On higher rank Cuntz-Pimsner algebras

Christina Margarete Cerny

Thesis submitted to the University of Nottingham

for the degree of Doctor of Philosophy,

July 2012

VON GUTEN MÄCHTEN STILL UND TREU UMGEBEN

meinem Nordlicht,
meinen Eltern,
meinem mathematischen Vater,
und Diwali, dem Fest des Lichts

Acknowledgements

I am indebted to the University of Nottingham for awarding me a PhD grant, to Madalin for introducing me to the department by inviting me to give a talk on my master's dissertation and to him an Florence for getting me started at Nottingham. The UKIERI network enabled me to visit India twice and I would like to thank Professor Bhat and his family, Priyanki, Sumita and Shion for making it such a rewarding experience, organising trips and welcoming me into their culture and homes. Because of funding provided by the UIMP, the departments of Copenhagen and Münster, the CRM and many other institutes, I was immersed into the C^*-community by participating in the relevant conferences which was a valuable experience to me. Professor Hofmann always gave me a warm welcome at my hometown university Darmstadt and Rolf, Suzan and Claus made me feel very appreciated for both my work and myself every time I went to Aberystwyth. Thanks to Robin and Olga for an enchanting afternoon in Southwell, to Martin for making the pre-Christmas trip to Belfast happen, to Savvas for a nice afternoon, and to Martin's family for having me and showering me with kindness as did everyone else named above.

Aidan's talk in Nottingham made me realise that a higher rank gauge-invariant uniqueness theorem can only hold for the covariant Toeplitz algebra and Nadja provided me with an insight into Cuntz-Pimsner algebras of more general discrete product systems and delivered a great lecture on the subject in Copenhagen. Thanks to the secretaries of the School of Mathematics for their thorough work and to Krystyna for attending to me beyond necessity. Thanks to my supervisor for providing me with my topic and loads of learning opportunities throughout my PhD and to Michael, Stephanie, both Stefans and Malte for providing me with a mathematical home. Steffi's, Malte's and Stefan's support, proof-reading and helpful comments made my last weeks of writing up a lot easier.

Thank you, Mirka, for making me feel that I am not the only one and for helping me to keep both my compassion and my sanity. To Kathryn, Paul and my friends, office mates and house mates Sarah, Adam, Mike, Jen, Jorma, Susanne, Steffi, Thomas, Priyanki, Julia, Walter, Martin, Lisa, Matthew, Alberto, Davide, Jonathan, Steve, James, Olli, Dave, Toni, Louis, Roz, Hang and everyone who helped my week of finding closure, tribute for this should be paid as well.

Last but not least, I would like to thank both my parents and Stefan for supporting me, loving me no matter what and providing me with a homing beacon. My time in Nottingham will have a great impact on who I am and will turn out to be. This I acknowledge being well aware of where I stand. I was lucky enough to have known a great deal of kindness, friendship, compassion and love and received just the right lessons at the right time which I am grateful for and humbled by. May others be as lucky as I am and may people get what they give.

Contents

Chapter 1

Introduction

In 1977, Joachim Cuntz introduced Cuntz algebras in [Cun77] which were the first example of separable, unital, simple and purely infinite C^*-algebras. For a given natural number $n \in \mathbb{N} \setminus \{0\}$, the Cuntz algebra $\mathcal{O}_n$ is the C^*-algebra generated by n isometries with orthogonal ranges summing up to identity. From the same paper it follows that these C^*-algebras can be obtained by factorising the C^*-algebra generated by the creation operators on the full Fock space over an n-dimensional Hilbert space, which is often called Toeplitz algebra, by the compact operators on the Fock space. Together with Wolfgang Krieger, Cuntz generalised these algebras to Cuntz-Krieger algebras in [CK80]. Given a matrix $A \in \mathbb{M}_n(\{0,1\})$, the Cuntz-Krieger algebra $\mathcal{O}_A$ is generated by n partial isometries with orthogonal ranges whose relations are encoded in A. Shortly afterwards, Masatoshi Enomoto and Yasou Watatani provided an intuitive framework for Cuntz-Krieger algebras in [EW80] using finite dimensional directed graphs. The study of C^*-algebras that are induced by graphs has henceforth been pursued and is presented well in [Rae05]. The advantage of this theory is that some structural information of a graph C^*-algebra is contained in and easier to extract from the underlying graph.

Another rich class of examples of C^*-algebras are crossed products, which were originally motivated by the study of dynamical systems. Given a C^*-algebra $\mathcal{A}$ and an automorphism $\alpha \in \mathrm{Aut}(\mathcal{A})$, the crossed product of $\mathcal{A}$ with the integers with respect to α is a C^*-algebra containing $\mathcal{A}$ such that α is inner. Since the construction of a crossed product with some more general group only requires the existence of a Haar measure, crossed products can be defined for any locally compact group. However, the corresponding crossed product does not contain a copy of that group in general. Since the crossed product of the complex numbers with any locally compact group G with respect to the trivial action is isomorphic to the (full) group C^*-algebra $C^*(G)$, crossed products can be regarded as generalisations of group C^*-algebras. A chapter on discrete crossed products can be found in [Dav96] whereas [Wil07] provides a concise overview of the more general theory.

To a Hilbert $\mathcal{A}$-correspondence (E, φ), that is a Hilbert $\mathcal{A}$-module E together with a *-homomorphism $\varphi : \mathcal{A} \to \mathcal{L}(E)$ from the coefficient algebra of E into its adjointable operators, Michael Pimsner associated two C^*-algebras, the Toeplitz algebra $\mathcal{T}_E$ and the Cuntz-Pimsner algebra $\mathcal{O}_E$. The latter one incorporates both Cuntz-Krieger algebras and crossed products by the integers, the former one generalises the Toeplitz algebra $\mathcal{T}_n$ generated by creation operators on the full Fock space of an n-dimensional Hilbert space mentioned above. Pimsner fashioned uniqueness theorems for $\mathcal{T}_E$ and $\mathcal{O}_E$, a semisplit extension of $\mathcal{O}_E$ and a KK-equivalence between $\mathcal{T}_E$ and $\mathcal{A}$ in [Pim97]. Having started by considering the concrete version of $\mathcal{T}_E$, that is the C^*-algebra $\widetilde{\mathcal{T}}_E$ generated by creation operators on the Fock module, the Hilbert module equivalent of the full Fock space, one defines a universal C^*-algebra $\mathcal{T}_E$ and shows it can be represented faithfully on the Fock module. The theorem ensuring this is called the gauge-invariant uniqueness theorem, since a vital part of the proof involves the construction of a gauge action on both algebras. The task of proving the desired isomorphism between $\mathcal{T}_E$ and $\widetilde{\mathcal{T}}_E$ then reduces to the question if the fixed point algebras of these actions are isomorphic. One benefits from the study of Cuntz-Pimsner algebras, since it provides structural information for a wide class of examples of C^*-algebras.

In [KP99], Alex Kumjian and David Pask introduced a higher dimensional analogue of directed graphs called higher rank graphs. More precisely, a higher rank graph or graph of rank k is a countable category Λ together with a functor $d : \Lambda \to \mathbb{N}^k$ with the factorisation property that for every morphism λ and every decomposition $d(\lambda) = \mathbf{m} + \mathbf{n}$ with $\mathbf{m}, \mathbf{n} \in \mathbb{N}^k$ there exist unique morphisms μ and ν such that $d(\mu) = \mathbf{m}$, $d(\nu) = \mathbf{n}$ and $\lambda = \mu\nu$. Rank one graphs correspond one to one to directed graphs by identifying a directed graph with the category of its paths and choosing $d(\mu) = |\mu|$. They are based on Robertson's and Steger's higher rank Cuntz-Krieger algebras which are associated to a collection of $\{0,1\}$-matrices with certain commutation relations. Robertson and Steger studied actions on the boundary of triangle buildings in [RS99] and noticed that crossed products by these actions are generated by interacting Cuntz-Krieger families.

Certain higher rank graph C^*-algebras are examples of higher rank Cuntz-Pimsner algebras associated to discrete product systems of Hilbert correspondences which were introduced by Fowler in [Fow02]. Given a discrete semigroup $(P,+)$, a product system is a family of Hilbert $\mathcal{A}$-correspondences $\mathbf{X} := (X_p, \varphi_p)_{p\in P}$ such that the internal tensor products are compatible with the semigroup structure of P, that is

$$X_s \otimes_{\varphi_t} X_t \cong X_{s+t} \text{ for all } s,t \in P.$$

This thesis focusses on product systems $\mathbf{X}$ over the semigroup $\mathbb{N}^k$ and provides a gauge-invariant uniqueness theorem for the higher rank Toeplitz algebra $\mathcal{T}_{\mathbf{X}}$ and a semisplit extension of the higher rank Cuntz-Pimsner algebra $\mathcal{O}_{\mathbf{X}}$.

There is an algebraic analogue of Pimsner's work, started by Leavitt's study of isomorphism classes of algebraic modules over rings in [Lea62]. For $1 \leq m \leq n$ and an arbitrary field K, the Leavitt algebra $L_K(m,n)$ is an algebra with a universal isomorphism between free modules of rank m and n respectively. In particular, $L_{\mathbb{C}}(1,n)$ is isomorphic to a dense *-subalgebra of the Cuntz algebra $\mathcal{O}_n$ and both algebras are simple and purely infinite for $n \leq 2$. Similar to graph C^*-algebras generalising Cuntz-Krieger algebras, there is an algebraic equivalent generalising Leavitt algebras, called the Leavitt algebra of a directed graph or Leavitt path algebra. Given a ring R, Carlsen and Ortega present a method of associating two rings to an R-system (P,Q,ψ) in [CO09], where P and Q are R-bimodules and $\psi : P\otimes Q \to R$ is an R-bimodule homomorphism. In some respects, these rings behave similar to the Toeplitz and Cuntz-Pimsner algebra of a Hilbert correspondence. There exists an algebraic version of the gauge-invariant uniqueness theorem for the Toeplitz ring in this article and examples include Leavitt path algebras. The Leavitt path algebras of separated graphs introduced by Ara and Goodearl in [AG10] may be the algebraic analogue of higher rank Cuntz-Pimsner algebras.

As mentioned above, this thesis generalises Pimsner's results on Toeplitz and Cuntz-Pimsner algebras of a single Hilbert correspondence to Toeplitz and Cuntz-Pimsner algebras of product systems over $\mathbb{N}^k$ in the sense of Fowler. The second chapter provides a concise overview of the theory the results discussed in chapters three and four are based upon. We will start by recalling some facts about C^*-algebras, some general constructions such as universal C^*-algebras, inductive limits and crossed products and some examples. For a more detailed description of the foundations of this subject, we refer to [Mur90]. The next section revises the theory of Hilbert C^*-modules. On first sight, they appear to behave almost like Hilbert spaces since they are equipped with a map that apart from taking values in a more general C^*-algebra than $\mathbb{C}$ closely resembles a scalar product. However, the consequences of this generalisation are dire. A section on the main definitions and results in Kasparov's KK-theory concludes the second chapter.

Pimsner's paper [Pim97] is investigated in the third chapter. We start by introducing his construction of the concrete Toeplitz algebra $\widetilde{\mathcal{T}}_E$ and the Cuntz-Pimsner algebra $\mathcal{O}_E$ associated to a given Hilbert $\mathcal{A}$-correspondence E and consider the most common examples of Cuntz-Pimsner algebras, namely Cuntz-Krieger algebras, graph C^*-algebras and crossed products by the integers. In the next section, we define the abstract Toeplitz algebra $\mathcal{T}_E$ and prove that it is isomorphic to $\widetilde{\mathcal{T}}_E$. This theorem is called the gauge-invariant uniqueness theorem, since the proof makes heavy use of the gauge actions existing on both $\widetilde{\mathcal{T}}_E$ and $\mathcal{T}_E$. Moreover, we retrace Pimsner's construction of a semisplit extension of $\mathcal{O}_E$ the section after. It is based upon the insight that a small modification of the coefficient algebra of E provides us with a bimodule, which enables us to define a two-sided Fock module, that is a Fock module over $\mathbb{Z}$ rather than $\mathbb{N}$. Since $\mathcal{O}_E$ is isomorphic to the C^*-algebra generated by creation operators on this two-sided Fock module we get a completely positive map from this algebra onto the Toeplitz algebra involved which is induced by the compression of the two-sided Fock module onto the Fock module over $\mathbb{N}$. This map lifts the quotient map from the Toeplitz algebra into the Cuntz-Pimsner algebra. In order to get into the right mindset for this section, this idea is made more precise in the motivating example 3.1.3. We conclude the chapter by working through Pimsner's proof of the KK-equivalence between $\mathcal{T}_E$ and $\mathcal{A}$ and mention results on approximation properties of $\mathcal{O}_E$ including a novel proof for the fact that $\mathcal{O}_E$ inherits nuclearity from the coefficient algebra of E.

After introducing the notion of product systems over discrete semigroups in the sense of Fowler we associate to a product system $\mathbf{X}$ the higher rank Toeplitz and Cuntz-Pimsner algebras $\mathcal{T}_{\mathbf{X}}$ and $\mathcal{O}_{\mathbf{X}}$ in the final chapter which contain $\mathcal{T}_E$ and $\mathcal{O}_E$ from the previous chapter as the special case when considering a product system over $\mathbb{N}$. We then restrict our attention to the semigroup $\mathbb{N}^k$ and present the main results of this thesis, namely a gauge-invariant uniqueness theorem of the corresponding higher rank covariant Toeplitz algebra and a semisplit extension of the higher rank Cuntz-Pimsner algebra. This time, we are going around the other way for the gauge-invariant uniqueness theorem, that is we start with the universal object, the abstract Toeplitz algebra $\mathcal{T}_{\mathbf{X}}$ which is generated by all Toeplitz representations of $\mathbf{X}$ and afterwards define the concrete Toeplitz algebra $\widetilde{\mathcal{T}}_{\mathbf{X}}$ by associating a Fock module to $\mathbf{X}$ which yields creation operators generating that algebra. When fashioning gauge actions by $\mathbb{T}^k$, we realise that this imposes an extra condition on the Toeplitz representations which for $\mathbf{X}$ being a product system over $\mathbb{N}^k$ boils down to them being doubly commuting. We will then see that the isomorphism between the Toeplitz algebra $\mathcal{T}_{\mathbf{X}}^{\mathrm{cov}}$ generated by all doubly commuting Toeplitz representations and the concrete Toeplitz algebra $\widetilde{\mathcal{T}}_{\mathbf{X}}$ can be obtained by almost the same methods that were employed in the previous chapter. Due to the structure of $\mathbb{N}^k$ the fixed point algebras of the gauge actions carry a different structure compared to the ones in chapter three, which we will need to account for. The major obstacles in this chapter arise because $\mathbb{N}^k$ has k generators rather than one generating $\mathbb{N}$. They can still be overcome because of the component-wise linear order of $\mathbb{N}^k$ and the isomorphisms $\chi_{i,j}$ relating the different base fibres of the product system. From this point of view, the proof of the semisplit exact Toeplitz extension is quite straightforward. Since those two results turn out nicely, one may hope to answer the questions if $\mathcal{T}_{\mathbf{X}}^{\mathrm{cov}}$ and $\mathcal{A}$ are KK-equivalent as well and if $\mathcal{O}_{\mathbf{X}}$ inherits approximation properties from $\mathcal{A}$. For this, we provide an overview of Deaconu's iteration of Pimsner's construction from [Dea07], which explains how the algebras $\mathcal{T}_{\mathbf{X}}^{\mathrm{cov}}$ and $\mathcal{O}_{\mathbf{X}}$ fit into the framework of the third chapter. This will make the reason why the approximation results "generalise" immediately apparent and enable us to both iterate Pimsner's KK-equivalence and fully understand the ideal structure of $\mathcal{T}_{\mathbf{X}}^{\mathrm{cov}}$ for a product system $\mathbf{X}$ over $\mathbb{N}^k$ with finitely generated full base fibres and non-degenerate left actions.

Chapter 2

C^*-Algebras, Hilbert Modules and Kasparov's KK-Theory

2.1 A very short guide to C^*-algebras

We start by giving a brief overview of the theory of C^*-algebras, providing the necessary definitions in order to establish a common language and stating the results that are essential for the rest of this thesis.

2.1.1 Some definitions

The study of C^*-algebras originated in the theory of von Neumann algebras. An abstract classification was first given by Gelfand and Naimark around 1943, the term "C^*-algebra" was introduced by Segal in [Seg47]. See [GN94] for a corrected reprint of the original paper.

2.1.1.1 Definition. Given an algebra $\mathcal{A}$, an *involution* is a conjugate-linear map ${}^* : \mathcal{A} \to \mathcal{A},\ a \mapsto a^*$ such that

$$(a^*)^* = a$$
$$(ab)^* = b^* a^*$$

for all $a, b \in \mathcal{A}$. An algebra equipped with an involution is called a **-algebra.* A Banach algebra with involution is said to be a B^*-algebra. It is a *C^*-algebra* if it satisfies the

C^*-equation, that is

$$\|a^*a\| = \|a\|^2 \text{ for all } a \in \mathcal{A}.$$

We will call a norm with this property a C^**-norm.*

2.1.1.2 Remark. On any *-algebra there exists at most one C^*-norm. See e.g. [Mur90, corollary 2.1.2].

From now on $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ will always denote C^*-algebras.

2.1.1.3 Definition. A **-homomorphism* is a linear map $\varphi : \mathcal{A} \to \mathcal{B}$ such that

$$\varphi(ab) = \varphi(a)\varphi(b)$$
$$\varphi(a^*) = \varphi(a)^*.$$

2.1.1.4 Remark. Given a B^*-algebra $\mathcal{B}$ and a C^*-algebra $\mathcal{A}$, any *-homomorphism $\varphi : \mathcal{B} \to \mathcal{A}$ is contractive, that is

$$\|\varphi(b)\| \leq \|b\| \text{ for all } b \in \mathcal{B}.$$

We assume without loss of generality that $\mathcal{A}$ and $\mathcal{B}$ are unital and that $\varphi(\mathbb{1}_{\mathcal{B}}) = \mathbb{1}_{\mathcal{A}}$. Given an invertible element $b \in \mathcal{B}$ its image $\varphi(b)$ is invertible in $\mathcal{A}$. Therefore, $\sigma(\varphi(b)) \subseteq \sigma(b)$. Hence, we know that $\rho(\varphi(b)) \leq \rho(b)$, where ρ denotes the spectral radius. Since the norm of a selfadjoint element is greater or equal to its spectral radius in any Banach*-algebra and equality holds if and only if, in addition, the algebra is a C^*-algebra, we get that

$$\|\varphi(b)\|^2 = \|\varphi(b^*b)\| = \rho(\varphi(b^*b)) \leq \rho(b^*b) \leq \|b^*b\| \leq \|b\|^2.$$

Should $\mathcal{A}$, $\mathcal{B}$ or φ not be unital consider the unitisations of $\mathcal{A}$ and $\mathcal{B}$ instead and extend φ by $\varphi(a, \lambda) = \varphi(a) + \lambda\mathbb{1}$.

2.1.1.5 Definition. A subset $\mathcal{B} \subseteq \mathcal{A}$ is called a **-subalgebra* of $\mathcal{A}$ if it is closed with respect to scalar multiplication, multiplication, addition and involution on $\mathcal{A}$ and C^**-subalgebra* if in addition to this it is norm-closed. Given some subset $F \subseteq \mathcal{A}$, we denote by $C^*(F)$ the smallest C^*-subalgebra of $\mathcal{A}$ containing F.

2.1.1.6 Remark. Given a C^*-algebra $\mathcal{A}$ and a subset $F \subseteq \mathcal{A}$, let $F^* := \{a \in \mathcal{A} : a^* \in F\}$. Then $C^*(F)$ is the closure of the set spanned by all finite products of elements in F and

F^*, that is

$$C^*(F) := \overline{\operatorname{span}}\Big\{ \prod_{k=1}^{n} a_k : n \in \mathbb{N},\ a_n \in F \cup F^* \text{ for all } k = 1, \ldots, n\Big\}.$$

2.1.1.7 Remark. Given a *-homomorphism $\varphi : \mathcal{A} \to \mathcal{B}$, $\ker(\varphi)$ is an ideal in $\mathcal{A}$ and $\varphi(\mathcal{A})$ is a C^*-subalgebra of $\mathcal{B}$. See [Mur90, 3.1.6] for details.

2.1.1.8 Remark. If both $\mathcal{A}$ and $\mathcal{B}$ are C^*-algebras and $\varphi : \mathcal{B} \to \mathcal{A}$ is injective, $\varphi : \mathcal{B} \to \varphi(\mathcal{B})$ is invertible. Since the inverse is a *-homomorphism between C^*-algebras, it is contractive. Therefore, φ is a *-isometry.

2.1.1.9 Remark. Given a norm-closed two-sided ideal $\mathcal{I}$ that is also closed under the involution, we define

$$\mathcal{A}/\mathcal{I} := \{a + \mathcal{I} : a \in \mathcal{A}\}.$$

This is a C^*-algebra with respect to the norm $\|a + \mathcal{I}\|_{\mathcal{A}/\mathcal{I}} := \inf\{\|a + x\| : x \in \mathcal{I}\}$. From now on, we always assume an ideal of a C^*-algebra to be closed under norm and involution and two-sided unless stated otherwise.

2.1.1.10 Theorem. *Let $\mathcal{I} \lhd \mathcal{A}$ be an ideal and $\mathcal{B} \subseteq \mathcal{A}$ be a C^*-subalgebra of $\mathcal{A}$. Then $\mathcal{B} + \mathcal{I}$ is a C^*-subalgebra of $\mathcal{A}$ and*

$$\mathcal{B}/(\mathcal{B} \cap \mathcal{I}) \cong (\mathcal{B} + \mathcal{I})/\mathcal{I}.$$

A proof can be found in [Lin01, corollary 1.5.16].

2.1.1.11 Definition. A C^*-algebra $\mathcal{A}$ is called *simple*, if any ideal of $\mathcal{A}$ is trivial, that is equal to either $\{0\}$ or $\mathcal{A}$.

2.1.1.12 Definition. A C^*-algebra $\mathcal{A}$ is called *separable*, if it has a countable, norm-dense subset.

2.1.1.13 Definition. If $\mathcal{A}$ contains a neutral element with respect to its multiplication, $\mathcal{A}$ is called *unital*. We denote this element by $\mathbb{1}$.

Note that if such an element exists, it is unique. In this case, we call $\mathbb{1}$ a *unit*. If $\mathcal{A}$ does not contain a unit, we can adjoin one.

2.1.1.14 Remark. Given a non-unital C^*-algebra $\mathcal{A}$, set $\mathcal{A}^\sim := \{(a,z) : a \in \mathcal{A}, z \in \mathbb{C}\}$ with coordinate-wise addition and scalar multiplication and define multiplication and involution as follows:

$$(a,y)(b,z) := (ab + za + yb, yz)$$
$$(a,z)^* := (a^*, \bar{z}).$$

With respect to the norm

$$\|(a,\lambda)\| := \sup\{\|ab + \lambda b\| : \|b\| = 1\}$$

$\mathcal{A}^\sim$ is a C^*-algebra, contains $\mathcal{A}$ as closed two-sided ideal by the inclusion

$$\iota : \mathcal{A} \hookrightarrow \mathcal{A}^\sim, \ \iota(a) := (a, 0)$$

and $\mathcal{A}^\sim \big/ \mathcal{A} \cong \mathbb{C}$. Therefore, $\mathcal{A}^\sim$ is the smallest unital C^*-algebra containing $\mathcal{A}$ as an ideal and is called the *unitisation* of $\mathcal{A}$. If $\mathcal{A}$ is unital, $\mathcal{A}^\sim := \mathcal{A}$.

2.1.1.15 Example. *Given a locally compact Hausdorff space X which is not compact,*

$$\mathcal{C}_0(X)^\sim \cong \mathcal{C}(X^+),$$

where X^+ denotes the one-point compactification of X.

2.1.1.16 Definition. An element $a \in \mathcal{A}$ is called

- ⋆ *selfadjoint* if $a = a^*$,
- ⋆ an *orthogonal projection* if $a^* = a = a^2$,
- ⋆ *normal* if $aa^* = a^*a$.

If $\mathcal{A}$ is unital, $a \in \mathcal{A}$ is called

- ⋆ an *isometry* if $a^*a = \mathbb{1}$,
- ⋆ a *coisometry* if $aa^* = \mathbb{1}$,
- ⋆ *unitary* if $aa^* = \mathbb{1} = a^*a$.

2.1.1.17 Theorem. *For $a \in \mathcal{A}$ self-adjoint the following conditions are equivalent:*

- ⋆ *The spectrum of a is positive, that is $\sigma(a) \subseteq \mathbb{R}_+$.*
- ⋆ *There exists a self-adjoint element $b \in \mathcal{A}$ such that $b^2 = a$.*
- ⋆ *There exists an element $b \in \mathcal{A}$ such that $b^*b = a$.*

2.1.1.18 Definition. In this case, a is called *positive*. By $\mathcal{A}_+$ we denote the set of all positive elements of $\mathcal{A}$. This is actually a cone.

2.1.1.19 Definition. By $\mathbb{M}_n(\mathcal{A})$, we denote $n \times n$ matrices with $\mathcal{A}$-valued entries. A linear map $\varphi : \mathcal{A} \to \mathcal{B}$ is called *positive* if $\varphi(a) \in \mathcal{B}_+$ for every $a \in \mathcal{A}_+$, *n-positive* if $(\varphi(a_{i,j}))_{1 \leq i,j \leq n} \in \mathbb{M}_n(\mathcal{B})_+$ for all $(a_{i,j})_{1 \leq i,j \leq n} \in \mathbb{M}_n(\mathcal{A})_+$ and *completely positive* if φ is n-positive for all $n \in \mathbb{N}$.

2.1.1.20 Definition. A *linear functional* on $\mathcal{A}$ is a linear map $\varphi\colon \mathcal{A} \to \mathbb{C}$. A functional φ is called *positive* if $\varphi(a) \geq 0$ for all $a \in \mathcal{A}_+$. A *state* is a positive linear functional of norm one. We denote by $S(\mathcal{A})$ the set of all states on $\mathcal{A}$.

2.1.1.21 Definition. Let $\mathcal{B}$ be a C^*-subalgebra of $\mathcal{A}$. A *conditional expectation* of $\mathcal{A}$ onto $\mathcal{B}$ is a completely positive contractive map $\psi : \mathcal{A} \to \mathcal{B}$ such that

$$\begin{aligned} \psi(b) &= b, \\ \psi(ba) &= b\psi(a), \\ \psi(ab) &= \psi(a)b \end{aligned}$$

for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$.

2.1.1.22 Theorem. *Let $\mathcal{B}$ be a C^*-subalgebra of $\mathcal{A}$ and $p\colon \mathcal{A} \to \mathcal{B}$ a projection of norm one. Then p is a conditional expectation.*

For a proof and the following examples, see [Bla06, theorem II.6.10.2, ff.].

2.1.1.23 Example. *Let $(X, \mathfrak{A}, \mu)$ be a probability space, $\mathfrak{B}$ a sub σ-algebra of $\mathfrak{A}$, $\mathcal{A} := L^\infty(X, \mathfrak{A}, \mu)$ and $\mathcal{B} := L^\infty(X, \mathfrak{B}, \mu)$. Then a conditional expectation in the sense of probability theory is a conditional expectation in the sense of the previous definition.*

2.1.1.24 Example. *For a unital C^*-algebra $\mathcal{A}$ and $\mathcal{B} := \mathbb{C}\mathbb{1}$ a conditional expectation of $\mathcal{A}$ onto $\mathcal{B}$ is a state on $\mathcal{A}$.*

2.1.1.25 Example. *If a compact topological group G acts on a C^*-algebra $\mathcal{A}$ by automorphisms $\alpha = \{\alpha_g\}_{g\in G}$ and $g \mapsto \alpha_g$ is continuous,*

$$\psi(x) := \int_{g\in G} \alpha_g(x) d\mu(g)$$

defines a conditional expectation of $\mathcal{A}$ onto $\mathcal{F}$, where μ denotes the normalised Haar measure on G and

$$\mathcal{F} := \{a \in \mathcal{A} \colon \alpha_g(a) = a \text{ for all } g \in G\}$$

the fixed point algebra of the action α.

By the unitisation, we have considered the smallest way to embed a C^*-algebra $\mathcal{A}$ into a unital one. We will now consider the largest algebra containing $\mathcal{A}$ as an essential ideal. For a commutative C^*-algebra $\mathcal{A} = \mathcal{C}_0(X)$, where X is a locally compact Hausdorff space which is not compact, this task corresponds to the question how X can be embedded into a compact space. Having seen how $\mathcal{A}^\sim$ corresponds to the one-point compactification of X in 2.1.1.15, we will now consider the analogue of the Stone-Čech compactification.

2.1.1.26 Definition. For an ideal $\mathcal{I} \lhd \mathcal{A}$, the set

$$\mathcal{I}^\perp := \{x \in \mathcal{A} : \mathcal{I}x = \{0\}\}$$

is called the *annihilator* of $\mathcal{I}$. The ideal $\mathcal{I}$ is *essential* in $\mathcal{A}$, if $\mathcal{I}^\perp = \{0\}$.

2.1.1.27 Remark. The annihilator is the orthogonal complement of $\mathcal{I}$ in $\mathcal{A}$ considered a Hilbert module over itself. An ideal $\mathcal{I}$ is essential if and only if $\mathcal{I} \cap \mathcal{J} \neq \{0\}$ for every closed non-zero ideal $\mathcal{J}$.

2.1.1.28 Example. *Let X be a compact Hausdorff space and $Y \subset X$ open. Then $\mathcal{C}_0(Y)$ is an ideal in $\mathcal{C}(X)$ and $\mathcal{C}_0(Y)^\perp = \mathcal{C}_0(Z)$, where Z denotes the interior of $X \setminus Y$. Therefore, $\mathcal{C}_0(Y)$ is an essential ideal in $\mathcal{C}(X)$ if and only if Y is dense in X.*

2.1.1.29 Definition. The *multiplier algebra* of a C^*-algebra $\mathcal{A}$ is the largest unital C^*-algebra containing $\mathcal{A}$ as an essential ideal and is denoted by $\mathcal{M}(\mathcal{A})$.

2.1.1.30 Example. *For the commutative non-unital C^*-algebra $\mathcal{C}_0(X)$, where X is a locally compact Hausdorff space which is not compact, the following holds:*

$$\mathcal{M}\big(\mathcal{C}_0(X)\big) = \mathcal{C}\big(\beta(X)\big),$$

where $\beta(X)$ denotes the Stone-Čech compactification of X.

There is an alternative approach towards the multiplier algebra.

2.1.1.31 Definition. For $\mathcal{A}$, a *double centraliser* is a pair (L, R) of bounded linear maps on $\mathcal{A}$ such that

$$L(ab) = L(a)b$$
$$R(ab) = aR(b)$$
$$R(a)b = aL(b)$$

for all $a, b \in \mathcal{A}$.

Any element $c \in \mathcal{A}$ induces a double centraliser (L_c, R_c), where

$$L_c(a) := ca$$
$$R_c(a) := ac.$$

One can check that for any double centraliser (L, R), $\|L\| = \|R\|$ and that the set of all double centralisers of $\mathcal{A}$ is a C^*-algebra with respect to multiplication, involution and norm

$$(L_1, R_1)(L_2, R_2) := (L_1L_2, R_2R_1)$$
$$(L, R)^* := (R^*, L^*)$$
$$\|(L, R)\| := \|L\| = \|R\|$$

where $L^*(a) := (L(a^*))^*$. It coincides with the multiplier algebra $\mathcal{M}(\mathcal{A})$.

2.1.1.32 Remark. Any nondegenerate representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ extends uniquely to a representation $\tilde{\pi} : \mathcal{M}(\mathcal{A}) \to \mathcal{B}(\mathcal{H})$.

2.1.2 Some important properties

2.1.2.1 Definition. An *approximate unit* of a C^*-algebra $\mathcal{A}$ is a net $\{u_\lambda\}_{\lambda\in\Lambda} \subseteq \mathcal{A}_+$ of positive elements in the closed unit ball of $\mathcal{A}$ such that

$$a = \lim_\lambda u_\lambda a \text{ for all } a \in \mathcal{A}.$$

2.1.2.2 Theorem. *Every C^*-algebra has an approximate unit. Indeed, considering the upwards-directed set Λ of all positive elements $a \in \mathcal{A}$ with $\|a\| < 1$ and setting $u_\lambda = \lambda$, $\{u_\lambda\}_{\lambda\in\Lambda}$ is an approximate unit for $\mathcal{A}$.*

For a proof see e.g. [Mur90, 3.1.1.].

2.1.2.3 Remark. Hence, every separable C^*-algebra has a countable approximate unit.

2.1.2.4 Definition. A C^*-algebra is called *σ-unital* if it has a countable approximate unit.

2.1.2.5 Definition. A positive element $h \in \mathcal{A}$ is called *strictly positive* if $\rho(h) > 0$ for all states $\rho \in S(\mathcal{A})$.

2.1.2.6 Definition. A C^*-subalgebra $\mathcal{B}$ of $\mathcal{A}$ is called *hereditary* if for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$, $0 \leq a \leq b$ implies that $a \in \mathcal{B}$. The subalgebra $\mathcal{B}$ is called *full* if it is not contained in any proper closed two-sided ideal of $\mathcal{A}$.

2.1.2.7 Remark. For any projection $p \in \mathcal{M}(\mathcal{A})$, $p\mathcal{A}p$ is a hereditary subalgebra of $\mathcal{A}$ and will be called a *corner of* $\mathcal{A}$.

2.1.2.8 Lemma. *For any $a \in \mathcal{A}_+$, $\overline{a\mathcal{A}a}$ is the hereditary C^*-algebra generated by a, that is the smallest hereditary C^*-subalgebra of $\mathcal{A}$ containing a.*

See [Lin01, lemma 1.5.9].

2.1.2.9 Theorem. *Every hereditary C^*-subalgebra of a simple C^*-algebra is simple.*

See [Mur90, 3.2.8] for a proof.

The first equivalence in the following proposition is derived from [Ped79, proposition 3.10.5], the latter one is due to [Lin01, theorem 1.5.10]

2.1.2.10 Proposition. *For a C^*-algebra $\mathcal{A}$ the following are equivalent:*

- $\star$ *$\mathcal{A}$ contains a strictly positive element h.*
- $\star$ *$\mathcal{A}$ is σ-unital.*
- $\star$ *There exists a positive element $h \in \mathcal{A}_+$ such that*

$$\mathcal{A} = \overline{h\mathcal{A}h}$$

We will now construct a C^*-algebra, which is not σ-unital.

2.1.2.11 Example. *Let $\mathcal{H}$ be a Hilbert space with orthonormal base $\{e_\lambda\}_{\lambda\in\Lambda}$, where Λ is an uncountable set. Choosing a countably infinite subset $\Lambda_0 \subsetneq \Lambda$ and letting $p \in \mathcal{B}(\mathcal{H})$ be the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}_0 := \overline{\operatorname{span}}\{e_\lambda : \lambda \in \Lambda_0\}$, we consider the ideal $\mathcal{I}_p$ that is generated by p inside $\mathcal{B}(\mathcal{H})$. It holds that*

$$\mathcal{K}(\mathcal{H}_0) \subsetneq \mathcal{I}_p \subsetneq \mathcal{B}(\mathcal{H}).$$

In order to see that $\mathcal{I}_p = \overline{\mathcal{B}(\mathcal{H})p\mathcal{B}(\mathcal{H})}$ is not σ-unital, we show that it does not contain a strictly positive element. For $h \in \mathcal{B}(\mathcal{H})$, we define the range projection $\mathrm{RP}(h) := P_{\overline{h\mathcal{H}}}$ to be the orthogonal projection onto the closed range of h. If $\mathrm{RP}(h) \lneq \mathrm{id}_{\mathcal{B}(\mathcal{H})}$, we can find a nontrivial one-dimensional projection $\mathcal{B}(\mathcal{H}) \ni q \leq (\mathrm{id}_{\mathcal{H}} - \mathrm{RP}(h))$. But then we can define the linear map $\rho_0 : \operatorname{span}\{q\} \to \mathbb{C}$ by $\rho_0(q) := 1$ which yields a non-trivial state $\rho : \mathcal{B}(\mathcal{H}) \to \mathbb{C}$, where $\rho(a) := \rho_0(qaq)$. Since ρ vanishes on h, h cannot be strictly positive. We are left to check that $\mathrm{RP}(h) \lneq \mathrm{id}_{\mathcal{H}}$ holds for all $h \in \mathcal{I}_p$. For this, note that every $h \in \mathcal{I}_p$ is a limit

$$h = \lim_{n\to\infty} h_n,$$

where $h_n = \sum_{i\in I} x_i p y_i$ are finite sums and $x_i, y_i \in \mathcal{B}(\mathcal{H})$. Since the range of $p = \mathrm{RP}(p)$ is countably generated, so are the ranges of $\mathrm{RP}(h_n)$. This limit converges only if the following sum converges

$$h = h_1 + (h_2 - h_1) + (h_3 - h_2) + \ldots.$$

But since this is a countable sum, the range of $\mathrm{RP}(h)$ is countably generated as well. Since the range of $\mathrm{id}_{\mathcal{H}}$ is uncountably generated, $\mathrm{RP}(h)$ is a proper subprojection of $\mathrm{id}_{\mathcal{H}}$.

The next corollary can be found in [Lan, lemma 6.1] and follows from the equivalence of the first and the third assertion in the previous proposition.

