

I've Got the World on a Brane

John Omotani

Thesis submitted to the University of Nottingham
for the degree of Doctor of Philosophy

July 2012

Abstract

This thesis treats several topics in the study of extra-dimensional models of the world, concerning Heterotic M-Theory and the dynamics of branes.

We describe a reduction to five dimensions, over a Calabi-Yau manifold, of an improved version of Heterotic M-Theory, which is valid to all orders in the gravitational coupling. This provides a starting point for considering the consequences of the improved theory for the very fruitful phenomenology of the original.

We investigate the singularities formed by the collision of gravitating branes in scalar field theory. By considering the asymptotic structure of the spacetime, the properties of the horizons formed and the growth of the curvature we argue that the singularity is not a black brane, as one might have expected, but rather a big crunch.

Finally, we construct a restricted class of multi-galileon theories as braneworld models with codimension greater than one, developing in the process some of the formalism needed for the general construction.

Acknowledgements

Firstly I would like to express my gratitude to my parents for their love and support throughout my life. Thanks are due of course to my supervisor Paul Saffin for his guidance and his invaluable help and also to Ian Moss for his assistance and insight during our collaboration. Finally I would like to thank everyone in the Particle Theory and Astronomy groups for making Nottingham such a friendly place to work and especially Rob Chuter and Duncan Buck, my office mates for much of the journey. My PhD Studentship was funded by STFC.

Contents

1	Introduction	3
1.1	Supergravity	3
1.2	Quantum Anomalies	4
1.3	M-Theory	6
1.4	Hořava-Witten Theory	7
1.4.1	Phenomenology	8
1.5	Gravity with Boundaries	9
1.6	Improved Heterotic M-Theory	9
1.7	Dimensional Reduction of the Improved Theory	11
1.8	Braneworlds	12
2	The Reduction to Five Dimensions of Heterotic M-Theory	15
2.1	The Eleven Dimensional Theory	18
2.1.1	Supergravity With Boundaries	18
2.1.2	Gravitational Anomaly Cancellation	20
2.1.3	Supersymmetric Boundary Conditions	22
2.1.4	Gauge Anomaly Cancellation	22
2.1.5	The Lorentz Multiplet	23
2.1.6	Heterotic M-Theory	25
2.2	Kaluza-Klein Reduction Over a Calabi-Yau Threefold	26
2.3	Reduction Ansatz	27
2.3.1	The Metric	28
2.3.2	The Three-Form C	29
2.3.3	The Gravitino	29
2.3.4	The E_8 Boundary	30
2.3.5	The E_6 Boundary	31
2.4	Five Dimensional Multiplets	32
2.5	The Bulk	34
2.5.1	Einstein-Hilbert Term	34
2.5.2	The Five Dimensional Gravitino	35
2.5.3	Three-Form Terms	35
2.5.4	Aside on Auxiliary Fields	37
2.5.5	Dualization	37
2.5.6	Hypermultiplet Quaternionic Structure	40
2.6	The Boundaries	42
2.6.1	Boundary Action for the Gravity Multiplet	42
2.6.2	Boundary Sources for G and the Embedding of the Spin Connection	43

2.6.3	Reducing the Lorentz Multiplet	43
2.6.4	Yang-Mills Action on the E_8 Boundary	44
2.6.5	Chiral Multiplets	45
2.6.6	Yang-Mills Action on the E_6 Boundary	46
2.6.7	Boundary Conditions	47
2.7	Gaugino Condensation	48
2.8	Conclusions	49
3	Big Crunch from Colliding Branes	51
3.1	Static Kinks	52
3.2	Asymptotic Structure	55
3.3	Dynamical Solutions	56
3.4	Simulation Methods	58
3.5	Simulation Results	61
3.6	Horizon Structure	62
3.7	Closing Off the Spacetime	63
3.8	Conclusions	64
4	Braneworlds and Galileons	66
4.1	Galileons in Codimension One	68
4.2	Perturbative Formalism	69
4.3	Possible Actions	71
4.3.1	Volume Term $\mathcal{L}_{(2)}$	72
4.3.2	Einstein-Hilbert Term $\mathcal{L}_{(4)}$	72
4.3.3	Tadpole Term $\mathcal{L}_{(1)}$	73
4.4	Conclusions	73
5	Conclusions	75
A	Heterotic M-Theory Conventions	77
B	Calabi-Yau Geometry	80
C	E_8 Gauge Theory	83
D	Gauging a Quaternionic Isometry	85
E	Summary of the Reduced Heterotic M-Theory	87
E.1	Action	87
E.2	Boundary Conditions	90
E.3	Supersymmetry	92
F	Galileons: Further Details	93
	Bibliography	97

Chapter 1

Introduction

1.1 Supergravity

The story of particle physics in the 20th century was one of finding symmetries and exploiting them to construct quantum field theories, culminating in the construction of the standard model of particle physics with (exact) colour gauge symmetry and (broken) electroweak gauge symmetry. However, the theory of spacetime symmetry, that is the other great pillar of 20th century physics, general relativity, stubbornly refuses formulation as a quantum theory, which our experience of particle physics teaches us that it would have to be at short enough distances or high enough energies.

In the spirit of the search in particle physics for ever more symmetries to exploit, it is natural to ask whether there can be further symmetries of spacetime, in the hope that they might help to reconcile the incompatibility of gravity and quantum theory. The initial answer to this question is that there can be no further symmetries due to the theorem of Coleman and Mandula [1] which states that, under reasonable assumptions, the most general Lie algebra of symmetry operators consists of the Poincaré algebra plus internal symmetry algebras. However, this is not the end of the story as the Coleman-Mandula theorem can be evaded by relaxing its restrictions.

In particular, supersymmetry is a spacetime symmetry which circumvents the theorem by including anti-commutators as well as commutators in its algebra: it has a *graded* Lie algebra rather than just a Lie algebra. In addition to the bosonic generators of Poincaré symmetry there are fermionic generators of the supersymmetry. These fermionic generators transform bosonic fields to fermionic ones and vice versa. We can then group particles into ‘supermultiplets’ of ‘superpartners’ which form a closed cycle under the supersymmetry. If supersymmetry were unbroken, particles in a supermultiplet would all have the same mass and so it is clear that if the world is supersymmetric then that supersymmetry is broken somehow. Nevertheless, supersymmetry as a theory of physics beyond the standard model predicts that the observed particles have superpartners, albeit ones made heavy enough by the breaking of supersymmetry that they are unobservable. Indeed it is possible to construct phenomenological extensions to the standard model such as the ‘minimal supersymmetric standard model’ (MSSM) which reproduce the same results for experiments to date but predict superpartners observable by the Large Hadron Collider at CERN. One motivation for such models is that

they may solve the hierarchy problem, which is the puzzle of why the scale of the electroweak theory (i.e. the mass of the Higgs boson) is so much less than the natural scale of some more fundamental theory: $\sim 10^2\text{GeV}$ as opposed to $\sim 10^{16} - 10^{17}\text{GeV}$ for a grand unified theory or perhaps even $\sim 10^{19}\text{GeV}$ (the Planck scale). Spontaneously broken supersymmetry provides a mechanism for a Higgs mass (which would be zero with unbroken supersymmetry) which is exponentially suppressed, thus solving the problem (if one has such a supersymmetry breaking mechanism in hand).

As local Poincaré symmetry gives us general relativity, so local supersymmetry gives us supergravity. It is an interesting fact that there is an upper limit to the number of supersymmetries in a given number of spacetime dimensions, and this limit decreases as the number of dimensions increases. Indeed there is a maximum number of spacetime dimensions in which it is possible to construct supersymmetric theories. The reason is as follows: For the graviton to be the unique spin two particle, it must be the highest spin member of some supermultiplet. The supersymmetry generators are the components of \mathcal{N} spinors, where \mathcal{N} is the number of supersymmetries. In $2n$ or $2n + 1$ dimensions a spinor has 2^n components, so there are $\mathcal{N} \cdot 2^n$ supersymmetry generators. Consider their actions on a single particle state. For a massless particle we cannot choose the zero momentum state, but for a single particle state it suffices to consider only states with momentum in a particular direction, call it the 1 direction. States then have $p_1 = \pm p_0$ and we can choose the basis of the supersymmetry generators so that half annihilate the states with $p_1 = +p_0$ and the other half annihilate the states with $p_1 = -p_0$. Furthermore, when a generator acts on a state with some helicity h , it changes h by $\frac{1}{2}$ with half of the generators raising, and the other half lowering, h . Since the generators are fermionic and anti-commute with themselves, if we act twice with any one generator we destroy the state. So for states with say $p_1 = +p_0$ we have $\mathcal{N} \cdot 2^{n-1}$ generators which do not annihilate the states. The lowest helicity state of a supermultiplet is given by acting once on the highest helicity state with each of the $\mathcal{N} \cdot 2^{n-2}$ of these generators which also lower helicity. If the graviton, which has spin two, is the highest spin state then we start from helicity $+2$ and we can only have at most 8 generators, each taking us down by $\frac{1}{2}$, or the lowest helicity state would be below -2 (which would contradict our assertion that the graviton is the highest spin state). Thus we have a limit on the number of spacetime dimensions in which we can have a supergravity, for we must have $\mathcal{N} \cdot 2^{n-2} \leq 8$ in order for there to be a supermultiplet without any state having spin higher than the graviton. The highest dimension possible is eleven, with also $\mathcal{N} = 1$. This maximal eleven dimensional supergravity, which was first described by Cremmer, Julia and Scherk [2], is unique because if there is only one graviton then there can be only one supermultiplet, whose structure is entirely dictated by supersymmetry.

1.2 Quantum Anomalies

It may happen that a theory which is classically invariant under some symmetry is not invariant quantum mechanically, for instance it may be that it is impossible to construct a regularization scheme for divergent Feynman diagrams which respects that symmetry. This quantum variation is called the anomaly. In the case

of a global symmetry this may simply indicate that certain processes which are forbidden classically are allowed quantum mechanically, which is not a problem. For a local symmetry however, states related by a gauge transformation are supposed to be identified with each other and the presence of a quantum anomaly would render the theory inconsistent. Therefore one must ensure that if there are sources of such local anomalies, then their contributions cancel each other exactly.

The quantum anomaly is the variation under gauge transformations or diffeomorphisms of the effective action describing the gauge fields or the metric which is given by integrating out the fermions in the path integral. Anomalies in d dimensions are associated to certain $(d+2)$ -forms. The anomaly must obey what is known as the Wess-Zumino consistency condition. Let us call the effective action Γ . Then the anomaly, $G(\epsilon)$, is the variation by an amount ϵ of Γ : $G(\epsilon) = \delta_\epsilon \Gamma$. If we consider another variation, this time by η , then $\delta_\eta G(\epsilon) - \delta_\epsilon G(\eta) = \delta_\eta \delta_\epsilon \Gamma - \delta_\epsilon \delta_\eta \Gamma$, but this is just the Lie bracket acting on Γ , giving $G([\eta, \epsilon])$, so we have the consistency condition $\delta_\eta G(\epsilon) - \delta_\epsilon G(\eta) = G([\eta, \epsilon])$. This condition can be fulfilled straightforwardly if we can describe the anomaly, in a d -dimensional theory, by starting from a (formal) $(d+2)$ -form $I_{(d+2)}$ which is an invariant polynomial of the 2-form (gauge or gravitational) curvature. We only need even rank forms because quantum anomalies only occur in even dimensions. According to the Chern-Weil theorem invariant polynomials of the curvature are closed so $dI_{(d+2)} = 0$. Therefore it can be written as the derivative of a $(d+1)$ -form, $I_{(d+2)} = dI_{(d+1)}$, at least locally. Since $I_{(d+2)}$ is gauge invariant, the variation of $I_{(d+1)}$ must be a total derivative $\delta_\epsilon I_{(d+1)} = dI_d(\epsilon)$. Now if the quantum anomaly is given by the integral of such a d -form, $G(\epsilon) = \int_{\mathcal{M}_d} I_d(\epsilon)$ and if we regard \mathcal{M}_d (formally) as the boundary of $\mathcal{M}_{(d+1)}$ then $G(\epsilon) = \int_{\mathcal{M}_{(d+1)}} dI_d(\epsilon) = \int_{\mathcal{M}_{(d+1)}} \delta_\epsilon I_{(d+1)}$ and so it is easy to see that the Wess-Zumino consistency condition is obeyed: $\delta_\eta G(\epsilon) - \delta_\epsilon G(\eta) = \int_{\mathcal{M}_{(d+1)}} (\delta_\eta \delta_\epsilon I_{(d+1)} - \delta_\epsilon \delta_\eta I_{(d+1)}) = \int_{\mathcal{M}_{(d+1)}} \delta_{[\eta, \epsilon]} I_{(d+1)} = G([\eta, \epsilon])$. This illustrates why $(d+2)$ -form polynomials might be useful for calculating quantum anomalies. In fact the connection is deeper, all the gauge and gravitational anomalies can be so calculated and the coefficients of the polynomial can be determined. The reason is that the anomaly can be related to the index of the Dirac operator in $(d+2)$ dimensions, which can in turn be calculated using the Atiyah-Singer index theorem [3]. A careful discussion useful to our purposes later is [4]. We can ensure anomaly cancellation just by requiring that the sum of the polynomials from all the sources of variation cancel; this is helpful because there is an ambiguity in the definition of the anomalies themselves since their explicit form depends upon a choice of gauge. Anomalies can occur where we have chiral fermions (the index of the Dirac operator is the difference in the number of zero modes with positive and negative chirality): in particular Yang-Mills gauge theory has an anomaly in any even dimension and gravity has an anomaly in dimension $d = 4k + 2$. In ten dimensions, where there are both gauge and gravitational anomalies, enforcing anomaly cancellation is a strong constraint on the possible theories, such as those discussed below, which are formulated either in ten dimensions (superstring theories) or in spaces with ten dimensional boundaries (Heterotic M-Theory).

1.3 M-Theory

String theory promises to be a quantum theory of gravity. Since the fundamental objects have a finite size, the traditional problem for quantum gravity of how to deal with ultraviolet divergences is ameliorated since the fundamental objects have finite size rather than being point particles. The questions are how to formulate string theory without inconsistencies such as tachyons and quantum anomalies and, that being accomplished, whether and how such a theory can describe the real world. Tachyons can be banished by incorporating supersymmetry and it is possible to construct such superstring theories which are free of anomalies. However, this puts rather stringent constraints on the theories, in particular they are all ten dimensional, but still there are five consistent superstring theories known.

From the point of view of discovering the correct description of quantum gravity for the real world, this might be viewed as an embarrassment of riches, for how are we to decide which of these theories ought to apply to us? However, it seems that the various superstring theories are not in fact independent. They are related by duality transformations which imply that they are merely different limits of a single underlying theory, which is known as M-Theory. Though the complete structure of M-Theory remains unknown, still we can say some things about it: in particular its low energy

limit is eleven dimensional. This was first observed [5, 6] in the type IIA superstring theory, which in the limit of large string coupling ‘decompactifies’: its spacetime is seen to be $R^{10} \times S^1$ rather than R^{10} and the size of the circle is set by the string coupling, so that both tend to infinity in this limit. We have then, in the low energy limit of M-Theory, an eleven dimensional, supersymmetric theory of gravity, so in light of the uniqueness of the eleven dimensional supergravity mentioned above it should not be surprising that this is, in fact, eleven dimensional supergravity. The strong coupling limit of the $E_8 \times E_8$ heterotic superstring is also eleven dimensional supergravity, as discussed below. The other superstring theories, the type IIB, $SO(32)$ heterotic and the type I do not have clear eleven dimensional limits themselves but are continuously related to the type IIA and $E_8 \times E_8$ heterotic theories and so must be part of the same picture.

