Stochastic modelling and optimization with applications to actuarial models

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Abstract

This thesis is devoted to Ruin Theory which sometimes referred to the collective ruin theory. In Actuarial Science, one of the most important problems is to determine the finite time or infinite time ruin probability of the risk process in an insurance company. To treat a realistic economic situation, the random interest factor should be taken into account.

We first define the model with the interest rate and approximate the ruin probability for the model by the Brownian motion and develop several numerical methods to evaluate the ruin probability.

Then we construct several models which incorporate possible investment strategies. We estimate the parameters from the simulated data. Then we find the optimal investment strategy with a given upper bound on the ruin probability.

Finally we study the ruin probability for our class of models with the Heavy- Tailed claim size distribution.

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CHAPTER 1

Introduction

1.1 Introduction and Literature Review

1.1.1 Ruin Probability

Ruin theory, sometimes called as collective ruin theory, is the branch of actuarial science that studies an insurer's insolvency based on mathematical modelling of the insurer's policy. In Actuarial Science, one of the most important problems is determining the finite time or ultimate ruin probability of the risk process in an insurance or investment company, for independent and dependent claims, see e.g. Cheng and Pai (2003, for independent claims); Dufresne and Gerber (1989, for general methods); Embrechts et al (1997, for modelling); Albrecher (1998); Schmidli (2008, for stochastic techniques) and Laven *et al.* (2005, for dependent claims).

Probability of ruin is the probability that liabilities will exceed assets on a present value basis at a given future valuation date, resulting in ruin. It is considered as the measure of risk of insolvency for an insurance company. Hence the ruin probability is a crucial parameter for assessing the risk exposure of companies.

The theoretical foundation of ruin theory/risk probability, known as the classical risk model in the literature, is introduced in 1903 by Lundberg (1934). The model is stated below.

Let $U(u,t) > 0$ be a classical continuous time surplus process, then

$$U(u,t) = u + ct - \sum_{i=1}^{N_t} X_i,$$

where

- u is the insurer's initial surplus,
- c is the insurer's rate of premium income per unit time,
- N_t is the number of claims in the time interval (0, t] and has a Poisson distribution with mean λt and,
- $\{X_i\}$ is a sequence of independent and identically distributed (i.i.d.) random variables representing the individual claim amount.

In Lundberg's model, the company does not have any investment return on its assets. Ruin occurs when $U(t)$ falls below 0, this may equate to insolvency. The probability of ruin is

$$\Psi(u) = P[U(t) < 0 \text{ for some t } 0 < t < \infty].$$

Lundberg establishes an explicit formula for the ruin probability when claim sizes are exponentially distributed and another main result he obtained is known as Lundberg's inequality (1930), which gives an upper bound on the probability of ultimate ruin. It states that

$$\Psi(u) \leq exp(-Ru),$$

where the parameter R known as the adjustment coefficient. Assenter and Nielsen (1995) get a similar result when the premium rate is a right continuous function of the reserve. Taylor (1976) derives an improvement to the above inequality function.

Later, the corresponding ideas are developed by Thorin (1973). He studies the classical model where claim size has gamma distribution. Grandell (1991) derives Lundberg inequalities for the finite time horizon ruin probability in the Cox-Ingersoll-Ross Model (Ross, 2000). Another method uses the connection between the probability of ruin and the maximal aggregate random variable which is suggested by Goovaters and Vylder (1984a, 1984b).

The classical model is widely studied and developed in the actuarial literature. Anderson (1957) extends the model in which claims occur as a general renewal process.

In this new model, the inter-claim times form a sequence of independent and identically distributed (i.i.d.) random variables with common distribution function and the claim sizes are also i.i.d. He obtains an equation for the surviving probability and the equation has an exact solution when claims sizes are exponentially distributed. In addition, an explicit result for the ultimate ruin probability is derived for a particular case. Then, much of the study of this model has concentrated on numerical procedures for calculating ruin probabilities (for example, Dickson, 1998).

In the classical model, when the claim size is exponentially distributed (or closely related to it), simple analytic results for the probability of ruin in infinite time exist. For more general claim amount distributions, e.g. heavy-tailed, the Laplace transform technique does not work and one needs some approximations. There are some common approximations.

Cramer-Lundberg Approximation (Grandell, 1991) yields quite accurate results, however it requires the adjustment coefficient to exist. Vylder (1996) derives Exponential Approximation. The Beekman-Bowers approximation (Burnecki, *et al*. 2004) gives the better results, it even becomes an exact formula in some cases.

The drawback of the above methods is that they ignore the interest rate of surplus of insurance company. In more recent developments, the ruin probability of a risk process with interest rate has received considerable attention. Brekelmans and Waegenaere (2002) split the time horizon into small intervals of equal length and consider ruin probability in the case when the premium income (reserve) in a time interval is received at the beginning of that interval, instead of assuming claims are paid at the end of an interval, and derive lower and upper bounds of the ruin probability. The combined results of two bounds converge to the actual ruin probability with the high accuracy.

Recently, research has focused on calculating the ruin probability modelled when a stochastic interest rate is used. Cai (2002) develops two models. The first model assumes that the interest rates form an i.i.d. sequence. The second model assumes that the interest rates form an autoregressive time series model. He obtains Lundberg type inequalities for the ruin probability. Yang and Zhang (2006) use martingale techniques (Hall and Heyde, 1980) to prove the convergence of the discounted surplus process and to obtain an expression for the ruin probability of a discrete time risk model with random interest rate.

Markov Model

Paulsen (2008) shows that since 1998, there has been three particular new developments in risk theory. These are

- The emphasis on heavy tailed claim distribution;
- The application of Gerber-Shiu penalty function;
- By control of the risky investments and possibly reinsurance, the possibility to influence the ruin probability.

He also introduces the risk process by means of two basic processes: A basic process P with $P_0 = 0$; and a return on investment generating process V with $V_0 = 0$. By assumption, P and V have the forms

$$P_t = pt + \sigma_P W_{P,t} - \sum_{i=1}^{N_t} S_i, \qquad V_t = vt + \sigma_V W_{V,t}.$$

Here W_P and W_V are Brownian motion, N is a Poisson process with rate λ and the $\{S_i\}$ are positive i.i.d. random variables with distribution function F. Moreover, W_P, W_V, N and $\{S_i\}$ are all independent, p is the premium rate. W_P is the small claim. The return on process V is the standard Black Scholes return process. Under these assumptions, Y_t is a strong homogeneous Markov Process and defined as

$$Y_t = Y_0 + P_t + \int_0^t Y_j dV_j.$$

The probabilities of ruin in finite versus infinite time is defines as

$$\Psi(t, y) = P(T \le t | Y_0 = y) \ \ and \ \ \Psi y = P(T < \infty | Y_0 = y)$$

respectively where T is the time of ruin as $T = inf\{t : Y_t < 0\}$.

Some Results for the Markov Model with heavy tails

Following the above definition and assumption, then $\Psi(y)$<1 if $v > \frac{1}{2}\sigma_V^2$ and Ψ is twice differentiable on $(0, \infty)$ and is a solution of the equation, $L\Psi(y) = -\lambda\overline{F}(y)$, with boundary conditions $\lim_{y\to\infty}\Psi(y) = 0$ and $\Psi(0) = 1$ if $\sigma_P > 0$. L is the integral-differential operator and $\overline{F} = 1 - F(y)$.

Asymptotic Results

Asymptotic results for heavy tail distributions is a popular topic (e.g. Embrechts *et al.* (1997), Mikosch and Nagaev (2001), Biard *et al.* (2008)). To present the results, let us introduce some definitions.

- $F \in R_\alpha$ if $\overline{F} = x^{-\alpha}f(x)$ where f is a slowly varying function (see Chapter 4 for more notation).
- $F \in ERV(\alpha, \beta)$ if for $0 < \alpha \le \beta < \infty$ and all $t > 1$,

$$t^{-\beta} \le \lim_{x\to\infty} inf\frac{\overline{F}(tx)}{\overline{F}(x)} \le \lim_{x\to\infty} sup\frac{\overline{F}(tx)}{\overline{F}(x)} \le t^{-\alpha}.$$

- $F \in S$ if $\lim_{x\to\infty}\frac{\overline{F*2}(x)}{\overline{F}(x)} = 2$.
- $F \in L$ if $\lim_{x\to\infty}\frac{\overline{F}(x+1)}{\overline{F}(x)}$=1.
- $F \in D$ if $\lim_{x\to\infty} sup\frac{\overline{F}(x/2)}{\overline{F}(x)} < \infty$.

It is obvious that $R_{-\alpha} = ERV(\alpha, \alpha)$. Then when $\sigma_V = 0$ and $F \in S$, Jiang and Yan (2006) prove that

$$\Psi(t, y) \sim \frac{\lambda}{\alpha v}\overline{F}(y)(1 - e^{\alpha v t}), \quad 0 < t \le \infty,$$

when N is the Poisson process and otherwise

$$\Psi(t,y) \sim \overline{F}(y) \int_0^t e^{-\alpha v s} dm(s),$$

where m is the renewal measure of N.

For the light tailed case with $\sigma_V > 0$ and $p = \frac{2v}{\sigma_V^2 \alpha}$, the most precise results are (Grandits, 2004):

- If $F \in R_{-\alpha}$ where $\alpha < p$, then $\Psi(y) \sim \frac{2\lambda}{\sigma_V^2 \alpha(p-\alpha)} \overline{F}(y)$;
- If $F \in R_{-}pand\overline{F}(y) = x^{-p} f(x)$ and $M < \infty$ where $M = \int_1^\infty \frac{f(x)}{x} dx$, then $\Psi(y)$ has the similar result as above. If $M = \infty$ then

  $$\Psi(y) \sim \frac{2\lambda}{\sigma_V^2 p} \int_1^y \frac{f(x)}{x} dx.$$

The Finite Time Model and ruin probability

Let S_n be the surplus of the company at the end of time n, $n = 1, 2, \cdots$. Let V_n be the income at the end of time n, we can construct a recurrence equation

$$S_n = (1 + \delta_n) S_{n-1} + V_n, \quad n = 1, 2, \cdots,$$

where δ_n denotes the interest rate between [n-1, n] and constitute a sequence of i.i.d. random variables. Tang and Tsitsiashvili (2003) prove that, if $F \in L \bigcap D$ and $EY^p < \infty$ for some positive number p then

$$\Psi(g,n) \sim \sum_{j=1}^{n} P(G \prod_{i=1}^{j} Y_i > g), \quad n = 1, 2, \cdots,$$

where G_n is premium minus claim between time $n-1$ and n, $Y_n = \frac{1}{1+\delta_n}$.

Except the above surplus model we state, there are other classical risk processes, such as, for example,

Classical Binomial Risk Model defined by

- The claim-number process is a binomial process N_t, t=0,1,2,... In each time period, the probability of a claim is q, $0 < q < 1$ and the probability of no claim is $p = 1 - q$,
- Let I_t be a sequence of i.i.d. Bernoulli random variables denoting occurrences of claims,
- Let X_t be a sequence of positive i.i.d. r.v.'s representing the claim amounts.

For t=1, 2, . . . , the surplus at time t is

$$U_t = u + t - [X_1 I_1 + X_2 I_2 + \cdots + X_t I_t],$$

where $u > 0$ is the initial surplus.

This model is called classical compound binomial risk model which was first proposed by Gerber (1988), and extended by Cossette *et al*. (2003) to the so-called compound Markov binomial model. They derive a more general model by using a Markovian environment in 2004. Yang, *et al*. (2008) consider a Markov risk model, in which the claim occurrence and the claim amount are both regulated by a Markov process which is a discrete time finite space homogeneous and irreducible Markov chain.

When the initial surplus $u = 0$, they derive an explicit expression for the discounted joint probability function of the surplus before ruin.

Relationship to the Time-Series, Modelling and Statistical Analysis

In statistics and mathematical finance, a time-series is considered as an ordered sequence of values of a variable at equally spaced time points
For example, Autoregressive (AR) model, the autoregressive process of order p is denoted AR(p), and defined as

$$X_t = c + \sum_{i=1}^{p} \alpha_i X_{t-i} + \varepsilon_t,$$

where $\alpha_1, \cdots, \alpha_p$ are the fixed constants, ε_t is a sequence of independent (or uncorrelated) random variables with mean 0 and variance σ^2, c is constant.

Maximum likelihood estimation. Bayesian and other computation-intensive statistical methods.

To fit the model to the data, the estimation of parameters is an important problem. There are two powerful estimation methods: Bayesian and Maximum likelihood estimation (MLE). Although, MCMC methods approved to be more flexible (e.g. McCulloch and Tsay (1993) use Gibbs sampler in time-series analysis), in economics, however, MLE is traditionally considered the most appropriate method (Tanner and Wong (1987), Tsay (1989), Tong, (1990), Magdalinos (2005)).

Parameter and Model Uncertainty.

In actuarial science, the estimation is often not robust due to the model uncertainty. In insurance cases, model and parameter uncertainty are often ignored. However, recently the parameter uncertainty in general insurance problems is acknowledged (see Klugman, (1992) for overview, see Scollnik, (1998) for a case study, see Harris (1999) for MCMC for regime switching models, see Parker (1997) for the uncertainty in the pension and life insurance problems, see Draper (1995) for theoretical assessments of uncertainty and see Cairns (2000) for discussions of the process of parameter and model uncertainty.)

Other developments of the actuarial sciences

For other directions in actuarial sciences, we mention Engle (1982), Maravall (1983) (for linear); Tong, (1990) (for non-linear) time-series modelling; Tong (1978), Tsay (1989) and Tiao and Tsay (1994) for the threshold auto regressive (TAR) models, Hosking (1981), Tiao and Tsay (1994) for ARMA model ; Tiao and Xu (1993), Tiao and Tsay (1994) for adaptive Forecasting and Philips and Solo (1992) for asymptotics.

1.1.2 Optimal investment Strategy and Gambling

A series of papers discussed the optimal strategy in the gamble game. It provides main motivation for our approach to the Optimal Investment Strategy under Ruin probability constraint.

Formulation of problem There is a primitive casino where one can stake any amount in one's possession, gaining r times the stake with probability w and losing the stake with probability $1 - w$ ($r > 0, 0 < w < 1$). The casino is subfair or fair if $w(1 + r) \leq 1$. α is the inflation rate and if $r = 1$, the casino is called red-and-black casino. We formulate the above game as a Dubins-Savage gambling game (1965).

For each integer $n \geq 1$, let f_{n-1} be the gambler's fortune before the nth play (with f_0 denoting the initial fortune). A strategy $\theta = \{y_1, y_2, \cdots\}$ is a sequence of stakes, where $0 \leq y_n \leq f_{n-1}$ is the gambler's stake on the nth play. The value of the strategy θ is defined as $V_\theta(f)$=P[$f_n \geq 1$ for some $n \geq 0 \mid f_0 = f$] for $f \geq 0$.

The gambler is said to use the bold strategy (B) if he takes the bold stake $b(f)$ such that $b(f)$=min { f, $(1 + \alpha - f)/r$ } if $0 \leq f \leq 1$ and $b(f) = 0$ if $f \geq 1$. Then by considering one play, we have

$$V_B(f) = (1 - w)V_B(\frac{f - b(f)}{1 + \alpha}) + wV_B(\frac{f + rb(f)}{1 + \alpha}).$$

Dubins and Savage (1967) show that an optimal strategy is a bold strategy if the primitive casino is subfair or fair and the α is 0 since there is no other strategy that provides gambler with a higher probability of reaching the goal. Klugman (1977) shows all optimal strategies are characterised when $w \leq \frac{1}{2}$ for the discounted red-and-black casinos. Chen (1977, 1978) prove that the bold strategy is optimal in the subfair red-and-black casino. But Chen et al. (2004, 2005) find that the bold strategy is not optimal for subfair primitive casinos with inflation if both $r > 1$ and $1/r \leq \alpha < r$. They conjectured that the bold strategy is optimal for subfair primitive casinos if $r < 1$.

Simulation approach.

We applied the simulation approach to find that our results support the Chen Shepp theoretical results (Chen 1977). This motivated us to use the simulation approach to find the optimal investment strategy with the given upper bound on the ruin probability.

1.2 Aims and objectives

Throughout this thesis, we attempt to determine the finite or infinite time ruin probability which employed in several different risk processes. In this way, we model the every risk process we list. We take into account the random interest factor to treat a realistic economic situation. For example, we assume that the interest rates are the random variables distributed with normal distribution given in Section 2.2. We define the model with the interest rate and approximate the ruin probability at fixed time point for the models by the Brownian motion in Chapter 2 and 3. Then we also construct several models which incorporate possible investment policies. We not only use the simulations to estimate the parameters in the models from the simulated data, but also test the different approximations. In Section 3.5, we hope to find the optimal investment policy with a given upper bound on the approximated ruin probability.

Finally we study the ruin probability for our class of models with the Heavy-Tailed claim size distribution.

1.3 Structure of the Thesis

In Chapter 2, we first define a model with a continuous interest rate and show how to approximate the ruin probability for the model by Brownian motion. We develop several numerical methods to evaluate the ruin probability. The model is introduced in Section 2.2.

In Section 2.3 the first approach via the Approximation of Sum of Correlated Lognormal Random Variables is reviewed.

In Sections 2.4 and 2.5, the second approach through the Approximation by Brownian Motion is analysed. Numerical calculations support the claim that ruin probability is increasing function of the variance of interest rate. However it appears that the approximation is not stable. In addition, it numerically discovered a surprising threshold such that the rate of increasing of the ruin probability much higher after the threshold point. Also the results for some parameters agreed with the Gaussian approximation suggested in Matsumoto and Yor (2005) (Proposition 2.6).

In Chapter 3, we construct several models which incorporate possible investment strategies. We estimate the parameters from the simulated data. Then we find the optimal investment strategy with a given upper bound on the ruin probability. The models, likelihood functions and MLE's are introduced in Sections 3.2 and 3.3. Numerical analysis with stochastic simulation for the estimation of the parameters is conducted in Section 3.4. In Section 3.5 we find via the random search algorithm the optimal investment strategy with a given upper bound on the ruin probability. Finally, in Section 3.6 we approximate the ruin probability by the associated integrals of the Brownian motion.

Again, the numerical threshold discovered in Chapter 2 was supported in all the examples. Also, it was discovered that although the estimation of interest rate was stable in all the models, the estimation of the exponential claim was unstable in many models, which was partly inspired by Magdalinos (2007).

Finally, in Chapter 4, motivated by Biard *et al* (2008), we study the ruin probability for our class of models with the Heavy - Tailed claim size distribution. Two new realistic models with interest rate factors were suggested and treated in Sections 4.3 and 4.4. The derived results extend Biard *et al*. (2008) to models (discrete and continuous time) with interest rate.

CHAPTER 2

The Ruin Probability with Brownian Motion

In this chapter we define a model with a continuous interest rate and show how to approximate the ruin probability for the model by Brownian motion. We develop several numerical methods to evaluate the ruin probability. The model is introduced in Section 2.2.

In Section 2.3 the first approach via the Approximation of Sum of Correlated Lognormal Random Variables is reviewed.

In Sections 2.4 and 2.5, the second approach through the Approximation by Brownian Motion is analysed. Numerical calculations support the claim that ruin probability is increasing function of the variance of interest rate.

2.1 Introduction

In order to obtain a good approximation for the ruin probability in a realistic economic situation, it should be taken into account that random interest factor may fluctuate stochastically with time. That means it is more realistic to use **continuous interest rates** in the modelling of the ruin probability and consider them as stochastic process. **Continuous interest rate** is a form of compound interest rate. With the continuous interest the length of the compounding period is assumed to be infinitely small. Therefore the interest is continuously compounded. In addition it is possible to measure how fast the sum of money growing at specific time point as the force of interest (e.g. per annum).

To match the continuous interest rate (δ) and the discrete interest rate i we assume that

$$\exp(\delta) = 1 + i.$$

In this chapter, we define the model with interest rate and discuss two approaches *Approximation by Sum of Correlated Lognormal Random Variables* and *Approximation by Brownian motion* to evaluate the ruin probability. For the second approach, we also develop several numerical methods.

2.2 The Model and Assumptions

Brekelmans and Waegenaere (2002) develop a model for the ruin probability with a stochastic interest rate. Here we develop a similar model which is based upon the following assumptions and notation.

- the rate of claims occurrence is constant;
- the amount of each claim, denoted by C, is constant and is paid by the insurance company just after the end of each time interval;
- the premium is a single payment and this premium is paid just after the beginning of first time interval;
- the reserve of the insurance company will be denoted by A at time n=0;
- δ_k is the interest rate at the time k;
- the capital and the claim of the company are increasing with continuous stochastic interest rate δ.

At time n the accumulated capital is given by $Ae^{\sum_{i=1}^{n}\delta_i}$. The accumulated claims at time n is $C\sum_{i=1}^{n} e^{\sum_{k=i+1}^{n}\delta_k}$. Therefore the surplus Z_n of the company at time n is given by

$$Z_n = Ae^{\sum_{i=1}^{n}\delta_i} - C\sum_{i=1}^{n} e^{\sum_{k=i+1}^{n}\delta_k}, \ \ n = 0, 1, 2, \ldots.$$

We note that this simple model does not include insurance risk. This will be incorpriated with later models. To calculate the ruin probability, we need to know the present value of Z_n. The present value of a single or multiple future payments known as cash flows is the nominal amounts of money to change hands at some future date, discounted to account for the time value of money. The *present value* of Z_n, denoted by $\bar{Z_n}$, is given by

$$\begin{aligned}\bar{Z_n} &= Z_n / e^{\sum_{i=1}^{n} \delta_i} \\ &= A - C\sum_{i=1}^{n} e^{-\sum\limits_{k=1}^{i} \delta_k}.\end{aligned}$$

Here the $\{\delta_k\}$ are i.i.d. and have a normal distribution with mean 0 and variance σ^2. Therefore for the above model, the ruin probability is

$$\Psi_n = P[\bar{Z_n} < 0] = P[(A - C\sum_{i=1}^{n} e^{-\sum\limits_{k=1}^{i} \delta_k}) < 0]. \tag{2.2.1}$$

Here the ruin probability we defined in Chapters 2 and 3 deal with ruin at a fixed time n only, which is different form the classical European ruin definition we stated in Chapter 4.

2.2.1 Approaches to calculating Ψ_n

To calculate the ruin probability as given in (2.2.1), Ψ_n can be rewritten in the following form

$$\begin{aligned}P[(A - C\textstyle\sum_{i=1}^{n} e^{-\sum\limits_{k=1}^{i} \delta_k}) < 0] &= P[(A - C\textstyle\sum_{i=1}^{n} \beta_i) < 0] \\ &= P[(\sum_{i=1}^{n} \beta_i) > \frac{A}{C}]. \end{aligned} \tag{2.2.2}$$

We note that $E(e^{-\sum\limits_{k=1}^{i} \delta_k}) = e^{\frac{i\sigma^2}{2}}$ and $Var(e^{-\sum\limits_{k=1}^{i} \delta_k}) = (e^{i\sigma^2} - 1)e^{i\sigma^2}$. In addition $-\sum\limits_{k=1}^{i} \delta_k$ has a limit Normal distribution. Therefore β_i has a Log-normal distribution with mean $e^{\frac{i\sigma^2}{2}}$ and variance $(e^{i\sigma^2} - 1)e^{i\sigma^2}$. We proposed two approaches calculating Ψ_n.

First approach is to use an approximation for sum of correlated lognormal random

variables. It is difficult to obtain an explicit formula for above probability, because the $\{\beta_i\}$ are not independent. Moreover, it is tempting to use the Central limit theorem (CLT) applied to $\sum_{i=1}^{n} \beta_i$. Because firstly the complicated sum can be treated as a single log normal random variable; secondly $\sum_{i=1}^{n} \beta_i$ could be considered as a linear process; and finally CLT for i.i.d sequences can be extended to general linear processes (Peligrad and Utev, 2006).
However, as we show later (Section 2.3) CLT is not appropriate and we need its stronger version called functional central limit theorem (FCLT) or invariance principle.

Second approach is to use an approximation based upon Brownian motion. If the continuous interest rates have other common distributions rather than normal, they could be transformed to the normal random variables followed by FCLT. In addition based upon our simulation, it seems that the transformation does not effect accuracy of the approximation for the considered model.

2.3 First approach: Approximating the sum of correlated log-normal random variables

Mehta *et al*. (2006) use a method based on matching a low-order Gauss-Hermite approximation of the moment generating function (MGF) of the sum of random variables with that of a lognormal distribution at a small number of points to evaluate an approximation to the sum of the correlated lognormal random variables. Therefore, the $\sum_{i=1}^{n} \beta_i$ in (2.2.2) can be approximated as a single log-normal random variable, and consequently we can obtain an approximation to the ruin probability (see Gauss Hermite formula in APPENDIX 1). Applying the above method, firstly there exists a lognormal random variable B, whose MGF can be written as

$$\begin{aligned}
\Phi_B(s) &= \int_0^\infty exp(-sb)P_B(b)db \\
&= \int_0^\infty exp(-sb)\frac{1}{\sigma b\sqrt{2\pi}}exp[-\frac{(lnb-u)^2}{2\sigma^2}]db \\
&= \sum_{n=1}^{N} \frac{W_n}{\sqrt{\pi}} \exp[-s \times exp(\sqrt{2}\sigma a_n + u)] + R_N.
\end{aligned}$$

In the above $P_B(b)$ is the probability density function of B, N is the Hermite integration order and R_N is a reminder term that decreases as N increases. W_n, a_n are weights and abscissas respectively, which can be both derived by using an appropriate computer package (e.g. Matlab).

For our model as shown in (2.2.1), we want to find the MGF of the sum of N correlated lognormal random variables, $\{\beta_i\}_{i=1}^N$ (same random variables in (2.2.2)) with the corresponding Gaussian, $\{X_i\}_{i=1}^N$, $X_i = 10\log_{10}\beta_i$, $i = 1, \ldots, N$, follow the joint distribution below

$$P_X(x) = \frac{1}{(2\pi)^{N/2}|\mathbf{C}|^{1/2}} exp(-\frac{(\mathbf{x}-\mu_x)^T\mathbf{C}^{-1}(\mathbf{x}-\mu_x)}{2}). \tag{2.3.1}$$

Where $\mathbf{x}$ is a vector, $\mathbf{C}$ is the covariance matrix, μ_x is the vector of means, $|.|$ denotes the determinant, and $(.)^T$ denotes the Hermitian transpose. By using (2.3.1) and the Gauss-Hermite approximation, the MGF. of $\beta_1 + \cdots + \beta_N$ can be written as

$$\Phi^{(c)}_{(\sum_{i=1}^N \beta_k)}(s) = \int_{-\infty}^{\infty} \frac{1}{(2\pi)^{N/2}|C|^{1/2}} \prod_{i=1}^{N} \exp(-s[\exp(\tfrac{x_i}{\xi})]) \times \exp(-\frac{(x-u_x)^T C^{-1}(x-u_x)}{2} dx$$

$$\triangleq \sum_{n_1=1}^{N} \cdots \sum_{n_N=1}^{N} \Big[\prod_{i=1}^{N} \frac{w_{n_i}}{\sqrt{\pi}}\Big] \times \exp\Big(-s\sum_{i=1}^{N}\Big[\exp\Big(\frac{\sqrt{2}}{\xi}\sum_{j=1}^{N} c'_{ij}a_{nj} + \frac{u_i}{\xi}\Big)\Big]\Big),$$

where $\xi = 0.1ln10$. The sum of $\beta_1 + \cdots + \beta_N$ (N correlated lognormal random variables) is approximated by a simple lognormal random variable, B. We assume that there exists an equation

$$\sum_{n=1}^{N} \frac{W_n}{\sqrt{\pi}} \exp[-s \times exp(\sqrt{2}\sigma a_n + u)] = \Psi^{(c)}_{(\sum_{i=1}^N \beta_k)}(s),$$

where the u and σ can be calculated numerically using a computer package such as Matlab. However this method only works for small n, so we are going to show another method for calculating the ruin probability in next section.

