Boundary Element Formulations for Elastoplastic Stress Analysis Problems

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Thesis submitted to the University of Nottingham for the degree of Doctor of Philosophy

August 1997

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ABSTRACT

This thesis presents an advanced quadratic formulation of the boundary element (BE) method for two-dimensional elasto-plastic analysis in which 3-node isoparametric quadratic elements are used to model the boundary and 8-node isoparametric quadrilateral quadratic elements are used to model the interior domain.

The main objectives of the research are to present a comprehensive review of the many different BE approaches in elasto-plasticity, to investigate the potential accuracy, robustness and reliability of each approach, and to implement the favoured approach in a comprehensive computer program for use by engineers. Full details of the elasto-plastic analytical formulations and numerical implementations are presented without ambiguity or omission of details.

A brief review of the basic principles of plasticity is presented followed by the expressions for elasto-plastic flow rules and the numerical implementations which treat mixed hardening material behaviour. The analytical BE formulation in linear elastic applications is presented. Full details of its expansion to elasto-plastic problems are shown.

Two main BE approaches in elasto-plasticity are presented in detail in this work; the initial strain displacement gradient approach with its compulsory modelling of the partial or full interior domain, and the particular integral approach which can be applied exclusively to the surface avoiding any modelling of the interior. It was decided that the initial strain displacement gradient approach is more robust than the particular integral approach and is more likely to be favoured by an inexperienced user of a BE program, despite its main

disadvantage of interior modelling.

The initial strain displacement gradient formulation as well as other alternative formulations are presented. The values of stress and strain rates at interior points are calculated via the numerical differentiation of the displacement rates in an element-wise manner; an approach similar to that used in Finite Element (FE) formulations. Full details of the numerical implementation algorithm which uses incremental and/or iterative procedures are presented.

The details of the particular integral approach which circumvents the strongly singular integrals arising in domain integrals are also discussed in detail. A computer program for the particular integral approach was written, but, due to the constraints of time and the added complexity of this approach, it was not possible to fully test the program on practical elasto-plastic cases within this project.

A full computer program, in Fortran, based on the initial strain displacement gradient elasto-plastic BE formulation is written and applied to several practical test problems. The program is written with emphasis on clarity at the expense of efficiency in order to provide a foundation for extension to three-dimensional applications and more complex plastic behaviour. The BE solutions are compared with the corresponding FE solutions provided by the commercially available FE package, ABAQUS, and, where appropriate, exact analytical solutions. The BE solutions are shown to be in very good agreement with other analytical and numerical solutions. It is concluded that the numerical differentiation of displacement rates in an element-wise manner is an accurate and numerically efficient technique which enables the strongly singular integrals to be performed.

NOTATION

Some of the key variables used in this work are listed below. All other symbols are defined when first introduced.

A	:	Area of solution domain
[A]	:	Matrix containing the kernels multiplying [U]
[A*]	:	Solution matrix multiplying the unknown vector $[\dot{x}]$
[B]	:	Matrix containing the kernels multiplying [t]
[B*]	:	Modified form of [B] after application of the boundary condition
[C]	:	Vector containing known quantities on the right-hand side of the equation
[C*]	:	Modified form of [B] after application of the effect of the plastic strain rates
dS_{ij}^{e}	:	Deviatoric stress increment
$d\varepsilon_{ij}$	:	Total strain increment
$d\varepsilon^{e}_{ij}$	:	Elastic strain increment
$d\varepsilon^{p}_{ij}$	:	Plastic strain increment
$d\lambda$	:	Non-negative constant in Prandtt-Reuss relation
$d\sigma_{ij}$	:	Stress increment
$d\sigma^{e}_{ij}$	:	Elastic stress increment
$d\sigma^{i}_{ij}$	:	Initial stress increment
[D]	:	Matrix including elastic material properties
D^{ep}_{ijkl}	:	Fourth-order elastoplastic tensor
D_{kij}	:	Third-order displacement tensor for stress
D^{ε}_{kij}	:	Third-order displacement tensor for strain

$D^{Pl}_{iml}(Q, P_m)$		The displacement tensor for particular solution
e_i	:	Unit vector in one of the Cartesian directions
E	:	Young's modulus
f_i	:	Body force vector
f	:	Loading function
F^{σ}_{ij}	:	Free-term arising in integral equations for stress rate
F^{ε}_{ij}	:	Free-term arising in integral equations for strain rate
G_i	:	Galarkin vector
H	:	Hardening parameter
$J(\xi)$	:	Jacobian of transformation in two-dimension problem
J'_i	:	Stress invariants defined in terms of principal deviatoric stresses
J_i	:	Stress invariants
$J(\xi_1, \xi_2)$		Jacobian of transformation for evaluation of elasto-plastic kernel
$L_c(\eta_1, \eta_2)$		Linear shape function for evaluation of elasto-plastic kernels
m	:	Tangential vector
n	:	Unit outward normal at the boundary
$N_c(\xi)$	:	Quadratic shape functions for boundary
$N_c(\xi_1, \xi_2)$		Quadratic shape functions for domain kernels
P	:	Load point moved to the boundary or surface of solution domain
q	:	Field point inside the solution domain
Q	:	Field point on the boundary
r (P, Q)		Distance between point P and Q
R	:	Factor for elastic stress increments

R_{cor} : Correction factor for the calculation of plastic strain rates

S_i : Principal deviatoric stresses

S_{ij} : Deviatoric stresses

S_{kij} : Third-order traction tensor for stresses

$S^{\varepsilon}{}_{kij}$: Third-order traction tensor for strain

$S^{Pl}{}_{ijlm}\ (Q, P_m)$ The stress tensor for particular solution

t_i : Traction vector

t_i : Traction rate

$T_{ij}\ (P, Q)$ Traction kernel

$T^{Pl}{}_{iml}\ (Q, P_m)$ Traction tensor for particular solution

u_i : Displacement vector

$\dot{u}_i$: Displacement rates

$U_{ij}\ (P, Q)$ Displacement kernel

$V^{\varepsilon}{}_{ijkh}\ (P, q)$, $\bar{v}^{\varepsilon}{}_{ijkh}(P, q)$ Fourth-order strain tensor for strain

$v^{\sigma}{}_{ijkh}\ (P, q)$, $\bar{v}^{\sigma}{}_{ijkh}\ (P, q)$ Fourth-order strain tensor for stress

$v_{kij}\ (P, Q)$ Third-order strain tensor for displacement

$w^{\varepsilon}{}_{ijkh}\ (P, q)$, $\bar{w}^{\varepsilon}{}_{ijkh}\ (P, q)$ Fourth-order stress tensor for strain

$w^{\sigma}{}_{ijkh}\ (P, q)$, $\bar{w}^{\sigma}{}_{ijkh}\ (P, q)$ Fourth-order stress tensor for stress

$w_{kij}\ (P, Q)$ Third-order stress tensor

$[\dot{x}]$: Vector containing unknowns

$[\dot{y}]$: Vector containing known variables

α : Convergence accelerator

α_i : Principal translation of the centre of a yield surface

α_{ij} : Translation of the centre

$\dot{\alpha}_{ij}$	:	Rate of translation of the centre of a yield surface
r	:	Boundary of solution domain
δ_{ij}	:	Kroneker delta
ε_{eq}	:	Equivalent strain
ε_{ij}	:	Total strain
$\dot{\varepsilon}_{ij}$	:	Total strain rate
$\dot{\varepsilon}^{p}_{ij}$	:	Plastic strain rate
$\dot{\varepsilon}^{e}_{ij}$	:	Elastic strain rate
λ	:	Lame's constant
μ	:	Shear modulus
ν	:	Poisson's ratio
ξ	:	Local intrinsic coordinate in two-dimensional boundary elements
ξ_1, ξ_2	:	Local intrinsic coordinates for internal elements (cells)
σ_{eq}	:	Equivalent stress
$\dot{\sigma}^{i}_{ij}$	:	Plastic stress rate
$\dot{\sigma}_{eq}$	:	Equivalent stress rate
σ_y, Y	:	Yield stress
σ_{yp}	:	Yield stress of a virgin material
ϕ_{ij}	:	Tensor quantities for particular solution

ACKNOWLEDGEMENTS

I wish to express my sincere and deepest gratitude to Dr A A Becker for his invaluable guidance and encouragement during the course of this work. I would also like to thank Dr K H Lee who provided me with helpful discussions and programming support. Finally, I wish to thank Miss S Harrison for her excellent typing of this thesis.

CHAPTER 1

INTRODUCTION

Due to advances in computer technology in recent years, numerical methods have been powerful tools in analysing problems arising in engineering applications. It can be said that it is almost impossible to solve practical problems analytically, without the aid of computational approaches. In such techniques the main aim is to reach a compromise between efficiency and accuracy.

Most numerical techniques in solid mechanics are based on the basic idea that it is possible to obtain some equations and relationships so as to establish accurately the behaviour of an infinitesimally small 'differential element' of a solution domain. It can be seen that it is possible to obtain a solution matrix which gives an admissible accurate prediction of the values of variables such as displacement, strain and stress in the solution domain (which is generally of a complex shape, made up of a number of different materials and subjected to complex loads) by dividing the whole domain into a large number of these smaller elements and using the required relationships to assemble these elements together.

The Finite Element (FE) method which is a powerful tool for engineering applications is a comparatively slow computational technique, because the solution domain must be completely represented as a collection of finite element and a remeshing process is sometimes required. In recent years the Boundary Element (BE) approach has emerged as a powerful computational alternative to the FE approach, especially when better accuracy is required in practical engineering problems with rapidly changing variables such as stress concentration, contact problems or where a solution domain becomes infinite. The BE method has sound mathematical foundations; it is based on several theorems suggested by mathematicians such as Betti, Somigliana and Fredholm, in elasticity and potential theory. However, the boundary element method is not without disadvantages. Its implementation to engineering applications results in non-symmetric and fully-populated solution matrices. For non-linear applications, the partial differential equations are non-linear and not convertible to the surface of the domain to be solved. Therefore, the main drawback is that its extension to non-linear problems requires an increase in both numerical and analytical work.

Rizzo [1967] provided the first direct integral equation approach in which displacements and tractions were assumed to be constant over each straight line element. Cruse [1969] presented the extension of the direct integral equation approach to three-dimensional problems where displacements and tractions were assumed to be constant over each triangular element. From 1967 to 1972, the boundary integral equation method was extended to cover complex engineering problems including elastodynamic problems (Cruse [1968] and Cruse and Rizzo [1968]), three-dimensional fracture mechanics (Cruse and Van Buren [1971]) and anisotropic materials (Cruse and Swedlow [1971]).

The initial elasto-plastic implementation of boundary integral equations was presented in the early seventies (see, for example, Ricardella [1973] and Rzasnicki [1973]). Some authors such as Banerjee and Cathie [1980] and Telles and Brebbia [1980] presented its implementation with improvements in both accuracy and efficiency. The concept of higher-order elements used in FE formulation was employed by Lachat [1975] and improved by Lachat and Watson [1975, 1976]. Some authors such as Cruse and Wilson [1978] and Tan and Fenner [1978, 1979] presented numerical implementation of the boundary element method by using isoparametric quadratic elements which allow both geometry and variables to behave quadratically over each element.

Since the early seventies the development of the boundary integral equation method have been significant and it has been applied to a very wide range of engineering applications, e.g. continuum mechanics, potential problems and fluid mechanics, including advanced non-linear applications. In published literature the integral equation formulations have been referred to as both the boundary integral equation (BIE) method and the boundary element (BE) method. It is worth noting that the hyper-singular boundary integral equations (HBIE) which can be generated by taking the gradients of displacement (in continuum mechanics) appearing in the BIE has emerged as a contemporary treatment of BE method.

Although the elasto-plastic boundary element formulation has been covered in a number of publications, the details of the formulation and numerical treatment have been either omitted or ambiguous. In this thesis the main aim is to clarify the two dimensional elasto-plastic boundary element formulation which needs special attention for incremental-iterative process. Full details of the analytical and numerical formulations are presented.

In chapter 2 a brief review of the basic principles of plasticity is presented. For numerical analysis, flow rules which are based on the von Mises yield criterion are expressed and mixed hardening behaviour is taken into account. In the following chapter the analytical foundation of the direct BE method and then both the initial strain and the initial stress approaches for elasto-plastic analysis are presented.

The numerical treatment of the integral equation are presented in chapter 4 in which the stress rates at the boundary and internal points are treated separately. To evaluate the stress and strain rates at internal points, element-wise numerical differentiation of displacement rates obtained from the boundary integral equation are performed.

The particular integral approach is discussed in detail in chapter 5. This approach is used in order to circumvent the strong singular integrals arising in the domain kernels for elasto-plastic analysis, as an alternative approach to the element-wise numerical differentiation of displacement rates.

In chapter 6 the full details of the incremental-iterative procedures used for the evaluation of the plastic-strain rates which employs the flow rules in consistent manner are explained. The computational solution algorithm is presented and implemented in a Fortran program.

In chapter 7 the BE formulation presented in this thesis is applied to some standard test problems and then the solutions are compared with the corresponding finite element and exact results in order to assess its accuracy and efficiency.

CHAPTER 2

BASIC PRINCIPLES OF PLASTICITY

Plasticity, a material behaviour defined in the mechanics of solids, is a type of permanent (irreversible) deformation of materials caused by external loads. It is known that the subject of the plasticity theory in solid mechanics is very complex. There are several books published in this field such as Hill [1950], Mendelson [1968], Kachanov [1974] and Lubliner [1990]. It can be observed that the theory of plasticity was not applied widely to practical engineering applications before the implementation of the computational approaches.

By using a uniaxial tension test of a metal specimen the yield point in its stress-strain curve can be defined. However, in most practical engineering applications, such as power plant components, the stress state is quite different when the structure is subjected to multiaxial loads which result in a multiaxial stress state. Hence a criterion is required to define the onset of plastic deformation in such external loading conditions.

In the multiaxial (triaxial) material behaviour the yield conditions can be defined by the yield criterion. Furthermore, to analyse the elastic-plastic deformation, not only the explicit strain-stress relationship for the elastic behaviour, but also a relationship between strain and stress after the onset of plastic deformation have to be defined.

2.1 ELASTIC BEHAVIOUR

When a loaded material displays an elastic behaviour which is governed by Hooke's law, in a two-dimensional case (for both plane strain and stress cases) the strain-stress relationship can be written as follows:

$$\varepsilon_{zz} = \frac{1}{E} [\sigma_{zz} - \nu \ (\sigma_{xx} + \sigma_{yy}]$$

$$\varepsilon_{yy} = \frac{1}{E} [\sigma_{yy} - \nu \ (\sigma_{xx} + \sigma_{zz}]$$

$$\varepsilon_{xx} = \frac{1}{E} [\sigma_{xx} - \nu \ (\sigma_{yy} + \sigma_{zz}] \qquad \textbf{(2.1)}$$

$$\varepsilon_{xy} = \frac{1}{2\mu} \sigma_{xy}$$

where E is Young's modulus, ν is Poisson's ratio and μ is the shear modulus and defined as follows:

$$\mu = \frac{E}{2 \ (1 + \nu)} \qquad \textbf{(2.2)}$$

By using 'effective' material properties (E*, ν^* and μ^*) , the following expressions of the stress-strain relationship can be written in order to cover both plane strain ($\varepsilon_{zz} = 0$) and plane stress ($\sigma_{zz} = 0$) situations:

$$\varepsilon_{xx} = \left(\frac{1 - \nu^{*2}}{E^*}\right)\sigma_{xx} + \left(\frac{-\nu^*(1+\nu^*)}{E^*}\right)\sigma_{yy}$$

$$\varepsilon_{yy} = \left(\frac{-\nu^*(1+\nu^*)}{E^*}\right)\sigma_{xx} + \left(\frac{1 - \nu^{*2}}{E^*}\right)\sigma_{yy} \tag{2.3}$$

$$\varepsilon_{xy} = \frac{1}{2\,\mu^*}\,\sigma_{xy}$$

where

$$E^* = E \qquad \nu^* = \nu \qquad \mu^* = \mu \text{ (for plane strain)}$$

$$E^* = \frac{E\,(1+2\nu)}{(1+\nu)^2} \qquad \nu^* = \frac{\nu}{1+\nu} \qquad \mu^* = \mu \text{ (for plane stress)} \tag{2.4}$$

By rearranging equation (2.1), the stress-strain expressions can be written as follows:

$$\sigma_{xx} = \frac{2\,\mu\,\nu}{1-2\nu}(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}) + 2\mu\,\varepsilon_{xx}$$

$$\sigma_{yy} = \frac{2\,\mu\,\nu}{1-2\nu}(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}) + 2\mu\,\varepsilon_{yy} \tag{2.5}$$

$$\sigma_{zz} = \frac{2\,\mu\,\nu}{1-2\nu}(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}) + 2\mu\,\varepsilon_{zz}$$

$$\sigma_{xy} = 2\mu \, \varepsilon_{xy}$$

or, in tensor notation

$$\sigma_{ij} = \frac{2 \mu \nu}{1 - 2\nu} \delta_{ij} \, \varepsilon_{mm} + 2\mu \, \varepsilon_{ij} \qquad \textbf{(2.6)}$$

where δ_{ij} is the Kroneker delta defined as follows:

$$\begin{aligned} \delta_{ij} &= 1 \qquad \text{if} \qquad i = j \\ &= 0 \qquad \qquad \quad i \neq j \end{aligned} \qquad \textbf{(2.7)}$$

By considering a small differential area of a body, subjected to loads, in two dimensions, the stress equilibrium equations are given as follows:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + f_x = 0$$

$$\textbf{(2.8)}$$

$$\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + f_y = 0$$

or in tensor notation

$$\frac{\partial \sigma_{ij}}{\partial x_j} + f_i = 0 \qquad \textbf{(2.9)}$$

The strain-displacement relationships are given as follows:

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} \; ; \qquad \varepsilon_{yy} = \frac{\partial u_y}{\partial y} \; ;$$

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) \tag{2.10}$$

or in tensor notation

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{2.11}$$

By substituting the strain-displacement expressions of equation (2.11) into the stress-strain equation (2.6) and using the equilibrium equation, a differential equation containing displacement, called the Navier equations, can be obtained as follows:

$$\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} + \frac{1}{1 - 2\nu}\left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_y}{\partial x \, \partial y}\right) = \frac{-f_x}{\mu}$$

$$\frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2} + \frac{1}{1 - 2\nu}\left(\frac{\partial^2 u_y}{\partial y^2} + \frac{\partial^2 u_x}{\partial x \, \partial y}\right) = \frac{-f_y}{\mu} \tag{2.12}$$

or in tensor notation

$$\frac{\partial^2 u_i}{\partial x_j \, \partial x_j} + \frac{1}{(1 - 2\nu)} \frac{\partial^2 u_j}{\partial x_i \, \partial x_j} = \frac{-f_i}{\mu} \tag{2.13}$$

2.2 YIELD CRITERIA

In order to define the onset of plastic deformation some theoretical criteria generally based on strains, stresses or strain energy in a triaxial material behaviour can be used. These theorems are discussed below.

2.2.1 Maximum principal stress theory

According to this theory, attributed to Rankine, yielding commences when the maximum principal stress exceeds a value equal to the yield stress in a simple tension test, σ^t_{yp}, at the onset of yielding. This criteria can be mathematically defined as follows:

$$
\begin{aligned}
|\sigma_1| &= \sigma^t_{yp} \\
|\sigma_3| &= \sigma^c_{yp}
\end{aligned}
\qquad \textbf{(2.14)}
$$

where the subscript yp stands for yield point, and the superscript t and c stand for tension and compression respectively.

2.2.2 Maximum principal strain theory

In this theory, also known as St. Venant theory, yielding commences when the principal strain is equal to the tensile yield strain, ε^t_{yp}, in a simple tensile test, or when the minimum principal strain is equal to the compressive yield strain, ε^c_{yp}. By using Hookes's law, this criterion can be expressed as follows:

$$\begin{aligned} |\sigma_1 - \nu\ (\sigma_2 + \sigma_3)| &= \sigma_{yp}^t \\ |\sigma_3 - \nu\ (\sigma_1 + \sigma_2)| &= \sigma_{yp}^c \end{aligned} \tag{2.15}$$

If the yield stresses are equal to each other, for plane stress conditions ($\sigma_3 = 0$) the criterion has to be rewritten as follows:

$$\begin{aligned} \sigma_1 - \nu\ \sigma_2 &= \sigma_{yp} = \sigma_{yp}^t \\ \nu\ (\sigma_1 + \sigma_2) &= \sigma_{yp} = \sigma_{yp}^c \end{aligned} \tag{2.16}$$

2.2.3 Maximum shear stress theory

This theory, proposed by C A Coulomb, often known as Tresca's criterion, is based on the idea that yielding commences when the maximum shear stress is equal to the absolute value of the shear stress at the yield point in a simple tension test. This criterion is expressed as follows:

$$|\sigma_1 - \sigma_3| = \sigma_{yp} = 2\ \tau_{yp} \tag{2.17}$$

It can be seen that the maximum shear stress is given by half the absolute value of the difference between the maximum and the minimum stresses. The maximum shear stress at yielding point in a simple tension test is $\sigma_{yp}/2$.

2.2.4 Maximum distortion energy theory

This theory proposed by Maxwell, Von Mises and Hencky, usually known as the Von Mises criterion, states that yielding starts when the shear strain energy per unit volume in a multi-axial stress state is equal to the strain energy per unit volume at the yield point in a simple tension test. In this theory, the following mathematical definition can be given in terms of principle stresses as follows:

$$\tfrac{1}{2}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right] = (\sigma_{yp})^2 \tag{2.18}$$

or for plane stress conditions ($\sigma_3 = 0$):

$$\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2 = \sigma_{yp}^2 \tag{2.19}$$

2.2.5 Experimental support

From experimental evidence the onset of plastic deformation begins because of shear stresses. Therefore, it can be said that a suitable yield criterion should be based on the shear stresses. The maximum shear stress (Tresca) and shear strain energy (Von Mises) criteria are two yield criteria in common use. Both Tresca and Von Mises criteria have been shown to correlate well with experimental results. The latter shows generally better correlation with experimental results.

It is possible to represent these two yield criterion in the σ_1-σ_2-σ_3 'stress space', as shown in Figure 2.1(a) by considering all shear stress values. A representation of these two yield criterion in two-dimensional stress states is given in Figure 2.1(b). It is also possible to represent the yield surface geometrically by using 'π-plane', the plane in σ_1 - σ_2 - σ_3 space defined by $\sigma_1 + \sigma_2 + \sigma_3 = 0$, shown in Figure 2.2, which passes through the origin and subtending equal angles with the coordinate axes.

2.3 PRINCIPAL AND EQUIVALENT STRESSES

It can be shown that there are three planes (called principal planes) where shear stresses are zero, in multi-axial loading states. The stresses acting on these planes are termed principal stresses (σ_1, σ_2, σ_3) and in treating the principal stresses, the usual convection is that $\sigma_1 > \sigma_2 > \sigma_3$.

Stresses, in any stress state, can be divided in two separate parts which are a hydrostatic components, σ_m, and a deviatoric components, S_{ij}. These components can be given as follows:

$$\sigma_m = \frac{1}{3} (\sigma_{xx} + \sigma_{yy} + \sigma_{zz})$$
$$S_{ij} = \sigma_{ij} - \frac{1}{3} \delta_{ij} \sigma_{kk} \tag{2.20}$$

It is known that the deviatoric stresses are responsible for the plastic flow, while the hydrostatic stresses are responsible only for the change in the volume of the materials. Therefore, these given expressions are important in plasticity (because of experimental evidence). One way of describing the effect of the general stress or strain state of the material subjected to the complex loading state is to define the equivalent stress, σ_{eq} (which is equal to the uniaxial yield stress) or the equivalent effective strain.

For a material obeying the Von Mises yield flow, the following expression of the equivalent stress is given in terms of the principal stresses:

$$\sigma_{eq} = \frac{1}{\sqrt{2}} \{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_3)^2\}^{\frac{1}{2}} \tag{2.21}$$

or in terms of the deviatoric stresses

$$\sigma_{eq} = \{\frac{2}{3} S_{ij} S_{ij}\}^{\frac{1}{2}} \tag{2.22}$$

and the equivalent strain can be expressed in terms of principal strains as follows:

$$\varepsilon_{eq} = \frac{\sqrt{2}}{3} \{(\varepsilon_1 - \varepsilon_2)^2 + (\varepsilon_2 - \varepsilon_3)^2 + (\varepsilon_1 - \varepsilon_2)^2\}^{\frac{1}{2}} \tag{2.23}$$

During the plastic deformation, the incremental equivalent plastic strain can be expressed in terms of principal plastic strains as follows:

$$d\,\varepsilon^p_{eq} = \frac{\sqrt{2}}{3} \left[(d\,\varepsilon^p_1 - d\,\varepsilon^p_2)^2 + (d\,\varepsilon^p_2 - d\,\varepsilon^p_3)^2 + (d\,\varepsilon^p_1 - d\,\varepsilon^p_3)^2\right]^{\frac{1}{2}} \tag{2.24}$$

2.4 STRAIN HARDENING

When a loaded material reaches its elastic limit, yielding starts and theoretically it commences to flow without additional loads. Even so, most engineering materials do not lose their stiffness completely after the onset of plastic deformation. Therefore, for this type of material additional loads are required for further plastic deformation. Another plastic behaviour, which may occur is that the metal becomes 'harder' after yielding. For these types of materials the applied load must be increased in order to deform plasticity again after each complete previous loading cycle. Kinematic and isotropic hardening are two hardening material behaviours in common use. Furthermore, most engineering materials display a combined form (usually called mixed hardening).

If at each plastic deformation state the yield surface is a uniform expansion of the original yield surface, without any rigid motion, this strain hardening is called isotropic hardening. If at each plastic deformation state the yield surface keeps its shape and its size (but translates in the stress space as a rigid body motion), this strain hardening is called kinematic. The latter hardening model takes into account the Bauschinger effect observed experimentally in which the yield stress in compression is less than the yield stress in tension (see, for example, Owen and Hinton [1980])

2.5 YIELD FUNCTION

In simple tensile tests when the stress level is equal to the stress level at the yield point, the following expression can be used by:

$$F(\sigma) = \sigma - \sigma_{yp} = 0 \tag{2.25}$$

where F(σ) is referred to as a yield function. This expression can be extended to triaxial stress states. Yield will commence if the following expression is valid:

$$F(\sigma_{ij}) = 0 \tag{2.26}$$

Since the onset of the plastic deformation is independent of hydrostatic pressure, the yield, which is independent of the orientation of the coordinate system considered, is commonly defined as characteristic values of the stress field. The following expression of the stress invariants can be written:

$$
\begin{aligned}
J_1 &= \sigma_{xx} + \sigma_{yy} + \sigma_{zz} \\
J_2 &= \sigma_{xx}\,\sigma_{yy} + \sigma_{yy}\,\sigma_{zz} + \sigma_{zz}\,\sigma_{xx} - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2 \\
J_3 &= \sigma_{xx}\,\sigma_{yy}\,\sigma_{zz} - \sigma_{xx}\,\tau_{yz}^2 - \sigma_{yy}\,\tau_{zx}^2 - \sigma_{zz}\,\tau_{xy}^2 + 2\,\tau_{xy}\,\tau_{yz}\,\tau_{zx}
\end{aligned} \tag{2.27}
$$

When the employed coordinates systems coincides with the principle directions, the stress invariants in terms of the principal stresses can be defined as follows:

$$
\begin{aligned}
J_1 &= \sigma_1 + \sigma_2 + \sigma_3 \\
J_2 &= \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_2 \\
J_3 &= \sigma_1\,\sigma_2\,\sigma_3
\end{aligned} \tag{2.28}
$$

It is known that the onset of plastic deformation depends only on the magnitudes of the three principal stresses and can be defined as follows:

$$
F\,(J_1, J_2, J_3) = 0 \tag{2.29}
$$

As mentioned earlier, stresses at any point a loaded body can be divided into a hydrostatic component and a deviatoric component. Here the principal deviatoric stresses can be written as follows:

$$
\begin{aligned}
S_1 &= \sigma_1 - \sigma_m = \frac{(\sigma_1 - \sigma_2) + (\sigma_1 - \sigma_3)}{3} \\
S_2 &= \sigma_2 - \sigma_m = \frac{(\sigma_2 - \sigma_3) - (\sigma_1 - \sigma_2)}{3} \\
S_3 &= \sigma_3 - \sigma_m = -\frac{(\sigma_2 - \sigma_3) + (\sigma_1 - \sigma_3)}{3}
\end{aligned} \tag{2.30}
$$

The invariants, J_i' can be defined in terms of principal deviatoric stress, S_i, as follows:

$$\begin{aligned} J_1' &= S_1 + S_2 + S_3 = 0 \\ J_2' &= -(S_1 S_2 + S_2 S_3 + S_3 S_1) \\ J_3' &= S_1 S_2 S_3 = \frac{1}{3}(S_1^3 + S_2^3 + S_3^3) \end{aligned} \tag{2.31}$$

Equation (2.29) can be written in terms of principal deviatoric stresses as follows:

$$F(J_2', J_3') = 0 \tag{2.32}$$

During plastic deformation, the material yield strength may be not be constant, but a function of strain and stresses. In general scalar form, the yield criterion (surface) can be rewritten as follows:

$$f(\sigma_{ij}, \varepsilon_{ij}^p, k) = F(\sigma_{ij}, \varepsilon_{ij}^p) - Y(k) = 0 \tag{2.33}$$

In this expression the yield function is a function of the stress, σ_{ij} and the plastic strain, ε_{ij}^p respectively. The yield stress may be defined as a function of a hardening parameter, k, which governs the change of the yield surface.

In isotropic plastic deformation (for simplicity) the following expression can be used.

$$f(\sigma_{ij}, k) = F(\sigma_{ij}) - Y(k) = 0 \tag{2.34}$$

For a perfectly plastic material, the yield stress, Y(k) is constant.

2.6 MATERIAL BEHAVIOUR AT YIELD

2.6.1 Stress-strain relations

When the onset of plastic deformation commences, the behaviour is no longer linear elastic and only an incremental relationship between stresses and strain can be defined. The following assumptions are required to derive the relationship between plastic stress and strain.

(i) The plastic strain increments are linearly proportional to the stress increments.

(ii) The yield surface in the stress space is convex with respect to the origin.

For an incremental (infinitesimally small) strain, the total strain increment (or rate) can be expressed as follows:

$$d\,\varepsilon = d\,\varepsilon^{e} + d\,\varepsilon^{p} \tag{2.35}$$

in which the superscripts e and p indicate elastic and plastic components, respectively.