2.1.2.12 Corollary. *A positive element $h \in \mathcal{A}_+$ is strictly positive if and only if*

$$\overline{h\mathcal{A}} = \mathcal{A}.$$

2.1.2.13 Lemma. *For a projection $p \in \mathcal{M}(\mathcal{A})$ the following are equivalent*

- ⋆ *$\tilde{\pi}(p) \neq 0$ for all nondegenerate representations π of $\mathcal{A}$.*
- ⋆ *$\tilde{\pi}(p) \neq 0$ for all irreducible representations π of $\mathcal{A}$.*
- ⋆ *$p\mathcal{A}p$ is full.*
- ⋆ *$p\mathcal{A}$ is not contained in any proper closed two-sided ideal of $\mathcal{A}$.*
- ⋆ *$\mathcal{A}p$ is not contained in any proper closed two-sided ideal of $\mathcal{A}$.*
- ⋆ *The norm-closed two-sided ideal in $\mathcal{M}(\mathcal{A})$ generated by p includes $\mathcal{A}$.*
- ⋆ *p is not contained in any proper strictly closed two-sided ideal of $\mathcal{M}(\mathcal{A})$.*

Check [Bro77] for the proof. We also quote corollary 2.9 from the same paper below.

2.1.2.14 Theorem (L. Brown)**.** *If the C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ both contain a strictly positive element, then the following assertions are equivalent:*

- ⋆ *$\mathcal{A}$ and $\mathcal{B}$ are stably isomorphic.*
- ⋆ *There is a C^*-algebra $\mathcal{C}$ with a strictly positive element such that each $\mathcal{A}$ and $\mathcal{B}$ is isomorphic to a full corner of $\mathcal{C}$.*
- ⋆ *There is a C^*-algebra $\mathcal{C}$ such that both $\mathcal{A}$ and $\mathcal{B}$ are isomorphic to full hereditary subalgebras of $\mathcal{C}$.*

Note that when $\mathcal{A}$ and $\mathcal{B}$ are stably isomorphic, they are Morita equivalent as well. In this case, the C^*-algebra $\mathcal{C}$ from the previous theorem is a special case of the linking algebra in Theorem 2.2.4.6.

2.1.3 Universal C^*-algebras

We now want to build a C^*-algebra in a canonical way from a given set $G := \{x_i : i \in I\}$ of generators and a set R of relations between the generators and their adjoints. A relation has to be realisable as bounded linear operators on some Hilbert space and impose a norm bound on each generator when realised as an operator on a Hilbert space. The general form of a relation is

$$\|p(x_{i_1}, \dots, x_{i_n}, x_{i_1}^*, \dots, x_{i_n}^*)\| \leq m$$

for some polynomial p and $m \geq 0$. A representation of (G, R) is a set of bounded linear operators $\{T_i : i \in I\}$ satisfying the relations R. It defines a *-representation of the free *-algebra $\mathcal{F}_G$ over G. Define

$$\|x\|_{\text{universal}} := \sup\{\|\pi(x)\| : \ \pi \text{ representation of } (G, R)\}.$$

If the supremum is finite for all $x \in \mathcal{F}_G$, this is a seminorm on $\mathcal{A}$. Let $\mathcal{N} \subseteq \mathcal{A}$ be the set of elements with seminorm equal to zero. The completion of $\mathcal{A}/\mathcal{N}$ with respect to $\|\cdot\|_{\text{universal}}$ is called *universal C^*-algebra* of (G, R) and denoted by $C^*(G, R)$.

2.1.3.1 Example. *For any $n \in \mathbb{N}$, the Toeplitz algebra $\mathcal{T}_n$ is the universal C^*-algebra generated by isometries $\{T_1, \dots, T_n\}$ subject to relations*

$$T_i^* T_j = \delta_{i,j} \ \text{id}$$
$$\sum_{i=1}^{n} T_i T_i^* \leq \text{id}$$

In [Cun77] Cuntz introduced the following class of C^*-algebras.

2.1.3.2 Example. *For any $n \in \mathbb{N}$, the Cuntz algebra $\mathcal{O}_n$ is the universal C^*-algebra generated by isometries $\{S_1, \dots, S_n\}$ subject to relations*

$$S_i^* S_j = \delta_{i,j} \, \text{id}$$
$$\sum_{i=1}^{n} S_i S_i^* = \text{id}$$

2.1.3.3 Lemma. *The ideal generated inside $\mathcal{T}_n$ by $\text{id} - \sum_{i=1}^{n} T_i T_i^*$ is isomorphic to the compact operators of the full Fock space over an n-dimensional Hilbert space $\mathcal{H}_n := \mathbb{C}^n$*

and the following sequence is exact

$$0 \to \mathcal{K}\big(\Gamma(\mathcal{H}_n)\big) \to \mathcal{T}_n \to \mathcal{O}_n \to 0.$$

2.1.3.4 Example. *For any $\theta \in [0,1]$ we define A_θ to be the universal C^*-algebra generated by two unitaries S_1, S_2 subject to the relation*

$$S_1 S_2 = e^{2\pi i\theta} S_2 S_1.$$

For $\theta \in [0,1] \setminus \mathbb{Q}$, the algebra A_θ is called irrational rotation algebra *and possesses some particularly nice properties. For example, for irrational θ every $\mathcal{A}_\theta$ is simple and equipped with a unique trace. For further discussion, see [Dav96].*

2.1.3.5 Example. *A directed graph $G = (G_0, G_1, r, s)$ is called countable if both the set G_0 of vertices and the set G_1 of edges are countable. For such a graph G, we define the* graph C^*-algebra *$C^*(G)$ to be the universal C^*-algebra generated by pairwise orthogonal projections $\{P_v\}_{v\in G_0}$ and partial isometries $\{s_e\}_{e\in G_1}$ with orthogonal ranges such that*

$$\begin{aligned} s_e^* s_e &= P_{r(e)} \\ s_e s_e^* &\leq P_{s(e)} \\ P_v &= \sum_{s(e)=v} s_e s_e^* \text{ if } 0 < |s^{-1}(e)| < \infty. \end{aligned}$$

These algebras are interesting because of the interplay between properties of the graph and structural properties of the associated C^*-algebra.

2.1.4 Inductive limits of C^*-algebras

The direct or inductive limit construction provides a notion for the union of a collection of objects that are not contained in the same space. We follow the approach from [WO].

2.1.4.1 Definition. A *directed system* of algebraic objects in the same category, be it monoids, groups, algebras or *-algebras, consists of a family $\{\mathcal{A}_i\}_{i\in I}$ over a directed index set I of objects in the respective category and a collection of morphisms $\phi_{i,j} : \mathcal{A}_j \to \mathcal{A}_i$

for $j < i$ satisfying the coherence condition

$$\phi_{i,j} = \phi_{i,k} \circ \phi_{k,j} \text{ for } j < k < i.$$

Given a directed system $\{A_i, \phi_{i,j}\}$ of algebraic objects, there exists a universal algebraic object $\mathcal{A}_\infty^{\mathrm{alg}} = \lim \mathcal{A}_i = \lim\{A_i, \phi_{i,j}\}$ in the same category as the $\mathcal{A}_i$, called the *algebraic direct limit* of $\{A_i, \phi_{i,j}\}$ and *canonical morphisms* $\phi_i : \mathcal{A}_i \to \mathcal{A}_\infty$ such that

$$\mathcal{A}_\infty^{\mathrm{alg}} = \bigcup_{i \in I} \phi_i(A_i)$$

holds and the following diagram commutes for $j < i$:

$$\begin{tikzcd}
\mathcal{A}_j \arrow[r, "\phi_j"] \arrow[d, "\phi_{i,j}"'] & \mathcal{A}_\infty^{\mathrm{alg}} \\
\mathcal{A}_i \arrow[ur, "\phi_i"'] &
\end{tikzcd}$$

2.1.4.2 Remark. The algebraic direct limit can be considered the subset of the algebraic direct sum of the algebras $\mathcal{A}_i$ consisting of elements with "predictable tails", that is

$$\mathcal{A}_\infty^{\mathrm{alg}} = \bigoplus\nolimits_i \mathcal{A}_i \big/ \langle (0, \dots, 0, x_i, 0, \dots, 0, \phi_j(x_i), 0, \dots) : i \le j \in I \rangle,$$

where $x_i \in \mathcal{A}_i$. We would like to point out that this object differs from the one introduced in [WO] right after theorem L.1.1.

2.1.4.3 Theorem (universal property)**.** *Let $\{A_i, \phi_{i,j}\}$ be a directed family of algebraic objects, $\mathcal{A}_\infty^{alg} = \lim A_i$ and $\mathcal{N}$ another algebraic object in the same category such that there exists morphisms $\psi_i : \mathcal{A}_i \to \mathcal{N}$ with $\psi_j = \psi_i \circ \phi_{i,j}$ for $j < i$. Then there exists a unique morphism $\Psi : \mathcal{A}_\infty \to \mathcal{N}$ such that the following diagram commutes for all i, j where $j < i$.*

$$\begin{tikzcd}
\mathcal{A}_j \arrow[r, "\phi_j"] \arrow[d, "\phi_{i,j}"'] \arrow[dr, "\psi_j" near start] & \mathcal{A}_\infty^{alg} \arrow[d, "\Psi"] \\
\mathcal{A}_i \arrow[ur, "\phi_i" near start] \arrow[r, "\psi_i"'] & \mathcal{N}
\end{tikzcd}$$

2.1.4.4 Remark. Additionally, if all $\mathcal{A}_i$ are topological spaces, we equip $\mathcal{A}_\infty^{\mathrm{alg}}$ with the final topology with respect to $\{\phi_i\}_{i\in I}$. If $\mathcal{N}$ is a topological space and all $\psi_i : \mathcal{A}_i \to \mathcal{N}$ are continuous, then Ψ is continuous as well.

Starting with a directed system $\{\mathcal{A}_i, \phi_{i,j}\}$ of C^*-algebras and *-homomorphisms, we know that $\mathcal{A}_\infty^{\mathrm{alg}}$ is a *-algebra and all canonical homomorphisms $\phi_i : \mathcal{A}_i \to \mathcal{A}_\infty$ are *-homomorphisms.

2.1.4.5 Remark. There is a C^*-seminorm on $\mathcal{A}_\infty^{\mathrm{alg}}$ given by

$$\alpha(x) := \limsup\{\|\phi_{ij}(a_j)\| : i \in I\},$$

where $x = \phi_j(a_j)$.

If every $\phi_{i,j}$ is injective, this is a C^*-norm. In general, we define the equivalence relation

$$x \sim y \;\Leftrightarrow\; \alpha(x-y) = 0 \;\Leftrightarrow\; x - y \in \mathcal{N}_\alpha,$$

where $\mathcal{N}_\alpha := \{x \in \mathcal{A}_\infty^{\mathrm{alg}} : \alpha(x) = 0\}$ is a two-sided *-ideal in $\mathcal{A}_\infty^{\mathrm{alg}}$. The quotient $\mathcal{A}_\infty^{\mathrm{alg}}/\mathcal{N}_\alpha$ is a pre C^*-algebra with respect to $\|x + \mathcal{N}_\alpha\| := \alpha(x)$.

2.1.4.6 Definition. The completion of $\mathcal{A}_\infty^{\mathrm{alg}}/\mathcal{N}_\alpha$ with respect to the norm above is called the *C^*-algebraic direct limit* of $\{A_i, \phi_{i,j}\}$ and will be denoted by $\varinjlim \mathcal{A}_i$ or $\mathcal{A}_\infty$.

From now on, a limit of C^*-algebras denoted by $\lim \mathcal{A}_i$ will always refer to this construction as well.

2.1.4.7 Example. *Let $I := \mathbb{N}$, $\mathcal{A}_i := \mathbb{M}_{n^i}$ and*

$$\phi_{i,i+1} : A \mapsto A \otimes \mathrm{id} = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}.$$

Then $\varinjlim \mathcal{A}_i$ is the UHF-algebra corresponding to the supernatural number n^∞. More general, one may consider an arbitrary union of matrix algebras, that is $\mathcal{A}_i = \mathbb{M}_{n_i}$ for some sequence n_i of natural numbers with respect to embeddings $A \mapsto A \otimes \mathrm{id}$, a so-called UHF algebra, which is short for uniformly hyperfinite. These embeddings force every number n_i

to divide its successor n_{i+1}. For each prime number p there is a number

$$\mathbb{N} \cup \infty \ni \epsilon_p := \sup\{n \in \mathbb{N} : p^n \textit{ divides } n_i \textit{ for arbitrarily large } i\}.$$

The supernatural number associated to the directed system $\{A_i\}$ is defined to be the formal product

$$\prod_{p \textit{ prime}} p^{\epsilon_p}$$

and can be considered the limit of the prime factorisations of larger and larger n_i. A UHF algebra is determined by their supernatural number up to isomorphism by Glimm's theorem, see for example [Dav96, theorem III.5.2]. We will denote the UHF-algebra that corresponds to the supernatural number ϵ by $\mathrm{UHF}(\epsilon)$.

2.1.5 Discrete crossed products

For a given C^*-algebra $\mathcal{A}$ and an automorphism $\alpha \in \mathrm{Aut}(\mathcal{A})$, the crossed product construction supplies a C^*-algebra such that α is inner.

2.1.5.1 Definition. A *C^*-algebraic dynamical system* is a triple $(\mathcal{A}, G, \alpha)$ consisting of a C^*-algebra $\mathcal{A}$, a discrete group G and a group homomorphism $\alpha \colon G \to \mathrm{Aut}(\mathcal{A})$. We denote $\alpha_s := \alpha(s)$. A *covariant representation* of such a triple is a pair (U, π), where $\pi \colon \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is a *-representation and $U \colon G \to \mathcal{B}(\mathcal{H}), s \mapsto U_s$ is a unitary representation of G such that

$$U_s \pi(a) U_s^* = \pi\big(\alpha_s(a)\big) \text{ for all } a \in \mathcal{A}, s \in G.$$

We now consider the space of finite formal sums

$$\mathcal{A}G := \mathcal{C}_{\mathrm{C}}(G, \mathcal{A}) := \{f = \sum_{\text{finite}} a_s s : a_s \in \mathcal{A}\}$$

and aim to complete it to a C^*-algebra. With

$$tat^{-1} := \alpha_t(a)$$

we get the following multiplication rule

$$\begin{aligned} f * g &= \sum_{t\in G}\sum_{u\in G} a_t t b_u u &&= \sum_{t\in G}\sum_{u\in G} a_t t b_u t^{-1} t u \\ &= \sum_{t\in G}\sum_{u\in G} a_t \alpha_t(b_u) t u &&= \sum_{t\in G}\sum_{s\in G} a_t \alpha_t(b_{t^{-1}s}) s\ , \end{aligned}$$

where $f, g \in \mathcal{A}G$. Since $s^* = s^{-1}$, we get that

$$(as)^* = \alpha_s^{-1}(a^*)s^{-1}$$

and hence an involution

$$f^* = \sum_{t\in G} \alpha_t(a_{t^{-1}}^*)t.$$

With this multiplication and involution, $\mathcal{A}G$ is a *-algebra. Given a covariant representation (π, U) of $(\mathcal{A}, G, \alpha)$ we obtain a *-representation of $\mathcal{A}G$ by

$$\sigma(f) := \sum_{t\in G} \pi(a_t)U_t.$$

Given a *-representation σ, the neutral element $e \in G$ and an approximate unit $\{e_\lambda\}_{\lambda\in\Lambda}$ of $\mathcal{A}$, the restrictions

$$\pi(a) := \sigma(ae) \text{ and } U_t := \lim_{\lambda\in\Lambda} \sigma(e_\lambda t)$$

are a covariant representation of $(\mathcal{A}, G, \alpha)$.

2.1.5.2 Definition. The *(full) crossed product* $\mathcal{A} \ltimes_\alpha G$ is the enveloping C^*-algebra of $\mathcal{A}G$, that is the completion of $\mathcal{A}G$ with respect to the norm

$$\|f\| := \sup\{\|\sigma(f)\| : \sigma\ ^*\text{-rep. of } \mathcal{A}G\}.$$

2.1.5.3 Remark. For the sake of completeness, we mention that crossed products can be defined for any locally compact group. In the more general case, one additionally requires the unitary representation $G \to \mathcal{B}(\mathcal{H})$ to be strongly continuous. The completion of the continuous, compactly supported functions from G to $\mathcal{A}$ with respect to the norm

$$\|f\|_1 := \int_{s\in G} \|f(s)\| d\mu_G(s)$$

is denoted by $L^1(G, \mathcal{A})$ and is a *-algebra with respect to multiplication and involution

$$f * g := \int_{s,t \in G} f(t)\alpha_t\big(g(t^{-1}s)\big),$$
$$f^*(s) := \Delta(s)^{-1}\alpha_s\big(f(s^{-1})^*\big),$$

where Δ denotes the modular function that relates left and right Haar measure of G. The crossed product $\mathcal{A} \ltimes_\alpha G$ is the completion of $L^1(G, \mathcal{A})$ with respect to the norm

$$\|f\| := \sup\{\|\pi(f)\| : \pi \text{ non-degenerate } ^*\text{-representation of } L^1(G, \mathcal{A})\}.$$

2.1.5.4 Remark. The irrational rotation algebra from 2.1.3.4 is a crossed product by the integers. For this we define the rotation automorphism

$$R_\theta : \mathbb{T} \to \mathbb{T},\ e^{2\pi i t} \mapsto e^{2\pi i(\theta + t)}$$

for $\theta \in [0, 1] \setminus \mathbb{Q}$. This induces an automorphism ρ_θ of the continuous functions on the complex unit circle by $\rho_\theta(f)(z) := f\big(R_\theta(z)\big)$ and

$$A_\theta = \mathcal{C}(\mathbb{T}) \ltimes_{\rho_\theta} \mathbb{Z}.$$

2.1.5.5 Remark. Given a nondegenerate faithful representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$, there always exists a covariant representation (π_α, λ) of $(\mathcal{A}, G, \alpha)$ given by

$$\big(\pi_\alpha(x)\xi\big)(t) := \pi\big(\alpha_{t^{-1}}(x)\big)\big(\xi(t)\big)$$
$$\big(\lambda(t)\xi\big)(s) := \xi(t^{-1}s).$$

The corresponding *-representation $\pi_\alpha \times \lambda$ of $\mathcal{A}G$ is faithful and

$$\|f\|_r := \|\pi_\alpha \times \lambda(f)\|$$

is independent from the choice of π and defines a norm on $\mathcal{A}G$. The completion of $\mathcal{A}G$ with respect to $\|\cdot\|_r$ is called *reduced crossed product* and denoted by $\mathcal{A} \ltimes_\alpha^r G$. The reduced and the full crossed product are isomorphic whenever G is amenable.

The following cyclic six-term exact sequence known as *Pimsner-Voiculescu sequence* is an important tool to calculate the K-theory of a full crossed product of a unital C^*-algebra $\mathcal{A}$ by the integers with respect to any automorphism α and was originally developed in [PV80] to calculate the K-groups of $\mathcal{A}_\theta$. For the proof, see theorem 2.4 in this paper.

2.1.5.6 Theorem. *For a unital C^*-algebra $\mathcal{A}$ and an automorphism α of $\mathcal{A}$ the following sequence is exact:*

$$\begin{tikzcd} K_0(\mathcal{A}) \arrow[r, "\mathrm{id}-\alpha_*"] & K_0(\mathcal{A}) \arrow[r, "\iota"] & K_0(\mathcal{A} \ltimes_\alpha \mathbb{Z}) \arrow[d] \\ K_1(\mathcal{A} \ltimes_\alpha \mathbb{Z}) \arrow[u] & K_1(\mathcal{A}) \arrow[l, "\iota"'] & K_1(\mathcal{A}) \arrow[l, "\mathrm{id}-\alpha_*"'] \end{tikzcd} ,$$

where $\iota : \mathcal{A} \hookrightarrow \mathcal{A} \ltimes_\alpha \mathbb{Z}$.

2.2 Hilbert C^*-modules

A Hilbert C^*-module is a Hilbert space with the scalar product taking values in an arbitrary C^*-algebra rather than $\mathbb{C}$. The concept was introduced by Kaplansky 1953 in [Kap53] for commutative unital C^*-algebras and generalised to arbitrary C^*-algebras by Paschke in [Pas73] and Rieffel in [Rie74] independently. A good overview about the topic can be found in [Bla06], a more detailed treatment in [Lan]. After reviewing the definition and some examples we will give an overview about the main results we require and see how Hilbert C^*-modules form a category between Banach spaces and Hilbert spaces.

2.2.1 Definition and examples

When attempting to generalise the Hilbert space notion of a complex-valued scalar product to a function mapping into an arbitrary C^*-algebra, one requires a suitable notion of positivity. It turns out that the positive elements of the C^*-algebra are the suitable equivalent to $\mathbb{R}_+$ inside $\mathbb{C}$.

2.2.1.1 Definition. Let $\mathcal{A}$ be a C^*-algebra. A *(right) pre-Hilbert $\mathcal{A}$-module*, or *(right) pre-Hilbert module over $\mathcal{A}$* is a right $\mathcal{A}$-module E equipped with a sesquilinear map $\langle \cdot, \cdot \rangle : E \times E \to \mathcal{A}$ such that

- $\star$ $\langle x, ya \rangle = \langle x, y \rangle a$,

- $\star$ $\langle x, y\rangle^* = \langle y, x\rangle$,
- $\star$ $\langle x, x\rangle \geq 0$,
- $\star$ $\langle x, x\rangle = 0 \;\Rightarrow\; x = 0$,

where $x, y \in E$ and $a \in \mathcal{A}$. We assume sesquilinear maps to be conjugate linear in the first variable and call a map with these properties an *inner product.*

2.2.1.2 Remark. It is easy to check that $\langle\cdot,\cdot\rangle$ induces a norm on E by

$$\|x\| := \|\langle x, x\rangle\|^{\frac{1}{2}}.$$

See for example [Pas73, proposition 2.3].

2.2.1.3 Definition. If E is complete with respect to this norm, it is called a *(right) Hilbert $\mathcal{A}$-module.* The Hilbert module E is called *full*, if

$$\overline{\operatorname{span}}\{\langle x, y\rangle : x, y \in E\} = \mathcal{A}.$$

A subset $Z \subseteq E$ is called a *generating set* of E, if the closed submodule generated by Z in E is E. A Hilbert module E is *countably generated* if it has a countable generating set.

2.2.1.4 Remark. One may just as well define left Hilbert modules. We prefer to work with right modules so operators can act on the left hand side.

From now on, E and F and G will denote Hilbert $\mathcal{A}$-modules.

2.2.1.5 Example. *Hilbert spaces are Hilbert $\mathbb{C}$-modules and vice versa.*

2.2.1.6 Example. *Any C^*-algebra $\mathcal{A}$ is a Hilbert $\mathcal{A}$-module with respect to the inner product $\langle a, b\rangle := a^*b$ and right action of $\mathcal{A}$ on itself by the algebra multiplication.*

2.2.1.7 Example. *Given a Hilbert space $\mathcal{H}$ and a C^*-algebra $\mathcal{A}$, $\mathcal{H} \otimes \mathcal{A}$ is a Hilbert $\mathcal{A}$-module with $(x \otimes a)b := x \otimes (ab)$ and inner product*

$$\langle x \otimes a, y \otimes b\rangle_{\mathcal{H}\otimes\mathcal{A}} := \langle x, y\rangle_{\mathcal{H}} a^* b.$$

This is a special case of the internal tensor product. See 2.2.3.4 for the definition.

2.2.1.8 Example. *For n Hilbert $\mathcal{A}$-modules E_k, $(k = 1, \ldots, n)$, with inner products $\langle \cdot, \cdot \rangle_k$, $E_1 \oplus \cdots \oplus E_n$ is a Hilbert $\mathcal{A}$-module with right action*

$$(x_1 \oplus \cdots \oplus x_n)a := (x_1 a) \oplus \cdots \oplus (x_n a)$$

and inner product

$$\langle (x_1, \ldots, x_n), (y_1, \ldots, y_n) \rangle := \sum_{k=1}^{n} \langle x_k, y_k \rangle_k$$

for all $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in E_1 \oplus \cdots \oplus E_n$.

2.2.1.9 Example. *For an arbitrary collection $\{E_i\}_{i \in I}$ of Hilbert $\mathcal{A}$-modules with inner products $\langle \cdot, \cdot \rangle_i$ one can define*

$$\bigoplus_{i \in I} E_i := \{(x_i)_{i \in I} \in \prod_{i \in I} E_i : \sum_i \langle x_i, x_i \rangle_i \text{ converges in } \mathcal{A}\}.$$

In particular,

$$H_{\mathcal{A}} := \bigoplus_{i \in \mathbb{N}} \mathcal{A}$$

is called the standard Hilbert module *of $\mathcal{A}$.*

2.2.1.10 Lemma. *A C^*-algebra $\mathcal{A}$ considered a Hilbert module over itself is countably generated if and only if it is σ-unital.*

Proof. Let us assume that $\mathcal{A}$ is σ-unital. By 2.1.2.10 it has a strictly positive element $h \in \mathcal{A}$. But then $\{h\}$ is a generating set.

Conversely, if $\mathcal{A}$ is countably generated with generating set $\{a_n : n \geq 1\}$, we can assume that $a_n \geq 0$ and $\|a_n\| \leq 1$ and define

$$h := \sum_n 2^{-n} a_n.$$

Since the a_n form a generating set of $\mathcal{A}$, for every non-trivial state ρ there exists at least one n such that $\rho(a_n) \gneq 0$. Therefore, h is strictly positive. $\square$

2.2.2 Operators on Hilbert modules

In order to adjust the concept of bounded linear operators to the Hilbert module setting, we will demonstrate a pitfall inherent to this setting before providing the definition that generalises bounded linear operators.

2.2.2.1 Remark. Hilbert modules resemble Hilbert spaces in many respects. For example,

$$\|x\| = \sup\{\|\langle x, y\rangle\| : y \in E, \|y\| \leq 1\}.$$

However, unlike Hilbert spaces Hilbert modules are not necessarily spanned by a closed linear subspace and its orthogonal complement. For example, consider the C^*-algebra $\mathcal{C}([0,1])$ of continuous functions over the unit interval a Hilbert module over itself with closed linear subspace $\mathcal{C}_0((0,1])$, that is the set of all continuous functions vanishing at 0. In this case, the orthogonal complement of this subspace is trivial but the subspace itself is a proper ideal of $\mathcal{C}([0,1])$. As an immediate consequence not every bouded linear operator needs to have an adjoint. Check out 2.2.2.4 for the respective example.

2.2.2.2 Definition. A map $T : E \to F$ is called *adjointable*, if there exists a map $T^* : F \to E$ such that

$$\langle Tx, y\rangle = \langle x, T^*y\rangle$$

for all $x \in E, y \in F$. We denote by $\mathcal{L}(E,F)$ the set of all adjointable maps from E to F and write $\mathcal{L}(E)$ for $\mathcal{L}(E,E)$.

2.2.2.3 Remark. Every $T \in \mathcal{L}(E,F)$ is linear and bounded. It is easy to see that T has to be $\mathcal{A}$-linear. To see that it has to be bounded, we denote by E_1 the unit ball of E and define a map $T_x : F \to \mathcal{A}$ for every $x \in E_1$ by

$$T_x(y) := \langle Tx, y\rangle$$

for all $y \in F$. Since $\|T_x(y)\| \leq \|T^*(y)\|$ for all $x \in E_1$, the set $\{\|T_x\| : x \in E_1\}$ is bounded by Banach-Steinhaus. Therefore T is bounded.

2.2.2.4 Remark. To see that a bounded linear map need not be adjointable, consider the Hilbert modules $E := \mathcal{C}_0((0,1]) := \{f \in \mathcal{C}([0,1]) \colon f(0) = 0\}$ and $F := \mathcal{C}([0,1])$ together with inclusion map $\iota \colon E \to F$ and the constant one function $\mathbb{1} \in F$. If ι was adjointable, then $\iota^*\mathbb{1} = \mathbb{1} \in E$. Since this contradicts the definition of E, the bounded linear map ι is not adjointable.

2.2.2.5 Remark. For every Hilbert $\mathcal{A}$-module E, $\mathcal{L}(E)$ is a C^*-algebra. For this, see [Bla06, proposition 13.2.2].

The generalisation of finite rank operators and compact operators in this setting resembles the Hilbert space case as well.

2.2.2.6 Definition. For every $x \in E$ and $y \in F$ we define a map $\theta_{x,y} \colon F \to E$ by

$$\theta_{x,y}(z) := x\langle y, z\rangle$$

for all $z \in F$ and set

$$\mathcal{K}(E, F) := \overline{\operatorname{span}}\{\theta_{x,y} : x \in E, y \in F\}.$$

We call $\theta_{x,y}$ a *rank one operator* and $\mathcal{K}(E, F)$ the *compact operators* and denote $\mathcal{K}(E, E)$ by $\mathcal{K}(E)$.

2.2.2.7 Remark. For a unital C^*-algebra $\mathcal{A}$, the identity $\mathrm{id}_{\mathcal{A}} = \theta_{\mathbb{1},\mathbb{1}}$ is a rank one operator, but $\mathrm{id}_{\mathcal{A}}$ is not a compact operator from the Banach space $\mathcal{A}$ into itself unless $\mathcal{A}$ is finite dimensional.

2.2.2.8 Remark. It is easy to check that $\theta_{x,y} \colon F \to E$ is adjointable, $\theta^*_{x,y} = \theta_{y,x}$ and that the following relations hold

$$\begin{aligned}
\theta_{x,y}\theta_{u,v} &= \theta_{x\langle y,u\rangle,v} = \theta_{x,v\langle u,y\rangle} \\
T\theta_{x,y} &= \theta_{Tx,y} \\
\theta_{x,y}S &= \theta_{x,S^*y}
\end{aligned}$$

for $x \in E$, $y, u \in F$, $v \in G$, $T \in \mathcal{L}(E, G)$ and $S \in \mathcal{L}(G, F)$.
In particular, $\mathcal{K}(E)$ is an ideal in $\mathcal{L}(E)$.

2.2.2.9 Remark. It holds that $\mathcal{L}(E) \cong \mathcal{M}(\mathcal{K}(E))$ ([Lan, theorem 2.4]).

2.2.2.10 Theorem (Kasparov's stabilisation theorem)**.** *Let $\mathcal{A}$ be σ-unital. For every countably generated Hilbert $\mathcal{A}$-module X it holds that*

$$X \oplus H_{\mathcal{A}} \cong H_{\mathcal{A}}.$$

Consult [Lan, theorem 6.2] for the proof.

The following definition and theorem can be found in [FL02, definition 2.1, theorem 4.1].

2.2.2.11 Definition. Let $\mathcal{A}$ be unital. A finite or countably infinite set $\{x_i\}_{i\in I}$ is said to be a *frame* of the Hilbert $\mathcal{A}$-module X if there exist two real constants $C, D > 0$ such that

$$C\langle x, x\rangle \leq \sum_{i\in I}\langle x, x_i\rangle\langle x_i, x\rangle \leq D\langle x, x\rangle \text{ for all } x \in X.$$

The frame is called *standard* if the sum in the middle converges in $\mathcal{A}$ in norm and *normalised tight* if $C = 1 = D$. A sequence is called a *(generalised) Riesz basis* if $\{x_i\}_{i\in I}$ is a generating set and a frame with

$$\sum_{i\in S} x_i a_i = 0 \text{ iff } x_i a_i = 0 \text{ for all } i \in S,$$

where $S \subseteq I$ and $a_i \in \mathcal{A}$.

The following theorems and their proofs can be found in [FL02].

2.2.2.12 Theorem. *Every countably generated Hilbert module has a standard normalised tight frame.*

2.2.2.13 Theorem (frame transform and reconstruction formula)**.** *For a unital C^*-algebra $\mathcal{A}$ and a finitely or countably generated Hilbert $\mathcal{A}$-module X that possesses a standard normalised tight frame $\{x_i\}_{i\in I}$ the corresponding frame transform*

$$\theta : X \to H_{\mathcal{A}},\ \theta(x) := \{\langle x, x_i\rangle\}_{i\in I}$$

is isometric and adjointable. The adjoint θ^* is surjective and $\theta^*(e_i) = x_i$ for every $i \in I$. For every $x \in X$,

$$x = \sum_i x_i\langle x_i, x\rangle,$$

where the sum converges in norm.

2.2.3 Tensor products of Hilbert modules

Let E be a Hilbert $\mathcal{A}$-module and F be a Hilbert $\mathcal{B}$-module. We start by turning the module tensor product $E \otimes F$ into an $\mathcal{A} \otimes \mathcal{B}$-module. Considering E and F vector spaces, their algebraic tensor product $E \odot F$ is an $\mathcal{A} \odot \mathcal{B}$ module with right action

$$(x \otimes y)(a \otimes b) = xa \otimes yb.$$

Extending

$$\langle x_1 \otimes y_1, x_2 \otimes y_2 \rangle_{\mathcal{A} \otimes \mathcal{B}} := \langle x_1, x_2 \rangle_{\mathcal{A}} \odot \langle y_1, y_2 \rangle_{\mathcal{B}}$$

yields an $\mathcal{A} \odot \mathcal{B}$-valued inner product on $E \odot F$.

2.2.3.1 Definition. The Hilbert $\mathcal{A} \otimes \mathcal{B}$-module we get by closing $E \odot F$ with respect to $\langle \cdot, \cdot \rangle_{\mathcal{A} \otimes \mathcal{B}}$ is called *outer* or *external tensor product* of E and F and is denoted by $E \hat{\otimes} F$.

When, in addition, given a *-homomorphism $\varphi \colon \mathcal{A} \to \mathcal{L}(F)$, we can construct a Hilbert $\mathcal{B}$-module from E and F. The *-homomorphism implements an additional left action of $\mathcal{A}$ on the right Hilbert $\mathcal{B}$-module F.

2.2.3.2 Definition. For two C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ a *(Hilbert) $\mathcal{A}$-$\mathcal{B}$-correspondence* is a pair (E, φ), where E is a Hilbert $\mathcal{B}$-module and $\varphi : \mathcal{A} \to \mathcal{L}(E)$ is a *-homomorphism. It is called *non-degenerate* if

$$\varphi(\mathcal{A})E = E$$

and *faithful*, if φ is injective. An $\mathcal{A}$-correspondence refers to an $\mathcal{A}$-$\mathcal{A}$-correspondence.

2.2.3.3 Remark. Due to Cohen's factorisation theorem it holds that

$$\varphi(\mathcal{A})E = \overline{\varphi(\mathcal{A})E}.$$

It can be found in [Bla06, theorem II.5.3.7].

Given a Hilbert $\mathcal{A}$-module E and a Hilbert $\mathcal{A}$-$\mathcal{B}$-correspondence F, the (algebraic) module tensor product $E \odot F$ of E and F is a $\mathcal{B}$-module with $(\xi \otimes \eta)b = \xi \otimes (\eta b)$. On $E \odot F$, we define a $\mathcal{B}$-valued pairing by

$$\langle \xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2 \rangle_{\mathcal{B}} := \langle \eta_1, \varphi(\langle \xi_1, \xi_2 \rangle_E)\eta_2 \rangle_F.$$

This need not be an inner product, since it is not necessarily definite. One checks that it vanishes on elements of the form

$$\xi a \otimes \eta - \xi \otimes \varphi(a)\eta.$$

In fact, these elements span the kernel of this pairing [Lan, proof of proposition 4.5]. Hence, this pairing is an inner product on the quotient

$$E \odot F \big/ \mathrm{span}\{\xi a \otimes \eta - \xi \otimes \varphi(a)\eta\} \, .$$

2.2.3.4 Definition. The Hilbert $\mathcal{B}$-module we get by completing the above quotient with respect to $\langle \cdot, \cdot \rangle_{\mathcal{B}}$ is called the *internal* or *inner tensor product* of E and F or the tensor product of E and F which is *balanced over* $\mathcal{A}$. We denote it by $E \otimes_\varphi F$. Given an $\mathcal{A}$-correspondence E, we write

$$E^{\otimes n} := \underbrace{E \otimes_\varphi E \otimes_\varphi \cdots \otimes_\varphi E}_{n \text{ times}}$$

for any $n \in \mathbb{N}$ with the convention that $E^0 := \mathcal{A}$.

2.2.3.5 Remark. For any adjointable operator $T \in \mathcal{L}(E)$ we can define $T \otimes \mathrm{id} \in \mathcal{L}(E \otimes_\varphi F)$ by

$$T \otimes \mathrm{id}(x \otimes y) := T(x) \otimes y.$$

Therefore, given an $\mathcal{A}$-$\mathcal{B}$-correspondence (E, φ) and a $\mathcal{B}$-$\mathcal{C}$-correspondence (F, ψ), $(E \otimes_\psi F, \varphi \otimes \mathrm{id})$ is an $\mathcal{A}$-$\mathcal{C}$-correspondence with left action

$$(\varphi \otimes \mathrm{id})(a)(x \otimes y) := \big(\varphi(a)(x)\big) \otimes y$$

for all $a \in \mathcal{A}$, $x \in E$ and $y \in F$. In particular, $E^{\otimes n}$ is an $\mathcal{A}$-correspondence with this left action, which we will denote by φ_n for all $n \geq 1$. For $n = 0$, $\varphi_0(a)b := ab$.

2.2.3.6 Lemma. *For E and F as above and for every $x \in E$ the equation*

$$T_x(y) := x \otimes y,$$

defines an adjointable operator $T_x \in \mathcal{L}(F, E \otimes_\varphi F)$ *with*

$$T_x^*(z \otimes y) = \varphi(\langle x, z\rangle)y,$$

where $y \in F$. *We call* T_x *creation operator by* x *and* T_x^* *annihilation operator by* x. *They satisfy the following relations:*

$$\begin{aligned} T_x T_z^*(e \otimes f) &= T_x\big(\varphi(\langle z, e\rangle)f\big) = x\langle z, e\rangle \otimes f \\ &= \theta_{x,z}(e) \otimes f \\ T_z^* T_x &= \varphi(\langle z, x\rangle)\ . \end{aligned}$$

In particular, $\|T_x\|^2 = \|\varphi(\langle x, x\rangle)\|$.

See [Lan, lemma 4.6] for proof.