2.4 Second approach: Approximation by Brownian Motion

Brownian motion, $\{B_t, t \geq 0\}$, is a continuous-time stochastic process with continuous sample paths, $B(0) = 0$ and B_t has independent increments with distribution $B_t - B_s$ ∼N (0, t-s) for $0 \leq s \leq t$. $N(\mu, \sigma^2)$ denotes the Normal distribution with mean μ and variance σ^2. (Ross 2000)

The approach we applied uses a key lemma as below.

Lemma 2.1: *For any continuous sequence $\{a_i\}$, there is the following equation*

$$\sum_{j=1}^{[nx]} a_{j-1} = n \int_0^x a_{[ny]} dy. \tag{2.4.1}$$

[ny] in (2.4.1) is integer part of ny and has formula

$$\frac{[ny]}{n} = y + o(\frac{1}{n}). \tag{2.4.2}$$

Proof: Consider an integral

$$\int_{\frac{k-1}{n}}^{\frac{k}{n}} a_{[ny]} dy.$$

Since $k-1 < yn < k$, so if we consider integer part of yn, $[yn] = k-1$, so $a_{[ny]} = a_{k-1}$, hence

$$\int_{\frac{k-1}{n}}^{\frac{k}{n}} a_{[ny]} dy = \frac{1}{n} a_{k-1}.$$

This leads to the following equality

$$n \sum_{k=1}^{M} \int_{\frac{k-1}{n}}^{\frac{k}{n}} a_{[ny]} dy = \sum_{k=1}^{M} a_{k-1}.$$

It is same as

$$n \int_0^{\frac{M}{n}} a_{[ny]} dy = \sum_{k=1}^{M} a_{k-1}.$$

Now if M=[nx], we complete the proof of (2.4.1). Next, let $e^{-\sum_{t=1}^{i} \delta_t}$ in (2.2.1) equal to

a_{j-1} and applying Lemma 2.1 to (2.2.1) we have

$$\begin{aligned}
P[(A - C\sum_{i=1}^{n} e^{-\sum_{k=1}^{i}\delta_k}) < 0] &= P[(A - C \cdot n\int_0^{\frac{[nx]}{n}} a_{[ny]}dy) < 0] \\
&= P[(A - C \cdot n\int_0^{\frac{[nx]}{n}} exp(-\sum_{k=1}^{[ny]+1}\delta_k)dy) < 0] \\
&\approx P[(A - C \cdot n\int_0^{x} exp(-h \cdot B_y)dy) < 0] \\
&= P[\int_0^{x} exp(-h \cdot B_y)dy > \frac{A}{Cn}],
\end{aligned}$$

where $h = \sqrt{n+1}\sigma_k$. We use approximation $\sum_{k=1}^{[ny]} \delta_k \overset{d}{=} hB_y$, since $\{\delta_k\}$ is i.i.d. random variables. Given x=1, our approximation for ruin probability is

$$\Psi_n \approx P[\int_0^{1} exp(-h \cdot B_y)dy > \frac{A}{Cn}]. \tag{2.4.3}$$

The probability does not have a simple form, and will be found by numerical approximation.

2.5 Integral of Exponential Brownian motion

Let $B^{\mu} = \{B_t^{\mu} = B_t + \mu t\}$ be the corresponding Brownian motion, B_t, with constant drift $\mu \in R$. The exponential functional $A^{\mu} = \{A_t^{\mu}\}$ is defined by

$$A_t^{\mu} = \int_0^t exp(2B_s^{\mu})\, ds, \quad t \geq 0.$$

When $\mu = 0$, A_t^0 is written as A_t.

Matsumoto and Yor (2005) in a survey state that Bougerol (1983) obtains an interesting and important identity in probability, which plays an important role in the results that flow. Here is the simplest form of it.

Proposition 2.1: *Define W_t as a one-dimensional Brownian motion starting from 0, independent of B. Then, $\int_0^t exp(B_s)\, dW_s$, $sinh(B_t)$ and W_{A_t} are identical in law for every fixed t.*

From the above proposition, Matsumoto and Yor (2005) derive the following by considering the corresponding densities and characteristic functions.

Proposition 2.2: *For every $a \in \mathbf{R}$ and $t > 0$, there is*

$$E[\frac{1}{\sqrt{A_t}}exp(-\frac{a^2}{2A_t})] = \frac{1}{\sqrt{(1+a^2)t}}exp(-\frac{(Arcsinh(a))^2}{2t}) \tag{2.5.1}$$

and

$$E[exp(-\frac{a^2}{2}A_t)] = \sqrt{\frac{2}{\pi t}}\int_0^\infty cos(a \times sinh(s))exp(-\frac{s^2}{2t})\, ds. \tag{2.5.2}$$

2.5.1 The Law of A_t^μ at fixed times

Dufrense (2001), Alili and Gruet (1997) have some results that show the integral representations for the density of A_t^μ by somewhat complicated forms, even A_t^μ is simplified. Many specialists have tried to obtain simpler expressions. Yor (1992) proves the following formula:

$$P(A_t^\mu \in d\mu,\ B_t^\mu \in dx) = e^{\mu x - \mu^2 t/2} exp(-\frac{1+e^{2x}}{2\mu})\theta(e^x/\mu, t)\frac{d\mu dx}{\mu}, \tag{2.5.3}$$

where for t > 0 and r > 0,

$$\theta(r,t) = \frac{r}{(2\pi^3 t)^{1/2}} e^{\pi^2/2t} \int_0^\infty e^{-s^2/2t} e^{-r\times\cosh(s)} \sinh(s)\sin(\frac{\pi s}{t})\, ds. \tag{2.5.4}$$

The function $\theta(r,t)$ represents the density of the Hartman-Watson distribution (Hartman and Watson, 1974) and satisfies

$$\int_0^\infty e^{-a^2 t/2}\theta(r,t)\, dt = I_a(r), \ \ a > 0, \tag{2.5.5}$$

where I_a is the modified Bessel function.

As following the identity in law, when $\mu > 0$, Dufrense (2001) shows that

$$A_\infty^{-\mu} = \int_0^\infty exp(2B_s^{-\mu})\, ds = \frac{1}{2\gamma_\mu},$$

where γ_μ is a gamma random variable with parameter μ, that is,

$$P(\gamma_\mu \in dx) = \frac{1}{\Gamma(\mu)} x^{\mu-1} e^{-x} dx, \ \ x \geq 0.$$

Let the function $f^{\mu}(a,t)$ be the density function of $1/2A_t^{\mu}$, then following integral representation (2.5.4) for $\theta(r,t)$, Matsumoto and Yor (2005) obtain

$$f^{\mu}(a,t) = \frac{2}{(2\pi^3 t)^{1/2}} e^{\pi^2/2t-\mu^2 t/2} e^{-a} a^{-(\mu+1)/2} \int_0^{\infty} \xi^{\mu} e^{-\xi^2}\, d\xi$$

$$\times \int_0^{\infty} e^{-s^2/2t} e^{-2\sqrt{a}\xi\cosh(s)} \sinh(s)\sin(\frac{\pi s}{t})\, ds$$

if μ > -1. When $\mu = 0$, Dufresne (2001) carries out the complex integral and obtains the a simple expression for $f^{\mu}(a,t)$.

Proposition 2.3: *For t >0 and a > 0, there is*

$$f^0(a,t) = \frac{2e^{\pi^2/8t}}{\pi\sqrt{2ta}} \int_o^{\infty} e^{-y^2/2t} e^{-a(\cosh(y))^2} \cosh(y)\cos(\frac{\pi y}{2t})\, dy \qquad (2.5.6)$$

and

$$f^1(a,t) = \frac{2e^{\pi^2/8t-t/2}}{\pi\sqrt{2ta}} \int_o^{\infty} e^{-y^2/2t} e^{-a(\cosh(y))^2} \sin(y)\cosh(y)\cos(\frac{\pi y}{2t})\, dy. \qquad (2.5.7)$$

By using the Hermite function we also can obtain the function for general value μ.

2.5.2 Moments

We present some results about positive moments of A_t.

Proposition 2.4: *For any t > 0, there is*

$$E[(A_t)^n] = \frac{1}{E[(B_1)^2 n} \int_{\mathbf{R}} (\sinh(x))^2 n \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}\, dx, \quad n = 1,2,\ldots.$$

From above proposition, there is

$$E[(A_t)^n] = \frac{\sqrt{\pi}}{\Gamma(n+1/2)2^{3n-1}} e^{2n^2 t}(1+o(1)) \;\; as\; t \to \infty.$$

For a general value of μ, $E[(A_t^{\mu})^n]$ can be written as

$$E[(A_t^{\mu})^n] = e^{-\mu^2 t/2} E[exp(\mu B_t)(A_t)^n],$$

which is can be derived by the Cameron-Martin theorem (Cameron and Martin (1944)).

When $\mu \geq 0$, $A_t^{\mu} \to \infty$ as $t \to \infty$, and there is the limit theorem for $log(A_t^{\mu})$.

Proposition 2.5: *N is defined as a standard normal random variable. Then, as t $\to \infty$:*

1. *if* $\mu = 0$, $t^{-1}/2log(A_t^0)$ *converges in law to* $2\,|\,N\,|$;

2. *if* $\mu > 0$, $t^{-1}/2(log(A_t^0) - 2\mu t)$ *converges in law to 2N,*

Remark,

$$A_t^0 = t\int_0^1 e^{2\sqrt{t}B_s}\,ds.$$

2.5.3 Application

The exponential type functional always plays an important role. For example, In ruin probability, Mathematics Finance, diffusion processes in random environments. Matsumoto and Yor (2005) shows an example about pricing formula for call options for the Asian options. Asian options were first introduced by Boyle and Emanuel (1980). Its pay off is determined by the average underlying price over some pre-set period of time. In the recent years, Asian options are so popular due to lots of economic reasons.

By the Black-Scholes model (Black and Scholes (1973)), there is a risk-less bond $b = b_t$ with a constant interest rate and risky asset $S = S_t$ with a constant rate and volatility. When r > 0, $\mu \in \mathbf{R}$ and $\sigma > 0$ are constants, there is the stochastic differential equation

$$\frac{db_t}{b_t} = rdt, \ \ \frac{dS_t}{S_t} = \mu dt + \sigma dB_t,$$

where $B = B_t$ is a one dimensional Brownian motion with $B_0 = 0$ on a complete probability space. In this example, $b_0 = 1$, then we have

$$b_t = exp(rt), \quad S_t = S_0 exp(\sigma B_t + (\mu - \sigma^2/2)t).$$

With fixed strike price k > 0 and maturity T, the pay-offs of the European and Asian call option are given by

$$(S_T - k)_+ \ \ and \ \ (\zeta(T) - k)_+,$$

respectively, where f_+=max $[f, 0]$ and for $0 < t \leq T$

$$\zeta(t) = \frac{1}{t}\int_0^t S_u\, du.$$

By the non-arbitrage argument, the theoretical price C(k, T) of Asian call options at time t=0 is

$$C(k, T) = e^{-rT} E^{Q}[(\zeta(T) - k)_+],$$

where E^Q denotes the expectation with respect to the martingale measure Q. we can obtain several integral representations for C(k, T) by using explicit expressions for the density of A_t^u showed in last section. Geman and Yor (1993) use the Laplace transform of C(k, T) in T to obtain the simpler form.

Proposition 2.6: *For all $\mu \in \mathbf{R}$, $\lambda > max\{2(1+\mu), 0\}$ and $k > 0$, there is*

$$\lambda \int_0^\infty e^{-\lambda T} E^Q[(\zeta(T) - k)_+]\, dt$$
$$= \frac{1}{(\lambda - 2(1+\mu))\Gamma(b-1)} \int_0^{1/2k} e^{-t} t^{b-2} (1-2kt)^{a+1} dt,$$

where $\sigma = 2$ and $A_t^\mu = \int_0^\infty exp(2B_s^\mu)\, ds$.

Linetsky (2004) revisit the continuously sampled arithmetic Asian option problem and consider the constant dividend yield. He used the similar method to obtain the spectral representation of the expected value of pay-offs of Asian options.

2.5.4 Numerical evaluation of the stochastic integral

Even implicit expressions of integrals of a stochastic process exists, normally the forms of expressions are very complicated and it is hard to evaluate it. Therefore numerical methods are important. (see Badr, 2011).

Definition: Let $B = B_t, t \geq 0$ be a one dimensional Brownian motion starting from 0 defined on a probability space (Ω, F, P). Let $t_n, n \geq 0$ be a partition of the interval [0, T] and let $h(B(t), t) = h(t)$ be a continuous function on [0,T]. The stochastic integral $\int_0^T h(t)dB(t)$, which satisfies $E(\int_0^T h(t)dB(t)) = 0$, is defined as

$$I(h) = \int_0^T h(t)dB(t) = \lim_{n\to\infty} \sum_{i=0}^{n} h(t_i)(B(t_{i+1}) - B(t_i)).$$

Badr (2011) applied quadrature rules to obtain the following approximation for the stochastic integral.

Proposition 2.7: *Let M_N be the roots of a legendre polynomial of degree N and let τ_k, $k = 1, 2, \cdots, N$ be the corresponding weights. A quadrature approximation to the stochastic integral is*

$$\int_0^T h(t)dB(t) = \sum_{k=1}^{N} Q_k g(M_k) + R(f),$$

where R(f) is a remainder term.

Now back to our case, we set $y = a^2 s$ in (2.4.3) then we can rewrite

$$\int_0^1 exp(-h \cdot B_y)dy = a^2 \int_0^{\frac{1}{a^2}} exp(-haB_s)ds = \frac{2}{h}^2 \int_0^{\frac{h^2}{4}} exp(2B_s)ds,$$

where we set $a^2 = \frac{2^2}{h}$. Then by Proposition 2.3, it is possible to derive the analytic expressions for the density function of

$$\int_0^1 exp(-h \cdot B_y)dy \ .$$

However, the analytic expressions still require numerical computations. In the next section, we will introduce numerical methods to approximate the ruin probability which agreed with Proposition 2.5.

2.6 Approximation of the Integral

In this section, we establish three approximations for the integral

$$\int_0^1 exp(-h \cdot B_y)dy$$

in (2.4.3) by exploring different methods, so that the ruin probability can be calculated by simulation that will be shown in next section. The basic idea of every approximation is to transform $\int_0^1 exp(-h \cdot B_y)dy$ to the suitable expression, similar to Badr.

2.6.1 First Approximation: Classical approach

The classical approach for evaluating approximation of the integral of any function is

$$\int_0^1 f(s)ds \approx \sum_{k=1}^{m} \left(\frac{f(\frac{k-1}{m}) + f(\frac{k}{m})}{2}\right)\frac{1}{m}.$$

Let

$$f(y) = exp(-h \cdot B_y),$$

then we get

$$\int_0^1 exp(-h \cdot B_y)dy \approx \sum_{k=1}^{m} \frac{[exp(-h \cdot B_{\frac{k-1}{m}}) + exp(-h \cdot B_{\frac{k}{m}})]}{2}\frac{1}{m}. \tag{2.6.1}$$

2.6.2 Second Approximation: via Taylor Formula

We first write

$$\int_0^1 exp(-h \cdot B_y)\, dy = \sum_{k=1}^{m} \int_{\frac{k-1}{m}}^{\frac{k}{m}} exp(-h \cdot B_s)\, ds.$$

Since $exp(h \cdot B_s) \stackrel{d}{=} exp(-h \cdot B_s)$, we use the Taylor expansion to expand $exp(h \cdot B_s)$ as

$$exp(h \cdot B_s) \approx exp(h \cdot B_{\frac{k-1}{m}}) + \tfrac{h^2}{2} exp(h \cdot B_{\frac{k-1}{m}})(B_s - B_{\frac{k-1}{m}})^2$$

$$+h \cdot exp(h \cdot B_{\frac{k-1}{m}})(B_s - B_{\frac{k-1}{m}}),$$

by computing integral of $exp(h \cdot B_s)$ respect to s from $\frac{k-1}{m}$ to $\frac{k}{m}$ we have the following equation

$$\int_{\frac{k-1}{m}}^{\frac{k}{m}} exp(h \cdot B_s)\, ds \approx exp(h \cdot B_{\frac{k-1}{m}})\frac{1}{m} + \frac{h^2}{2} exp(h \cdot B_{\frac{k-1}{m}}) \int_{\frac{k-1}{m}}^{\frac{k}{m}} (s - \frac{k-1}{m})\, ds.$$

$$+ h \cdot exp(h \cdot B_{\frac{k-1}{m}}) \int_{\frac{k-1}{m}}^{\frac{k}{m}} (B_s - B_{\frac{k-1}{m}})\, ds, \qquad (2.6.2)$$

since

$$E(B_s - B_{\frac{k-1}{m}})^2 = (s - \frac{k-1}{m}) \text{ for } \frac{k-1}{m} \leq s \leq \frac{k}{m}.$$

Let

$$\begin{aligned} Y_k &= \int_{\frac{k-1}{m}}^{\frac{k}{m}} (B_s - B_{\frac{k-1}{m}}) ds \\ X_k &= B_{\frac{k}{m}} - B_{\frac{k-1}{m}}. \end{aligned}$$

Then

$$E(X_k) = E(Y_k) = 0,$$

$$Var(X_k) = E({X_k}^2) = E({B_{\frac{k}{m}}}^2 - 2B_{\frac{k}{m}} B_{\frac{k-1}{m}} + {B_{\frac{k-1}{m}}}^2) = \frac{k}{m} - 2\frac{k-1}{m} + \frac{k-1}{m} = \frac{1}{m},$$

and

$$Var(Y_k) = E({Y_k}^2) = \frac{1}{2m^3}.$$

Moreover

$$\begin{aligned} E(X_k Y_k) &= \int_{\frac{k-1}{m}}^{\frac{k}{m}} E[(B_s B_{\frac{k}{m}} - B_s B_{\frac{k-1}{m}} - B_{\frac{k-1}{m}} B_{\frac{k}{m}} + {B_{\frac{k-1}{m}}}^2)] ds \\ &= \int_{\frac{k-1}{m}}^{\frac{k}{m}} (s - \frac{k-1}{m})\, ds \\ &= \frac{1}{2m^2}. \end{aligned}$$

It is obvious that $[X_k, Y_k]$ are i.i.d Gaussian random vectors. Now take

$$Y_k = aX_k + bZ_k,$$

where $Z_k \perp X_k$ and $\{Z_k\}$ are normal distributed with mean 0 and variance 1. Using $E({X_k}^2)$, $E(Y_k^2)$ and $E(X_kY_k)$, we obtain

$$E(X_kY_k) = E[X_k(aX_k + bZ_k)] = E[(a{X_k}^2) + bE(X_kZ_k)] = aE({X_k}^2),$$

since X_k and Z_k are independent, hence we deduce that $a = \frac{1}{2m}$. Using a similar idea, but working with $V(Y_k)$ we deduce that $b = \frac{1}{2\sqrt{m^3}}$. Hence

$$Y_k = \frac{1}{2m}X_k + \frac{1}{2\sqrt{m^3}}Z_k,$$

so that

$$\int_{\frac{k-1}{m}}^{\frac{k}{m}} (B_s - B_{\frac{k-1}{m}})ds \overset{d}{=} \frac{1}{2m}(B_{\frac{k}{m}} - B_{\frac{k-1}{m}}) + \frac{1}{2\sqrt{m^3}}Z_k.$$

This gives the following

$$\begin{aligned}\int_0^1 exp(h \cdot B_y)dy \quad &= \quad \sum_{k=1}^{m} exp(hB_{\frac{k-1}{m}})\frac{1}{m} + \sum_{k=1}^{m} \frac{h^2}{2}exp(hB_{\frac{k-1}{m}}) \int_{\frac{k-1}{m}}^{\frac{k}{m}} (s - \frac{k-1}{m})ds \\ &\quad + \sum_{k=1}^{m} hexp(hB_{\frac{k-1}{m}}) \int_{\frac{k-1}{m}}^{\frac{k}{m}} (B_s - B_{\frac{k-1}{m}})\, ds.\end{aligned}$$

After rearrangement, we get the 2nd approximation

$$\int_0^1 exp(h \cdot B_y)dy = \frac{1}{m}\{[\frac{1+h^2}{4m} + \frac{1}{2}(B_{\frac{k}{m}} - B_{\frac{k-1}{m}}) + \frac{1}{\sqrt{m}Z_k}] \sum_{k=1}^{m} exp(h \cdot B_{\frac{k-1}{m}})\}. \quad (2.6.3)$$

2.6.3 Third Approximation: Itô Formula

In this approximation we employ the Itô formula (Itô (1951))

$$f(B_1) = f(B_0) + \int_0^1 f'(B_s)dB_s + \frac{1}{2}\int_0^1 f''(B_s)ds \quad (2.6.4)$$

We let $f''(B_s) = exp(h \cdot B_s)$. So

$$f'(B_s) = \frac{1}{h}exp(h \cdot B_s) \text{ and } f(B_s) = \frac{1}{h^2}exp(h \cdot B_s).$$

In order to find $\int_0^1 exp(-h \cdot B_s)ds$, it should be known the value of $\int_0^1 f'(B_s)dB_s$.

Applying a Taylor expansion to $\int_0^1 f'(B_s)dB_s$ in a similar way as in the second approximation, we get

$$\begin{aligned}
\int_0^1 f'(B_s)dB_s &= \sum_{k=1}^{m} \int_{\frac{k-1}{m}}^{\frac{k}{m}} exp(h \cdot B_s)dB_s \\
&\approx \sum_{k=1}^{m} exp(h \cdot B_{\frac{k-1}{m}})[(B_{\frac{k}{m}} - B_{\frac{k-1}{m}}) + h \int_{\frac{k-1}{m}}^{\frac{k}{m}} (B_s - B_{\frac{k-1}{m}})dBs] \\
&= \sum_{k=1}^{m} exp(h \cdot B_{\frac{k-1}{m}})[(B_{\frac{k}{m}} - B_{\frac{k-1}{m}}) + h(\int_{\frac{k-1}{m}}^{\frac{k}{m}} B_s dB_s - \frac{1}{m} B_{\frac{k-1}{m}})],
\end{aligned}$$

where

$$\int_{\frac{k-1}{m}}^{\frac{k}{m}} B_s dB_s = \int_0^{\frac{k}{m}} B_s dB_s - \int_0^{\frac{k-1}{m}} B_s dB_s. \tag{2.6.5}$$

Now we apply Itô formula to $\int_0^{\frac{k}{m}} B_s dB_s$. We get

$$\int_0^{\frac{k}{m}} B_s dB_s = \frac{B^2{}_{\frac{k}{m}} - \frac{k}{m}}{2}, \text{ and } \int_0^{\frac{k-1}{m}} B_s dB_s = \frac{B^2{}_{\frac{k-1}{m}} - \frac{k-1}{m}}{2},$$

which leads (2.6.5) is to be

$$\int_{\frac{k-1}{m}}^{\frac{k}{m}} B_s dB_s = \frac{1}{2}[(B^2{}_{\frac{k}{m}} - B^2{}_{\frac{k-1}{m}}) - \frac{1}{m}]. \tag{2.6.6}$$

By combining equations (2.6.4) and (2.6.6) for $\sum_{k=1}^{m} \int_{\frac{k-1}{m}}^{\frac{k}{m}} exp(h \cdot B_s)dB_s$, we obtain

$$\begin{aligned}
&\int_0^1 exp(h \cdot B_s)dB_s \\
= &\sum_{k=1}^{m} exp(h \cdot B_{\frac{k-1}{m}})[(B_{\frac{k}{m}} - B_{\frac{k-1}{m}}) + h(\frac{1}{2}[(B^2{}_{\frac{k}{m}} - B^2{}_{\frac{k-1}{m}}) - \frac{1}{m}] - \frac{1}{m} B_{\frac{k-1}{m}})].
\end{aligned}$$

Substituting above equation for the corresponding term in (2.6.4) and after rearrangement, we have the third approximation:

$$\begin{aligned}
\int_0^1 exp(h \cdot B_y)\, dy &= 2\{exp(h \cdot B_1) - \frac{1}{h^2} - \frac{1}{h} \sum_{k=1}^{m} exp(h \cdot B_{\frac{k-1}{m}})[(B_{\frac{k}{m}} - B_{\frac{k-1}{m}}) \\
&\quad + h(\frac{1}{2}[(B^2{}_{\frac{k}{m}} - B^2{}_{\frac{k-1}{m}}) - \frac{1}{m}] - \frac{1}{m} B_{\frac{k-1}{m}})]\}
\end{aligned} \tag{2.6.7}$$

2.7 Simulation and Comparison

The objective of simulation is

- To compare different approximations of $\int_0^1 exp(-h \cdot B_y)dy$;
- To compare three approximations with ordinary model we considered;
- To find the relationship, if any exists, between variance of the interest rate and ruin probability.

2.7.1 Basic Idea and Method

We use the R language to carry out the simulation and derive corresponding values and graphs.

Firstly in the simulation of every approximation for $\int_0^1 exp(-h \cdot B_y)dy$, we consider the key transformation:

$$B_{\frac{k}{m}} \stackrel{d}{=} \frac{S_k}{\sqrt{m}} \qquad\qquad S_k = \sum_{i=1}^{k} Z_i,$$

which is a typical way to simulate Brownian Motion by random walk in discrete time, where the Z_k are i.i.d. normal random variables with mean 0 and variance 1.

Therefore the first approximation, (2.6.1) becomes:

$$\begin{aligned}
\int_0^1 exp(-h \cdot B_y)dy &\approx \sum_{k=1}^{m} \frac{[exp(-h \cdot B_{\frac{k-1}{m}}) + exp(-h \cdot B_{\frac{k}{m}})]}{2} \frac{1}{m} \\
&= \sum_{k=1}^{m} \frac{[exp(-h\frac{S_{k-1}}{\sqrt{m}}) + exp(-h\frac{S_k}{\sqrt{m}})]}{2} \frac{1}{m}.
\end{aligned}$$

The second approximation, (2.6.3) becomes:

$$\begin{aligned}
\int_0^1 exp(-h \cdot B_y)\,dy &\approx \frac{1}{m}\{[1 + \frac{h^2}{4m} + \frac{1}{2}(B_{\frac{k}{m}} - B_{\frac{k-1}{m}}) + \frac{1}{\sqrt{m}Z_k}] \sum_{k=1}^{m} exp(h \cdot B_{\frac{k-1}{m}})\} \\
&= \frac{1}{m}\{[1 + \frac{h^2}{4m} + \frac{1}{2}(\frac{S_k}{\sqrt{m}} - \frac{S_{k-1}}{\sqrt{m}}) + \frac{1}{\sqrt{m}}Z_k] \sum_{k=1}^{m} exp(h\frac{S_{k-1}}{\sqrt{m}})\}.
\end{aligned}$$

The third approximation (2.6.7) becomes:

$$\int_0^1 exp(-h \cdot B_y)\, dy \approx 2\{exp(h \cdot B_1) - \frac{1}{h^2} - \frac{1}{h}\sum_{k=1}^{m} exp(h{\cdot}B_{\frac{k-1}{m}})[(B_{\frac{k}{m}} - B_{\frac{k-1}{m}})$$

$$+h(\tfrac{1}{2}[(B^2{}_{\frac{k}{m}} - B^2{}_{\frac{k-1}{m}}) - \tfrac{1}{m}] - \tfrac{1}{m}B_{\frac{k-1}{m}})]\}$$

$$\approx 2\{exp(h\frac{S_m}{\sqrt{m}}) - \frac{1}{h^2}$$

$$-\frac{1}{h}\sum_{k=1}^{m} exp(h{\cdot}B_{\frac{k-1}{m}})[\frac{S_k}{\sqrt{m}} + \frac{h}{m}(\frac{S^2{}_k - S^2{}_{k-1} - 1}{2} - \frac{S_{k-1}}{\sqrt{m}})]\}.$$

Next, for each approximation we use R language to generate Z_k and calculate corresponding expression above, the simulation of $\int_0^1 exp(-h \cdot B_y)dy$. We do this K times. Hence we can calculate the sample mean and variance of the simulations, and compare them with mean and variance of $\int_0^1 exp(-h \cdot B_y)dy$ for the specific variance of δ_k. Finally the most important step is that for each approximation, recording how many times the simulated approximation of $\int_0^1 exp(-h \cdot B_y)dy$ is bigger than $\frac{A}{Cn}$ in (2.4.3), but we should check the situation that $\sum_{i=1}^{n} \beta_i > \frac{C}{A}$. We end up the process with dividing this time described above by K, which is the simulated ruin probability we are looking for.