The plastic strain increment is given by (see, for example, Owen and Hinton [1950])

$$d\,\varepsilon_{ij}^{p} = d\lambda\,\frac{\partial Q}{\partial \sigma_{ij}} \tag{2.36}$$

where $d\varepsilon_{ij}^{p}$ is an equivalent plastic strain increment and $d\lambda$ (sometimes mentioned in the literature as a load factor) is a proportionality constant determined by the stress state. Q is known as a plastic potential function. If Q = F (the yield function) the elasto-plastic behaviour is defined as associative plasticity, otherwise it is non-associative. For associative plasticity, equation (2.36) can be given as follows:

$$d\,\varepsilon_{ij}^{p} = d\,\lambda\,\frac{\partial F}{\partial \sigma_{ij}} \tag{2.37}$$

According to Von Mises yield criteria, the onset of plastic deformation takes place when the second deviatoric stress invariants reach the yield value. Therefore, for a material obeying the Von Mises yield criteria, the following expression applies (for details see, for example, Owen and Hinton [1980]):

$$\frac{\partial F}{\partial \sigma_{ij}} = S_{ij} \tag{2.38}$$

and equation (2.37) becomes

$$d\,\lambda = \frac{d\,\varepsilon_{xx}^{p}}{S_{xx}} = \frac{d\,\varepsilon_{yy}^{p}}{S_{yy}} = \frac{d\,\varepsilon_{zz}^{p}}{S_{zz}} = \frac{d\,\varepsilon_{yy}^{p}}{S_{xy}} = \frac{d\varepsilon_{xz}^{p}}{S_{xz}} = \frac{d\,\varepsilon_{yz}^{p}}{S_{yz}} \tag{2.39}$$

in which $S_{xx} = \sigma_{xx} - \sigma_{m}$, $S_{xy} = \sigma_{xy}$ etc

In tensor notation

$$d\,\varepsilon_{ij}^{p} = d\,\lambda\,S_{ij} \tag{2.40}$$

which is known as the Prandtt-Reuss equation. The parameter $d\lambda$ can be defined in uniaxial conditions using equation (2.22) and (2.40) as follows:

$$d\,\lambda = \frac{3}{2}\left(\frac{d\,\varepsilon_{eq}}{\sigma_{eq}}\right) \tag{2.41}$$

Equation (2.40) can be expanded to obtain the following equations for the principal plastic strains using equation (2.40), (2.41) and (2.20).

$$
\begin{aligned}
d\,\varepsilon_1^p &= \frac{d\,\varepsilon_{eq}^p}{\sigma_{eq}}\left[\sigma_1 - \frac{1}{2}(\sigma_2 + \sigma_3)\right] \\
d\,\varepsilon_2 &= \frac{d\,\varepsilon_{eq}^p}{\sigma_{eq}}\left[\sigma_2 - \frac{1}{2}(\sigma_1 + \sigma_3)\right] \\
d\,\varepsilon_2^p &- \frac{d\,\varepsilon_{eq}^p}{\sigma_{eq}}\left[\sigma_3 - \frac{1}{2}(\sigma_2 + \sigma_1)\right]
\end{aligned}
\tag{2.42}
$$

During the plastic deformation, the total dissipation can be defined as follows:

$$
k = \int_0^{\varepsilon_{ij}^p} \sigma_{ij}\, d\,\varepsilon_{ij}^p = \int_0^t \sigma_{ij}\, \dot{\varepsilon}_{ij}^p\, dt \tag{2.43}
$$

This scalar quantity is considered to characterize the material hardening during the permanent deformation.

By differentiating equation (2.34), the following equation is obtained.

$$
df = \frac{\partial F}{\partial \sigma}\, d\sigma - \frac{dY}{dk}\, dk = 0 \tag{2.44}
$$

or

$$
a^T\, d\sigma - A\, d\,\lambda = 0 \tag{2.45}
$$

where the vector a , called flow vector, is defined by

$$a = \frac{\partial F}{\partial \sigma} \tag{2.46}$$

and A is given by

$$A = \frac{1}{d\lambda} \frac{dY}{dk} dk \tag{2.47}$$

By using equation (2.37), the following expression can be written

$$d\sigma = [D] \, (d\varepsilon - d\varepsilon^p) = [D] \left(d\varepsilon - d\lambda \frac{\partial F}{\partial \sigma} \right) \tag{2.48}$$

or

$$d\varepsilon = [D]^{-1} \, d\sigma + d\lambda \frac{\partial F}{\partial \sigma} \tag{2.49}$$

By multiplying both side of equation (2.49) by $a^T D$ and using equation (2.45), the expression for the plastic multiplier $d\lambda$ can be obtained.

$$d\lambda = \frac{a^T \, [D] \, d\varepsilon}{A + a^T \, [D] \, a} \tag{2.50}$$

where D is the elastic constants matrix. Nayak et al [1972] pointed out that work and strain hardening coincide only for materials obeying the Von Mises yield criterion. For work hardening hypothesis $dk = \sigma d\varepsilon^p$, the following expression for A which appears in equation (2.50) can be written (for details, see Owen and Hinton [1980] and Marques [1984])

$$d\lambda = \frac{a^T [D] d\varepsilon}{H + a^T [D] a} \tag{2.51}$$

where H is the slope of the stress-strain a curve in the plastic range and defined for linearly hardening behaviour, represented in Figure 2.3 as follows:

$$H = \frac{d\sigma_{eq}}{d\varepsilon^p} \tag{2.52}$$

Therefore, equation (2.48) becomes

$$d\sigma = [D] d\varepsilon - [D] \frac{a a^T [D] d\varepsilon}{H + a^T [D] a} \tag{2.53}$$

For a material obeying the Von Mises yield criterion, by treating some of the terms appearing in this expression, the following useful expression in notation can be obtained (for details, see Kane [1994])

$$d\sigma_{ij} = 2\mu \left[\frac{\nu}{1 - 2\nu} \delta_{ij} d\varepsilon_{kk} + d\varepsilon_{ij} - \frac{3 S_{ij} S_{kl}}{2 \sigma_{eq}^2 \left(1 + \frac{H}{3\mu}\right)} d\varepsilon_{kl} \right] \tag{2.54}$$

The following incremental stress-strain relationship can be written

$$d\sigma_{ij} = D_{ijkl}^{ep} d\varepsilon_{kl} \tag{2.55}$$

in which

$$D_{ijkl}^{ep} = 2\mu \left[\frac{\nu}{1 - 2\nu} \delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} - \frac{3 \dot{S}_{ij} \dot{S}_{kl}}{2 \sigma_{eq}^2 \left(1 + \frac{H}{3\mu}\right)} \right] \tag{2.56}$$

As Figure 2.4 represents on the plane of the Haigh-Westergaard stress space, for a material displaying a mixed hardening and obeying Von Mises yield criterion, the yield surface is assumed both to expand and to translate (see, for example, Hodge [1957] and Lee [1983]). In Figure 2.4, σ_i and α_i indicate the principal current stresses, σ_{ij} and current translation, α_{ij} of the centre of the yield surface.

For a mixed hardening behaviour, the yield criterion can be written as follows:

$$f = F\ (\sigma_{ij}, \alpha_{ij}) - Y\ (\varepsilon_{eq}^p) \tag{2.57}$$

Axelsson and Samuelson [1979] proposed that the plastic strain rate is decomposed into its isotropic and kinematic parts as follows:

$$\begin{aligned} d\varepsilon_{ij}^{p(i)} &= M\ d\varepsilon_{ij}^p \\ d\varepsilon_{ij}^{p(k)} &= (1 - M)\ d\varepsilon_{ij}^p \end{aligned} \tag{2.58}$$

in which M is defined the mixed hardening parameter which is equal to -1 for isotropic softening, 0 for kinematic hardening and 1 for isotropic hardening respectively.

The translation rate, α_{ij}, of the yield surface is defined as follows:

$$\alpha_{ij} = \frac{2}{3} H \, d\varepsilon^{p(k)} \tag{2.59}$$

The slope of the stress-plastic strain curve in uniaxial tensile test, H, is given as follows:

$$H = \frac{d\sigma_{eq}}{d\varepsilon_{eq}^{p(i)}} \tag{2.60}$$

As given in equation (2.54), for a material obeying the Von Mises yield criterion, the following expression of plastic strain increments can be written (see Lee [1983]).

$$\dot{\varepsilon}_{ij}^{p} = \frac{3}{2} \left(\frac{\dot{S}_{kl} \, \dot{\varepsilon}_{kl}}{1 + H/3\mu} \right) \frac{\dot{S}_{ij}}{(\sigma_{eq})^2} \tag{2.61}$$

The expression of the plastic strain increments in terms of the stress increments is given by

$$\dot{\varepsilon}_{ij}^{p} = \frac{9}{4} \left(\frac{\dot{S}_{kl} \, \dot{\sigma}_{kl}}{H} \right) \frac{\dot{S}_{ij}}{(\sigma_{eq})^2} \tag{2.62}$$

In which S_{kl} and σ_{eq} are current deviatoric stresses and equivalent stress respectively.

2.6.2 Navier equation in incremental form

It is possible to define the elastic strain rate in terms of the total strain rate and the plastic strain rate as follows:

$$\varepsilon_{ij}^{e} = \dot{\varepsilon}_{ij} - \varepsilon_{ij}^{p} \tag{2.63}$$

By substituting this expression into equation (2.6), the elastic stress-strain relationships (Hookes law) of equation (2.6) can be written as follows:

$$\dot{\sigma}_{ij} = \frac{2\mu\nu}{1 - 2\nu}\, \delta_{ij}\, \dot{\varepsilon}_{kk} + 2\mu\, \dot{\varepsilon}_{ij} - \left(\frac{2\mu\nu}{1 - 2\nu}\, \delta_{ij}\, \varepsilon_{kk}^{p} + 2\mu\, \dot{\varepsilon}_{ij}^{p} \right) \tag{2.64}$$

In this equation the second part (given in brackets) can be referred to as the 'initial' stress rate as follows:

$$\dot{\sigma}_{ij}^{i} = \left[\frac{2\mu\nu}{1 - 2\,\nu}\, \delta_{ij}\, \dot{\varepsilon}_{kk}^{p} + 2\mu\, \dot{\varepsilon}_{ij}^{p} \right] \tag{2.65}$$

The total strain-displacement relationship is given by

$$\dot{\varepsilon}_{ij} = \frac{1}{2} \left(\frac{\partial\, \dot{u}_i}{\partial\, x_j} + \frac{\partial\, \dot{u}_j}{\partial\, x_i} \right) \tag{2.66}$$

The equilibrium equation in incremental form is given as follows:

$$\frac{\partial\, \dot{\sigma}_{ij}}{\partial\, x_j} + \dot{f}_i = 0 \tag{2.67}$$

The following Navier equation can be obtained by following the procedure in a manner similar to the linear case (see, for example, Lee [1983]).

$$\frac{\partial^2 \dot{u}_i}{\partial x_j \, \partial x_j} + \frac{1}{(1 - 2\,\nu)} \frac{\partial^2 \dot{u}_j}{\partial x_i \, \partial x_j} - \left(k_1 \frac{\partial \dot{\varepsilon}^p_{jj}}{\partial x_i} + 2 \frac{\partial \dot{\varepsilon}^p_{ij}}{\partial x_j} \right) = \frac{-\dot{f}_i}{\mu} \tag{2.68}$$

in which the parameter k_1 is given by

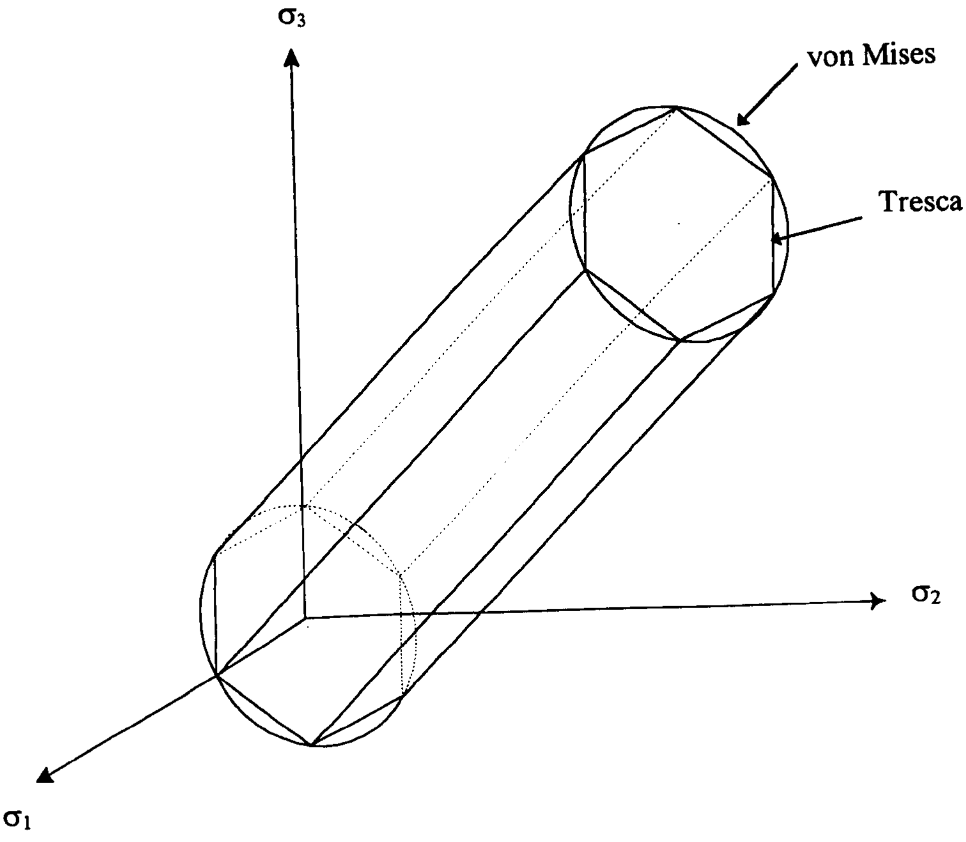
$$k_1 = 0 \text{ for plane strain case } (\varepsilon_{zz} = 0) \tag{2.69}$$

$$k_1 = \frac{2\nu}{1 - 2\nu} \text{ for plane stress case } (\sigma_{zz} = 0)$$

The term 'time' defined in computational approach for the elasto-plastic analysis represents the iterations process through load increments. The size of the time step is commonly taken as a unity and defined for example for the strain rate, as follows:

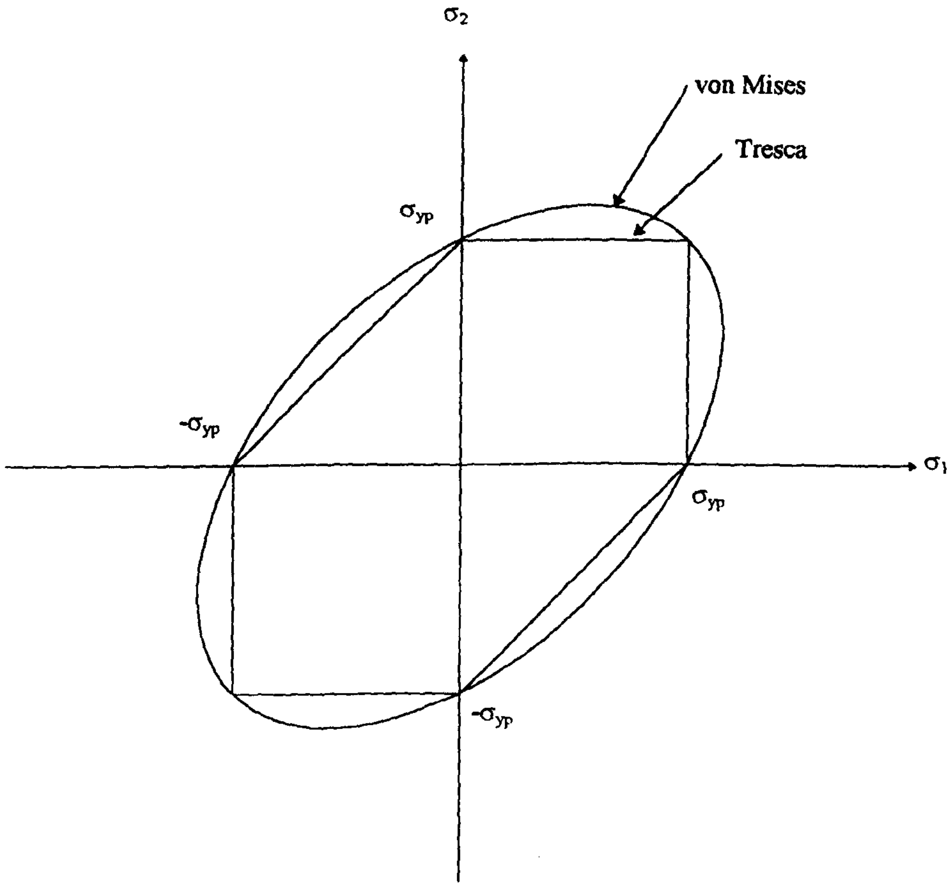
$$\Delta\, \varepsilon^p_{ij} = \Delta\, \dot{\varepsilon}^p_{ij} \times \Delta t \tag{2.70}$$

The final stress state, or strain rate, in the structure to be analysed can be obtained by the accumulation of the deformation and stresses over each of the external applied load increments. The load increments should be reasonably small.



(a) Three-dimensional representation

Fig.2.1 : Representation of the Von Mises and the Tresca yield criteria.



(b) Two-dimensional representation

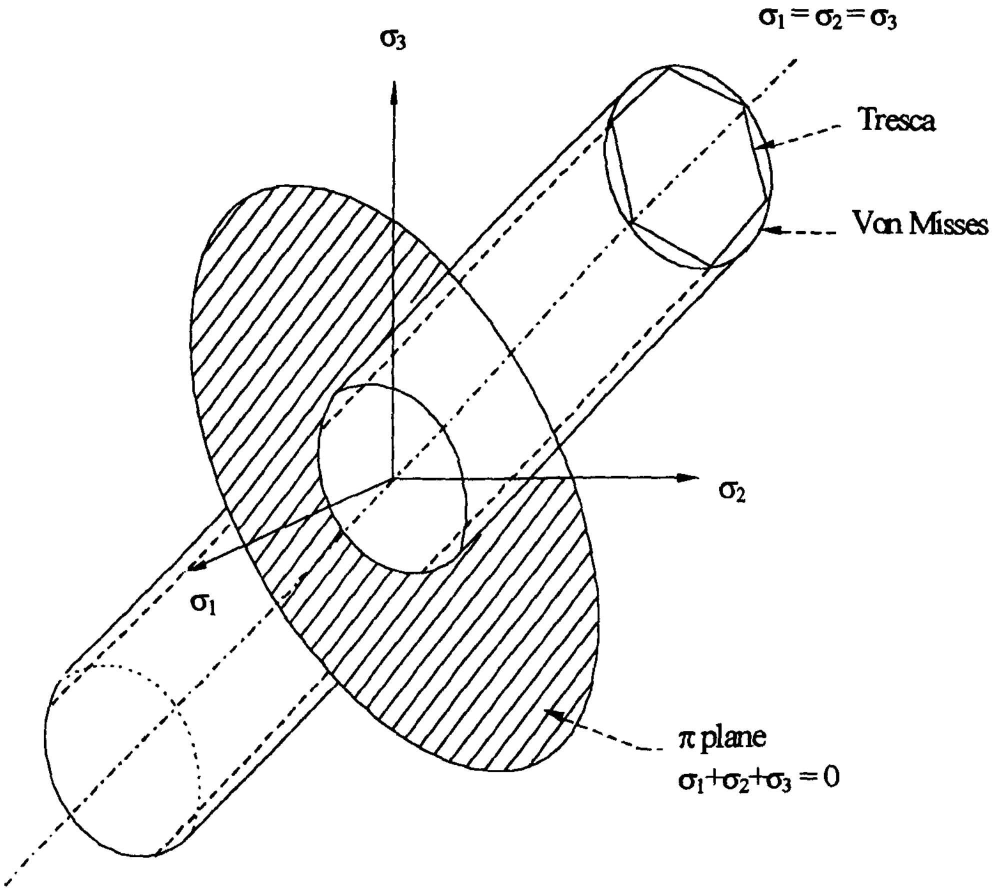


Fig. 2.2 : Geometrical representation of the von Mises and Tresaca yield surfaces on the π plane.

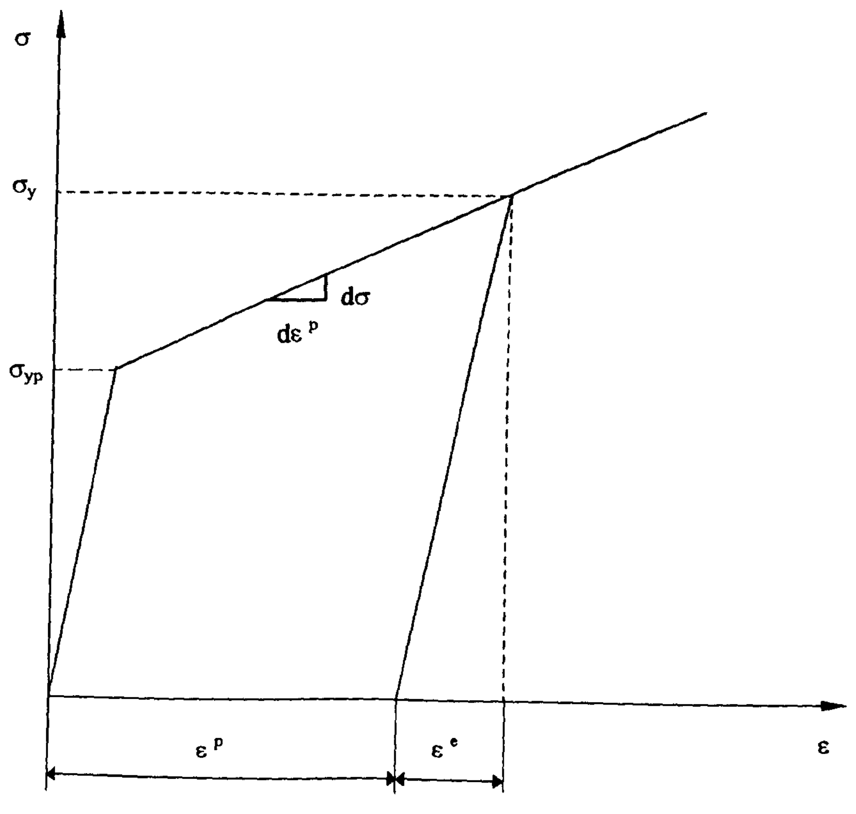


Fig.2.3 :Linearly hardening material behaviour for the uniaxial case

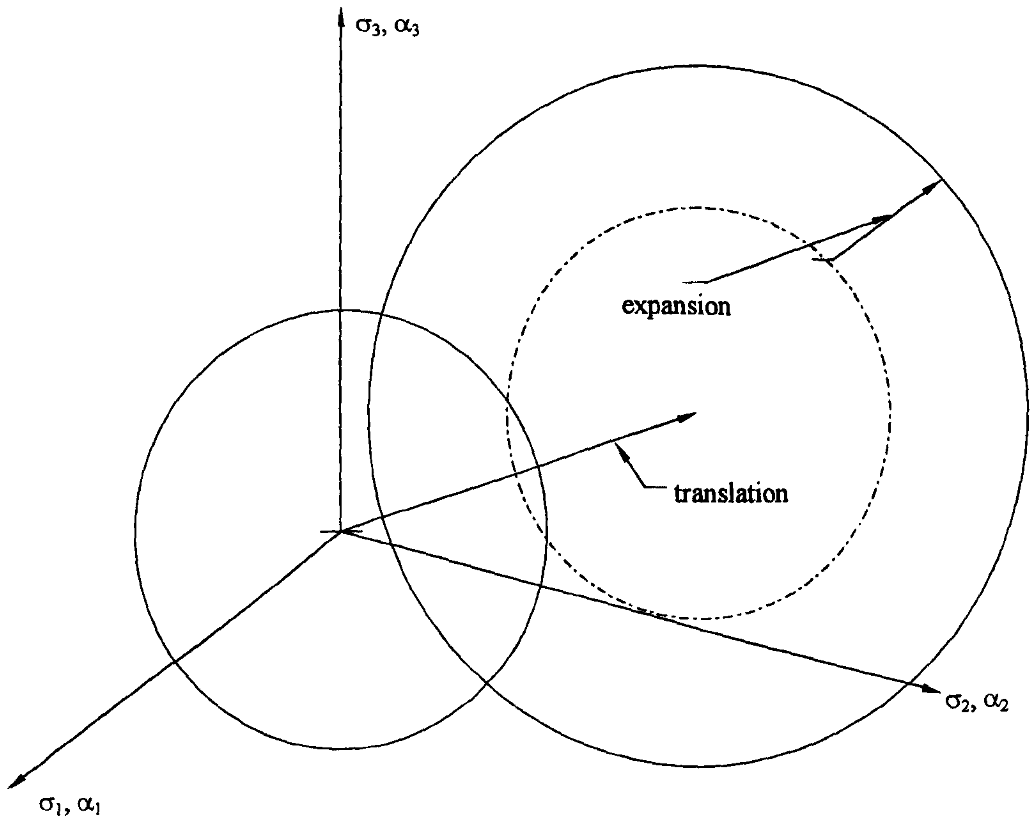


Fig.2.4 : Subsequent yield loci as described on the π plane in $\sigma_1\ \sigma_2\ \sigma_3$ -space for a material displaying a mixed hardening behaviour and obeying von Mises yield criterion.

CHAPTER 3

THE ANALYTICAL FORMULATION OF THE BOUNDARY ELEMENT METHOD IN 2D ELASTOPLASTICITY

It is known that the foundations of the BE formulation are the fundamental solution to the governing differential equation of a given problem and Betti's work theorem. Therefore, it may be defined as the combination of the Betti's work theorem with the fundamental solution, which leads to a singular solution to a given governing differential equation. It is also possible to formulate the boundary integral equation using indirect (also known as the source potential technique), semi-direct or direct approaches. The direct approach is based on a formulation in terms of physical quantities such as tractions or displacements on the boundary, or surface, of the solution domain. Therefore, this technique has been much more developed. In the BE formulation, the governing differential equations of a given problem are put into integral expressions in order to be applicable over boundary of the solution domain. Hence, when applying the BE approach to linear problems the main advantage is that a complete domain meshing, or remeshing, process is not required.

For non-linear problems, such as elasto-plastic problems, the extension of the BE method requires the evaluation the domain, or volume, integrals. One of the main difficulties encountered in almost all BE formulation is to perform the singular integrals affecting both computational efficiency and accuracy, but there are well-established techniques to evaluate them accurately.

In this chapter, the analytical formulation of the elasto-plastic BE formulation is presented. Before proceeding to the elasto-plastic case, the mathematical basis of the BE method in two-dimensional elasticity is presented.

3.1 BOUNDARY ELEMENT METHOD IN 2D ELASTICITY

3.1.1 The Galerkin Vector

Navier equation can be transformed into biharmonic differential equations for which solutions exist. To do this the following expression can be used.

$$\begin{aligned} u_x &= \frac{\partial^2 G_x}{\partial x^2} + \frac{\partial^2 G_x}{\partial y^2} - \frac{1}{2(1-\nu)}\left(\frac{\partial^2 G_x}{\partial x^2} + \frac{\partial^2 G_y}{\partial x \partial y}\right) \\ u_y &= \frac{\partial^2 G_y}{\partial x^2} + \frac{\partial^2 G_y}{\partial y^2} - \frac{1}{2(1-\nu)}\left(\frac{\partial^2 G_y}{\partial y^2} + \frac{\partial^2 G_x}{\partial x \partial y}\right) \end{aligned} \qquad \textbf{(3.1)}$$

Or in tensor notation

$$u_i = \frac{\partial^2 G_i}{\partial x_j \, \partial x_j} - \frac{1}{2(1-\nu)} \frac{\partial^2 G_j}{\partial x_i \, \partial x_j} \qquad \textbf{(3.2)}$$

in which the vector, G, is called the Galerkin Vector.

By substitution of the equation (3.1) in Navier equations, equation (2.12), the following biharmonic equations are obtained.

$$\nabla^4 G_x = \nabla^2 (\nabla^2 G_x) = \frac{-f_x}{\mu}$$
$$\nabla^4 G_y = \nabla^2 (\nabla^2 G_y) = \frac{-f_y}{\mu} \tag{3.3}$$

The fundamental solution is based on the three-dimensional classical solution of a point force in an infinite medium called the Kelvin solution.

3.1.2 The Kelvin Solution

The problem of a single concentrated force applied in the interior of an infinite domain is known as the Kelvin problem. It is assumed that a unit force is applied on an interior point P with coordinates X_p, Y_p, Z_p and the effect of this force on another point Q with coordinates x_Q, y_Q and z_Q anywhere in the domain can be examined. Capital letters signify fixed coordinates while lower case letters signify variable coordinates. The solution has to satisfy two conditions:

(i) All stresses must vanish as the distance between P and Q tends to infinity.

(ii) The stresses must be 'singular' at P itself (i.e. tend to infinity as the distance between P and Q tends to zero).

For two-dimensional problems, the Kelvin solution can be interpreted as a line load, whereas for axisymmetric problems its interpretation is a ring load.

It can be verified (Cruse [1977]) that the following solutions satisfy the biharmonic equation (3.3):

$$G_x = G_y = \frac{1}{8\pi\mu} r^2(P,Q) \ln\left[\frac{1}{r(P,Q)}\right] \tag{3.4}$$

where r(P,Q) is the distance between P, and Q, defined as follows:

$$r\ (P,Q) = \sqrt{(X_p - x_Q)^2 + (Y_p - y_Q)^2} \tag{3.5}$$

by substituting G_X and G_Y into equations (3.1) the following expression is obtained.

$$u_i = \frac{1}{8\pi\mu(1-\nu)} \left\{ (3-4\upsilon) \ln\left[\frac{1}{r(P,Q)}\right] \delta_{ij} + \frac{\partial r(P,Q)}{\partial x_i} \frac{\partial r(P,Q)}{\partial x_j} \right\} \tag{3.6}$$

In order to divide the displacement vector components into tensor functions the following expression can be written

$$u_i = U_{ij}\ (P,Q)\ e_j \tag{3.7}$$

in which the functions $U_{ij}(P,Q)$ are defined as

$$\begin{aligned} U_{xx}\ (P,Q) &= \frac{1}{8\pi\mu(1-\nu)} \left[(3-4\nu) \ln\left(\frac{1}{r}\right) + \left(\frac{\partial r}{\partial x}\right)^2 \right] \\ U_{xy}\ (P,Q) &= U_{yx}\ (P,Q) = \frac{1}{8\pi\mu(1-\nu)} \frac{\partial r}{\partial x} \frac{\partial r}{\partial y} \\ U_{yy}\ (P,Q) &= \frac{1}{8\pi\mu(1-\nu)} \left[(3-4\nu) \ln\left(\frac{1}{r}\right) + \left(\frac{\partial r}{\partial y}\right)^2 \right] \end{aligned} \tag{3.8}$$

Or in tensor notation

$$U_{ij}\ (P,Q) = \frac{1}{8\pi\mu(1-\nu)} \left[(3-4\nu)\ \ln\left(\frac{1}{r\ (P,Q)}\right)\delta_{ij} + \frac{\partial r(P,Q)}{\partial x_i}\ \frac{\partial r(P,Q)}{\partial x_j}\right] \tag{3.9}$$

Those functions are called displacement kernels. By differentiating the displacement vector and substituting in the Hooke's law equations (2.5), the traction vector can be obtained as follows (Becker [1992]):

$$\begin{aligned} t_i = & \frac{-1}{4\pi(1-\nu)\ r(P,Q)} \left(\frac{\partial r(P,Q)}{\partial n}\right)\left[(1-2\nu)\ \delta_{ij} + 2\ \frac{\partial r(P,Q)}{\partial x_i}\ \frac{\partial r(P,Q)}{\partial x_j}\right] \\ & + \frac{1-2\nu}{4\pi(1-\nu)r(P,Q)} \left[\frac{\partial r(P,Q)}{\partial x_j}\ n_i - \frac{\partial r(P,Q)}{\partial x_i}\ n_j\right] \end{aligned} \tag{3.10}$$

In order to divide the traction vector components into tensor functions the following expression can be used:

$$t_i = T_{ij}\ (P,Q)\ e_j$$

In this expression the functions T_{ij}(P,Q)are called the traction kernels and given as follows:

$$\begin{aligned} T_{xx}\ (P,Q) &= \frac{-1}{4\pi(1-\nu)r}\left(\frac{\partial r}{\partial n}\right)\left[(1-2\nu) + 2\left(\frac{\partial r}{\partial x}\right)^2\right] \\ T_{xy}\ (P,Q) &= \frac{-1}{4\pi(1-\nu)r}\left[2\ \frac{\partial r}{\partial x}\ \frac{\partial r}{\partial y}\ \frac{\partial r}{\partial n} + (1-2\nu)\left(\frac{\partial r}{\partial y}\ n_x - \frac{\partial r}{\partial x}\ n_y\right)\right] \\ T_{yx}\ (P,Q) &= \frac{-1}{4\pi(1-\nu)r}\left[2\ \frac{\partial r}{\partial x}\ \frac{\partial r}{\partial y}\ \frac{\partial r}{\partial n} - (1-2\nu)\left(\frac{\partial r}{\partial y}\ n_x - \frac{\partial r}{\partial x}\ n_y\right)\right] \\ T_{yy}\ (P,Q) &= \frac{-1}{4\pi(1-\nu)r}\left(\frac{\partial r}{\partial n}\right)\left[(1-2\nu) + 2\left(\frac{\partial r}{\partial y}\right)^2\right] \end{aligned} \tag{3.11}$$

or in tensor notation

$$T_{ij}(P,Q)=\frac{-1}{4\pi(1-\nu)r}\left(\frac{\partial r}{\partial n}\right)\left[(1-2\nu)\ \delta_{ij}\ +2\frac{\partial r}{\partial x_i}\frac{\partial r}{\partial x_j}\ +(1-2\nu)\left(\frac{\partial r}{\partial x_j}n_i\ -\frac{\partial r}{\partial x_i}n_j\right)\right] \tag{3.12}$$

In this expression, the derivative $\partial r/\partial n$ is given by

$$\frac{\partial r}{\partial n} = \frac{\partial r}{\partial x}\frac{\partial x}{\partial n} + \frac{\partial r}{\partial y}\frac{\partial y}{\partial n} \tag{3.13}$$

The components of the unit outward normal in the x and y direction, n_x and n_y are given by

$$n_x = \frac{\partial x}{\partial n}\ ;\ n_y = \frac{\partial y}{\partial n} \tag{3.14}$$

The derivatives of the distance r(P,Q) can be written as follows:

$$\frac{\partial r(P,Q)}{\partial x} = \frac{x_Q - X_P}{r(P,Q)}\ ;\ \frac{\partial r(P,Q)}{\partial y} = \frac{y_Q - Y_P}{r(P,Q)} \tag{3.15}$$

3.1.3 The Boundary Integral Equation

It is possible to consider a body under equilibrium with two different sets of stresses and strains, as follows:

i) A set (a) of applied stresses $\sigma^{(a)}_{ij}$ that gives rise to a set of strains $\varepsilon^{(a)}_{ij}$.

ii) A set (b) of applied stresses $\sigma^{(b)}_{ij}$ that gives rise to a set of strains $\varepsilon^{(b)}_{ij}$.