With the unification of the various superstring theories into M-Theory, we then have progress of a sort towards contact with the real world. Though we have lost (at least for now) a complete description of our theory of the world, at least there is only the one and we know its low energy limit; which limit would seem to be necessarily an important part of any phenomenological model, even if not absolutely the whole story. Clearly however further effort is needed to describe the real world. We have an eleven dimensional theory (or ten dimensional

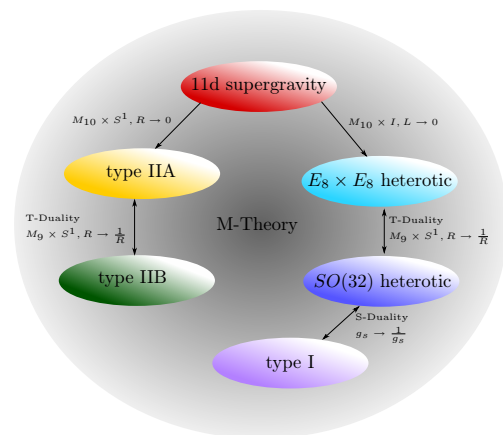


Figure 1.1: M-Theory Dualities

in some limits) and a four dimensional world. One way to progress is to adopt a Kaluza-Klein picture: we compactify the extra dimensions so that we have four visible, extended dimensions and a tower of massive excited states which are too massive for us to have yet observed due to the small size of the extra dimensions. A particularly attractive option for a supersymmetric theory is to compactify on a Calabi-Yau manifold. This is a six dimensional space (though they can also be defined in other even dimensions) whose essential property for this purpose is that it has a single covariantly constant spinor. This means that the single supersymmetry parameter of the eleven (or ten) dimensional theory gives only a single supersymmetry parameter in five (or four) dimensions and so we are left after compactification with an $\mathcal{N} = 1$ supersymmetric theory. If we started in eleven dimensions we still have one yet to compactify in order to describe the real world, which might be useful phenomenologically; alternatively we could compactify the eleven dimensional model directly over a G_2 manifold, which is a seven dimensional analogue of a Calabi-Yau manifold sharing the property of having a single covariantly constant spinor. The demand of supersymmetry after compactification allows us to make contact with the phenomenology of supersymmetric field theory and thus gives us a route to physically plausible scenarios, which with no such restriction on the possible compactifications would be a challenge of daunting complexity. It also allows us to retain attractive features of supersymmetric extensions to the standard model such as resolving the hierarchy problem with spontaneous breaking of supersymmetry.

1.4 Hořava-Witten Theory

Heterotic M-Theory, which was invented by Hořava and Witten [7, 8], is eleven dimensional supergravity with two boundaries, one at each end of the eleventh dimension. It represents the limit of the heterotic $E_8 \times E_8$ string theory which is both low energy and strongly coupled; dimensional reduction of the interval gives the supergravity which is the low energy limit of that theory. The construction of the theory depends intimately on anomaly cancellation. The supergravity in the bulk induces gravity on the boundaries and the bulk gravitino becomes chiral on the boundaries. Thus the boundaries are ten dimensional spacetimes with gravity and chiral fermions, and so they have a gravitational quantum anomaly. In order to cancel this anomaly, one is forced to introduce further chiral fermions on the boundary (as well as some Green-Schwarz terms in the bulk). This can be done by adding an $\mathcal{N} = 1$ Yang-Mills supermultiplet (which in ten dimensions is the maximal supersymmetric theory that is not a supergravity) with the gauge group chosen to give the right number of fermions to cancel the gravity anomaly. This requirement dictates that the gauge group be E_8 , and so we have two independent E_8 gauge theories, one on each boundary, which gives us the $E_8 \times E_8$ gauge group of the string theory on compactification of the eleventh dimension. The Yang-Mills theory also has a gauge anomaly in ten dimensions which must be cancelled by coupling the three-form of the bulk supergravity to the boundary so that a classical variation of the three-form can cancel the gauge quantum anomaly on the boundary. This structure rigidly determines the theory since we have the maximal supersymmetric theory in the bulk and the maximal (non-gravitational) supersymmetric theory on the boundary with the gauge group and couplings

between the two dictated by anomaly cancellation.

1.4.1 Phenomenology

The heterotic theory is an attractive corner of M-Theory for phenomenological model building. Since it contains E_8 gauge theory as an intrinsic part, it is a natural place to build particle physics models by breaking the E_8 gauge symmetry to give a grand unified theory or even a theory with the gauge group of the standard model itself. This is of course true of both $(E_8 \times E_8)$ heterotic string theory and Heterotic M-Theory and much phenomenological work was done on the string theory before the advent of M-Theory (as well as after). Since models of the real world (at least the real world today) are necessarily low energy ones though, the limit taken in Heterotic M-Theory is more or less forced upon us and so this is the natural place to build particle physics models as we do not need then to assume weak string coupling. On the other hand in cosmological contexts it might well prove not to be the case that the low energy limit is sufficient, and since the full M-Theory is not known the string theory is the only way to relax this limit and gain insight into such situations.

To turn an eleven dimensional theory into a model of the real world, we need to hide seven dimensions somehow. As mentioned above one way of doing this is to take those seven dimensions to be small, and to get an $\mathcal{N} = 1$ supersymmetric theory in four dimensions the extra dimensions must be a G_2 manifold. In the context of Heterotic M-Theory one of those seven must be the interval since the standard model particles are supposed to come from the Yang-Mills fields on one of the boundaries, meaning that the boundaries must span the extended dimensions. The other six dimensions are curled up into some small manifold, for which an attractive choice is a Calabi-Yau manifold. For a given topology of the Calabi-Yau manifold, we can describe the geometry of the compact space by some (geometrical) ‘moduli’ which measure the size of the eleventh dimension (the interval), the overall volume of the Calabi-Yau and the size of any topologically non-trivial cycles of the Calabi-Yau. A particular model will be given by specifying a background with a particular compactification manifold having some values of these moduli and possibly also non-zero configurations of the fields (which might give some additional moduli) which solve the eleven dimensional equations of motion. The effective four dimensional theory is then the theory of the fluctuations around this background. For a realistic model we obviously need ultimately to have an effective theory which resembles the standard model; models close to the MSSM are a useful intermediate step in string or M-Theory phenomenology. We also need a stable background, which means that we need some mechanism for making sure that the moduli which specify the particular model we are looking at are fixed.

A substantial amount of work has been done building particle physics models with increasing degrees of similarity to the standard model. For example, fairly soon after the invention of Heterotic M-Theory, models were constructed with the E_8 on the visible brane broken to the standard model gauge group [9] by choosing a particular compactification background. Recently techniques have been developed for computational searches of heterotic models [10, 11] and a large class of ‘heterotic standard models’ with ‘the precise matter spectrum of

the MSSM, at least one pair of Higgs doublets, the standard model gauge group and no exotics charged under the standard model of any kind' have been identified [12]. There has also been recent progress on the problem of moduli stabilization in the context of particle physics model building where [13, 14] it has been shown that it is possible to stabilize all the geometric moduli using the gauge bundle plus non-perturbative effects, for some choices of gauge bundle.

Given the successes of Heterotic M-Theory for particle physics model building, it is desirable also to have cosmological models constructed in the same framework. This motivated the first cosmological solutions [15] and continues to motivate, for example, models of assisted inflation driven by M5-brane dynamics [16, 17]. Heterotic M-Theory is also a natural home for braneworld cosmological models, about which more below, the famous example being the ekpyrotic model [18] in which the big bang is generated by a collision of branes in a five dimensional bulk, which is of continued interest [19, 20] as an alternative to inflation.

1.5 Gravity with Boundaries

The question of how to construct consistently a theory of gravity which includes boundaries is one of direct relevance for Heterotic M-Theory and for braneworld models in general. For a theory of gravity without boundaries, the variation of the Einstein-Hilbert action vanishes for metrics that solve the Einstein equations. However in the presence of a boundary this is no longer true. Even when the variation of the metric is constrained to vanish on the boundary, there is still a non-vanishing boundary term involving the normal derivative of the variation. In order to make the variational principle consistent, it is necessary to introduce boundary terms into the action. The Einstein-Hilbert action in the bulk is made consistent by the inclusion of the Gibbons-Hawking action on the boundary [21]. More complex gravity theories in the bulk also require additional surface terms. Of relevance for us will be supergravity, where a gravitino boundary contribution is required by similar considerations [22], and Lovelock gravity, where the higher curvature terms require corresponding Myers terms on the boundary [23].

1.6 Improved Heterotic M-Theory

The original Hořava-Witten formulation of Heterotic M-Theory has a serious problem. It is only entirely well defined up to first order in the eleven-dimensional gravitational coupling, $\mathcal{O}(\kappa^{2/3})$. It was described as an orbifold $M_{10} \times S^1/\mathbb{Z}_2$, which is equivalent to an interval with boundaries at the fixed points of the \mathbb{Z}_2 symmetry and which was considered to be technically convenient. In this description the coupling of the bulk three-form to the boundary is accomplished with δ -function sources in the Bianchi identity of its field strength. Imposition of supersymmetry of the action at the orbifold fixed points then brought terms proportional to $\delta(0)$ into the action at $\mathcal{O}(\kappa^{4/3})$. As Heterotic M-Theory is only supposed to be an effective theory of a particular low-energy limit of a complete (albeit still unknown) quantum M-Theory this inconsistency need not necessarily be considered problematic as long as one is content to work within the approximations it entails. In practice however, this may in fact cause difficulties: taking

reasonable values for the GUT scale and Newton's constant, it has been found [24] that the expansion parameter of the theory, $\epsilon = \kappa^{2/3} \rho V^{-2/3}$ is of order one (where ρ is the size of S^1/\mathbb{Z}_2 and V is the volume of the Calabi-Yau manifold used in the compactification to a four dimensional theory). This calls into question the validity of the first order expansion in $\kappa^{2/3}$. More recently, terms second order in $\kappa^{2/3}$ have been found to be important in calculating the back-reaction of anti-branes [25] and the contribution of gaugino condensation in the presence of anti-branes [26], using such $\mathcal{O}(\kappa^{4/3})$ terms involving gauge matter fields as were considered able to be computed reliably [27].

These phenomenological considerations give one motivation for relaxing the restriction of Heterotic M-Theory to first order in $\kappa^{2/3}$. On the other hand, if one considers Heterotic M-Theory as supergravity on a manifold with boundary it is an attractive theory: eleven-dimensional supergravity is the maximal supergravity and is unique, and Heterotic M-Theory is the only way to include boundaries without introducing gravitational anomalies. However, considered just as a supergravity theory it is highly unsatisfactory that it should be well-defined only up to first order in the gravitational coupling, providing another motivation for improvement.

The problem has been addressed by Ian Moss who has constructed an improved version of Heterotic M-Theory [28–31] which is consistent and supersymmetric to all orders in $\kappa^{2/3}$. This construction is performed taking the view of the eleventh dimension as an interval with boundaries, with careful attention paid to the boundary conditions which must be satisfied by the bulk fields. The structure of the eleven-dimensional improved theory is described in Section 2.1. It is constructed as an expansion in the curvature up to R^2 terms and at each order is rigidly constrained by the requirements of anomaly cancellation at the boundaries and supersymmetry, leaving only one free parameter: the gravitational coupling, κ .

The problems in the Hořava-Witten version of the theory seem to arise principally from its failure to fully account for the effect of the energy-momentum density localized on the orbifold fixed planes. Its effect as a source for the flux G was carefully taken into account by modifying the Bianchi identity for G , but it must also be considered as a source for the spacetime curvature of the bulk, and possibly also for the gravitino, modifying the junction conditions across the brane. In particular, given some energy-momentum localized on a codimension one surface, the Israel junction conditions [32], which relate the extrinsic curvature of the surface to its energy-momentum, must apply. In order for these to be imposed by the action, the Gibbons-Hawking term must be included on the boundary. One might wonder whether it could be possible to impose the junction conditions in the upstairs picture by modifying the Bianchi identity for the Riemann curvature to $\nabla_{[I} R^J{}_{K|LM]} = \delta(x^{11}) \dots$ in analogy with the modification of the Bianchi identity for G . However, the boundary action approach seems more natural to us and we work here in the manifold with boundary picture everywhere. This picture also does not require carrying around a redundant, and physically irrelevant, copy of the bulk as the orbifold picture does. As for the gravitino, it will turn out (Section 2.1.3) that supersymmetry of the boundary forces us to include the Yang-Mills fields in its boundary condition as well.

1.7 Dimensional Reduction of the Improved Theory

We will describe in Chapter 2 the reduction to five dimensions of the improved Heterotic M-Theory over a Calabi-Yau manifold, to find the model analogous to that found in [33] for the original Hořava-Witten theory but with the inclusion here of terms with up to two fermions, which have not been presented either for the original or for the improved theory before our work [34]. The five dimensional reduction is a natural intermediate step to phenomenological models since the scales in the problem suggest [24] that the size of interval (the eleventh dimension) is about ten times the length scale of the Calabi-Yau, and the latter can therefore be integrated out before the former. It is also a useful starting point for the introduction of five-branes [35], which seem to be vital for many phenomenological applications, and anti-five-branes [25], and also for the study of gaugino condensation [26]. The five dimensional reduction of the improved version of Heterotic M-Theory is then a first step in assessing the impact on phenomenology of a theory consistent to all orders in $\kappa^{2/3}$.

The topic of gaugino condensation has received some attention already, as discussed in Section 2.7. It was considered in [36, 37] in a much simpler reduction, in which attention was focused on the gravitino and the Calabi-Yau volume modulus. It was found there that the condensate gives a contribution to the flux through the boundary condition on the supergravity three-form and also induces a twist in the chirality condition on the gravitino between the two boundaries. The latter is a particular effect of the improved theory where the gauginos appear in the boundary condition of the gravitino. The twisted boundary conditions break supersymmetry and give a Casimir contribution to the vacuum energy which can lift the cosmological constant to give de Sitter vacua. Since in this scenario both moduli stabilization and uplift depend on the gaugino condensate they naturally have similar scales, though fine tuning of a parameter in the superpotential is still required to give a small four dimensional cosmological constant. Since the improved theory is valid to all orders in the gravitational coupling warping of the bulk metric which is not small can be consistently accommodated when considering the gaugino condensate, though for the Casimir energy calculations in [37] small warping was still assumed.

The other motivation for this work, apart from any possible consequences for the phenomenology of Heterotic M-Theory, is the investigation of supergravity on a manifold with boundary per se. Such supergravities with boundary matter have recently been constructed; the most detailed are in three dimensions [38, 39], where an off-shell formulation is available, but there are also models in five [40] and seven [41] dimensions. Five dimensional theories are obviously of particular interest since their four dimensional boundaries might correspond to the physical universe. The contribution of the present work in this context is that by dimensionally reducing a consistent eleven-dimensional supergravity with boundaries we find an explicit example of a consistent five-dimensional supergravity on a manifold with boundary, which includes boundary matter, and we do so without having to include distributions in the theory. In the absence of a general formalism in five dimensions, hopefully such an example is of some interest.

1.8 Braneworlds

In order to hide extra dimensions, rather than postulating that the extra dimensions are just too small to be observed à la Kaluza-Klein, we could allow them to be arbitrarily large if we suppose that we are constrained to live on a four dimensional brane and so cannot see them. This braneworld scenario is further motivated by the possibility that something like Heterotic M-Theory might be the fundamental theory of the universe, since in Heterotic M-Theory the fields which give rise to standard model matter are perforce confined to branes.

The simplest model of a braneworld is a scalar field model, which can be arranged to have topological defects representing the branes and the fluctuations of the scalar field (the ‘matter’ in the theory) can be confined to the defect. Even if such models do not give a realistic picture of the universe, their simplicity makes them amenable to analysis, in particular to numerical simulation, and one can hope that studying such models may give insight into some generic features of braneworld scenarios.

Generically, one would expect gravity to propagate away from the brane into the bulk spacetime, and so the effects of large extra dimensions would be visible in gravitational effects even if they are hidden from standard model type processes. However, if the bulk spacetime is appropriately warped then gravity can be confined to the brane as well, as was demonstrated by the Randall-Sundrum model [42], which fact makes it plausible that the universe today might be described as a braneworld with large extra dimensions even though gravity appears in experiments to date to be just as four dimensional as the other forces of nature (at least up to solar system scales).

The braneworld picture also provides an alternative to inflation as a description of the very early universe. In the ekpyrotic scenario [18] the big bang is caused by a collision of branes in a five dimensional bulk. The horizon and flatness problems are solved not by superluminal expansion, as in inflationary theories, but rather by the dynamics of the branes, by letting the branes approach each other rather slowly before the collision so that areas widely separated relative to the Hubble scale after collision were in causal contact before it.

In light of this ekpyrotic cosmological scenario, it is interesting to study the topic of brane collision in scalar field theory models of braneworlds, with an eye on learning about the generic features of brane collisions [43–48]. A contribution to this effort is described in Chapter 3, concerning the nature of the singularities which may be formed by the collision of scalar field theory domain walls.