Value of parameter in simulation. We consider

- the replications number $K = 500$
- the term for policy $n = 5$
- in every approximation, $m = 40$
- the capital $A = 10000$ and the claim size $C = 200$
- using R language to simulate 100 realisations from a $U(0, 1)$ distributed to use as the variance of interest rate, σ_k^2. (the exact value is given in Appendix 2)

2.7.2 Comparison of mean and variance for different approximations

In this section, we will compare mean and variance between the model stated in Section 2.2 and three approximations, for this purpose, we take σ_k^2, the variance of interest rate, equal to 1.

So in the ordinary example, for comparing with $\int_0^1 exp(-h \cdot B_y)dy$, we compute the expectation of $(\sum_{i=1}^n e^{-\sum_{k=1}^{i} \delta_k})/n$

$$E[(\sum_{i=1}^{n} e^{-\sum_{k=1}^{i} \delta_k})/n] = (\sum_{i=1}^{n} e^{\frac{i\sigma^2}{2}})/n.$$

The variance is

$$Var[(\sum_{i=1}^{n} e^{-\sum_{k=1}^{i} \delta_k})/n] = \sum_{i=1}^{n}[e^{i\sigma^2} - 1)e^{i\sigma^2}]/n^2,$$

where $\sigma = \sigma_k$, since the δ_k are i.i.d. random variables.

We can also obtain the mean and variance of $\int_0^1 exp(-h \cdot B_y)dy$ as

$$E[\int_0^1 exp(-h \cdot B_y)dy] = \frac{1}{h^2}2[e^{\frac{h^2}{2}} - 1]$$

and

$$Var[\int_0^1 exp(-h \cdot B_y)dy] = \frac{4}{3h^4}[\frac{1}{2}(e^{2h^2} - 1) - 2(e^{\frac{h^2}{2}} - 1)] - \frac{1}{h^2}2[e^{\frac{h^2}{2}} - 1]^2,$$

where we take $h = \sqrt{(n+1)}\sigma$.

Now replace σ by 1, the mean and variance of $(\sum_{i=1}^n e^{-\sum_{k=1}^{i} \delta_k})/n$ and its approximations are given in the Table 2.1. As we can see from the Table 2.1, the results of simulation show the expected values of $(\sum_{i=1}^n e^{-\sum_{k=1}^{i} \delta_k})/n$ and $\int_0^1 exp(-h \cdot B_y)dy$. The first and second approximation are quite close, in fact the biggest difference is just 0.35. Otherwise although the variance of $\int_0^1 exp(-h \cdot B_y)dy$ and the second approximation are nearly 1.3 times than $(\sum_{i=1}^n e^{-\sum_{k=1}^{i} \delta_k})/n$, by taking into account the large variance for ordinary model, 1057.71 and 1009.42 in Table 2.1 are acceptable. In contrast either the mean or variance of third approximation is too big compare with others, especially

Table 2.1: Comparison of mean and variance for different approximations

$\sigma = 1$	Mean / Sample Mean	Variance /Sample Variance
$(\sum_{i=1}^{n} e^{-\sum_{k=1}^{i} \delta_k})/n$	6.01	776.30
$\int_0^1 exp(-h \cdot B_y)dy$	6.37	1057.71
1st Approx.	6.02	780.03
2nd Approx.	6.23	1009.42
3rd Approx.	17.93	18949.45

variance of it is almost 19000. In consideration of mean and variance, it is reasonably believe that $\int_0^1 exp(-h \cdot B_y)dy$ is a good approximation of $(\sum_{i=1}^{n} e^{-\sum_{k=1}^{i} \delta_k})/n$, so are the first and second approximation.

2.7.3 Comparison of Simulated Ruin Probabilities

For every simulation, we use the same value of the parameters as described before and derive two kinds of plots using R.

Figure 2.1 shows the four histograms that display the tabulated frequencies of simulated ruin probability for (2.2.1) and other three approximations. In every histogram, there are one hundred simulated values of ruin probability corresponding to one hundred different values of variance of interest rate. Even though the summaries of ruin probability in each simulation are quite close (see Appendix 3), the first and second histograms (from the left to the right) are more analogous whereas the third and fourth plots are also looked similar. It is clear from the histograms that over one half of the simulated ruin probabilities are equal to 0 in the ordinary model and the first approximation, but less 50 % in the second and the third approximations.

As similar as Section 2.7.2, the first approximation works well here. The performance of the second approximation is better in this section. The large sample variance does not effect the value of simulation, especially in consideration of similar results given by the third approximation. To sum up the distribution of simulated ruin probabilities in ordinary model and each approximation are nearly same on the whole.

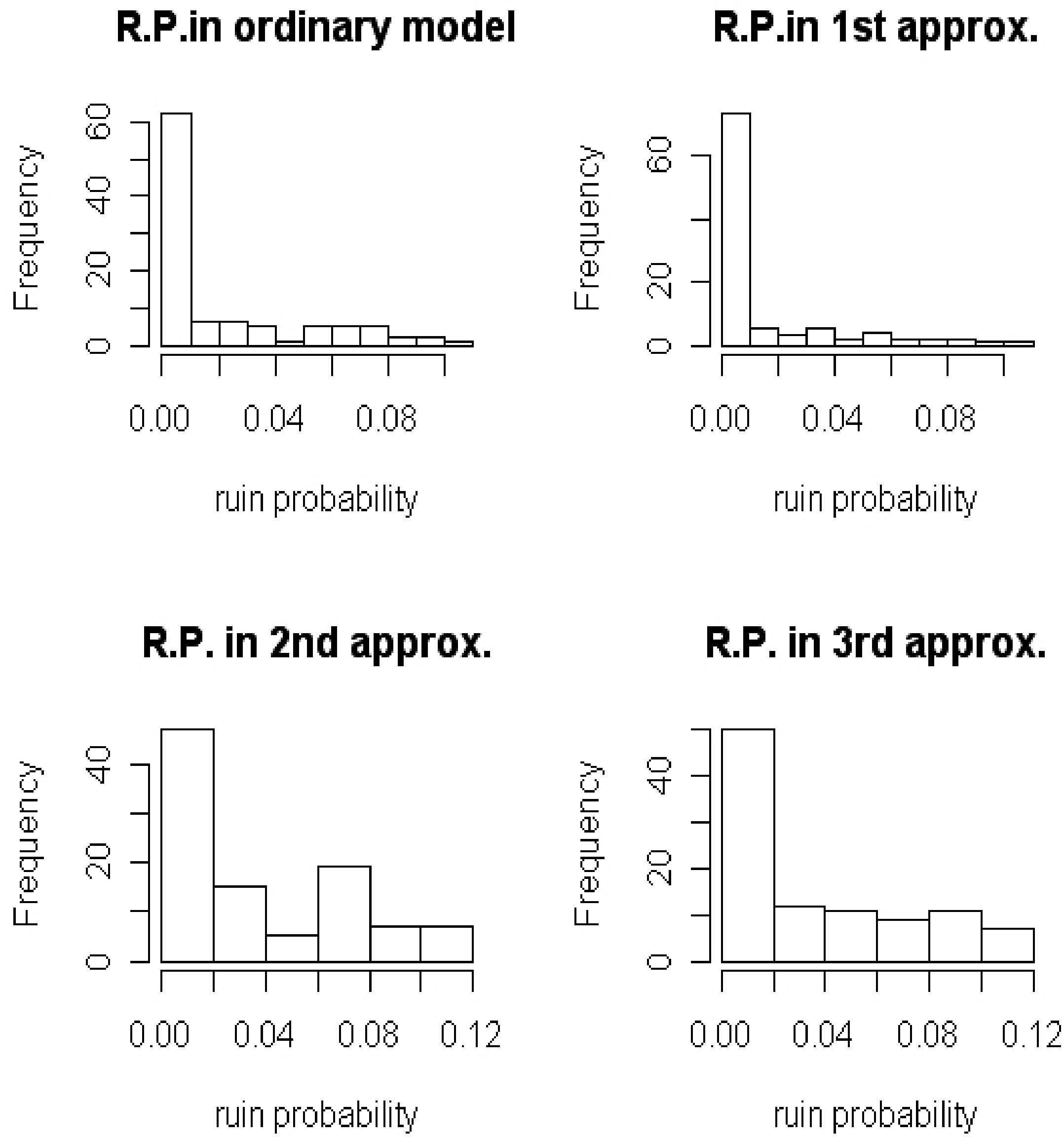

Figure 2.1 Histograms of ruin probabilities in simulation where R.P. denotes ruin probability.

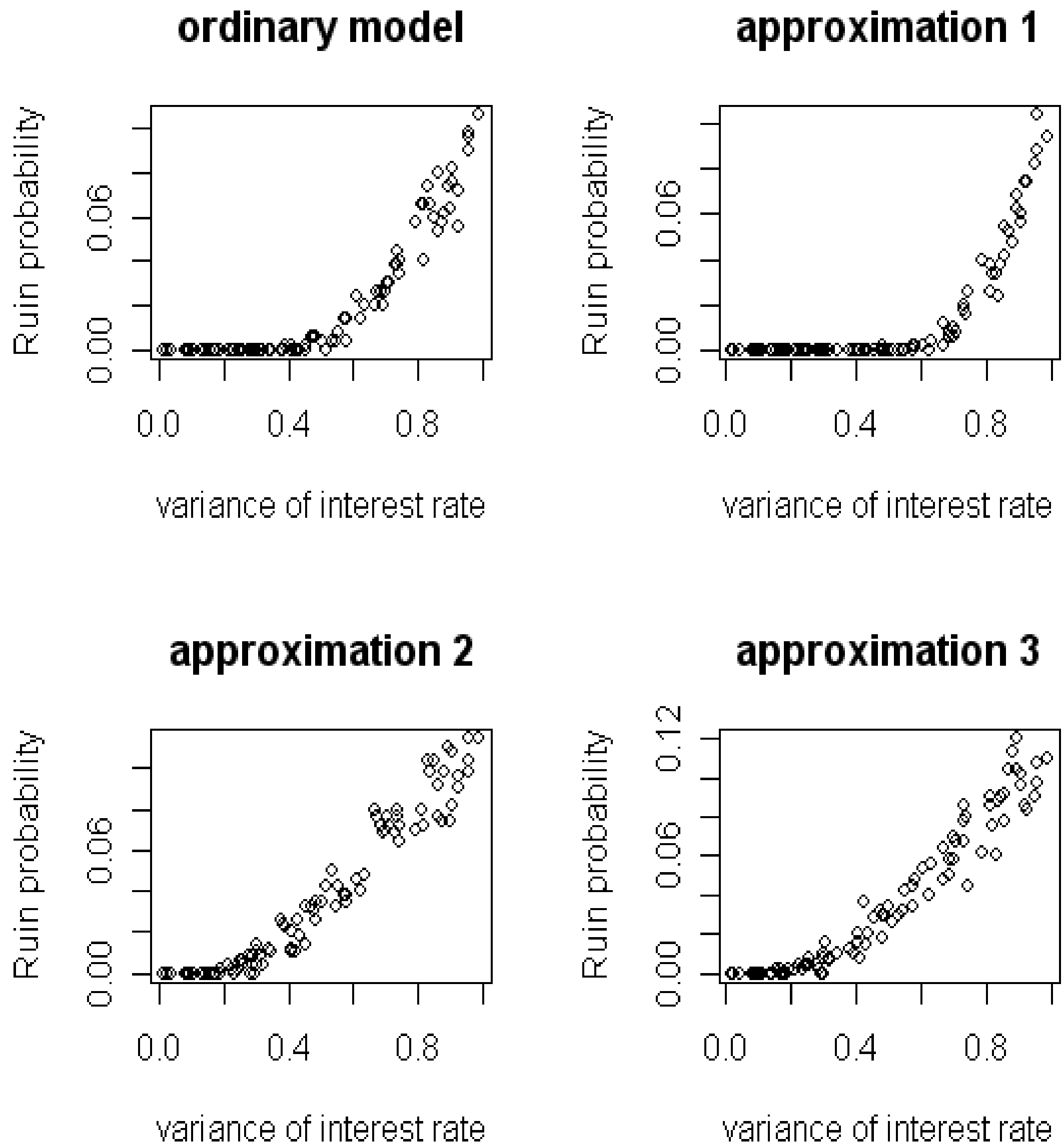

Figure 2.2 Plots of simulated ruin probability against variance of interest rate for ordinary model and each approximation.

Figure 2.2 is the combination of four plots which show the relationships between the simulated ruin probability and the variance of the interest rate. The surprise is that there is a "threshold" in every plot which was empirically discovered. In first and second plot, the "threshold" is around nearly 0.5 compare with 0. 3 in third and fourth plot. The probability of ruin raises significantly from beyond "threshold" values. But the appearance and position of "threshold" is not fixed, they both depend on the value of parameter in Section 2.7.1, For example, if the n change from 5 to 15 and the rest parameters change less, then there should be no "threshold" any more. Besides this, a noticeable trend in every plot is the simulated ruin probability which is increasing as the variance is increasing without "threshold". In addition in top right of the second plot the points form a distinct curve, but in other plots this is less so

In summary, from the simulation, the approximation of $(\sum_{i=1}^{n} e^{-\sum\limits_{k=1}^{i}\delta_k})/n$, $\int_0^1 exp(-h \cdot B_y)dy$, is no problem. Among three approximations, the first one works very well, it does not only has the similar expected value and variance compare with $(\sum_{i=1}^{n} e^{-\sum\limits_{k=1}^{i}\delta_k})/n$, but has also quite similar simulated ruin probabilities with our ordinary model. Although the second and third approximation do not perform as well as first one, they are still valuable approximations.

2.8 The Advanced Model

2.8.1 Construction of Model

In real insurance world, our previous ruin probability models has too many restrictions. Normally the claim size of an insurance is not fixed but random, and it is not possible that there is a claim in every time interval. As a consequence many researchers consider the claims as a stochastic process. Therefore the model of ruin probability will be more complicated.

For constructing this model, we firstly make same assumption for the continuous interest rate, then the surplus of insurance company at time n, Z_n, changes to be

$$Z_n = A_n - \sum_{i=1}^{n} D_i e^{\sum_{k=i+1}^{n} \delta_k}, \tag{2.8.1}$$

where D_i is claim size at time i, A_n is the capital of insurance company at time n and

$$A_n = A_0 + \sum_{i=1}^{n} K_i e^{\sum_{k=i}^{n} \delta_k},$$

where A_0 is the reserve at the beginning, K_i is the premium received at time i. Next by considering the present valuation of Z_n, (2.8.1) becomes:

$$\bar{Z}_n = A_0 - \sum_{i=1}^{n} C_i e^{-\sum_{k=1}^{i} \delta_k},$$

where

$$C_i = D_i - K_i.$$

Now, our new ruin probability for finite time point M is

$$\Psi'_M = P[\min_{0 \le n \le M} (A_0 - \sum_{i=1}^{n} C_i e^{-\sum_{k=1}^{i} \delta_k}) < 0]. \tag{2.8.2}$$

We call it the *earliest ruin probability*.

For the purpose of approximation, we rewrite (2.8.2) as

$$\Psi'_M = P[\max_{0\le n\le M}(\sum_{i=1}^{n} C_i e^{-\sum\limits_{k=1}^{i}\delta_k}) > A_0].$$

Let $n = [Mx]$ and $I_{[Mx]} = \sum_{i=1}^{n} C_i e^{-\sum\limits_{k=1}^{i}\delta_k}$. Then by applying Lemma 2.1 to $I_{[Mx]}$ we obtain

$$I_{[Mx]} \approx M\int_0^x e^{-h\cdot B_y-\Gamma y}dC(y).$$

We use a new transformation as shown as below

$$\sum_{i=1}^{[My]} \delta_i = h\cdot B_y + \Gamma y,$$

where Γ is fixed and > 0.

We treat C_i as a jump and set $C_i = dC(y)$ on $[\frac{i}{M}, \frac{i+1}{M}]$. So, we have derived the following result.

Property: *Under conditions in above*

$$\Psi'_M \approx\ P(\max_{0\le n\le M}\int_0^x e^{-h\cdot B_y-\Gamma y}dC(y) > \frac{A_0}{M}). \tag{2.8.3}$$

Remark: Let us explain how this property can be used together with stochastic simulation to compute numerically the ruin probability. We set

$$C(y) = -Cy + JP_o\lambda(y)$$

where C and J are constant, $P_o\lambda(y)$ is a Possion process with rate λ. Hence

$$\int_0^x e^{-h\cdot B_y-\Gamma y}dC(y) = -C\int_0^x e^{-h\cdot B_y-\Gamma y}dy + J\int_0^x e^{-h\cdot B_y-\Gamma y}dP_o\lambda(y), \tag{2.8.4}$$

note:

$$\int_0^x e^{-h\cdot B_y-\Gamma y}dP_o\lambda(y) = \sum_{i=1}^{q-1} e^{-h\cdot B_{T_i}-\Gamma T_i}.$$

where T_i is the waiting time until ith event for a Possion process with mean rate λ and q is the minimum number for $T_k > x$. We also consider the approximation for $-C \int_0^x e^{-h\cdot B_y - \Gamma y} dy$:

$$-C \int_0^x e^{-h\cdot B_y - \Gamma y} dy \approx -C \frac{1}{mx} \sum_{k=1}^{mx} e^{-h\cdot B_{\frac{k}{m}} - \Gamma \frac{k}{m}} \tag{2.8.5}$$

from (2.8.4) and (2.8.5) we have:

$$\int_0^x e^{-h\cdot B_y - \Gamma y} dC(y) \approx -C \frac{1}{mx} \sum_{k=1}^{mx} e^{-h\cdot B_{\frac{k}{m}} - \Gamma \frac{k}{m}} + \sum_{i=1}^{q-1} e^{-h\cdot B_{T_i} - \Gamma T_i}. \tag{2.8.6}$$

By using the same methods as in section 2.7.1, to do the simulation for (2.8.6), but we consider here:

$$B_y \longrightarrow \frac{S_{[ym]}}{\sqrt{m}} \text{ instead of } \frac{S_{ym}}{\sqrt{m}}$$

for T_i,

$$T_i = E_1 + \cdots + E_i,$$

where $\{E_i\}$ are i.i.d. exponential random variables with parameter λ. So we have described the simulation process for (2.8.6). Similar techniques will be suggested and applied in Chapter 3.

2.9 Conclusion

In this chapter, firstly we tried to use simple model suggested by Dr. Neil Butler to describe the risk process of the insurance company. we approximated the ruin probability for the model by Brownian motion and develop several numerical methods to evaluate the approximated ruin probability. Numerical calculations is applied on the approximation and show that the ruin probability is increasing function of the variance of stochastic interest rate. A surprising threshold is found such that the rate of increasing of the ruin probability much higher after the threshold point. Also the results for some parameters agreed with the Gaussian approximation suggested in Matsumoto and Yor (2005) (Proposition 2.6). We applied three approximations in Section 2.6 to the ruin probability. Using simulation, they works reasonably well. The approximations reflect the same relationship between the variance of interest rate and the ruin probability.

Although there is no explicit way to derive the ruin probability with the stochastic continuous interest rate, by approximation and simulation we can find how the variance of the interest rate effects the probability of ruin in finite time. Then, in Chapter 3 we will extend our model with incorporating investment policy.

CHAPTER 3

Optimal constant fraction policies under the ruin probability constraints

In Chapter 3, we construct several models which incorporate possible investment strategies. We estimate the parameters from the simulated data. Then we find the optimal investment policy with a given upper bound on the ruin probability. The models, likelihood functions and MLE's are introduced in Sections 3.2 and 3.3. Numerical analysis with stochastic simulation for the estimation of the parameters is conducted in Section 3.4. In Section 3.5 we find via the random search algorithm the optimal investment strategy with a given upper bound on the ruin probability. Finally, in Section 3.6 we approximate the ruin probability by the associated integrals of the Brownian motion.

3.1 Approach

In this chapter, we want to investigate the optimal investment strategy of the insurance company if there is a given upper bound on the risk or ruin probability for the insurance policies (e.g. Paulsen, 2008). When the insurance companies receive the premium from policy holders, they may use that to invest. Therefore how to choose the investment policy is an important objective for them. As we know, the more profitable and risky investment, such as stocks are, the more the uncertainty is. That means that these types of investments will have high expected return rate however the variation is also high. In contrast, the less profitable and risky investments, such as bonds, saving accounts and so on, have a low expected return but less variation. In Chapter 2, we have found that the high variance of investment/interest rate will result in high ruin probability. In this situation, it is reasonable for the insurance company to construct an optimal investment strategy to find the balance between the two kinds of investments.

We apply our analysis to the simulated data. Then we find the optimal investment strategy by the random search algorithm. The method is as follows:

- Construct the stochastic model with unknown parameters which are to be estimated from data;
- Estimate the parameters from the data;
- Use the estimated parameter to calculate the ruin probability or obtain an upper bound for it;
- Find the optimal constant fraction policies.

To estimate the parameters, we evaluate the likelihood functions, then whenever it is possible, we find the maximum likelihood estimation (MLE) of the parameter by direct calculations or by using the numerical or stochastic simulation techniques.

3.2 The Construction of Model

At the beginning, we use the simple model as which is based upon the following assumptions:

• the claim size, denoted by C_i, is random variable and is paid by the insurance company just after the end of each time interval, the rate claim occurrence is constant;

• the premium is one payment and paid just after the beginning of the first time interval;

• A_0 is the reserve at the beginning, A_n is the surplus of insurance company at time n;

• δ_k is the i.i.d. continuous interest rate at time k, and has normal distribution with mean 0 and variance σ^2;

• the capital and the claim payment of the company are increasing with continuous stochastic interest rate δ_n.

In our model, at time n the accumulated surplus is given by

$$\begin{aligned} A_n &= A_{n-1}e^{\delta_n} - C_n \\ &= A_0 e^{\sum_{i=1}^{n}\delta_i} - \sum_{i}^{n} C_n e^{\sum_{k=i+1}^{n}\delta_k}. \end{aligned}$$

Thus A_n is a function of A_{n-1}, δ_n, C_n, i.e $A_n = f(A_{n-1}; \delta_n, C_n)$, where we assume $\{\delta_n\}$ and $\{C_n\}$ are independent. Therefore $\{A_n\}$ is homogeneous Markov Chain.

As you can see, the above model we constructed is very simple. There is just one kind of investment policy that means there is just one kind of interest rate and we do not need any investment strategy. In the following section, we add the second interest/investment rate and make different assumptions about claim sizes and their occurrence.

In statistics, maximum likelihood estimation is a well-known method of estimation. The estimation begins with finding the likelihood function of the sample data. The values of these parameters that maximize the sample likelihood are known as the maximum likelihood estimates or MLE's.

Suppose we have a set $D = \{x_1, \cdots, x_n\}$ of i.i.d. realisations from the density $p(x|\theta)$. $L(\theta|D)$ is defined as the likelihood function of θ, unknown parameter, with respect to D as

$$L(\theta|D) = p(D|\theta) = p(x_1, \cdots, x_n|\theta) = \prod_{i=1}^{n} p(x_i|\theta),$$

where θ is a parameter. It is often easier to work with the logarithm of the likelihood function, called the log-likelihood, or its scaled version, called the average log-likelihood:

$$\ln L(\theta \mid x_1, \ldots, x_n) = \sum_{i=1}^{n} \ln p(x_i|\theta), \qquad \hat{\ell} = \frac{1}{n}\ln L,$$

and the log-likelihood of θ is

$$l(\theta) = l(\theta \mid x_1, \ldots, x_n).$$

The MLE of θ is, by definition, the value of $\hat{\theta}$ that maximise $L(\theta|D)$ and can be computed as

$$\hat{\theta}_{\text{mle}} = \arg\max \hat{\ell}(\theta \mid x_1, \ldots, x_n).$$

A MLE estimate is the same regardless of whether maximum the likelihood or the log-likelihood function, since log is a monotone transformation.

To estimate the parameters from the data, we apply the max-likelihood method.

The conditional probability density function(pdf) for A_n is found as follows if we let $f(A_n|A_{n-1})$ to be the conditional pdf of $A_n|A_{n-1}$ then we can express the conditional likelihood function of $A_1, \ldots, A_n$:

$$L(A_1, \ldots, A_n) = f(A_1) \times f(A_2|A_1) \cdots f(A_n|A_{n-1}).$$

For the single parameter model we may calculate the estimate directly, but for the two or more parameters it is not too easy to obtain the likelihood function. So we use computer to find the maximum of the likelihood. We also use MLE to test our model and assumptions.

3.3 Estimation: Likelihood Functions

In this section, we evaluate the likelihood functions, then whenever its possible, find the MLE by direct calculations. In the next section, we apply numerical methods.

3.3.1 Model 1

We keep all the assumptions as in Section 3.2 except we assume the claim size is constant at C. Hence our conditional probability for the first model is

$$\begin{aligned} P[A_n < t | A_{n-1} = a_{n-1}] &= P[A_{n-1} e^{\delta_n} - C < t | A_{n-1} = a_{n-1}] \\ &= P[\delta_n < ln(\frac{t+C}{a_{n-1}})]. \end{aligned}$$

Then the conditional pdf of A_n is

$$\frac{d}{dt} P[\delta_n < ln(\frac{t+C}{a_{n-1}})] = \frac{1}{t+C} \times \frac{1}{\sigma\sqrt{2\pi}} \times exp(-\frac{(ln(\frac{t+C}{a_{n-1}}))^2}{2\sigma^2}),$$

since δ_n has normal distribution with mean 0 and variance σ^2.

By considering the product of conditional pdfs and then taking the log-likelihood function of σ, $l(\sigma)$, is

$$l(\sigma) \propto -n \times ln(\sigma) - \frac{\sum_{i=1}^{n} ln(\frac{t+C}{a_{i-1}})^2}{2\sigma^2}.$$

By solving $\frac{\partial l}{\partial \sigma} = 0$, we obtain the MLE of θ, denoted by $\hat{\sigma}$, is gien by

$$\hat{\sigma} = \sqrt{\frac{\sum_{i=1}^{n} (ln(\frac{t+C}{a_{i-1}}))^2}{n}},$$

where the t, C and a_i are known constants.

3.3.2 Model 2

In our second case we assume the C_n are exponentially distributed with mean λ^{-1} instead of constant in the first model. Furthermore we assume that the $\{\delta_k\}$ are independent of the $\{C_k\}$. So we have two unknown parameters, σ and λ, to estimate.

