^p$ is the space of functions with continuous X-derivative of order p, then $u_X^{(n-1)} \varepsilon \mathcal{L}^p$, we also assume that $\ell^{(n-1)}, c^{(n-1)} \varepsilon \mathcal{L}^p$, hence equation (2.1.21) implies $u^{(n)} + \sigma^{(n)} \varepsilon \mathcal{L}^p$, which gives $u_X^{(n)} + \sigma_X^{(n)} \varepsilon \mathcal{L}^{p-1}$. Equation (2.1.22) then implies that $\sigma_X^{(n)} \varepsilon \mathcal{L}^{p-1}$, which gives $\sigma^{(n)} \varepsilon \mathcal{L}^p$ and hence $u^{(n)} \varepsilon \mathcal{L}^p$. We therefore lose one order of differentiability per iteration. Hence if the differentiability of A is finite, for example if A is continuous, but A' is discontinuous, then the first iterate $u^{(o)}$ will be continuous but not differentiable, so that no more iterations can be performed to produce finite iterates. We therefore cannot carry out an iterative procedure by integrating along X = 0 and $\alpha = 0$. This shows the limitations of the 'modulated simple wave' formalism as already mentioned in connection with the work of Varley and Cumberbatch. We must modify the process to obtain bounded solutions at subsequent iterations.

Equation (2.1.19) may be written as

$$(u+\sigma)_\alpha = \ell c (u+\sigma)_X - \frac{\ell c}{2} \left[ (u+\sigma)_X - \frac{A'}{A} s\Sigma \right]$$

hence

$$(u+\sigma)_\alpha = \frac{\ell c}{2} (u+\sigma)_X + \frac{\ell}{2} \frac{A'}{A} \Sigma \; .$$

We now write $u+\sigma = w$ and obtain the following equations

$$w_\alpha - \frac{1}{2}\ell c w_X = \frac{1}{2}\frac{A'}{A}\ell\Sigma \tag{2.1.24}$$

$$2\sigma_X + \frac{A'}{A}S = w_X \tag{2.1.25}$$

$$\ell_X = s_\alpha \tag{2.1.26}$$

where $S(\sigma) = s\Sigma$, these may be used to set up an iterative procedure starting from $\Sigma^{(o)}$:

$$w_\alpha^{(n)} - \frac{1}{2}\ell^{(n-1)}c^{(n-1)}w_X^{(n)} = \frac{1}{2}\frac{A'}{A}\ell^{(n-1)}\Sigma^{(n-1)} \tag{2.1.27}$$

$$2\sigma_X^{(n)} + \frac{A'}{A}S(\sigma^{(n)}) = w_X^{(n)} \tag{2.1.28}$$

$$\ell_X^{(n)} = s_\alpha^{(n)} \tag{2.1.29}$$

where $s^{(n)} = s(\sigma^{(n)})$.

In equation (2.1.24) the left-hand side is a total derivative along $\frac{dX}{d\alpha} = -\frac{1}{2}\ell c$, which is one of the exact characteristics of the system (2.1.3) and (2.1.4), so that (2.1.27) determines $w^{(n)}$ by integrating along successive approximations to $\frac{dX}{d\alpha} = -\frac{1}{2}\ell c$. Equation (2.1.28) then determines $\sigma^{(n)}$, which gives $u^{(n)}$ and equation (2.1.29) gives $\ell^{(n)}$ as a function of $s(\sigma^{(n)})$. The process is repeated iteratively, hence in general the solution to (2.1.27) is found by solving

$$\frac{dw^{(n)}}{\frac{1}{2}\frac{A'}{A}\ell^{(n-1)}\Sigma^{(n-1)}} = \frac{d\alpha}{1} = \frac{dX}{-\frac{1}{2}\ell^{(n-1)}c^{(n-1)}}.$$

Starting the iterative procedure from $\Sigma^{(o)} = 0$, $\ell^{(o)} = 1$ gives $w^{(1)} = 0$. Equation (2.1.28) then gives

$$2\sigma_X^{(1)} + \frac{A'}{A}\frac{\Sigma^{(1)}}{\Sigma^{(1)'}} = 0$$

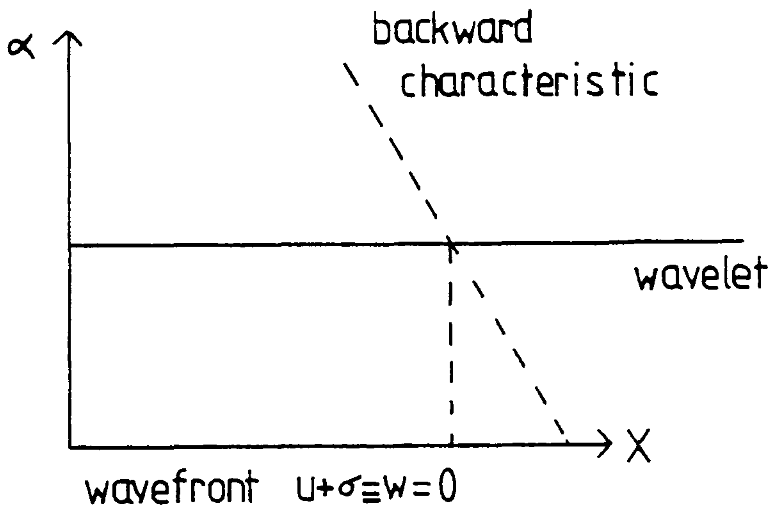



Figure 2.2 The backward characteristics in the $(X,\alpha)$ plane

which implies

$$\Sigma^{(1)}A^{\frac{1}{2}} = A_0^{\frac{1}{2}}\, r(\alpha) \quad , \qquad \text{say} \tag{2.1.30}$$

and also we obtain $\ell^{(1)}$ from (2.1.29). This is the 'modulated simple wave' approximation.

We shall show that this procedure converges under some conditions, but first we illustrate the practical process by considering the special case of the 'small amplitude, finite rate' (42) theory. This gives $m'(X)$ proportional to $(A/A_0)^{\frac{1}{2}}$ so that

$$s^{(1)} = s_0 - m'(X)r(\alpha)$$

$$\ell^{(1)} = 1 - m(X)\; r'(\alpha)$$

where $s_0$ is independent of $X$. Equation (2.1.29) is then satisfied.
To solve equation (2.1.27) we need to integrate along the $\beta$-characteristics, where

$$\frac{dX}{d\alpha} = -\frac{1}{2}\frac{\ell^{(1)}}{s^{(1)}}$$

$$= -\frac{1}{2}\left(\frac{1 - m(X)r'(\alpha)}{s_- - m'(X)r(\alpha)}\right) \quad . \tag{2.1.31}$$

Integrating this we obtain a relationship between X and $\beta$. Then along $\beta$=constant equation (2.1.27) gives

$$\frac{dw^{(2)}}{d\alpha} = \frac{1}{2}\frac{A'}{A}\ell^{(1)}\Sigma^{(1)} ,$$

however

$$\Sigma^{(1)} = \left(\frac{k}{A}\right)^{\frac{1}{2}} \qquad \text{from (2.1.30)} ,$$

hence

$$\frac{dw^{(2)}}{d\alpha} = \frac{1}{2}k^{\frac{1}{2}}\frac{A'}{A^{3/2}}\{1 - m(X)r'(\alpha)\} ,$$

the right-hand side of this is a known function of X and $\alpha$, we substitute for X using the relationship obtained from (2.1.31) and then integrate along $\beta$ = constant.

We now show that the iterative procedure converges in some region behind the front $X = X^{+}(\alpha)$ of a wave moving into a static region. Equation (2.1.24) implies that

$$w(X,\alpha) = \int_{X}^{X^{+}} \frac{A'(\bar{X})}{A(\bar{X})} P(\bar{X},X^{+})d\bar{X} , \qquad (2.1.32)$$

along $\beta$ = constant, that is along $\beta(\bar{X},\bar{\alpha}) = \beta(X,\alpha)$, so that $\beta(X^{+},0) = \beta(X,\alpha)$,

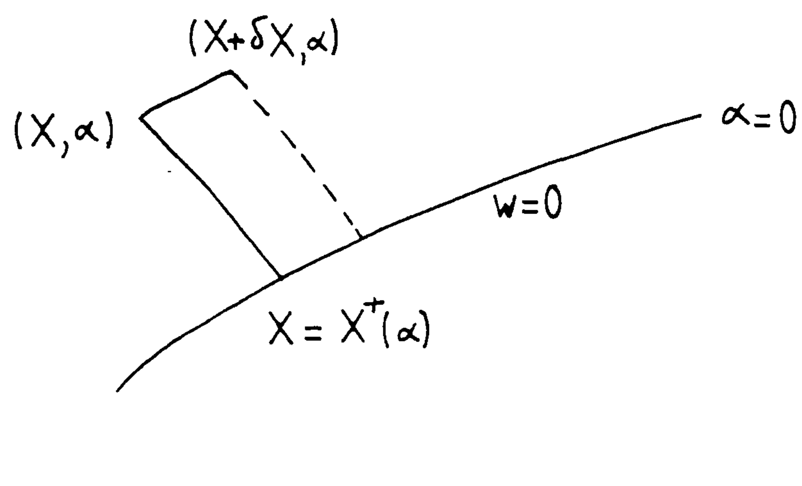



Figure 2.3 The region behind the wavefront

where $P(\overline{X},X^+) = S(\sigma)$.

Then

$$w_X = \left.\frac{\partial w}{\partial X}\right|_\alpha = \int_X^{X^+} \left(\frac{A'}{A} P\right)_X d\overline{X} + \frac{A'(X^+)}{A(X^+)} S(X^+,0) - \frac{A'(X)}{A(X)} S(X,\alpha) .$$

Now $S(X^+,0) = 0$ and equation (2.1.25) gives

$$2\sigma_X + \frac{A'}{A} S = \int_X^{X^+} \frac{\partial}{\partial X}\left(\frac{A'S}{A}\right) d\overline{X} - \frac{A'}{A} S$$

therefore

$$\sigma_X + \frac{A'S}{A} = \frac{1}{2}\int_X^{X^+} \frac{\partial}{\partial X}\left(\frac{A'S}{A}\right) d\overline{X} ,$$

where $\sigma$ is specified on $X = 0$.

To determine the region for which the procedure converges we consider two successive iterations:

$$\sigma_X^{(n-1)} + \frac{A'}{A} S(\sigma^{(n-1)}) = \frac{1}{2}\int_X^{X^+} \frac{\partial}{\partial X}\left[\frac{A'}{A} S(\sigma^{(n-2)})\right] d\overline{X} \qquad (2.1.33)$$

$$\sigma_X^{(n)} + \frac{A'}{A} S(\sigma^{(n)}) = \frac{1}{2}\int_X^{X^+} \frac{\partial}{\partial X}\left[\frac{A'}{A} S(\sigma^{(n-1)})\right] d\overline{X} \qquad (2.1.34)$$

We assume in the following analysis that $A' > 0$, although we note that only $A'/A$ appears in the analysis and the case of $A' < 0$ may be considered by writing $A = B^{-1}$ since this gives $-\frac{A'}{A} = \frac{B'}{B}$ .

Now

$$S = \Sigma s = \frac{T}{c} = \frac{T}{\sqrt{T'(\lambda)}}$$

and for a typical non-linear stress-strain law we can choose a Lipschitz constant $N > 0$ such that $|S(\sigma^{(n)}) - S(\sigma^{(n-1)})| \leq N|\sigma^{(n)} - \sigma^{(n-1)}|$ for a range of $\sigma$.

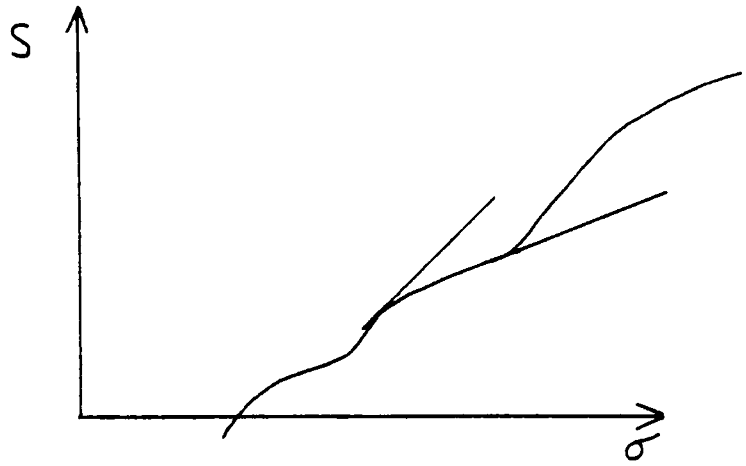



Figure 2.4 The function $S(\sigma)$

Subtracting (2.1.33) from (2.1.34) we obtain

$$\sigma_X^{(n)} - \sigma_X^{(n-1)} + \frac{A'}{A}\left[S(\sigma^{(n)}) - S(\sigma^{(n-1)})\right]$$

$$= \frac{1}{2}\int_X^{X^+} \frac{\partial}{\partial X}\left(\frac{A'}{A}\left[S(\sigma^{(n-1)}) - S(\sigma^{(n-2)})\right]\right) d\bar{X} .$$

$$|\sigma_X^{(n)} - \sigma_X^{(n-1)}| \le \frac{1}{2}\left|\int_X^{X^+} \frac{\partial}{\partial X}\left(\frac{A'}{A}\left[S(\sigma^{(n-1)}) - S(\sigma^{(n-2)})\right]\right) d\bar{X}\right|$$

$$+ \left|\frac{A'}{A}\right|\left|S(\sigma^{(n)}) - S(\sigma^{(n-1)})\right|$$

$$\le \frac{1}{2}\int_X^{X^+} \left|\frac{\partial}{\partial X}\left(\frac{A'}{A}\left[S(\sigma^{(n-1)}) - S(\sigma^{(n-2)})\right]\right)\right| d\bar{X}$$

$$+ \frac{|A'|}{A} N|\sigma^{(n)} - \sigma^{(n-1)}| . \tag{2.1.35}$$

Now for any function $f(X)$, $\left|\frac{\partial f}{\partial X}\right| \ge \frac{\partial}{\partial X}|f|$ ,

hence

$$\frac{\partial}{\partial X}\left|\sigma^{(n)} - \sigma^{(n-1)}\right| \le \frac{1}{2}\int_X^{X^+}\left|\frac{\partial}{\partial X}\left(\frac{A'}{A}\left[S(\sigma^{(n-1)}) - S(\sigma^{(n-2)})\right]\right)\right| d\bar{X}$$

$$+ \frac{|A'|}{A} N|\sigma^{(n)} - \sigma^{(n-1)}|$$

which gives

$$\frac{\partial}{\partial X}\left|\sigma^{(n)} - \sigma^{(n-1)}\right| - \frac{|A'|}{A} N|\sigma^{(n)} - \sigma^{(n-1)}|$$

$$\leq \frac{1}{2}\int_X^{X^+} \left|\frac{\partial}{\partial X}\left(\frac{A'}{A}\left[S(\sigma^{(n-1)}) - S(\sigma^{(n-2)})\right]\right)\right| d\bar{X} .$$

Hence for $A' > 0$

$$\frac{\partial}{\partial X}\left(|\sigma^{(n)} - \sigma^{(n-1)}| A^{-N}\right) \leq \frac{A^{-N}}{2}\int_X^{X^+}\left|\frac{\partial}{\partial X}\left(\frac{A'}{A}\left[S(\sigma^{(n-1)}) - S(\sigma^{(n-2)})\right]\right)\right| d\bar{X}$$

and

$$|\sigma^{(n)}(\tilde{X},\alpha) - \sigma^{(n-1)}(\tilde{X},\alpha)|A^{-N}(\tilde{X})$$

$$\leq \int_0^{\tilde{X}} \frac{A^{-N}}{2}\int_X^{X^+}\left|\frac{\partial}{\partial X}\left(\frac{A'}{A}\left[S(\sigma^{(n-1)}) - S(\sigma^{(n-2)})\right]\right)\right| d\bar{X}dX ,$$

since $\sigma^{(n)} = \sigma^{(n-1)}$ on $X = 0$ .

Hence

$$|\sigma^{(n)}(\tilde{X},\alpha) - \sigma^{(n-1)}(X,\alpha)|A^{-N}(\tilde{X})$$

$$\leq \frac{\tilde{X}}{2}\,||A^{-N}(X)||_2||X^+ - X||_2\left\{\left|\left|\frac{\partial}{\partial X}\left(\frac{A'}{A}\right)\right|\right|_1 N||\sigma^{(n-1)}-\sigma^{(n-2)}||_1\right.$$

$$\left. + K\left|\left|\frac{A'}{A}\right|\right|_1\left|\left|\sigma_X^{(n-1)} - \sigma_X^{(n-2)}\right|\right|_1\right\}$$

in which $K$ is an upper bound for $\left|\frac{dS}{d\sigma}\right|$ , we choose $K = N$, and the norms are given by .

$$||f(Y)||_2 = \max_{0 \leq Y \leq \tilde{X}} |f(Y)|$$

$$||f(Y)||_1 = \max_{X \leq Y \leq X^+} |f(Y)| \ .$$

We now let $||f(Y)|| = \max |f(Y)|$ over the region considered, then $||f(Y)||_i \leq ||f(Y)||$, $i = 1,2$,

and

$$||\sigma^{(n)} - \sigma^{(n-1)}|| \leq \frac{1}{2}||\tilde{X}(X^+ - X)A^N(\tilde{X})|| \ ||A^{-N}(X)||_2 \ N \times$$

$$\left\{\left|\left|\frac{\partial}{\partial X}\left(\frac{A'}{A}\right)\right|\right| \ ||\sigma^{(n-1)} - \sigma^{(n-2)}|| + \left|\left|\frac{A'}{A}\right|\right| \ ||\sigma_X^{(n-1)} - \sigma_X^{(n-2)}||\right\} \ . \tag{2.1.36}$$

We now seek an upper bound on $||\sigma_X^{(n-1)} - \sigma_X^{(n-2)}||$. From (2.1.35) we have the inequality

$$||\sigma_X^{(n)} - \sigma_X^{(n-1)}|| \leq \left|\left|\frac{A'}{A}\right|\right| \ N||\sigma^{(n)} - \sigma^{(n-1)}||$$

$$+ \frac{N}{2} \ ||X^+ - X||\left\{\left|\left|\frac{\partial}{\partial X}\left(\frac{A'}{A}\right)\right|\right| \ ||\sigma^{(n-1)} - \sigma^{(n-2)}||\right.$$

$$\left. + \left|\left|\frac{A'}{A}\right|\right| \ ||\sigma_X^{(n-1)} - \sigma_X^{(n-2)}||\right\} \ .$$

Repeated application of this inequality for $||\sigma_X^{(i)} - \sigma_X^{(i-1)}||$ gives

$$||\sigma_X^{(n)} - \sigma_X^{(n-1)}|| \leq \left|\left|\frac{A'}{A}\right|\right| N||\sigma^{(n)} - \sigma^{(n-1)}||$$

$$+ \frac{1}{2} \ ||X^+ - X||NU||\sigma^{(n-1)} - \sigma^{(n-2)}||$$

$$+ \frac{1}{2^2} \ ||X^+ - X||^2 N^2 U\left|\left|\frac{A'}{A}\right|\right| \ ||\sigma^{(n-2)} - \sigma^{(n-3)}||$$

$$+ \ldots + \frac{1}{2^{n-2}}||X^+-X||^{n-2}N^{n-2}U\left|\left|\frac{A'}{A}\right|\right|^{n-3}||\sigma^{(2)}-\sigma^{(1)}||$$

$$+ \frac{1}{2^{n-1}}||X^+-X||^{n-1}N^{n-1}U\left|\left|\frac{A'}{A}\right|\right|^{n-2}||\sigma^{(1)}|| \ ,$$

where

$$U \equiv \left|\left|\frac{\partial}{\partial X}\left(\frac{A'}{A}\right)\right|\right| + N\left(\left|\left|\frac{A'}{A}\right|\right|\right)^2 \quad .$$

Substituting this inequality into (2.1.36) gives

$$||\sigma^{(n)}-\sigma^{(n-1)}|| \leq ||\tilde{X}(X^+-X)A^N(\tilde{X})|| \ \ ||A^{-N}(X)||_2 NU\{||\sigma^{(n-1)}-\sigma^{(n-2)}||$$

$$+ M||\sigma^{(n-2)}-\sigma^{(n-3)}|| + M^2||\sigma^{(n-3)}-\sigma^{(n-4)}||$$

$$+ \ldots + M^{n-3}||\sigma^{(2)}-\sigma^{(1)}|| + M^{n-2}||\sigma^{(1)}||\} \ ,$$

where

$$M \equiv \frac{1}{2}||X^+-X||N\left|\left|\frac{A'}{A}\right|\right| \quad .$$

We define

$$L = \frac{1}{2}||\tilde{X}(X^+-X)A^N(\tilde{X})|| \ \ ||A^{-N}(X)||_2 NU$$

and

$$\tau^{(n)} = ||\sigma^{(n)} - \sigma^{(n-1)}|| \ .$$

The $\tau^{(n)}$ then form a positive sequence which must tend to zero as $n \to \infty$ if the iterative process is to converge. Repeated application of the inequality

$$\tau^{(n)} \leq L\{\tau^{(n-1)} + M\tau^{(n-2)} + \ldots + M^{n-2}\tau^{(1)}\}$$

gives

$$\tau^{(n)} \leq L(L+M)\{\tau^{(n-2)} + \ldots + M^{n-3}\tau^{(1)}\}$$

$$\leq \ldots \leq L(L+M)^{n-2}\tau^{(1)} \quad .$$

Therefore $\tau^{(n)} \to 0$ as $n \to \infty$ if $(L+M) < 1$, which may be rewritten as

$$||X^{+}-X||(||\tilde{X}|| + G) < H \quad ,$$

where H and G are constants given by

$$H \;=\; 2N^{-1}U^{-1}||A^{N}(\tilde{X})||^{-1}||A^{-N}(X)||_{2}^{-1}$$

and

$$G \;=\; \frac{1}{2}\,N\left|\left|\frac{A'}{A}\right|\right|H \quad .$$

Now

$$X^{+} - X \leq \alpha \max\left|\frac{dX}{d\alpha}\right|$$

and

$$\frac{dX}{d\alpha} \;=\; -\frac{1}{2}\,\ell c \;,$$

hence

$$X^{+} - X \leq \frac{\alpha}{2}\max\,(\ell c) \;.$$

Also

$$\ell \;=\; 1 + \int_{0}^{X} s_{\alpha}dX$$

so that

$$\ell \leq 1 + |X|\;||s_{\alpha}|| \quad .$$

This leads to an upper bound for $X^{+}-X$ in terms of $X$ and $\alpha$.

For this non-linear analysis we have assumed that the behaviour of $S(\sigma)$, $\ell$, $c$ is such that there exists a Lipschitz constant $N$ where

$$|S(\sigma^{(n)}) - S(\sigma^{(n-1)})| \leq N|\sigma^{(n)} - \sigma^{(n-1)}|$$

$$\left|\frac{dS}{d\sigma}\right| \leq N$$

and $\ell c$ is bounded. The value of $N$ depends on the range of $\sigma$ occurring in the solution, this difficulty does not arise in the linear theory and so we now consider the linearized version, where the speed is constant $c = C$, say, hence $s = \frac{1}{C} = \frac{1}{\Sigma'}$ and $S = \sigma$. Then from equation (2.1.32) we find

$$w = \int_X^{X^+} \frac{A'}{A}\,\sigma d\overline{X}\ .$$

As before substituting into equation (2.1.25) for $w_X$ we obtain the equation for $\sigma$:

$$\sigma_X + \frac{A'}{A}\sigma = \frac{1}{2}\int_X^{X^+} \frac{\partial}{\partial X}\left(\frac{A'\sigma}{A}\right) d\overline{X}\ .$$

which gives

$$\left|\frac{\partial}{\partial X}(A\sigma)\right| \leq \frac{1}{2} A|X^+ - X| \left|\left|\frac{\partial}{\partial X}\left(\frac{A'\sigma}{A}\right)\right|\right| \quad .$$

In this linearized version $s$ is constant, hence $\ell_X = s_\alpha = 0$, but $\ell(\alpha,0) = 1$, therefore $\ell(\alpha,X) \equiv 1$. The function $w$ is obtained by integrating along the lines where $\frac{dX}{d\alpha} = -\frac{1}{2}\frac{\ell}{s}$, which in this situation becomes $\frac{dX}{d\alpha} = -\frac{C}{2}$. We may therefore find the curves explicitly and independently of the signal $r(\alpha)$ as $X = -\frac{C}{2}\alpha + D$, where $D$ is a constant. Now $X^+$ is the point at which this line crosses the line $\alpha = 0$, hence $X^+ = D$

and $X^{+} - X = \frac{C\alpha}{2}$. The condition for convergence then becomes

$$\frac{C}{2}\,||\tilde{\alpha}||(||\tilde{X}|| + G) < H\ , \tag{2.1.37}$$

where the constants G and H will be the same as in the non-linear theory with N being 1.

In the following sections we consider an explicit example for the linear theory, which may be used to check the predictions of this section.

## 2.2 SOLUTION OF THE WAVE EQUATION WITH WAVE SPEED HAVING DISCONTINUOUS FIRST DERIVATIVE

In this section we consider a linear problem for wave propagation through an inhomogeneous region, which can be solved exactly using a Liouville-Green transformation (32). This allows the predictions of section 1.2 to be checked and also those of other procedures to be considered in later sections. We determine the exact solution to the problem of solving the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial x^2} = k^2(x)\frac{\partial^2 u}{\partial t^2}, \qquad (2.2.1)$$

where $k(x)$ is chosen so that solutions of the form $u = v(x)e^{i\omega t}$ may be determined simply. The function $v(x)$ then satisfies $v'' + k^2(x)\omega^2 v = 0$. We choose

$$k(x) = \begin{cases} 1 & , \quad x < 0 \\ \dfrac{1}{1+\varepsilon x} & , \quad 0 < x < \varepsilon^{-1} \\ \dfrac{1}{2} & , \quad x > \varepsilon^{-1} \end{cases}$$

so that $k(x)$ is a continuous function, but has discontinuous first derivative. The choice of $k(x)$ for the interval $0 < x < \varepsilon^{-1}$ was motivated by considering the transformation

$$dy = k(\varepsilon x)dx, \qquad w = [k(\varepsilon x)]^{\frac{1}{2}}v(x)$$

from which we obtain

$$\frac{d^2w}{dy^2} + k_1^2 w = 0,$$

where $k_1^2 = 1 - \dfrac{1}{2k^3}\dfrac{d^2k}{dx^2} + \dfrac{3}{4}\dfrac{(k')^2}{k^4}$ .

By choosing

$$k^{-3}k'' + \frac{3}{2}\,k'^2k^{-4} \;=\; A \quad , \tag{2.2.2}$$

where A is constant, we obtain

$$k_1^2 \;=\; 1 - \frac{1}{2}\,A \tag{2.2.3}$$

so that solutions w(x) are sinusoidal. To solve (2.2.3) we write $k'(x) = z$, we obtain from (2.2.2)

$$\frac{dk}{dz} \;=\; k^{3/2}(2Ak + B)^{\frac{1}{2}} \quad , \tag{2.2.4}$$

where B is an arbitrary constant.

Special cases of this arise for $A = 0$ or $B = 0$.

(i) $B = 0$. From (2.2.4) we deduce that $k = \frac{c}{x+d}$ , for some constants c and d and from (2.2.3)

$$k_1^2 \;=\; 1 - \frac{1}{4c^2} \; .$$

(ii) $A = 0$. We deduce that $k = \frac{c}{(x+d)^2}$ and $k_1^2 = 1$. Therefore choosing $k(x) = \frac{1}{1+\varepsilon x}$ is an example of case (i) and so leads to an equation for w having constant coefficients.

To solve equation (2.2.1) we consider the three regions $x < 0$, $0 < x < \varepsilon^{-1}$ and $x > \varepsilon^{-1}$ and solve the equation in these regions, matching the solutions on $x = 0$ and $x = \varepsilon^{-1}$.

Therefore in I $$\frac{\partial^2 u}{\partial x^2} \;=\; \frac{\partial^2 u}{\partial t^2} \qquad , \quad x < 0 \tag{2.2.5}$$

in II $$\frac{\partial^2 u}{\partial x^2} \;=\; \frac{1}{(1+\varepsilon x)^2}\,\frac{\partial^2 u}{\partial t^2} \qquad , \quad 0 < x < \varepsilon^{-1} \tag{2.2.6}$$

in III $$\frac{\partial^2 u}{\partial x^2} \;=\; \frac{1}{4}\,\frac{\partial^2 u}{\partial t^2} \qquad , \quad x > \varepsilon^{-1} \quad . \tag{2.2.7}$$

We assume that for $x < 0$ we have an incoming sinusoidal wave $\sin(x-t)$. The required solutions to (2.2.5) and (2.2.7) may then be written as:

$$\text{in I} \qquad u = \sin(x-t) + B_1\sin(x+t) + B_2\cos(x+t) \tag{2.2.8}$$

$$\text{and in III} \quad u = A_3\sin\left(\frac{x}{2} - t\right) + A_4\cos\left(\frac{x}{2} - t\right) , \tag{2.2.9}$$

where $A_3$, $A_4$, $B_1$ and $B_2$ are arbitrary constants, the B's being the constants for the reflected wave and the A's those for the transmitted wave. For region II we introduce the variable y as above where $\frac{dy}{dx} = \frac{1}{1+\varepsilon x}$ , which gives $y = \frac{1}{\varepsilon}\log(1+\varepsilon x)$, where $0 < y < \frac{1}{\varepsilon}\log 2$. Defining $v = k^{\frac{1}{2}}u = \frac{1}{(1+\varepsilon x)^{\frac{1}{2}}}u$, (2.2.6) may be rewritten as

$$\frac{\partial^2 v}{\partial y^2} - \frac{\partial^2 v}{\partial t^2} = \frac{1}{4}\varepsilon^2 v , \tag{2.2.10}$$

the telegraph equation for which the general solution having period $2\pi$ in t is

$$v = A_1\sin\left[\left(1 - \frac{1}{4}\varepsilon^2\right)^{\frac{1}{2}}y - t\right] + A_2\cos\left[\left(1 - \frac{1}{4}\varepsilon^2\right)^{\frac{1}{2}}y - t\right]$$

$$+ B_3\sin\left[\left(1 - \frac{1}{4}\varepsilon^2\right)^{\frac{1}{2}}y + t\right] + B_4\cos\left[\left(1 - \frac{1}{4}\varepsilon^2\right)^{\frac{1}{2}}y - t\right] \tag{2.2.11}$$

Equation (2.2.10) may also be obtained by considering a linearization of (2.1.3) and (2.1.4), writing $\lambda = 1+e$, $T = c^2e$ gives

$$u_x = e_t ,$$

$$c^2e_x + \frac{A'}{A}c^2e = u_t .$$

We then introduce y and v by writing $X = cy$, $u = v_t$, $ce = v_y$ and obtain

$$u_y = v_{ty} = (ce)_t$$

and

$$v_{yy} + \frac{d}{dy}(\log A)v_y = v_{tt} \quad .$$

Making the transformation $v(y,t) = A^{-\frac{1}{2}} w(y,t)$, the function $w(y,t)$ satisfies

$$w_{yy} - w_{tt} = \tfrac{1}{2}\left[\frac{d^2}{dy^2}(\log A) + \left(\frac{d}{dy}(\log A)\right)^2\right]w$$

which is of the same form as (2.2.10), when

$$\left[\frac{d^2}{dy^2}(\log A) + \left(\frac{d}{dy}(\log A)\right)^2\right] = \text{constant.} \qquad (2.2.12)$$

The results to be obtained for equations (2.2.5)-(2.2.7) will therefore also apply for the linearization mentioned above of the example considered in section 2.1, when the cross-sectional area A satisfies (2.2.12).

Now $u = (1+\varepsilon x)^{\frac{1}{2}}v$, $y = \frac{1}{\varepsilon}\log(1+\varepsilon x)$ so that from (2.2.11) in region II we have

$$\begin{aligned} u = (1+\varepsilon x)^{\frac{1}{2}}\Big\{&A_1\sin\left[\left(1 - \tfrac{1}{4}\varepsilon^2\right)^{\frac{1}{2}}\tfrac{1}{\varepsilon}\log(1+\varepsilon x) - t\right] \\ &+ A_2\cos\left[\left(1 - \tfrac{1}{4}\varepsilon^2\right)^{\frac{1}{2}}\tfrac{1}{\varepsilon}\log(1+\varepsilon x) - t\right] \\ &+ B_3\sin\left[\left(1 - \tfrac{1}{4}\varepsilon^2\right)^{\frac{1}{2}}\tfrac{1}{\varepsilon}\log(1+\varepsilon x) + t\right] \\ &+ B_4\cos\left[\left(1 - \tfrac{1}{4}\varepsilon^2\right)^{\frac{1}{2}}\tfrac{1}{\varepsilon}\log(1+\varepsilon x) + t\right]\Big\} \end{aligned} \qquad (2.2.13)$$

The constants $A_1$, $A_2$, $A_3$, $A_4$, $B_1$, $B_2$, $B_3$ and $B_4$ are found by matching the solutions at $\varepsilon x = 0$ and 1. For all x, u and $u_x$ are continuous, so that u and $u_x$ must be continuous at $\varepsilon x = 0$ and 1.

In I,

$$u_x = \cos(x-t) + B_1\cos(x+t) - B_2\sin(x+t) \ , \qquad (2.2.14)$$

in II

$$\begin{aligned} u_x = \ & \frac{\varepsilon}{2}(1+\varepsilon x)^{-\frac{1}{2}}\Big\{A_1\sin\Big[\Big(1-\frac{1}{4}\varepsilon^2\Big)^{\frac{1}{2}}\frac{1}{\varepsilon}\log(1+\varepsilon x) - t\Big] \\ & + A_2\cos\Big[\Big(1-\frac{1}{4}\varepsilon^2\Big)^{\frac{1}{2}}\frac{1}{\varepsilon}\log(1+\varepsilon x) - t\Big] \\ & + B_3\sin\Big[\Big(1-\frac{1}{4}\varepsilon^2\Big)^{\frac{1}{2}}\frac{1}{\varepsilon}\log(1+\varepsilon x) + t\Big] \\ & + B_4\cos\Big[\Big(1-\frac{1}{4}\varepsilon^2\Big)^{\frac{1}{2}}\frac{1}{\varepsilon}\log(1+\varepsilon x) + t\Big]\Big\} \\ & + (1+\varepsilon x)^{-\frac{1}{2}}\Big(1-\frac{1}{4}\varepsilon^2\Big)^{\frac{1}{2}}\Big\{A_1\cos\Big[\Big(1-\frac{1}{4}\varepsilon^2\Big)^{\frac{1}{2}}\frac{1}{\varepsilon}\log(1+\varepsilon x) - t\Big] \\ & - A_2\sin\Big[\Big(1-\frac{1}{4}\varepsilon^2\Big)^{\frac{1}{2}}\frac{1}{\varepsilon}\log(1+\varepsilon x) - t\Big] \\ & + B_3\cos\Big[\Big(1-\frac{1}{4}\varepsilon^2\Big)^{\frac{1}{2}}\frac{1}{\varepsilon}\log(1+\varepsilon x) + t\Big] \\ & - B_4\sin\Big[\Big(1-\frac{1}{4}\varepsilon^2\Big)^{\frac{1}{2}}\frac{1}{\varepsilon}\log(1+\varepsilon x) + t\Big]\Big\}, \qquad (2.2.15) \end{aligned}$$

in III

$$u_x = \frac{1}{2}A_3\cos\Big(\frac{x}{2} - t\Big) - \frac{1}{2}A_4\sin\Big(\frac{x}{2} - t\Big) \ . \qquad (2.2.16)$$

We make u continuous at x = 0, using equations (2.2.8) and (2.2.13) and equate coefficients of sin t and cos t to yield the conditions

$$B_1 - 1 = B_3 - A_1 \qquad (2.2.17)$$

$$B_2 = A_2 + B_4 \ . \qquad (2.2.18)$$

Similarly making $u_x$ continuous at $x = 0$ and substituting for $B_3$ and $B_4$ from (2.2.17) and (2.2.18) we obtain

$$1 + B_1 = \frac{\varepsilon}{2} B_2 + \left(1 - \frac{1}{4}\varepsilon^2\right)^{\frac{1}{2}}(2A_1 + B_1 - 1) \tag{2.2.19}$$

$$- B_2 = \frac{\varepsilon}{2}(B_1 - 1) + \left(1 - \frac{1}{4}\varepsilon^2\right)^{\frac{1}{2}}(2A_2 - B_2) \quad . \tag{2.2.20}$$

We write $\left(1 - \frac{1}{4}\varepsilon^2\right)^{\frac{1}{2}} \frac{1}{\varepsilon} \log 2 = \gamma$, then $u$ and $u_x$ being continuous at $\varepsilon x = 1$ gives, on substituting for $B_3$ and $B_4$ from (2.2.16) and (2.2.17):

$$\sqrt{2}\{(B_1-1)\cos\gamma + (2A_2-B_2)\sin\gamma\} = -A_3\cos\frac{1}{2\varepsilon} + A_4 \sin\frac{1}{2\varepsilon} \tag{2.2.21}$$

$$\sqrt{2}\{(2A_1+B_1-1)\sin\gamma + B_2\cos\gamma\} = A_3\sin\frac{1}{2\varepsilon} + A_4\cos\frac{1}{2\varepsilon} \tag{2.2.22}$$

$$\begin{aligned}&\frac{\varepsilon}{2}\{(B_1-1)\cos\gamma + (2A_2-B_2)\sin\gamma\}\\ &\quad + \left(1 - \frac{1}{4}\varepsilon^2\right)^{\frac{1}{2}}\{(1-B_1)\sin\gamma + (2A_2-B_2)\cos\gamma\}\\ &\quad = \frac{\sqrt{2}}{2}\left(A_3\sin\frac{1}{2\varepsilon} + A_4\cos\frac{1}{2\varepsilon}\right)\end{aligned} \tag{2.2.23}$$

and

$$\begin{aligned}&\frac{\varepsilon}{2}\{(2A_1+B_1-1)\sin\gamma + B_2\cos\gamma\}\\ &\quad + \left(1 - \frac{1}{4}\varepsilon^2\right)^{\frac{1}{2}}\{(2A_1+B_1-1)\cos\gamma - B_2\sin\gamma\}\\ &\quad = \frac{\sqrt{2}}{2}\left(A_3\cos\frac{1}{2\varepsilon} - A_4\sin\frac{1}{2\varepsilon}\right)\end{aligned} \tag{2.2.24}$$

Equations (2.2.17) - (2.2.24) give eight equations for the eight unknowns $A_1, \ldots, B_4$. We write $\eta = \left(1 - \frac{1}{4}\varepsilon^2\right)^{\frac{1}{2}}$, hence $\gamma = \frac{\eta}{\varepsilon}\log 2$. Equations

(2.2.18) and (2.2.19) may then be written

$$2\eta A_1 + (\eta-1)B_1 + \frac{\varepsilon}{2} B_2 = 1 + \eta \qquad (2.2.25)$$

and

$$2\eta A_2 + \frac{\varepsilon}{2} B_1 - (\eta-1)B_2 = \frac{\varepsilon}{2} . \qquad (2.2.26)$$

Eliminating $A_3$ and $A_4$ from equations (2.2.21) - (2.2.24) gives

$$A_1\{\varepsilon \sin\gamma + 2\eta\cos\gamma\} + 2A_2\sin\gamma + B_1\{\cos\gamma + \frac{\varepsilon}{2}\sin\gamma + \eta\cos\gamma\}$$

$$+ B_2\{- \sin\gamma + \frac{\varepsilon}{2}\cos\gamma - \eta\sin\gamma\} = \cos\gamma + \frac{\varepsilon}{2}\sin\gamma + \eta\cos\gamma \qquad (2.2.27)$$

and

$$-2A_1\sin\gamma + A_2\{\varepsilon \sin\gamma + 2\eta\cos\gamma\} + B_1\{\frac{\varepsilon}{2}\cos\gamma - (\eta+1)\sin\gamma\}$$

$$+ B_2\{- \frac{\varepsilon}{2}\sin\gamma - (\eta+1)\cos\gamma\} = - \sin\gamma + \frac{\varepsilon}{2}\cos\gamma - \eta\sin\gamma . \qquad (2.2.28)$$

$A_1$, $A_2$, $B_1$ and $B_2$ may therefore be determined from equations (2.2.25) - (2.2.28) and then $B_3$, $B_4$, $A_3$ and $A_4$ determined by (2.2.17), (2.2.18), (2.2.23) and (2.2.24).

We now consider the asymptotic solution to these algebraic equations, for $\varepsilon << 1$, $|\sin\gamma|$, $|\cos\gamma| << 1$ and $\eta = 1 - \frac{1}{8}\varepsilon^2 + \ldots$ . Solving the equations to zero order we find that $A_1 = 1$, $A_2 = 0$, $B_1 = 0$, $B_2 = 0$, $A_3 = \sqrt{2}\cos\left(\gamma - \frac{1}{2\varepsilon}\right)$, $A_4 = \sqrt{2}\sin\left(\gamma - \frac{1}{2\varepsilon}\right)$, $B_3 = 0$ and $B_4 = 0$. Hence to this order there is no reflected wave in region I or II and the phase of the transmitted wave in region III is $\gamma = \frac{1}{\varepsilon}\log 2$. We expand $A_1,\ldots,B_4$ as $A_1 = 1 + \varepsilon A_{11} + \ldots, \ldots, B_4 = \varepsilon B_{41} + \ldots$, and equating powers of $\varepsilon$ we obtain

$$A_1 = 1 + \frac{\varepsilon^2}{16}\cos 2\gamma + \ldots$$

$$A_2 = \frac{1}{4}\varepsilon + \frac{\varepsilon^2}{16}\sin 2\gamma + \ldots$$

$$A_3 = \sqrt{2}\cos\left(\gamma - \frac{1}{2\varepsilon}\right) + \varepsilon^2 \frac{\sqrt{2}}{8}\sin\gamma \sin\left(\frac{1}{2\varepsilon} - 2\gamma\right) + \ldots$$

$$A_4 = \sqrt{2}\sin\left(\gamma - \frac{1}{2\varepsilon}\right) + \varepsilon^2 \frac{\sqrt{2}}{8}\sin\gamma \cos\left(\frac{1}{2\varepsilon} - 2\gamma\right) + \ldots$$

$$B_1 = -\frac{\varepsilon}{4}\sin 2\gamma + \ldots$$

$$B_2 = \frac{\varepsilon}{4}(1 - \cos 2\gamma) + \ldots$$

$$B_3 = -\frac{\varepsilon}{4}\sin 2\gamma + \frac{\varepsilon^2}{16}\cos 2\gamma + \ldots$$

$$B_4 = -\frac{\varepsilon}{4}\cos 2\gamma - \frac{\varepsilon^2}{16}\sin 2\gamma + \ldots$$

The magnitude of the reflected wave in region I is $\sqrt{B_1^2 + B_2^2} \sim \frac{\varepsilon}{2}|\sin\gamma|$, which depends crucially on the phase relationship, that is the travel time $\gamma = \frac{1}{\varepsilon}\log 2$. The magnitude of the transmitted wave in region II is $\sqrt{A_1^2 + A_2^2}$ and that of the reflected wave in that region is $\sqrt{B_3^2 + B_4^2}$. In region III the transmitted wave has magnitude $\sqrt{A_3^2 + A_4^2}$. The reflected waves are due solely to the discontinuities at the interfaces. In the next section we consider the solution near a wavefront.

## 2.3 CONSIDERATION OF THE PROBLEM INTRODUCED IN SECTION 2.2 IN THE REGION WHERE THE WAVE SPEED IS VARYING

In this section we again consider equation (2.2.6) and make the transformation $u = v(1+\varepsilon x)^{\frac{1}{2}}$, $y = \log(1+\varepsilon x)$ to obtain as in (2.2.10)

$$v_{yy} - v_{tt} = \frac{1}{4}\varepsilon^2 v \quad . \tag{2.3.1}$$

Solutions to this equation may be found using various methods.

First we observe that for sinusoidal wavetrains $v = \sin(Cy-t)$ satisfies (2.3.1) if $C^2 = 1 - \frac{1}{4}\varepsilon^2$, so that this solution may be written as

$$v = \sin\left[\left(1 - \frac{1}{4}\varepsilon^2\right)^{\frac{1}{2}} y - t\right] \quad . \tag{2.3.2}$$

More generally, any solution to (2.3.1) may be obtained by writing $v = \sum_{p=0}^{\infty} \varepsilon^{2p} v_p$. The choice $v_0 = \sin(y-t)$ then allows us to write v as

$$v = \sin(y-t) - \frac{1}{8}\varepsilon^2 y(1-\cos(y-t)) - \frac{1}{128}\varepsilon^4(y^2\sin(y-t)$$
$$+ y(\cos(y-t)-1)) + \ldots , \tag{2.3.3}$$

where we have not included a reflected wave, and we have imposed the condition $v = 0$ on $y = t$. It is readily shown on expanding (2.3.2) in powers of $\varepsilon$ that the $O(1)$, $O(\varepsilon^2)$ and $O(\varepsilon^4)$ terms of (2.3.2) and (2.3.3) agree.

The method may also be used to construct solutions $v(y,t)$ satisfying more general boundary conditions on $y = 0$ and at a wavefront $y = t$. For this purpose it is easier to change coordinates to $y$ and $\zeta$, where $\zeta = t - y$ so that equation (2.3.1) becomes

$$v_{yy} - 2v_{y\zeta} = \frac{1}{4}\varepsilon^2 v \quad . \tag{2.3.4}$$

We are interested in the region $t \geq y$, that is $\zeta \geq 0$ and impose the boundary conditions

$$v(y,0) = 0, \qquad v(0,\zeta) = v_0(\zeta) . \qquad (2.3.5)$$

We now solve this problem using Laplace transforms. The Laplace transform of v with respect to $\zeta$ is denoted by

$$\bar{v}(y,s) = \int_0^\infty v(y,\zeta)e^{-s\zeta}\, d\zeta .$$

Hence, using the boundary condition on $\zeta = 0$, equation (2.3.4) is transformed into

$$\bar{v}_{yy} - 2s\bar{v}_y - \frac{1}{4}\varepsilon^2\bar{v} = 0 ,$$

which may be solved using the boundary condition on $y = 0$ and imposing the restriction that $\bar{v}$ is bounded as $y \to \infty$. The solution is

$$\bar{v} = \frac{-1}{1+s^2} e^{ys} e^{-\sqrt{s^2+\varepsilon^2/4}\, y} ,$$

which may be inverted using Bessel functions to give the solution

$$v(y,\zeta) = v_0(\zeta) - \frac{\varepsilon y}{2}\int_0^\zeta \frac{J_1[\varepsilon/2\{r(r+2y)\}^{\frac{1}{2}}]}{[r(r+2y)]^{\frac{1}{2}}} v_0(\zeta-r)dr .$$

Now the Bessel function $J_1(x)$ may be expanded as

$$J_1(x) = \frac{x}{2}\{1 - \frac{x^2}{8} + \ldots\} .$$

Substituting this into the above expression for $v(y,\zeta)$ and performing the integrations gives an expansion for v:

$$v(y,\zeta) = v_0(\zeta) - \frac{\varepsilon^2 y}{8}\int_0^\zeta v_0(\zeta-r)dr + O(\varepsilon^4) .$$

Considering the special case $v_0 = -\sin\zeta$ we obtain

$$v(y,\zeta) = -\sin\zeta + \frac{\varepsilon^2 y}{8}(1 - \cos\zeta) + \ldots \quad . \tag{2.3.6}$$

We see that the terms in (2.3.6) agree with those in (2.3.3). We may also compare the solution with that obtained in Section 2.2, however in that section we impose boundary conditions on $y = 0$ and $\varepsilon y = \log 2$. The solutions are of a similar form if we neglect the reflected part in region II which is due to the boundary $\varepsilon y = \log 2$.

We may use the above method of Laplace transforms imposing a boundary condition at $\varepsilon y = \log 2$ and lifting the restriction that $\bar{v}$ is bounded as $y \to \infty$. We then obtain an additional term in $\bar{v}$ proportional to $e^{(s+\sqrt{s^2+\varepsilon^2/4}\, y}$, the arbitrary constants being determined from the boundary conditions.

Equation (2.3.4) may also be solved by Riemann's method, see Section 2.4, where the Riemann function u is the Bessel function

$$J_0\left(\frac{\varepsilon}{2}\sqrt{(\zeta-\zeta_0)\{2(y-y_0) + (\zeta-\zeta_0)\}}\right) \quad .$$

## 2.4 FURTHER METHODS OF SOLVING THE PROBLEM IN SECTION 2.3

We consider an alternative method of solution of equation (2.3.1) using Riemann's method (7). First we transform the equation using characteristic coordinates $\zeta$, $\eta$, where $\zeta = \frac{t-y}{4}$, $\eta = \frac{t+y}{4}$ and we are interested in the region $t \geq y$, $\zeta \geq 0$. Equation (2.3.1) then becomes

$$v_{\zeta\eta} = -\varepsilon^2 v \quad . \tag{2.4.1}$$

We now define the linear operator L, by

$$L(v) = v_{\zeta\eta} + \varepsilon^2 v \ . \tag{2.4.2}$$

This operator is self-adjoint with the property

$$vL(u)-uL(v) = \frac{\partial}{\partial\zeta}(vu_\eta) - \frac{\partial}{\partial\eta}(uv_\zeta) \ .$$

If D is a domain whose boundary is a regular closed curve C, it follows by Green's theorem that

$$\iint_D (U_\eta + V_\zeta)d\eta d\zeta = \int_C (Ud\zeta - Vd\eta)$$

for functions U, $V(\zeta,\eta)$. Now $L(v) = 0$ when v satisfies (2.4.1) and we choose u such that $L(u) = 0$, then $vL(u)-uL(v) = 0$, hence

$$\int_C \{(vu_\eta)d\eta + (v_\zeta u)d\zeta\} = 0 \ . \tag{2.4.3}$$

We choose C to be the contour PQRS shown in figure 2.5 and we choose u such that u = 1 on PQ and SP, hence $u_\eta = 0$ on PQ, $u_\zeta = 0$ on SP and $u(P) = 1$.

Equation (2.4.3) then gives

$$\int_{RS} vu_\eta d\eta + \int_{SP} v_\zeta d\zeta + \int_{QR} (vu_\eta + v_\zeta u)d\zeta = 0 \quad ,$$

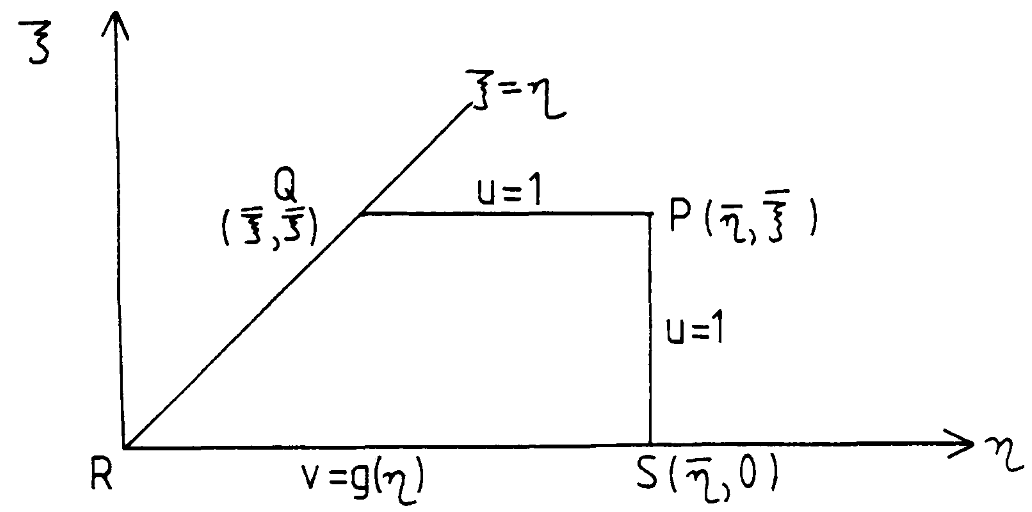



Figure 2.5 The region of integration for Riemann's method

hence

$$v(P) \;=\; v(S) - \int_0^{\bar{\eta}} v(\eta,0)u_\eta(\eta,0)d\eta$$

$$- \int_{\bar{\zeta}}^{0} (vu_\eta + v_\zeta u)d\zeta \;. \qquad (2.4.4)$$

We impose the boundary conditions

$$v(\eta,0) \;=\; g(\eta), \qquad v(\zeta,\zeta) \;=\; h(\zeta) \;.$$

The problem is therefore reduced to that of finding a function u such that $L(u) = 0$ with $u = 1$ on $\zeta = \bar{\zeta}$ and $\eta = \bar{\eta}$, this function is known as the Riemann function for L.

Following Copson (7) we write u as the summation $u = \sum_{j=0}^{\infty} \frac{u_j \Gamma^j}{j!j!}$, where $\Gamma = (\bar{\zeta}-\zeta)(\bar{\eta}-\eta)$. Choosing $u_0 = 1$ the boundary conditions are satisfied and we find that $L(u) = 0$ if $u_j = (-\varepsilon^2)^j$, hence

$$u \;=\; \sum_{j=0}^{\infty} \frac{(-1)^j}{j!j!} (\Gamma\varepsilon^2)^j \;.$$

However, the Bessel function $J_0(z)$ may be written in the form (61)

$$J_0(z) = \sum_{j=0}^{\infty} \frac{(-1)^j(\frac{1}{2}z)^{2j}}{j!j!}$$

so comparing this with the expression for u, we find that u may be written as

$$u = J_0(2\varepsilon\sqrt{(\bar{\zeta}-\zeta)(\bar{\eta}-\eta)}) .$$

Substituting this into (2.4.4) we obtain

$$v(P) = v(S) - \int_0^{\bar{\eta}} g(\eta)u_\eta(\eta,0)d\eta$$

$$+ \int_0^{\bar{\zeta}} \left[\frac{\varepsilon h(\zeta)\ (\bar{\zeta}-\zeta)}{\sqrt{(\bar{\zeta}-\zeta)(\bar{\eta}-\eta)}}\ J_1(2\varepsilon\sqrt{(\bar{\eta}-\zeta)(\bar{\zeta}-\zeta)})\right.$$

$$\left. + h'(\zeta)\ J_0(2\varepsilon\sqrt{(\bar{\eta}-\zeta)(\bar{\zeta}-\zeta)})\right] d\zeta ,$$

where we have used

$$J_0'(z) = -J_1(z) .$$

Now the Bessel functions may be expanded as

$$J_0(z) = 1 - \frac{z^2}{4} + \frac{z^4}{2^2 4^2} + \dots$$

$$J_1(z) = \frac{z}{2} - \frac{z^3}{2^2 4} + \dots$$

We now choose $g(\eta) = 0$ and expanding the Bessel functions we obtain

$$v(\bar{\zeta},\bar{\eta}) = h(\bar{\zeta}) - \varepsilon^2\int_0^{\bar{\zeta}} h(\zeta)(\bar{\eta}-\zeta)d\zeta + O(\varepsilon^4) . \qquad (2.4.5)$$

The problem could also be solved with boundary conditions applied on $y = 0$ and $y = 1$ that is $\eta - \zeta = 0$ and $\eta - \zeta = \frac{1}{4}$, the region of integration is then shown in figure 2.6.

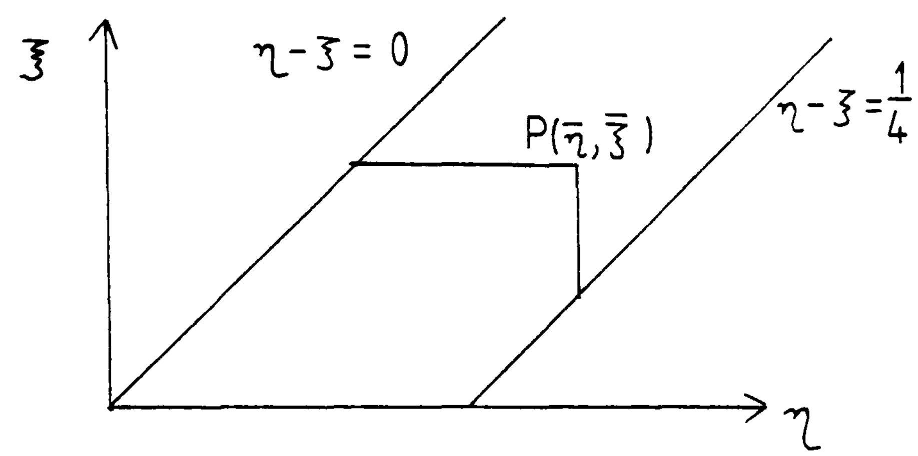



Figure 2.6 The region of integration with boundary conditions on $y = 0$ and $y = 1$

We now examine a direct iterative procedure for solving (2.4.1) analogous to that of Section 2.1. We assume that v is given along $y = 0$ and along $\zeta = 0$.

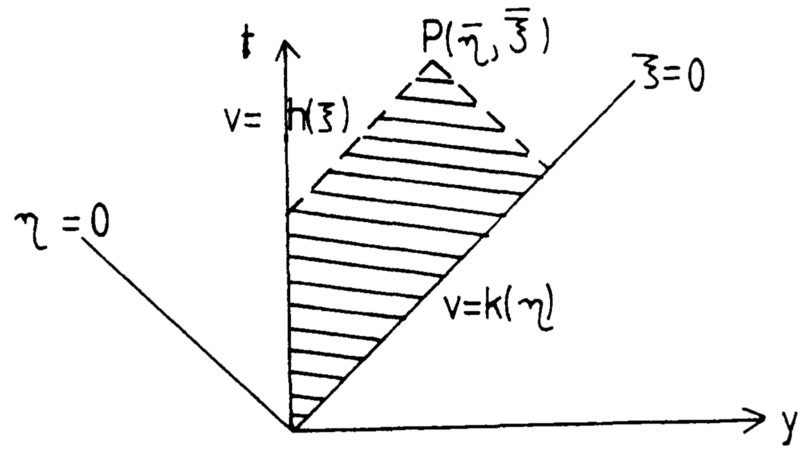



Figure 2.7 The region of integration with boundary conditions on $y = 0$ and $\zeta = 0$

The solution may then be written in the form

$$v = -\varepsilon^2 \int_0^{\bar{\zeta}} \int_{\zeta}^{\bar{\eta}} v \, d\eta d\zeta + f(\bar{\zeta}) + g(\bar{\eta}) \tag{2.4.6}$$

We let $v = k(\eta)$ along $\zeta = 0$

and $\quad v = h(\zeta)$ along $\zeta = \eta$,

hence $f(\bar{\zeta})$ and $g(\bar{\eta})$ are given by

$$k(\bar{\eta}) = f(0) + g(\bar{\eta}) \tag{2.4.7}$$

$$h(\bar{\zeta}) = f(\bar{\zeta}) + g(\bar{\zeta}) \ .$$

We may set up an iterative procedure

$$v^{(n)} = -\varepsilon^2 \int_0^{\bar{\zeta}} \int_{\zeta}^{\bar{\eta}} v^{(n-1)} d\eta d\zeta + f(\bar{\zeta}) + g(\bar{\eta}) \ ,$$

where since f and g are determined from the boundary conditions, they are the same for each iteration. Hence

$$v^{(n)} - v^{(n-1)} = -\varepsilon^2 \int_0^{\bar{\zeta}} \int_{\zeta}^{\bar{\eta}} (v^{(n-1)} - v^{(n-2)}) d\eta d\zeta \ .$$

We define $\tau^{(n)} = |v^{(n)} - v^{(n-1)}|$

then

$$\tau^{(n)} \leq \varepsilon^2 (\bar{\zeta}\bar{\eta} - \frac{1}{2} \bar{\zeta}^2) ||\tau^{(n-1)}||$$

where

$$||f(\eta,\zeta)|| = \max_{\substack{0 \leq \zeta \leq \bar{\zeta} \\ 0 \leq \eta \leq \bar{\eta}}} |f(\eta,\zeta)| \ ,$$

therefore

$$||\tau^{(n)}|| \leq \varepsilon^2 \bar{\zeta} (\bar{\eta} - \frac{1}{2} \bar{\zeta}) ||\tau^{(n-1)}|| \ .$$

The iterative procedure will therefore converge if $\varepsilon^2\bar{\zeta}(\bar{\eta} - \frac{1}{2}\bar{\zeta}) < 1$, which may be written in terms of y and t as

$$\varepsilon^2(t-y)(t+3y) < 32 \ , \qquad (2.4.9)$$

this has asymptotes $t = y$ and $t + 3y = 0$. Therefore as in Section 2.1, we have the limit of the region of convergence in a region bounded by a curve crossing the t axis at a finite value of t, the region being shown in figure 2.8.

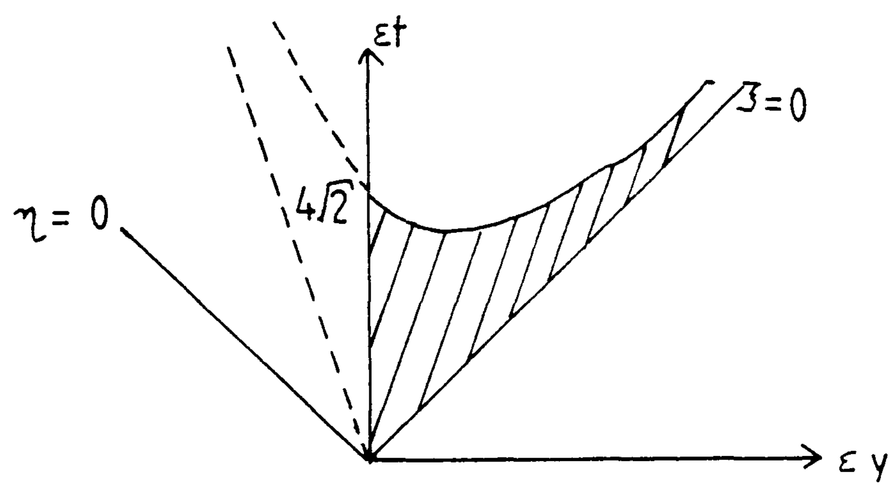



Figure 2.8 The region of convergence for the iterative procedure

The estimates of the regions of convergence obtained for the different procedures in this section and the previous ones are underestimates and the procedures probably converge everywhere. It may be shown that the condition (2.1.37) obtained in Section 2.1 is satisfied if (2.4.9) is satisfied when A is chosen to fulfil the condition (2.2.12), for example by choosing $A = e^{1/\sqrt{2}\varepsilon\, y}$.

We now give the first few terms of the iterative procedure described above, choosing $g(\eta) = 0$:

$$v^{(1)} \ = \ f(\bar{\zeta})$$

$$
\begin{aligned}
v^{(2)} &= -\varepsilon^2 \int_0^{\bar{\zeta}} \int_{\zeta}^{\bar{\eta}} f(\zeta)\,d\eta\,d\zeta + f(\bar{\zeta}) \\
&= -\varepsilon^2 \int_0^{\bar{\zeta}} (\bar{\eta} - \zeta) f(\zeta)\,d\zeta + f(\bar{\zeta}) \; .
\end{aligned} \tag{2.4.10}
$$

From (2.4.8) we see that $h(\zeta) = f(\zeta)$ and comparing (2.4.10) with (2.4.5) we see that the solutions obtained are the same.

We have also solved the equation

$$v_{\zeta\eta} = -\varepsilon^2 v$$

with boundary condition on $\zeta = 0$ and $\eta = 0$, namely

$$v(\zeta,0) = \sin\zeta \quad , \qquad v(0,\eta) = 0 \; .$$

We have considered the solution using Laplace transforms, a series solution, a solution using Riemann's method and using an iterative procedure as in Section 2.3. Using Riemann's method and the method of Laplace transforms the solution involves a Bessel function, however expanding this we find that the solutions agree with

$$v = \sin\zeta + \varepsilon^2\eta(\cos\zeta - 1) + \ldots$$

and certainly converge for $\varepsilon^2|\zeta||\eta| < 1$.

## 2.5 AN ALTERNATIVE APPROACH TO THE PROBLEM INTRODUCED IN SECTION 2.2

We consider the equation (2.2.1) in the three regions with incoming wave $\sin(x-t)$.

In region I, for $x < 0$, $\quad u_{xx} = u_{tt}$ (2.5.1)

which has solution $u = \sin(x-t) + k(x+t)$.

In region II, for $0 < x < \varepsilon^{-1}$, $\quad (1+\varepsilon x)^2 u_{xx} = u_{tt}$ . (2.5.2)

In region III, for $x > \varepsilon^{-1}$, $\quad 4u_{xx} = u_{tt}$ . (2.5.3)

for which the general outgoing wave is $u = h(\tfrac{1}{2}x-t)$, where $k$ and $h$ are arbitrary functions.