2.2.3.7 Lemma. *Given a countably generated Hilbert* $\mathcal{A}$*-module* F *and a* C^**-algebra* $\mathcal{B}$ *with* $\mathcal{K}(F) \subseteq \mathcal{B} \subseteq \mathcal{L}(F)$. *Then*

$$\mathcal{B} \otimes F \cong F.$$

Proof. We will show that the map $\mathcal{B} \otimes F \to F$, $b \otimes \xi \mapsto b(\xi)$ is an isomorphism. Let $\{\xi_i\}_{i \in I}$ be a standard normalised tight frame of F from 2.2.2.12 and $\Lambda = \{\lambda \subset I : \lambda \text{ finite}\}$. Then $e_\lambda := (\sum_{i \in \lambda} \theta_{\xi_i, \xi_i})$, $(\lambda \in \Lambda)$ is an approximate unit of $\mathcal{K}(F)$. Since $e_\lambda(\xi) \to \xi$ for all $\xi \in F$, the map is surjective. Furthermore,

$$\begin{array}{rcccl} \|b \otimes \xi\|^2 &=& \|\langle \xi, \varphi(b^*b)\xi\rangle\| &=& \|\langle \varphi(b)\xi, \varphi(b)\xi\rangle\| \\ &=& \|\langle b(\xi), b(\xi)\rangle\| &=& \|b(\xi)\|^2, \end{array}$$

so the map is isometric and hence an isometry. □

2.2.4 Hilbert bimodules

2.2.4.1 Definition. A *(Hilbert)* $\mathcal{A}$-$\mathcal{B}$*-bimodule* is a right Hilbert $\mathcal{B}$-module E with right $\mathcal{B}$-linear $\mathcal{B}$-valued inner product $\langle \cdot, \cdot \rangle_{\mathcal{B}}$ that is at the same time a left Hilbert $\mathcal{A}$-module with left $\mathcal{A}$-linear $\mathcal{A}$-valued inner product ${}_{\mathcal{A}}\langle \cdot, \cdot \rangle$ such that

$${}_{\mathcal{A}}\langle x, y\rangle z = x\langle y, z\rangle_{\mathcal{B}}.$$

for all $x, y, z \in E$. An $\mathcal{A}$-$\mathcal{B}$-bimodule is said to be *full* if it is full both as a left Hilbert $\mathcal{A}$-module and a right Hilbert $\mathcal{B}$-module.

2.2.4.2 Remark. In the literature, full $\mathcal{A}$-$\mathcal{B}$-bimodules are called imprimitivity bimodules.

2.2.4.3 Example. *$\mathcal{A}$ is a full $\mathcal{A}$-$\mathcal{A}$-bimodule with left $\mathcal{A}$-linear inner product ${}_{\mathcal{A}}\langle a, b\rangle := ab^*$ and right $\mathcal{A}$-linear inner product $\langle a, b\rangle_{\mathcal{A}} := a^*b$.*

2.2.4.4 Example. *Every Hilbert $\mathcal{A}$-module E is a full $\mathcal{K}(E)$-$\mathcal{A}$ bimodule with left $\mathcal{K}(E)$-linear inner product*

$$ {}_{\mathcal{K}(E)}\langle x, y\rangle := \theta_{x,y}. $$

2.2.4.5 Definition. The C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ are called *Morita equivalent*, if a full $\mathcal{A}$-$\mathcal{B}$-bimodule exists.

2.2.4.6 Theorem. *The C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ are Morita equivalent if and only if they are complementary full corners in some C^*-algebra $\mathcal{C}$, meaning there is a projection $p \in \mathcal{M}(\mathcal{C})$ such that $\mathcal{A} \cong p\mathcal{C}p$, $\mathcal{B} \cong (1-p)\mathcal{C}(1-p)$ and*

$$ \overline{\mathcal{C}p\mathcal{C}} = \mathcal{C} = \overline{\mathcal{C}(1-p)\mathcal{C}} \ . $$

In this case, $\mathcal{C}$ is called linking algebra.

In general, since $\mathcal{A}$ is Morita equivalent to $\mathcal{A} \otimes \mathcal{K}$, if $\mathcal{A}$ and $\mathcal{B}$ are stably isomorphic, then $\mathcal{A}$ and $\mathcal{B}$ are Morita equivalent. In general, the converse implication does not hold. For example, for a nonseparable Hilbert space $\mathcal{H}$, $\mathcal{K}(\mathcal{H})$ is Morita equivalent to $\mathbb{C}$ but $\mathcal{K}(\mathcal{H})$ and $\mathbb{C}$ are not stably isomorphic.

2.2.4.7 Theorem. *If $\mathcal{A}$ and $\mathcal{B}$ are σ-unital, they are stably isomorphic if and only if they are Morita equivalent.*

For this, see [Bla06, II.7.6].

2.2.4.8 Definition. For an $\mathcal{A}$-$\mathcal{B}$-bimodule E, we set $E^* := \{x^* \colon x \in E\}$ and define

$$ \begin{aligned} x^* + y^* &:= (x+y)^* \\ bx^*a &:= (a^*xb^*)^* \\ {}_{\mathcal{B}}\langle x^*, y^*\rangle &:= \langle x, y\rangle_{\mathcal{B}} \\ \langle x^*, y^*\rangle_{\mathcal{A}} &:= {}_{\mathcal{A}}\langle x, y\rangle \ . \end{aligned} $$

With these operations, E^* is a Hilbert $\mathcal{B}$-$\mathcal{A}$-bimodule called the *opposite module of* E.

2.2.4.9 Remark. In the previous definition, we think of E^* equal to E as a set but equipped with different module operations and respective inner products as a Hilbert bimodule. There is no involution on an arbitrary Hilbert bimodule in general, so x^* is just a symbol reminding us to consider the element x as a member of E^* rather than E which entails a different multiplication with elements in the coefficient algebras $\mathcal{A}$ and $\mathcal{B}$. However, when considering a C^*-algebra $\mathcal{A}$ as an $\mathcal{A}$-bimodule with left $\mathcal{A}$-linear inner product ${}_{\mathcal{A}}\langle a, b\rangle := ab^*$ and right $\mathcal{A}$-linear inner product $\langle a, b\rangle_{\mathcal{A}} := a^*b$, all the stars occuring when forming the opposite module of $\mathcal{A}$ as above coincide with the involution of $\mathcal{A}$.

2.2.4.10 Remark. With the obvious modifications, we can define the opposite module of a Hilbert correspondence as well. Unlike the opposite module of a bimodule, the opposite module of a correspondence lies in a different category, since it turns a right module into a left module and vice versa. Obviously, $E \cong E^*$ by $x \mapsto x^*$. Moreover, $\mathcal{K}(E) \cong E \otimes E^*$ by $\theta_{x,y} \mapsto x \otimes y^*$ and $E^* \otimes E \cong \overline{\operatorname{span}}\{\langle x, y\rangle_{\mathcal{B}} : x, y \in E\}$ by $x^* \otimes y \mapsto \langle x, y\rangle$. So, $E^* \otimes E \cong \mathcal{B}$ holds if and only if E is full as a right Hilbert $\mathcal{B}$-module.

2.3 Kasparov's KK-theory

We will now mention some basics on KK-theory in order to introduce the six-term cyclic exact sequences in KK-theory arising from semisplit exact sequences of C^*-algebras, since this result is crucial for what is to come in the following chapters.

2.3.1 The Kasparov groups $\mathbf{KK}_0(\mathcal{A}, \mathcal{B})$ and $\mathbf{KK}^1(\mathcal{A}, \mathcal{B})$

2.3.1.1 Definition. A *graded C^*-algebra* is a C^*-algebra $\mathcal{B}$ equipped with an order two automorphism $\beta_{\mathcal{B}}$, that is a *-automorphism with $(\beta_{\mathcal{B}})^2 = \mathrm{id}$. We call $\beta_{\mathcal{B}}$ grading automorphism and say that $\mathcal{B}$ is graded by $\beta_{\mathcal{B}}$.

2.3.1.2 Remark. A graded C^*-algebra $\mathcal{B}$ decomposes into the eigenspaces of $\beta_{\mathcal{B}}$, that is $\mathcal{B} = \mathcal{B}_0 \oplus \mathcal{B}_1$, where

$$\mathcal{B}_0 := \{b \in \mathcal{B} : \beta_{\mathcal{B}}(b) = b\} \text{ and } \mathcal{B}_1 := \{b \in \mathcal{B} : \beta_{\mathcal{B}}(b) = -b\}.$$

This is just a Banach space decomposition (unlike $\mathcal{B}_0$, $\mathcal{B}_1$ is not a C^*-algebra). Elements of $\mathcal{B}_i$ are called *homogeneous* of degree i.

For the rest of the section, $\mathcal{A}$ and $\mathcal{B}$ will denote σ-unital graded C^*-algebras unless stated otherwise.

2.3.1.3 Definition. A *graded homomorphism* $\phi : \mathcal{A} \to \mathcal{B}$ between the graded C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ is a *-homomorphism, that commutes with the gradings, that is $\phi \circ \beta_{\mathcal{A}} = \beta_{\mathcal{B}} \circ \phi$. In other words, considering $\phi : \mathcal{A}_0 \oplus \mathcal{A}_1 \to \mathcal{B}_0 \oplus \mathcal{B}_1$ an operator matrix, ϕ is diagonal.

A *graded Hilbert $\mathcal{B}$-module* over the graded C^*-algebra $\mathcal{B}$ is a Hilbert $\mathcal{B}$-module E equipped with a linear bijection $S_E : E \to E$, called *grading operator*, satisfying

$$S_E(\xi b) = S_E(\xi)\beta_{\mathcal{B}}(b) \qquad \text{for } \xi \in E, b \in \mathcal{B}, \tag{2.3.1}$$

$$\langle S_E(\xi), S_E(\eta)\rangle = \beta_{\mathcal{B}}(\langle \xi, \eta\rangle) \qquad \text{for } \xi, \eta \in E, \tag{2.3.2}$$

$$(S_E)^2 = \mathrm{id} \tag{2.3.3}$$

2.3.1.4 Remark. Again we have $E = E_0 \oplus E_1$, where E_i are the eigenspaces of S_E. Equations (2.3.1) and (2.3.2) imply that $E_i\mathcal{B}_j \subseteq E_{i+j}$ and $\langle E_i, E_j\rangle \subseteq \mathcal{B}_{i+j}$ for $i, j \in \{0, 1\}$. Equation (2.3.3) implies that $\|S_E\| \leq 1$.

2.3.1.5 Example. *Any C^*-algebra $\mathcal{B}$ is graded by $\beta_{\mathcal{B}} = \mathrm{id}_{\mathcal{B}}$ and any Hilbert $\mathcal{B}$-module E is graded by $S_E = \mathrm{id}_E$. This is called the trivial grading.*

2.3.1.6 Example. *Given a C^*-algebra $\mathcal{B}$,*

$$\beta_{\mathcal{B}\oplus\mathcal{B}} : \mathcal{B} \oplus \mathcal{B}, (a, b) \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} (a, b) = (b, a)$$

defines a grading on $\mathcal{B} \oplus \mathcal{B}$, called the odd grading. We denote $\mathcal{B} \oplus \mathcal{B}$ with this grading by $\mathcal{B}_{(1)}$.

2.3.1.7 Example. *Any graded C^*-algebra $\mathcal{B}$ is a graded Hilbert $\mathcal{B}$-module $E = \mathcal{B}_{\mathcal{B}}$ with $S_E = \beta_B$.*

2.3.1.8 Example. *The grading S_E of a Hilbert module E induces an order two *-auto-*

morphism of $\mathcal{L}(E)$ *by*

$$\beta_{\mathcal{L}(E)} : \mathcal{L}(E) \to \mathcal{L}(E), T \mapsto S_E T S_E^{-1}.$$

The grading that $\mathcal{L}(E)$ *inherits from* E *is called induced grading. Since* $S_E(\mathcal{K}(E)) \subseteq \mathcal{K}(E)$, *we obtain a grading of the compact operators on* E *as well. This is the default grading on* $\mathcal{L}(E)$ *and* $\mathcal{K}(E)$.

2.3.1.9 Example. *Given two graded Hilbert* $\mathcal{B}$*-modules* E *and* F*, their direct sum* $E \oplus F$ *is graded by* $S_E \oplus S_F$*. This is the common grading on direct sums.*

2.3.1.10 Definition. A *Kasparov* $\mathcal{A}$*-*$\mathcal{B}$*-module* is a triple $\mathcal{E} = (E, \phi, F)$, where E is a countably generated Hilbert $\mathcal{B}$-module, $\phi : \mathcal{A} \to \mathcal{L}(E)$ is compatible with the respective gradings, that is $\phi \circ \beta_{\mathcal{A}} = \beta_{\mathcal{B}} \circ \phi$, and $F \in \mathcal{L}(E)$ is an element of degree 1, such that

$$\begin{aligned} [F, \phi(a)] = (F\phi(a) - \phi(a)F) &\in \mathcal{K}(E) && \text{for all } a \in \mathcal{A} \\ (F^2 - \mathrm{id})\phi(a) &\in \mathcal{K}(E) && \text{for all } a \in \mathcal{A} \\ (F^* - F)\phi(a) &\in \mathcal{K}(E) && \text{for all } a \in \mathcal{A} \end{aligned}$$

We will denote the set of all Kasparov $\mathcal{A}$-$\mathcal{B}$-modules by $\mathbb{E}(\mathcal{A}, \mathcal{B})$. A Kasparov module is called *degenerate*, if

$$[F, \phi(a)] = (F^2 - \mathrm{id})\phi(a) = (F - F^*)\phi(a) = 0 \text{ for all } a \in \mathcal{A}.$$

The set of all degenerate Kasparov $\mathcal{A}$-$\mathcal{B}$-modules is denoted by $\mathbb{D}(\mathcal{A}, \mathcal{B})$.

We will now define three equivalence relations on $\mathbb{E}(\mathcal{A}, \mathcal{B})$.

2.3.1.11 Definition. Two Kasparov $\mathcal{A}$-$\mathcal{B}$-modules $\mathcal{E}_1 = (E_1, \phi_1, F_1)$ and $\mathcal{E}_2 = (E_2, \phi_2, F_2)$ are *isomorphic* when there is a Hilbert $\mathcal{B}$-module isomorphism $\psi : E_1 \to E_2$ such that

$$\begin{aligned} \psi \circ S_{E_1} &= S_{E_2} \circ \psi, \\ \psi \circ F_1 &= F_2 \circ \psi, \\ \psi \circ \phi_1(a) &= \phi_2(a) \circ \psi \end{aligned}$$

for all $a \in \mathcal{A}$. We write $\mathcal{E}_1 \cong \mathcal{E}_2$ in this case.

Clearly, $\cong$ defines an equivalence relation on $\mathbb{E}(\mathcal{A},\mathcal{B})$. For $\mathcal{E}_1,\dots,\mathcal{E}_n \in \mathbb{E}(\mathcal{A},\mathcal{B})$ and 0 the trivial Kasparov module $(0,0,0)$ it holds that

$$\begin{aligned}(\mathcal{E}_1\oplus\mathcal{E}_2)\oplus\mathcal{E}_3 \cong \mathcal{E}_1\oplus(\mathcal{E}_2\oplus\mathcal{E}_3) &\cong \mathcal{E}_1\oplus\mathcal{E}_2\oplus\mathcal{E}_3,\\ \mathcal{E}_{\sigma(1)}\oplus\cdots\oplus\mathcal{E}_{\sigma(n)} &\cong \mathcal{E}_1\oplus\cdots\oplus\mathcal{E}_n,\\ \mathcal{E}_1\oplus 0 &\cong \mathcal{E}_1,\end{aligned}$$

where $\sigma : \{1,\dots,n\}\to\{1,\dots,n\}$ denotes any permutation. For the next definition we need to introduce some notation. Denote by

$$I\mathcal{B} := \mathcal{C}(I,\mathcal{B}) = \mathcal{C}(I)\otimes\mathcal{B}$$

all continuous $\mathcal{B}$-valued functions over the unit interval $I=[0,1]$ and by $\pi_t : I\mathcal{B}\to\mathcal{B}$ the point evaluation at $t\in I$. If $\mathcal{B}$ is graded by $\beta_{\mathcal{B}}$ we consider $I\mathcal{B}$ graded by $\mathrm{id}\otimes\beta_{\mathcal{B}}$. Then every π_t is a graded homomorphism and for $\mathcal{E}=(E,\phi,F)\in\mathbb{E}(\mathcal{A},I\mathcal{B})$ we obtain

$$\mathcal{E}_{\pi_t} := (E\otimes_{\pi_t}\mathcal{B}, \pi_t\circ\phi, \pi_t\circ F)\in\mathbb{E}(\mathcal{A},\mathcal{B})$$

for every $t\in I$.

2.3.1.12 Definition. Two Kasparov modules $\mathcal{E}$ and $\mathcal{F}\in\mathbb{E}(\mathcal{A},\mathcal{B})$ are called *homotopic* if there exists a Kasparov module $\mathcal{G}\in\mathbb{E}(\mathcal{A},I\mathcal{B})$ such that $\mathcal{G}_{\pi_0}\cong\mathcal{E}$ and $\mathcal{G}_{\pi_1}\cong\mathcal{F}$. We write $\mathcal{E}\sim\mathcal{F}$, if there exists a finite sequence of Kasparov modules $\mathcal{E}_1,\dots,\mathcal{E}_d\in\mathbb{E}(\mathcal{A},\mathcal{B})$ such that $\mathcal{E}_1=\mathcal{E}$, $\mathcal{E}_d=\mathcal{F}$ and $\mathcal{E}_i$ is homotopic to $\mathcal{E}_{i+1}$ for all $1\le i\le d-1$.

2.3.1.13 Remark. $\sim$ defines an equivalence relation on $\mathbb{E}(\mathcal{A},\mathcal{B})$.

2.3.1.14 Definition. Two Kasparov modules $\mathcal{E}$ and $\mathcal{F}\in\mathbb{E}(\mathcal{A},\mathcal{B})$ are called *operator homotopic* if there exist a graded Hilbert $\mathcal{B}$-module E, a graded homomorphism $\phi:\mathcal{A}\to\mathcal{L}(E)$ and a norm-continuous path $(F_t)_{t\in I}$ of adjointable operators on E such that

$$\begin{aligned}&\mathcal{F}_0\cong\mathcal{E},\\ &\mathcal{F}_1\cong\mathcal{F},\\ &\mathcal{F}_t=(E,\phi,F_t)\in\mathbb{E}(\mathcal{A},\mathcal{B})\end{aligned}$$

for all $t \in I$. We write $\mathcal{E} \approx \mathcal{F}$ when there are degenerate Kasparov modules $\mathcal{E}_d$ and $\mathcal{F}_d \in \mathbb{D}(\mathcal{A}, \mathcal{B})$ such that $\mathcal{E} \oplus \mathcal{E}_d$ is operator homotopic to $\mathcal{F} \oplus \mathcal{F}_d$.

2.3.1.15 Remark. $\approx$ defines an equivalence relation on $\mathbb{E}(\mathcal{A}, \mathcal{B})$.

2.3.1.16 Definition. For two graded C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ we now define

$$\mathrm{KK}(\mathcal{A}, \mathcal{B}) := \mathrm{KK}_0(\mathcal{A}, \mathcal{B}) := \mathbb{E}(\mathcal{A}, \mathcal{B})/_{\sim}$$

and denote the class of $\mathcal{E} \in \mathbb{E}(\mathcal{A}, \mathcal{B})$ by $[\mathcal{E}]$. Furthermore,

$$\widehat{\mathrm{KK}}(\mathcal{A}, \mathcal{B}) := \widehat{\mathrm{KK}}_0(\mathcal{A}, \mathcal{B}) := \mathbb{E}(\mathcal{A}, \mathcal{B})/_{\approx}.$$

In this quotient, the class of $\mathcal{E} \in \mathbb{E}(\mathcal{A}, \mathcal{B})$ is denoted by $\{\mathcal{E}\}$.

2.3.1.17 Remark. Some useful facts:

(a) Any degenerate Kasparov module $\mathcal{E} \in \mathbb{D}(\mathcal{A}, \mathcal{B})$ is homotopic to 0, see [Bla06, proposition 17.2.3].

(b) For $\mathcal{E}$, $\mathcal{F} \in \mathbb{E}(\mathcal{A}, \mathcal{B})$, $\mathcal{E} \approx \mathcal{F}$ implies that $\mathcal{E} \sim \mathcal{F}$, see [JT91, lemma 2.1.21].

(c) $\mathrm{KK}(\mathcal{A}, \mathcal{B})$ and $\widehat{\mathrm{KK}}(\mathcal{A}, \mathcal{B})$ are abelian groups with the respective additions

$$[\mathcal{E}] + [\mathcal{F}] := [\mathcal{E} \oplus \mathcal{F}]$$
$$\{\mathcal{E}\} + \{\mathcal{F}\} := \{\mathcal{E} \oplus \mathcal{F}\}$$

The inverse element of some $(E, \phi, F) \in \mathrm{KK}(\mathcal{A}, \mathcal{B})$ is $(-E, \tilde{\phi}, -F)$, where $-E$ is identical to E as a Hilbert $\mathcal{B}$-module but graded by $-S_E$ and $\phi_- := \phi \circ \beta_{\mathcal{A}}$, for details see [Bla06, proposition 17.3.3] or [JT91, theorem 2.1.23].

2.3.1.18 Definition. We define

$$\mathrm{KK}^1(\mathcal{A}, \mathcal{B}) := \mathrm{KK}(\mathcal{A}, \mathcal{B}_{(1)}),$$

where $\mathcal{B}_{(1)}$ denotes $\mathcal{B} \oplus \mathcal{B}$ with the odd grading from example 2.3.1.6.

2.3.1.19 Theorem. *If $\mathcal{A}$ is separable and $\mathcal{B}$ is σ-unital $\mathrm{KK}^1(\mathcal{A}, \mathcal{B})$ is isomorphic to $Ext^{-1}(\mathcal{A}, \mathcal{B})$, the group of invertible extensions of $\mathcal{A}$ by $\mathcal{B}$.*

For this, see [JT91, corollary 3.3.11].

2.3.1.20 Remark. Both $\mathrm{KK}_0(\cdot\,,\cdot)$ and $\mathrm{KK}^1(\cdot\,,\cdot)$ are bifunctors from pairs of C^*-algebras to the category of abelian groups which are contravariant in the first variable and covariant in the second one.

2.3.1.21 Remark. KK-theory incorporates K-homology within its first and K-theory within its second variable. In particular,

$$\mathrm{KK}_0(\mathbb{C},\mathcal{A}) = \mathrm{K}_0(\mathcal{A}) \text{ and } \mathrm{KK}^1(\mathbb{C},\mathcal{A}) = \mathrm{K}_1(\mathcal{A}).$$

For this see [Bla98, 17.5.4].

2.3.1.22 Remark. The following definition is equivalent to the one given above.

$$\mathrm{KK}^1(\mathcal{A},\mathcal{B}) := \mathrm{KK}_0(\mathcal{A},\mathcal{B}\,\hat{\otimes}\,\mathbb{C}_1),$$

where $\hat{\otimes}$ as in [Bla98, 14.4.1 and 14.4.2] and $\mathbb{C}_1$ denotes $\mathbb{C}^2$ with the odd grading, which is a complex Clifford algebra. For example, Blackadar follows this approach. It allows a more elegant formulation of Bott periodicity for KK-theory. For this, note that for graded C^*-algebras $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{D}$ the map

$$\begin{aligned} \mathbb{E}(\mathcal{A},\mathcal{B}) &\to \mathbb{E}(\mathcal{A}\,\hat{\otimes}\,\mathcal{D},\mathcal{B}\,\hat{\otimes}\,\mathcal{D}) \\ (E,\phi,F) &\mapsto (E\,\hat{\otimes}\,\mathcal{D},\phi\otimes \mathrm{id},F\otimes \mathrm{id}) \end{aligned}$$

induces a homomorphism $\tau_{\mathcal{D}}\colon\ \mathrm{KK}(\mathcal{A},\mathcal{B}) \to \mathrm{KK}(\mathcal{A}\,\hat{\otimes}\,\mathcal{D},\mathcal{B}\,\hat{\otimes}\,\mathcal{D})$.

2.3.1.23 Theorem. *For any graded, σ-unital C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, the map*

$$\tau_{\mathbb{C}_1}\colon\ \mathrm{KK}(\mathcal{A},\mathcal{B}) \to \mathrm{KK}(\mathcal{A}\,\hat{\otimes}\,\mathbb{C}_1,\mathcal{B}\,\hat{\otimes}\,\mathbb{C}_1)$$

is an isomorphism. In particular,

$$\mathrm{KK}^1(\mathcal{A},\mathcal{B}) \cong \mathrm{KK}_0(\mathcal{A}\,\hat{\otimes}\,\mathbb{C}_1,\mathcal{B})$$

and

$$\mathrm{KK}_0(\mathcal{A},\mathcal{B}) \cong \mathrm{KK}^1(\mathcal{A},\mathcal{B}\,\hat{\otimes}\,\mathbb{C}_1) \cong \mathrm{KK}^1(\mathcal{A}\,\hat{\otimes}\,\mathbb{C}_1,\mathcal{B}) \cong \mathrm{KK}_0(\mathcal{A}\,\hat{\otimes}\,\mathbb{C}_1,\mathcal{B}\,\hat{\otimes}\,\mathbb{C}_1)$$

2.3.2 The Kasparov product

The Kasparov product provides a map

$$\hat{\otimes}_{\mathcal{D}} : \mathrm{KK}(\mathcal{A},\mathcal{D}) \times \mathrm{KK}(\mathcal{D},\mathcal{B}) \to \mathrm{KK}(\mathcal{A},\mathcal{B}).$$

For fixed elements $x \in \mathrm{KK}(\mathcal{A},\mathcal{D})$ and $y \in \mathrm{KK}(\mathcal{D},\mathcal{B})$ choose the respective representatives $(E_1,\phi_1,F_1) \in \mathbb{E}(\mathcal{A},\mathcal{D})$ and $(E_2,\phi_2,F_2) \in \mathbb{E}(\mathcal{D},\mathcal{B})$. We will associate the product with the class of the module $(E,\phi,F_1\sharp F_2) \in \mathbb{E}(\mathcal{A},\mathcal{B})$, where $E := E_1 \otimes_{\phi_2} E_2$ and $\phi := \phi_1 \otimes \mathrm{id}$. All the work is put into finding a suitable operator $F_1\sharp F_2$. For this, we want to form a combination of $F_1 \otimes \mathrm{id}$ and $\mathrm{id} \otimes F_2$. It is easy to make sense of the first expression by 2.2.3.5, whereas the second one is troublesome. Given a countably generated, graded Hilbert $\mathcal{D}$-module E_1, a countably generated, graded Hilbert $\mathcal{B}$-module E_2, a graded *-homomorphism $\psi : \mathcal{D} \to \mathcal{L}(E_2)$ and $F_2 \in \mathcal{L}(E_2)$ such that $[F_2,\psi(\mathcal{D})] \subseteq \mathcal{K}(E_2)$, we are looking for $F \in \mathcal{L}(E)$, where $E = E_1 \otimes_\psi E_2$ which acts like $\mathrm{id} \otimes F_2$ up to compacts. Connections were introduced by Connes and Skandalis in [CS84].

2.3.2.1 Definition. An operator $F \in \mathcal{L}(E)$ is called an F_2-*connection* for E_1 if for any $x \in E_1$,

$$T_x \circ F_2 - (-1)^{\delta x \cdot \delta F_2} F \circ T_x \subseteq \mathcal{K}(E_2,E),$$
$$F_2 \circ T_x^* - (-1)^{\delta x \cdot \delta F_2} T_x^* \circ F \subseteq \mathcal{K}(E,E_2),$$

where $T_x \in \mathcal{L}(E_2,E)$ denotes the creation operator as in 2.2.3.6.

2.3.2.2 Remark. For $x \in E_1$ let

$$\tilde{T}_x := \begin{pmatrix} 0 & T_x^* \\ T_x & 0 \end{pmatrix}$$

and for $F \in \mathcal{L}(E)$ let $\tilde{F} := F_2 \oplus F \in \mathcal{L}(E_2 \oplus E)$. Then, F is an F_2-connection if and only if

$$[\tilde{T}_x,\tilde{F}] \in \mathcal{K}(E_2 \oplus E) \text{ for all } x \in E_1.$$

2.3.2.3 Definition. Let F be an F_2-connection for E. The triple (E,ϕ,F) is called a *Kasparov product* of (E_1,ϕ_1,F_1) and (E_2,ϕ_2,F_2) if (E,ϕ,F) is a Kasparov $\mathcal{A}$-$\mathcal{B}$-module and

$$\phi(a)\,[F_1 \otimes \mathbb{1}, F]\,\phi(a) \geq 0 \mod \mathcal{K}(E)$$

for all $a \in \mathcal{A}$. The set of all F such that (E, ϕ, F) is a Kasparov product is denoted by $F_1 \sharp_{\mathcal{D}} F_2$.

2.3.2.4 Theorem. *If $\mathcal{A}$ is separable and $\mathcal{D}$ is σ-unital, the Kasparov product for (E_1, ϕ_1, F_1) and (E_2, ϕ_2, F_2) exists and is unique up to operator homotopy. Actually, the Kasparov product defines a bilinear function*

$$\hat{\otimes}_{\mathcal{D}} \colon \mathrm{KK}(\mathcal{A}, \mathcal{D}) \times \mathrm{KK}(\mathcal{D}, \mathcal{B}) \to \mathrm{KK}(\mathcal{A}, \mathcal{B}).$$

See [Bla98, theorems 18.4.3 and 18.4.4]

2.3.3 Six-term cyclic exact sequences in KK-theory

There is a version of the six-term cyclic exact sequence for both C^*-algebraic K-theory and K-homology in the setting of KK-theory.

2.3.3.1 Definition. A short exact sequence $0 \to \mathcal{I} \to \mathcal{A} \xrightarrow{q} \mathcal{A}/\mathcal{I} \to 0$ is called *semisplit*, if there exists a linear, completely positive, contractive map $p : \mathcal{A}/\mathcal{I} \to \mathcal{A}$ such that

$$q \circ p = \mathrm{id}_{\mathcal{A}/\mathcal{I}}\,.$$

In this case, p is called completely positive *cross section* of q. An ideal $\mathcal{I} \lhd \mathcal{A}$ is said to be *semisplit*, if the respective short exact sequence $0 \to \mathcal{I} \to \mathcal{A} \to \mathcal{I}/\mathcal{A} \to 0$ is semisplit.

2.3.3.2 Remark. There is a 1:1-correspondence between semisplit stable ideals and invertible extensions. If $\mathcal{A}$ is nuclear, then every ideal of $\mathcal{A}$ is semisplit. For this see [Bla98, theorems 15.7.1 and 15.8.4].

2.3.3.3 Theorem. *For a semisplit exact sequence*

$$0 \to \mathcal{I} \to \mathcal{A} \xrightarrow{q} {}^{\mathcal{A}}\!/_{\mathcal{I}} \to 0$$

of σ-unital C^*-algebras and for any separable graded C^*-algebra $\mathcal{D}$, there exist connecting *maps such that the following diagram is exact:*

$$\begin{tikzcd} \mathrm{KK}_0(\mathcal{D},\mathcal{I}) \arrow[r] & \mathrm{KK}_0(\mathcal{D},\mathcal{A}) \arrow[r] & \mathrm{KK}_0(\mathcal{D},\mathcal{A}/\mathcal{I}) \arrow[d] \\ \mathrm{KK}^1(\mathcal{D},\mathcal{A}/\mathcal{I}) \arrow[u] & \mathrm{KK}^1(\mathcal{D},\mathcal{A}) \arrow[l] & \mathrm{KK}^1(\mathcal{D},\mathcal{I}) \arrow[l] \end{tikzcd}$$

If $\mathcal{A}$ is separable, then there exist connecting maps such that for every separable graded C^*-algebra $\mathcal{D}$ the diagram below is exact as well.

$$\begin{tikzcd} \mathrm{KK}_0(\mathcal{I},\mathcal{D}) \arrow[d] & \mathrm{KK}_0(\mathcal{A},\mathcal{D}) \arrow[l] & \mathrm{KK}_0(\mathcal{A}/\mathcal{I},\mathcal{D}) \arrow[l] \\ \mathrm{KK}^1(\mathcal{A}/\mathcal{I},\mathcal{D}) \arrow[r] & \mathrm{KK}^1(\mathcal{A},\mathcal{D}) \arrow[r] & \mathrm{KK}^1(\mathcal{I},\mathcal{D}) \arrow[u] \end{tikzcd}$$

See [Bla98, 19.5.7] or [Ska91].

2.3.3.4 Remark. Recall from 2.3.1.19 that invertible extensions $\mathrm{Ext}(\mathcal{A},\mathcal{B})^{-1}$ are isomorphic to $\mathrm{KK}_1(\mathcal{A},\mathcal{B})$ and let $x \in \mathrm{KK}^1(\mathcal{A}/\mathcal{I},\mathcal{A})$ be the element that corresponds to a given semisplit exact sequence

$$0 \to \mathcal{I} \to \mathcal{A} \xrightarrow{q} \mathcal{A}/\mathcal{I} \to 0.$$

Then for any separable C^*-algebra $\mathcal{D}$, the connecting maps in the above diagrams are given by the Kasparov products

$$\begin{array}{rcl} \mathrm{KK}_i(\mathcal{D},\mathcal{A}/\mathcal{I}) & \to & \mathrm{KK}_{i+1}(\mathcal{D},\mathcal{I}) \\ y & \mapsto & yx \end{array}$$

and

$$\begin{array}{rcl} \mathrm{KK}_i(\mathcal{A}/\mathcal{I},\mathcal{D}) & \to & \mathrm{KK}_{i+1}(\mathcal{I},\mathcal{D}) \\ y & \mapsto & xy\ , \end{array}$$

where $i, i+1 \in \{0,1\}$ due to Bott periodicity.

Chapter 3

The Toeplitz and Cuntz-Pimsner algebra of a Hilbert correspondence

3.1 Construction of $\widetilde{\mathcal{T}}_E$ and $\mathcal{O}_E$

In [Pim97], Pimsner associated C^*-algebras to Hilbert C^*-correspondences that generalise Cuntz-Krieger algebras and crossed products of C^*-algebras by the integers. In this section, we will give Pimsner's definition and study the main examples.

3.1.1 Definitions

We start by constructing a Hilbert correspondence that is a suitable domain for creation and annihilation operators.

3.1.1.1 Definition. Given an $\mathcal{A}$-correspondence (E, φ), we define the *Fock module* of E by

$$\Gamma(E) := \bigoplus_{n \in \mathbb{N}_0} E^{\otimes n},$$

where $E^{\otimes n}$ is the Hilbert $\mathcal{A}$-module from 2.2.3.4. The Fock module $\Gamma(E)$ is a Hilbert $\mathcal{A}$-correspondence with left action

$$\varphi_{\Gamma(E)} := \oplus \varphi_n,$$

where φ_n as in 2.2.3.5. We will denote this action by $\varphi \otimes \mathrm{id}$.

3.1.1.2 Remark. For every $x \in E$ and $F := E^{\otimes n}$ let $T_x \in \mathcal{L}(E^{\otimes n}, E^{\otimes n+1})$ denote the creation operator from 2.2.3.6. By extending those operators to the Fock module linearly, we get adjointable operators on the Fock module and denote them by T_x as well.

3.1.1.3 Definition. We will call the adjointable operator $T_x \in \mathcal{L}(\Gamma(E))$ from the previous remark *creation operator on the Fock module.* The concrete C^*-algebra

$$\widetilde{\mathcal{T}}_E := C^*\{T_x : x \in E\} \subseteq \mathcal{L}\big(\Gamma(E)\big),$$

which is generated by all creation operators inside the adjointable operators on the Fock module is called *concrete Toeplitz algebra* of E. In addition, let P_n denote the orthogonal projection of $\Gamma(E)$ onto $\bigoplus_{i=0}^n E^{\otimes i}$ and

$$\mathcal{J}_{\Gamma(E)} := \overline{\bigcup_{n,m\in\mathbb{N}} P_n \mathcal{L}\big(\Gamma(E)\big) P_m}$$

the C^*-algebra generated by all P_n.

3.1.1.4 Definition. The C^*-algebra

$$\mathcal{O}_E := C^*\{S_x : x \in E\} \subseteq \mathcal{M}(\mathcal{J}_{\Gamma(E)}) \big/ \mathcal{J}_{\Gamma(E)}$$

is called *Cuntz-Pimsner algebra* of E, where S_x denotes the class of T_x in the corona algebra $\mathcal{M}(\mathcal{J}_{\Gamma(E)}) \big/ \mathcal{J}_{\Gamma(E)}$.

3.1.1.5 Remark. Therefore,

$$\mathcal{O}_E = \big(\widetilde{\mathcal{T}}_E + \mathcal{J}_{\Gamma(E)}\big) \big/ \mathcal{J}_{\Gamma(E)} \cong \widetilde{\mathcal{T}}_E \big/ \widetilde{\mathcal{T}}_E \cap \mathcal{J}_{\Gamma(E)} .$$

If φ is injective, $\|S_\xi\| = \|\xi\|$. If $\mathcal{L}(E) = \mathcal{K}(E)$ which happens precisely when E is finitely generated, then $\mathcal{J}_{\Gamma(E)}$ coincides with $\mathcal{K}\big(\Gamma(E)\big)$.