If $\{a_k\} \geq 0$, then for each n the conditional probability of A_n given $A_{n-1} = a_{n-1}$ is

$$\begin{aligned} P[A_n < t | A_{n-1} = a_{n-1}] &= P[A_{n-1}e^{\delta_n} - C_n < t | A_{n-1} = a_{n-1}] \\ &= P[a_{n-1}e^{\delta_n} - C_n < t] \\ &= P[C_n > a_{n-1}e^{\delta_n} - t] \\ &= E[I(C_n > a_{n-1}e^{\delta_n} - t)]. \end{aligned} \tag{3.3.1}$$

Since C_n has an exponential distribution with mean λ^{-1} we have

$$P[C_n > x] = e^{-\lambda x} I(x \geq 0) + I(x < 0).$$

This leads (3.3.1) to be

$$E[e^{-\lambda(a_{n-1}e^{\delta_n} - t)} I(a_{n-1}e^{\delta_n} - t \geq 0) + I(a_{n-1}e^{\delta_n} - t < 0)]$$

$$= E[e^{-\lambda(a_{n-1}e^{\delta_n} - t)} I(a_{n-1}e^{\delta_n} - t \geq 0)] + E[I(a_{n-1}e^{\delta_n} - t < 0)].$$

Since δ_n is a normal variable with mean 0 and variance σ^2, then we obtain

$$P[A_n < t | A_{n-1} = a_{n-1}] = P(\delta_n < ln(\frac{t}{a_{n-1}})) + \frac{1}{\sigma\sqrt{2\pi}} e^{\lambda t} \int_{ln(\frac{t}{a_{n-1}})}^{+\infty} e^{-\lambda a_{n-1} e^x - \frac{x^2}{2\sigma^2}} \, dx.$$

Then we can express the conditional pdf of $A_n < t | A_{n-1} = a_{n-1}$ as

$$\begin{aligned} & \frac{d}{dt} P[A_n < t | A_{n-1} = a_{n-1}] \\ = & \frac{d}{dt} P(\delta_n < ln(\frac{t}{a_{n-1}})) + \frac{d}{dt} \frac{1}{\sigma\sqrt{2\pi}} e^{\lambda t} \int_{ln(\frac{t}{a_{n-1}})}^{+\infty} e^{-\lambda a_{n-1} e^x - \frac{x^2}{2\sigma^2}} \, dx. \end{aligned}$$

The first part is

$$\frac{d}{dt} P(\delta_n < ln(\frac{t}{a_{n-1}})) = \frac{1}{t} \frac{1}{\sqrt{2\pi}\sigma} e^{-(\frac{ln(\frac{t}{a_{n-1}})}{\sigma\sqrt{2}})^2}.$$

The second part is

$$\begin{aligned} & \frac{d}{dt}\frac{1}{\sigma\sqrt{2\pi}}e^{\lambda t}\int_{ln(\frac{t}{a_{n-1}})}^{+\infty} e^{-\lambda a_{n-1}e^{x}-\frac{x^2}{2\sigma^2}}\,dx \\ = & \frac{\lambda}{\sigma\sqrt{2\pi}}e^{\lambda t}\int_{ln(\frac{t}{a_{n-1}})}^{+\infty} e^{-\lambda a_{n-1}e^{x}-\frac{x^2}{2\sigma^2}}\,dx - \frac{1}{t}\frac{1}{\sqrt{2\pi}\sigma}e^{-(\frac{ln(\frac{t}{a_{n-1}})}{\sigma\sqrt{2}})^2}. \end{aligned}$$

By combining the first and second parts, we have that the conditional pdf of $A_n|A_{n-1} = a_{n-1}$

$$\frac{d}{dt}P[A_n < t|A_{n-1} = a_{n-1}] = \frac{\lambda}{\sigma\sqrt{2\pi}}e^{\lambda t}\int_{ln(\frac{t}{a_{n-1}})}^{+\infty} e^{-\lambda a_{n-1}e^{x}-\frac{x^2}{2\sigma^2}}\,dx.$$

Since the integral does not have a simple form, we write

$$\frac{d}{dt}P[A_n < t|A_{n-1} = a_{n-1}] = \lambda e^{\lambda t}E[Q(y)I(y \geq ln(\frac{t}{a_{n-1}}))],$$

where $Q(y) = e^{-\lambda a_{n-1}e^{y}}$, y being a realisation from a normal random variable with mean 0 and variance σ^2. We apply Monte Carlo methods (Ross 2001). We generate i.i.d. random variables y_i with conditional pdf of $A_n|A_{n-1} = a_{n-1}$ and by the law of large numbers and we obtain the approximation to the conditional pdf of $A_n|A_{n-1} = a_{n-1}$

$$\frac{d}{dt}P[A_n < t|A_{n-1=a_{n-1}}] \approx \lambda e^{\lambda t}\frac{1}{M}\sum_{i=1}^{M} Q(y_i)I(y_i \geq ln(\frac{t}{a_{n-1}})).$$

3.3.3 Model 3-Binomial Model

In this model we assume the claim size is a constant instead of random variable, but we do not know when claim occurs. So the surplus of the company at time n is

$$A_n = A_{n-1}e^{\delta_n} - Cy_n,$$

where we set constant C as claim size, and y_n is a realisation of a random variable Y_n with probability p of a claim being made. So the conditional probability for t, $a_{n-1} \geq 0$

is

$$
\begin{aligned}
P[A_n < t|A_{n-1} = a_{n-1}] &= P[a_{n-1}e^{\delta_n} - Cy_n < t] \\
&= P[a_{n-1}e^{\delta_n} < t \mid y_n = 0]P[y_n = 0] \\
&\quad +P[a_{n-1}e^{\delta_n} - C < t \mid y_n = 1]P[y_n = 1] \\
&= (1-p)P[\delta_n < ln(\frac{t}{a_{n-1}})] + pP[\delta_n < ln(\frac{t+C}{a_{n-1}})].
\end{aligned}
$$

Then the conditional pdf of $A_n|A_{n-1} = a_{n-1}$ is given by

$$
\begin{aligned}
\frac{d}{dt}P[A_n < t|A_{n-1} = a_{n-1}] = \frac{1}{t}\frac{1}{\sqrt{2\pi}\sigma}e^{-(\frac{ln(\frac{t}{a_{n-1}})}{\sigma\sqrt{2}})^2} \times (1-p) \\
+\frac{1}{t+C}\frac{1}{\sqrt{2\pi}\sigma}e^{-(\frac{ln(\frac{t+C}{a_{n-1}})}{\sigma\sqrt{2}})^2} \times p.
\end{aligned}
$$

3.3.4 Model 4- Model with investment strategy with constant claim size

So far we have introduced three different models. In these models, we do just think about only one investment/interest rate, so there is nothing to do with strategy. However in next three models, an investment strategy will be modelled. Hence there are two interest rates in our model, one is bond interest rate and the other is the investment rate. The new time series model for surplus A_n is

$$
A_n = A_{n-1}(\theta e^{\delta_n{}^{(1)}} + (1-\theta)e^{\delta_n{}^{(2)}}) - C,
$$

where

- The C is claim size and it is constant by assumption.
- The $\delta_n{}^{(1)}$ and $\delta_n{}^{(2)}$ define the return rates of two different investment policies. In this case $\delta_n{}^{(1)}$, the investment interest, is a normal random variable with mean 0 and variance σ^2 and $\delta_n{}^{(2)}$, the bond interest rate, is a constant, and we let $v = e^{\delta_n{}^{(2)}}$.
- The θ is the most interesting parameter which is the proportion to invest, we assume that θ' is fixed.

Firstly, we are looking for the conditional probability

$$\begin{aligned} & P[A_n < t \mid A_{n-1} = a_{n-1}] \\ = \;& P[a_{n-1}(\theta e^{\delta_n{}^{(1)}} + (1-\theta)e^{\delta_n{}^{(2(}}) - C < t] \\ = \;& P[a_{n-1}\theta e^{\delta_n{}^{(1(}} + a_{n-1}v - a_{n-1}\theta v - C < t] \\ = \;& P[\delta_n{}^{(1)} < ln(\frac{t + C - a_{n-1}v(1-\theta)}{a_{n-1}\theta})]. \end{aligned}$$

So the conditional pdf of $A_n \mid A_{n-1} = a_{n-1}$ is

$$\begin{aligned} & \frac{d}{dt}P[A_n < t | A_{n-1} = a_{n-1}] \\ = \;& \frac{1}{t + C - a_{n-1}v(1-\theta)} \frac{1}{\sigma\sqrt{2\pi}} exp(-\frac{(ln(\frac{t+C-a_{n-1}v(1-\theta)}{a_{n-1}\theta}))^2}{2\sigma^2}), \end{aligned}$$

where $\delta_n{}^{(1)}$ has normal distribution with mean 0 and variance σ^2. Hence log-likelihood function of σ, $l(\sigma)$ is

$$l(\sigma) \propto -nln(\sigma) - \frac{\sum_{i=1}^{n} ln(\frac{t+C-a_{i-1}v(1-\theta)}{a_{i-1}\theta})}{2 \times \sigma^2}.$$

By solving $\frac{\partial l}{\partial \sigma} = 0$, we obtain, $\hat{\sigma}$, the MLE of σ as

$$\hat{\sigma} = \sqrt{\frac{\sum_{i=1}^{n}(ln(\frac{t+C-a_{i-1}v(1-\theta)}{a_{i-1}\theta}))^2}{n}},$$

since the t, C and a_{i-1} are known constants. The estimate of $\hat{\sigma}$ depends on θ.

3.3.5 Model 5-Model with investment strategy with exponential claims

Assumption:

- $\{C_n\}$ are iid exponentially distributed with mean $\frac{1}{\lambda}$;
- $\{\delta_k\}$ are independent;
- The novelty is that the claims $\{C_n\}$ are not constants as in Model 4. This Model

extends Model 4 by allowing the claims to be exponentially distributed.

The surplus A_n will be

$$A_n = A_{n-1}(\theta e^{\delta_n{}^{(1)}} + (1-\theta)e^{\delta_n{}^{(2)}}) - C_n,$$

where $C_n$has an exponential distribution with mean λ^{-1}. The conditional probability $\mathrm{P}[A_n < t \mid A_{n-1} = a_{n-1}]$ can be expressed as follows

$$\begin{aligned} & P[A_n < t \mid A_{n-1} = a_{n-1}] \\ = \; & P[a_{n-1}(\theta e^{\delta_n{}^{(1)}} + (1-\theta)e^{\delta_n{}^{(2)}}) - C_n < t] \\ = \; & P[a_{n-1}\theta e^{\delta_n{}^{(1)}} + a_{n-1}v - a_{n-1}\theta v - C_n < t]. \end{aligned}$$

Let $\theta e^{\delta_n{}^{(1)}} = e^{ln(\theta)+\delta_n{}^{(1)}}$ and let $ln(\theta) + \delta_n{}^{(1)} = \delta_n$, so that

$$\begin{aligned} & P[A_n < t \mid A_{n-1} = a_{n-1}] \\ = \; & P[a_{n-1}e^{\delta_n} - C_n < t - a_{n-1}v + a_{n-1}\theta v] \\ = \; & P[C_n > a_{n-1}e^{\delta_n} - t + a_{n-1}v - a_{n-1}\theta v] \\ = \; & E[I(C_n > a_{n-1}e^{\delta_n} - t + a_{n-1}v - a_{n-1}\theta v)] \\ = \; & E[I(a_{n-1}e^{\delta_n} - t + a_{n-1}v - a_{n-1}\theta v < 0) \\ & \quad + e^{-(\lambda a_{n-1}e^{\delta_n} - t + a_{n-1}v - a_{n-1}\theta v)} I(a_{n-1}e^{\delta_n} - t + a_{n-1}v - a_{n-1}\theta v > 0)] \\ = \; & E[I(\delta_n < ln(\frac{t - a_{n-1}v + a_{n-1}\theta v}{a_{n-1}})) \\ & \quad + I(\delta_n > ln(\frac{t - a_{n-1}v + a_{n-1}\theta v}{a_{n-1}}))e^{-\lambda(a_{n-1}e^{\delta_n} - t + a_{n-1}v - a_{n-1}\theta v)}] \\ = \; & P(\delta_n < ln(\frac{t - a_{n-1}v + a_{n-1}\theta v}{a_{n-1}})) \\ & \quad + \int_{ln(\frac{t-a_{n-1}v+a_{n-1}\theta v}{a_{n-1}})}^{+\infty} e^{-\lambda(a_{n-1}e^x - t + a_{n-1}v - a_{n-1}\theta v)} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-ln(\theta))^2}{2\sigma^2}}\, dx, \end{aligned}$$

where δ_n is a normal random variable with mean $ln(\theta)$ and variance σ^2. By using

$$\frac{d}{dt}\int_{f(t)}^{\infty} g(x)\, dx = [\frac{d}{dt}(-f(t))]g(f(t)) + \int_{f(t)}^{\infty} \frac{dg}{dt}\, dx,$$

with $f(t) = ln(\frac{t-a_{n-1}v+a_{n-1}\theta v}{a_{n-1}})$, we derive

$$g(t,x) = e^{-\lambda(a_{n-1}e^x - t + a_{n-1}v - a_{n-1}\theta v)} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-ln(\theta))^2}{2\sigma^2}}.$$

Then

$$\begin{aligned}
& \frac{d}{dt}P[A_n < t | A_{n-1} = a_{n-1}] \\
= \quad & \frac{1}{t - a_{n-1}v + a_{n-1}\theta v} \times \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(ln(\frac{t-a_{n-1}v+a_{n-1}\theta v}{a_{n-1}\theta}))^2}{2\sigma^2}} \\
& - \frac{1}{t - a_{n-1}v + a_{n-1}\theta v} \times \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(ln(\frac{t-a_{n-1}v+a_{n-1}\theta v}{a_{n-1}\theta}))^2}{2\sigma^2}} \\
& + \int_{ln(\frac{t-a_{n-1}v+a_{n-1}\theta v}{a_{n-1}})}^{+\infty} \lambda e^{-\lambda(a_{n-1}e^x - t + a_{n-1}v - a_{n-1}\theta v)} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-ln(\theta))^2}{2\sigma^2}} \, dx \\
= \quad & \int_{ln(\frac{t-a_{n-1}v+a_{n-1}\theta v}{a_{n-1}})}^{+\infty} \lambda e^{-\lambda(a_{n-1}e^x - t + a_{n-1}v - a_{n-1}\theta v)} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-ln(\theta))^2}{2\sigma^2}} \, dx.
\end{aligned}$$

For the purpose of approximation, we rewrite the above expression as

$$\lambda e^{-\lambda(a_{n-1}v - a_{n-1}\theta v - t)} \int_{ln(\frac{t-a_{n-1}v+a_{n-1}\theta v}{a_{n-1}})}^{+\infty} e^{-\lambda(a_{n-1}e^x)} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-ln(\theta))^2}{2\sigma^2}} \, dx.$$

$$= \lambda e^{-\lambda(a_{n-1}v - a_{n-1}\theta v - t)} E(Q(y) I(y > ln(\frac{t-a_{n-1}v+a_{n-1}\theta v}{a_{n-1}})),$$

where $Q(y) = e^{-\lambda a_{n-1}e^y}$, and y is normal random variable with mean 0 and variance σ^2. Then by using the law of large numbers, we obtain the approximation

$$\frac{d}{dt}P[A_n < t | A_{n-1} = a_{n-1}] \approx$$

$$\lambda e^{-\lambda(a_{n-1}v - a_{n-1}\theta v - t)} \frac{1}{M} \sum_{i=1}^{M} Q(y) I(y > ln(\frac{t-a_{n-1}v+a_{n-1}\theta v}{a_{n-1}})).$$

3.3.6 Model 6

This is the last model we consider. It is quite similar to the previous one, but we let ${\delta_n}^{(2)}$ be a random variable instead of being a constant. The surplus of policy at time n is

$$A_n = A_{n-1}(\theta e^{\delta_n{}^{(1)}} + (1-\theta)e^{\delta_n{}^{(2)}}) - C_n.$$

In this case ${\delta_n}^{(1)}$ is a normal random variable with mean 0 and variance ${\sigma_1}^2$. We let ${\delta_n}^{(2)}$ also be a normal random variable with mean 0 and variance ${\sigma_2}^2$. The conditional cdf is

$$\begin{aligned} P[A_n < t \mid A_{n-1} = a_{n-1}] &= P[A_{n-1}(\theta e^{\delta_n{}^{(1)}} + (1-\theta)e^{\delta_n{}^{(2)}}) - C_n < t \mid A_{n-1} = a_{n-1}] \\ &= P[a_{n-1}(\theta e^{\delta_n{}^{(1)}} + (1-\theta)e^{\delta_n{}^{(2)}}) - C_n < t] \end{aligned}$$

We let $\theta e^{\delta_n{}^{(1)}} = e^{\delta_n}$, where δ_n is the normal random variable with mean $ln(\theta)$ and variance σ^2. We let

$$(1-\theta)e^{\delta_n{}^{(2)}} = e^{\delta_n{}^{(2)}{}_{\hat{\theta}}},$$

where ${\delta_n}^{(2)}{}_{\hat{\theta}}$ is the normal random variable with mean $ln((1-\theta))$ and variance σ^2. Therefore by standard simplifications

$$\begin{aligned} & P[A_n < t \mid A_{n-1} = a_{n-1}] \\ = \;& P[a_{n-1}(e^{\delta_n} + e^{\delta_n{}^{(2)}{}_{\hat{\theta}}}) - C_n < t] \\ = \;& P[C_n > a_{n-1}(e^{\delta_n} + e^{\delta_n{}^{(2)}{}_{\hat{\theta}}}) - t] \\ = \;& E(I(C_n > a_{n-1}(e^{\delta_n} + e^{\delta_n{}^{(2)}{}_{\hat{\theta}}}) - t)) \\ = \;& E(I(a_{n-1}(e^{\delta_n} + e^{\delta_n{}^{(2)}{}_{\hat{\theta}}})) - t < 0) \\ & \quad + e^{-\lambda(a_{n-1}(e^{\delta_n} + e^{\delta_n{}^{(2)}{}_{\hat{\theta}}})) - t)} I(a_{n-1}(e^{\delta_n} + e^{\delta_n{}^{(2)}{}_{\hat{\theta}}})) - t > 0)) \\ = \;& E[I(e^{\delta_n} + e^{\delta_n{}^{(2)}{}_{\hat{\theta}}} < \frac{t}{a_{n-1}}) + e^{-\lambda(a_{n-1}(e^{\delta_n} + e^{\delta_n{}^{(2)}{}_{\hat{\theta}}})) - t)} I(e^{\delta_n} + e^{\delta_n{}^{(2)}{}_{\hat{\theta}}} > \frac{t}{a_{n-1}})]. \end{aligned}$$

By using Gaussian density (Ross 2001), we have

$$
\begin{aligned}
& P[A_n < t \mid A_{n-1} = a_{n-1}] \\
= \ & \int_0^\infty \int_0^{\frac{t}{a_{n-1}}-y} \frac{1}{x\sigma\sqrt{2\pi}} e^{\frac{(ln(x)-ln(\theta))^2}{2\sigma^2}} \times \frac{1}{y\sigma\sqrt{2\pi}} e^{-\frac{(ln(\frac{y}{1-\theta}))^2}{2\sigma 2}} \, dx \, dy + \\
& \int_0^\infty \int_{\frac{t}{a_{n-1}}-y}^\infty \frac{1}{x\sigma\sqrt{2\pi}} e^{\frac{(ln(x)-ln(\theta))^2}{2\sigma^2}} \times \frac{1}{y\sigma\sqrt{2\pi}} e^{-\frac{(ln(\frac{y}{1-\theta}))^2}{2\sigma 2}} e^{-\lambda(a_{n-1}(x+y)-t)} \, dx \, dy.
\end{aligned}
$$

Now, let

$$g(y) = \frac{t}{a_{n-1}} - y,$$

$$h(x) = \frac{1}{x\sigma\sqrt{2\pi}} e^{\frac{(ln(x)-ln(\theta))^2}{2\sigma^2}},$$

and

$$f(y) = \frac{1}{y\sigma\sqrt{2\pi}} e^{-\frac{(ln(\frac{y}{1-\theta}))^2}{2\sigma^2}},$$

$$q(x,y) = e^{-\lambda(a_{n-1}(x+y)-t)}.$$

With these notations then

$$
\begin{aligned}
& P[A_n < t \mid A_{n-1} = a_{n-1}] \\
= \ & \int_0^\infty \int_0^{g(y)} h(x) f(y) \, dx \, dy + \int_0^\infty \int_{g(y)}^\infty h(x) f(y) q(x,y) \, dx \, dy \\
= \ & \int_0^\infty [H(x) f(y)]_0^{g(y)} \, dy + \int_0^\infty \int_{g(y)}^\infty h(x) f(y) q(x,y) \, dx \, dy \\
= \ & \int_0^\infty H(g(y)) f(y) - H(0) f(y) \, dy + \int_0^\infty \int_{g(y)}^\infty h(x) f(y) q(x,y) \, dx \, dy,
\end{aligned}
$$

where $H(x) = \int h(x)dx$. Hence by standard calculation and differentiating integrals,

$$
\begin{aligned}
& \frac{d}{dt}P[A_n < t|A_{n-1} = a_{n-1}] \\
= & \int_0^\infty \frac{dg(y)}{dt}\frac{dH(g(y))}{dg(y)}f(y)\,dy - \int_0^\infty \frac{dg(y)}{dt}h(g)f(y)q(g(y),y,t)\,dy \\
& + \int_0^\infty \int_{g(y)}^\infty \frac{dq(x,y)}{dt}h(x)f(y)q(x,y)\,dx\,dy \\
= & \int_0^\infty \frac{dg(y)}{dt}h(g(y))f(y)\,dy - \int_0^\infty \frac{dg(y)}{dt}h(g(y))f(y)q(g(y),y)\,dy \\
& - \int_0^\infty \int_{g(y)}^\infty \frac{dq(x,y)}{dt}h(x)f(y)q(x,y)\,dx\,dy.
\end{aligned}
$$

Because $q(g(y),y) = 1$ and $\frac{dq(x,y)}{dt} = -\lambda$, then

$$
\begin{aligned}
& \frac{d}{dt}P[A_n < t|A_{n-1} = a_{n-1}] \\
= & \int_0^\infty \int_{g(y)}^\infty \lambda h(x)f(y)q(x,y)\,dx\,dy \\
= & \lambda \int_0^\infty \int_{\frac{t}{a_{n-1}}-y}^\infty \frac{1}{x\sigma\sqrt{2\pi}}e^{\frac{(ln(\frac{x}{\theta}))^2}{2\sigma^2}} \times \frac{1}{y\sigma\sqrt{2\pi}}e^{-\frac{(ln(\frac{y}{1-\theta}))^2}{2\sigma^2}}e^{-\lambda(a_{n-1}(x+y)-t)}\,dx\,dy.
\end{aligned}
$$

For the purpose of approximation, we rewrite the above expression as

$$
\begin{aligned}
& \lambda \int_0^\infty \int_{\frac{t}{a_{n-1}}-y}^\infty \frac{1}{x\sigma\sqrt{2\pi}}e^{\frac{(ln(\frac{x}{\theta}))^2}{2\sigma^2}} \times \frac{1}{y\sigma\sqrt{2\pi}}e^{-\frac{(ln(\frac{y}{1-\theta}))^2}{2\sigma^2}}e^{-\lambda(a_{n-1}(x+y)-t)}\,dx\,dy \\
= & \lambda E[L(x,y)I(x+y > \frac{t}{a_{n-1}}],
\end{aligned}
$$

where

$$L(x,y) = e^{-\lambda(a_{n-1}(x+y)-t)},$$

x is a log-normal random variable with mean $e^{\frac{\sigma_1^2}{2}}$ and variance $(e^{\frac{\sigma_1^2}{2}} - 1)e^{\frac{\sigma_1^2}{2}}$ and y is another log-normal random variable with mean $e^{\frac{\sigma_2^2}{2}}$ and variance $(e^{\frac{\sigma_2^2}{2}} - 1)e^{\frac{\sigma_2^2}{2}}$.

As before, at the last step, by the law of large numbers to obtain the approximation of $\frac{d}{dt}P[A_n < t|A_{n-1} = a_{n-1}]$ in the following way

$$\frac{d}{dt}P[A_n < t|A_{n-1} = a_{n-1}] \approx \lambda\frac{1}{M}\sum_{i=1}^{M} L(x_i,y_i)I(x_i + y_i > \frac{t}{a_{n-1}}).$$

3.4 Estimation of the parameters: Numerical Analysis with Stochastic Simulation

For a particular model, we simulated data for given values of the parameters, and then we use the simulated data to estimate the parameters. The reason to do the simulation is we can compare the estimate with the true values of parameters. Next, we use the estimated parameters and simulation approach to find the optimal investment strategy. Now the question is how we construct data.

Since in Model 1, we can estimate the parameter directly, we begin with Model 2. In Model 2, the surplus of policy at time n is

$$A_n = A_{n-1}e^{\delta_n} - C_n.$$

In this model the σ^2, the variance of δ_n, and the λ, the inverse of the mean of C_n, are both the parameters. So if we set the value of A_0, σ, and λ, we can simulate the whole sequence of $\{A_n\}$. There are some assumptions on our data.

- our data $\{a_n\}$ from a time series model

$$a_n = a_{n-1}e^{\delta_n} - C_n;$$

- we use given σ and λ to simulate $\{\delta_n\}$ and $\{C_n\}$;
- $\{a_n\} > 0$ in the estimation.
- The number of replications is 1000

3.4.1 How to choose A_0 in the simulation

Since by assumption $\{A_n\} > 0$ so we should choose A_0 carefully. $A_n > 0$ means

$$A_0 e^{\sum_{i=1}^{n} \delta_i} - \sum_{i=1}^{n} C_i e^{\sum_{i=k+1}^{i} \delta_i} > 0.$$

If we replace the C_i by $\frac{1}{\lambda} = E(C_i)$ then

$$A_0 e^{\sum_{i=1}^{n} \delta_i} > \frac{1}{\lambda} \sum_{i=1}^{n} e^{\sum_{i=k+1}^{i} \delta_i}.$$

Next we use present valuation instead of future valuation then

$$A_0 > \frac{1}{\lambda} \sum_{i=1}^{n} e^{-\sum_{i=k+1}^{i} \delta_i},$$

hence there is a lower bound for A_0 in our simulation. However we should not set large value for A_0 if λ^{-1} is not large.

We do estimations twice. Once is for short term using $n = 20$. Another is for long term using $n = 60$. We use two different computer packages, namely R-language and Maple, carry out the simulation.

3.4.2 The results of estimation

In this section, we will do the estimation for every model. Maple is a tool we used in this section, because it can calculate the integral in the model directly and carry out the whole process quicker than using R.

Experiment 1, $\lambda = 1$, $\sigma = 0.05$.

In this experiment, we set $\lambda = 1$ and $\sigma = 0.05$. But we use different initial value, A_0, to construct different data. In Model 1, we consider claim size, $C = 1$, over time. In Model 3, we consider the probability of occurrences of claims, p, as 0.5 in our data. Since there are two interest/investment rates in Model 5 and 6, we take constant rate, $v = 1.05$, in Model 4, and additional variable rate, $\sigma_1 = 0.1$. In addition we use $\theta = 0.5$ to construct data in Model 5 and 6.

The Table 3.1 shows that the estimation of σ works well. For different initial value the results of estimation of σ are good. Because of the way we construct data, both $\hat{\sigma}$ for $n = 20$ and $n = 60$ are same.

a_0	C	σ	$n = 20$	$n = 60$
			$\hat{\sigma}$	$\hat{\sigma}$
50	1	0.05	0.052	0.052
60	1	0.05	0.054	0.054
100	1	0.05	0.048	0.048
200	1	0.05	0.058	0.058

Table 3.1: The results of estimation for Model 1 (Experiment 1)

a_0	λ	σ	$n = 20$		$n = 60$	
			$\hat{\lambda}$	$\hat{\sigma}$	$\hat{\lambda}$	$\hat{\sigma}$
50	1	0.05	1.0	0.05	1.2	0.05
60	1	0.05	0.7	0.05	1.4	0.06
100	1	0.05	0.9	0.05	1.7	0.05
200	1	0.05	0.7	0.05	0.7	0.05

Table 3.2: The results of estimation for Model 2 (Experiment 1)

As shown in the Table 3.2, for different sample data, we always get the good estimation of σ. In contrast, the estimation of λ is very unstable. When $n = 20$, the estimated value for λ is better than when $n = 60$. And we find when n is 20, the estimated value of λ is decreasing when a_0 is increasing from 50 to 100 but jump to 0.5 with $a_0 = 200$. When $n = 60$, the estimated value of λ is increasing when a_0 is increasing from 50 to 100 but is just 0.5 when a_0 is 200.

a_0	p	σ	$n = 20$		$n = 60$	
			$\hat{p}$	$\hat{\sigma}$	$\hat{p}$	$\hat{\sigma}$
50	0.5	0.05	1.0	0.05	0.2	0.05
60	0.5	0.05	0.95	0.05	0.05	0.05
100	0.5	0.05	0.45	0.05	0.05	0.05
200	0.5	0.05	0.7	0.04	1	0.05

Table 3.3: The results of estimation for Model 3 (Experiment 1)

The Table 3.3 shows that the results of estimation of p is bad and very unstable in contrast the simulation on σ is good and stable over different sample data. And the values of $\hat{p}$ at $n = 60$ is much less than at $n = 20$ except the last simulation in the above figure.

a_0	θ	C	σ	$n = 20$	$n = 60$
				$\hat{\sigma}$	$\hat{\sigma}$
50	0.5	1	0.05	0.036	0.046
60	0.5	1	0.05	0.053	0.051
100	0.5	1	0.05	0.056	0.049
200	0.5	1	0.05	0.044	0.046

Table 3.4: The results of estimation for Model 4 (Experiment 4)

a_0	λ	σ	$n = 20$ $(\theta = 0.5)$		$n = 60$ $(\theta = 0.5)$	
			$\hat{\lambda}$	$\hat{\sigma}$	$\hat{\lambda}$	$\hat{\sigma}$
50	1	0.05	1.1	0.05	0.8	0.04
60	1	0.05	2.0	0.06	0.7	0.05
100	1	0.05	0.6	0.04	0.9	0.05
200	1	0.05	2.0	0.06	1.3	0.05

Table 3.5: The results of estimation for Model 5 (Experiment 1)

a_0	λ	σ_1	σ_2	$n = 20$ ($\theta = 0.5$)			$n = 60$ ($\theta = 0.5$)		
				$\hat{\lambda}$	$\hat{\sigma_1}$	$\hat{\sigma_2}$	$\hat{\lambda}$	$\hat{\sigma_1}$	$\hat{\sigma_2}$
50	1	0.1	0.05	0.03	0.10	0.06	0.03	0.11	0.04
60	1	0.1	0.05	0.02	0.10	0.06	0.02	0.10	0.05
100	1	0.1	0.05	0.01	0.10	0.05	0.02	0.11	0.06
200	1	0.1	0.05	0.01	0.10	0.05	0.01	0.08	0.04

Table 3.6: The results of estimation for Model 6 (Experiment 1)

The Table 3.4, 3.5 and 3.6 show the good and stable estimation of σ, standard deviation of interest rate, again. However, the estimation of λ in the last two models is not good, especially in the last model. The possible explanations are that in short term, the initial value does not effect the estimation too much. Also when the initial value is large, the whole model does not strongly depend on λ, so the estimation is unstable.