Braneworld models can also provide a means of modifying (four dimensional) gravity on large scales, as in the DGP model [49]. Here the bulk spacetime is flat, rather than warped as it is in the Randall-Sundrum case, and four dimensional gravity is provided by having an Einstein-Hilbert term on the brane, built from the induced metric, as well as one in the bulk built from the full metric. In this theory gravity appears four dimensional at short distances but five dimensional at long distances. There is an extra scalar degree of freedom, but this has only second order equations of motion even though the Lagrangian contains higher derivative terms. This property allows the construction of solutions in which the scalar field is strongly coupled at small scales and its fluctuations are suppressed by that strong coupling. This sort of hiding of scalar fields in modified gravity

by strong coupling at small scales is known generically as Vainshtein screening. The importance of this phenomenologically is that the scalar field can be hidden from observation on solar system scales, which have been well tested, while being active on larger scales to affect the expansion of the universe.

The attractive properties of this scalar when considered in the four dimensional effective theory inspired a generalization to find a broader class of theories having a scalar with similar couplings to gravity and to itself as the one in the DGP model and also retaining second order equations of motion [50]. The outcome was a set of theories of gravity coupled to a scalar having a ‘Galilean’ symmetry, leading the scalar field to be christened the ‘galileon’. The restriction of Galilean symmetry and second order equations of motion resulted in an action with just five free parameters. As the DGP model is a particular example of these galileon theories, it is natural to ask whether there is a braneworld description of the whole class. Indeed there is, as first described in [51]. The picture is similar to DGP, with a four dimensional probe brane in a five dimensional spacetime and the scalar field being just the displacement of the brane in the extra dimension. Now, however, we include in the brane action all the curvature terms that give second order equations of motion, that is the Lovelock terms in the intrinsic curvature [52] and the Myers terms in the extrinsic curvature [23]. In the number of dimensions here, in addition to the Einstein-Hilbert term there is a cosmological constant; the first two Myers terms; and a tadpole term. One can also consider such theories with de Sitter or anti-de Sitter rather than Minkowski spaces, as was done in [53]. On the other hand from the four dimensional field theory perspective it is also natural to consider how one might generalize to more than one galileon field [54, 55], though in such theories one encounters a rapidly proliferating number of terms in the action as more galileon fields are added. To describe such theories in a braneworld picture we need a higher codimension bulk, as each galileon field corresponds to the fluctuations in one of the extra dimensions. This is more challenging than in the codimension one case as we can no longer describe the branes as boundaries and the toolkit for building the actions is less well developed. A particular model has been found for the most symmetric case with a Minkowski brane in a Minkowski bulk [56]. In Chapter 4 we develop the higher codimension construction allowing for other maximally symmetric spaces. In order to do so we develop some machinery that may be useful for a further generalization to the full class of models which can be constructed in the field theory approach.

Statement of Original Research

Chapter 2 describes work previously reported in [34] done by the author in collaboration with Ian Moss and Paul Saffin. The original research is described in Sections 2.3 to 2.6 and 2.8, with the results summarized in Appendix E; the work relating to the bosonic fields obviously has strong parallels with [33] which started from Hořava and Witten’s eleven dimensional theory [7, 8], whereas the work here starts from Ian Moss’s improved version [28–31]. Section 2.7 describes some work by Ahmed and Moss [36, 37], and not by the author, to illustrate a consequence of the reduction of the improved theory.

Chapter 3 describes work previously reported in [57] done by the author in col-

laboration with Jorma Louko and Paul Saffin. The original results are described in Sections 3.2, 3.3 and 3.5 to 3.8.

Chapter 4 describes previously unpublished work done by the author in collaboration with Ian Moss, Antonio Padilla and Paul Saffin. The original research is described in Sections 4.2 to 4.4 with further details of the calculations given in Appendix F.

Chapter 2

The Reduction to Five Dimensions of Heterotic M-Theory

The improved version introduced by Ian Moss [28–31] cured Heterotic M-Theory of the inconsistency of the original formulation of Hořava and Witten [7, 8] beyond leading order in the gravitational coupling. Here we give a self-contained description of the improved theory and its reduction to five dimensions. The aim of the reduction is to provide a starting point for exploring the consequences for the phenomenology of the theory, which has been extensively investigated in the original version.

We begin in Section 2.1 with a derivation of the improved version of the theory. The purpose of this is to show how its structure is entirely determined by the requirements of anomaly cancellation and supersymmetry and also to make explicit the form of the eleven dimensional theory from which our reduction begins. We will perform the reduction restricted to terms with two fermions or less and so for simplicity we neglect higher terms in the eleven dimensional description as well, though they were given in the original presentation of the improved theory [29]. The difference between the Hořava-Witten theory and the improved version is that in the manifold-with-boundaries picture, which we use everywhere, the coupling of the bulk three-form of the eleven dimensional theory to the boundaries can be simply given by a boundary condition, in which distributions do not appear, rather than by modifying its Bianchi identity, as was done by Hořava and Witten working on the orbifold S^1/\mathbb{Z}_2 , where distributions do appear. The importance of avoiding distributions is that in the Hořava-Witten theory they contrive to appear in the boundary action as $\delta(0)$ terms, which can be swept under the carpet at leading order in the gravitational coupling but make the theory inconsistent at higher orders. In the improved theory the other bulk fields, the metric and the gravitino, are given a consistent description by the inclusion of boundary terms in the action and have non-trivial boundary conditions which include contributions from the gauge fields on the boundary.

We perform a reduction to find the equivalent, for the improved version, of the five dimensional theory found by Lukas et al. from the Hořava-Witten version of Heterotic M-Theory [33]. Here we include everywhere the fermion sector (up to terms with two fermions), but the major difference of course is in the treatment of the boundaries. Thus the main results here are the boundary action and the boundary conditions on the fields derived from the eleven dimensional three-form.

These include fermion bilinear terms, which are particularly important if there is a gaugino condensate on the boundary as they give the coupling of the condensate to the bulk fields. Like [33] we allow the Hodge number $h^{1,1} \geq 1$ to be arbitrary but take $h^{2,1} = 0$ ($h^{p,q}$ is the number of independent, covariantly constant (p, q) -forms). A full reduction to include $h^{2,1} > 0$ would be a logical future extension of the work described here. The reason we include the $h^{1,1} > 1$ sector first is that there is a ‘non-zero mode’ induced by boundary sources whose charges are related to the $h^{1,1}$ basis elements of $(1,1)$ -forms on the Calabi-Yau space and so to fully describe this effect we must include the whole $h^{1,1} > 1$ sector. One might also hope that the complex structure moduli, which are associated with the $h^{2,1}$ $(2,1)$ -cycles of the Calabi-Yau, are stabilized at a higher scale than the Kähler moduli associated with the $h^{1,1}$ $(2,2)$ -cycles. In that case, which seems to be plausible both in the IIB case [58] and in Heterotic M-Theory [13], the low energy theory without a $(2,1)$ sector and the stabilization of the complex structure moduli could be treated independently.

We perform the reduction entirely directly (apart from the potential of the chiral multiplets on the boundary, where we use the structure of the general four dimensional supersymmetric theory to help us: Section 2.6.5) rather than by making use of the known five dimensional supergravity theories and four dimensional super-Yang-Mills theories to construct the reduced theory once the field content has been ascertained, as in [33]. The reason for this is that although the general forms of the five dimensional supergravity and the four dimensional Yang-Mills theory are known the supersymmetric coupled theory with boundary matter and bulk, five dimensional supergravity is not known. Therefore we must derive the coupling from that in the eleven dimensional theory, which we do know. As we will see in Section 2.1 the eleven dimensional theory is rigidly constructed: there is only one way of coupling the bulk supergravity to the boundary matter and the gauge coupling constant is fixed in terms of the gravitational coupling. If we want to make sure that the reduced theory is coupled according to the dictates of the eleven dimensional theory then we need to know not just the form of the five and four dimensional theories but also their relative normalizations. In addition to the explicit boundary conditions, there are bulk fields in the boundary action which we would not be able to find in the reduced theory other than directly. The direct reduction will give us, by virtue of the eleven dimensional supersymmetry and the choice of a Calabi-Yau as the reduction manifold, a supersymmetric supergravity-with-boundary-matter theory whose bulk-boundary couplings we could not find exactly by comparison with a ‘bottom-up’ approach, because the general coupled theory is not known in five dimensions. Thus performing a direct reduction of the bulk and boundary actions and of the boundary conditions from eleven to five dimensions is necessary to give us a reduced theory which respects the rigid constraints which anomaly cancellation and supersymmetry place upon the bulk-boundary coupling in eleven dimensions.

We perform the reduction with the standard embedding of the spin connection into the $SU(3)$ subgroup of the E_8 gauge group on $\partial\mathcal{M}_1$, which breaks it to E_6 . If both gauge groups are left unbroken then the sources for the non-zero mode on either boundary have opposite signs and so we cannot match both with a constant background flux (as would occur in Section 2.6.2 if we set $\text{tr}F^2 = 0$ on $\partial\mathcal{M}_1$ instead of using the standard embedding). Therefore some measure needs

to be taken to give us equal sources on either boundary (unless perhaps one has five-branes in the bulk to provide extra sources, but we do not consider these here). The standard embedding gives us an explicit and fairly simple way of achieving this which is familiar from the literature, thus facilitating comparison and possible extension to non-standard embeddings that have been considered before (in [59] for example).

The reduction splits an eleven dimensional spacetime, with boundaries, into the product of a five dimensional spacetime with boundaries and a Calabi-Yau threefold and we perform a Kaluza-Klein reduction to five dimensions. The reduction proceeds in several stages. First, in order to obtain the five dimensional theory we need to find a suitable ansatz for the split of the eleven dimensional fields into their five dimensional spacetime and their Calabi-Yau components. The ansatz we use and the reasons for choosing it are described in Section 2.3. Having found a suitable ansatz, we substitute it into the eleven dimensional action, boundary conditions and supersymmetry transformations, and then

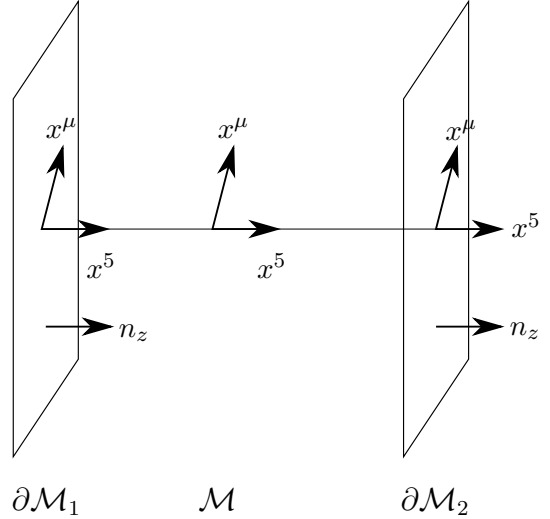


Figure 2.1: The five dimensional spacetime

turn the expressions we thus find into products of spacetime and Calabi-Yau parts. We integrate out the Calabi-Yau space in the action and identify the parts of boundary conditions and supersymmetry transformations which share identical Calabi-Yau components, in order to identify the boundary conditions and supersymmetry transformations of the five dimensional fields. We omit the details of this set of rather lengthy calculations, just giving the results through Sections 2.5 and 2.6, to spare the reader the large number of terms resulting from the product of combinations of spacetime and Calabi-Yau components of tensors with the various combinations of fermions. Finally we organize the results we have found to show the structure of the five dimensional supergravity in the bulk and the four dimensional super-Yang-Mills on the boundaries. We find the supergravity structure by dualizing the four-form field strength $G_{\alpha\beta\gamma\delta}$ resulting from the reduction to a scalar σ (Section 2.5.5) and identifying a hypermultiplet with quaternionic structure (Section 2.5.6). The identification, on the boundary $\partial\mathcal{M}_1$, of the scalars and fermions in the fundamental representation of E_6 as chiral multiplets is described in Section 2.6.5.

A comprehensive list of the conventions used in this chapter for the metric, gamma matrices, naming of indices, etc. is given in Appendix A. The model we find as a result of the reduction is summarized in Appendix E where the full action, boundary conditions and supersymmetry transformations are gathered together.

2.1 The Eleven Dimensional Theory

2.1.1 Supergravity With Boundaries

If we introduce boundaries into a theory described by an action we must take care that the variational principle that gives us the equations of motion from the action remains well defined. Let us first consider the simple example of a free scalar field on a manifold \mathcal{M} . In the absence of a boundary the actions $S = -\int_{\mathcal{M}} dv \nabla_I \phi \nabla^I \phi$ and $S' = \int_{\mathcal{M}} dv \phi \nabla^2 \phi$ are equivalent: they are related by the total derivative $-\int_{\mathcal{M}} dv \nabla_I (\phi \nabla^I \phi)$ which vanishes if the boundary is empty. In either case $\delta S = \delta S' = 2 \int dv \delta \phi \nabla^2 \phi$ giving us the equation of motion $\nabla^2 \phi = 0$. However, consider now the variation of S' *with* a boundary, $\partial\mathcal{M}$. Now $\delta S' = \int_{\mathcal{M}} dv \delta \phi \nabla^2 \phi + \int_{\partial\mathcal{M}} dv (\phi \nabla_N \delta \phi - \delta \phi \nabla_N \phi)$ where $\nabla_N = n^I \nabla_I$ is the component of the derivative parallel to the outward unit normal vector to the boundary, n^I . This is not consistent since given an arbitrary variation $\delta \phi$, the derivatives of the variation are determined, and so $\nabla_N \delta \phi$ is not independent. Since ∇_N is perpendicular to the boundary $\partial\mathcal{M}$, we cannot get rid of it by integrating by parts. Instead we must introduce a boundary term $S'_b = -\int_{\partial\mathcal{M}} dv \phi \nabla_N \phi$ whose variation, $\delta S'_b = -\int_{\partial\mathcal{M}} dv (\delta \phi \nabla_N \phi + \phi \nabla_N \delta \phi)$ cancels the $\nabla_N \delta \phi$ part of the variation of S' and leaves us with a consistent variational principle for $S' + S'_b$. In this simple case we can see that in fact S'_b is just the total derivative $S'_b = -\int_{\mathcal{M}} dv \nabla^I (\phi \nabla_I \phi)$ which is the difference between S and S' . Thus $S = -\int_{\mathcal{M}} dv \nabla_I \phi \nabla^I \phi$ is in this case a ‘better’ choice of action in the presence of a boundary. In a less simple case the boundary term will not generally be a total derivative in this way and we will have to augment the bulk action with a boundary term, as we will now see for gravity.

Gravity is described by the Einstein-Hilbert action $S_{EH} = \frac{1}{2\kappa^2} \int_{\mathcal{M}} dv R$. The variational principle gives us the equation of motion

$$\begin{aligned} 0 = \delta S_{EH} &= \frac{1}{2\kappa^2} \int_{\mathcal{M}} dv \left(\frac{1}{2} g^{IJ} R - R^{IJ} \right) \delta g_{IJ} \\ &\Rightarrow R_{IJ} - \frac{1}{2} g_{IJ} R = 0 \end{aligned}$$

which is the Einstein equation in vacuum. Once again this variation includes a total derivative. If we introduce a boundary, $\partial\mathcal{M}$ to the manifold \mathcal{M} the boundary terms arising from the total derivative mean that the variational problem is no longer well posed, in a similar way to that which we just saw for the scalar field.

$$\begin{aligned} \delta S_{EH} &= \frac{1}{2\kappa^2} \int_{\mathcal{M}} dv \left(\frac{1}{2} g^{IJ} R - R^{IJ} \right) \delta g_{IJ} \\ &\quad + \frac{1}{2\kappa^2} \int_{\partial\mathcal{M}} dv \left(-K^{IJ} \delta h_{IJ} - n^I h^{JK} \nabla_I \delta h_{JK} + 2K n^I \delta n_I \right) \end{aligned}$$

where h_{IJ} is the induced metric on the boundary, n^I is the outward pointing unit normal vector on the boundary and $K_{IJ} = h_I^K h_J^L \nabla_K n_L$ is the extrinsic curvature. We only have freedom to vary independently δg_{IJ} and δh_{IJ} , not also $n^I \nabla_I \delta h_{JK}$ or δn_I . The resolution is to introduce the Gibbons-Hawking-York boundary term, that is the extrinsic curvature scalar, K , into the boundary

action. $\delta K = \frac{1}{2}h^{IJ}n^K\nabla_K\delta h_{IJ} - Kn^K\delta n_K$ and so the variation of the action $\frac{1}{2\kappa^2}\int_{\mathcal{M}}dvR + \frac{1}{2\kappa^2}\int_{\partial\mathcal{M}}dv(2K)$ depends only on δg_{IJ} in the bulk and δh_{IJ} on the boundary and we can consistently apply the variational principle. Here $\int_{\partial\mathcal{M}}dvK$ is not equivalent to a total derivative term in the bulk, so we necessarily have an action with both a bulk and a boundary term.