3.1.1.6 Remark. In fact, there is an explicit description of the intersection $\widetilde{\mathcal{T}}_E \cap \mathcal{J}_{\Gamma(E)}$. Denote by $\mathcal{I}_\varphi := \varphi^{-1}(\mathcal{K}(E))$. Since this is an ideal in $\mathcal{A}$, we get that $\Gamma(E)\mathcal{I}_\varphi$ is a submodule of $\Gamma(E)$. For this submodule, $\mathcal{K}\big(\Gamma(E)\mathcal{I}_\varphi\big) \cong \widetilde{\mathcal{T}}_E \cap \mathcal{J}_{\Gamma(E)}$ holds because the operators in the Toeplitz algebra with finite-dimensional image originate from the relation

$T_\xi^* T_\eta = \varphi \otimes \mathrm{id}(\langle \xi, \eta \rangle)$. Therefore,

$$\mathcal{O}_E \cong \widetilde{\mathcal{T}}_E / \mathcal{K}\big(\Gamma(E)\mathcal{I}_\varphi\big) \ .$$

In other words, the following short sequence is exact:

$$0 \to \mathcal{K}\big(\Gamma(E)\mathcal{I}_\varphi\big) \to \widetilde{\mathcal{T}}_E \to \mathcal{O}_E \to 0 \ .$$

See [MS00] for this approach.

3.1.2 Examples

Cuntz-Pimsner algebras generalise some well-known C^*-algebras.

3.1.2.1 Example (Cuntz algebras)**.** *Consider a d-dimensional Hilbert space $\mathcal{H}_d = \mathbb{C}^d$ with orthonormal basis $\{e_1, \ldots, e_d\}$ a Hilbert $\mathbb{C}$-correspondence with trivial action of $\mathbb{C}$ from the left as in example 2.2.1.5. Then $\Gamma(\mathcal{H}_d)$ is the full Fock space over $\mathcal{H}_d$, which is spanned by complex polynomials in d non commuting variables $e_1, \ldots, e_d$. Since*

$$C^*\{T_\xi : \xi \in \mathcal{H}_d\} = C^*\{T_j : 1 \leq j \leq d\} \subseteq \mathcal{L}\big(\Gamma(\mathcal{H}_d)\big),$$

where $T_j := T_{e_j}$, $\mathcal{T}_{\mathcal{H}_d}$ is the Toeplitz algebra on $\Gamma(\mathcal{H}_d)$ in the sense of [Pop89]. The Hilbert $\mathbb{C}$-module $\mathcal{H}_d$ is finitely generated, so $\mathcal{I}_\varphi = \mathbb{C}$ and hence $\mathcal{K}\big(\Gamma(\mathcal{H}_d)\mathcal{I}_\varphi\big) = \mathcal{K}\big(\Gamma(\mathcal{H}_d)\big)$. Since

$$\mathcal{K}\big(\Gamma(\mathcal{H}_d)\big) \ni P_\Omega = \mathrm{id}_{\Gamma(\mathcal{H}_d)} - \sum_{j=1}^{d} T_j T_j^*,$$

where P_Ω is the orthogonal projection onto the vacuum vector Ω of $\Gamma(\mathcal{H}_d)$, we know that

$$\mathrm{id}_{\Gamma(\mathcal{H}_d)/\mathcal{K}} = \sum_{j=1}^{d} S_j S_j^*,$$

where $S_j = [T_j]$. Since S_j are isometries with orthogonal ranges, $\mathcal{O}_{\mathcal{H}_d}$ equals the d-dimensional Cuntz algebra $\mathcal{O}_d$.

3.1.2.2 Example (Cuntz-Krieger algebras)**.** *Starting with a Cuntz-Krieger matrix A, that* *is* $A = (a_{ij})_{i,j=1}^d \in \mathbb{M}_d(\{0,1\})$ *such that no row and no column vanishes, we define*

$$E_A := \operatorname{span}\{e_i \otimes e_j : a_{ij} = 1\} \subseteq \mathbb{C}^d \otimes \mathbb{C}^d,$$

where $\{e_j\}_{j=1}^d$ *is the standard orthonormal base of* $\mathbb{C}^d$*. Extending*

$$\langle e_i \otimes e_j, e_k \otimes e_l \rangle := \delta_{ik}\delta_{jl}e_j$$

to E_A *linearly yields a* $\mathbb{C}^d$*-valued inner product. Together with component wise multiplication,* E_A *is a Hilbert* $\mathbb{C}^d$*-module. A left action of* $\mathbb{C}^d$ *on* E_A *is defined by*

$$\varphi(x_1, \dots, x_d)(y_1, \dots, y_d) = (x_1y_1, \dots, x_dy_d).$$

The respective Cuntz-Pimsner algebra $\mathcal{O}_{E_A}$ *coincides with the Cuntz-Krieger algebra* $\mathcal{O}_A$*.*

3.1.2.3 Example (Graph algebras)**.** *In order to generalise the previous example, let* $G = (G_0, G_1, r, s)$ *be a directed countable graph with vertices* G_0*, edges* G_1*, range map* $r\colon G_1 \to G_0$ *and source map* $s\colon G_1 \to G_0$*. Let* $\mathcal{A} := \mathcal{C}_0(G_0)$ *and complete* $E := \mathcal{C}_C(G_1)$ *with respect to the norm induced by the* $\mathcal{A}$*-valued inner product*

$$\langle \xi, \eta \rangle(v) := \sum_{r(e)=v} \overline{\xi(e)}\eta(e).$$

The right action is given by

$$(\xi f)(e) := \xi(e) f\big(r(e)\big)$$

and the left action $\varphi : \mathcal{A} \to \mathcal{L}(E)$ *by*

$$\varphi(f)\xi(e) := f\big(s(e)\big)\xi(e).$$

3.1.2.4 Remark. Since $\ker(\varphi) = \mathcal{C}_0\big(\{v \in G_0 : |s^{-1}(v)| = 0\}\big)$ need not be trivial, in [Kat04], [Kat06a], [Kat06b] and [Kat08] Katsura refined Pimsner's original definition dropping the assumption that φ is injective and postulating Cuntz-Pimsner covariance for elements in $\mathcal{I}_\varphi \cap \ker(\varphi)^\perp$ rather than in $\mathcal{I}_\varphi$ in order to include examples of graphs with sources. A short motivation for why this is desirable can be found in [Rae05, example 8.13].

3.1.2.5 Example (Crossed products)**.** *As Pimsner states in [Pim97], crossed products* *are examples of Cuntz-Pimsner algebras as well. We will now check the details, which are omitted in the paper. Given a C^*-algebra $\mathcal{A}$ with unit $\mathbb{1}$ together with a *-automorphism π, consider $E := \mathcal{A}$ a Hilbert $\mathcal{A}$-module as in 2.2.1.6 with left action $a \cdot b := \pi(a)b$. Then $E \otimes_\pi E$ is isomorphic to E as a right Hilbert $\mathcal{A}$-module via $a \otimes b \mapsto \pi(a)b$. Since*

$$a(x \otimes y) = \big(\pi(a)x\big) \otimes y \cong \pi^2(a)\pi(x)y,$$

this isomorphism does not preserve the left action of $\mathcal{A}$. By the same argument it holds for any $n \in \mathbb{N}$ that $E^{\otimes n}$ is isomorphic to the Hilbert $\mathcal{A}$-correspondence E equipped with the left action

$$a(x_1 \otimes \cdots \otimes x_n) = \pi^n(a)\pi^{n-1}(x_1)\ldots\pi(x_{n-1})x_n.$$

Hence, $\Gamma(E) \cong \bigoplus_{n\in\mathbb{N}_0} \mathcal{A}$ with left action

$$\boldsymbol{\pi}(a)(x_0, x_1, x_2, x_3, \ldots) := (ax_0, \pi(a)x_1, \pi^2(a)x_2, \pi^3(a)x_3, \ldots)\ .$$

One checks that $T_{\mathbb{1}}^* a T_{\mathbb{1}} = \pi(a)$ and that $[T_{\mathbb{1}}]$ is unitary in $\mathcal{O}_E$, so $\mathcal{O}_E$ is a quotient of $\mathcal{A} \rtimes_\pi \mathbb{Z}$. In other words, there exists an ideal $\mathcal{I} \lhd \mathcal{A} \rtimes_\pi \mathbb{Z}$ such that

$$\mathcal{O}_E \cong {}^{\mathcal{A} \rtimes_\pi \mathbb{Z}}\big/_{\mathcal{I}}.$$

We will now see that $\mathcal{I}$ is trivial. Any faithful representation $\varphi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ yields a representation $(\tilde{\varphi}, U)$ of $\mathcal{A} \rtimes_\pi \mathbb{Z}$ on $l^2(\mathbb{Z}; \mathcal{H})$, where

$$\tilde{\varphi}(a) = \begin{pmatrix} \ddots & & & \\ & \varphi(\pi^{-1}(a)) & & & \\ & & \varphi(a) & & \\ & & & \varphi(\pi(a)) & \\ & & & & \ddots \end{pmatrix} \text{ and } U := \begin{pmatrix} \ddots & & & & \\ \ddots & 0 & & & \\ & 1 & 0 & & \\ & & 1 & 0 & \\ & & & \ddots & \ddots \end{pmatrix}.$$

Actually, this is a covariant representation, that is $U^*\tilde{\varphi}(a)U = \tilde{\varphi}(\pi(a))$. By compression with the orthogonal projection V of $l^2(\mathbb{Z}; \mathcal{H})$ onto $l^2(\mathbb{N}; \mathcal{H})$ we obtain the completely positive map

$$x \mapsto VxV$$

for which $V\tilde{\varphi}(a)V = \boldsymbol{\pi}(a)$, where the left action by a on the Fock module is componentwisely represented on $\mathcal{H}$ via φ, and $VUV = T_{\mathbb{1}}$. We notice that this map is injective and that the intersection of its image with the compacts is trivial. Hence

$$\mathcal{A} \rtimes_\pi \mathbb{Z} \to \mathcal{T}_E \to \mathcal{T}_E/\mathcal{I} \cong \mathcal{A} \rtimes_\pi \mathbb{Z}/\mathcal{I}$$

is injective, so the ideal $\mathcal{I}$ is trivial. In other words, the Cuntz-Pimsner algebra $\mathcal{O}_E$ coincides with the full crossed product $\mathcal{A} \rtimes_\pi \mathbb{Z}$.

3.1.2.6 Remark. One may just as well define the Cuntz-Pimsner algebra for the $\mathcal{A}$-correspondence $(\mathcal{A}, \pi)$, where π is an endomorphism. Therefore, Pimsner's construction provides a canonical way to define crossed products of C^*-algebras by endomorphisms.

3.1.3 Motivating example

We now consider example 3.1.2.1 with $d := 1$ that is, the Hilbert $\mathbb{C}$-correspondence $E := \mathbb{C}$ with trivial left action $\varphi(z_1)(z_2) := z_1 \cdot z_2$. The full Fock module over $\mathbb{C}$ is $l^2(\mathbb{N})$, the space of square-summable complex sequences over $\mathbb{N}$ and the Toeplitz algebra $\mathcal{T}_E$ which is generated by the right shift $S(a_0, a_1, a_2, \dots) := (0, a_0, a_1, a_2, \dots)$ on $l^2(\mathbb{N})$ coincides with the regular Toeplitz algebra $\mathcal{T}_1$ from 2.1.3.1. Since $\mathcal{I}_\varphi = \mathbb{C}$, 3.1.1.6 implies that

$$\mathcal{O}_E = \mathcal{T}_1/\mathcal{K}\big(l^2(\mathbb{N})\big) = \mathcal{O}_1,$$

which is isomorphic to $\mathcal{C}(\mathbb{T})$. On the other hand, $\mathcal{O}_1$ is isomorphic to any C^*-algebra generated by a unitary with full spectrum so, in particular, to the C^*-algebra which is generated by the right shift $\hat{S}$ on $l^2(\mathbb{Z})$. This yields a completely positive map from $\mathcal{O}_1$ to $\mathcal{T}_1$ by compression. More precisely, the map

$$\begin{aligned} l^2(\mathbb{Z}) &\to l^2(\mathbb{N}) \\ (\dots, a_{-1}, a_0, a_1, \dots) &\mapsto (a_0, a_1, a_2, \dots) \end{aligned}$$

induces a map $\mathcal{B}(l^2(\mathbb{Z})) \to \mathcal{B}(l^2(\mathbb{N}))$ and its restriction to $\mathcal{O}_1$ is a completely positive lift of the quotient map $\mathcal{T}_1 \to \mathcal{O}_1$. In other words, the short exact sequence

$$0 \to \mathcal{K}\big(l^2(\mathbb{N})\big) \to \mathcal{T}_1 \to \mathcal{O}_1 \to 0$$

is semisplit, hence induces two six-term cyclic exact sequences in KK-theory by theorem 2.3.3.3.

When analysing Pimsner's construction of a semisplit extension of the Cuntz-Pimsner algebra $\mathcal{O}_E$, where (E, φ) is any Hilbert $\mathcal{A}$-correspondence such that $\varphi : \mathcal{A} \to \mathcal{L}(E)$ is injective, one comes to realise that Pimsner's proof generalises this example. He first showed that the Toeplitz algebra is a universal object by the gauge-invariant uniqueness theorem, which we will state and prove in the following section. This is particularly helpful for the fourth chapter, since we will then start with the universal object and faithfully represent it on the respective Fock module by generalising this theorem to our new setting. The intention of defining a two-sided Fock module generalising $l^2(\mathbb{Z})$ is hindered by the fact that for a natural number n there is no obvious notion of a negative tensor power $E^{\otimes -n}$ of an $\mathcal{A}$-correspondence E. However, if $E^{\otimes n}$ was an $\mathcal{A}$-bimodule, that is possessed a left Hilbert $\mathcal{A}$-module structure compatible with the right Hilbert $\mathcal{A}$-module structure of $E^{\otimes n}$, one remains in the same category when taking the opposite module of $E^{\otimes n}$. In this case, we may define $E^{\otimes -n}$ to be the the opposite module of $E^{\otimes n}$. Recall from 2.2.4.4 that every $E^{\otimes n}$ is a $\mathcal{K}(E^{\otimes n})$ - $\mathcal{A}$ bimodule. For this reason, Pimsner sets $\mathcal{F}_E := C^*\{\mathcal{K}(E^{\otimes n}) : n \in \mathbb{N}\}$ and extends the scalars of the Hilbert $\mathcal{A}$-correspondence E to $\mathcal{F}_E$ by considering the correspondence $E_\infty := E \otimes_{\mathcal{A}} \mathcal{F}_E$ instead. After checking that E_∞ is in fact a Hilbert $\mathcal{F}_E$-bimodule, one defines a two-sided Fock module $\Gamma_{\mathbb{Z}}(E_\infty)$ as suggested above. The Cuntz-Pimsner algebra $\mathcal{O}_{E_\infty}$ is then isomorphic to both $\mathcal{O}_E$ and the C^*-algebra generated by creation operators on $\Gamma_{\mathbb{Z}}(E_\infty)$. The completely positive map $\mathcal{L}\big(\Gamma_{\mathbb{Z}}(E_\infty)\big) \to \mathcal{L}\big(\Gamma(E_\infty)\big)$ gained by compressing $\Gamma_{\mathbb{Z}}(E_\infty)$ onto $\Gamma(E_\infty)$ lifts the quotient map $\mathcal{T}_{E_\infty} \to \mathcal{O}_E$ yielding six-term cyclic exact sequences in KK-theory for the slightly modified Toeplitz-extension of the Cuntz-Pimsner algebra.

3.2 A gauge-invariant uniqueness theorem for $\widetilde{\mathcal{T}}_E$

In this section, we will reproduce Pimsner's uniqueness theorem for the Toeplitz algebra $\widetilde{\mathcal{T}}_E$ associated to a Hilbert $\mathcal{A}$-correspondence (E, φ). In other words, we will see that $\widetilde{\mathcal{T}}_E$, the C^*-algebra generated by creation operators on the Fock module of E, is a universal C^*-algebra.

3.2.1 The abstract Toeplitz algebra

3.2.1.1 Definition. A *Toeplitz representation* of an $\mathcal{A}$-correspondence (E, φ) on a C^*-algebra $\mathcal{B}$ is a tuple (τ, σ), where $\tau : E \to \mathcal{B}$ is a linear map and $\sigma : \mathcal{A} \to \mathcal{B}$ is a *-homomorphism such that

$$\begin{array}{lrcl} \text{(TR1)} & \tau(xa) & = & \tau(x)\sigma(a), \\ \text{(TR2)} & \tau(x)^*\tau(y) & = & \sigma(\langle x, y\rangle), \\ \text{(TR3)} & \tau(\varphi(a)x) & = & \sigma(a)\tau(x) \end{array}$$

for all $x, y \in E$ and $a \in \mathcal{A}$. The C^*-algebra generated by all Toeplitz representations of (E, φ) is called the *abstract Toeplitz algebra* of E and is denoted by $\mathcal{T}_E$.

See for example [Dea07] for this definition.

3.2.1.2 Remark. The Fock representation $(T, \varphi \otimes \mathrm{id})$, where $T(x) := T_x$ is a Toeplitz representation of (E, φ).

3.2.1.3 Remark. Alternatively, we can consider $\mathcal{T}_E$ the universal C^*-algebra with generators $\{t_x : x \in E\} \cup \{a : a \in \mathcal{A}\}$, where the latter set retains the structure of $\mathcal{A}$, and subject to relations

$$\begin{array}{lrcl} \text{(TR1')} & t_x a & = & t_{xa}, \\ \text{(TR2')} & t_x^* t_y & = & \langle x, y\rangle, \\ \text{(TR3')} & a t_x & = & t_{\varphi(a)x}\ . \end{array}$$

3.2.2 The abstract and the concrete gauge action

We start by noticing that every product of generators of the universal Toeplitz algebra can be written in the following way.

3.2.2.1 Lemma. *Any finite product with factors in $\{t_x : x \in E\} \cup \{t_x^* : x \in E\} \cup \mathcal{A}$ inside $\mathcal{T}_E$ reduces to*

$$t_{x_1} \dots t_{x_k} t_{y_1}^* \dots t_{y_l}^*, \tag{3.2.1}$$

where $x_1, \dots, x_k, y_1, \dots, y_l \in E$, $k, l \in \mathbb{N}$ with the convention that this represents elements in $\mathcal{A}$ if $k = 0 = l$.

Proof. Given a product $t = t_1 t_2 t_3 \dots t_n$, where $t_i \in \{t_x : x \in E\} \cup \{t_x^* : x \in E\}$, which is not in the required shape already, we use relation (TR2) to eliminate any t_y^* on the left hand side of a t_x and then use relations (TR1) and (TR3) to absorb elements of $\mathcal{A}$ if necessary. □

Since $\mathcal{T}_E$ is spanned by products in $\{t_x : x \in E\} \cup \{t_x^* : x \in E\} \cup \mathcal{A}$, by the previous lemma products of the form (3.2.1) already span $\mathcal{T}_E$. The same holds for $\widetilde{\mathcal{T}}_E$ as well.

3.2.2.2 Definition. For any finite product $t = t_{x_1} \dots t_{x_k} t_{y_1}^* \dots t_{y_l}^* \in \mathcal{T}_E$, we define the *degree* of t to be $k - l$. We write $\deg(t) = k - l$. In particular, $\deg(a) = 0$, $\deg(t_x) = 1$ and $\deg(t_y^*) = -1$.

3.2.2.3 Remark. If $\{t_x : x \in E\} \cup \mathcal{A}$ satisfies relations (TR1') - (TR3'), for every $z \in \mathbb{T}$ the set $\{z \cdot t_x : x \in E\} \cup \mathcal{A}$ does as well. Defining

$$\lambda(z)(t) := z^{\deg(t)} t$$

for any finite product t and linearly extending this to $\mathcal{T}_E$ yields an action

$$\lambda : \mathbb{T} \to \operatorname{Aut}(\mathcal{T}_E).$$

We define another action $\tilde{\lambda} : \mathbb{T} \to \operatorname{Aut}(\widetilde{\mathcal{T}}_E)$ by linearly extending

$$\tilde{\lambda}(z)(T_{x_1} \dots T_{x_k} T_{y_1}^* \dots T_{y_l}^*) := z^{k-l}(T_{x_1} \dots T_{x_k} T_{y_1}^* \dots T_{y_l}^*).$$

3.2.2.4 Definition. The action $\lambda : \mathbb{T} \to \operatorname{Aut}(\mathcal{T}_E)$ from the previous remark is called the *abstract gauge action*, $\tilde{\lambda} : \mathbb{T} \to \operatorname{Aut}(\widetilde{\mathcal{T}}_E)$ is called *concrete gauge action.*

3.2.3 The theorem

We will now prove the following theorem.

3.2.3.1 Proposition. *It holds that*

$$\mathcal{T}_E \cong \widetilde{\mathcal{T}}_E.$$

In other words, the Fock representation of E is faithful.

We denote by $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ the fixed point algebras of the abstract gauge action λ and concrete gauge action $\tilde{\lambda}$ as above. These actions yield conditional expectations ψ_λ and $\psi_{\tilde{\lambda}}$ as in example 2.1.1.25 by

$$\psi_\lambda(t) := \int_{z\in\mathbb{T}} \lambda_z(t)d\mu(z) \ ,$$
$$\psi_{\tilde{\lambda}}(T) := \int_{z\in\mathbb{T}} \tilde{\lambda}_z(T)d\mu(z)$$

and the following diagram commutes:

$$\begin{tikzcd}
\mathcal{T}_E \arrow[r, "\pi"] \arrow[d, "\psi_\lambda"'] & \widetilde{\mathcal{T}}_E \arrow[d, "\psi_{\tilde{\lambda}}"] \\
\mathcal{F} \arrow[r, "\pi|_{\mathcal{F}}"] & \widetilde{\mathcal{F}},
\end{tikzcd}$$

Here, π denotes the quotient homomorphism from $\mathcal{T}_E$ onto $\widetilde{\mathcal{T}}_E$ and $\pi|_{\mathcal{F}}$ its restriction to $\mathcal{F}$.

Suppose $\pi|_{\mathcal{F}}$ was injective. Since ψ_λ is faithful, this implies that $\psi_{\tilde{\lambda}} \circ \pi$ is injective. Since $\psi_{\tilde{\lambda}}$ is faithful, π has to be injective. Obviously, π is surjective, so $\mathcal{T}_E$ and $\widetilde{\mathcal{T}}_E$ are isometric. We will now show that $\pi|_{\mathcal{F}}$ is injective, that is the fixed point algebras of the abstract and the concrete gauge action coincide.

For this, we take a closer look at the fixed point algebras $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ of the actions λ and $\tilde{\lambda}$. They are spanned by all products of degree zero respectively, so by Lemma 3.2.2.1 we know that

$$\begin{aligned}
\mathcal{F} &= \mathcal{A} \cup \overline{\operatorname{span}}\{t_{x_1}\dots t_{x_n}t_{y_1}^*\dots t_{y_n}^* : n\in\mathbb{N},\ x_i,y_i\in E\} \\
&= \mathcal{A} \cup \bigcup_{n\in\mathbb{N}} \overline{\operatorname{span}}\{t_{x_1}\dots t_{x_n}t_{y_1}^*\dots t_{y_n}^* : x_i,y_i\in E\}, \\
\widetilde{\mathcal{F}} &= \mathcal{A} \cup \overline{\operatorname{span}}\{T_{x_1}\dots T_{x_n}T_{y_1}^*\dots T_{y_n}^* : n\in\mathbb{N},\ x_i,y_i\in E\} \\
&= \mathcal{A} \cup \bigcup_{n\in\mathbb{N}} \overline{\operatorname{span}}\{T_{x_1}\dots T_{x_n}T_{y_1}^*\dots T_{y_n}^* : x_i,y_i\in E\} \ .
\end{aligned}$$

3.2.3.2 Lemma. *Let A and $\mathcal{B}$ be C^*-algebras, $\mathcal{A} \subseteq \mathcal{B}$ and $E \subseteq \mathcal{B}$ a closed subspace such that $xa \in E$ and $x^*y \in \mathcal{A}$ for every $x, y \in E$ and $a \in \mathcal{A}$. Then the following holds:*

1. *E with inner product $\langle x, y\rangle := x^*y$ is a right Hilbert $\mathcal{A}$-module with $\|x\|_E = \|x\|_{\mathcal{B}}$.*

2. *Elements $t \in \mathcal{B}$ with $tx \in E$ and $t^*x \in E$ for all $x \in E$ define elements in $\mathcal{L}(E)$ by left multiplication. In particular, $xy^* \in \mathcal{K}(E)$ for all $x, y \in E$.*

3. *Moreover, $\overline{\operatorname{span}}\{xy^* : x, y \in E\} \cong \mathcal{K}(E)$.*

Here, 1. and 2. are obvious and a proof of 3. can be found in [Pim97, lemma 3.2]. We now continue investigating the structure of the fixed point algebras $\mathcal{F}$ and $\widetilde{\mathcal{F}}$.

Let $E_1 := \overline{\operatorname{span}}\{t_x : x \in E\} \subseteq \mathcal{T}_E$ and $E_1' := \overline{\operatorname{span}}\{T_x : x \in E\} \subseteq \widetilde{\mathcal{T}}_E$. By the previous lemma, we get

$$\overline{\operatorname{span}}\{t_x t_y^* : x, y \in E\} \cong \mathcal{K}(E) \cong \overline{\operatorname{span}}\{T_x T_y^* : x, y \in E\}.$$

Applying the same argument to

$$E_n := \overline{\operatorname{span}}\{t_x : x \in E^{\otimes n}\} \cong \overline{\operatorname{span}}\{t_{x_1} \dots t_{x_n} : x_1, \dots, x_n \in E\}$$

and

$$E_n' := \overline{\operatorname{span}}\{T_x : x \in E^{\otimes n}\} \cong \overline{\operatorname{span}}\{T_{x_1} \dots T_{x_n} : x_1, \dots, x_n \in E\}$$

we get that

$$\overline{\operatorname{span}}\{t_x t_y^* : x, y \in E^{\otimes n}\} \cong \mathcal{K}(E^{\otimes n}) \cong \overline{\operatorname{span}}\{T_x T_y^* : x, y \in E^{\otimes n}\}.$$

In other words, we saw that the span of products of length $2n$ in and degree zero inside $\mathcal{T}_E$ and $\widetilde{\mathcal{T}}_E$, which are the building blocks of $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ respectively, are isomorphic to $\mathcal{K}(E^{\otimes n})$. It is now left to show that the respective ways they are put together in $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ are compatible.

We know that the fixed point algebra $\mathcal{F}$ is the universal C^*-algebra generated by $\{\mathcal{K}(E^{\otimes n}) : n \in \mathbb{N}\}$ subject to relations

$$\begin{aligned} ab &= (a \otimes \mathrm{id})b \ , \\ ba &= b(a \otimes \mathrm{id}) \end{aligned}$$

where $a \in \mathcal{K}(E^{\otimes n})$, $b \in \mathcal{K}(E^{\otimes m})$ and $n < m$. For any $T_x T_y^* \in \mathcal{K}(E^{\otimes n}) \subseteq \widetilde{\mathcal{F}}$ and $z \in E^{\otimes k}$ we know that

$$T_x T_y^*(z) = \begin{cases} 0, & \text{if } k < n \\ \theta_{x,y}(z), & \text{if } k = n \\ \theta_{x,y} \otimes \mathrm{id}(z), & \text{if } k > n \ . \end{cases}$$

So $\widetilde{\mathcal{F}} = \bigoplus_{n \in \mathbb{N}} \mathcal{K}(E^{\otimes n})$ with

$$(a, b) \cdot (a', b') = (aa', (a \otimes \mathrm{id})b' + b(a' \otimes \mathrm{id}) + bb'),$$

where $(a, b),\ (a', b') \in \mathcal{K}(E^{\otimes n}) \oplus \mathcal{K}(E^{\otimes m})$ and $n < m$.

3.2.3.3 Lemma. *Given two C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ and a *-homomorphism $\eta : \mathcal{A} \to \mathcal{M}(\mathcal{B})$ the universal C^*-algebra $\mathcal{C}_1$ generated by $\mathcal{A}$ and $\mathcal{B}$ subject to relations*

$$\begin{aligned} ab &= \eta(a)b, \\ ba &= b\ \eta(a) \end{aligned}$$

is isomorphic to $\mathcal{C}_2 := \mathcal{A} \oplus \mathcal{B}$ with

$$(a, b) \cdot (a', b') := (aa', \eta(a)b' + b\eta(a') + bb').$$

Proof. For the elements $(a, 0)$ and $(0, b) \in \mathcal{C}_2$ it holds that $(a, 0) \cdot (0, b) = (0, \eta(a)b)$ and $(0, b) \cdot (a, 0) = (0, b\ \eta(a))$, so $\mathcal{C}_2$ is a quotient of $\mathcal{C}_1$. On the other hand, for $(a + b)$, $(a' + b') \in \mathcal{C}_1$ the following equation holds:

$$(a + b)(a' + b') = (aa' + \eta(a)b' + b\ \eta(a') + bb')$$

Therefore, $\mathcal{C}_1$ is a quotient of $\mathcal{C}_2$. Since both are universal C^*-algebras, they are isomorphic.

$\square$

We will now prove that $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ are isomorphic by induction.
Let $\mathcal{A} := \mathcal{K}(E)$, $\mathcal{B} := \mathcal{K}(E^{\otimes 2})$ and $\eta : \mathcal{A} \to \mathcal{L}(E^{\otimes 2}), \theta_{x,y} \mapsto \theta_{x,y} \otimes \mathrm{id}$. By the previous lemma, we know that $C^*\{\mathcal{K}(E), \mathcal{K}(E^{\otimes 2})\}$ with respect to the above relations is isomorphic to $\mathcal{K}(E) \oplus \mathcal{K}(E^{\otimes 2})$ with the above product.
Assume now that $\mathcal{A}_n := C^*\{\mathcal{K}(E^{\otimes k}) : 1 \leq k \leq n\}$ is isomorphic to $\mathcal{A}_n' := \bigoplus_{1 \leq k \leq n} \mathcal{K}(E^{\otimes k})$ and let $\mathcal{B} := \mathcal{K}(E^{\otimes n+1})$. We define

$$\begin{array}{rccl} \eta_{k,n} : & \mathcal{K}(E^{\otimes k}) & \to & \mathcal{L}(E^{\otimes n+1}) \\ & a & \mapsto & a \otimes \underbrace{\mathrm{id} \otimes \cdots \otimes \mathrm{id}}_{n+1-k \text{ times}} . \end{array}$$

and

$$\begin{array}{rccl} \eta_n : & \mathcal{A}_n & \to & \mathcal{M}(\mathcal{B}) \cong \mathcal{L}(E^{\otimes n+1}) \\ & (a_k)_{k=1}^n & \mapsto & \sum_{k=1}^n \eta_{k,n}(a_k) \end{array}$$

where $a_k \in \mathcal{K}(E^{\otimes k})$. By applying the lemma again, we know that

$$C^*\{\mathcal{A}_n, \mathcal{B}\} = C^*\{\mathcal{K}(E^{\otimes k}) : a \leq k \leq n+1\} \cong \mathcal{A}_n \oplus \mathcal{B} \ .$$

The latter is isomorphic to $\mathcal{A}_n' \oplus \mathcal{B} = \bigoplus_{k=1}^{n+1} \mathcal{K}(E^{\otimes k})$ by induction hypothesis. We hence know that

$$\mathcal{F} = C^*\{\mathcal{K}(E^{\otimes k}) : k \in \mathbb{N}\} \cong \bigoplus_{k \in \mathbb{N}} \mathcal{K}(E^{\otimes k}) = \widetilde{\mathcal{F}}.$$

This finishes the proof of theorem 3.2.3.1.

Theorem 3.2.3.1 enables us to formulate the universal property of $\mathcal{T}_E$ in the following way.

3.2.3.4 Corollary. *For any countably generated full isometric Hilbert $\mathcal{A}$-correspondence (E, φ), any C^*-algebra $\mathcal{B}$ and any *-homomorphism $\sigma : \mathcal{A} \to \mathcal{B}$ with the property that there* *exist elements $\tilde{t}_x \in \mathcal{B}$ such that*

$$\begin{aligned} \alpha\tilde{t}_x + \beta\tilde{t}_y &= \tilde{t}_{\alpha x+\beta y}, \\ \tilde{t}_{xa} &= \tilde{t}_x \cdot \sigma(a), \\ \tilde{t}_x^* \cdot \tilde{t}_y &= \sigma(\langle x, y\rangle), \\ \tilde{t}_{\varphi(a)x} &= \sigma(a) \cdot \tilde{t}_x \end{aligned}$$

for all $\alpha, \beta \in \mathbb{C}$, $x, y \in E$ and $a \in \mathcal{A}$ there exists a unique extension $\tilde{\sigma} : \mathcal{T}_E \to \mathcal{B}$ of σ such that the following diagram commutes:

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\sigma"] \arrow[d, "\varphi\otimes\mathrm{id}"'] & \mathcal{B} \\ \mathcal{T}_E \arrow[ur, "\tilde{\sigma}"'] & \end{tikzcd}$$

3.2.3.5 Remark. If E is finitely generated, it possesses a normalised standard tight frame $\{u_i\}_{i=1,\dots,n}$ with

$$x = \sum_{i=1}^{n} u_i \langle u_i, x\rangle$$

and $\mathcal{O}_E$ is the universal C^*-algebra generated by $\{S_i\}_{i=1,\dots,n}$ and $\mathcal{A}$ subject to relations

$$\begin{aligned} \sum_{i=1}^{n} S_i S_i^* &= \mathrm{id}, \\ S_i^* S_j &= \langle u_i, u_j\rangle, \\ a \cdot S_j &= \sum_{i=1}^{n} S_i \langle u_i, a u_i\rangle. \end{aligned}$$

The Toeplitz algebra is the universal C^*-algebra generated by $\{S_i\}_{i=1,\dots,n}$ and $\mathcal{A}$ satisfying the second and third relation. This approach can be found in [KPW98].

3.3 A semisplit Toeplitz extension of $\mathcal{O}_E$

In this section, we will recall Pimsner's construction of a semisplit Toeplitz extension of a Cuntz-Pimsner algebra. In general, one does not know if the Toeplitz extension from 3.1.1.6 is semisplit or not, but by extending the coefficient algebra $\mathcal{A}$ of the underlying Hilbert module E to $\mathcal{F}_E$ and associating a new Hilbert $\mathcal{F}_E$-module E_∞, $\mathcal{T}_E$ embeds into $\mathcal{T}_{E_\infty}$. This embedding yields an isomorphism between the Cuntz-Pimsner algebras of E and E_∞. Since E_∞ is in fact a Hilbert bimodule, one can define a two-sided Fock module $\Gamma_\mathbb{Z}(E_\infty)$. Then the Cuntz-Pimsner algebra $\mathcal{O}_{E_\infty}$ is isomorphic to the C^*-algebra that is generated by multiplication operators on $\Gamma_\mathbb{Z}(E_\infty)$ and the completely positive map $\mathcal{L}\big(\Gamma_\mathbb{Z}(E_\infty)\big) \to \mathcal{L}\big(\Gamma(E_\infty)\big)$ induced by compression yields the desired lift of the quotient map.

3.3.1 Extending the scalars

3.3.1.1 Definition. For any $\mathcal{A}$-correspondence (E, φ), we define the C^*-algebra

$$\mathcal{F}_E := \overline{\bigcup_{n\in\mathbb{N}_0} \mathcal{K}(E^{\otimes n})}$$

inside $\varinjlim \mathcal{L}(E^{\otimes n})$ with respect to the inclusions $T \mapsto T \otimes \mathrm{id}$ with the convention that $\mathcal{K}(E^{\otimes 0}) := \mathcal{A}$. In addition, we define the Hilbert $\mathcal{F}_E$-module

$$E_\infty := E \otimes_i \mathcal{F}_E,$$

where $i : \mathcal{A} \hookrightarrow \mathcal{F}_E$ and set

$$\mathcal{F}_E^k := \overline{\bigcup_{n\geq k} \mathcal{K}(E^{\otimes n})}$$

3.3.1.2 Remark. If $\varphi(\mathcal{A})$ contains $\mathcal{K}(E)$, it follows that $\mathcal{K}(E^{\otimes n}) \supseteq \mathcal{K}(E^{\otimes n+1})$. In this case, $\mathcal{F}_E = \mathcal{A}$.

3.3.1.3 Example. *If $E := \mathbb{C}$ as in example 3.1.3, $\varphi(\mathbb{C}) = \mathbb{C} = \mathcal{K}(E)$, so $\mathcal{F}_E = \mathbb{C}$. In this special case the construction below produces exactly the completely positive lift we already considered.*

3.3.1.4 Example. *Consider now the slightly more general Hilbert $\mathbb{C}$-module $E := \mathcal{H}_d$ as in* *example 3.1.2.1 with* $d \geq 2$. *Since* $\mathcal{K}(\mathbb{C}^d) \cong \mathbb{M}_d(\mathbb{C})$, *we have* $\mathcal{K}(E^{\otimes n}) \cong \underbrace{\mathbb{M}_d(\mathbb{C}) \otimes \cdots \otimes \mathbb{M}_d(\mathbb{C})}_{n \text{ times}}$ *and*

$$\bigotimes_n \mathbb{M}_d(\mathbb{C}) \to \bigotimes_{n+1} \mathbb{M}_d(\mathbb{C}), \; A \mapsto A \otimes \mathrm{id} \; .$$

So in this case, $\mathcal{F}_E = \mathrm{UHF}(d^\infty) \neq \mathrm{UHF}(1^\infty) = \mathbb{C}$.

3.3.1.5 Lemma. *Let* E *be a Hilbert* $\mathcal{A}$*-module and* (F, φ) *an* $\mathcal{A}$*-*$\mathcal{B}$*-correspondence. Then the map*

$$\psi : \mathcal{L}(E \otimes_\varphi F) \ni T \mapsto \widetilde{T} \in \mathcal{L}\big(E \otimes \mathcal{K}(F)\big),$$

which acts on elementary tensors by $\widetilde{T}(\xi \otimes (\mu \otimes \eta^*)) := (T(\xi \otimes \mu)) \otimes \eta^*$, *is an isomorphism. In other words, the adjointable operators and the compact operators on* $E \otimes_\varphi F$ *are isomorphic to the ones on* $E \otimes \mathcal{K}(F)$.

In this proof, we omit the indices of the internal tensor product.