The reciprocal work theorem, also knows as Betti's theorem, states that the work done by the stresses of system (a) on the displacement of system (b) is equal to the work done by the stresses of system (b) on the displacements of system (a). Therefore, the following relationship can be written:

$$\int_v \sigma_{ij}^{(a)} \, \varepsilon_{ij}^{(b)} \, dV = \int_v \sigma_{ij}^{(b)} \, \varepsilon_{ij}^{(a)} \, dV \tag{3.16}$$

From equation (3.16), the following expression for Betti's theorem can be derived, (See Becker [1992]):

$$\int_s t_i^{(a)} \, u_i^{(b)} \, dS + \int_v f_i^{(a)} \, u_i^{(b)} \, dV = \int_s t_i^{(b)} \, u_i^{(a)} \, dS + \int_v f_i^{(b)} \, u_i^{(a)} \, dV \tag{3.17}$$

This integral equation can be transformed into a boundary integral equation (BIE) by using two distinct sets of displacements and tractions as follows:.

set (a): This is the actual problem to be solved in which the displacement, $u_i^{(a)}$, and the traction $t_i^{(a)}$ which satisfy the boundary conditions of the problem to be solved are unknown.

set (b): The displacement $u_i^{(b)}$ and the traction $t_i^{(b)}$ which have to be valid for any geometry in equilibrium to be solved are known

Hence the following expression can be written

$$u_i^{(a)} = u_i\,(Q) \;; \quad t_i^{(a)} = t_i\,(Q) \;; \quad f_i^{(a)} = f_i(q)$$

(3.18)

$$u_i^{(b)} = U_{ij}\,(P,Q)\,e_j \;; \quad t_i^{(b)} = T_{ij}\,(P,Q)\,e_j \;; \quad f_i^{(b)} = 0$$

By substituting the above expression into Betti's equation (3.17) without considering body forces, the following boundary integral equation, known as Somigliana identity, can be written:

$$C_{ij}(P)\ u_i(P) + \int_S T_{ij}(P,Q)\ u_j(Q)\ dS(Q) = \int_S U_{ij}(P,\ Q)\ t_j(Q)\ dS(Q) \tag{3.19}$$

where U_{ij} and T_{ij} are displacements and tractions respectively at field point, Q, in the j[th] direction due to a unit load acting at the load point, P, or the interior point. S indicates the boundary of the domain to be solved.

The free-term C_{ij} can be calculated by surrounding the point P by a small circle of radius 'ε' and defined in the limit as $\varepsilon \rightarrow 0$ by

$$C_{ij}(P) = \delta_{ij} + \lim_{\varepsilon \rightarrow 0} \int_{\varepsilon(P)} T_{ij}(P,\ Q)\ dS(Q) \tag{3.20}$$

By differentiating this expression for the displacements at load point P and substitution in equation (2.5), for stresses at load point P, the following integral equation can be written:

$$\sigma_{ij}\ (P) + \int_S S_{kij}\ (P,\ Q)\ u_k\ (Q)\ dS(Q) = \int_S D_{kij}\ (P,\ Q)\ t_k\ (Q)\ dS\ (Q) \tag{3.21}$$

In this expression the kernels S_{kij} and D_{kij} are given as follows:

$$S_{kij}(P,Q) = \frac{\mu}{2\pi\ (1-\nu)}\left(\frac{1}{r^2}\right)n_i\left[2\nu\ \frac{\partial r}{\partial x_j}\frac{\partial r}{\partial x_k} + (1-2\nu)\delta_{jk}\right]$$

$$+\frac{\mu}{2\pi\,(1-\nu)}\left(\frac{1}{r^2}\right)n_j\left[2\nu\frac{\partial r}{\partial x_i}\frac{\partial r}{\partial x_k}+(1-2\nu)\,\delta_{ik}\right]$$

(3.22)

$$+\frac{\mu}{2\pi\,(1-\nu)}\left(\frac{1}{r^2}\right)n_k\left[2\,(1-2\nu)\,\frac{\partial r}{\partial x_i}\frac{\partial r}{\partial x_j}-(1-4\nu)\,\delta_{ij}\right]$$

$$+\frac{\mu}{\pi(1-\nu)}\left(\frac{1}{r^2}\right)\left(\frac{\partial r}{\partial n}\right)\left[(1-2\nu)\delta_{ij}\frac{\partial r}{\partial x_k}+\nu\left(\delta_{jk}\frac{\partial r}{\partial x_i}+\delta_{ik}\frac{\partial r}{\partial x_j}\right)-4\frac{\partial r}{\partial x_i}\frac{\partial r}{\partial x_j}\frac{\partial r}{\partial x_k}\right]$$

$$D_{kij}(P,Q)=\frac{1}{4\pi(1-\nu)}\left(\frac{1}{r}\right)\left[(1-2\nu)\left(\delta_{jk}\frac{\partial r}{\partial x_i}+\delta_{ik}\frac{\partial r}{\partial x_j}-\delta_{ij}\frac{\partial r}{\partial x_k}\right)+2\frac{\partial r}{\partial x_i}\frac{\partial r}{\partial x_j}\frac{\partial r}{\partial x_k}\right] \quad \textbf{(3.23)}$$

Further details of the elastic BIE formulation can be found in boundary element textbooks (see, for example Becker [1992], Banerjee [1994]).

3.2 ANALYTICAL ELASTO-PLASTIC BE FORMULATION

One of the main difficulties encountered in almost all BE formulations is the integration of the singular integrals which affects both computational efficiency and accuracy. There are, however, well-established techniques to evaluate them accurately.

For elasto-plastic problems it is known that the direct BE formulation is commonly treated using both initial strain approach and initial stress approach. Therefore, in this chapter both approaches are reviewed.

3.2.1 A brief review of the elasto-plastic BE formulation

The first elasto-plastic BE formulation presented by Swedlow and Cruse [1971] was based on a direct analytical formulation. Riccardella [1973] presented the initial strain formulation based on a constant plastic strain over each internal cell with a non-iterative procedure. Mendelson [1973] provided a review of the BE formulations which is based on indirect, semi-direct and direct approaches in two and three-dimensional problems. Mukherjee [1977] presented a correct direct BIE formulation in plane strain analysis. Telles and Brebbia [1980] presented a direct BE formulation based on the initial strain approach with corrections for the internal stresses and a semi-analytical approach for the efficient evaluation of the strongly singular integrals appearing in the domain kernels by using linear elements.

It is possible to analyse the elasto-plastic problems using either the indirect BE approach used by some authors such as Banerjee and Mustoe [1978], Kobayashi and Nishimura [1980], Morjaria and Mukherjee [1981], or the direct BIE approach. The latter formulation is a much more developed approach.

Faria et al [1981] performed the singular integrals in a manner similar to that of Telles and Brebbia [1981] by using quadratic elements. The elasto-plastic BE formulation was discussed in detail by Lee [1983] who presented an accelerated convergence procedure using an initial strain approach and quadratic elements. Some authors, such as Tan and Lee [1983] and Lee and Fenner [1986], used this approach to analyse practical problems such as fracture problems.

There are other BIE formulations which are applicable to other non-linear problems, such as viscoplasticity and time-dependent problems (see, for example, Kumar and Mukherjee [1977], Telles and Brebbia [1982], Banerjee and Davies [1984] and Ahmad and Banerjee [1988]).

One of the most significant difficulties of all non-linear BE analysis is the evaluation of the singular integrals (defined only in the Cauchy Principal values sense) arising in the solution domain, or volume kernels. Henry and Banerjee [1988] presented an particular integral approach to circumvent the singular volume integrals. Okada et al [1990] presented another approach, which handles geometric and material non-linearity problems, based on the interpolation of the basic variables to be computed in solution domain.

Banerjee and Ravendra [1986] presented a direct approach to evaluate the strongly singular integrals by excluding a small sphere, where load point is located, from the integration of volume cell. Banerjee et al [1989] presented an indirect approach, initial stress expansion technique, which is based on the admissible stress states for evaluation of the strongly singular integrals.

Lu and Ye [1990] presented a direct technique by the use of coordinate transformation and a form of Stokes' theorem with numerical examples which are two-dimensional elasto-plastic and three-dimensional elastic problems using quadratic internal cell and quadratic boundary element. Guiggiani and Gigante [1990] presented a direct approach to evaluate the strongly singular integrals for arbitrary cells by using Taylor series expansion and local polar coordinates. The study of Guiggiani et al [1992] provided a general algorithm in order to treat numerically the hyper-singular integrals arising in BE formulation. This work can be considered as extension of the study presented by Guiggiani and Gigante [1990]. Dallner and Kuhn [1993] presented a direct approach for the efficient evaluation of the strongly singular integrals appearing in the solution domain, or volume, kernels in non-linear BE formulations with three-dimensional examples by using a regularised formulation based on the Gauss theorem. This approach is capable of handling viscoplasticity and large deformation problems.

3.2.2 The initial strain approach

To include the effect of the elasto-plastic material behaviour, by modifying Betti's work theorem, the direct BE formulation has an additional term based on the work done by the strain rate, ε^p_{ij}, multiplied by the stress at load point (a variable point in the i^{th} direction (see, for example, Lee [1983]). This plastic work term can be defined as follows:

$$\int_A \dot{\varepsilon}^p_{ij}\ W_{kij}(P,\ q)\ dA \tag{3.24}$$

where A indicates surface of the solution domain and the kernel, W_{kij}, the stress of corresponding fundamental solution can be written as follows:

$$W_{kij}(P,q)=\frac{-1}{4\pi\ (1-\nu)}\left(\frac{1}{r}\right)\left[(1-2\nu)\left(\delta_{jk}\frac{\partial r}{\partial x_i}+\delta_{ik}\frac{\partial r}{\partial x_j}-k\delta_{ij}\frac{\partial r}{\partial x_k}\right)+2\frac{\partial r}{\partial x_i}\frac{\partial r}{\partial x_j}\frac{\partial r}{\partial x_k}\right] \tag{3.25}$$

where the parameter k is either equal to 1 (plane stress) or $1/(1-2\nu)$ (plane strain).

As mentioned earlier, for elasto-plastic problems, without considering the thermal loads and body forces, by adding the additional term the BE equation can be written as follows:

$$C_{ij}(P)\,\dot{u}_i(P) + \int_S T_{ij}(P,Q)\,\dot{u}_j(Q)\,dS(Q) = \int_S U_{ij}(P,Q)\,\dot{t}_j(Q)\,dS(Q) + \int_A W_{kij}(P,q)\,\dot{\varepsilon}^p_{ij}\,dA(q) \tag{3.26}$$

In this expression U_{ij} and T_{ij} are fundamental displacement and traction at x in the j^{th} direction due to a unit load at load point P acting in i^{th} direction. Note that the equation (3.26) is expressed in rate form.

To include body forces, in quasi behaviour, the modified BIE in the initial strain approach can be written as follows (see Lee [1983]).

$$C_{ij}(P)\,\dot{u}_i(P) + \int_S T_{ij}(P,Q)\,\dot{u}_j(Q)\,dS(Q) = \int_S U_{ij}\,(P,Q)\,\dot{t}_j(Q)\,dS(Q) + \int_A U_{ij}\,(P,Q)\,\dot{f}_j(q)\,dA(q) + \int_A W_{kij}\,(P,q)\,\dot{\varepsilon}^p_{ij}(q)\,dA(q) \tag{3.27}$$

In order to obtain the correct expression of the plastic deformation rate in the solution domain, differentiation can be employed. At internal points the total plastic rate can then be given as follows:

$$
\begin{aligned}
\dot{\varepsilon}_{ij}(P) + \int_S S^{\varepsilon}_{kij}(P,Q)\, \dot{u}_k(Q)\, dS(Q) &= \int_S D^{\varepsilon}_{kij}(P,Q)\, \dot{t}_k\, dS(Q) \\
&+ \int_A D^{\varepsilon}_{kij}(P,q)\, \dot{f}_k(q)\, dA(q) \\
&+ \int_A \left[W^{\varepsilon}_{ijkh}(P,q) + \bar{W}^{\varepsilon}_{ijkh}(P,q) \right] \dot{\varepsilon}^{P}_{kh}\, dA(q) + F^{\varepsilon}_{ij}\left(\dot{\varepsilon}^{P}_{kh}(P)\right)
\end{aligned}
\tag{3.28}
$$

where the third-order kernels are given as follows:

$$
D^{\varepsilon}_{kij}(P,Q) = \frac{1}{8\pi\mu(1-\nu)} \left(\frac{1}{r}\right) \left[(1-2\nu)\left(\delta_{jk}\frac{\partial r}{\partial x_i} + \delta_{ik}\frac{\partial r}{\partial x_j} \right) - \delta_{ij}\frac{\partial r}{\partial x_k} + 2\frac{\partial r}{\partial x_i}\frac{\partial r}{\partial x_j}\frac{\partial r}{\partial x_k} \right]
\tag{3.29}
$$

$$
S^{\varepsilon}_{kij}(P,Q) = \frac{1}{4\pi(1-\nu)} \left(\frac{1}{r^2}\right) \left(\frac{\partial r}{\partial n}\right) \left[2\nu\left(\delta_{jk}\frac{\partial r}{\partial x_i} + \delta_{ik}\frac{\partial r}{\partial x_j} \right) + 2\delta_{ij}\frac{\partial r}{\partial x_k} - 8\frac{\partial r}{\partial x_i}\frac{\partial r}{\partial x_j}\frac{\partial r}{\partial x_k} \right]
$$

$$
+ \frac{1}{4\pi\,(1-\nu)} \left(\frac{1}{r^2}\right) n_i \left[(1-2\nu)\,\delta_{jk} + 2\nu\frac{\partial r}{\partial x_j}\frac{\partial r}{\partial x_k} \right]
$$

$$
+ \frac{1}{4\pi\,(1-\nu)} \left(\frac{1}{r^2}\right) n_j \left[(1-2\nu)\,\delta_{ik} + 2\nu\frac{\partial r}{\partial x_i}\frac{\partial r}{\partial x_k} \right]
$$

$$- \frac{(1-2\nu)}{4\pi\ (1-\nu)} \left(\frac{1}{r^2}\right) n_k \left[\delta_{ij} - 2\ \frac{\partial r}{\partial x_i}\frac{\partial r}{\partial x_j}\right]$$

(3.30)

$$W^{\varepsilon}_{ijkh}\ (P,q) = \frac{1}{4\pi\ (1-\nu)} \left(\frac{1}{r^2}\right) \left[\ 2(1-2\nu)\ \delta_{ij}\frac{\partial r}{\partial x_k}\frac{\partial r}{\partial x_h} + 2\delta_{kh}\frac{\partial r}{\partial x_i}\frac{\partial r}{\partial x_j}\right.$$

$$- 8\ \frac{\partial r}{\partial x_i}\frac{\partial r}{\partial x_j}\frac{\partial r}{\partial x_k}\frac{\partial r}{\partial x_h}$$

$$+ 2\nu\left(\ \delta_{jh}\ \frac{\partial r}{\partial x_i}\frac{\partial r}{\partial x_k} + \delta_{jk}\ \frac{\partial r}{\partial x_i}\frac{\partial r}{\partial x_h}\right.$$

$$\left. + \delta_{ih}\ \frac{\partial r}{\partial x_j}\frac{\partial r}{\partial x_k} + \delta_{ik}\ \frac{\partial r}{\partial x_j}\frac{\partial r}{\partial x_h}\right)$$

$$\left. + (1-2\nu)\ (\ \delta_{jk}\delta_{ih} + \delta_{ik}\delta_{jh} - \delta_{ij}\delta_{kh})\right]$$

(3.31)

$$\bar{W}^{\varepsilon}_{ijkh}\ (P,q) = -\ \frac{\nu}{4\pi(1-\nu)} \left(\frac{1}{r^2}\right) \left(\delta_{ij}\ \delta_{kh} - 2\delta_{ij}\ \frac{\partial r}{\partial x_k}\ \frac{\partial r}{\partial x_h}\right) \quad \textit{(plane strain)}$$

$$= 0 \quad \textit{(plane stress)}$$

(3.32)

The free-term F_{ij}^{ε} is given as

$$F_{ij}^{\varepsilon}\left(\dot{\varepsilon}_{kh}^{P}(P)\right) = \frac{3-4\nu}{4(1-\nu)}\,\dot{\varepsilon}_{kh}^{P}(P) - \frac{1}{8(1-\nu)}\,\delta_{kh}\,\dot{\varepsilon}_{mm}^{P}(P) \quad (plane\ strain)$$

$$= \frac{3-4\nu}{4(1-\nu)}\,\dot{\varepsilon}_{kh}^{P}(P) - \frac{1-4\nu}{8(1-\nu)}\,\delta_{kh}\,\dot{\varepsilon}_{mm}^{P}(P) \quad (plane\ stress) \tag{3.33}$$

By using both equation (3.28) and the stress-strain relationship, the stress rate at domain points can be defined as follows:

$$\sigma_{ij}(P) + \int_S S_{kij}(P,Q)\,\dot{u}_k(Q)\,dS(Q) = \int_S D_{kij}(P,Q)\,\dot{t}_k(Q)\,dS(Q)$$
$$+ \int_A D_{kij}(P,Q)\,\dot{f}_k(q)\,dA(q) + \int_A \left[\,W_{ijkh}^{\sigma}(P,q) + \bar{W}_{ijkh}^{\sigma}(P,q)\,\right]\dot{\varepsilon}_{kh}^{P}\,dA(q)$$
$$+ F_{ij}^{\sigma}\left(\dot{\varepsilon}_{kh}^{P}(P)\right) \tag{3.34}$$

where $S_{kij}(P,Q)$ and $D_{kij}(P,Q)$ are given by equations (3.22) and (3.33), and the area (domain) kernels are given by:

$$W_{ijkh}^{\sigma}(P,q) = \frac{\mu}{2\pi(1-\nu)}\left(\frac{1}{r^2}\right)\left[\,2(1-2\nu)\left(\delta_{kh}\frac{\partial r}{\partial x_i}\frac{\partial r}{\partial x_j} + \delta_{ij}\frac{\partial r}{\partial x_k}\frac{\partial r}{\partial x_h}\right)\right.$$
$$- 8\,\frac{\partial r}{\partial x_i}\frac{\partial r}{\partial x_j}\frac{\partial r}{\partial x_k}\frac{\partial r}{\partial x_h}$$
$$+ 2\nu\left(\delta_{jh}\frac{\partial r}{\partial x_i}\frac{\partial r}{\partial x_k} + \delta_{jk}\frac{\partial r}{\partial x_i}\frac{\partial r}{\partial x_h} + \delta_{ih}\frac{\partial r}{\partial x_j}\frac{\partial r}{\partial x_k} + \delta_{ik}\frac{\partial r}{\partial x_j}\frac{\partial r}{\partial x_h}\right)$$
$$\left. + (1-2\nu)\left(\delta_{jk}\delta_{ih} + \delta_{ik}\delta_{jh}\right) - (1-4\nu)\,\delta_{ij}\delta_{kh}\,\right] \tag{3.35}$$

$$\bar{W}^{\sigma}_{ijkh}(P,q) = \frac{-\mu\nu}{\pi(1-\nu)}\left(\frac{1}{r^2}\right)\left(\delta_{ij}\,\delta_{kh} - 2\delta_{ij}\,\frac{\partial r}{\partial x_k}\,\frac{\partial r}{\partial x_h}\right) \quad (plane\ strain)$$

$$\bar{W}^{\sigma}_{ijkh}(P,q) = 0 \quad (plane\ stress) \tag{3.36}$$

The free-term $F^{\sigma}_{ij}\left(\dot{\varepsilon}^{P}_{kh}\,(P)\right)$ is given as follows:

$$F^{\sigma}_{ij}\left(\dot{\varepsilon}^{P}_{kh}(P)\right) = -\frac{\mu}{2(1-\nu)}\,\dot{\varepsilon}^{P}_{kh}(P) - \frac{\mu(1-4\nu)}{4(1-\nu)}\,\delta_{kh}\,\dot{\varepsilon}^{P}_{mm}(P) \quad (plane\ strain)$$

$$F^{\sigma}_{ij}\left(\dot{\varepsilon}^{P}_{kh}(P)\right) = -\frac{\mu}{2(1-\nu)}\,\dot{\varepsilon}^{P}_{kh}(P) - \frac{\mu}{4(1-\nu)}\,\delta_{kh}\,\dot{\varepsilon}^{P}_{mm}(P) \quad (plane\ stress) \tag{3.37}$$

3.2.3 Initial Stress Approach

It is obvious from the basic idea based on the Betti's work theorem that the initial stress rate becomes a primary unknown relating to the solution domain in the elasto-plastic BE formulation. Hence, by using the given relationship between the initial stress increments (rates) and the initial strain increments, in quasi-static behaviour, the modified BE integral equation including body forces (but not thermal effects) in initial strain approach can be rewritten as follows:

$$C_{ij}(P)\,\dot{u}_j(P) + \int_S T_{ij}(P,Q)\,u_j(Q)\,dS(Q) = \int_S U_{ij}(P,Q)\,t_j(Q)\,dS(Q)$$

$$+\int_A U_{ij}(P,Q)\,f_j(q)\,dA(q) + \int_A V_{kij}(P,q)\,\dot{\sigma}^{i}_{ij}\,(q)\,dA(q) \tag{3.38}$$

In this expression, the kernel, V_{kij}, the stress of corresponding fundamental solution, can be defined as follows:

$$V_{kij}(P,q)= \frac{-1}{8\pi\mu(1-\nu)}\left(\frac{1}{r}\right)\left[(1-2\nu)\left(\delta_{jk}\frac{\partial r}{\partial x_i}+\delta_{ik}\frac{\partial r}{\partial x_j}\right)-k\delta_{ij}\frac{\partial r}{\partial x_k}+2\frac{\partial r}{\partial x_i}\frac{\partial r}{\partial x_j}\frac{\partial r}{\partial x_k}\right] \tag{3.39}$$

where the parameter k is given as follows:

$$k = 1 + 2\nu(1-2\nu) \qquad \text{plane strain} \tag{3.40}$$

$$k = 1 \qquad \text{plane stress}$$

Following a similar procedure to the strain approach the initial strain increments can be written:

$$\begin{aligned}\dot{\varepsilon}_{ij}(P) + \int_S S^{\varepsilon}_{kij}(P,Q)\,\dot{u}_k(Q)\,dS(Q) &= \int_S D^{\varepsilon}_{kij}(P,Q)\,\dot{t}_k(Q)\,dS(Q)\\ +\int_A D^{\varepsilon}_{kij}(P,q)\,\dot{f}_k(q)\,dA(q) &+ \int_A\left[V^{\varepsilon}_{ijkh}(P,q)+\bar{V}^{\varepsilon}_{ijkh}(P,q)\right]\dot{\sigma}^i_{kh}\,dA(q)\\ &+ F^{\varepsilon}_{ij}\left(\dot{\sigma}^i_{kh}\,(q)\right)\end{aligned} \tag{3.41}$$

where

$$\begin{aligned}V^{\varepsilon}_{ijkh}(P,q) = \frac{1}{8\pi\mu(1-\nu)}\left(\frac{1}{r^2}\right)\Bigg[\,&2\delta_{ij}\frac{\partial r}{\partial x_k}\frac{\partial r}{\partial x_h}+2\delta_{kh}\frac{\partial r}{\partial x_i}\frac{\partial r}{\partial x_j}\\ &-8\frac{\partial r}{\partial x_i}\frac{\partial r}{\partial x_j}\frac{\partial r}{\partial x_k}\frac{\partial r}{\partial x_h}\\ &+2\nu\left(\delta_{jh}\frac{\partial r}{\partial x_i}\frac{\partial r}{\partial x_k}+\delta_{jk}\frac{\partial r}{\partial x_i}\frac{\partial r}{\partial x_h}+\delta_{ih}\frac{\partial r}{\partial x_j}\frac{\partial r}{\partial x_k}+\delta_{ik}\frac{\partial r}{\partial x_j}\frac{\partial r}{\partial x_h}\right)\\ &+(1-2\nu)\left(\delta_{jk}\delta_{ih}+\delta_{ik}\delta_{jh}\right)-\delta_{ij}\delta_{kh}\Bigg]\end{aligned} \tag{3.42}$$

and

$$\bar{V}^{\varepsilon}_{ijkh}(P,q)= -\frac{\nu(1-2\nu)}{4\pi\mu\,(1-\nu)}\left(\frac{1}{r^2}\right)\left(\delta_{ij}\delta_{kh}-2\frac{\partial r}{\partial x_k}\frac{\partial r}{\partial x_h}\right) \quad (\textit{plane strain})$$

$$\bar{V}^{\varepsilon}_{ijkh}\,(P,q) = 0 \quad (\textit{plane stress})$$

(3.43)

The free-term F^{ε}_{ij} $(\dot{\sigma}_{ij}^{i}$ (P)) is defined by

$$F^{\varepsilon}_{ij}(\dot{\sigma}^{i}_{kh}\ (P))= \frac{3-4\nu}{8\mu(1-\nu)}\dot{\sigma}^{i}_{kh}(P)+ \frac{1+4\nu-8\nu^{2}}{16\mu(\nu-1)}\delta_{kh}\dot{\sigma}^{i}_{mm}(P) \quad (plane\ strain)$$

$$F^{\varepsilon}_{ij}(\dot{\sigma}^{i}_{kh}(P))= \frac{3-4\nu}{8\mu\ (1-\nu)}\dot{\sigma}^{i}_{kh}(P)+ \frac{1}{16\mu\ (\nu-1)}\ \delta_{kh}\dot{\sigma}^{i}_{mm}(P) \quad (plane\ stress) \qquad \mathbf{(3.44)}$$

As in the initial strain approach the stress rate at the solution domain point can be given as follows (for details, see Lee [1983]):

$$\dot{\sigma}_{ij}(P) + \int_S S_{kij}(P,Q)\ \dot{u}_k(Q)\ dS(Q) + \int_S D_{kij}(P,Q)\ \dot{t}_k(Q)\ dS(Q)$$

$$+ \int_A D_{kij}(P,q)\ \dot{f}_k(q)\ dA(q) + \int_A \left[V^{\sigma}_{ijkh}(P,q) + \bar{V}^{\sigma}_{ijkh}(P,q)\right]\ \dot{\sigma}^{i}_{kh}(q)\ dA(q)$$

$$+ f^{\sigma}_{ij}\ (\dot{\sigma}^{i}_{kh}(q)) \qquad \mathbf{(3.45)}$$

where

$$V^{\sigma}_{ijkh}(P,q)= \frac{1}{4\pi(1-\nu)}\left(\frac{1}{r^{2}}\right)\ [\ (2-4\nu)\delta_{kh}\frac{\partial r}{\partial x_i}\frac{\partial r}{\partial x_j}+ 2\delta_{ij}\frac{\partial r}{\partial x_k}\frac{\partial r}{\partial x_h}$$

$$-8\frac{\partial r}{\partial x_i}\frac{\partial r}{\partial x_j}\frac{\partial r}{\partial x_k}\frac{\partial r}{\partial x_h}$$

$$+2\nu\left((\delta_{jh}\frac{\partial r}{\partial x_i}\frac{\partial r}{\partial x_k}+ \delta_{jk}\frac{\partial r}{\partial x_i}\frac{\partial r}{\partial x_h}+ \delta_{ih}\frac{\partial r}{\partial x_j}\frac{\partial r}{\partial x_k}+ \delta_{ik}\frac{\partial r}{\partial x_j}\frac{\partial r}{\partial x_h}\right)$$

$$+ (1-2\nu)(\delta_{jk}\delta_{ih}+ \delta_{ik}\delta_{jh}- \delta_{ij}\delta_{kh})] \qquad \mathbf{(3.46)}$$

and

$$\bar{V}^{\sigma}_{ijkh}(P,q) = \frac{\nu\ (2\nu-1)}{2\pi\ (1-\nu)}\left(\frac{1}{r^{2}}\right)\left[\delta_{ij}\ \delta_{kh}- 2\delta_{ij}\frac{\partial r}{\partial x_k}\frac{\partial r}{\partial x_h}\right] \quad (plane\ strain)$$

$$\bar{V}^{\sigma}_{ijkh}(P,q) = 0 \quad (plane\ stress)$$

$$(3.47)$$

Finally, the free-term $f_{ij}^{\sigma}(\dot{\sigma}_{kh}^{i}(P))$ is defined as follows:

$$f_{ij}^{\sigma}\ \dot{\sigma}_{kh}^{i}(P)) = \frac{1}{4(\nu-1)}\ \dot{\sigma}_{kh}^{i}(P)\ 1\frac{1-8\nu+8\nu^2}{8(1-\nu)}\ \delta_{kh}\dot{\sigma}_{mm}^{i}\ (P)\quad (plane\ strain)$$

$$f_{ij}^{\sigma}(\dot{\sigma}_{kh}^{i}(P)) = \frac{1}{4(\nu-1)}\dot{\sigma}_{kh}^{i}(P) + \frac{4\nu-1}{8(1-\nu)}\ \delta_{kh}\dot{\sigma}_{mm}^{i}(P)\quad (plane\ stress) \tag{3.48}$$

3.2.4 The incremental solution procedure

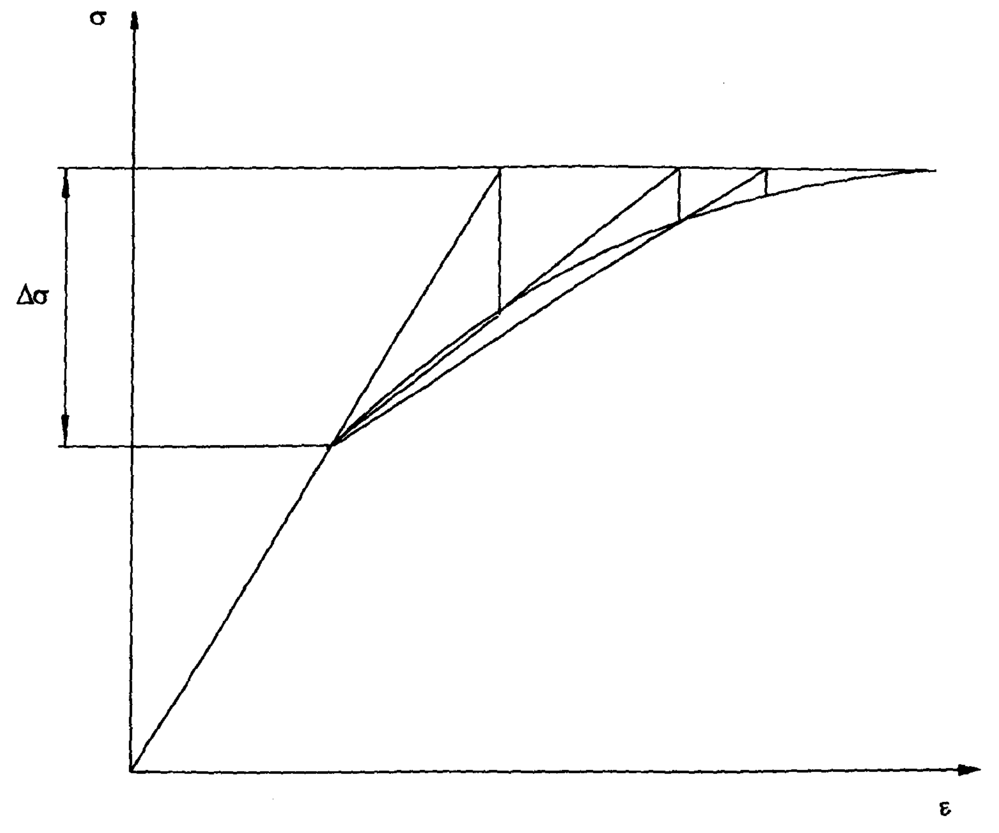
In the elasto-plastic finite element analysis, depending on the formulation of the stiffness matrix, either the tangential stiffness technique or the initial stiffness technique can be employed. Both the initial and final stiffnesses are sometimes employed to compute an average elasto-plastic stiffness matrix (see Figure 3.1(a)), but this procedure leads to a computational burden. In the tangential stiffness matrix formulation, the stiffness matrix can be interpreted as a function of the tangent of the equivalent stress-strain curve of the material at the stress state being analysed (see Figure 3.1(b)). In the initial stiffness method, the stiffness matrix is computed only once at the beginning of each new load increment. Therefore, this results in a constant stiffness matrix during the iteration (see Figure 3.1(c)). On the other hand, more iterations may be required in order to reach a convergence.