There is a new parameter, θ, in the last two models. So we will do some simulations to examine if the estimation of parameters is stable over different value of θ.

The Table 3.7 shows that the estimation of σ and λ for different value of θ. We construct data by considering $\sigma = 0.1$, $\lambda = 1$, $v = 1.05$, and $A_0 = 220$. We obtain good and stable estimations of σ in short and long term. The results of $\hat{\lambda}$ is not bad in long term for different θ. Therefore we believe that the estimation of σ is stable over θ

θ	0.2	0.3	0.4	0.6	0.8	0.9
$\hat{\lambda}$ ($n = 20$)	0.6	2.0	0.4	0.4	0.4	0.3
$\hat{\sigma}$ ($n = 20$)	0.1	0.08	0.10	0.08	0.09	0.10
$\hat{\lambda}$ ($n = 60$)	1.1	1.1	1.2	0.5	0.6	0.7
$\hat{\sigma}$ ($n = 60$)	0.09	0.12	0.10	0.10	0.10	0.11

Table 3.7: The estimation for Model 4 using different values of θ

Experiment 2 $\lambda = 1, \sigma = 0.1$.

For this experiment, we set $\lambda = 1$ and $\sigma = 0.1$. We still use different initial value, a_0, to construct different data. In addition we consider $p = 0.02$ in Model 3, and we take one of two rates in Model 5 as 0.15 instead of 0.05.

a_0	C	σ	$n = 20$	$n = 60$
			$\hat{\sigma}$	$\hat{\sigma}$
150	1	0.1	0.1206	0.1206
200	1	0.1	0.1279	0.1279
300	1	0.1	0.1013	0.1013
400	1	0.1	0.0949	0.0949

Table 3.8: The results of estimation for Model 1 (Experiment 2)

a_0	λ	σ	$n = 20$		$n = 60$	
			$\hat{\lambda}$	$\hat{\sigma}$	$\hat{\lambda}$	$\hat{\sigma}$
150	1	0.1	1.0	0.1	1.2	0.1
200	1	0.1	0.9	0.1	1.4	0.1
300	1	0.1	1	0.1	1.5	0.1
420	1	0.1	1	0.1	1	0.1

Table 3.9: The results of estimation for Model 2 (Experiment 2)

a_0	p	σ	$n = 20$		$n = 60$	
			$\hat{p}$	$\hat{\sigma}$	$\hat{\lambda}$	$\hat{\sigma}$
150	0.2	0.1	0.05	0.11	0.05	0.09
200	0.2	0.1	0.3	0.11	0.45	0.11
300	0.2	0.1	0.4	0.12	0.1	0.1
420	0.2	0.1	1	0.09	0.05	0.1

Table 3.10: The results of estimation for Model 3 (Experiment 2)

a_0	θ	C	σ	$n = 20$	$n = 60$
				$\hat{\sigma}$	$\hat{\sigma}$
150	0.5	1	0.1	0.080	0.106
200	0.5	1	0.1	0.100	0.106
300	0.5	1	0.1	0.106	0.104
400	0.5	1	0.1	0.099	0.091

Table 3.11: The results of estimation on Model 4 (Experiment 2)

A_0	λ	σ	$n = 20$ $(\theta = 0.5)$		$n = 60$ $(\theta = 0.5)$	
			$\hat{\lambda}$	$\hat{\sigma}$	$\hat{\lambda}$	$\hat{\sigma}$
150	1	0.1	0.3	0.09	0.3	0.09
200	1	0.1	2.0	0.12	1.1	0.11
300	1	0.1	2.0	0.08	0.7	0.08
420	1	0.1	0.9	0.11	0.2	0.11

Table 3.12: The results of estimation on Model 5 (Experiment 2)

As can be seen from Table 3.8 to 3.12, the results are similar to those of Experiment 1.

a_0	λ	σ_1	σ_2	$n=20$ $(\theta=0.5)$			$n=60$ $(\theta=0.5)$		
				$\hat{\lambda}$	$\hat{\sigma_1}$	$\hat{\sigma_2}$	$\hat{\lambda}$	$\hat{\sigma_1}$	$\hat{\sigma_2}$
150	1	0.15	0.10	0.10	0.13	0.11	0.01	0.16	0.11
200	1	0.15	0.10	0.1	0.13	0.07	0.2	0.15	0.09
300	1	0.15	0.10	0.2	0.17	0.12	0.20	0.16	0.08
420	1	0.15	0.10	0.10	0.23	0.11	0.11	0.18	0.06

Table 3.13: The results of estimation for Model 6 (Experiment 2)

The Table 3.13 shows that results of estimation for Model 6 in Experiment 1 are worse than the results for Experiment 2. Especially, $\hat{\lambda}$, is very small in every simulation.

In conclusion, in both two Experiments, the estimated value of σ is stable over the time and initial value. However the estimated value of λ is very unstable in Model 5 and 6 and in Model 2 when the n is 60. It's worthy to notice that, for same value of A_0, the $\hat{\lambda}$ is bigger when n is 20 than n is 60 in the most cases.

Since the estimated value for λ is unstable, we will try to find the reason why we do not obtain the stable estimation for λ and the relationship between the estimated value and the initial value.

3.4.3 Discussion: Analysis of estimation

We have found that it is possible to obtain the robust (stable) estimation of σ but the estimation of the parameter λ is unstable.

Now we want to know if the estimation of σ is robust when λ varies, i.e. over a range different values of λ. In addition, we also wish to ascertain if there are reasons why we always obtain the robust estimation for σ.

Let us focus on Model 1. As described before

$$\hat{\sigma} = \sqrt{\frac{\sum_{i=1}^{n}(ln(\frac{a_i+C}{a_{i-1}}))^2}{n}},$$

after rearrangement we have

$$\hat{\sigma}^2 = \frac{\sum_{i=1}^{n}(ln(\frac{a_i+C}{a_{i-1}}))^2}{n}.$$

We assume we do not know the value of the constant C and use random variable instead of constant, so the $\hat{\sigma}$ will be the conditional MLE estimator:

$$\hat{\sigma}^2 = \frac{\sum_{i=1}^{n}(ln(\frac{a_i+\hat{C}_k}{a_{i-1}}))^2}{n}.$$

Here we will do different assumptions for $\hat{C}_k$ and check the value of $\hat{\sigma}$. We do many simulations, but we only simulate our data once.

Firstly we set $E(C_k) = \hat{C}_k$, then

$$(\hat{\sigma})^2 = \frac{\sum_{k=1}^{n}(ln(\frac{A_k+E(C_k)}{A_{k-1}}))^2}{n}.$$

By our simulation, we know the stable estimation value of σ for different $E(C_k)$.

The next question is whatever distribution of C_k we consider, whether or not we will get stable estimation of σ.

Now we assume that C_k is i.i.d exponentially distributed. But we do not know the observation of C_k so

$$\begin{aligned}(\hat{\sigma})^2 &= \frac{\sum_{k=1}^{n}(ln(\frac{A_k+\hat{C}_k}{A_{k-1}}))^2}{n} \\ &= \frac{\sum_{k=1}^{n}(ln(\frac{A_k}{A_{k-1}})+ln(1+\frac{\hat{C}_k}{A_{k-1}}))^2}{n}.\end{aligned} \tag{3.4.1}$$

Here we apply the following Lemma.

Lemma 3.1: *Let $A_n \to \infty$ let $\{C_n\}$ be i.i.d then if $E(C_1) < \infty$ then:*

1. $\frac{1}{N}\sum_{n=1}^{N}\frac{C_n}{A_n} \to 0$ *as* $N \to \infty$
2. $\frac{1}{N}\sum_{n=1}^{N}ln(1+\frac{C_n}{A_n}) \to 0$ *as* $N \to \infty$

Proof:

(1) If $A_n \to \infty$ then $\frac{C_n}{A_n} \to 0$ almost surely in probability. Hence

$$E(\frac{C_n}{A_n}) = \frac{E(C_n)}{A_n} = \frac{E(C_1)}{A_n} \to 0,$$

by Convergence theorem (see Ross (2001)).

(2) Because

$$0 \leq ln(1+x) \leq x \text{ for } x \geq 0,$$

then

$$0 \leq E(ln(1+\frac{C_n}{A_n})) \leq E(\frac{C_n}{A_n}).$$

It follows from the (1), that $E(ln(1+\frac{C_n}{A_n})) \to 0$.

By the Lemma 3.1, if $\hat{C}_k$ is asymptotically smaller than A_k then the equation (3.4.1) will be

$$\frac{\sum_{k=1}^{n}(ln(\frac{A_k}{A_{k-1}}) + ln(1+\frac{\hat{C}_k}{A_{k-1}}))^2}{n} \approx \frac{1}{n}\sum_{k=1}^{n}(ln(\frac{A_k}{A_{k-1}})^2 + o(1),$$

o(1) here means the omitted value. Since $A_k = A_{k-1}e^{\delta_k} - C_k$. then

$$\frac{1}{n}\sum_{k=1}^{n}(ln(\frac{A_k}{A_{k-1}})^2 \approx \frac{1}{n}\sum_{k=1}^{n}\delta_k^2,$$

so the estimated value of σ^2 will approximately equal to

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{k=1}^{n}\delta_k^2.$$

It shows that the estimation of $\hat{\sigma}$ is stable if claims are relatively small.

However if the value of C_k is close to the value of A_k, ie if the claim is similar to the asset at time k, then the value of $\sum_{k=1}^{n} ln(1+\frac{\hat{C}_k}{A_{k-1}})$ is not stable. As a result the value of $\hat{\sigma}$ will be unstable. The simulation process supports that. If we use $\lambda = 1$, $\sigma = 0.1$ and $a_0 = 600$ to construct the data by the same way described in Section 3.4.2. In addition we assume $\{\hat{C}_k\}$ is exponentially distributed with mean λ^{-1}. We do ten times estimations with ten different rates of $\{\hat{C}_k\}$ at $n = 20$ and $n = 60$. Table 3.14 shows the results of simulation. As we see, the value of $\hat{\sigma}$ is very unstable when λ is small. When the value of $\frac{1}{\lambda}$ is bigger than 10, we can not obtain good estimation.

$\frac{1}{\lambda}$	1	5	10	15	20
$\hat{\sigma}(n = 20)$	0.112	0.1216	0.0907	0.1133	0.1009
$\hat{\sigma}(n = 60)$	0.10	0.1086	0.2492	0.1752	0.1996
$\frac{1}{\lambda}$	25	30	35	40	45
$\hat{\sigma}(n = 20)$	0.1898	0.4484	1.832	9.909	0.278
$\hat{\sigma}(n = 60)$	4.6198	0.2838	0.2237	0.4897	2.5061

Table 3.14: The results of estimation of σ

Now, we consider a similar model as Model 1, but where $\{C_k\}$ are being i.i.d exponentially distributed with mean 1 so that

$$A_k = A_{k-1}v - C_k,$$

where v is constant.

Further assume that *av*-*t*>0 for observed variables. The conditional probability

$$\begin{aligned} P(A_n < t | A_{n-1} = a) &= p(av - C_n < t) \\ &= P(C_n > av - t) \\ &= e^{-\lambda(av-t)} I(av - t > 0) + I(av - t < 0) \\ &= e^{-\lambda(av-t)}. \end{aligned}$$

Then

$$\frac{d}{dt} P[A_n < t | A_{n-1} = a] = \lambda e^{-\lambda(av-t)},$$

and the conditional likelihood function $L(\lambda|a_1, \cdots a_n)$ is

$$\lambda^n e^{-\lambda \sum_{k=1}^{n}(a_{k-1}v - a_k)}.$$

The log-likelihood function is

$$l(\lambda) = nlog(\lambda) - \lambda \sum_{k=1}^{n}(a_{k-1}v - a_k).$$

Therefore by the equation belo

$$\frac{dl(\lambda)}{d\lambda} = n\frac{1}{\lambda} - \sum_{k=1}^{n}(a_{k-1}v - a_k) = 0,$$

and so the MLE of λ is given by

$$\hat{\lambda} = \frac{n}{\sum_{k=1}^{n} a_{k-1}\hat{v} - a_k}.$$

If we know the value of $\hat{v}$, we should obtain good and stable estimation of λ. A simple simulation can support that. If we use $\lambda = 1$, $\sigma = 0.01$ to 0.1 and $A_0 = 200$ to construct the data by the same way described in section 3.4. We assume that we know the value of $\hat{v}$ as 0.1 here. We simulate the data and estimate λ. Then Table 3.15 shows that the estimation of λ is good and stable over σ.

σ	0.01	0.02	0.03	0.04	0.05
$\hat{\lambda}(n = 20)$	0.94	0.0.96	1.05	0.90	0.80
σ	0.06	0.07	0.08	0.09	0.10
$\hat{\lambda}(n = 20)$	1.10	0.96	1.22	1.01	1.02

Table 3.15: The results of estimation of λ with known value of $\hat{v} = 1.1$

However if we do not know the value of $\hat{v}$, we consider $\hat{v}$ as a random variable. and we use same data as in the last simulation to estimate λ.

σ	0.01	0.02	0.03	0.04	0.05
$\hat{\lambda}(n=20)$	0.85	0.0.73	0.50	0.90	1.80
σ	0.06	0.07	0.08	0.09	0.10
$\hat{\lambda}(n=20)$	-0.66	0.96	1.62	-0.71	0.62

Table 3.16: The results of estimation of λ with unknown value of $\hat{v}$

As shown in Table 3.16, the estimation on λ is unstable and even give negative values in some cases. The above three simulations show the reason why we always obtain good and stable estimation of σ, but unstable estimation of λ.

The research was partly motivated by Magdalinos (2007). The simulation results and the discussion support his theoretical discoveries. The unrestricted estimation of the parameter is shown to be inconsistent for an autoregressive process.

3.5 Optimal investment policy

By now we have constructed models to describe the process of insurance policy, and how to estimate the parameters of the models. Although the estimated value of parameter of the claim is not stable, we understood the reason why and we will try to avoid the bad estimation by simulating the proper initial value. The next step is to investigate the how to obtain an optimal investment with respect to constant fraction policies. We should notice the optimality discussed in this section is not among all admissible strategies.

3.5.1 Simulation process

The simple and fast way is that we simulate the whole process of policy and then find the best investment strategy with given upper bound on ruin probability. We assume we have known the estimated value of parameters in the model based on the real data. Then we do the four different experiments for Model 6 by using averaging process. For both experiments. We set one hundred different values of θ from 0 to 1. For every θ we use the value of parameters to simulate the value of surplus M times and record the value of surplus at the end of time. Finally we calculate the average value of it for every θ.

For **experiment 3**: Initial value $A_0 = 200$, the standard deviation of the risk-investment interest rate $\sigma_1 = 0.15$, the standard deviation of the bond interest rate $\sigma_2 = 0.1$, the mean of claim size is 1, the term of policy $n = 60$ and the recycle times M is 5000.

For **experiment 4**, we keep the same value for all the parameters except we consider new $\sigma_1 = 0.45$.

The Figures 3.1 and 3.2 show that the average value of A_{60} against θ, the value of A_{60} is increasing when θ is increasing in both plots. In the Figure 3.1, the points distributed as a straight line, we believe the reason is that the value of $\sigma_1 = 0.15$ is quite close to $\sigma_2 = 0.1$. However in the Figure 3.2, because $\sigma_1 = 0.45$ is much bigger than $\sigma_2 = 0.1$, so the points distributed as a curve. The value of A_{60} on the Figure 3.2 is much bigger than on the Figure 3.1. The simulation tell us for the same investment strategy, the investments with high variance will take also high expected return.

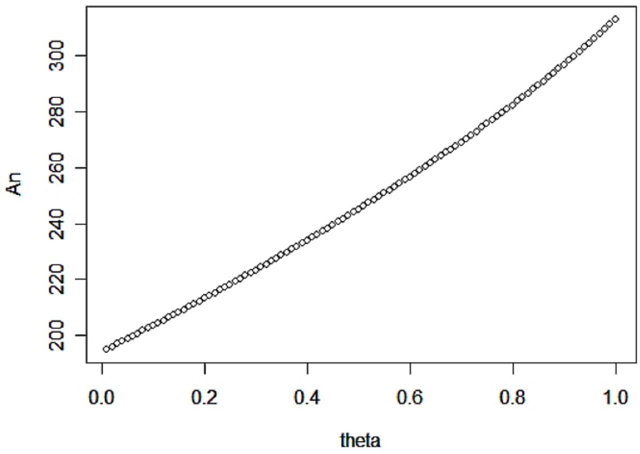

Figure 3.1: Plot of the average value of $A_{n=60}$ against θ ($\sigma_1 = 0.15$, $\sigma_2 = 0.1$)

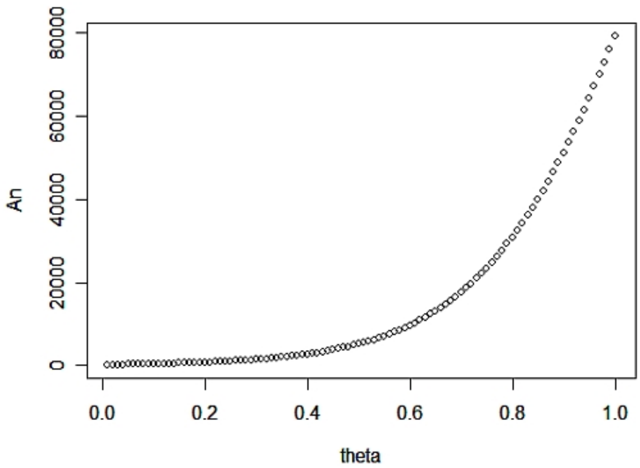

Figure 3.2: Plot of the average value of $A_{n=60}$ against θ ($\sigma_1 = 0.45$, $\sigma_2 = 0.1$)

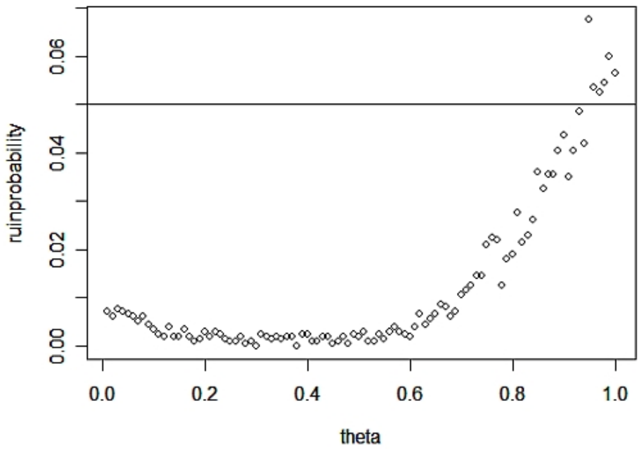

Figure 3.3: Plot of the ruin probability for Model 6 against θ ($\sigma_1 = 0.15$, $\sigma_2 = 0.1$)

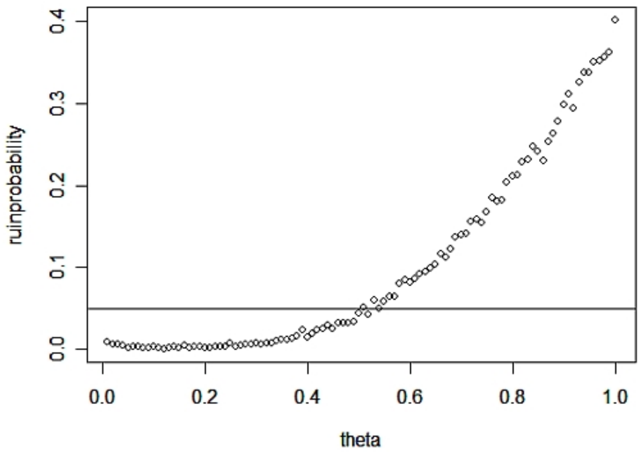

Figure 3.4: Plot of the ruin probability for Model 6 against θ ($\sigma_1 = 0.45$, $\sigma_2 = 0.1$)

In experiments 5 and 6, we still set one hundred values of θ from 0 to 1. For every θ, we use R language to generate M=5000 times value of A_n and record how many times the simulated $A_n < 0$ in the Model 6. We end up the process with dividing this time described above by M, which is the simulated ruin probability we are looking for.

For **experiment 5**, we take $\sigma_1 = 0.15$, $\sigma_2 = 0.1$, $A_0 = 200$ and n=60.

For **experiment 6**, we take $\sigma_1 = 0.45$, $\sigma_2 = 0.1$, $A_0 = 200$ and n=60.

The Figures 3.3 and 3.4 show the relationship between the simulated ruin probability and θ. Here we use the same value of parameters as in the first simulation. The surprising result is that there is a threshold in every plot which was empirically discovered. In the first plot, the threshold is around $\theta = 0.5$ compare with $\theta = 0.4$ in the second plot. After the probability of ruin raise significantly. Another noticeable point is the value of simulated ruin probability on the Figure 3.4 plot is much higher than on the Figure 3.3 because of high value of σ_1. The horizontal lines on both plots show $A_{60} = 0.05$ against different value of θ. It shows that the investment policy take more percentage of the first investment with higher variance by considering same ruin probability.

Discussion

In conclusion, the simulations support intuitive results. When the value of θ is increasing which means that the percentage of high risk investments is increasing, as a result, the ruin probability is increasing. However the rising of θ will make higher profit at time $n = 60$. Therefore if there is an upper bound, on the ruin probability, for example 0.05 on the Figure 3.3 and 3.4, then the optimal investment policy is that the policy makes the ruin probability close to this value. Figure 3.3 shows that the θ as required is 0.93, then A_{60} with 0.93 on the Figure 3.1 is 298.6. Figure 3.4 shows that the θ we are looking for is 0.54, then A_{60} with this value on the Figure 3.2 is 6641. This optimal policy is the balance, we are looking for, between the high return and the low risk.

Now we extend our simulation process. For **experiment 7**, we still use the same initial value $A_0 = 200$ and the variance of bond interest rate, ${\sigma_2}^2$, is 0.01. Because σ_1 should be bigger than σ_2 , then we set one hundred different values of σ_1 from 0.15 to 1.14. For every σ_1, we find the optimal policy as described by considering an upper bound of ruin probability (0.5 here) above with the simulation. Figure 3.5 shows that the plot of the optimal θ against σ_1. The value of θ is decreasing until σ_1 is about 0.45. For $\sigma_1 > 0.45$, θ appears to vary about the value value 0.5.

For every σ_1, we have determined the optimal strategy, hence the optimal investment (surplus at time n) will be calculated. The Figure 3.6 is the plot of $log(A_{60})$ against σ_1, and 1e+03 means 10^3 on the figure. As we can see, the surplus is increasing when σ_1 is increasing. Therefore, the insurance company will receive higher return with the optimal strategy if they make the more risky investment.

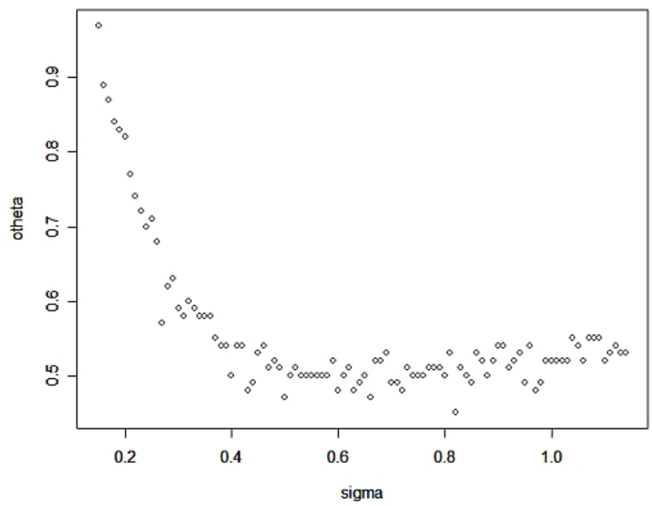

Figure 3.5: Plot of the value of θ that gives the optimal policy against the standard deviation of interest rate optimal (with $\sigma_2 = 0.1$, n=60, A_{60}=0.05,) experiment 7.

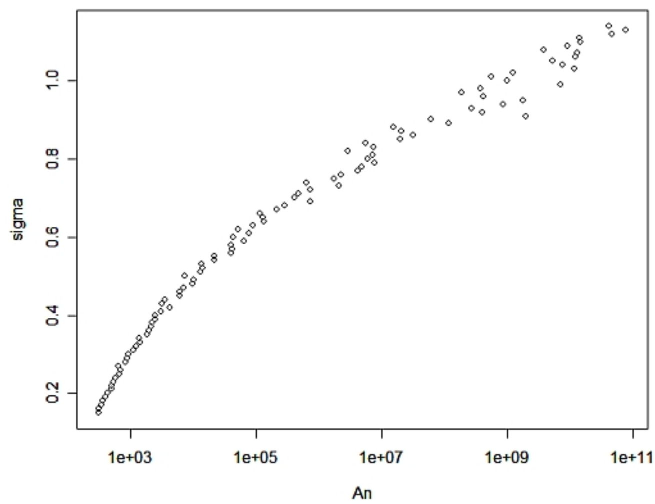

Figure 3.6: Plot of the standard deviation of the interest rate against the optimal investment A_n (with $\sigma_2 = 0.1$, $A_0 = 200$, n=60,) experiment 7

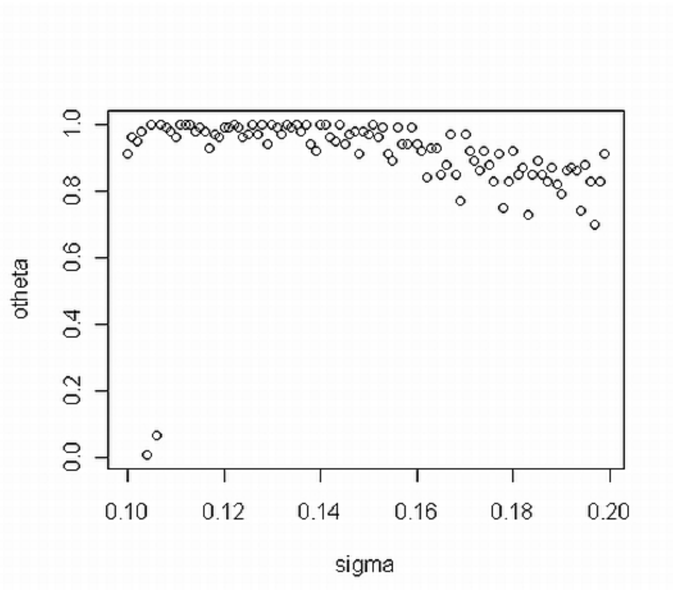

Figure 3.7: Plot of the value of θ that gives the optimal policy against the standard deviation of interest rate optimal (with $\sigma_2 = 0.1$, n=60, A_{60}=0.05,) experiment 8

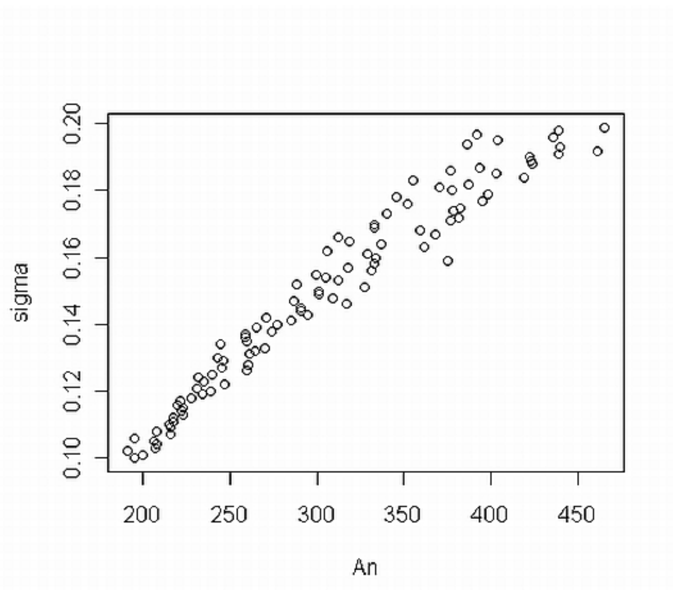

Figure 3.8: Plot of the standard deviation of the interest rate against the optimal investment A_n (with $\sigma_2 = 0.1$, $A_0 = 200$, n=60,) experiment 8

In **experiment 8**, for more realistic assumption, we set one hundred different values of σ_1 from 0.10 to 0.19. And we repeat same simulation approach. The Figure 3.7 and 3.8 show the results of simulation. It is notable that there are two points on the Figure 3.7 close to 0. Because at that point, the value of σ_1 and σ_2 are quite close.