Proof. It holds that $E \otimes \mathcal{K}(F) \cong E \otimes F \otimes F^*$, where F^* denotes the opposite module of F. With this identification, the above map ψ becomes

$$\begin{gathered} \mathcal{L}(E \otimes F) \to \mathcal{L}(E \otimes F \otimes F^*), \\ T \mapsto T \otimes \mathrm{id} \; . \end{gathered}$$

In order to see that the inverse of this map is of the same form again, note that

$$E \otimes F \cong (E \otimes F) \otimes \underbrace{(F^* \otimes F)}_{=\mathrm{span}\{\langle x,y\rangle : x,y \in F\}} \cong (E \otimes (F \otimes F^*)) \otimes F \cong (E \otimes \mathcal{K}(F)) \otimes F.$$

The map

$$\begin{gathered} \mathcal{L}\big(E \otimes \mathcal{K}(F)\big) \to \mathcal{L}\big(E \otimes \mathcal{K}(F) \otimes F\big), \\ T \mapsto T \otimes \mathrm{id} \; . \end{gathered}$$

is the inverse of the previous one. $\square$

3.3.1.6 Theorem. *It holds that*

1. $\mathcal{K}(E_\infty)$ *is isomorphic to* $\mathcal{F}_E^1$.

2. $\mathcal{K}(E_\infty^{\otimes n})$ *is isomorphic to* $\mathcal{F}_E^n$.

3. $E_\infty^{\otimes n}$ *is isomorphic to* $E^{\otimes n} \otimes \mathcal{F}_E$.

4. $\Gamma(E_\infty)$ *is isomorphic to* $\Gamma(E) \otimes \mathcal{F}_E$.

Proof. We will proof the above assertions step by step.

1. : By the preceding lemma,

$$\mathcal{K}\big(E \otimes \mathcal{K}(E^{\otimes n})\big) = \mathcal{K}(E^{\otimes n+1}) \text{ for all } n \in \mathbb{N}_0.$$

   Since $\mathcal{K}(E_\infty) = \lim \mathcal{K}\big(E \otimes \mathcal{K}(E^{\otimes n})\big)$, this implies the claim.

2. : Apply (1) to $E^{\otimes n}$.

3. : $E_\infty^{\otimes n} = E \otimes \mathcal{F}_E \otimes E \otimes \cdots \otimes E \otimes \mathcal{F}_E = E^{\otimes n} \otimes \mathcal{F}_E$. This is true, since $\mathcal{K}(E) \subseteq \mathcal{F}_E \subseteq \mathcal{L}(E)$ implies that $\mathcal{F}_E \otimes E = E$.

4. follows from (3).

□

3.3.1.7 Remark. E_∞ is actually an $\mathcal{F}_E$ bimodule with right $\mathcal{F}_E$-linear scalar product

$$\langle \xi \otimes a, \eta \otimes b \rangle_{\mathcal{F}_E} := a^* \langle \xi, \eta \rangle b,$$

left $\mathcal{F}_E$-linear scalar product

$${}_{\mathcal{F}_E}\langle \xi \otimes a, \eta \otimes b \rangle := \theta_{\xi,\eta} \otimes ab^*$$

and left action $\varphi_\infty : \mathcal{F}_E = \mathcal{K}(E) \otimes \mathcal{F}_E \to \mathcal{L}(E_\infty)$, where $\varphi_\infty(a \otimes b)(\xi \otimes c) := a(\xi) \otimes bc$.

By this remark, E_∞^* is an $\mathcal{F}_E$-bimodule as well, although it is not full in general.

3.3.2 The two-sided Fock module

After establishing that E_∞ is a Hilbert bimodule we can make the following definitions.

3.3.2.1 Definition. $E_\infty^{\otimes -n} := E_\infty^{*\otimes n}$ for every $n \in \mathbb{N}$ and

$$\Gamma_{\mathbb{Z}}(E_\infty) := \bigoplus_{n\in\mathbb{Z}} E_\infty^{\otimes n}$$

We also write $\Gamma_{\mathbb{Z}}(E_\infty)^n := \Gamma_{\mathbb{Z}}(E_\infty) \otimes \mathcal{F}_E^n$.

With the above notation we observe that

$$\begin{aligned}\Gamma_{\mathbb{Z}}(E_\infty)^1 \otimes E_\infty &= \Gamma_{\mathbb{Z}}(E_\infty) \otimes \mathcal{F}_E^1 \otimes E \otimes \mathcal{F}_E \cong \Gamma_{\mathbb{Z}}(E_\infty)\ ,\\ \Gamma_{\mathbb{Z}}(E_\infty) \otimes E_\infty^* &= \cdots \oplus (E_\infty^* \otimes E_\infty^*) \oplus (\mathcal{F}_E \otimes E_\infty^*) \oplus (E_\infty \otimes E_\infty^*) \otimes (E_\infty^2 \otimes E_\infty^*) \otimes \dots\\ &= \Gamma_{\mathbb{Z}}(E_\infty) \otimes \mathcal{K}(E_\infty) = \Gamma_{\mathbb{Z}}(E_\infty)^1\end{aligned}$$

which provides us with the following two *-homomorphisms.

3.3.2.2 Definition. We set

$$\begin{array}{rrcll} \alpha\colon & \mathcal{L}\big(\Gamma_{\mathbb{Z}}(E_\infty)^1\big) & \to & \mathcal{L}\big(\Gamma_{\mathbb{Z}}(E_\infty)\big) & \cong \ \mathcal{L}\big(\Gamma_{\mathbb{Z}}(E_\infty)^1 \otimes E_\infty\big), \\ & T & \mapsto & T \otimes \mathrm{id} & \\ \beta\colon & \mathcal{L}\big(\Gamma_{\mathbb{Z}}(E_\infty)\big) & \to & \mathcal{L}\big(\Gamma_{\mathbb{Z}}(E_\infty)^1\big) & \cong \ \mathcal{L}\big(\Gamma_{\mathbb{Z}}(E_\infty) \otimes E_\infty^*\big), \\ & T & \mapsto & T \otimes \mathrm{id} & \end{array}$$

3.3.2.3 Remark. The *-homomorphisms α and β have the following properties:

(i) $\alpha\beta = \mathrm{id}_{\Gamma_{\mathbb{Z}}(E_\infty)}$ and $\beta\alpha = \mathrm{id}_{\Gamma_{\mathbb{Z}}(E_\infty)^1}$.

(ii) $\{\alpha^n\}_{n\in\mathbb{N}}$ is an action of $\mathbb{N}$ as isometric endomorphisms of $\mathcal{L}\big(\Gamma_{\mathbb{Z}}(E_\infty)\big)$, where

$$\begin{array}{rrcll} \alpha^n\colon & \mathcal{L}\big(\Gamma_{\mathbb{Z}}(E_\infty)^1\big) & \to & \mathcal{L}\big(\Gamma_{\mathbb{Z}}(E_\infty)\big) & \cong \ \mathcal{L}\big(\Gamma_{\mathbb{Z}}(E_\infty)^1 \otimes \underbrace{E_\infty \otimes \cdots \otimes E_\infty}_{n \text{ times}}\big), \\ & T & \mapsto & T \otimes \underbrace{\mathrm{id} \otimes \cdots \otimes \mathrm{id}}_{n \text{ times}} & . \end{array}$$

(iii) For elements $T \in \alpha^n\Big(\mathcal{L}\big(\Gamma_{\mathbb{Z}}(E_\infty)\big)\Big)$, $\alpha^{-n}(T) := \beta^n(T)$ makes sense.

3.3.2.4 Definition. For every $\xi \in E_\infty$ and every $\xi^* \in E_\infty^*$ we define operators M_ξ and

$M_{\xi^*} \in \mathcal{L}\big(\Gamma_{\mathbb{Z}}(E_\infty)\big)$ by

$$M_\xi(\eta) := \xi \otimes \eta \ ,$$
$$M_{\xi^*}(\eta) := \xi^* \otimes \eta \ ,$$

where $\xi \otimes \eta^* \in \mathcal{F}_E^1 \cong \mathcal{K}(E_\infty)$ and $\xi^* \otimes \eta \in \mathcal{F}_E$.

The proof of the following lemma is straightforward.

3.3.2.5 Lemma. *For those operators the following properties hold*

1. $M_\xi M_{\zeta^*} = \xi \otimes \zeta^*$, $M_{\zeta^*} M_\xi = \zeta^* \otimes \xi$ *and* $M_\xi^* = M_{\xi^*}$.
2. M_ξ *is an element of the fixed point algebra of the action* $(\alpha^n)_{n\in\mathbb{Z}}$.
3. *Denoting by* $p_m \in \mathcal{L}\big(\Gamma_{\mathbb{Z}}(E_\infty)\big)$ *the orthogonal projection onto* $\bigoplus_{n\geq m} E_\infty^{\otimes n}$ *it holds that* $\alpha^n(p_m) = p_{n+m}$.

3.3.3 A semisplit extension

3.3.3.1 Proposition. *The Toeplitz extension of* $\mathcal{O}_{E_\infty}$ *is semisplit. Moreover,*

1. *the map*
$$\mathcal{L}\big(\Gamma(E)\big) \to \mathcal{L}\big(\Gamma(E_\infty)\big) \cong \mathcal{L}\big(\Gamma(E) \otimes \mathcal{F}_E\big), T \mapsto T \otimes \mathrm{id}$$
induces an inclusion $\mathcal{T}_E \hookrightarrow \mathcal{T}_{E_\infty}$ *and an isomorphism* $\mathcal{O}_E \cong \mathcal{O}_{E_\infty}$.
2. *the map* $S_\xi \mapsto M_\xi$ *for every* $\xi \in E$ *extends to an isomorphism*
$$\mathcal{O}_E \cong C^*\{M_\xi : \xi \in E\} \subseteq \mathcal{L}\big(\Gamma_{\mathbb{Z}}(E_\infty)\big).$$
3. *the compression map* $\mathcal{L}\big(\Gamma_{\mathbb{Z}}(E_\infty)\big) \ni T \mapsto p_0 T p_0 \in \mathcal{L}\big(\Gamma(E_\infty)\big)$ *defines a completely positive map* $\phi : \mathcal{O}_E \to \mathcal{T}_{E_\infty}$*, which is a cross section to the quotient map* $\mathcal{T}_{E_\infty} \to \mathcal{O}_E$.

3.3.3.2 Remark (ad 2.). E is a subset of $E_\infty = E \otimes \mathcal{F}_E$, so M_ξ with $\xi \in E$ makes sense.

Proof. 1. : The isometry $\mathcal{L}(E) \ni T \mapsto T \otimes \mathrm{id} \in \mathcal{L}(E \otimes \mathcal{F}_E)$ induces an isomorphism

$$\mathcal{T}_E \cong C^*\{T_\xi : \xi \in E\} \subseteq \mathcal{T}_{E_\infty} \subseteq \mathcal{L}\big(\Gamma(E_\infty)\big).$$

Moreover, it sends $\{P_n\}_{n\in\mathbb{N}_0}$, which is the approximate unit of $\mathcal{J}_{\Gamma(E)}$ consisting of the orthogonal projections on the first n direct summands of $\Gamma(E)$ to $\{p_0 - p_n\}_{n\in\mathbb{N}}$ with p_n as in the previous lemma, which is an approximate unit for $\mathcal{J}_{\Gamma(E_\infty)}$. Hence

$$\mathcal{O}_E = \underbrace{C^*\{S_\xi : \xi \in E\}}_{=\mathcal{O}_{E_\infty}} \subseteq \mathcal{M}(\mathcal{J}_{\Gamma(E_\infty)})\big/\mathcal{J}_{\Gamma(E_\infty)} \ .$$

To prove assertions 2. and 3. note that

$$M_\xi p_n = p_{n+1} M_\xi p_n \quad \text{and} \quad p_n M_\xi = p_n M_\xi p_{n-1}$$

for every $\xi \in E_\infty$ and every $n \in \mathbb{Z}$. This implies on the one hand that

$$\phi(T) = p_0 T p_0 \in \mathcal{M}(\mathcal{J}_{\Gamma(E_\infty)}) \text{ for all } T \in C^*\{M_\xi\}$$

and on the other hand that $\pi \circ \phi : C^*\{M_\xi \colon \xi \in E_\infty\} \to \mathcal{O}_{E_\infty} \cong \mathcal{O}_E$ is a *-homomorphism that maps M_ξ to S_ξ, where $\pi : \mathcal{M}(\mathcal{J}_{\Gamma(E_\infty)}) \to \mathcal{M}(\mathcal{J}_{\Gamma(E_\infty)})\big/\mathcal{J}_{\Gamma(E_\infty)}$ is the quotient map. We now want to construct an inverse map for $\pi \circ \phi$.

For this, note that

$$\{T \in \mathcal{L}\big(\Gamma(E_\infty)\big) : \text{sot}^*\text{-} \lim_{n\to\infty} \alpha^{-n}(T) \text{ exists } \}$$

is a C^*-algebra. Of course, T being in this set this implies that $T \in \operatorname{im}(\alpha^n)$ for every $n \in \mathbb{N}$. Since

$$\begin{aligned}
\lim_{n\to\infty} \alpha^{-n}(T_\xi)p_i &= \lim_{n\to\infty} \alpha^{-n}(p_0 M_\xi p_0)p_i \\
&= \lim_{n\to\infty} p_{-n} M_\xi p_{-n} p_i = M_\xi p_i \ , \\
\lim_{n\to\infty} \alpha^{-n}(T_\xi^*)p_i &= \lim_{n\to\infty} \alpha^{-n}(p_0 M_\xi^* p_0)p_i \\
&= \lim_{n\to\infty} p_{-n} M_\xi^* p_{-n} p_i = M_\xi^* p_i,
\end{aligned}$$

this C^*-algebra contains $\mathcal{T}_{E_\infty}$. Since the above limit vanishes for elements of $\mathcal{J}_{\Gamma(E_\infty)}$ that lie in $\operatorname{im}(\alpha^n)$ for every $n \in \mathbb{N}$, the above limit defines a *-homomorphism from the quotient algebra to $\mathcal{L}(E_\infty)$ that maps S_ξ to M_ξ, which shows assertion 2.

We are left to show that $\mathrm{im}(\phi) \subseteq \mathcal{T}_{E_\infty}$. It holds that

$$\begin{aligned}
p_0 M_{\xi_1} M_{\xi_2} \dots M_{\xi_n} p_0 &= p_0 M_{\xi_1} p_0 M_{\xi_2} p_0 \dots p_0 M_{\xi_n} p_0 \\
&= T_{\xi_1} T_{\xi_2} \dots T_{\xi_n}, \\
p_0 M^*_{\xi_1} M^*_{\xi_2} \dots M^*_{\xi_n} p_0 &= p_0 M^*_{\xi_1} p_0 M^*_{\xi_2} p_0 \dots p_0 M^*_{\xi_n} p_0 \\
&= T^*_{\xi_1} T^*_{\xi_2} \dots T^*_{\xi_n} \text{ and} \\
p_0 M^*_\zeta M_\xi p_0 &= T^*_\zeta T_\xi
\end{aligned}$$

for all $\xi_1, \dots, \xi_n, \zeta \in E_\infty$. The linear combinations of words in $S_{\xi_1}, \dots, S_{\xi_n}, S^*_{\zeta_1}, \dots, S^*_{\zeta_m}$ and $a \in \mathcal{F}_E$ are dense in $\mathcal{O}_{E_\infty}$. Since the operators $M^*_\zeta M_\xi$ generate $\mathcal{F}_E$, the above argument concludes the proof.

□

3.3.3.3 Corollary. *For any graded, separable C^*-algebra $\mathcal{B}$ the following cyclic sequence is exact:*

$$\begin{tikzcd}
\mathrm{KK}_0\big(\mathcal{B}, \mathcal{K}(\Gamma(E_\infty)\mathcal{I}_\varphi)\big) \arrow[r] & \mathrm{KK}_0(\mathcal{B}, \mathcal{T}_{E_\infty}) \arrow[r] & \mathrm{KK}_0(\mathcal{B}, \mathcal{O}_E) \arrow[d] \\
\mathrm{KK}^1(\mathcal{B}, \mathcal{O}_E) \arrow[u] & \mathrm{KK}^1(\mathcal{B}, \mathcal{T}_{E_\infty}) \arrow[l] & \mathrm{KK}^1\big(\mathcal{B}, \mathcal{K}(\Gamma(E_\infty)\mathcal{I}_\varphi)\big) \arrow[l]
\end{tikzcd}$$

If $\mathcal{T}_{E_\infty}$ is separable, then the following cyclic sequence is exact as well:

$$\begin{tikzcd}
\mathrm{KK}_0\big(\mathcal{K}(\Gamma(E_\infty)\mathcal{I}_\varphi), \mathcal{B}\big) \arrow[d] & \mathrm{KK}_0(\mathcal{T}_{E_\infty}, \mathcal{B}) \arrow[l] & \mathrm{KK}_0(\mathcal{O}_E, \mathcal{B}) \arrow[l] \\
\mathrm{KK}^1(\mathcal{O}_E, \mathcal{B}) \arrow[r] & \mathrm{KK}^1(\mathcal{T}_{E_\infty}, \mathcal{B}) \arrow[r] & \mathrm{KK}^1\big(\mathcal{K}(\Gamma(E_\infty)\mathcal{I}_\varphi), \mathcal{B}\big) \arrow[u]
\end{tikzcd}$$

3.3.3.4 Remark. Note that $\mathcal{T}_E$ is separable if and only if E is countably generated and its coefficient algebra $\mathcal{A}$ is separable.

Proof. By, 2.3.3.3 the short semisplit exact sequence

$$0 \to \mathcal{K}(\Gamma(E_\infty)\mathcal{I}_{E_\infty}) \to \mathcal{T}_{E_\infty} \to \mathcal{O}_{E_\infty} \cong \mathcal{O}_E \to 0$$

gained in the previous theorem induces the above sequences. □

3.4 Other properties

In this section, we discuss which properties $\mathcal{T}_E$ and $\mathcal{O}_E$ inherit from the coefficient algebra $\mathcal{A}$ of the Hilbert $\mathcal{A}$-correspondence E.

3.4.1 A KK-equivalence between $\mathcal{T}_E$ and $\mathcal{A}$

Given a separable C^*-algebra $\mathcal{A}$ and a Hilbert $\mathcal{A}$-correspondence (E, φ) such that E is countably generated and full, Pimsner shows in [Pim97, theorem 4.4] that $\mathcal{A}$ and $\mathcal{T}_E$ are KK-equivalent, that is there exist elements $\alpha \in \mathrm{KK}(\mathcal{A}, \mathcal{T}_E)$ and $\beta \in \mathrm{KK}(\mathcal{T}_E, \mathcal{A})$ for which the identities $\alpha \circ \beta = \mathrm{id}_{\mathcal{A}}$ and $\beta \circ \alpha = \mathrm{id}_{\mathcal{T}_E}$ hold. Note that if two C^*-algebras are KK-equivalent, in particular all their KK-groups are isomorphic. Hence after proving the KK-equivalence between $\mathcal{T}_E$ and $\mathcal{A}$, we can consider $\mathrm{KK}(\mathcal{A}, \mathcal{B})$ instead of $\mathrm{KK}(\mathcal{T}_E, \mathcal{B})$ and $\mathrm{KK}(\mathcal{B}, \mathcal{A})$ instead of $\mathrm{KK}(\mathcal{B}, \mathcal{T}_E)$ for every graded separable C^*-algebra $\mathcal{B}$ in the previous cyclic exact sequences, which is why 3.4.1.3 follows from the previous corollary once the KK-equivalence is shown. We now present Pimsner's construction and proof.

There is a natural inclusion $i : \mathcal{A} \to \mathcal{T}_E$ via $i(a)(T_\xi) := T_{\varphi(a)\xi}$ for all $\xi \in E$. This induces the element $[i] := (\mathcal{T}_E, i, \mathrm{id}_{\mathcal{T}_E} \oplus 0) \in \mathrm{KK}(\mathcal{A}, \mathcal{T}_E)$, where we consider $\mathcal{T}_E$ a Hilbert $\mathcal{T}_E$-module equipped with the trivial grading. We will denote this element by α.

Now consider the graded Hilbert $\mathcal{A}$-module $\Gamma(E) \oplus \Gamma(E)$, where the first summand has degree zero and the second one degree one and define the flip $T \in \mathcal{L}\big(\Gamma(E) \oplus \Gamma(E)\big)$ via $T(\xi \otimes \zeta) = (\zeta \otimes \xi)$. For the left action of $\mathcal{T}_E$ on this module let $\pi_0 : \mathcal{T}_E \to \mathcal{L}\big(\Gamma(E)\big)$ be the natural action of $\mathcal{T}_E$ on $\Gamma(E)$ and $\pi_1 : \mathcal{T}_E \to \mathcal{L}\big(\Gamma(E)\big)$ be the map that restricts both the left action of $\mathcal{A}$ and the operators in $\mathcal{T}_E$ to tensors of length greater or equal to one, that is

$$\begin{aligned} \pi_1(T_\xi)(\eta_0, \eta_1, \eta_2, \dots) &= \pi_0(T_\xi)(0, \eta_1, \eta_2, \dots) &= (0, 0, \xi \otimes \eta_1, \xi \otimes \eta_2, \dots) \\ \pi_1(a)(\eta_0, \eta_1, \eta_2, \dots) &= \pi_0(a)(0, \eta_1, \eta_2, \dots) &= (0, \varphi(a)\eta_1, \varphi(a)\eta_2, \dots) \end{aligned}$$

We set $\pi : \mathcal{T}_E \to \mathcal{L}\big(\Gamma(E) \oplus \Gamma(E)\big),\ t \mapsto \pi_1(t) \oplus \pi_2(t)$.

3.4.1.1 Lemma. *For all $t \in \mathcal{T}_E$, $\pi_0(t) - \pi_1(t) \in \mathcal{K}\big(\Gamma(E)\big)$ holds.*

Proof. Since every $t \in \mathcal{T}_E$ is a sum of products of the form $T_{\xi_1} \dots T_{\xi_k} T^*_{\zeta_l} \dots T^*_{\zeta_1}$ it suffices to prove the claim for the creation operators $\{T_\xi : \xi \in E\}$. But

$$\begin{aligned} \pi_0(T_\xi)(a, 0, \dots) - \pi_1(T_\xi)(a, 0, \dots) &= (0, \xi a, 0, \dots), \\ \pi_0(T_\xi)(\eta) - \pi_1(T_\xi)(\eta) &= 0 \end{aligned}$$

for all $a \in \mathcal{A}$ and all $\eta \in E^{\otimes n}$, where $n \in \mathbb{N}$. □

This lemma implies that

$$\begin{aligned} [T, \pi(a)](\xi, \zeta) &= \big(\pi_1(a)\zeta, \pi_0(a)\xi\big) - \big(\pi_0(a)\zeta, \pi_1(a)\xi\big) \\ &= \Big(-\big(\pi_0(a) - \pi_1(a)\big)\zeta, \big(\pi_0(a) - \pi_1(a)\big)\xi\Big) \end{aligned}$$

is an element of $\mathcal{K}\big(\Gamma(E) \oplus \Gamma(E)\big)$. Moreover, $(T^2 - \mathrm{id}_{\Gamma(E)\oplus\Gamma(E)})\pi(a) = 0$ and T is selfadjoint. Therefore, $\big(\Gamma(E) \oplus \Gamma(E), \pi, T\big)$ defines a Kasparov module. We denote its class in $\mathrm{KK}(\mathcal{T}_E, \mathcal{A})$ by β.

3.4.1.2 Theorem. *For a separable C^*-algebra $\mathcal{A}$, a countably generated Hilbert $\mathcal{A}$-module E and $\alpha \in \mathrm{KK}(\mathcal{A}, \mathcal{T}_E)$, $\beta \in \mathrm{KK}(\mathcal{T}_E, \mathcal{A})$ as above the following equations hold:*

$$\begin{aligned} \alpha \otimes_{\mathcal{T}_E} \beta &= \mathbb{1}_{\mathcal{A}}, \\ \beta \otimes_{\mathcal{A}} \alpha &= \mathbb{1}_{\mathcal{T}_E}. \end{aligned}$$

Proof. Let us start with the first equality.

$\alpha \otimes_{\mathcal{T}_E} \beta = \mathbb{1}_{\mathcal{A}}$: Since α is induced by the *-homomorphism $i : \mathcal{A} \to \mathcal{T}_E$, it holds that

$$\alpha \otimes_{\mathcal{T}_E} \beta = [\Gamma(E) \oplus \Gamma(E), \pi \circ i, T] \in \mathrm{KK}(\mathcal{A}, \mathcal{A}).$$

Since π decomposes into a direct sum by definition so does $\pi \circ i$, and $\pi_0 \circ i$ is the natural representation of $\mathcal{A}$ on the first factor $\Gamma(E) \oplus 0$ whereas $\pi_1 \circ i = (\mathrm{id} - q_0)\pi_0 \circ i$, where $q_0 \in \mathcal{L}\big(\Gamma(E)\big)$ is the orthogonal projection onto the zero component, that is $\mathcal{A} \subseteq \Gamma(E)$. Hence the Kasparov module $\alpha \otimes \beta$ decomposes into the direct sum of Kasparov modules corresponding to $\mathcal{A} \oplus \mathcal{A}$ and $\bigoplus_{n=1}^\infty E^{\otimes n} \oplus \bigoplus_{n=1}^\infty E^{\otimes n}$. The first module represents the class of $\mathbb{1}_{\mathcal{A}}$ whereas the second one is degenerate.

$\beta \otimes_{\mathcal{A}} \alpha = \mathbb{1}_{\mathcal{T}_E}$**:** Note that $\beta \otimes_{\mathcal{A}} \alpha$ is given by the Kasparov module

$$[\Gamma(E) \otimes_i \mathcal{T}_E \oplus \Gamma(E) \otimes_i \mathcal{T}_E, (\pi_0 \otimes \mathbb{1}) \oplus (\pi_1 \otimes \mathbb{1}), T \otimes \mathbb{1}].$$

We will now prove the second equality by showing that the Kasparov module $\beta \otimes_{\mathcal{A}} \alpha - \mathbb{1}_{\mathcal{T}_E}$ is degenerate. For this, represent $\beta \otimes_{\mathcal{A}} \alpha - \mathbb{1}_{\mathcal{T}_E}$ by

$$[\Gamma(E) \otimes_i \mathcal{T}_E \oplus \Gamma(E) \otimes_i \mathcal{T}_E, (\pi_0 \otimes \mathbb{1}) \oplus \pi_1', T \otimes \mathbb{1}],$$

where $\pi_1' : \mathcal{T}_E \to \mathcal{L}\big(\Gamma(E) \otimes \mathcal{T}_E\big),\ t \mapsto \tau_1(t) + \pi_1 \otimes \mathbb{1}(t)$, where for $t \in \mathcal{T}_E$, $\tau_1(t)$ multiplies with t from the left on $\mathcal{T}_E \cong \mathcal{A} \otimes \mathcal{T}_E \subseteq \Gamma(E) \otimes \mathcal{T}_E$ and equals zero on $\bigoplus_{n=1}^{\infty} E^{\otimes n} \otimes \mathcal{T}_E$. We are left to prove that $\pi_0 \otimes \mathbb{1}$ and π_1' are connected by a one-parameter family of *-homomorphisms $\pi_t' : \mathcal{T}_E \to \mathcal{L}\big(\Gamma(E) \otimes \mathcal{T}_E\big)$, where $t \in [0,1]$, such that both of the following assertions hold:

1. the map $[0,1] \ni t \mapsto \pi_t'(x)$ is continuous for every $t \in \mathcal{T}_E$
2. $\pi_t'(x) - \pi_0'(x) \in \mathcal{K}\big(\Gamma(E) \otimes \mathcal{T}_E\big)$ for every $x \in \mathcal{T}_E$.

For this, note that

$$\pi_0 \otimes \mathbb{1}(T_\xi) = \tau_0(T_\xi) + \pi_1 \otimes_i \mathbb{1}(T_\xi),$$

for all $\xi \in E$, where $\tau_0(T_\xi)(t) = \xi \otimes t$ for $t \in \mathcal{T}_E \cong \mathcal{A} \otimes \mathcal{T}_E$ and zero for $t \in \bigoplus_{n=1}^{\infty} E^{\otimes n} \otimes \mathcal{T}_E$. The images of $\tau_0(T_\xi)$, $\tau_1(T_\zeta)$ and $\pi_1 \otimes \mathbb{1}(T_\mu)$ are pairwise orthogonal for $\xi, \zeta, \mu \in E$. So for fixed $t \in I$ the operators defined by

$$t_\xi := \cos(\frac{\pi}{2}t)\tau_0(T_\xi) + \sin(\frac{\pi}{2}t)\tau_1(T_\xi) + \pi_1 \otimes \mathbb{1}(T_\xi)$$

satisfy the universal relations (TR1), (TR2) and (TR3) of the Toeplitz algebra from definition 3.2.1.1. By the universal property of the Toeplitz algebras we discussed in 3.2.3.4, there exists a *-representations $\pi_t' : \mathcal{T}_E \to \mathcal{L}\big(\Gamma(E) \otimes \mathcal{T}_E\big)$ which maps the creation operators T_ξ to t_ξ. For every $x \in \mathcal{T}_E$, the continuity of $[0,1] \ni t \mapsto \pi_t'(x)$ is obvious since it is continuous for all creation operators T_ξ, which generate $\mathcal{T}_E$. This proves 1. Since, $\pi_t'(T_\xi) - \pi_0'(T_\xi) = \cos(\frac{\pi}{2}t)\tau_0(T_\xi) + \sin(\frac{\pi}{2}t)\tau_1(T_\xi) \in \mathcal{K}\big(\Gamma(E) \otimes \mathcal{T}_E\big)$, 2. holds as well. We have shown that the Kasparov module above is operator homotopic and therefore homotopic to a degenerate one. $\square$

Combining the previous theorem with 3.3.3.3, we get the following Corollary.

3.4.1.3 Corollary. *For any graded, separable C^*-algebra $\mathcal{B}$ the following cyclic sequence is exact:*

$$\begin{tikzcd}
\mathrm{KK}_0(\mathcal{B},\mathcal{K}(\Gamma(E_\infty)\mathcal{I}_\varphi)) \arrow[r] & \mathrm{KK}_0(\mathcal{B},\mathcal{F}_E) \arrow[r] & \mathrm{KK}_0(\mathcal{B},\mathcal{O}_E) \arrow[d] \\
\mathrm{KK}^1(\mathcal{B},\mathcal{O}_E) \arrow[u] & \mathrm{KK}^1(\mathcal{B},\mathcal{F}_E) \arrow[l] & \mathrm{KK}^1(\mathcal{B},\mathcal{K}(\Gamma(E_\infty)\mathcal{I}_\varphi)) \arrow[l]
\end{tikzcd}$$

If $\mathcal{T}_{E_\infty}$ is separable, then the following cyclic sequence is exact as well:

$$\begin{tikzcd}
\mathrm{KK}_0(\mathcal{K}(\Gamma(E_\infty)\mathcal{I}_\varphi),\mathcal{B}) \arrow[d] & \mathrm{KK}_0(\mathcal{F}_E,\mathcal{B}) \arrow[l] & \mathrm{KK}_0(\mathcal{O}_E,\mathcal{B}) \arrow[l] \\
\mathrm{KK}^1(\mathcal{O}_E,\mathcal{B}) \arrow[r] & \mathrm{KK}^1(\mathcal{F}_E,\mathcal{B}) \arrow[r] & \mathrm{KK}^1(\mathcal{K}(\Gamma(E_\infty)\mathcal{I}_\varphi),\mathcal{B}) \arrow[u]
\end{tikzcd}$$

3.4.2 Approximation properties of $\mathcal{O}_E$

Given a Hilbert $\mathcal{A}$-module E, [SZ10] contains a more general result quoted below which implies that the Cuntz-Pimsner algebra $\mathcal{O}_E$ inherits nuclearity from $\mathcal{A}$. We now give a novel proof of this fact.

3.4.2.1 Theorem. *Let $\mathcal{A}$ be a nuclear C^*-algebra and E a countably generated Hilbert $\mathcal{A}$-module. Then $\mathcal{O}_E$ is nuclear.*

Proof. Let $\mathcal{B}$ be a C^*-algebra. Then, by [BO08, theorem 5.3.5] the gauge action λ of $\mathbb{T}$ on $\mathcal{O}_E$ extends to an action $\lambda\otimes\mathrm{id}$ of $\mathbb{T}$ on $\mathcal{O}_E\otimes_{\max}\mathcal{B}$ and $\mathcal{O}_E\otimes_{\min}\mathcal{B}$ with fixed point algebras $\mathcal{F}_{\max}$ and $\mathcal{F}_{\min}$ respectively, yielding conditional expectations $\theta_{\max}:\mathcal{O}_E\otimes_{\max}\mathcal{B}\to\mathcal{F}_{\max}$ and $\theta_{\min}:\mathcal{O}_E\otimes_{\min}\mathcal{B}\to\mathcal{F}_{\min}$ by

$$\begin{aligned}
\theta_{\min}(S\otimes b) &:= \int_{t\in\mathbb{T}}\lambda_t\otimes_{\min}\mathrm{id}(S\otimes_{\min} b)d\mu(t)\ , \\
\theta_{\max}(S\otimes b) &:= \int_{t\in\mathbb{T}}\lambda_t\otimes_{\max}\mathrm{id}(S\otimes_{\max} b)d\mu(t)
\end{aligned}$$

for $S \in \mathcal{O}_E$ and $b \in \mathcal{B}$. From this integral representation, we learn that $\mathcal{F}_{\max} = \mathcal{F}_E \otimes_{\max} \mathcal{B}$ and $\mathcal{F}_{\min} = \mathcal{F}_E \otimes_{\min} \mathcal{B}$.

To see that $\theta_{\min}$ is faithful, fix an element $0 \leq x \in \mathcal{O}_E \otimes_{\min} \mathcal{B}$. The positivity of x implies that $\lambda_t \otimes_{\min} \mathrm{id}(x) \geq 0$ for all $t \in \mathbb{T}$. If x is strictly positive, there exists a state $\varphi : \mathcal{O}_E \otimes \mathcal{F}_{\min} \to \mathbb{C}$ with $\varphi(x) \neq 0$. We get

$$\varphi\left(\int_{t\in\mathbb{T}} \lambda_t \otimes_{\min} \mathrm{id}(x) d\mu(t)\right) = \int_{t\in\mathbb{T}} \varphi\big((\lambda_t \otimes_{\min} \mathrm{id})(x)\big) d\mu(t).$$

In particular, $\varphi\big((\lambda_1 \otimes_{\min} \mathrm{id})(x)\big) \gneq 0$. Since $t \mapsto \varphi\big((\lambda_t \otimes_{\min} \mathrm{id})(x)\big)$ is continuous, there exists an $\varepsilon > 0$ such that $\varphi\big((\lambda_t \otimes_{\min} \mathrm{id})(x)\big) \gneq 0$ for $t \in (1-\varepsilon, 1]$. Therefore,

$$\theta_{\min}(x) := \int_{t\in\mathbb{T}} \lambda_t \otimes_{\min} \mathrm{id}(x) d\mu(t) \gneq 0.$$

The conditional expectation $\theta_{\max}$ is faithful for the same reasons.

By the universal property of the maximal tensor product we therefore get the following commutative diagram:

$$\begin{tikzcd}
\mathcal{O}_E \otimes_{\max} \mathcal{B} \arrow[r, "\pi"] \arrow[d, "\theta_{\max}"'] & \mathcal{O}_E \otimes_{\min} \mathcal{B} \arrow[d, "\theta_{\min}"] \\
\mathcal{F}_E \otimes_{\max} \mathcal{B} \arrow[r, "\pi|_{\mathcal{F}_{\max}}"] & \mathcal{F} \otimes_{\min} \mathcal{B}
\end{tikzcd}$$

Since $\mathcal{A}$ is nuclear, $\mathcal{K}(E^{\otimes n})$ is as well for all $n \in \mathbb{N}$ and therefore so is $\mathcal{F}_E = \lim_{n\to\infty} \mathcal{K}(E^{\otimes n})$. Hence, $\pi|_{\mathcal{F}_{\max}}$ is an isomorphism. Since both $\theta_{\max}$ and $\theta_{\min}$ are faithful, so is π. $\square$

Nuclearity is one of many approximation properties a C^*-algebra may possess. The following definition lists some others.

3.4.2.2 Definition. We say that a C^*-algebra $\mathcal{A}$ possesses

1. the *completely positive approximation property* if there exist nets of completely pos-

itive contractions $\varphi_\lambda : \mathcal{A} \to \mathbb{M}_{n_\lambda}$ and $\psi_\lambda : \mathbb{M}_{n_\lambda} \to \mathcal{A}$ such that

$$\lim_\lambda \psi_\lambda \circ \varphi_\lambda(x) = x \text{ for all } x \in \mathcal{A}.$$

2. the *completely bounded approximation property* if there exists a net $(\varphi_\lambda : \mathcal{A} \to \mathcal{A})$ of finite rank maps such that

$$\varphi_\lambda(x) \to x \text{ for all } x \in \mathcal{A} \text{ and } \sup_\lambda \|\varphi_\lambda\| < \infty.$$

3. the *strong operator approximation property* if there exists a net $\varphi_\lambda : \mathcal{A} \to \mathcal{A}$ of finite rank maps such that

$$(\varphi_\lambda \otimes \mathrm{id})(x) \to x \text{ for all } x \in \mathcal{A} \otimes \mathcal{B}(l^2(\mathbb{N})) \ ,$$

where $\otimes$ denotes the minimal tensor product.

4. *nuclear embeddability* if for every faithful representation of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ there exist nets of completely positive contractions $\varphi_\lambda : \mathcal{A} \to \mathbb{M}_{n_\lambda}$ and $\psi_\lambda : \mathbb{M}_{n_\lambda} \to \mathcal{B}(\mathcal{H})$ such that

$$\psi_\lambda \circ \varphi_\lambda(x) \to x \text{ for all } x \in \mathcal{A}.$$

5. the *operator approximation property* if there exists a net $\varphi_\lambda : \mathcal{A} \to \mathcal{A}$ of finite rank maps such that

$$(\varphi_\lambda \otimes \mathrm{id})(x) \to x \text{ for all } x \in \mathcal{A} \otimes \mathcal{K}\big(l^2(\mathbb{N})\big) \ .$$

The following result was achieved by Skalski and Zacharias in [SZ10].