To determine the solution in region II, we consider a two-timing approach and write $X = \varepsilon x$, $Y = g(X)/\varepsilon$. Derivatives are then transformed as

$$\frac{d}{dX} = \frac{dY}{dX}\frac{\partial}{\partial Y} + \frac{\partial}{\partial X} ,$$

so that equation (2.5.2) becomes

$$(1+X)^2\{g'^2u_{YY} + \varepsilon g''u_Y + 2\varepsilon g'u_{YX} + \varepsilon^2u_{XX}\} - u_{tt} = 0 . \qquad (2.5.4)$$

Now $u = u(X,Y,t;\varepsilon)$ and we seek an asymptotic solution $u = u_0 + \varepsilon u_1 + \ldots$ . Substituting this expansion into (2.5.4) gives to zero order an equation for $u_0$:

$$(1+X)^2g'^2\, u_{0,YY} - u_{0,tt} = 0 . \qquad (2.5.5)$$

We choose $g(X)$ such that $(1+X)^2g'^2 = 1 + O(\varepsilon)$, where the $O(\varepsilon)$ term is identically zero on a wavefront. It is not necessary to choose the particular form of wave speed $1+\varepsilon x$, any function $F(\varepsilon x)$ may replace $(1+\varepsilon x)$ and then $g(X)$ chosen to satisfy

$$F(X)^2 g'(X)^2 \;=\; 1 + O(\varepsilon) \;. \tag{2.5.6}$$

Equation (2.5.5) then has solution

$$u_0 \;=\; A(X)f(Y-t) + C(X)\ell(Y+t)$$

and the equation for $u_1$ becomes

$$u_{1,YY} - u_{1,tt} \;=\; A(X)f'(Y-t) + C(X)\ell'(Y+t)$$

$$12(1+X)\{A'(X)f'(Y-t) + C'(X)\ell'(Y+t)\} \;.$$

This may be solved formally treating X as independent of Y and t. In general this gives 'secular terms' (32), which grow with Y-t and Y+t so that the expansion will not converge at large distances behind a wavefront. To eliminate secular terms we choose

$$A(X) - 2(1+X)A'(X) \;=\; 0$$

$$C(X) - 2(1+X)C'(X) \;=\; 0$$

which give $A(X) \;=\; a_0(1+X)^{\frac{1}{2}}$

and $C(X) \;=\; c_0(1+X)^{\frac{1}{2}}$ ,

where $a_0$ and $c_0$ are constants.

Hence

$$u_0 \;=\; (1+X)^{\frac{1}{2}}\{a_0 f_0(Y-t) + c_0 \ell_0(Y+t)\} \;.$$

The function $u_1$ may then be written

$$u_1 \;=\; A_1(X)f_1(Y-t) + C_1(X)\ell_1(Y+t) \;,$$

which could be absorbed into $u_0$ by perturbing A(X) as $A = a_0(1+X)^{\frac{1}{2}} + O(\varepsilon)$.

We now suppose that $u_0$ has the specific form

$$u_0 = (1+X)^{\frac{1}{2}}\{a_0\sin(Y-t) + b_0\cos(Y-t) + c_0\sin(Y+t) + d_0\cos(Y+t)\},$$

where the choice of sines and cosines is crucial for the elimination of further secular terms, which is to be performed. We write $u_1$ as

$$u_1 = A_1\sin(Y-t) + B_1\cos(Y-t) + C_1\sin(Y+t) + D_1\cos(Y+t).$$

The function $u_2$ then satisfies

$$\begin{aligned} u_{2,YY} - u_{2,tt} = {} & A_1\cos(Y-t) - B_1\sin(Y-t) + C_1\cos(Y+t) \\ & - D_1\sin(Y+t) - 2(1+X)\{A_1'\cos(Y-t) \\ & - B_1'\sin(Y-t) + C_1'\cos(Y+t) - D_1'\sin(Y+t)\} \\ & - (1+X)^2\{-\tfrac{1}{4}(1+X)^{-3/2}\}\{a_0\sin(Y-t) \\ & + b_0\cos(Y-t) + c_0\sin(Y+t) + d_0\cos(Y+t)\} . \end{aligned}$$

Secular terms are eliminated by choosing

$$A_1 - 2(1+X)A_1' = \frac{1}{4}(1+X)^{\frac{1}{2}} b_0$$

which implies

$$A_1 = (1+X)^{\frac{1}{2}}\{a_1 - \frac{1}{8} b_0\log(1+X)\}$$

and similarly

$$B_1 = (1+X)^{\frac{1}{2}}\{b_1 + \frac{1}{8} a_0\log(1+X)\}$$

$$C_1 = (1+X)^{\frac{1}{2}}\{c_1 - \frac{1}{8} d_0\log(1+X)\}$$

$$D_1 = (1+X)^{\frac{1}{2}}\{d_1 + \frac{1}{8} c_0\log(1+X)\} .$$

Now in region I from (2.2.8) we may write

$$u = \sin(x-t) + R\sin(x+t) + S\cos(x+t)$$

and in region III from (2.2.9) we write

$$u = K \sin\left(\frac{x}{2} - t\right) + L \cos\left(\frac{x}{2} - t\right) .$$

We now, as in Section 2.2, make u and $u_x$ continuous at $x = 0$ and $x = \varepsilon$ and obtain the arbitrary constants in the solutions. In region I we find that

$$u = \sin(x-t) - \frac{\varepsilon}{2} \sin\nu \sin(x+t-\nu)$$

and in region III

$$u = \sqrt{2} \left(\cos\left(\nu - \frac{1}{2\varepsilon}\right) + \frac{\varepsilon}{8} \log 2 \sin\left(\frac{1}{2\varepsilon} - \nu\right) + \ldots\right) \sin\left(\frac{x}{2} - t\right)$$

$$+ \sqrt{2}\left(\sin\left(\nu - \frac{1}{2\varepsilon}\right) + \frac{\varepsilon}{8} \log 2 \cos\left(\frac{1}{2\varepsilon} - \nu\right) + \ldots\right) \cos\left(\frac{x}{2} - t\right),$$

where $\nu = \frac{1}{\varepsilon} \log 2$. Therefore, the reflected part in I depends crucially on $1/\varepsilon$ in amplitude and in phase change, which is what is expected as the timing of the arrival of the reflected wave in II at the I - II boundary determines whether or not the wave is in phase and so reinforces the existing wave. In region II we find

$$u = (1+\varepsilon x)^{\frac{1}{2}}\Bigg[\sin\left(\frac{1}{\varepsilon} \log(1+\varepsilon x) - t\right)$$

$$+ \varepsilon\Big\{\left(\frac{1}{8} \log(1+\varepsilon x) + \frac{1}{4}\right) \cos\left(\frac{1}{\varepsilon} \log(1+\varepsilon x) - t\right)$$

$$- \frac{1}{4} \sin\left(\frac{2}{\varepsilon} \log 2\right) \sin\left(\frac{1}{\varepsilon} \log(1+\varepsilon x) + t\right)$$

$$- \frac{1}{4} \cos\left(\frac{2}{\varepsilon} \log 2\right) \cos\left(\frac{1}{\varepsilon} \log(1+\varepsilon x) + t\right)\Big\}$$

$$+ O(\varepsilon^2) \Bigg] .$$

This method has the advantage over those considered in Sections 2.3 and 2.4 in that since we are working directly on the wave equation the method is not restricted to the particular inhomogeneity considered here. The function $g(X)$ may be chosen to satisfy (2.5.6) for any function $F(X)$. However, the method does depend crucially on the choice of $u_0$ as a linear combination of sine and cosine functions of the variables $Y-t$ and $Y+t$ and so is not appropriate for signals behind the front of a wave advancing into a quiet region.

## CONCLUSION

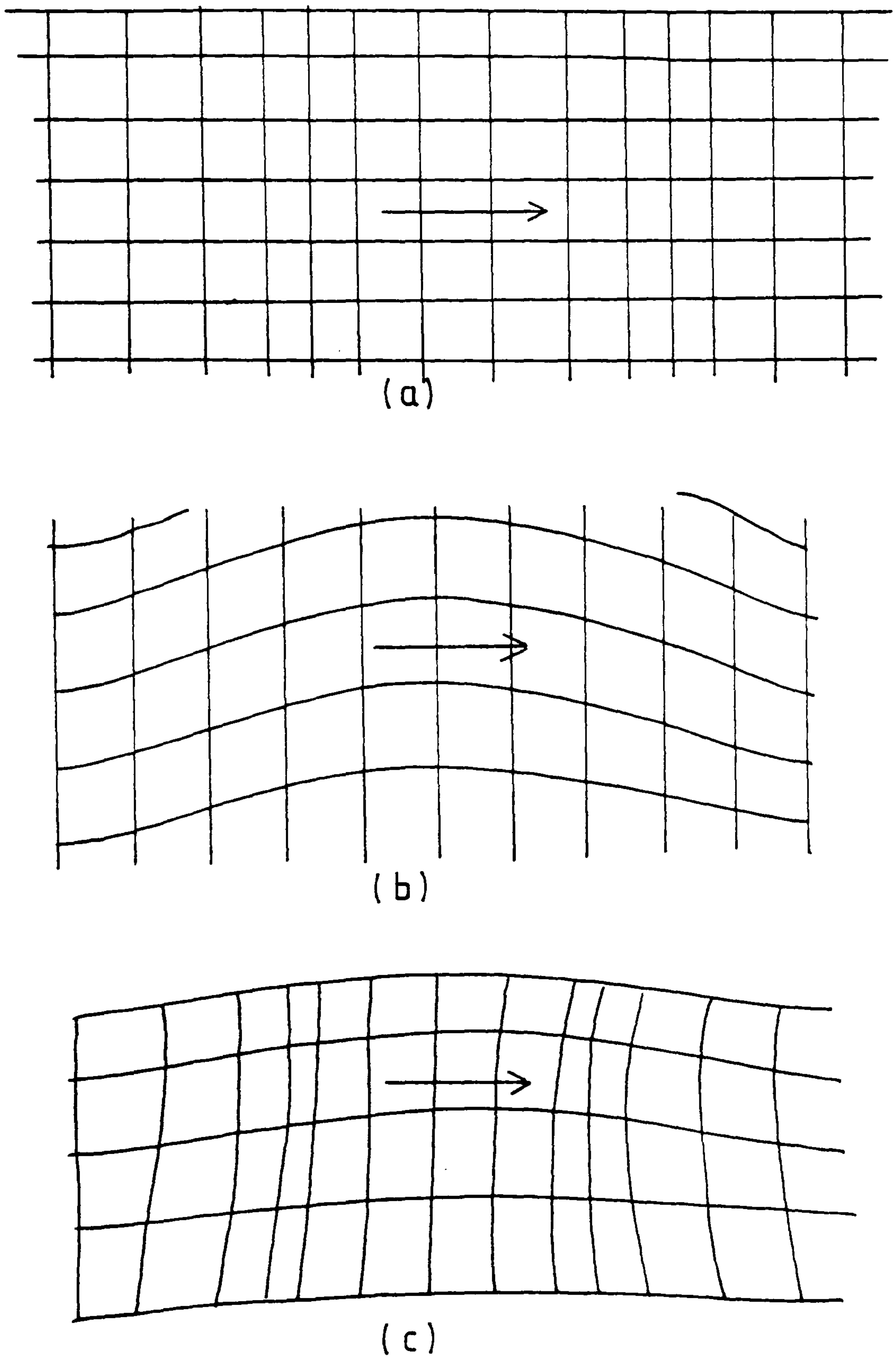
In this chapter we have considered an iterative procedure for determining a non-linear wave propagating into a non-uniform region. We have shown that by using a straight-forward iterative process, integrating along $X = 0$ and the forward characteristic only a finite number of iterations could be performed while obtaining bounded iterates. We therefore integrated along $\frac{dX}{d\alpha} = -\frac{1}{2}\ell c$, a backward characteristic and along a forward characteristic and found a bound on the region for which an iterative solution to these equations converges. To check this prediction we have analysed some linear problems, which are special cases of the general non-linear theory, since a variety of methods for solution of these linear equations are available.

In Section 2.2 the wave equation with speed having a discontinuous first derivative was considered. An analytical solution was obtained, the coefficients in this being found from asymptotic expansions. In the next section we considered other methods of solving this problem for the region where the wave speed is varying, first transforming the equation. In Section 2.4 we used characteristic coordinates and considered Riemann's method of solution and also an iterative procedure similar to that used in Section 2.1. As in that section we obtained a bound on the region of convergence for this iterative procedure. In Section 2.5 we returned to the same formulation of the problem as in Section 2.2 and developed an asymptotic method which could be applied to other examples of inhomogeneity, although this method is restricted to a sinusoidal solution, since it depends on the elimination of secular terms.

# PART II

## INTRODUCTION

The simplest type of acoustic wave that can travel through an isotropic elastic material is a longitudinal wave, in which as the wave passes, the material is alternately compressed and expanded. A second kind of wave is the transverse, or shear wave, in which material particles oscillate from side to side at right angles to the direction of the acoustic signal. A third kind of wave for an elastic half-space with a free surface is one which moves without change of shape adjacent to the traction-free surface. The amplitude of the wave decays rapidly with depth so that the effect of the wave is confined to the vicinity of the free surface. Such waves incorporate both shear and longitudinal components, which are combined together in such a way as to satisfy the boundary conditions and travel along the surface of a solid, much like ripples on the surface of a pond, see Figure 1 (22). In linear elasticity they are known as Rayleigh waves, after Lord Rayleigh, who in 1885 demonstrated theoretically their existence (46). Rayleigh waves form a principal part of a seismic signal. The other waves are attenuated throughout a three-dimensional volume, while Rayleigh waves only spread their energy in a two-dimensional region near the earth's surface. Seismograms show that longitudinal waves appear first, followed by shear waves and then Rayleigh waves in accord with the ordering of speeds $c_R^2 \leq c_S^2 \leq c_L^2$ where $c_R$ is the Rayleigh wave speed, $c_S$ the shear wave speed and $c_L$ the longitudinal wave speed (50). This ordering is to be expected, since the reaction of the material to the volume change in compression should be more vigorous than that to orientation change in shear, while the free surface permits a still easier distortion and hence a still lower speed.



(a)

(b)

(c)

Figure 1  Longitudinal waves (a), shear waves (b) and Rayleigh waves (c)

More recently acoustic surface waves have been studied for a different application, to process signals in communication systems. In this application electrical signals are used to excite Rayleigh waves on the surface of a crystal a few centimetres long and one or two millimetres thick. The waves excited in this way are of frequency one million hertz or more and are employed to store and recognise electronic signals. The advantage provided by Rayleigh waves as opposed to longitudinal or shear waves which were used initially in electronic applications is that the waves may be detected at the surface and so are well adapted to the technology for creating microcircuits in thin flat structures. In typical applications most of the acoustic energy is contained within a distance of a few hundredths of a millimetre of the crystal surface. The waves can be excited easily anywhere on the surface and readily collected elsewhere on the same 'chip'. In recent years the technology of acoustic waves has expanded rapidly with the development of the interdigital transducer, an efficient type of transducer for converting the electrical signal into an acoustic surface wave and reconverting the acoustic wave back into an electrical signal. An interdigital transducer is normally placed in a piezoelectric material. Many applications of surface waves are described in the book by Viktorov (60).

A considerable amount of theoretical work has been done on Rayleigh waves. Lapin has considered in (25) the reflection of a Rayleigh wave from periodic corrugations of a surface in oblique incidence, in (24) he examines the reflections from periodic irregularities of a liquid-solid interface and in (23) the scattering of surface waves propagating over an uneven liquid-solid interface. Barnett and Lothe (2) have examined the questions of existence and uniqueness of surface waves in an anisotropic elastic body, which is not subjected to pre-stress and they have produced a theoretical framework within which these questions can be answered. Surface waves have also been investigated by, amongst others, Taylor and Crampin. In (55) Taylor and Crampin examine the propagation of a wave in a homogeneous piezo-electric half-space and in (54) Taylor considers the secular equation for the propagation velocity and its numerical solution. In (6) Chadwick and Jarvis investigate to what extent the theory of Barnett and Lothe is applicable when the reference state is not stress-free.

The propagation of surface waves of small amplitude in an elastic body which is not stress-free in its undisturbed state was first studied by Hayes and Rivlin (17). They applied the theory of superposition of infinitesimal deformations in an isotropic elastic material to the study of the propagation of surface waves in a semi-infinite body which is subjected to a static pure homogeneous deformation, and is traction-free at its plane boundary. The same simplifications have been adopted by all subsequent writers on the subject. Hayes and Rivlin derived the secular equation determining the speed of propagation of a surface wave travelling along a principal axis of stretch. In (14) Flavin considered the problem of determining conditions on the pre-strain which are necessary and sufficient for the existence of a progressive surface wave. He considered incompressible materials of the neo-Hookean and Mooney types, materials which are described in Chapter Five, and in the case of a neo-Hookean material showed that there are three possibilities: a surface wave can propagate in every in-plane direction, in some directions only, forming two opposite sectors or in no direction. Willson (65) has investigated properties of surface waves for a variety of isotropic, elastic materials, compressible and incompressible and for different states of pre-stress and pre-strain. In (65) Willson indicated that the possibilities exhibited by Flavin also apply when the transmitting material is compressible and characterised by a restricted form of the Hadamard strain-energy function. Braun (4) has also considered Rayleigh waves in a pre-stressed neo-Hookean material and Iwashimizu and Kobori (18) the Rayleigh wave in a finitely deformed elastic material.

In standard analyses Rayleigh waves are sought in the form of disturbances having displacements whose horizontal variation is sinusoidal and which propagate without distortion at some speed c. Since the governing equations and boundary conditions involve no natural scale of length or time, this speed is independent of wavelength so that Rayleigh waves are non-dispersive and all wavelengths travel at the same speed $c_R$, the 'Rayleigh wave speed'. Superposing these periodic solutions as a Fourier integral yields a representation of a disturbance having surface elevation of arbitrary profile. Each such surface wave propagates without distortion or attenuation at the speed $c_R$.

The existence of such a wide variety of Rayleigh waves in linear elasticity suggests the possibility that certain non-linear elastic surface waves may also propagate without distortion. Such waves with surface elevation profiles of permanent form may be expected to appear when a disturbance travels for large distances near the traction-free surface of a homogeneous elastic half-space. It might seem that such disturbances may be analysed by perturbation methods designed to elicit a relationship between amplitude, wavelength and propagation speed; however, it will be shown that such methods fail. If the leading approximation to the disturbance is taken as the standard Rayleigh wave having wavenumber k, the next approximation satisfies linear equations and boundary conditions which involve terms corresponding to the wavenumbers 2k and zero. Solutions to these equations are readily constructed by separation of variables, but the boundary conditions can be satisfied only if the solutions contain terms which grow exponentially with depth. A similar situation arises at each later stage of the perturbation process, essentially because there is no dispersive effect to balance the non-linear effects.

The shortcoming of the process lies not in the expansion procedure, but in the assumption that the surface elevation is close to a sinusoidal wavetrain. This assumption must be relaxed by allowing the leading approximation to the disturbance to be an arbitrary non-distorting solution of the linear elastic equations. We anticipate then that certain of these solutions are the infinitesimal-strain limits of non-linear disturbances which can travel without distortion at speeds close to $c_R$. For this reason we represent the general linear Rayleigh wave in terms of a pair of conjugate harmonic functions, which like the representations in (5), does not unduly emphasise sinusoidal wave profiles. Using this representation as the leading term approximation in an expansion in terms of strain amplitude, it is found that the surface elevation profiles of non-distorting waveforms must satisfy a certain non-linear functional equation. In the small-strain limit this reduces to a quadratic functional equation, solutions of which are computed for a compressible and an incompressible material. Methods are presented for the analysis of both periodic, but non-sinusoidal, waveforms and non-periodic waveforms.

For periodic waveforms the quadratic functional equation reduces to a system of quadratic algebraic equations for the coefficients in a series representation. Solutions to this system of equations have been obtained by successively solving truncated systems of N equations involving the first N coefficients. Two solutions were found for waves travelling at a speed different from that of the standard Rayleigh wave and one for a wave travelling at the same speed. Particle paths are plotted at various depths. In the search for non-periodic waveforms the problem reduces to that of solving a quadratic integral equation.

# CHAPTER 3

# NON-LINEAR ELASTIC SURFACE WAVES

## 3.1 LINEAR THEORY

The basic theory has been developed in Parker (39) and (41), but it is outlined here for completeness. Two-dimensional plane-strain disturbances are considered in the elastic half-space $X_2 \geq 0$, where $(x_1, x_2, x_3)$ are the current rectangular cartesian coordinates and $(X_1, X_2, X_3)$ are Lagrangian coordinates taken in an unstressed reference state. The current configuration may then be specified by $x_3 = X_3$ and

$$x_i = x_i(X_j,t) = X_i + u_i(X_j,t) \quad , \quad i,j = 1,2.$$

The associated components of velocity $\underset{\sim}{v}$, deformation gradient $\underset{\sim}{F}$ and Piola-Kirchhoff (engineering) stress $\underset{\sim}{\tau}$ are:

$$\begin{aligned} v_i &= \frac{\partial x_i}{\partial t} = \frac{\partial u_i}{\partial t} , \quad F_{ij} = \frac{\partial x_i}{\partial X_j} = \delta_{ij} + \frac{\partial u_i}{\partial X_j} \\ \tau_{ij} &= \frac{\partial W}{\partial F_{ij}} , \qquad i,j = 1,2 , \end{aligned} \tag{3.1.1}$$

where $W = W(\underset{\sim}{F})$ is the strain-energy density. The momentum equations are:

$$\frac{\partial \tau_{ij}}{\partial X_j} = \rho \frac{\partial v_i}{\partial t} \quad \text{in } X_2 > 0, \quad i,j = 1,2, \tag{3.1.2}$$

where $\rho$ is the density in the reference configuration, assumed uniform. Vanishing of the tractions on the surface $X_2 = 0$ then imposes the boundary condition

$$\tau_{i2} = 0 \qquad \text{on } X_2 = 0 \quad . \tag{3.1.3}$$

Disturbances which propagate without change of form, travelling at some speed c in the $X_1$ direction, are described by the two-dimensional quantities $\underset{\sim}{u} = \underset{\sim}{u}(X,X_2)$, $\underset{\sim}{v} = \underset{\sim}{v}(X,X_2)$, $\underset{\sim}{F} = \underset{\sim}{F}(X,X_2)$ and $\underset{\sim}{\tau} = \underset{\sim}{\tau}(X,X_2)$, where $X = X_1 - ct$. Hence using the relationships:

$$\underset{\sim}{v} \;=\; -c\,\frac{\partial \underset{\sim}{u}}{\partial X} \quad , \quad \frac{\partial \underset{\sim}{v}}{\partial t} \;=\; -c\,\frac{\partial \underset{\sim}{v}}{\partial X} \;=\; c^2\,\frac{\partial^2 \underset{\sim}{u}}{\partial X^2}$$

in (3.1.3) gives

$$\frac{\partial}{\partial X}\left\{\tau_{i1} - \rho\, c^2\,\frac{\partial u_i}{\partial X}\right\} + \frac{\partial \tau_{i2}}{\partial X_2} \;=\; 0 \quad \text{in } X_2 > 0. \tag{3.1.4}$$

Solutions to (3.1.4) may be represented in terms of stress functions $\alpha_i(X,X_2)$ such that, using the comma notation to denote partial derivatives $f_{,1} = \frac{\partial f}{\partial X}$ , $f_{,2} = \frac{\partial f}{\partial X_2}$ of a function $f(X,X_2)$,

$$\begin{aligned} \tau_{11} - \rho\, c^2\, u_{1,1} &= \alpha_{1,2} \quad , \quad \tau_{12} = -\alpha_{1,1} \, , \\ \tau_{21} - \rho\, c^2\, u_{2,1} &= \alpha_{2,2} \quad , \quad \tau_{22} = -\alpha_{2,1} \, , \end{aligned} \tag{3.1.5}$$

whilst, without loss of generality, the boundary conditions (3.1.3) may be taken as

$$\alpha_1(X,0) \;=\; 0 \;=\; \alpha_2(X,0) \quad . \tag{3.1.6}$$

In linear elasticity, $\tau_{ij}$ is indistinguishable from the Cauchy stress and the constitutive law for an isotropic material is

$$\tau_{ij} \;=\; \lambda\, \delta_{ij} u_{k,k} + \mu(u_{i,j} + u_{j,i}) \;=\; \tau_{ji} \; . \tag{3.1.7}$$

Substituting this into equation (3.1.5) yields a set of four linear, homogeneous partial differential equations for $u_1$, $u_2$, $\alpha_1$ and $\alpha_2$. These equations may be combined into the form

$$\frac{\partial}{\partial X}\left[(2\mu - \rho c^2)u_1 - \alpha_2\right] - \frac{\partial}{\partial X_2}\left[2\mu u_2 + \alpha_1\right] = 0 ,$$

$$\mu \frac{\partial}{\partial X_2}\left[(2\mu - \rho c^2)u_1 - \alpha_2\right] - (\mu - \rho c^2)\frac{\partial}{\partial X}\left[2\mu u_2 + \alpha_1\right] = 0 ,$$

$$\frac{\partial}{\partial X}\left[(2\mu - \rho c^2)u_2 + \alpha_1\right] + \frac{\partial}{\partial X_2}\left[2\mu u_1 - \alpha_2\right] = 0 ,$$

$$(\lambda + 2\mu)\frac{\partial}{\partial X_2}\left[(2\mu - \rho c^2)u_2 + \alpha_1\right] - (\lambda + 2\mu - \rho c^2)\frac{\partial}{\partial X}\left[2\,\mu u_1 - \alpha_2\right] = 0,$$

which are similar to two pairs of Cauchy-Riemann equations. Solutions which decay sufficiently rapidly as $X_2 \to \infty$ and which also satisfy the boundary condition (3.1.8) are called non-distorting progressive surface waves.

We define positive A and B by

$$A^2 = 1 - \frac{\rho c^2}{\mu} , \quad B^2 = 1 - \frac{\rho c^2}{\lambda+2\mu} > A^2 \tag{3.1.8}$$

and introduce $\phi_1$, $\phi_2$, $\psi_1$ and $\psi_2$ by

$$\begin{aligned} \rho c^2\phi_1 &= (2\mu - \rho c^2)u_1 - \alpha_2, & \rho c^2\psi_1 &= A(2\mu u_2 + \alpha_1) \\ \rho c^2\phi_2 &= B(2\mu u_1 - \alpha_2) , & \rho c^2\psi_2 &= (2\mu - \rho c^2)u_2 + \alpha_1 \end{aligned} \tag{3.1.9}$$

so that

$$\frac{\partial\phi_1}{\partial X} - \frac{1}{A}\frac{\partial\psi_1}{\partial X_2} = 0 , \quad \frac{\partial\psi_1}{\partial X} + \frac{1}{A}\frac{\partial\phi_1}{\partial X_2} = 0 \tag{3.1.10}$$

$$\frac{\partial\phi_2}{\partial X} - \frac{1}{B}\frac{\partial\psi_2}{\partial X_2} = 0 , \quad \frac{\partial\psi_2}{\partial X} + \frac{1}{B}\frac{\partial\phi_2}{\partial X_2} = 0 . \tag{3.1.11}$$

Equations (3.1.10) show that $\phi_1$ and $\psi_1$ are conjugate harmonic functions of the coordinate pair $(X, AX_2)$, while (3.1.11) show that $\phi_2$ and $\psi_2$ are conjugate harmonic functions of $(X, BX_2)$. Moreover since $u_1$, $u_2$, $\alpha_1$ and $\alpha_2$ are given by

$$u_1 = B^{-1}\phi_2 - \phi_1 \quad , \qquad u_2 = A^{-1}\psi_1 - \psi_2 \quad ,$$

$$\alpha_1 = 2\mu(\psi_2 - \frac{1+A^2}{2A}\psi_1), \qquad \alpha_2 = 2\mu\left(\frac{1+A^2}{2B}\phi_2 - \phi_1\right),$$

the boundary conditions (3.1.6) become

$$\psi_2(X,0) = \frac{1+A^2}{2A}\psi_1(X,0) \ , \quad \phi_2(X,0) = \frac{2B}{1+A^2}\phi_1(X,0) \ .$$

We then write $y = AX_2$, $Y = BX_2$ and define the functions

$$u(X,y) = \phi_1(X,X_2) \ , \qquad v(X,y) = \psi_1(X,X_2) \ , \tag{3.1.12}$$

$$U(X,Y) = \frac{1+A^2}{2B}\phi_2(X,X_2) \ , \quad V(X,Y) = \frac{1+A^2}{2B}\psi_2(X,X_2) \ . \tag{3.1.13}$$

Equations (3.1.10) and (3.1.11) then simplify as

$$\frac{\partial u}{\partial X} - \frac{\partial v}{\partial y} = 0 \ , \quad \frac{\partial v}{\partial X} + \frac{\partial u}{\partial y} = 0 \qquad \text{in } y > 0 \tag{3.1.14}$$

$$\frac{\partial U}{\partial X} - \frac{\partial V}{\partial Y} = 0 \ , \quad \frac{\partial V}{\partial X} + \frac{\partial U}{\partial Y} = 0 \qquad \text{in } Y > 0 \ , \tag{3.1.15}$$

whilst the boundary conditions reduce to

$$U(X,0) = u(X,0) \quad , \quad V(X,0) = Kv(X,0) \quad , \tag{3.1.16}$$

where
$$K \equiv \frac{(1+A^2)^2}{4AB} \tag{3.1.17}$$

and $u(X,0)$ , $v(X,0)$ are related to the surface displacements by

$$u_1(X,0) = \frac{1-A^2}{1+A^2}u(X,0) \quad , \quad u_2(X,0) = \frac{1-A^2}{2A}v(X,0) \ . \tag{3.1.18}$$

Equations (3.1.14) , (3.1.16) show that both $v(X,\eta)$ and $V(X,\eta)$ are harmonic in the half-plane $\eta > 0$ and take identical values on $\eta = 0$.

Additionally they must remain bounded as $\eta \to \infty$ so that $u_1$ and $u_2$ decay as $\eta \to \infty$. In (41) Parker makes precise the manner of decay for (A) periodic waveforms and (B) general waveforms.

For case (A) he restricts attention to functions $u(X,y)$, $v(X,y)$, $U(X,Y)$ and $V(X,Y)$ which belong to the space $\mathcal{B}_p(\bar{\Omega}|\mathbb{R})$, where $\bar{\Omega}$ is the subset of real two dimensional Euclidean space $\{(X,\eta): -\infty < X < \infty, \eta \geq 0\}$ and functions belonging to $\mathcal{B}_p(\bar{\Omega}|\mathbb{R})$ are periodic in X with period P and decay uniformly in X as $\eta \to \infty$. General waveforms, for case (B), are restricted by the requirements that $u(X,y)$, $v(X,y)$, $U(X,Y)$ and $V(X,Y)$ belong to $\mathcal{B}_*(\bar{\Omega}|\mathbb{R})$, where $\mathcal{B}_*(\bar{\Omega}|\mathbb{R})$ is the set of bounded real, infinitely differentiable functions $\phi(X,\eta)$ for which $(X^2+\eta^2)^{\frac{1}{2}}\phi(X,\eta)$ remains bounded as $(X^2+\eta^2)^{\frac{1}{2}} \to \infty$ for all $\eta \geq 0$. The boundary values are correspondingly once continuously differentiable functions, the spaces for which are denoted by $\mathcal{B}_p'(\mathbb{R}|\mathbb{R})$, $\mathcal{B}_*'(\mathbb{R}|\mathbb{R})$ in cases (A) and (B) respectively. Then restricting the harmonic function $v(X,\eta) - V(X,\eta)$ to either $\mathcal{B}_p(\bar{\Omega}|\mathbb{R})$ or $\mathcal{B}_*(\bar{\Omega}|\mathbb{R})$ is sufficient together with (3.1.16) to ensure that $V(X,\eta) = v(X,\eta)$ throughout $\bar{\Omega}$. Thus we may write

$$v(X,y) = \gamma(X,y) = \gamma(X,AX_2), \quad V(X,Y) = \gamma(X,Y) = \gamma(X,BX_2),$$

where $\gamma(X,\eta)$ is any harmonic function belonging to $\mathcal{B}_p(\bar{\Omega}|\mathbb{R})$ or $\mathcal{B}_*(\bar{\Omega}|\mathbb{R})$ in cases (A) or (B) respectively. Each such function has a unique harmonic conjugate $\beta(X,\eta)$ belonging to the same subspace, so we may write

$$u(X,y) = \beta(X,y) = \beta(X,AX_2), \quad U(X,y) = \beta(X,y) = \beta(X,BX_2).$$

Then provided that

$$K = 1 \tag{3.1.19}$$

the boundary condition $(3.1.16)_2$ puts no further restrictions on $\beta(X,\eta)$

and $\gamma(X,\eta)$. Condition (3.1.19) after substitution from equations (3.1.8) can be rewritten as the standard 'secular equation'

$$(\rho c^2)^3 - 8\mu(\rho c^2)^2 + 8\mu^2\left(\frac{3\lambda+4\mu}{\lambda+\mu}\right)\rho c^2 - 16\mu^3\left(\frac{\lambda+\mu}{\lambda+2\mu}\right) = 0 \tag{3.1.20}$$

defining the possible propagation speeds c of sinusoidal Rayleigh waves. For all realistic Poisson ratios (5), it defines only one real positive speed, which we denote by $c = c_R$ and which satisfies $0 < \rho c_R^2 < \mu$. The corresponding quantities

$$A_R = \left(1 - \frac{\rho c_R^2}{\mu}\right)^{\frac{1}{2}}, \quad B_R = \left(1 - \frac{\rho c_R^2}{\lambda+2\mu}\right)^{\frac{1}{2}} > A_R$$

defined by (3.1.8) are then real, and X, y and Y become $y = A_R X_2$, $Y = B_R X_2$, $X = X_1 - c_R t$.

Each pair $\beta(X,\eta)$, $\gamma(X,\eta)$ of conjugate harmonic functions belonging to either $\mathcal{B}_p(\bar{\Omega}|\mathbb{R})$ or $\mathcal{B}_*(\bar{\Omega}|\mathbb{R})$ may be used to describe an elastic surface wave travelling at speed $c_R$. The corresponding displacements are

$$\begin{aligned} u_1(X,X_2) &= (A_R B_R)^{-\frac{1}{2}}\beta(X,B_R X_2) - \beta(X,A_R X_2) \\ u_2(X,X_2) &= -(B_R/A_R)^{\frac{1}{2}}\gamma(X,B_R X_2) - \gamma(X,A_R X_2) \\ u_3(X,X_2) &= 0 . \end{aligned} \tag{3.1.21}$$

The vertical displacement at the surface may be written in the form

$$u_2(X,0) = \frac{1 - A_R^2}{2A_R}\,\sigma(X) \tag{3.1.22}$$

where $\sigma(X) = \gamma(X,0)$ .

For periodic waves (case (A)) $\sigma(X)$ may be any function in $\mathcal{B}_p'(\mathbb{R}|\mathbb{R})$, while general waves (case (B)) may have surface elevation given by any $\sigma(X)\in\mathcal{B}_*'(\mathbb{R}|\mathbb{R})$.

Conversely, if any $\sigma(X)$ belonging to either of these spaces is given, a unique linear Rayleigh wave may be determined. In case (B), $\beta(X,\eta)$ and $\gamma(X,\eta)$ may be constructed from the boundary values $\gamma(X,0) = \sigma(X)$ either by the Poisson integral formulae or by Fourier transformation. In case (A) they may be represented as Fourier series in X. The special case $\sigma(X) = \cos kX$ gives

$$\beta(X,\eta) = -e^{-k\eta} \sin kx, \quad \gamma(X,\eta) = e^{-k\eta} \cos kx,$$

which represents the standard sinusoidal wavetrain having wavenumber k. It has period P whenever $kP/2\pi$ is an integer.

## 3.2 NON-LINEAR THEORY

The constitutive law $\underset{\sim}{\tau} = \underset{\sim}{\tau}(\underset{\sim}{F})$ describing an isotropic elastic material may be approximated for small $||u_{i,j}||$, by the linear relation (3.1.7). In general, it may be split into its linear and non-linear parts as

$$\tau_{ij} = \lambda\delta_{ij}u_{k,k} + \mu(u_{i,j} + u_{j,i}) + N_{ij}(u_{k,l}) \, , \tag{3.2.1}$$

where the non-linear terms satisfy $|N_{ij}| = o(||u_{ij}||)$ as $||u_{ij}|| \to 0$. Consequently the $N_{ij}$ terms describe small corrections to (3.1.7) when the displacement gradient is small.

As in the previous section stress functions $\alpha_i(X,X_2)$ are introduced. Substituting the constitutive law (3.2.1) into the equations (2.1.5) gives

$$(\lambda + 2\mu - \rho c^2)u_{1,1} + \lambda u_{2,2} - \alpha_{1,2} = -N_{11} \, ,$$

$$\mu(u_{1,2} + u_{2,1}) + \alpha_{1,1} = -N_{12} \, ,$$

$$\mu u_{1,2} + (\mu - \rho c^2)u_{2,1} - \alpha_{2,2} = -N_{21} \, ,$$

$$\lambda u_{1,1} + (\lambda + 2\mu)u_{2,2} + \alpha_{2,1} = -N_{22} \, .$$

Taking linear combinations of these and introducing $\phi_1$, $\phi_2$, $\psi_1$ and $\psi_2$ as in (3.1.9) gives

$$\frac{\partial\phi_1}{\partial X} - \frac{1}{A}\frac{\partial\psi_1}{\partial X_2} = \frac{N_{22} - N_{11}}{\rho c^2} \, , \quad \frac{\partial\psi_1}{\partial X} + \frac{1}{A}\frac{\partial\phi_1}{\partial X_2} = -\frac{(A^2N_{12} + N_{21})}{A\rho c^2}, \tag{3.2.2}$$

$$\frac{\partial\phi_2}{\partial X} - \frac{1}{B}\frac{\partial\psi_2}{\partial X_2} = \frac{B^2N_{22} - N_{11}}{B\rho c^2} \, , \quad \frac{\partial\psi_2}{\partial X} + \frac{1}{B}\frac{\partial\phi_2}{\partial X_2} = -\frac{(N_{12} + N_{21})}{\rho c^2} \tag{3.2.3}$$

which are coupled, non-linear versions of (3.1.10) and (3.1.11). Since the boundary conditions remain as $\psi_2(X,0) = \frac{1+A^2}{2A}\psi_1(X,0)$, $\phi_2(X,0) = \frac{2B}{1+A^2}\phi_1(X,0)$ we introduce u, v, U and V as in (3.1.12) and (3.1.13). Equations (3.2.2) and (3.2.3) then become

$$u_x - v_y = \frac{N_{22} - N_{11}}{\rho c^2}, \quad v_x + u_y = -\frac{(AN_{12} + A^{-1}N_{21})}{\rho c^2}, \tag{3.2.4}$$

$$U_X - V_Y = \frac{1+A^2}{2B}\left(\frac{BN_{22} - B^{-1}N_{11}}{\rho c^2}\right), \quad V_X + U_Y = -\frac{1+A^2}{2B}\left(\frac{N_{12} + N_{21}}{\rho c^2}\right), \tag{3.2.5}$$

whilst the boundary conditions (3.1.16) remain unchanged as

$$U(X,0) = u(X,0), \quad V(X,0) = Kv(X,0) . \tag{3.2.6}$$

Here $y = AX_2$ and $Y = BX_2$, so that $Y = \Lambda y$ ,

$$\text{where} \quad \Lambda \equiv \frac{B}{A} = \left(\frac{\lambda+2\mu-\rho c^2}{\mu-\rho c^2}\right)^{\frac{1}{2}} = \Lambda(c^2)$$

$$\text{and} \quad K \equiv \frac{(1+A^2)^2}{4AB} = \frac{(2\mu-\rho c^2)}{4\mu^2}\left\{\frac{\mu(\lambda+2\mu)}{(\mu-\rho c^2)(\lambda+2\mu-\rho c^2)}\right\}^{\frac{1}{2}} = K(c^2)$$

are functions of the square of the propagation speed c, as are A and B. Consequently, if $c^2 = \hat{c}^2(\simeq\frac{1}{2}\mu/\rho)$ is the value for which $\frac{dK}{dc^2} = 0$, $c^2$ may be expressed as a single-valued function of K in the interval $\hat{c} < c < (\mu/\rho)^{\frac{1}{2}}$ which includes $c = c_R$ and within this range we may regard c, A, B and $\Lambda$ as functions of K. In terms of the solutions to (3.2.4) - (3.2.6) the displacements are:

$$\begin{aligned} u_1(X,X_2) &= \frac{2}{1+A^2}U(X,Y) - u(X,y) , \\ u_2(X,X_2) &= \frac{1}{A}v(X,y) - \frac{2B}{1+A^2}V(X,Y) , \end{aligned} \tag{3.2.7}$$

and the surface displacements as in (3.1.18) are

$$u_1(X,0) = \frac{1-A^2}{1+A^2}\, u(X,0) \ , \quad u_2(X,0) = \frac{1-A^2}{2A}\, v(X,0) \ .$$

In the infinitesimal limit the right-hand sides of (3.2.4) and (3.2.5) are neglected. As was shown in Section 3.1, the linearization (3.1.14) - (3.1.16) then possesses solutions only for $K = 1$. However, corresponding to this 'eigenvalue', $v(X,0)$ or equivalently $u(X,0)$, may be chosen to be any function belonging to $\mathcal{L}_p^{\prime}(\mathbb{R}|\mathbb{R})$ or $\mathcal{L}_*^{\prime}(\mathbb{R}|\mathbb{R})$. Once non-linearity is included, linear superposition is lost. We expect (3.2.4) - (3.2.6) still to possess solutions, but only for certain families of functions $v(X,0)$. Generally, corresponding to each such family we should expect the 'non-linear eigenvalue' $K$ to depend on the amplitude $|v(X,0)|_{max}$ of the surface elevation.

We now seek to examine which profiles out of the superabundance of linear Rayleigh wave profiles are the small-strain limits of non-linear surface waves of permanent form. This we do by considering the system

$$u_X - v_y = \ell(X,y) \ , \qquad v_X + u_y = m(X,y) \tag{3.2.8}$$

$$U_X - V_Y = L(X,Y) \ , \qquad V_X + U_Y = M(X,Y) \tag{3.2.9}$$

in $y > 0$, $Y = \Lambda y > 0$, subject to the boundary conditions (3.2.6) with

$$\left.\begin{array}{ll} u(X,\eta) \to 2\mu_1 \ , & v(X,\eta) \to 2AB\mu_2 \ , \\ U(X,\eta) \to (1+A^2)\mu_1 \ , & V(X,\eta) \to (1+A^2)\mu_2 \ , \end{array}\right\} \text{as } \eta \to \infty$$

for some constants $\mu_1$ and $\mu_2$. (This is equivalent to replacing $N_{ij}$ by specified functions of $(X,y)$ in (3.2.4) and of $(X,Y)$ in (3.2.5) respectively.) We introduce the functions $w(X,\eta)$ and $\nu(X,\eta)$ defined for $(X,\eta)\in\overline{\Omega}$ by

$$U(X,\eta) \equiv u(X,\eta) + w(X,\eta) \;, \quad V(X,\eta) \equiv v(X,\eta) + \nu(X,\eta) \;. \tag{3.2.10}$$

Then (3.2.8) and (3.2.9) become

$$w_X - \nu_\eta = L(X,\eta) - \ell(X,\eta) \;, \quad \nu_X + w_\eta = M(X,\eta) - m(X,\eta) \tag{3.2.11}$$

whilst the boundary conditions (3.2.6) give

$$w(X,0) \;=\; 0 \tag{3.2.12}$$

and

$$\nu(X,0) \;=\; (K-1)v(X,0) \tag{3.2.13}$$

with

$$w(X,\eta) \to (A^2 - 1)\mu_1, \quad \nu(X,\eta) \to (1 + A^2 - 2AB)\mu_2 \text{ as } \eta \to \infty \;.$$

The values of $\mu_1$ and $\mu_2$ in case (A) are obtained from (3.2.8) - (3.2.13) by using Green's theorem.

We recall that $w$, $\nu$, $M$, $m$, $L$ and $\ell$ are P-periodic in X and apply Green's theorem to $(3.2.11)_2$, using the boundary condition (3.2.12) then gives

$$\int_0^\infty \int_0^P \{M(X,\eta) - m(X,\eta)\}\, dX d\eta \;=\; P \lim_{\eta\to\infty} w \;=\; -P(1-A^2)\mu_1 \;. \tag{3.2.14}$$

Provided that this double integral exists, that $L-\ell$ and $M-m$ have continuous pth-order derivatives in $\Omega$ and that they decay sufficiently rapidly for (3.2.11) and (3.2.12) to possess a solution such that $w(X,\eta) + (1-A^2)\mu_1 \varepsilon \mathcal{B}_p^{p+1}(\bar{\Omega}|\mathbb{R})$, then this solution is unique. The solution $w(X,\eta)$ defines $\nu_X$ and $\nu_\eta$ uniquely, so that there is a unique solution $\nu(X,\eta)$ to (3.2.11) having $\nu(X,\eta) - (1 + A^2 - 2AB)\mu_2 \varepsilon \mathcal{B}_p^{p+1}(\bar{\Omega}|\mathbb{R})$. This function is a linear functional of $L-\ell$ and $M-m$, so that its boundary value may be expressed as

$$\nu(X,0) - (1 + A^2 - 2AB)\mu_2 = \mathcal{L}(L-\ell, M-m)$$

$$= \mathcal{L}(L,M) - \mathcal{L}(\ell,m). \qquad (3.2.15)$$

Equation (3.2.13) then gives

$$(K-1)v(X,0) = \mathcal{L}(L,M) - \mathcal{L}(\ell,m) + (1 + A^2 - 2AB)\mu_2 \quad , \qquad (3.2.16)$$

where $\mathcal{L}$ is a linear functional which may be constructed (see section 3.4) by Fourier series procedures. The constant $\mu_2$ is found by applying (3.2.13) to the two identities

$$\int_0^\infty \int_0^P \{L(X,\eta) - \ell(X,\eta)\} dX d\eta = - \int_0^P (1 + A^2 - 2AB)\mu_2 dX$$

$$+ \int_0^P \nu(X,0) dX$$

$$\int_0^\infty \int_0^P \ell(X,\eta) dX d\eta = - \int_0^P 2AB\mu_2 dX + \int_0^P v(X,0) dX$$

which follow from $(3.2.11)_1$ and $(3.2.8)_1$. This leads to

$$\int_0^\infty \int_0^P \{L(X,\eta) - K\ell(X,\eta)\} dX d\eta = -P\mu_2(1 + A^2 - 2KAB)$$

$$= -\tfrac{1}{2} P(1 - A^4)\mu_2 \quad . \qquad (3.2.17)$$

Similarly, in case (B) Green's theorem may be used to show that if L, $\ell$, M and m are integrable over the half-plane $\eta > 0$, then $\mu_1 = 0$ and $\mu_2 = 0$. Provided that L-$\ell$ and M-m have continuous pth-order derivatives in $\Omega$ and decay as $X^2 + \eta^2 \to \infty$ sufficiently rapidly for (3.2.11) and (3.2.12) to possess a solution, then there exist unique functions $w(X,\eta)$, $\nu(X,\eta) \in \mathcal{B}_*^{P+1}(\bar{\Omega}|\mathbb{R})$. Equations (3.2.15) and (3.2.16) still hold, but with $\mu_1 = \mu_2 = 0$, and where the linear functional $\mathcal{L}$ may be constructed either by Fourier transforms or by Green's function methods.

From (3.2.15) we see that three situations may arise:

1. $\mathcal{L}(L-\ell, M-m) = \mathcal{L}(L,M) - \mathcal{L}(\ell,m) \neq -(1 + A^2 - 2AB)\mu_2$. Apart from the translation $\mu_2(4A^2 - 2AB)/(K-1)$, the surface profile $v(X,0)$ may then be any scalar multiple of $\mathcal{L}(L-\ell, M-m)$. Each choice of multiplier determines K-1, and so specifies how the propagation speed c differs from the linear Rayleigh wave speed $c_R$. Also it provides the boundary condition $v(X,0)$ which determines unique solutions $u(X,y)$ and $v(X,y)$ to equation (3.2.8) and also determines $U(X,\eta) = u(X,\eta) + w(X,\eta)$ and $V(X,\eta) = v(X,\eta) + \nu(X,\eta)$.
2. $\mathcal{L}(L,M) - \mathcal{L}(\ell,m) = -(1 + A^2 - 2AB)\mu_2$, with $K = 1$. In this case the speed c remains equal to the linear Rayleigh speed $c_R$ and equation (3.2.16) imposes no restriction on $v(X,0)$. Each choice of profile $v(X,0)$ allows u, v, U and V to be uniquely constructed.
3. $\mathcal{L}(L,M) - \mathcal{L}(\ell,m) = \text{constant}$, $v(X,0) = 0$. In any such disturbance the surface $X_2 = 0$ would remain level and (3.2.14) would not restrict K. We shall show in section 3.3 that this possibility does not arise for weakly non-linear waves.

Returning to equations (3.2.4) - (3.2.6) we see that $\ell$, m, L and M must be treated as non-linear functions of the first-order partial derivatives of u, v, U and V. Equation (3.2.13) is in reality a relation between $v(X,0)$ and a certain non-linear functional of the first-order partial derivatives of $u(X,y)$, $v(X,y)$, $U(X,Y)$ and $V(X,Y)$. Since these functions are themselves solutions of equations (3.2.4) and (3.2.5), with boundary conditions (3.2.6) which include the data $v(X,0)$, equation (3.2.14) is a non-linear functional equation for $v(X,0)$. The manner in which the various branches of its solution evolve from the infinitesimal limit $||v(X,0)|| = 0$ is investigated in section 3.3 by a perturbation analysis.

## 3.3 PERTURBATION ANALYSIS

To study non-linear effects when the displacement gradients remain small, we regard displacement gradients $u_{i,j}$ as $O(\varepsilon)$ quantities, where $\varepsilon$ is a small parameter and we suppose that the length scale has been chosen so that typical magnitudes of $u_i$ and $u_{i,j}$ are comparable. Then, provided that the strains remain within the range for which the stress-strain law is analytic, the terms $(\rho c^2)^{-1}N_{ij}$ are $O(\varepsilon^2)$ quantities. Equation (3.2.13) suggests that K-1 is $O(\varepsilon)$. Following Parker (41) we may write formal expansions

$$u_{i,j} = \varepsilon\overset{1}{u}_{i,j} + \varepsilon^2\overset{2}{u}_{i,j} + \ldots \quad , \quad K = 1 + \varepsilon\overset{1}{K} + \ldots$$

which then would allow us to write

$$N_{ij}(u_{k,\ell}) = \mu\varepsilon^2 M_{ij}(\overset{1}{u}_{k,\ell} + \varepsilon^2\overset{2}{u}_{k,\ell} + \ldots; \varepsilon)$$

$$= \varepsilon^2\mu M_{ij}(\overset{1}{u}_{k,\ell}; 0) + O(\varepsilon^3\mu) \; ,$$

where comparison with (3.2.1) shows that

$$\mu M_{ij}(\overset{1}{u}_{k,\ell}; 0) = c_{ijk\ell mn}\overset{1}{u}_{k,\ell}\overset{1}{u}_{m,n} \tag{3.3.1}$$

and the coefficients $c_{ijk\ell mn}$ are the second-order elastic moduli

$$c_{ijk\ell mn} = \left. \frac{\partial^2\tau_{ij}}{\partial u_{k,\ell}\partial u_{m,n}} \right|_0$$

To leading order equation (3.2.14) then becomes a quadratic functional equation involving the $O(\varepsilon)$ terms $\overset{1}{u}_{i,j}$ and $\overset{1}{K}$.

This formal expansion procedure involves cumbersome notation which we avoid by writing u, v, U and V in a form suitable for iterative solution of (3.2.4) - (3.2.6). Then anticipating that u, v, U and V will be close to a solution of the linearized problem, we write

$$u(X,y) = \varepsilon\beta(X,y) + \varepsilon^2\bar{u}(X,y;\varepsilon), \quad v(X,y) = \varepsilon\gamma(X,y) + \varepsilon^2\bar{v}(X,y;\varepsilon)$$

$$(3.3.2)$$

$$U(X,Y) = \varepsilon\beta(X,Y) + \varepsilon^2\bar{U}(X,Y;\varepsilon), \quad V(X,Y) = \varepsilon\gamma(X,Y) + \varepsilon^2\bar{V}(X,y;\varepsilon)$$

$$K - 1 = \varepsilon\kappa, \quad \mu_1 = \varepsilon^2\bar{\mu}_1(\varepsilon), \quad \mu_2 = \varepsilon^2\bar{\mu}_2(\varepsilon). \qquad (3.3.3)$$

The substitution

$$(\rho c^2)^{-1}N_{ij}(u_{k,\ell}) = \frac{\varepsilon^2}{1-A^2} M_{ij}(\varepsilon^{-1}u_{k,\ell};\varepsilon) \qquad (3.3.4)$$

shows that all $O(\varepsilon)$ terms cancel from equations (3.2.4) - (3.2.6) when (3.3.2) and (3.3.3) are inserted. The surviving terms give the exact equations

$$\bar{u}_X - \bar{v}_y = \bar{\ell}(X,y;\varepsilon), \quad \bar{v}_X + \bar{u}_y = \bar{m}(X,y;\varepsilon) \qquad (3.3.5)$$

$$\bar{U}_X - \bar{V}_y = \bar{L}(X,Y;\varepsilon), \quad \bar{V}_X + \bar{U}_y = \bar{M}(X,Y;\varepsilon) \qquad (3.3.6)$$

where

$$\bar{\ell} = \frac{M_{22} - M_{11}}{1-A^2}, \qquad \bar{m} = -\frac{(AM_{12} + A^{-1}M_{21})}{1-A^2}$$

$$(3.3.7)$$

$$\bar{L} = \left(\frac{1+A^2}{1-A^2}\right)\frac{BM_{22} - B^{-1}M_{11}}{2B}, \quad \bar{M} = -\left(\frac{1+A^2}{1-A^2}\right)\frac{M_{12} + M_{21}}{2B}.$$

These are analogous to (3.2.8) and (3.2.9), so that in a similar way to (3.2.10) we introduce the functions $\bar{w}$ and $\bar{\nu}$ where

$$\bar{w}(X,\eta;\varepsilon) = \bar{U}(X,\eta;\varepsilon) - \bar{u}(X,\eta;\varepsilon)$$

$$(3.3.8)$$

$$\bar{\nu}(X,\eta;\varepsilon) = \bar{V}(X,\eta;\varepsilon) - \bar{v}(X,\eta;\varepsilon)$$

which satisfy

$$\bar{w}_X - \bar{\nu}_\eta = \bar{L}(X,\eta) - \bar{\ell}(X,\eta), \quad \bar{\nu}_X + \bar{w}_\eta = \bar{M}(X,\eta) - \bar{m}(X,\eta) \qquad (3.3.9)$$

with

$$\bar{w}(X,0) \;=\; 0 \tag{3.3.10}$$

$$\bar{\nu}(X,0) \;=\; \kappa\left[\gamma(X,0) + \varepsilon\bar{v}(X,0)\right] \tag{3.3.11}$$

and $\bar{w} \to (A^2 - 1)\bar{\mu}_1$, $\bar{\nu} \to (1 + A^2 - 2AB)\bar{\mu}_2$ as $\eta \to \infty$. The compatibility condition for this system may be derived directly from (3.2.16) as the exact equation

$$\kappa\left[\gamma(X,0) + \varepsilon\bar{v}(X,0)\right] - \bar{\mu}_2(1 + A^2 - 2AB) \;=\; \mathcal{L}(\bar{L},\bar{M}) - \mathcal{L}(\bar{\ell},\bar{m}) \tag{3.3.12}$$

We recall that A, B, c and $\Lambda = B/A = Y/y$ are known functions of $\varepsilon\kappa = K-1$ and the expressions of $\varepsilon^{-1}u_{k,\ell}$ required in $M_{ij}$ for the right-hand side of (3.3.12) are the X and $X_2$ derivatives of

$$\varepsilon^{-1}u_1(X,X_2) \;=\; \frac{2}{1+A^2}\left[\beta(X,Y) + \varepsilon\bar{U}(X,Y)\right] - \beta(X,y) - \varepsilon\bar{u}(X,y)$$

$$\varepsilon^{-1}u_2(X,X_2) \;=\; A^{-1}\gamma(X,y) + \varepsilon A^{-1}\bar{v}(X,y) - \frac{2B}{1+A^2}\left[\gamma(X,Y) + \varepsilon\bar{V}(X,Y)\right] .$$

Similarly the formulae for $\bar{\mu}_1$ and $\bar{\mu}_2$ may be obtained from (3.2.14) and (3.2.17). Equation (3.3.12) may be solved iteratively. The iterates $w^{(n+1)}$, $\nu^{(n+1)}$ to w and $\nu$ are obtained by solving (3.3.9) and (3.3.10) with $u^{(n)}$, $v^{(n)}$, ..., the iterative approximations to u, v, ..., on the right-hand sides. The condition (3.3.11) then requires that

$$v^{(n+1)}(X,0) \;=\; \frac{\kappa^{-1}\nu^{(n+1)}(X,0) - \gamma(X,0)}{\varepsilon} , \tag{3.3.13}$$

from which we deduce that $\nu^{(n+1)}(X,0)$ must differ from $\kappa\gamma(X,0)$ only by $O(\varepsilon)$ terms. Clearly not all families of harmonic functions will satisfy this criterion. To identify those which do, the limit $\varepsilon \to 0$ must be considered.

Parker (41) considers this limit and shows how the infinitely differentiable functions $\ell^{(o)}$, $m^{(o)}$, $L^{(o)}$, $M^{(o)}$, the first approximations to $\bar{\ell}$, $\bar{m}$, $\bar{L}$ and $\bar{M}$, may be regarded as linear functionals of $\gamma(X,0)$. He writes this formally as

$$\begin{aligned} \ell^{(o)}(X,\eta) &= T_1\gamma(X,0) \quad , \quad m^{(o)}(X,\eta) = T_2\gamma(X,0) \\ L^{(o)}(X,\eta) &= T_3\gamma(X,0) \quad , \quad M^{(o)}(X,\eta) = T_4\gamma(X,0) \end{aligned} \tag{3.3.14}$$

where the $T_i$ are transformations, so that equation (3.3.11) may be written to this approximation as

$$\begin{aligned} \kappa\gamma(X,0) &= \mathcal{L}(T_3\gamma(X,0),\ T_4\gamma(X,0)) \\ &\quad -\mathcal{L}(T_1\gamma(X,0),\ T_2\gamma(X,0)) + \tfrac{1}{2}(1 - A^4)\mu_2^{(o)} \ . \end{aligned} \tag{3.3.15}$$

This equation is a quadratic functional equation for $\gamma(X,0)$, where $\gamma(X,0)\varepsilon\mathcal{B}_p'(\mathbb{R}|\mathbb{R})$ for case (A) and $\gamma(X,0)\varepsilon\mathcal{B}_*'(\mathbb{R}|\mathbb{R})$ for case (B).

The right-hand side of (3.3.15) is homogeneous of second degree in $\gamma(X,0)$ so that the equation is unchanged by the mapping $\gamma(X,0) \to \tau\gamma(X,0)$, $\kappa \to \tau\kappa$, where $\tau$ is any real parameter. Hence, any profile $\gamma(X,0)$ which satisfies (3.3.15) for some value of $\kappa$ may be scaled to give another profile, provided that $\kappa$ is multiplied by the same factor. To this approximation we may take $c = c(\kappa)$ as $c = c_R + \varepsilon\kappa c'(1)$, therefore the perturbation speed is proportional to $\kappa$.

Following Parker (41) we now examine the three possibilities introduced in Section 3.2.

1. $\kappa \neq 0$. Without loss of generality we may choose $\gamma(X,0) = \sigma(X)$, where $\sigma(X)$ is a solution to (3.3.5) with $\kappa = 1$, then for any real $\kappa$ the multiple $\gamma(X,0) = \kappa\sigma(X)$ is a solution to (3.3.15) as has been mentioned above. The disturbances with

$$u(X,y) \sim \varepsilon\kappa\beta(X,y), \qquad v(X,y) \sim \varepsilon\kappa\gamma(X,y) \ ,$$

$$U(X,Y) \sim \varepsilon\kappa\beta(X,Y), \qquad V(X,Y) \sim \varepsilon\kappa\gamma(X,Y)$$

are then the limits of non-distorting progressive surface waves with propagation speed $c \sim c_R + \varepsilon\kappa c'(1)$. Clearly $\varepsilon\kappa$ is a measure of the disturbance amplitude and may be regarded as either positive or negative. Thus if one non-distorting profile propagates faster than the standard Rayleigh wave the inverted profile travels more slowly.

2. $\kappa = 0$. In this case (3.3.15) reduces to a homogeneous second degree equation. The amplitude of the solutions is then arbitrary and a normalization may be introduced. To this approximation profiles which are any multiple of a solution to the homogeneous equation all travel at the standard Rayleigh wave speed. In keeping with the terminology of hyperbolic waves these profiles will be called exceptional waveforms.

3. The case $\gamma(X,0) = 0$ gives $\ell^{(o)} = m^{(o)} = L^{(o)} = M^{(o)} = 0$, so that $u^{(1)} = U^{(1)} = V^{(1)} = v^{(1)} = 0$ and we only obtain the trivial solution, so that this is not considered further.

## 3.4 CONSTRUCTION OF THE OPERATOR $\mathcal{L}(L,M)$

In (41) Parker shows how to construct the operator $\mathcal{L}(L,M)$ required in the solution of (3.3.14). He treats the two cases of periodic and non-periodic waveforms separately.

For periodic waveforms, case (A); the X-period P may be taken to be $2\pi$, without loss of generality, since the governing equations (3.1.4) and (3.2.1) involve no intrinsic length scale. Any other wavelength may then be obtained by scaling $X_1$, $X_2$, t and $\underset{\sim}{u}$ by the factor $P/2\pi$, which leaves the amplitudes of the strain, the velocity and the deformation gradient unchanged, but scales the displacement amplitude, the wavelength and the period by the same factor $P/2\pi$. Parker obtains solutions using Fourier series and states that equation (3.3.15) reduces to an infinite set of algebraic equations in the Fourier coefficients of $\gamma(X,\eta)$.

For general waveforms, case (B), he derives two different representations for $\mathcal{L}(L,M)$. The first one uses Green's functions; however, the formulae obtained are cumbersome, so that a second representation using Fourier transforms is preferable for computational purposes. This has the additional advantage of being a generalisation of the method based on Fourier series for the periodic case. Parker obtains the following quadratic integral equation for the complex function $\Gamma(k,0)$:

$$\kappa\Gamma(k,0) = \int_{-\infty}^{\infty} \kappa(k,s)\Gamma(k-s,0)\Gamma(s,0)ds \tag{3.4.1}$$

where

$$\Gamma(k,\eta) = \int_{-\infty}^{\infty} e^{-ikX}\gamma(X,\eta)dX$$

is the Fourier transform of $\gamma(X,\eta)$. Equation (3.4.1) is analogous to the infinite set of algebraic equations to be solved for the Fourier coefficients in case (A). Parker notes two properties of the kernel

$\kappa(k,s)$, that the range of integration must be split at $s = 0$ and $s = k$ and that since each contribution is homogeneous of first degree in k and s, then if $\Gamma(k,0)$ is one solution to (3.4.1) so also is $\alpha^{-2}\Gamma(\alpha k,0)$, for all $\alpha > 0$. This situation arises because the original problem defines no length scale. The function $\alpha^{-2}\Gamma(\alpha k,0)$ is the Fourier transform of $\alpha\gamma(X/\alpha,0)$, which is obtained from the profile $\gamma(X,0)$ by rescaling the amplitude and lengths by the factor $\alpha$.

Parker also shows by investigating the mappings $T_1$, $T_2$, $T_3$, $T_4$ and $\mathcal{L}$ that the iterates $u^{(n)} - 2\mu_1^{(n)}$, $v^{(n)} - 2AB\mu_2^{(n)}$, $V^{(n)} - (1+A^2)\mu_1^{(n)}$ and $U^{(n)} - (1+A^2)\mu_2^{(n)}$ belong to either $\mathcal{B}_p(\overline{\Omega}|\mathbb{R})$ or $\mathcal{B}_*(\overline{\Omega}|\mathbb{R})$, when the initial function $\gamma(X,0)$ belongs to $\mathcal{B}_p'(\mathbb{R}|\mathbb{R})$ or $\mathcal{B}_*'(\mathbb{R}|\mathbb{R})$ in cases (A) and (B) respectively.