3.6 Approximation of the ruin probability

Simulation is a relatively simple way to estimate ruin probability. But are there any theoretical approaches that could be used?

In this section, we use the weak convergence results (Billingsley, 1968) together with traditional Taylor-type analysis to find appropriate approximations for the ruin probabilities to be applied with the further numerical analysis. In this section, proofs precede results.

3.6.1 Approximation to Model 1

To explain our approach, we start with Model 1 described in Section 3.1. The accumulated surplus at time n in this model is given by

$$\begin{aligned} A_n &= A_{n-1}e^{\delta_n} - C \\ &= A_0 e^{\sum_{i=1}^{n}\delta_i} - \sum_{i=1}^{n} C e^{\sum_{k=i+1}^{n}\delta_k}. \end{aligned}$$

Hence the ruin probability for this model is

$$P[(A_0 e^{\sum_{i=1}^{n}\delta_i} - \sum_{i=1}^{n} C e^{\sum_{k=i+1}^{n}\delta_k}) < 0] = P[(A_0 - \sum_{i=1}^{n} C e^{-\sum_{k=1}^{i}\delta_k}) < 0]. \tag{3.6.1}$$

Next, let $e^{-\sum_{t=1}^{i}\delta_t}$ in (3.6.1) equal to a_{j-1} and applying Lemma 2.1 to (3.6.1) we have similar result as (2.4.3), namely

$$\begin{aligned} P[(A_0 - C\sum_{i=1}^{n} e^{-\sum\limits_{k=1}^{i}\delta_k}) < 0] &= P[(A - C\cdot n\int_0^{\frac{[nx]}{n}} a_{[ny]}dy) < 0] \\ &\approx P[\int_0^x exp(-h\cdot B_y)dy > \frac{A_0}{Cn}]\ , \end{aligned}$$

where $h = \sqrt{n+1}\sigma_k$ and B_y represents Brownian motion with distribution $B_t - B_s \sim$N (0, t-s) for $0 \le s \le t$. $N(\mu,\ \sigma^2)$ denotes the Normal distribution with mean μ and variance σ^2. Here, by the Donsker invariance principle (see below Fact (DIP)) we use approximation

$$\sum_{k=1}^{[ny]} \delta_k \overset{d}{\approx} hB_y,$$

since $\{\delta_k\}$ are i.i.d.random variables.

Given $x = 1$, our approximation for ruin probability is

$$P[\int_0^1 exp(-h \cdot B_y)dy > \frac{A_0}{Cn}]. \tag{3.6.2}$$

Fact (DIP) (Donsker Invariance Principle, (Billingsley, 1968)):

Let X be i.i.d. with E(X)=0, $E(X^2) = 1$. Define $S_n(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} X_i$, Then, $S_n \to B$ in C[0,1], where C[0.1] is a space of continuous functions on interval [0,1]. In particular, if L is continuous, then

$$f(S_n) = \int_0^1 L(S_n(t))dt \to \int_0^1 L(B(t))dt$$

weakly, where B is a standard Brownian motion.

Therefore we derive the following theorem.

Theorem 3.2: *Under the same assumption in Section 3.2, as $n \to \infty$, the ruin probability is approximated by*

$$P[A_n < 0] \approx P[\int_0^1 exp(-h \cdot B_y)dy > \frac{A_0}{Cn}].$$

As you can see, the probability in (3.6.2) can not be calculated directly because there is the integral of the Brownian motion. However we will calculate the approximated ruin probability in (3.6.2) by using the simulation approach. Approximations to Models 3 and 4 can be found in a similar fashion.

3.6.2 Approximation to Model 4

Now we apply the similar method to Model 4. We still need to know the surplus at time n under assumption which given by, (see Section 3.3)

$$\begin{aligned} A_n &= A_{n-1}(\theta e^{\delta_n{}^{(1)}} + (1-\theta)e^{\delta_n{}^{(2)}}) - C_n \\ &= A_{n-1}(\theta e^{\delta_n} + (1-\theta)v) - C. \end{aligned}$$

If we let $f(n) = \theta e^{\delta_n} + (1-\theta)v$, then

$$\begin{aligned} A_n &= A_{n-1}f(n) - C \\ &= A_0 \prod_{i=1}^{n} f(i) - C\sum_{i=1}^{n}\prod_{j=i+1}^{n} f(j). \end{aligned}$$

Hence

$$\begin{aligned} P(A_n < 0) &= P(A_0 \prod_{i=1}^{n} f(i) - C\sum_{i=1}^{n}\prod_{j=i+1}^{n} f(j) < 0) \\ &= P(\sum_{i=1}^{n}\prod_{j=1}^{i} f(j)^{-1} > \frac{A_0}{C}). \end{aligned}$$

The method we applied uses Lemma 2.1.

Let $\prod_{j=1}^{i} f(j)^{-1} = a_{i-1}$, then

$$\begin{aligned} \sum_{i=1}^{n}\prod_{j=1}^{n} f(j)^{-1} &= n\int_0^1 a_{[ny]}dy = n\int_0^1 \prod_{j=1}^{[ny]} f(j)^{-1}dy \\ &= n\int_0^1 e^{ln(\prod_{j=1}^{[ny]} f(j)^{-1})}dy. \end{aligned}$$

We write

$$ln(\prod_{j=1}^{[ny]} f(j)^{-1}) = -\sum_{j=1}^{[ny]} ln(\theta e^{\delta_j} + (1-\theta)v).$$

When n$\to\infty$, we consider

$$\delta_k = \frac{h}{\sqrt{n}}Z_k,$$

where the Z_k are i.i.d standard normal random variables. Using a Taylor expansion we have

$$\begin{aligned}
& ln(\theta e^{\delta_k} + (1-\theta)v) \\
= \; & ln[\theta(1 + \frac{h}{\sqrt{n}} Z_k + \frac{h^2}{2}\frac{Z_k^2}{n} + o(\frac{1}{n})) + (1-\theta)v] \\
= \; & ln[(\theta + (1-\theta)v) + \theta(\frac{h}{\sqrt{n}} Z_k + \frac{h^2}{2}\frac{Z_k^2}{n} + o(\frac{1}{n}))] \\
= \; & ln[v + \theta(1-v) + \theta(\frac{h}{\sqrt{n}} Z_k + \frac{h^2}{2}\frac{Z_k^2}{n} + o(\frac{1}{n}))].
\end{aligned}$$

If we let $v = e^{\frac{\gamma}{n}} = 1 + \frac{\gamma}{n} + o(\frac{1}{n})$, where γ is a constant then

$$ln(\theta e^{\delta_k} + (1-\theta)v)] = ln[1 + \frac{\gamma}{n} - \theta(\frac{\gamma}{n}) + \theta(\frac{h}{\sqrt{n}} Z_k + \frac{h^2}{2}\frac{Z_k^2}{n} + o(\frac{1}{n}))].$$

By considering

$$ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + o(x^4),$$

when n is $n \to \infty$, we can write

$$\begin{aligned}
& ln[1 + \frac{\gamma}{n} - \theta(\frac{\gamma}{n}) + \theta(\frac{h}{\sqrt{n}} Z_k + \frac{h^2}{2}\frac{Z_k^2}{n} + o(\frac{1}{n}))] \\
= \; & (1-\theta)\frac{\gamma}{n} + \theta(\frac{h}{\sqrt{n}} Z_k + \frac{h^2}{2}\frac{Z_k^2}{n}) - \frac{\theta^2 h^2}{n}\frac{Z_k^2}{2} + o(\frac{1}{n}).
\end{aligned}$$

Therefore

$$\begin{aligned}
& \sum_{j=1}^{[ny]} ln(\theta e^{\delta_j} + (1-\theta)v) \\
= \; & \sum_{j=1}^{[ny]} \frac{\gamma(1-\theta)}{n} + \sum_{j=1}^{[ny]} \frac{h\theta}{\sqrt{n}} Z_j + \sum_{j=1}^{[ny]} (1-\theta)\frac{\theta h^2}{n}\frac{Z_j^2}{2} + o(1) \\
= \; & \gamma y(1-\theta) + h\theta B_y + (1-\theta)\frac{\theta h^2}{2}\sum_{j=1}^{[ny]}\frac{Z_j^2}{n} + o(1) \\
\to \; & \gamma y(1-\theta) + h\theta B_y + (1-\theta)\frac{\theta h^2}{2} y,
\end{aligned}$$

by the Donsker invariance principle (see Fact (DIP)), where B_y denotes Brownian Motion. This leads to the following theorem

Theorem 3.3: *Under assumption in the Model 4, as $n \to \infty$, the ruin probability is approximated by*

$$P(A_n < 0) \approx P\left(\int_0^1 e^{-[\gamma y(1-\theta)+h\theta B_y+(1-\theta)\frac{\theta h^2}{2}y]}dy\right) > \frac{A_0}{Cn}).$$

It is difficult to explicitly calculate this probability directly. Therefore, we consider the following key transformation

$$B_{\frac{k}{m}} \stackrel{d}{=} \frac{S_k}{\sqrt{m}} \qquad\qquad S_k = \sum_{i=1}^{k} Z_i.$$

Then we have the approximation

$$\begin{aligned}
& \int_0^1 e^{-[\gamma y(1-\theta)+h\theta B_y+(1-\theta)\frac{\theta h^2}{2}y]}dy \\
\approx\ & \sum_{k=1}^{m}\left[e^{-[\gamma\frac{k-1}{m}(1-\theta)+h\theta B_{\frac{k-1}{m}}+(1-\theta)\frac{\theta h^2}{2}\frac{k-1}{m}]}+e^{-[\gamma\frac{k}{m}(1-\theta)+h\theta B_{\frac{k}{m}}+(1-\theta)\frac{\theta h^2}{2}\frac{k}{m}]}\right]\left(\frac{1}{2m}\right) \\
=\ & \sum_{k=1}^{m}\left[e^{-[\gamma\frac{k-1}{m}(1-\theta)+h\theta\frac{S_{k-1}}{\sqrt{m}}+(1-\theta)\frac{\theta h^2}{2}\frac{k-1}{m}]}+e^{-[\gamma\frac{k}{m}(1-\theta)+h\theta\frac{S_k}{\sqrt{m}}+(1-\theta)\frac{\theta h^2}{2}\frac{k}{m}]}\right]\left(\frac{1}{2m}\right).
\end{aligned}$$

So with the help of a computer package, we can estimate the ruin probability.

We do two types of the simulations on the same model, the first simulation is same as the one in Section 3.5.1, and the second one is that we calculate the above approximation M times, and record how many times the simulated value is bigger than $\frac{A_0}{Cn}$. The simulated probability will be this value divided by M.

We set initial value $A_0 = 200$, the standard deviation of the first interest rate $\sigma_1 = 0.65$, the rate of the second interest rate is $v = 1.02$, the claim size, C, is 1, the term of policy $n = 60$, the recycle times M is 2000 and number of grids $m = 2000$ also.

The Figure 3.9 and 3.10 show the simulated ruin probability plotted against θ. The two plots are similar. The ruin probability increases as θ increases. There are 'threshold' in both plots. In conclusion, the simulation supports the theoretical approximation.

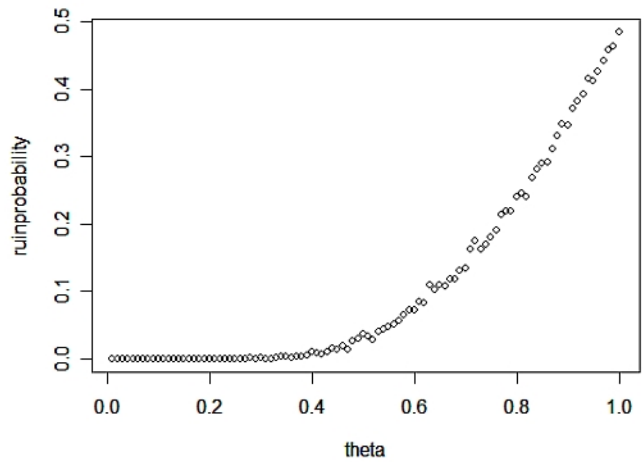

Figure 3.9: Plot of the tuin probability against θ produced by simulation of the approximation ($\sigma_1 = 0.65$, v=1.02)

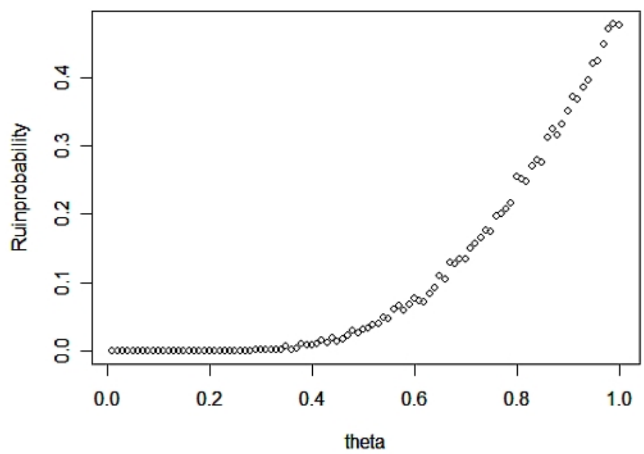

Figure 3.10: Plot of the ruin probability against θ produced by simulation of the process ($\sigma_1 = 0.65$, v=1.02)

3.6.3 Approximation to Model 5

However in Model 5, the answer is a little different from the previous ones because the claim size is random variable but not a constant. The surplus at time n is

$$\begin{aligned} A_n &= A_{n-1}(\theta e^{\delta_n{}^{(1)}} + (1-\theta)e^{\delta_n{}^{(2)}}) - C_n \\ &= A_0 \prod_{i=1}^{n} f(i) - \sum_{i=1}^{n} C_i \prod_{j=i+1}^{n} f(j), \end{aligned}$$

where

$$f(n) = \theta e^{\delta_n{}^{(1)}} + (1-\theta)v.$$

Hence the ruin probability is

$$P(A_n < 0) = P(\sum_{i=1}^{n} C_i \prod_{j=1}^{i} f(j)^{-1} > A_0).$$

We let

$$C_i \prod_{j=1}^{i} f(j)^{-1} = a_{i-1},$$

by using Lemma 3.1, we get

$$\begin{aligned} \sum_{i=1}^{n} C_i \prod_{j=1}^{n} f(j)^{-1} &= n\int_0^1 a_{[ny]}dy = n\int_0^1 C_{[ny]} \prod_{j=1}^{[ny]} f(j)^{-1}dy \\ &= n\int_0^1 C_{[ny]} e^{ln(\prod_{j=1}^{[ny]} f(j)^{-1})}dy. \end{aligned}$$

We have known the approximation of $ln(\prod_{j=1}^{\{ny\}} f(j)^{-1})$, so we could have the theorem below.

Theorem 3.4: *Under the assumptions of Model 5,*

$$P(A_n < 0) \approx P(\int_0^1 C_{[ny]} e^{-[\gamma y(1-\theta) + h\theta B_y + (1-\theta)\frac{\theta h^2}{2} y]}dy > \frac{A_0}{n}).$$

as $n \to \infty$.

Moreover, we have the approximation

$$\begin{aligned}
&\int_0^1 C_{[ny]} e^{-[\gamma y(1-\theta)+h\theta B_y+(1-\theta)\frac{\theta h^2}{2}y]}dy \\
\approx\ & \sum_{k=1}^{m}(2m)^{-1}\Big(C_{[\frac{n(k-1)}{m}]}e^{-[\gamma\frac{k-1}{m}(1-\theta)+h\theta B_{\frac{k-1}{m}}+(1-\theta)\frac{\theta h^2}{2}\frac{k-1}{m}]} \\
&\quad +C_{[\frac{nk}{m}]}e^{-[\gamma\frac{k}{m}(1-\theta)+h\theta B_{\frac{k}{m}}+(1-\theta)\frac{\theta h^2}{2}\frac{k}{m}]}\Big) \\
=\ & \sum_{k=1}^{m}\Big[C_{[\frac{n(k-1)}{m}]}e^{-[\gamma\frac{k-1}{m}(1-\theta)+h\theta\frac{S_{k-1}}{\sqrt{m}}+(1-\theta)\frac{\theta h^2}{2}\frac{k-1}{m}]}+C_{[\frac{nk}{m}]}e^{-[\gamma\frac{k}{m}(1-\theta)+h\theta\frac{S_k}{\sqrt{m}}+(1-\theta)\frac{\theta h^2}{2}\frac{k}{m}]}\Big]\Big(\frac{1}{2m}\Big).
\end{aligned}$$

where $[ny]$ *is an integer part of* ny *and has the formula*

$$\frac{[ny]}{n} = y + o(1/n).$$

We use the same method and assumptions as on Section 3.5.2 to do the simulation except we consider that the claim size is exponentially distributed with mean 1 here. The Figure 3.11 shows the results of simulation. The the plot on the left produced by the simulation of the approximation, and another plot is from the simulation of the approximation. We have the similar conclusion as the Figures 3.9 and 3.10.

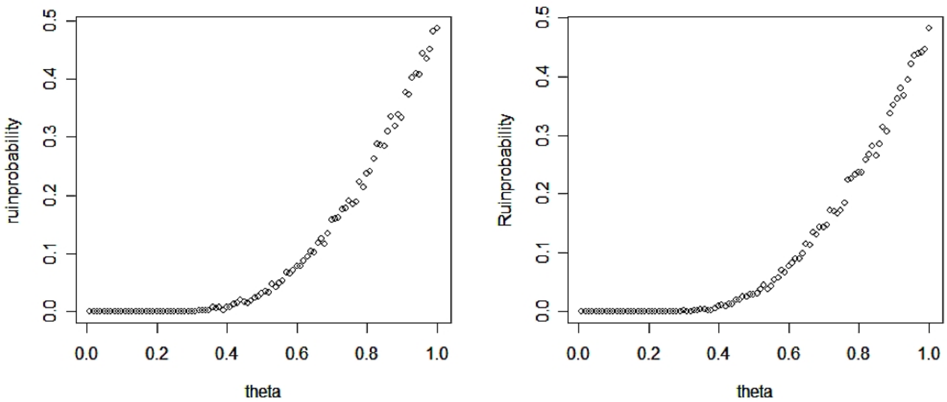

Figure 3.11: Plots of the ruin probability against θ produced by two different simulations ($\sigma_1 = 0.65$, v=1.02)

3.6.4 Approximation to Model 6

The same method will be applied to Model 6 which has random claim size and two random return rates. The surplus at time n in this model is

$$A_n = A_{n-1}(\theta e^{\delta_n{}^{(1)}} + (1-\theta)e^{\delta_n{}^{(2)}}) - C_n.$$

Finally, to find the similar approximation to Model 6, we start with

$$\begin{aligned} P(A_n < 0) &= P(\sum_{i=1}^{n} C_i \prod_{j=1}^{i} f(j)^{-1} > A_0) \\ &= P(n\int_0^1 C_{[ny]} e^{ln(\prod_{j=1}^{[ny]} f(j)^{-1})} dy > A_0), \end{aligned}$$

where

$$ln(\prod_{j=1}^{[ny]} f(j)^{-1}) = -\sum_{j=1}^{[ny]} ln(\theta e^{\delta_n{}^{(1)}} + (1-\theta)e^{\delta_n{}^{(2)}}).$$

For n large, we consider

$$\delta_k{}^{(1)} = \frac{h_1}{\sqrt{n}} Z_k \text{ and } \delta_k{}^{(2)} = \frac{h_2}{\sqrt{n}} \hat{Z}_k,$$

where $h_1 = \sqrt{n}\sigma_1$ and $h_2 = \sqrt{n}\sigma_2$. Z_k, $\hat{Z}_k$ are both i.i.d standard normal random variables. Then by a Taylor expansion we have

$$\begin{aligned} & ln(\theta e^{\delta_k{}^{(1)}} + (1-\theta)\delta_k{}^{(2)}) \\ = \;& ln(\theta e^{\frac{h_1}{\sqrt{n}} Z_k} + (1-\theta)e^{\frac{h_2}{\sqrt{n}} \hat{Z}_k}) \\ = \;& ln[\theta(1 + \frac{h_1}{\sqrt{n}} Z_k + \frac{h_1^2}{2}\frac{Z_k^2}{n} + o(\frac{1}{n})) + (1-\theta)(1 + \frac{h_2}{\sqrt{n}}\hat{Z}_k + \frac{h_2^2}{2}\frac{\hat{Z}_k{}^2}{n} + o(\frac{1}{n}))] \\ = \;& ln[1 + \theta(\frac{h_1}{\sqrt{n}} Z_k + \frac{h_1^2}{2}\frac{Z_k^2}{n} + o(\frac{1}{n})) + (1-\theta)(\frac{h_2}{\sqrt{n}}\hat{Z}_k + \frac{h_2^2}{2}\frac{\hat{Z}_k{}^2}{n} + o(\frac{1}{n}))]. \end{aligned}$$

When n$\rightarrow \infty$,

$$
\begin{aligned}
& ln[1+\theta(\frac{h_1}{\sqrt{n}}Z_k + \frac{h_1^2}{2}\frac{Z_k^2}{n} + o(\frac{1}{n})) + (1-\theta)(\frac{h_2}{\sqrt{n}}\hat{Z}_k + \frac{h_2^2}{2}\frac{\hat{Z}_k^{\,2}}{n} + o(\frac{1}{n}))] \\
= \ & \theta(\frac{h_1}{\sqrt{n}}Z_k + \frac{h_1^2}{2}\frac{Z_k^2}{n}) + (1-\theta)(\frac{h_2}{\sqrt{n}}\hat{Z}_k + \frac{h_2^2}{2}\frac{\hat{Z}_k^{\,2}}{n}) \\
& \quad -\frac{1}{2}(\theta^2 h_2^2 \frac{Z_k^2}{n} + 2\theta(1-\theta)\frac{h_1 h_2}{n} Z_k \hat{Z}_k + (1-\theta)^2 h_2^2 \frac{\hat{Z}_k^2}{n}) + o(\frac{1}{n}).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
& \sum_{j=1}^{[ny]} ln(\theta e^{\delta_j{}^{(1)}} + (1-\theta)e^{\delta_j{}^{(2)}}) \\
= \ & \sum_{j=1}^{[ny]} \theta(\frac{h_1}{\sqrt{n}}Z_k + \frac{h_1^2}{2}\frac{Z_k^2}{n}) + \sum_{j=1}^{[ny]}(1-\theta)(\frac{h_2}{\sqrt{n}}\hat{Z}_k + \frac{h_2^2}{2}\frac{\hat{Z}_k^{\,2}}{n}) \\
& \quad - \sum_{j=1}^{[ny]} -\frac{1}{2}(\theta^2 h_2^2 \frac{Z_k^2}{n} + 2\theta(1-\theta)\frac{h_1 h_2}{n} Z_k \hat{Z}_k + (1-\theta)^2 h_2^2 \frac{\hat{Z}_k^2}{n}) + o(1) \\
\to \ & \sqrt{(\theta h_1)^2 + ((1-\theta)h_2)^2} B_y + \theta(1-\theta)\frac{h_1^2 + h_2^2}{2} y.
\end{aligned}
$$

We obtain the following result.

Theorem 3.5: *By the construction and assumption of the Model 6 described in Section 3.3.6, as $n \to \infty$, then*

$$
P(A_n < 0) \approx P(\int_0^1 C_{[ny]} e^{-[\sqrt{(\theta h_1)^2 + ((1-\theta)h_2)^2} B_y + \theta(1-\theta)\frac{h_1^2+h_2^2}{2} y]} dy) > \frac{A_0}{n}),
$$

$$
\begin{aligned}
& \int_0^1 C_{[ny]} e^{-[\sqrt{(\theta h_1)^2 + ((1-\theta)h_2)^2} B_y + \theta(1-\theta)\frac{h_1^2+h_2^2}{2} y]} dy \\
\approx \ & \sum_{k=1}^{m} \left[C_{[\frac{n(k-1)}{m}]} e^{-[\sqrt{(\theta h_1)^2 + ((1-\theta)h_2)^2}\frac{S_{k-1}}{\sqrt{m}} + \theta(1-\theta)\frac{h_1^2+h_2^2}{2}\frac{k-1}{m}]} \right] \left(\frac{1}{2m}\right) \\
& \quad + \sum_{k=1}^{m} \left[C_{[\frac{nk}{m}]} e^{-[\sqrt{(\theta h_1)^2 + ((1-\theta)h_2)^2}\frac{S_k}{\sqrt{m}} + \theta(1-\theta)\frac{h_1^2+h_2^2}{2}\frac{k}{m}]} \right] \left(\frac{1}{2m}\right).
\end{aligned}
$$

By the same simulation approach, we set initial value $A_0 = 200$, the standard deviation of the first interest rate $\sigma_1 = 0.45$, the standard deviation of the second rate is $\sigma_1 = 0.1$, the claim size, C_k, is an exponential random variable with men 1, the term of policy $n = 60$ and Both M and m are set to 2000.

The Figure 3.12 shows the results of simulation. The left side of figure is the plot produced by the simulation of the approximation we introduced in section 3.5.1, and another plot is from the simulation of the risk process. Again, the two plots are quite similar that supports that the approximation works very well.