3.4.2.3 Theorem. *Let (E, φ) be a finitely generated $\mathcal{A}$-correspondence. If $\mathcal{A}$ possesses one of the five approximation properties above, so does $\mathcal{O}_E$.*

Chapter 4

Higher rank Toeplitz and Cuntz-Pimsner algebras

Having recalled how to associate the C^*-algebras $\mathcal{T}_E$ and $\mathcal{O}_E$ to a Hilbert $\mathcal{A}$-correspondence E and the results Pimsner achieved for these algebras, we will now start with certain families of Hilbert correspondences, so-called discrete product systems, and associate a Toeplitz and a Cuntz-Pimsner type algebra by imitating Pimsner's construction. This was introduced in [Fow02]. Thereafter, we generalise the main theorems from the previous chapter to product systems over $\mathbb{N}^k$ in the second and third section of this chapter. In the final section, we investigate the implications of [Dea07] to the special case we considered. This chapter contains the main theorems of this thesis in 4.2.3.1, 4.3.3.1 and 4.4.1.4.

4.1 Definitions

4.1.1 Discrete product systems

Given a discrete semigroup P, we will now formalise what we consider a P-family of Hilbert $\mathcal{A}$-correspondences respecting the semigroup structure of P by the following definition. It first appeared in [Fow02].

4.1.1.1 Definition. Let $(P, +)$ be a countable monoid, meaning P is a countable semigroup with identity e. A *(discrete) product system* is a triple $(\mathbf{X}, p, (\varphi_s)_{s\in P})$ consisting of a semigroup $\mathbf{X}$, a semigroup homomorphism $p : \mathbf{X} \to P$ and a family $(\varphi_s)_{s\in P}$ of

*-homomorphisms such that

1. $X_s := p^{-1}(s)$ is a Hilbert $\mathcal{A}$-module for all $s \in P$.

2. φ_s implements a left action of $\mathcal{A}$ on X_s, that is $\varphi_s : \mathcal{A} \to \mathcal{L}(X_s)$.

3. the fibre of the neutral element is trivial, meaning $X_e = \mathcal{A}$ and $\varphi_e(a)b = ab$.

4. the multiplications $X_e \times X_s \to X_s$ and $X_s \times X_e \to X_s$ satisfy

$$ax = \varphi_s(a)x \text{ and } xa = x \cdot a,$$

   where $a \in X_e$ and $x \in X_s$.

5. the map $\psi_{s,t} : X_s \times X_t \ni (x, y) \mapsto xy \in X_{s+t}$ extends to an isomorphism of the Hilbert $\mathcal{A}$-correspondences $X_s \otimes_{\varphi_t} X_t$ and X_{s+t} for all $s, t \in P \setminus \{e\}$.

4.1.1.2 Remark. If P is an abelian semigroup, the last condition yields isomorphisms $\chi_{s,t} : X_s \otimes X_t \to X_t \otimes X_s$ by

$$X_s \otimes X_t \overset{\psi_{s,t}}{\cong} X_{s+t} = X_{t+s} \overset{\psi_{t,s}^{-1}}{\cong} X_t \otimes X_s.$$

4.1.1.3 Remark. In this case, the condition that $\psi_{s,t}$ in definition 4.1.1.1 is an isomorphism of Hilbert correspondences together with 4.1.1.2 implies that $\chi_{s,t}$ intertwines φ_s and φ_t, that is

$$\big((\varphi_t(a) \otimes \mathrm{id}_{X_s}) \circ \chi_{s,t}\big)(X_s \otimes X_t) = \big(\chi_{s,t} \circ (\varphi_s(a) \otimes \mathrm{id}_{X_t})\big)(X_s \otimes X_t)$$

for all $a \in \mathcal{A}$ and $s, t \in P$.

For the rest of the chapter, we will consider the semigroup $P = \mathbb{N}^k$ unless stated otherwise. By $\mathbf{e_i} := (\delta_{i,j})_{j=1,\ldots,k} \in \mathbb{N}^k$ we denote the standard generators of $\mathbb{N}^k$ and

$$X_i := \mathbf{X}_{\mathbf{e_i}} = p^{-1}(\mathbf{e_i}).$$

4.1.1.4 Definition. For a product system $\mathbf{X}$ over $\mathbb{N}^k$ and $(n_1, \ldots, n_k) = \mathbf{n} \in \mathbb{N}^k$ we define the $\mathbf{n}$-th generalised power of $\mathbf{X}$ to be

$$\mathbf{X}^{\otimes \mathbf{n}} := X_1^{\otimes n_1} \otimes \cdots \otimes X_k^{\otimes n_k}$$

with the notation from the third chapter that $X_i^{\otimes n} := X_i \otimes_{\varphi_i} \cdots \otimes_{\varphi_i} X_i$ denotes the n-fold internal tensor product of the correspondence (X_i, φ_i) and the convention that $X_i^{\otimes 0} := \mathcal{A}$.

4.1.1.5 Remark. It holds that

$$\mathbf{X_n} = \mathbf{X}_{n_1 e_1 + \cdots + n_k e_k} \cong X_{n_1 e_1} \otimes \cdots \otimes X_{n_k e_k} \cong \mathbf{X}^{\otimes \mathbf{n}}.$$

In other words, there is a 1:1-correspondence between product systems $\mathbf{X}$ over $\mathbb{N}^k$ and a collection of Hilbert correspondences $(X_1, \ldots, X_k)$ together with *-isomorphisms $\chi_{i,j} : X_i \otimes X_j \longrightarrow X_j \otimes X_i$.

4.1.1.6 Remark. By the definition of product systems and the previous remark, we know that for every $\mathbf{n} = (n_1, \ldots, n_k)$, $(\mathbf{X}^{\otimes \mathbf{n}}, \varphi_{\mathbf{n}})$ is an $\mathcal{A}$-correspondence. By recalling the induced left action on the internal tensor product of Hilbert correspondences from 2.2.3.5, we see that $\varphi_{\mathbf{n}}$ acts by $\varphi_{n_{i_0}}$ on $X_{i_0}^{\otimes n_{i_0}}$ and trivially elsewhere, where

$$i_0 := \min\{i : 1 \leq i \leq k, n_i \neq 0\}.$$

Hence given an elementary tensor in $\mathbf{X}^{\otimes \mathbf{n}}$, $a \in \mathcal{A}$ acts on its first component by the respective left action and by the identity everywhere else.

4.1.1.7 Remark. Given two generalised powers $\mathbf{X}^{\otimes \mathbf{n}}$ and $\mathbf{X}^{\otimes \mathbf{m}}$ of a product system $\mathbf{X}$ over $\mathbb{N}^k$, $\mathbf{X}^{\otimes \mathbf{n}} \otimes_{\varphi_{\mathbf{m}}} \mathbf{X}^{\otimes \mathbf{m}}$ is isomorphic to a generalised power. In other words, we can use the isomorphisms $\chi_{i,j} : X_i \otimes X_j \to X_j \otimes X_i$ to rearrange a word $x \in \mathbf{X}^{\otimes \mathbf{n}} \otimes \mathbf{X}^{\otimes \mathbf{m}}$ such that

$$x \cong x_1^{(1)} \otimes \cdots \otimes x_{n_1+m_1}^{(1)} \otimes x_1^{(2)} \otimes \cdots \otimes x_{n_2+m_2}^{(2)} \otimes \cdots \otimes x_1^{(k)} \otimes \cdots \otimes x_{n_k+m_k}^{(k)},$$

where $x_j^{(i)}$ is the empty word if $n_j + m_j = 0$ and an element of X_i for all $1 \leq j \leq n_j + m_j$ otherwise. For now, we denote by χ the appropriate composition of $\chi_{i,j}$ that identifies a word in $\mathbf{X}^{\otimes \mathbf{n}} \otimes \mathbf{X}^{\otimes \mathbf{m}}$ with a word in the generalised power $\mathbf{X}^{\otimes \mathbf{n}+\mathbf{m}}$ subject to the above isomorphism.

4.1.1.8 Example. *For an $\mathcal{A}$-correspondence (E, φ), the set $\{\xi \in E^{\otimes n} : n \in \mathbb{N}\}$ of ele-*

ments in arbitrary tensor powers of E with semigroup operation $\otimes$ and *-homomorphisms $\varphi_n := \varphi \otimes \mathrm{id}$ as in 2.2.3.5 is a product system over $\mathbb{N}$. By remark 4.1.1.5, a product system $\mathbf{X}$ over $\mathbb{N}$ corresponds 1:1 to the Hilbert correspondence $\mathbf{X}_1 =: E$.

4.1.2 Representations of product systems

To associate C^*-algebras to a product system, Fowler made the following definitions.

4.1.2.1 Definition. For a product system $\mathbf{X}$ over P, a *Toeplitz representation* of $\mathbf{X}$ on a C^*-algebra $\mathcal{B}$ is a tuple (t, σ), where $\sigma : \mathcal{A} \to \mathcal{B}$ is a *-homomorphism and $t : \mathbf{X} \to \mathcal{B}$ is a linear contractive map such that

1. For all $s \in P$, (t_s, σ) is a Toeplitz representation of X_s in the sense of definition 3.2.1.1, i.e.

   (i) $t_s(x_s a) = t_s(x_s)\sigma(a)$,

   (ii) $t_s(x_s)^* t_s(y_s) = \sigma(\langle x_s, y_s \rangle)$,

   (iii) $t_s(\varphi_s(a)x_s) = \sigma(a)t_s(x_s)$

   for all $x_s, y_s \in X_s$ and $a \in \mathcal{A}$ and

2. $t(xy) = t(x)t(y)$ for all $x, y \in \mathbf{X}$,

where t_s denotes the restriction of t to X_s for $s \in P$.

4.1.2.2 Remark. The fibres $t_s : X_s \to \mathcal{B}$ of a Toeplitz representation t of X induce *-homomorphisms $\tilde{\sigma}_s : \mathcal{K}(X_s) \to \mathcal{B}$ by

$$\tilde{\sigma}_s(\theta_{\xi,\eta}) := t_s(\xi)t_s(\eta)^*.$$

4.1.2.3 Definition. Let $\mathcal{I}_s := \varphi_s^{-1}(\mathcal{K}(X_s))$ denote the ideal in $\mathcal{A}$ one gets by forming the pre-image of the compacts of X_s under φ_s. A Toeplitz representation of $\mathbf{X}$ is said to be *Cuntz-Pimsner covariant* if every (t_s, σ) is Cuntz-Pimsner covariant, that is

$$\sigma(a) = (\tilde{\sigma}_s \circ \varphi)(a) \text{ for all } a \in \mathcal{I}_s.$$

In other words, the diagram

$$\begin{tikzcd}
& \mathcal{I}_s \arrow[dl, "\varphi_s"'] \arrow[dr, "\sigma"] & \\
\mathcal{K}(X_s) \arrow[rr, "\tilde{\sigma}_s"] & & \mathcal{B}
\end{tikzcd}$$

commutes for all $s \in P$.

4.1.2.4 Definition. The universal C^*-algebra generated by all Toeplitz representations of $\mathbf{X}$ is called *Toeplitz algebra* of $\mathbf{X}$ and denoted by $\mathcal{T}_{\mathbf{X}}$. The universal C^*-algebra generated by all Cuntz-Pimsner covariant representations of $\mathbf{X}$ is called *Cuntz-Pimsner algebra* of $\mathbf{X}$ and denoted by $\mathcal{O}_{\mathbf{X}}$.

4.1.2.5 Example. *For product systems $\mathbf{X}$ over $\mathbb{N}$, we get that $\mathcal{T}_{\mathbf{X}} = \mathcal{T}_{X_1}$ and $\mathcal{O}_{\mathbf{X}} = \mathcal{O}_{X_1}$. In this sense, the higher rank Toeplitz and Cuntz-Pimsner algebras generalise Pimsner's original construction. See [Fow02, proposition 2.11] for details.*

4.1.2.6 Example. *Given a k-graph Λ, one can associate to it a product system E_Λ of directed graphs over $\mathbb{N}^k$, see [SY10, example 1.5, (4)]. To any directed graph $E = (E^0, E^1, s, r)$ one can associate a Hilbert $\mathcal{C}_0(E^0)$-correspondence as in [FS02, 2.3]. Putting those two together, one obtains for the given k-graph a product system $X(E_\Lambda)$ of Hilbert correspondences as in [FS02, proposition 3.2]. If Λ is row-finite and has no sources, the graph C^*-algebra $C^*(\Lambda)$ is isomorphic to $\mathcal{O}_{X(E_\Lambda)}$ as stated in [FS02, corollary 4.4].*

4.2 A gauge-invariant uniqueness theorem for $\mathcal{T}_{\mathbf{X}}^{\mathrm{cov}}$

4.2.1 The Fock representation of X

Similar to the one-dimensional case, there always exists at least one Toeplitz representation of $\mathbf{X}$, namely the higher rank Fock representation. It is obtained by associating elements of the product system with creation operators on the higher rank analogue of the Fock module.

4.2.1.1 Definition. For a product system $\mathbf{X}$ over $\mathbb{N}^k$ we define the *(higher rank) Fock module* of $\mathbf{X}$ by

$$\Gamma(\mathbf{X}) := \bigoplus_{\mathbf{n}\in\mathbb{N}^k} X^{\otimes\mathbf{n}} = \left\{ (x_{\mathbf{n}}) \in \prod_{\mathbf{n}\in\mathbb{N}^k} X^{\otimes\mathbf{n}} : \sum_{\mathbf{n}\in\mathbb{N}^k} \langle x_{\mathbf{n}}, x_{\mathbf{n}}\rangle_{\mathbf{X}^{\otimes\mathbf{n}}} \text{ converges in } \mathcal{A} \right\}.$$

4.2.1.2 Remark. This is an $\mathcal{A}$-correspondence with inner product

$$\langle x, y\rangle_{\Gamma(\mathbf{X})} = \sum_{\mathbf{n}\in\mathbb{N}^k} \langle x_{\mathbf{n}}, y_{\mathbf{n}}\rangle_{\mathbf{X}^{\otimes\mathbf{n}}}$$

and left action $\varphi_{\Gamma(\mathbf{X})} = \oplus\varphi_{\mathbf{n}}$, with $\varphi_{\mathbf{n}}$ as in 4.1.1.6, which we will just denote by φ.

We will now define creation operators on $\Gamma(\mathbf{X})$.

4.2.1.3 Definition. For every $1 \leq i \leq k$ and $x^{(i)} \in X_i$ we define

$$T_{x^{(i)}}(x_{\mathbf{n}}) := \chi(x^{(i)} \otimes x_{\mathbf{n}}) \in \mathbf{X}^{\otimes\mathbf{e_i}+\mathbf{n}}$$

for all $x_{\mathbf{n}} \in \mathbf{X}^{\otimes\mathbf{n}}$ and $\mathbf{n} \in \mathbb{N}^k$ and extend this linearly to $\Gamma(\mathbf{X})$. This operator $T_{x^{(i)}}$ is called a *creation operator by* $x^{(i)}$.

4.2.1.4 Remark. In fact, $T_{x^{(i)}}$ is adjointable with $T^*_{x^{(i)}}$ vanishing on all $X^{\otimes\mathbf{n}}$ with $n_i = 0$. If $n_i \neq 0$ we know that

$$T^*_{x^{(i)}}(x) = \varphi(\langle x^{(i)}, x_1^{(i)}\rangle)\tilde{x},$$

where $X_i \otimes \mathbf{X}^{\otimes(\mathbf{n}-\mathbf{e_i})} \ni (x_1^{(i)} \otimes \tilde{x}) \stackrel{\chi}{=} x \in \mathbf{X}^{\otimes\mathbf{n}}$. We call $T^*_{x^{(i)}}$ *annihilation operator* by $x^{(i)}$.

4.2.1.5 Definition. The *concrete Toeplitz algebra associated to* $\mathbf{X}$ is the concrete C^*-algebra

$$\widetilde{\mathcal{T}}_{\mathbf{X}} := C^*\{T_{x^{(i)}} : 1 \leq i \leq k\} \subseteq \mathcal{L}\big(\Gamma(\mathbf{X})\big)$$

generated by the creation operators inside the adjointable operators on the Fock module.

4.2.1.6 Remark. Fowler made these definitions for much more arbitrary product systems. In this remark, we quickly recall his argument: If $\mathbf{X}$ is a product system over some semigroup P, the Fock module $\Gamma(\mathbf{X}) = \bigoplus_{s\in P}\mathbf{X}_s$ as above makes sense. If P is left-cancellative then for any $x \in \mathbf{X}$ and $\oplus x_s \in \Gamma(\mathbf{X})$, $p(xx_s) = p(xx_t)$ if and only if $s = t$

so

$$y_t = \begin{cases} xx_s & \text{if } t = p(x)s \\ 0 & \text{else} \end{cases}$$

defines an element in $\Pi_{t \in P} X_t$. Since $\langle xx_t, xx_t \rangle \leq \|x\|^2 \langle x_t, x_t \rangle$ for each $t \in P$, $\oplus y_t$ is an element of the Fock module and one checks that $T(x)(\oplus x_t) := \oplus y_t$ defines an adjointable operator. The map $t : \mathbf{X} \to \mathcal{L}\big(\Gamma(\mathbf{X})\big)$ is a Toeplitz representation, called the *Fock representation of* $\mathbf{X}$.

4.2.1.7 Lemma. *The Fock representation*

$$\begin{array}{rrcll} T_{\mathbf{n}}\colon & \mathbf{X}^{\otimes \mathbf{n}} & \to & \mathcal{L}\big(\Gamma(\mathbf{X})\big), \;\; x_1^{(1)} \otimes x_2^{(1)} \otimes \cdots \otimes x_{l_k}^{(k)} & \mapsto & T_{x_1^{(1)}} T_{x_2^{(1)}} \ldots T_{x_{l_k}^{(k)}} \\ \sigma\colon & \mathcal{A} & \to & \mathcal{L}\big(\Gamma(\mathbf{X})\big), \qquad\qquad\qquad\qquad a & \mapsto & \varphi(a) = \oplus \varphi_{\mathbf{n}}(a) \end{array}$$

is a Toeplitz representation of the product system $\mathbf{X}$ *over* $\mathbb{N}^k$.

Proof. Let $x_{\mathbf{n}}, y_{\mathbf{n}} \in \mathbf{X}^{\otimes \mathbf{n}}$ and $z_{\mathbf{m}} \in \mathbf{X}^{\otimes \mathbf{m}}$. Then

1. $(T_{\mathbf{n}}, \sigma)$ is a Toeplitz representation of the Hilbert module $X^{\otimes \mathbf{n}}$, since

   (i) $T_{\mathbf{n}}(x_{\mathbf{n}} a)(z_{\mathbf{m}}) = x_{\mathbf{n}} a \otimes z_{\mathbf{m}} = x_{\mathbf{n}} \otimes \varphi_m(a) z_{\mathbf{m}} = T_{\mathbf{n}}(x_{\mathbf{n}}) \circ \sigma(a)(x_{\mathbf{m}})$,

   (ii) $T_{\mathbf{n}}(x_{\mathbf{n}})^* T_{\mathbf{n}}(y_n)(z_{\mathbf{m}}) = T_{\mathbf{n}}(x_{\mathbf{n}})^*(y_{\mathbf{n}} \otimes x_{\mathbf{m}}) = \varphi_{\mathbf{m}}(\langle x_{\mathbf{n}}, y_{\mathbf{n}} \rangle)(z_{\mathbf{m}}) = \sigma(\langle x_{\mathbf{n}}, y_{\mathbf{n}} \rangle)(z_{\mathbf{m}})$,

   (iii) $T_{\mathbf{n}}\big(\varphi_{\mathbf{n}}(a) x_{\mathbf{n}}\big)(z_{\mathbf{m}}) = \varphi_{\mathbf{n}+\mathbf{m}}(a)(x_{\mathbf{n}} \otimes z_{\mathbf{m}}) = (\sigma(a) \circ T_{\mathbf{n}}(x_{\mathbf{n}})(z_{\mathbf{m}})$,

2. $T_{\mathbf{n}+\mathbf{m}}(x_{\mathbf{n}} \otimes z_{\mathbf{m}}) = T_{\mathbf{n}}(x_{\mathbf{n}}) T_{\mathbf{m}}(z_{\mathbf{m}})$.

□

4.2.1.8 Remark. For the rest of this section, we require the range of the representation of a product system to be bounded linear operators on some Hilbert space rather than an abstract C^*-algebra, since we are about to employ the evaluation maps

$$\mathcal{B}(\mathcal{H}) \to \mathcal{H}, \; T \mapsto T(\xi)$$

for $\xi \in \mathcal{H}$. However, since we can represent any C^*-algebra on its GNS Hilbert space faithfully this does not pose a problem.

4.2.2 The abstract and the concrete gauge action

In this setting, we take an approach towards defining the gauge-action on the abstract Toeplitz algebra which differs a little from the one in the previous chapter because it makes things easier. Instead of defining the action we require on the abstract Toeplitz algebra right away, we first define an action of the complex k-torus on the product system over $\mathbb{N}^k$. The idea behind this is to multiply each base fibre by a complex phase. Note that this makes sense, since every Hilbert module carries a complex vector space structure.

4.2.2.1 Remark. Given a product system $\mathbf{X}$ over $\mathbb{N}^k$, every

$$\mathbf{z} = (z_1, \ldots, z_k) \in \mathbb{T}^k = \Big\{\mathbf{z} \in \mathbb{C}^k : |z_i| = 1 \text{ for all } i = 1, \ldots, k\Big\}$$

in the complex k-torus induces an automorphism of $\mathbf{X}$. By this, we mean a bijective semigroup homomorphism $h\colon \mathbf{X} \to \mathbf{X}$ under which the fibres of the product system and the respective left actions and inner products remain invariant, that is h restricted to each fibre is an automorphism of Hilbert correspondences. For this, we make the following definition: Given an element $\mathbf{z} \in \mathbb{T}^k$, let

$$\lambda_{\mathbf{z}}(x) := \mathbf{z}^{\mathbf{n}} x := z_1^{n_1} \cdot \ldots \cdot z_k^{n_k} x$$

for $x \in \mathbf{X}^{\otimes \mathbf{n}}$. Since this extends to an automorphism of $\mathbf{X}$, we get an action

$$\lambda : \mathbb{T}^k \to \mathrm{Aut}(\mathbf{X}) \tag{4.2.1}$$

of the k-torus on the product system. For any Toeplitz representation (t, σ), the tuple $(t \circ \lambda, \sigma)$ is a Toeplitz representation as well. Therefore, λ induces an action of $\mathbb{T}^k$ on $\mathcal{T}_{\mathbf{X}}$ which we will also denote by λ. In particular, it induces an action of $\mathbb{T}^k$ on the concrete Toeplitz algebra $\widetilde{\mathcal{T}}_{\mathbf{X}}$ by composition of λ with the Fock representation.

4.2.2.2 Remark. The restriction $t_i : X_i \to \mathcal{B}(\mathcal{H})$ of a Toeplitz representation t to a fibre X_i induces a contraction $\tilde{t}_i : X_i \otimes \mathcal{H} \to \mathcal{H}$ by

$$\tilde{t}_i(x \otimes h) := t_i(x)(h)$$

for all $x \in X_i$ and all $h \in \mathcal{H}$.

4.2.2.3 Definition. A Toeplitz representation (σ, t) of $\mathbf{X}$ is called *doubly commuting*, if

$$\tilde{t}_j^* \tilde{t}_i = (\mathrm{id}_{X_j} \otimes \tilde{t}_i)(\chi_{i,j} \otimes \mathrm{id}_{\mathcal{H}})(\mathrm{id}_{X_i} \otimes \tilde{t}_j^*).$$

In other words, the following diagram commutes:

$$\begin{tikzcd}
& \mathcal{H} \arrow[dr, "\tilde{t}_j^*"] & \\
X_i \otimes \mathcal{H} \arrow[ur, "\tilde{t}_i"] \arrow[d, "\mathrm{id}_{X_i} \otimes \tilde{t}_j^*"'] & & X_j \otimes \mathcal{H} \\
Xi \otimes X_j \otimes \mathcal{H} \arrow[rr, "\chi_{i,j} \otimes \mathrm{id}_{\mathcal{H}}"] & & X_j \otimes X_i \otimes \mathcal{H} \arrow[u, "\mathrm{id}_{X_j} \otimes \tilde{t}_i"']
\end{tikzcd}$$

We will call the universal C^*-algebra generated by all doubly commuting Toeplitz representations of $\mathbf{X}$ the *covariant Toeplitz algebra* of $\mathbf{X}$ and denote it by $\mathcal{T}_{\mathbf{X}}^{\mathrm{cov}}$.

4.2.2.4 Remark. By [Sol08, lemma 3.11], a Toeplitz representation is doubly commuting if and only if

$$\tilde{t}_{\mathbf{n}} \tilde{t}_{\mathbf{n}}^* \tilde{t}_{\mathbf{m}} \tilde{t}_{\mathbf{m}}^* = \tilde{t}_{\mathbf{n} \vee \mathbf{m}} \tilde{t}_{\mathbf{n} \vee \mathbf{m}}^*$$

for all $\mathbf{n}, \mathbf{m} \in \mathbb{N}^k$, where $\mathbf{n} \vee \mathbf{m}$ denotes the component-wise maximum of $\mathbf{n}$ and $\mathbf{m}$. We write $i \vee j := \mathbf{e_i} \vee \mathbf{e_j}$ Note that this means that doubly commuting Toeplitz representations in the sense of Solel are Nica-covariant representations in the sense of [Fow02, definition 5.1] for the quasi-lattice ordered semigroup $(\mathbb{Z}^k, \mathbb{N}^k)$, as Solel stated in this lemma. Moreover, for $a \in \mathcal{K}(X_i)$ and $b \in \mathcal{K}(X_j)$ it holds that $(a \otimes \mathrm{id}_{X_j})(\mathrm{id}_{X_i} \otimes b) \in \mathcal{K}(X_{i \vee j})$, so any product system over $\mathbb{N}^k$ is automatically compactly aligned in the sense of [Fow02, definition 5.7]. Of course, all the left actions are by compact operators since we assumed all base fibres to be finitely generated, so one can alternatively use [Fow02, proposition 5.8] to see that all the product systems we are considering in this chapter are compactly aligned. Therefore, the covariant Toeplitz algebra $\mathcal{T}_{\mathbf{X}}^{\mathrm{cov}}$ from the previous definition coincides with Fowler's covariant Toeplitz algebra.

Because of the equivalent definition of doubly commuting discussed above, two things become apparent: Firstly, the Fock representation is doubly commuting since for $x^{(i)} \in X_i$ and $y^{(j)} \in X_j$ an annihilation operator $T_{y^{(j)}}^*$ does not interfere with a creation operator $T_{x^{(i)}}$ when $i \neq j$. Therefore, $\widetilde{\mathcal{T}}_{\mathbf{X}}$ can be considered as a quotient of $\mathcal{T}_{\mathbf{X}}^{\mathrm{cov}}$. Secondly, since there is a one-to one map between doubly commuting Toeplitz representations of Hilbert correspondences (σ, t) and their associated contractions $\tilde{t}$ by [Sol08, lemma 2.4] it is immediate that every finite product of generators $t(x^{(i)})$ and their adjoints $t(y^{(j)})^*$ in $\mathcal{T}_{\mathbf{X}}^{\mathrm{cov}}$ has a canonical form analogous to 3.2.2.1. For $x^{(i)} \in X_i$, $y^{(j)} \in X_j$ we will call $t(x^{(i)})$ *(abstract) creation operator* and $t(y^{(j)})^*$ *(abstract) annihilation operator*. We now provide the explicit calculation of this shape for the Fock representation to get an idea how the Nica-covariance condition works.

4.2.2.5 Lemma. *If all X_i, $1 \leq i \leq k$, are finitely generated, then every finite product of creation operators and their adjoints inside $\widetilde{\mathcal{T}}_{\mathbf{X}}$ is a linear combination of products of the form*

$$T(x_1^{(1)}) \dots T(x_{l_1}^{(1)}) T(x_1^{(2)}) \dots T(x_{l_k}^{(k)}) T(y_1^{(1)})^* \dots T(y_{m_1}^{(1)})^* T(y_1^{(2)})^* \dots T(y_{m_k}^{(k)})^*,$$

where $x_j^{(i)}, y_j^{(i)} \in X_i$ and $l_i, m_i \in \mathbb{N}_0$ for $1 \leq i \leq k$.

Proof. Since terms of the form $T(x^{(i)})^* T(y^{(i)}) = \varphi(\langle x^{(i)}, y^{(i)} \rangle)$ are $\mathcal{A}$-scalar, our only concern are mixed terms, that is terms of the form $T(x^{(i)})^* T(y^{(j)})$, where $1 \leq i,\ j \leq k$, $i \neq j$, $x^{(i)} \in X_i$, and $y^{(j)} \in X_j$.

Let $\{x_k^{(i)}\}_{k=1}^N$ be a normalised tight frame of X_i and $\{y_l^{(j)}\}_{l=1}^M$ be a normalised tight frame of X_j. Then

$$\sum_{k=1}^N T(x_k^{(i)}) T(x_k^{(i)})^*$$

is the orthogonal projection onto the submodule of $\Gamma(\mathbf{X})$ spanned by all elements with non-trivial X_i component. In particular,

$$\left(\sum_{k=1}^N T(x_k^{(i)}) T(x_k^{(i)})^* \right) T(x^{(i)}) = T(x^{(i)}).$$

Furthermore, $T(y^{(j)})$ with $i \neq j$ commutes with this projection, so we know that

$$\begin{aligned}
T(x^{(i)})^*T(y^{(j)}) &= T(x^{(i)})^* \left(\sum_{k=1}^{N} T(x_k^{(i)})T(x_k^{(i)})^* \right) T(y^{(j)}) \\
&= T(x^{(i)})^*T(y^{(j)}) \left(\sum_{k=1}^{N} T(x_k^{(i)})T(x_k^{(i)})^* \right) \\
&= T(x^{(i)})^* \sum_{k=1}^{N} T(y^{(j)})T(x_k^{(i)})T(x_k^{(i)})^*
\end{aligned}$$

$$\begin{aligned}
&\cong T(x^{(i)})^* \sum_{k=1}^{N} T(\chi_{j,i}(y^{(j)} \otimes x_k^{(i)}))T^*_{x_k^{(i)}} \\
&= T(x^{(i)})^* \sum_{k=1}^{N} \sum_{l=1}^{M} T(x_k^{(i)})T(\varphi_j(a_k)y_l^{(j)}b_l)T^*_{x_k^{(i)}} \\
&= \sum_{k=1}^{N} \sum_{l=1}^{M} \underbrace{T(x^{(i)})^*T(x_k^{(i)})}_{=\varphi\left(\langle x^{(i)}, x_k^{(i)}\rangle\right)} T(\varphi_j(a_k)y_l^{(j)}b_l)T^*_{x_k^{(i)}} \\
&= \sum_{k=1}^{N} \sum_{l=1}^{M} T\left(\varphi_j\left(\langle x^{(i)}, x_k^{(i)}\rangle a_k\right) y_l^{(j)} b_l\right) T^*_{x_k^{(i)}}
\end{aligned}$$

where

$$\chi_{j,i}(y^{(j)} \otimes x_k^{(i)}) = \sum_{k=1}^{N} \sum_{l=1}^{M} x_k^{(i)} a_k \otimes y_l^{(j)} b_l = \sum_{k=1}^{N} \sum_{l=1}^{M} x_k^{(i)} \otimes \varphi_j(a_k) y_l^{(j)} b_l$$

for some $a_k, b_l \in \mathcal{A}$. Hence $T(x^{(i)})^*T(y^{(j)})$ has the desired form. $\square$

We now check that the composition of $\lambda : \mathbb{T}^k \to \mathbf{X}$ and a doubly commuting Toeplitz representation is a doubly commuting Toeplitz representation.

4.2.2.6 Lemma. *Given a doubly commuting Toeplitz representation $\{\sigma, t\}$ of $\mathbf{X}$, it holds that $\{\sigma, t \circ \lambda_{\mathbf{z}}\}$ is a doubly commuting representation of $\mathbf{X}$ as well.*

Proof. It holds that

$$\begin{aligned}
(\widetilde{t^{(j)} \circ \lambda_{\mathbf{z}}})^*(\widetilde{t^{(i)} \circ \lambda_{\mathbf{z}}}) &= \overline{\mathbf{z}}\mathbf{z}\tilde{t}^{(j)*}\tilde{t}^{(i)} \\
&= \overline{\mathbf{z}}\mathbf{z}(\mathrm{id}_{X_j} \otimes \tilde{t}^{(i)})(\chi_{i,j} \otimes \mathrm{id}_{\mathcal{H}})(\mathrm{id}_{X_i} \otimes \tilde{t}^{(j)*}) \\
&= (\mathrm{id}_{X_j} \otimes \widetilde{t^{(i)} \circ \lambda_{\mathbf{z}}})(\chi_{i,j} \otimes \mathrm{id}_{\mathcal{H}})(\mathrm{id}_{X_i} \otimes (\widetilde{t^{(j)} \circ \lambda_{\mathbf{z}}})^*).
\end{aligned}$$

□

4.2.2.7 Remark (Gauge action on $\mathcal{T}_{\mathbf{X}}^{\mathrm{cov}}$). By the previous lemma, λ as in equation 4.2.1 induces an action of $\mathbb{T}^k$ on $\mathcal{T}_{\mathbf{X}}^{\mathrm{cov}}$ which we call the *abstract gauge action* and denote by λ as well. It yields a conditional expectation ψ from $\mathcal{T}_{\mathbf{X}}^{\mathrm{cov}}$ onto the fixed point algebra $\mathcal{F}$ of λ which is spanned by products of degree zero of abstract creation and annihilation operators in $\mathcal{T}_{\mathbf{X}}^{\mathrm{cov}}$ by

$$\psi(t) := \int_{\mathbf{z} \in \mathbb{T}^k} \lambda_{\mathbf{z}}(t) d\mu(\mathbf{z}),$$

where μ is the normalised Haar measure on $\mathbb{T}^k$.

4.2.2.8 Remark (Gauge action on $\widetilde{\mathcal{T}}_{\mathbf{X}}$). Since $\widetilde{\mathcal{T}}_{\mathbf{X}}$ is a λ-invariant quotient of $\mathcal{T}_{\mathbf{X}}^{\mathrm{cov}}$, λ induces an action of $\mathbb{T}^k$ on $\widetilde{\mathcal{T}}_{\mathbf{X}}$ as well. It can even be unitarily implemented: For $(z_1, \ldots, z_k) = \mathbf{z} \in \mathbb{T}^k$, define the unitary operator $U_{\mathbf{z}} \in \mathcal{L}(\Gamma_{\mathbb{N}^k}\mathbf{X})$ by

$$U_{\mathbf{z}}(x) = \mathbf{z}^{\mathbf{n}} x$$

for every $x \in \mathbf{X}^{\otimes \mathbf{n}}$. Then

$$\tilde{\lambda}_{\mathbf{z}}(t) := U_{\mathbf{z}}^* t U_{\mathbf{z}}$$

defines an action of $\mathbb{T}^k$ on $\widetilde{\mathcal{T}}_{\mathbf{X}}$ which we will call it the *concrete gauge action.* It yields a second faithful conditional expectation $\tilde{\psi}$ by

$$\tilde{\psi}(t) := \int_{\mathbf{z} \in \mathbb{T}^k} \tilde{\lambda}_{\mathbf{z}}(t) d\mu(\mathbf{z})$$

onto the fixed point algebra $\widetilde{\mathcal{F}}$ of $\tilde{\lambda}$ which is spanned by products of degree zero in $\widetilde{\mathcal{T}}_{\mathbf{X}}$.

4.2.3 The theorem

We use the rest of this section to prove the gauge-invariant uniqueness theorem for the covariant Toeplitz algebra of a discrete product system. Despite of the fact that this theorem is a special case of [Fow02, theorem 6.3] it deserves separate consideration, since we can apply Pimsner's methods to our setting, which is not possible in greater generality.

4.2.3.1 Proposition (gauge-invariant uniqueness theorem)**.** *For a product system* $\mathbf{X}$ *over* $\mathbb{N}^k$ *with finitely generated base fibres* $X_1, \ldots, X_k$ *it holds that*

$$\mathcal{T}_{\mathbf{X}}^{\mathrm{cov}} \cong \widetilde{\mathcal{T}}_{\mathbf{X}}.$$

As mentioned above, since the Fock representation is a doubly commuting Toeplitz representation of $\mathbf{X}$, we know that $\widetilde{\mathcal{T}}_{\mathbf{X}}$ is a quotient of $\mathcal{T}_{\mathbf{X}}^{\mathrm{cov}}$. Together with the faithful conditional expectations ψ and $\widetilde{\psi}$ constructed from the abstract and the concrete gauge action in remarks 4.2.2.8 and 4.2.2.7 the following diagram commutes.

$$\begin{tikzcd}
\mathcal{T}_{\mathbf{X}}^{\mathrm{cov}} \arrow[r, "f"] \arrow[d, "\psi"'] & \widetilde{\mathcal{T}}_{\mathbf{X}} \arrow[d, "\widetilde{\psi}"] \\
\mathcal{F} \arrow[r, "g"] & \widetilde{\mathcal{F}}
\end{tikzcd}$$

Since ψ and $\widetilde{\psi}$ are faithful, showing that $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ are isomorphic implies that $g \circ \psi$ is faithful. Since the diagram commutes, $\widetilde{\psi} \circ f$ must be faithful as well, which implies that f is injective. But since $\widetilde{\mathcal{T}}_{\mathbf{X}}$ is a quotient of $\mathcal{T}_{\mathbf{X}}^{\mathrm{cov}}$, the two algebras will then be isomorphic. It therefore remains to be seen that the fixed point algebras $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ of the abstract and the concrete gauge action are isomorphic.