## 3.5 AN ALTERNATIVE METHOD

In this section we present another method of obtaining the condition (3.3.15) on the relationship between $\kappa$ and $\gamma(X,0)$. This is based on Fredholm's alternative theory (8). We consider the differential equations for $\bar{u}$, $\bar{v}$, $\bar{U}$, $\bar{V}$ (3.3.5) and (3.3.6) with the boundary conditions

$$\bar{U}(X,0) = \bar{u}(X,0)$$

$$\bar{V}(X,0) = \bar{v}(X,0) + \kappa(\gamma(X,0) + \varepsilon\bar{v}(X,0)).$$

Instead of using the two functions $w(X,\eta)$, $\nu(X,\eta)$ defined in (3.2.10), we introduce two new functions $\sigma(X,\eta)$ and $\rho(X,\eta)$ where $\sigma(X,\eta)$ - a and $\rho(X,\eta)$ - b belong to either $\mathcal{B}_p(\bar{\Omega}|\mathbb{R})$ or $\mathcal{B}_*(\bar{\Omega}|\mathbb{R})$ in cases (A) and (B) respectively where a and b are constants and

$$\sigma_X + \rho_\eta = 0, \qquad \sigma_\eta - \rho_X = 0 . \tag{3.5.1}$$

Now

$$\sigma\bar{\ell} + \rho\bar{m} = \frac{\partial}{\partial X}(\sigma\bar{u} + \rho\bar{v}) + \frac{\partial}{\partial y}(-\sigma\bar{v} + \rho\bar{u}),$$

where

$$\sigma = \sigma(X,y) \qquad \text{and} \qquad \rho = \rho(X,y) ,$$

so that in case (A)

$$\int_0^\infty\int_{-P/2}^{P/2}(\sigma\bar{\ell} + \rho\bar{m})dXdy = \int_0^\infty\int_{-P/2}^{P/2}\{\frac{\partial}{\partial X}(\sigma\bar{u} + \rho\bar{v}) + \frac{\partial}{\partial y}(-\sigma\bar{v} + \rho\bar{u})dXdy$$

$$= \int_0^\infty\left[\sigma\bar{u} + \rho\bar{v}\right]_{-P/2}^{P/2} dy + \int_{-P/2}^{P/2}(\sigma\bar{v} - \rho\bar{u})\Big|_{y=0} dX$$

$$= \int_{-P/2}^{P/2}(\sigma\bar{v} - \rho\bar{u})\Big|_{y=0} dX .$$

Similarly, for $\sigma = \sigma(X,Y)$ and $\rho = \rho(X,Y)$,

$$\int_0^\infty\int_{-P/2}^{P/2}(\sigma\bar{L} + \rho\bar{M})\,dXdY = \int_{-P/2}^{P/2}(\sigma\bar{V} - \rho\bar{U})\Big|_{Y=0}\,dX$$

$$= \int_{-P/2}^{P/2}(\sigma\bar{v} + \sigma\kappa(\gamma + \varepsilon\bar{v}) - \rho\bar{u})\Big|_{y=0}\,dX$$

$$= \int_{-P/2}^{P/2}(\sigma\bar{v} - \rho\bar{u})\Big|_{y=0}\,dX + \kappa\int_{-P/2}^{P/2}(\sigma(\gamma + \varepsilon\bar{v}))\Big|_{y=0}\,dX.$$

Therefore,

$$\int_0^\infty\int_{-P/2}^{P/2}(\sigma\bar{L} + \rho\bar{M})\,dXdY = \int_0^\infty\int_{-P/2}^{P/2}(\sigma\bar{\ell} + \rho\bar{m})\,dXdy + \kappa\int_{-P/2}^{P/2}(\sigma(\gamma + \varepsilon\bar{v}))\Big|_{y=0}\,dX \qquad (3.5.2)$$

Similarly in case (B) we obtain

$$\int_0^\infty\int_{-\infty}^\infty(\sigma\bar{L} + \rho\bar{M})\,dXdY = \int_0^\infty\int_{-\infty}^\infty(\sigma\bar{\ell} + \rho\bar{m})\,dXdy + \kappa\int_{-\infty}^\infty(\sigma(\gamma + \varepsilon\bar{v}))\Big|_{y=0}\,dX\ . \qquad (3.5.3)$$

Equations (3.5.2) and (3.5.3) must then be satisfied for all functions $\sigma$ and $\rho$ satisfying the stated conditions. In each case we choose a basis for $\{\sigma,\rho\}$ and find that we obtain the same condition as (3.3.15).

## 3.6 A MODEL EQUATION

In this section we illustrate some of the features of the solution by considering a simple system of equations, having a form similar to those obtained in the previous sections. Initially we consider a system of equations with a forcing function on the boundary $Y = 0$:

$$\nabla^2\phi = f(\phi_X, \phi_Y, \psi_X, \psi_Y)$$

$$\nabla^2\psi = 0 \tag{3.6.1}$$

$$\left.\begin{aligned} \phi &= \varepsilon \cos X \\ \frac{\partial\psi}{\partial Y} &= \kappa \frac{\partial\phi}{\partial Y} \end{aligned}\right\} \text{ on } Y = 0$$

where $\phi$ and $\psi$ are $2\pi$-periodic in $X$ and decay as $Y \to \infty$. We are interested in the region $Y \geq 0$.

Now

$$\int_0^\infty \int_{-\pi}^{\pi} \nabla^2\phi dXdY = \int_{-\pi}^{\pi} \left.\frac{\partial\phi}{\partial Y}\right|_{Y=0} dX$$

that is

$$\int_0^\infty \int_{-\pi}^{\pi} f\ dXdY = \int_{-\pi}^{\pi} \left.\frac{\partial\phi}{\partial Y}\right|_{Y=0} dX$$

and also

$$\int_0^\infty \int_{-\pi}^{\pi} \nabla^2\psi\ dXdY = \int_{-\pi}^{\pi} \left.\frac{\partial\psi}{\partial Y}\right|_{Y=0} dX$$

$$= \kappa\int_{-\pi}^{\pi} \left.\frac{\partial\phi}{\partial Y}\right|_{Y=0} dX = \kappa\int_0^\infty \int_{-\pi}^{\pi} f\ dXdY.$$

However $\nabla^2\psi = 0$ so that we obtain the condition

$$\int_0^\infty\int_{-\pi}^{\pi} f \, dXdY = 0 \tag{3.6.2}$$

Choosing $f = \psi_X\psi_Y$ and assuming that $\phi,\psi$ are even functions of X implies that $\psi_X\psi_Y$ is an odd function so that condition (3.6.2) is satisfied. Writing $\phi = \varepsilon\bar{\phi}$, $\psi = \varepsilon\bar{\psi}$, the system (3.6.1) becomes

$$\begin{aligned} \nabla^2\bar{\phi} &= \varepsilon\bar{\psi}_X\bar{\psi}_Y \\ \nabla^2\bar{\psi} &= 0 \\ \left.\begin{aligned} \bar{\phi} &= \cos X \\ \frac{\partial\bar{\psi}}{\partial Y} &= \kappa\frac{\partial\bar{\phi}}{\partial Y} \end{aligned}\right\} &\text{ on } Y = 0 \end{aligned} \tag{3.6.3}$$

a solution to this may be written in the form

$$\bar{\phi} = e^{-Y}\cos X + \varepsilon\bar{\phi}_1 + \ldots$$

$$\bar{\psi} = \kappa e^{-Y}\cos X + \varepsilon\bar{\psi}_1 + \ldots \quad .$$

Equating terms of $O(\varepsilon)$ gives

$$\nabla^2\bar{\phi}_1 = \tfrac{1}{2}\kappa^2 e^{-2Y}\sin 2X$$

with $\bar{\phi}_1 = 0$ on $Y = 0$ , which has solution

$$\bar{\phi}_1 = -\frac{1}{8}\kappa^2 Y\, e^{-2Y}\sin 2X.$$

The function $\bar{\psi}_1$ must then satisfy

$$\nabla^2\bar{\psi}_1 = 0$$

$$\frac{\partial\bar{\psi}_1}{\partial Y} = -\frac{1}{8}\kappa^3\sin 2X \quad \text{on } Y = 0,$$

which has a solution

$$\bar{\psi}_1 = \frac{1}{16}\kappa^3 e^{-2Y} \sin 2X .$$

In this example there is no difficulty in the perturbation scheme, the terms being obtained in order without any further constraint being imposed.

To obtain a model equation in which we have to solve a system of algebraic equations to obtain the coefficients in $\bar{\phi}$ and $\bar{\psi}$ we consider the system:

$$\nabla^2\phi = (\psi^2)_{XY}$$

$$\nabla^2\psi = 0$$

$$\left.\begin{aligned} \phi &= 0 \\ \frac{\partial\psi}{\partial Y} &= \kappa\frac{\partial\phi}{\partial Y} \end{aligned}\right\} \text{ on } Y = 0$$

and we assume that $\phi$, $\psi \to 0$ as $Y \to \infty$. The compatibility condition (3.6.2) is satisfied since

$$\int_0^\infty\int_{-\pi}^{\pi} (\psi^2)_{XY}\, dXdY = 0 .$$

In this system we need to solve first for $\psi$ before we can obtain $\phi$. We try a solution for $\psi$ in the form

$$\psi = \sum_{n=1}^{\infty} (A_n e^{-nY} \cos nX + B_n e^{-nY} \sin nX)$$

and find that the solution for $\phi$ may be written in the form

$$\phi = \sum_{n=1}^{\infty} \{C_n e^{-nY} \cos nX + D_n e^{-nY} \sin nX$$

$$+ \alpha_n Y e^{-nY} \cos nX + \beta_n Y e^{-nY} \sin nX$$

$$+ \sum_{s=1}^{\infty} \frac{n(n+2s)}{4s(n+s)} (B_{n+s}B_s + A_{n+s}A_s)e^{-(n+2s)Y} \sin nX$$

$$- \sum_{s=1}^{\infty} \frac{n(n+2s)}{4s(n+s)} (A_sB_{n+s} - A_{n+s}B_s)e^{-(n+2s)Y} \cos nX\} \quad ,$$

where

$$\alpha_n = \frac{n}{4} \sum_{r=1}^{n-1} (A_rB_{n-r} + A_{n-r}B_r) \ ,$$

$$\beta_n = -\frac{n}{4} \sum_{r=1}^{n-1} (A_rA_{n-r} - B_rB_{n-r}) \ .$$

The condition $\phi(X,0) = 0$ implies

$$C_n = \sum_{s=1}^{\infty} \frac{n(n+2s)}{4s(n+s)} (A_sB_{n+s} - A_{n+s}B_s) \ ,$$

$$D_n = -\sum_{s=1}^{\infty} \frac{n(n+2s)}{4s(n+s)} (B_{n+s}B_s + A_{n+s}A_s) \ .$$

Substituting for $\phi$ and $\psi$ in the condition $\frac{\partial\psi}{\partial Y} = \kappa \frac{\partial\phi}{\partial Y}$ on $Y = 0$ and equating coefficients of cos nX and sin nX yields the conditions

$$nA_n = n\kappa C_n + \kappa\alpha_n + \kappa \sum_{s=1}^{\infty} \frac{n(n+2s)^2}{4s(n+s)} (A_sB_{n+s} - A_{n+s}B_s)$$

$$nB_n = n\kappa D_n + \kappa\beta_n + \kappa \sum_{s=1}^{\infty} \frac{n(n+2s)^2}{4s(n+s)} (B_{n+s}B_s + A_{n+s}A_s) \ ,$$

where $C_n$, $D_n$ $\alpha_n$ and $\beta_n$ are given above.

These equations are a simpler version of the type encountered in treating the Rayleigh wave. As in that problem, the system may be solved by a numerical procedure, see Section 4.2, in which we truncate the sequences $A_n$ and $B_n$. The two equations may also be combined by writing $F_n = A_n + i\ B_n$, the transformation $F_n \rightarrow e^{in\theta}F_n$ then corresponds to a translation in X. We seek a solution in which $\psi$ is an odd function, that is $A_n = 0$ for all n. We then have to solve the system of equations

$$-B_n + \kappa \sum_{s=1}^{\infty} \frac{(n+2s)}{2(n+s)} B_s B_{n+s} + \frac{\kappa}{4} \sum_{r=1}^{n-1} B_r B_{n-r} = 0 .$$

We solve this by making the transformation $B_n \to 1/\kappa Q_n$. To obtain starting values to the numerical procedure we truncate the sequence $B_n$ after n = 3 and solve the equations

$$-Q_1 + \frac{3}{4} Q_1 Q_2 + \frac{5}{6} Q_2 Q_3 = 0$$

$$-Q_2 + \frac{2}{3} Q_1 Q_3 + \frac{1}{4} Q_1^2 = 0$$

$$-Q_3 + \frac{1}{4}(Q_1 Q_2 + Q_1 Q_2) = 0$$

which have real analytical solutions

$$Q_1 = 1.28,\ Q_2 = 0.89,\ Q_3 = 0.57 ;$$
$$Q_1 = -1.28,\ Q_2 = 0.89,\ Q_3 = -0.57 ;$$
$$Q_1 = 2.04,\ Q_2 = -2.69,\ Q_3 = 2.75 ;$$
$$Q_1 = -2.04,\ Q_2 = -2.69,\ Q_3 = -2.75 .$$

We may then solve the system for n = 4, say, using as starting values the exact solution obtained for $Q_1$, $Q_2$ and $Q_3$ and setting $Q_4 = 0$. This process may be repeated iteratively in the hope that it converges as n increases.

# CHAPTER 4

# RAYLEIGH WAVES ON A COMPRESSIBLE MATERIAL WITH NON-LINEAR CONSTITUTIVE LAW

## 4.1 THE FORM OF THE EQUATIONS

In this chapter, as an example of the general theory, we consider a compressible material of the harmonic type introduced by John (19) and (20). Initially we consider periodic functions, that is case (A) of Chapter Three and examine the motion of particles in the medium, and the dependence of the wave speed on the form and amplitude of non-distorting wave profiles. In Section 4.4 we attempt to obtain solutions for non-periodic waveforms.

The material we consider is a harmonic material as defined by John (19) and (20). In (19) he considers plane strain problems for a perfectly elastic material and seeks to find forms of the strain-energy function for which the equations of equilibrium and stress-strain relations simplify considerably. He finds these simplifications for materials of harmonic type, in which the strain-energy function in plane strain reduces to

$$W(r,s) = 2\mu(G(r) - s) \tag{4.1.1}$$

where $$r = \{(F_{11} + F_{22})^2 + (F_{12} - F_{21})^2\}^{\frac{1}{2}} ,$$

$$s = F_{11}F_{22} - F_{12}F_{21}$$

and G(r) is an arbitrary function. For consistency with the classical linear theory the following restrictions need to be imposed

$$G'(2) = 1, \qquad G''(2) = \frac{\lambda+2\mu}{2\mu} , \tag{4.1.2}$$

$\lambda$ and $\mu$ being the Lamé constants of the linear theory. Without loss of generality it is assumed that $G(2) = 1$. John then reduces elastic equilibrium problems for materials with a strain-energy function of the form (4.1.1) to potential problems for which the existence of solutions is easily established on the basis of known properties of harmonic functions. For this reason materials with strain-energy function of the form (4.1.1) are called 'harmonic' materials. 'Harmonic' materials give a mathematical model for which a qualitative discussion of problems involving large deformations can be carried out. For the behaviour of solutions of the equilibrium equations to conform with physical intuition the function $G(r)$ must satisfy certain conditions. For example, if we assume that the tension increases as the material is expanded uniformly, then $G'(r)/r$ must increase monotonically with $r$. In (59) Varley and Cumberbatch use and extend some of the results obtained by John to analyse plane strain or plane stress deformations of finitely strained blocks or sheets of materials which contain holes, notches or inclusions. They show that the predictions of the theory agree well with experimental evidence.

In this chapter we consider materials with the standard energy function

$$W = \left(\frac{\lambda+2\mu}{2\mu}\right)(r-2)^2 + 2\mu(r-s-1) \ , \qquad (4.1.3)$$

which are examples of harmonic materials (4.1.1) satisfying (4.1.2) where

$$G(r) = \left(\frac{\lambda+2\mu}{2\mu}\right)(r-2)^2 + (r-2) + 1 \ .$$

The strain-energy function may be written as

$$W = 2\mu\Bigg[\left(\frac{\lambda+2\mu}{4\mu}\right)\Big[(F_{11} + F_{22})^2 + (F_{12} - F_{21})^2$$

$$\overline{[illegible])^2 + (F_{12} - F_{21})^2 + 4}\Big] - F_{11}F_{22} + F_{12}F_{21} - 1$$

$$+ \sqrt{(F_{11} + F_{22})^2 + (F_{12} - F_{21})^2}\Big]$$

where $F_{11} = 1 + u_{1,1}$ , $F_{12} = u_{1,2}$ ,

$F_{21} = u_{2,1}$ , $F_{22} = 1 + u_{2,2}$ ,

and the $u_{i,j}$ are $0(\varepsilon)$ for $i,j = 1,2$. In the analysis leading to (3.3.13) and (3.3.14) only the limiting behaviour as $u_{i,j} \to 0$ is required so that it is sufficient to express W as

$$W = 2\mu\Big\{\left(\frac{\lambda+2\mu}{4\mu}\right)(u_{1,1} + u_{2,2} + \frac{(u_{1,2} - u_{2,1})^2}{4} + \ldots)$$

$$- u_{1,1}u_{2,2} + u_{1,2}u_{2,1} + \frac{(u_{1,2} - u_{2,1})^2}{4} + \ldots \Big\}$$

from which we obtain the stress components

$$\tau_{11} = (\lambda+2\mu)u_{1,1} + \lambda u_{2,2} + \frac{\lambda+\mu}{4}(u_{1,2} - u_{2,1})^2 + 0(\varepsilon^3)$$

$$\tau_{12} = \mu(u_{1,2} + u_{2,1}) + \frac{\lambda+\mu}{2}(u_{1,1} + u_{2,2})(u_{1,2} - u_{2,1}) + 0(\varepsilon^3)$$

$$\tau_{21} = \mu(u_{1,2} + u_{2,1}) - \frac{\lambda+\mu}{2}(u_{1,1} + u_{2,2})(u_{1,2} - u_{2,1}) + 0(\varepsilon^3)$$

$$\tau_{22} = \lambda u_{1,1} + (\lambda+2\mu)u_{2,2} + \frac{\lambda+\mu}{4}(u_{1,2} - u_{2,1})^2 + 0(\varepsilon^3) .$$

Hence the non-linear parts $N_{ij}$ of these $\tau_{ij}$ $i,j = 1,2$ may be written as

$$N_{11} = \frac{\lambda+\mu}{4}(u_{1,2} - u_{2,1})^2 + 0(\varepsilon^3)$$

$$N_{12} = \frac{\lambda+\mu}{2}(u_{1,1} + u_{2,2})(u_{1,2} - u_{2,1}) + 0(\varepsilon^3)$$

$$N_{21} = \frac{\lambda+\mu}{2}(u_{1,1} + u_{2,2})(u_{2,1} - u_{1,2}) + 0(\varepsilon^3)$$

$$N_{22} = \frac{\lambda+\mu}{4}(u_{1,2} - u_{2,1})^2 + 0(\varepsilon^3) .$$

From (3.3.4) - (3.3.7) it is clear that we need to write $N_{11}$, $N_{12}$, $N_{21}$ and $N_{22}$ in terms of u, v, U and V, this we do using (3.3.11) to give

$$N_{11} = \left(\frac{\lambda+\mu}{4}\right)\varepsilon^2\{\frac{2}{1+A^2}\left[B\beta_Y(X,Y)+\varepsilon B\bar{U}_Y(X,Y)\right] - A\beta_y(X,y)+\varepsilon A\bar{u}_y(X,y)$$

$$- \left[\frac{1}{A}\gamma_X(X,y) + \frac{\varepsilon}{A}\bar{v}_X(X,y) - \frac{2B}{1+A^2}\left[\gamma_X(X,Y)+\varepsilon\bar{V}_X(X,Y)\right]\right]\}^2$$

$$+ 0(\varepsilon^3) ,$$

$$N_{12} = \left(\frac{\lambda+\mu}{2}\right)\varepsilon^2\{\frac{2}{1+A^2}\left[\beta_X(X,Y)+\varepsilon\bar{U}_X(X,Y)\right] - \beta_X(X,y)+\varepsilon\bar{u}_X(X,y)$$

$$+ \gamma_y(X,y) + \varepsilon\bar{v}_y(X,y) - \frac{2B^2}{1+A^2}\left[\gamma_Y(X,Y)+\varepsilon\bar{V}_Y(X,Y)\right]\}x$$

$$\{\frac{2}{1+A^2}\left[B\beta_Y(X,Y)+\varepsilon B\bar{U}_Y(X,Y)\right] - A\beta_u(X,y)+\varepsilon A\bar{u}_y(X,y)$$

$$- \left[\frac{1}{A}\gamma_X(X,y) + \frac{\varepsilon}{A}\bar{v}_X(X,y) - \frac{2B}{1+A^2}\left[\gamma_X(X,Y)+\varepsilon\bar{V}_X(X,Y)\right]\right]\}$$

$$+ 0(\varepsilon^3) ,$$

$$N_{21} = -N_{12} + 0(\varepsilon^3) ,$$

$$N_{22} = N_{11} + 0(\varepsilon^3) . \tag{4.1.4}$$

It can be shown that for a general isotropic material in plane strain, second order elasticity theory involves only two material moduli other than $\lambda$ and $\mu$ and that condition (4.1.4) results from the special choice of these parameters consistent with (4.1.3). From (3.3.4) we have

$$M_{ij} = \frac{(1-A^2)}{\rho c^2\varepsilon^2} N_{ij}$$

and from the definitions of A and B

$$\frac{\lambda+\mu}{\rho c^2} = \frac{(B^2 - A^2)}{(1-A^2)(1-B^2)}$$

from which we obtain M in terms of A, B, $\beta$, $\gamma$, $\bar{u}$, $\bar{v}$, $\bar{U}$ and $\bar{V}$. In this and the following sections A and B refer to the approximations $A_R$ and $B_R$. Equations (3.3.7) give

$$\ell^{(o)} = 0 \quad , \qquad m^{(o)} = \frac{M_{12}}{A} \quad ,$$

$$L^{(o)} = \frac{(1+A^2)(B^2-1)}{(1-A^2)2B^2} M_{11} \quad , \qquad M^{(o)} = 0 \; .$$

The fact that $\ell^{(o)}$ and $M^{(o)}$ are zero is a direct consequence of (4.1.4). We therefore write $M_{12}$ as a function of (X,y) and $M_{11}$ as a function of (X,Y) for use in $m^{(o)}$ and $L^{(o)}$ respectively, so that we obtain the following expressions for $\ell^{(o)}$, $m^{(o)}$, $L^{(o)}$ and $M^{(o)}$:

$$\begin{aligned} \ell^{(o)}(X,y) &= 0 \; , \\ m^{(o)}(X,y) &= -\frac{(B^2-A^2)(1-A^2)}{A^2(1+A^2)} \gamma_Y(X, \tfrac{B}{A} y) \; \gamma_X(X,y) \quad , \\ L^{(o)}(X,Y) &= -\frac{(B^2-A^2)(1-A^2)}{2AB} \gamma_X^2(X, \tfrac{A}{B} Y) \; , \\ M^{(o)}(X,Y) &= 0 \; , \end{aligned} \tag{4.1.5}$$

which may then be used in the equations for $u^{(1)}$, $v^{(1)}$, $U^{(1)}$ and $V^{(1)}$.

## 4.2 PERIODIC SOLUTIONS

In this section we construct a $2\pi$-periodic waveform for the harmonic material, using a variant of the method outlined in section 3.3. In general $\gamma(X,\eta)$ may be represented as

$$\sum_{n=1}^{\infty}\{C_n e^{-n\eta}\cos nX + D_n e^{-n\eta}\sin nX\} \quad ,$$

where the coefficients $C_n$ and $D_n$ are obtained by applying the condition (3.3.13). First we look for symmetric waveforms so $\gamma(X,\eta)$ is an even function and is represented by

$$\sum_{n=1}^{\infty} C_n e^{-n\eta}\cos nX \ .$$

In section 4.3 we do consider the more general $\gamma(X,\eta)$ although the solutions obtained to this more general problem are reproductions of the case of even $\gamma(X,\eta)$ after a translation in X. Then $\ell^{(o)}$, $m^{(o)}$, $L^{(o)}$ $M^{(o)}$ are given by (4.1.5).

The functions $w^{(1)}(X,\eta)$, $\nu^{(1)}(X,\eta)$ introduced in Section 3.3 must satisfy the equations

$$w_X^{(1)} - \nu_\eta^{(1)} = L^{(o)} - \ell^{(o)} \quad ,$$

$$\nu_X^{(1)} + w_\eta^{(1)} = M^{(o)} - m^{(o)} \quad ,$$

$$w^{(1)}(X,0) = 0 \quad ,$$

$$\nu^{(1)}(X,0) = \kappa\gamma(X,0) \ ,$$

$$w^{(1)} \to (A-1)\bar{\mu}_1 \quad , \qquad \nu^{(1)} \to (1+A-2AB)\bar{\mu}_2 \quad \text{as } \eta \to \infty \ .$$

Eliminating the function $\nu^{(1)}(X,\eta)$ from these equations gives

$$w_{XX}^{(1)} + w_{\eta\eta}^{(1)} = L_X^{(o)} - \ell_X^{(o)} + M_\eta^{(o)} - m_\eta^{(o)} \tag{4.2.1}$$

$$w^{(1)}(X,0) = 0 \tag{4.2.2}$$

$$-w_\eta^{(1)}(X,0) + M^{(o)}(X,0) - m^{(o)}(X,0) = \kappa\gamma_X(X,0) \tag{4.2.3}$$

We solve (4.2.1) and (4.2.2) for $w^{(1)}$ and then use (4.2.3) to provide us with a condition on the relationship between $\kappa$ and $\gamma(X,\eta)$. The condition $w^{(1)} \to (A-1)\bar{\mu}_1$ as $\eta \to \infty$ will then give $\bar{\mu}_1$.

In the example under consideration

$$w_{XX}^{(1)} + w_{\eta\eta}^{(1)} = -\frac{(B^2-A^2)(1-A^2)}{2AB(1+A^2)}\frac{\partial}{\partial X}\{\gamma_X^2(X, \tfrac{A}{B}\eta)\}$$

$$+\frac{(B^2-A^2)(1-A^2)}{A^2(1+A^2)}\frac{\partial}{\partial \eta}\{\tfrac{A}{B}\gamma_\eta(X, \tfrac{B}{A}\eta)\gamma_X(X,\eta)\} \tag{4.2.4}$$

$$w^{(1)}(X,0) = 0 \tag{4.2.5}$$

$$-w_\eta^{(1)}(X,0) + \frac{(B^2-A^2)(1-A^2)}{A^2(1+A^2)}\frac{A}{B}\gamma_\eta(X, \tfrac{B}{A}\eta)\Big|_{\eta=0}\gamma_X(X,0) = \kappa\gamma_X(X,0) \tag{4.2.6}$$

On substituting for $\gamma(X,\eta)$ as $\sum_{n=1}^{\infty} C_n e^{-n\eta} \cos nX$, we find that $w^{(1)}(X,\eta)$ may be written in the form

$$w^{(1)}(X,\eta) = W_o(\eta) + \sum_{n=1}^{\infty} W_n(\eta) \sin nX .$$

The functions $W_n$ may be written for $n \geq 1$ as

$$W_n(\eta) = g_n e^{-n\eta} + a_n e^{-n\frac{A}{B}\eta} + \sum_{s=1}^{\infty} b_n e^{-(n+2s)\frac{A}{B}\eta}$$

$$+ \sum_{r=1}^{n-1} c_n e^{-(r\frac{B}{A} + n-r)\eta} + \sum_{s=1}^{\infty} d_n e^{-((n+s)\frac{B}{A} + s)\eta}$$

$$+ \sum_{s=1}^{\infty} f_n e^{-(s\frac{B}{A} + n+s)\eta}$$

where $g_n$, $a_n$, $b_n$, $c_n$, $d_n$ and $f_n$ are constants given by

$$n^2\left(-1 + \frac{A^2}{B^2}\right)a_n = -\frac{(B^2-A^2)(1-A^2)}{2AB(1+A^2)} \sum_{r=1}^{n-1} \frac{nr(n-r)}{2} C_r C_{n-r}$$

$$\left(-n^2 + (n+2s)^2\frac{A^2}{B^2}\right)b_n = \frac{(B^2-A^2)(1-A^2)}{2AB(1+A^2)}\, 2 \sum_{s=1}^{\infty} \frac{ns(n+s)}{2} C_s C_{n+s}$$

$$\left(-n^2 + (r\frac{B}{A} + n-r)^2\right)c_n = -\frac{(B^2-A^2)(1-A^2)}{A^2(1+A^2)} \sum_{r=1}^{n-1} \frac{r(n-r)}{2}(r\frac{B}{A} + n-r)C_r C_{n-r}$$

$$\left(-n^2 + ((n+s)\frac{B}{A} + s)^2\right)d_n = \frac{(B^2-A^2)(1-A^2)}{A^2(1+A^2)} \sum_{s=1}^{\infty} ((n+s)\frac{B}{A} + s)\frac{s(n+s)}{2} C_s C_{n+s}$$

$$\left(-n^2 + (s\frac{B}{A} + n+s)^2\right)f_n = -\frac{(B^2-A^2)(1-A^2)}{A^2(1+A^2)} \sum_{s=1}^{\infty} (s\frac{B}{A} + n+s)\frac{s(n+s)}{2} C_s C_{n+s}$$

and from (4.2.5) $g_n + a_n + b_n + c_n + d_n + f_n = 0$.

The function $W_0(\eta)$ is readily found to be of the form $a\eta$, where $a$ is a constant, so that the condition of $w^{(1)}$ being bounded as $\eta \to \infty$ implies that $a = 0$ and hence $\bar{\mu}_1 = 0$. The constant $\bar{\mu}_2$ may be found by solving for $\nu^{(1)}$ from the terms independent of X and it is found to be zero. Equating coefficients of sin nX in condition (4.2.6) then gives the following algebraic equation:

$$n(1-\frac{A}{B})a_n + b_n(n-(n+2s)\frac{A}{B}) + c_n r(1-\frac{B}{A})$$

$$+ d_n(n-(n+s)\frac{B}{A}-s) - f_n s(1+\frac{B}{A})$$

$$+ \frac{(B^2-A^2)(1-A^2)}{A^2(1+A^2)} \sum_{r=1}^{n-1} \frac{r(n-r)}{2} C_r C_{n-r} = -\kappa n C_n$$

which may be written as

$$-2\kappa n C_n + \sum_{r=1}^{n-1} \alpha_{rn} r(n-r) C_r C_{n-r} + \sum_{s=1}^{\infty} \beta_{sn} s(n+s) C_s C_{n+s} = 0 \qquad (4.2.7)$$

where

$$\alpha_{rn} = \frac{(B^2-A^2)(1-A^4)}{8A^2B} \left\{\frac{1}{A+B} - \frac{2n}{(2n-r)A+rB}\right\}$$

and

$$\beta_{sn} = \frac{(B^2-A^2)(1-A^4)}{4A^2B} \left\{\frac{n}{(n+s)(A+B)} - \frac{n}{nB+(n+2s)A} - \frac{n}{(2n+s)A+sB}\right\}$$

Equation (4.2.7) is to be satisfied for $n = 1,2,\ldots$ and so provides an infinite set of quadratic algebraic equations, which will need to be solved numerically after suitable truncation of the sequence $\{C_n\}$. As $s \to \infty$ for fixed n, $\beta_{sn} \to \infty$ so that we may hope to obtain solutions with $C_n \to 0$ as $n \to \infty$. We first need to obtain values for A and B, this we achieve by specifying Poisson's ratio and solving the equation $\frac{(1+A^2)^2}{4AB} = 1$ to determine $c_R$ and $A(=A_R)$, $B(=B_R)$. Taking Poisson's ratio to be 1/4 we find that $A = 0.3933$ and $B = 0.8475$, we use these values to specify the coefficients in (4.2.7).

We are seeking solutions for waves which may travel at a speed different from the linear Rayleigh speed $c_R$, the difference in the wave speeds may be written as a function of $\kappa$ and hence, for $\kappa \neq 0$, as a function of the amplitude. We have

$$\frac{(1+A^2)^2}{4AB} = 1 + \kappa\varepsilon \tag{4.2.8}$$

where

$$A^2 = 1 - \frac{\rho c^2}{\mu} , \qquad B^2 = 1 - \frac{\rho c^2}{\lambda+2\mu}$$

and we write $c = c_R + \varepsilon\bar{c}$, where $c_R$, the linear Rayleigh speed satisfies equation (4.2.8) with $\varepsilon\kappa = 0$.

Then

$$\varepsilon\kappa = \varepsilon\bar{c}\left(\frac{dK}{dc^2} \quad 2c\right)_{c=c_R}$$

hence

$$\varepsilon\kappa = \frac{\varepsilon\bar{c}}{c_R}\left\{\frac{(1-A_R^2)(1-3A_R^2)}{A_R^2(1+A_R^2)} + \frac{(1-B_R^2)}{B_R^2}\right\} .$$

For the harmonic material under consideration with Poisson's ratio 1/4 this reduces to

$$\frac{\bar{c}}{c_R} = 0.3413\kappa .$$

Equations (3.3.11) give the horizontal and vertical displacements $u_1(X,X_2)$, $u_2(X,X_2)$:

$$u_1(X,X_2) = \varepsilon\left[\frac{2}{1+A^2}\,\beta(X,Y) - \beta(X,y)\right] + O(\varepsilon^2)$$

$$u_2(X,X_2) = \varepsilon\left[\frac{1}{A}\,\gamma(X,y) - \frac{2B}{1+A^2}\,\gamma(X,Y)\right] + O(\varepsilon^2) ,$$

therefore to first order $u_1$ and $u_2$ are given by

$$u_1(X,X_2) = -\varepsilon\left[\frac{2}{1+A^2}\sum_{n=1}^{\infty} C_n e^{-nBX_2}\sin nX - \sum_{n=1}^{\infty} C_n e^{-nAX_2}\sin nX\right] ,$$

$$u_2(X,X_2) = \varepsilon\left[\frac{1}{A}\sum_{n=1}^{\infty} C_n e^{-nAX_2}\cos nX - \frac{2B}{1+A^2}\sum_{n=1}^{\infty} C_n e^{-nBX_2}\cos nX\right] . \qquad (4.2.9)$$

These displacements describe a Rayleigh wave of linear elasticity having even surface elevation proportional to $\gamma(X,0) = \sum_{n=1}^{\infty} C_n \cos nX$. However, in linear elasticity theory the coefficients $\{C_n\}$ are arbitrary whereas the present theory shows that a wave profile of moderate amplitude propagates without distortion at speed $c \simeq c_R(1 + \varepsilon 0.3413\kappa)$ only if the coefficients satisfy (4.2.7).

We now consider the two possibilities outlined in Section 3.3 for $\kappa \neq 0$ and $\kappa = 0$.

1. $\kappa \neq 0$. Clearly $\kappa$ may be scaled out of equation (4.2.7) by making the substitution $nC_n = \kappa P_n$, the equation to be satisfied by $P_n$ becoming

$$-2P_n + \sum_{r=1}^{n-1} \alpha_{rn} P_r P_{n-r} + \sum_{s=1}^{\infty} \beta_{sn} P_s P_{n+s} = 0 \quad , \quad n=1,2,\ldots, \qquad (4.2.10)$$

This gives an infinite system of quadratic equations to be satisfied by the $P_n$. Solutions to this system of equations have been obtained by successively solving truncated systems of N equations involving only the unknowns $P_1, P_2, \ldots, P_N$. The process was started for $N = 3$ since in this case the system may be solved analytically, the problem reducing to that of solving quadratic equations for $P_1$ and $P_2$. These solutions are then used as the starting values for the numerical procedure in a similar way to that described for the model equation in Section 3.6. The numerical procedure used for solving this was obtained from the NAG library and was based on Newton's method but with the important feature that if the full Newton-Raphson correction was too large, then the displacement from $\underset{\sim}{X}^{(k)}$ was biased towards the steepest descent direction of $\underset{\sim}{F}(\underset{\sim}{X})$, where $\underset{\sim}{F}(\underset{\sim}{X})$ is the sum of squares of the residuals found by the routine, (28) and (45). The number of equations being solved was increased in steps of r, the starting values being the values found for the previous system for $P_1, \ldots, P_{N-r}$. Starting value zero was given to $P_{N-r+1}, \ldots, P_N$. Solutions were sought for r being one or two, but the convergence was found to be slow, for $\kappa \neq 0$, so that in this case r was taken to be five, for which the solution was found to converge. This is similar to the method used in (36) by Olfe and Rottmann for obtaining solutions for deep-water waves of permanent form.

The truncated system of three equations for $P_1$, $P_2$ and $P_3$ gives

$$\beta_{21}(\alpha_{13} + \alpha_{23})P_2^2 + 2\beta_{11}P_2 - 4 = 0 \tag{4.2.11}$$

$$P_1^2 = \frac{4P_2}{2\alpha_{12} + \beta_{12}(\alpha_{13} + \alpha_{23})P_2} \tag{4.2.12}$$

$$P_3 = \frac{(\alpha_{13} + \alpha_{23})}{2} P_1 P_2 \, . \tag{4.2.13}$$

Solving equation (4.2.11) we find two values for $P_2$, equation (4.2.12) may then be solved for each of these to give two values for $P_1$, without loss of generality we choose $P_1 > 0$ since changing the sign of $P_1$ is the same as changing the displacement by half a wavelength. $P_3$ is then obtained from (4.2.13). Different solutions to these equations were used as starting values. As has already been mentioned, solutions were sought for which as N, the number of unknowns was increased the $P_i$ converged to a non-trivial solution. Two such solutions were found, although it seems probable that more could be found by considering different starting values to the system of equations.

Solution 1. Table 4.1 shows the first 18 values of the $P_n$, together with the values of $C_n/\kappa$ which are given by $C_n/\kappa = P_n/n$, obtained by increasing N to the value 43. At the 8th iteration, the last one performed, $P_1$ changes by less than $10^{-3}$ and for $n > 18$, $|C_n/\kappa| < 2.10^{-2}$.

Solution 2. The first 25 values of the $P_n$ are given in Table 4.2 for $N = 48$ with the corresponding values of $C_n/\kappa$. At the 9th iteration $P_1$ changes by less than $2.10^{-3}$ and for $n > 26$, $|C_n/\kappa| < 5.10^{-2}$.

For solution 1 we see that $|C_1/C_{18}| \simeq 60$, whereas for solution 2, $|C_1/C_{18}| \simeq 31$ from which we deduce that the decay of the oscillations in the $C_n$'s is faster for solution 1.

In figures 4.1 and 4.5 the horizontal and vertical displacements at the surface are plotted for solutions 1 and 2. For the standard Rayleigh wave with $\gamma \propto e^{-\eta} \cos X$ these would clearly be sine and cosine curves. As in that case the particle motion is retrograde, that is the sense of the rotary motion of a surface particle is such that the motion is in the opposite direction to that of the propagation of the wave at the peaks of the profile (9). In figure 4.2 the vertical displacement at various depths for solution 1 is shown. Figures 4.3, 4.4, 4.6-4.8 compare the particle paths at various depths of the non-linear waves and the standard wave. For the standard wave the motion at any depth is elliptical with the sense of rotation being reversed at depth $X_2 \simeq 1.21$. The largest displacements do not occur at the surface but the vertical motions remain approximately of constant magnitude down to the depth $X_2 \simeq 1.21 \simeq \frac{1}{6} \times$ wavelength then dying away rapidly. For the non-linear waves the direction of rotation is again reversed, although the reversal is notably different from that of the standard wave.

2. $\kappa = 0$. The case when there is no perturbation in the wave speed from the Rayleigh speed is now discussed. Returning to equation (4.2.7) we make the substitution $nC_n = Q_n$ to obtain the system of non-linear algebraic equations:

$$\sum_{r=1}^{n-1} \alpha_{rn} Q_r Q_{n-r} + \sum_{s=1}^{\infty} \beta_{sn} Q_s Q_{n+s} = 0 \quad , \qquad n = 1,2,\ldots \; .$$

This is then solved by the same numerical method considered for the case $\kappa \neq 0$. In this problem we have an arbitrary scaling factor, that is if $Q_n$ is a solution, so also is $\tau Q_n$, where $\tau$ is arbitrary. So also, it is plausible to look for solutions in which $Q_n = 0$, for n even, so that the even function $\gamma(X,\eta)$ has the property $\gamma(X+\pi,\eta) = -\gamma(X,\eta)$. An analytical solution is found by truncating the sequence after $Q_5$, with $Q_2 = Q_4 = 0$, we choose $Q_1 = 1$ and use this solution as a starting value for the numerical procedure, where again we increase the number of unknowns in steps of five. To ensure that the solution does not converge to the trivial solution we rescale the $Q_n$'s after each iteration so that $\sum_{n=1}^{N} {Q_n}^2 = 1$ and these rescaled values are used as the starting values for the next step of the iteration procedure. The values of the first 20 $Q_n/\tau$ and corresponding $C_n/\tau$ are shown in Table 4.3 when $N = 31$, $\tau$ being an arbitrary parameter. At the 13th iteration, the last one performed $Q_1/\tau$ changed by less than $10^{-3}$ and for $n > 20$, $|C_n/\tau| < 2.10^{-2}$. The displacements are again calculated from equation (4.2.9) and the horizontal and vertical displacements at the surface are shown in figure 4.9, as are the particle paths at various depths in figure 4.10.

For all cases we may scale the solution in the X-direction by choosing a different period for the wave. For $\kappa > 0$ the wave travels faster than the standard Rayleigh wave, whereas for $\kappa < 0$ the wave travels more slowly. We have assumed $\kappa > 0$ in the figures, for $\kappa < 0$ the surface elevation profiles will be inverted and the particle paths rotated through $180^{\circ}$. For $\kappa = 0$ we may still scale in the X-direction, but the inclusion of the arbitrary parameter $\tau$ implies that the amplitude of the wave may also be scaled independently, since the speed c is independent of the amplitude in this case.

We have attempted to find different periodic solutions by examining the case of a forced wave in which the vertical traction was a specified periodic load of a small specified amplitude travelling at a speed close to the standard Rayleigh wave speed. We chose the load to be sinusoidal. Solutions were obtained for different values of the load amplitude and different values of K-1, which is proportional to the deviation of the speed from the linear Rayleigh speed. Solutions were sought for which the magnitude of the forcing function could be reduced to zero while the amplitude of the wave remained O(1). However, in the procedures attempted it was not found possible to decrease the loading amplitude without a corresponding decrease in the wave amplitude.

Table 4.1 Solution 1 for $\kappa \neq 0$

| $n$ | $P_n$ | $C_n/\kappa$ |
|---|---|---|
| 1 | 1.2673 | 1.2673 |
| 2 | -1.5236 | -0.7618 |
| 3 | 1.2415 | 0.4198 |
| 4 | -0.6711 | -0.1678 |
| 5 | 0.0425 | 0.0085 |
| 6 | 0.4508 | 0.0751 |
| 7 | -0.6873 | -0.0982 |
| 8 | 0.6403 | 0.0800 |
| 9 | -0.3731 | -0.0415 |
| 10 | 0.0095 | 0.0009 |
| 11 | 0.3122 | 0.0284 |
| 12 | -0.4845 | -0.0404 |
| 13 | 0.4629 | 0.0356 |
| 14 | -0.2744 | -0.0196 |
| 15 | 0.0019 | 0.0001 |
| 16 | 0.2495 | 0.0156 |
| 17 | -0.3905 | -0.0230 |
| 18 | 0.3776 | 0.0210 |

Table 4.2 Solution 2 for $\kappa \neq 0$

| n | $P_n$ | $C_n/\kappa$ |
|---|---|---|
| 1 | 2.2565 | 2.2565 |
| 2 | 6.0555 | 3.0277 |
| 3 | -5.1611 | -1.7203 |
| 4 | -0.8027 | -0.2007 |
| 5 | 0.1972 | 0.0392 |
| 6 | 0.7890 | 0.1315 |
| 7 | 2.6560 | 0.3794 |
| 8 | -2.8137 | -0.3517 |
| 9 | -0.3836 | -0.0426 |
| 10 | 0.1528 | 0.0153 |
| 11 | 0.5758 | 0.0523 |
| 12 | 1.7775 | 0.1481 |
| 13 | -2.1077 | -0.1621 |
| 14 | -0.2217 | -0.0158 |
| 15 | 0.1547 | 0.0103 |
| 16 | 0.4860 | 0.0304 |
| 17 | 1.3460 | 0.0792 |
| 18 | -1.7652 | -0.0981 |
| 19 | -0.1263 | -0.0066 |
| 20 | 0.1641 | 0.0082 |
| 21 | 0.4378 | 0.0208 |
| 22 | 1.0877 | 0.0494 |
| 23 | -1.5741 | -0.0684 |
| 24 | -0.0581 | -0.0024 |
| 25 | 0.1763 | 0.0071 |

Table 4.3 Solution for $\kappa = 0$

| $n$ | $Q_n/\tau$ | $C_n/\tau$ |
|---|---|---|
| 1 | 0.5632 | 0.5632 |
| 3 | -0.2324 | -0.0775 |
| 5 | 0.0849 | 0.0170 |
| 7 | 0.0273 | 0.0039 |
| 9 | -0.1200 | -0.0133 |
| 11 | 0.1948 | 0.0177 |
| 13 | -0.2505 | -0.0193 |
| 15 | 0.2853 | 0.0190 |
| 17 | -0.3012 | -0.0177 |
| 19 | 0.2970 | 0.0156 |