Because of the nature of the elasto-plastic BE formulation, neither tangential stiffness method nor the initial stiffness method can be used. However the initial strain and the initial stress techniques which are used in the finite element approach (see, for example, Zienkiewicz et al [1969]) can be modified in order to be applicable to the elasto-plastic BE analysis.

In the elasto-plastic BE analysis, by using either equation (2.61) or (2.62) the plastic strain increments can be determined. It is clear from the equation (2.62) that it is necessary to know the actual stress increments in order to obtain the initial strain increments (see Figure 3.2). In this approach (initial strain approach) it should be noted that the initial load increments are obtained from the initial strain increments.

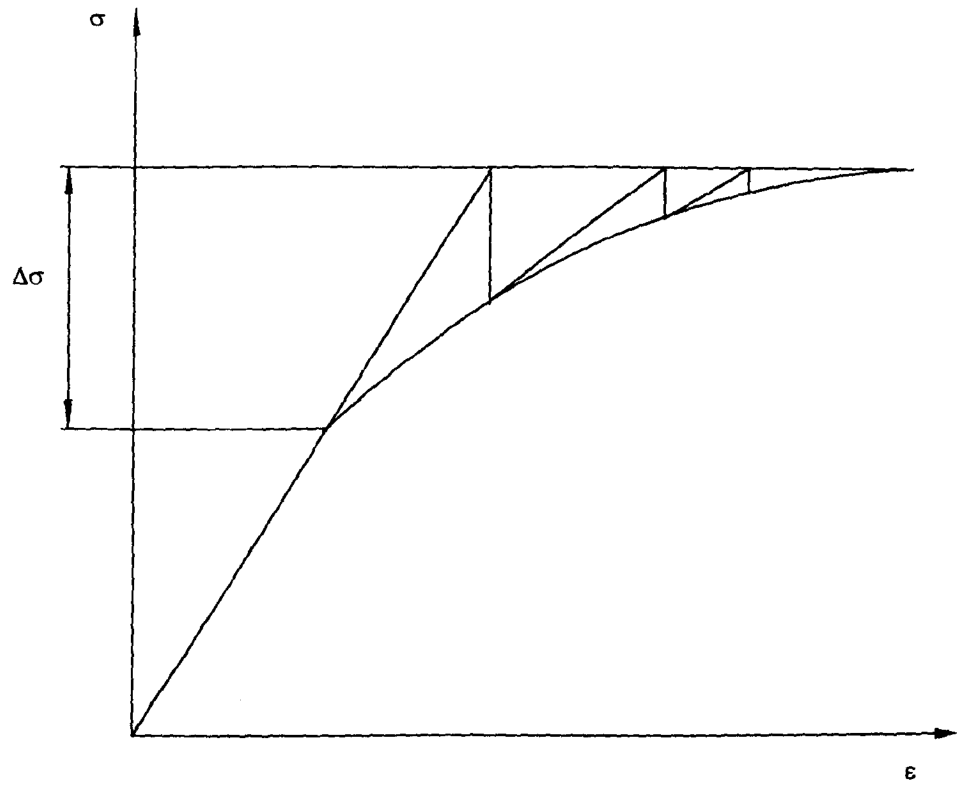
The initial stresses can be obtained by using equation (2.61) in which the total strain increments are assumed to be known (see Figure 3.3). In this approach (initial stress approach) it is obvious that the initial stresses are obtained from the total strain increment in order to determine initial loads. In the initial strain approach it is possible to obtain the plastic strain increments by using equation (2.61), which can handle the perfectly plastic material behaviour, in which total strain increments are assumed to be known.

There is no significant difference between initial stress approach and initial strain approach, because integral equations in both approaches include the effect of plasticity. It is known that the first approximation for the stress increments are usually reasonably accurate. Therefore the initial strain formulation is suitable for traction-controlled problems.

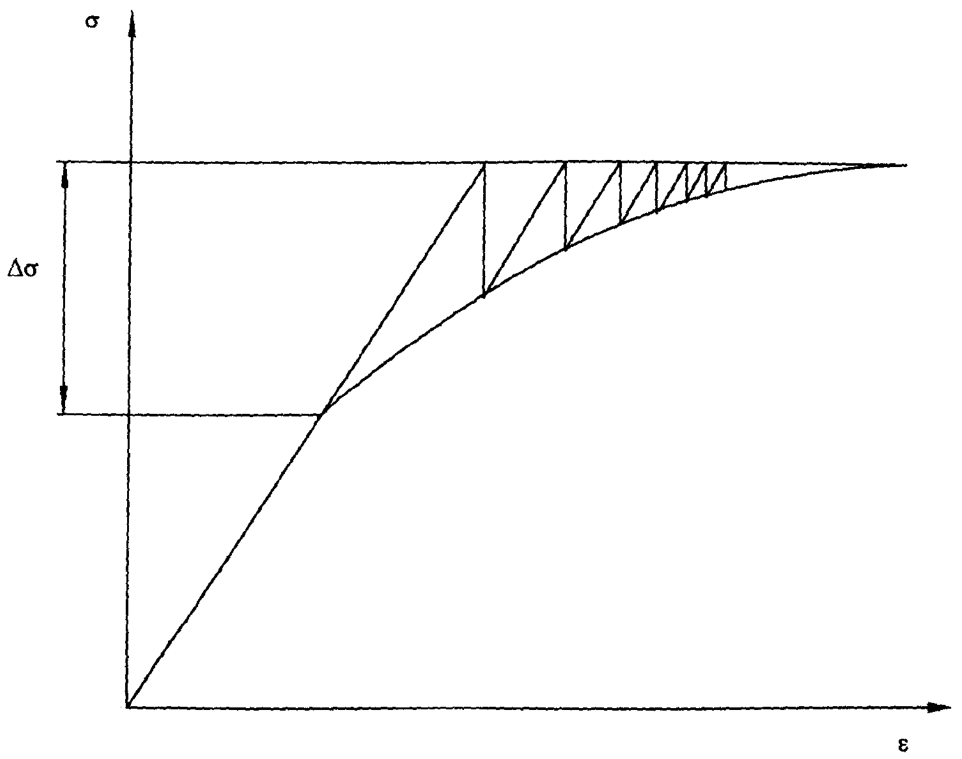


(a) Average stiffness matrix approach

Fig.3.1 : Tangential stiffness solution technique used in FE analysis.



(b) Updated stiffness matrix approach



(c) Constant stiffness matrix approach

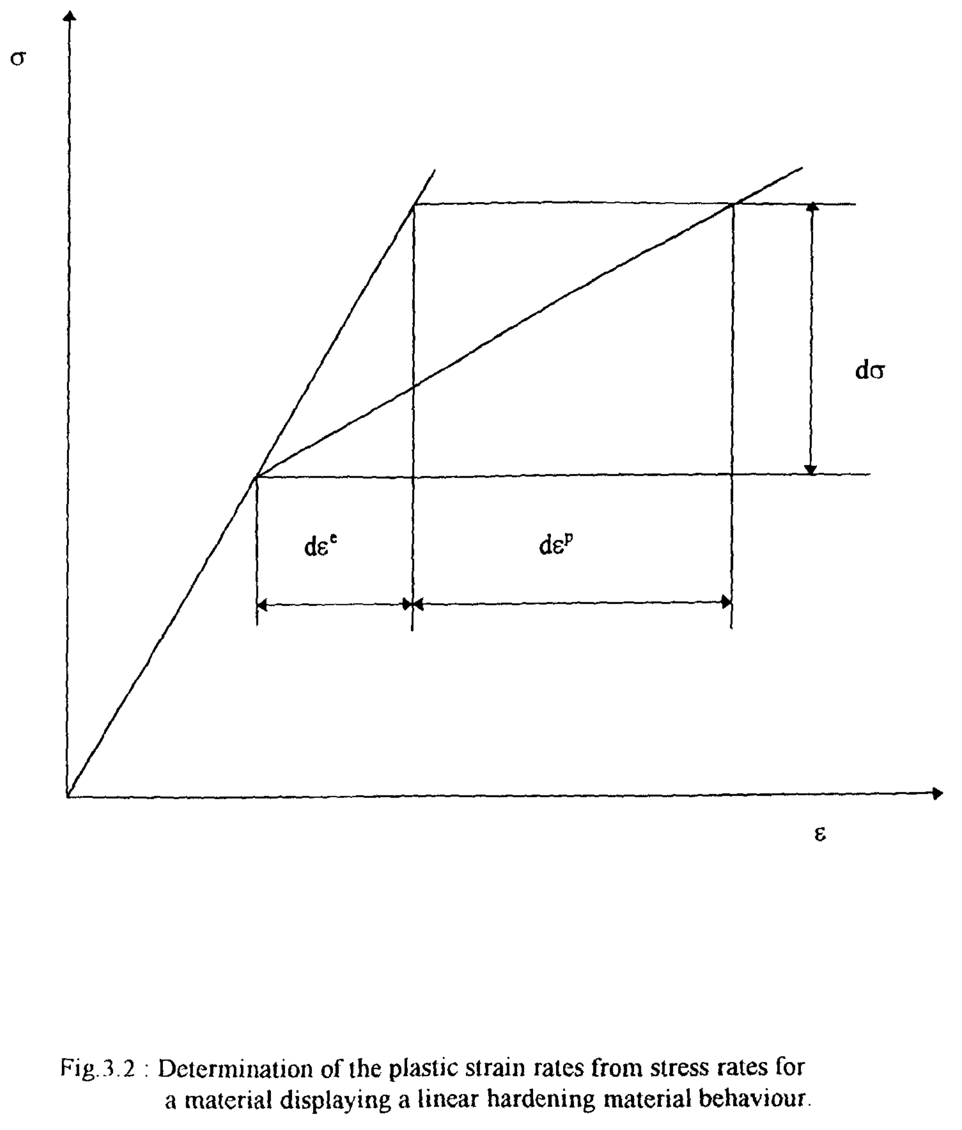


Fig.3.2 : Determination of the plastic strain rates from stress rates for a material displaying a linear hardening material behaviour.

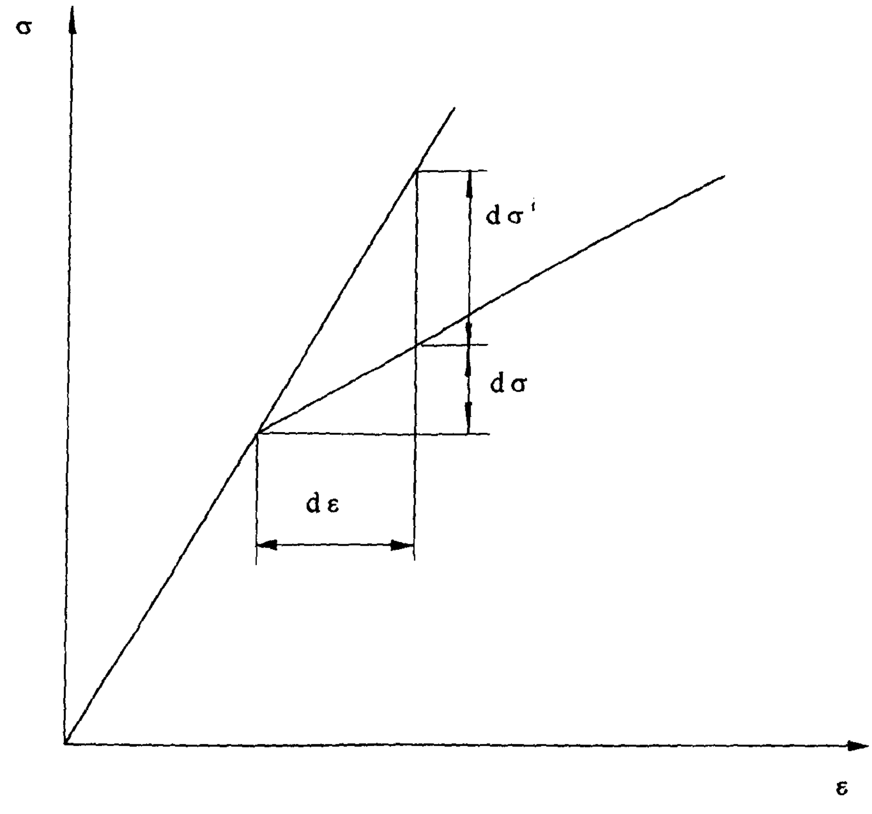


Fig.3.3 : Determination of initial stress rates from total strain increments for a material displaying a linearly hardening behaviour .

CHAPTER 4

THE NUMERICAL IMPLEMENTATION OF THE BOUNDARY ELEMENT METHOD IN 2D ELASTO-PLASTICITY

In this chapter, the numerical implementation of the boundary element method in two dimensional elasto-plasticity using the initial strain approach is presented. Isoparametric quadratic elements (three-noded boundary elements and eight-noded domain cells) are used.

4.1 NUMERICAL IMPLEMENTATION OF THE INTEGRAL EQUATION

It is obvious from the elasto-plastic BE formulation discussed previously that both boundary elements and domain cells (internal cells) are necessary in order to perform the integrals arising in the BE formulation. Both the boundary elements and the domain cells are used in two-dimensional elasto-plastic BE analysis are illustrated in Figure 4.1. In a manner similar to the elastostatic BE analysis, the boundary is represented as a collection of boundary elements. The solution domain is discretised into domain cells in order to perform the integrals relating to the domain kernels. The geometry of boundary and the solution variables (traction and displacement) can be described in terms of quadratic shape functions in a local coordinates (see, for example, Becker [1992]). The coordinates of a boundary element can be described as follows:

$$x(\xi) = \sum_{c=1}^{3} N_c(\xi)\, x_c$$

$$y(\xi) = \sum_{c=1}^{3} N_c(\xi)\, y_c \tag{4.1}$$

where N_c is the quadratic shape function and ξ is the local intrinsic coordinate. The displacements and tractions can be similarly defined as follows:

$$\dot{u}_i(\xi) = \sum_{c=1}^{3} N_c(\xi)\, \dot{u}_i$$

$$\dot{t}_i(\xi) = \sum_{c=1}^{3} N_c(\xi)\, \dot{t}_i \tag{4.2}$$

The Jacobian, due to transformation from the local coordinate, ξ, to the Cartesian coordinates is given by

$$J(\xi) = \left[\left(\frac{d\, x(\xi)}{d\, \xi} \right)^2 + \left(\frac{d\, y(\xi)}{d\, \xi} \right)^2 \right]^{1/2} \tag{4.3}$$

In order to perform the domain integrals, the domain cell coordinates, traction and displacement increments and the plastic strain increments can be defined in terms of quadratic shape functions described in local (intrinsic) coordinates, ξ_1, ξ_2 (see Becker [1992]) as follows:

$$x_i(\xi_1, \xi_2) = \sum_{c=1}^{8} N_c(\xi_1, \xi_2)\, (x_i)_c$$

$$\dot{u}_i(\xi_1, \xi_2) = \sum_{c=1}^{8} N_c(\xi_1, \xi_2)\, (\dot{u}_i)_c \tag{4.4}$$

$$\dot{\varepsilon}_{ij}(\xi_1, \xi_2) = \sum_{c=1}^{8} N_c(\xi_1, \xi_2)\, (\dot{\varepsilon}_{ij})_c$$

where the shape functions $N_c(\xi_1, \xi_2)$ are defined in Appendix A.

The Jacobian is given in terms of new local coordinates, ξ_1, ξ_2 (for quadrilateral elements) as follows:

$$J\ (\xi_1, \xi_2) = \frac{\partial x}{\partial \xi_1}\frac{\partial y}{\partial \xi_2} - \frac{\partial x}{\partial \xi_2}\frac{\partial y}{\partial \xi_1} = \frac{\partial\ (x,y)}{\partial(\xi_1\ \xi_2)} \tag{4.5}$$

The elasto-plastic BE equation in the initial strain approach (without considering body forces), in discretised form, can be given as follows:

$$\begin{aligned} C_{ij}\ \dot{u}_i(P) &+ \sum_{m=1}^{M} \sum_{c=1}^{3} \dot{u}_j(Q) \int_{-1}^{+1} T_{ij}(P,Q)\ N_c(\xi)\ J(\xi)\ d\xi \\ &= \sum_{m=1}^{M} \sum_{c=1}^{3} \dot{t}_j(Q) \int_{-1}^{+1} U_{ij}(P,Q)\ N_c(\xi)\ J(\xi)\ d\xi \\ &+ \sum_{m=1}^{D} \sum_{c=1}^{8} \dot{\varepsilon}^p_{ij}(q) \int_{-1}^{+1}\int_{-1}^{+1} W_{ijk}(P,q)\ N_c(\xi_1,\xi_2)\ J(\xi_1,\xi_2)\ d\xi_1\ d\xi_2 \end{aligned} \tag{4.6}$$

where P denotes the node where the integration is performed, Q indicates the c^{th} node of the m^{th} boundary element and q indicates the c^{th} node of the m^{th} domain cell.

4.1.1 Evaluation of equation coefficients

The integrals appearing in equation (4.6) have to be calculated in order to obtain the coefficients for the set of algebraic equations. It is obvious from the nature of the kernels that the integrals become singular when P (load point) coincides with either Q or q. Hence, it is very important to examine the numerical evaluation of the integrals in such cases.

When P is not a node of S or A, the singularity does not exist. Therefore, the standard Gaussian quadrature formulae can be used. When P is a node of S, there are two situations, to be considered. When P and Q are different nodes (in the same element) the Gaussian quadrature formulae can also be used, because there is no weak (logarithmic) or strongly singular integrand in the kernels.

For the case when P coincides with Q, the integrals appearing in the tensor U_{ij} can be evaluated by using logarithmic quadrature formulae. To perform this evaluation the displacement tensor U_{ij} is divided into three parts which are a singular logarithmic part, a non-singular logarithmic part and a non-singular part. To deal with a singular logarithmic part logarithmic quadrature formulae are used. Gaussian quadrature is applied to the latter two parts (see, Becker [1992]). In the same case the integrals relating to the T_{ij} kernel display the singularity defined only in the Cauchy principle value sense. These integrals and the free-term coefficients, C_{ij} (P) can be calculated using rigid body motion (see, for example, Becker [1992]).

When P is a node of the domain cell, the domain cell must be divided into sub-elements in order to perform the integrations, as shown in Figure 4.2. When P is a node of the domain cell, the domain kernel, W_{ijk} is a singular of the order of $1/r$. To deal with this type of singularity, there is an integration scheme which can be adopted by subdividing the quadrilateral element into two or three triangular sub elements (see Lee [1983]). As shown in Figure 4.2, a quadrilateral element must be divided into two or three sub-elements, depending on the location of the load point P. As shown in Figure 4.2 at the vertex where P is located, the points 1 and 2 of the rectangle are joined together as follows:

$$(\xi_1)_1 = (\xi_1)_2 \quad , \quad (\xi_2)_1 = (\xi_2)_2 \tag{4.7}$$

In this approach, the sub-element can be considered as a four-noded quadrilateral element by using new local coordinates, η_1, η_2, which vary from -1 to 1 (for details see, for example, Lee [1983] and Becker [1992]).

The new linear shape function can be given as follows:

$$\begin{aligned}
\xi_1 (\eta_1, \eta_2) &= L_1 (\eta_1, \eta_2) (\xi_2)_1 + L_2 (\eta_1, \eta_2) (\xi_1)_2 \\
&\quad + L_3 (\eta_1, \eta_2) (\xi_1)_3 + L_4 (\eta_1, \eta_2) (\xi_1)_4 \\
\xi_2 (\eta_1, \eta_2) &= L_1 (\eta_1, \eta_2) (\xi_2)_1 + L_2 (\eta_1, \eta_2) (\xi_2)_2 \\
&\quad + L_3 (\eta_1, \eta_2) (\xi_2)_3 + L_4 (\eta_1, \eta_2) (\xi_2)_4
\end{aligned} \tag{4.8}$$

In this expression, L_1, L_2, L_3 and L_4 are the linear rectangular shape functions given by

$$\begin{aligned}
L_1 (\eta_1, \eta_2) &= \frac{1}{4} (1 - \eta_1) (1 - \eta_2) \\
L_2 (\eta_1, \eta_2) &= \frac{1}{4} (1 + \eta_1) (1 - \eta_2) \\
L_3 (\eta_1, \eta_2) &= \frac{1}{4} (1 + \eta_1) (1 + \eta_2) \\
L_4 (\eta_1, \eta_2) &= \frac{1}{4} (1 - \eta_1) (1 + \eta_2)
\end{aligned} \tag{4.9}$$

Finally, the new Jacobian is defined in terms of the new local coordinates η_1, η_2 as follows:

$$J\ \eta_1, \eta_2) = \frac{\partial(\xi_1, \xi_2)}{\partial(\eta_1, \eta_2)}$$

$$= \frac{\partial \xi_1(\eta_1, \eta_2)}{\partial \eta_1} \frac{\partial \xi_2\ (\eta_1, \eta_2)}{\partial \eta_2} - \frac{\partial \xi_2\ (\eta_1, \eta_2)}{\partial \eta_1} \frac{\partial \xi_1\ (\eta_1, \eta_2)}{\partial \eta_2} \tag{4.10}$$

In this expression the differentials of local coordinates ξ_1, ξ_2, with respect to the coordinates η_i can be written as follows:

$$\frac{\partial\ \xi_1\ (\eta_1, \eta_2)}{\partial\ \eta_1} = \frac{\partial\ L_1\ (\eta_1, \eta_2)}{\partial\ \eta_1} (\xi_1)_1 + \frac{\partial\ L_2\ (\eta_1, \eta_2)}{\partial\ \eta_1} (\xi_1)_2$$
$$+ \frac{\partial\ L_3\ (\eta_1, \eta_2)}{\partial\ \eta_1} (\xi_1)_3 + \frac{\partial\ L_4\ (\eta_1, \eta_2)}{\partial\ \eta_1} (\xi_1)_4$$

$$\frac{\partial\ \xi_2\ (\eta_1, \eta_2)}{\partial\ \eta_1} = \frac{\partial\ L_1\ (\eta_1, \eta_2)}{\partial\ \eta_1} (\xi_2)_1 + \frac{\partial\ L_2\ (\eta_1, \eta_2)}{\partial\ \eta_1} (\xi_2)_2$$
$$+ \frac{\partial\ L_3\ (\eta_1, \eta_2)}{\partial\ \eta_1} (\xi_2)_3 + \frac{\partial\ L_4\ (\eta_1, \eta_2)}{\partial\ \eta_1} (\xi_2)_4 \tag{4.11}$$

In a similar manner, the differentials of the coordinates ξ_1 and ξ_2 with respect to the coordinate η_2 can be obtained. The differentials of the linear shape functions are shown in Appendix C.

4.1.2 Evaluation of the system equations

The linear algebraic equations obtained from the discretised integral equation can be formed as follows:

$$[A]\ [\dot{u}] = [B]\ [\dot{t}] + [W]\ [\dot{\varepsilon}^p] \tag{4.12}$$

where the matrices [A], [B] and [W] indicate the displacements, tractions and domain kernel integrals, respectively. For two-dimensional problems, if the total number of nodes defined on the boundary and the total number of the domain cell points defined on the solution domain are n and h respectively, then the solution matrices [A] and [B] will be square matrices of size 2n x 2n, whereas the matrix [W] will be a matrix of size 2n x 3h. [W] is not a square matrix and the solution matrices are fully populated.

So far, all the coefficients of the matrices [A], [B] and [W] have been determined, but the problem is not yet unique until given boundary conditions are imposed. Boundary conditions are specified values of the solution variables which are displacement rates, u_i and tractions rates $\dot{t}_i$ on the boundary of the domain to be solved. For simple solution domains, the following three types of boundary conditions are possible:

(i) Prescribed tractions

(ii) Prescribed displacement

(iii) linear relationship between traction and displacement

The boundary conditions are prescribed over each element (rather than node) and they are considered to be incremental form. For two-dimensional problems it is clear that each node must have two of the four variables ($\dot{u}_x$, $\dot{u}_y$, $\dot{t}_x$ and $\dot{t}_y$) prescribed. The treatment boundary conditions are discussed in detail in the textbook by Becker [1992].

To be able to implement a standard equation solver, the matrices [A] and [B] of equation(4.12) must be arranged as follows:

$$[A^*]\ [\dot{x}] = [B^*]\ [\dot{y}] + [W]\ [\dot{\varepsilon}^p] \tag{4.13}$$

In this expression the unknown vector $[\dot{x}]$ includes the unknown traction and displacement increments and the vector [y] includes the prescribed values of displacement and traction which gives a new known vector [C]. Therefore, the equations can be formed as follows:

$$[A^*]\ [\dot{x}] = [C] + [W]\ [\dot{\varepsilon}^p] \tag{4.14}$$

The plastic strain increments which are defined as a function of current stress state are unknown. The approximate values of the plastic strains can be only calculated by consulting the flow rules. Therefore, iterations have to be performed. To do this, by using their approximate values the following solution equation can be used.

$$[A^*]\ [\dot{x}] = [C^*] \tag{4.15}$$

where the known vector $[C^*]$ includes the effect of the plastic strain increments in the domain to be solved.

It is known that elasto-plastic problems give well-conditioned solution matrices, unless there is a mistake in the computational steps. The obtained solution matrix $[A^*]$ is not symmetric and fully populated. Therefore, the Gaussian elimination technique must be used.

4.2 EVALUATION OF STRESS AND TOTAL STRAIN RATES AT THE BOUNDARY

After solving the solution equation (4.15) the stress and total strain increments at the boundary nodes can be obtained by using the values of the nodal tractions and the displacement increments. As shown in Figure 4.1(a), the local coordinates of any point on the boundary can be defined by the unit tangential vector, m(ξ), and normal vector, n(ξ). The local tangential displacement rate $\dot{u}_1(\xi)$ in terms of the Cartesian displacement is given by

$$\dot{u}_1 (\xi) = \dot{u}_x (\xi) m_x + \dot{u}_y (\xi) m_y \tag{4.16}$$

By differentiating the above expression the tangential strain increment can be obtained as follows:

$$\dot{\varepsilon}_{11} (\xi) = \frac{1}{J (\xi)} \frac{\partial \dot{u}_1 (\xi)}{\partial \xi} \tag{4.17}$$

where the Jacobian J(ξ) is defined in equation (4.3) and the components m_x and m_y of the unit tangential vector m appearing in equation (4.16) are given by

$$m_x = \frac{1}{J (\xi)} \frac{d x (\xi)}{d \xi}$$

$$m_y = \frac{1}{J (\xi)} \frac{dy (\xi)}{d \xi} \tag{4.18}$$

The unit outward normal n is perpendicular to m, therefore, the components of the unit outward normal are given by

$$n_x = m_y$$
$$n_y = - m_x \tag{4.19}$$

The local shear stress rate and the local stress rate in the normal direction are directly derived from tractions rates as follows:

$$\dot{\sigma}_{12} = \dot{t}_1$$
$$\dot{\sigma}_{22} = \dot{t}_2 \tag{4.20}$$

The local stress can be transformed to global Cartesian stresses, by using the transformation matrix as follows:

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \begin{bmatrix} \sin^2\alpha & \cos^2\alpha & -\sin\alpha\ \cos\alpha \\ \cos^2\alpha & \sin^2\alpha & 2\sin\alpha\ \cos\alpha \\ -\sin\alpha\ \cos\alpha & \sin\alpha\ \cos\alpha & \cos^2\alpha - \sin^2\alpha \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} \tag{4.21}$$

By using the inverse of the transformation matrix the global Cartesian stresses can be transformed to the local stresses.

The plastic strain rates can be written in terms of the stress rates in local coordinates as follows:

$$\dot{\varepsilon}^p_{11} = \dot{\varepsilon}_{11} - \frac{1}{E}\left[\dot{\sigma}_{11} - \nu\ (\dot{\sigma}_{22} + \dot{\sigma}_{33})\right]$$
$$\dot{\varepsilon}^p_{33} = \dot{\varepsilon}_{33} - \frac{1}{E}\left[\dot{\sigma}_{33} - \nu\ (\dot{\sigma}_{22} + \dot{\sigma}_{11})\right] \tag{4.22}$$

In this expression the local components of the plastic strains are used. By using the inverse of the transformation matrix given in equation (4.21), the plastic strain components in the global coordinates can be transformed to the local coordinates.

For ductile material, it is known that the total plastic deformation is incompressible. Therefore the following expression can be written

$$\dot{\varepsilon}^p_{11} + \dot{\varepsilon}^p_{22} + \dot{\varepsilon}^p_{33} = 0 \tag{4.23}$$

In the plane stress case ($\dot{\sigma}_{33} = 0$), and by using both equations (4.22) and (4.23). The local tangential stress rate can be derived as follows:

$$\dot{\sigma}_{11} = \nu\ \dot{\sigma}_{22} + E\ (\dot{\varepsilon}_{11} - \dot{\varepsilon}^p_{11}) \tag{4.24}$$

Similarly for the plane strain case ($\dot{\varepsilon}_{33} = 0$) and the local tangential stress rate and the local stress rate in third direction can be derived as follows:

$$\begin{aligned} \dot{\sigma}_{11} &= \frac{\nu}{1+\nu}\ \dot{\sigma}_{22} + \frac{E}{1-\nu^2}\ (\dot{\varepsilon}_{11} - \dot{\varepsilon}^p_{11}) + \frac{\nu E}{1-\nu^2}\ (\dot{\varepsilon}^p_{11} + \dot{\varepsilon}^p_{22}) \\ \dot{\sigma}_{33} &= \nu(\dot{\sigma}_{22} + \dot{\sigma}_{11}) + E\ (\dot{\varepsilon}^p_{11} + \dot{\varepsilon}^p_{22}) \end{aligned} \tag{4.25}$$

By using the transformation defined in equation (4.21) the stresses obtained in the local coordinate system can be transformed to the global Cartesian stress rates and then the corresponding strain rates at the boundary can be obtained by using Hooke's law. It should be noted that the average values of the stress rates have to be used at nodes shared between two or more elements.