From 4.2.2.4 and 4.2.2.5 we know that a product of (abstract or concrete) creation operators and annihilation operators can be reduced to a linear combination of products in which all the creation operators go to the left and all the annihilation operators go to the right. Such a product is an element of the fixed point algebra of the respective gauge action if and only if for every $i = 1, \ldots, k$ there are as many creation operators by an element in X_i as there are annihilation operators. If this is the case, we call the product balanced. Note that compared to the previous chapter we no longer think of the length of such a product as a single integer but as a k-tuple of integers. So the building blocks of both $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ are balanced products of $\mathbf{n}$ creation and equally many annihilation operators for all $\mathbf{n} \in \mathbb{N}^k$ inside $\mathcal{T}_{\mathbf{X}}^{\mathrm{cov}}$ and $\widetilde{\mathcal{T}}_{\mathbf{X}}$ respectively. From 3.2.3.2 we conclude that for any fixed $\mathbf{n} = (n_1, \dots, n_k) \in \mathbb{N}^k$, the closure of the span of products of the form

$$t(x_1^{(1)}) \dots t(x_{n_1}^{(1)}) \dots t(x_1^{(k)}) \dots t(x_{n_k}^{(k)}) t(y_1^{(1)})^* \dots t(y_{n_1}^{(1)})^* \dots t(y_1^{(k)})^* \dots t(y_{n_k}^{(k)})^*$$

is isomorphic to $\mathcal{K}(\mathbf{X}^{\otimes \mathbf{n}})$ inside $\mathcal{F}$ and the closure of the span of products of the form

$$T(x_1^{(1)}) \dots T(x_{n_1}^{(1)}) \dots T(x_1^{(k)}) \dots T(x_{n_k}^{(k)}) T(y_1^{(1)})^* \dots T(y_{n_1}^{(1)})^* \dots T(y_1^{(k)})^* \dots T(y_{n_k}^{(k)})^*$$

is isomorphic to $\mathcal{K}(\mathbf{X}^{\otimes \mathbf{n}})$ inside $\widetilde{\mathcal{F}}$. All that is left to check if the way those building blocks are put together in the respective algebras is compatible. Given any $\mathbf{n}, \mathbf{m} \in \mathbb{N}^k$ they are either comparable or they are not. If $\mathbf{m}$ and $\mathbf{n}$ are comparable, one of them majorises the other one, say $\mathbf{n} \leq \mathbf{m}$. In this case, the product of two elements $a \in \mathcal{K}(\mathbf{X}^{\otimes \mathbf{n}})$ and $b \in \mathcal{K}(\mathbf{X}^{\otimes \mathbf{m}})$ makes sense with

$$\begin{aligned} ab &:= (a \otimes \mathrm{id}) b \\ ba &:= b(a \otimes \mathrm{id}). \end{aligned}$$

because of the embedding

$$\mathcal{K}(\mathbf{X}^{\otimes \mathbf{n}}) \ni a \mapsto a \otimes \mathrm{id} \in \mathcal{M}\big(\mathcal{K}(\mathbf{X}^{\otimes \mathbf{n}+\mathbf{p}})\big) \cong \mathcal{L}(\mathbf{X}^{\otimes \mathbf{n}+\mathbf{p}})$$

for any $\mathbf{n}, \mathbf{p} \in \mathbb{N}^k$, where $\mathbf{n} + \mathbf{p} = \mathbf{m}$. In fact, if $\mathbf{n}$ and $\mathbf{m}$ are comparable, by 3.2.3.3 we know that the ways that we are required to put $\mathcal{K}(\mathbf{X}^{\otimes \mathbf{n}})$ and $\mathcal{K}(\mathbf{X}^{\otimes \mathbf{m}})$ together inside $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ respectively are compatible, so we are left to show they are compatible if $\mathbf{n}$ and $\mathbf{m}$ are not comparable.

For $\mathbf{n}$ and $\mathbf{m}$ not comparable we define $\mathbf{n}\vee\mathbf{m} := \max(n_i, m_i)_{i=1}^k$ as in 4.2.2.4. By using the isomorphisms $\chi_{i,j}$ appropriately, we get that

$$\mathbf{X}^{\otimes\mathbf{n}} \otimes \mathbf{X}^{\otimes\mathbf{p}_1} \cong \mathbf{X}^{\otimes\mathbf{n}\vee\mathbf{m}} \cong \mathbf{X}^{\otimes\mathbf{m}} \otimes \mathbf{X}^{\otimes\mathbf{p}_2}$$

with $\mathbf{p}_1 = \mathbf{n}\vee\mathbf{m} - \mathbf{n}$ and $\mathbf{p}_2 = \mathbf{n}\vee\mathbf{m} - \mathbf{m}$. Denoting the former isomorphism by χ_{p_1} and the latter one by χ_{p_2}, we can embed both $\mathcal{K}(\mathbf{X}^{\otimes\mathbf{n}})$ and $\mathcal{K}(\mathbf{X}^{\otimes\mathbf{m}})$ into $\mathcal{M}\big(\mathcal{K}(\mathbf{X}^{\otimes\mathbf{n}\vee\mathbf{m}})\big)$ by

$$a \mapsto \chi_{p_1}(a\otimes \mathrm{id})\chi_{p_1}^{-1}$$

and

$$b \mapsto \chi_{p_2}(b\otimes \mathrm{id})\chi_{p_2}^{-1},$$

respectively. For example, if $k = 2$, $\mathbf{n} = (1,0)$ and $\mathbf{m} = (0,1)$,

$$\mathcal{K}(\mathbf{X}^{\otimes(1,0)}) \ni a \mapsto a\otimes \mathrm{id} \in \mathcal{M}\big(\mathcal{K}(\mathbf{X}^{\otimes(1,1)})\big)$$

and

$$\mathcal{K}(\mathbf{X}^{\otimes(0,1)}) \ni b \mapsto \chi_{2,1}(b\otimes \mathrm{id})\chi_{1,2} \in \mathcal{M}\big(\mathcal{K}(\mathbf{X}^{\otimes(1,1)})\big).$$

4.2.3.2 Lemma. *Given C^*-algebras $\mathcal{A},\mathcal{B},\mathcal{D}$ and *-homomorphisms $\pi_1 : \mathcal{A}\to\mathcal{M}(\mathcal{D})$ and $\pi_2 : \mathcal{B}\to\mathcal{M}(\mathcal{D})$, the universal C^*-algebra $\mathcal{C}_1$ generated by $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{D}$ subject to relations*

$$\begin{aligned} ab &= ba = 0 \\ ad &= \pi_1(a)d \\ da &= d\pi_1(a) \\ bd &= \pi_2(b)d \\ db &= d\pi_2(b) \end{aligned}$$

is isomorphic to the direct sum $\mathcal{C}_2 := \mathcal{A}\oplus\mathcal{B}\oplus\mathcal{D}$ with

$$(a_1, b_1, d_1)(a_2, b_2, d_2) := (a_1a_2, b_1b_2, \pi_1(a_1)d_2 + \pi_2(b_1)d_2 + d_1\pi_1(a_2) + d_1\pi_2(b_2) + d_1d_2).$$

Proof. For $(a,0,0),(0,b,0),(0,0,d) \in \mathcal{C}_2$ the equalities below hold.

$$
\begin{aligned}
(a,0,0)(0,b,0) &= (0,0,0), & (0,b,0)(a,0,0) &= (0,0,0)\\
(a,0,0)(0,0,c) &= (0,0,\pi_1(a)c), & (0,0,c)(a,0,0) &= (0,0,c\pi_1(a))\\
(0,b,0)(0,0,c) &= (0,0,\pi_2(b)c), & (0,0,c)(0,b,0) &= (0,0,c\pi_2(b))
\end{aligned}
$$

Since $\mathcal{C}_1$ is the universal C^*-algebra with these relations, $\mathcal{C}_2$ is a quotient of $\mathcal{C}_1$. Conversely,

$$
\begin{aligned}
(a+b+c)(a'+b'+c') &= aa' + ab' + ac' + ba' + bb' + bc' + ca' + cb' + cc'\\
&= aa' + bb' + \pi_1(a)c' + \pi_2(b)c' + c\pi_1(a') + c\pi_2(b') + cc',
\end{aligned}
$$

for $(a+b+c),(a'+b'+c') \in \mathcal{C}_1$, so $\mathcal{C}_1$ is a quotient of $\mathcal{C}_2$. Hence, $\mathcal{C}_1$ and $\mathcal{C}_2$ are isomorphic. □

We are now ready to see that the respective fixed point algebras of the abstract and the concrete gauge action are isomorphic: Take any two k-tuples $\mathbf{m},\mathbf{n} \in \mathbb{N}^k$. If $\mathbf{m}$ and $\mathbf{n}$ are comparable, we use lemma 3.2.3.3 to see that $C^*\{\mathcal{K}(\mathbf{X}^{\otimes\mathbf{m}}),\mathcal{K}(\mathbf{X}^{\otimes\mathbf{n}})\}$ and $\mathcal{K}(\mathbf{X}^{\otimes\mathbf{m}})\oplus\mathcal{K}(\mathbf{X}^{\otimes\mathbf{n}})$ are isomorphic. If $\mathbf{m}$ and $\mathbf{n}$ are not comparable, $C^*\{\mathcal{K}(\mathbf{X}^{\otimes\mathbf{m}}),\mathcal{K}(\mathbf{X}^{\otimes\mathbf{n}}),\mathcal{K}(\mathbf{X}^{\otimes\mathbf{m}\vee\mathbf{n}})\}$ and $\mathcal{K}(\mathbf{X}^{\otimes\mathbf{m}})\oplus\mathcal{K}(\mathbf{X}^{\otimes\mathbf{n}})\oplus\mathcal{K}(\mathbf{X}^{\otimes\mathbf{m}\vee\mathbf{n}})$ are isomorphic by the previous lemma. When adding the next building block $\mathcal{K}(\mathbf{X}^{\otimes\mathbf{p}})$ we either apply 3.2.3.3 if $\mathbf{p}$ and $\mathbf{m}\vee\mathbf{n}$ are comparable or the previous lemma if they are not in order to see that the resulting C^*-algebras are isomorphic again. Iterating this procedure finishes the proof of 4.2.3.1, since $\mathcal{F} = C^*\{\mathcal{K}(\mathbf{X}^{\otimes\mathbf{n}}) : \mathbf{n} \in \mathbb{N}^k\}$ and $\widetilde{\mathcal{F}} = \bigoplus_{\mathbf{n}\in\mathbb{N}^k}\mathcal{K}(\mathbf{X}^{\otimes\mathbf{n}})$.

4.3 A semisplit Toeplitz extension of $\mathcal{O}_{\mathbf{X}}$

4.3.1 Extending the scalars

We now construct a semisplit Toeplitz extension of a slightly modified higher rank Cuntz-Pimsner algebra. For this, we start by modifying the coefficient algebra of the product system $\mathbf{X}$.

4.3.1.1 Definition. Inside the direct limit $\varinjlim\{\mathcal{L}(\mathbf{X}^{\otimes\mathbf{n}}) : \mathbf{n} \in \mathbb{N}^k\}$ with respect to the embeddings

$$\begin{array}{rcl} \mathcal{L}(\mathbf{X}^{\otimes \mathbf{n}}) & \to & \mathcal{L}(\mathbf{X}^{\otimes (\mathbf{n}+\mathbf{m})}) \\ T & \mapsto & T \otimes \mathrm{id}_{\mathbf{X}^{\otimes \mathbf{m}}} \end{array}$$

we define

$$\mathcal{F}_{\mathbf{X}} := C^*\{\mathcal{K}(\mathbf{X}^{\otimes \mathbf{n}}) : \mathbf{n} \in \mathbb{N}^k\}$$

with the convention that $\mathcal{K}(\mathbf{X}^{\otimes 0}) := \mathcal{A}$. For every $\mathbf{m} \in \mathbb{N}^k$, we set

$$\begin{aligned} \mathcal{F}_{\mathbf{X}}^{\mathbf{m}} &:= C^*\{\mathcal{K}(\mathbf{X}^{\otimes \mathbf{n}}) : \mathbf{m} \leq \mathbf{n}\} \ , \\ \mathbf{X}_\infty &:= (X_1 \otimes \mathcal{F}_{\mathbf{X}}, \ldots, X_k \otimes \mathcal{F}_{\mathbf{X}}) \ . \end{aligned}$$

In order to see that $\mathbf{X}_\infty$ is a product system over $\mathcal{F}_{\mathbf{X}}$, we require left actions of $\mathcal{F}_{\mathbf{X}}$ on $X_i \otimes \mathcal{F}_{\mathbf{X}}$ and isomorphisms

$$\chi_{i,j} : (X_i \otimes \mathcal{F}_{\mathbf{X}}) \otimes (X_j \otimes \mathcal{F}_{\mathbf{X}}) \to (X_j \otimes \mathcal{F}_{\mathbf{X}}) \otimes (X_i \otimes \mathcal{F}_{\mathbf{X}}).$$

Not only $X_i \otimes \mathcal{F}_{\mathbf{X}}$ but all mixed powers are $\mathcal{F}_{\mathbf{X}}$-bimodules as we shall see in 4.3.2.1. This is why we do not investigate the left action on the $X_i \otimes \mathcal{F}_{\mathbf{X}}$. The isomorphisms are given in 4.3.2.6.

4.3.2 Tools

4.3.2.1 Lemma. *With* $\mathbf{e_i} = (\delta_{i,j})_{j=1}^k \in \mathbb{N}^k$ *we get that*

1. $\mathcal{K}(\mathbf{X}_\infty^{\otimes \mathbf{e_i}}) = \mathcal{F}_{\mathbf{X}}^{\mathbf{e_i}}$
2. $\mathbf{X}_\infty^{\otimes \mathbf{n}} = \mathbf{X}^{\otimes \mathbf{n}} \otimes \mathcal{F}_{\mathbf{X}}$
3. $\mathcal{K}(\mathbf{X}_\infty^{\otimes \mathbf{n}}) = \mathcal{F}_{\mathbf{X}}^{\mathbf{n}}$
4. *In particular,* $\Gamma_{\mathbb{N}^k}(\mathbf{X}_\infty) = \Gamma_{\mathbb{N}^k}(\mathbf{X}) \otimes \mathcal{F}_{\mathbf{X}}$.

Proof. We will proof the claims of this lemma step by step.

1. : By 3.3.1.5, we know that $\mathcal{K}\big(\mathbf{X}^{\otimes \mathbf{e_i}} \otimes \mathcal{K}(\mathbf{X}^{\otimes \mathbf{n}})\big) \cong \mathcal{K}(\mathbf{X}^{\otimes(\mathbf{e_i}+\mathbf{n})})$. The fact that $\mathcal{K}(\mathbf{X}_\infty^{\otimes \mathbf{e_i}}) \cong \mathcal{K}(\mathbf{X}_{\mathbf{e_i}} \otimes \mathcal{F}_{\mathbf{X}}) \cong \lim \mathcal{K}\big(\mathbf{X}^{\otimes \mathbf{e_i}} \otimes \mathcal{K}(\mathbf{X}^{\otimes \mathbf{n}})\big)$ implies the claim.

2. : Since $\mathcal{K}(X_i) \cong \mathcal{K}(\mathbf{X}^{\otimes \mathbf{e_i}}) \subseteq \mathcal{F}_{\mathbf{X}}$ for all $1 \leq i \leq k$, we get that

$$\begin{aligned} \mathbf{X}_\infty^{\otimes \mathbf{n}} &= \underbrace{X_1 \otimes \mathcal{F}_{\mathbf{X}} \otimes \cdots \otimes X_1 \otimes \mathcal{F}_{\mathbf{X}}}_{n_1 \text{ times}} \otimes \cdots \otimes \underbrace{X_k \otimes \mathcal{F}_{\mathbf{X}} \otimes \cdots \otimes X_k \otimes \mathcal{F}_{\mathbf{X}}}_{n_k \text{ times}} \\ &\cong X_1^{\otimes n_1} \otimes \cdots \otimes X_k^{\otimes n_k} \otimes \mathcal{F}_{\mathbf{X}} = \mathbf{X}^{\otimes \mathbf{n}} \otimes \mathcal{F}_{\mathbf{X}}, \end{aligned}$$

where 2.2.3.7 yields the penultimate isomorphism.

3. : By 2., we get $\mathcal{K}(\mathbf{X}_\infty^{\otimes \mathbf{n}}) \cong \mathcal{K}(\mathbf{X}_\infty^{\otimes \mathbf{n}} \otimes \mathcal{F}_{\mathbf{X}})$. Now, apply the same argument as in 1.

4. : Apply 2. to $\mathbf{X}^{\otimes \mathbf{n}}$ for all $\mathbf{n} \in \mathbb{N}^k$.

□

4.3.2.2 Remark. By 4.3.2.1 1., we get that for all $1 \leq i \leq k$, $X_i \otimes \mathcal{F}_{\mathbf{X}}$ is an $\mathcal{F}_{\mathbf{X}}$-bimodule with left $\mathcal{F}_{\mathbf{X}}$-linear inner product

$${}_{\mathcal{F}_{\mathbf{X}}}\langle \xi_i \otimes a, \eta_i \otimes b \rangle = \theta_{\xi_i \otimes a, \eta_i \otimes b},$$

where $\xi_i, \eta_i \in X_i$ and $a, b \in \mathcal{F}_{\mathbf{X}}$. The same holds for every generalised power of $\mathbf{X}_\infty$, so the opposite module of $\mathbf{X}_\infty^{\otimes \mathbf{n}}$ is again an $\mathcal{F}_{\mathbf{X}}$-bimodule for every $\mathbf{n} \in \mathbb{N}^k \setminus \{0\}$.

4.3.2.3 Definition. For all $\mathbf{n} \in \mathbb{N}^k \setminus \{0\}$ we set

$$\mathbf{X}_\infty^{\otimes -\mathbf{n}} := (\mathbf{X}_\infty^{\otimes \mathbf{n}})^*.$$

So we can define a two-sided Fock module by

$$\Gamma_{\mathbb{Z}^k}(\mathbf{X}_\infty) := \bigoplus_{\mathbf{n} \in \mathbb{Z}^k} \mathbf{X}_\infty^{\otimes \mathbf{n}}.$$

Furthermore, we set

$$\Gamma_{\mathbb{Z}^k}^{\mathbf{n}}(\mathbf{X}_\infty) := \Gamma_{\mathbb{Z}^k}(\mathbf{X}_\infty) \otimes \mathcal{F}_{\mathbf{X}}^{\mathbf{n}}.$$

4.3.2.4 Lemma. *It holds that*

1. $\Gamma_{\mathbb{Z}^k}(\mathbf{X}_\infty) \cong \Gamma_{\mathbb{Z}^k}(\mathbf{X}_\infty) \otimes \mathbf{X}_\infty^{\otimes \mathbf{e_i}}$
2. $\Gamma_{\mathbb{Z}^k}^{\mathbf{e_i}}(\mathbf{X}_\infty) \otimes \mathbf{X}_\infty^{\otimes e_i} \cong \Gamma_{\mathbb{Z}^k}(\mathbf{X}_\infty)$
3. $\Gamma_{\mathbb{Z}^k}(\mathbf{X}_\infty) \otimes \mathbf{X}_\infty^{\otimes -\mathbf{e_i}} \cong \Gamma_{\mathbb{Z}^k}^{\mathbf{e_i}}(\mathbf{X}_\infty)$

Proof. We show these claims one by one as well.

1. Considering the generalised powers of $\mathbf{X}_\infty$ appearing in the Fock module individually, we see that

$$\mathbf{X}_\infty^{\otimes \mathbf{n}} \otimes \mathbf{X}_\infty^{\otimes \mathbf{e_i}} = \begin{cases} \mathbf{X}_\infty^{\otimes(\mathbf{n}+\mathbf{e_i})} & \text{for } \mathbf{n} \neq -\mathbf{e_i}, \\ \overline{\operatorname{span}}\{\langle \xi, \eta \rangle : \xi, \eta \in \mathbf{X}_\infty^{\otimes \mathbf{e_i}}\} = \mathcal{F}_{\mathbf{X}} & \text{for } \mathbf{n} = -\mathbf{e_i} \end{cases}$$

2. By 4.3.2.1 we know that $\mathcal{F}_{\mathbf{X}}^{\mathbf{e_i}} \cong \mathcal{K}(\mathbf{X}_\infty^{\otimes \mathbf{e_i}})$, so by 2.2.3.7, $\mathcal{F}_{\mathbf{X}}^{\mathbf{e_i}} \otimes \mathbf{X}_\infty^{\otimes \mathbf{e_i}} \cong \mathbf{X}_\infty^{\otimes \mathbf{e_i}}$. The first assertion of the lemma now implies the claim.

3. It holds that

$$\begin{aligned} \Gamma_{\mathbb{Z}^k}(\mathbf{X}_\infty) \otimes \mathbf{X}_\infty^{\otimes -\mathbf{e_i}} &\overset{1.}{\cong} \Gamma_{\mathbb{Z}^k}(\mathbf{X}_\infty) \otimes \mathbf{X}_\infty^{\otimes \mathbf{e_i}} \otimes (\mathbf{X}_\infty^{\otimes \mathbf{e_i}})^* \cong \Gamma_{\mathbb{Z}^k}(\mathbf{X}_\infty) \otimes \mathcal{K}(\mathbf{X}_\infty^{\otimes \mathbf{e_i}}) \\ &\overset{4.3.2.1}{\cong} \Gamma_{\mathbb{Z}^k}(\mathbf{X}_\infty) \otimes \mathcal{F}_{\mathbf{X}}^{\mathbf{e_i}}. \end{aligned}$$

□

4.3.2.5 Definition. For every $1 \leq i \leq k$, we define *-homomorphisms α_i and β_i by

$$\begin{aligned} \alpha_i : \quad \mathcal{L}\big(\Gamma_{\mathbb{Z}^k}^{\mathbf{e_i}}(\mathbf{X}_\infty)\big) &\to \mathcal{L}\big(\Gamma_{\mathbb{Z}^k}^{\mathbf{e_i}}(\mathbf{X}_\infty)\big) \otimes (X_i \otimes \mathcal{F}_{\mathbf{X}}) \cong \mathcal{L}\big(\Gamma_{\mathbb{Z}^k}(\mathbf{X}_\infty)\big) \\ T &\mapsto T \otimes \mathrm{id} \ , \\ \beta_i : \quad \mathcal{L}\big(\Gamma_{\mathbb{Z}^k}(\mathbf{X}_\infty)\big) &\to \mathcal{L}\big(\Gamma_{\mathbb{Z}^k}(\mathbf{X}_\infty)\big) \otimes (X_i \otimes \mathcal{F}_{\mathbf{X}}) \cong \mathcal{L}\big(\Gamma_{\mathbb{Z}^k}^{\mathbf{e_i}}(\mathbf{X}_\infty)\big) \\ T &\mapsto T \otimes \mathrm{id} \end{aligned}$$

4.3.2.6 Remark. It holds that $\alpha_i \circ \beta_i = \mathrm{id}_{\mathcal{L}\left(\Gamma_{\mathbb{Z}^k}(\mathbf{X}_\infty)\right)}$ and $\beta_i \circ \alpha_i = \mathrm{id}_{\mathcal{L}\left(\Gamma_{\mathbb{Z}^k}^{\mathbf{e_i}}(\mathbf{X}_\infty)\right)}$.

Moreover,

$$\begin{aligned} X_i \otimes \mathcal{F}_{\mathbf{X}} \otimes X_j \otimes \mathcal{F}_{\mathbf{X}} &\overset{2.2.3.7}{\cong} X_i \otimes X_j \otimes \mathcal{F}_{\mathbf{X}} \overset{\chi_{i,j}\otimes \mathrm{id}}{\cong} X_j \otimes X_i \otimes \mathcal{F}_{\mathbf{X}} \\ &\overset{2.2.3.7}{\cong} X_j \otimes \mathcal{F}_{\mathbf{X}} \otimes X_i \otimes \mathcal{F}_{\mathbf{X}}. \end{aligned}$$

In other words, the isomorphisms $\chi_{i,j} : X_i \otimes X_j \to X_j \otimes X_i$ induce isomorphisms

$$(\mathbf{X}_\infty^{\otimes \mathbf{e_i}} \otimes \mathbf{X}_\infty^{\otimes \mathbf{e_j}}) \to (\mathbf{X}_\infty^{\otimes \mathbf{e_j}} \otimes \mathbf{X}_\infty^{\otimes \mathbf{e_i}}).$$

We can hence define

$$\alpha_{\mathbf{n}} := \alpha_1^{n_1} \circ \cdots \circ \alpha_k^{n_k}.$$

In the same manner, we can define $\beta_{\mathbf{n}}$. Then $\alpha_{\mathbf{n}} \circ \beta_{\mathbf{n}}$ and $\beta_{\mathbf{n}} \circ \alpha_{\mathbf{n}}$ are the respective identities as well. We hence get an action of $\mathbb{Z}^k$ via the endomorphisms $\alpha_{\mathbf{n}}$.

We now define two-sided creation operators on $\Gamma_{\mathbb{Z}^k}(\mathbf{X}_\infty)$.

4.3.2.7 Definition. For every $x^{(i)} \in X_i \otimes \mathcal{F}_{\mathbf{X}}$, every $(y^{(i)})^* \in (X_i \otimes \mathcal{F}_{\mathbf{X}})^*$ and every word $\mu \in \mathbf{X}^{\otimes \mathbf{n}}$ we define

$$\begin{aligned} M_{x^{(i)}}(\mu) &:= x^{(i)} \otimes \mu \ , \\ M_{(y^{(i)})^*}(\mu) &:= (y^{(i)})^* \otimes \mu \end{aligned}$$

and extend both of these linearly to operators on the full Fock module $\Gamma_{\mathbb{Z}^k}(\mathbf{X}_\infty)$.

4.3.2.8 Remark. In the second equation, the first X_i-component of μ is moved to the front by appropriate use of the isomorphisms $\chi_{i,j}$. The inner product of $y^{(i)}$ with it then acts on the next component via φ. If μ has trivial X_i-component, $M_{(y^{(i)})^*}(\mu) = 0$.

4.3.2.9 Lemma. *Let $x^{(i)} \in X_i \otimes \mathcal{F}_{\mathbf{X}}$, $(y^{(i)})^* \in (X_i \otimes \mathcal{F}_{\mathbf{X}})^*$.*

1. *The two-sided creation operators satisfy the following properties:*

$$\begin{aligned} M_{x^{(i)}} M_{(y^{(i)})^*} &= x^{(i)} \otimes (y^{(i)})^* \in \mathcal{F}_{\mathbf{X}}^{\mathbf{e_i}} \\ M_{(y^{(i)})^*} M_{x^{(i)}} &= \langle y^{(i)}, x^{(i)} \rangle \in \mathcal{F}_{\mathbf{X}} \\ M^*_{x^{(i)}} &= M_{(x^{(i)})^*} \end{aligned}$$

2. *$M_{x^{(i)}}$ belongs to the fixed point algebra of $(\alpha_{\mathbf{n}})_{\mathbf{n}\in\mathbb{Z}^k}$.*

3. *Let $P_{\mathbf{m}}$ be the orthogonal projection of the two-sided Fock module onto $\bigoplus_{\mathbf{n}\geq\mathbf{m}}\mathbf{X}_\infty^{\otimes\mathbf{n}}$. Then*
$$\alpha_{\mathbf{n}}(P_{\mathbf{m}}) = P_{\mathbf{n}+\mathbf{m}}.$$

Proof. The first two claims are clear by definition. To see the third one, we check that for every $\eta_i\otimes\eta\in X_i\otimes X^{\otimes\mathbf{n}}, x^{(i)}\in X_i$, $\nu\in\mathbf{X}^{\otimes\mathbf{n}}$ the following equation holds:

$$\begin{array}{rclcl}\langle M^*_{x^{(i)}}(\eta_i\otimes\eta),\nu\rangle & = & \langle\eta_i\otimes\eta, M_{x^{(i)}}(\nu)\rangle & = & \langle\eta,\varphi\left(\langle\eta_i,x^{(i)}\rangle\right)\nu\rangle\\ & = & \langle\eta,\varphi\left(\langle x^{(i)},\eta_i\rangle\right)^*\nu\rangle & = & \langle\varphi\left(\langle x^{(i)},\eta_i\rangle\right)\eta,\nu\rangle\\ & = & \langle M_{(x^{(i)})^*}(\eta_i\otimes\eta),\nu\rangle & & \end{array}$$

□

4.3.2.10 Definition. Inside $\mathcal{T}^{\mathrm{cov}}_{\mathbf{X}}\cong\widetilde{\mathcal{T}}_{\mathbf{X}}$, we denote by

$$\mathcal{I}_i := \langle \mathrm{id}_{\Gamma_{\mathbb{N}^k}(\mathbf{X})} - P_{\mathbf{e_i}}\rangle \text{ for all } 1\leq i\leq k$$

the ideal generated by the projection onto $\bigoplus_{\substack{\mathbf{n}\in\mathbb{N}^k\\ n_i\neq 0}}\mathbf{X}^{\otimes\mathbf{n}}$ for all $1\leq i\leq k$ and by

$$\mathcal{I} := \mathcal{I}_1+\cdots+\mathcal{I}_k = \langle\mathrm{id}_{\Gamma_{\mathbb{N}^k}(\mathbf{X})} - P_{\mathbf{e_i}} : 1\leq i\leq k\rangle$$

the sum ideal.

4.3.2.11 Lemma. *Let $\mathbf{X}=(X_1,\ldots,X_k)$ be a product system over $\mathbb{N}^k$. If X_i is finitely generated for all $1\leq i\leq k$ it holds that*

$$\mathcal{T}^{\mathrm{cov}}_{\mathbf{X}}/\mathcal{I}\;\cong\;\mathcal{O}_{\mathbf{X}}.$$

Proof. First we have to check that $\mathcal{I}\subseteq\mathcal{T}^{\mathrm{cov}}_{\mathbf{X}}$ if all the X_i are finitely generated. Let $\{x_j^{(i)}\}_{j=1}^{l_i}$ be a frame of X_i. Then

$$\mathrm{id}-P_{\mathbf{e_i}} = \sum_{j=1}^{l_i} T_i(x_j^{(i)})T_i(x_j^{(i)})^*,$$

hence $\mathcal{T}^{\mathrm{cov}}_{\mathbf{X}}$ contains $(\mathrm{id}-P_{\mathbf{e_i}})$ for all $1\leq i\leq k$ and therefore the ideal they generate.

To see that $\mathcal{O}_{\mathbf{X}}$ contains the quotient $\mathcal{T}_{\mathbf{X}}^{\mathrm{cov}}/\mathcal{I}$, we recall that a Toeplitz representation $\{T_i : X_i \to \mathcal{B}(\mathcal{H})\}_{i=1}^{k}$ of $\mathbf{X}$ is Cuntz-Pimsner covariant if and only if

$$\sum_{j=1}^{l_i} T_i(x_j^{(i)}) T_i(x_j^{(i)})^* = \mathrm{id}_{\mathcal{H}} \tag{4.3.1}$$

for all $1 \leq i \leq k$. Since this relation holds in the quotient and $\mathcal{O}_{\mathbf{X}}$ is universal for Cuntz-Pimsner covariant Toeplitz representations, $\mathcal{T}_{\mathbf{X}}^{\mathrm{cov}}/\mathcal{I}$ is a quotient of $\mathcal{O}_{\mathbf{X}}$.

To see the other inclusion, consider the following diagram.

$$\begin{tikzcd}
\mathcal{T}_{\mathbf{X}}^{\mathrm{cov}}/\mathcal{I} \arrow[d] & \mathcal{T}_{\mathbf{X}}^{\mathrm{cov}} \arrow[l] \arrow[dl, "\pi"] \arrow[d] \\
\mathcal{O}_{\mathbf{X}} \arrow[r] & \mathcal{B}(\mathcal{H})
\end{tikzcd}$$

By a diagram chasing argument, we will now see that the quotient $\mathcal{T}_{\mathbf{X}}^{\mathrm{cov}}/\mathcal{I}$ is universal for Cuntz-Pimsner covariant Toeplitz representations of $\mathbf{X}$.

Since all X_i are finitely generated, $\mathcal{A}$ acts by compact operators on all X_i. Hence every Cuntz-Pimsner covariant Toeplitz representation is Nica-covariant by [Fow02, proposition 5.4] and therefore doubly commuting by [Sol08, lemma 3.11, remark 3.12]. Since $\mathcal{T}_{\mathbf{X}}^{\mathrm{cov}}$ is universal for doubly commuting Toeplitz representations, $\mathcal{O}_{\mathbf{X}}$ is a quotient of $\mathcal{T}_{\mathbf{X}}^{\mathrm{cov}}$. In the diagram, π denotes the quotient homomorphism. By (4.3.1) we know that $\pi(P_{\mathbf{e_i}}) = 0$, hence π factors through $\mathcal{T}_{\mathbf{X}}^{\mathrm{cov}}/\mathcal{I}$. □

4.3.3 A semisplit extension

We are now equipped to prove that the Toeplitz extension of $\mathcal{O}_{\mathbf{X}}$ is semisplit.

4.3.3.1 Proposition. *The following assertions hold:*

1. *: The embedding*

$$\begin{aligned} \mathcal{L}\big(\Gamma_{\mathbb{N}^k}(\mathbf{X})\big) \to \quad & \mathcal{L}\big(\Gamma_{\mathbb{N}^k}(\mathbf{X}) \otimes \mathcal{F}_{\mathbf{X}}\big) \cong \quad \mathcal{L}\big(\Gamma_{N^k}(\mathbf{X}_\infty)\big) \\ T \mapsto \quad & T \otimes \mathrm{id} \end{aligned}$$

   induces an embedding $\mathcal{T}^{\mathrm{cov}}_{\mathbf{X}} \hookrightarrow \mathcal{T}^{\mathrm{cov}}_{\mathbf{X}_\infty}$ and an isomorphism $\mathcal{O}_{\mathbf{X}} \cong \mathcal{O}_{\mathbf{X}_\infty}$.

2. *$\mathcal{O}_{\mathbf{X}}$ is isomorphic to the C^*-algebra generated by the $M_{x^{(i)}} \in \mathcal{L}\big(\Gamma_{\mathbb{Z}}(\mathbf{X})\big)$.*

3. *The compression map $\psi : \mathcal{L}\big(\Gamma_{\mathbb{Z}}(\mathbf{X})\big) \ni T \mapsto P_0TP_0 \in \mathcal{L}\big(\Gamma_{\mathbb{N}}(\mathbf{X})\big)$ defines a completely positive map $\phi : \mathcal{O}_{\mathbf{X}} \to \mathcal{T}^{\mathrm{cov}}_{\mathbf{X}_\infty}$ which is a cross section to the quotient map $\pi : \mathcal{T}^{\mathrm{cov}}_{\mathbf{X}_\infty} \to \mathcal{O}_{\mathbf{X}}$. In other words, the short exact sequence*

$$0 \to \mathcal{I} \to \mathcal{T}^{\mathrm{cov}}_{\mathbf{X}_\infty} \to \mathcal{O}_{\mathbf{X}} \to 0$$

   is semisplit.

Proof. 1. : Let $(e_\lambda)_{\lambda\in\Lambda}$ be an approximate unit for $\mathcal{A}$. Since $\mathcal{A} \subseteq \mathcal{F}_{\mathbf{X}}$ it holds that

$$\mathcal{T}^{\mathrm{cov}}_{\mathbf{X}} \ni T \otimes e_\lambda \,\mathrm{id} = Te_\lambda \otimes \mathrm{id}\,.$$

The latter converges to $T \otimes \mathrm{id}$ in norm.

To see the second assertion, note that

$$\mathcal{O}_{\mathbf{X}} \cong C^*\{S_\xi \otimes \mathrm{id} : \xi \in \mathbf{X}\} \subseteq C^*\{S_{\xi\otimes a} : \xi \in \mathbf{X}, a \in \mathcal{F}_{\mathbf{X}}\} = \mathcal{O}_{\mathbf{X}_\infty}.$$

We now need to see that

$$S_{\xi\otimes a} = S_\xi a. \tag{4.3.2}$$

The latter expression in this equation makes sense when considering $\mathcal{F}_{\mathbf{X}}$ a subset of $\mathcal{O}_{\mathbf{X}}$ by identifying $\theta_{\xi,\eta}$ with $S_\xi S^*_\eta$. Due to this identification it holds that every

$a \in \mathcal{F}_{\mathbf{X}}$ can be approximated by a finite sum,

$$a \approx \sum_i S_{\xi_i} S^*_{\eta_i} \tag{4.3.3}$$

for some ξ_i and η_i in the appropriate generalised powers of $\mathbf{X}$.

Now, showing equation (4.3.2) is equivalent to showing that

$$T_{\xi \otimes a} - (T_\xi \otimes \mathrm{id}) \sum_i (T_{\xi_i} \otimes \mathrm{id})(T^*_{\eta_i} \otimes \mathrm{id}) \in \mathcal{I}. \tag{4.3.4}$$

We will show this equation for every finite approximation of a as in (4.3.3). Let $\sum_i S_{\xi_i} S^*_{\eta_i}$ be such an approximation with $\xi_i, \eta_i \in \mathbf{X}^{\otimes \mathbf{n}_i}$. Then $T_{\xi \otimes \sum_i S_{\xi_i} S^*_{\eta_i}}$ and $\sum_i (T_\xi \otimes \mathrm{id})(T_{\xi_i} \otimes \mathrm{id})(T_{\eta_i} \otimes \mathrm{id})^*$ agree on sufficiently large generalised powers of $\mathbf{X}$. In other words, it holds that

$$T_{\xi \otimes \sum_i S_{\xi_i} S^*_{\eta_i}} (\zeta \otimes b) = \sum_i (T_\xi \otimes \mathrm{id})(T_{\xi_i} \otimes \mathrm{id})(T_{\eta_i} \otimes \mathrm{id})^* (\zeta \otimes b)$$

for all $\zeta \otimes b \in \mathbf{X}^{\otimes \mathbf{m}} \otimes \mathcal{F}_{\mathbf{X}}$, where $\mathbf{m} \geq \mathbf{n} = \max\{\mathbf{n}_i\}$. This implies that

$$T_{\xi \otimes \sum_i S_{\xi_i} S^*_{\eta_i}} P_{\mathbf{n}} = \sum_i (T_\xi \otimes \mathrm{id})(T_{\xi_i} \otimes \mathrm{id})(T_{\eta_i} \otimes \mathrm{id})^* P_{\mathbf{n}},$$

so the difference between the elements in question can only be non-trivial on the range of $\mathrm{id} - P_{\mathbf{n}}$. But since this projection lies in the ideal $\mathcal{I}$, equality holds everywhere. Since equation (4.3.4) holds for all approximations of a, it holds for a as well.