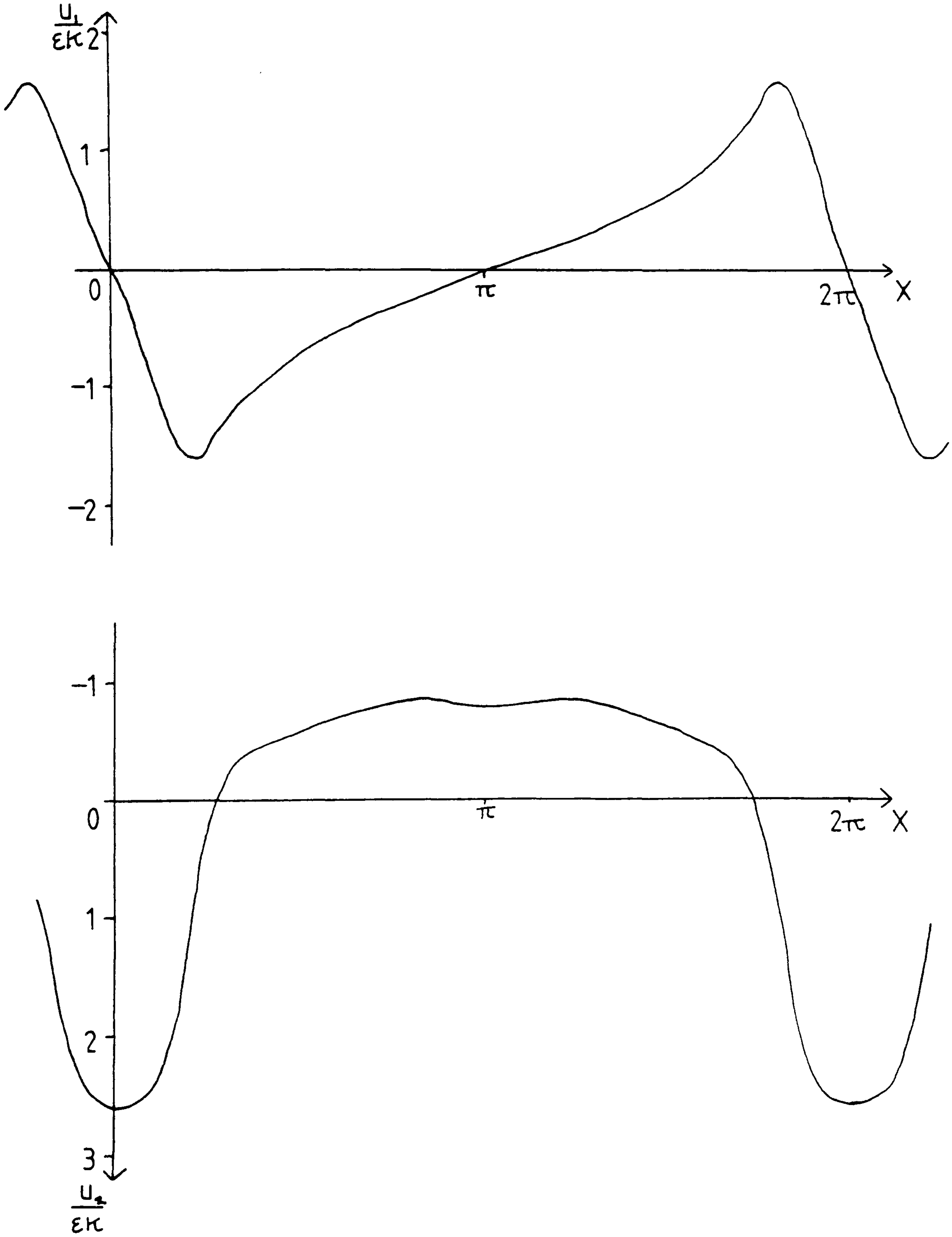



Figure 4.1 Horizontal and vertical displacements at the surface for solution 1

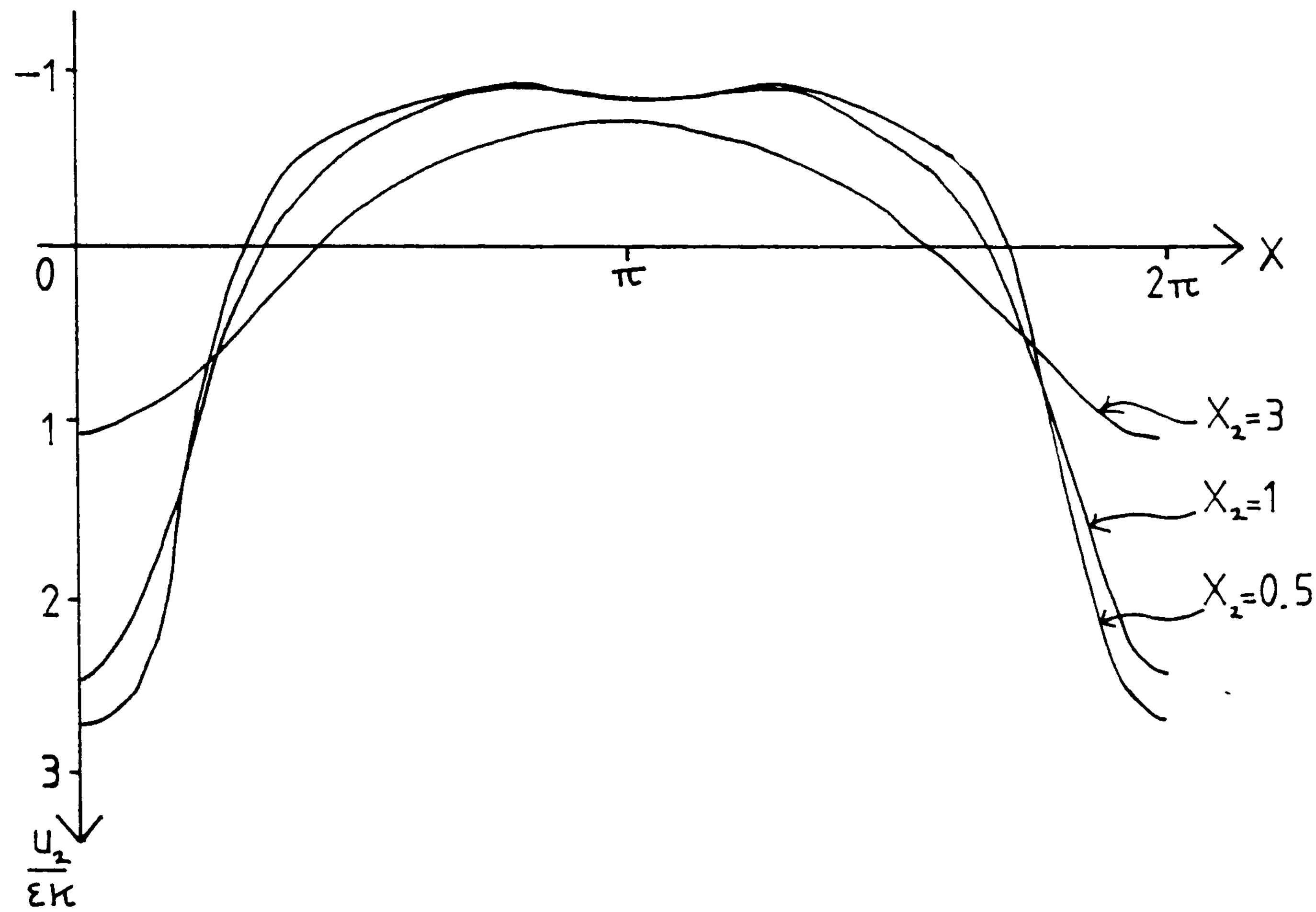



Figure 4.2 Vertical displacements at various depths for solution 1

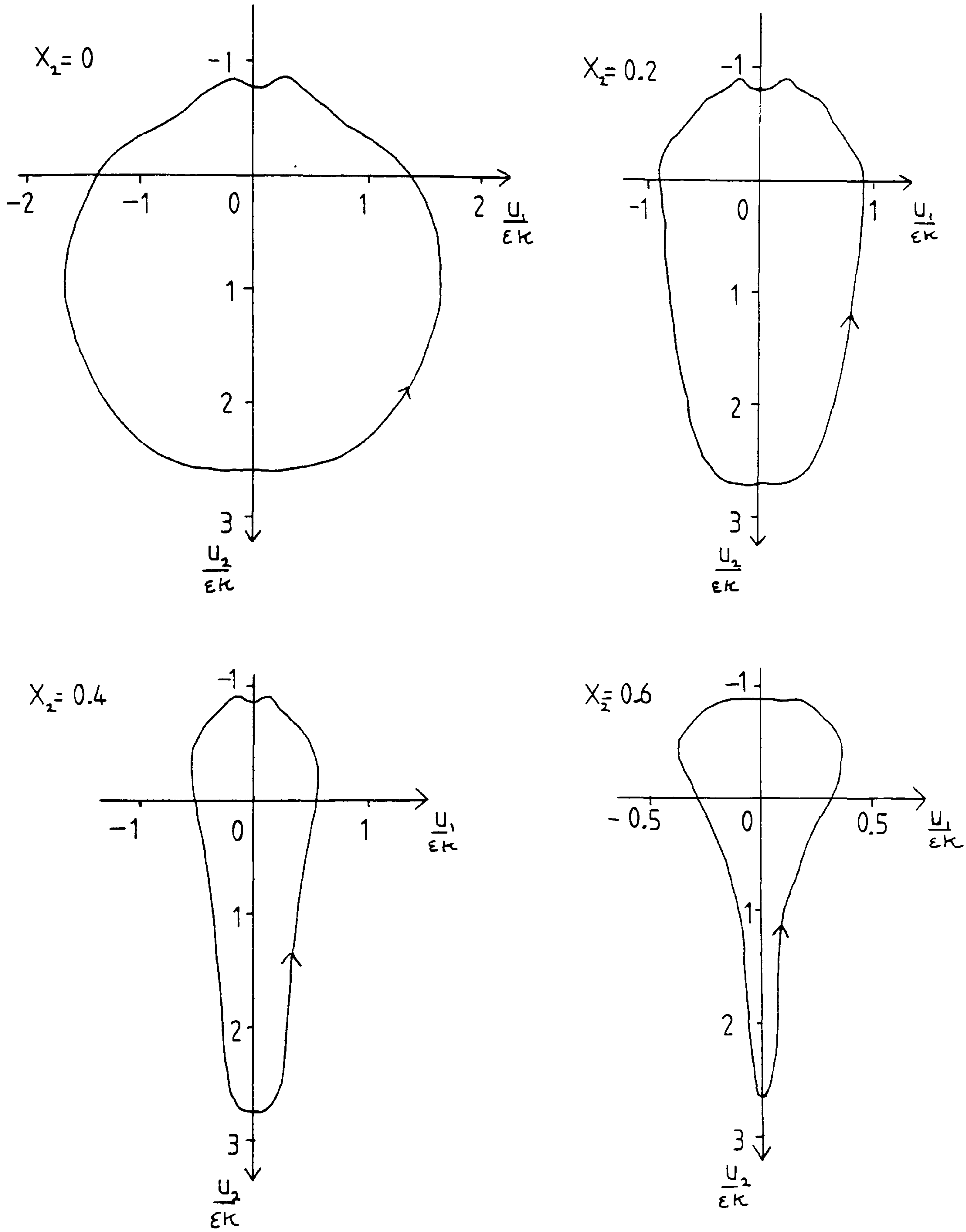



Figure 4.3 Particle paths for solution 1

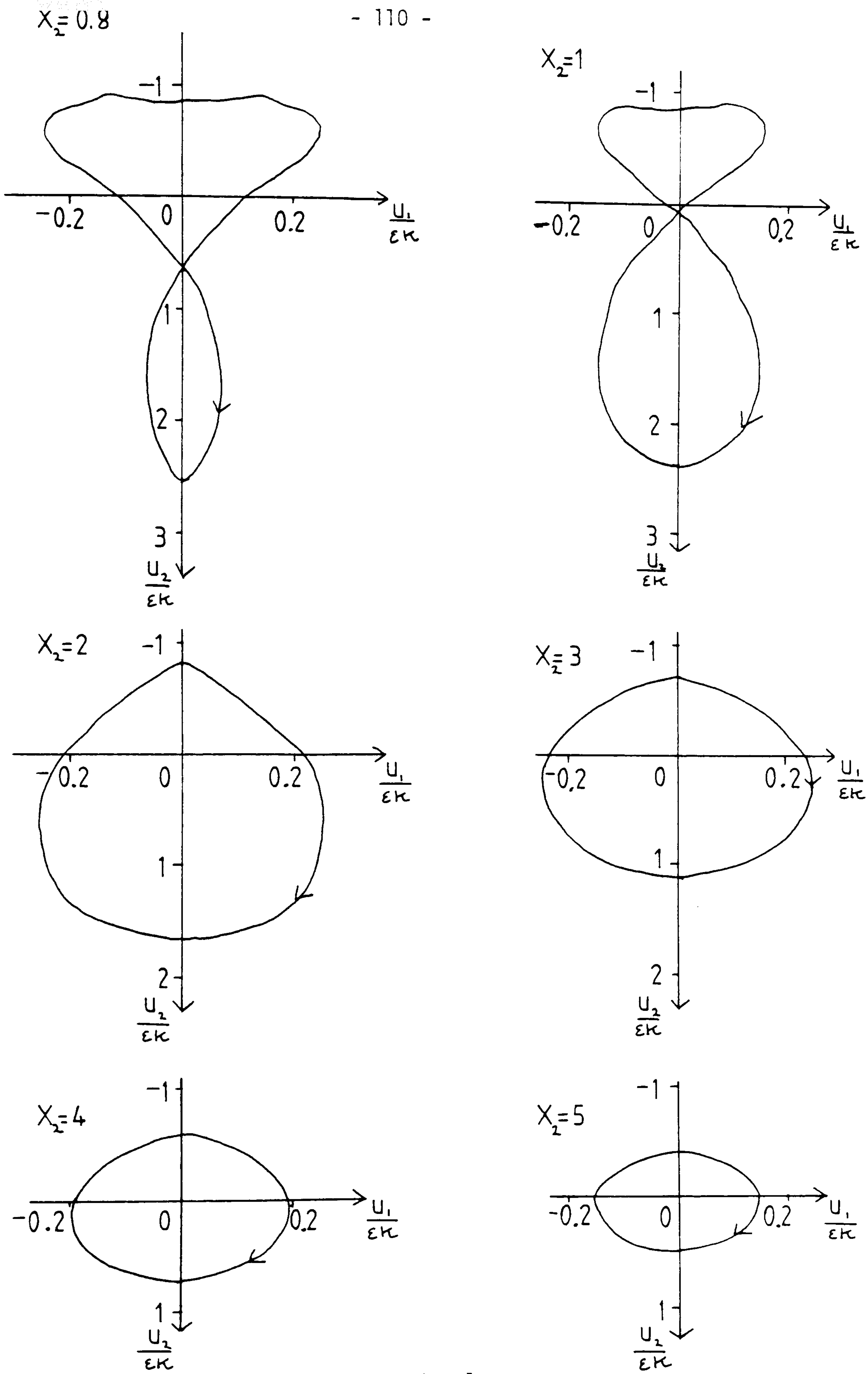



Figure 4.4 Particle paths for solution 1

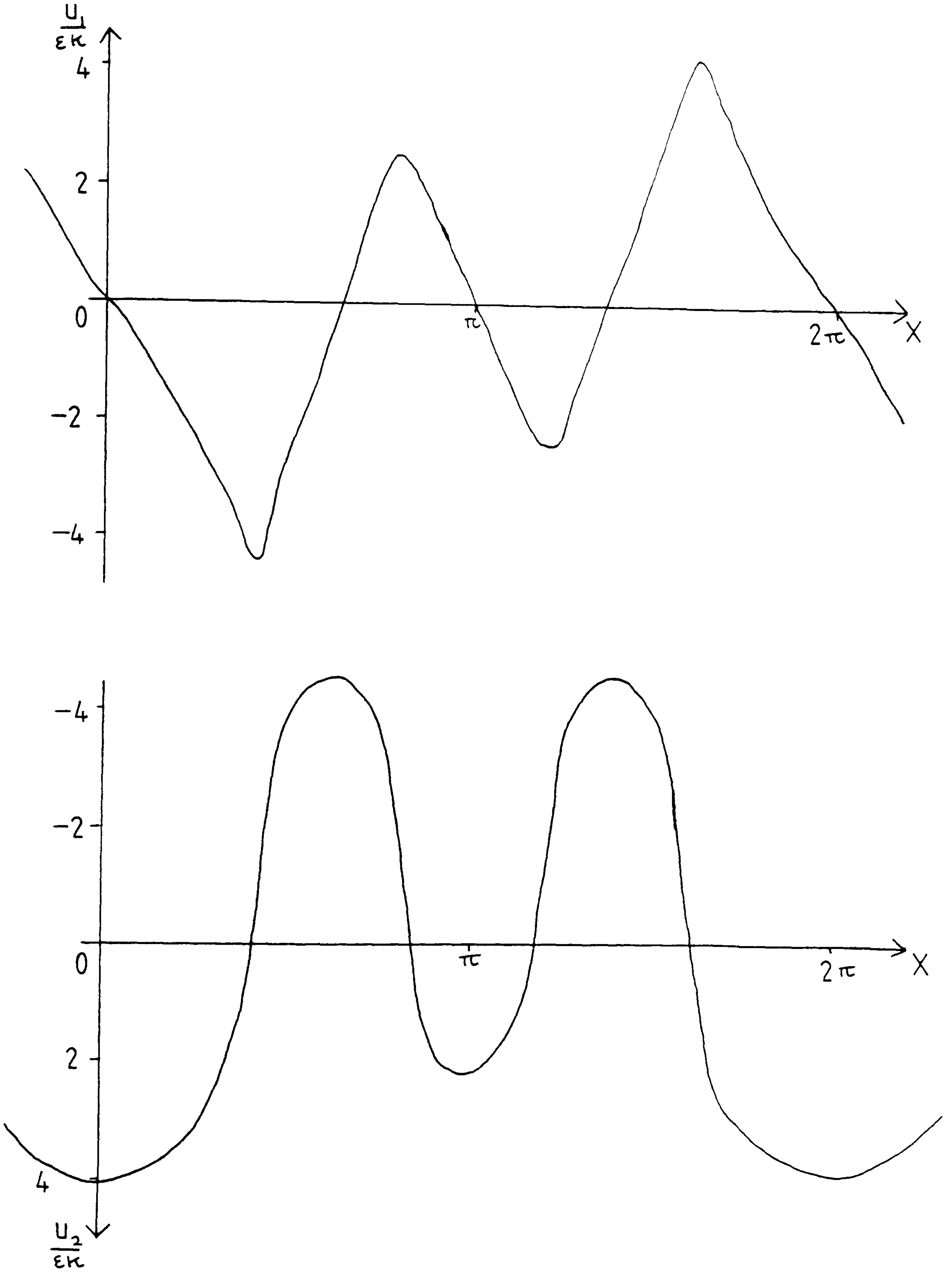



Figure 4.5 Horizontal and vertical displacements at the surface for solution 2

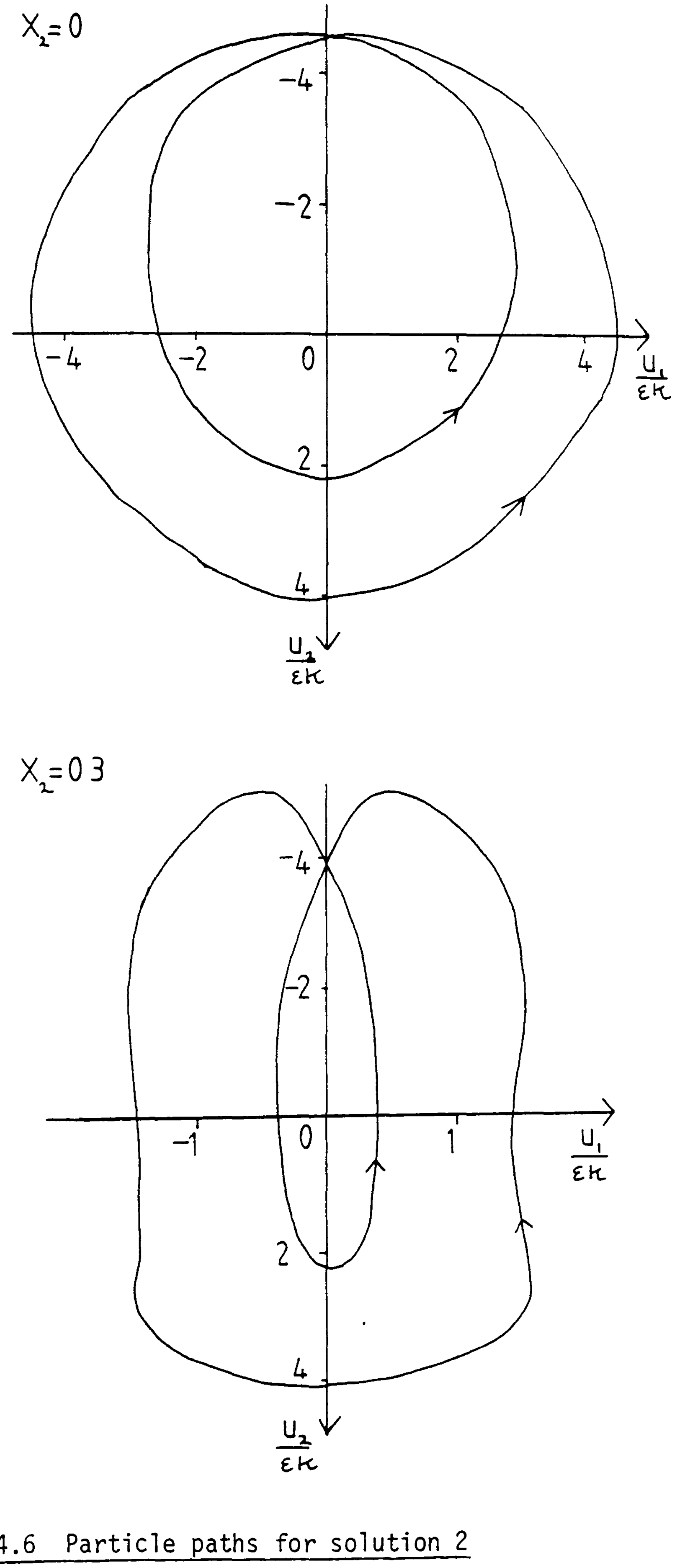



Figure 4.6 Particle paths for solution 2

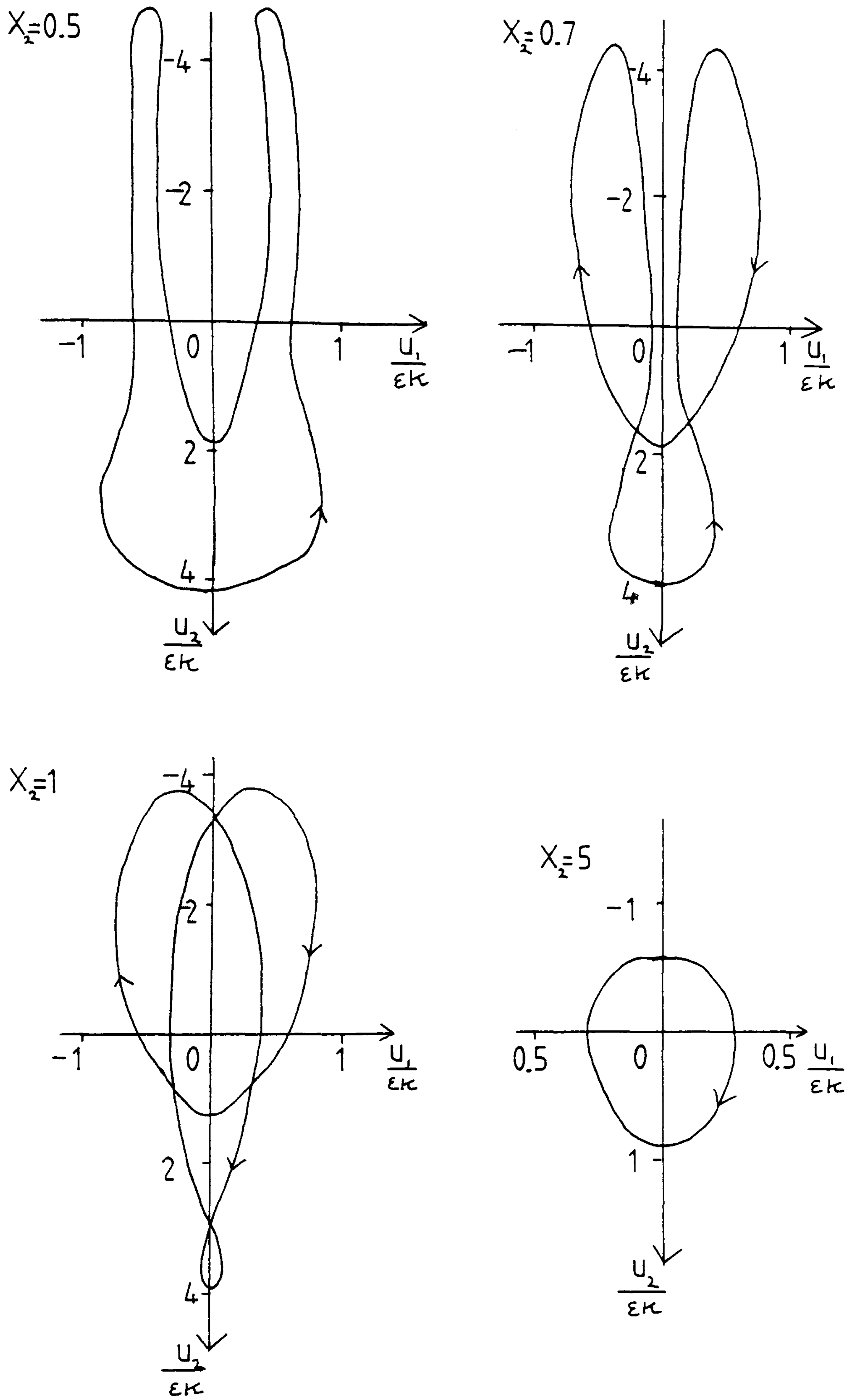



Figure 4.7 Particle paths for solution 2

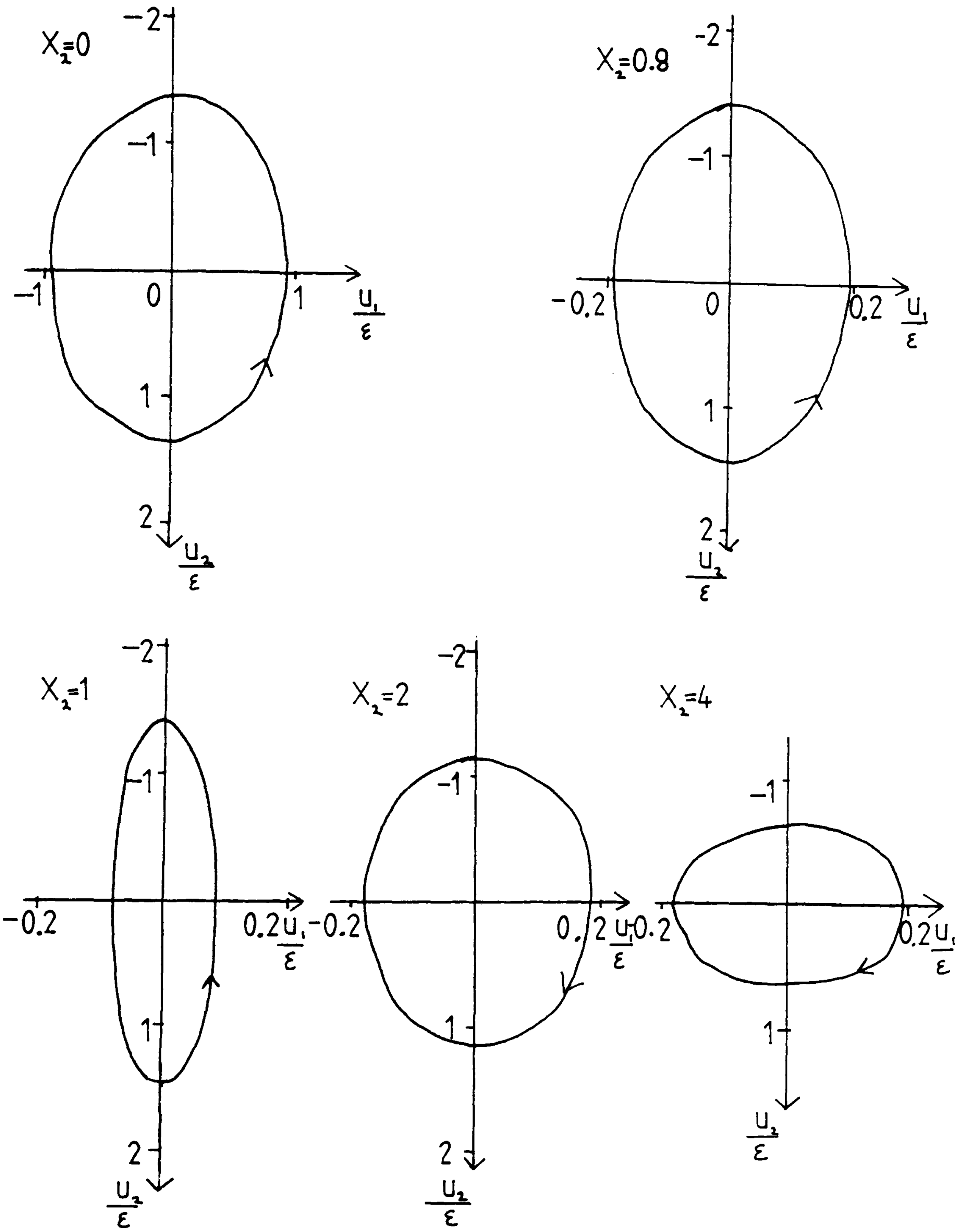



Figure 4.8 Particle paths for the standard harmonic wave with $C_1$ = 1.3

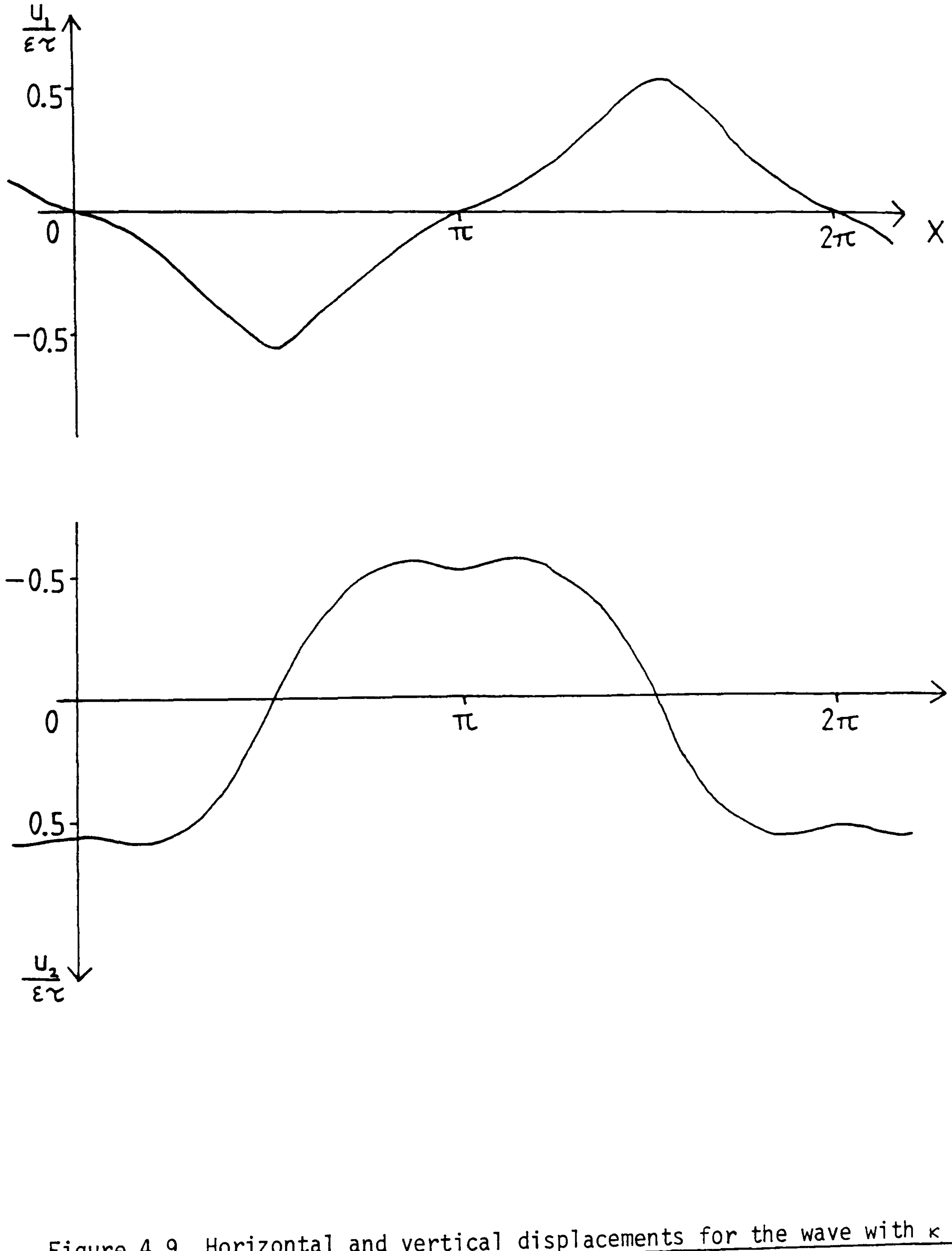



Figure 4.9 Horizontal and vertical displacements for the wave with $\kappa = 0$

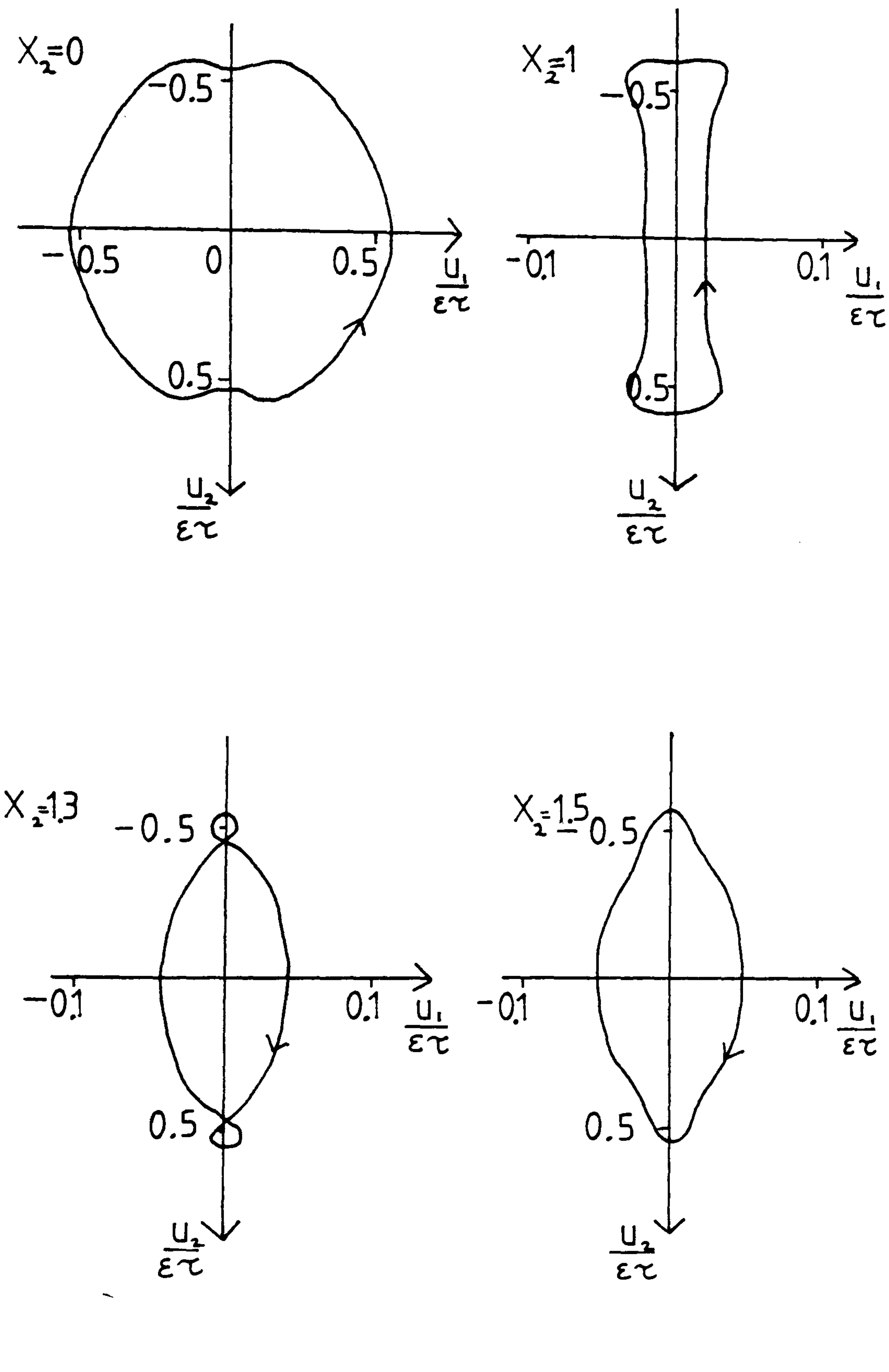



Figure 4.10 Particle paths for the solution with $\kappa = 0$

## 4.3 MORE GENERAL SOLUTIONS

We consider here the solution for more general $\gamma(X,\eta)$ so that we no longer restrict attention to even functions. As was stated at the beginning of the previous section we may then represent $\gamma(X,\eta)$ by $\sum_{n=1}^{\infty} \{C_n e^{-n\eta} \cos nX + D_n e^{-n\eta} \sin nX\}$. The equations to be satisfied by the $C_n$'s and $D_n$'s may then be obtained as above. Alternatively, the problem may be formulated using complex functions.

The equations to be solved (3.2.8) and (3.2.9) with boundary conditions (3.2.6) are

$$\begin{aligned} u_X - v_y &= \ell(X,y) \quad , \quad & v_X + u_y &= m(X,y) \\ U_X - V_Y &= L(X,Y) \quad , \quad & V_X + U_Y &= M(X,Y) \end{aligned} \tag{4.3.1}$$

with

$$\begin{aligned} U(X,0) &= u(X,0) \\ V(X,0) &= (1+\varepsilon\kappa)v(X,0) \ . \end{aligned} \tag{4.3.2}$$

We introduce the complex functions $w(z,\bar{z})$ and $W(Z,\bar{Z})$ where $z = X + iy$, $Z = X + iY$ and $w$, $W$ are defined by $w = u + iv$, $W = U + iV$, bars denoting complex conjugates. We also define functions $h(z,\bar{z})$ and $H(Z,\bar{Z})$ by $h = \ell + im$, $H = L + iM$. The equations (4.3.1) may then be written

$$\begin{aligned} w_{\bar{z}} &= \tfrac{1}{2}\, h(z,\bar{z}) \\ W_{\bar{Z}} &= \tfrac{1}{2}\, H(Z,\bar{Z}) \ . \end{aligned} \tag{4.3.3}$$

The surface boundary conditions are applied on $z = \bar{z} = Z = \bar{Z}$, conditions (4.3.2) therefore become

$$W(Z,Z) = w(z,z) + \varepsilon ik \ \mathrm{Im}(w(z,z)) \tag{4.3.4}$$

We let $\nu(z,\bar{z})$ be an analytic function so that

$$\nu_{\bar{z}} = 0 \qquad \text{gives} \qquad \nu w_{\bar{z}} = (\nu w)_{\bar{z}}$$

and

$$\iint \nu w_{\bar{z}} \, dz d\bar{z} = \iint (\nu w)_{\bar{z}} \, dz d\bar{z} \quad , \tag{4.3.5}$$

where the integrals are over the region where $\mathrm{Im}(z) \geq 0$.

The complex Stokes' theorem (34) states that if $f(z,\bar{z})$ is a function of $z = x + iy$, $\bar{z} = x - iy$ which is continuous and differentiable in an area S enclosed by a contour C then

$$\int_C f(z,\bar{z}) dz = 2i \iint_S \frac{\partial f}{\partial \bar{z}} \, dS$$

and

$$\int_C f(z,\bar{z}) d\bar{z} = -2i \iint_S \frac{\partial f}{\partial z} \, dS \ .$$

Using the first of these we have

$$\iint_S (\nu w)_{\bar{z}} \, dS = \frac{1}{2i} \int_C \nu w dz \ ,$$

where S is the region $z \geq \bar{z}$ , $0 \leq \mathrm{Re}(z) \leq 2\pi$ and C is the line $z = \bar{z}$. Similarly with $\nu(Z,\bar{Z})$ we obtain

$$\iint_S (\nu W)_{\bar{Z}} \, dS = \frac{1}{2i} \int_C \nu W dZ \ .$$

Hence using equation (4.3.5) we deduce

$$\tfrac{1}{2} \iint_S \nu h \, dS = \frac{1}{2i} \int_C \nu w \, dz \tag{4.3.6}$$

and

$$\tfrac{1}{2} \iint_S \nu H \, dS = \frac{1}{2i} \int_C \nu W \, dZ \ . \tag{4.3.7}$$

Subtracting (4.3.6) from (4.3.7) and using the boundary condition (4.3.4) gives

$$\iint_S \nu H \, dS - \iint_S \nu h \, dS = \frac{\varepsilon}{i} \int_C i\kappa \mathrm{Im}(w(z,z))dz \, . \qquad (4.3.8)$$

This provides a condition on the solution which is analogous to (3.3.14). The functions h and H are $O(\varepsilon)$, which implies that $w_{\bar{z}}$ and $W_{\bar{Z}}$ are $O(\varepsilon)$ and so we may write $w(z,\bar{z}) = \zeta(z) + \varepsilon\tilde{w}(z,\bar{z})$. However, h and H also decay so $(w-\zeta)/\varepsilon$ is bounded, hence $\zeta$ decays as $\mathrm{Im}(z) \to +\infty$, $\mathrm{Im}(z) \geq 0$ and is periodic in $\mathrm{Re}(z)$. Similarly we write $W(Z,\bar{Z}) = \zeta(Z) + \varepsilon\tilde{W}(Z,\bar{Z})$ since $W(Z,\bar{Z}) - w(z,\bar{z}) = O(\varepsilon)$ from equations (4.3.4). We choose $\zeta(z,\bar{z}) = \sum_{n=1}^{\infty} C_n e^{inz}$, $C_n$ is complex and substituting for $\zeta$ in equation (4.3.8) we obtain the system of complex equations for the $C_n$'s.

$$-2\kappa C_n + \sum_{r=1}^{n-1} \alpha_{rn} r(n-r) C_r C_{n-r} + \sum_{s=1}^{\infty} \beta_{sn} s(n+s) C_s C_{n+s} = 0 \, . \qquad (4.3.9)$$

In this equation the transformation $C_n \to e^{in\theta} C_n$ leaves the equation unchanged and corresponds merely to a change of phase of the wave or a translation of the coordinate axes.

Equation (4.3.9) may be solved by writing $C_n = D_n + iE_n$ and equating real and imaginary parts of equation (4.3.9). All the solutions found to these equations were reproductions of the case of even $\gamma(X,n)$ with a change of phase.

## 4.4 NON-PERIODIC WAVEFORMS

In this section we consider the problem of obtaining non-periodic waveforms, case (B), we use Fourier transforms as introduced in section 3.4. As in section 4.2, we look for symmetric waveforms so that we represent $\gamma(X,\eta)$ by the Fourier cosine transform $\int_0^\infty E(s)e^{-s\eta}\cos sX\,ds$, $\beta(X,\eta)$ may then be represented by the Fourier sine transform $\int_0^\infty [-E(s)e^{-s\eta}\sin sX]ds$. As in Section 4.2, we introduce the function $w^{(1)}$ which satisfies the equations (4.2.4) - 4.2.6), which in this case become:

$$w^{(1)}_{XX} + w^{(1)}_{\eta\eta}$$

$$= \frac{(B^2-A^2)(1-A^4)}{8A^2B^2}\int_0^\infty\int_0^\infty \frac{rsE(r)E(s)}{2}\, e^{-(r+s)A\eta/B} \; .$$

$$\{(r-s)\sin(r-s)X - (r+s)\sin(r+s)X\}drds$$

$$+ \frac{(B^2-A^2)(1-A^4)}{4A^3B}\int_0^\infty\int_0^\infty - \frac{rsE(r)E(s)}{2}\left(r\frac{B}{A} + s\right)e^{-(rB/A+s)\eta}$$

$$\{\sin(r+s)X - \sin(r-s)X\}drds \qquad (4.4.1)$$

$$w^{(1)}(X,0) = 0 \qquad (4.4.2)$$

$$-w^{(1)}_\eta(X,0) + \frac{(B^2-A^2)(1-A^4)}{4A^3B}\int_0^\infty\int_0^\infty \frac{rsE(r)E(s)}{2}\{\sin(r+s)X + \sin(r-s)X\}drds$$

$$= -\kappa\int^\infty sE(s)\sin sX\,ds \qquad (4.4.3)$$

Writing $w^{(1)}(X,\eta)$ in the form

$$w^{(1)}(X,\eta) = W_0(\eta) + \int_0^\infty W(r,\eta)\sin rX\,dr$$

and substituting this into the equation (4.4.1) and boundary conditions (4.4.2) and (4.4.3) we find that $W_0(\eta) \equiv 0$ and $W(r,\eta)$ satisfies the equation

$$\int_0^\infty \left[W_{\eta\eta}(r,\eta) - r^2W(r,\eta)\right] \sin rX\, dr$$

$$= \frac{(B^2-A^2)(1-A^4)}{8A^2B^2} \int_0^\infty\int_0^\infty \frac{rs(E(r)E(s)}{2} e^{-(r+s)A\eta/B}$$

$$\{(r-s)\sin(r-s)X - (r+s)\sin(r+s)X\}drds$$

$$- \frac{(B^2-A^2)(1-A^4)}{4A^3B} \int_0^\infty\int_0^\infty \frac{rsE(r)E(s)}{2} \left(\frac{rB}{A} + s\right)e^{-(rB/A+s)\eta} \;.$$

$$\{\sin(r+s)X - \sin(r-s)X\}drds$$

with $\int_0^\infty W(r,0)\sin rXdr = 0$ .

This must be true for all X so that we obtain the following differential equation for $W(r,\eta)$:

$$W_{\eta\eta} - r^2W = \frac{(B^2-A^2)(1-A^4)}{8A^3B^2} \left\{\int_0^\infty \frac{E(s)E(r+s)s(r+s)}{2} \right. \;.$$

$$\left[2Are^{-(r+2s)A\eta/B} + 2\left(\frac{(r+s)B+s}{A}\right)Be^{-(r+s)B/A+s)\eta}\right.$$

$$\left. -2\left(\frac{sB}{A} + r+s\right)Be^{-(sB/A+r+s)\eta}\right]ds$$

$$- \int_0^r \left[Are^{-rA\eta/B} + 2\left(\frac{sB^2}{A} + (r-s)B\right)e^{-(sB/A+r-s)\eta}\right] \;.$$

$$\left.\frac{E(s)E(r-s)s(r-s)}{2}\, ds\right\}$$

with $W(r,0) = 0$ .

Solving for W and substituting the result into condition (4.4.3) and again using the fact that (4.4.3) must be true for all X we obtain a condition to be satisfied by the function $E(r)$:

$$-r\kappa E(r) = \frac{(B^2-A^2)(1-A^4)}{8A^2B}\left\{2\int_0^\infty \left[\frac{r}{(r+2s)A+Br} - \frac{r}{(r+s)(A+B)} + \frac{r}{sB+(2r+s)A}\right] s(r+s)E(s)E(r+s)ds + \int_0^r \left[\frac{-1}{2(A+B)} + \frac{r}{sB+(2r-s)A}\right] s(r-s)E(s)E(r-s)ds\right\} \qquad (4.4.4)$$

which is similar to the equation (4.2.7) which the coefficients in the Fourier series must satisfy.

To solve equation (4.4.4) we look for waves travelling at a speed different from the linear Rayleigh speed $c_R$ and make the transformation $sE(s) = \kappa F(s)$, since $\kappa$ is non-zero. The equation for $F(s)$ may then be written in the form:

$$-2F(r) + \int_0^r \alpha(s,r)F(s)F(r-s)ds + \int_0^\infty \beta(s,r)F(s)F(r+s)ds = 0 \qquad (4.4.5)$$

where

$$\alpha(s,r) = \frac{(B^2-A^2)(1-A^4)}{8A^2B}\left[\frac{1}{A+B} - \frac{2r}{sB+(2r-s)A}\right],$$

$$\beta(s,r) = \frac{(B^2-A^2)(1-A^4)}{4A^2B}\left[\frac{r}{(r+s)(A+B)} - \frac{r}{(r+2s)A+Br} - \frac{r}{sB+(2r+s)A}\right]. \qquad (4.4.6)$$

Differentiating this equation with respect to $r$ we find

$$2F'(0) = -\frac{(B^2-A^2)(1-A^4)}{8A^3B}\int_0^\infty \frac{(F(s))^2}{s}\,ds,$$

and since $\kappa F(s) = sE(s)$

$$\kappa F'(0) = E(0)$$

which gives an expression for $E(0)$.

Baker (1) suggests for equations of type (4.4.5) reduction to a system of non-linear algebraic equations. We attempt to solve (4.4.5) using various discretizations. We consider the equally spaced discretization points $s_j$ and for $s\varepsilon(s_j,s_{j+1})$ we represent F(s) by the linear approximation

$$F(s) = F(s_j) + (s-s_j)\left(\frac{F(s_{j+1}) - F(s_j)}{s_{j+1} - s_j}\right) \quad . \tag{4.4.7}$$

Hence

$$\int_{s_j}^{s_{j+1}} \alpha(s,r)F(s)F(r-s)ds = \int_{s_j}^{s_{j+1}} \alpha(s,r)\left[F(s_j) + \left(\frac{F(s_{j+1}-F(s_j)}{s_{j+1} - s_j}\right)(s-s_j)\right]$$
$$\left[F(r-s_j) + \left(\frac{F(r-s_{j+1})-F(r-s_j)}{s_{j+1} - s_j}\right)(s-s_j)\right]ds$$

and

$$\int_{s_j}^{s_{j+1}} \beta(s,r)F(s)F(r+s)ds = \int_{s_j}^{s_{j+1}} \beta(s,r)\left[F(s_j) + \left(\frac{F(s_{j+1}-F(s_j)}{s_{j+1}-s_j}\right)(s-s_j)\right]$$
$$\left[F(r+s_j) + \left(\frac{F(r+s_{j+1})-F(r+s_j)}{s_{j+1} - s_j}\right)(s-s_j)\right]ds \ .$$

Defining $s_j \equiv \frac{j}{N}$, $r \equiv s_i \equiv \frac{i}{N}$ and $F(s_i) \equiv X_i$

then

$$\int_{s_j}^{s_{j+1}} \alpha(s,r)F(s)F(r-s)ds = C_{ij}X_jX_{i-j} + D_{ij}X_{i-j}X_{j+1}$$
$$+ E_{ij}X_jX_{i-j-1} + F_{ij}X_{j+1}X_{i-j-1} \tag{4.4.8}$$

and

$$\int_{s_j}^{s_{j+1}} \beta(s,r)F(s)F(r+s)ds = G_{ij}X_jX_{i+j} + H_{ij}X_{i+j}X_{j+1}$$
$$+ L_{ij}X_{i+j+1}X_j + M_{ij}X_{i+j+1}X_{j+1} \tag{4.4.9}$$

where

$$C_{ij} = \int_{s_j}^{s_{j+1}} \alpha(s,r)(1+2j+j^2-2sN-2jNs+s^2N^2)ds$$

$$D_{ij} = \int_{s_j}^{s_{j+1}} \alpha(s,r)(-j-j^2+sN+2jNs-s^2N^2)ds$$

$$E_{ij} = \int_{s_j}^{s_{j+1}} \alpha(s,r)(-j-j^2+sN+2jNs-s^2N^2)ds$$

$$= D_{ij}$$

$$F_{ij} = \int_{s_j}^{s_{j+1}} \alpha(s,r)(j^2-2sjN+s^2N^2)ds$$

$$G_{ij} = \int_{s_j}^{s_{j+1}} \beta(s,r)(1+2j+j^2-2sN-2sjN+s^2N^2)ds$$

$$H_{ij} = \int_{s_j}^{s_{j+1}} \beta(s,r)(-j-j^2+sN+2sjN-s^2N^2)ds$$

$$L_{ij} = \int_{s_j}^{s_{j+1}} \beta(s,r)(-j-j^2+sN+2sjN-s^2N^2)ds$$

$$= H_{ij}$$

$$M_{ij} = \int_{s_j}^{s_{j+1}} \beta(s,r)(j^2-2sjN+s^2N^2)ds \quad .$$

These are evaluated analytically using $\alpha$ and $\beta$ from (4.4.6) and the resulting algebraic equations substituted in (4.4.8) and (4.4.9) and these are then used in (4.4.5). We therefore have a system of algebraic quadratic equations for the $X_i$.

Initially, we assume that $F(s)$ is only non-zero on the interval $[0,1]$, which we may do without loss of generality since the $s$ scale is arbitrary. From the definition of $F(s)$ it is clear that $F(0) = 0$ and we are interested only in non-negative $s$, we therefore consider the points $s_i = i/N$ for $i = 1,2,\ldots,N$. Equation (4.4.5) then gives $N$ equations for the $N$ unknowns $X_i$. Solutions to these were sought using the same method as in the periodic case. However, we were unable to obtain solutions to these equations which converged as $N$ was increased.

We have also attempted to obtain a solution by approximating the function as above on $[0,1]$, but for $s > 1$ assuming that $F(s)$ can be approximated by some chosen function. Clearly we require $F(s)$ to decay as $s \to \infty$ so we choose $F(s) = k/s$ for $s > 1$, where the constant $k$ is chosen so that $F(s)$ is continuous at $s = 1$, that is $k = F(1)$. We then divide the integral from 0 to $\infty$ into three regions $(0, 1-r)$, $(1-r, 1)$ and $(1, \infty)$, so that:

$$\int_0^\infty \beta(s,r)F(s)F(r+s)ds = \int_0^{1-r} \beta(s,r)F(s)F(r+s)ds + \int_{1-r}^1 \beta(s,r)\frac{F(s)F(1)}{s+r}ds + \int_1^\infty \beta(s,r)\frac{F(1)^2}{s(s+r)}ds .$$

Again solving the system of equations for the $X_i$ as above we were unable to find solutions which converged.

A simpler method of solving (4.4.5) is to approximate the integrals by the trapezium rule. Initially we consider $F(s)$ to be zero outside the interval $[0,1]$, as before we are interested in the region where $s \geq 0$ and we have $F(0) = 0$. Dividing the interval $[0,1]$ into $N$ sections of length 1/N, we may approximate equation (4.4.5) by

$$-2F(r) + \frac{1}{2N}\left[\alpha(0,r)F(0)F(r) + 2\sum_{s=1}^{r-1} \alpha\left(\frac{s}{N}, r\right) F\left(\frac{s}{N}\right) F\left(r - \frac{s}{N}\right)\right.$$

$$\left. + \alpha(r,r)F(r)F(0)\right]$$

$$+ \frac{1}{2N}\left[\beta(0,r)F(0)F(r) + 2\sum_{s=1}^{N-1} \beta\left(\frac{s}{N}, r\right) F\left(\frac{s}{N}\right) F\left(\frac{s}{N} + r\right)\right.$$

$$\left. + \beta(1,r)F(1)F(1+r)\right] = 0 \quad . \qquad (4.4.10)$$

Solutions were found numerically to these equations, however the maximum value of $|F(s)|$ was occurring at $s = 1$. We therefore attempted to add an approximation to $F(s)$ for $s > 1$, the functions used were $\frac{F(1)}{s}$, $\frac{F(1)}{s^2}$, $F(1)e^{1-s}$, $F(1)e^{\frac{1}{2}(1-s)}$ and $F(1)e^{\frac{1}{4}(1-s)}$. With the function $F(s) = \frac{F(1)}{s}$, for $s > 1$. it was found that the solution for $0 \geq s \geq 1$ was changed considerably so that $F(s) \sim \frac{F(1)}{s}$ for $s > \frac{1}{2}$. The same situation was found to occur with $F(s) = \frac{F(1)}{s^2}$ for $s > 1$. Choosing $F(s)$ to be an exponential function $F(1)e^{-s}$ for $s > 1$ we found that the maximum value of $|F(s)|$ occurred at $s = 1$ and choosing $F(s) = F(1)e^{(1-s)/2}$ for $s > 1$ the magnitude of the solution decreased. For $F(s) = F(1)e^{(1-s)/4}$ for $s > 1$ the magnitude of the solution decreased further.

Equation (4.4.4) may be rewritten in a form where the kernel functions depend only on r,s and $\bar{\Lambda}$, with $\bar{\Lambda} = \frac{2A}{A+B}$. This is achieved by writing

$$\frac{(B^2-A^2)(1-A^4)}{8A^2B}\, sE(s) = \kappa F(s)$$

and obtaining

$$F(r) = \int_0^r \left(1 - \frac{2r}{s+\bar{\Lambda}(r-s)}\right) F(s)F(r-s)ds$$

$$+ \int_0^\infty 2\left(\frac{r}{r+s} - \frac{r}{r+\bar{\Lambda}s} - \frac{r}{s+\bar{\Lambda}r}\right)F(s)F(r+s)ds \ ,$$

where $\bar{\Lambda} < 1$.

Comparing equation (4.4.11) with the equation (4.2.7) obtained when using Fourier series we see that making the transformation $F(r/N) \rightarrow NP_i$ we obtain equation (4.2.7) with the infinite sum replaced by the sum from 1 to N-1. We therefore attempted to find a solution using that already obtained in the periodic case. Choosing N = 24 we found a solution for this value of N, however, proceeding to the case with N = 48 the solution changed a great deal and seemed to be tending to the trivial solution. Since the solution obtained for the Fourier series is oscillatory it seems probable that the approximation used for s > 1 should also be oscillatory.

# CHAPTER 5

# RAYLEIGH WAVES ON AN INCOMPRESSIBLE MATERIAL

## 5.1 INCOMPRESSIBLE MATERIALS

In this chapter we consider the theory for an incompressible material, the analysis of which is similar to that already considered for a compressible material, but differs because the existence of a hydrostatic pressure means that the constitutive law cannot be written in the form (3.2.1). First we discuss different constitutive laws for rubberlike materials and then in the following sections we use one of these in our analysis of waves on the surface of incompressible materials.

One of the fundamental problems in the theory of elasticity is to find an appropriate expression for the strain-energy of a body subjected to a homogeneous strain. Rubber and similar substances have a low shear modulus which means that a moderate strain can lead to large shears. A different approach from that of the previous chapter is therefore required for any adequate theory of the elasticity of rubber-like materials, that is materials which are nearly incompressible. We model their behaviour by considering incompressible materials.

There have been many attempts to reproduce theoretically the stress-strain curves obtained experimentally on the isothermal deformation of these types of materials. Most of these have followed Mooney (31) who expressed the strain-energy as a function of the principal stretches. He postulates that the elastic material considered, in addition to being homogeneous and free from hysteresis has the following properties:

(1) is isotropic in the undeformed state and also after a positive or negative stretch-squeeze it remains isotropic in the plane at right angles to the stretch,

(2) the deformations are isochoric, that is occur without change of volume and

(3) the traction in simple shear in any isotropic plane is proportional to the shear.

Mooney deduces that the most general choice of W, the strain-energy function consistent with the three postulates is

$$W(\lambda_1,\lambda_2,\lambda_3) = C_1(\lambda_1^2+\lambda_2^2+\lambda_3^2-3) + C_2\left(\frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2} + \frac{1}{\lambda_3^2} - 3\right) \qquad (5.1.1)$$

where $C_1$ and $C_2$ are constants and the principal stretch $\lambda_i$ is the ratio of final to initial length in the direction of the i-strain axis, the condition of constant volume requiring $\lambda_1\lambda_2\lambda_3 = 1$. Mooney shows that calculated forces agree closely with experimental data on soft rubber from 400% elongation to 50% compression.

In (48) Rivlin also considers the development of a theory of large elastic deformations and deduces from a generalization, to the case of large strain, of Hooke's law that the strain-energy function may be written as

$$W = \frac{1}{6} E(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3) \qquad (5.1.2)$$

where $\lambda_1$, $\lambda_2$, $\lambda_3$ are the principal stretches and E is Young's modulus. This is therefore a special case of the Mooney material with the constant $C_2$ being zero. Rivlin defines this as an incompressible neo-Hookean material.

The strain invariants $I_1$, $I_2$ and $I_3$ are defined by

$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \quad , \quad I_2 = \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2 + \lambda_1^2\lambda_2^2 \quad ,$$

$$I_3 = \lambda_1^2\lambda_2^2\lambda_3^2 \quad ,$$

so that both (5.1.1) and (5.1.2) may be written in terms of the strain invariants. In (47) Rivlin reports on experiments on a rubber cylinder and concludes that the form (5.1.2) gives a first approximation to the strain-energy function and that a second approximation is provided by the form (5.1.1). The differences between the forms being more or less accentuated depending on the type of deformation studied. Mooney (31) points out that the strain-energy function can be expanded as an infinite series of $I_1$ and $I_2$. Thus

$$W = \sum_{m,n=0}^{\infty} C_{mn}(I_1-3)^m(I_2-3)^n , \qquad (5.1.3)$$

with $C_{oo} = 0$, the $C_{mn}$'s being constant. The neo-Hookean and Mooney forms are special cases of this.

Any mathematical analysis based on the general expansion (5.1.3) or on more complicated forms of the strain-energy density as a function of the strain invariants, constructed to give closer approximations to the experimental data, tends to be cumbersome, particularly in relation to problems in which the principal axes of strain vary through the material. Thus an adequate correlation between theory and experiment for a wide range of strains has only been achieved at the expense of mathematical simplicity. In (34) Ogden seeks a strain-energy function which provides an adequate representation of the mechanical response of rubber-like materials for large ranges of deformation, whilst being simple enough to be amenable to mathematical analysis. He defines the function $\phi(\alpha)$ by

$$\phi(\alpha) = \begin{cases} (\lambda_1^\alpha + \lambda_2^\alpha + \lambda_3^\alpha - 3)/\alpha & , \quad \alpha \neq 0 \\ \ell n(\lambda_1\lambda_2\lambda_3) & , \quad \alpha = 0 \end{cases}$$

which is symmetric in the $\lambda_i$'s, the three principal stretches. He then proposes a strain-energy function $W = \sum_r \mu_r\phi(\alpha_r)$, where the $\mu_r$'s are constants. He compares results obtained with experimental results on rubber. For an incompressible material $\lambda_1\lambda_2\lambda_3 = 1$ and in (35) Ogden extends this theory to that for a compressible rubber-like material by adding to the strain-energy a function of the density ratio $\rho_0/\rho \equiv \lambda_1\lambda_2\lambda_3$, which is zero when $\lambda_1\lambda_2\lambda_3 = 1$, that is when the material is incompressible. Levinson and Burges (26) have also considered the case of slightly compressible rubber-like materials as have Blatz and Ko (3).

In the following sections we examine the case of Rayleigh waves on a Mooney-Rivlin material.

## 5.2 THE FORM OF THE EQUATIONS

For the case of plane strain which is under consideration the strain invariants $I_1$, $I_2$ and $I_3$ reduce to:

$$I_1 = F_{11}^2 + F_{12}^2 + F_{21}^2 + F_{22}^2 + 1 \ ,$$

$$I_2 = I_3 + I_1 - 1$$

$$I_3 = (F_{11}F_{22} - F_{12}F_{21})^2 \ ,$$

so that we have only two independent invariants $I_1$ and $I_3$, say. For an incompressible material $I_3 = 1$ from which we deduce that $I_1 = I_2$. Also for an incompressible material the strain-energy function in the stress-strain relations is replaced formally by

$$W(I_1, I_2) - \tfrac{1}{2}p(I_3 - 1) \ ,$$

where p represents an arbitrary hydrostatic pressure and plays the role of a Lagrange multiplier, which is the reaction to the constraint of incompressibility. The example we consider is the Mooney-Rivlin material for which, in this situation of plane strain, we may replace the strain-energy function W by

$$C(I_1 - 3) - \tfrac{1}{2}p(I_3 - 1) \ ,$$

so that using the incompressibility condition

$$F_{11}F_{22} - F_{12}F_{21} = 1 \ ,$$

we obtain the stress components $\tau_{ij}$ as

$$\tau_{11} = 2CF_{11} - pF_{22} \ ,$$

$$\tau_{12} = 2CF_{12} + pF_{21} \ ,$$

$$\tau_{21} = 2CF_{21} + pF_{12} \quad ,$$

$$\tau_{22} = 2CF_{22} - pF_{11} \quad .$$

Using $F_{ij} = \delta_{ij} + \frac{\partial u_i}{\partial X_j}$ the stress components may then be rewritten as

$$\tau_{11} = 2C - p + 2C\, u_{1,1} - p\, u_{2,2} \quad ,$$

$$\tau_{12} = 2C\, u_{1,2} + p\, u_{2,1} \quad ,$$

$$\tau_{21} = 2C\, u_{2,1} + p\, u_{1,2} \quad ,$$

$$\tau_{22} = 2C - p + 2C\, u_{2,2} - p\, u_{1,1} \quad .$$

We let $\varepsilon$ denote a typical magnitude of $u_{i,j}$, $i,j = 1,2$. As for the compressible material we substitute for the stress components in the momentum equations and introduce the stress functions $\alpha_i(X,X_2)$, $i = 1,2$, where

$$\begin{aligned} 2C - p + (2C - \rho c^2)u_{1,1} - p\, u_{2,2} &= \alpha_{1,2} \\ 2C\, u_{1,2} + p\, u_{2,1} &= -\alpha_{1,1} \\ p\, u_{1,2} + (2C - \rho c^2)u_{2,1} &= \alpha_{2,2} \\ 2C - p - p\, u_{1,1} + 2C\, u_{2,2} &= -\alpha_{2,1} \end{aligned} \qquad (5.2.1)$$

These may be combined to give two pairs of equations similar to those obtained in the compressible case:

$$\frac{\partial}{\partial X}\{(p+2C-\rho c^2)u_1 - \alpha_2\} - \frac{\partial}{\partial X_2}\{(p+2C)u_2 + \alpha_1\} = u_1 \frac{\partial p}{\partial X} - u_2 \frac{\partial p}{\partial X_2} ,$$

$$\frac{\partial}{\partial X_2}\{(p+2C-\rho c^2)u_1 - \alpha_2\} + \left(1 - \frac{\rho c^2}{2C}\right) \frac{\partial}{\partial X}\{(p+2C)u_2 + \alpha_1\}$$

$$= u_1 \frac{\partial p}{\partial X_2} + \left(1 - \frac{\rho c^2}{2C}\right) u_2 \frac{\partial p}{\partial X} \quad ,$$

$$\frac{\partial}{\partial X}\{(p+2C-\rho c^2)u_2 + \alpha_1\} + \frac{\partial}{\partial X_2}\{(p+2C)u_1 - \alpha_2\} \tag{5.2.2}$$

$$= u_2 \frac{\partial p}{\partial X} + u_1 \frac{\partial p}{\partial X_2} \quad ,$$

$$\frac{\partial}{\partial X_2}\{(p+2C-\rho c^2)u_2 + \alpha_1\} - \frac{\partial}{\partial X}\{(p+2C)u_1 - \alpha_2\}$$

$$= u_2 \frac{\partial p}{\partial X_2} - u_1 \frac{\partial p}{\partial X} - \rho c^2(u_{2,2} + u_{1,1}) \quad .$$

As in Section 3.3, we presuppose that the length scale has been chosen so that typical magnitudes of $u_i$ and $u_{i,j}$ are comparable. The pressure gradients $\frac{\partial p}{\partial X_j}$ must then be $O(\varepsilon)$ to balance with the $\frac{\partial \tau_{ij}}{\partial X_j}$. Also from the condition $I_3 = 1$ it follows that $u_{1,1} + u_{2,2} = -(u_{1,1}u_{2,2} - u_{1,2}u_{2,1})$ so that all the terms on the right hand side of equation (5.2.2) are second order. Again without loss of generality the boundary conditions may be taken as $\alpha_1(X,0) = 0 = \alpha_2(X,0)$.

The functions $\phi_1$, $\phi_2$, $\psi_1$ and $\psi_2$ are introduced as before and are given by

$$\begin{aligned}
\rho c^2\phi_1 &= (p + 2C - \rho c^2)u_1 - \alpha_2 \\
\rho c^2\phi_2 &= (p + 2C)u_1 - \alpha_2 \\
\rho c^2\psi_1 &= A[(p+2C)u_2 + \alpha_1] \\
\rho c^2\psi_2 &= (p + 2C - \rho c^2)u_2 + \alpha_1
\end{aligned} \tag{5.2.3}$$

where $$A^2 = 1 - \frac{\rho c^2}{2C} \quad . \tag{5.2.4}$$

Equations (5.2.2) may then be rewritten:

$$\frac{\partial\phi_1}{\partial X} - \frac{1}{A}\frac{\partial\psi_1}{\partial X_2} = \frac{1}{\rho c^2}\{u_1\frac{\partial p}{\partial X} - u_2\frac{\partial p}{\partial X_2}\}$$

$$\frac{\partial\psi_1}{\partial X} + \frac{1}{A}\frac{\partial\phi_1}{\partial X_2} = \frac{1}{A\rho c^2}\{u_1\frac{\partial p}{\partial X_2} + A^2 u_2\frac{\partial p}{\partial X}\}$$

$$\frac{\partial\psi_2}{\partial X} + \frac{\partial\phi_2}{\partial X_2} = \frac{1}{\rho c^2}\{u_2\frac{\partial p}{\partial X} + u_1\frac{\partial p}{\partial X_2}\} \qquad (5.2.5)$$

$$\frac{\partial\phi_2}{\partial X} - \frac{\partial\psi_2}{\partial X_2} = -\frac{1}{\rho c^2}\{u_2\frac{\partial p}{\partial X_2} - u_1\frac{\partial p}{\partial X}\} - (u_{1,1}u_{2,2} - u_{1,2}u_{2,1})$$

with the compatibility condition

$$u_{1,1} + u_{2,2} = -(u_{1,1}u_{2,2} - u_{1,2}u_{2,1}) \,. \qquad (5.2.6)$$

The boundary conditions become

$$\alpha_1(X,0) = (p(X,0)+2C)\psi_2(X,0) + A^{-1}(p(X,0)+2C-\rho c^2)\psi_1(X,0)$$
$$= 0$$

and (5.2.7)

$$\alpha_2(X,0) = (p(X,0)+2CA^2)\phi_2(X,0) - (p(X,0)+2C)\phi_1(X,0)$$
$$= 0$$

As in the compressible case, we introduce new variables $y = AX_2$, $Y = X_2$ and define

$$u(X,y) = \phi_1(X,X_2) \quad , \quad v(X,y) = \psi_1(X,X_2) \quad ,$$

$$U(X,Y) = \frac{(1+A^2)}{2}\phi_2(X,X_2) \quad , \quad V(X,Y) = \frac{(1+A^2)}{2}\psi_2(X,X_2) \, ,$$

where the multiplier $(1+A^2)/2$ has been chosen so that the boundary condition $\alpha_2(X,0) = 0$ implies $u(X,0) = U(X,0)$ to zero order. Equations (5.2.5) then become

$$\frac{\partial u}{\partial X} - \frac{\partial v}{\partial y} = \frac{2C\varepsilon}{\rho c^2}\left\{\left(\frac{2}{1+A^2}U - u\right)\frac{\partial\bar{p}}{\partial X} - \left(A^{-1}v - \frac{2}{1+A^2}V\right)\frac{\partial\bar{p}}{\partial X_2}\right\}$$

$$\frac{\partial v}{\partial X} + \frac{\partial u}{\partial y} = \frac{2C\varepsilon}{\rho c^2 A}\left\{\left(\frac{2}{1+A^2}U - u\right)\frac{\partial\bar{p}}{\partial X_2} + A^2\left(A^{-1}v - \frac{2}{1+A^2}V\right)\frac{\partial\bar{p}}{\partial X}\right\}$$

$$\frac{\partial V}{\partial X} + \frac{\partial U}{\partial Y} = \frac{2C(1+A^2)\varepsilon}{2\rho c^2}\left\{\left(A^{-1}v - \frac{2}{1+A^2}V\right)\frac{\partial\bar{p}}{\partial X} + \left(\frac{2}{1+A^2}U - u\right)\frac{\partial\bar{p}}{\partial X_2}\right\} \tag{5.2.8}$$

$$\frac{\partial U}{\partial X} - \frac{\partial V}{\partial Y} = \frac{2C(1+A^2)\varepsilon}{2\rho c^2}\left\{\left(\frac{2}{1+A^2}U - u\right)\frac{\partial\bar{p}}{\partial X} - \left(A^{-1}v - \frac{2}{1+A^2}V\right)\frac{\partial\bar{p}}{\partial X_2}\right\}$$

$$- \frac{(1+A^2)}{2}\left\{\left[\frac{2}{1+A^2}\frac{\partial U}{\partial X} - \frac{\partial u}{\partial X}\right]\left[\frac{\partial v}{\partial y} - \frac{2}{1+A^2}\frac{\partial V}{\partial Y}\right]\right.$$

$$\left. - \left[\frac{2}{1+A^2}\frac{\partial U}{\partial Y} - A\frac{\partial u}{\partial y}\right]\left[\frac{1}{A}\frac{\partial v}{\partial X} - \frac{2}{1+A^2}\frac{\partial V}{\partial X}\right]\right\}$$

with boundary conditions

$$(2+\varepsilon\bar{p}(X,0))\frac{2}{1+A^2}V(X,0) - \frac{1}{A}(1+A^2+\varepsilon\bar{p}(X,0))v(X,0) = 0$$

$$(1+A^2+\varepsilon\bar{p}(X,0))\frac{2}{1+A^2}U(X,0) - (2+\varepsilon\bar{p}(X,0))u(X,0) = 0 \quad . \tag{5.2.9}$$

The functions u, v, U and V are all $O(\varepsilon)$, so that to first order (5.2.8) and (5.2.9) give

$$\frac{\partial u}{\partial X} - \frac{\partial v}{\partial y} = 0 \quad , \quad \frac{\partial v}{\partial X} + \frac{\partial u}{\partial y} = 0 \quad ,$$

$$\frac{\partial U}{\partial X} - \frac{\partial V}{\partial Y} = 0 \quad , \quad \frac{\partial V}{\partial X} + \frac{\partial U}{\partial Y} = 0 \quad ,$$

with boundary conditions

$$U(X,0) = u(X,0)$$

$$V(X,0) = Kv(X,0) \quad ,$$

where $$K = \frac{(1+A^2)^2}{4A} \quad . \tag{5.2.10}$$

If $\gamma(X,\eta)$ is a harmonic function in $\eta > 0$ taking boundary values $\gamma(X,0) = \sigma(X)$, with harmonic conjugate $\beta(X,\eta)$, then a solution to the above system is

$$u(X,\eta) = U(X,\eta) = \beta(X,\eta)$$

$$v(X,\eta) = V(X,\eta) = \gamma(X,\eta)$$

with $\frac{(1+A^2)^2}{4A} = 1$ and boundary conditions $\gamma(X,0) = \sigma(X)$.

The displacements $u_1$ and $u_2$ are given by

$$u_1(X,X_2) = \frac{2}{1+A^2} U(X,Y) - u(X,y) = \varepsilon\{\frac{2}{1+A^2}\beta(X,Y) - \beta(X,y)\}$$

$$u_2(X,X_2) = \frac{1}{A} v(X,y) - \frac{2}{1+A^2} V(X,Y) = \varepsilon\{\frac{1}{A}\gamma(X,y) - \frac{2}{1+A^2}\gamma(X,Y)\}$$

Choosing $\sigma(X) = \cos X$ we find that

$$\beta(X,\eta) = -e^{-\eta}\sin X, \qquad \gamma(X,\eta) = e^{-\eta}\cos X$$

which gives

$$u_1 = \varepsilon\sin X\{e^{-y} - \frac{2}{1+A^2} e^{-Y}\}$$

$$u_2 = \varepsilon\cos X\{\frac{1}{A} e^{-y} - \frac{2}{1+A^2} e^{-Y}\}$$

from which we obtain

$$u_2(X,0) = \frac{\varepsilon\rho c^2}{4CA}\cos X \quad .$$

In this linear case the compatibility condition becomes

$$u_{1,1} + u_{2,2} = 0$$

and it is readily checked that this is satisfied for harmonic solutions $\beta(X,\eta)$ and $\gamma(X,\eta)$.

The condition $(1+A^2)^2/4A = 1$ gives on substituting for A an equation satisfied by $c^2$ namely

$$(\rho c^2)^3 - 8.2C(\rho c^2)^2 + 24(2C)^2\rho c^2 - 16(2C)^3 = 0 \qquad (5.2.11)$$

which is equation (3.1.20) when the Poisson ratio is $\frac{1}{2}$ and $\mu = 2C$, again this equation has only one real positive root for $c^2$. Linear Rayleigh waves for an incompressible material are regular limits of waves for a compressible material.

We now write

$$\begin{aligned} u(X,\eta) &= \varepsilon\beta(X,\eta) + \varepsilon^2\bar{u}(X,\eta;\varepsilon) \ , \\ v(X,\eta) &= \varepsilon\gamma(X,\eta) + \varepsilon^2\bar{v}(X,\eta;\varepsilon) \ , \\ U(X,\eta) &= \varepsilon\beta(X,\eta) + \varepsilon^2\bar{U}(X,\eta;\varepsilon) \ , \\ V(X,\eta) &= \varepsilon\gamma(X,\eta) + \varepsilon^2\bar{V}(X,\eta;\varepsilon) \ , \end{aligned} \qquad (5.2.12)$$

where

$$\bar{u} = \overset{1}{u} + \varepsilon\overset{2}{u} + \ldots, \quad \bar{v} = \overset{1}{v} + \varepsilon\overset{2}{v} + \ldots, \quad \bar{U} = \overset{1}{U} + \varepsilon\overset{2}{U} + \ldots,$$

$$\bar{V} = \overset{1}{V} + \varepsilon\overset{2}{V} + \ldots, \quad \bar{p} = \overset{1}{p} + \varepsilon\overset{2}{p} + \ldots \qquad (5.2.13)$$

and also write $K = \dfrac{(1+A^2)^2}{4A} = 1 + \kappa\varepsilon$.

To solve (5.2.8) and (5.2.9) we need to find an expression for $\bar{p}$, this may be obtained from (5.2.1):

$$\varepsilon\bar{p} = \frac{2}{1+A^2}(A^2U_X - V_Y) + A^2(v_y - u_x) + \varepsilon^2\bar{p}_Y\left[\frac{V}{A} - \frac{2V}{1+A^2}\right] \qquad (5.2.14)$$

To $O(\varepsilon)$ this gives

$$\overset{1}{p} = -\frac{(1-A^4)}{2A}\frac{\partial\beta}{\partial X}$$

which may be substituted into (5.2.8) and (5.2.9). Choosing $\gamma(X,\eta) = e^{-\eta}\cos X$ implies $\beta(X,\eta) = -e^{-\eta}\sin X$ and taking the $O(\varepsilon^2)$ terms in (5.2.8) and (5.2.9) gives equations for $\overset{1}{u}$, $\overset{1}{v}$, $\overset{1}{U}$ and $\overset{1}{V}$. However, the solution to these equations satisfying the surface boundary conditions introduces terms in $e^{2Y}\cos 2X$ and $e^{2Y}\sin 2X$, which grow as $Y \to \infty$. We therefore cannot find an acceptable solution performing a straightforward expansion with $\gamma = e^{-\eta}\cos X$.

In the next section we consider perturbation procedures analogous to those in Chapter Three, for solving the problem with a general surface elevation. We consider a periodic waveform in Section 5.3.

## 5.3 SOLUTIONS FOR THE PERIODIC WAVEFORM

Substituting the expansions (5.2.13) into the equations (5.2.8) and the boundary conditions (5.2.9) gives equations of the form

$$\begin{aligned} \bar{u}_X - \bar{v}_y &= \bar{\ell}(X,y;\varepsilon) \qquad \bar{v}_X + \bar{u}_y = \bar{m}(X,y;\varepsilon) \\ \bar{U}_X - \bar{V}_Y &= \bar{L}(X,Y;\varepsilon) \qquad \bar{V}_X + \bar{U}_Y = \bar{M}(X,Y;\varepsilon) \end{aligned} \tag{5.3.1}$$

$$\begin{aligned} \bar{U}(X,0) &= \bar{u}(X,0) + r_1(X;\varepsilon) \\ \bar{V}(X,0) &= \bar{v}(X,0) + \kappa[\gamma(X,0) + \varepsilon\bar{v}(X,0)] + r_2(X;\varepsilon). \end{aligned} \tag{5.3.2}$$

As in Section 3.3 we introduce functions $\bar{w}(X,\eta;\varepsilon)$, $\bar{\nu}(X,\eta;\varepsilon)$ where

$$\bar{U}(X,\eta;\varepsilon) = \bar{u}(X,\eta;\varepsilon) + \bar{w}(X,\eta;\varepsilon)$$

and

$$\bar{V}(X,\eta;\varepsilon) = \bar{v}(X,\eta;\varepsilon) + \bar{\nu}(X,\eta;\varepsilon) .$$

From (5.3.1) and (5.3.2) it can be deduced that the equations to be satisfied by $\bar{w}$ and $\bar{\nu}$ are

$$\begin{aligned} \bar{w}_X(X,\eta;\varepsilon) - \bar{\nu}_\eta(X,\eta;\varepsilon) &= \bar{L}(X,\eta;\varepsilon) - \bar{\ell}(X,\eta;\varepsilon) \\ \bar{\nu}_X(X,\eta;\varepsilon) + \bar{w}_\eta(X,\eta;\varepsilon) &= \bar{M}(X,\eta;\varepsilon) - \bar{m}(X,\eta;\varepsilon) \end{aligned} \tag{5.3.3}$$

with

$$\begin{aligned} \bar{w}(X,0;\varepsilon) &= r_1(X;\varepsilon) \\ \bar{\nu}(X,0;\varepsilon) &= \kappa[\gamma(X,0) + \varepsilon\bar{v}(X,0;\varepsilon)] + r_2(X;\varepsilon) \end{aligned} \tag{5.3.4}$$

and

$$\left.\begin{aligned} \bar{w}(X,\eta;\varepsilon) &\to (A^2-1)\bar{\mu}_1 \\ \bar{\nu}(X,\eta;\varepsilon) &\to (1+A^2-2A)\bar{\mu}_2 \end{aligned}\right\} \quad \text{as } \eta \to \infty$$

We are considering solutions which are $2\pi$-periodic in X, we therefore consider the interval $-\pi \leq X \leq \pi$. The function $\bar{\nu}$ may be eliminated from the above equations to give

$$\bar{w}_{XX} + \bar{w}_{\eta\eta} = \bar{L}_X - \bar{\ell}_X + \bar{M}_\eta - \bar{m}_\eta$$

$$\bar{w}(X,0) = r_1(X) \tag{5.3.5}$$

$$\bar{M}(X,0) - \bar{m}(X,0) - \bar{w}_\eta(X,0) = \kappa\left[\gamma_X(X,0) + \varepsilon\bar{v}_X(X,0)\right] + r_{2,X}(X)$$

$$\bar{w}(X,\eta) \to (A^2-1)\bar{\mu}_1 \quad \text{as } \eta \to \infty .$$

Clearly there cannot be a solution to this problem for all harmonic conjugate pairs $\beta(X,\eta)$, $\gamma(X,\eta)$ and any value of $\kappa$. We therefore investigate what conditions $\beta$, $\gamma$ and $\kappa$ must satisfy for a solution of (5.3.5) to exist.