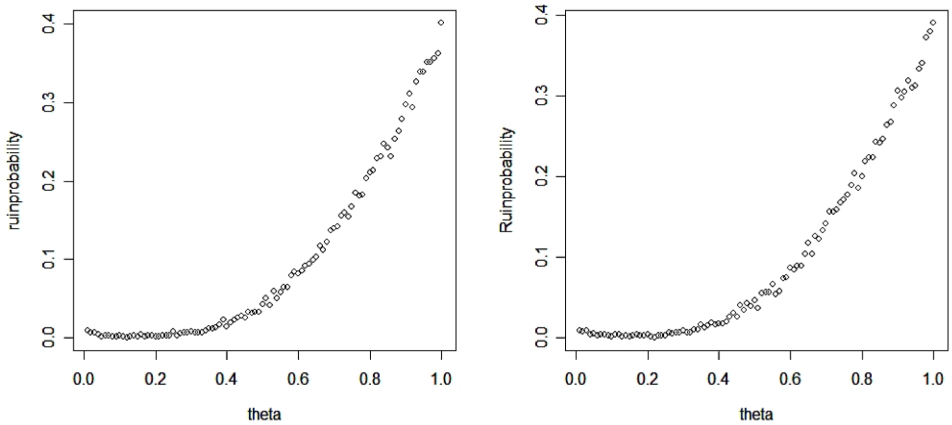

Figure 3.12: Plots of the Ruin probability against θ produced by two simulations ($\sigma_1 = 0.45$, $\sigma_2 = 0.1$), the plot on the left is produced by simulation of the approximation and the plot on the right is produced by simulation of the risk process

3.7 Remarks

In this chapter, we constructed the six stochastic risk models stated in Sections 3.2 and 3.3. with unknown parameters which are to be estimated from simulated data in Section 3.3. We simulated the several groups of data in different experiments and use to obtain the estimated parameter. As shown in Section 3.4.3, the experiments 1 to 3 show that we always obtain good and stable estimation of standard deviation of interest rate, σ, but unstable estimation of rate of the claim size, λ, even give negative values in some case. This found was partly motivated by Magdalinos (2007). The simulation results and the discussion support his theoretical discoveries.

We use the estimated parameter to approximate the ruin probability or obtain an upper bound for it and find the optimal constant fraction policies investment strategy with a given upper bound on the ruin probability. Again, the numerical threshold discovered in Chapter 2 was supported in all the examples. Numerical analysis with stochastic simulation for the estimation of the parameters is conducted in Section 3.4.

Finally, in Section 3.6 we approximate the ruin probability by the associated integrals of the Brownian motion. By using a Taylor expansion, Theorems 3.3 to 3.6 show the approximated ruin probability in Models 1, 4, 5 and 6 respectively. The simulations we applied support Theorems by compare the simulation of approximations and the simulation of risk processes.

CHAPTER 4

Ruin probability with Heavy-Tailed claim amounts

In this chapter, Sections 4.1 and 4.2 introduce the heavy-tailed distribution and corresponding definition. Two new realistic models with interest rate factors were suggested and treated in Sections 4.3 and 4.4. The derived results extend Biard *et al*. (2008) to models (discrete and continuous time) with interest rate.

4.1 Introduction

An insurance company wants to measure and manage risks and stay solvent with a high probability. On the other hand the company wants to make a high profitable portfolio to minimise the ruin probability. In the analysis of insurance processes, computing the optimal investment policy and capital requirement for claim often includes calculating the approximate ruin probabilities using an appropriate model.

We remind the reader of classic Anderson model (Anderson 1957). In this model, the risk process $U(u,\ t)$ is a continuous time surplus process and for $t \geq 0$ is defined as follows:

$$U(u,\ t) = u + pt - S(t),$$

where u is the insurer's initial surplus,

p is the insurer's rate of premium income per unit time,

$N(t)$ is the number of claims in the time interval [0, t] and has a renewal process,

$S(t)$ is the cumulated claim amount up to time t and $S(t) = \sum_{i=1}^{N(t)} X_i$,

$\{X_i\}$ is a sequence of independent and identically distributed (i.i.d), random variables representing the individual claim amounts.

Ruin occurs when U falls below 0, this may equate to insolvency. The probability of ruin with initial capital u is denoted by $\Psi(u, t)$

$$\Psi(u,\ t) = P[U(s) < 0 \ \text{ for some s },\ 0 < s < t].$$

Here we only consider the finite time ruin probabilities.
Many recent papers in insurance and finance, connected with heavy-tailed risks, have considered heavy-tailed distributions to model the risk such as individual claim amounts. Normally tail of distribution associated with large losses (Glasserman *et. al*.2002). Heavy trailed gives more risks and empirical data always show higher peaks and heavier tails than estimated by normal or log-normal distribution. For example, see Embrechts *et al*. (1997). In probability theory, Heavy-Tailed distributions (also known as power-law distributions) are probability distributions whose tails are not exponentially bounded: that is, they have heavier tails than the exponential distribution. In this chapter, we refer to a heavy-tailed distribution as having finite coefficient of (regular) variation as opposed to the exponential distribution which has infinite coefficient (Asmussen 1995). Motivated by Embrechts *et al*. (1997) and numerous papers (notably by Birad *et al*. 2008), in this chapter we study the ruin probability for our class of models but now with the heavy-tailed claim size distribution.

4.2 Definition

The distribution of a random variable X with cdf, F, is a heavy-tailed distribution (Asmussen 2003) if

$$\lim_{x\to\infty} e^{\lambda x} \Pr[X > x] = \infty \text{ , for all } \lambda > 0.$$

We rewrite it in terms of the tail distribution function

$$\lim_{x\to\infty} e^{\lambda x}\overline{F}(x) = \infty \quad \text{for all } \lambda > 0,$$

where

$$\overline{F}(x) = \Pr(X > x) = 1 - F(x).$$

We list three important sub classes of heavy-tailed distributions, the long-tailed distributions, the sub exponential distributions and the regular variation for the further study (Cai and Tang 2004):

- L (Long-tailed) : a distribution function F is long-tailed distribution if

$$\lim_{x\to\infty} \overline{F}(x+t) = \overline{F}(x) \text{ , for any } t > 0,$$

- S (sub exponential) : a distribution function F is sub exponential distribution if

$$\lim_{x\to\infty} \overline{F^{*n}}(x) = n\overline{F}(x) \text{ , for any } n \geq 2,$$

  where $\overline{F^{*n}}(x)$ refer to a common distribution function of a sum of n independent random variables $X_1, \ldots, X_n$. That is

$$\overline{F^{*n}}(x) = \Pr(X_1 + \cdots + X_n > x),$$

- R (regular variation) : A distribution function F belong to Regular variation for index number α (written $F \in R_{-\alpha}$) if

$$\lim_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = y^{-\alpha}$$

  for some $\alpha \geq 0$ and for any $y > 0$.

It is well known (Cai and Tang 2004) that these classes satisfy the following inclusions :

$$R \subset S \subset L.$$

Back to the risk process U(u, t), one way is to derive the asymptotic of ruin probability $\Psi(u,\ t)$ by analysing directly properties of the distribution functions that belong to $R_{-\alpha}$ of sums of independent random variables, when initial reserve is large enough (Lundberg 1934):

$$\Psi(u,\ t) \sim \frac{1}{E(T_1)} u^{-\alpha} \quad \text{as } u \to +\infty,$$

Here we define (T_k) $k \geq 1$ as the internal occurrence times and assume that they and claim amounts are mutually independent.

Note: Throughout this chapter f(x) $\sim$ h(x) means $\lim_{x\to\infty} \frac{f(x)}{h(x)} = 1$.

However in the real financial world, the mutual independence of claim amounts and occurrence times is not always reasonable. Normally, the claims amounts are not independent and they have strong positive dependence. Such as in car insurance. In addition, claim amounts have strong dependence in conditions such as economic crisis, natural calamity, war and so on. When the crisis happens, claim correlation rises. Furthermore the distribution or the parameter of the claim amounts could changes. (See Biard *et al*. 2008)

4.3 The Simple Model with Heavy-Tailed Claims

In this section we investigate the asymptotic of finite time ruin probability based upon the simple model we provided in Section 4.1. We consider the counting number of claims as a renewal process that is independent from the distribution of claim amounts

Firstly, we introduce a property for distributions named max-sum equivalent.

Distributions F_1 and F_2 are said to be max-sum-equivalent, written as $F_1 \sim_M F_2$, if

$$\overline{F_1 * F_2}(x) = P[X_1 + X_2 > x] \ \sim \ \overline{F_1}(x) + \overline{F_2}(x),$$

where * is defined as convolution and $\overline{F}(x)$ is defined as the tail distribution of F, $\overline{F}(x) = 1 - F(x)$.

For example, suppose there are two independent random variables X_1 and X_2. If they have distributions F_1 and F_2 respectively, then $F_1 \sim_M F_2$ is equivalent to

$$P[X_1 + X_2 \geq x] \ \sim \ P[\max(X_1, X_2) \geq x]$$

which means the maximum of the two random variables determines the tail probability of the sum of two independent random variables asymptotically.

This is a very important property for describing heavy-tailed distributions and sometimes also used in modelling extreme events. In addition, F is a sub exponential distribution if $F \sim_M F$. (Cai and Tang 2004)

Following the definition of regular variation and max-sum-equivalent, we state several lemmas for the later works (Bingham *et al*. 1989). Although the results are well known by now, we provide slightly modified proofs which make it easy for us to generalize.

Lemma 4.1: *Let random variables δ and v be independent, non-negative and their distribution are F_δ and F_v respectively regularly varying of index $-\gamma$ with $\gamma > 0$, then the distribution $F_{\delta+v}$ is regularly varying of index γ.*

Proof:

By definition of Regular Variation, we need to show:

$$\lim_{t\to\infty} \frac{\overline{F}_{\delta+v}(ta)}{\overline{F}_{\delta+v}(t)} = a^{-\gamma}, \quad \text{for some } \gamma \geq 0 \text{ and for any } y > 0.$$

Since F_δ and F_v belong to $R_{-\gamma}$ then max-sum-equivalent shows

$$\overline{F_\delta * F_v}(x) \ \sim \ \overline{F_\delta}(x) + \overline{F_v}(x).$$

This leads to

$$\begin{aligned} \frac{\overline{F}_{\delta+v}(ta)}{\overline{F}_{\delta+v}(t)} &\sim \frac{\overline{F_\delta}(ta) + \overline{F_v}(ta)}{\overline{F_\delta}(t) + \overline{F_v}(t)} \\ &= \frac{\overline{F_\delta}(t)\frac{\overline{F_\delta}(ta)}{\overline{F_\delta}(t)} + \overline{F_v}(t)\frac{\overline{F_v}(ta)}{\overline{F_v}(t)}}{\overline{F_\delta}(t) + \overline{F_v}(t)}. \end{aligned}$$

Then for large t,

$$\begin{aligned} \frac{\overline{F}_{\delta+v}(ta)}{\overline{F}_{\delta+v}(t)} &= \frac{\overline{F_\delta}(t)a^{-\gamma} + \overline{F_v}(t)a^{-\gamma}}{\overline{F_\delta}(t) + \overline{F_v}(t)} \\ &= a^{-\gamma}. \end{aligned}$$

Therefore we have proved of the Lemma. $\diamond$

Lemma 4.2 *Let $\{W_i\}_{0\leq i\leq n}$ be a sequence of i.i.d positive random variables with distribution F_W that is regularly varying index $-\gamma$ with $\gamma \geq 0$, then*

$$P[W_1 + \cdots + W_n > x] \sim P[W_1 >] + \cdots + P[W_n > x].$$

Proof:

We use induction to prove this lemma.

Firstly, when n=2, by the Lemma 4.1 and max-sum-equivalent, we obtain

$$P[W_1 + W_2 > x] \sim P[W_1 > x] + P[W_n > x].$$

Secondly, we assume the Lemma is true when $n = k$.

Finally, when $n = k + 1$

$$P[W_1 + \cdots + W_{k+1} > x] = P[(W_1 + \cdots + W_k) + W_{k+1} > x],$$

since $\{W_i\}_{0 \leq i \leq n}$ are i.i.d. random variables and $(W_1 + \cdots + W_k)$ and W_{k+1} are sub exponential. Hence

$$P[(W_1 + \cdots + W_k) + W_{k+1} > x] \sim P[W_1 + \cdots + W_k > x] + P[W_{k+1} > x].$$

With the assumption,

$$P[(W_1 + \cdots + W_k) + W_{k+1} > x] \sim P[W_1 > x] + \cdots + P[W_{k+1} > x],$$

which means the equation holds when $n = k + 1$. Therefore by induction, we have proved the lemma.$\diamond$

Lemma 4.3 *Let random variables δ and v be dependent, non-negative with distribution are F_δ and F_v respectively regularly varying of index $-\gamma$ with $\gamma > 0$. Then for any $p, q \geq 0$ with $p + q > 0$,*

$$\lim_{t \to \infty} \frac{\overline{F}_\delta(ta)p + \overline{F}_v(ta)q}{\overline{F}_\delta(t)p + \overline{F}_v(t)q} = a^{-\gamma}.$$

Proof:

For and $a > 0$

$$\begin{aligned}
& \lim_{t\to\infty} \frac{\overline{F}_\delta(ta)p + \overline{F}_v(ta)q}{\overline{F}_\delta(t)p + \overline{F}_v(t)q} \\
= & \lim_{t\to\infty} \frac{\frac{\overline{F}_\delta(ta)}{\overline{F}_\delta(t)} p\overline{F}_\delta(t) + \frac{\overline{F}_v(ta)}{\overline{F}_v(t)} q\overline{F}_v(t)}{\overline{F}_\delta(t)p + \overline{F}_v(t)q} \\
= & \frac{a^{-\gamma} p\overline{F}_\delta(t) + a^{-\gamma} q\overline{F}_v(t)}{\overline{F}_\delta(t)p + \overline{F}_v(t)q},
\end{aligned}$$

since F_δ and $F_v \in \Re_{-\gamma}$. Then

$$\lim_{t\to\infty} \frac{\overline{F}_\delta(ta)p + \overline{F}_v(ta)q}{\overline{F}_\delta(t)p + \overline{F}_v(t)q} = a^{-\gamma}.$$

This completes the proof of Lemma 4.3. $\diamond$

Moreover, in the mixtures case, p and q should be defined so that $p + q = 1$. Which means we can make $F_\delta p + F_v q$ as a mixture distribution. Therefore from lemma 4.3, we have

$$F_\delta p + F_v q \in \Re_{-\gamma}.$$

From Lemmas 4.2 and 4.3, we derive another lemma, which does not require the independence of random variables.

Lemma 4.4 *Let $\{W_i\}_{0\le i\le n}$ is a sequence of positive random variables with distribution F_{W_i} that is regularly varying index $-\gamma$ with $\gamma > 0$. Then for any real constants*

$$C_1 + C_2 + \cdots + C_n \neq 0 \quad C_i \ge 0,$$

we have

$$\lim_{t\to\infty} \frac{C_1 P(W_1 \ge ta) + \cdots + C_n P(W_n \ge ta)}{C_1 P(W_1 \ge t) + \cdots + C_n P(W_n \ge t)} = a^{-\gamma}.$$

Proof:

$$
\begin{aligned}
& \lim_{t\to\infty} \frac{C_1 P(W_1 \geq ta) + \cdots + C_n P(W_n \geq ta)}{C_1 P(W_1 \geq a) + \cdots + C_n P(W_n \geq a)} \\
= & \lim_{t\to\infty} \frac{C_1 \frac{P(W_1 \geq ta)}{P(W_1 \geq t)} P(W_1 \geq t) + \cdots + C_n \frac{P(W_n \geq ta)}{P(W_n \geq t)} P(W_n \geq t)}{C_1 P(W_1 \geq t) + \cdots + C_n P(W_n \geq t)} \\
= & \frac{C_1 a^{-\gamma} * P(W_1 \geq t) + \cdots + C_n a^{-\gamma} P(W_n \geq t)}{C_1 P(W_1 \geq t) + \cdots + C_n P(W_n \geq t)},
\end{aligned}
$$

Since F_{W_i} is regularly varying index $-\gamma$ with $\gamma > 0$. Hence

$$
\lim_{t\to\infty} \frac{C_1 P(W_1 \geq ta) + \cdots + C_n P(W_n \geq ta)}{C_1 P(W_1 \geq t) + \cdots + C_n P(W_n \geq t)} = a^{-\gamma}.
$$

This completes the proof. $\diamond$

If we assume

$$
C_1 + C_2 + \cdots + C_n = 1,
$$

then there is a mixture distribution F_W written as

$$
F_W(x) = \sum_{i=1}^{n} C_i F_{W_i}(x),
$$

we use Lemma 4.4 to get $F_W \in \Re_{-\gamma}$.

Corollary 4.5 *Let the random variable W be non-negative and its distribution be regularly varying of index $-\gamma$ with $\gamma > 0$, then for any $b \geq 0$, $Wb \in \Re_{-\gamma}$ with index $-\gamma$.*

By combining the Lemmas 4.1 to 4.5, we derive

Lemma 4.6 *Let $\{W_i\}_{0 \leq i \leq n}$ be a sequence of positive, independent random variables of distribution F_{W_i} that is regularly varying index $-\gamma$ with $\gamma > 0$, then for $b_1, \ldots, b_n \geq 0$, the distribution of $\sum_{i=1}^{n} W_i b_i$ is regularly varying with index $-\gamma$.*

For the purpose of future work, we review the proposition and its proof established by Biard, Lefevre and Loisel (2008). They consider a compound renewal process risk model with assumption of regular variation. They have established the asymptotic ruin probability. More exactly, suppose that premiums (income) arrive at a constant rate b

and claims occur according to some point process $N(t)_{t\geq 0}$. In addition, independently of this arrival claim process, the successive claim amounts $(X_n)_{n\geq 1}$ are defined as

$$X_n = I_N M_0 + (1 - I_n) M_n, \quad n \geq 1,$$

where $(M_n)_{n\geq 0}$ is a sequence of i.i.d. positive random variables with distribution function F_M where

$$F_M \in \Re_{-\gamma}, \ \ \gamma \geq 0,$$

and $(I_n)_{n\geq 1}$ is a sequence of i.i.d. Bernoulli random variables with

$$P[I_n = 1] = p \in [0, 1].$$

The two sequences M_n and I_n are assumed to be mutually independent. Let u be the initial reserve and let $\Psi_p(u, t)$ be the ruin probability over any fixed finite-time horizon (0,t).

Proposition 4.7: (Biard,Lefevre and Loisel (2008)): *When u is large enough,*

$$\begin{aligned}
\Psi_p(u,t) &\sim \{(1-p)E[N(t)] + E[Z_p(t)]^{\gamma}\}\overline{F_M}(u+ct), \\
&\sim \{(1-p)E[N(t)] + E[Z_p(t)]^{\gamma}\}\overline{F_M}(u).
\end{aligned}$$

where $Z_{(p)t}$ is a mixed binomial random variable $Bin[N(t), p]$.

We will present the modified version of their proof in order to use it in our case.

Proof:

Define the aggregate claim amount as

$$S_p(t) = \sum_{n=1}^{N(t)} X_n.$$

There are two steps to the proof. The first is deriving $P[S_P(t) \geq x]$ for large x, and second is finding an approximation of $\Psi_p(u, t)$ by $P[S_P(t) \geq u]$.

Step 1: The convolution closure (Cai and Tang 2004), states that

$$F_1 * F_2 \in \Re_{-\gamma},$$

when

$$F_1 \text{ and } F_2 \text{ belong to } \Re_{\gamma}, \;\; \gamma \geq 0.$$

The max-sum-equivalence shows that

$$\overline{F_1 * F_2}(x) \sim \overline{F_1} + \overline{F_2}.$$

We can use these properties and the Lemma 4.6 we developed to obtain that for any $k \geq 1$ and any pairwise distinct $n_1, \ldots, n_{k-j} \geq 1$ with $0 \leq j \leq k-1$,

$$\begin{aligned}
P(M_{n_1} + \cdots + M_{n_{k-j}} + jM_0 \geq x) &\sim (k-j)\overline{F_M}(x) + \overline{F_M}(\frac{x}{j}) \\
&\sim \left(k - j + \frac{\overline{F_M}(\frac{x}{j})}{\overline{F_M}(x)}\right)\overline{F_M}(x) \\
&\sim (k - j + j^{\gamma})\overline{F_M}(x).
\end{aligned}$$

Therefore, for any $k \geq 1$ and $0 \leq j \leq k-1$ we have

$$P\left(S_p(t) \geq x | N(t) = k, \sum_{i=1}^{k} I_i = j\right) \sim (k - j + j^{\gamma})\overline{F_M}(x).$$

For the case $j = k$ and $k \geq 1$ we have

$$\begin{aligned}
P\left(S_p(t) \geq x | N(t) = k, \sum_{i=1}^{k} I_i = k\right) &= P(kM_0 \geq x) \\
&= \frac{\overline{F_M}(\frac{x}{k})}{\overline{F_M}(x)}\overline{F_M}(x) \\
&\sim k^{\gamma}\overline{F_M}(x),
\end{aligned}$$

and for $k = j = 0$,

$$P[S_p(t) \geq x | N(t) = 0] = 0.$$

Combining three cases above,

$$P[S_p(t) \geq x] \sim \left\{ \sum_{k=1}^{\infty} P[N(t)=k] \sum_{j=0}^{k} \binom{k}{j} p^j (1-p)^{k-j} (k-j+j^\gamma) \right\} \overline{F_M}(x). \qquad (4.3.1)$$

Since where $Z_{(p)t}$ is a mixed binomial random variable $Bin[N(t), p]$, we define

$$P(Z_k = j) = \binom{k}{j} p^j (1-p)^{k-j},$$

this leads the right hand side of (4.3.1) to be

$$\begin{aligned} & \left\{ \sum_{k=1}^{\infty} P[N(t)=k] \sum_{j=0}^{k} \binom{k}{j} p^j (1-p)^{k-j} (k-j+j^\gamma) \right\} \overline{F_M}(x) \\ = & \left\{ \sum_{k=1}^{\infty} P[N(t)=k] \sum_{j=0}^{k} P(Z_k=j)(k-j+j^\gamma) \right\} \overline{F_M}(x), \end{aligned}$$

where

$$\sum_{j=0}^{k} P(Z_k=j) j^\gamma = E[Z_k^\gamma],$$

$$\sum_{j=0}^{k} P(Z_k=j) j = E[Z_k] = kp,$$

and

$$\sum_{j=0}^{k} P(Z_k=j) k = k.$$

Therefore, equation (4.3.1) can be rewritten as

$$\begin{aligned} & \left\{ \sum_{k=1}^{\infty} P[N(t)=k] \sum_{j=0}^{k} \binom{k}{j} p^j (1-p)^{k-j} (k-j+j^\gamma) \right\} \times \overline{F_M}(x) \\ = & \left\{ \sum_{k=1}^{\infty} P[N(t)=k][k(1-p) + E[Z_k^\gamma]] \right\} \times \overline{F_M}(x) \\ = & \left\{ \sum_{k=1}^{\infty} P[N(t)=k] k[(1-p)] + \sum_{k=1}^{\infty} P[N(t)=k] E[Z_k^\gamma] \right\} \times \overline{F_M}(x) \\ = & \left\{ (1-p)E[N(t)] + E[Z_p(t)]^\gamma \right\} \times \overline{F_M}(x), \end{aligned}$$

since

$$\sum_{k=1}^{\infty} P[N(t) = k]k = E[N(t)].$$

Hence

$$P[S_p(t) \geq x] \sim \left\{(1-p)E[N(t)] + E[Z_p(t)]^{\gamma}\right\} \times \overline{F_M}(x). \tag{4.3.2}$$

Step 2: As $u \to \infty$,

$$\begin{aligned} 0 &\leq \frac{\Psi_p(u,t) - P[S_P(t) \geq u + ct]}{\Psi_p(u,t)} \\ &\leq \frac{P[S_P(t) \geq u) - P[S_P(t) \geq u + ct]}{P[S_P(t) \geq u + ct} \\ &\sim \frac{\overline{F_M}(u)}{\overline{F_M}(u+ct)} - 1. \end{aligned}$$

For any $x \in R$, when $u \to \infty$,

$$\frac{\overline{F_M}(u)}{\overline{F_M}(u+ct)} \sim 1,$$

which is proved in Embrechts *et al*. (1997). So that

$$\Psi_p(u,t) \sim P[S_P(t) \geq u + ct]. \tag{4.3.3}$$

Finally, the proposition is proved by combining equations (4.3.2) and (4.3.3). $\diamond$

In particular, let $p = 1$ which means that

$$X_n = M_n.$$

and hence

$$\Psi_p(u,t) \sim E[N(t)]\overline{F_M}(u+ct).$$

If $p = 0$ then

$$X_n = W_0 \text{ for all n,}$$

hence

$$\Psi_p(u,t) \sim E[N(t)]^{\gamma}\overline{F_M}(u+ct).$$

We can extend Proposition 4.7 to different cases by considering the different types of claim amount $(M_n)_{n\geq 0}$ or we can add new factors to the model, for example, interest rate.

We again use all the definitions and assumptions as in Proposition 4.7 except the following. We let $(M_n)_{n\geq 1}$ a sequence of i.i.d. positive random variables with distribution function F_M with

$$F_M \in \Re_{-\gamma}, \;\; \gamma \geq 0.$$

M_0 is a positive random variables of distribution function F_{M_0} with

$$F_{M_0} \in \Re_{-\gamma}, \;\; \gamma \geq 0.$$

Here $(M_n)_{n\geq 1}$ and M_0 are independent. By this assumption there are two types of claim amounts, one has a a small amount of value, and another has a bigger amount of value.

Property: Based upon above definitions, let $\Psi_p(u,t)$ be the ruin probability over (0, t). As $u \to \infty$ we have the following result

$$\Psi_p(u,t) \sim \{(1-p)E[N(t)]\}\overline{F_M}(u+ct) + \{E[Z_p(t)]^\gamma\}\overline{F_{M_0}}(u). \tag{4.3.4}$$

Proof: We followed similar proof process as in Proposition 4.7. In first step, for any $k \geq 1$ and any pairwise distinct $n_1, \ldots, n_{k-j} \geq 1$ with $0 \leq j \leq k-1$,

$$\begin{aligned}
P(M_{n_1} + \cdots M_{n_{k-j}} + jM_0 \geq x) &\sim (k-j)\overline{F_M}(x) + \overline{F_{M_0}}(\frac{x}{j}) \\
&\sim (k-j)\overline{F_M}(x) + \frac{\overline{F_{M_0}}(\frac{x}{j})}{\overline{F_{M_0}}(x)} \\
&\sim (k-j)\overline{F_M}(x) + j^\gamma \overline{F_{M_0}}(x).
\end{aligned}$$

For the case $j = k$ and $k \geq 1$, we have

$$P\left(S_p(t) \geq x | N(t) = k, \sum_{i=1}^{k} I_i = k\right) \sim k^\gamma \overline{F_{M_0}}(x),$$

hence

$$P[S_p(t) \geq x] \sim \{(1-p)E[N(t)]\}\overline{F_M}(x) + \{E[Z_p(t)]^\gamma\}\overline{F_{M_0}}(x).$$

Then we have established equation (4.3.4) by usinga similar idea as in the step 2 of the proof process of Proposition 4.7. $\diamond$

In the Chapter 2 and 3, we constructed models which included the important factor of variable interest rate. Here we now model the case of Heavy-Tailed claim amounts together with the interest rate factor.

We begin with the following simple model. The risk process $U(u,\ t)$ is a surplus process and for $t \geq 0$ is defined as follows:

$$U(u,\ t) = u\Gamma_t - S(t),$$

where

- u is the insurer's initial surplus;
- $N(t)$ is the number of claims in the time interval [0, t], and has a renewal process;
- $S(t)$ is the accumulated claim amount up to time t and $S(t) = \sum_{i=1}^{N(t)} X_i\Gamma_i$;
- Γ_i is aggregate interest rate from time i to time t. $\Gamma_i = exp(\sum_{i+1}^{t} \delta_i)$, where δ_i is the continuous interest at time i;
- X_i is the claims amounts;
- Define the $(T_k)_{k\geq 1}$ as the inter-occurrence times of the claims and assumed to be mutually independent with claim amounts.

The probability of ruin with initial capital u denoted by $\Psi(u,t)$

$$\Psi(u,\ t) = P[U(s) < 0 \ \text{ for some s },\ 0 < s < t].$$

Notice that if $\Gamma_i = \Gamma$, and under the same definitions as in Proposition 4.7. Then

$$S_p(t) = \Gamma \sum_{i=1}^{N(t)} X_i$$

for $0 \leq j \leq k-1$,

$$P[S_p(t) \geq x] \sim \left\{(1-p)E[N(t)] + E[Z_p(t)]^{\gamma}\right\} \times \overline{F_M}(x) \times \Gamma^{\gamma},$$

which yield to

$$\Psi_p(u,\ t) \sim P[S_p(t) \geq u\Gamma],$$

as $u \to \infty$, thus

$$\Psi_p(u,\ t) \sim \left\{(1-p)E[N(t)] + E[Z_p(t)]^{\gamma}\right\} \times \overline{F_M}(u).$$

As the approximation shows whatever value of Γ we take, as $u \to \infty$, the constant aggregate interest factor does not affect the ruin probability. This result is as expected, since in the real economic world, this assumption is not strong.