4.3 EVALUATION OF THE INTERIOR VARIABLES

Similar to the boundary integral equation for displacement rates at a point P on the boundary, the integral equation for displacement rates at an interior point can be expressed in discretised form as follows:

$$\dot{u}_i(P) + \sum_{m=1}^{M} \sum_{c=1}^{3} \dot{u}_j\,(Q) \int_{-1}^{+1} T_{ij}(P,Q)\; N_c(\xi)\; J\,(\xi)\; d(\xi)$$

$$= \sum_{m=1}^{M} \sum_{c=1}^{3} \dot{t}_j(Q) \int_{-1}^{+1} U_{ij}(P,Q)\; N_c(\xi)\; J(\xi)\; d(\xi)$$

$$+ \sum_{m=1}^{D} \sum_{c=1}^{8} \dot{\varepsilon}^{p}_{ij}(q) \int_{-1}^{+1}\int_{-1}^{+1} W_{kij}(P,q)\; N_c\;(\xi_1,\xi_2)\; J(\xi_1,\xi_2)\; d\xi_1\; d\xi_2 \tag{4.26}$$

The differentials of the displacements vector components can be written as follows:

$$\frac{\partial \dot{u}_1}{\partial \xi_1} = \frac{\partial \dot{u}_1}{\partial x_1}\frac{\partial x_1}{\partial \xi_1} + \frac{\partial \dot{u}_1}{\partial x_2}\frac{\partial x_2}{\partial \xi_1}$$

$$\frac{\partial \dot{u}_1}{\partial \xi_2} = \frac{\partial \dot{u}_1}{\partial x_1}\frac{\partial x_1}{\partial \xi_2} + \frac{\partial \dot{u}_1}{\partial x_2}\frac{\partial x_2}{\partial \xi_2} \tag{4.27}$$

or in matrix form

$$\begin{bmatrix} \dfrac{\partial \dot{u}_1}{\partial \xi_1} \\[2ex] \dfrac{\partial \dot{u}_1}{\partial \xi_2} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial x_1}{\partial \xi_1} & \dfrac{\partial x_2}{\partial \xi_1} \\[2ex] \dfrac{\partial x_1}{\partial \xi_2} & \dfrac{\partial x_2}{\partial \xi_2} \end{bmatrix} \begin{bmatrix} \dfrac{\partial \dot{u}_1}{\partial x_1} \\[2ex] \dfrac{\partial \dot{u}_1}{\partial x_2} \end{bmatrix} \tag{4.28}$$

The first matrix on the right hand side is the Jacobian [J]. Therefore, the derivatives of displacement rate in x_1 and x_2 can be written as follows:

$$\begin{bmatrix} \dfrac{\partial \dot{u}_1}{\partial x_1} \\ \dfrac{\partial \dot{u}_1}{\partial x_2} \end{bmatrix} = [J^{-1}] \begin{bmatrix} \dfrac{\partial \dot{u}_1}{\partial \xi_1} \\ \dfrac{\partial \dot{u}_1}{\partial \xi_2} \end{bmatrix} \tag{4.29}$$

By inverting the Jacobian matrix, the following expression can be obtained

$$[J^{-1}] = \frac{1}{\left| \dfrac{\partial x_2}{\partial \xi_1} \dfrac{\partial x_1}{\partial \xi_2} - \dfrac{\partial x_1}{\partial \xi_1} \dfrac{\partial x_2}{\partial \xi_2} \right|} \begin{bmatrix} \dfrac{\partial x_2}{\partial \xi_2} & -\dfrac{\partial x_2}{\partial \xi_1} \\ -\dfrac{\partial x_1}{\partial \xi_2} & \dfrac{\partial x_1}{\partial \xi_1} \end{bmatrix} \tag{4.30}$$

The differentials of x_1 and x_2 with respect to the local coordinates ξ_1 and ξ_2 can be written as follows:

$$\begin{aligned} \frac{\partial x_1}{\partial \xi_1} &= \sum_{c=1}^{8} \frac{\partial N_c\,(\xi_1, \xi_2)}{\partial \xi_1}\,(x_1)_c \\ \frac{\partial x_1}{\partial \xi_2} &= \sum_{c=1}^{8} \frac{\partial N_c\,(\xi_1, \xi_2)}{\partial \xi_2}\,(x_1)_c \end{aligned} \tag{4.31}$$

Similarly the differentials of x_2 and u_2 with respect to the local coordinates ξ_1 and ξ_2 can be obtained. The derivatives of the shape functions are given in Appendix B.

By using the strain-displacement relationship in equation (2.10) the cell total strain rates can be obtained as follows:

$$\dot{\varepsilon}_{xx} = \frac{\partial \dot{u}}{\partial x_1} \quad , \quad \dot{\varepsilon}_{yy} = \frac{\partial \dot{u}}{\partial x_2}$$

$$\dot{\varepsilon}_{xy} = \frac{1}{2}\left(\frac{\partial \dot{u}_1}{\partial x_2} + \frac{\partial \dot{u}_2}{\partial x_1}\right) \tag{4.32}$$

The total strain rates in the third direction can be derived from equation (2.6) by considering plane stress case ($\dot{\sigma}_{33} = 0$) and can be written as follows:

$$\dot{\varepsilon}_{zz} = \frac{-\nu}{1 - \nu}\,(\dot{\varepsilon}_{xx} + \dot{\varepsilon}_{yy}) - \frac{1 - 2\nu}{1 - \nu}\,(\dot{\varepsilon}^p_{xx} + \dot{\varepsilon}^p_{yy}) \tag{4.33}$$

It should be noted that here the Poisson's ratio corresponds to the actual (not effective) Poisson's ratio. For plane strain case, the total plastic strain rates, $\dot{\varepsilon}_{zz}$, is zero.

For the plane strain case, cell stress rates in the global coordinates can be written as follows:

$$\dot{\sigma}_{xx} = 2\mu\,(\dot{\varepsilon}_{xx} - \dot{\varepsilon}^p_{xx}) + \frac{2\mu\nu}{1 - 2\nu}\,(\dot{\varepsilon}_{xx} + \dot{\varepsilon}_{yy})$$

$$\dot{\sigma}_{yy} = 2\mu\,(\dot{\varepsilon}_{yy} - \dot{\varepsilon}^p_{yy}) + \frac{2\mu\nu}{1 - 2\nu}\,(\dot{\varepsilon}_{yy} + \dot{\varepsilon}_{xx})$$

$$\dot{\sigma}_{zz} = 2\mu\,(\dot{\varepsilon}^p_{xx} + \dot{\varepsilon}^p_{yy}) + \frac{2\mu\nu}{1 - 2\nu}\,(\dot{\varepsilon}_{xx} + \dot{\varepsilon}_{yy})$$

$$\dot{\sigma}_{xy} = 2\mu\,(\dot{\varepsilon}_{xy} - \dot{\varepsilon}^p_{xy}) \tag{4.34}$$

For plane stress case, the stress rates in the interior are

$$\dot{\sigma}_{xx} = 2\mu\,(\dot{\varepsilon}_{xx} - \dot{\varepsilon}^p_{xx}) + \frac{2\mu\nu}{1 - 2\nu}\,(\dot{\varepsilon}_{xx} + \dot{\varepsilon}_{yy} - \dot{\varepsilon}^p_{xx} - \dot{\varepsilon}^p_{yy})$$

$$\dot{\sigma}_{yy} = 2\mu\,(\dot{\varepsilon}_{yy} - \dot{\varepsilon}^p_{yy}) + \frac{2\mu\nu}{1 - 2\nu}\,(\dot{\varepsilon}_{yy} + \dot{\varepsilon}_{yy} - \dot{\varepsilon}^p_{xy} - \dot{\varepsilon}^p_{yy})$$

$$\dot{\sigma}_{xy} = 2\mu\,(\dot{\varepsilon}_{xy} - \dot{\varepsilon}^p_{xy})$$

$$\dot{\sigma}_{zz} = 0 \tag{4.35}$$

In this expression it should be noted that Poisson's ratio corresponds to the effective Poisson's ratio.

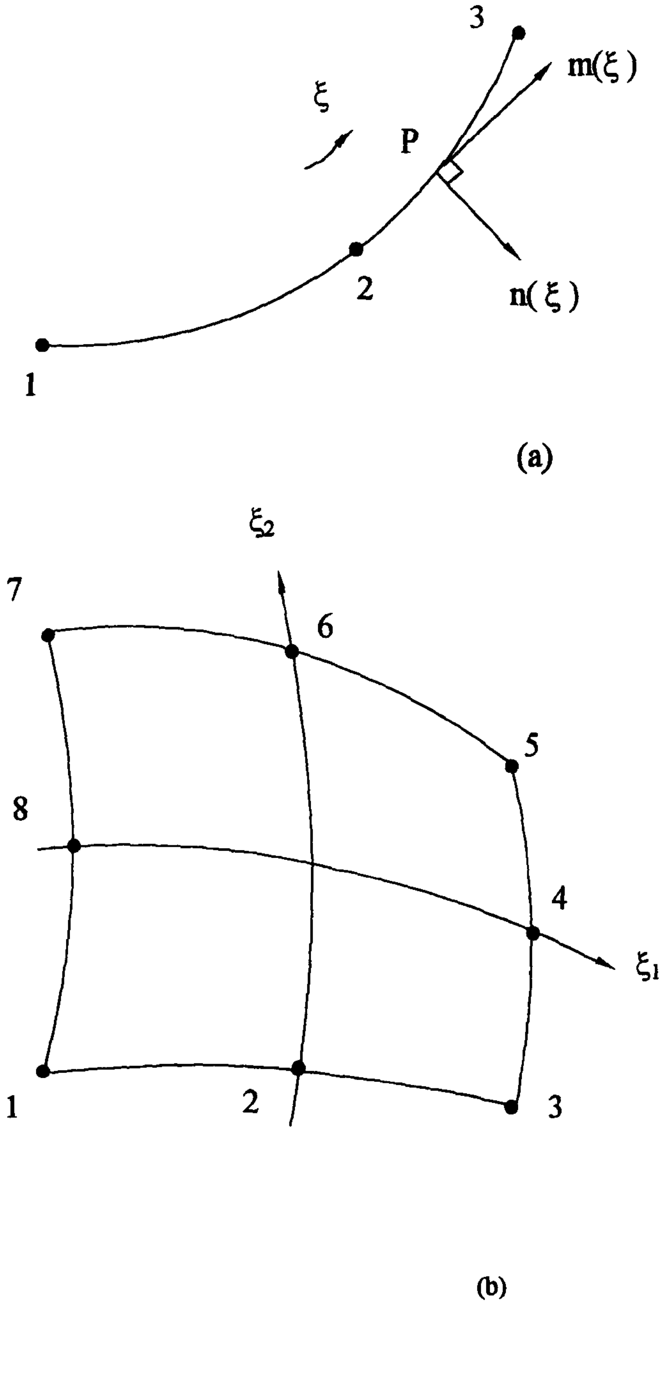


Fig.4.1 : Isoparametric quadratic elements used in elasto-plastic BIE analysis
(a) Three-noded boundary element
(b) Eight-noded quadratilateril domain cell

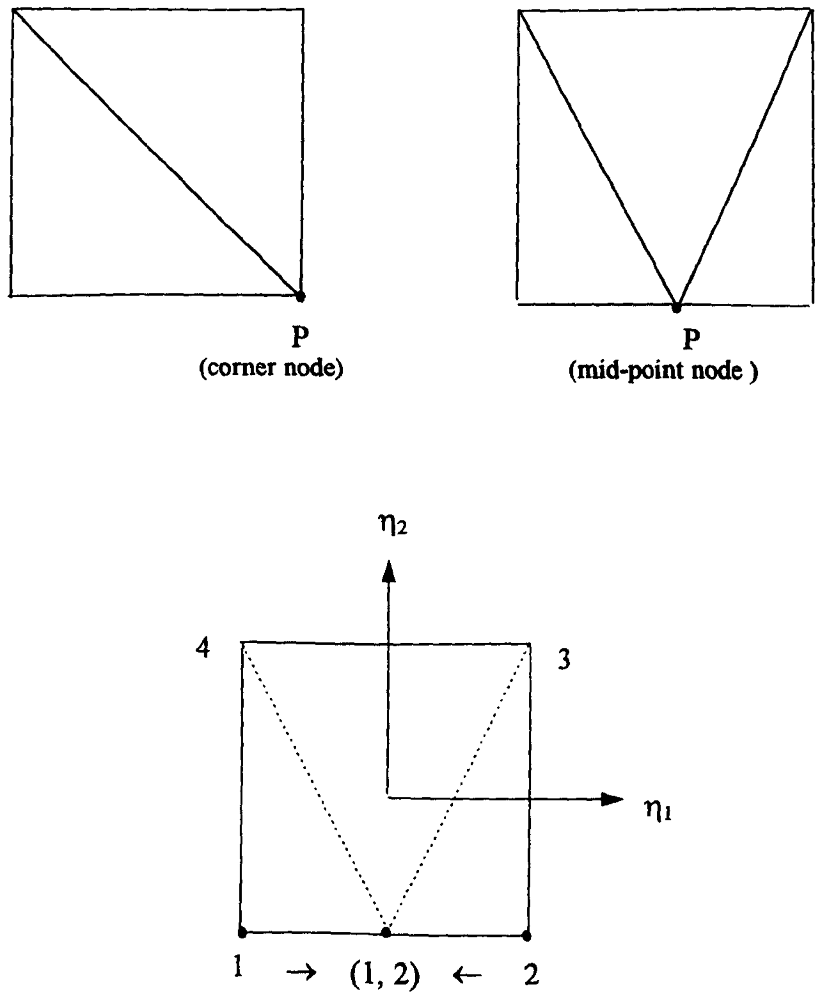


Fig.4.2 : Division of the quadrilaterial domain cell into sub-elements used in integration process as a self-element.

CHAPTER 5

ALTERNATIVE ELASTOPLASTIC BOUNDARY ELEMENT FORMULATION

In the application of BE method to elastic and elasto-plastic problems, the crucial point is to treat the singular integrals arising from the boundary integral equations. Therefore, techniques for evaluating the strongly singular integrals and both analytical and numerical formulations of the particular integral approach which can be performed without any additional integration in solution domain are discussed in this chapter.

5.1 EVALUATION OF STRONGLY SINGULAR INTEGRALS

As mentioned earlier, in order to include the effect of plasticity in the solution domain additional domain integrals appear in the boundary integral equations and they display strongly and weakly singular behaviour, of the order of $1/r^2$ and $1/r$ respectively in two-dimensional applications.

The techniques for evaluation of the strong singular integrals can be classified into two main groups which are called the indirect and direct approaches.

From the literature it can be seen that indirect techniques need to consider a known reference solution such as an admissible stress field (see, Henry and Benerjee [1988]) and a constant plastic strain field (see Brebbia et al [1984]). In the application of this approach, a subregion formulation can be used in order to avoid the discretisation of the entire domain to be solved.

The direct approaches which are commonly used are based on the regularisation of the strongly singular integrals, except one presented by Benerjee and Raveendra [1986] in which the strongly singular integrals are excluded from the solution domain using a small circle (or sphere for three-dimensional applications), where plastic strain is assumed to be constant, and calculated analytically. The regularisation procedure of the strongly singular integrals can be performed using polar coordinate transformation and Taylor series expansion (see Guiggiani et al [1993]). The advantage of this approach is to handle the hyper-singular integral equation (HSE) in which the singular integral is of higher order singularity than the strongly singular integral equation (SSE), with high accuracy for any arbitrary integration cell. It is possible to regularise the strong singular integral using Gauss theorem without consulting special coordinate transformation and the representation of Taylor series expansion (see Dallner and Kuhn [1993]). The advantage of this approach is that it is applicable to any arbitrary integration cell and location of the singular point in which the load point (source point) is located.

It can be concluded that the use of the integral identities gives more accurate solutions for the interior stress and strain rates, but it is obviously more tedious and consumes more computation time.

It is possible to circumvent the strongly singular integrals by differentiating the displacement rates via the shape functions in order to obtain the strain and the stress rates or by using the particular integral approach.

In the first approach which is well known and may be called the classical approach, the discretisation of the entire solution domain is not compulsory, because the stress rates at interior points and at boundary nodes can be treated separately, as discussed in detail in previous chapters.

In the particular integral approach, the effect of the plasticity in solution domain is treated as a special kind of body force and can be put into boundary variables over the global shape functions without performing any domain integration. In this method the internal (fictitious nodes) should be consistent with the boundary discretisation. Therefore, the entire solution domain has to be discretised by using both boundary nodes and fictitious nodes. In order to avoid the discretisation, whole domain subregion formulation must be implemented.

5.2 ANALYTICAL FORMULATION OF THE PARTICULAR INTEGRAL APPROACH

For elasto-plastic behaviour Navier's equation can be written as follows:

$$\frac{\partial^2 \dot{u}_i}{\partial x_j\, \partial x_j} + \frac{1}{(1-2\nu)} \frac{\partial^2 \dot{u}_j}{\partial x_i\, \partial x_j} = \frac{\partial \dot{\sigma}^i_{ij}}{\partial x_j} \qquad \textbf{(5.1)}$$

It is clear that the effects of plasticity can be treated as a kind of body force. It is well known that the solution of partial differential equations can be obtained using a complementary function and a particular integral (see, for example, Stephenson [1973]). Therefore, the displacements in the Navier differential equation can be defined as a combination of complementary and particular parts as follows:

$$u_i = (u_i)^{CE} + (u_i)^{PF} \tag{5.2}$$

In this expression the superscripts CF and PF stand for the complementary function and particular integral components respectively. Similarly the stress and tractions can be written in terms of complementary function and particular integral components as follows:

$$\begin{aligned} t_i &= (t_i)^{CF} + (t_i)^{PF} \\ \sigma_{ij} &= (\sigma_{ij})^{CF} + (\sigma_{ij})^{PF} \end{aligned} \tag{5.3}$$

Without considering non-homogenous terms which appear in Navier's equation (5.1), the elastic solution given by the boundary integral equation is the complementary solution. In order to obtain the complementary solution for displacement and stress rates, the boundary integral equation can be written as follows:

$$C_{ij}\, u_i^{CF}(P) + \int_S T_{ij}(P,Q)\, \dot{u}_j^{CF}(Q)\, dS = \int_S U_{ij}(P,Q)\, \dot{t}_j^{CF}(Q) dS \tag{5.4}$$

and the stress equation given by

$$\sigma_{ij}^{CF}(P) + \int_S S_{kij}(P,Q)\, \dot{t}_k^{CF}(Q) dS = \int_S D_{kij}(P,Q)\, \dot{u}_k^{CF}(Q) dS \tag{5.5}$$

It should be noted that the fundamental tensors $T_{ij}(P,Q)$, $U_{ij}(P,Q)$, $S_{kij}(P,Q)$ and $D_{kij}(P,Q)$ in the above expressions are exactly the ones given in equations (3.19) and (3.21) for elastic solution.

The following expressions can be written for the particular solutions for the stresses as follows:

$$\dot{\sigma}_{ij} = 2\mu \, (\dot{\varepsilon}_{ij}^{CF} + \dot{\varepsilon}_{ij}^{PF}) + \frac{2\,\mu}{1 - 2\,\nu} \, (\dot{\varepsilon}_{kk}^{CF} + \dot{\varepsilon}_{kk}^{PF}) - \dot{\sigma}_{ij}^{i} \tag{5.6}$$

or

$$\dot{\sigma}_{ij}^{PF} = 2\,\mu\,\dot{\varepsilon}_{ij}^{PF} + \frac{2\,\mu}{1 - 2\nu}\,\dot{\varepsilon}_{kk}^{PF} - \dot{\sigma}_{ij}^{i} \tag{5.7}$$

For strains, the strain-displacement definitions can be used as follows:

$$\dot{\varepsilon}_{ij}^{PF} = \frac{1}{2}\left(\frac{\partial \dot{u}_i^{PF}}{\partial x_j} + \frac{\partial \dot{u}_i^{PF}}{\partial x_i}\right) \tag{5.8}$$

The initial stress distribution can be expressed in terms of the global shape functions K(Q, P) as follows (see Henry [1987]):

$$\dot{\sigma}_{ij}^{i}\,(Q) = \sum_{m=1}^{M} K\,(Q, P_m)\;\phi_{ij}\,(P_m) \tag{5.9}$$

In which P_m is considered to be boundary nodes and internal (fictitious) nodes in the solution domain. It should be noted that fictitious nodes have to be consistent with the boundary discretisation of the domain to be solved. M is the total number of unknowns and can be changed as required.

The function $K(Q, P_m)$ is given by

$$K(Q,P_m) = \frac{\partial^4 C\ (Q,P_m)}{\partial^2 x_m \partial^2 x_n}$$

(5.10)

In this expression $C(Q, P_m)$ is the global shape function which represents the initial stress $\sigma_{ij}{}^i$ distribution in the solution domain. It should be noted that different types are possible by choosing an alternative different polynomials. For the most successful representation of the initial stress distribution, it has been suggested (see Henry [1987]) that the following expression can be used:

$$C\ (Q,\ P_m) = C_o^4\ (r^4 - b_n\ r^5)$$

(5.11)

In this expression C_o is the characteristic length related to the solution domain, which can be chosen as the longest distance in the solution domain. As usual, $r(Q, P_m)$ is the distance between the field point Q and load point P_m.

For two dimensional problems, the $K(Q, P_m)$ is given by (see Henry [1987])

$$K(Q,\ P_m) = 64 - 225\ b_n\ r\ (Q,\ P_m)$$

(5.12)

In this expression it should be noted that all distances are non-dimensionalised by using the characteristic length. The parameter b_n is chosen to minimise the solution error which may be caused by arbitrary ordering of the nodes, by scaling down each column of the matrix $K(Q,P_m)$ such that the lowest value is forced to be zero in order to optimize the solution matrix (see Henry [1987]).

For two dimensional problems, the particular integral for displacement rates can be expressed as follows (see Henry [1987] and Kane [1994]).

$$\dot{u}_i^{PF}(Q) = \sum_{m=1}^{M} D_{iml}^{PF}(Q, P_m)\, \dot{\phi}_{lm}(P_m) \tag{5.13}$$

where ϕ is a tensor quantity and the tensor D_{iml}^{PF} is given in Appendix D.

The particular solution for strain can be obtained by substituting equation (5.13) into (5.7) and using equation (5.8). The particular solution for stress rates can be obtained as follows (for details see Henry [1987] and Kane [1997]):

$$\sigma_{ij}^{PF} = \sum_{m=1}^{M} S_{ijlm}^{PF}(Q, P)\, \phi_{lm}(P_m) \tag{5.14}$$

In which the tensor $S^{PF}_{ijlm}(Q, P_m)$ is given in Appendix D.

Finally, the particular integral for traction rate is obtained using the Cauchy stress transformation as follows:

$$\dot{t}_i^{PF}(Q) = \dot{\sigma}_{ij}^{PF}(Q)\, n_j = \sum_{m=1}^{M} T_{iml}^{PF}(Q, P_m)\, \phi_{lm}(P_m) \tag{5.15}$$

In this expression the tensor T_{iml}^{PF} is given by

$$T_{iml}^{PF}\ (Q,\ P_m) = S_{ijlm}^{PF}\ (Q,\ P_m)\ n_j \tag{5.16}$$

5.3 NUMERICAL IMPLEMENTATION OF THE PARTICULAR INTEGRAL APPROACH

By considering a particular load step and using the expression in equation (5.2), the complementary displacement and traction rates at the boundary can be written as follows:

$$[\dot{u}^{CF}] = [\dot{u}] - [\dot{u}^{PF}] \tag{5.17}$$

$$[\dot{t}^{CF}] = [\dot{t}] - [\dot{t}^{PF}] \tag{5.18}$$

Similarly, the stress rates at interior point can be written as follows:

$$[\dot{\sigma}^{CF}] = [\dot{\sigma}] - [\dot{\sigma}^{PF}] \tag{5.19}$$

It should be noted that the complementary solution is simply the elastic solution. Therefore, by introducing the particular integrals into the system equations obtained by the discretised BE formulation for elastic applications, the elasto-plastic system equations can be formed as follows:

$$[A]\ [\dot{u}] - [A]\ [\dot{u}^{PF}] = [B]\ [\dot{t}] - [B]\ [\dot{t}^{PF}] \tag{5.20}$$

or

$$[A]\ [\dot{u}] = [B]\ [\dot{t}] - [B]\ [\dot{t}^{PF}] + [A]\ [\dot{u}^{PF}] \tag{5.21}$$

As discussed previously, the displacements and traction rates are arranged such that all unknown variables are placed in the left hand side, and all prescribed values in the right hand side. The system equation (5.20) can then be formed as follows:

$$[A^*]\ [\dot{x}] = [B^*]\ [\dot{y}] - [B]\ [\dot{t}^{PF}] + [A]\ [\dot{u}^{PF}] \tag{5.22}$$

where the matrices [A*] and [B*] are the modified forms of [A] and [B], while the vector $[\dot{x}]$ and $[\dot{y}]$ contain the unknown and known values respectively, of either tractions or displacement rates. In this computational step it should be noted that the particular integrals for traction and displacement rates are unknown. By using equations (5.17) and (5.18), the particular solutions for displacement and traction rates are given in matrix form as follows:

$$[\dot{u}^{PF}] = [D]\ [\dot{\phi}] \tag{5.23}$$

$$[\dot{t}^{PF}] = [T]\ [\dot{\phi}] \tag{5.24}$$

For two-dimensional applications it should be noted that the displacement and traction vectors are

$$[\dot{u}^{PF}]^T = [\dot{u}_x(P_1)\ \dot{u}_y(P_1)\ \dot{u}_x(P_2)\ \dot{u}_y(P_2)\ \ldots\ldots\ldots\ldots\ldots\ldots\ \dot{u}_x(P_N)\ \dot{u}_y(P_N)]^T \tag{5.25}$$

$$[\dot{t}^{PF}]^T = [\dot{t}_x(P_1)\ \dot{t}_y(P_1)\ \dot{t}_x(P_2)\ \dot{t}_y(P_2)\ \ldots\ldots\ldots\ldots\ldots\ \dot{t}_y(P_N)\ \dot{t}_y(P_N)]^T \tag{5.26}$$

and

$$[\dot{\phi}] = [\ \dot{\phi}_{xx}(P_1)\ \dot{\phi}_{xx}(P_2)\ \ldots\ldots\ldots\ldots\ldots\ldots\ \dot{\phi}_{xx}(P_m)$$
$$\dot{\phi}_{yy}(P_1)\ \dot{\phi}_{yy}(P_2)\ \ldots\ldots\ldots\ldots\ldots\ldots\ \dot{\phi}_{yy}(P_m)$$
$$\dot{\phi}_{xy}(P_1)\ \dot{\phi}_{xy}(P_2)\ \ldots\ldots\ldots\ldots\ldots\ldots\ \dot{\phi}_{xy}(P_m)\ ]^T \tag{5.27}$$

In this expression the rectangular matrices [D^{PF}] and [T^{PF}] have 2 N rows and 3M columns, N is the total number of boundary nodes and M is the total number of boundary and interior (fictitious) nodes in the solution domain.

It is clear that the tensor quantities ϕ_{ij} (the coefficients of the polynomial) have to be determined in order to evaluate vectors [u^{PF}] and [t^{PF}]. To do this equation (5.9) has to be formed as follows:

$$\begin{bmatrix} [\dot{\sigma}^i_{xx}] \\ [\dot{\sigma}^i_{yy}] \\ [\dot{\sigma}^i_{xy}] \end{bmatrix} = \begin{bmatrix} [k] & 0 & 0 \\ 0 & [k] & 0] \\ 0 & 0 & [k] \end{bmatrix} \begin{bmatrix} [\dot{\phi}^i_{xx}] \\ [\dot{\phi}^i_{yy}] \\ [\dot{\phi}^i_{xy}] \end{bmatrix} \tag{5.28}$$

or

$$[\dot{\sigma}^i] = [K^*]\ [\dot{\phi}] \tag{5.29}$$

The unknown quantity ϕ_{ij} can be obtained by using the following expression:

$$[\dot{\phi}] = [K^*]^{-1}\ [\dot{\sigma}^i] \tag{5.30}$$

In this expression the initial stresses can be obtained by using the plastic flow rules and the expression for initial stress in equation (2.65). The evaluation of plastic strain increment is explained in the next chapter. By substituting the known vector $[\phi]$ into equation (5.22) and (5.23), the particular solution for displacement and tractions are obtained and then the system equation (5.21) can be formed as follows:

$$[A^*]\ [\dot{x}] = [C^*] \tag{5.31}$$

Now in this expression the vector [C*] includes the particular solution for displacement and traction rates.

By using equation (5.14) the particular solution for stress rates can be written in matrix form as follows:

$$[\dot{\sigma}^{PF}] = [S^{PF}]\ [\dot{\phi}] \tag{5.32}$$

Finally, by using equation (5.19) and (5.32), the particular solution for the stress rates at interior nodes can be written in matrix form as follows:

$$[\dot{\sigma}] = [S^{CF}]\ [\dot{u}] + [D^{CF}]\ [\dot{t}] + [\dot{\sigma}^{PF}] - [S^{CF}]\ [\dot{u}^{PF}] - [D^{CF}]\ [\dot{t}^{PF}] \tag{5.33}$$

In this expression the matrices $[S^{CF}]$ and $[D^{CF}]$ are given for elastic solution by integrating numerically the kernels relating to the fundamental tensors S_{kij} and D_{kij} defined in equations (3.22) and (3.23) respectively. Therefore the stress rates at interior points can be obtained without performing any extra integration.

5.4 IMPLEMENTATION OF THE PARTICULAR INTEGRAL APPROACH IN A COMPUTER PROGRAM

The solution algorithm for adopting the particular integral approach in a computer program can be summarised as follows:

Step 1 - Elastic solution

- Calculate the kernels U_{ij} and T_{ij} and the coefficients of the particular integral matrices $[T^{PF}]$, $[D^{PF}]$ and $[K^{*}]$.

- Apply all the boundary conditions with full loading to get:

  $[A]\ [x] = [c] + 0$

- Solve the system equations to obtain a solution for the [x[unknowns.

- Sort [x] into [u] and [t].

- From [u] and [t], calculate internal variables.

- Using Hooke's law, calculate $[\varepsilon]$ from $[\sigma]$.

Step 2 - First yield

- Search for the highest value of the Von Mises stress and compare it to the yield stress.

- Determine first yield and scale down the load vector [c] and the prescribed displacements and tractions accordingly.

- Determine the remainder of the load and decide on the size of the load increment.

Step 3 - Load increment after yield

- Initialise all plastic strains at the first load increment after yield.

- Initialise all incremental values; [Δu], [Δt], [$\Delta\sigma$] and [$\Delta\varepsilon$].

- Apply the load increment (multiply the load vector [c] by the load size factor). This will also scale both the prescribed displacements and traction rates.

- Obtain [Δx].

- Sort [Δx] into [Δu] and [Δt].

- From the boundary values of [Δu] and [Δt], obtain the internal values of [Δu] and [$\Delta\sigma$]. All values of variable (boundary and internal) are now determined.

- For yielded nodes only, calculate the plastic strains using the flow rule, equation (2.61), and then obtain the plastic stress rates using equation (2.65). Note that this is the first approximation for the plastic stress increments $[\Delta\sigma^i]$.

Step 4 - Iteration ($\varepsilon^p > 0$)

- Initialise all incremental values of $[\Delta u]$, $[\Delta t]$, $[\Delta\sigma]$ and $[\Delta\varepsilon]$, but not $[\Delta\sigma^i]$.

- Use equation (5.30) to obtain the tensor quantity ϕ and the previously stored complimentary and particular solution matrices to get :

  $[A]\ [\Delta x] = [\Delta C] + (\ [A]\ [D^{PF}] - [B]\ [T^{PF}]\ )\ \ [\phi]$

- Solve for new $[\Delta x]$ which now includes the effect of plasticity.

- Sort $[\Delta x]$ into $[\Delta u]$ and $[\Delta t]$.

- Calculate the boundary values of stress from the boundary values of $[\Delta u]$ and $[\Delta t]$.

- Obtain the stress increment for the interior nodes using the following expression:

  $[\Delta\sigma] = [S^{CF}]\ [\Delta u] + [D^{CF}]\ [\Delta t] + \ \ (\ [S^{PF}] - [S^{CF}]\ [D^{PF}] - [D^{CF}]\ [T^{PF}]\ )\ \ [\phi]$

- Using Hooke's law, calculate the total strain increments.

- For yielded nodes, calculate the plastic strains using equation (2.61).

Step 5 - Convergence check

- For convergence, check that the largest percentage change in the plastic strain rate at any node $[\Delta\varepsilon^p]$ is below the tolerance (typically 0.1%).