Again, we are considering solutions periodic in X, for simplicity we suppose that $\gamma(X,\eta)$ is an even function of X and choose

$$\gamma(X,\eta) = \sum_{n=1}^{\infty} C_n e^{-n\eta} \cos nX, \quad \beta(X,\eta) = -\sum_{n=1}^{\infty} C_n e^{-n\eta} \sin nX.$$

The vertical displacement at the surface is given to first order by

$$u_2(X,0) = \frac{\varepsilon(1-A)^2}{A(1+A^2)} \gamma(X,0).$$

Equations (5.3.5) may be solved by writing

$$\bar{w}(X,\eta) = \sum_{n=1}^{\infty} W_n(\eta)\sin nX + W_0(\eta) ,$$

which gives

$$W_0''(\eta) + \sum_{n=1}^{\infty}\left\{\left[W_n''(\eta) - n^2 W_n(\eta)\right]\sin nX\right\} = \bar{L}_X - \bar{\ell}_X + \bar{M}_\eta - \bar{m}_\eta \tag{5.3.6}$$

$$W_0(0) + \sum_{n=1}^{\infty} W_n(0)\sin nX = r_1(X) \tag{5.3.7}$$

$$W_0(\eta) + \sum_{n=1}^{\infty} W_n(\eta)\sin nX \to (A^2-1)\bar{\mu}_1 \quad \text{as } \eta \to \infty \tag{5.3.8}$$

$$\overline{M}(X,0) - \bar{m}(X,0) - W_0'(0) - \sum_{n=1}^{\infty} W_n'(0)\sin nX$$

$$= \kappa\left[\gamma_X(X,0) + \varepsilon\bar{v}_X(X,0)\right] + r_{2,X}(X) . \tag{5.3.9}$$

We solve this to zero order. Solving (5.3.6) to (5.3.8) for $W_0(\eta)$ we find

$$W_0(\eta) = \frac{(1-A^2)^2}{4A} \sum_{s=1}^{\infty} \frac{sC_s^2}{2} ,$$

where

$$\bar{\mu}_1 = -\frac{(1-A^2)}{8A} \sum_{s=1}^{\infty} sC_s^2 .$$

This also satisfies (5.3.9). We solve (5.3.6) to (5.3.8) for $W_n(\eta)$ and equating the coefficients of sin nX in (5.3.9) gives a condition on the $C_n$'s of the form (4.2.7). In fact in this problem it is simpler to use the method described in Section 3.5 to obtain this condition on the coefficients $C_n$, the only difference from the theory of that section being in that the boundary conditions have the extra terms $r_1(X)$, $r_2(X)$.

As in Section 3.5 we choose two functions $\sigma(X,\eta)$, $\rho(X,\eta)$ which are $2\pi$-periodic in X, bounded in $\eta > 0$ and satisfy $\sigma_X + \rho_\eta = 0$, $\sigma_\eta - \rho_X = 0$. Then

$$\int_0^{\infty}\int_{-\pi}^{\pi} (\sigma\bar{\ell}+\rho\bar{m})\,dXdy = \int_{-\pi}^{\pi} (\sigma\bar{v}-\rho\bar{u})\Big|_{y=0} dX$$

and

$$\int_0^{\infty}\int_{-\pi}^{\pi} (\sigma\overline{L}+\rho\overline{M})\,dXdY = \int_{-\pi}^{\pi} (\sigma\overline{V}-\rho\overline{U})\Big|_{Y=0} dX$$

$$= \int_{-\pi}^{\pi} (\sigma\bar{v}+\sigma\bar{r}_2+\sigma\kappa(\gamma+\varepsilon\bar{v}) - \rho\bar{u} - \rho\bar{r}_1)\Big|_{y=0} dX .$$

Therefore

$$\int_0^\infty\int_{-\pi}^{\pi}(\sigma\overline{L}+\rho\overline{M})\,dXdY = \int_0^\infty\int_{-\pi}^{\pi}(\sigma\bar{\ell}+\rho\bar{m})\,dXdy$$

$$+\int_{-\pi}^{\pi}(\sigma\bar{r}_2-\rho\bar{r}_1+\kappa\sigma(\gamma+\varepsilon\bar{v}))\Big|_{y=0}\,dX. \qquad (5.3.10)$$

We choose a basis for $(\sigma,\rho)$ as

$$\sigma = e^{-n\eta}\sin nX = \sigma_n^-, \quad \rho = e^{-n\eta}\cos nX = \rho_n^-$$

and , $n = 0,1,2,\ldots$

$$\sigma = e^{-n\eta}\cos nX = \sigma_n^+, \quad \rho = -e^{-n\eta}\sin nX = \rho_n^+ .$$

Equation (5.3.10) is solved to zero order with the above bases and reduces to two equations which must be satisfied

$$\int_0^\infty\int_{-\pi}^{\pi} e^{-ny}(\ell^{(o)}\sin nX + m^{(o)}\cos nX)\,dXdy$$

$$= \int_0^\infty\int_{-\pi}^{\pi} e^{-nY}(L^{(o)}\sin nX + M^{(o)}\cos nX)\,dXdY$$

$$- \int_{-\pi}^{\pi} (r_2^{(o)}\sin nX - r_1^{(o)}\cos nX + \kappa\sin nX\,\gamma(X,0))\,dX,$$

$$n = 0,1,2,\ldots \qquad (5.3.11)$$

and

$$\int_0^\infty\int_{-\pi}^{\pi} e^{-ny}(\ell^{(o)}\cos nX - m^{(o)}\sin nX)\,dXdy$$

$$= \int_0^\infty\int_{-\pi}^{\pi} e^{-nY}(L^{(o)}\cos nX - M^{(o)}\sin nX)\,dXdY$$

$$- \int_{-\pi}^{\pi}(r_2^{(o)}\cos nX + r_1^{(o)}\sin nX + \kappa\cos nX\,\gamma(X,0))\,dX,$$

$$n = 0,1,2,\ldots \qquad (5.3.12)$$

For the example under consideration with $\gamma = \sum_{n=1}^{\infty} C_n e^{-n\eta} \cos nX$, (5.3.11) is trivially satisfied and (5.3.12) reduces to the same condition as that obtained from (5.3.9). The equation to be solved for the $C_n$'s is therefore

$$-2\kappa C_n + \sum_{r=1}^{n-1} \alpha_{rn} r(n-r) C_r C_{n-r} + \sum_{s=1}^{\infty} \beta_{sn} s(n+s) C_s C_{n+s} = 0 \qquad (5.3.13)$$

where as in the compressible case, for fixed n, $\beta_{sn} \to 0$ as $s \to \infty$, and

$$\alpha_{rn} = -\frac{(1-A^4)(1-A)^2}{8A^2}\frac{n}{n-r} - \frac{(1-A^2)^2}{4A}\frac{n}{r} + \frac{2(1+A)nr}{A(n-r)(n+r+A(n-r))}$$

$$+ \frac{2(1-A^2)n}{(1+A^2)(2n-r+rA)} + \frac{r}{n-r}(1-A) - \frac{rn(1+A)(1+A^2)}{A(n-r)(r+(2n-r)A)} ,$$

$$\beta_{sn} = \frac{(1-A^4)(1-A)^2}{8A^2}\frac{n^2}{s(n+s)} - \frac{(1-A^2)^2}{4A}\frac{(n+2s)n}{s(n+s)}$$

$$+ \frac{(1-A)}{A}\frac{(n+s)n}{s(2n+s+As)} - \frac{2(3+A^2)n}{1+A^2}\left(\frac{1}{(n+s)(1+A)} + \frac{1}{2n+s+sA}\right)$$

$$+ \frac{8A}{1+A^2}\left(\frac{n}{(n+2s)A+n}\right) + \frac{4n}{(1+A^2)(n+s)} + \frac{(1-A)(n+s)n}{As(2n+s+As)}$$

$$- \frac{(1+A^2)(1-A)}{A} n\left[\frac{1}{s(1+A)} + \frac{s}{(n+s)(s+(2n+s)A)}\right]$$

$$- \frac{(1-A^2)}{A}\frac{n(n+2s)}{s(An+n+2s)} + \frac{(1-A^2)}{A}\frac{sn}{(n+s)(n+2s+An)}$$

$$+ \frac{(1+A^2)(1-A)n}{2As(1+A)} - \frac{(1+A^2)(1-A)ns}{2A(s+n)(s+(2n+s)A)} .$$

The coefficients $\alpha_{rn}$, $\beta_{sn}$ depend on A, which is calculated from the condition K = 1, namely $(1+A^2)^2/4A = 1$, which has a solution A = 0.295598. Equation (5.3.13) may then be solved by the same numerical procedure as was used for the harmonic material.

As in the case of the harmonic material we can find the difference between the speed of a non-linear wave and the standard Rayleigh wave as a function of the amplitude. From equation (5.3.13) $K = \frac{(1+A^2)^2}{4A} = 1+\kappa\varepsilon$ with $A^2 = 1 - \frac{\rho c^2}{2C}$ and again writing $c = c_R + \varepsilon\bar{c}$ where $c_R$ is the standard Rayleigh wave speed, we find that

$$\varepsilon\kappa = \left(\frac{dK}{dc^2}\, 2c\right)_{c=c_R} \varepsilon\bar{c}$$

and hence

$$\varepsilon\kappa = \frac{\varepsilon\bar{c}}{c_R}\, \frac{(1-A_R^2)}{A_R}\left(1 + A_R^2 - \frac{1}{A_R}\right)$$

which implies $\frac{\bar{c}}{c_R} = 0.1411\ \kappa$.

The horizontal and vertical displacements $u_1(X,X_2)$, $u_2(X,X_2)$ are given by

$$u_1(X,X_2) = \varepsilon\left[\frac{2}{1+A^2}\,\beta(X,Y) - \beta(X,y)\right] + O(\varepsilon^2)$$

$$u_2(X,X_2) = \varepsilon\left[\frac{1}{A}\,\gamma(X,y) - \frac{2}{1+A^2}\,\gamma(X,Y)\right] + O(\varepsilon^2)$$

so that to first order $u_1$ and $u_2$ are given by

$$u_1(X,X_2) = -\varepsilon\left[\frac{2}{1+A^2}\sum_{n=1}^{\infty} C_n e^{-nX_2}\sin nX - \sum_{n=1}^{\infty} C_n e^{-nAX_2}\sin nX\right],$$

$$u_2(X,X_2) = \varepsilon\left[\frac{1}{A}\sum_{n=1}^{\infty} C_n e^{-nAX_2}\cos nX - \frac{2}{1+A^2}\sum_{n=1}^{\infty} C_n e^{-nX_2}\cos nX\right]. \tag{5.3.14}$$

As in Section 4.2, we consider the cases for $\kappa \neq 0$ and $\kappa = 0$.

1. For $\kappa \neq 0$ we again use the transformation $nC_n = \kappa P_n$ to obtain the equations

$$-2P_n + \sum_{r=1}^{n-1} \alpha_{rn} P_r P_{n-r} + \sum_{s=1}^{\infty} \beta_{sn} P_s P_{n+s} = 0 \quad , \quad n = 1,2,\ldots\ .$$

Proceeding as before, iterating in steps of five from the analytical solution for N = 3 we obtain two solutions.

Solution 1. The first 23 values of the $P_n$'s and corresponding $C_n/\kappa$'s are shown in Table 5.1 for N = 33. At the 6th iteration, the last one performed, $P_n$ changes by less than $2.10^{-3}$ and for $n > 23$, $|C_n/\kappa| < 2.10^{-2}$.

Solution 2. For n = 48, the first 23 values for the $P_n$'s and $C_n/\kappa$'s are shown in Table 5.2. At the 9th iteration, $P_1$ changes by less than $10^{-3}$ and for $n > 23$, $|C_n/\kappa| < 5.10^{-2}$.

For solution 1 we see that $|C_1/C_{23}| \simeq 87$, whereas for solution 2 $|C_1/C_{23}| \simeq 33$, so that as in the solutions obtained for the harmonic material the decay of the oscillations in the $C_n$'s is faster for solution 1.

In figures 5.1 and 5.4 the horizontal and vertical displacements at the surface for the two solutions are plotted, again the motion is retrograde as is expected but the displacements differ considerably from those for the standard Rayleigh wave, which are sine and cosine functions. The vertical displacements at various depths are shown in figure 5.5. The particle paths at various depths are compared for the non-linear waves and the standard wave in figures 5.2, 5.3 and 5.6-5.9.

2. We have also obtained solutions for which there is no change in the wave speed that is $\kappa = 0$. We then make the substitution $nC_n = Q_n$ and obtain the equations

$$\sum_{r=1}^{n-1} \alpha_{rn} Q_r Q_{n-r} + \sum_{s=1}^{\infty} \beta_{sn} Q_s Q_{n+s} = 0 \quad , \quad n = 1,2,\dots \quad .$$

This system of equations is solved in the same way as for the harmonic compressible material. Again any multiple of a solution is also a solution and the $Q_n$'s are rescaled after each iteration to make $\sum_{n=1}^{N} Q_n^2 = 1$. The $Q_n$'s were found to converge and the solution obtained when $N = 31$ is given in Table 5.3 for the first 20 $Q_n/\tau$ and $C_n/\tau$, $\tau$ being an arbitrary parameter. For even n $Q_n$ and hence $C_n$ are zero. At the 13th iteration $Q_1/\tau$ changed by less than $3.10^{-3}$ and for $n > 20$, $|C_n/\tau| < 5.10^{-3}$.

The displacements at the surface are shown as are the particle paths in figures 5.10 - 5.11, where we see that the sense of rotation changes as the depth increases and the disturbance then dies away. In this case where there is no change in the wave speed we can scale both the wavelength and the amplitude independently, as there is an arbitrary scaling parameter $\tau$ in the $C_n$'s.

Table 5.1 Solution 1 for $\kappa \neq 0$

| $n$ | $P_n$ | $C_n/\kappa$ |
|---|---|---|
| 1 | 1.0148 | 1.0148 |
| 2 | -1.2198 | -0.6099 |
| 3 | 0.9512 | 0.3171 |
| 4 | -0.4709 | -0.1177 |
| 5 | -0.0304 | -0.0061 |
| 6 | 0.4050 | 0.0675 |
| 7 | -0.5665 | -0.0809 |
| 8 | 0.5031 | 0.0639 |
| 9 | -0.2724 | -0.0303 |
| 10 | -0.0235 | -0.0023 |
| 11 | 0.2744 | 0.0249 |
| 12 | -0.3980 | -0.0360 |
| 13 | 0.3644 | 0.0257 |
| 14 | -0.2008 | -0.0143 |
| 15 | -0.0226 | -0.0015 |
| 16 | 0.2205 | 0.0138 |
| 17 | -0.3228 | -0.0193 |
| 18 | 0.2998 | 0.0166 |
| 19 | -0.1648 | -0.0087 |
| 20 | -0.0243 | -0.0012 |
| 21 | 0.1963 | 0.0096 |
| 22 | -0.2883 | -0.0131 |
| 23 | 0.2689 | 0.0117 |

Table 5.2 Solution 2 for $\kappa \neq 0$

| n | $P_n$ | $C_n/\kappa$ |
|---|---|---|
| 1 | 1.8180 | 1.8180 |
| 2 | 4.8745 | 2.4372 |
| 3 | -4.2058 | -1.4019 |
| 4 | -0.6153 | -0.1538 |
| 5 | 0.0314 | 0.0063 |
| 6 | 0.6379 | 0.1063 |
| 7 | 2.2932 | 0.3276 |
| 8 | -2.3504 | -0.2938 |
| 9 | -0.2944 | -0.0327 |
| 10 | 0.0699 | 0.0070 |
| 11 | 0.4646 | 0.0422 |
| 12 | 1.5247 | 0.1271 |
| 13 | -1.7541 | -0.1349 |
| 14 | -0.1698 | -0.0121 |
| 15 | 0.0892 | 0.0059 |
| 16 | 0.3921 | 0.0245 |
| 17 | 1.1502 | 0.0677 |
| 18 | -1.4594 | -0.0811 |
| 19 | -0.0954 | -0.0050 |
| 20 | 0.1022 | 0.0051 |
| 21 | 0.3521 | 0.0168 |
| 22 | 0.9241 | 0.0420 |
| 23 | -1.2871 | -0.0560 |

Table 5.3 Solution for $\kappa = 0$

| $n$ | $Q_n/\tau$ | $C_n/\tau$ |
|---|---|---|
| 1 | 0.6614 | 0.6614 |
| 3 | -0.3459 | -0.1153 |
| 5 | 0.2453 | 0.0491 |
| 7 | -0.1956 | -0.0279 |
| 9 | 0.1655 | 0.0184 |
| 11 | -0.1451 | -0.0132 |
| 13 | 0.1305 | 0.0100 |
| 15 | -0.1197 | -0.0080 |
| 17 | 0.1116 | 0.0066 |
| 19 | -0.1059 | -0.0056 |

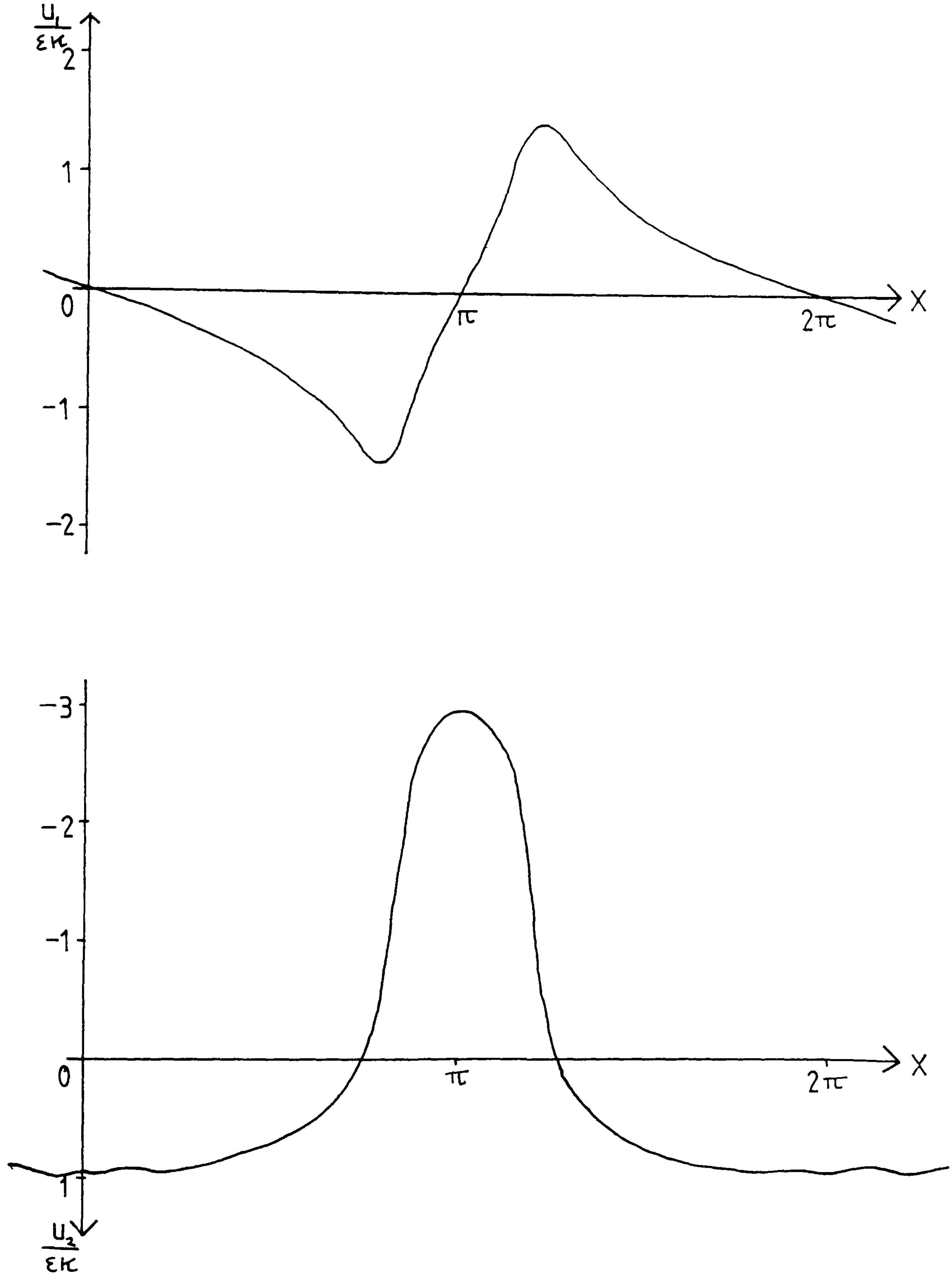



Figure 5.1 Horizontal and vertical displacements at the surface for solution 1

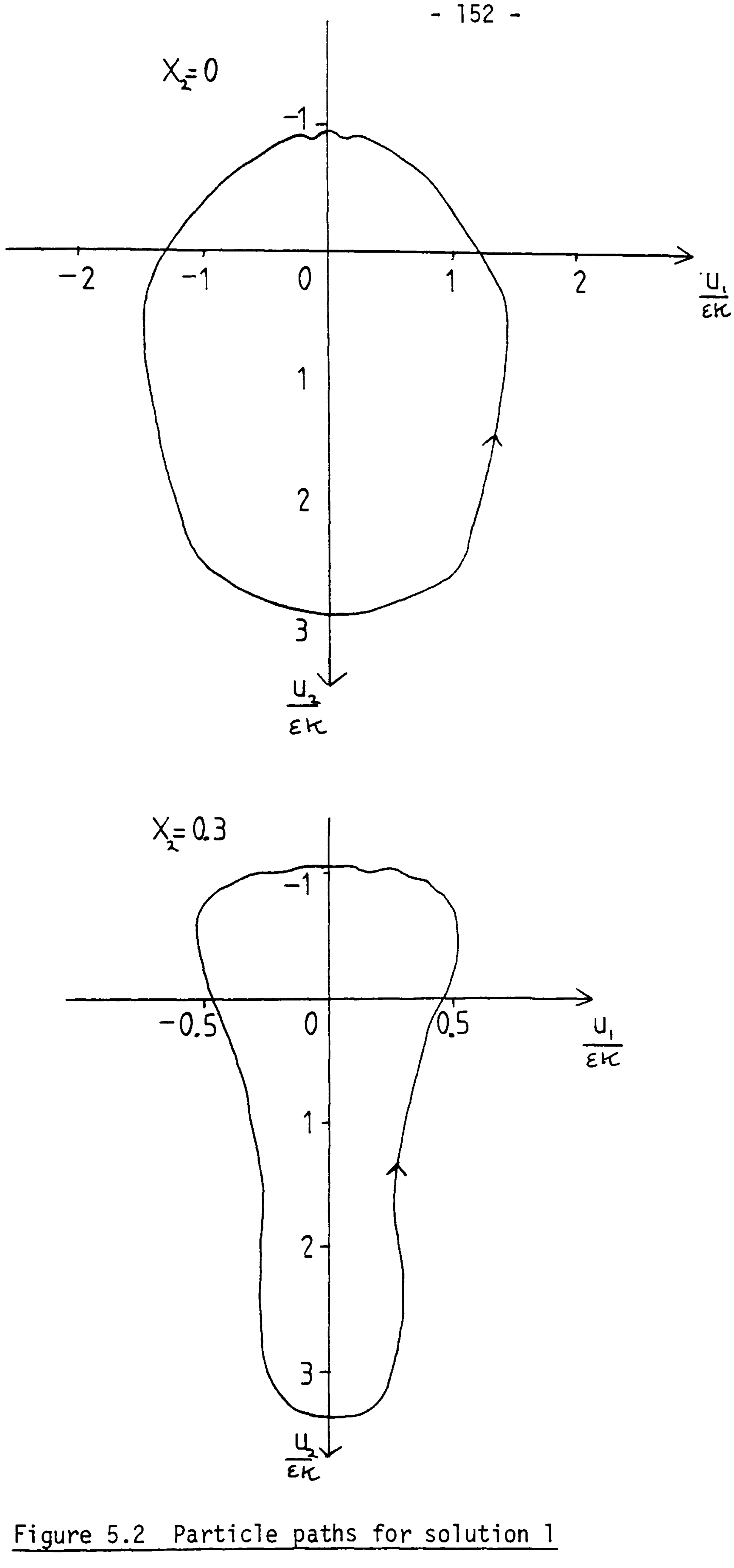



Figure 5.2 Particle paths for solution 1

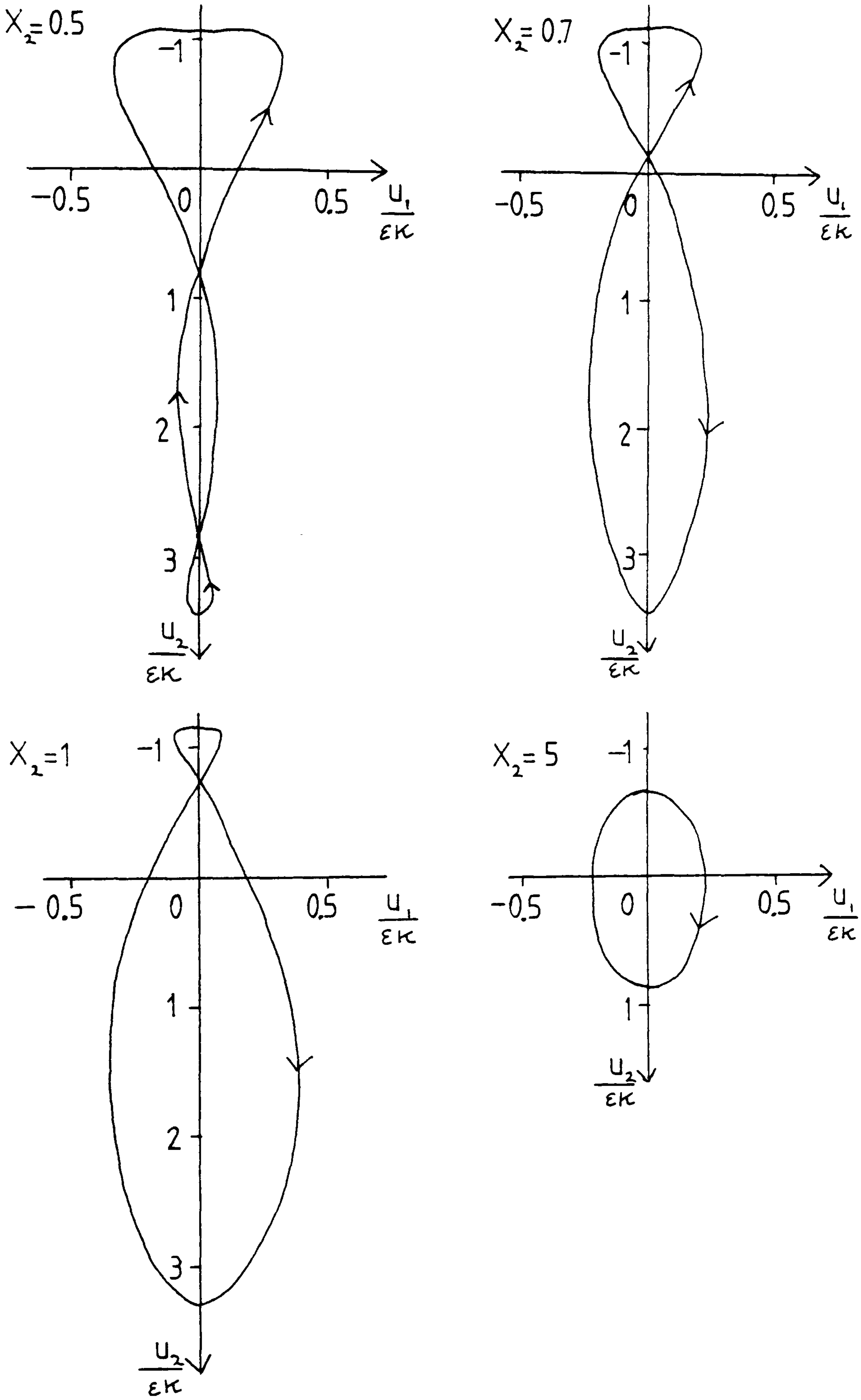



Figure 5.3 Particle paths for solution 1

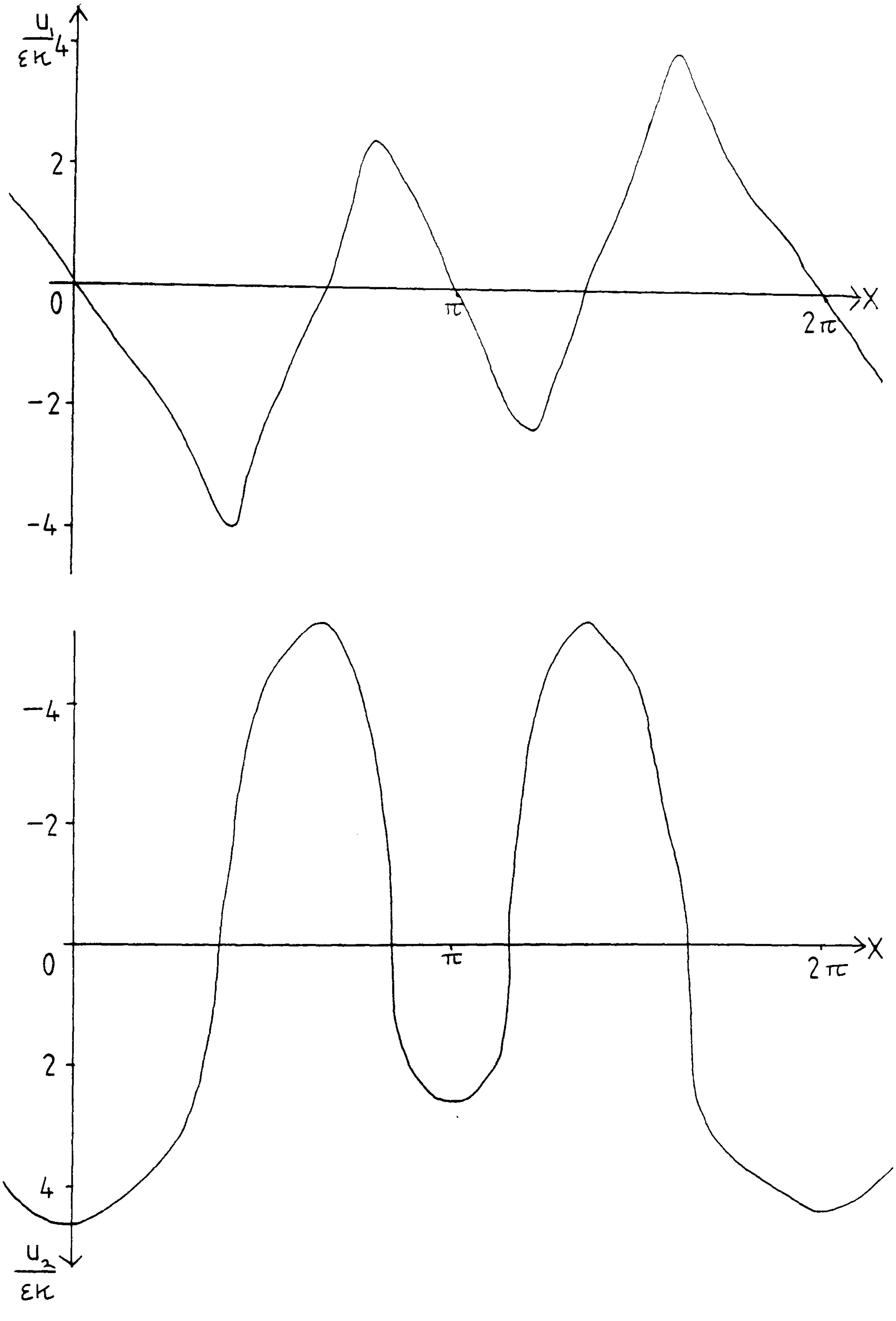



Figure 5.4 Horizontal and vertical displacements at the surface for solution 2

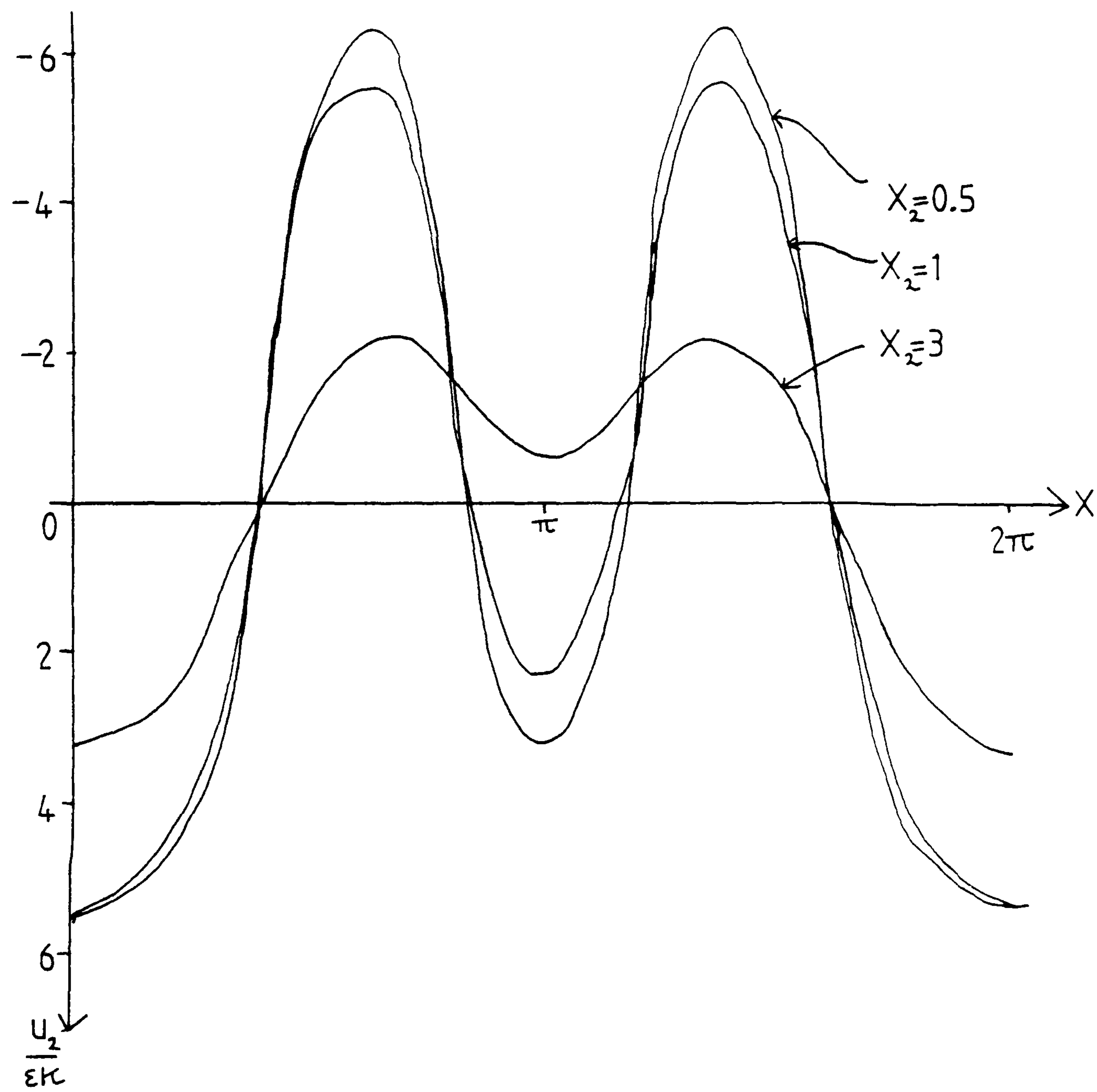



Figure 5.5 Vertical displacements at various depths for solution 2

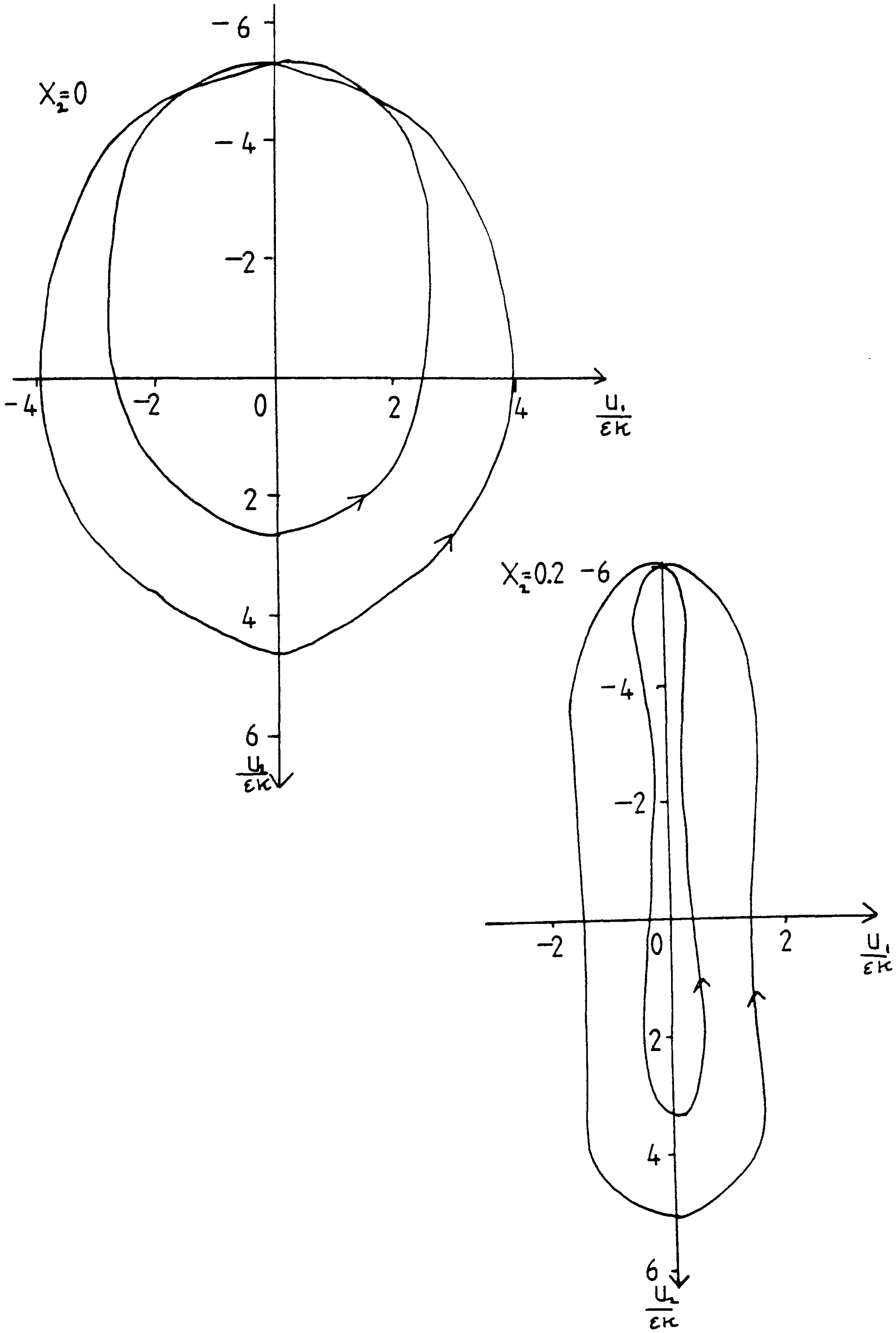



Figure 5.6 Particle paths for solution 2

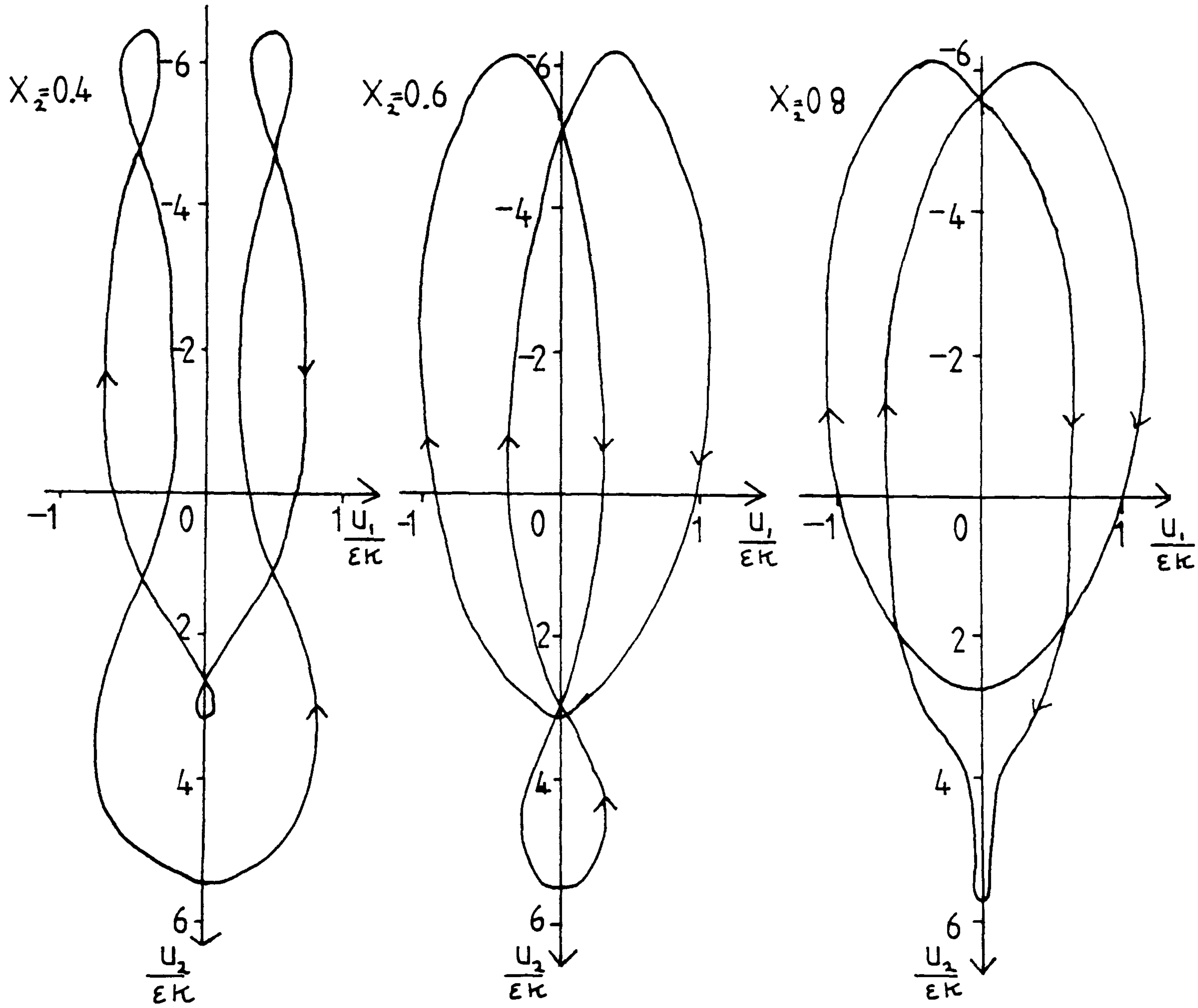



Figure 5.7 Particle paths for solution 2

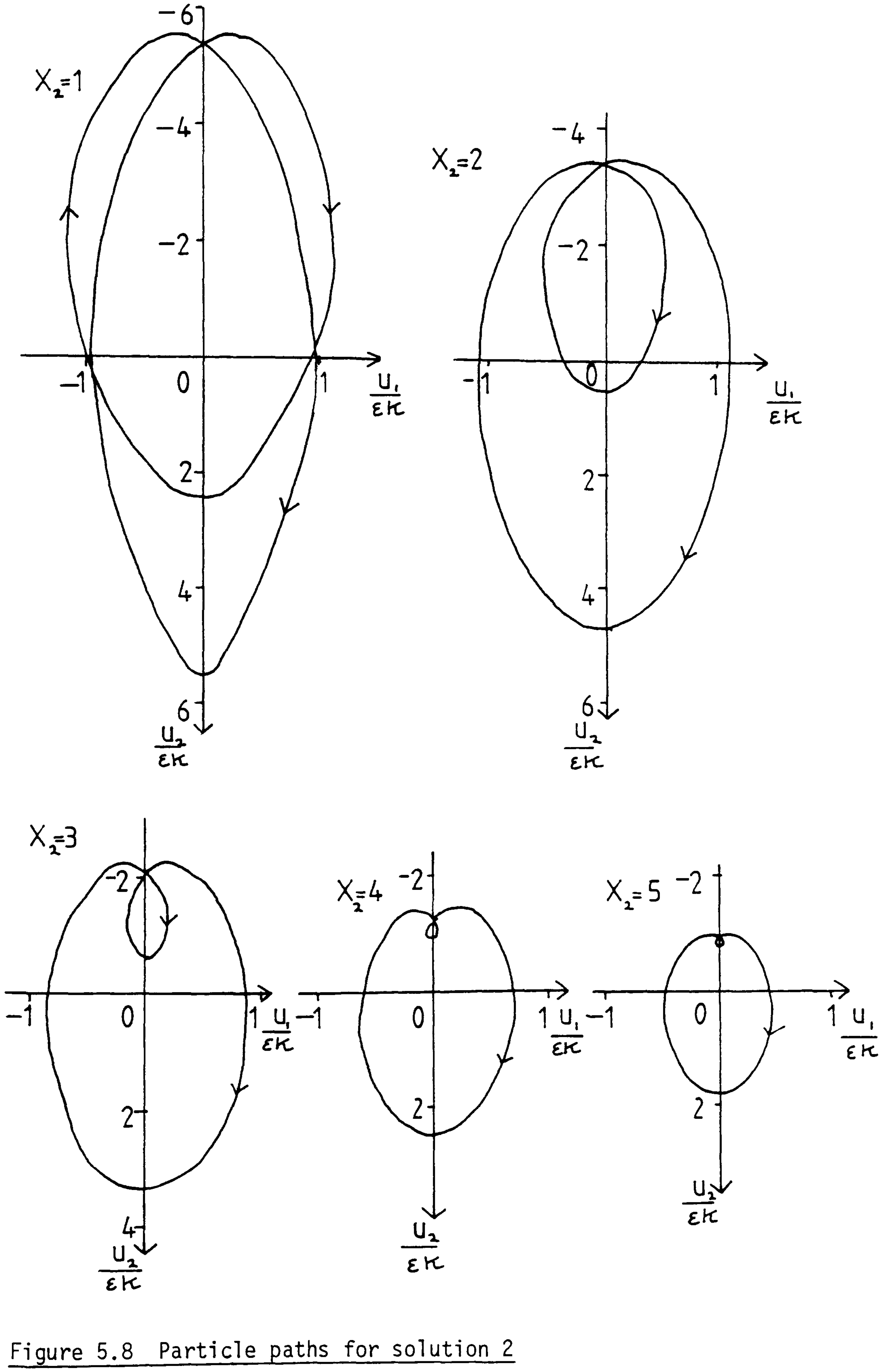



Figure 5.8 Particle paths for solution 2

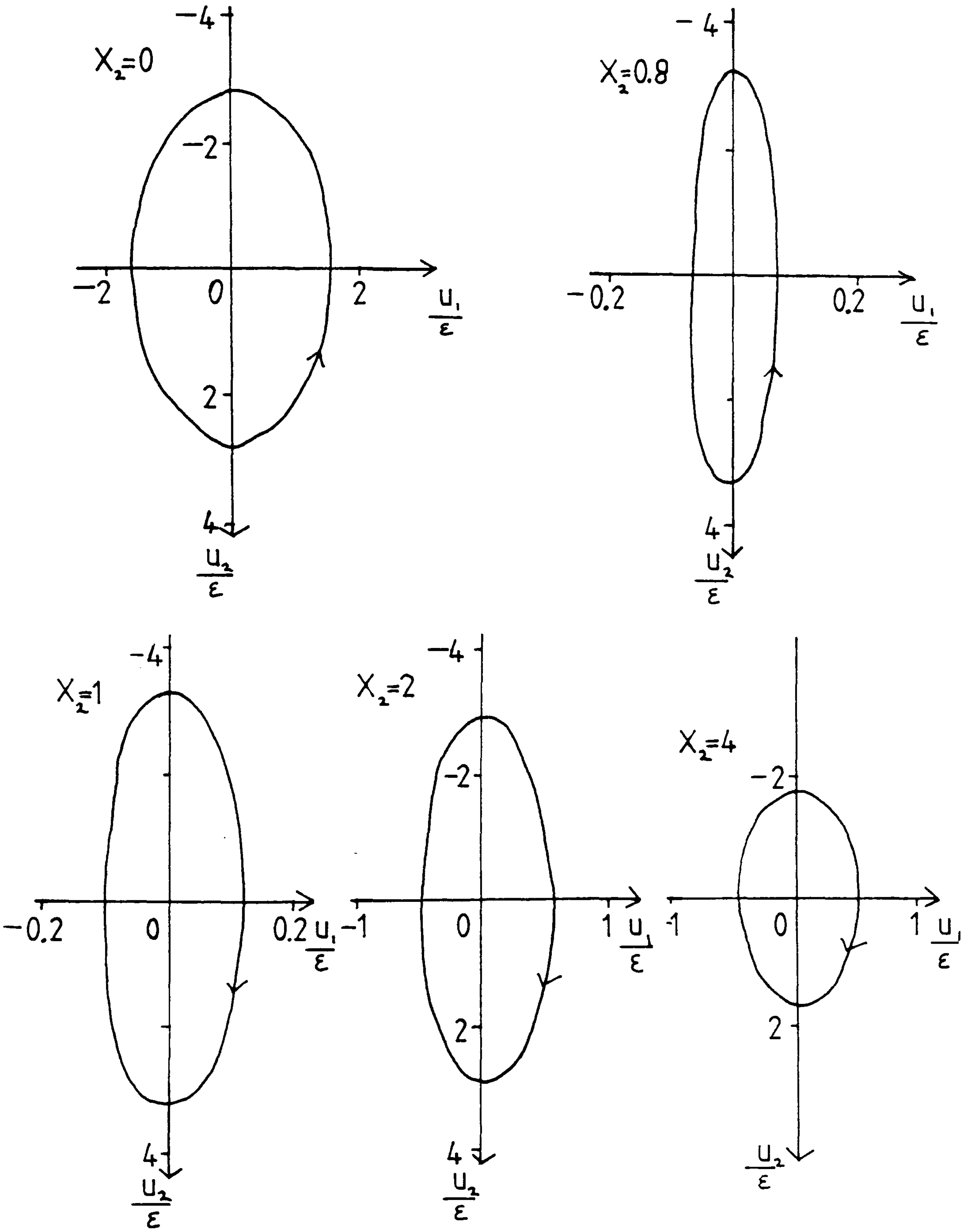



Figure 5.9 Particle paths for the standard Rayleigh wave with $C_1 = 1.8$

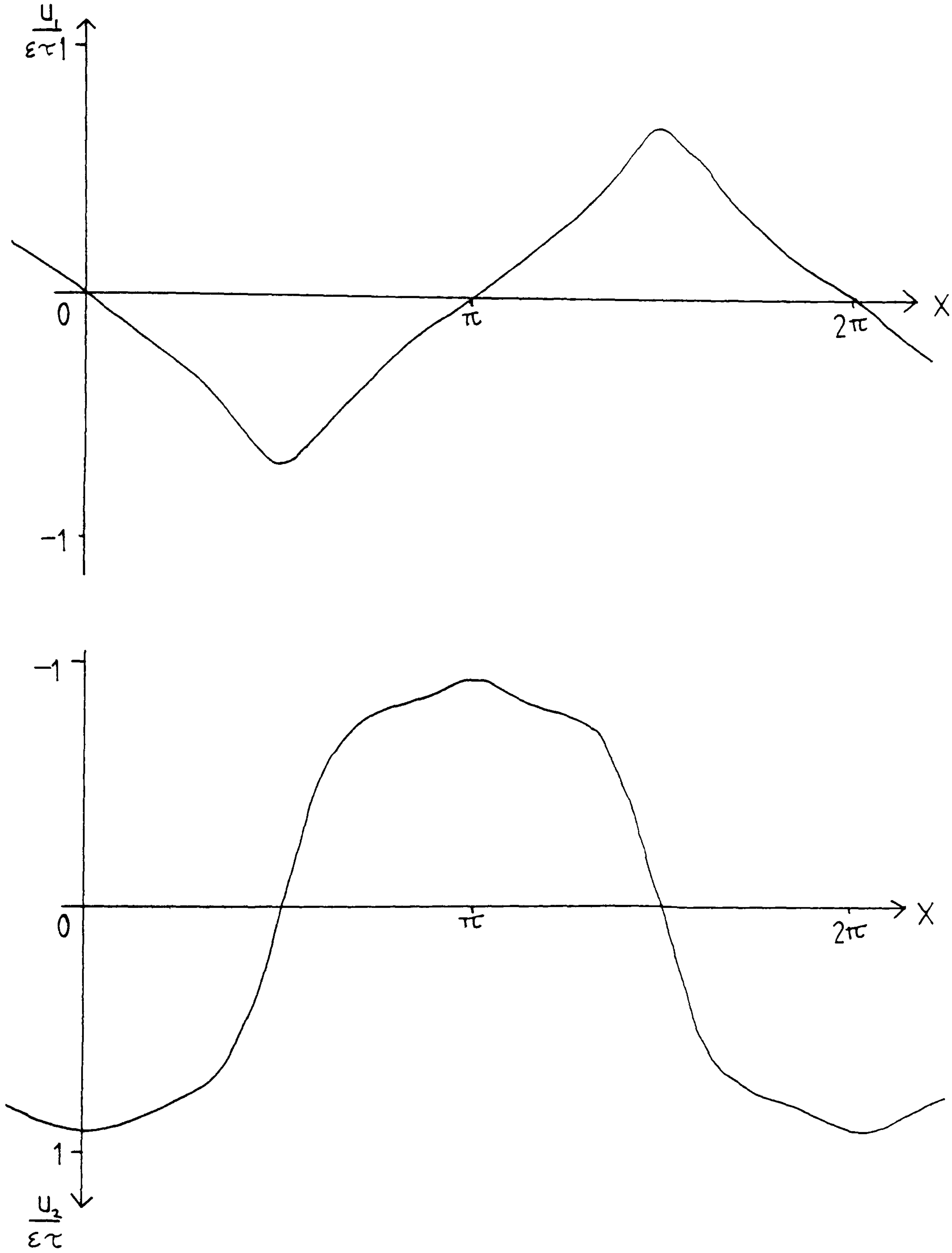



Figure 5.10 Horizontal and vertical displacements at the surface for the solution with $\kappa = 0$

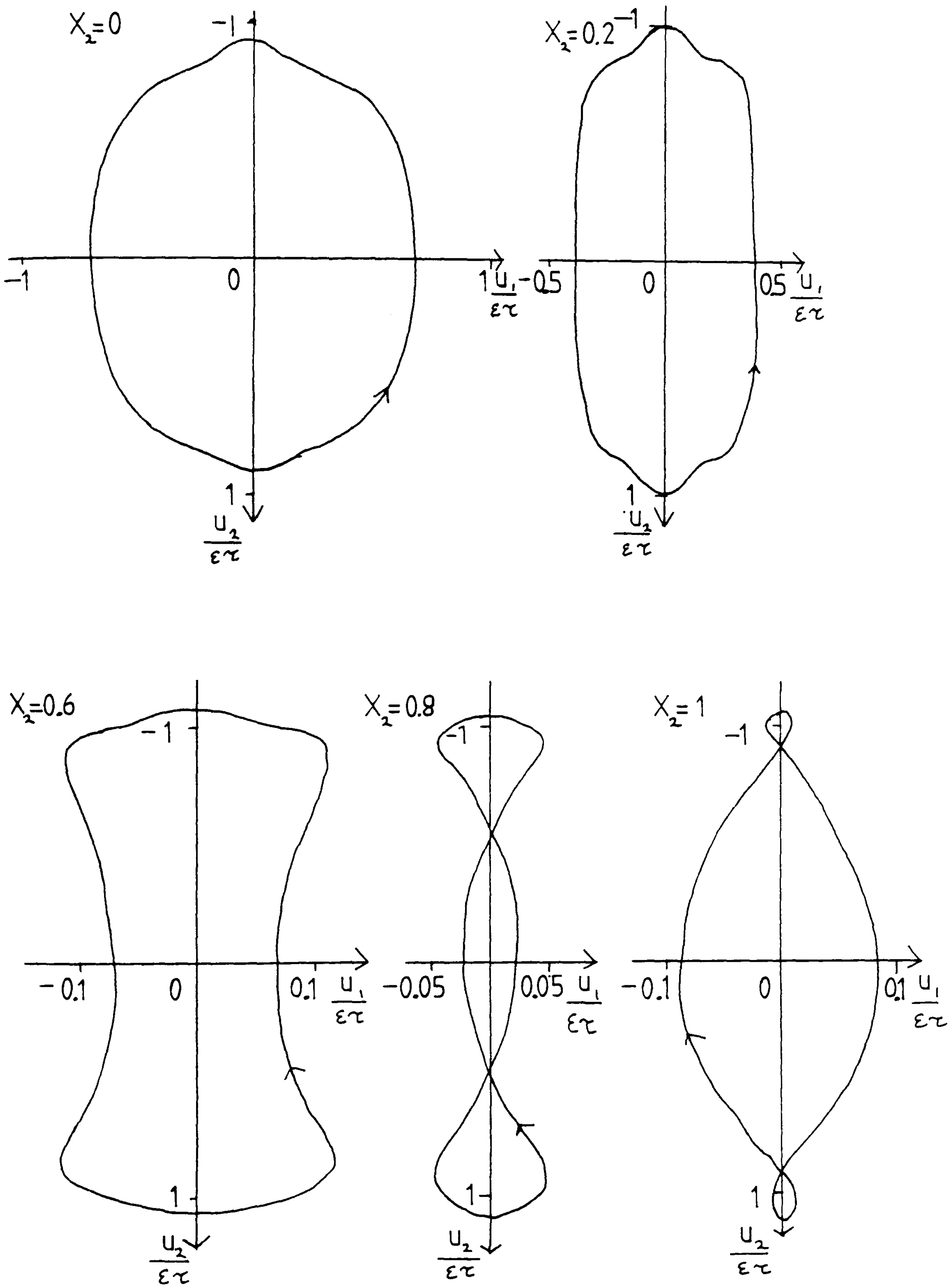



Figure 5.11 Particle paths for the wave with $\kappa = 0$

# CONCLUSION

In Chapter Five we have seen how to construct solutions for Rayleigh waves on an incompressible material using the method based on Fredholm's alternative theory introduced in Section 3.5. In Chapter Four we considered Rayleigh waves on the surface of a compressible, harmonic material. In both the compressible and incompressible materials we have obtained two periodic solutions travelling at a speed different from the standard Rayleigh wave speed and a solution travelling at the same speed. The solutions obtained for waves travelling at a speed different from the standard Rayleigh wave speed have very different elevations and particle paths from the standard wave, although as the depth increases the waveform tends to a sinusoidal form. Solutions 1 are similar for the two materials as are solutions 2. In Chapter Four we also attempted to find non-periodic waveforms, however we were unsuccessful in obtaining convergence of the numerical procedure.

For the solutions obtained the steeper slopes for the elevation suggests that for a given peak displacement the maximum surface accelerations are considerably greater than for a standard sinusoidal waveform having the same amplitude. The acceleration of the wave is significant for earthquakes.

In figure 2 we show how the speed varies with the strain amplitude for the different solutions obtained. The cases of waves travelling faster and more slowly than the standard wave are included.

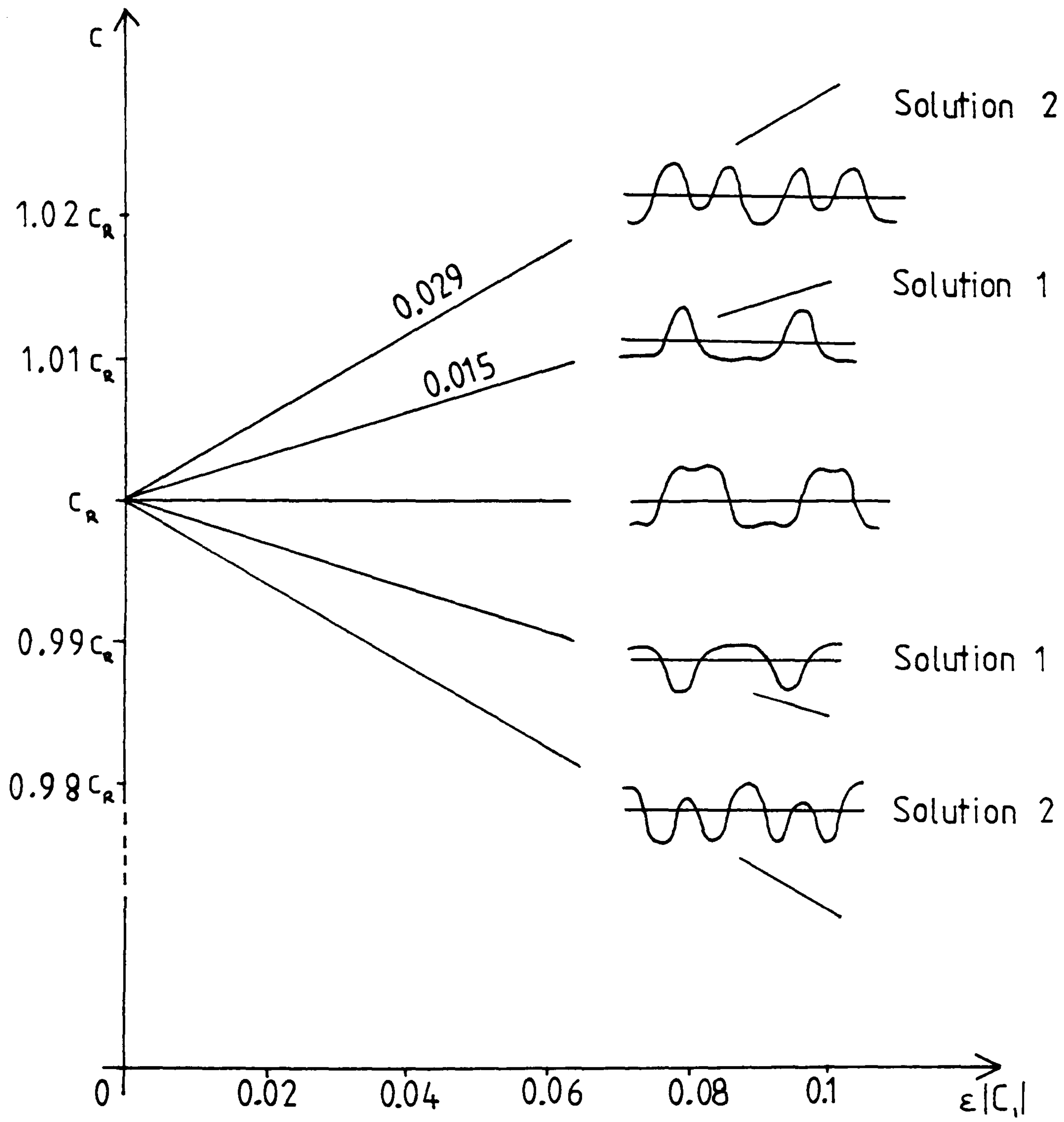



Figure 2  The dependence of the speed on the fundamental amplitude $|C_1|$

# PART III

# INTRODUCTION

In the following two chapters we consider the analysis of a fibre-reinforced material in the form of a belt stretched round a system of pulleys. A great deal of attention has been given in recent years to studying the mechanics of a material reinforced with strong fibres having high extensional modulus (52). There are many materials of this type both artificial and natural. The fibre-reinforced material is regarded as a material with certain properties which may be specified for the composite as a whole. Although we are concerned primarily with continuum theories and macroscopic models of material behaviour, the properties of the composite material derive ultimately from the properties and geometrical arrangement of its constituents, this is considered in Section 7.6.

Static deformations have been analysed using many different theories (44), whilst dynamic disturbances have been analysed mostly on the basis of small deformation theory (51) or of acceleration wave theory (16). Fibre-reinforced materials exhibit highly anisotropic elastic behaviour in the sense that their elastic moduli for extension in the fibre direction are frequently of the order of fifty or more times greater than their elastic moduli in transverse extension or shear. For this reason an approach commonly used for large deformations is to consider 'ideally reinforced' materials (44). In this Pipkin and Rogers assume that the material is incompressible, that the fibres are inextensible and are continuously distributed throughout the material. Although no real material satisfies these constraints, the application of this ideal theory gives a good description of materials for which the bulk modulus and the extensional modulus in the fibre direction are large in comparison to the shear moduli of the composite. Pipkin and Rogers show that plane strain static deformations with a reference configuration in which all fibres are straight and parallel are simply described. They show that every kinematically admissible deformation is also statically admissible. In (43) they consider a specific mixed boundary-value problem of a slab of fibre-reinforced material deformed so that one side parallel to the fibres is bonded to a rigid wall, in which case more than one kinematically admissible deformation satisfies the prescribed displacement boundary conditions. Pipkin and Rogers (43) show how to determine whether or not a given kinematically admissible deformation furnishes a solution, by using the prescribed traction boundary conditions. Pipkin and Rogers (43) also point out that solutions using the inextensible theory frequently predict the existence of singular fibres or sheets of fibres which carry infinite stress, but finite force. These singular fibres may occur either adjacent to the surface of a body or in its interior. In the case of incompressible plane strain, it is also possible for normal curves to be singular.

In (49), Rogers and Pipkin apply the general theory developed in (43) to the case in which the fibres are initially curved. The example they consider is that of a pressurized tube. They conclude that using the constraints of incompressibility and inextensibility the deformation of the tube can be completely specified when the shape of the deformed inner surface is known. A constitutive equation is not needed in determining the shape but only to evaluate the resultant shearing stress over a radial cross-section of the tube wall. A limitation of this theory using the constraints of incompressibility and inextensibility is that although the deformation is completely determined, the stress field is still ambiguous to the extent of a tension which is constant along each fibre and the pressure arising as a reaction to it. Rogers and Pipkin note that this tension could be determined by formulating a theory of small displacements for slightly extensible and compressible materials, which is superposed on the solution for large displacements already obtained.