Heterotic M-Theory is eleven dimensional supergravity on a manifold with boundary. The requirements of supersymmetry and anomaly cancellation dictate the structure of the theory completely. We have an eleven dimensional bulk with two disjoint, spatially separated, ten dimensional boundaries. In the bulk we have eleven dimensional supergravity [2]. In the presence of a boundary the bulk Einstein-Hilbert action must, as we have just seen, be supplemented by an extrinsic curvature term [21, 60] $\frac{1}{2\kappa_{11}^2}\int dv(2K)$. The bulk Rarita-Schwinger action also requires a boundary term [22] $\int dv(\frac{1}{2}\bar{\Psi}_A\Gamma^{AB}\Psi_B)$. Finally, in the supergravity action there is torsion whose variation gives a boundary term and so to cancel this we need an additional piece in the boundary action [29] $\int dv(\bar{\Psi}_A\Gamma^A\Psi_N)$. So we have as a starting point the action

$$S = \frac{1}{2\kappa_{11}^2}\int_{\mathcal{M}_{11}}dv\mathcal{L}(\mathcal{M}_{11}) + \frac{1}{2\kappa_{11}^2}\int_{\partial\mathcal{M}_{10,1}}dv\mathcal{L}(\partial\mathcal{M}_{10,1}) + \frac{1}{2\kappa_{11}^2}\int_{\partial\mathcal{M}_{10,2}}dv\mathcal{L}(\partial\mathcal{M}_{10,2}) \quad (2.1.1)$$

with

$$\begin{aligned} \mathcal{L}(\mathcal{M}_{11}) = & R(\Omega) + \bar{\Psi}_I\Gamma^{IJK}D_J(\Omega^*)\Psi_K \\ & - \frac{1}{96}(\bar{\Psi}_M\Gamma^{IJKLMO}\Psi_O + 12\bar{\Psi}^I\Gamma^{JK}\Psi^L)G_{IJKL}^* \\ & - \frac{1}{48}G_{IJKL}G^{IJKL} + \frac{1}{12^4}\epsilon^{I_1\dots I_{11}}G_{I_1I_2I_3I_4}G_{I_5I_6I_7I_8}C_{I_9I_{10}I_{11}} \end{aligned} \quad (2.1.2)$$

$$\mathcal{L}(\partial\mathcal{M}_{10,1}) = -2K + \frac{1}{2}\bar{\Psi}_A\Gamma^{AB}\Psi_B - \bar{\Psi}_A\Gamma^A\Psi_N \quad (2.1.3)$$

$$\mathcal{L}(\partial\mathcal{M}_{10,2}) = 2K - \frac{1}{2}\bar{\Psi}_A\Gamma^{AB}\Psi_B + \bar{\Psi}_A\Gamma^A\Psi_N \quad (2.1.4)$$

where I, J, K, \dots are eleven dimensional spacetime indices; A, B, C, \dots are ten dimensional spacetime indices; $K = g^{AB}K_{AB}$ is the trace of the extrinsic curvature $K_{AB} = h_A^I h_B^J \nabla_I n_J$; n^I is the unit normal vector to the boundary which is inward pointing on $\partial\mathcal{M}_{10,1}$ and outward pointing on $\partial\mathcal{M}_{10,2}$ (so that its definition can be consistently extended across the bulk); $\Omega_{IJK} = \omega_{IJK} - \frac{1}{4}(\bar{\Psi}_I\Gamma_J\Psi_K - \bar{\Psi}_I\Gamma_K\Psi_J + \bar{\Psi}_J\Gamma_K) - \frac{1}{8}\bar{\Psi}^L\Gamma_{IJKLM}\Psi^M$ where ω_{IJK} is the Levi-Civita spin connection, the supercovariant connection is $\hat{\Omega}_{IJK} = \omega_{IJK} - \frac{1}{4}(\bar{\Psi}_I\Gamma_J\Psi_K - \bar{\Psi}_I\Gamma_K\Psi_J + \bar{\Psi}_J\Gamma_K)$ and $\Omega^* = \frac{1}{2}(\Omega + \hat{\Omega})$; the supercovariant field strength is $\hat{G}_{IJKL} = G_{IJKL} + 3\bar{\Psi}_{[I}\Gamma_{JK}\Psi_{L]}$ and $G_{IJKL}^* = \frac{1}{2}(G_{IJKL} + \hat{G}_{IJKL})$. The supersymmetry transformations are (with supersymmetry parameter S)

$$\delta_S E^I{}_J = -\frac{1}{2}\bar{S}\Gamma^I\Psi_J \quad (2.1.5)$$

$$\delta_S \Psi_I = D_I S - \frac{1}{288}(\Gamma_I{}^{JKLM} - 8\delta_I^J\Gamma^{KLM})G_{JKLM}S \quad (2.1.6)$$

$$\delta_S C_{IJK} = -\frac{3}{2}\bar{S}\Gamma_{[IJ}\Psi_{K]} \quad (2.1.7)$$

where $E^{\hat{I}}_J$ is the eleven dimensional vielbein. This action gives boundary conditions (with the upper signs on $\partial\mathcal{M}_{10,1}$ and the lower signs on $\partial\mathcal{M}_{10,2}$)

$$K_{AB} = \mp\kappa_{11}^2 T_{AB} = 0 \quad (2.1.8)$$

$$P_- \Psi_A = 0 \quad (2.1.9)$$

while the boundary condition

$$C_{ABC} = 0 \quad (2.1.10)$$

must be imposed in addition to the action, similarly to a Bianchi identity which is not imposed by variation of an action. The supersymmetry parameter is chiral ($P_- S = 0$) on the boundary, with the chiral projection operator being $P_{\pm} = \frac{1}{2}(\mathbf{1} \pm \Gamma_N)$.

This action is supersymmetric but it has a quantum anomaly since the gravitino is a chiral fermion on the ten dimensional boundary [7].

2.1.2 Gravitational Anomaly Cancellation

The anomaly in ten dimensional may be found using a twelve dimensional index theorem [3] (I have found [4, 61] helpful expositions of anomaly cancellation and I follow here the notation of [4]). The anomalous variation of the Euclidean quantum effective action under diffeomorphisms, $\delta\Gamma_E$ may be calculated from a twelve-form polynomial in the curvature as

$$\delta\Gamma_E = -i \int_{\mathcal{M}_{10}} \hat{I}_{10}^1 \quad (2.1.11)$$

where \hat{I}_{10}^1 is given by

$$d\hat{I}_{10}^1 = \delta\hat{I}_{11}, \quad d\hat{I}_{11} = \hat{I}_{12} \quad (2.1.12)$$

and

$$\begin{aligned} \hat{I}_{12}^{gravitino} &= \frac{1}{4} \left(-\hat{I}_{12}^{spin\frac{3}{2}}(R) + \hat{I}_{12}^{spin\frac{1}{2}}(R) \right) \\ &= -\frac{1}{4} \cdot 2\pi \cdot \frac{1}{(4\pi)^6} \left(-\frac{495}{5670} \text{tr}R^6 + \frac{225}{4320} \text{tr}R^4 \text{tr}R^2 - \frac{63}{10368} (\text{tr}R^2)^3 \right) \\ &\quad + \frac{1}{4} \cdot 2\pi \cdot \frac{1}{(4\pi)^6} \left(\frac{1}{5670} \text{tr}R^6 + \frac{1}{4320} \text{tr}R^4 \text{tr}R^2 + \frac{1}{10368} (\text{tr}R^2)^3 \right) \\ &= \frac{1}{8(4\pi)^5} \left(\frac{496}{5670} \text{tr}R^6 - \frac{224}{4320} \text{tr}R^4 \text{tr}R^2 + \frac{64}{10368} (\text{tr}R^2)^3 \right) \end{aligned} \quad (2.1.13)$$

where R is the curvature two-form with two indices regarded as $SO(1,9)$ matrix indices so that, for example, $\text{tr}R^2 = \frac{1}{4} \text{tr}(R_{AB}R_{CD}) \mathbf{d}x^A \wedge \mathbf{d}x^B \wedge \mathbf{d}x^C \wedge \mathbf{d}x^D = \frac{1}{4} R_{ABE}{}^F R_{CDF}{}^E \mathbf{d}x^A \wedge \mathbf{d}x^B \wedge \mathbf{d}x^C \wedge \mathbf{d}x^D$.

A theory with local quantum anomalies is not well defined so we must find a mechanism to cancel them. The factorizable part of the anomaly, that is the $\text{tr}R^4 \text{tr}R^2$ and $(\text{tr}R^2)^3$ terms, can be cancelled by a Green-Schwarz type mechanism (discussed below) but to cancel the irreducible part, $\text{tr}R^6$, it is necessary to introduce some gauge fields on the ten dimensional boundary so that the irreducible part of the quantum anomaly of the Yang-Mills theory can cancel the

irreducible part of the anomaly from the gravitino. The gauge theory has gauge, gravity and mixed anomalies. The pure gravity part of the Yang-Mills anomaly polynomial is

$$\begin{aligned}\hat{I}_{12}^{YM}(R) &= -\frac{1}{2}\hat{I}_{12}^{\hat{spin}\frac{1}{2}}(R) \\ &= -\frac{1}{4(4\pi)^5}\left(\frac{1}{5670}\text{tr}R^6 + \frac{1}{4320}\text{tr}R^4\text{tr}R^2 + \frac{1}{10368}(\text{tr}R^2)^3\right)\end{aligned}\quad (2.1.14)$$

so we can see that we need 248 gauginos on each boundary so that $248\hat{I}_{12}^{YM}(R)$ cancels the $\text{tr}R^6$ part of $\hat{I}_{12}^{\hat{gravitino}}$, thus cancelling the irreducible part of the anomaly. In the ten dimensional string theory anomaly cancellation, the possible groups (having rank 496) are $SO(32)$ and $E_8 \times E_8$. Since $SO(32)$ is irreducible it cannot be split between the two boundaries and so we can only have $E_8 \times E_8$ with an E_8 gauge theory on each boundary giving us the 248 gauginos we need. The criterion for choosing acceptable gauge groups is that $\text{tr}F^6$ must factorize so that the Green-Schwarz mechanism can be used to cancel all of the remaining anomalies.¹ For E_8 , $\text{tr}F^6 = \frac{1}{8}\text{tr}(F^2)^3$ and also $\text{tr}F^4 = \frac{3}{10}\text{tr}(F^2)^2$.

The overall anomaly is

$$\begin{aligned}\hat{I}_{12} &= \hat{I}_{12}^{\hat{gravitino}} + \hat{I}_{12}^{YM}(R) + \hat{I}_{12}^{YM}(R, F) + \hat{I}_{12}^{YM}(F) \\ &= \frac{1}{8(4\pi)^5}\left(\frac{496}{5670}\text{tr}R^6 - \frac{224}{4320}\text{tr}R^4\text{tr}R^2 + \frac{64}{10368}(\text{tr}R^2)^3\right) \\ &\quad - \frac{1}{4(4\pi)^5}\text{tr}(\mathbb{1}_{248})\left(\frac{1}{5670}\text{tr}R^6 + \frac{1}{4320}\text{tr}R^4\text{tr}R^2 + \frac{1}{10368}(\text{tr}R^2)^3\right) \\ &\quad - \frac{2\pi}{2}\left(\frac{-30}{2(2\pi)^2}\right)\text{tr}F^2\frac{1}{(4\pi)^4}\left(\frac{1}{360}\text{tr}R^4 + \frac{1}{288}(\text{tr}R^2)^2\right) \\ &\quad - \frac{2\pi}{2}\cdot\frac{30}{24(2\pi)^4}\text{tr}F^4\frac{1}{(4\pi)^2}\frac{1}{12}\text{tr}R^2 \\ &\quad - \frac{2\pi}{2}\cdot\left(\frac{-30}{720(2\pi)^6}\text{tr}F^6\right) \\ &= \frac{1}{(4\pi)^5}\left(-\frac{1}{48}\text{tr}R^2\text{tr}R^4 + \frac{1}{24}\text{tr}F^2\text{tr}R^4 - \frac{1}{192}(\text{tr}R^2)^3 + \frac{5}{96}\text{tr}F^2(\text{tr}R^2)^2\right. \\ &\quad \left.- \frac{1}{8}(\text{tr}F^2)^2\text{tr}R^2 + \frac{1}{12}(\text{tr}F^2)^3\right) \\ &= \frac{1}{24(4\pi)^5}\left(\text{tr}F^2 - \frac{1}{2}\text{tr}R^2\right)\left(\text{tr}R^4 + \frac{1}{4}(\text{tr}R^2)^2 - 2\text{tr}F^2\text{tr}R^2 + 2(\text{tr}F^2)^2\right) \\ &= \frac{1}{12(4\pi)^5}I_4\left(I_4^2 + \frac{1}{2}\text{tr}R^4 - \frac{1}{8}(\text{tr}R^2)^2\right) \\ &= \frac{1}{12(4\pi)^5}(I_4^3 + I_4X_8)\end{aligned}\quad (2.1.15)$$

where $I_4 = \text{tr}F^2 - \frac{1}{2}\text{tr}R^2$ and $X_8 = \frac{1}{2}\text{tr}R^4 - \frac{1}{8}(\text{tr}R^2)^2$

¹Note that for E_8 we define $\text{tr} = \frac{1}{30}\text{Tr}$ where Tr is the trace in the adjoint of E_8 . For E_8 there is no fundamental representation smaller than the adjoint in which to define tr . This definition is given by analogy with $SO(n)$ where a distinct fundamental does exist and gives the correct relation for the fundamental of the $SO(16)$ subgroup of E_8 .

We have cancelled the irreducible part of the gravity anomaly, so that the anomaly is now factorizable. Therefore it can be cancelled by classical inflow from the bulk fields, but first we need to find the boundary conditions consistent with supersymmetry now that we have a Yang-Mills gauge theory on the boundary and thus a non-zero energy-momentum localized on the boundary.

2.1.3 Supersymmetric Boundary Conditions

Since there is a gauge theory on the boundary, the stress-energy tensor is non-vanishing and so the boundary has extrinsic curvature, from (2.1.8). This makes the boundary condition on Ψ_A and G , (2.1.9) and 2.1.10, inconsistent with supersymmetry since

$$\begin{aligned} \delta_S(P_- \Psi_A) &= P_- \delta_S \Psi_A \\ &= P_- D_A S - \frac{1}{288} (\Gamma_A^{JKLM} - 8\delta_A^J \Gamma^{KLM}) G_{JKLM} S \end{aligned} \quad (2.1.16)$$

but since $P_- S = 0$, $D_A(P_- S) = 0$ and so

$$\begin{aligned} P_- D_A S &= (D_A P_-) S \\ &= \mp \frac{1}{2} K_{AB} \Gamma^B S \neq 0 \end{aligned} \quad (2.1.17)$$

which does not vanish when $T^{AB} \neq 0$ as (2.1.8) shows. Since the remaining part of the variation does vanish if $G = 0$, we see that the non-vanishing energy-momentum now present on the boundary breaks the supersymmetry of the theory, which we need to restore. We accomplish this by extending the boundary conditions (2.1.9) and (2.1.10). By writing down all the terms in the boundary fields that have the right tensor structure and requiring the variation of the boundary conditions to vanish under the supersymmetry transformations (2.1.34-2.1.38) it can be shown that the unique way in which we may recover supersymmetry is to have

$$P_- \Psi_A = \mp \frac{\kappa_{11}^2}{\lambda^2} (\Gamma_A^{BC} - 10\delta_A^B \Gamma^C) \text{tr}(F_{BC} \chi) \quad (2.1.18)$$

$$C = \mp \frac{\kappa_{11}^2}{\lambda^2} (\omega_3^Y + \omega_3^X) \quad (2.1.19)$$

where $\frac{1}{\lambda^2}$ is the coupling constant of the Yang-Mills theory, $d\omega_3^Y = \text{tr}(F \wedge F)$ and $(\omega_3^X)_{ABC} = -\frac{1}{4} \text{tr}(\bar{\chi} \Gamma_{ABC} \chi)$.