- If convergence is achieved, accumulate the incremental values of $[\Delta u]$, $[\Delta t]$, $[\Delta\sigma]$, $[\Delta\varepsilon]$ and $[\Delta\varepsilon^p]$ and go to step 3 for the next load increment, if required.

- If convergence is not achieved:, check the number of iterations. If the maximum limit is not exceeded, go to step 4. If exceeded, terminate the analysis.

5.5 DISCUSSION OF THE SUITABILITY OF THE PARTICULAR INTEGRAL APPROACH

As mentioned earlier, in order to include the effect of plasticity in the BE formulation, interior discretisation is required, which results in strongly singular integrals. A significant increase in analytical and numerical work is required to maintain the accuracy of the Be formulation. The particular integral approach is an alternative formulation which uses fictitious nodes (internal nodes) in the solution domain to evaluate the internal variables. Comparing the numerical implementation of the particular integral approach to the interior discretisation approach in which the numerical differentiation is performed over internal cells via the shape functions, several drawbacks of the particular integral approach can be identified, as follows:

(i) A greater degree of mathematical complexity which makes the BE approach even less attractive to engineers.

(ii) It needs a great deal of effort in producing a robust computer program and may be unreliable for use in general-purpose BE packages.

(iii) Inexperienced users are more likely to arrive at the wrong solution.

(iv) The distribution of the fictitious nodes needs to be consistent with the boundary discretisation of the solution domain.

Therefore, it is concluded that, despite the need for interior discretisation, the numerical differentiation of the displacements is a more attractive option for general-purpose reliable BE software.

CHAPTER 6

THE SOLUTION PROCEDURE FOR THE DISPLACEMENT GRADIENT ELASTO-PLASTIC BOUNDARY ELEMENT FORMULATION

This chapter presents the numerical evaluation of the plastic strain increments and the numerical algorithms for the present solution procedure which is based on the initial strain displacement gradient approach in which the displacement vector is differentiated via the shape function in order to evaluate the strain rates and the stress rates at interior points.

6.1 EVALUATION OF PLASTIC STRAIN INCREMENT

In the first step of the computational procedure, the plastic strain increments are zero and the basic unknown vector [x] is determined in solution domain. Then the second approximation for the plastic strain increments can be obtained by using either equation (2.61) or equation (2.62). The process is repeated until a suitable convergence criterion is satisfied.

Both the flow rules and integral equations are given in rate form. Therefore, at the end of the iteration process for each load step, a suitable time step is chosen and multiplied by the incremental variables in order to determine the actual incremental variables as shown in equation (2.70). As mentioned earlier, the time step can be chosen as unity and the prescribed boundary conditions defined in an incremental form are treated in the same manner.

The computation of the scalar factor R is necessary in order to define the yield surface. Therefore, the yield criterion (surface) in incremental form can be written as follows:

$$f\ (\sigma_{ij} + R\ d\sigma^e_{ij}) = F\ (\sigma_{ij} + R\ d\sigma^e_{ij}) - Y = 0 \tag{6.1}$$

In this expression $d\sigma_{ij}^e$ is the incremental estimated elastic stress vector and σ_{ij} is the stress state at the beginning of the load step.

For the iterative process at a particular load step the above expression for the yield criterion can be written as follows:

$$F\ (\sigma^n) + R\ [\ F\ (\sigma^{n+1}) - f\ (\sigma^n)\ ] - Y = 0 \tag{6.2}$$

In this expression n denotes the iterative number at each load increment. By using the above expression, the factor R which is based on linear interpolation (see Owen and Hinton [1980]) can be defined as follows:

$$R = \frac{Y - F\ (\sigma^n)}{F\ (\sigma^{n+1}) - F\ (\sigma^n)} \tag{6.3}$$

Nayak and Zienkiewicz [1972] presented a procedure for a refined evaluation of the factor R with iterative process.

For a material obeying the Von Mises yield criterion, the yield surface can be defined as follows:

$$f = F\,(\sigma_{ij} + R\,d\sigma^{e}_{ij}) - Y = 0 \tag{6.4}$$

or

$$Y = (3\,J_2)^{\frac{1}{2}} = \left[\frac{3}{2}\,(S_{ij} + R\,dS^{e}_{ij})\,(S_{ij} + R\,dS^{e}_{ij})\right]^{\frac{1}{2}} \tag{6.5}$$

In quadratic form, the yield surface is expressed as

$$dS^{e}_{ij}\,dS^{e}_{ij}\,R^2 + 2S_{ij}\,dS^{e}_{ij}\,R + S_{ij}\,S_{ij} - \frac{2}{3}\,Y^2 = 0 \tag{6.6}$$

In this expression S_{ij} is the deviatoric stress of the stress state σ_{ij} and dS_{ij}^{e} is the deviatoric stress of the stress increments $d\sigma_{ij}^{e}$. By solving the above quadratic equation, the factor R can be obtained as follows:

$$R = \frac{1}{dS^{e}_{ij}dS^{e}_{ij}}\left\{- S_{ij}\,dS^{e}_{ij} \pm \left[\left(S_{ij}\,dS^{e}_{ij}\right)^2 + dS^{e}_{ij}dS^{e}_{ij}\left(\frac{2}{3}Y^2 - S_{ij}S_{ij}\right)\right]^{\frac{1}{2}}\right\} \tag{6.7}$$

It is known that the factor R varies from 0 to 1. Therefore, the positive sign of the above expression for the factor R can be taken. It is worth mentioning that Bicanic [1989] presented the evaluation of the factor for most common yield conditions by solving quadratic equations.

Before the onset of yield has taken place, the uniaxial equivalent stress Y is equal to the yield stress of virgin material σ_{yp}. After the onset of yield has occurred, it is considered as the initial Von Mises stress (equivalent stress) σ_{vm} for the current load step.

During the iteration process, by using the value of the yield criterion (surface) on the n^{th} iteration number f^n and the value of the yield criterion on the $(n+1)^{th}$ iteration number f^{n+1}, the following possible cases have to be examined.

$$
\begin{array}{llcl}
\text{(a)} & f^n < 0 & , & f^{n+1} < 0 \\
\text{(b)} & f^n < 0 & , & f^{n+1} \leq 0 \\
\text{(c)} & f^n = 0 & , & f^{n+1} \geq 0 \\
\text{(d)} & f^n = 0 & , & f^{n+1} < 0
\end{array}
\qquad \textbf{(6.8)}
$$

In case (a) there is no plastic deformation and the solution in purely elastic. In case (b) the transition from elastic to plastic conditions occurs; in other words the factor R is larger than zero, but less than one. Therefore, the evaluation of the elastic stress increments is compulsory in order to bring the stress state to stress state at which the onset of yield commences. The plastic strains are obtained by using the flow rule and the remaining part, (1 - R) $d\sigma_{ij}^e$, which occurs beyond the yield point. In the case (c) the stress increments $d\sigma^e$ occur completely beyond the yield point. Hence it requires only the computation of the plastic strain increment by using either equation (2.61) or equation (2.62). In case (d) unloading takes place and it is assumed that there is no plastic deformation for the node being considered.

It should be noted that the stress increments in the third direction which are obtained by taking the actual stresses $d\sigma_{ij}$ as the imaginary elastic stresses $d\sigma_{ij}^e$ are given as follows:

$$d\sigma_{33} = \nu \ (d\sigma_{11} + d\sigma_{22}) \qquad \textit{plane strain}$$

$$d\sigma_{33} = 0 \qquad \textit{plane stress} \tag{6.9}$$

By using equation (6.7), the factor R can be obtained and then the elastic stress increments which have to be brought to stress state at which the onset of yield takes place are obtained as follows:

$$d\sigma_{ij}^e = R \ d\sigma_{ij} \tag{6.10}$$

In this expression it should be noted that the stress increments $d\sigma_{ij}$ include the stress increments in third direction. The elastic strain increment can be obtained by using equation (6.10) and (2.1).

6.2 CORRECTION FACTOR

As discussed earlier, for the plane strain case the stress increment in third direction can be written as follows:

$$\dot{\sigma}_{33} = \nu \ (\dot{\sigma}_{22} + \dot{\sigma}_{11}) + E \ (\dot{\varepsilon}^{p}_{11} + \dot{\varepsilon}^{p}_{22}) \tag{6.11}$$

or

$$\dot{\sigma}_{33} = \nu \ (\dot{\sigma}_{22} + \dot{\sigma}_{11}) - E \ \dot{\varepsilon}^{p}_{33} \tag{6.12}$$

It can be seen that the approximation for $\dot{\varepsilon}_{33}{}^{p}$ is computed by considering that the plastic deformation is incompressible; in other words it is computed indirectly and employed in the flow rule. Therefore, it may be assumed that the new value obtained using the flow rules will not agree with the old values.

Lee [1983] presented correction factors depending on either prescribed traction or displacement control problems, in order to consistently employ the flow rules during the iteration. For problems given with prescribed traction boundary conditions, the correction factor is given as follows:

$$R_{cor} = \frac{(\dot{S}_{kl} \ \dot{\sigma}_{kl})^{n-1} + E \ S_{33} \ (\varepsilon^{p}_{33})^{n}}{(\dot{S}_{kl} \ \dot{\sigma}_{lk})^{n-1} + E \ S_{33} \ (\dot{\varepsilon}^{p}_{33})^{n-1}} \tag{6.13}$$

where n donates the number of the iteration step. It should be noted that the correction factor makes it possible to obtain exact expressions for the plastic strain increments in the solution domain without stress redistribution. In case stress redistribution takes place, Lee [1983] presented an acceleration process which can be used in combination with the correction factor and is defined by introducing accelerators to the plastic strain increments as follows:

$$\dot{\varepsilon}^{p}_{ij} = (\dot{\varepsilon}^{p}_{ij})^{n} + \alpha \left[(\dot{\varepsilon}^{p}_{ij})^{n-1} - (\dot{\varepsilon}^{p}_{ij})^{n}\right] \tag{6.14}$$

Based on experience, it is known that it is sufficient to specify a fixed value of the accelerator, α, for the problem to be solved. Its value varies from 1.4 to 1.7 for plane stress problems and form 1.2 to 1.5 for plane strain problems (see Lee [1983]).

6.3 CONVERGENCE CRITERION

In order to ensure that a prescribed standard accuracy is satisfied in the iterative process, a convergence criterion can be used. There are mainly four practical convergence criteria :

(i) The norm of changes in the primary unknown vector $[\dot{x}]$

(ii) The norm of changes in the plastic strain increments $\dot{\varepsilon}_{ij}{}^{p}$ (for initial strain approach)

(iii) The magnitude of changes in the primary unknowns vector $[\dot{x}]$ or

(iv) The magnitude of changes in the plastic strain increments which are considered as secondary unknown.

The first and second criteria are based on the square root of the sum of the squares of the components of either the primary unknowns or the secondary unknowns. For this type of convergence criterion, typical values of percentage changes range between 0.001% and 0.01%. The third and fourth criteria are based on the absolute values of the largest term in either the primary unknowns or the secondary unknown to the largest percentage change in the each term of either the primary unknown or the secondary unknown. For this convergence criterion, the convergence values for the percentage changes range between 0.1 and 1%.

6.4 COMPUTER SOLUTION PROCEDURE

The solution algorithm, based on the initial strain approach, used in the present work can be summarised as follows:

Step 1 - Elastic Solution

- Calculate the kernels: U_{ij}, T_{ij} and W_{kij}.

- Assemble the matrices.

- Apply all the boundary conditions with full load to get:

$[A]\ [x] = [C] + 0$

- Store [A] and [W], which will not be calculated again.

- Solve the system equations to get the [x] known.

- Sort [x] into [u] and [t].

- From [u] and [t], calculate boundary stresses $[\sigma]$.

- From the boundary values of [u] and [t], obtain internal values.

- Using Hook's law, calculate $[\varepsilon]$ from $[\sigma]$.

Step 2 - First yield

- Search for highest nodal value of the Von Mises stress and compare it to the yield stress.

- Determine first yield and scale down the load [C] and the prescribed displacements and prescribed tractions.

- Determine the remainder of load and decide on the size of load increment.

- Store the scaled down first-yield values of [u], [t], [σ] and [ε]. These are the initial values.

Step 3 - Load Increment After Yield

- Initialise all plastic strains at the first load increment after yield.

- Initialise all incremental values of variables [Δu], [Δt], [$\Delta\sigma$], [$\Delta\varepsilon$].

- Apply the load increment (multiply the load vector [C] by load size factor). This will also scale both prescribed stresses and displacements.

- Solve the equations to obtain [Δx].

- Sort [Δx] into displacement [Δu] and traction [Δt] increment.

- From [Δu] and [Δt] calculate boundary stress [$\Delta\sigma$].

- From the boundary values of [Δu] and [Δt], obtain internal values of [Δu] and [$\Delta\sigma$].

- Use Hooke's law to calculate [$\Delta\varepsilon$] from [$\Delta\sigma$]. All values of all variables (boundary and internal) are now known.

- Calculate plastic strains using flow rules (either equation (2.61) or (2.62)) for yielded nodes.

- For first load increment after yield this is the first approximation for $[\Delta\varepsilon^p]$.

Step 4 - Iteration ($\Delta\varepsilon^p > 0$)

- Initialise all incremental values of $[\Delta u]$, $[\Delta t]$, $[\Delta\sigma]$, $[\Delta\varepsilon]$ but not $[\Delta\varepsilon^p]$.

- Use previously stored [A] and [W] to get $[A]\,[\Delta x] = [\Delta C] + [W]\,[\Delta\varepsilon^p]$.

- Solve the equations to obtain the new $[\Delta x]$ which now includes the effect of plasticity.

- Sort $[\Delta x]$ into $[\Delta u]$ and $[\Delta t]$.

- From $[\Delta u]$ and $[t]$, calculate boundary stresses $[\Delta\sigma]$.

- From the boundary values of $[\Delta u]$ and $[\Delta t]$ and $[\Delta\varepsilon^p]$ over cells, obtain internal $[\Delta u]$.

- Differentiate numerically (via the shape functions) the internal $[\Delta u]$ to get differentials of $[\Delta u]$.

- Use the strain-displacement relationships to calculate internal $[\Delta\varepsilon]$. This is the total

$[\Delta\varepsilon]$ which is equal to $[\Delta\varepsilon^e] + [\Delta\varepsilon^p]$.

- Use stress-strain relationship to calculate internal $[\Delta\sigma]$.

- For yielded nodes calculate the plastic strains using flow rules.

Step 5 - Convergence check

- Check the largest percentage change in the plastic strain rates $\Delta\varepsilon^p$ at each node to see whether the percentage change is equal or less than the tolerance (typically 0.1%).

- If convergence has occurred, accumulate the incremental values to obtain values of [u], [t], $[\sigma]$, $[\varepsilon]$, $[\varepsilon^p]$. Go to step 3 for next load increment, if required

- If convergence is not achieved:, check the number of iterations. If the maximum limit is not exceeded, go to step 4. If exceeded, terminate the analysis.

6.5 SOME REMARKS ABOUT THE INCREMENTAL-ITERATIVE PROCEDURE

In the application of the integral identities throughout the elasto-plastic analysis it is not compulsory to employ an extensive iterative process (see Banerjee [1984]), but for both initial strain and particular integral approaches it is necessary to employ an iterative process. This iterative process can be performed over both the load vector due to the applied load increment and the plastic load vector due to the plastic strain increment. In this process any stress increment is not stored until the convergence criterion is satisfied. The advantages of this procedure is that it is easy to check whether the iterative procedure is performed in a consistent manner. On the other hand, the iterative process can be performed by storing both the elastic and the actual stress increments at the end of the each iteration step. Kane [1994] presented an iterative process in which the actual stresses is stored during the iteration. In this process the actual stresses $\dot{\sigma}_{ij}$, are computed by using the following expression

$$\dot{\sigma}_{ij} = R \left\{ d\,\sigma_{ij} - \frac{3\, S_{ij}\, S_{kl}}{2\, \sigma_{eq}^{2} \left(1 + \dfrac{H}{3\mu} \right)}\, d\,\sigma_{kl} \right\} \tag{6.15}$$

The factor R defined in equation (6.2) is used in this expression. By considering a simple case (uniaxial behaviour) it can be easily seen that this expression for the actual stresses causes significant accumulative errors in loading. It can be observed that the iterative process where the stress increments are not added to the total stresses until convergence is satisfied is much more convenient.

The boundary integral equation and the flow rules are given in rate form. Therefore, in the iterative process the average values of the plastic strain rates should be used in order to stabilize the iterative process.

CHAPTER 7

APPLICATIONS IN TWO-DIMENSIONAL ELASTO-PLASTIC PROBLEMS

In this chapter, the present initial strain formulation in which the displacement rates are obtained from the integral equations are differentiated via the shape functions in order to obtain the stress and strain increments at interior points is applied to some classical test problems and the results are compared with either analytical solutions or the corresponding FE solutions. In order to evaluate of integrals six Gaussian points for integrals over boundary and interior elements are employed.

For the FE analysis, the commercially available finite element package ABAQUS [1995] was used. For data files required by ABAQUS, the commercially available graphical pre-processor FEMGEN [1994] was used. ABAQUS is a general purpose FE package with many non-linear capabilities. ABAQUS input file is prepared in terms of elements, nodes material properties, boundary conditions, load steps and output control. In modelling elasto-plastic problems using ABAQUS, the user can either define the number of load steps and incrementations within each step or let the program use automatic incrementation by dividing the total applied load into small steps and analysing each load-step. Isoparametric quadratic quadrilateral 8-noded elements a nd 4-noded elements with reduced (2x2) integration points were used in all the applications.

7.1 UNIAXIAL TENSILE PROBLEM

7.1.1 Plane Stress Case

This test problem consists of a square plate subjected to uniform tension in the x-direction. The geometry and loading conditions are presented in Figure 7.1. The BE discretisation is shown in Figure 7.2 where two meshes were used; mesh A (4 boundary elements with no internal points) and mesh B (8 boundary elements and 4 internal cells). The material is to be loaded in tension up to 587 N/mm^2 and it is to be solved using 24 increments. The following properties are assumed for the linear strain hardening material

$\sigma_y = 483\ N/mm^2$

$E = 200 \times 10^3\ N/mm^2$

$\nu = 0.3$

$H = 4223.8267$

In the FE model, this problem is represented by 4 quadratic plane stress element under reduced (2x2) integration. As depicted in Figure 7.3(a), the stress-strain curve given by mesh A, which has no internal points, is in excellent agreement with the corresponding FE results. Figure 7.3(b) shows a comparison of the two BE meshes, which indicates that there is no significant changes in stress levels both at interior and boundary nodes used in mesh B.

7.2.2 Plane Strain Case

In this problem, the square plate is assumed to be sufficiently thick for plane strain conditions to be applicable. The boundary conditions and material properties are exactly the same as the plane stress problem, as shown in Figure 7.1. The same two meshes are employed in this problem. The plate is to be loaded in tension up to 595 N/mm^2 and it is to be solved using 10 load increments. For the FE analysis it is represented by 4 quadratic plane strain element under reduced (2x2) integration. As represented in Figure 7.4(a), the BE stress-strain solutions obtained using mesh A are in very good agreement with the corresponding FE solutions. The BE results obtained by using mesh B show that there is no significant changes in stress levels at interior and boundary nodes, as shown in Figure 7.4(b).

7.2 THICK CYLINDER UNDER INTERNAL PRESSURE

The geometry and loading conditions of this test problem are given in Figure 7.5. The radius ratio b/a is taken to be 2. The analytical solution of this problem was presented by Hodge and White [1950]. The material is assumed to be elastic-perfectly plastic with the following material properties.

σ_y = 200 N/mm^2

E = 200 x 10^3 N/mm^2

ν = 0.33

The boundary element discretisation is shown in Figure 7.6 in which two meshes are used;

mesh A with 10 boundary element and 4 cells occupying a 15° sector and mesh B occupying a 5° sector with the same number of elements. For the FE analysis, four 8-noded axisymmetric element with reduced (2x2) integration are employed.

The variation of the load factor P/σ_y with the non-dimensionalised displacement ($4\mu\, u_b/\sigma_y a$) at the outer radius for both BE meshes and the corresponding FE solutions are depicted in Figure 7.7. In the BE analysis 5 load increments were used to reach the load factor, P/σ_y, value of 0.79. The BE solutions show very good agreement with the analytical and FE results.

Figure 7.8 shows the hoop stress distribution along the radius for the load factor, P/σ_y, value of 0.76. The BE results, computed using 5 load increments are again in good agreement with the analytical and FE results.

7.3 PERFORATED PLATE IN TENSION

This problem is included to demonstrate the applicability of the BE formulation in a stress concentration situation in which a sharp stress gradient occurs near the plastic region. This plane stress problem was also investigated experimentally by Theocaris and Marketos [1964]. The geometry and loading conditions are given in Figure 7.9. The following material properties are assumed for the linear strain hardening material.

$\sigma_y = 24.3\ \text{N/mm}^2$

$E = 7000$ N/mm^2

$H = 224$

$\nu = 0.2$

For the BE analysis, three meshes were employed, as represented in Figure 7.10; mesh A with 12 cells and 19 boundary elements, mesh B with 16 cells and 22 boundary elements and mesh C with 20 cells and 25 boundary elements. All BE meshes employ partial rather than full interior modelling in order to demonstrate this capability of the initial strain approach. For the FE analysis, two meshes are also used; mesh A with 120 quadratic plane stress elements and 905 nodes, and mesh B with 500 first-order plane stress elements and 546 nodes, as shown in Figure 7.11.

Figure 7.12 shows the development of maximum strain at the first yielding point with the mean stresses, σ_m, at the root of the plate, for the BE, FE and previously published experimental results. The BE results have been computed to reach the non-dimensionalised mean stresses, $2\sigma_m/\sigma_y$, values of 0.9897. Both the BE and FE solutions are a little lower than experimental results. All three meshes used in the BE analysis gave consistent results until near the collapse load point. At the collapse load point mesh A did not give the correct approximation because the boundary element discretisations were not enough to cover the plastic response of the structure. Meshes B and C gave slightly better results than ABAQUS at the collapse load point. Figure 7.13 shows that the BE and FE results are in a satisfactory agreement on the variation of non-dimensionalised stresses σ_{yy}/σ_y at the root of plate near the collapse load in which the non-dimensionalised stresses, σ_{yy}/σ_y is equal to 0.91.

Figures 7.14 and 7.15 show comparisons of the BE solutions using perfectly plastic and linear strain hardening material models with the experimental result, for the mean non-dimensionalised stresses $2\sigma_m/\sigma_y$ and σ_{yy}/σ_y, respectively. Good agreement is reached between the BE solutions from meshes B and C and the experimental results. For the BE analysis, 6, 4 and 4 load increments were employed for meshes A, B, C, respectively. For the perfectly-plastic BE analysis, 4 and 3 load increments were employed for meshes B and C.

7.4 NOTCHED PLANE PLATE

The geometry and loading conditions of this plane stress problem is given in Fig. 7.16. The following material properties are considered for linear strain hardening material

σ_y = 24.3 N/mm^2

E = 7000 N/mm^2

H = 224.0

ν = 0.2

For the BE analysis two meshes are employed, as represented in Figure 7.17; mesh A has 20 boundary elements and 15 cells, while Mesh B has 22 boundary elements and 16 cells. For the FE analysis, two meshes are used, mesh A with 120 quadratic plane stress elements and 405 nodes, and mesh B with 980 first-order plane stress elements and 525 nodes, as shown in Figure 7.18. For the BE analysis the structure is loaded to reach the non-dimensional stresses $(2\sigma_m/\sigma_y)$ =1.144.

The stress-strain response at the root of notched plate for both the BE and FE methods is represented in Figure 7.19. Both BE meshes gave consistently good agreement with the corresponding FE solutions. For the BE analysis, 12 and 4 load increments were employed for meshes A and B respectively.

It is worth mentioning that it is possible to use a non-iterative incremental procedure. In this procedure it is clear that load increments should be reasonably small which make this approach impractical.

In the iterative incremental procedure the most efficient way is to reduce the number of load steps as much as possible and , when possible, to use average values for strain and stress increments which represent reasonably the particular load step in order to avoid accumulative errors caused by insufficient number of the load-steps.

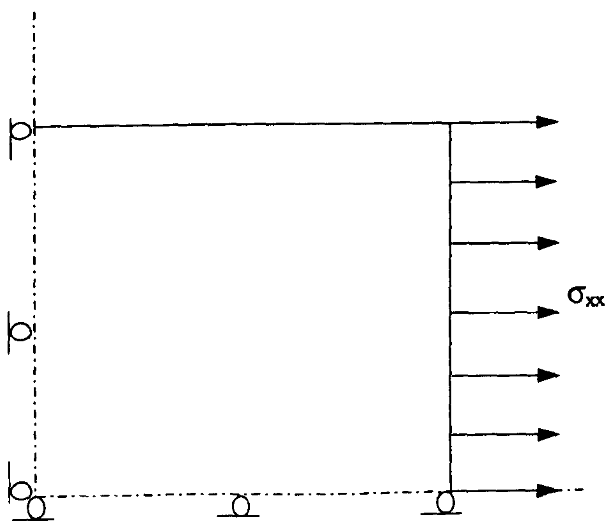
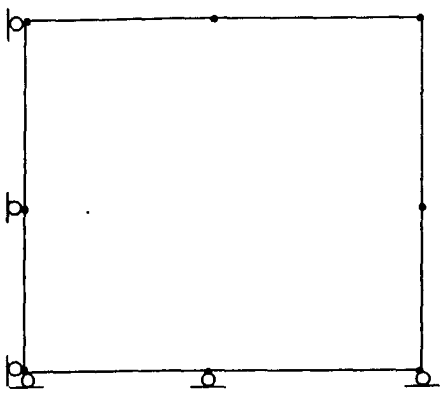
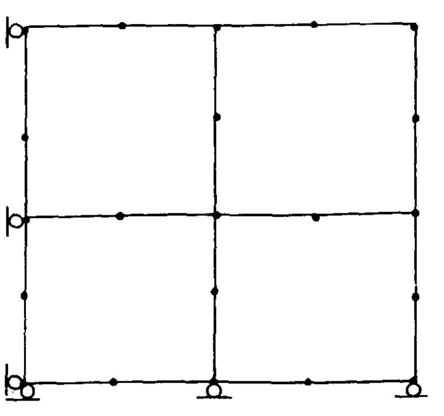


Fig.7.1 : Geometry and loading condition for uniaxial test problem



(a)



(b)

Fig.7.2 :Boundary element discretization for uniaxial test problem

(a) Mesh design for mesh A

(b) Mesh design for mesh B

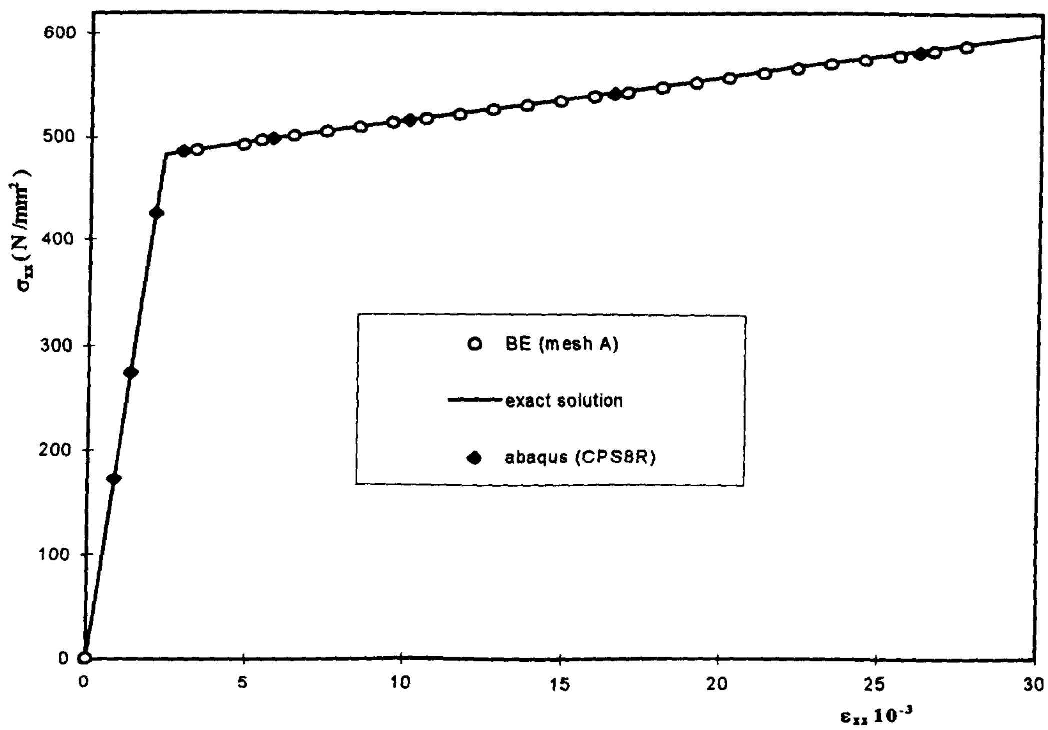


Fig.7.3(a) : Results for uniaxial problem under the plane stress condition (BE ,using mesh A , and ABAQUS).

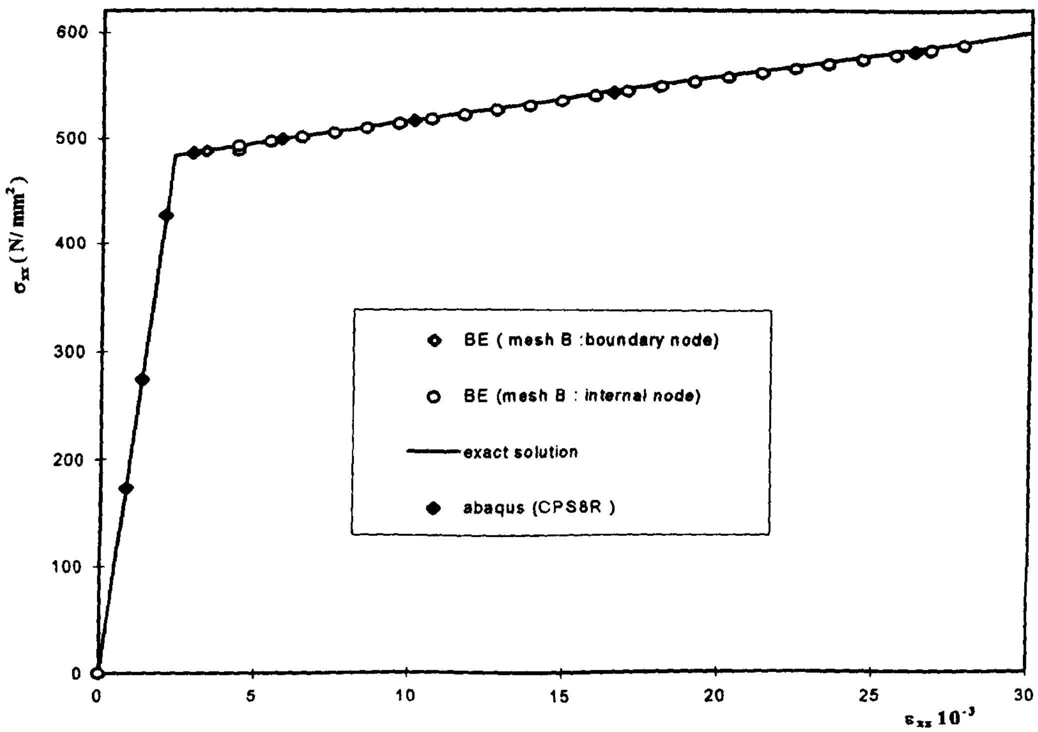


Fig.7.3(b) : Results for uniaxial problem under the stress condition (BE ,using mesh B , and ABAQUS).