As has already been mentioned, a mathematical consequence of considering inextensible fibres is the occurrence of singular sheets of fibres which carry infinite stress but finite force. Everstine and Pipkin (12) demonstrate, by considering some simple examples, that these singular sheets of fibres represent narrow bands of intense stress concentration. Everstine and Pipkin show that if $\ell$ is a characteristic length of a problem, then these bands have width of order $(\mu_L/E)^{\frac{1}{2}}\ell$ and along them the stress decays in a length of order $(\mu_L/E)^{-\frac{1}{2}}\ell$, where $\mu_L$ is the shear modulus and E the extensional modulus. The inextensible theory corresponds to the limit $\mu_L/E \to 0$. Thus for $\mu_L/E \ll 1$ the singular fibres represent boundary layers across which certain components vary rapidly. Everstine and Pipkin point out that the equations are of a suitable form for the application of a boundary layer and singular perturbation analysis. In (13) they develop such an analysis and apply it to the problem of the deflection of a cantilever beam under end load. Spencer (53) further develops this theory and compares its predictions with some exact solutions in anisotropic elasticy. He does not assume that the material is incompressible as well as inextensible in the fibre direction, and includes the case of plane stress as well as plane strain. He shows that the problem reduces to the solution of Laplace's equation for the two displacement components in appropriately scaled coordinates. Spencer remarks that it is often possible to analyse the stress and deformation in boundary layers without requiring a complete solution elsewhere, and that since boundary layers are usually the regions of greatest stress, it is often sufficient to be able to analyse these regions.

In (40) Parker considers plane strain disturbances of a slab of ideal fibre-reinforced material of uniform thickness with a reference configuration in which all fibres are straight and parallel. He notes that at each point the tangent and normal to the current 'fibre direction' are important and changes to 'fibre-normal' coordinates $(s, X_2)$, where at each instant the curves of constant $X_2$ are fibres, whilst $s$ = constant denotes that plane cross-section which is normal to the fibres and for which $s$ measures length along some reference fibre. Parker shows that all kinematically admissible plane strain disturbances are also mechanically admissible, which is a generalization of the result obtained by Pipkin and Rogers (43) for static deformations.

In the work presented here a fibre-reinforced material is considered in which the fibres are initially concentric circles. The problem of a fan belt stretched round pulleys is that considered by Everatt (11), but the approach here differs from that used by him in that in a similar way to the method used by Parker (40) we change coordinates to $(R, s)$ where $R$ measures distance along fibre normals and $s$ is constant along fibre normals. In Chapter 6 we consider the theory for a belt stretched round an arbitrary number of pulleys, before the particular case of a belt round two pulleys is analysed in more detail in Chapter 7.

Everatt only considers the ideal theory and hence does not determine the arbitrary contribution to the tension which is constant along each fibre, which has already been mentioned in connection with the work of Rogers and Pipkin (49). To determine this function we consider the case of fibres which are slightly extensible. We also consider this case to examine the boundary layers in more detail. Within these the fibres are subjected to large tractions which are required in order to balance the large gradients of shear which occur near boundaries of highly anisotropic materials. It is found that the large stresses are indeed confined to narrow layers near the surfaces for a range of geometrical parameters. This situation, where the fibres are slightly extensible, may be regarded as the start of a perturbation process.

# CHAPTER 6

# THE GENERAL THEORY FOR A FIBRE-REINFORCED FAN-BELT STRETCHED ROUND A SYSTEM OF PULLEYS

## 6.1 THE EQUATIONS OF EQUILIBRIUM IN POLAR COORDINATES

The belt is assumed to be reinforced by virtually inextensible fibres lying in closed curves parallel to the surface of the belt. The fibres are assumed to be continuously distributed throughout the cross-section. The belt is assumed to be initially circular with inner radius $R_0$, moreover we assume that it deforms only in plane strain. To analyse the configuration of the fan-belt, we consider an element which is deformed as shown in figure 6.1, where $\theta$ is the inclination of a fibre to the reference direction, A is the fibre elongation, AB = $\Delta$ is the dilation and $\gamma$ is the shearing.

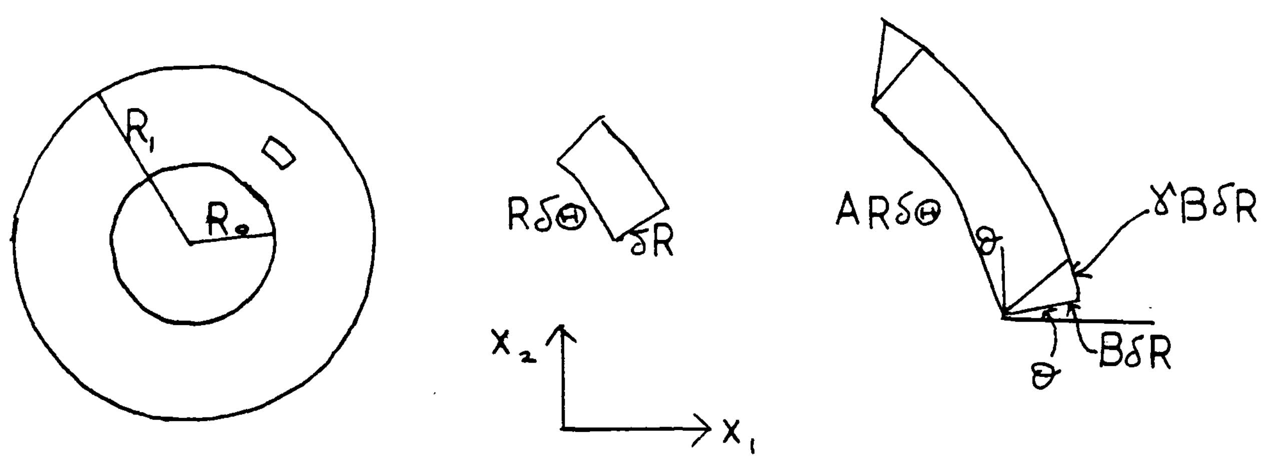



Figure 6.1 An element in a typical state of deformation

In plane strain, the deformation gradient has components

$$F_{ij} = \frac{\partial x_i}{\partial X_j}, \qquad i, j = 1, 2,$$

where $(x_1, x_2)$ are Eulerian coordinates and $(X_1, X_2)$ are Lagrangian coordinates. Making the transformation to polar coordinates $(R, \Theta)$ in the reference configuration where

$$X_1 = R\cos\Theta , \qquad X_2 = R\sin\Theta$$

leads to

$$\frac{\partial x_i}{\partial X_1} = \cos\Theta\frac{\partial x_i}{\partial R} - \frac{\sin\Theta}{R}\frac{\partial x_i}{\partial\Theta} ,$$

$$\frac{\partial x_i}{\partial X_2} = \sin\Theta\frac{\partial x_i}{\partial R} + \frac{\cos\Theta}{R}\frac{\partial x_i}{\partial\Theta} ,$$

for $i = 1, 2$.

Therefore, the deformation gradient $\underset{\sim}{F}$ is given by

$$\underset{\sim}{F} = \begin{pmatrix} \dfrac{\partial x_1}{\partial R} & \dfrac{1}{R}\dfrac{\partial x_1}{\partial\Theta} \\ \dfrac{\partial x_2}{\partial R} & \dfrac{1}{R}\dfrac{\partial x_2}{\partial\Theta} \end{pmatrix}\begin{pmatrix} \cos\Theta & \sin\Theta \\ -\sin\Theta & \cos\Theta \end{pmatrix}$$

Now $\left.\dfrac{\partial x_1}{\partial R}\right|_\Theta = B\cos\theta - \gamma B\sin\theta$, $\left.\dfrac{1}{R}\dfrac{\partial x_1}{\partial\Theta}\right|_R = -A\sin\theta$,

$$\left.\frac{\partial x_2}{\partial R}\right|_\Theta = B\sin\theta + \gamma B\cos\theta, \quad \left.\frac{1}{R}\frac{\partial x_2}{\partial\Theta}\right|_R = A\cos\theta,$$

where A and B are as shown in figure 6.1.

Hence

$$\underset{\sim}{F} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} B & 0 \\ \gamma B & A \end{pmatrix}\begin{pmatrix} \cos\Theta & \sin\Theta \\ -\sin\Theta & \cos\Theta \end{pmatrix},$$

which we may write in the form $\underset{\sim}{K}\underset{\sim}{B}\underset{\sim}{H}^T$,

where

$$\underset{\sim}{B} = \begin{pmatrix} B & 0 \\ \gamma B & A \end{pmatrix}$$

and $\underset{\sim}{K}$, $\underset{\sim}{H}$ are the orthogonal matrices

$$\underset{\sim}{K} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \qquad \underset{\sim}{H} = \begin{pmatrix} \cos\Theta & -\sin\Theta \\ \sin\Theta & \cos\Theta \end{pmatrix}$$

The compatibility conditions

$$\frac{\partial}{\partial\Theta}\left(\frac{\partial x_i}{\partial R}\right) = \frac{\partial}{\partial R}\left(\frac{\partial x_i}{\partial\Theta}\right)$$

must be satisfied for i = 1, 2.

For i = 1, this gives

$$\frac{\partial}{\partial\Theta}\left(B\cos\theta - \gamma B\sin\theta\right) = \frac{\partial}{\partial R}\left(-AR\sin\theta\right)$$

that is

$$\frac{\partial B}{\partial\Theta}(\cos\theta - \gamma\sin\theta) + B\left(-\sin\theta\frac{\partial\theta}{\partial\Theta} - \gamma\cos\theta\frac{\partial\theta}{\partial\Theta} - \frac{\partial\gamma}{\partial\Theta}\sin\theta\right)$$

$$= -A\sin\theta - \frac{\partial A}{\partial R}R\sin\theta - AR\cos\theta\frac{\partial\theta}{\partial R},$$

and for i = 2

$$\frac{\partial B}{\partial\Theta}(\sin\theta + \gamma\cos\theta) + B\left(\cos\theta\frac{\partial\theta}{\partial\Theta} - \gamma\sin\theta\frac{\partial\theta}{\partial\Theta} + \frac{\partial\gamma}{\partial\Theta}\cos\theta\right)$$

$$= A\cos\theta + \frac{\partial A}{\partial R}R\cos\theta - AR\sin\theta\frac{\partial\theta}{\partial R}.$$

Eliminating $\sin\theta$ and $\cos\theta$ from these gives

$$\frac{\partial B}{\partial \Theta} - \gamma B \frac{\partial \theta}{\partial \Theta} = -AR \frac{\partial \theta}{\partial R} \tag{6.1.1}$$

and $$\gamma \frac{\partial B}{\partial \Theta} + B \frac{\partial \theta}{\partial \Theta} + B \frac{\partial \gamma}{\partial \Theta} = A + R \frac{\partial A}{\partial R} \tag{6.1.2}$$

We let $T_{ij}$ denote the components of the Piola-Kirchhoff stress, $T_{ij}N_j$ the i component of the traction on an element of surface having unit normal $\underset{\sim}{N}$ in the reference configuration. Then for equilibrium of plane stress deformations we have

$$\frac{\partial T_{ij}}{\partial X_j} = 0 \ , \qquad i,j = 1, 2,$$

which can be written in the form

$$\frac{\partial}{\partial R}\Big[R(T_{i1} \cos \Theta + T_{i2} \sin \Theta)\Big] + \frac{\partial}{\partial \Theta}\Big[- T_{i1} \sin \Theta + T_{i2} \cos \Theta\Big] = 0 \ . \tag{6.1.3}$$

We define $\underset{\sim}{\tau} \equiv \underset{\sim}{T}\underset{\sim}{H}$, where $\underset{\sim}{H}$ is defined as above,

then $\tau_{i1} = T_{i1} \cos \Theta + T_{i2} \sin \Theta$

and $\tau_{i2} = -T_{i1} \sin \Theta + T_{i2} \cos \Theta$

so (6.1.3) gives

$$\frac{\partial}{\partial R} (R\, \tau_{i1}) + \frac{\partial}{\partial \Theta} (\tau_{i2}) = 0 \ . \tag{6.1.4}$$

If W is the strain-energy function, then

$$T_{ik} = \frac{\partial W}{\partial F_{ik}}$$

hence $\tau_{ij} = \dfrac{\partial W}{\partial F_{ik}} H_{kj}$

We define $\underset{\sim}{C} = \underset{\sim}{F}\underset{\sim}{H} = \underset{\sim}{K}\underset{\sim}{B}$

$$= \begin{pmatrix} B\cos\theta - \gamma B\sin\theta & -A\sin\theta \\ B\sin\theta + \gamma B\cos\theta & A\cos\theta \end{pmatrix} \tag{6.1.5}$$

$$= \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}.$$

Hence $\underset{\sim}{F} = \underset{\sim}{C}\underset{\sim}{H}^T$,

which may be written in component form

$$F_{\ell k} = C_{\ell r}H_{kr} .$$

Then
$$\frac{\partial W}{\partial C_{ij}} = \frac{\partial W}{\partial F_{\ell k}}\frac{\partial F_{\ell k}}{\partial C_{ij}}$$

$$= \frac{\partial W}{\partial F_{ik}} H_{kj}$$

$$= \tau_{ij}$$

is equivalent to $\dfrac{\partial W}{\partial F_{ij}} = T_{ij}$ .

Hence $\underset{\sim}{\tau} = \dfrac{\partial W}{\partial \underset{\sim}{C}} = \dfrac{\partial W}{\partial A}\dfrac{\partial A}{\partial \underset{\sim}{C}} + \dfrac{\partial W}{\partial B}\dfrac{\partial B}{\partial \underset{\sim}{C}} + \dfrac{\partial W}{\partial \gamma}\dfrac{\partial \gamma}{\partial \underset{\sim}{C}}$ .

To find expressions for $\tau_{ij}$ we therefore need to write A, B and $\gamma$ as functions of $C_{ij}$, these we obtain from (6.1.5) as

$$A = \left(C_{12}^2 + C_{22}^2\right)^{\frac{1}{2}} , \qquad B = \frac{C_{11}C_{22} - C_{12}C_{21}}{\left(C_{12}^2 + C_{22}^2\right)^{\frac{1}{2}}} ,$$

$$\gamma = \frac{C_{21}C_{22} + C_{11}C_{12}}{C_{11}C_{22} - C_{12}C_{21}} .$$

Hence

$$\tau_{11} = \frac{\partial W}{\partial C_{11}} = \frac{\partial W}{\partial B}\cos\theta + \frac{\partial W}{\partial \gamma}\frac{1}{B}(-\sin\theta - \gamma\cos\theta) ,$$

$$\tau_{12} = \frac{\partial W}{\partial C_{12}} = -\frac{\partial W}{\partial A}\sin\theta - \frac{\partial W}{\partial B}\frac{\gamma B}{A}\cos\theta + \frac{\partial W}{\partial \gamma}\frac{(1+\gamma^2)}{A}\cos\theta ,$$

$$\tau_{21} = \frac{\partial W}{\partial C_{21}} = \frac{\partial W}{\partial B}\sin\theta + \frac{\partial W}{\partial \gamma}\frac{1}{B}(\cos\theta - \gamma\sin\theta)$$

$$\tau_{22} = \frac{\partial W}{\partial C_{22}} = \frac{\partial W}{\partial A}\cos\theta - \frac{\partial W}{\partial B}\frac{\gamma B}{A}\sin\theta + \frac{\partial W}{\partial \gamma}\frac{(1+\gamma^2)}{A}\sin\theta .$$

These may usefully be expressed by resolving tractions along and normal to the fibres, defining $\underset{\sim}{\Sigma} = \underset{\sim}{K}^T\underset{\sim}{\tau}$, which implies that

$$\underset{\sim}{\Sigma} = \begin{pmatrix} \frac{\partial W}{\partial B} - \frac{\gamma}{B}\frac{\partial W}{\partial \gamma} & -\frac{\gamma B}{A}\frac{\partial W}{\partial B} + \frac{1+\gamma^2}{A}\frac{\partial W}{\partial \gamma} \\ \frac{1}{B}\frac{\partial W}{\partial \gamma} & \frac{\partial W}{\partial A} \end{pmatrix}$$

$$= \frac{\partial W}{\partial A}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\partial W}{\partial B}\begin{pmatrix} 1 & \frac{-\gamma B}{A} \\ 0 & 0 \end{pmatrix}$$

$$+ \frac{\partial W}{\partial \gamma}\begin{pmatrix} -\frac{\gamma}{B} & \frac{1+\gamma^2}{A} \\ \frac{1}{B} & 0 \end{pmatrix} . \qquad (6.1.6)$$

Substituting the expressions for $\tau_{ij}$ into the equilibrium conditions (6.1.4) we obtain

$$\frac{\partial}{\partial R}\left(\frac{\partial W}{\partial B} R\cos\theta + R\frac{\partial W}{\partial \gamma}\frac{1}{B}(-\sin\theta - \gamma\cos\theta)\right)$$

$$+ \frac{\partial}{\partial \Theta}\left(-\frac{\partial W}{\partial A}\sin\theta - \frac{\partial W}{\partial B}\frac{\gamma B}{A}\cos\theta + \frac{\partial W}{\partial \gamma}\frac{(1+\gamma^2)}{A}\cos\theta\right) = 0$$

and $\frac{\partial}{\partial R}\left(\frac{\partial W}{\partial B} R \sin\theta + \frac{\partial W}{\partial \gamma}\frac{R}{B}(\cos\theta - \gamma\sin\theta)\right)$

$$+ \frac{\partial}{\partial \Theta}\left(\frac{\partial W}{\partial A}\cos\theta - \frac{\partial W}{\partial B}\frac{\gamma B}{A}\sin\theta + \frac{\partial W}{\partial \gamma}\frac{(1+\gamma^2)}{A}\sin\theta\right) = 0 .$$

Eliminating $\sin\theta$ and $\cos\theta$ from these we obtain

$$\frac{\partial}{\partial R}\left(R\left(\frac{\partial W}{\partial B} - \frac{\gamma}{B}\frac{\partial W}{\partial \gamma}\right)\right) + \frac{\partial}{\partial \Theta}\left(-\frac{\gamma B}{A}\frac{\partial W}{\partial B} + \frac{(1+\gamma^2)}{A}\frac{\partial W}{\partial \gamma}\right)$$

$$-\frac{R}{B}\frac{\partial W}{\partial \gamma}\frac{\partial \theta}{\partial R} - \frac{\partial W}{\partial A}\frac{\partial \theta}{\partial \Theta} = 0 \qquad (6.1.7)$$

and

$$\frac{\partial}{\partial R}\left(\frac{R}{B}\frac{\partial W}{\partial \gamma}\right) + \frac{\partial}{\partial \Theta}\left(\frac{\partial W}{\partial A}\right) + R\left(\frac{\partial W}{\partial B} - \frac{\gamma}{B}\frac{\partial W}{\partial \gamma}\right)\frac{\partial \theta}{\partial R}$$

$$+ \left(-\frac{\gamma B}{A}\frac{\partial W}{\partial B} + \frac{(1+\gamma^2)}{A}\frac{\partial W}{\partial \gamma}\right)\frac{\partial \theta}{\partial \Theta} = 0 \qquad (6.1.8)$$

We may regard these as the equilibrium equations resolved along and normal to the fibres.

For an ideal material, which is incompressible and inextensible, we have the constraints $A = B = 1$ and we replace $\frac{\partial W}{\partial A}$, $\frac{\partial W}{\partial B}$ by undetermined multipliers, so (6.1.1) and (6.1.2) give

$$\gamma\frac{\partial \theta}{\partial \Theta} = R\frac{\partial \theta}{\partial R}$$

that is $\theta$ is constant along $\frac{d\Theta}{dR} = -\frac{\gamma}{R}$

and $\frac{\partial}{\partial \Theta}(\theta + \gamma) = A = 1$

so that $\theta + \gamma = \Theta + \Gamma(R)$ , (6.1.9)

where $\Gamma(R)$ is an arbitrary function. This suggests the change of coordinates introduced in the next section.

## 6.2 THE CHANGE TO CURVILINEAR COORDINATES ALONG AND NORMAL TO THE FIBRES

As was stated in the previous section, for the ideal case $A = B = 1$ and also $\theta$ is constant along the lines $\frac{d\Theta}{dR} = -\frac{\gamma B}{AR}$. Therefore, as was mentioned in the introduction to part III, we introduce a label s which is constant along the fibre normals and is such that $R_0 s$ measures the unstretched distance along the reference fibre $R = R_0$; then $s = s(R, \Theta)$ satisfies

$$AR\frac{\partial s}{\partial R} - \gamma\frac{B\partial s}{\partial \Theta} = 0 .$$

We now change to independent variables (R, s) where $\Theta = \tilde{\Theta}(R, s)$ is determined by $\frac{\partial\tilde{\Theta}}{\partial R} = -\frac{\gamma B}{AR}$, $\tilde{\Theta}(R_0, s) = s$. The other first derivatives of $\tilde{\Theta}$, denoted by $\ell \equiv \frac{\partial\tilde{\Theta}}{\partial s}$, satisfies the compatibility condition

$$\frac{\partial \ell}{\partial R} = \frac{\partial}{\partial R}\left(\frac{\partial\tilde{\Theta}}{\partial s}\right) = \frac{\partial}{\partial s}\left(\frac{\partial\tilde{\Theta}}{\partial R}\right) = \frac{\partial}{\partial s}\left(-\frac{\gamma B}{AR}\right)$$

that is $\frac{\partial \ell}{\partial R} + \frac{\partial}{\partial s}\left(\frac{\gamma B}{AR}\right) = 0$ , with $\ell(R_0, s) = 1.$

We are changing coordinates from $(R, \Theta)$ to $(R, s)$ so that derivatives are transformed as:

$$\left.\frac{\partial}{\partial R}\right|_{\Theta} = \left.\frac{\partial}{\partial R}\right|_{s} + \frac{\gamma B}{AR\ell}\left.\frac{\partial}{\partial s}\right|_{R}$$

$$\left.\frac{\partial}{\partial \Theta}\right|_{R} = \frac{1}{\ell}\left.\frac{\partial}{\partial s}\right|_{R} .$$

The compatibility conditions (6.1.1) and (6.1.2) become

$$\frac{1}{\ell}\frac{\partial B}{\partial s} - \frac{\gamma B}{\ell}\frac{\partial \theta}{\partial s} = -AR\frac{\partial \theta}{\partial R} - \frac{\gamma B}{\ell}\frac{\partial \theta}{\partial s}$$

and

$$\frac{\gamma}{\ell}\frac{\partial B}{\partial s} + \frac{B}{\ell}\frac{\partial \theta}{\partial s} + \frac{B}{\ell}\frac{\partial \gamma}{\partial s} = A + R\frac{\partial A}{\partial R} + \frac{\gamma B}{A\ell}\frac{\partial A}{\partial s}$$

which imply

$$\frac{\partial\theta}{\partial R} = -\frac{1}{AR\ell}\frac{\partial B}{\partial s} \tag{6.2.1}$$

and

$$\frac{B}{\ell}\frac{\partial\theta}{\partial s} = A - \frac{B}{\ell}\frac{\partial\gamma}{\partial s} + R\frac{\partial A}{\partial R} + \frac{\gamma B}{A\ell}\frac{\partial A}{\partial s} - \frac{\gamma}{\ell}\frac{\partial B}{\partial s} \tag{6.2.2}$$

After transformation to coordinates (R, s), (6.1.7) and (6.1.8) become

$$\frac{\partial}{\partial R}\left\{R\left(\frac{\partial W}{\partial B} - \frac{\gamma}{B}\frac{\partial W}{\partial\gamma}\right)\right\} + \frac{\gamma B}{AR\ell}\frac{\partial}{\partial s}\left\{R\frac{\partial W}{\partial B} - \frac{\gamma}{B}\frac{\partial W}{\partial\gamma}\right\}$$

$$+ \frac{1}{\ell}\frac{\partial}{\partial s}\left\{-\frac{\gamma B}{A}\frac{\partial W}{\partial B} + \frac{(1+\gamma^2)}{A}\frac{\partial W}{\partial\gamma}\right\} - \frac{R}{B}\frac{\partial W}{\partial\gamma}\left\{\frac{\partial\theta}{\partial R} + \frac{\gamma B}{AR\ell}\frac{\partial\theta}{\partial s}\right\}$$

$$-\frac{1}{\ell}\frac{\partial W}{\partial A}\frac{\partial\theta}{\partial s} = 0 \tag{6.2.3}$$

and

$$\frac{\partial}{\partial R}\left(\frac{R}{B}\frac{\partial W}{\partial\gamma}\right) + \frac{\gamma B}{AR\ell}\frac{\partial}{\partial s}\left(\frac{R}{B}\frac{\partial W}{\partial\gamma}\right) + \frac{1}{\ell}\frac{\partial}{\partial s}\left(\frac{\partial W}{\partial A}\right)$$

$$+ R\left(\frac{\partial W}{\partial B} - \frac{\gamma}{B}\frac{\partial W}{\partial\gamma}\right)\left(\frac{\partial\theta}{\partial R} + \frac{\gamma B}{AR\ell}\frac{\partial\theta}{\partial s}\right) + \left(-\frac{\gamma B}{A}\frac{\partial W}{\partial B} + \frac{(1+\gamma^2)}{A}\frac{\partial W}{\partial\gamma}\right)\frac{1}{\ell}\frac{\partial\theta}{\partial s} = 0 \tag{6.2.4}$$

Now

$$t_{ij} = (\det \underset{\sim}{F})^{-1}\, T_{ik}F_{jk}$$

and

$$\det \underset{\sim}{F} = \det(\underset{\sim}{K}\,\underset{\sim}{B}\,\underset{\sim}{H}^T) = \det \underset{\sim}{B} = AB$$

so that

$$t_{ij} = \frac{1}{AB}\, T_{ik}\, F_{jk}\,.$$

The Cauchy stress $\underset{\sim}{t}$ can therefore be written in the form

$$\underset{\sim}{t} = \underset{\sim}{K}\,\underset{\sim}{\sigma}\,\underset{\sim}{K}^T$$

where

$$\underset{\sim}{\sigma} = \frac{1}{AB}\,\underset{\sim}{\Sigma}\,\underset{\sim}{B}^{T} ,$$

that is, from (6.1.6)

$$\underset{\sim}{\sigma} = \frac{1}{B}\frac{\partial W}{\partial A}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{A}\frac{\partial W}{\partial B}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$+ \frac{1}{AB}\frac{\partial W}{\partial \gamma}\begin{pmatrix} -\gamma & 1 \\ 1 & \gamma \end{pmatrix} .$$

We now define two new functions T and p by $-p = \sigma_{11}$, $T = \sigma_{22}$, where p is the transverse pressure and T is the tension in the fibre direction, these imply

$$-p = \frac{1}{A}\left(\frac{\partial W}{\partial B} - \frac{\gamma}{B}\frac{\partial W}{\partial \gamma}\right) \tag{6.2.5}$$

$$T = \frac{1}{B}\left(\frac{\partial W}{\partial A} + \frac{\gamma}{A}\frac{\partial W}{\partial \gamma}\right) \tag{6.2.6}$$

which give

$$\underset{\sim}{\sigma} = \begin{pmatrix} -p & \frac{1}{AB}\frac{\partial W}{\partial \gamma} \\ \frac{1}{AB}\frac{\partial W}{\partial \gamma} & T \end{pmatrix} .$$

Substituting the expressions (6.2.5) and (6.2.6) into the equations (6.2.3) and (6.2.4) gives the following equations

$$\frac{\partial}{\partial R}(-RAp) + \frac{\gamma B}{A\ell}\frac{\partial}{\partial s}(Ap) + \frac{1}{\ell}\frac{\partial}{\partial s}\left(\gamma Bp + \frac{1}{A}\frac{\partial W}{\partial \gamma}\right) - \frac{R}{B}\frac{\partial W}{\partial \gamma}\frac{\partial \theta}{\partial R}$$

$$- \frac{B}{\ell}T\frac{\partial \theta}{\partial s} = 0 \tag{6.2.7}$$

and

$$\frac{\partial}{\partial R}\left(\frac{R}{B}\frac{\partial W}{\partial \gamma}\right) + \frac{\gamma B}{A\ell}\frac{\partial}{\partial s}\left(\frac{1}{B}\frac{\partial W}{\partial \gamma}\right) + \frac{1}{\ell}\frac{\partial}{\partial s}\left(BT - \frac{\gamma}{A}\frac{\partial W}{\partial \gamma}\right)$$

$$- RAp\left(\frac{\partial \theta}{\partial R} + \frac{\gamma B}{AR\ell}\frac{\partial \theta}{\partial s}\right) + \frac{1}{\ell}\frac{\partial \theta}{\partial s}\left(\gamma Bp + \frac{1}{A}\frac{\partial W}{\partial \gamma}\right) = 0 \quad , \qquad (6.2.8)$$

Now $\ell$ satisfies the compatibility condition

$$\frac{\partial \ell}{\partial R} + \frac{\partial}{\partial s}\left(\frac{\gamma B}{AR}\right) = 0 \quad . \qquad (6.2.9)$$

Substituting for $\frac{\partial \gamma}{\partial s}$ from (6.2.2) gives

$$A\frac{\partial}{\partial R}(R\ell) - B\frac{\partial \theta}{\partial s} = - R\ell\frac{\partial A}{\partial s} \quad . \qquad (6.2.10)$$

Using (6.2.1), (6.2.2) and (6.2.9) equations (6.2.7) and (6.2.8) become

$$\frac{1}{A}\frac{\partial}{\partial s}\left(\frac{\partial W}{\partial \gamma}\right) - A\frac{\partial}{\partial R}(R\ell p) - AT\frac{\partial}{\partial R}(\ell R)$$

$$= (T+p)\,R\ell\frac{\partial A}{\partial R} + \frac{1}{A}\frac{\partial W}{\partial \gamma}\left(\frac{1}{A}\frac{\partial A}{\partial s} - \frac{1}{B}\frac{\partial B}{\partial s}\right) \qquad (6.2.11)$$

and

$$\frac{\partial}{\partial R}\left(R\ell\frac{\partial W}{\partial \gamma}\right) + \frac{\partial W}{\partial \gamma}\frac{\partial}{\partial R}(\ell R) + B^2\frac{\partial T}{\partial s}$$

$$= \left(\frac{R\ell}{B}\frac{\partial W}{\partial \gamma} - (T+p)B\right)\frac{\partial B}{\partial s} - \frac{R\ell}{A}\frac{\partial W}{\partial \gamma}\frac{\partial A}{\partial R} \quad . \qquad (6.2.12)$$

We therefore have to solve equations (6.2.1), (6.2.2), (6.2.10), (6.2.11) and (6.2.12) for $\gamma$, p, T, $\ell$ and $\theta$. In the next section we consider solutions to the problem with A and B close to 1, that is solutions for a material which is nearly ideal.

## 6.3 DETERMINATION OF THE GEOMETRICAL CONFIGURATION USING THE IDEAL THEORY

We are seeking solutions for a material which is nearly ideal, so we write $A = 1 + \varepsilon^2 A_1$ and $B = 1 + \varepsilon^2 B_1$. We also define $\frac{\partial W}{\partial \gamma} \equiv G$. We regard equations (6.2.1), (6.2.2), (6.2.10), (6.2.11) and (6.2.12) as of the form

$$\frac{\partial \theta}{\partial R} = f_1(R, s) , \qquad (6.3.1)$$

$$\frac{\partial}{\partial R}(\ell R) - \frac{\partial \theta}{\partial s} = f_2(R, s) , \qquad (6.3.2)$$

$$\frac{\partial}{\partial s}(\gamma + \theta) - \ell = f_3(R, s) , \qquad (6.3.3)$$

$$\frac{\partial G}{\partial s} - \frac{\partial}{\partial R}(R\ell p) - T\frac{\partial}{\partial R}(\ell R) = f_4(R, s) , \qquad (6.3.4)$$

$$\frac{\partial}{\partial R}(\ell R G) + G\frac{\partial}{\partial R}(\ell R) + \frac{\partial T}{\partial s} = f_5(R, s) . \qquad (6.3.5)$$

For the outer solution, that is the solution away from the boundaries, the $f_i$'s are $O(\varepsilon^2)$ and hence vanish in the ideal theory. The functions $f_i(R, s)$ are given by

$$f_1(R, s) = -\frac{1}{AR\ell}\frac{\partial B}{\partial s} ,$$

$$f_2(R, s) = (1-A)\frac{\partial}{\partial R}(R\ell) - (1-B)\frac{\partial \theta}{\partial s} - R\ell\frac{\partial A}{\partial R} ,$$

$$f_3(R, s) = (1-B)\frac{\partial}{\partial s}(\theta+\gamma) - (1-A)\ell + R\ell\frac{\partial A}{\partial R} + \gamma\frac{B}{A}\frac{\partial A}{\partial s} - \gamma\frac{\partial B}{\partial s}$$

$$f_4(R, s) = -(1-A^2)\frac{\partial}{\partial R}(R\ell p) - (1-A^2)\,T\frac{\partial}{\partial R}(\ell R) + (T+p)R\ell A\frac{\partial A}{\partial R}$$

$$+ G\left(\frac{1}{A}\frac{\partial A}{\partial s} - \frac{1}{B}\frac{\partial B}{\partial s}\right)$$

$$f_5(R,\, s) \;=\; (1-B^2)\,\frac{\partial T}{\partial s} + \left(\frac{R\ell}{B}\,G - (T+p)B\right)\frac{\partial B}{\partial s} - \frac{R\ell G}{A}\,\frac{\partial A}{\partial R}\,.$$

We seek solutions to the system of equations (6.3.1) - (6.3.5) with $\gamma$, $p$, $T$, $\ell$ and $\psi$-s periodic in s of period $2\pi$ for all R in $R_0 \le R \le R_1$, where $R_1$ is the initial external radius of the belt. Clearly, from the definition of the $f_i$'s they will then be periodic in s for all R in $R_0 \le R \le R_1$. The compatibility conditions ensuring that this is possible give the information needed to determine the arbitrary function which the ideal theory contains. This is similar to the way in which compatibility conditions in the Rayleigh wave problem singled out certain waveforms from an arbitrary function.

Integrating equation (6.3.1) with respect to R gives:

$$\theta \;=\; \psi(s) + F_1(R,\, s)\,, \tag{6.3.6}$$

where $F_1(R,\, s) \;=\; \int_{R_0}^{R} f_1 d\tilde{R}$

and $\psi(s)$ is an arbitrary function. The function $F_1(R,\, s)$ is periodic in s, since $f_1(R,\, s)$ is periodic in s, hence $\psi$-s is periodic with period $2\pi$.

Substituting for $\frac{\partial\theta}{\partial s}$ in equation (6.3.2) and integrating with respect to R gives

$$\ell R \;=\; \phi'(s) + R\psi'(s) + RF_2(R,\, s) \tag{6.3.7}$$

where $\phi'(s)$ is an arbitrary function and $F_2(R,\, s)$ is defined by

$$F_2 \;=\; \frac{1}{R}\int_{R_0}^{R}\left(\frac{\partial F_1}{\partial s} + f_2\right)d\tilde{R}\,.$$

We have chosen $F_2(R_0,\, s) = 0$, so that the boundary condition $\ell(R_0,\, s) = 1$ then yields the condition $1 = \psi'(s) + \frac{\phi'(s)}{R_0}$. We therefore choose:

$$\phi(s) \;=\; R_0(s - \psi(s))$$

and rewrite equation (6.3.7) as

$$\ell R = R_0 + (R - R_0)\psi'(s) + RF_2(R, s). \tag{6.3.8}$$

Substituting for $\ell$ in equation (6.3.3) and integrating with respect to s gives:

$$\gamma + \theta = \frac{R_0}{R} s + \left(\frac{R - R_0}{R}\right)\psi(s) + \Gamma(R) + F_3(R, s), \tag{6.3.9}$$

where $\Gamma(R)$ is an arbitrary function and $F_3$ is given by:

$$F_3 = \int_0^s (f_3 + F_2)d\tilde{s} .$$

From (6.3.9) we find that

$$\gamma(R, s + 2\pi) - \gamma(R, s) + \theta(R, s + 2\pi) - \theta(R, s)$$

$$= \frac{R_0}{R} 2\pi + \left(\frac{R - R_0}{R}\right)(\psi(R, s + 2\pi) - \psi(R, s))$$

$$+ F_3(R, s + 2\pi) - F_3(R, s)$$

However, $\gamma$, $s-\psi$ and $s-\theta$ are $2\pi$-periodic in s, hence we must have

$$F_3(R, s + 2\pi) = F_3(R, s).$$

This requires that:

$$\int_s^{2\pi+s} (f_3 + F_2)d\tilde{s} = 0 \qquad \text{for all } s,$$

which is satisfied whenever

$$\int_0^{2\pi} (f_3 + F_2)d\tilde{s} = 0. \tag{6.3.10}$$

We now consider the boundary conditions. We let $P^+(s)$ and $Q^+(s)$ denote the tangential and normal tractions applied over the surface $R = R_1$

and $P^-(s)$ and $Q^-(s)$ denote those over the surface $R = R_0$. Hence:

$$G(R_0, s) = P^-(s) , \quad G(R_1, s) = P^+(s)$$

$$p(R_0, s) = -Q^-(s) , \quad p(R_1, s) = -Q^+(s)$$

as is shown in figure 6.2.

Now $W = W(\gamma,A,B)$, and in a similar way to that used by Green (16) we introduce $\hat{W}$ the Legendre transform of the strain energy W by

$$\hat{W} \equiv W(\gamma,A,B) - W_A A - W_B B$$

and define

$$W_A = \hat{T}_1 , \quad W_B = \hat{T}_2 \tag{6.3.11}$$

Then

$$\hat{W}_{\hat{T}_1} = -A, \quad \hat{W}_{\hat{T}_2} = -B \quad \text{and} \quad \hat{W}_\gamma = W_\gamma . \tag{6.3.12}$$

The assumptions $A = 1 + \varepsilon^2 A_1$, $B = 1 + \varepsilon^2 B_1$ imply that:

$$\hat{W} = \int g(\gamma)d\gamma - \hat{T}_1 - \hat{T}_1 - \frac{\varepsilon^2}{2}(\alpha_{11}\hat{T}_1 + 2\alpha_{12}\hat{T}_1\hat{T}_2 + \alpha_{22}\hat{T}_2) + O(\varepsilon^4) , \tag{6.3.13}$$

where $\alpha_{11}$, $\alpha_{12}$ and $\alpha_{22}$ are the compliances, which are assumed to be constant. From (6.3.12) we therefore deduce that:

$$A_1 = \alpha_{11}\hat{T}_1 + \alpha_{12}\hat{T}_2$$

$$B_1 = \alpha_{12}\hat{T}_1 + \alpha_{22}\hat{T}_2.$$

Substituting for $\hat{T}_1$, $\hat{T}_2$ from (6.3.11) using (6.3.5) and (6.3.6), we obtain expressions for $A_1$ and $B_1$:

$$A_1 = \alpha_{11}BT - \alpha_{12}Ap + \frac{\gamma G}{AB}(\alpha_{12}A - \alpha_{11}B) \tag{6.3.14}$$

$$B_1 \;=\; \alpha_{12}BT - \alpha_{22}Ap + \frac{\gamma G}{AB}(\alpha_{22}A - \alpha_{12}B) \qquad (6.3.15)$$

From equation (6.3.13) for $\hat{W}$, we see that $\frac{\partial \hat{W}}{\partial \gamma} = g(\gamma) + O(\varepsilon^2)$. Now, $\frac{\partial \hat{W}}{\partial \gamma} = \frac{\partial W}{\partial \gamma} \equiv G$, hence we see that the dominant dependence of G is on $\gamma$ and we choose $G = G(\gamma)$.

In general, using the formula already obtained for $\gamma$ in equation (6.3.9), the calculated shear stress G is not compatible with the applied shear tractions. As has already been mentioned in connection with the work of Pipkin and Rogers (43), this is resolved by including concentrated loads acting along the boundaries. The consequent high tensions T along the fibres are balanced by large values of $\frac{\partial G}{\partial R}$. The severe gradients of shear allow G to adjust to the boundary conditions through a boundary layer as has been demonstrated for linearised static deformations by Everstine and Pipkin (13) and by Spencer (53).

We define $L^-(s)$ and $L^+(s)$ as the concentrated loads carried by the layers immediately adjacent to the surfaces $R = R_0$, $R_1$ respectively:

$$L^-(s) \;=\; \int_{R_0}^{R_0^+} T\,dR, \qquad L^+(s) \;=\; \int_{R_1^-}^{R_1} T\,dR\ ,$$

where the boundary layers are of thickness $R_0^+ - R_0$ and $R_1 - R_1^-$ respectively. For the ideal theory $L^-$ and $L^+$ are concentrated loads and for the boundary layer analysis $L^-$ and $L^+$ are given by 'inner' expansions. Similarly, we let $G^+(s)$, $G^-(s)$, $p^+(s)$ and $p^-(s)$ denote the shear stress and pressure adjacent to each of these layers; they are predicted by the outer solution, being the inner limits of the outer behaviour. These are shown in figure 6.2.

Integrating equation (6.3.4) through the boundary layer $R^-_1$ to $R_1$ we obtain the equilibrium equation:

$$R_1\ell(R_1, s)\left[Q^+(s) + p^+(s)\right] - L^+(s)\frac{d\psi}{ds} \;=\; \int_{R_1^-}^{R_1}\left[f_4 + Tf_2 + T\frac{\partial F_1}{\partial s}\right] dR\ , \qquad (6.3.16)$$

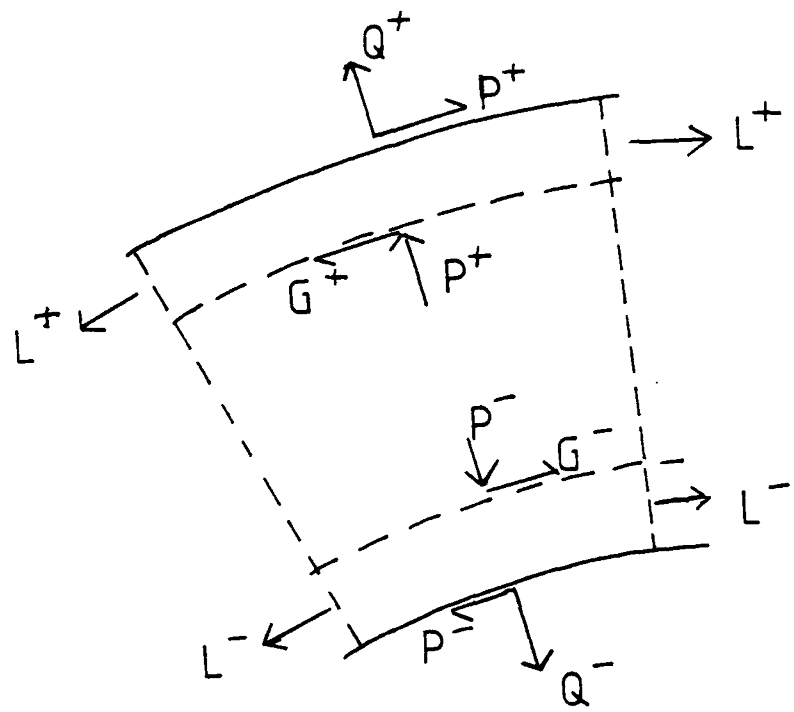



Figure 6.2 Forces acting on the boundary layers at $R = R_0$ and $R = R_1$ and the resultant loads $L^+$ and $L^-$ within these layers.

and similarly integrating equation (6.3.4) through the boundary layer $R_0$ to $R_0^+$ we obtain

$$-R_0\ell(R_0, s)\left[p^-(s) + Q^-(s)\right] - L^-(s)\,\frac{d\psi}{ds}$$

$$= \int_{R_0}^{R_0^+}\left[f_4 + Tf_2 + T\,\frac{\partial F_1}{\partial s}\right]dR\ . \qquad (6.3.17)$$

Integrating equation (6.3.5) from $R^-_1$ to $R_1$ gives

$$\frac{dL^+}{ds} + \ell(R_1, s)R_1\left[P^+(s) - G^+(s)\right] = \int_{R_1^-}^{R_1}\left[f_5 - Gf_2 - G\,\frac{\partial F_1}{\partial s}\right]dR \quad (6.3.18)$$

and integrating this equation from $R_0$ to $R^+_0$ gives:

$$\frac{dL^-}{ds} + \ell(R_0, s)R_0\left[G^-(s) - P^-(s)\right] = \int_{R_0}^{R_0^+}\left[f_5 - Gf_2 - G\,\frac{\partial F_1}{\partial s}\right]dR. \quad (6.3.19)$$

Integrating equation (6.3.4) from $R_0^+$ to $R_1^-$ through the belt and defining the resultant load

$$U(s) = \int_{R_0^+}^{R_1^-} TdR + L^+(s) + L^-(s) \qquad (6.3.20)$$

and the resultant shear load

$$S(s) = \int_{R_0^+}^{R_1^-} GdR$$

gives

$$\frac{dS}{ds} - R_1\ell(R_1,s)p^+(s) + R_0\ell(R_0,s)p^-(s) - \frac{d\psi}{ds}\left[U(s) - L^+ - L^-\right]$$

$$= \int_{R_0^+}^{R_1^-} \left[f_4 + Tf_2 + T\frac{\partial F_1}{\partial s}\right] dR$$

and

$$\frac{dU}{ds} - \frac{dL^+}{ds} - \frac{dL^-}{ds} + R_1\ell(R_1,s)G^+ - R_0\ell(R_0,s)G^- + S(s)\frac{d\psi}{ds}$$

$$= \int_{R_0^+}^{R_1^-} \left[f_5 - Gf_2 - G\frac{\partial F_1}{\partial s}\right] dR \ .$$

Substituting for $p^+$, $p^-$, $\frac{dL^+}{ds}$ and $\frac{dL^-}{ds}$ from equations (6.3.16) - (6.3.19), these equations become:

$$\frac{dS}{ds} + R_1\ell(R_1,s)Q^+(s) - R_0\ell(R_0,s)Q^-(s) - U\frac{d\psi}{ds}$$

$$= \int_{R_0}^{R_1} (f_4 + Tf_2 + T\frac{\partial F_1}{\partial s})ds \qquad (6.3.21)$$

and

$$\frac{dU}{ds} + R_1\ell(R_1,s)P^+(s) - R_0\ell(R_0,s)P^-(s) + S\frac{d\psi}{ds}$$

$$= \int_{R_0}^{R_1} (f_5 - Gf_2 - G\frac{\partial F_1}{\partial s})dR \qquad (6.3.22)$$

The right-hand sides of these equations are small, since in the 'outer' region the functions $f_i$ and $F_i$ are $O(\varepsilon^2)$ so that the integrands are small and the region of integration is small in the 'inner' regions.

In the ideal case the $f_i$ are zero, $i=1,\ldots,5$, so that in this case (6.3.21) and (6.3.22) become:

$$\frac{dS}{ds} + R_1\ell(R_1,s)Q^+(s) - R_0\ell(R_0,s)Q^-(s) - U\frac{d\psi}{ds} = 0 \qquad (6.3.23)$$

and

$$\frac{dU}{ds} + R_1\ell(R_1,s)P^+(s) - R_0\ell(R_0,s)P^-(s) + S\frac{d\psi}{ds} = 0. \qquad (6.3.24)$$

On the unloaded portions, that is the free sections of the belt between the pulleys, $P^+$, $P^-$, $Q^+$ and $Q^-$ are zero and hence (6.3.23) and (6.3.24) reduce to:

$$\frac{dS}{ds} - W\frac{d\psi}{ds} = 0$$

and (6.3.25)

$$\frac{dU}{ds} + S\frac{d\psi}{ds} = 0$$

on these portions, also since we are considering the ideal case $\theta = \psi(s)$. We therefore obtain the solutions

$$\begin{aligned} S &= Y\cos\theta + Z\sin\theta \\ U &= -Y\sin\theta + Z\cos\theta \end{aligned} \qquad (6.3.26)$$

on the unloaded portions. These show that the resultant load is constant with components Y and Z.

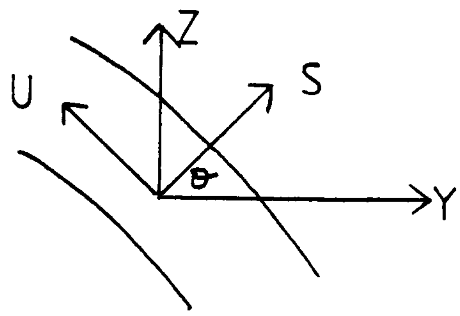



Figure 6.3  The resultant loads acting on the belt

We now turn our attention to the equation for the tension. Considering this for the ideal theory and applying the periodicity constraint yields an equation for $\Gamma(R)$ the arbitrary function introduced in the expression (6.3.9) for $\gamma$ which in the ideal case reduces to

$$\gamma = \frac{R_0}{R}(s-\psi) + \Gamma(R) . \qquad (6.3.27)$$

Integrating equation (6.3.5) with respect to s gives for the ideal case:

$$T + \int^{s}\left[\frac{\partial}{\partial R}(\ell RG) + G\frac{\partial}{\partial R}(\ell R)\right]ds = T_0(R) .$$

Now we are restricting attention to $G = G(\gamma)$ and Treloar (56) shows that G proportional to $\gamma$ is a good approximation to the experimental data, where the constant of proportionality is the shear modulus. We may therefore non-dimensionalize all stresses and loads with respect to this modulus and take

$$G = \gamma . \qquad (6.3.28)$$

Substituting for $\gamma$ from equation (6.3.27), the above equation reduces to:

$$T + \frac{1}{R}\int(s-\psi)\psi' ds + \Gamma(R)\psi(s) + \frac{\partial}{\partial R}\int\frac{(s-\psi)}{R}ds + \Gamma'(R)s$$

$$+ \frac{1}{R^2}\int(s-\psi)\psi' ds + \frac{\partial}{\partial R}\left[(R-1)\Gamma(R)\right]\int\psi'(s)ds = T_0(R) \qquad (6.3.29)$$

Now T must be periodic in s with period $2\pi$ and hence writing $T(R,s_0) = T(R,s_0 + 2\pi)$ and using the fact that $\psi(s_0 + 2\pi) - \psi(s_0) = 2\pi$ for all s , we may obtain from (6.3.29) an equation which $\Gamma(R)$ must satisfy, namely:

$$\frac{d}{dR}(R\Gamma(R)) = -(s_0 + \pi - \frac{1}{2\pi}\int_{s_0}^{s_0+2\pi}\psi ds) . \tag{6.3.30}$$

It may easily be shown that the right-hand side of equation (6.3.30) is independent of $s_0$. We therefore choose $s_0 = 0$ and obtain

$$\Gamma(R) = -\frac{1}{R}(\pi - \frac{1}{2\pi}\int_0^{2\pi}\psi ds) + \frac{k}{R^2} , \tag{6.3.31}$$

where k is a constant.

In the ideal case (6.3.18) and (6.3.19) become:

$$\frac{dL^+}{ds} = R_1\ell(R_1,s)\left[G^+(s) - P^+(s)\right]$$

and

$$\frac{dL^-}{ds} = R_0\ell(R_0,s)\left[P^-(s) - G^-(s)\right] .$$

If we are considering a belt which is stretched round the pulleys so that the same surface is always in contact with the pulleys, the pressure applied on the outside of the belt may be taken to be zero, that is $P^+ = 0$ for all s. Hence, substituting $G = \gamma$ in the above, we obtain:

$$\frac{dL^+}{ds} = \left[R_0 + (R_1-R_0)\psi'(s)\right]\left[(s-\psi)\frac{R_0}{R_1} + \Gamma(R_1)\right] \tag{6.3.32}$$

and

$$\frac{dL^-}{ds} = R_0\ell(R_0,s)\left[P^-(s) - (s-\psi) - \Gamma(R_0)\right] . \tag{6.3.33}$$

Now $L^+$ and $L^-$ are $2\pi$-periodic in s, so that

$$\int_{s_0}^{s_0+2\pi}\frac{dL^+}{ds}ds = 0 \tag{6.3.34}$$

and

$$\int_{s_0}^{s_0+2\pi}\frac{dL^-}{ds}ds = 0 \tag{6.3.35}$$

for all $s_0$. Equation (6.3.34) therefore implies that

$$2\pi s_0 + 2\pi^2 - \int_{s_0}^{s_0+2\pi} \psi ds + 2\pi R_1 \Gamma(R_1) = 0 , \quad \text{for all } s_0,$$

choosing $s_0 = 0$ and substituting for $\Gamma(R)$ from (6.3.31) this reduces to:

$$\frac{2\pi k}{R_1} = 0 \qquad \text{hence} \quad k = 0$$

and (6.3.31) becomes

$$\Gamma(R) = -\frac{1}{R}\left(\pi - \frac{1}{2\pi}\int_0^{2\pi} \psi ds\right) . \tag{6.3.36}$$

Equation (6.3.35) implies that

$$\int_{s_0}^{s_0+2\pi} P^-(s)ds = 0 \tag{6.3.37}$$

Therefore, the integral round the belt of the shear loading on the internal surface of the belt must be zero.

We now consider a portion of the belt connecting two pulleys, the $i^{th}$ and $(i+1)^{th}$, say. In general, we let the $i^{th}$ pulley be centred at $(x_i, y_i)$ with radius $r_i$, the belt joining the pulley at an angle $\theta_{2i-1}$ to the reference direction and leaving it at an angle $\theta_{2i}$.

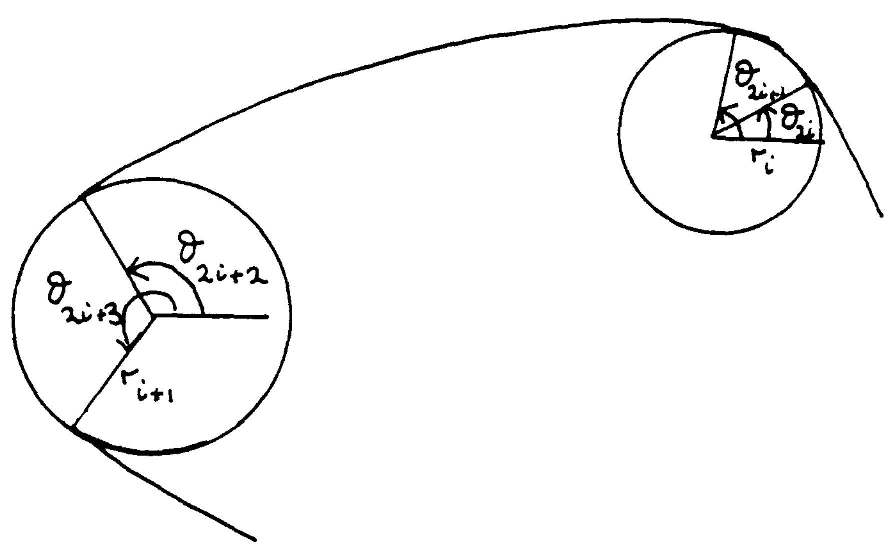



Figure 6.4 The geometrical arrangement of the belt round two consecutive pulleys

The pulley centres are fixed and the horizontal and vertical components of the distance between them must be the same as those obtained from integrating round the inner fibre of the belt. These restraints give two relations between the angles and the distances:

$$x_{i+1} - x_i = r_i \cos \theta_{2i} - r_{i+1} \cos \theta_{2i+1} - \int_{\theta_{2i}}^{\theta_{2i+1}} \frac{\sin \theta}{\theta'(s)} d\theta \qquad (6.3.38)$$

$$y_{i+1} - y_i = r_i \sin \theta_{2i} - r_{i+1} \sin \theta_{2i+1} + \int_{\theta_{2i}}^{\theta_{2i+1}} \frac{\cos \theta}{\theta'(s)} d\theta \qquad (6.3.39)$$

On the pulleys the radius of curvature of the belt is fixed, so that on the pulley radius $r_i$,

$$\psi'(s) = \frac{1}{r_i} \qquad (6.3.39)$$

therefore

$$\psi(s) = \frac{s}{r_i} + \lambda_i \quad . \qquad (6.3.40)$$

The portions of the belt between the pulleys are unloaded so that $P^+$, $P^-$, $Q^+$ and $Q^-$ are all zero on these sections, and we have shown (6.3.25) that in this case, for the ideal material

$$S = Y_i \cos \theta + Z_i \sin \theta \qquad (6.3.41)$$

for the portion of the belt between the pulleys of radii $r_i$ and $r_{i+1}$. However,

$$S = \int_{R_0}^{R_1} G \, dR$$

$$= \int_{R_0}^{R_1} \left[ \frac{R_0}{R} (s-\psi) + \Gamma(R) \right] dR$$

Hence:

$$S = R_0(s-\psi) \; \ell n \frac{R_1}{R_0} + \int_{R_0}^{R_1} \Gamma(R) dR \; . \qquad (6.3.42)$$

Equating the two expressions for S in (6.3.41) and (6.3.42) we obtain

$$R_0(s-\psi)\ \ell n\,\frac{R_1}{R_0} + \int_{R_0}^{R_1}\Gamma(R)dR \;=\; Y_i\cos\theta + Z_i\sin\theta\ . \tag{6.3.43}$$

Differentiating (6.3.43) with respect to s gives:

$$R_0(1-\psi')\ \ell n\,\frac{R_1}{R_0} \;=\; (-Y_i\sin\theta + Z_i\cos\theta)\psi'\ .$$

Hence

$$\frac{d\psi}{ds} \;=\; \frac{R_0\ell n(R_1/R_0)}{R_0\ell n(R_1/R_0) - Y_i\sin\psi + Z_i\cos\psi}\ . \tag{6.3.44}$$

which may be integrated to give:

$$R_0\ell n\,\frac{R_1}{R_0}\,(\psi-s) + Y_i\cos\psi + Z_i\sin\psi \;=\; \text{constant} \tag{6.3.45}$$

on the free portion of the belt between the pulleys radii $r_i$ and $r_{i+1}$.

We now consider the pulley radius $r_i$ and suppose that we apply a torque of moment $M_i$ on this pulley. Taking moments about the centre of the pulley, $O_i$, we obtain:

$$M_i \;=\; r_i\Big[U(\theta_{2i}) - U(\theta_{2i-1})\Big]\ .$$

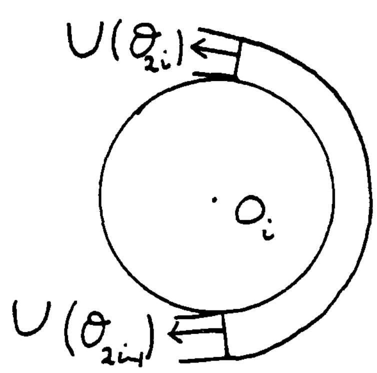



Figure 6.5 The load acting on the belt round the $i^{th}$ pulley

However, from (6.3.26):

$$U = -Y_i \sin\theta + Z_i \cos\theta \ .$$

Hence

$$M_i = r_i \left[ - Y_i \sin\theta_{2i} + Z_i \cos\theta_{2i} + Y_i \sin\theta_{2i-1} - Z_i \cos\theta_{2i-1} \right] \qquad (6.3.46)$$

and also

$$M_i = -r_i \int_{s_{2i-1}}^{s_{2i}} P_i^-(s)ds \ . \qquad (6.3.47)$$

In this section we have, therefore, shown how to obtain the geometrical configuration using the ideal theory. The periodicity constraints have been employed to determine the arbitrary function $\Gamma(R)$ introduced in the solution for the shear $\gamma$ and the tension determined to within an arbitrary function $T_0(R)$, which is constant along the fibres.

## 6.4 CALCULATION OF THE TENSION FUNCTION $T_0(R)$, WHICH IS CONSTANT ALONG THE FIBRES

The configurations in the previous section, described by the ideal theory involve an arbitrary function $T_0(R)$, which can be obtained only by considering the effect of small extensibility and compressibility. We are considering a regular expansion and returning to equation (6.3.10) we find on substituting for $F_2$ that:

$$\int_0^{2\pi}(f_3 + \frac{1}{R}\int^R f_2 dR)ds = 0 . \tag{6.4.1}$$

Now,

$$f_3 = (1-B)\frac{\partial}{\partial s}(\theta+\gamma) - (1-A)\ell + R\ell\frac{\partial A}{\partial R} + \frac{\gamma B}{A}\frac{\partial A}{\partial s} - \gamma\frac{\partial B}{\partial s}$$

and $$f_2 = (1-A)\frac{\partial}{\partial R}(R\ell) - (1-B)\frac{\partial\theta}{\partial s} - R\ell\frac{\partial A}{\partial R}$$

where $A = 1 + \varepsilon^2 A_1$

and $B = 1 + \varepsilon^2 B_1$

so that, within the belt:

$$f_3 = \left[(A_1-B_1)\ell + R\ell\frac{\partial A_1}{\partial R} + \gamma\frac{\partial A_1}{\partial s} - \gamma\frac{\partial B_1}{\partial s}\right] + O(\varepsilon^4)$$

and

$$f_2 = \left[(B_1-A_1)\psi'(s) - R\ell\frac{\partial A_1}{\partial R}\right] + O(\varepsilon^4) .$$

Substituting these into (6.4.1) gives

$$\int_0^{2\pi}\left[(A_1-B_1)\ell + R\ell\frac{\partial A_1}{\partial R} + \gamma\frac{\partial A_1}{\partial s} - \gamma\frac{\partial B_1}{\partial s} + \frac{1}{R}\psi'(s)\int^R(B_1-A_1)dR\right.$$

$$\left. - \frac{1}{R}\int^R R\ell\frac{\partial A_1}{\partial R}dR\right]ds = O(\varepsilon^2) . \tag{6.4.2}$$

Substituting for A and B in (6.4.2) gives the equation:

$$\int_0^{2\pi}\Big\{\Big[(\alpha_{11}-\alpha_{12})T - (\alpha_{12}-\alpha_{22})p + \gamma G(2\alpha_{12}-\alpha_{11}-\alpha_{22})\Big]\ell$$

$$+ R\ell\Big[\alpha_{11}\frac{\partial T}{\partial R} - \alpha_{12}\frac{\partial p}{\partial R} + (\alpha_{12}-\alpha_{11})\frac{\partial}{\partial R}(\gamma G)\Big]$$

$$+ \gamma\Big[(\alpha_{11}-\alpha_{12})\frac{\partial T}{\partial s} - (\alpha_{12}-\alpha_{22})\frac{\partial p}{\partial s} + (2\alpha_{12}-\alpha_{11}-\alpha_{22})\frac{\partial}{\partial s}(\gamma G)\Big]$$

$$+ \frac{1}{R}\psi'(s)\int^R\Big[-(\alpha_{11}-\alpha_{12})T + (\alpha_{12}-\alpha_{22})p - \gamma G(2\alpha_{12}-\alpha_{11}-\alpha_{22})\Big]dR$$

$$- \frac{1}{R}\int^R R\ell\Big[\alpha_{11}\frac{\partial T}{\partial R} - \alpha_{12}\frac{\partial p}{\partial R} + (\alpha_{12}-\alpha_{11})\frac{\partial}{\partial R}(\gamma G)\Big]dR\Big\}ds$$

$$= O(\varepsilon^2) \quad . \tag{6.4.3}$$

We therefore need to substitute expressions for T and p in terms of known functions and $T_0(R)$ to determine an equation for $T_0(R)$.

Integrating equation (6.3.4) for $R_0^+$ to R gives:

$$\frac{\partial}{\partial s}\int_{R_0^+}^R G dR - R\ell p + R_0 p^-(s) - \psi'(s)\int_{R_0^+}^R T dR = O(\varepsilon^2) \quad .$$

We now substitute $G = \gamma$ from (6.3.25) into this to give:

$$\Big[1 - \psi'(s)\Big]R_0\ \ell n\frac{R}{R_0} - R\ell p + R_0 p^-(s) - \psi'(s)\int_{R_0^+}^R T dR = O(\varepsilon^2) \ .$$

Now from (6.3.17):

$$R_0 p^-(s) = - R_0 Q^-(s) - L^-(s)\psi'(s),$$

hence

$$R\ell p = (1 - \psi'(s))R_0\ \ell n\frac{R}{R_0} - R_0 Q^-(s) - \psi'(s)\Big[L^-(s) + \int_{R_0^+}^R T dR\Big]$$

$$+ O(\varepsilon^2) \ .$$

Also from equation (6.3.29):

$$T = T_0(R) - \frac{R_0}{R}\int(s-\psi)\psi' ds - \Gamma(R)\int\psi'(s)ds$$
$$- \int\frac{\partial}{\partial R}\left[(R_0 + (R - R_0)\psi')\left(\frac{R_0}{R}(s - \psi) + \Gamma(R)\right)\right]ds$$
$$+ O(\varepsilon^2) . \qquad (6.4.4)$$

In order to obtain an equation for T(R) we therefore write $Q^-(s)$ and $L^-(s)$ in terms of functions already known.