Let $(T_k)_{k\geq 1}$ be the inter-occurrence times between successive claims and in the point process of claims, $N(t)$. And let

$$W_n = \sum_{i=1}^{n} T_i, \qquad \text{n} \geq 1,$$

We refer to W_n as the waiting time until the nth event or the arrival time of the nth event.

Important assumption.

We consider the compound interest rate as the random value and changing over time that is given by

$$\Gamma_j = \Gamma(W_j),$$

where $N(t)_{t\geq 0}$, $X(t)_{t\geq 0}$ and $\Gamma(t)_{t\geq 0}$ are mutually independent. Then we consider the similar definition of claim amount as in Proposition 4.7 and use similar proof to find asymptotic on $\Psi_p(u,\ t)$ as $u \to \infty$.

Theorem 4.8: *Let claim amounts $(M_i)_{i\geq 0}$ have the distribution F_M with $F \in \Re_{-\gamma}$, $\gamma \geq 0$. Consider the aggregate interest rate as the random variable at time j and changing over time that is given by*

$$\Gamma_j = \Gamma(\sum_{i=1}^{j} T_i),$$

In addition, let $N(t)_{t\geq 0}$, $X(t)_{t\geq 0}$ and $\Gamma(t)_{t\geq 0}$ be mutually independent. Then, for u large enough,

$$\Psi_p(u,t) \sim \{(1-p)E[\sum_{i=1}^{N(t)} \Gamma_{n_i}^{\gamma}] + E[(\sum_{h=k-j+1}^{N(t)} \Gamma_{n_h})^{\gamma}]\} \times \overline{F_M}(u). \tag{4.3.5}$$

Example. We can apply this approximation to

$$\Gamma_j = 1, \qquad 1 \leq j \leq t.$$

Then we have

$$\Psi_p(u,\ t) \sim \left\{(1-p)E[N(t)] + E[Z_p(t)]^{\gamma}\right\} \times \overline{F_M}(u), \tag{4.3.6}$$

same as the result of Proposition 4.7.

Proof of Theorem 4.8:

Firstly, for any $k \geq 1$ and $0 \leq j \leq k-1$,

$$P(\Gamma_{n_1} M_{n_1} + \cdots + \Gamma_{n_{k-j}} M_{n_{k-j}} + \Gamma_{n_{k-j+1}} M_0 + \cdots + +\Gamma_{n_k} M_0)$$

$$\sim E[\sum_{i=1}^{k-j} \Gamma_{n_i}^{\gamma} + (\sum_{h=k-j+1}^{k} \Gamma_{n_h})^{\gamma}]\overline{F_M}(x). \tag{4.3.7}$$

Therefore for any $k \geq 1$ and $0 \leq j \leq k-1$,

$$P\left(S_p(t) \geq x | N(t) = k,\ \sum_{i=1}^{k} I_i = j\right)$$

$$\sim E[\sum_{i=1}^{k-j} \Gamma_{n_i}^{\gamma} + (\sum_{h=k-j+1}^{k} \Gamma_{n_h})^{\gamma}]\overline{F_M}(x). \tag{4.3.8}$$

For $k = j = 0$ we have

$$P\left(S_p(t) \geq x | N(t)\right) = 0. \tag{4.3.9}$$

For $k \geq 1$ and $j = k$, we have

$$\begin{aligned} P\left(S_p(t) \geq x | N(t) = k, \sum_{i=1}^{k} I_i = k\right) &= P(\sum_{i=1}^{k} \Gamma_i M_0 \geq x) \\ &\sim E(\sum_{h=1}^{k} \Gamma_{n_h})^{\gamma} \overline{F_M}(x). \end{aligned} \tag{4.3.10}$$

By combining (4.3.8) to (4.3.10), and since $N(t)_{t\geq 0}$, $X(t)_{t\geq 0}$, $I(t)_{t\geq 0}$ and $\Gamma(t)_{t\geq 0}$ are mutually independent, we obtain
$P[S_p(t) \geq x]$

$$\begin{aligned} \sim & \left\{\sum_{k=1}^{\infty} P[N(t) = k] \sum_{j=0}^{k} \binom{k}{j} p^j (1-p)^{k-j} E[\sum_{i=1}^{k-j} \Gamma_{n_i}^{\gamma} + (\sum_{h=k-j+1}^{k} \Gamma_{n_h})^{\gamma}]\right\} \\ & \times \overline{F_M}(x). \end{aligned} \tag{4.3.11}$$

Now, let us divide $[0\ ,\ t]$ into the m intervals such as

$$[0\ \frac{t}{m}], \quad (\frac{t}{m}\ ,\ \frac{2t}{m}],\ \cdots,$$

Since probability $P(N(t) = 0) = 1$, overlaps over 1 point is not important. Then equation (4.3.8) can be rewritten as

$$\begin{aligned} & P[S_p(t) \geq x] \\ \sim \ & \{\sum_{k=1}^{\infty} \sum_{j=0}^{k} \binom{k}{j} p^j (1-p)^{k-j} \\ & \times \sum_{n_1 \neq \cdots \neq n_k} P[N(t) = k, \text{ k points occur in intervals } [\frac{tn_i}{m}, \frac{tn_{i+1}}{m}]\ \ i = 1, \cdots k] \\ & \qquad \times E[\sum_{i=1}^{k-j} \Gamma_{n_i}^{\gamma} + (\sum_{h=k-j+1}^{k} \Gamma_{n_h})^{\gamma}] \times \overline{F_M}(x). \end{aligned} \tag{4.3.12}$$

Let $(U_i)_{1\le i\le k}$ be a sequence of uniform variables U[0, T_i], then

$$
\begin{aligned}
& P[N(t)=k,\ \text{k points occur in intervals } [\frac{tn_i}{m},\ \frac{tn_{i+1}}{m}]\ \ i=1,\cdots k] \\
= \ & P[N(t)=k]\times P[\bigcap_{i=1}^{k} U_i \in [\frac{tn_i}{m},\ \frac{tn_{i+1}}{m}]\,] \\
= \ & P[N(t)=k]\times \prod_{i=1}^{k} P[U_i \in [\frac{tn_i}{m},\ \frac{tn_{i+1}}{m}]\,] \\
= \ & P[N(t)=k]\prod_{i=1}^{k}\frac{1}{m} \\
= \ & P[N(t)=k]m^{-k}, \qquad (4.3.13)
\end{aligned}
$$

where t is the time of whole process, n_i represent how many intervals that the i points occur in and since for $(U_i)_{1\le i\le k}$

$$P(U_i \in [a,b]) = \frac{b-a}{t},$$

and for $n_1 = 0$

$$\sum_{0\le n_1\neq\cdots\neq n_k\le m-1} = \binom{m}{k}k!. \qquad (4.3.14)$$

Using equations (4.3.13) and (4.3.14), equation (4.3.12) can be rewritten as

$$
\begin{aligned}
& P[S_p(t)\ge x] \\
\sim \ & \{\sum_{k=1}^{\infty} P[N(t)=k]\sum_{j=0}^{k}\binom{k}{j}p^j(1-p)^{k-j}\times m^{-k}\binom{m}{k}k! \\
& \qquad \times E[\sum_{i=1}^{k-j}\Gamma_{n_i}^{\gamma} + (\sum_{h=k-j+1}^{k}\Gamma_{n_h})^{\gamma}]\} \\
& \times\overline{F_M}(x), \qquad (4.3.15)
\end{aligned}
$$

Notice that choosing time intervals does not effect results, because probability of a jump at a fixed a point is 0.

when $m \to \infty$. We have similar condition as in Proposition 4.7, and

$$\begin{aligned}
\lim_{m\to\infty} k! m^{-k}\binom{m}{k} &= \frac{m!}{(m-k)!m^k} \\
&= \frac{m-k+1}{m} \times \frac{m-k+2}{m} \times \cdots \times \frac{m}{m} \\
&= (1-\frac{k+1}{m}) \times (1-\frac{k+2}{m}) \times \cdots \times 1 \\
&= 1,
\end{aligned}$$

which leads equation (4.3.15) to be written

$$\begin{aligned}
P[S_p(t) \geq x] \sim\ & \{\sum_{k=1}^{\infty} P[N(t)=k] \sum_{j=0}^{k} \binom{k}{j} p^j (1-p)^{k-j} \\
& \times E[\sum_{i=1}^{k-j} \Gamma_{n_i}^{\gamma} + (\sum_{h=k-j+1}^{k} \Gamma_{n_h})^{\gamma}]\} \\
& \times \overline{F_M}(x). \qquad (4.3.16)
\end{aligned}$$

Further, $Z_{(p)t}$ is a mixed binomial random variable $Bin[N(t), p]$ same as in Proposition 4.7, hence

$$P[S_p(t) \geq x] \sim \{(1-p)E[\sum_{i=1}^{N(t)} \Gamma_{n_i}^{\gamma}] + E[\sum_{h=k-j+1}^{N_t} \Gamma_{n_h})^{\gamma}]\} \times \overline{F_M}(x). \qquad (4.3.17)$$

As $u \to \infty$,

$$\Psi_p(u,t) \sim P[S_P(t) \geq u]. \qquad (4.3.18)$$

Followed by (4.3.17) and (4.3.18), we derive the formula

$$\Psi_p(u,t) \sim \{(1-p)E[\sum_{i=1}^{N(t)} \Gamma_{n_i}^{\gamma}] + E[\sum_{h=k-j+1}^{N_t} \Gamma_{n_h})^{\gamma}]\} \times \overline{F_M}(u).$$

The proof of Theorem 4.8 is complete. $\diamond$ Exact asymptotic for finite time ruin probability is traditionally on possible claims, on the other hand it is impossible in general. Our results gives a particular example imply the results by Biard *et al*. (2008) who give illustration and numerical examples.

4.4 Sum of Two Compound Poisson Processes

4.4.1 Construction of new model

In the last section, we use a compound renewal process to model the accumulated claim amount up to a finite time, and we obtained some asymptotics for heavy-tailed claim amounts under suitable assumptions. The homogeneous Poisson process (HPP) or called by classical Poisson process (Ross 2000) is the most common and best known claim arrival point process with stationary and independent increments. Poisson law is used to model the number of claims in a given time interval. Poisson process is usually appropriate in connection with life insurance modelling. For example, the number of deaths in the hospital, car accidents or birth defects and genetic mutations are often modelled by Poisson process.

We define the accumulated claim amount until time t as follows:

$$S(t) = \sum_{j=1}^{N^{'}(t)} M_j\Gamma_j + \sum_{j=1}^{N^{''}(t)} M_0\Gamma_j,$$

where the claims occur at the point of the Poisson process, N(t), with parameter λ. With each claim occurrence point, there are associated

probability p with claim size $(M_n)_{n\geq 0}$

probability q with claim size M_0,

where p+q=1. The number of claims $(M_n)_{n\geq 0}$ until time t is modelled by Poisson process $N^{'}(t)$ with parameter λq and the number of claims M_0 is modelled by Poisson Process $N^{''}(t)$ with parameter λp.

The assumptions of the model are

- the i.i.d positive inter-occurrence times of claims occur. $(T_k)_{k\geq 1}$, has exponential distribution F_T assumed to be mutually independent with claim amounts, $W_n = \sum_{i=1}^{n} T_i$ is defined as the arrival time of the nth amount;
- the Laplace transform of T_1 exists over subset of R;

- The $\Gamma_j = \Gamma(W_j)$ is aggregate interest rate time of jth claim to time t;
- For claim amounts $(M_i)_{i\geq 0}$ that have the distribution F_M with $F \in \Re_{-\gamma}$, $\gamma \geq 0$;
- $N^{'}(t), N^{''}(t)$, and $X(t)_{t\geq 0}$ are mutually independent.

Obviously in this model, we can split the claim process up to two components

4.4.2 Model Expression

We again use the similar process to find the approximate ruin probability under the Poisson assumption, i.e., we calculate $P[S(t) \geq x]$ for large x. Then

$$\begin{aligned} P[S(t) \geq x] &= P[(\sum_{j=1}^{N^{'}(t)} M_j\Gamma_j + \sum_{j=1}^{N^{''}(t)} M_0\Gamma_j) \geq x] \\ &= P[\int_0^t M_s\Gamma(W_s)dN^{'}(s) \geq x] + P[\int_0^t M_0\Gamma(W_s)dN^{''}(s) \geq x]. \end{aligned}$$

If we consider the first term we obtain

$$\begin{aligned} P[\int_0^t M_s\Gamma(s)dN^{'}(s) \geq x] &\sim E[\sum_{i=1}^{N^{'}(t)} \Gamma(W_i)^{\gamma}] \times \overline{F}_M(x) \\ &= E[\int_0^t \Gamma(W_s)dN^{'}(s)] \times \overline{F}_M(x). \end{aligned}$$

From Fubini's theorem (Kudryavtsev 2001), there is

$$E[dN^{'}(s)] = \lambda^{'}ds,$$

hence

$$P[\int_0^t M_s\Gamma(s)dN^{'}(s) \geq x] \sim \lambda^{'} \int_0^t \Gamma(s)^{\gamma}ds \times \overline{F}_M(x).$$

The second part is:

$$P[\int_0^t M_0\Gamma(W_s)dN^{''}(s) \geq x] \sim E[K(t)^{\gamma}] \times \overline{F}_M(x),$$

where

$$K(t) = \int_0^t \Gamma(W_s)dN^{''}(s).$$

Combining the two parts, we have

$$P[S(t) \geq x] \sim \{\lambda' \int_0^t \Gamma(s)^\gamma ds + E[K(t)^\gamma\} \times \overline{F}_M(x). \tag{4.4.1}$$

Then followed by similar calculation as in Theorem 4.8, we have

Theorem 4.9: *Let claim amounts $(M_i)_{i\geq 0}$ have the distribution F_M with $F \in \Re_{-\gamma}$, $\gamma \geq 0$. Let the aggregate rate be defined by*

$$\Gamma_j = \Gamma(\sum_{i=1}^{j} T_i),$$

The number of claims $(M_n)_{n\geq 0}$ until time t is modelled by Poisson process $N^{'}(t)$ with parameter λq and the number of claims M_0 is modelled by Poisson Process $N^{''}(t)$ with parameter λp. In addition $N(t)_{t\geq 0}$, $X(t)_{t\geq 0}$ and $\Gamma(t)_{t\geq 0}$ are mutually independent. Then, for u large enough

$$\Psi_p(u,t) \sim \left\{\lambda^{'} E[\int_0^t \Gamma(W_s)^\gamma ds] + E[K(t)^\gamma]\right\} \times \overline{F_M}(u). \tag{4.4.2}$$

Example. As before, for $\Gamma(i) = 1$ up to time t, this leads (4.4.2) to

$$P[S(t) \geq x] \sim \{\lambda^{'} t + E[\sum_{0}^{N^{''}(t)}]\} \times \overline{F}_M(x),$$

which is same as

$$P[S_p(t) \geq x] \sim \left\{(1-p)E[N(t)] + E[Z_p(t)]^\gamma\right\} \times \overline{F_M}(x),$$

when we assume $N(t)$ is Poisson process with parameter λ.

4.4.3 Moments of MGF

For purpose of future work, we use aggregate discounted value of the claim at time 0 over the time interval $[0,\ t]$. We let

$$\Gamma(W_j) = e^{-\delta W_j},$$

where δ is the stochastic interest rate factor and considered as constant until time t, then the present value of $S(t)$, $\hat{S}(t)$ is

$$\hat{S}(t) = \sum_{j=1}^{N^{'}(t)} M_j e^{-\delta W_j} + \sum_{j=1}^{N^{''}(t)} M_0 e^{-\delta W_j}.$$

Then

$$P[\hat{S}(t) \geq x] \sim \{\lambda^{'} \int_0^t e^{-\delta s \gamma} ds + E[\hat{K}(t)^{\gamma}]\} \times \overline{F}_M(x), \tag{4.4.3}$$

where

$$\hat{K}(t) = \int_0^t e^{-\delta W_j} dN^{''}(s).$$

We can derive the expression of $\int_0^t e^{-\delta s \gamma} ds$, but it is not possible to obtain the distribution of $\hat{K}(t)^{\gamma}$, the accumulated aggregate claims. We could assume the γ as integer value to make approximation computable, then we could find the value of $E[\hat{K}(t)^{\gamma}]$ by calculated moment generating function of $\hat{K}(t)$.

We note that

$$\begin{aligned} E[\hat{K}(t)] &= E[\sum_{j=1}^{N^{''}(t)} e^{-\delta W_j}] \\ &= \int_0^t e^{-\delta s}\, dF_T(s) + \int_0^t e^{-\delta s} E[\hat{K}(t-s)]\, dF_T(s) \\ &= \int_0^t e^{-\delta s}\, dE[N^{''}(s)] = \lambda p \int_0^t e^{-\delta s}\, ds. \end{aligned}$$

Hence we have

$$P[\hat{S}(t) \geq x] \sim \int_0^t e^{-\delta s}\, ds \times \overline{F}_M(x) = \frac{1 - e^{-t\delta}}{\delta} \times \overline{F}_M(x).$$

We note that

$$
\begin{aligned}
E[\hat{K}(t)^2] &= E\{E[(\sum_{j=1}^{N''(t)} e^{-\delta W_j})^2 | N''(t)]\} \\
&= E\{E[\sum_{j=1}^{N''(t)} e^{-2\delta W_j}] + \sum_{j=1}^{N''(t)} \sum_{j=1, i\neq j}^{N''(t)} e^{-\delta(W_j+W_i)} | N''(t)]\} \\
&= E[\sum_{j=1}^{N''(t)} e^{-2\delta W_j}] + \sum_{j=1}^{N''(t)} \sum_{j=1, i\neq j}^{N''(t)} e^{-\delta(W_j+W_i)},
\end{aligned}
$$

which can be derived by renewal argument.

Lemma 4.10 *For any $t \geq 0$,*

$$
\begin{aligned}
E[\hat{K}(t)^2] &= \int_0^t e^{-2\delta s}\, dE[N''(s)] + \int_0^t \int_0^{t-s} e^{-\delta(2s+v)}\ dE[N''(v)]dE[N''(s)] \\
&= \lambda p \int_0^t e^{-2\delta s}\, ds + 2(\lambda p)^2 \int_0^t \int_0^{t-s} e^{-\delta(2s+v)}\, dv\, ds \\
&= \lambda p \frac{1 - e^{-2t\delta}}{2\delta} + (\lambda p)^2 (\frac{1 - e^{-t\delta}}{\delta})^2.
\end{aligned}
$$

Hence equation (4.4.3) leads to

$$
P[\hat{S}(t) \geq x] \sim [\lambda \frac{1 - e^{-2t\delta}}{2\delta} + (\lambda p)^2 (\frac{1 - e^{-t\delta}}{\delta})^2] \times \overline{F}_M(x).
$$

We have calculated the first two moments of $\hat{K}(t)^\gamma$. For $\gamma > 2$, we then derive the moment generating function of $\hat{K}(t)$, $M_{\hat{K}(t)}(s)$.

Below we use the Cauchy principal integral (Frederick 2009) with singularity point x=0, defined by

$$
\int_{-\infty}^{\infty} f(x)dx = \lim_{a \downarrow 0} \Big(\int_{-\infty}^{-a} f(x)dx + \int_a^{\infty} f(x)dx \Big),
$$

when $\int_0^a f(x)dx = 0$.

Theorem 4.11: *Following the definition of $\hat{K}(t)^{\gamma}$, then for any $s \in \mathbb{R}$, and $\gamma \in \mathbb{Z}$, the MGF of $\hat{K}(t)^{\gamma}$ is given by*

$$M_{\hat{K}(t)}(s) = exp(-\lambda p \frac{Ei(1,-s) - Ei(1,-\mathrm{s}\mathrm{e}^{-\delta\, t}) + \delta\, t}{\delta}),$$

where Ei is Cauchy principal value exponential integral function and defined as

$$Ei(x) = -\int_{-x}^{\infty} \frac{e^{-t}}{t}\, dt,$$

for real, non-zero value of x.

Proof:

Jang (2004) found a suitable martingale to derive the Laplace transform of the distribution of accumulated aggregate claims at time t. Such a martingale is

$$f(\hat{K}, t) = exp(-s\hat{K}(t)e^{\delta t})exp\left\{\lambda p \int_0^t [1 - \hat{f}_x(se^{\delta v})]dv\right\},$$

is a martingale where f_x is the density function of claim size and

$$\hat{f}_x(s) = \int_0^{\infty} e^{-sy}\, dF_x(y).$$

From the above equation, we can obtain the Laplace transform of distribution of $\hat{K}_t$ at time t

$$E(e^{-s\hat{K}_t}|\hat{K}_0) = exp(-s\hat{K}(0)e^{-\delta t})exp\left\{\lambda p \int_0^t [1 - \hat{f}_x(se^{\delta v})]dv\right\}.$$

In our case, the initial value of $\hat{K}_t$, $\hat{K}_0$ is 0 so the MGF of $\hat{K}_t$ is

$$M_{\hat{K}(t)}(s) = exp\{\lambda p \int_0^t [\hat{f}_x(se^{-\delta v}) - 1]dv\},$$

where $X_k = 1$ up to time t. This leads to

$$\begin{aligned} M_{\hat{K}(t)}(s) &= exp\{\lambda p \int_0^t [(e^{se^{-\delta v}} - 1]dv\} \\ &= exp(-\lambda p \frac{Ei(1,-s) - Ei(1,-\mathrm{s}\mathrm{e}^{-\delta\, t}) + \delta\, t}{\delta}). \end{aligned}$$

Since it is hard to obtain the simple calculable expression of this M.G.F., the numerical method maybe applied to find the moments. $\diamond$

4.5 Conclusion

In this chapter, we have investigated the different risk models form the ones in Chapters 2 and 3. To treat realistic economic situation and fit for empirical data, heavy-tailed distribution is more used to model individual claim amounts. In Sections 3.1 and 3.2, we introduced this distribution and applied lemmas.

We study the ruin probability for our class of models with the Heavy - Tailed claim size distribution motivated by Biard *et al* (2008). Theorems 4.8 and 4.11 extend Biard *et al*. (2008) to models (discrete and continuous time) with interest rate

CHAPTER 5

Concluding and further work

In Chapter 1 of thesis, we introduced ruin probability, the probability that liabilities will exceed assets on a present value basis at a given future valuation date, resulting in ruin. It is the crucial parameter for assessing the risk exposure of companies and the measure of risk of insolvency for an insurance company. There are many risk models, for example Markov model, finite time model and binomial risk model.

For approximating the ruin probability, we start with a simple risk model suggested by Dr. Neil Butler in Chapter 2. By using Brownian motion , we approximated the ruin probability for the model and develop several numerical methods such as Taylor expansion and Black-Scholes model to evaluate the approximated ruin probability. Numerical calculations is applied on the approximation and show that the ruin probability is increasing function of the variance of stochastic interest rate. A surprising threshold is found such that the rate of increasing of the ruin probability much higher after the threshold point. This threshold probably merits further investigation to discover how it can be explained. Also the results for some parameters agreed with the Gaussian approximation suggested in Matsumoto and Yor (2005) (Proposition 2.6). We applied three approximations in Section 2.6 to the ruin probability. Using simulation, they works reasonably well. The approximations reflect the same relationship between the variance of interest rate and the ruin probability.

Although there is no explicit way to derive the ruin probability with the stochastic continuous interest rate, by approximation and simulation we can find how the variance of the interest rate effects the probability of ruin in finite time. Insurance world, our ruin models stated in Section 2.1 has too many restrictions. Normally the claim size of an insurance is not fixed but random. As a consequence many researchers consider the claims as a stochastic process. Therefore we develop an advanced model in Section 2,8. The more simulations will be studied in the future.

In Chapter 3, the risk models were developed in Sections 3.2 and 3.3 with unknown parameters which are to be estimated We simulated the several groups of data in different experiments and use to obtain the estimated parameter. As shown in Sections 3.4 to 3.6, Numerical analysis supports the claim that ruin probability is increasing function of the variance of interest rate. It numerically discovered a surprising threshold such that the rate of increasing of the ruin probability much higher after the threshold point. We always obtain good and stable estimation of standard deviation of interest rate, σ, but unstable estimation of rate of the claim size, λ, even give negative values in some case. Theorems 3.3 to 3.6 show the approximated ruin probability for Models 1, 4, 5 and 6 respectively. The simulations support these theorems.

We found the optimal constant fraction policies investment strategy with a given upper bound on the ruin probability. Again, the numerical threshold discovered in Chapter 2 was supported in all the examples. In the future, more works about dynamic optimal investment strategy should be considered.

Finally, in Chapter 4, we developed model in Chapters 2 and 3 by make more realistic assumption on interest rate and claims motivated by Biard *et al*. (2008). Hevay-tailed distribution is introduced and used to model the claim size. The two new risk models were build and Theorems 4.8 and 4.11 are the biggest achievements we obtain throughout the work. Even the MGF is hard to calculate in Theorem 4.11 when $\gamma > 2$, however we could use Maple to obtain more results. And the explicit formula could be expressed by using induction. This work will leave for the future.

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Appendix 1

Gauss-Hermite Formula, (Abramowitz 1970)

The formula uses the Hermite polynomials $H_n(x)$ to deal with the integration of $(-\infty, \infty)$.

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)dx &= \int_{-\infty}^{\infty} e^{-x^2}[e^{x^2} f(x)]dx \\ &= \sum_{k=1}^{n} w(x_k)[e^{x^2} f(x_k)] + R_n(x)\end{aligned}$$

where X_k is kth zero of $H_n(x)$,

$$w(x_k) = \frac{2^{n-1} n! \sqrt{\pi}}{n^2 [H_{n-1}(x_k)]^2}$$

and

$$R_n x = \frac{n! \sqrt{\pi} f^{(2n)}(x)}{2^n (2n)!}$$

H.P are solution of Hermite's Differential Equation

$$H_n x = (1)^n e^{x^2} \frac{d^n}{dx^n}(e^{-x^2})$$

when $n = 0, 1, 2, 3$

Appendix 2 (Section 2.7.1)

100 variance of continuous interest rate used in simulation

0.25826142	0.28196884	0.38056092	0.73379036	0.27992473
0.69031010	0.45596998	0.16809908	0.48333321	0.57925480
0.17107121	0.83717329	0.68685754	0.39312954	0.29171645
0.90578313	0.57274508	0.29797457	0.25786461	0.15742122
0.50140450	0.92092187	0.03908921	0.74042513	0.43082837
0.88079442	0.30414883	0.86055642	0.03959701	0.85914356
0.09825043	0.51445155	0.41006651	0.22429632	0.69829572
0.63322432	0.31853664	0.95329933	0.81460351	0.34068674
0.42312630	0.10879914	0.83224278	0.66680408	0.81215068
0.17022953	0.01699669	0.92429300	0.25056953	0.73283787
0.38285285	0.72968434	0.55214482	0.79146790	0.14384940
0.28866631	0.70765106	0.60944191	0.53375864	0.57642442
0.18855107	0.87076188	0.74425246	0.54577662	0.84693434
0.30166596	0.09180047	0.34487122	0.17820724	0.48614927
0.11067378	0.95718657	0.18979385	0.95704486	0.13525687
0.08381547	0.62340073	0.47487325	0.81743876	0.89783200
0.8951250	0.47930268	0.08092672	0.90664777	0.70357941
0.67254306	0.21283616	0.98634889	0.43533704	0.40823803
0.31513155	0.13342939	0.24477215	0.45545881	0.23368945
0.08694565	0.02539103	0.30113375	0.40851137	0.10627077

Appendix 3

Summary of simulated ruin probabilities of ordinary example and approximations 1 to 3.

	ordinary ex.	1st approx.	2nd approx.	3rd approx.
Min	0.000	0.000	0.000	0.000
1st Qu.	0.00000	0.00000	0.00200	0.00200
Median	0.00100	0.00000	0.02600	0.02300
Mean	0.02064	0.01354	0.036920	0.03576
3rd Qu.	0.03450	0.01650	0.07200	0.06500
Max	0.11400	0.10400	0.11400	0.12000