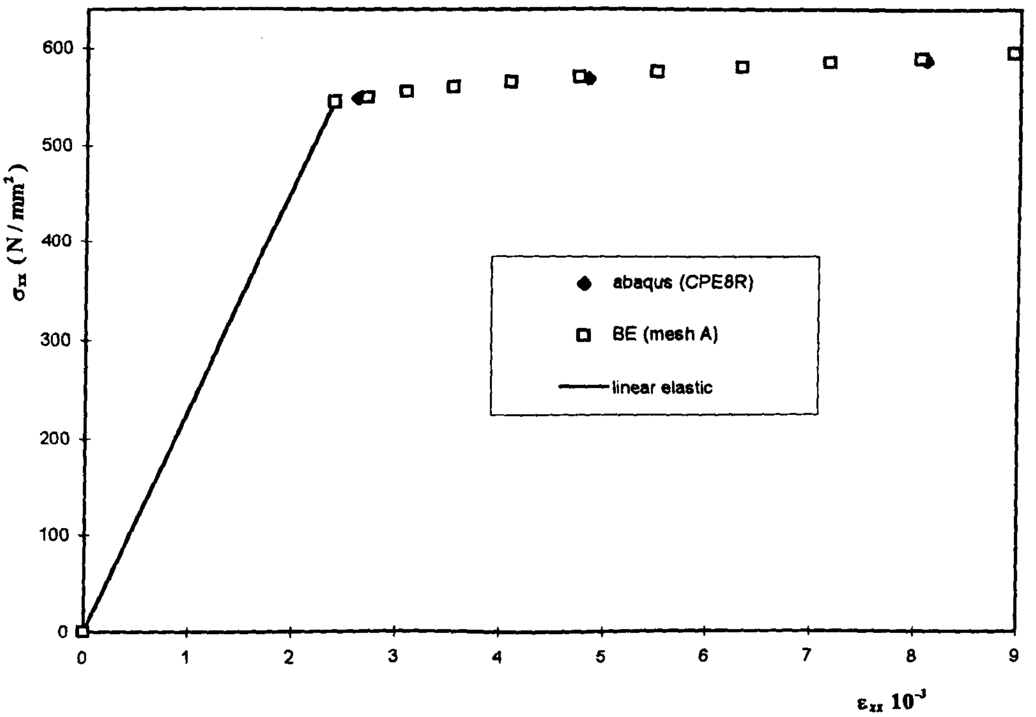


Fig.7.4(a) : Results for uniaxial test problem under the plane strain condition (BE ,using mesh A , and ABAQUS).

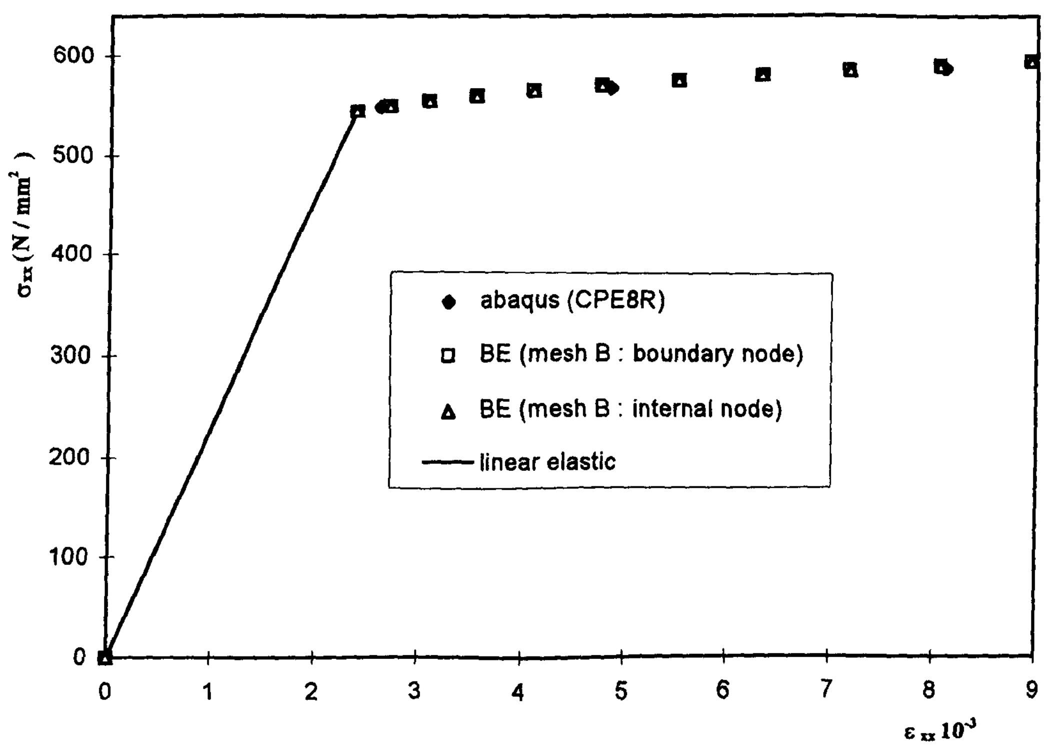


Fig.7.4(b) : Results for uniaxial test problem under the plane strain condition (BE ,using mesh B , and ABAQUS).

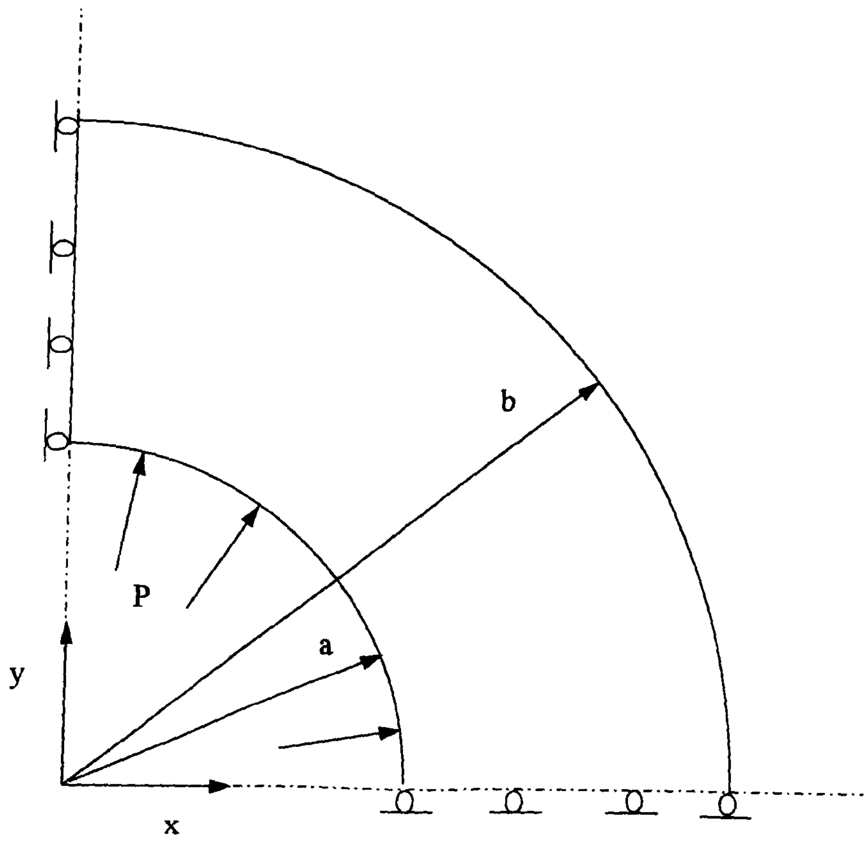
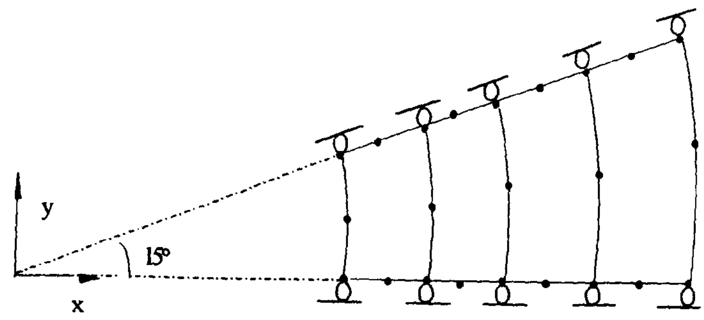
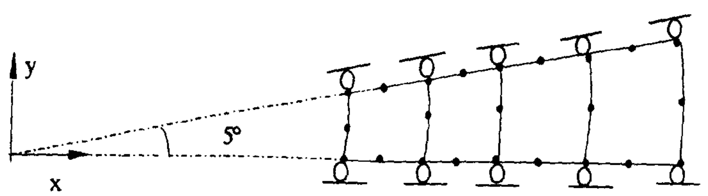


Fig.7.5 : Thick cylinder subjected to internal pressure



(b)



(a)

Fig.7.6 : The boundary element discretization for internally pressurized cylinder
(a) Mesh design for mesh A
(b) Mesh design for mesh B

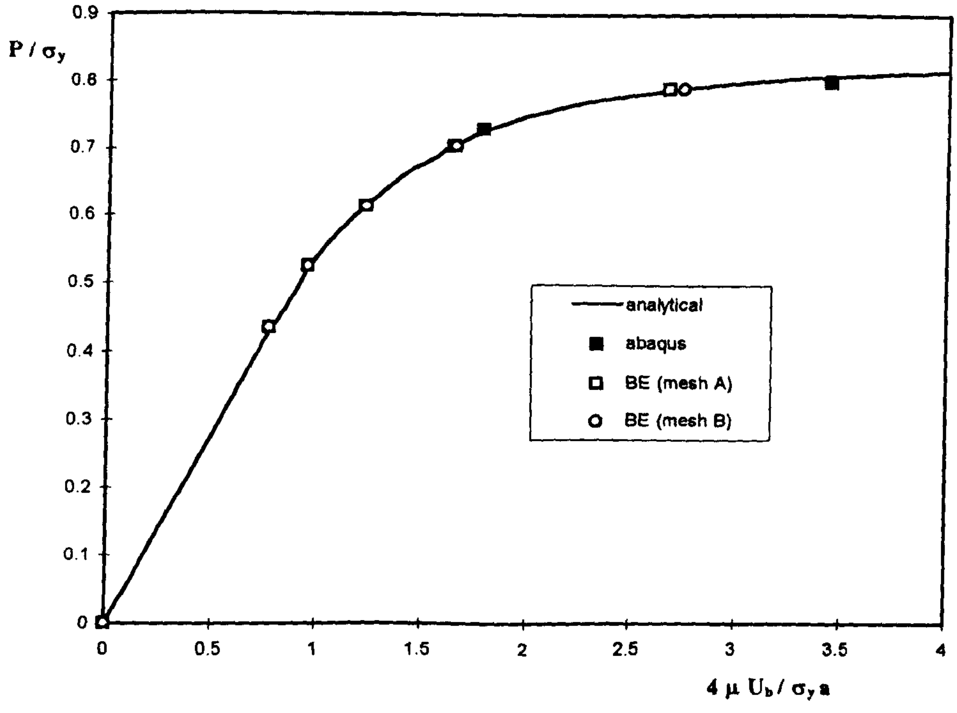


Fig.7.7 : Non-dimensionalized radial displacement at outer radius of thick cylinder for BE,using mesh A and mesh B , ABAQUS and analytical results from Hodge and White[1950].

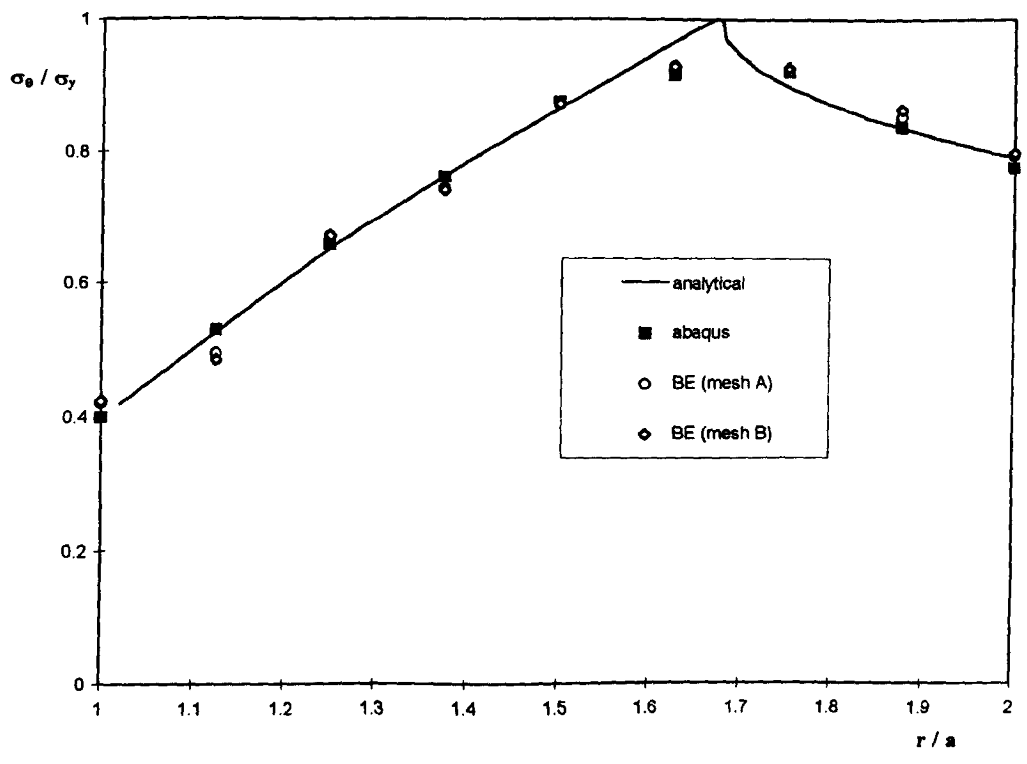


Fig.7.8 : Hoop stresses with radius at the load factor $P / \sigma_y = 0.76$ for BE,using mesh A and mesh B, ABAQUS and analytical results from Hodge and White[1950].

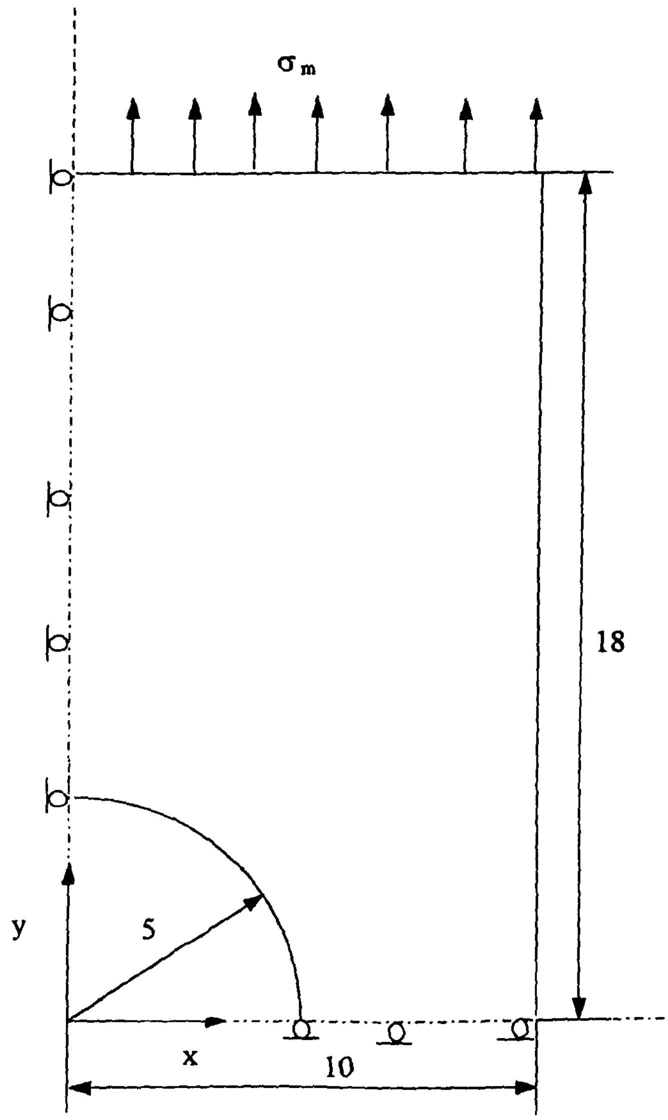
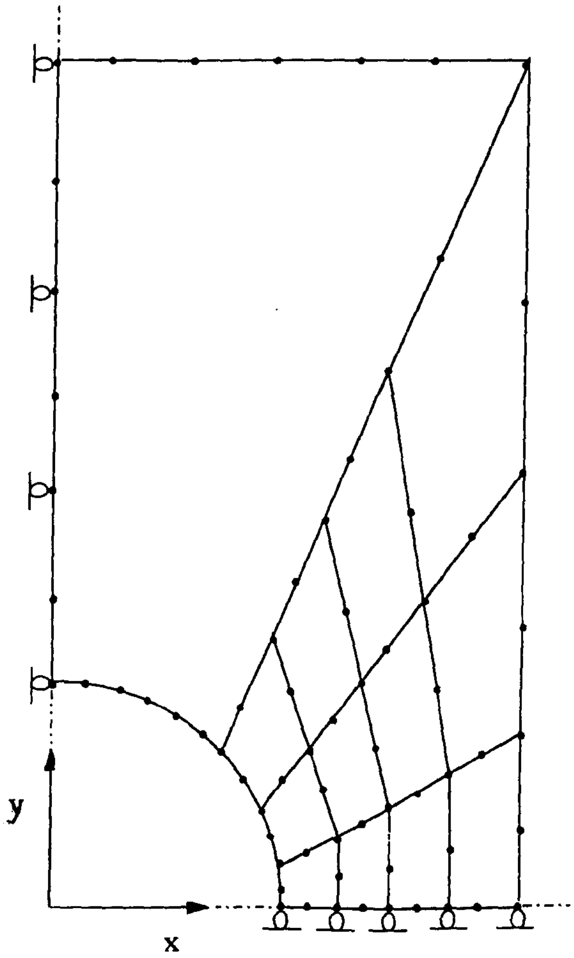
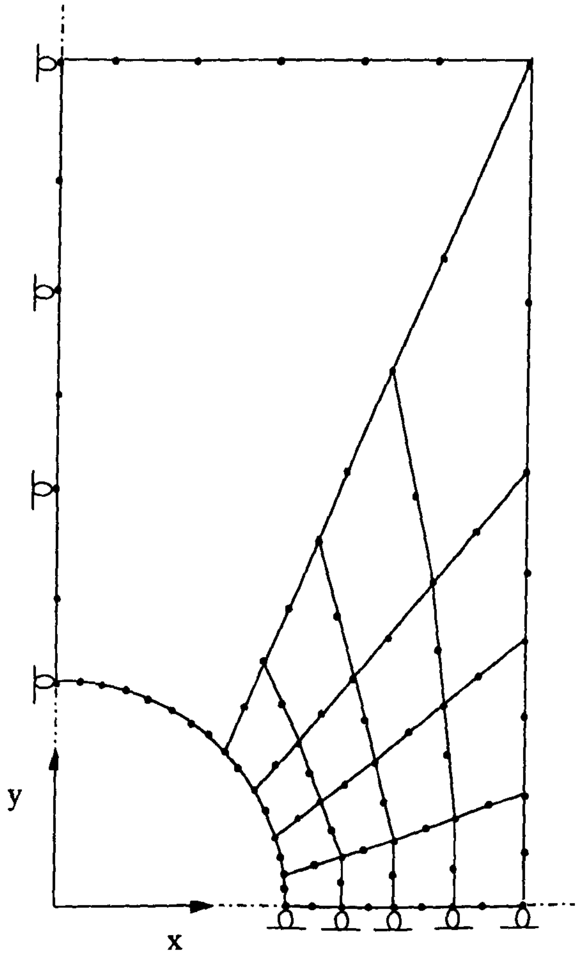


Fig.7.9 : Perforated plate problem under the plane stess condition



(a)

Fig.7.10 : The boundary element discretization of the perforated plate
(a) Mesh design for mesh A
(b) Mesh design for mesh B
(c) Mesh design for mesh C



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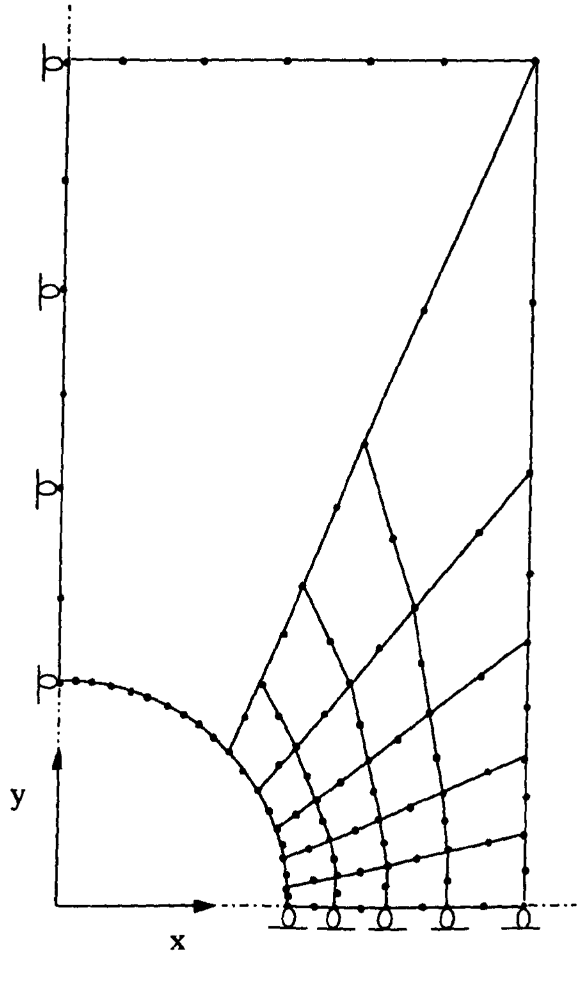
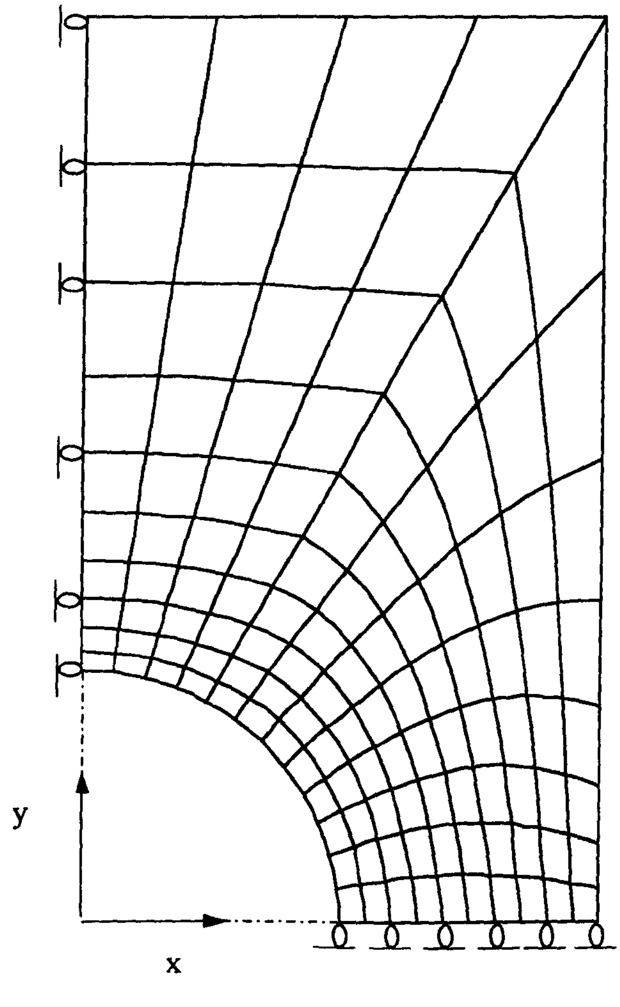
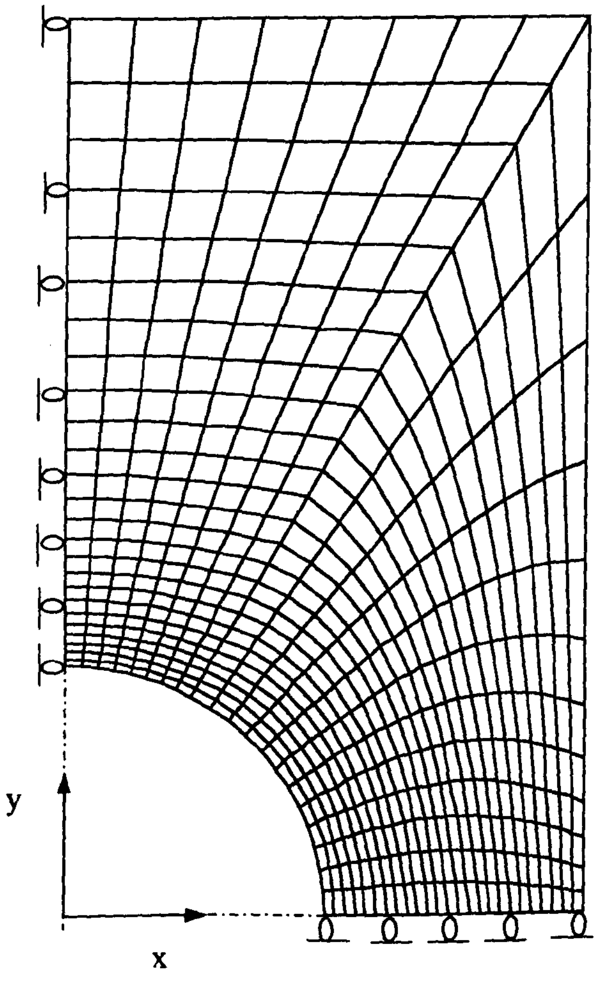


(c)



(a)

Fig.7.11 : The finite element discretization of the perforated plate for ABAQUS.
(a) Mesh design,mesh A, for 8-noded element (CPS8R)
(b) Mesh design,mesh B, for 4-noded element (CPS4R)



(b)

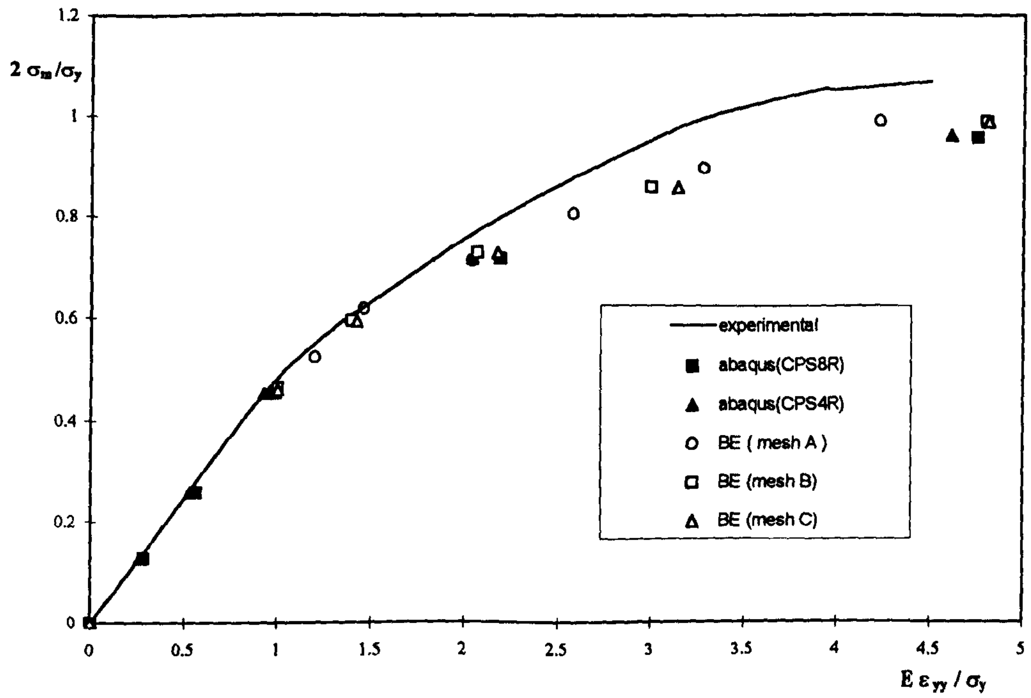


Fig.7.12 : Stress-strain response at the first yielding point for BE ,using mesh A , mesh B and mesh C , ABAQUS and experimental results from Theocaris and Marketos[1964].

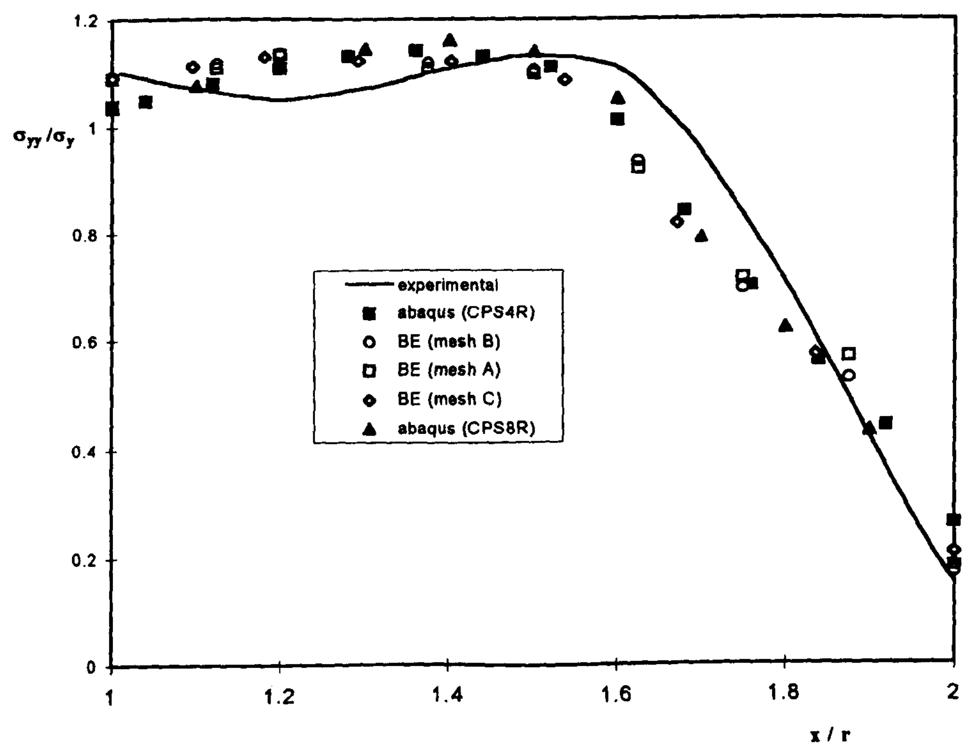


Fig.7.13 : Stress distribution at the root of the plate near the collapse load for BE, using mesh A ,mesh B and mesh C ,ABAQUS and experimental results from Theocaris and Marketos[1964].

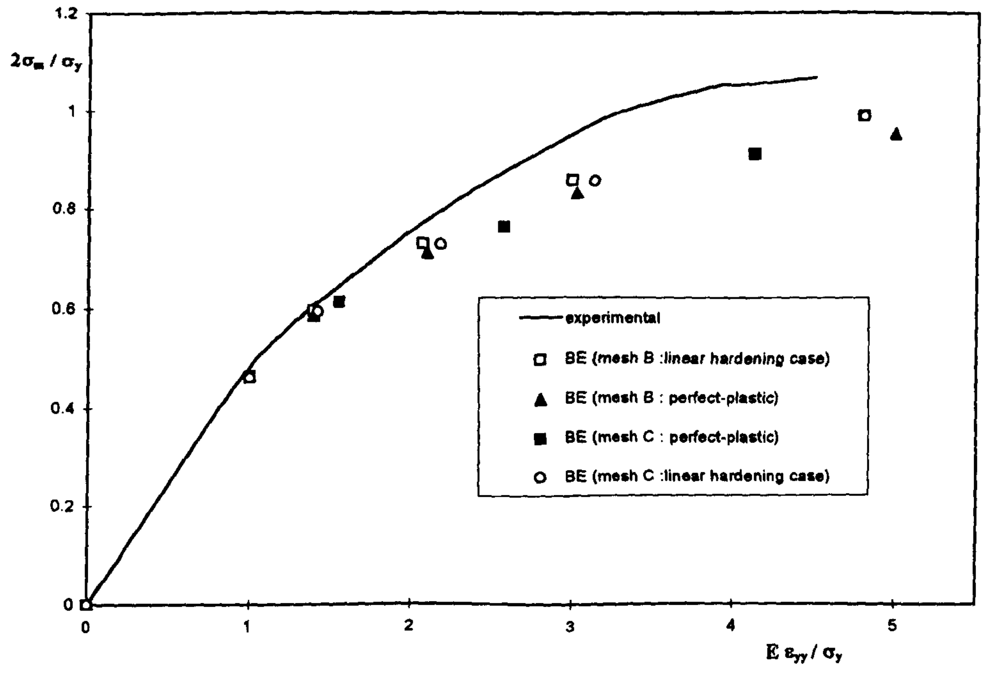


Fig.7.14 : Stress-strain response at the first yielding point for BE ,using mesh B, mesh C and assuming linear strain hardening and perfect-plastic cases, and experimental results from Theocaris and Marketos[1964].