From equations (6.3.23) and (6.3.24) setting $P^-(s)$ and $Q^+(s)$ zero since the outer surface of the belt is unloaded we obtain:

$$\frac{dS}{ds} - U\frac{d\psi}{ds} = R_0 Q^-(s) \qquad (6.4.5)$$

$$\frac{dU}{ds} + S\frac{d\psi}{ds} = R_0 P^-(s) \qquad (6.4.6)$$

On the free portions of the belt $Q^-(s)$ and $P^-(s)$ are zero, since these sections are unloaded. We consider the situation on the pulleys, where $\frac{d\psi}{ds} = \frac{1}{r_i}$. Then, eliminating U(s) from equation (6.4.5) and (6.4.6) we find

$$\frac{d^2S}{ds^2} + \frac{1}{r_i^{\,2}}\frac{dS}{ds} = R_0\frac{d\,Q_i^-(s)}{ds} + \frac{R_0}{r_i}P_i^-(s)$$

where $Q = Q_i$, $P = P_i$ on the pulley radius $r_i$. However:

$$S(s) = R_0(s - \psi)\,\ell n\,\frac{R_1}{R_0} + \int_{R_0}^{R_1}\Gamma(R)dR ,$$

which implies that:

$$\frac{1}{r_i^{\,2}}\left\{R_0(s - \psi)\,\ell n\,\frac{R_1}{R_0} + \int_{R_0}^{R_1}\Gamma(R)dR\right\} = R_0\frac{dQ_i^-}{ds} + \frac{R_0}{r_i}P_i^-, \quad i = 1, 2. \qquad (6.4.7)$$

Now equation (6.3.47) gives

$$\int_{s_{2i-1}}^{s_{2i}} P_i^-(s)ds = -\frac{M_i}{r_i} \qquad (6.4.8)$$

and since from equation (6.3.37) $\int_0^{2\pi} P^-(s)ds = 0$, we have

$$\sum_{\substack{j \\ j\neq i}} \int_{s_{2j-1}}^{s_{2j}} P_j^-(s)ds = \frac{M_i}{r_i} \quad . \tag{6.4.9}$$

Hence, specifying a friction law will give one relationship between $Q_i$ and $P_i$, which will make (6.4.7) into a differential equation for $P_i$. The conditions (6.4.8) and (6.4.9) determine the arbitrary constants introduced in the solution of these equations.

From equation (6.3.33),

$$\frac{dL^-}{ds} = R_0\left(P^-(s) - \frac{(s-\psi)}{R_0} - \Gamma(R_0)\right) \quad ,$$

hence

$$L^- = R_0 \int_0^s P^-(s)ds - \frac{s^2}{2} + \int_0^s \psi ds - s\, R_0\Gamma(R_0) + \mu^-$$

where $L^-(0) = \mu^-$.

Also, from equation (6.3.22),

$$\frac{dL^+}{ds} = \left[R_0 + (R_1 - R_0)\psi'(s)\right]\left[(s - \psi)\,\frac{R_0}{R_1} + \Gamma(R_1)\right]$$

hence

$$L^+ = \frac{R_0^2}{R_1}\int_0^s (s-\psi)ds + (R_1 - R_0)\,\frac{R_0}{R_1}\int_0^s \psi'(s-\psi)ds$$

$$+ R_0\Gamma(R_1)s + (R_1 - R_0)\Gamma(R_1)\psi(s) + \psi^+$$

where $L^+(0) = \mu^+$.

The constants $\mu^-$ and $\mu^+$ are obtained by considering the analysis of the boundary layers on $R = R_0$ and $R = R_1$, respectively.

Substituting these functions into equation (6.4.3) we find that the equation for $T_0(R)$ is of the form

$$\int_0^{2\pi}\int_{R_0}^{R} b(R,s)\left(\int_{R_0}^{R} T_0(\tilde{R})d\tilde{R}\right)dR\ ds + \frac{1}{R}\int_{R_0}^{R} T_0(R)dR \int_0^{2\pi} c(R,s)ds$$

$$+ d(R)T_0(R) + e(R)T_0{}'(R) = f(R). \tag{6.4.10}$$

This equation is analysed by defining

$$\tau(R) = \int_{R_0}^{R} T_0(R)dR ,$$

which gives

$$\int_0^{2\pi}\int_{R_0}^{R} b(R,s)\tau(R)dR + \frac{1}{R}\tau(R)\int_0^{2\pi} c(R,s)ds + d(R)\ \tau'(R)$$

$$+ e(R)\ \tau''(R) = f(R) . \tag{6.4.11}$$

From the definition of $\tau(R)$ we see that

$$\tau(R_0) = 0 ,$$

also $\tau'(R_0)$may be obtained by specifying the load, which is equivalent to specifying the separation L of the pulleys, see section 7.3. Solving (6.4.11) with these boundary conditions then gives us the function $\tau(R)$ from which we may obtain the arbitrary function $T_0(R)$ in the tension.

## 6.5 BOUNDARY LAYER ANALYSIS

In this section we determine the constants $\mu^-$ and $\mu^+$ introduced in the ideal solution for the concentrated loads $L^-(s)$ and $L^+(s)$ on the surfaces $R = R_0$ and $R = R_1$.

In the boundary layers we write $R = R_i + \lambda_1(\varepsilon)\rho$, $i = 0,1$, with $\rho > 0$ for the boundary layer near the inner surface $R = R_0$ and $\rho < 0$ for the boundary layer near the outer surface $R = R_1$ and where $\lambda_1$ is a small parameter with $\lambda_1 \to 0$ as $\varepsilon \to 0$. The exact equations written in the boundary layer variables then become:

$$\frac{1}{\lambda_1(\varepsilon)}\frac{\partial\theta}{\partial\rho} = -\frac{1}{A(R_i + \lambda_1(\varepsilon)\rho)\ell}\,\varepsilon^2\frac{\partial B_1}{\partial s} \tag{6.5.1}$$

$$\frac{1}{\lambda(\varepsilon)_1}\frac{\partial}{\partial\rho}(\ell(R_i + \lambda_1(\varepsilon)\rho)) - \frac{\partial\theta}{\partial s} = -\frac{\varepsilon^2 A_1}{\lambda_1(\varepsilon)}\frac{\partial}{\partial s}(\ell(R_i + \lambda_1(\varepsilon)\rho))$$

$$+\ \varepsilon^2 B_1\frac{\partial\theta}{\partial s} + \frac{(R_i + \lambda_1(\varepsilon)\rho)}{\lambda_1(\varepsilon)}\ell\varepsilon^2\frac{\partial A_1}{\partial\rho} \tag{6.5.2}$$

$$\frac{\partial}{\partial s}(\theta + \gamma) - \ell = \varepsilon^2 A_1\ell - \varepsilon^2 B_1\frac{\partial}{\partial s}(\theta + \gamma)$$

$$+\ \frac{(R_i + \lambda_1(\varepsilon)\rho)\ell}{\lambda_1(\varepsilon)}\varepsilon^2\frac{\partial A_1}{\partial\rho} + \frac{\gamma B}{A}\varepsilon^2\frac{\partial A_1}{\partial s} - \varepsilon^2\gamma\frac{\partial B_1}{\partial s} \tag{6.5.3}$$

$$\frac{\partial G}{\partial s} - \frac{1}{\lambda_1(\varepsilon)}\frac{\partial}{\partial\rho}((R_i + \lambda_1(\varepsilon)\rho)\ell p) - \frac{(T-p)}{\lambda_1(\varepsilon)}\frac{\partial}{\partial\rho}(\ell(R_i + \lambda_1(\varepsilon)\rho))$$

$$= \frac{\varepsilon^2 A_1}{\lambda_1(\varepsilon)}\frac{\partial}{\partial\rho}((R_i + \lambda_1(\varepsilon)\rho)\ell p) + \frac{\varepsilon^2 A_1(T-p)}{\lambda_1(\varepsilon)}\frac{\partial}{\partial\rho}(\ell(R_i + \lambda_1(\varepsilon)\rho))$$

$$+\ (R_i + \lambda_1(\varepsilon)\rho)\,T\ell\,\frac{A\varepsilon^2}{\lambda_1(\varepsilon)}\frac{\partial A_1}{\partial\rho} + G\varepsilon^2\left(\frac{1}{A}\frac{\partial A_1}{\partial\rho} - \frac{1}{B}\frac{\partial B_1}{\partial\rho}\right) \tag{6.5.4}$$

$$\frac{1}{\lambda_1(\varepsilon)} \frac{\partial}{\partial \rho} (\ell(R_i + \lambda_1(\varepsilon)\rho)G) + \frac{G}{\lambda_1(\varepsilon)} \frac{\partial}{\partial \rho} (\ell(R_i + \lambda_1(\varepsilon)\rho))$$

$$+ \frac{\partial}{\partial s} (T-p) = - \varepsilon^2 B_1 \frac{\partial}{\partial s} (T-p) + ((R_i + \lambda_1(\varepsilon)\rho) \frac{\ell G}{B} - TB)\varepsilon^2 \frac{\partial B_1}{\partial s}$$

$$- \frac{\ell}{A} (R_i + \lambda_1(\varepsilon)\rho) \frac{\varepsilon^2}{\lambda_1(\varepsilon)} \frac{\partial A_1}{\partial \rho} \tag{6.5.5}$$

Now within the boundary layers the fibres are subjected to large stresses in order to balance the large gradients of shear. From equation (6.5.5) we observe that $T = O(1/\lambda_1(\varepsilon))$, we therefore write:

$$T = \frac{1}{\lambda_1(\varepsilon)} \tilde{T} ,$$

then from (6.2.14) and (6.2.15) $A_1$ and $B_1$ may be written as:

$$A_1 = \frac{\alpha_{11}}{\lambda_1(\varepsilon)} \tilde{T} - \alpha_{12}p + (\alpha_{12} - \alpha_{11})\gamma G ,$$

and

$$B_1 = \frac{\alpha_{12}}{\lambda_1(\varepsilon)} \tilde{T} - \alpha_{22}p + (\alpha_{22} - \alpha_{12})\gamma G .$$

We also write:

$$\theta = \theta_0 + \lambda_1(\varepsilon)\theta_1 + \ldots .$$

$$\ell = \ell_0 + \lambda_1(\varepsilon)\ell_1 + \ldots .$$

$$\gamma = \gamma_0 + \lambda_1(\varepsilon)\gamma_1 + \ldots .$$

$$p = p_0 + \lambda_1(\varepsilon)p_1 + \ldots ..$$

and substitute $G = \gamma$. If we consider leading order terms, then $L^- \sim \int_0^\infty \tilde{T} d\rho$ and $L^+ \sim \int_{-\infty}^0 \tilde{T} d\rho$, hence:

$$\mu^- \sim \int_0^\infty T_{-1}(\rho,0)d\rho \tag{6.5.6}$$

and

$$\mu^+ \sim \int_{-\infty}^0 T_{-1}(\rho,0)d\rho , \tag{6.5.7}$$

where $\tilde{T} = T_{-1} + \lambda_1(\varepsilon)T_0 + \ldots$ .

The solutions in the boundary layers must also match with the solutions in the main part of the belt. We choose $\lambda_1(\varepsilon) = \varepsilon$ and obtain from equations (6.5.1) - (6.5.5) the following equations:

$$O(1) \text{ in } (6.5.1) \qquad \frac{\partial \theta_0}{\partial \rho} = 0 \tag{6.5.8}$$

$$O(1/\varepsilon) \text{ in } (6.5.2) \qquad \frac{\partial \ell_0}{\partial \rho} = 0 \tag{6.5.9}$$

$$O(1) \text{ in } (6.5.2) \qquad R_i \frac{\partial \ell_1}{\partial \rho} + \ell_0 - \frac{\partial \theta_0}{\partial s} = \alpha_{11} R_i \ell_0 \frac{\partial T_{-1}}{\partial \rho} \tag{6.5.10}$$

$$O(1) \text{ in } (6.5.3) \qquad \frac{\partial}{\partial s}(\theta_0 + \gamma_0) - \ell_0 = R_i \ell_0 \alpha_{11} \frac{\partial T_{-1}}{\partial \rho} \tag{6.5.11}$$

$O(1/\varepsilon)$ in (6.5.4)

$$-R_i \frac{\partial}{\partial \rho}(\ell_0 p_0) - T_{-1}\left(\frac{\partial \ell_0}{\partial \rho} + R_i \frac{\partial \ell_1}{\partial \rho}\right) = R_i \ell_0 (T_{-1} + p_{-1}) \alpha_{11} \frac{\partial T_{-1}}{\partial \rho} \tag{6.5.12}$$

$$O(1/\varepsilon) \text{ in } (6.5.5) \qquad \frac{\partial}{\partial \rho}(\ell_0 R_i \gamma_0) + \frac{\partial T_{-1}}{\partial s} = 0 \tag{6.5.13}$$

Equation (6.5.8) gives $\theta_0 = \psi_0(s)$ and equation (6.5.9) gives $\ell_0 = \ell_0(s)$ matching with the solution (6.3.8) for $\ell$ in the main part of the belt we obtain

$$\ell_0 = \frac{1}{R_i}(R_0 + (R_i - R_0)\gamma_0').$$

Equations (6.5.11) and (6.5.12) may then be written in the form:

$$\frac{\partial}{\partial s}\left[\gamma_0 - \frac{R_0}{R_i}(s - \psi_0)\right] = (R_0 + (R_i - R_0)\psi_0')\alpha_{11} \frac{\partial T_{-1}}{\partial \rho} \tag{6.5.14}$$

and

$$(R_0 + (R_i - R_0)\gamma_0') \frac{\partial}{\partial \rho}\left[\gamma_0 - \frac{R_0}{R_i}(s - \psi_0)\right] = -\frac{\partial T_{-1}}{\partial s} \tag{6.5.15}$$

from which we may obtain $T_{-1}$ with $T_{-1} \to 0$ as $\rho \to \infty$ for the boundary layer on $R = R_0$ and $T_{-1} \to 0$ as $\rho \to -\infty$ for the boundary layer on $R = R_1$. The boundary conditions on $\gamma_0$ are $\gamma_0 = 0$ on $R = R_0$ or $R = R_1$, that is on $\rho = 0$ and $\gamma_0 \to (s-\gamma_0) + \Gamma(R_0)$ as $\rho \to \infty$ for the boundary layer on $R = R_0$ and $\gamma \to \frac{R_0}{R_1}(s-\psi_0) + \Gamma(R_1)$ as $\rho \to -\infty$ for the boundary layer on $R = R_1$. To solve equations (6.5.14) and (6.5.15) we make the transformation of variables:

$$y = \frac{\rho}{\sqrt{\alpha_{11}}}, \quad x = R_0 s + (R_i - R_0)\psi_0(s)$$

and write

$$u = \gamma_0 - \frac{R_0}{R_i}(s-\psi_0)$$

$$v = \sqrt{\alpha_{11}}\, T_{-1},$$

then equations (6.5.14) and (6.5.15) become

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \tag{6.5.16}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{6.5.17}$$

with

$$u = -\frac{R_0}{R_i}(s-\psi_0) + \Gamma(R_i) \text{ on } y = 0$$

and

$$v \to 0 \qquad \text{as } y \to \infty$$

$$u \to \Gamma(R_0) \qquad \text{as } y \to \infty$$

for the boundary layer on $R = R_0$,

while $v \to 0$ as $y \to -\infty$

$$u \to \Gamma(R_1) \qquad \text{as } y \to -\infty$$

for the boundary layer on $R = R_1$.

From the solution to equations (6.5.16) and (6.5.17) we may determine the functions $T_{-1}(\rho, s)$ and hence from equations (6.5.6) and (6.5.7) the constants $\mu^-$ and $\mu^+$ required for the loads $L^-(s)$ and $L^+(s)$.

In this chapter we have used the ideal theory to derive equations to determine the geometrical configuration of a fibre-reinforced belt stretched round a system of pulleys. The ideal theory may be used to determine the tension in the belt to within an arbitrary function $T_0(R)$. We have shown how this function may be calculated by considering a belt of small extensibility and compressibility. In the next chapter we obtain solutions for the case of a belt stretched round two pulleys.

# CHAPTER 7

# APPLICATION OF THE GENERAL THEORY OF A FAN-BELT TO THE SPECIFIC EXAMPLE OF A BELT STRETCHED ROUND TWO PULLEYS

## 7.1 THE EQUATIONS OF THE GEOMETRICAL CONFIGURATION

In this chapter we apply the theory developed in the previous chapter to the case of a belt stretched round two pulleys. The configuration is shown below where $r_1$ and $r_2$ the radii of the pulleys, the coordinates of their centres $(x_1,0)$, $(x_2,0)$, $R_0$ and $R_1$ the initial inner and outer radii of the belt and $M_1$ the applied torque are all specified. We let $L = |x_1 - x_2|$.

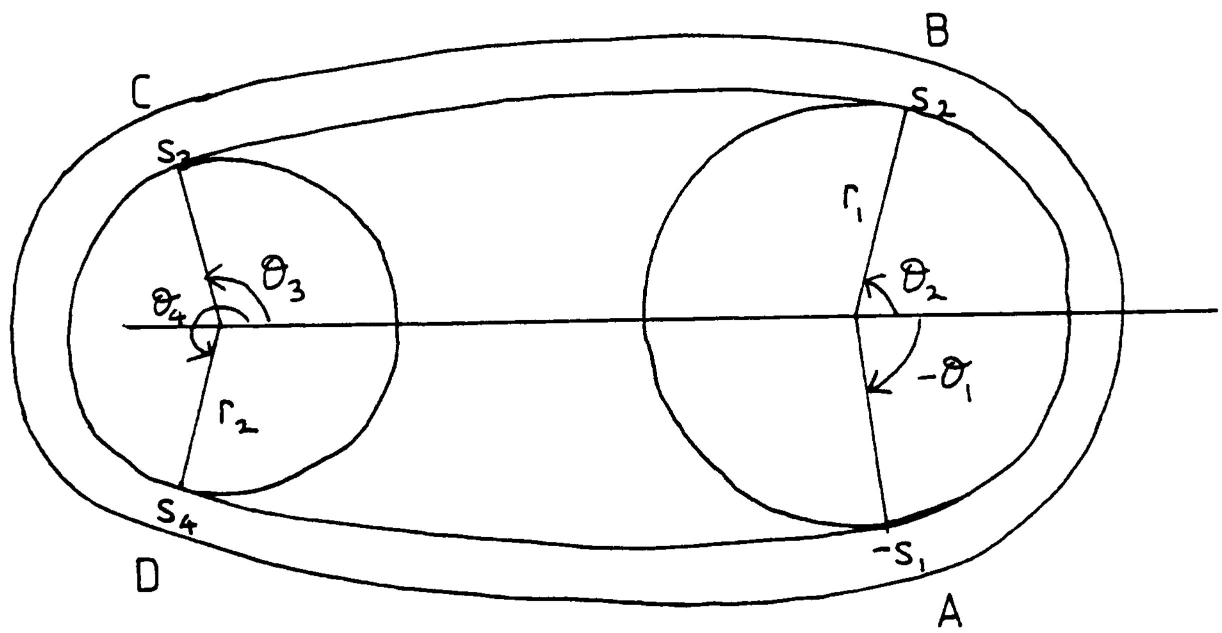



Figure 7.1 The geometrical arrangement of the belt round two pulleys

We consider initially the ideal theory to determine the parameters $\theta_1$, $\theta_2$, $\theta_3$, $\theta_4$, $s_1$, $s_2$, $s_3$, $s_4$, $Y_1$, $Y_2$, $Z_1$, $Z_2$ and the function $\Gamma(R)$. Equation (6.3.40) applied at the points A, B, C and D gives relationships between $s_i$ and $\theta_i$. We suppose without loss of generality, that $\psi(0) = 0$ so that on AB $\qquad \psi(s) = \dfrac{s}{r_1}$ (7.1.1)

hence $\qquad \theta_1 = \dfrac{s_1}{r_1}$ , (7.1.2)

and $\qquad \theta_2 = \dfrac{s_2}{r_1}$ , (7.1.3)

on CD $\qquad \psi(s) = \dfrac{s}{r_2} + \lambda_2$ (7.1.4)

hence $\qquad \theta_3 = \dfrac{s_3}{r_2} + \lambda_2$ (7.1.5)

and $\qquad \theta_4 = \dfrac{s_4}{r_4} + \lambda_2$ . (7.1.6)

Equations (7.1.1) and (7.1.4) therefore give us the function $\psi(s)$ round the pulleys in terms of the constant $\lambda_2$. On the free portions of the belt BC and DA equation (6.3.44) may be applied to find $d\psi/ds$:

$$\frac{d\psi}{ds} = \frac{R_0 \ell n(R_1/R_0)}{R_0 \ell n(R_1/R_0) - Y_i \sin\psi + Z_i \cos\psi}$$

on integration this yields:

$$R_0 \ell n \frac{R_1}{R_0} \psi + Y_i \cos\psi + Z_i \sin\psi = R_0 \ell n \frac{R_1}{R_0} s + \text{constant},$$

where the constant is chosen to make $\psi(s)$ continuous. For BC making $\psi$ continuous at B, we obtain

$$R_0 \ell n \frac{R_1}{R_0} (\psi-\theta_2) + Y_1 \cos\psi + Z_1 \sin\psi - Y_1 \cos\theta_2 - Z_1 \sin\theta_2$$
$$= R_0 \ell n \frac{R_1}{R_0} (s-s_2)$$

and for DA, choosing the constant to make $\psi$ continuous at D gives

$$R_0 \ell n \frac{R_1}{R_0} (\psi-\theta_4) + Y_2 \cos\psi + Z_2 \sin\psi - Y_2 \cos\theta_4 - Z_2 \sin\theta_4$$
$$= R_0 \ell n \frac{R_1}{R_0} (s-s_4) \ .$$

With the solutions obtained $\psi(s)$ is also continuous at C and A.

In order to determine $\Gamma(R)$ from equation (6.3.46) we need to integrate $\psi(s)$ round the belt, this we do by splitting the region of integration into the different sections of the belt:

$$\int_0^{2\pi} \psi ds = \int_{s_1}^{s_2} \psi ds + \int_{s_2}^{s_3} \psi ds + \int_{s_3}^{s_4} \psi ds + \int_{s_4}^{s_1} \psi ds.$$

The integrals round the pulleys are determined from equations (7.1.1) and (7.1.4), whilst the integrals over the free portions may be rewritten as

$$\int_{s_{2i}}^{s_{2i+1}} \psi ds = \int_{\theta_{2i}}^{\theta_{2i+1}} \psi \frac{ds}{d\psi} d\psi = \int_{\theta_{2i}}^{\theta_{2i+1}} \frac{\psi}{\frac{d\psi}{ds}} d\psi \qquad \text{for } i = 1 \text{ and } 2.$$

Equation (6.3.44) is then used for $\frac{d\psi}{ds}$ and hence $\Gamma(R)$ is determined explicitly in terms of $\theta_1$, $\theta_2$, $\theta_3$, $\theta_4$, $\lambda_2$, $Y_1$, $Y_2$, $Z_1$ and $Z_2$. Equation (6.3.43) which was deduced from equating the two expressions obtained for the shear load S, may be applied at each of the points A, B, C and D giving:

$$\text{at A:} \quad R_0\theta_1(r_1-1) \ln \frac{R_1}{R_0} + \int_{R_0}^{R_1} \Gamma(R)dR = Y_2\cos\theta_1 + Z_2\sin\theta_1 \tag{7.1.7}$$

$$\text{at B:} \quad R_0\theta_2(r_1-1) \ln \frac{R_1}{R_0} + \int_{R_0}^{R_1} \Gamma(R)dR = Y_1\cos\theta_2 + Z_1\sin\theta_2 \tag{7.1.8}$$

$$\text{at C:} \quad R_0(\theta_3(r_2-1) - r_2\lambda_2) \ln \frac{R_1}{R_0} + \int_{R_0}^{R_1} \Gamma(R)dR = Y_1\cos\theta_3 + Z_1\sin\theta_3 \tag{7.1.9}$$

$$\text{at D:} \quad R_0(\theta_4(r_2-1) - r_2\lambda_2) \ln \frac{R_1}{R_0} + \int_{R_0}^{R_1} \Gamma(R)dR = Y_2\cos\theta_4 + Z_2\sin\theta_4 \tag{7.1.10}$$

We now apply equations (6.3.38) and (6.3.39), obtained from equating the two expressions for the horizontal and vertical distances between the pulley centres, to the two free sections of the belt BC and DA, using the two expressions already found for $d\psi/ds$. This yields the four conditions:

$$0 = \ln\frac{R_1}{R_0}(r_2\sin\theta_3 - r_1\sin\theta_2) - \ln\frac{R_1}{R_0}(\sin\theta_3 - \sin\theta_2)$$
$$+ \frac{Y_1}{2}(\sin^2\theta_3 - \sin^2\theta_2) - \frac{Z_1}{4}(\sin 2\theta_3 - \sin 2\theta_2)$$
$$- \frac{Z_1}{2}(\theta_3 - \theta_2) \qquad (7.1.11)$$

$$0 = \ln\frac{R_1}{R_0}(-r_1\sin\theta_1 - r_2\sin\theta_4) + \ln\frac{R_1}{R_0}(\sin\theta_1 + \sin\theta_4)$$
$$+ \frac{Y_2}{2}(\sin^2\theta_1 - \sin^2\theta_4) + \frac{Z_2}{4}(\sin 2\theta_1 + \sin 2\theta_4)$$
$$- \frac{Z_2}{2}(2\pi + \theta_1 - \theta_4) \qquad (7.1.12)$$

$$L\ln\frac{R_1}{R_0} = \ln\frac{R_1}{R_0}(r_2\cos\theta_3 - r_1\cos\theta_2) - \ln\frac{R_1}{R_0}(\cos\theta_3 - \cos\theta_2)$$
$$- \frac{Y_1}{2}(\theta_3 - \theta_2) + \frac{Y_1}{4}(\sin 2\theta_3 - \sin 2\theta_2)$$
$$+ \frac{Z_1}{2}(\sin^2\theta_3 - \sin^2\theta_2) \qquad (7.1.13)$$

$$L\ln\frac{R_1}{R_0} = \ln\frac{R_1}{R_0}(r_2\cos\theta_4 - r_1\cos\theta_1) + \ln\frac{R_1}{R_0}(\cos\theta_1 - \cos\theta_4)$$
$$+ \frac{Y_2}{2}(2\pi + \theta_1 - \theta_4) + \frac{Y_2}{4}(\sin 2\theta_1 + \sin 2\theta_4)$$
$$- \frac{Z_2}{2}(\sin^2\theta_1 - \sin^2\theta_4) \qquad (7.1.14)$$

Equation (6.3.46) gives:

$$\frac{M_1}{r_1} = -Y_1\sin\theta_2 + Z_1\cos\theta_2 - Y_2\sin\theta_1 - Z_2\cos\theta_1 \qquad (7.1.15)$$

Substituting the analytical expression for $\Gamma(R)$ in terms of $\theta_1$, $\theta_2$, $\theta_3$, $\theta_4$, $Y_1$, $Y_2$, $Z_1$ and $Z_2$ into (7.1.7) - (7.1.10), we therefore have nine equations (7.1.7) - (7.1.15) for the nine unknowns $\theta_1$, $\theta_2$, $\theta_3$, $\theta_4$, $\lambda_2$, $Y_1$, $Y_2$, $Z_1$ and $Z_2$. These are solved numerically and the solution is presented in Section 7.2.

## 7.2 THE SOLUTION OF THE EQUATIONS IN THE IDEAL CASE

In this section we determine the solution to the nine equations (7.1.7) - (7.1.15), obtained from the ideal theory, which give the configuration of the pulleys and belt. The equations were solved numerically for different values of the specified quantities: radii of the pulleys, their separation, the applied torque and the thickness of the belt, that is the initial inner and outer radii. We assume, without loss of generality, that the inner radius of the belt, $R_0$ is 1. The distance round the inner fibre in the reference configuration is then $2\pi$, thus to be physically realistic the radii and separation of the pulleys must be chosen so that the minimum distance round the inner fibre in the new configuration is not much greater than $2\pi$. The numerical procedure used to solve these equations was based on Newton's method.

When no torque is applied and the radii of the pulleys are equal, we have two axes of symmetry, one being the line joining the centres of the pulleys and the other being the perpendicular bisector to this line. We can then solve the equations analytically and this solution is used as a starting value for the cases of unequal pulley radii and when there is an applied torque. The results obtained are shown in Table 7.1. A consequence of the symmetry for equal pulley radii is that $Z_1 = Z_2 = 0$, as is shown in the table. We also observe that for the cases when there are two axes of symmetry, that is when the pulley radii are equal (0.3) and there is no applied torque the angles are the same in each case and we have $Z_1 = Z_2 = 0$ and $Y_1 = -Y_2$.

Table 7.1 Solution of the equations in the ideal theory

| | $R_1$=1.1, L=1.8<br>$r_1$=$r_2$=0.3<br>M=0 | $R_1$=1.1, L=1.8<br>$r_1$=0.3,$r_2$=0.35<br>M=0 | $R_1$=1.1, L=1.8<br>$r_1$=0.3,$r_2$=0.3<br>M=0.1 | $R_1$=1.1, L=1.8<br>$r_1$=0.3,$r_2$=0.35<br>M=0.1 | $R_1$=1.1, L=2.04<br>$r_1$=$r_2$=0.34<br>M=0 | $R_1$=1.3, L=1.8<br>$r_1$=$r_2$=0.3<br>M=0 | $R_1$=1.5, L=1.8<br>$r_1$=$r_2$=0.3<br>M=0 |
|---|---|---|---|---|---|---|---|
| $\theta_1$ | -0.4650 | -0.5516 | -0.0923 | -0.2101 | -1.2572 | -0.4650 | -0.4650 |
| $\theta_2$ | 0.4650 | 0.5516 | 1.3587 | 1.3324 | 1.2572 | 0.4650 | 0.4650 |
| $\theta_3$ | 2.6766 | 2.5568 | 1.7829 | 1.7548 | 1.8843 | 2.6766 | 2.6766 |
| $\theta_4$ | 3.6066 | 3.7264 | 3.2339 | 3.3699 | 4.3988 | 3.6066 | 3.6066 |
| $Y_1$ | -0.0347 | -0.0433 | -0.3433 | -0.3472 | -0.2564 | -0.0955 | -0.1477 |
| $Y_2$ | 0.0347 | 0.0433 | 0.0246 | 0.0294 | 0.2564 | 0.0955 | 0.1477 |
| $Z_1$ | 0 | 0.0002 | 0 | 0.0081 | 0 | 0 | 0 |
| $Z_2$ | 0 | 0.0002 | 0 | -0.0002 | 0 | 0 | 0 |
| $\lambda_2$ | -7.3304 | -5.8344 | -6.0450 | -4.9587 | -6.0984 | -7.3304 | -7.3304 |

## 7.3 THE EQUATION FOR $T_0(R)$

Having determined the geometrical configuration of the belt and the function $\Gamma(R)$ we now turn our attention to the calculation of $T_0(R)$. We need to know the applied torque M and some details of the loads. For simplicity we consider a belt stretched round smooth pulleys so that $P^- = 0$ on the pulleys and since it is already zero on the free sections of the belt, $P^- \equiv 0$ round the belt, which implies that $M = 0$. A consequence of M being zero is that $\Gamma(R)$ is zero.

From equation (6.4.3) we obtain the following equation for $T_0(R)$:

$$
\begin{aligned}
&2\pi\Big\{\alpha_{12}T_0(R)\left(1+\frac{1}{R^2}\right)+\alpha_{11}RT_0{}'(R)+\frac{\alpha_{22}}{R}\int_{R_0}^{R}T_0(R)dR\Big\}\\
&+(\alpha_{12}-\alpha_{22})\frac{R_0{}^2}{R}\int_{R_0}^{R}T_0(R)dR\int_0^{2\pi}\frac{(1-\psi')\,ds}{(R-R_0)(R_0+(R-R_0)\psi')}\\
&+\frac{\alpha_{22}}{R}\int_0^{2\pi}\psi'(s)\Big\{\ell n(R_0+(R-R_0)\psi')\int_{R_0}^{R}T_0(R)dR\\
&-\int_{R_0}^{R}\ell n(R_0+(R-R_0)\psi')T_0(R)dR\Big\}ds\\
&=\frac{a_1}{R}+\frac{a_2}{R^2}+\frac{a_3}{R^3}+a_4\frac{\ell nR}{R}+\ell nR\int_0^{2\pi}\frac{a_5(s)ds}{R_0+(R-R_0)\psi'}\\
&+\int_0^{2\pi}\frac{a_6(s)ds}{R_0+(R-R_0)\psi'}+\int_0^{2\pi}\frac{a_7(s)\ \ell nR\ ds}{(R_0+(R-R_0)\psi')^2}\\
&+\int_0^{2\pi}\frac{a_8(s)\ ds}{(R_0+(R-R_0)\psi')^2}+\int_0^{2\pi}\frac{a_9(s)ds}{R(R_0+(R-R_0)\psi')^2}\\
&+\int_0^{2\pi}\frac{a_{10}(s)ds}{R(R_0+(R-R_0)\psi')}+\int_0^{2\pi}\frac{a_{11}(s)}{R}\int^{R}\frac{\ell nR\ dR\ ds}{(R_0+(R-R_0)\psi')}\\
&+\int_0^{2\pi}\frac{a_{12}(s)\ \ell n(R_0+(R-R_0)\psi')ds}{R}\\
&=g(R)\quad,\quad\text{say}
\end{aligned}
\tag{7.3.1}
$$

where

$$a_1 = \int_0^{2\pi}\{(\alpha_{12} - 2\alpha_{11})\psi'\int_0^s(s-\psi)\psi'ds - \alpha_{22}(Q^- + \psi'L^-)$$

$$- \alpha_{12}(1-\psi')\}ds \ ,$$

$$a_2 = \int_0^{2\pi}\{-(\alpha_{11} - \alpha_{12})\left[\left[\int_0^s(s-\psi)\psi'ds - \int_0^s(s-\psi)ds\right]\psi'\right.$$

$$\left. + R_0(1-\psi')\int_0^s(s-\psi)\psi'ds\right] + (2\alpha_{12} - \alpha_{11} - \alpha_{22})(s-\psi)^2R_0{}^2\psi'$$

$$- \alpha_{11}\left[2\psi'\left(\int_0^s(s-\psi)\psi'ds + \int_0^s(s-\psi)ds\right)\right.$$

$$\left. + R_0(1-\psi')\int_0^s(s-\psi)\psi'ds\right]$$

$$+ (\alpha_{12} - \alpha_{22})(-2\psi'(s-\psi)R_0{}^2) + (\alpha_{11} - \alpha_{12})(-\psi'(s-\psi)^2)$$

$$- (\alpha_{11} - \alpha_{12})\psi'\left(\int_0^s(s-\psi)\psi'ds - \int_0^s(s-\psi)ds\right)$$

$$+ (2\alpha_{12} - \alpha_{11} - \alpha_{22})\psi'(s-\psi)\ R_0{}^2$$

$$- \alpha_{11}((1-\psi')R_0 + 2\psi')\left[\int_0^s(s-\psi)\psi'ds + 2\psi'\int_0^s(s-\psi)ds\right]$$

$$- (\alpha_{12} - \alpha_{11})2\psi'R_0{}^2(s-\psi)^2\}ds \ ,$$

$$a_3 = \int_0^{2\pi}\{- (\alpha_{11} - \alpha_{12})R_0(1-\psi')\left[\int_0^s(s-\psi)\psi'ds - \int_0^s(s-\psi)ds\right]$$

$$+ (2\alpha_{12} - \alpha_{11} - \alpha_{22})(s-\psi)^2R_0{}^3(1-\psi')$$

$$- 2\alpha_{11}R_0(1-\psi')\left[\int_0^s(s-\psi)\psi'ds - \int_0^s(s-\psi)ds\right.$$

$$+ (\alpha_{12} - \alpha_{11})(-2R_0{}^3(s-\psi)^2(1-\psi'))$$

$$+ (\alpha_{11} - \alpha_{12})(s-\psi)^2 R_0(1-\psi') + (2\alpha_{12} - \alpha_{11} - \alpha_{22})2(s-\psi)^2 R_0{}^3(1-\psi')$$

$$- \alpha_{11}R_0(1-\psi')\left\{\int_0^s (s-\psi)\psi' ds + \int_0^s (s-\psi)ds\right.$$

$$\left. - (\alpha_{12} - \alpha_{11})R_0{}^3(s-\psi)^3(1-\psi')\right\} ds \ ,$$

$$a_4 = \int_0^{2\pi} \{- \alpha_{22}(1-\psi') + \alpha_{12}\psi' \int_0^s (s-\psi)\psi' ds\} ds \ ,$$

$$a_5 = \alpha_{12}(1-\psi')\psi' \ ,$$

$$a_6 = - \alpha_{12}(Q^- + \psi' L^-)\psi' \ ,$$

$$a_7 = (\alpha_{12} - \alpha_{22})(s-\psi)R\psi'' \ ,$$

$$a_8 = -(\alpha_{12} - \alpha_{22})(s-\psi)R_0 Q^- \psi'' \ ,$$

$$a_9 = (\alpha_{12} - \alpha_{22})(s-\psi)R_0{}^2\psi''(Q^- + L^-) \ ,$$

$$a_{10} = (\alpha_{12} - \alpha_{22})(s-\psi)R_0(Q^{-1} + \psi' L^{-1}) \ ,$$

$$a_{11} = -\alpha_{22}\psi'(1-\psi') \ ,$$

$$a_{12} = \alpha_{22}(Q^- + \psi' L^-) \ .$$

We now write $\tau(R) = \int_{R_0}^{R} T_0(R)dR$, which implies $\tau'(R) = T_0(R)$ and $\tau(R_0) = 0$, without loss of generality as has been mentioned, we let $R_0 = 1$ and we also assume that the pulleys are of equal radii $r_1 = r_2 = r$, say. We then have, as was stated in §7.2, two axes of symmetry. Equation (7.3.1) then becomes:

$$2\pi\alpha_{11}R\tau''(R) + 2\pi\alpha_{12}\left(1 + \frac{1}{R}\right)\tau'(R)$$

$$+ \left[\frac{2\pi\alpha_{22}}{R} + (\alpha_{12} - \alpha_{22})\frac{1}{R(R-1)}\left\{\frac{4(1-1/r)s_2}{1+(R-1)1/r} - \frac{4Y_1\cos\theta_2}{\ell nR}\right\}\right.$$

$$\left. - \frac{2(\alpha_{12} - \alpha_{22})(\pi - 2\theta_2)}{R} + 2(\alpha_{12} - \alpha_{22})\ell nR_1G(R)\right]\tau(R)$$

$$+ \alpha_{22}\int_1^R \tau(R)\left\{\frac{4s_2}{r^2 + (R-1)1/r} + 2\ell nR_1G(R)\right\}dR = g(R) \qquad (7.3.2)$$

where

$$G(R) = \int_{\theta_2}^{\pi-\theta_2} \frac{d\psi}{R\ell nR_1 - Y_1\sin\psi} .$$

We write (7.3.2) in the form

$$A(R)\tau''(R) + B(R)\tau'(R) + C(R)\tau(R)$$

$$+ \int_1^R D(\tilde{R})\tau(\tilde{R})d\tilde{R} = g(R) . \qquad (7.3.3)$$

To solve this we first solve the homogeneous problem for $\bar{\tau}$:

$$A(R)\bar{\tau}''(R) + B(R)\bar{\tau}'(R) + C(R)\bar{\tau}(R) + \int_1^R D(\tilde{R})\bar{\tau}(\tilde{R})d\tilde{R} = 0 \qquad (7.3.4)$$

with boundary conditions $\bar{\tau}'(1) = 1$, $\bar{\tau}(1) = 0$ and then solve the inhomogeneous problem for $\hat{\tau}$:

$$A(R)\hat{\tau}''(R) + B(R)\hat{\tau}'(R) + C(R)\hat{\tau}(R) + \int_1^R D(\tilde{R})\hat{\tau}(\tilde{R})d\tilde{R} = g(R) \qquad (7.3.5)$$

with boundary conditions $\hat{\tau}'(1) = 0$, $\hat{\tau}(1) = 0$. We then write $\tau = \hat{\tau} + \lambda\bar{\tau}$

so $\tau(1) = 0$ as required by the definition of $\tau$ and $\tau'(1) = \lambda$, which implies that $T_0(1) = \lambda$, so that $\lambda$ is fixed by specifying the load, this is equivalent to specifying the separation of the pulleys, which we have already done. The constant $\lambda$ is found by equating the two expressions we have for the load on the free sections.

A consequence of the assumption that the pulleys are smooth is that $p^- = 0$, hence from equations (6.4.5) and (6.4.7):

$$R_0 Q_i^-(s) = \frac{dS}{ds} - U\frac{d\psi}{ds}$$

and

$$R_0 \frac{dQ_i^-}{ds} = \frac{1}{r_i^2}\left\{R_0(s-\psi)\ln\frac{R_1}{R_0} + \int_{R_0}^{R_1}\Gamma(R)dR\right\}.$$

Now equation (6.3.20) gives

$$U(s) = \int_{R_0^+}^{R_1^-}(T-p)dR + L^+(s) + L^-(s)$$

and equations (6.3.32) and (6.3.33) give:

$$L^+ = \frac{R_0^2}{R_1}\int_0^s(s-\psi)ds + (R_1-R_0)\frac{R_0}{R_1}\int_0^s(s-\psi)\psi' ds$$

$$+ R_0\Gamma(R_1)s + (R_1-R_0)\Gamma(R_1)\psi(s) + \mu^+$$

and

$$L^- = -R_0\int_0^s\left[s - \psi + \Gamma(R_0)\right]ds + \mu^-$$

Using these expressions and equation (6.4.4) for T, we find:

$$U(s) = \tau(R_1) - R_0\ln\frac{R_1}{R_0}\int_0^s(s-\psi)\psi' ds - \int_{R_0}^{R_1}\Gamma(R)dR\psi(s)$$

$$- \int_0^s\left[(R_0 + (R_1-R_0)\psi')\left(\frac{R_0}{R_1}(s-\psi) + \Gamma(R_1)\right)\right.$$

$$\left. - (R_0(s-\psi) + R_0\Gamma(R_0))\right]ds$$

$$+ \frac{R_0^2}{R_1} \int_0^s (s-\psi) ds + (R_1-R_0) \frac{R_0}{R_1} \int_0^s (s-\psi)\psi' ds$$

$$+ R_0 \Gamma(R_1) s + (R_1-R_0)\Gamma(R_1)\psi(s) + \mu^+$$

$$- R_0 \int_0^s (s - \psi + \Gamma(R_0)) ds + \mu^-$$

$$= \tau(R_1) - R_0 \ln \frac{R_1}{R_0} \int_0^s (s-\psi)\psi' ds - \int_{R_0}^{R_1} \Gamma(R) dR \psi(s) + \mu^+ + \mu^- .$$

From equation (6.3.42) we find that

$$\frac{dS}{ds} = R_0(1-\psi') \ln \frac{R_1}{R_0} ,$$

hence,

$$R_0 Q^-(s) = R_0(1-\psi') \ln \frac{R_1}{R_0} - \psi' \{ \tau(R_1) - R_0 \ln \frac{R_1}{R_0} \int_0^s (s-\psi)\psi' ds$$

$$- \int_{R_0}^{R_1} \Gamma(R) dR \psi(s) + \mu^+ + \mu^- \} .$$

We are considering the case where no torque is applied that is $M = 0$, a consequence of which is that $\Gamma(R) \equiv 0$. Hence:

$$R_0 Q^-(s) = R_0(1-\psi') \ln \frac{R_1}{R_0} - \psi' \{ \tau(R_1) - R_0 \ln \frac{R_1}{R_0} \int_0^s (s-\psi)\psi' ds + \mu^+ + \mu^- \}$$

and

$$U(s) = \tau(R_1) - R_0 \ln \frac{R_1}{R_0} \int_0^s (s-\psi)\psi' ds + \mu^+ + \mu^- . \qquad (7.3.6)$$

Hence on the free section BC:

$$U(s) = \tau(R_1) - R_0 \ln \frac{R_1}{R_0} \left( \int_0^{s_2} (s-\psi)\psi' ds + \int_{s_2}^{s} (s-\psi)\psi' ds \right) + \mu^+ + \mu^-$$

$$= \tau(R_1) - R_0 \ln \frac{R_1}{R_0} \frac{\theta_2^2}{2} (r-1) + Y_1 \sin\theta_2 - Y_1 \sin\psi + \mu^+ + \mu^- .$$

However from (6.3.26)

$$U(s) = -Y_1 \sin\psi \qquad \text{on } BC$$

which implies that

$$\tau(R_1) - R_0 \ell n \frac{R_1}{R_0} \frac{\theta_2^2}{2} (r-1) + Y_1 \sin\theta_2 + \mu^+ + \mu^- = 0$$

that is

$$\tau(R_1) + \mu^+ + \mu^- = -(1-r) \frac{\theta_2^2}{2} R_0 \ell n \frac{R_1}{R_0} - Y_1 \sin\theta_2 \; .$$

The right hand side of this is positive, which shows that the total load U is positive at s = 0, which is what is expected physically. The equation may be rewritten as:

$$\hat{\tau}(R_1) + \lambda\bar{\tau}(R_1) + \mu^+ + \mu^- = -(1-r) \frac{\theta_2^2}{2} R_0 \ell n \frac{R_1}{R_0} - Y_1 \sin\theta_2$$

which gives

$$\lambda = \frac{-(1-r) \frac{\theta_2^2}{2} R_0 \ell n \frac{R_1}{R_0} - Y_1 \sin\theta_2 - \mu^+ - \mu^- - \hat{\tau}(R_1)}{\bar{\tau}(R_1)} \; . \qquad (7.3.7)$$

Having determined $\lambda$ the function $\tau(R)$ may be calculated as $\hat{\tau}(R) + \lambda\bar{\tau}(R)$ and hence the function $T_0(R)$ obtained.

We have been considering the case when the radii of the pulleys are equal; however, the analysis for the solution of equation (7.3.1) is similar when different sized pulleys are considered. In the remaining analysis we consider pulleys with the same radius, r.

## 7.4 THE BOUNDARY LAYER EQUATIONS

As has already been stated, the constants $\mu^+$ and $\mu^-$ in the function $L^+$ and $L^-$ are found from the boundary layer equations. We consider first the boundary layer on $R = R_0$, equations (6.5.16) and (6.5.17) give

$$\frac{\partial u}{\partial s} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial s}$$

with $v \to 0$ as $y \to +\infty$

$u \to 0$ as $y \to +\infty$

$u = -(s-\psi_0)$ on $y = 0$.

Since the function $\psi_0 - s$ is $2\pi$-periodic it may be written as a Fourier series,

$$\psi_0 - s = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} (A_n \cos ns + B_n \sin ns)$$

where $A_n = \frac{1}{\pi}\int_0^{2\pi} (\psi_0 - s)\cos ns \, ds$

and $B_n = \frac{1}{\pi}\int_0^{2\pi} (\psi_0 - s)\sin ns \, ds$

so that $A_0 = 0$, then

$$u(y,s) = \sum_{n=1}^{\infty} e^{-ny}(A_n \cos ns + B_n \sin ns)$$

$$v(y,s) = \sum_{n=1}^{\infty} e^{-ny}(-B_n \cos ns + A_n \sin ns)$$

which satisfy the required boundary conditions.

Now:

$$L^-(s) \sim \int_0^{\infty} T_{-1} d\rho = \int_0^{\infty} v \, dy = \sum_{n=1}^{\infty} \left(-\frac{B_n}{n}\cos ns + \frac{A_n}{n}\sin ns\right)$$

so that

$$\mu^- = L^-(0) \sim \sum_{n=1}^{\infty} - \frac{B_n}{n} \tag{7.4.1}$$

For the boundary layer on $R = R_1$ the equations to be satisfied are:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = - \frac{\partial v}{\partial x}$$

where u and v are periodic in x of period $2\pi R_1$, with $u \to 0$ as $y \to -\infty$

$$u = - \frac{1}{R_1} (s-\psi_0) \text{ on } y = 0$$

$$= \chi(x)$$

and $v \to 0$ as $y \to -\infty$.

Now the function $\chi(x)$ is periodic in x of period $2\pi R_1$, it may therefore be written as a Fourier series

$$\chi(x) = \frac{1}{2} C_0 + \sum_{n=1}^{\infty} (C_n \cos \frac{nx}{R_1} + D_n \sin \frac{nx}{R_1}) ,$$

where

$$C_n = \frac{1}{\pi R_1} \int_0^{2\pi R_1} \chi(x) \cos \frac{nx}{R_1} dx$$

and

$$D_n = \frac{1}{\pi R_1} \int_0^{2\pi R_1} \chi(x) \sin \frac{nx}{R_1} dx$$

which implies $C_0 = 0$.

Then:

$$u(y,x) = \sum_{n=1}^{\infty} e^{ny/R_1} (C_n \cos \frac{nx}{R_1} + D_n \sin \frac{nx}{R_1})$$

$$v(y,x) = \sum_{n=1}^{\infty} e^{ny/R_1} (D_n \cos \frac{nx}{R_1} - C_n \sin \frac{nx}{R_1}) .$$

Also:

$$L^+(s) \sim \int_{-\infty}^{0} T_{-1} d\rho \;=\; \int_{-\infty}^{0} v\, dy \;=\; \sum_{n=1}^{\infty} \frac{R_1}{n} \left(D_n \cos \frac{nx}{R_1} - C_n \sin \frac{nx}{R_1}\right)$$

hence

$$\mu^+ \;=\; L^+(0) \sim \sum_{n=1}^{\infty} \frac{R_1}{n} D_n \;. \tag{7.4.2}$$

In this symmetric case $A_n$ and $C_n$ are zero for all n and $B_n$ and $D_n$ are zero when n is odd. When n is even, n = 2m say

$$B_{2m} \;=\; \frac{(-1)^m}{m} + \frac{4}{\pi} \int_0^{\pi/2} \phi(x) \sin 2mx \, dx$$

and $$D_{2m} \;=\; \frac{(-1)^m}{m} + \frac{4}{\pi} \int_0^{\pi R_1/2} \phi(x) \sin \frac{2mx}{R_1} \, dx \;,$$

where $\psi(s) = \phi(x)$.

The computed values of $B_{2m}$ and $D_{2m}$ are shown in Table 7.2.

To solve the equations for $\hat{\tau}$ and $\bar{\tau}$ we need to specify the material to obtain values for the constants $\alpha_{11}$, $\alpha_{12}$ and $\alpha_{22}$, this we do in Section 7.6. First we examine the function $T_0(R)$ to discover where it takes maximum and minimum values. The numerical solution to (7.3.4) and (7.3.5) is then presented in Section 7.7.

Table 7.2 The Fourier coefficients in the boundary layers

| | r=0.3 | | | | r=0.34 | |
|---|---|---|---|---|---|---|
| | R =1.1, 1.3, 1.5 | R =1.1 | R =1.3 | R =1.5 | R =1.1 | |
| m | $B_{2m}$ | $D_{2m}$ | $D_{2m}$ | $D_{2m}$ | $B_{2m}$ | $D_{2m}$ |
| 1 | 0.2296 | 0.2107 | 0.1804 | 0.1572 | 0.6386 | 0.5917 |
| 2 | 0.1013 | 0.0910 | 0.0750 | 0.0633 | 0.2103 | 0.1595 |
| 3 | 0.0630 | 0.0552 | 0.0434 | 0.0352 | 0.0528 | 0.0112 |
| 4 | 0.0437 | 0.0369 | 0.0269 | 0.0204 | -0.0135 | -0.0332 |
| 5 | 0.0317 | 0.0253 | 0.0164 | 0.0108 | -0.0306 | -0.0272 |
| 6 | 0.0232 | 0.0171 | 0.0089 | 0.0042 | -0.0221 | -0.0056 |
| 7 | 0.0168 | 0.0109 | 0.0035 | -0.0003 | -0.0058 | 0.0095 |
| 8 | 0.0118 | 0.0061 | -0.0004 | -0.0031 | 0.0067 | 0.0111 |
| 9 | 0.0077 | 0.0024 | -0.0029 | -0.0044 | 0.0105 | 0.0037 |
| 10 | 0.0044 | -0.0004 | -0.0042 | -0.0044 | 0.0070 | -0.0040 |
| 11 | 0.0017 | -0.0023 | -0.0045 | -0.0035 | 0.0005 | -0.0061 |
| 12 | -0.0003 | -0.0035 | -0.0040 | -0.0021 | -0.0043 | -0.0027 |
| 13 | -0.0019 | -0.0041 | -0.0030 | -0.0004 | -0.0053 | 0.0019 |
| 14 | -0.0030 | -0.0041 | -0.0016 | 0.0010 | -0.0028 | 0.0038 |
| 15 | -0.0036 | -0.0037 | -0.0002 | 0.0019 | 0.0008 | 0.0022 |
| 16 | -0.0039 | -0.0029 | 0.0009 | 0.0023 | 0.0031 | -0.0009 |
| 17 | -0.0038 | -0.0020 | 0.0018 | 0.0020 | 0.0030 | -0.0026 |
| 18 | -0.0034 | -0.0009 | 0.0022 | 0.0013 | 0.0010 | -0.0018 |
| 19 | -0.0028 | 0.0001 | 0.0021 | 0.0004 | -0.0012 | 0.0004 |
| 20 | -0.0021 | 0.0009 | 0.0016 | -0.0005 | -0.0023 | 0.0019 |

## 7.5 MAXIMA AND MINIMA FOR THE TENSION T

For the case under examination $\Gamma(R) = 0$ and the equation for T becomes, from (6.3.29):

$$T = T_0(R) - \left(\frac{1}{R} + \frac{1}{R^2}\right) \int_0^s (s-\psi)\psi' ds + \frac{1}{R^2} \int_0^s (s-\psi) ds \ , \qquad (7.5.1)$$

where $T(R,0) = T_0(R)$.

Equation (7.5.1) may be rewritten in the form

$$T(R,s) = T_0(R) + \frac{1}{2R^2}(s-\psi)^2 - \frac{1}{R}\int_0^s (s-\psi)\psi' ds \ . \qquad (7.5.2)$$

Hence

$$T(R,s_2) = T_0(R) + \frac{\theta_2^2}{2}\left(\frac{(1-r)^2}{R^2} + \frac{(1-r)}{R}\right)$$

and it is also found that

$$T(R,s_1) = T(R,s_2) = T(R,s_3) = T(R,s_4) \ ,$$

which we would expect in this case with two axes of symmetry.

We now examine the function T to see where it takes its maximum and minimum values. Differentiating (7.5.2) with respect to s gives:

$$\frac{\partial T}{\partial s} = \frac{1}{R^2}(s-\psi)(1-\psi') - \frac{1}{R}(s-\psi)\psi' \ .$$

Hence $\frac{\partial T}{\partial s} = 0$ when (a) $s = \psi$

or when (b) $\frac{1}{R}(1-\psi') = \psi'$

which implies $\psi' = \frac{1}{R}$.

Since we are considering the symmetric case, we need only consider $0 \leq s \leq \pi/2$. On the pulley with $0 \leq s \leq s_2$, $\psi' = 1/r$ and $1/R \neq 1/r$ so that case (b) cannot be satisfied. Also $\psi = s/r$ and $r \neq 1$ so case (a) cannot be satisfied unless $s = 0$. On the free section

$$s_2 \leq s \leq \pi/2 \ , \qquad \frac{d\psi}{ds} = \frac{\ell n R_1}{\ell n R_1 - Y_1 \sin\psi}$$

so that in case (b) $R \, \ell n \, R_1 = -Y_1 \sin\psi$ , which may be rewritten:

$$R = \frac{(1-r)\theta_2}{\cos\theta_2} \sin \psi \quad . \tag{7.5.3}$$

However, $0 < \dfrac{(1-r)\theta_2}{\cos\theta_2} < 1$ and $|\sin\psi| \leq 1$, whereas $R \in [1,1.1]$, hence (7.5.3) cannot be satisfied on the free section. On the free section $s_2 \leq s \leq \pi/2$, $s = \psi$ only if $s = \pi/2$. Hence the only values of s for which there might be a maximum or minimum value of T are 0, $\pi/2$, $\pi$, $3\pi/2$, $2\pi$. We need to consider $\frac{\partial T}{\partial R}$ to discover whether there are any maximum or minimum values for these values of s. Now

$$\frac{\partial T}{\partial R} = T_0'(R) - \frac{1}{R}(s-\psi)^2 + \frac{1}{R^2}\int_0^s (s-\psi)\psi' \, ds \tag{7.5.4}$$

and on $s = \psi$, $\frac{\partial T}{\partial R} = T_0'(R)$, hence the maximum and minimum values will occur only on $s = 0, \pi/2, \ldots.$ at radii where $T_0'(R) = 0$ or $R = R_0, R_1$.

## 7.6 THE CALCULATION OF CONSTANTS FOR THE COMPOSITE MATERIAL

In order to obtain the constants $\alpha_{11}$, $\alpha_{12}$ and $\alpha_{22}$ we need to consider the composition of the material in more detail. We are considering a matrix material reinforced by strong continuous fibres. We suppose that the matrix is rubber and the fibres steel. The suffix R is used to denote quantities associated with the rubber and the suffix S to denote quantities associated with the steel. The elastic constants of the materials are denoted by $\lambda$ and $\mu$ with $\nu$ being Poisson's ratio. We assume that the volume concentrations of the two materials are $c_R$ and $c_S$, where $c_R + c_S = 1$. Both materials are assumed to be isotropic.

We consider an element of the composite and recall from the definitions of $A_1$ and $B_1$ that:

$$A-1 = \varepsilon^2\left[\alpha_{11}T - \alpha_{12}p + (\alpha_{12} - \alpha_{11})\gamma G\right]$$

and

$$B-1 = \varepsilon^2\left[\alpha_{12}T - \alpha_{22}p + (\alpha_{22} - \alpha_{12})\gamma G\right] \quad ,$$

defining $T^{(1)} \equiv T - \gamma G$ and $T^{(2)} \equiv - p + \gamma G$, these may be rewritten as

$$A-1 = \varepsilon^2\left[\alpha_{11}T^{(1)} + \alpha_{12}T^{(2)}\right] \tag{7.6.1}$$

and

$$B-1 = \varepsilon^2\left[\alpha_{12}T^{(1)} + \alpha_{22}T^{(2)}\right] \quad . \tag{7.6.2}$$

We non-dimensionalise G by taking $G = \gamma$, so that for the case of $\gamma = 0$ the stresses acting on the element are $\mu_L T^{(1)}$ and $\mu_T T^{(2)}$ as is shown in the diagram, where $\mu_L$ and $\mu_T$ are the axial and transverse shear moduli of the composite.

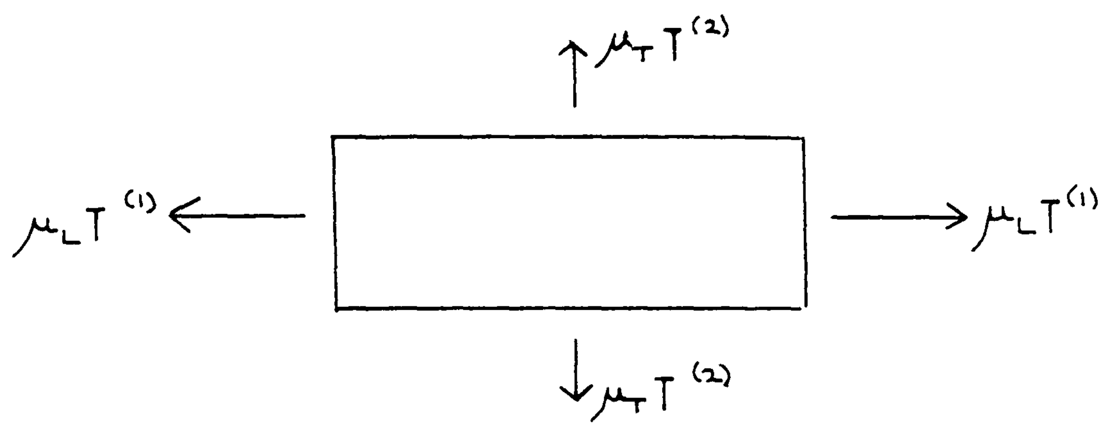



Figure 7.2 The stresses acting on an element

To determine the axial extensional modulus $E_L$ and $\mu_L$ we first consider axial extension and suppose that both constituents undergo an axial extension e under a mean axial stress $\sigma$ as shown in figure 7.3(a). Clearly:

$$\sigma = (c_R E_R + c_S E_S)e \text{ , from which we obtain}$$

$$E_L = c_R E_R + c_S E_S \quad . \tag{7.6.3}$$

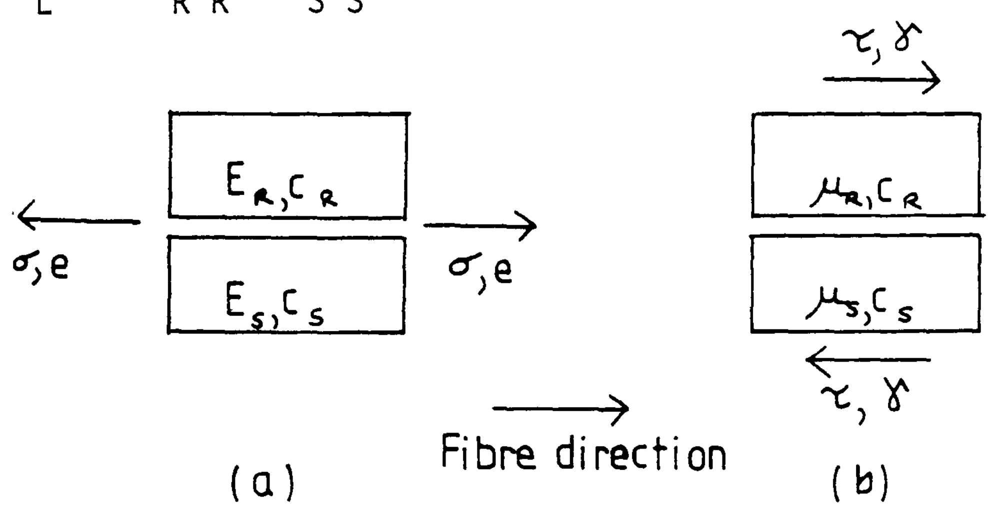



Figure 7.3 Extension and shear of the constituents

In axial shear shown in figure 7.3(b), the shear stress $\tau$ is approximately related to the mean shear strain $\gamma$ by:

$$\gamma = \left(\frac{c_R}{\mu_R} + \frac{c_S}{\mu_S}\right)\tau$$

hence

$$\frac{1}{\mu_L} = \frac{c_R}{\mu_R} + \frac{c_S}{\mu_S} \quad . \tag{7.6.4}$$

For the case when $T^{(2)} = 0$, we only have axial extensions and

$$\mu_L T^{(1)} = E_L(A-1) \ . \tag{7.6.5}$$

However, from (7.6.1) with $T^{(2)} = 0$ ,

$$A-1 = \varepsilon^2\alpha_{11}T^{(1)} \tag{7.6.6}$$

therefore from (7.6.5) and (7.6.6) we obtain an expression for $\alpha_{11}$:

$$\varepsilon^2\alpha_{11} = \frac{\mu_L}{E_L} \quad . \tag{7.6.7}$$

Also from (7.6.2) with $T^{(2)} = 0$ we find that

$$B-1 = \varepsilon^2\alpha_{12}T^{(1)} \ . \tag{7.6.8}$$

Eliminating $T^{(1)}$ from (7.6.6) and (7.6.8) we obtain

$$\frac{B-1}{A-1} = \frac{\alpha_{12}}{\alpha_{11}} \ . \tag{7.6.9}$$

To determine $\alpha_{12}$ from (7.6.9), having already found $\alpha_{11}$, we therefore need to find an expression for (B-1)/(A-1) in terms of known physical constants of the materials. The case being considered is shown in figure 7.4, where $f_R$ and $f_S$ are the transverse extensions of the constituents, which will be negative in the case under consideration with $T^{(2)} = 0$.

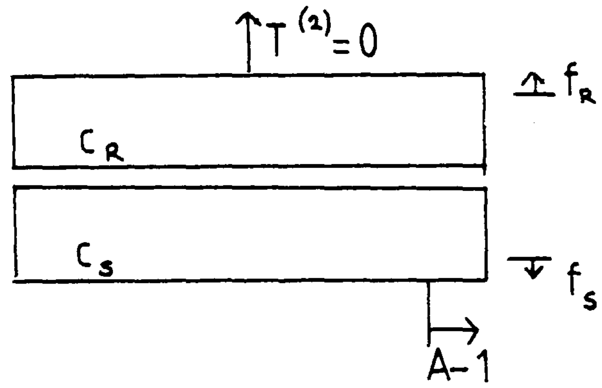



Figure 7.4  Axial extension of an element

Clearly the total transverse extension B-1 = $c_R f_R + c_S f_S$

hence

$$\frac{B-1}{A-1} = \frac{c_R f_R + c_S f_S}{A-1} . \tag{7.6.10}$$

Now for each constituent we have $\sigma_{22} = 0$, $e_{33} = 0$, $e_{11} = A-1$, $e_{22} = f$ and $\sigma_{22}$ is defined as

$$\sigma_{22} = \lambda e_{11} + (\lambda + 2\mu)e_{22}$$

hence

$$\frac{-e_{22}}{e_{11}} = \frac{\lambda}{\lambda+2\mu} = \frac{\nu}{1-\nu}$$

and substituting for $e_{22}$ and $e_{11}$ we obtain

$$-\frac{f_R}{A-1} = \frac{\nu_R}{1-\nu_R}$$

and

$$-\frac{f_S}{A-1} = \frac{\nu_S}{1-\nu_S} .$$

Substituting these expressions into (7.6.10) we obtain

$$\frac{B-1}{A-1} = -\left(\frac{c_R\nu_R}{1-\nu_R} + \frac{c_S\nu_S}{1-\nu_S}\right)$$

and using this in (7.6.9) gives

$$\frac{\alpha_{12}}{\alpha_{11}} = -\left(\frac{c_R\nu_R}{1-\nu_R} + \frac{c_S\nu_S}{1-\nu_S}\right). \tag{7.6.11}$$

We therefore have obtained expressions for $\alpha_{11}$ and $\alpha_{12}$ in terms of known physical constants. We now calculate $\alpha_{22}$.