2.1.4 Gauge Anomaly Cancellation

We start by noting that the three-form field in supergravity has (in the bulk) an Abelian symmetry $C \rightarrow C + da$ since under this transformation G is invariant and $C \wedge G \wedge G$ is a total derivative. If we associate this transformation of C to the gauge and diffeomorphism transformations of the boundary we can arrange classical inflow from that total derivative in the bulk to cancel the quantum anomaly on the boundary.

Consider first just the gauge part of the anomaly, i.e. just the $\text{tr}F^2$ parts. We choose a so that on the boundary $da = \mp \frac{\kappa_{11}^2}{\lambda^2} \delta\omega_3^Y$, consistent with the boundary

condition (2.1.19). Now we note that the $C \wedge G \wedge G$ term in the bulk action contributes a total derivative which gives a variation $\frac{1}{12\kappa_{11}^2} \int_{\partial\mathcal{M}_1} a \wedge G \wedge G - \frac{1}{12\kappa_{11}^2} \int_{\partial\mathcal{M}_2} a \wedge G \wedge G$ and since on the boundary $G = \mp \frac{\kappa_{11}^2}{\lambda^2} \text{tr} F^2 + \dots = \mp \frac{\kappa_{11}^2}{\lambda^2} I_4 + \dots$ this gives a contribution to the anomaly polynomial of $-\frac{1}{12} \frac{\kappa_{11}^2}{\lambda^4} I_4^3$. The gauge part of the anomaly is then cancelled if the gauge coupling $\frac{1}{\lambda^2}$ is given by

$$\frac{\kappa_{11}^2}{\lambda^2} = \frac{1}{4\pi} \left(\frac{\kappa_{11}}{4\pi} \right)^{\frac{2}{3}} \quad (2.1.20)$$

It is interesting to note that one might, as we did initially, take the supersymmetry transformations of the Yang-Mills theory, (2.1.37) and (2.1.38), to both have the opposite sign, as the supersymmetry of the Yang-Mills theory itself is obviously unaffected by the sign of the arbitrary parameter. In that case one finds that the necessary boundary condition on G has the opposite sign. Then the gauge anomaly would be cancelled by $\frac{\kappa_{11}^2}{\lambda^2}$ having the same magnitude, but negative sign. Hence we see that the requirements of anomaly cancellation give us extra constraints on the supersymmetry: the relative sign of the supersymmetry parameter in the transformations of the bulk fields and the transformations of the boundary fields cannot be freely chosen.

To cancel the full I_4^3 part of the anomaly, we use the same mechanism with the boundary condition on C extended to

$$C = \mp \frac{\kappa_{11}^2}{\lambda^2} \left(\omega_3^Y - \frac{1}{2} \omega_3^L \right) + \text{fermion terms} \quad (2.1.21)$$

where $d\omega_3^L = \text{tr} R^2$. In order to do this supersymmetrically we must have R^2 terms in the boundary action. These can be included [31] by constructing a Yang-Mills like ‘Lorentz multiplet’, inspired by a method introduced by Bergshoeff and de Roo [62, 63] but since we will not use the fermion terms it suffices to say that the $\text{tr}(F_{AB}F^{AB})$ term in the boundary Lagrangian is extended to $\text{tr}(F_{AB}F^{AB}) - \frac{1}{2} \text{tr}(R_{AB}R^{AB})$. Since we now have (2.1.21) instead of just (2.1.19), the inflow from the $C \wedge G \wedge G$ term completely cancels the I_4^3 part of the anomaly.

We can cancel the $I_4 X_8$ part of the anomaly by introducing a Green-Schwarz term in the bulk $S_{GS} = -\frac{1}{\kappa_{11}^2 12(4\pi)^5} \int_{\mathcal{M}} C \wedge X_8$. Since X_8 is invariant and $dX_8 = 0$, $\delta S_{GS} = \frac{1}{\kappa_{11}^2 12(4\pi)^5} \int_{\partial\mathcal{M}_1} a \wedge X_8 - \frac{1}{\kappa_{11}^2 12(4\pi)^5} \int_{\partial\mathcal{M}_2} a \wedge X_8$. Hence the contribution of δS_{GS} to the anomaly polynomial is $-\frac{1}{12(4\pi)^5} I_4 \wedge X_8$ cancelling the second term in the quantum anomaly. Since S_{GS} is $\mathcal{O}(R^4)$ and we will be working in the low curvature regime, keeping terms at most $\mathcal{O}(R^2)$, it will not appear henceforth.

2.1.5 The Lorentz Multiplet

We have just seen that in order to cancel the gravitational quantum anomaly on the boundaries, we must introduce R^2 terms into the boundary action. The question then is how to do this in a way that respects supersymmetry. The R^2 term appears in the boundary condition of G as part of $I_4 = \text{tr}(F_{AB}F^{AB}) - \frac{1}{2} \text{tr}(R_{AB}R^{AB})$; it appears in exactly the same way as F^2 except for a factor of $-\frac{1}{2}$. The way to include the R^2 terms on the boundary is, as was explained in [31], to exploit this analogy between the Yang-Mills curvature and the gravitational

curvature by considering the latter as the curvature of a gauge field of the Lorentz symmetry group $SO(1, 9)$. This is done by identifying a ‘Lorentz multiplet’ built to give supersymmetry transformations as close as possible to an exact analogy of the transformations of the Yang-Mills multiplet. Since we know already the supersymmetric action for a Yang-Mills multiplet we can exploit the analogy to construct a supersymmetric action for the Lorentz multiplet. The construction in eleven dimensions is complicated, as compared to the ten dimensional case, by the components of the spacetime curvature and the gravitino normal to the boundary. These are necessary for the consistency of the construction, and are included in [31], but for simplicity we will ignore them for this overview.

The supersymmetry variation of the gravitino, (2.1.6), is the covariant derivative of the supersymmetry parameter S plus some terms depending on the flux G . If we use this to define a new derivative $D_I^{(L)}$ so that $\delta\Psi_I = D_I^{(L)}S$ then we can construct a gravitino curvature which has two ‘ $SO(1, 9)$ gauge indices’

$$\Psi_{AB} = 2D_{[A}^{(L)}\Psi_{B]} \quad (2.1.22)$$

and which varies as

$$\delta\Psi_{AB} = \left[D_A^{(L)}, D_B^{(L)} \right] S = \frac{1}{4} R_{CDAB}^{(L)} \Gamma^{CD} S \quad (2.1.23)$$

and so is our analogue of the gaugino χ .

Naively one would expect the analogue of the gauge field to be just the spin connection Ω_{ABC} , but this does not transform in the proper way. The field which does transform properly contains an additional G -flux term: it is

$$\Omega_{ABC}^- = \Omega_{ABC} - \frac{1}{2} G_{NABC} \quad (2.1.24)$$

This pair of fields then transforms almost as a Yang-Mills multiplet

$$\delta\Omega_{ABC}^- = \frac{1}{2} \bar{S} \Gamma_A \Psi_{BC} + y_{ABC} \quad (2.1.25)$$

$$\delta\Psi_{AB} = \frac{1}{4} R_{CDAB}^- \Gamma^{CD} S + y_{AB} \quad (2.1.26)$$

where y_{ABC} and y_{AB} are correction terms which depend on the G -flux and are higher order in the derivative expansion used in [31]. To construct the supersymmetric R^2 action we then add this Lorentz multiplet to the action and boundary conditions just as the Yang-Mills multiplet appears but with a coupling constant which is $-\frac{1}{2}$ of the Yang-Mills one. The R^2 action is

$$S_{R^2} = -\frac{1}{2\lambda^2} \int_{\partial\mathcal{M}_{10}} \left(-\frac{1}{4} R_{ABCD}^- R^{-ABDC} + \frac{1}{2} \bar{\Psi}_{BC} \Gamma^A D_A \Psi^{CB} - \frac{1}{4} \bar{\Psi}_C \Gamma^{AB} \Gamma^C R_{ABDE} \Psi^{ED} \right) \quad (2.1.27)$$

with R^- the curvature of the connection Ω^- . The boundary condition on C is

$$C_{ABC} = \mp \frac{\kappa_{11}^2}{\lambda^2} (\omega_{ABC}^Y + \omega_{ABC}^X) \pm \frac{\kappa_{11}^2}{2\lambda^2} (\omega_{ABC}^L + \omega_{ABC}^\Psi) \quad (2.1.28)$$

where $d\omega^L = \text{tr}(R^- \wedge R^-)$ and $\omega_{ABC}^\Psi = -\frac{1}{4}\bar{\Psi}_{DE}\Gamma_{ABC}\Psi^{ED}$.

In the reduction to five dimensions of this theory described below we will assume that the spacetime curvature is small enough that we can neglect almost all of this structure, except for the contribution of the Calabi-Yau components of R^2 as a boundary source of G -flux (Section 2.6.2). This approximation was not our first thought, but as explained in Section 2.6.3 it proved impractical to include more than just this piece.

2.1.6 Heterotic M-Theory

In summary, the complete eleven dimensional theory, which is the starting point for our reduction to five dimensions, is supergravity on a manifold with boundary

$$\begin{aligned}
 S_{SG} = & \frac{1}{2\kappa_{11}^2} \int_{\mathcal{M}_{11}} dv \left(R(\Omega) + \bar{\Psi}_I \Gamma^{IJK} D_J (\Omega^*) \Psi_K \right. \\
 & - \frac{1}{96} (\bar{\Psi}_M \Gamma^{IJKLMO} \Psi_O + 12 \bar{\Psi}^I \Gamma^{JK} \Psi^L) G_{IJKL}^* \\
 & \left. - \frac{1}{48} G_{IJKL} G^{IJKL} + \frac{1}{12^4} \epsilon^{I_1 \dots I_{11}} G_{I_1 I_2 I_3 I_4} G_{I_5 I_6 I_7 I_8} C_{I_9 I_{10} I_{11}} \right) \\
 & + \frac{1}{2\kappa_{11}^2} \int_{\partial\mathcal{M}_{10,1}} dv \left(-2K + \frac{1}{2} \bar{\Psi}_A \Gamma^{AB} \Psi_B - \bar{\Psi}_A \Gamma^A \Psi_N \right) \\
 & + \frac{1}{2\kappa_{11}^2} \int_{\partial\mathcal{M}_{10,2}} dv \left(2K - \frac{1}{2} \bar{\Psi}_A \Gamma^{AB} \Psi_B + \bar{\Psi}_A \Gamma^A \Psi_N \right) \quad (2.1.29)
 \end{aligned}$$

with an E_8 Yang-Mills multiplet and a Lorentz multiplet on each boundary (though here we neglect the fermion terms of the Lorentz multiplet)

$$\begin{aligned}
 S_{g1} = & \frac{1}{\lambda^2} \int_{\partial\mathcal{M}_{10,1}} dv \left(-\frac{1}{4} \left(\text{tr}(F_{AB} F^{AB}) - \frac{1}{2} \text{tr}(R_{AB} R^{AB}) \right) \right. \\
 & \left. + \frac{1}{2} \text{tr}(\bar{\chi} \Gamma^A D_A (\Omega^{**}) \chi) - \frac{1}{4} \bar{\Psi}_C \Gamma^{AB} \Gamma^C \text{tr}(F_{AB}^* \chi) \right) \quad (2.1.30)
 \end{aligned}$$

where $\Omega^{**} = \frac{1}{2}(\Omega + \Omega^*)$, the supercovariant field strength is $\hat{F}_{AB} = F_{AB} - 2\bar{\Psi}_{[A}\Gamma_{B]}\chi$ and $F_{AB}^* = \frac{1}{2}(F_{AB} + \hat{F}_{AB})$. S_{g2} is identical to S_{g1} but for $\partial\mathcal{M}_{10,1} \rightarrow \partial\mathcal{M}_{10,2}$. The theory is completed by the specification of the boundary condition on C . In a similar way to the Bianchi identity which cannot be imposed by an action, this boundary condition must be imposed in addition to the action. It is

$$C = \begin{cases} -\frac{\kappa_{11}^2}{\lambda^2} (\omega_3^Y + \frac{1}{2}\omega_3^L + \omega_3^X) & \text{on } \partial\mathcal{M}_{10,1} \\ \frac{\kappa_{11}^2}{\lambda^2} (\omega_3^Y + \frac{1}{2}\omega_3^L + \omega_3^X) & \text{on } \partial\mathcal{M}_{10,2} \end{cases} \quad (2.1.31)$$

The remaining boundary conditions, which do come from the action but are given here for completeness, are

$$K_{AB} - g_{AB}K = \begin{cases} -\kappa_{11}^2 T_{AB} & \text{on } \partial\mathcal{M}_{10,1} \\ \kappa_{11}^2 T_{AB} & \text{on } \partial\mathcal{M}_{10,2} \end{cases} \quad (2.1.32)$$

$$P_- \Psi_A = \begin{cases} -\frac{\kappa_{11}^2}{12\lambda^2} (\Gamma_A^{BC} \chi - 10\delta_A^B \Gamma^C) \text{tr}(F_{BC}\chi) & \text{on } \partial\mathcal{M}_{10,1} \\ \frac{\kappa_{11}^2}{12\lambda^2} (\Gamma_A^{BC} \chi - 10\delta_A^B \Gamma^C) \text{tr}(F_{BC}\chi) & \text{on } \partial\mathcal{M}_{10,2} \end{cases} \quad (2.1.33)$$

Finally the supersymmetry transformations are

$$\delta_S E^{\hat{I}}{}_J = -\frac{1}{2} \bar{S} \Gamma^{\hat{I}} \Psi_J \quad (2.1.34)$$

$$\delta_S \Psi_I = D_I(\hat{\Omega}) S - \frac{1}{288} (\Gamma_I^{JKLM} - 8\delta_I^J \Gamma^{KLM}) \hat{G}_{JKLM} S \quad (2.1.35)$$

$$\delta_S C_{IJK} = -\frac{3}{2} \bar{S} \Gamma_{[IJ} \Psi_{K]} \quad (2.1.36)$$

$$\delta_S A_A = \frac{1}{2} \bar{S} \Gamma_A \chi \quad (2.1.37)$$

$$\delta_S \chi = \frac{1}{4} \Gamma^{AB} \hat{F}_{AB} S \quad (2.1.38)$$

2.2 Kaluza-Klein Reduction Over a Calabi-Yau Threefold

To perform a dimensional reduction we consider the spacetime of a full theory in some higher dimension as the product of the lower dimensional spacetime of the reduced theory with the extra dimensions which we are hiding. The essence of a Kaluza-Klein reduction is that we take the extra dimensions to be compact and further to be small. Since the extra dimensions are compact the momenta of the fields of the full theory in those directions are quantized and the fields of the full theory can be expressed as a Fourier series: a sum over the possible momenta in the extra dimensions of fields depending only on the co-ordinates of the reduced theory. From the point of view of the reduced theory these Fourier modes are independent fields with various masses, which depend on their momenta in the extra directions. As the extra dimensions are small, the mass associated with a quantum of extra dimensional momentum is large and so at low energies compared to that mass scale we can truncate the reduced theory and consider only the less massive fields. Indeed we will take the size of the Calabi-Yau three-fold which describes the extra dimensions to be small enough that we need consider only the leading terms, i.e. those which have zero momentum in the Calabi-Yau directions.

The momentum operator is just the spacetime derivative, so zero momentum modes are those which are covariantly constant. The fields of the reduced theory then are those given by the fields of the full theory whose extra dimensional part is a constant tensor (for bosonic fields) or spinor (for fermionic fields). To identify the reduced fields we need to know all the constant tensors and spinors of the extra dimensions and then to see which of those can contribute to each of the fields of the full theory.

Calabi-Yau manifolds are those which have a single covariantly constant spinor, which we call u_A . This is the reason for choosing them for the reduction, since the fermionic supersymmetry parameter of the eleven dimensional theory then has a single zero mode given by that single constant spinor and so we have a single supersymmetry parameter in the five dimensional theory, which therefore possesses $\mathcal{N} = 1$ supersymmetry. One can also construct constant spinors by acting on u_A with gamma matrices, $\gamma_a u_A$ for example. The tensor structure precludes such objects from contributing to the supersymmetry parameter (there are no indices there to contract with those of the gamma matrices) but they can, and do, give contributions to the other fermionic fields.