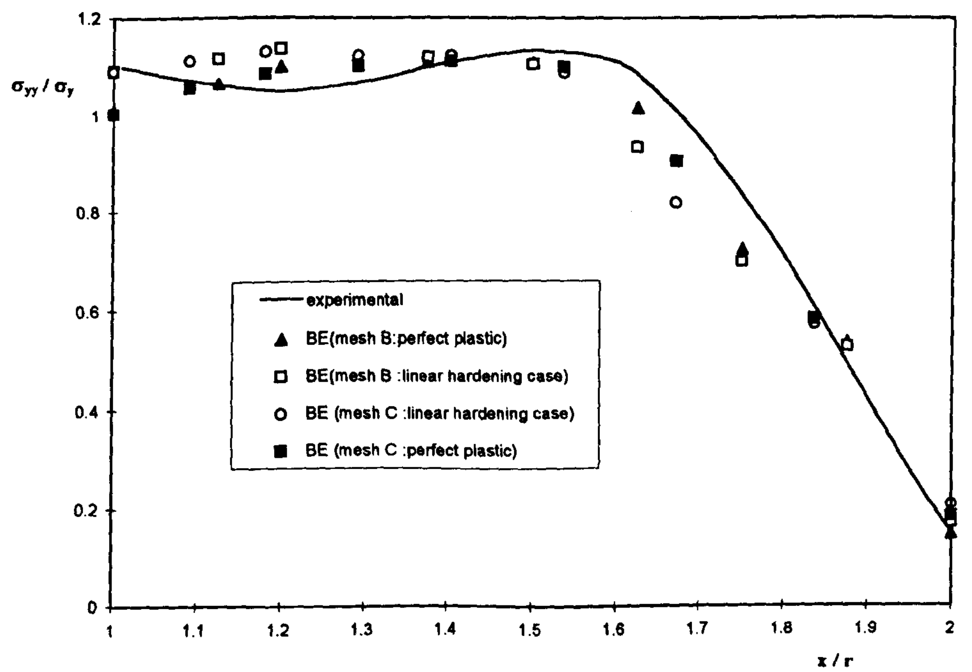


Fig.7.15 : Stress distribution at the root of the plate near the collapse load for BE , using mesh B ,mesh C and assuming linear hardening and perfect-plastic cases, and experimental results from Theocaris and Marketos[1964].

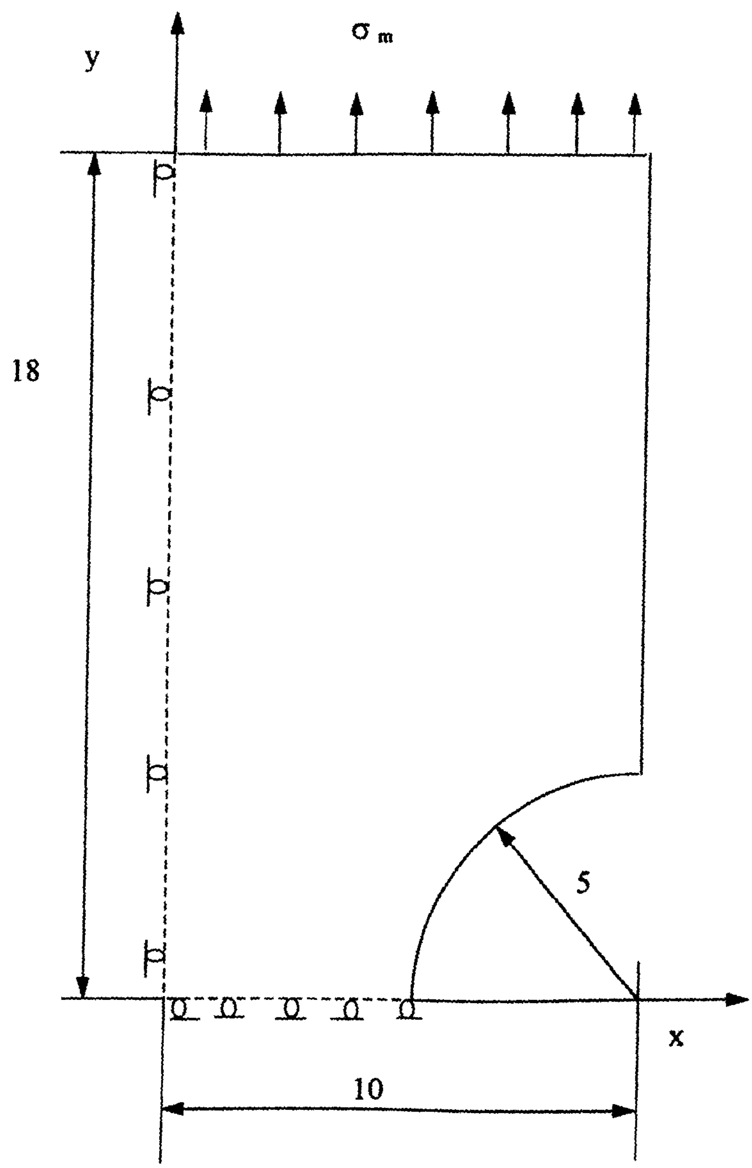
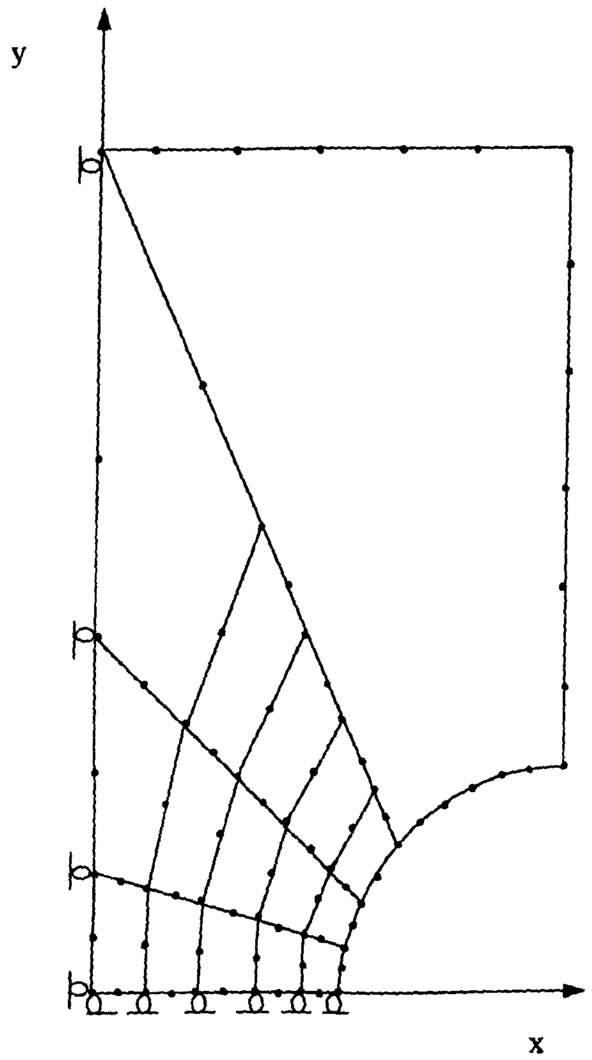
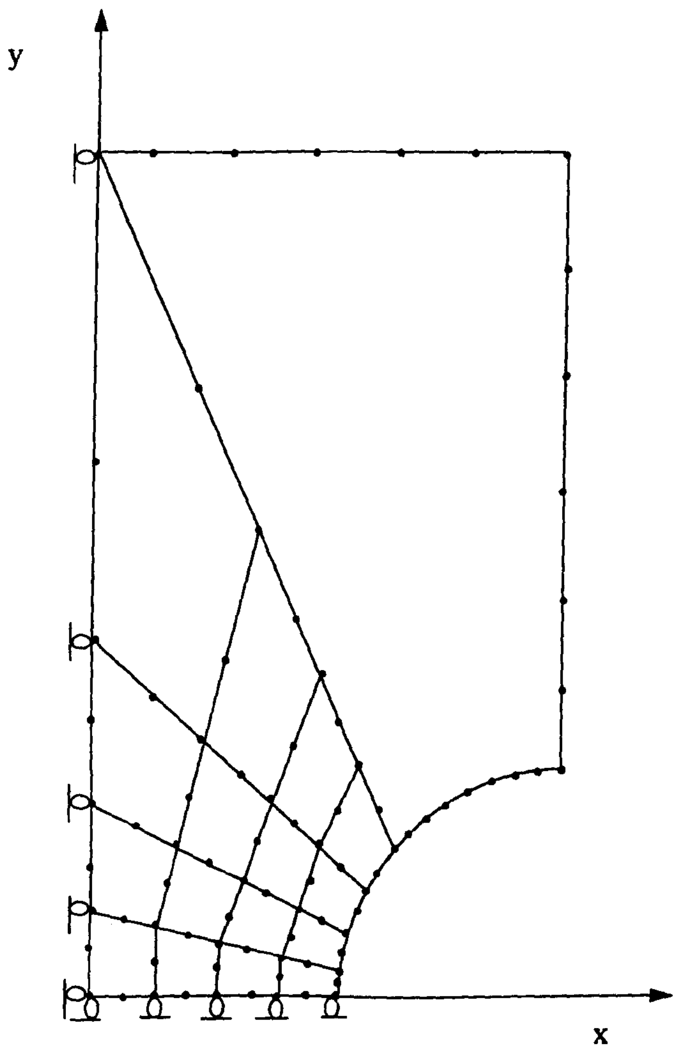


Fig. 7.16 : Notched plate problem under the plane stress condation

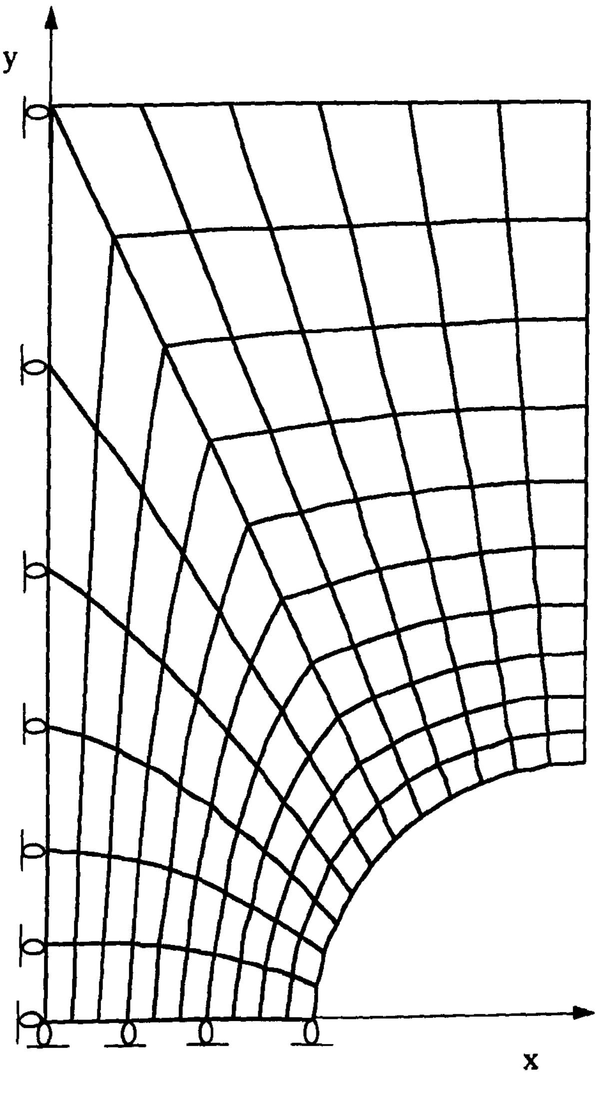


(a)

Fig.7.17 : The boundary element discretization of the notched plate
(a) Mesh design for mesh A
(b) Mesh design for mesh B

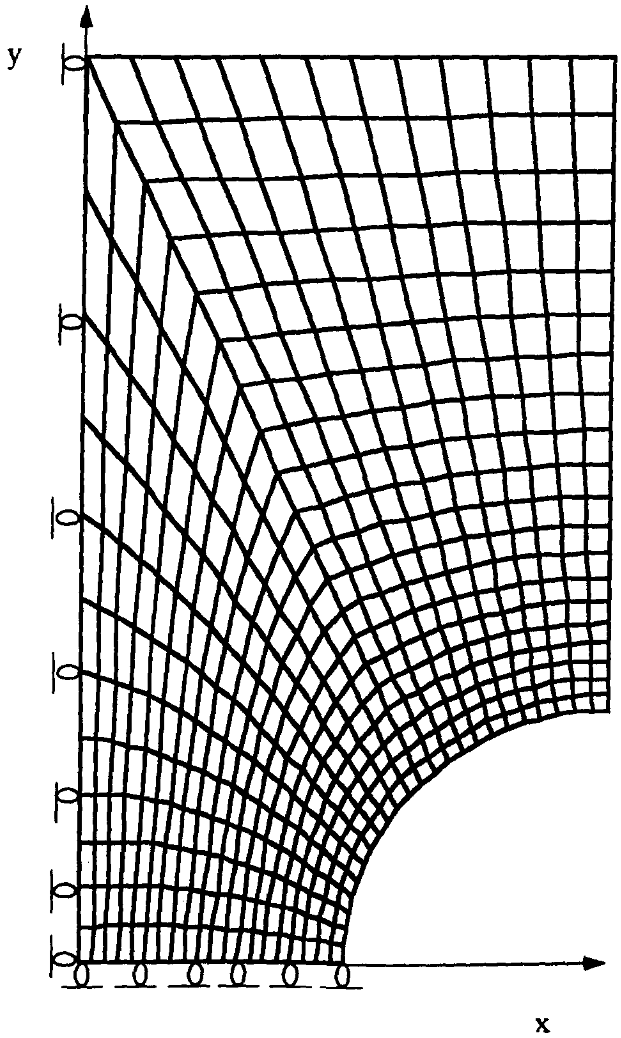


(b)



(a)

Fig.7.18 : The finite element discretization of the notched plate for ABAQUS.
(a) Mesh design,mesh A, for 8-noded element (CPS8R)
(b) Mesh design,mesh B, for 4-noded element (CPS4R)



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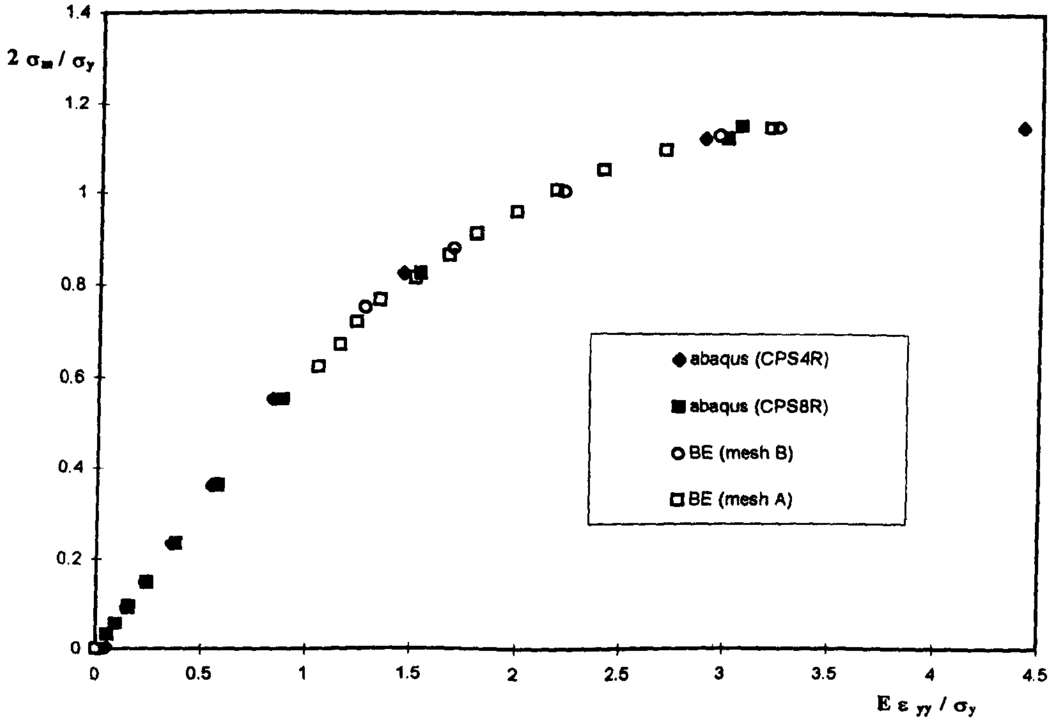

Fig.7.19 : Stress-strain response at the root of the notched plate for BE ,using mesh A and mesh B , and ABAQUS.

CHAPTER 8

CONCLUSIONS AND FURTHER STUDIES

A brief review of the basic principles of plasticity and expressions for Von Mises elasto-plastic flow rules which consider mixed hardening material behaviour was presented. The analytical formulation of the direct formulation of BE approach in linear elastic applications was briefly reviewed. Critical review of previous work in elasto-plastic BE formulation was presented in order to choose an accurate formulation easy to implement in a computer program and reliable enough for an inexperienced user to apply the program to practical engineering problems involving plasticity.

Previous published work on BE elasto-plasticity was studied and it became evident that many important details of the analytical and numerical details were not available. Furthermore, a degree of ambiguity was present in some implementations, which may explain the reason why the BE method is lagging behind the FE method in nonlinear applications. In this work, full details of the extension of the quadratic elastic BE formulation to elasto-plastic problems was presented without ambiguity or omission of important details. Integration schemes, incremental and iterative approaches, accurate calculation of stress and strain and details of the algebraic expressions are explained.

The FE method, which is now a well-established computational approach in elasto-plastic applications, is based on the use of stiffness matrices which provide a direct relationship between the forces and displacements at any point in domain to be analysed. The BE approach is different in that it does not use the concept of stiffness matrices. However, the concept of initial strain and initial stress approaches used in the finite element modelling of elasto-plasticity can be implemented into the BE method for elasto-plastic analysis. In the initial strain approach, by considering plastic strain rates as initial strain rates and modifying Betti's reciprocal energy theorem the integral equations are obtained. The initial stress approach is very similar to initial strain approach except that the initial plastic stress rates is used as a primary domain unknown in the integral equation. The choice between the use of initial strain and initial stress formulation is not critical because the effect of plasticity is catered for in the integral equations. The initial strain approach, however, is more suitable for traction-controlled problems because the first approximation for the stress increments are usually accurate. For the numerical implementation of BE method in this work, the initial strain formulation is adopted.

In the presence of plastic deformation in the domain to be analysed, these two approaches do not allow the Navier rate equation to be completely convertible to the boundary (or surface for three-dimensional applications) of the solution domain, because of the nature of elasto-plastic analysis. Therefore, the resulting integral equations include not only boundary integrals but also domain (or volume) integrals which consist of strongly singular integrands. This results in a significant increase in both numerical and analytical effort. Therefore, in order to circumvent the strongly singular integrals arising in domain kernels, the stress and strain rates inside the domain are calculated by numerical differentiation of the displacement rates obtained form the boundary integral equations in an elementwise manner.

As an alternative formulation to the numerical differentiation of displacement rates over the shape functions, the particular integral formulation is also discussed in detail. The main advantage of this method is that, it is possible to eliminate the domain integrals that are necessary in the initial strain method to include the effects of plasticity within the solution domain. However, the main disadvantage of the particular integral approach is that it requires a sub-region formulation in order to avoid the discretisation of the entire solution domain, which makes it implementation in a general-purpose computer program very complex. Furthermore, a great deal of effort is required to ensure the reliability of the numerical formulation in order that a minimum interaction from the program user is required. A computer program was written to implement the particular integral approach, but, due to the constraint of time and the effort involved in deriving and working out all the full details of all the numerical algorithms, the program was not fully tested on practical applications. The amount of effort required to devise and check the numerical algorithms in the particular integral approach is considerably more than that required for the initial strain approach.

For two-dimensional applications, isoparametric quadratic elements which allow both the geometry and variables to behave quadratically over each element were employed in numerical formulation because they provide a reasonable accurate integration of both the boundary and domain integrals. 3-node isoparametric quadratic elements were used to model the boundaries, whereas 8-node quadratic quadrilateral domain cells were used to model the interior. The treatment of boundary conditions, evaluation of algebraic equations and equation solving based on Gaussian elimination were discussed in detail. The initial strain displacement gradient BE formulation was implemented in a Fortran computer program BEPLAST which was written with emphasis on clarity. The program listing, of over 10,000

Fortran lines, contains full details of all the numerical algorithms and carefully explains the book-keeping adopted in the incremental-iterative algorithms. The intention was to provide a foundation for further extension into three-dimensional and more complex plasticity material behaviour in order to enable other researchers to continue this work. A self-contained program manual was also written.

The computer program was applied to several classical test problems in order to assess its accuracy and reliability. The applications included uniaxial tensile loading, pressurised thick cylinder, perforated plate under tension and a notched plate under tension. The BE solutions were shown to be in very good agreement with the corresponding analytical and FE solutions provided by the commercially available FE package, ABAQUS.

It was important to ensure that a prescribed accuracy in the convergence of solutions in each iteration at each load-step was satisfied. In the iterative process either initial values or average values of stress and plastic strain increments can be employed. In this work, initial values were used, because it was observed that it could be difficult to reach convergence in the results by using the average values. It can be concluded that the numerical differentiation of displacement rates in an elementwise manner is a powerful method which enables the strongly singular integrals to be circumvented. Other ways of obtaining the interior stresses and strain rates would require integration of such integrals, a difficult task which does not guarantee accuracy in all geometries.

It is worth mentioning that, for a comparable level of accuracy, the BE mesh discretisation, both on the boundary and interior, can be much coarser than that required for the FE method.

It is clear that BE requires both boundary and interior elements which are not dependant of each other. Therefore, unlike the FE method, interior elements (or cells) can be employed only in subregions where plastic deformation was expected. However, with the availability of modern FE interior mesh generators and the increasingly enhanced computer power available to analysts, it was concluded that partial modelling of the interior was not an attractive option since it required a pre-requisite knowledge of the possible plastic zones within the domain. Furthermore, selective interior modelling is not attractive to an inexperienced program user of a general-purpose package.

The range of applicability of the present BE formulation can be extended to cover multi-domain and contact mechanic problems. This would require a carefully designed robust numerical algorithm to incorporate nested iterations and load increments that are capable of monitoring contact development as well as marching the solution along the elasto-plastic material path. Introducing frictional stick-slip behaviour would further complicate the numerical algorithms.

Further work on the particular integral approach can deal with the robustness and reliability of this approach and compare it to the existing initial strain formulation in terms of execution time, convergence and accuracy.

The work presented in this thesis can be extended to three-dimensional elastoplastic problems utilising the initial strain approach. The present two-dimensional formulation treats the third (out of plane) stress rates indirectly, utilising a correction factor. In a three-dimensional implementation, this correction factor is not required, which means that the iteration process can result in quicker convergence. Interior discretisation of the solution domain in three-dimensional applications would require volume cells over which the internal displacements can be differentiated via the shape functions to obtain the internal stress and strain rates. In three-dimensional applications, however, a mesh generator would be required to model practical geometries. An existing FE mesh generator, such as FEMGEN, can be easily used to create BE meshes via a relatively simple translator program.

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APPENDIX A : THE QUADRATIC SHAPE FUNCTIONS

$$N_1(\xi_1,\xi_2) = \frac{-1}{4}\,(1 - \xi_1)\,(1 - \xi_2)\,(1 + \xi_1 + \xi_2)$$

$$N_2\,(\xi_1,\,\xi_2) = \frac{1}{2}\,(1 - \xi_1^2)\,(1 - \xi_2)$$

$$N_3\,(\xi_1,\,\xi_2) = \frac{-1}{4}\,(1 + \xi_1)\,(1 - \xi_2)\,(1 - \xi_1 + \xi_2)$$

$$N_4\,(\xi_1,\,\xi_2) = \frac{1}{2}\,(1 + \xi_1)\,(1 - \xi_2^2)$$

$$N_5\,(\xi_1,\,\xi_2) = \frac{-1}{4}\,(1 + \xi_1)\,(1 + \xi_2)\,(1 - \xi_1 - \xi_2)$$

$$N_6\,(\xi_1,\,\xi_2) = \frac{1}{2}\,(1 - \xi_1^2)\,(1 + \xi_2)$$

$$N_7\,(\xi_1,\,\xi_2) = \frac{-1}{4}\,(1 - \xi_1)\,(1 + \xi_2)\,(1 + \xi_1 - \xi_2)$$

$$N_8\,(\xi_1,\,\xi_2) = \frac{1}{2}\,(1 - \xi_1)\,(1 - \xi_2^2)$$

APPENDIX B : DIFFERENTIALS OF THE QUADRATIC SHAPE FUNCTIONS

$$\frac{\partial N_1(\xi_1, \xi_2)}{\partial \xi_1} = \frac{1}{4}(1 - \xi_2)(2\xi_1 + \xi_2) \quad ; \quad \frac{\partial N_1(\xi_1, \xi_2)}{\partial \xi_2} = \frac{1}{4}(1 - \xi_1)(2\xi_2 + \xi_1)$$

$$\frac{\partial N_2(\xi_1, \xi_2)}{\partial \xi_1} = -\xi_1(1 - \xi_2) \quad ; \quad \frac{\partial N_2(\xi_1, \xi_2)}{\partial \xi_2} = \frac{-1}{2}(1 - \xi_1^2)$$

$$\frac{\partial N_3(\xi_1,\xi_2)}{\partial \xi_1} = \frac{1}{4}(1 - \xi_2)(2\xi_1 - \xi_2) \quad ; \quad \frac{\partial N_3(\xi_1, \xi_2)}{\partial \xi_2} = \frac{1}{4}(1 + \xi_1)(2\xi_2 - \xi_1)$$

$$\frac{\partial N_4(\xi_1, \xi_2)}{\partial \xi_1} = \frac{1}{2}(1 - \xi_2^2) \quad ; \quad \frac{\partial N_4(\xi_1, \xi_2)}{\partial \xi_2} = -\xi_2(1 + \xi_1)$$

$$\frac{\partial N_5(\xi_1, \xi_2)}{\partial \xi_1} = \frac{1}{4}(1 + \xi_2)(2\xi_1 + \xi_2) \quad ; \quad \frac{\partial N_5(\xi_1, \xi_2)}{\partial \xi_2} = \frac{1}{4}(1 + \xi_1)(2\xi_2 + \xi_1)$$

$$\frac{\partial N_6(\xi_1, \xi_2)}{\partial \xi_1} = -\xi_1(1 + \xi_2) \quad ; \quad \frac{\partial N_6(\xi_1, \xi_2)}{\partial \xi_2} = \frac{1}{2}(1 - \xi_1^2)$$

$$\frac{\partial N_7(\xi_1, \xi_2)}{\partial \xi_1} = \frac{1}{4}(1 + \xi_2)(2\xi_1 - \xi_2) \quad ; \quad \frac{\partial N_7(\xi_1, \xi_2)}{\partial \xi_2} = \frac{1}{4}(1 - \xi_1)(2\xi_2 - \xi_1)$$

$$\frac{\partial N_8(\xi_1, \xi_2)}{\partial \xi_1} = \frac{-1}{2}(1 - \xi_2^2) \quad ; \quad \frac{\partial N_8(\xi_1, \xi_2)}{\partial \xi_2} = -\xi_2(1 - \xi_1)$$

APPENDIX C : DIFFERENTIALS OF THE LINEAR SHAPE FUNCTIONS

$$\frac{\partial L_1(\eta_1, \eta_2)}{\partial \eta_1} = \frac{-1}{4}(1 - \eta_2) \quad ; \quad \frac{\partial L_1(\eta_1, \eta_2)}{\partial \eta_2} = \frac{-1}{4}(1 - \eta_1)$$

$$\frac{\partial L_2(\eta_1, \eta_2)}{\partial \eta_1} = \frac{1}{4}(1 - \eta_2) \quad ; \quad \frac{\partial L_2(\eta_1, \eta_2)}{\partial \eta_2} = \frac{-1}{4}(1 + \eta_1)$$

$$\frac{\partial L_3(\eta_1, \eta_2)}{\partial \eta_1} = \frac{1}{4}(1 + \eta_2) \quad ; \quad \frac{\partial L_3(\eta_1, \eta_2)}{\partial \eta_2} = \frac{1}{4}(1 + \eta_1)$$

$$\frac{\partial L_4(\eta_1, \eta_2)}{\partial \eta_1} = \frac{-1}{4}(1 + \eta_2) \quad ; \quad \frac{\partial L_4(\eta_1, \eta_2)}{\partial \eta_2} = \frac{1}{4}(1 - \eta_1)$$

APPENDIX D : TENSORS FOR THE PARTICULAR INTEGRAL APPROACH

The particular solution tensor D^{PF}_{iml} is given by

$$D^{PF}_{iml}\,(Q, P_m) = C_o\,[(x_i\,\delta_{lm} + x_m\,\delta_{il})\,(c_1 + d_1\,r\,(Q, P_m)$$
$$+ (c_2 + d_2\,r\,(Q, P_m)\,x_l\,\delta_{im} + \frac{d_1}{r\,(Q, P_m)}\,x_i\,x_l\,x_m]$$

in which

$$x_i = x_i - (P_m)_i$$

$$c_1 = \frac{-8}{2\mu\,(1 - \nu)} \;;\; d_1 = \frac{b_n\,15}{2\mu\,(1 - \nu)}$$

$$c_2 = c_1 + \frac{32}{\mu} \;;\; d_2 = d_1 - \frac{75\,b_n}{\mu}$$

The particular solution tensor S^{PF}_{ijlm} (Q, P) is given by

$$
\begin{aligned}
S^{PF}_{ijlm}(Q, P_m) = {} & (e_2 + f_2\, r\,(Q, P_m))\, \delta_{jm}\, \delta_{il} \\
& + (e_3 + f_3\, r\,(Q, P_m))\, \delta_{ij}\, \delta_{lm} \\
& + (e_4 + f_4\, r\,(Q, P_m))\, \delta_{im}\delta_{jl} \\
& + \frac{f_1}{r\,(Q, P_m)} \left(x_j\, x_m\, \delta_{il} + x_i\, x_j\, \delta_{lm} + x_i\, x_m\, \delta_{jl}\right) \\
& + \frac{f_2}{r\,(Q, P_m)} (x_i\, x_l\, \delta_{jm} + x_j\, x_l\, \delta_{im}) \\
& + \frac{f_3}{r\,(Q, P_m)} \left(x_l\, x_m\, \delta_{ij}\right) + \frac{f_5}{r^3\; Q,\, P_m} \left(x_i\, x_j\, x_l\, x_m\right)
\end{aligned}
$$

where

$$e_1 = 2\,\mu\, c_1 \;\; ; \;\; f_1 = -\, f_5 = 2\mu\, d_1$$

$$e_2 = \mu\,(c_1 + c_2) \;\; ; \;\; f_2 = \mu\,(d_1 + d_2)$$

$$e_3 = e_1 + \frac{2\mu\nu}{1 - 2\,\nu}\,(3\, c_1 + c_2) \;\; ; \;\; f_3 = f_1 + \frac{2\mu\nu}{1 - 2\,\nu}\,[4\, d_1 + d_2]$$

$$e_4 = e_2 - 64 \;\; ; \;\; f_4 = f_2 + 225\, b_n$$

It should be noted that this formulation can handle both plane stress and plane strain cases by using the effective value of Poisson's ratio.