We suppose that the constituents undergo an axial extension $e_2$ with the axial stress in the rubber being $\sigma_R$ and that in the steel being $\sigma_S$. We also assume that the rubber undergoes an extension $e_R$ in the direction transverse to the fibre and the steel undergoes an extension $e_S$ in that direction, the stress in this transverse direction being the same, $\sigma$ say, for both constituents, as shown in figure 7.5.

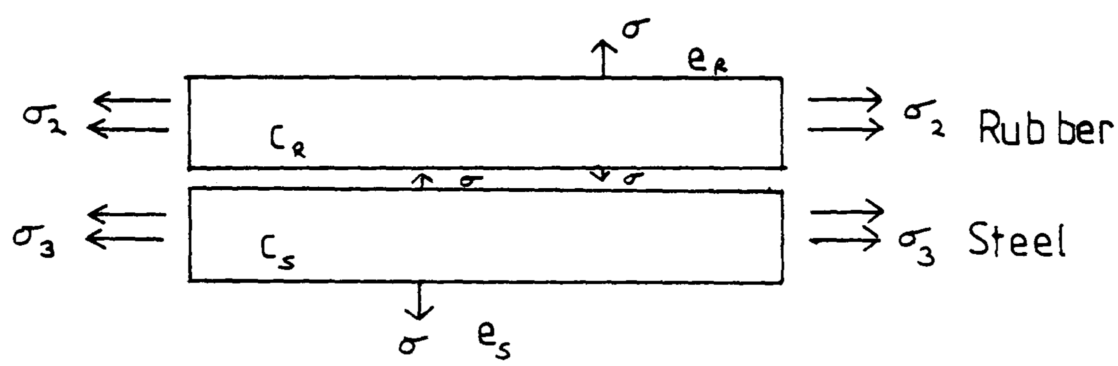



Figure 7.5 The constituents in axial and transverse extension

Now $\quad c_R\sigma_2 + c_S\sigma_3 = 0 .$ $\hfill$ (7.6.12)

For the rubber

$$\sigma = (\lambda_R + 2\mu_R)e_R + \lambda_R e_2 \tag{7.6.13}$$

and

$$\sigma_2 = \lambda_R e_R + (\lambda_R + 2\mu_R)e_2 . \tag{7.6.14}$$

For the steel

$$\sigma = (\lambda_S + 2\mu_S)e_S + \lambda_S e_2 \tag{7.6.15}$$

and

$$\sigma_3 = \lambda_S e_S + (\lambda_S + 2\mu_S)e_2 . \tag{7.6.16}$$

The extension in the transverse direction e is given by

$$e = c_S e_S + c_R e_R = \frac{\varepsilon^2 \alpha_{22} \sigma}{\mu_L} , \tag{7.6.17}$$

where as before $\mu_L$ is the shear modulus of the composite. From (7.6.13) and (7.6.15) we obtain

$$(\lambda_S - \lambda_R)\sigma = \lambda_S(\lambda_R + 2\mu_R)e_R - \lambda_R(\lambda_S + 2\mu_S)e_S . \tag{7.6.18}$$

From (7.6.13) and (7.6.14)

$$(\lambda_R + 2\mu_R)\sigma - \lambda_R\sigma_2 = ((\lambda_R + 2\mu_R)^2 - \lambda_R{}^2)e_R . \tag{7.6.19}$$

From (7.6.15) and (7.6.16)

$$(\lambda_S + 2\mu_S)\sigma - \lambda_S\sigma_3 = ((\lambda_S + 2\mu_S)^2 - \lambda_S{}^2)e_S$$

which becomes on using (7.6.12)

$$(\lambda_S + 2\mu_S)\sigma + \lambda_S \frac{c_R}{c_S} \sigma_2 = ((\lambda_S + 2\mu_S)^2 - \lambda_S{}^2)e_S . \tag{7.6.20}$$

Eliminating $\sigma_2$ between (7.6.19) and (7.6.20) gives:

$$\left[\lambda_S \frac{c_R}{c_S} (\lambda_R + 2\mu_R) + \lambda_R(\lambda_S + \mu_S)\right]\sigma = \lambda_S \frac{c_R}{c_S} ((\lambda_R + 2\mu_R)^2 - \lambda_R{}^2)e_R$$

$$+ \lambda_R((\lambda_S + 2\mu_S)^2 - \lambda_S{}^2)e_S . \tag{7.6.21}$$

Eliminating $e_s$ between (7.6.18) and (7.6.21) gives:

$$((\lambda_S + 2\mu_S)^2 - \lambda_S{}^2)(\lambda_S - \lambda_R)\sigma + (\lambda_S + 2\mu_S)\left[\lambda_S \frac{c_R}{c_S} (\lambda_R + 2\mu_R)\right.$$

$$\left. + \lambda_R(\lambda_S + 2\mu_S)\right]\sigma = \lambda_S(\lambda_R + 2\mu_R)((\lambda_S + 2\mu_S)^2 - \lambda_S{}^2)e_R$$

$$+ \lambda_s \frac{c_R}{c_S} (\lambda_S + 2\mu_S)((\lambda_R + 2\mu_R)^2 - \lambda_R{}^2)e_R . \tag{7.6.22}$$

Eliminating $e_R$ between (7.6.18) and (7.6.21) gives:

$$\frac{c_R}{c_S}(\lambda_S - \lambda_R)((\lambda_R + 2\mu_R)^2 - \lambda_R{}^2)\sigma$$

$$- (\lambda_R + 2\mu_R)\left[\lambda_S \frac{c_R}{c_S}(\lambda_R + 2\mu_R) + \lambda_R(\lambda_S + 2\mu_S)\right]\sigma$$

$$= \; - \lambda_R \frac{c_R}{c_S}((\lambda_R + 2\mu_R)^2 - \lambda_R{}^2)(\lambda_S + 2\mu_S)e_S$$

$$- \lambda_R(\lambda_R + 2\mu_R)((\lambda_S + 2\mu_S)^2 - \lambda_S{}^2)e_S \; . \qquad (7.6.23)$$

Substituting the expressions for $e_R$ and $e_S$ obtained in (7.6.22) and (7.6.23) into equation (7.6.17) gives:

$$\varepsilon^2 \frac{\alpha_{22}}{\mu_L} = \kappa_1 c_R + \kappa_2 c_S \qquad (7.6.24)$$

where

$$\kappa_1 = \frac{\lambda_R\lambda_S + 4\mu_S(\mu_S+\lambda_S) + \frac{c_R}{c_S}(\lambda_S+2\mu_S)(\lambda_R+2\mu_R)}{4\mu_S(\lambda_S+\mu_S)(\lambda_R+2\mu_R) + \frac{c_R}{c_S}4\mu_R(\lambda_S+2\mu_S)(\lambda_R+\mu_R)} \qquad (7.6.25)$$

and

$$\kappa_2 = \frac{\frac{c_R}{c_S}\lambda_R\lambda_S + 4\frac{c_R}{c_S}\mu_R(\lambda_R+\mu_R) + (\lambda_R+2\mu_R)(\lambda_S+2\mu_S)}{4\frac{c_R}{c_S}\mu_R(\lambda_S+2\mu_S)(\lambda_R+\mu_R) + 4\mu_S(\lambda_S+\mu_S)(\lambda_R+2\mu_R)} \qquad (7.6.26)$$

Now $\lambda$ and $\mu$ are defined by:

$$\lambda = \frac{E\nu}{(1-2\nu)(1+\nu)} \qquad (7.6.27)$$

$$\mu = \frac{E}{2(1+\nu)} \qquad (7.6.28)$$

where E is the extensional modulus.

We therefore have expressions for $\alpha_{11}$, $\alpha_{12}$ and $\alpha_{22}$ in terms of physical constants of the material, which we obtain from (15) and (21).

For steel we take $E_S = 2.119 \times 10^{11}\ Nm^{-2}$

and $\nu_S = 0.291$

and for rubber $E_R = 1.18 \times 10^{6}\ Nm^{-2}$

and $\nu_R = 0.49997$ .

Using these values in (7.6.27) and (7.6.28) the elastic constants are calculated to be:

$$\lambda_S = 1.143 \times 10^{11}\ Nm^{-2}$$

$$\mu_S = 8.207 \times 10^{10}\ Nm^{-2}$$

$$\lambda_R = 6.556 \times 10^{9}\ Nm^{-2}$$

$$\mu_R = 3.93 \times 10^{5}\ Nm^{-2} .$$

Using these in the expressions for $\kappa_1$ and $\kappa_2$ (7.6.25) and (7.6.26) we find that:

$$\kappa_1 = 6.345 \times 10^{-10}\ N^{-1}m^{2}$$

and $\kappa_2 = 6.091 \times 10^{-12}\ N^{-1}m^{2}$ .

We now choose the volume concentrations of the rubber and the steel to be equal, that is $c_R = c_S = \frac{1}{2}$ , then from (7.6.3):

$$E_L = \frac{1}{2}(E_R + E_S)$$

$$= 1.059 \times 10^{11}\ Nm^{-2}$$

and from (7.6.4):

$$\frac{1}{\mu_L} = \frac{1}{2}\left(\frac{1}{\mu_R} + \frac{1}{\mu_S}\right)$$

which gives:

$$\mu_L = 7.86 \times 10^5 \ \ \mathrm{Nm}^{-2} \ .$$

Therefore, from (7.6.7):

$$\varepsilon^2\alpha_{11} = \frac{\mu_L}{E_L} = 7.422 \times 10^{-6}$$

and from (7.6.11)

$$\frac{\alpha_{12}}{\alpha_{11}} = -\frac{1}{2}\left(\frac{\nu_S}{1-\nu_S} + \frac{\nu_R}{1-\nu_R}\right)$$

$$= -0.705$$

which gives:

$$\varepsilon^2\alpha_{12} = -5.233 \times 10^{-6}$$

and from (7.6.24):

$$\varepsilon^2\alpha_{22} = \frac{\mu_L}{2}(\kappa_1 + \kappa_2)$$

$$= 2.493 \times 10^{-4} \ .$$

Choosing $\varepsilon = 10^{-2}$ we find that

$$\alpha_{11} = 0.074$$

$$\alpha_{12} = -0.052$$

$$\alpha_{22} = 2.493 \ .$$

In section 7.7 we also consider an example with $\varepsilon = 10^{-1}$ so that we use

$$\alpha_{11} = 7.4 \times 10^{-4}$$

$$\alpha_{12} = -5.2 \times 10^{-4}$$

$$2.5 \times 10^{-2}$$

In the above analysis we have assumed that the volume concentrations of the rubber and steel are equal; however, choosing $c_R = \frac{2}{3}$, $c_S = \frac{1}{3}$ , we find that

$$\alpha_{11} = 0.083$$

$$\alpha_{12} = -0.067$$

$$\alpha_{22} = 2.5$$

which are not very different from those obtained above. The remaining calculations are performed with $c_R = c_S = \frac{1}{2}$ .

## 7.7 THE SOLUTION FOR $T_0(R)$

In this section we compute the functions $\bar{\tau}(R)$ and $\hat{\tau}(R)$ introduced in equations (7.3.3) and (7.3.4). We approximate the integrals involved in these equations using the trapezium rule and divide the width of the belt, that is $R_1 - 1$ into M equal intervals and divide the length of the free portion of the belt, that is BC into N equal intervals and similarly for DA. We discretize $\tau(R)$, writing $\tau_n = \tau(1+nh)$, where $Mh = R_1 - 1$, that is $h = (R_1 - 1)/M$. From the definition of $\tau$ we have $\tau_0 = 0$. We also define $A_n = A(1 + nh)$, $B_n = B(1 + nh)$, $C_n = C(1 + nh)$, $D_n = D(1 + nh)$ and $g_n = g(1 + nh)$. Equation (6.3) then discretizes to

$$A_n\left(\frac{\tau_{n+1} - 2\tau_n + \tau_{n-1}}{h^2}\right) + B_n\left(\frac{\tau_{n+1} - \tau_{n-1}}{2h}\right) + C_n\tau_n$$

$$+ \frac{h}{2}(2\tau_1 D_1 + 2\tau_2 D_2 + \ldots + 2\tau_{n-1}D_{n-1} + \tau_n D_n) = g_n$$

which may be rearranged to give:

$$\left(\frac{A_n}{h^2} + \frac{B_n}{2h}\right)\tau_{n+1} + \left(C_n - \frac{2A_n}{h^2} + \frac{hD_n}{2}\right)\tau_n + \left(\frac{A_n}{h} - \frac{B_n}{2h} + hD_{n-1}\right)\tau_{n-1}$$

$$+ hD_{n-2}\tau_{n-2} + \ldots hD_1\tau_1 = g_n \qquad n=1,\ldots,M-1 \tag{7.7.1}$$

We have the initial condition on $\tau'(1)$ for the two cases $\bar{\tau}$ and $\hat{\tau}$.

For the homogeneous problem $g_n = 0$ for all n and $\bar{\tau}'(1) = 1$, which we approximate by

$$\frac{\bar{\tau}_1 - \bar{\tau}_0}{h} = 1 ,$$

which implies $\bar{\tau}_1 = h$ . (7.7.2)

Equation (7.7.1) therefore gives, on setting $r = 1$,

$$\left(\frac{A_1}{h^2} + \frac{B_1}{2h}\right)\bar{\tau}_2 + \left(C_1 - \frac{2A_1}{h^2} + \frac{D_1 h}{2}\right)\bar{\tau}_1 = 0$$

from which we obtain $\bar{\tau}_2$. The process is repeated iteratively to find the remaining $\bar{\tau}_n$'s. The solution obtained was found to have derivative almost 1 for all R, so that the approximation of $\bar{\tau}'(1)$ was accurate enough. However, for the inhomogeneous problem the boundary condition $\hat{\tau}'(1) = 0$ and a higher order approximation for $\tau'(1)$ was used.

Using forward differences:

$$\tau'(x) = \frac{1}{h}\left[\Delta\tau(x) - \frac{1}{2}\Delta^2\tau(x)\right] + R_2'(x) , \qquad (7.7.3)$$

where $\Delta\tau(x) = \tau(x+h) - \tau(x)$

$$\Delta^2\tau(x) = \tau(x+2h) - 2\tau(x+h) + \tau(x)$$

and $R_2'(x) = \frac{h}{3}\tau^{(3)}(\eta)$

from which we obtain

$$\tau'(1) \simeq -\frac{1}{2h}\left[\tau(1+2h) - 4\tau(1+h) + 3\tau(1)\right] .$$

The condition $\hat{\tau}'(1) = 0$ therefore implies that

$$\hat{\tau}_2 = 4\hat{\tau}_1 .$$

From equation (7.7.1) we then deduce that

$$\hat{\tau}_1 = \frac{g_1h^2}{\{4\left(A_1 + \frac{B_1h}{2}\right) + C_1h^2 - 2A_1 + D_1\frac{h^3}{2}\}}$$

$$\hat{\tau}_2 = 4\hat{\tau}_1$$

and the remaining $\hat{\tau}_n$'s are obtained iteratively.

We consider solutions for the radius of the pulleys r = 0.3, the separation of the centres of the pulleys L = 1.8 and the thickness of the belt, $R_1$ - 1, 0.1, 0.3 and 0.5, where as has already been stated, we have taken the initial inner radius of the belt to be 1. We have also examined the case where $r = 0.34$, $R_1$-1=0.1 and L is chosen so that r/L stays the same as in the above mentioned cases, that is we choose $L = 2.04$, this is equivalent to changing the initial inner radius of the belt.

The solutions obtained for $\hat{\tau}$ and $\bar{\tau}$ are shown in Tables 7.3-7.7. Solutions were also computed for different values of M and N, for example $N = 40$, $M = 40$; $N = 40$, $M = 50$; $N = 30$, $M = 30$ and the results were found to differ from those tabulated by less than 1%.

$$\text{Now } T_0(R) = \tau'(R)$$

$$= \hat{\tau}'(R) + \lambda\bar{\tau}'(R)$$

where $\lambda$ is given by equation (7.3.7) with $R_0 = 1$

$$\lambda = \frac{-(1-r)\frac{\theta_2^2}{2}\ln R - Y_1\sin\theta_2 - \mu^+ - \mu^- - \hat{\tau}(R_1)}{\bar{\tau}(R_1)} .$$

For each of the cases under consideration we calculate $\mu^+$ and $\mu^-$ from (7.4.1) and (7.4.2) and hence obtain $\lambda$. The values of $\hat{\tau}'(R)$ and $\bar{\tau}'(R)$ are calculated from the functions $\hat{\tau}(R)$ and $\bar{\tau}(R)$, using central differences for all except the end points. For $R = 1$ the forward difference formula (7.7.3) is used and the similar backward difference formula for $R = R_1$. The function $T_0(R)$ can then be computed and is shown in Tables 7.3-7.7.

Following Nayfeh (32) who defines the composite solution $y^c$ in terms of the outer solution $y^o$, the inner solution $y^i$ and the inner limit of the outer solution $(y^o)^i$ as

$$y^c = y^o + y^i - (y^o)^i ,$$

we write the total function T as:

$$T \sim (T)_{\text{centre}} + \left(\frac{T-1}{\varepsilon}\right)_{\substack{\text{on } R=1 \text{ boundary}\\ \text{layer}}} + \left(\frac{T-1}{\varepsilon}\right)_{\substack{\text{on } R=R_1 \text{ boundary}\\ \text{layer}}}$$

Hence

$$T \sim T_0(R) - \frac{1}{R}\int_0^s (s-\psi)\psi' ds + \frac{1}{R^2}\frac{(s-\psi)^2}{2}$$

$$+ \frac{1}{\varepsilon\sqrt{\alpha_{11}}} \sum_{n=1}^{\infty} e^{-n(R-1)/\varepsilon\sqrt{\alpha_{11}}} (-B_n \cos ns)$$

$$+ \frac{1}{\varepsilon\sqrt{\alpha_{11}}} \sum_{n=1}^{\infty} e^{n(R-R_1)/R_1\varepsilon\sqrt{\alpha_{11}}} D_n \cos\left(\frac{n}{R_1}(s + (R_1-1)\psi)\right) .$$

We consider this at $s = 0$, $s_2$ and $\pi/2$, remembering that we have two axes of symmetry in the problem:

$$T(R,0) \sim T_0(R) - \frac{1}{\varepsilon\sqrt{\alpha_{11}}} \sum_{m=1}^{\infty} B_{2m}\, e^{-2m(R-1)/\varepsilon\sqrt{\alpha_{11}}}$$

$$+ \frac{1}{\varepsilon\sqrt{\alpha_{11}}} \sum_{m=1}^{\infty} D_{2m} e^{2m(R-R_1)/R_1\varepsilon\sqrt{\alpha_{11}}} \quad ,$$

$$T(R,s_2) \sim T_0(R) - \frac{\theta_2^2}{2R}(r-1) + \frac{1}{2R_1^2}\theta_2^2(r-1)^2$$

$$- \frac{1}{\varepsilon\sqrt{\alpha_{11}}} \sum_{m=1}^{\infty} B_{2m} \cos 2mr\theta_2 e^{-2m(R-1)/\varepsilon\sqrt{\alpha_{11}}}$$

$$+ \frac{1}{\varepsilon\sqrt{\alpha_{11}}} \sum_{m=1}^{\infty} D_{2m} \cos\left(\frac{2m}{R_1}\theta_2(r+R_1-1)\right) e^{2m(R-R_1)/\varepsilon\sqrt{\alpha_{11}}R_1} \quad ,$$

$$T(R,\tfrac{\pi}{2}) \sim T_0(R) - \frac{\theta_2^2(r-1)}{2R} - \frac{Y_1}{R\ell n\, R_1}(1 - \sin\theta_2)$$

$$- \frac{1}{\varepsilon\sqrt{\alpha_{11}}} \sum_{m=1}^{\infty} B_{2m}(-1)^m\, e^{-2m(R-1)/\varepsilon\sqrt{\alpha_{11}}}$$

$$+ \frac{1}{\varepsilon\sqrt{\alpha_{11}}} \sum_{m=1}^{\infty} D_{2m}(-1)^m\, e^{2m(R-R_1)/\varepsilon\sqrt{\alpha_{11}}R_1} \quad .$$

These are plotted in the various cases considered in figures 7.6 - 7.10 and the values are given in Tables 7.8 - 7.12.

We have shown in section 7.5 that the maximum and minimum values of $T(R,s)$ occur on $s = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$ .

Now $$\frac{dL^-}{ds} = -(s-\psi) \quad ,$$

$$\frac{dL^+}{ds} = (1 + (R_1-1)\psi')\frac{1}{R_1}(s-\psi)$$

and since it can easily be shown that

$$1 + (R_1-1)\psi' \neq 0 \quad \text{on the belt,}$$

the maximum and minimum values of $L^-(s)$ and $L^+(s)$ also occur at $s = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$.

From the graphs of the tension T for $\varepsilon = 10^{-2}$, through the belt we see that for $R_1 = 1.1$, that is when the thickness of the belt is 0.1, the boundary layer thickness is less than 1/15th of the thickness of the belt and in the other cases, $R_1 = 1.3, 1.5$, the thickness of the boundary layer is less than 1/30th of the thickness of the belt. We also observe that for the main part of the belt the inner part is in compression and the outer part is in tension. At $s = 0$ for all cases considered the boundary layer on the inner surface is in compression and the one on the outer surface is in tension. This is also the case at $\theta = \theta_2$, $s = s_2$, the point where the belt leaves the pulley, although here the magnitude of the tension is less than that at $s = 0$. For the case where the thickness of the belt is 1.1, with $\varepsilon = 10^{-2}$ and the radius of the pulley is 0.3, we see that at $s = \pi/2$ the tension in the boundary layer has changed sign, so that the inner boundary layer is in tension and the outer boundary layer in compression. For the other cases the magnitude of the tension in the boundary layer is less than that in the main part of the belt, but the inner boundary is still in compression and the outer boundary in tension.

We deduce that the maximum and minimum values of the tension occur on or close to the inner and outer surfaces of the belt at $s = n\pi/2$, $n = 0,1,2,\ldots$ .

We have also considered the case where we choose the small parameter, $\varepsilon$, to be $10^{-1}$, the tension was calculated in this case for $R_1 = 1.1$ and $r = 0.3$. Again the boundary layer is reasonably thin and the tension does not vary much round the belt.

Table 7.3 Solution for $\hat{\tau}$, $\bar{\tau}$, $T_o$ with M=30, N=40 at radius $R = 1+n(R_1-1)/M$ and $\varepsilon = 10^{-2}$, r=0.3, R =1.1

| n | $\hat{\tau}$ | $\bar{\tau}$ | $T_o$ |
|---|---|---|---|
| 0 | 0.0000 | 0.0000 | -30.44 |
| 1 | 0.0042 | 0.0033 | -27.92 |
| 2 | 0.0167 | 0.0067 | -25.49 |
| 3 | 0.0373 | 0.0100 | -23.06 |
| 4 | 0.0660 | 0.0134 | -21.27 |
| 5 | 0.1026 | 0.0168 | -18.93 |
| 6 | 0.1469 | 0.0202 | -16.62 |
| 7 | 0.1988 | 0.0236 | -14.67 |
| 8 | 0.2582 | 0.0271 | -12.45 |
| 9 | 0.3248 | 0.0305 | -9.99 |
| 10 | 0.3986 | 0.0339 | -8.19 |
| 11 | 0.4793 | 0.0374 | -6.70 |
| 12 | 0.5670 | 0.0409 | -4.05 |
| 13 | 0.6613 | 0.0443 | -2.07 |
| 14 | 0.7622 | 0.478 | -0.73 |
| 15 | 0.8696 | 0.0513 | 1.19 |
| 16 | 0.9832 | 0.0548 | 3.05 |
| 17 | 1.1030 | 0.0583 | 4.88 |
| 18 | 1.2288 | 0.0618 | 6.65 |
| 19 | 1.3605 | 0.0653 | 8.39 |
| 20 | 1.4978 | 0.0688 | 10.68 |
| 21 | 1.6408 | 0.0722 | 12.36 |
| 22 | 1.7892 | 0.0757 | 13.34 |
| 23 | 1.9429 | 0.0792 | 14.93 |
| 24 | 2.1018 | 0.0827 | 16.46 |
| 25 | 2.2657 | 0.0862 | 17.93 |
| 26 | 2.4345 | 0.0897 | 19.37 |
| 27 | 2.6080 | 0.0932 | 20.78 |
| 28 | 2.7861 | 0.0967 | 22.16 |
| 29 | 2.9688 | 0.1002 | 24.09 |
| 30 | 3.1558 | 0.1036 | 25.69 |

Table 7.4 Solution for $\hat{\tau}$, $\bar{\tau}$, $T_0$ with M=30, N=40 at radius $R = 1+n(R_1-1)/M$ and $\varepsilon = 10^{-2}$, r=0.3, $R_1$ =1.3

| n | $\hat{\tau}$ | $\bar{\tau}$ | $T_0$ |
|---|---|---|---|
| 0 | 0.0000 | 0.0000 | -232.10 |
| 1 | 0.1047 | 0.0100 | -211.16 |
| 2 | 0.4186 | 0.0201 | -195.24 |
| 3 | 0.9347 | 0.0304 | -177.71 |
| 4 | 1.6456 | 0.0407 | -158.61 |
| 5 | 2.5438 | 0.0511 | -142.58 |
| 6 | 3.6217 | 0.0615 | -124.99 |
| 7 | 4.8715 | 0.0719 | -108.19 |
| 8 | 6.2856 | 0.0824 | -92.18 |
| 9 | 7.8556 | 0.0928 | -71.98 |
| 10 | 9.5736 | 0.1032 | -60.29 |
| 11 | 11.4311 | 0.1135 | -44.44 |
| 12 | 13.4197 | 0.1237 | -29.44 |
| 13 | 15.5308 | 0.1338 | -15.29 |
| 14 | 17.7559 | 0.1437 | -4.34 |
| 15 | 20.0860 | 0.1535 | 10.37 |
| 16 | 22.5125 | 0.1681 | 28.85 |
| 17 | 25.0264 | 0.1724 | 41.79 |
| 18 | 27.6189 | 0.1816 | 53.83 |
| 19 | 30.2809 | 0.1904 | 67.30 |
| 20 | 33.0036 | 0.1990 | 79.89 |
| 21 | 35.7780 | 0.2073 | 91.58 |
| 22 | 38.5952 | 0.2153 | 102.38 |
| 23 | 41.4464 | 0.2229 | 114.63 |
| 24 | 44.3228 | 0.2302 | 123.67 |
| 25 | 47.2156 | 0.2371 | 128.65 |
| 26 | 50.0161 | 0.2436 | 143.80 |
| 27 | 53.0161 | 0.2497 | 159.92 |
| 28 | 55.9070 | 0.2553 | 162.88 |
| 29 | 58.7804 | 0.2606 | 170.02 |
| 30 | 61.6284 | 0.2654 | 172.13 |

Table 7.5 Solution for $\hat{\tau}$, $\bar{\tau}$, $T_0$ with M=30, N=40 at radius $R = 1+n(R_1-1)/M$ and $\varepsilon = 10^{-2}$, r=0.3, $R_1$=1.5

| n | $\hat{\tau}$ | $\bar{\tau}$ | $T_0$ |
|---|---|---|---|
| 0 | 0.0000 | 0.0000 | -434.89 |
| 1 | 0.3192 | 0.0167 | -400.93 |
| 2 | 1.2769 | 0.0336 | -368.16 |
| 3 | 2.8336 | 0.0508 | -342.14 |
| 4 | 4.9486 | 0.0682 | -314.23 |
| 5 | 7.5801 | 0.0855 | -280.20 |
| 6 | 10.6850 | 0.1028 | -248.78 |
| 7 | 14.2189 | 0.1198 | -215.69 |
| 8 | 18.1366 | 0.1366 | -185.35 |
| 9 | 22.3918 | 0.1529 | -147.46 |
| 10 | 26.9377 | 0.1687 | -124.39 |
| 11 | 31.7272 | 0.1839 | -93.80 |
| 12 | 36.7128 | 0.1983 | -61.71 |
| 13 | 41.8473 | 0.2119 | -32.43 |
| 14 | 47.0837 | 0.2246 | -1.62 |
| 15 | 52.3756 | 0.2363 | 26.44 |
| 16 | 57.6775 | 0.2470 | 56.15 |
| 17 | 62.9449 | 0.2564 | 83.21 |
| 18 | 68.1343 | 0.2647 | 107.71 |
| 19 | 73.2037 | 0.2717 | 134.10 |
| 20 | 78.1129 | 0.2773 | 158.11 |
| 21 | 82.8229 | 0.2816 | 179.85 |
| 22 | 87.2971 | 0.2845 | 203.78 |
| 23 | 91.5004 | 0.2859 | 225.68 |
| 24 | 95.3999 | 0.2859 | 245.68 |
| 25 | 98.9649 | 0.2844 | 263.90 |
| 26 | 102.1669 | 0.2814 | 276.12 |
| 27 | 104.9795 | 0.2770 | 291.17 |
| 28 | 107.3787 | 0.2710 | 304.86 |
| 29 | 109.3429 | 0.2636 | 317.32 |
| 30 | 110.8527 | 0.2548 | 307.45 |

Table 7.6 Solution for $\hat{\tau}$, $\bar{\tau}$, $T_0$ with M=30, N=40 at radius $R = 1+n(R_1-1)/M$ and $\varepsilon = 10^{-2}$, r=0.34, $R_1$=1.1

| n | $\hat{\tau}$ | $\bar{\tau}$ | $T_0$ |
|---|---|---|---|
| 0 | 0.0000 | 0.0000 | -222.67 |
| 1 | 0.0289 | 0.0033 | -205.33 |
| 2 | 0.1157 | 0.0067 | -188.05 |
| 3 | 0.2598 | 0.0100 | -170.92 |
| 4 | 0.4607 | 0.0134 | -158.39 |
| 5 | 0.7180 | 0.0168 | -143.76 |
| 6 | 1.0313 | 0.0203 | -127.05 |
| 7 | 1.4001 | 0.0237 | -108.23 |
| 8 | 1.8240 | 0.0271 | -93.99 |
| 9 | 2.3025 | 0.0306 | -82.12 |
| 10 | 2.8352 | 0.0341 | -65.92 |
| 11 | 3.4217 | 0.0376 | -52.07 |
| 12 | 4.0617 | 0.0412 | -36.11 |
| 13 | 4.7546 | 0.0447 | -20.27 |
| 14 | 5.5002 | 0.0483 | -8.97 |
| 15 | 6.2981 | 0.0519 | 6.66 |
| 16 | 7.1478 | 0.0555 | 22.14 |
| 17 | 8.0489 | 0.0591 | 35.30 |
| 18 | 9.0012 | 0.0628 | 46.11 |
| 19 | 10.0041 | 0.0665 | 65.72 |
| 20 | 11.0574 | 0.0701 | 76.32 |
| 21 | 12.1607 | 0.0739 | 89.00 |
| 22 | 13.3135 | 0.0776 | 106.05 |
| 23 | 14.5155 | 0.0813 | 118.52 |
| 24 | 15.7664 | 0.0851 | 128.68 |
| 25 | 17.0657 | 0.0889 | 143.14 |
| 26 | 18.4130 | 0.0927 | 157.51 |
| 27 | 19.8081 | 0.0965 | 169.53 |
| 28 | 21.2504 | 0.1004 | 183.66 |
| 29 | 22.7397 | 0.1042 | 197.67 |
| 30 | 24.2754 | 0.1081 | 207.14 |

Table 7.7 Solution for $\hat{\tau}$, $\bar{\tau}$, $T_o$ with M=30, N=40 at radius $R = 1+n(R_1-1)/M$ and $\varepsilon = 10^{-1}$, r=0.3, $R_1$=1.1

| n | $\hat{\tau}$ | $\bar{\tau}$ | $T_o$ |
|---|---|---|---|
| 0 | 0.0000 | 0.0000 | -30.06 |
| 1 | 0.0042 | 0.0033 | -27.54 |
| 2 | 0.0166 | 0.0067 | -25.10 |
| 3 | 0.0373 | 0.0100 | -22.67 |
| 4 | 0.0659 | 0.0134 | -20.90 |
| 5 | 0.1024 | 0.0168 | -18.56 |
| 6 | 0.1466 | 0.0202 | -16.28 |
| 7 | 0.1983 | 0.0236 | -14.36 |
| 8 | 0.2573 | 0.0271 | -12.17 |
| 9 | 0.3236 | 0.0305 | -9.72 |
| 10 | 0.3969 | 0.0339 | -7.92 |
| 11 | 0.4772 | 0.0374 | -5.47 |
| 12 | 0.5642 | 0.0409 | -3.87 |
| 13 | 0.6578 | 0.0443 | -1.91 |
| 14 | 0.7579 | 0.0478 | -0.60 |
| 15 | 0.8642 | 0.0513 | 1.27 |
| 16 | 0.9768 | 0.0548 | 3.10 |
| 17 | 1.0953 | 0.0583 | 4.87 |
| 18 | 1.2197 | 0.0618 | 6.61 |
| 19 | 1.3498 | 0.0653 | 8.31 |
| 20 | 1.4855 | 0.0688 | 10.54 |
| 21 | 1.6265 | 0.0722 | 12.15 |
| 22 | 1.7729 | 0.0757 | 13.12 |
| 23 | 1.9244 | 0.0792 | 14.64 |
| 24 | 2.0809 | 0.827 | 16.11 |
| 25 | 2.2422 | 0.0862 | 17.43 |
| 26 | 2.4082 | 0.0897 | 18.93 |
| 27 | 2.5788 | 0.0932 | 20.26 |
| 28 | 2.7537 | 0.0967 | 21.57 |
| 29 | 2.9330 | 0.1002 | 23.44 |
| 30 | 3.1164 | 0.1036 | 24.97 |

Table 7.8 The tension T at radius $R = 1+n(R_1-1)/30$ with $\varepsilon = 10^{-2}$, $r=0.3$, $R_1=1.1$

| n | $\theta = 0$ | $\theta = \theta_2$ | $\theta = \frac{\pi}{2}$ |
|---|---|---|---|
| 0 | -215.79 | -155.98 | 26.11 |
| 1 | -34.99 | -34.56 | -21.10 |
| 2 | -26.08 | -25.93 | -24.63 |
| 3 | -23.11 | -22.98 | -22.74 |
| 4 | -21.27 | -21.15 | -20.99 |
| 5 | -18.93 | -18.80 | -18.66 |
| 6 | -16.62 | -16.49 | -16.35 |
| 7 | -14.67 | -14.54 | -14.40 |
| 8 | -12.45 | -12.33 | -12.18 |
| 9 | -9.99 | -9.87 | -9.72 |
| 10 | -8.19 | -8.07 | -7.92 |
| 11 | -6.70 | -6.58 | -6.43 |
| 12 | -4.05 | -3.93 | -3.78 |
| 13 | -2.07 | -1.95 | -1.80 |
| 14 | 0.73 | 0.85 | 0.99 |
| 15 | 1.19 | 1.31 | 1.45 |
| 16 | 3.05 | 3.17 | 3.31 |
| 17 | 4.88 | 5.00 | 5.14 |
| 18 | 6.65 | 6.77 | 6.91 |
| 19 | 8.39 | 8.51 | 8.65 |
| 20 | 10.68 | 10.80 | 10.94 |
| 21 | 12.36 | 12.48 | 12.62 |
| 22 | 13.34 | 13.46 | 13.60 |
| 23 | 14.93 | 15.05 | 15.19 |
| 24 | 16.46 | 16.58 | 16.72 |
| 25 | 17.93 | 18.05 | 18.18 |
| 26 | 19.38 | 19.49 | 19.61 |
| 27 | 20.88 | 20.99 | 20.94 |
| 28 | 23.07 | 23.13 | 21.51 |
| 29 | 32.87 | 32.40 | 16.34 |
| 30 | 186.59 | 130.15 | -30.04 |

Table 7.9 The tension T at radius $R = 1+n(R_1-1)/30$ with $\varepsilon = 10^{-2}$, $r=0.3$, $R_1=1.3$

| n | $\theta = 0$ | $\theta = \theta_2$ | $\theta = \frac{\pi}{2}$ |
|---|---|---|---|
| 0 | -420.79 | -360.84 | -170.81 |
| 1 | -211.21 | -211.09 | -210.83 |
| 2 | -195.24 | -195.11 | -194.97 |
| 3 | -177.71 | -177.59 | -177.44 |
| 4 | -158.61 | -158.49 | -158.34 |
| 5 | -142.58 | -142.46 | -142.32 |
| 6 | -124.99 | -124.87 | -124.73 |
| 7 | -108.19 | -108.07 | -107.93 |
| 8 | -92.18 | -92.06 | -91.92 |
| 9 | -71.98 | -71.87 | -71.73 |
| 10 | -60.29 | -60.18 | -60.04 |
| 11 | -44.44 | -44.33 | -44.19 |
| 12 | -29.44 | -29.33 | -29.19 |
| 13 | -15.29 | -15.18 | -15.05 |
| 14 | -4.34 | -4.23 | -4.10 |
| 15 | 10.37 | 10.48 | 10.61 |
| 16 | 28.85 | 28.96 | 29.09 |
| 17 | 41.79 | 41.89 | 42.03 |
| 18 | 53.83 | 53.93 | 54.06 |
| 19 | 67.30 | 67.40 | 67.53 |
| 20 | 79.89 | 79.99 | 80.12 |
| 21 | 91.58 | 91.68 | 91.81 |
| 22 | 102.38 | 102.48 | 102.61 |
| 23 | 114.63 | 114.73 | 114.85 |
| 24 | 123.67 | 123.77 | 123.89 |
| 25 | 128.65 | 128.74 | 128.87 |
| 26 | 143.80 | 143.89 | 144.02 |
| 27 | 159.92 | 160.01 | 160.14 |
| 28 | 162.88 | 162.97 | 163.10 |
| 29 | 170.25 | 170.32 | 170.00 |
| 30 | 296.62 | 246.42 | 123.63 |

Table 7.10 The tension T at radius $R = 1+n(R_1-1)/30$ with $\varepsilon = 10^{-2}$, $r=0.3$, $R_1=1.5$

| n | $\theta = 0$ | $\theta = \theta_2$ | $\theta = \frac{\pi}{2}$ |
|---|---|---|---|
| 0 | -623.62 | -563.67 | -373.56 |
| 1 | -400.93 | -400.80 | -400.66 |
| 2 | -368.16 | -368.03 | -367.89 |
| 3 | -342.14 | -342.02 | -341.88 |
| 4 | -314.23 | -314.11 | -313.97 |
| 5 | -280.20 | -280.09 | -279.95 |
| 6 | -248.69 | -248.67 | -248.53 |
| 7 | -215.69 | -215.58 | -215.44 |
| 8 | -185.35 | -185.24 | -185.11 |
| 9 | -147.46 | -147.35 | -147.22 |
| 10 | -124.39 | -124.29 | -124.15 |
| 11 | -93.80 | -93.70 | -93.57 |
| 12 | -61.71 | -61.61 | -61.48 |
| 13 | -32.43 | -32.33 | -32.21 |
| 14 | -1.62 | -1.52 | -1.40 |
| 15 | 26.44 | 26.53 | 26.66 |
| 16 | 56.15 | 56.24 | 56.37 |
| 17 | 83.21 | 83.30 | 83.43 |
| 18 | 107.71 | 107.80 | 107.93 |
| 19 | 134.10 | 134.19 | 134.31 |
| 20 | 158.11 | 158.20 | 158.32 |
| 21 | 179.85 | 179.94 | 180.05 |
| 22 | 203.78 | 203.86 | 203.98 |
| 23 | 225.68 | 225.76 | 225.88 |
| 24 | 245.68 | 245.76 | 245.88 |
| 25 | 263.90 | 263.98 | 264.10 |
| 26 | 276.12 | 276.20 | 276.31 |
| 27 | 291.17 | 291.25 | 291.36 |
| 28 | 304.86 | 304.94 | 305.05 |
| 29 | 317.34 | 317.41 | 317.49 |
| 30 | 409.73 | 363.93 | 265.12 |

Table 7.11 The tension T at radius $R = 1+n(R_1-1)/30$ with $\varepsilon = 10^{-2}$, $r=0.34$, $R_1=1.1$

| n | $\theta = 0$ | $\theta = \theta_2$ | $\theta = \frac{\pi}{2}$ |
|---|---|---|---|
| 0 | -533.79 | -354.03 | -46.21 |
| 1 | -226.20 | -217.69 | -184.96 |
| 2 | -189.81 | -188.34 | -185.65 |
| 3 | -171.07 | -170.17 | -170.12 |
| 4 | -158.40 | -157.55 | -157.73 |
| 5 | -143.76 | -142.91 | -143.12 |
| 6 | -127.05 | -126.21 | -126.41 |
| 7 | -108.23 | -107.39 | -107.59 |
| 8 | -93.99 | -93.16 | -93.35 |
| 9 | -82.12 | -81.29 | -81.49 |
| 10 | -65.92 | -65.09 | -65.29 |
| 11 | -52.07 | -51.25 | -51.44 |
| 12 | -36.11 | -35.29 | -35.48 |
| 13 | -20.27 | -19.45 | -19.64 |
| 14 | -8.97 | -8.16 | -8.35 |
| 15 | 6.66 | 7.47 | 7.28 |
| 16 | 22.14 | 22.95 | 22.76 |
| 17 | 35.30 | 36.10 | 35.92 |
| 18 | 46.11 | 46.91 | 46.73 |
| 19 | 65.72 | 66.52 | 66.33 |
| 20 | 76.32 | 77.11 | 76.93 |
| 21 | 89.00 | 89.79 | 89.61 |
| 22 | 106.05 | 106.83 | 106.66 |
| 23 | 118.52 | 119.30 | 119.13 |
| 24 | 128.68 | 129.46 | 129.28 |
| 25 | 143.14 | 143.92 | 143.74 |
| 26 | 157.54 | 158.30 | 158.08 |
| 27 | 169.80 | 170.45 | 169.85 |
| 28 | 186.21 | 185.78 | 181.73 |
| 29 | 221.83 | 210.71 | 175.47 |
| 30 | 468.19 | 298.91 | 40.52 |

Table 7.12 The tension T at radius $R = 1+n(R_1-1)/30$ with $\varepsilon = 10^{-1}$, $r=0.3$, $R_1=1.1$

| n | $\theta = 0$ | $\theta = \theta_2$ | $\theta = \frac{\pi}{2}$ |
|---|---|---|---|
| 0 | -48.58 | -42.49 | -24.17 |
| 1 | -38.40 | -36.07 | -22.59 |
| 2 | -32.19 | -31.16 | -20.99 |
| 3 | -27.62 | -27.05 | -19.27 |
| 4 | -24.49 | -24.11 | -18.10 |
| 5 | -21.22 | -20.93 | -16.27 |
| 6 | -18.27 | -18.04 | -14.40 |
| 7 | -15.86 | -15.66 | -12.82 |
| 8 | -13.30 | -13.12 | -10.91 |
| 9 | -10.56 | -10.40 | -8.70 |
| 10 | -8.54 | -8.39 | -7.09 |
| 11 | -5.91 | -5.77 | -4.80 |
| 12 | -4.15 | -4.02 | -3.34 |
| 13 | -2.06 | -1.94 | -1.50 |
| 14 | -0.64 | -0.51 | -0.30 |
| 15 | 1.35 | 1.45 | 1.46 |
| 16 | 3.29 | 3.39 | 3.18 |
| 17 | 5.19 | 5.28 | 4.83 |
| 18 | 7.07 | 7.15 | 6.45 |
| 19 | 8.93 | 9.00 | 8.00 |
| 20 | 11.36 | 11.42 | 10.06 |
| 21 | 13.21 | 13.25 | 11.47 |
| 22 | 14.49 | 14.51 | 12.20 |
| 23 | 16.41 | 16.39 | 13.43 |
| 24 | 18.40 | 18.33 | 14.55 |
| 25 | 20.41 | 20.25 | 15.46 |
| 26 | 22.85 | 22.55 | 16.47 |
| 27 | 25.51 | 24.94 | 17.23 |
| 28 | 28.82 | 27.70 | 17.87 |
| 29 | 33.93 | 31.52 | 18.97 |
| 30 | 41.06 | 35.51 | 19.63 |

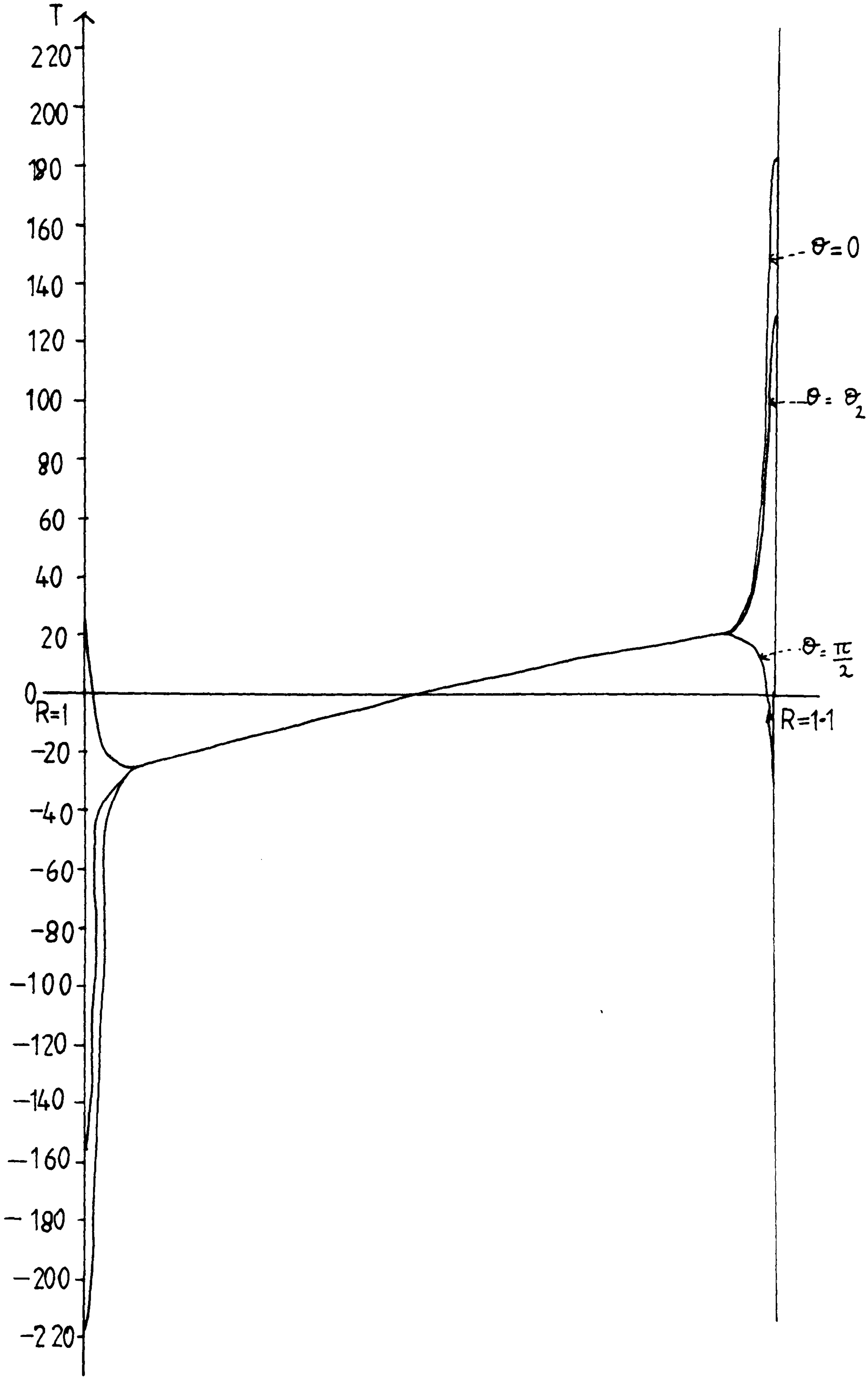



Figure 7.6 The tension with $\varepsilon = 10^{-2}$, $r = 0.3$, $R_1 = 1.1$

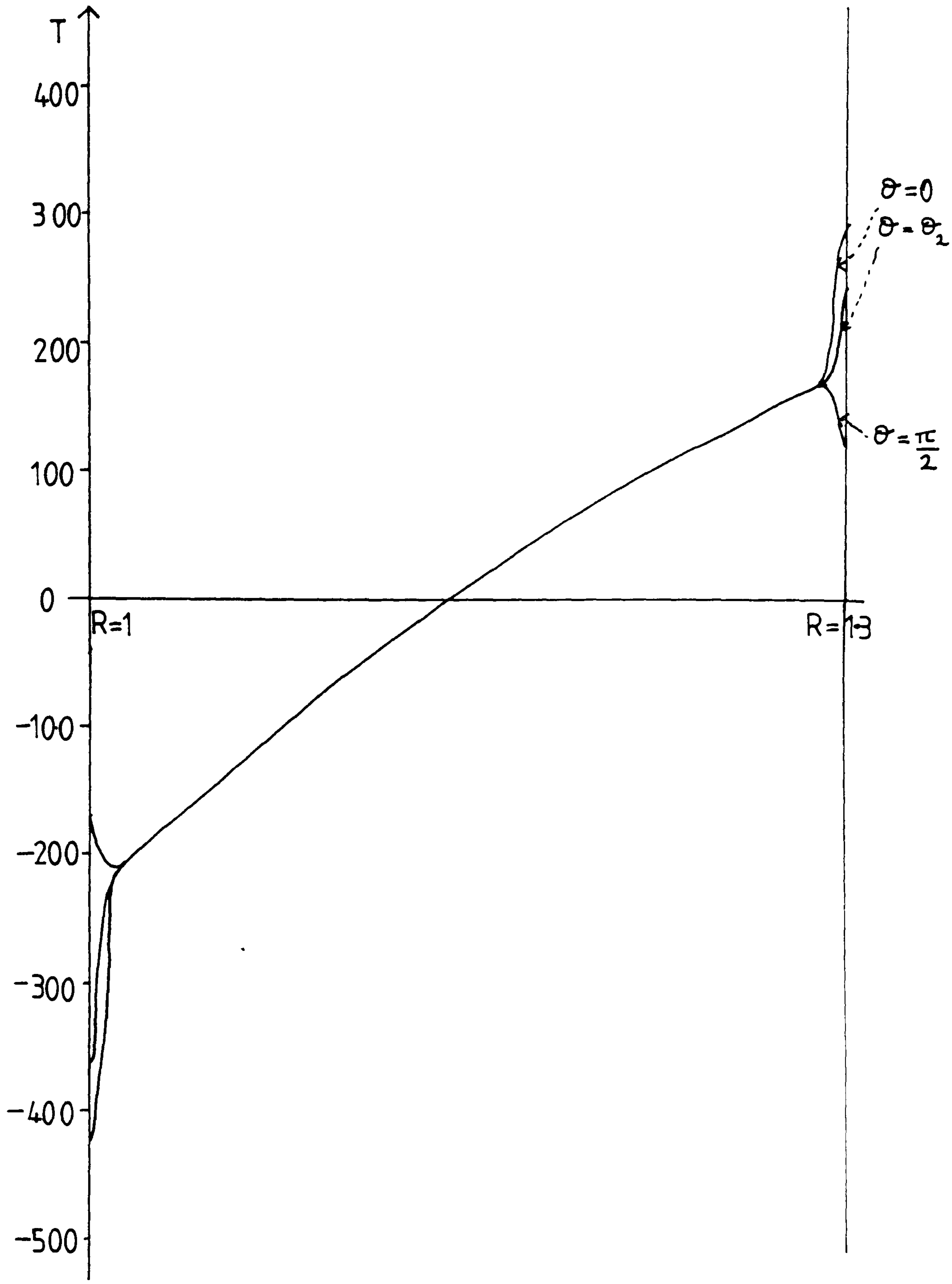



Figure 7.7 The tension with $\varepsilon = 10^{-2}$, $r = 0.3$, $R_1 = 1.3$

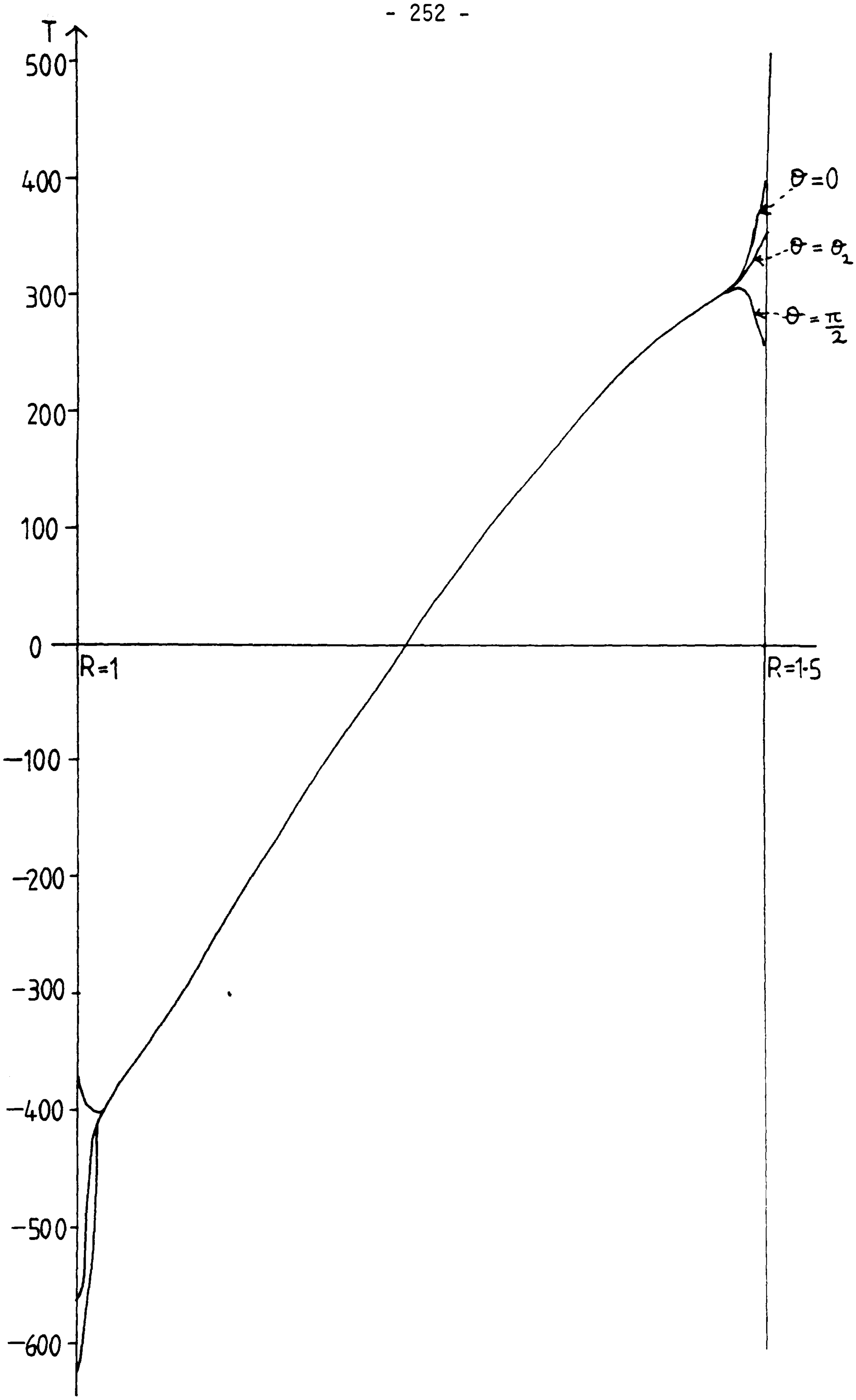



Figure 7.8 The tension with $\varepsilon = 10^{-2}$, $r = 0.3$, $R_1 = 1.5$

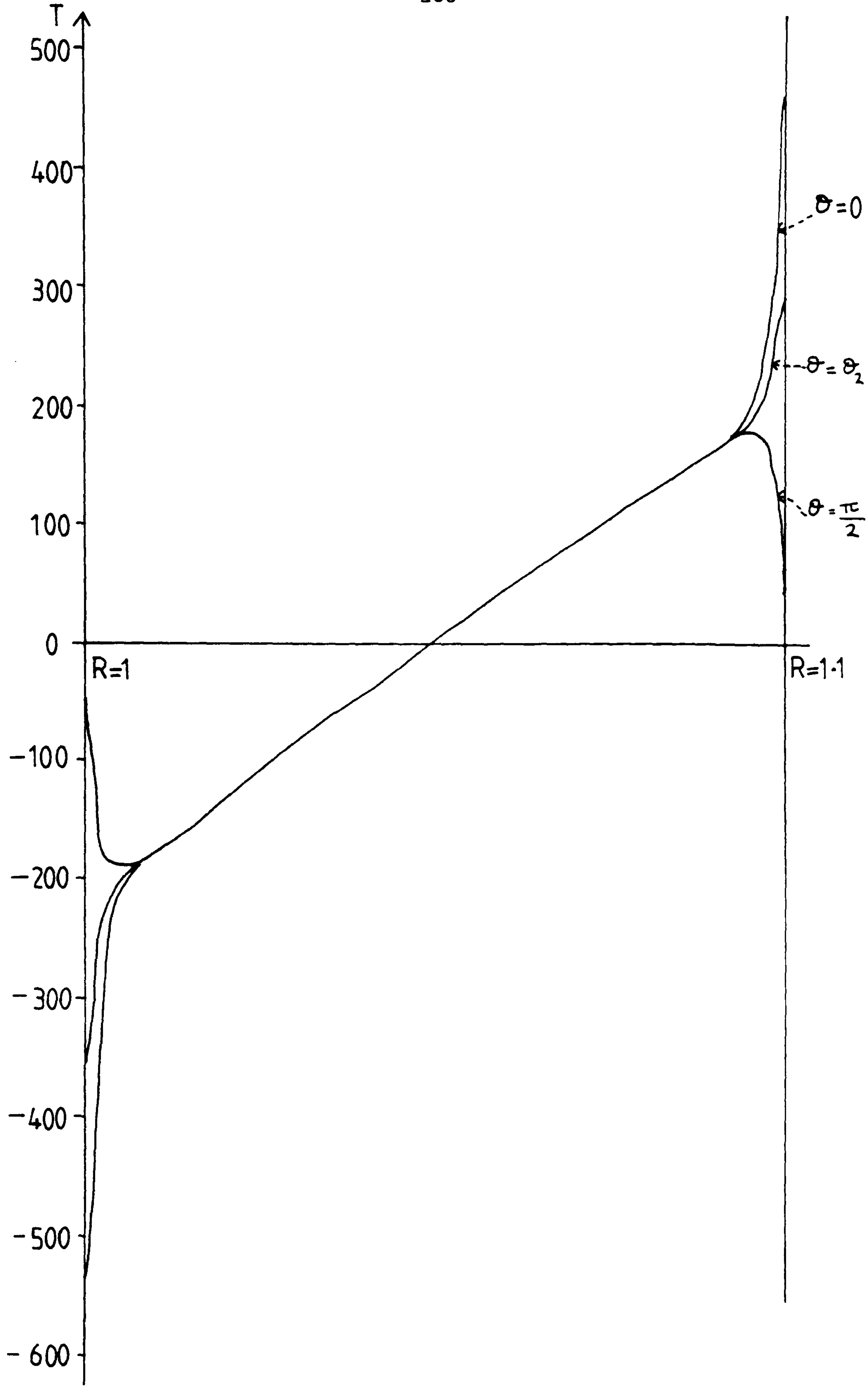



Figure 7.9 The tension with $\varepsilon = 10^{-2}$, $r = 0.34$, $R_1 = 1.1$

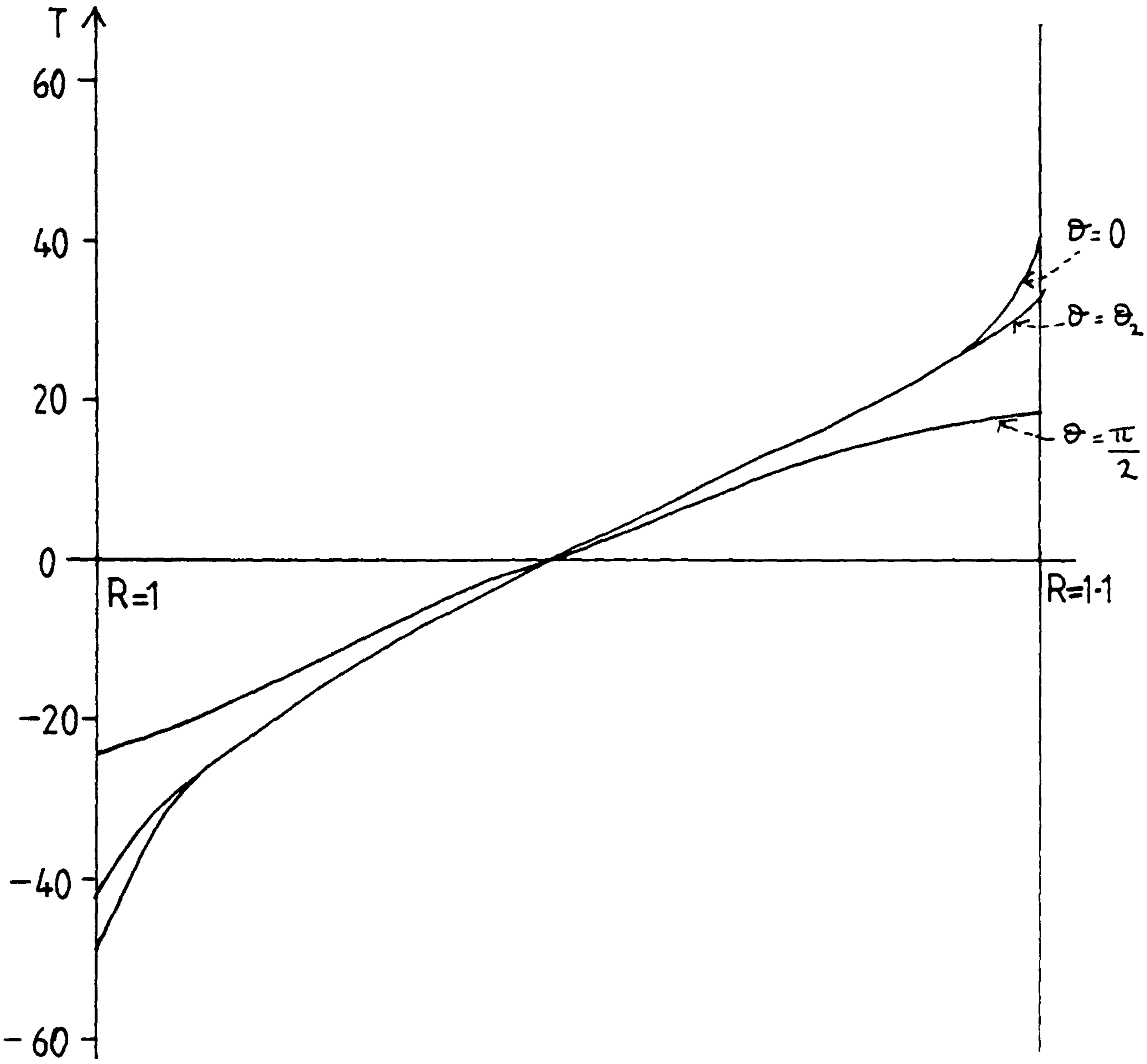



Figure 7.10 Tne tension with $\varepsilon = 10^{-1}$, $r = 0.3$, $R_1 = 1.1$

# CONCLUSION

In the previous two chapters we have shown how to calculate the geometrical configuration of a fan-belt using the ideal theory and then considered the theory for a slightly compressible and extensible material in order to obtain the tension in the belt. We have shown that the boundary layer is confined to the vicinity of the surfaces of the belt, that the tension does not vary greatly round the length of the belt and that the function $T_0(R)$ determined by the non-ideal theory forms the major contribution to the tension. The use of compatibility conditions to determine the function $T_0(R)$ is similar to their use in the Rayleigh wave problem of Chapters Three to Five to single out certain functions from the arbitrary waveforms.

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