2. : We start by noticing that the following equations hold

$$M_{x^{(i)}} P_{\mathbf{n}} = P_{\mathbf{m}} M_{x^{(i)}} P_{\mathbf{n}} \quad \text{for all } \mathbf{m} \leq \mathbf{n} + \mathbf{e_i} \tag{4.3.5}$$

$$P_{\mathbf{n}} M_{x^{(i)}} = P_{\mathbf{n}} M_{x^{(i)}} P_{\mathbf{m}} \quad \text{for all } \mathbf{m} \geq \mathbf{n} - \mathbf{e_i} \tag{4.3.6}$$

and define the set

$$\mathcal{L}\big(\Gamma_{\mathbb{N}}(\mathbf{X})\big)_\alpha := \{T \in \mathcal{L}\big(\Gamma_{\mathbb{N}}(\mathbf{X})\big) \cong P_0 \mathcal{L}\big(\Gamma_{\mathbb{Z}}(\mathbf{X})\big) P_0 : \operatorname*{sot\text{-}lim}_{n_1,\dots,n_k \to \infty} \alpha^{-\mathbf{n}}(T) \text{ exists}\}.$$

Note, that this is actually a C^*-algebra. We check that the Toeplitz-algebra is contained in this C^*-algebra.

For this, let $T_{x^{(i)}} \in \mathcal{T}^{\mathrm{cov}}_{\mathbf{X}_\infty}$, where $x^{(i)} \in X_i \otimes \mathcal{F}_{\mathbf{X}}$.

$$\begin{aligned}
\lim_{\mathbf{n}\to\infty} \alpha^{-\mathbf{n}}(T_{x^{(i)}})P_{\mathbf{m}} = \ & \lim\nolimits_{\mathbf{n}\to\infty} \alpha^{-\mathbf{n}}(P_0 M_{x^{(i)}} P_0) P_{\mathbf{m}} \\
= \ & \lim\nolimits_{\mathbf{n}\to\infty} P_{-\mathbf{n}} M_{x^{(i)}} P_{-\mathbf{n}} P_{\mathbf{m}} = \ M_{x^{(i)}} P_{\mathbf{m}} \text{ and} \\
\lim_{\mathbf{n}\to\infty} \alpha^{-\mathbf{n}}(T^*_{x^{(i)}})P_{\mathbf{k}} = \ & \lim\nolimits_{\mathbf{n}\to\infty} \alpha^{-\mathbf{n}}(P_0 M^*_{x^{(i)}} P_0) P_{\mathbf{k}} \\
= \ & \lim\nolimits_{\mathbf{n}\to\infty} P_{-\mathbf{n}} M^*_{x^{(i)}} P_{-\mathbf{n}} P_{\mathbf{k}} = \ M^*_{x^{(i)}} P_{\mathbf{k}}
\end{aligned}$$

where $\mathbf{m} \leq \mathbf{n} + \mathbf{e_i}$ and $\mathbf{k} \leq \mathbf{n} - \mathbf{e_i}$. We hence know that $\mathcal{T}^{\mathrm{cov}}_{\mathbf{X}} \subseteq \mathcal{L}\big(\Gamma_{\mathbb{N}}(\mathbf{X})\big)_\alpha$. Furthermore, the above limit vanishes for elements in $\mathcal{I}$, so $\lim_{\mathbf{n}\to\infty} \alpha^{-\mathbf{n}}$ defines a linear map from $\mathcal{T}^{\mathrm{cov}}_{\mathbf{X}} / \mathcal{I} \cong \mathcal{O}_{\mathbf{X}} \to C^*\{M_{x^{(i)}}\}$ mapping $S_{x^{(i)}}$ to $M_{x^{(i)}}$.

To see that it is a *-homomorphism, we show that $T \mapsto \alpha^{-\mathbf{n}}(T)$ is a *-homomorphism for all $\mathbf{n} \in \mathbb{N}^k$:

$$\begin{aligned}
\alpha^{-\mathbf{n}}(T_{x^{(i)}})\alpha^{-\mathbf{n}}(T_{y^{(i)}}) =& \alpha^{-\mathbf{n}}(P_0 M_{x^{(i)}} P_0)\alpha^{-\mathbf{n}}(P_0 M_{y^{(i)}} P_0) = (P_{-\mathbf{n}} M_{x^{(i)}} P_{-\mathbf{n}}) M_{y^{(i)}} P_{-\mathbf{n}} \\
\overset{(6)}{=}& P_{-\mathbf{n}} M_{x^{(i)}} M_{y^{(i)}} P_{-\mathbf{n}} = \alpha^{-\mathbf{n}}(T_{x^{(i)}} T_{y^{(i)}}) \\
\alpha^{-\mathbf{n}}(T^*_{x^{(i)}})\alpha^{-\mathbf{n}}(T_{y^{(i)}}) =& \alpha^{-\mathbf{n}}(P_0 M^*_{x^{(i)}} P_0)\alpha^{-\mathbf{n}}(P_0 M_{y^{(i)}} P_0) = P_{-\mathbf{n}} M^*_{x^{(i)}} (P_{-\mathbf{n}} M_{y^{(i)}} P_{-\mathbf{n}}) \\
\overset{(5)}{=}& P_{-\mathbf{n}} M^*_{x^{(i)}} M_{y^{(i)}} P_{-\mathbf{n}} = \alpha^{-\mathbf{n}}(M^*_{x^{(i)}} M_{y^{(i)}})
\end{aligned}$$

$$\begin{aligned}
\alpha^{-\mathbf{n}}(T_{x^{(i)}})\alpha^{-\mathbf{n}}(T^*_{y^{(i)}}) =& \alpha^{-\mathbf{n}}(P_0 M_{x^{(i)}} P_0)\alpha^{-\mathbf{n}}(P_0 M^*_{y^{(i)}} P_0) = (P_{-\mathbf{n}} M_{x^{(i)}} P_{-\mathbf{n}}) M^*_{y^{(i)}} P_{-\mathbf{n}} \\
\overset{(6)}{=}& P_{-\mathbf{n}} M_{x^{(i)}} M^*_{y^{(i)}} P_{-\mathbf{n}} = \alpha^{-\mathbf{n}}(M_{x^{(i)}} M^*_{y^{(i)}}) \\
\alpha^{-\mathbf{n}}(T_{x^{(i)}})^* =& P_{-\mathbf{n}} T^*_{x^{(i)}} P_{-\mathbf{n}} = \alpha^{-\mathbf{n}}(T^*_{x^{(i)}})
\end{aligned}$$

Furthermore, the map $\pi \circ \psi : C^*\{M_{x^{(i)}}\} \to \mathcal{O}_{\mathbf{X}}$ is a *-homomorphism, since it is an inverse map of $\lim_{\mathbf{n}\to\infty} \alpha^{-\mathbf{n}}$. Because it sends generators of $C^*\{M_{x^{(i)}}\}$ to generators of $\mathcal{O}_{\mathbf{X}}$, i.e.

$$(\pi \circ \psi)(M_{x^{(i)}}) = \pi(T_{x^{(i)}}) = S_{x^{(i)}} \text{ for all } x^{(i)} \in X_i, 1 \leq i \leq k,$$

it is surjective, so $C^*\{M_{x^{(i)}}\}$ and $\mathcal{O}_{\mathbf{X}}$ are isomorphic.

3. : follows from 2.

□

4.3.3.2 Corollary. *From 2.3.3.3, we get the following six-term cyclic exact sequence in* KK*-theory:*
For any separable graded C^*-algebra $\mathcal{D}$,

$$\begin{tikzcd}
\mathrm{KK}_0(\mathcal{D},\mathcal{I}) \arrow[r] & \mathrm{KK}_0(\mathcal{D},\mathcal{T}^{\mathrm{cov}}_{\mathbf{X}_\infty}) \arrow[r] & \mathrm{KK}_0(\mathcal{D},\mathcal{O}_{\mathbf{X}}) \arrow[d] \\
\mathrm{KK}^1(\mathcal{D},\mathcal{O}_{\mathbf{X}}) \arrow[u] & \mathrm{KK}^1(\mathcal{D},\mathcal{T}^{\mathrm{cov}}_{\mathbf{X}_\infty}) \arrow[l] & \mathrm{KK}^1(\mathcal{D},\mathcal{I}) \arrow[l]
\end{tikzcd}$$

is exact. If $\mathcal{T}^{\mathrm{cov}}_{\mathbf{X}_\infty}$ is separable, then

$$\begin{tikzcd}
\mathrm{KK}_0(\mathcal{I},\mathcal{D}) \arrow[d] & \mathrm{KK}_0(\mathcal{T}^{\mathrm{cov}}_{\mathbf{X}_\infty},\mathcal{D}) \arrow[l] & \mathrm{KK}_0(\mathcal{O}_{\mathbf{X}},\mathcal{D}) \arrow[l] \\
\mathrm{KK}^1(\mathcal{O}_{\mathbf{X}},\mathcal{D}) \arrow[r] & \mathrm{KK}^1(\mathcal{T}^{\mathrm{cov}}_{\mathbf{X}_\infty},\mathcal{D}) \arrow[r] & \mathrm{KK}^1(\mathcal{I},\mathcal{D}) \arrow[u]
\end{tikzcd}$$

is exact for every separable graded $\mathcal{D}$ as well.

4.4 Other properties

After proving results 4.2.3.1 and 4.3.3.1 which generalise 3.2.3.1 and 3.3.3.1 respectively it is a valid question to wonder if there is an adaption of 3.4.1.2 and 3.4.2.3 to this setting as well.

4.4.1 Are $\mathcal{T}_{\mathbf{X}}^{\mathrm{cov}}$ and $\mathcal{A}$ KK-equivalent?

In [Dea07], Deaconu shows how to iterate Pimsner's construction of the Toeplitz and Cuntz-Pimsner algebra which we will now sketch. Note that the entire construction is due to Deaconu.

Given a Hilbert $\mathcal{A}$-module E, a C^*-algebra $\mathcal{B}$ and a *-homomorphism $\rho : \mathcal{A} \to \mathcal{B}$ which is assumed to be injective, the internal tensor product $E \otimes_\rho B$ is a Hilbert $\mathcal{B}$-module with inner product

$$\langle \xi_1 \otimes b_1, \xi_2 \otimes b_2 \rangle = b_1^* \rho(\langle \xi_1, \xi_2 \rangle) b_2$$

and right action

$$(\xi \otimes b_1) b_2 = \xi \otimes (b_1 b_2).$$

Moreover, one knows that

$$\mathcal{K}(E \otimes_\rho \mathcal{B}) \cong (E \otimes \mathcal{B}) \otimes (E \otimes \mathcal{B})^* \cong E \otimes \mathcal{B} \otimes E^* \ ,$$

so if E is full, $\mathcal{K}(E \otimes \mathcal{B}) \cong \mathcal{B}$ holds. If $\mathcal{B}$ is unital, one gets an embedding $\mathcal{K}(E) \hookrightarrow \mathcal{K}(E \otimes \mathcal{B})$ by $x \otimes y^* \mapsto x \otimes \mathbb{1}_{\mathcal{B}} \otimes y^*$. If, in addition, E is an $\mathcal{A}$-correspondence together with some *-homomorphism $\varphi : \mathcal{A} \to \mathcal{L}(E)$ and there is a *-homomorphism $\varphi_{\mathcal{B}} : \mathcal{B} \to \mathcal{L}(E)$ extending φ, $(E \otimes_\rho \mathcal{B}, \varphi_{\mathcal{B}})$ is a $\mathcal{B}$-correspondence, so one can form $(E \otimes \mathcal{B})^{\otimes n}$ for $n \in \mathbb{N}$. If $(E \otimes \mathcal{B}, \varphi_{\mathcal{B}})$ is non-degenerate, one have $(E \otimes \mathcal{B})^{\otimes n} \cong E^{\otimes n} \otimes \mathcal{B}$. Therefore, $\mathcal{T}_{E \otimes \mathcal{B}}$ is represented faithfully by creation operators on $\Gamma(E \otimes \mathcal{B}) \cong \Gamma(E) \otimes \mathcal{B}$.

Now, let finitely generated Hilbert $\mathcal{A}$-modules X_1 and X_2 together with non-degenerate left actions $\varphi_i : \mathcal{A} \to \mathcal{L}(X_i)$, $i = 1, 2$ and a *-isomorphism $\chi_{1,2} : X_1 \otimes_{\varphi_2} X_2 \to X_2 \otimes_{\varphi_1} X_1$ be given. Since $\mathcal{T}_{X_1} \supseteq \{\varphi_1(\langle \xi, \eta \rangle) : \xi, \eta \in X_1\} = \mathcal{A}$, one knows that $X_2 \otimes_i \mathcal{T}_{X_1}$ is a Hilbert $\mathcal{T}_{X_1}$-module with respect to the embedding $i : \mathcal{A} \hookrightarrow \mathcal{T}_{X_1}$ with inner product and right action by $\mathcal{T}_{X_1}$ as above. One gets a left action of X_1 on $X_2 \otimes \mathcal{T}_{X_1}$ by

$$X_1 \otimes_{\varphi_2} X_2 \otimes_i \mathcal{T}_{X_1} \xrightarrow{\chi_{1,2} \otimes \mathrm{id}} X_2 \otimes_{\varphi_1} X_1 \otimes_i \mathcal{T}_{X_1} \longrightarrow X_2 \otimes \mathcal{T}_{X_1},$$

where the latter map is the absorption of X_1 into $\mathcal{T}_{X_1}$. Since X_1 generates $\mathcal{T}_{X_1}$, a left action of X_1 on $X_2 \otimes \mathcal{T}_{X_1}$ extends to a left action of $\mathcal{T}_{X_1}$ on $X_2 \otimes \mathcal{T}_{X_1}$. Moreover, this action is non-degenerate and extends φ_2 because of 4.1.1.3, so $X_2 \otimes \mathcal{T}_{X_1}$ is a $\mathcal{T}_{X_1}$-correspondence. By the same argument, $X_1 \otimes \mathcal{T}_{X_2}$ is a $\mathcal{T}_{X_2}$-correspondence. So one might as well consider the Toeplitz algebras $\mathcal{T}_{X_2 \otimes \mathcal{T}_{X_1}}$ and $\mathcal{T}_{X_1 \otimes \mathcal{T}_{X_2}}$ and investigate how they relate to each other. Note, that these algebras are special cases of Pimsner's original construction. Regardless of the additional structure on $X_1 \otimes \mathcal{T}_{X_2}$ and $X_2 \otimes \mathcal{T}_{X_1}$ either one of these modules is still a Hilbert C^*-correspondence. Moreover,

$$\Gamma(X_2 \otimes \mathcal{T}_{X_1}) \cong \Gamma(X_2) \otimes \mathcal{T}_{X_1},$$
$$\Gamma(X_2 \otimes \mathcal{T}_{X_1}) \cong \Gamma(X_2) \otimes \mathcal{T}_{X_1}.$$

After introducing this construction, Deaconu then states the following theorem in [Dea07, lemma 4.1].

4.4.1.1 Theorem. *Given two finitely generated, full and non-degenerate Hilbert $\mathcal{A}$-correspondences (X_1, φ_1) and (X_2, φ_2) one gets*

$$\mathcal{T}_{X_1 \otimes \mathcal{T}_{X_2}} \cong \mathcal{T}_{X_2 \otimes \mathcal{T}_{X_1}} \ .$$

Proof. By applying 3.2.3.1 twice, we know that both algebras are represented faithfully on $\mathcal{L}(\bigoplus_{n,m \in \mathbb{N}} X_1^{\otimes n} \otimes X_2^{\otimes m})$. Moreover, both algebras are generated by $\mathcal{T}_{X_1}$ and $\mathcal{T}_{X_2}$ subject to the commutation relation given by $\chi_{1,2}$. □

Having looked at the proof, we realise that the C^*-algebra considered in the previous theorem is isomorphic to the higher rank Toeplitz algebra of the product system over $\mathbb{N}^2$ determined by $X_{(1,0)} := X_1$ and $X_{(0,1)} := X_2$ and *-isomorphism $\chi_{1,2}$ as in 4.1.1.5. Therefore, the examples 4.4, 4.5 and 4.6 from Deaconu's paper are examples of higher-rank Cuntz-Pimsner algebras as well. Deaconu deduces this in [Dea07, remark 4.3] from applying the more general gauge-invariant uniqueness theorem [Fow02, theorem 6.3]. Our contribution is to use the previous gauge-invariant uniqueness theorem for product systems from 4.2.3.1 instead, which is sufficient for this purpose and a lot easier to prove.

4.4.1.2 Corollary. *Let $\mathbf{X}$ be a product system over $\mathbb{N}^2$ such that the fibres $X_1 := X_{(1,0)}$ and $X_2 := X_{(0,1)}$ are finitely generated, full and the respective left actions are non-degenerate. Then*

$$\mathcal{T}_{\mathbf{X}}^{\mathrm{cov}} \cong \mathcal{T}_{X_1 \otimes \mathcal{T}_{X_2}} \ .$$

By iterating this construction we get the following theorem.

4.4.1.3 Theorem. *Let $\mathbf{X}$ be a product system of Hilbert $\mathcal{A}$-correspondences over $\mathbb{N}^k$ and $X_i := X_{\mathbf{e_i}}$ for $1 \leq i \leq k$. It holds that*

$$\mathcal{T}_{\mathbf{X}}^{\mathrm{cov}} \cong \mathcal{T}_{X_1 \otimes \mathcal{T}_{X_2 \otimes \cdots \otimes \mathcal{T}_{X_k}}} .$$

4.4.1.4 Corollary. *Under the same prerequisites as in the previous theorem we get that*

$$\mathcal{T}_{\mathbf{X}}^{\mathrm{cov}} \sim_{\mathrm{KK}} \mathcal{A} .$$

Proof. By the previous identification and the KK-equivalence between the Toeplitz algebra of a Hilbert module and its coefficient algebra by 3.4.1.2 we know that

$$\mathcal{T}_{X_1 \otimes \mathcal{T}_{X_2 \otimes \cdots \otimes \mathcal{T}_{X_k}}} \sim_{\mathrm{KK}} \mathcal{T}_{X_2 \otimes \mathcal{T}_{X_3 \otimes \cdots \otimes \mathcal{T}_{X_k}}} \sim_{\mathrm{KK}} \cdots \sim_{\mathrm{KK}} \mathcal{A}.$$

□

When combining the previous corollary with 4.3.3.2, the following corollary in obvious.

4.4.1.5 Corollary. *For a product system $\mathbf{X}$ over $\mathbb{N}^k$ with finitely generated, full base fibres $X_1, \ldots, X_k$ and non-degenerate left actions the following two six-term cyclic sequences are exact for every separable graded C^*-algebra $\mathcal{D}$:*

$$\begin{tikzcd}
\mathrm{KK}_0(\mathcal{D},\mathcal{I}) \arrow[r] & \mathrm{KK}_0(\mathcal{D},\mathcal{F}_{\mathbf{X}}) \arrow[r] & \mathrm{KK}_0(\mathcal{D},\mathcal{O}_{\mathbf{X}}) \arrow[d] \\
\mathrm{KK}^1(\mathcal{D},\mathcal{O}_{\mathbf{X}}) \arrow[u] & \mathrm{KK}^1(\mathcal{D},\mathcal{F}_{\mathbf{X}}) \arrow[l] & \mathrm{KK}^1(\mathcal{D},\mathcal{I}) \arrow[l]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{KK}_0(\mathcal{I},\mathcal{D}) \arrow[d] & \mathrm{KK}_0(\mathcal{F}_{\mathbf{X}},\mathcal{D}) \arrow[l] & \mathrm{KK}_0(\mathcal{O}_{\mathbf{X}},\mathcal{D}) \arrow[l] \\
\mathrm{KK}^1(\mathcal{O}_{\mathbf{X}},\mathcal{D}) \arrow[r] & \mathrm{KK}^1(\mathcal{F}_{\mathbf{X}},\mathcal{D}) \arrow[r] & \mathrm{KK}^1(\mathcal{I},\mathcal{D}) \arrow[u]
\end{tikzcd}$$

4.4.1.6 Remark. When defining $\mathcal{T}_{X_1 \otimes \mathcal{O}_{X_2}}$, $\mathcal{O}_{X_2 \otimes \mathcal{T}_{X_1}}$, $\mathcal{O}_{X_1 \otimes \mathcal{O}_{X_2}}$ and $\mathcal{O}_{X_2 \otimes \mathcal{O}_{X_1}}$ the same way, one can show that $\mathcal{O}_{X_2 \otimes \mathcal{T}_{X_1}} \cong \mathcal{T}_{X_1 \otimes \mathcal{O}_{X_2}}$ and $\mathcal{O}_{X_1 \otimes \mathcal{O}_{X_2}} \cong \mathcal{O}_{X_2 \otimes \mathcal{O}_{X_1}} \cong \mathcal{O}_{\mathbf{X}}$. Iterating this result again, we learn that Toeplitz and Cuntz-Pimsner algebras of product systems over $\mathbb{N}^k$ with finitely generated, non-degenerate base fibres $X_1, \dots, X_k$ are isomorphic to Toeplitz and Cuntz-Pimsner algebras generated by a single Hilbert correspondence. Therefore, the approximation results from the previous chapter hold as well.

4.4.2 The irrational rotation algebra revisited

In the final section, we will have a look at the simplest example of the Toeplitz and Cuntz-Pimsner algebra of a product system over $\mathbb{N}^2$ and take it a little step further in order to understand the role of the isomorphism $\chi_{1,2} : X_1 \otimes X_2 \to X_2 \otimes X_1$ and the ideal structure inside $\mathcal{T}_{\mathbf{X}}^{\mathrm{cov}}$. For this, trivial base fibres $X_1 = \mathbb{C} = X_2$ will suffice.

4.4.2.1 Example. *Consider the product system* $\mathbf{X}$ *with base fibres* $X_1 = \mathbb{C} = X_2$ *and isomorphism* $\chi_{1,2} \colon X_1 \otimes X_2 \to X_2 \otimes X_1,\ x \otimes y \mapsto y \otimes x$. *Then* $\Gamma(\mathbf{X}) = l^2(\mathbb{N}) \otimes l^2(\mathbb{N})$, *each base fibre generates a copy of the Toeplitz algebra* $\mathcal{T}_1$ *and* $\mathcal{T}_{\mathbf{X}}^{\mathrm{cov}} = \mathcal{T}_1 \otimes \mathcal{T}_1$ *since the respective creation operators commute on the Fock module. Inside* $\mathcal{T}_{\mathbf{X}}^{\mathrm{cov}}$ *we get the ideals* $\mathcal{I}_1 = \mathcal{K} \otimes \mathcal{T}_1$ *and* $\mathcal{I}_2 = \mathcal{T}_1 \otimes \mathcal{K}$. *Now, starting with the Toeplitz extension* $0 \to \mathbb{K} \to \mathcal{T}_1 \to \mathcal{C}(\mathbb{T}) \to 0$ *of* $\mathcal{O}_1 \cong \mathcal{C}(\mathbb{T})$ *from motivating example 3.1.3 we get the following exact sequences*

$$0 \to \mathbb{K} \otimes \mathcal{T} \to \mathcal{T} \otimes \mathcal{T} \to \mathcal{C}(\mathbb{T}) \otimes \mathcal{T} \to 0$$
$$0 \to \mathcal{T} \otimes \mathbb{K} \to \mathcal{T} \otimes \mathcal{T} \to \mathcal{T} \otimes \mathcal{C}(\mathbb{T}) \to 0$$

yielding the following diagram with exact rows and columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & . \\
0 \arrow[r] & \mathbb{K} \otimes \mathbb{K} \arrow[r] \arrow[d] & \mathcal{T} \otimes \mathbb{K} \arrow[r] \arrow[d] & \mathcal{C}(\mathbb{T}) \otimes \mathbb{K} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{K} \otimes \mathcal{T} \arrow[r] \arrow[d] & \mathcal{T} \otimes \mathcal{T} \arrow[r] \arrow[d] & \mathcal{C}(\mathbb{T}) \otimes \mathcal{T} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{K} \otimes \mathcal{C}(\mathbb{T}) \arrow[r] \arrow[d] & \mathcal{T} \otimes \mathcal{C}(\mathbb{T}) \arrow[r] \arrow[d] & \mathcal{C}(\mathbb{T}) \otimes \mathcal{C}(\mathbb{T}) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Exactly the same diagram shows up in [DH71], where Douglas and Howe investigate

Toeplitz operators on the quarter plane. The preceding diagram yields the following short exact sequences:

$$0 \to \mathcal{T} \otimes \mathbb{K} + \mathbb{K} \otimes \mathcal{T} \to \mathcal{T} \otimes \mathcal{T} \to \mathcal{C}(\mathbb{T}) \otimes \mathcal{C}(\mathbb{T}) \to 0$$
$$0 \to \mathbb{K} \otimes \mathbb{K} \to \mathcal{T} \otimes \mathcal{T} \to \mathcal{C}(\mathbb{T}) \otimes \mathbb{K} + \mathbb{K} \otimes \mathcal{C}(\mathbb{T}) \to 0.$$

Note that the former sequence corresponds to 4.3.2.11 and $\mathcal{O}_{\mathbf{X}} \cong \mathcal{C}(\mathbb{T}) \otimes \mathcal{C}(\mathbb{T})$.

4.4.2.2 Example. *We get a more complicated object by slightly modifying the previous isomorphism. When considering the product system $X_1 = \mathbb{C} = X_2$ together with the isomorphism*

$$\begin{array}{rccc} \chi_{1,2}: & X_1 \otimes X_2 & \xrightarrow{\cong} & X_2 \otimes X_1, \\ & x_1 \otimes x_2 & \mapsto & e^{2\pi i \theta} x_2 \otimes x_1 \end{array}$$

for some $\theta \in [0,1] \setminus \mathbb{Q}$, the corresponding higher rank Cuntz-Pimsner algebra $\mathcal{O}_{\mathbf{X}}$ is the irrational rotation algebra $\mathcal{A}_\theta$. The explicit calculation of the K-groups of $\mathcal{A}_\theta$ can be found in [Dav96, example VIII.5.2].

Finally, we quote [Dea07, lemma 5.2] to deal with the ideal structure of $\mathcal{T}_{\mathbf{X}}^{\mathrm{cov}}$ for a product system $\mathbf{X}$ over $\mathbb{N}^2$.

4.4.2.3 Lemma. *Two closed two-sided ideals $\mathcal{I}$ and $\mathcal{J}$ in $\mathcal{A}$ yield the following commutative diagram with exact rows and columns*

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{I} \cap \mathcal{J} \arrow[r] \arrow[d] & \mathcal{I} \arrow[r] \arrow[d] & \mathcal{I}/(\mathcal{I} \cap \mathcal{J}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{J} \arrow[r] \arrow[d] & \mathcal{A} \arrow[r] \arrow[d] & \mathcal{A}/\mathcal{J} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{J}/(\mathcal{I} \cap \mathcal{J}) \arrow[r] \arrow[d] & \mathcal{A}/\mathcal{I} \arrow[r] \arrow[d] & \mathcal{A}/(\mathcal{I} + \mathcal{J}) \arrow[r] \arrow[d] & 0 \\
& 0 & 0 & 0 &
\end{tikzcd}$$

from which we get

$$0 \to \mathcal{I} + \mathcal{J} \to \mathcal{A} \to \mathcal{A}/(\mathcal{I} + \mathcal{J}) \to 0$$

and

$$0 \to \mathcal{I} \cap \mathcal{J} \to \mathcal{I} + \mathcal{J} \to \mathcal{I}/(\mathcal{I} \cap \mathcal{J}) \oplus \mathcal{J}/(\mathcal{I} \cap \mathcal{J}) \to 0.$$

4.4.2.4 Corollary. *For a product system* $\mathbf{X}$ *over* $\mathbb{N}^2$ *with finitely generated, full base fibres* X_1 *and* X_2 *and non-degenerate left actions we get the following diagram with exact rows and columns:*

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{I}_1 \cap \mathcal{I}_2 & \longrightarrow & \mathcal{I}_1 & \longrightarrow & \mathcal{I}_1/(\mathcal{I}_1 \cap \mathcal{I}_2) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{I}_2 & \longrightarrow & \mathcal{T}_{\mathbf{X}}^{\mathrm{cov}} & \longrightarrow & \mathcal{T}_{X_1} \otimes \mathcal{O}_{X_2} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{I}_2/(\mathcal{I}_1 \cap \mathcal{I}_2) & \longrightarrow & \mathcal{T}_{X_2} \otimes \mathcal{O}_{X_1} & \longrightarrow & \mathcal{O}_{\mathbf{X}} \cong \mathcal{O}_{X_1} \otimes \mathcal{O}_{X_2} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

4.4.2.5 Remark. For a product system $\mathbf{X}$ over $\mathbb{N}^k$, we get the k-dimensional analogue of this diagram.

Bibliography

[AG10] Pere Ara and Ken R. Goodearl, *Leavitt path algebras of separated graphs*, arXiv:1004.4979v1 [math.RA] (2010).

[Bla98] Bruce Blackadar, *K-theory for operator algebras*, second ed., Mathematical Sciences Research Institute Publications, vol. 5, Cambridge University Press, Cambridge, 1998.

[Bla06] ______, *Operator algebras*, Encyclopaedia of Mathematical Sciences, vol. 122, Springer-Verlag, Berlin, 2006, Theory of C^*-algebras and von Neumann algebras, Operator Algebras and Non-commutative Geometry, III.

[BO08] Nathanial P. Brown and Narutaka Ozawa, *C^*-algebras and finite-dimensional approximations*, Graduate Studies in Mathematics, vol. 88, American Mathematical Society, Providence, RI, 2008.

[Bro77] Lawrence G. Brown, *Stable isomorphism of hereditary subalgebras of C^*-algebras*, Pacific J. Math. **71** (1977), no. 2, 335–348.

[CK80] Joachim Cuntz and Wolfgang Krieger, *A class of C^*-algebras and topological Markov chains*, Invent. Math. **56** (1980), no. 3, 251–268.

[CO09] Toke M. Carlsen and Eduard Ortega, *Algebraic cuntz pimsner rings*, arXiv: 0810.3254v4 [math.RA] (2009).

[CS84] A. Connes and G. Skandalis, *The longitudinal index theorem for foliations*, Publ. Res. Inst. Math. Sci. **20** (1984), no. 6, 1139–1183.

[Cun77] Joachim Cuntz, *Simple C^*-algebras generated by isometries*, Comm. Math. Phys. **57** (1977), no. 2, 173–185.

[Dav96] Kenneth R. Davidson, *C^*-algebras by example*, Fields Institute Monographs, vol. 6, American Mathematical Society, Providence, RI, 1996.

[Dea07] Valentin Deaconu, *Iterating the Pimsner construction*, New York J. Math. **13** (2007), 199–213.

[DH71] R. G. Douglas and Roger Howe, *On the C^*-algebra of Toeplitz operators on the quarterplane*, Trans. Amer. Math. Soc. **158** (1971), 203–217.

[EW80] Masatoshi Enomoto and Yasuo Watatani, *A graph theory for C^*-algebras*, Math. Japon. **25** (1980), no. 4, 435–442.

[FL02] Michael Frank and David R. Larson, *Frames in Hilbert C^*-modules and C^*-algebras*, J. Operator Theory **48** (2002), no. 2, 273–314.

[Fow02] Neal J. Fowler, *Discrete product systems of Hilbert bimodules*, Pacific J. Math. **204** (2002), no. 2, 335–375.

[FS02] Neal J. Fowler and Aidan Sims, *Product systems over right-angled Artin semigroups*, Trans. Amer. Math. Soc. **354** (2002), no. 4, 1487–1509 (electronic). MR 1873016 (2002j:18006)

[GN94] I. Gel′fand and M. Neumark, *On the imbedding of normed rings into the ring of operators in Hilbert space*, C^*-algebras: 1943–1993 (San Antonio, TX, 1993), Contemp. Math., vol. 167, Amer. Math. Soc., Providence, RI, 1994, Corrected reprint of the 1943 original [MR **5**, 147], pp. 2–19.

[JT91] Kjeld Knudsen Jensen and Klaus Thomsen, *Elements of KK-theory*, Mathematics: Theory & Applications, Birkhäuser Boston Inc., Boston, MA, 1991.

[Kap53] Irving Kaplansky, *Modules over operator algebras*, Amer. J. Math. **75** (1953), 839–858.

[Kat04] Takeshi Katsura, *A class of C^*-algebras generalizing both graph algebras and homeomorphism C^*-algebras. I. Fundamental results*, Trans. Amer. Math. Soc. **356** (2004), no. 11, 4287–4322.

[Kat06a] ———, *A class of C^*-algebras generalizing both graph algebras and homeomorphism C^*-algebras. II. Examples*, Internat. J. Math. **17** (2006), no. 7, 791–833.

[Kat06b] ———, *A class of C^*-algebras generalizing both graph algebras and homeomorphism C^*-algebras. III. Ideal structures*, Ergodic Theory Dynam. Systems **26** (2006), no. 6, 1805–1854.

[Kat08] ———, *A class of C^*-algebras generalizing both graph algebras and homeomorphism C^*-algebras. IV. Pure infiniteness*, J. Funct. Anal. **254** (2008), no. 5, 1161–1187.

[KP99] Alex Kumjian and David Pask, *C^*-algebras of directed graphs and group actions*, Ergodic Theory Dynam. Systems **19** (1999), no. 6, 1503–1519.

[KPW98] Tsuyoshi Kajiwara, Claudia Pinzari, and Yasuo Watatani, *Ideal structure and simplicity of the C^*-algebras generated by Hilbert bimodules*, J. Funct. Anal. **159** (1998), no. 2, 295–322.

[Lan] E. C. Lance, *Hilbert C^*-modules*, London Mathematical Society Lecture Note Series, vol. 210.

[Lea62] Bill Leavitt, *The module type of a ring*, Trans. Amer. Math. Soc. **42** (1962), 113–130.

[Lin01] Huaxin Lin, *An introduction to the classification of amenable C^*-algebras*, World Scientific Publishing Co. Inc., River Edge, NJ, 2001.

[MS00] Paul S. Muhly and Baruch Solel, *On the Morita equivalence of tensor algebras*, Proc. London Math. Soc. (3) **81** (2000), no. 1, 113–168.

[Mur90] Gerard J. Murphy, *C^*-algebras and operator theory*, Academic Press Inc., Boston, MA, 1990.

[Pas73] William L. Paschke, *Inner product modules over B^*-algebras*, Trans. Amer. Math. Soc. **182** (1973), 443–468.

[Ped79] Gert K. Pedersen, *C^*-algebras and their automorphism groups*, London Mathematical Society Monographs, vol. 14, Academic Press Inc. [Harcourt Brace Jovanovich Publishers], London, 1979.

[Pim97] Michael V. Pimsner, *A class of C^*-algebras generalizing both Cuntz-Krieger algebras and crossed products by $\mathbb{Z}$*, Free probability theory (Waterloo, ON, 1995), Fields Inst. Commun., vol. 12, Amer. Math. Soc., Providence, RI, 1997, pp. 189–212.

[Pop89] Gelu Popescu, *Multi-analytic operators and some factorization theorems*, Indiana Univ. Math. J. **38** (1989), no. 3, 693–710.

[PV80] M. Pimsner and D. Voiculescu, *Exact sequences for K-groups and Ext-groups of certain cross-product C^*-algebras*, J. Operator Theory **4** (1980), no. 1, 93–118.

[Rae05] Iain Raeburn, *Graph algebras*, CBMS Regional Conference Series in Mathematics, vol. 103, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 2005.

[Rie74] Marc A. Rieffel, *Induced representations of C^*-algebras*, Advances in Math. **13** (1974), 176–257.

[RS99] Guyan Robertson and Tim Steger, *Affine buildings, tiling systems and higher rank Cuntz-Krieger algebras*, J. Reine Angew. Math. **513** (1999), 115–144.

[Seg47] I. E. Segal, *Irreducible representations of operator algebras*, Bull. Amer. Math. Soc. **53** (1947), 73–88.

[Ska91] Georges Skandalis, *Kasparov's bivariant K-theory and applications*, Exp. Math. **9** (1991), 193–249.

[Sol08] Baruch Solel, *Regular dilations of representations of product systems*, Math. Proc. R. Ir. Acad. **108** (2008), no. 1, 89–110.

[SY10] Aidan Sims and Trent Yeend, *C^*-algebras associated to product systems of Hilbert bimodules*, J. Operator Theory **64** (2010), no. 2, 349–376. MR 2718947 (2011j:46092)

[SZ10] Adam Skalski and Joachim Zacharias, *On approximation properties of pimsner algebras and crossed products by hilbert bimodules*, Rocky Mount. Jorn. Math. **40** (2010).

[Wil07] Dana P. Williams, *Crossed products of C^*-algebras*, Mathematical Surveys and Monographs, vol. 134, American Mathematical Society, Providence, RI, 2007.

[WO] N. E. Wegge-Olsen, *K-theory and C^*-algebras*, Oxford Science Publications.