A Calabi-Yau manifold is a complex manifold, so we categorize differential forms by the number of holomorphic and anti-holomorphic components: a (p, q) -form has p holomorphic and q anti-holomorphic components. There is no covariantly constant vector on a Calabi-Yau space by definition. There is a single holomorphic three-form, Ω_{abc} : there can only be one since a holomorphic index can only take the three values 1, 2, 3 and so a totally anti-symmetric object with three indices only has one independent component; thus all constant holomorphic three-forms can only differ by an overall factor. There are no constant (2,0)-forms because if there were we could contract them the three-form Ω to get a constant vector, which there cannot be. The number of independent, constant (2,1)- and (1,1)-forms are properties of particular Calabi-Yau manifolds, known as the Hodge numbers $h^{2,1}$ and $h^{1,1}$ respectively. We consider only cases with $h^{2,1} = 0$ and so we have no (2,1)-forms, nor do we have their complex conjugates, (1,2)-forms. We do however allow $h^{1,1}$ to be arbitrary and so we have a basis of (1,1) forms, $\omega_{i\bar{a}\bar{b}}$ with $h^{1,1}$ elements labelled by i . (3,1)-forms are Hodge duals of (0,2)-forms so there are none. (2,2)-forms are dual to (1,1)-forms so the basis $\omega_{i\bar{a}\bar{b}}$ gives us a dual basis $\nu^i_{\bar{a}\bar{b}\bar{c}\bar{d}}$ with $h^{1,1}$ elements. (3,2)-forms are dual to vectors and thus absent. (3,3)-forms are proportional to the volume element of the Calabi-Yau and dual to scalars. In summary the constant tensors we have available to construct the Calabi-Yau parts of our eleven dimensional fields are scalars, Ω_{abc} and $\omega_{i\bar{a}\bar{b}}$.

In performing the reduction we need to consider two types of object. The action (or parts thereof), which is an integral expression, and the boundary conditions and supersymmetry transformations which are tensor or spinor equations. Once we have truncated to just the zero energy modes, the eleven dimensional action is just a sum of scalars which are constant over the Calabi-Yau, so once we have contracted all the Calabi-Yau tensor indices and spinors the only part remaining is the volume element, which gives $\int_{CY} dv = v$ with v the volume of the reference Calabi-Yau whose spacetime dependence has been scaled out into the volume modulus V and the $h^{1,1}$ moduli b^i . On the other hand for the boundary condition and supersymmetry transformations, each eleven dimensional equation will generally give us several five dimensional equations since it must be satisfied separately for each independent Calabi-Yau tensor or spinor which is present.

2.3 Reduction Ansatz

We now move on to find the appropriate ansatz for our reduction of the improved Heterotic M-Theory from an eleven dimensional bulk with ten dimensional boundaries to a five dimensional bulk, \mathcal{M} , with four dimensional boundaries, $\partial\mathcal{M}_1$ and $\partial\mathcal{M}_2$. The reduction is performed over a Calabi-Yau threefold, X , which is taken to have $h^{1,1}$ (1, 1) moduli and no (2, 1) moduli. As in 11 dimensions the normal vector is inward pointing on $\partial\mathcal{M}_1$ and outward pointing on $\partial\mathcal{M}_2$. The spin connection is embedded in the E_8 gauge group on $\partial\mathcal{M}_1$, breaking it to $E_6 \otimes SU(3)$. To find the appropriate ansatz, essentially we need to find all the five dimensional fields that we can build from the eleven dimensional ones in this framework.

2.3.1 The Metric

Since we split the eleven dimensional spacetime into a direct product of the five dimensional spacetime and the Calabi-Yau space, the metric must be a sum of spacetime and Calabi-Yau parts, with no cross terms. The Calabi-Yau moduli appear in the ansatz in the Calabi-Yau components of the metric. In particular, we wish to scale out the Volume modulus, V , so that we integrate over a Calabi-Yau with a fixed reference volume v and so that the Calabi-Yau components of the metric are given by the metric of the reference Calabi-Yau multiplied by an appropriate power of the volume modulus. In order for the five dimensional action to be in the Einstein frame and for the volume modulus to have a correctly normalized kinetic term, the spacetime components of the metric must also contain a power of the volume modulus. Einstein frame is the choice of scaling of the metric which gives us the five dimensional Einstein-Hilbert action with no conformal factor, $S_{EH,5} = \int_{\mathcal{M}_5} dv R(g_{\alpha\beta})$. We then have a line element of the form

$$ds^2 = V^n g_{\alpha\beta} dx^\alpha dx^\beta + V^m \left(g_{a\bar{b}} dx^a dx^{\bar{b}} + g_{\bar{a}b} dx^{\bar{a}} dx^b \right) \quad (2.3.1)$$

where we must find the values of n and m which satisfy these requirements. $\alpha, \beta, \gamma, \dots = 0, \dots, 4$ are five dimensional spacetime indices. $a, b, c, \dots = 1, 2, 3$ and $\bar{a}, \bar{b}, \bar{c}, \dots = \bar{1}, \bar{2}, \bar{3}$ are respectively holomorphic and anti-holomorphic indices on X . $g_{a\bar{b}}$ is given in terms of the (1,1) moduli, b^i , as $ig_{a\bar{b}} = b^i \omega_{i a\bar{b}}$ where $\omega_{i a\bar{b}}$ is the basis of (1,1)-forms on X . With this form the metric determinant is

$$\sqrt{-g_{(11)}} = V^{\frac{5}{2}n + \frac{6}{2}m} \sqrt{-g_{(5)}} \sqrt{g_{CY}} \quad (2.3.2)$$

and the eleven dimensional Ricci scalar gives

$$\begin{aligned} R(g_{(11)}) &= V^{-n} R(g_{(5)}) + \left(4n - 3n^2 - \frac{3}{2}m^2 \right) V^{-2-n} \partial_\alpha V \partial^\alpha V \\ &+ \left(3m^2 - \frac{15}{2}nm \right) V^{-1-m} \partial_\alpha V \partial^\alpha V \\ &- (4n + 3m) V^{-1-n} \nabla^2 V - 3m V^{-m} \nabla^2 V + \dots \end{aligned} \quad (2.3.3)$$

The extra terms involve ∂b^i and we do not need them here. We must have $m = n + 1$ for the V factors to be consistent and, after integrating by parts, the kinetic term for V is

$$\left(36n^2 + \frac{99}{2}n + \frac{33}{2} \right) V^{\frac{9}{2}n+1} \partial_\alpha V \partial^\alpha V \quad (2.3.4)$$

In order for the coefficient to be $-\frac{1}{2}$, either $n = -\frac{2}{3}$ or $n = -\frac{17}{24}$. Finally, to be in Einstein frame, $V^{-n+(\frac{5}{2}n+\frac{6}{2}m)} = 1$ and so we must take $n = -\frac{2}{3}$ and $m = \frac{1}{3}$. So the metric ansatz is

$$ds^2 = V^{-\frac{2}{3}} g_{\alpha\beta} dx^\alpha dx^\beta + V^{\frac{1}{3}} \left(g_{a\bar{b}} dx^a dx^{\bar{b}} + g_{\bar{a}b} dx^{\bar{a}} dx^b \right) \quad (2.3.5)$$

Once we allow $h^{1,1} > 1$ the metric, $g_{a\bar{b}}$, of the Calabi-Yau is no longer constant since the moduli b^i are spacetime dependent scalars: $\partial_\alpha g_{a\bar{b}} = \omega_{i a\bar{b}} \partial_\alpha b^i$. This

results in many extra terms in the action, etc., but we find that almost all of them contribute just to connection terms in the derivatives of fields with i, j, k, \dots indices as $D_\alpha \phi^i = \partial_\alpha \phi^i + \partial_\alpha b^j \Gamma_{jk}^i \phi^k$. The exceptions are a single term from the kinetic term of the eleven dimensional gravitino, $\frac{i}{2} G_{ij} \bar{\lambda}_A^i \gamma^\alpha \gamma^\beta \psi_\alpha^A \partial_\beta b^j$, and the kinetic term for the moduli, $-G_{ij} \partial_\alpha b^i \partial^\alpha b^j$.

2.3.2 The Three-Form C

Each part of the ansatz for C is the product of a spacetime tensor with a tensor on the Calabi-Yau space. The available tensors are those given at the end of Section 2.2.

The constant scalar field over the Calabi-Yau, gives us a three-form in five dimensions from the pure spacetime component of C . We will see in Section 2.5.5 how to dualize this three-form to a scalar σ which will form part of the hypermultiplet of the five dimensional supergravity, Section 2.5.6. After this dualization we will have

$$G_{\alpha\beta\gamma\delta} = V^{-2} \epsilon_{\alpha\beta\gamma\delta}{}^\epsilon \mathcal{D}_\epsilon \sigma \quad (2.3.6)$$

We get a spacetime scalar from the Calabi-Yau holomorphic three-form Ω_{abc} , and another from its complex conjugate, the anti-holomorphic three-form $\bar{\Omega}_{\bar{a}\bar{b}\bar{c}} = -i\epsilon_{\bar{a}\bar{b}\bar{c}}$, which will both be part of the hypermultiplet

$$C_{abc} = \frac{i}{2} \xi \epsilon_{abc} \quad (2.3.7)$$

$$C_{\bar{a}\bar{b}\bar{c}} = -\frac{i}{2} \bar{\xi} \epsilon_{\bar{a}\bar{b}\bar{c}} \quad (2.3.8)$$

The basis of (1,1)-forms is $\omega_{i\bar{a}\bar{b}}$ with $i = 1, \dots, h^{1,1}$ and the Kähler form is picked out by the Kähler moduli b^i as $ig_{\bar{a}\bar{b}} = \omega_{\bar{a}\bar{b}} = b^i \omega_{i\bar{a}\bar{b}}$. These (1,1)-forms give us $h^{1,1}$ spacetime vectors

$$C_{\alpha\bar{a}\bar{b}} = \frac{1}{\sqrt{2}} \mathcal{A}_\alpha^i \omega_{i\bar{a}\bar{b}} \quad (2.3.9)$$

These vectors gauge the shift isometry of σ , as we will see in Section 2.5.6. The graviphoton $b_i \mathcal{A}_\alpha^i$, which belongs to the gravity multiplet, is the part corresponding to the Kähler form and the remaining $h^{1,1} - 1$ vectors belong to vector multiplets.

These are all the constant tensors on the Calabi-Yau. However, the boundary conditions on C contain (2,2)-form sources for its field strength G and so, as explained in Section 2.6.2, we also have a ‘non-zero mode’

$$G_{\bar{a}\bar{b}\bar{c}\bar{d}} = -\sqrt{2} \alpha_i \nu_{\bar{a}\bar{b}\bar{c}\bar{d}}^i = -\frac{\sqrt{2}}{2} \alpha_i * (\omega^i)_{\bar{a}\bar{b}\bar{c}\bar{d}} \quad (2.3.10)$$

where ν^i are the basis of (2,2)-forms dual to ω_i (B.0.7).

2.3.3 The Gravitino

As the metric is just the sum of the five dimensional metric and the Calabi-Yau metric, the gamma matrices split into direct products of a square root of

unity with the five dimensional and Calabi-Yau gamma matrices. There are volume factors given by the square root of the volume factors in the metric so that the eleven dimensional Dirac algebra, $\{\Gamma_I, \Gamma_J\} = 2g_{(11)IJ}$, reduces to give $\{\gamma_\alpha, \gamma_\beta\} = 2g_{\alpha\beta}$ and $\{\gamma_a, \gamma_{\bar{b}}\} = 2g_{a\bar{b}}$. Thus we have

$$\Gamma_\alpha = V^{-\frac{1}{3}} \gamma_\alpha \otimes \gamma_7 \quad (2.3.11)$$

$$\Gamma_a = V^{\frac{1}{6}} \mathbf{1} \otimes \gamma_a \quad (2.3.12)$$

The choice of γ_7 in Γ_α and $\mathbf{1}$ in Γ_a is made so that $\gamma_\mu^* = \gamma_\mu$, $\gamma_5^* = -\gamma_5$ and $\gamma_7^* = -\gamma_7$ where γ_5 is the chirality matrix for the 4d spacetime manifold and γ_7 is the chirality matrix for X .

An eleven dimensional spinor gives us a direct product of a five dimensional spinor with a Calabi-Yau spinor, so to find the ansatz for Ψ_I we use the constant spinor, u_A , and the vector-spinors, $\gamma_a u_A$ and $\gamma_{\bar{a}} u_A$, on the Calabi-Yau. So at first glance Ψ_I just gives us a five dimensional vector-spinor from u_A

$$\Psi_\alpha = \theta_\alpha^A \otimes u_A \quad (2.3.13)$$

and a spinor from $\gamma_a u_A$

$$\Psi_a = \zeta^A \otimes \gamma_a u_A \quad (2.3.14)$$

However, this only gives us enough fermions for the gravity multiplet and hypermultiplet. There are also $h^{1,1} - 1$ vector multiplets which need fermion components. In fact we have one more tool in the basis of (1,1)-forms, ω_i , which we can use to construct $h^{1,1}$ vector-spinors on the Calabi-Yau instead of just one:

$$\Psi_a = \frac{1}{2} V^{\frac{1}{3}} \lambda^{Ai} \otimes \omega_{i\bar{a}\bar{b}} \gamma^{\bar{b}} u_A \quad (2.3.15)$$

There are no more possibilities since the other distinct tensor we have available, Ω_{abc} does not give us anything new since $\Omega_{abc} \gamma^{bc} u_A = -2\gamma_a u_A$. As we explain in Section 2.5.2 the fermion, ζ^A , which comes from the Kähler form part of the basis, $\omega_{\bar{a}\bar{b}} \gamma^{\bar{b}}$, and is part of the hypermultiplet, is $\sqrt{2} b_i \lambda^{Ai}$, while θ_α^A is in fact not the five dimensional gravitino, ψ_α^A , but contains ζ^A as well. We will show there that the ansatz in terms of the five dimensional gravitino, ψ_α^A , is

$$\Psi_\alpha = V^{-\frac{1}{6}} \left(\psi_\alpha^A - \frac{\sqrt{2}i}{3} \gamma_\alpha \zeta^A \right) \otimes u_A \quad (2.3.16)$$

It might seem more natural to maintain everywhere the distinction between ζ^A and the other $h^{1,1} - 1$ components, $\lambda^{\perp Ai}$, rather than grouping them together into λ^{Ai} , as they do belong to different multiplets. However, we ultimately decided that the benefits of more concise presentation outweighed this desire.

2.3.4 The E_8 Boundary

On $\partial\mathcal{M}_2$, where the E_8 symmetry is not broken, the story is simple. There is no constant vector on the Calabi-Yau and the only constant spinor is u_A , so the gauge fields and gauginos just give their four dimensional counterparts

$$A_\mu^{(10)} = A_\mu \quad (2.3.17)$$

$$A_a^{(10)} = 0 \quad (2.3.18)$$

and

$$\chi = \chi^A \otimes u_A \quad (2.3.19)$$

with A_μ and χ^A belonging to the adjoint representation of E_8 just as the eleven dimensional fields do.

2.3.5 The E_6 Boundary

On $\partial\mathcal{M}_1$ the gauge group E_8 is broken to E_6 by the embedding of the spin connection of the Calabi-Yau into the $SU(3)$ part of the subgroup $E_6 \times SU(3)$. Here situation is somewhat more complicated than on $\partial\mathcal{M}_2$. We have a gauge multiplet just as on $\partial\mathcal{M}_2$, though in the adjoint of E_6 instead of E_8 . The part of the gauge field proportional to the generators of the $SU(3)$ subgroup is identified with the spin connection of the Calabi-Yau, which is a one-form (though not globally defined) giving $A_a^{(10)} \Big|_{SU(3)} = \omega_a$. However, there are also generators of the off-block-diagonal components of E_8 . The decomposition of the E_8 adjoint representation **248** is $\mathbf{248} = (\mathbf{8}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{78}) \oplus (\mathbf{3}, \mathbf{27}) \oplus (\bar{\mathbf{3}}, \bar{\mathbf{27}})$ where $(\mathbf{8}, \mathbf{1})$ is the adjoint representation of $SU(3)$ and $(\mathbf{1}, \mathbf{78})$ that of E_6 while $(\mathbf{3}, \mathbf{27})$ and $(\bar{\mathbf{3}}, \bar{\mathbf{27}})$ are the off-block-diagonal parts which are in the fundamental and anti-fundamental respectively of both $SU(3)$ and E_6 . These off-diagonal generators, T_{ap} and T^{ap} , have an E_6 gauge index, p , and also an $SU(3)$ index which is equivalent to a holomorphic or anti-holomorphic index on the Calabi-Yau space. We can have an additional contribution to $A_a^{(10)}$ by contracting the $SU(3)$ index with one index of a (1,1)-form, $A_a^{(10)} \sim \omega_{ia}{}^b T_{bp}$. This gives us a set of $h^{1,1}$ spacetime scalars which are charged under the E_6 , C^{ip} , and their complex conjugates, $\bar{C}^i{}_p$. The complete ansatz for the gauge field is then

$$A_\mu^{(10)} = A_\mu \quad (2.3.20)$$

$$A_a^{(10)} = \omega_a + \omega_{ia}{}^b T_{bp} C^{ip} \quad (2.3.21)$$

where A_μ belongs to the adjoint representation of E_6 and ω_a is the spin connection of the Calabi-Yau space considered as an $SU(3)$ adjoint-valued (1,0)-form.

As for the gaugino, we can introduce an extra fermion in the fundamental representation of E_6 by contracting the $SU(3)$ index of T_{ap} with $\gamma^a u_A$. However, in a similar way to the gravitino, we have $h^{1,1}$ multiplets (chiral multiplets this time) which need fermions, not just one. We find these in much the same way, by using the basis of (1,1)-forms to give us more vector-spinors on the Calabi-Yau, $\omega_{ia\bar{b}} \gamma^{\bar{b}} u_A$. In this way we have $h^{1,1}$ Majorana fermions η^{Aip} , in the fundamental of E_6 to go along with the five dimensional gaugino in the adjoint:

$$\chi = \chi^A \otimes u_A + \frac{1}{2} \omega_{ia\bar{b}} T^{ap} \eta_R^i{}_p \otimes \gamma^{\bar{b}} u_1 + \frac{1}{2} \omega_i{}^a{}_b T_{ap} \eta_L^{ip} \otimes \gamma^b u_2 \quad (2.3.22)$$

η_L^i and η_R^i are the left and right handed components of η^{Ai} : $\eta_L^i = \eta^{1i}$ and $\eta_R^i = \eta^{2i}$. We label them thus because the chirality in ten dimensions of χ , $P_-^{(11)} \chi = \frac{1}{2} (\mathbf{1} - \Gamma_N) \chi = 0$, tells us that if we define $P_L = \frac{1}{2} (\mathbf{1} + \gamma_5)$ and $P_R = \frac{1}{2} (\mathbf{1} - \gamma_5)$ then $P_R \eta_L^i = 0$ and $P_L \eta_R^i = 0$ (since $\gamma_7 u_1 = u_1$ and $\gamma_7 u_2 = -u_2$) and so, for example, $\bar{\eta}_A^i P_L \eta^{Aj} = \bar{\eta}_L^i \eta_L^j$. Defining $P_+^A{}_B = \begin{pmatrix} P_L & 0 \\ 0 & P_R \end{pmatrix}^A{}_B$ and $P_-^A{}_B =$

$\begin{pmatrix} P_R & 0 \\ 0 & P_L \end{pmatrix}^A_B$ the chirality of the 4d Majorana fermions is $P_-^A B \chi^B = 0$ and $P_+^A B \eta^{iB} = 0$.

2.4 Five Dimensional Multiplets

Now that we have the field content of the five dimensional theory, we need to know the supersymmetry structure. In any supersymmetric theory, the component fields are grouped into multiplets whose fields transform among themselves under supersymmetry. The supersymmetry transformations of the five dimensional fields are fixed by the eleven dimensional supersymmetry transformations (2.1.34-2.1.38) and the ansatz described above. (We do need to do some extra work to obtain the supersymmetry transformation of σ which is found as part of the dualization process by which we turn the three-form $C_{\alpha\beta\gamma}$ into the scalar σ .)

Let us start by considering the gravity multiplet. By definition it contains the five dimensional vielbein $e^{\hat{\alpha}}_{\alpha}$ whose supersymmetry transformation is

$$\delta e^{\hat{\alpha}}_{\alpha} = -\frac{1}{2} \bar{s}_A \gamma^{\hat{\alpha}} \psi_{\alpha}^A \quad (2.4.1)$$

ψ_{α}^A appears and must therefore be the second component of the gravity multiplet, confirming its identification as the gravitino which comes from examining the kinetic terms in the action (Section 2.5.2). The transformation of the gravitino is

$$\delta \psi_{\alpha}^A = \mathcal{D}_{\alpha} s^A - \frac{\sqrt{2}i}{12} (\gamma_{\alpha}^{\beta\gamma} - 4\delta_{\alpha}^{\beta} \gamma^{\gamma}) b_i \mathcal{F}_{\beta\gamma}^i s^A - \frac{\sqrt{2}}{6} V^{-1} \alpha \tau^A_B \gamma_{\alpha} s^B \quad (2.4.2)$$

in which there is one new field $b_i \mathcal{A}_{\alpha}^i \equiv \mathcal{A}_{\alpha}$, appearing as its field strength $b_i \mathcal{F}_{\alpha\beta}^i = 2b_i \partial_{[\alpha} \mathcal{A}_{\beta]}^i \equiv 2\partial_{[\alpha} \mathcal{A}_{\beta]}$. \mathcal{A}_{α} is the graviphoton, which is the final component of the gravity multiplet as we confirm by its supersymmetry transformation

$$\delta \mathcal{A}_{\alpha} = -\frac{3\sqrt{2}i}{4} \tau^A_B \bar{s}_A \psi_{\alpha}^B \quad (2.4.3)$$

which contains no new fields. Thus the set $\{e^{\hat{\alpha}}_{\alpha}, \psi_{\alpha}^A, \mathcal{A}_{\alpha}\}$ is closed under supersymmetry; it is the gravity multiplet.

With the $b_i \mathcal{A}_{\alpha}^i$ component taken care of by the gravity multiplet, the $h^{1,1} - 1$ remaining vectors, $\mathcal{A}_{\alpha}^{\perp i}$ form multiplets which we can see from the variations

$$\delta \mathcal{A}_{\alpha}^{\perp i} = \frac{\sqrt{2}}{2} \tau^A_B \bar{s}_A \gamma_{\alpha} \lambda^{\perp Bi} \quad (2.4.4)$$

$$\delta \lambda^{\perp iA} = -\frac{i}{2} \partial_{\alpha} b^i \gamma^{\alpha} s^A + \frac{\sqrt{2}}{4} \mathcal{F}_{\alpha\beta}^{\perp i} \gamma^{\alpha\beta} s^A - \frac{\sqrt{2}i}{4} V^{-1} G^{\perp ij} \alpha_j \tau^A_B s^B \quad (2.4.5)$$

$$\delta b^i = -\frac{i}{2} \tau^A_B \bar{s}_A \lambda^{\perp iB} \quad (2.4.6)$$

Varying $\mathcal{A}_{\alpha}^{\perp i}$ brings in $\lambda^{\perp Ai}$ which brings b^i and then the set closes, so we have $h^{1,1} - 1$ vector multiplets $\{\mathcal{A}_{\alpha}^{\perp i}, \lambda^{\perp iA}, b^{\perp i}\}$.

There is one more fermion (in the bulk) and so we have one more multiplet, which includes ζ^A .

$$\delta\zeta^A = \begin{pmatrix} \frac{i}{48\sqrt{2}}VG_{\alpha\beta\gamma\delta}\gamma^{\alpha\beta\gamma\delta} - \frac{i}{2\sqrt{2}}V^{-1}\partial_\alpha V\gamma^\alpha & \frac{i}{8\sqrt{2}}V^{-\frac{1}{2}}\partial_\alpha\xi\gamma^\alpha \\ -\frac{i}{8\sqrt{2}}V^{-\frac{1}{2}}\partial_\alpha\bar{\xi}\gamma^\alpha & -\frac{i}{48\sqrt{2}}VG_{\alpha\beta\gamma\delta}\gamma^{\alpha\beta\gamma\delta} - \frac{i}{2\sqrt{2}}V^{-1}\partial_\alpha V\gamma^\alpha \end{pmatrix} \begin{matrix} A \\ B \end{matrix} s^B \quad (2.4.7)$$

$$\delta V = -\frac{i}{\sqrt{2}}\tau^A{}_B\bar{s}_A\zeta^B \quad (2.4.8)$$

$$\delta\xi = -\frac{\sqrt{2}i}{2}V^{\frac{1}{2}}\bar{s}_2\zeta^1 \quad (2.4.9)$$

$$\delta\bar{\xi} = \frac{\sqrt{2}i}{2}V^{\frac{1}{2}}\bar{s}_1\zeta^2 \quad (2.4.10)$$

$$\delta C_{\alpha\beta\gamma} = -\frac{3}{2}\bar{s}_A\gamma_{[\alpha\beta} \left(\psi_{\gamma]}^A - \frac{\sqrt{2}}{3}\gamma_{\gamma]} \zeta^A \right) \quad (2.4.11)$$

The variation of ζ^A connects it to V , ξ , $\bar{\xi}$ and $G_{\alpha\beta\gamma\delta} = 4\partial_{[\alpha}C_{\beta\gamma\delta]}$. The variations of V , ξ and $\bar{\xi}$ connect back to ζ^A , which is as expected. However, the variation of $C_{\alpha\beta\gamma}$ also involves ψ_α^A and in any case a three-form does not fit into the possible five dimensional supersymmetry multiplets. Both issues are resolved by dualizing the three-form C to a scalar σ which, as we will see in Section 2.5.5, has a variation

$$\delta_s\sigma = -\frac{1}{\sqrt{2}}\tau^A{}_B\bar{\eta}_A\zeta^B + \frac{1}{\sqrt{2}}V^{\frac{1}{2}} \begin{pmatrix} 0 & \xi \\ \bar{\xi} & 0 \end{pmatrix} \begin{matrix} A \\ B \end{matrix} \bar{\eta}_A\zeta^B \quad (2.4.12)$$

which does not involve ψ_α^A , and so $\{\zeta^A, V, \sigma, \xi, \bar{\xi}\}$ form a hypermultiplet.

The gauge fields A_μ on the E_8 boundary $\partial\mathcal{M}_2$ simply transform into the gauginos χ^A and vice versa,

$$\delta A_\mu^I = \frac{1}{2}V^{-\frac{1}{2}}\bar{s}_A\gamma_\mu\chi^A \quad (2.4.13)$$

$$\delta\chi^{IA} = \frac{1}{4}V^{\frac{1}{2}}\gamma^{\mu\nu}F_{\mu\nu}^I s^A \quad (2.4.14)$$

so we have a vector multiplet $\{A_\mu, \chi^A\}$.

On the E_6 boundary $\partial\mathcal{M}_1$ there are a set of $h^{1,1}$ complex scalars C^i and fermions η^{Aip} which transform as

$$\delta\eta_L^{ip} = -\mathcal{D}_\mu C^{ip}\gamma^\mu s^2 - \frac{3\sqrt{10}}{2}V^{-\frac{1}{2}}\mathcal{K}^{-1}\mathcal{K}^i{}_{jk}d^{pqr}\bar{C}^j{}_q\bar{C}^k{}_r s^1 \quad (2.4.15)$$

$$\delta\eta_R^i{}_p = \mathcal{D}_\mu\bar{C}^i{}_p\gamma^\mu s^1 - \frac{3\sqrt{10}}{2}V^{-\frac{1}{2}}\mathcal{K}^{-1}\mathcal{K}^i{}_{jk}d_{pqr}C^{jq}C^{kr} \quad (2.4.16)$$

$$\delta C^{ip} = -\frac{1}{2}\bar{s}_2\eta_L^{ip} - \frac{i}{4}\Gamma^i{}_{jk}C^{jp}\bar{s}_A\lambda^{\perp Ak} \quad (2.4.17)$$

$$\delta\bar{C}^i{}_p = -\frac{1}{2}\bar{s}_1\eta_R^i{}_p - \frac{i}{4}\Gamma^i{}_{kj}\bar{C}^j{}_p\bar{s}_A\lambda^{\perp Ak} \quad (2.4.18)$$

so we have $h^{1,1}$ chiral multiplets $\{\eta^{Ai}, C^i, \bar{C}^i\}$. The appearance of the fermions $\lambda^{\perp Ai}$ is due to the fact that the basis over which the i, j, k, \dots indices run depends on the Calabi-Yau moduli b^i . The variation of the b^i , (2.4.6), then gives rise to

these terms which would not be present were the basis fixed. There is also a vector multiplet, $\{A_\mu, \chi^A\}$, similar to that on the E_8 boundary but with gauge group E_6 whose transformations are

$$\delta A_\mu^I = \frac{1}{2} V^{-\frac{1}{2}} \bar{s}_A \gamma_\mu \chi^A \quad (2.4.19)$$

$$\delta \chi^{IA} = \frac{1}{4} V^{\frac{1}{2}} \gamma^{\mu\nu} F_{\mu\nu}^I s^A + V^{-\frac{1}{2}} G_{ij} \bar{C}^i \Lambda^j C^j \tau^A s^B \quad (2.4.20)$$

The scalar fields C^i enter here because in the four dimensional Yang-Mills theory coupled to chiral multiplets the equation of motion for the auxiliary field of the vector multiplet depends on the chiral multiplets. When the auxiliary field which appears ‘legitimately’ in the supersymmetry transformations of the vector multiplet is replaced by its (algebraic) equation of motion, it introduces this dependence on the scalars of the chiral multiplet.

2.5 The Bulk

We find the action in the bulk of the reduced theory by substituting the ansatzes from Section 2.3 into the eleven dimensional action (2.1.29) and also dualizing the three-form $C_{\alpha\beta\gamma}$ to a scalar σ (Section 2.5.5) to show the five dimensional supergravity structure.

2.5.1 Einstein-Hilbert Term

After integrating out the Calabi-Yau modes, the eleven dimensional Einstein-Hilbert term gives rise to the five dimensional Einstein-Hilbert term as well as the kinetic terms for the Calabi-Yau moduli. The eleven dimensional metric determinant is $\sqrt{-g_{(11)}} = V^{-\frac{2}{3}} \sqrt{-g_{(5)}} \sqrt{g_{CY}}$ which tells us the relation between the eleven and five dimensional volume elements: $dv_{(11)} = \sqrt{-g_{(11)}} d^{11}x = V^{-\frac{2}{3}} \sqrt{-g_{(5)}} d^5x \sqrt{g_{CY}} d^6x = V^{-\frac{2}{3}} dv_{(5)} dv_{CY}$. We also need the connection coefficients, the non-zero components of which are

$$\Gamma_{(11)}^{\alpha}{}_{\beta\gamma} = \Gamma_{(5)}^{\alpha}{}_{\beta\gamma} - \frac{1}{3} V^{-1} (\delta_\beta^\alpha \partial_\gamma V + \delta_\gamma^\alpha \partial_\beta V - g_{\beta\gamma} \partial^\alpha V) \quad (2.5.1)$$

$$\Gamma_{(11)}^{\alpha}{}_{b\bar{c}} = -\frac{1}{6} g_{b\bar{c}} \partial^\alpha V + \frac{i}{2} V \omega_{ib\bar{c}} \partial^\alpha b^i \quad (2.5.2)$$

$$\Gamma_{(11)}^a{}_{\beta c} = \frac{1}{6} V^{-1} \delta_c^a \partial_\beta V - \frac{i}{2} \omega_{ib}{}^a \partial_\beta b^i \quad (2.5.3)$$

$$\Gamma_{(11)}^{\bar{a}}{}_{\beta \bar{c}} = \frac{1}{6} V^{-1} \delta_{\bar{c}}^{\bar{a}} \partial_\beta V - \frac{i}{2} \omega_{i\bar{b}}{}^{\bar{a}} \partial_\beta b^i \quad (2.5.4)$$

and we find

$$\begin{aligned} \frac{1}{2\kappa_{11}^2} \int_{\mathcal{M}_{11}} dv R_{(11)} &= \frac{1}{2\kappa_5^2} \int_{\mathcal{M}} dv \left(R - \frac{1}{2} V^{-2} \partial_\alpha V \partial^\alpha V - G_{ij} \partial_\alpha b^i \partial^\alpha b^j \right) \\ &\quad - \frac{1}{2\kappa_5^2} \int_{\partial\mathcal{M}_1} dv \frac{2}{3} V^{-1} \partial_z V + \frac{1}{2\kappa_5^2} \int_{\partial\mathcal{M}_2} dv \frac{2}{3} V^{-1} \partial_z V \end{aligned} \quad (2.5.5)$$

where the five dimensional and eleven dimensional gravitational couplings are related by the volume of the reference Calabi-Yau, v , as $\kappa_5^2 = \frac{\kappa_{11}^2}{v}$. The boundary terms which depend on $\partial_z V$ will cancel against terms coming from the Gibbons-Hawking term, (2.6.1) and (2.6.2), as we would expect since the consistency of the variational principle in eleven dimensions implies its consistency in five and so we cannot have such a term, whose variation would include $\partial_z \delta V$.

2.5.2 The Five Dimensional Gravitino

The eleven dimensional gravitino gives rise to a number of five dimensional fermion fields from its spacetime and Calabi-Yau components. We can identify the proper ansatz by demanding that the five dimensional fields have canonical kinetic terms. Suppose we start with the straightforward ansatz

$$\Psi_\alpha = V^{-\frac{1}{6}} \theta_\alpha^A \otimes u_A \quad (2.5.6)$$

$$\Psi_a = \beta V^{\frac{1}{3}} \lambda^{Ai} \otimes \omega_{i\bar{a}\bar{b}} \gamma^{\bar{b}} u_A \quad (2.5.7)$$

then reducing the eleven dimensional kinetic term we get

$$\begin{aligned} V^{-\frac{2}{3}} \bar{\Psi}_I \Gamma^{IJK} D_J \Psi_K &= \left(\bar{\theta}_{\alpha A} + \frac{4i}{3} \beta b_i \bar{\lambda}_A^i \gamma_\alpha \right) \gamma^{\alpha\beta\gamma} D_\beta \left(\theta_\gamma^A + \frac{4i}{3} \beta b_i \gamma_\gamma \lambda^{iA} \right) \\ &+ 4\beta^2 G_{ij}^\perp \bar{\lambda}_A^i \gamma^\beta D_\beta \lambda^{jA} + 8\beta^2 b_i \bar{\lambda}_A^i \gamma^\beta D_\beta (b_j \lambda^{jA}) \\ &- 4i\beta b_i \bar{\lambda}_A^i \gamma^\alpha \gamma^\beta \left(\theta_\alpha^A + \frac{4i}{3} \beta b_i \gamma_\alpha \lambda^{iA} \right) \partial_\beta V \\ &+ \frac{i}{2} G_{ij} \bar{\lambda}_A^i \gamma^\alpha \gamma^\beta \psi_\alpha^A \partial_\beta b^j \end{aligned} \quad (2.5.8)$$

so we see that $\beta = \frac{1}{2}$ to correctly normalize the b^i superpartners, $\lambda^{\perp iA} = \delta^{\perp i}_j \lambda^{jA}$; $b_i \lambda^{Ai} = \frac{1}{\sqrt{2}} \zeta^A$ (ζ^A is the fermion in the hypermultiplet); and the 5d gravitino $\psi_\alpha^A = \theta_\alpha^A + \frac{\sqrt{2}i}{3} \zeta^A$. Thus we see the ansatz (2.3.15), (2.3.16) was indeed the correct choice.

2.5.3 Three-Form Terms

The kinetic term for C , $-\frac{1}{48} G_{IJKL} G^{IJKL}$, gives

$$S_{C1} = \frac{1}{2\kappa_5^2} \int_{\mathcal{M}} dv \left(-V^2 G_{\alpha\beta\gamma\delta} G^{\alpha\beta\gamma\delta} - 2V^{-1} \partial_\alpha \bar{\xi} \partial^\alpha \xi - \frac{1}{4} \mathcal{F}_{i\alpha\beta} \mathcal{F}^{i\alpha\beta} - \frac{1}{2} V^{-2} \alpha_i \alpha^i \right) \quad (2.5.9)$$

where \mathcal{F}^i is the curvature of \mathcal{A}^i , $\mathcal{F}_{\alpha\beta}^i = 2\partial_{[\alpha} \mathcal{A}_{\beta]}^i$.

The Chern-Simons term, $\frac{1}{124}\epsilon^{I_1\dots I_{11}}G_{I_1I_2I_3I_4}G_{I_5I_6I_7I_8}C_{I_9I_{10}I_{11}}$, gives

$$\begin{aligned}
 S_{C2} = & \frac{1}{2\kappa_5^2} \int_{\mathcal{M}} dv \left(\frac{1}{4!} \epsilon^{\alpha\beta\gamma\delta\epsilon} G_{\alpha\beta\gamma\delta} (i(\xi\partial_\epsilon\bar{\xi} - \bar{\xi}\partial_\epsilon\xi) + \alpha_i \mathcal{A}_\epsilon^i) \right. \\
 & \left. + \frac{1}{8\sqrt{2}} \mathcal{K}^{-1} \mathcal{K}_{ijk} \epsilon^{\alpha\beta\gamma\delta\epsilon} \mathcal{A}_\alpha^i \mathcal{F}_{\beta\gamma}^j \mathcal{F}_{\delta\epsilon}^k \right) \\
 & - \frac{1}{2\kappa_5^2} \int_{\partial\mathcal{M}_1} dv \frac{1}{6} \epsilon^{\mu\nu\rho\sigma} C_{\mu\nu\rho} (i(\xi\partial_\sigma\bar{\xi} - \bar{\xi}\partial_\sigma\xi) + \alpha_i \mathcal{A}_\sigma^i) \\
 & + \frac{1}{2\kappa_5^2} \int_{\partial\mathcal{M}_2} dv \frac{1}{6} \epsilon^{\mu\nu\rho\sigma} C_{\mu\nu\rho} (i(\xi\partial_\sigma\bar{\xi} - \bar{\xi}\partial_\sigma\xi) + \alpha_i \mathcal{A}_\sigma^i) \quad (2.5.10)
 \end{aligned}$$

and the gravitino bilinear term $-\frac{1}{96}(\bar{\Psi}_M \Gamma^{JKLMO} \Psi_O + 12\bar{\Psi}^I \Gamma^{JK} \Psi^L) G_{IJKL}$ gives

$$\begin{aligned}
 S_{C3} = & \frac{1}{2\kappa_5^2} \int_{\mathcal{M}} dv \left(-\frac{1}{96} V G^{\alpha\beta\gamma\delta} \epsilon_{\alpha\beta\gamma\delta} \left(2\sqrt{2} \tau^A \bar{\zeta}_A \psi_\epsilon^B + 3i \tau^A \bar{\zeta}_A \gamma_\epsilon \zeta^B \right. \right. \\
 & \left. \left. + i G_{ij}^\perp \tau^A \bar{\lambda}_A^i \gamma_\epsilon \lambda^{jB} \right) \right. \\
 & - \frac{1}{96} V G^{\alpha\beta\gamma\delta} \left(-8\sqrt{2} i \tau^A \bar{\zeta}_A \gamma_{\alpha\beta\gamma} \psi_\delta^B + 12 \tau^A \bar{\psi}_{\alpha A} \gamma_{\beta\gamma} \psi_\delta^B \right) \\
 & + \bar{\psi}_{A\alpha} \gamma^{\alpha\beta\gamma} \psi_\gamma^B \begin{pmatrix} 0 & -\partial_\beta \xi \\ \partial_\beta \bar{\xi} & 0 \end{pmatrix}_B^A \\
 & - \sqrt{2} i \bar{\zeta}_A \gamma^\alpha \gamma^\beta \psi_\alpha^B \begin{pmatrix} 0 & \partial_\beta \xi \\ -\partial_\beta \bar{\xi} & 0 \end{pmatrix}_B^A \\
 & + G_{ij}^\perp \bar{\lambda}_A^i \gamma^\beta \lambda^{Bj} \begin{pmatrix} 0 & -\partial_\beta \xi \\ \partial_\beta \bar{\xi} & 0 \end{pmatrix}_B^A \\
 & - \frac{\sqrt{2}i}{8} \left(\bar{\psi}_{A\gamma} \gamma^\alpha \gamma^{\gamma\delta} \gamma^\beta \psi_\delta^A - \bar{\zeta}_A \gamma^{\alpha\beta} \zeta^A - \frac{1}{3} G_{jk}^\perp \bar{\lambda}_A^j \gamma^{\alpha\beta} \lambda^{Ak} \right) b_i \mathcal{F}_{\alpha\beta}^i \\
 & - \frac{\sqrt{2}}{4} G_{ij}^\perp \bar{\lambda}_A^i \gamma^\gamma \gamma^{\alpha\beta} \psi_\gamma^A \mathcal{F}_{\alpha\beta}^j + \frac{3\sqrt{2}i}{8} \mathcal{K}^{-1} \mathcal{K}_{ijk}^\perp \bar{\lambda}_A^i \gamma^\alpha \lambda^{Aj} \mathcal{F}_{\alpha\beta}^k \\
 & - \frac{\sqrt{2}}{4} V^{-1} \alpha \tau^A \bar{\psi}_{A\alpha} \gamma^{\alpha\beta} \psi_\beta^B - \frac{\sqrt{2}i}{2} V^{-1} \alpha_i^\perp \tau^A \bar{\lambda}_A^i \gamma^\alpha \psi_\alpha^B \\
 & + i V^{-1} \alpha \tau^A \bar{\zeta}_A \gamma^\alpha \psi_\alpha^B \\
 & + \frac{3\sqrt{2}}{4} V^{-1} \left(\mathcal{K}^{-1} \mathcal{K}_{jk}^\perp + \frac{1}{9} b^i G_{jk}^\perp \right) \alpha_i \tau^A \bar{\lambda}_A^j \lambda^{kB} \\
 & \left. - 2V^{-1} \alpha_i^\perp \tau^A \bar{\zeta}_A \lambda^{Bi} + \frac{3\sqrt{2}}{4} V^{-1} \alpha \tau^A \bar{\zeta}_A \zeta^B \right) \quad (2.5.11)
 \end{aligned}$$

where we define $\alpha = \alpha_i b^i$. Much use is made in the calculation of S_{C3} of the identities in Appendix B.

We now have the ingredients in the action needed to dualize the four-form field strength G to a scalar field σ we turn our attention to this process first, followed by the elucidation of the quaternionic structure of the hypermultiplet fields which simplifies these terms in the action somewhat.

2.5.4 Aside on Auxiliary Fields

To manifest the hypermultiplet structure of the five dimensional theory we wish to dualize the three-form C to a scalar σ . I was puzzled by the fact that the equation relating σ and $G_{\alpha\beta\gamma\delta}$ is not supersymmetry invariant, yet one uses it to go from an action in terms of C to one in terms of σ and both actions are supersymmetric. Since this confusion has apparently manifested elsewhere (e.g. in Weinberg's book [64] following equation (26.4.7) "With the auxiliary fields eliminated in this way, the action is no longer invariant under the supersymmetry transformations...") it seems worth setting down the elegant resolution to this 'paradox', explained to me by Ian Moss, before going into the details of the present case.

Suppose we have a supersymmetric Lagrangian $\mathcal{L}_0(\phi^i, F)$ which depends on some fields ϕ^i and an auxiliary field F . We wish to replace F by its (algebraic) equation of motion. The supersymmetric variation of \mathcal{L}_0 vanishes

$$\delta_s \mathcal{L}_0(\phi^i, F) = \frac{\delta \mathcal{L}_0}{\delta \phi^i} \delta_s \phi^i + \frac{\delta \mathcal{L}_0}{\delta F} \delta_s F = 0 \quad (2.5.12)$$

and since F is an auxiliary field, its equation of motion does not contain derivatives of F and so it is just

$$\frac{\delta \mathcal{L}_0}{\delta F} = 0 \quad (2.5.13)$$

$$\Rightarrow F = F(\phi^i) \quad (2.5.14)$$

which we substitute into \mathcal{L}_0 to get a Lagrangian depending only on ϕ^i

$$\mathcal{L}_1(\phi^i) = \mathcal{L}_0(\phi^i, F)|_{F=F(\phi^i)} \quad (2.5.15)$$

Now the supersymmetric variation of \mathcal{L}_1 is

$$\delta_s \mathcal{L}_1(\phi^i) = \frac{\delta \mathcal{L}_0}{\delta \phi^i} \delta_s \phi^i \Big|_{F=F(\phi^i)} + \frac{\delta \mathcal{L}_0}{\delta F} \frac{\delta F(\phi^i)}{\delta \phi^i} \delta_s \phi^i \Big|_{F=F(\phi^i)} \quad (2.5.16)$$

but evaluating (2.5.12) on $F = F(\phi^i)$ tells us that

$$\frac{\delta \mathcal{L}_0}{\delta \phi^i} \delta_s \phi^i \Big|_{F=F(\phi^i)} = - \frac{\delta \mathcal{L}_0}{\delta F} \delta_s F \Big|_{F=F(\phi^i)} \quad (2.5.17)$$

and so we see that $\delta_s \mathcal{L}_1(\phi^i) = 0$ even though generally (2.5.14) is not supersymmetric, i.e. $\delta_s F \neq \frac{\delta F(\phi^i)}{\delta \phi^i} \delta_s \phi^i$.

In our case there are boundaries as well, so we cannot drop total derivatives when integrating out the auxiliary field. Rather the total derivatives contribute to the process of imposing the boundary condition of $G_{\alpha\beta\gamma\delta}$.

2.5.5 Dualization

As we saw in Section 2.4, in order to manifest the multiplet structure of the five dimensional theory we must dualize the three-form $C_{\alpha\beta\gamma}$ to a scalar σ . We introduce σ as a Lagrange multiplier enforcing the Bianchi identity on the field strength $dG = 0$ and the boundary conditions on $G_{\alpha\beta\gamma\delta}$.

σ is given in relation to the eleven dimensional fields by the equation of motion for $g_{\alpha\beta\gamma\delta}$, as we will describe below. However, as just mentioned this equation of motion is not supersymmetric and so we cannot use it to find the supersymmetry transformation of σ from the reduction. Instead, it arises when we impose supersymmetry on the action after adding σ as a Lagrange multiplier, which fixes the variation to be (2.5.27).

The terms with $G_{\alpha\beta\gamma\delta}$ in the five dimensional Lagrangian are

$$\begin{aligned} \mathcal{L}_G = & -\frac{1}{48}V^2 G_{\alpha\beta\gamma\delta} G^{\alpha\beta\gamma\delta} + \frac{1}{4!}\epsilon^{\alpha\beta\gamma\delta\epsilon} G_{\alpha\beta\gamma\delta} (i(\xi\partial_\epsilon\bar{\xi} - \bar{\xi}\partial_\epsilon\xi) + \alpha_i\mathcal{A}^i) \\ & -\frac{1}{96}VG^{\alpha\beta\gamma\delta}\epsilon_{\alpha\beta\gamma\delta}{}^\epsilon \left(2\sqrt{2}\tau^A{}_B\bar{\zeta}_A\psi_\epsilon^B + 3i\tau^A{}_B\bar{\zeta}_A\gamma_\epsilon\zeta^B + iG_{ij}^\perp\tau^A{}_B\bar{\lambda}_A^i\gamma_\epsilon\lambda^{jB}\right) \\ & -\frac{1}{96}VG^{\alpha\beta\gamma\delta} \left(-8\sqrt{2}i\tau^A{}_B\bar{\zeta}_A\gamma_{\alpha\beta\gamma}\psi_\delta^B + 12\tau^A{}_B\bar{\psi}_{\alpha A}\gamma_{\beta\gamma}\psi_\delta^B\right) \end{aligned} \quad (2.5.18)$$

the integration by parts to eliminate $C_{\alpha\beta\gamma}$ from the bulk action also gives boundary terms

$$\begin{aligned} S_{C_2}(\partial\mathcal{M}) = & -\frac{1}{2\kappa_5^2} \int_{\partial\mathcal{M}_1} dv \frac{1}{6}\epsilon^{\mu\nu\rho\sigma} C_{\mu\nu\rho} (i(\xi\partial_\sigma\bar{\xi} - \bar{\xi}\partial_\sigma\xi) + \alpha_i\mathcal{A}^i) \\ & + \frac{1}{2\kappa_5^2} \int_{\partial\mathcal{M}_2} dv \frac{1}{6}\epsilon^{\mu\nu\rho\sigma} C_{\mu\nu\rho} (i(\xi\partial_\sigma\bar{\xi} - \bar{\xi}\partial_\sigma\xi) + \alpha_i\mathcal{A}^i) \end{aligned} \quad (2.5.19)$$

and the boundary condition on G is

$$G_{\mu\nu\rho\sigma} = \mp f_{\mu\nu\rho\sigma} \quad (2.5.20)$$

where we find from the reduction of (2.1.31) that $f_{\mu\nu\rho\sigma}$ is given by

$$\begin{aligned} f_{\mu\nu\rho\sigma} = & \frac{\kappa_5^2}{g^2} (6F_{[\mu\nu}^I F_{\rho\sigma]}^I - \partial_{[\mu} (V^{-1}\bar{\chi}_A^I \gamma_{\nu\rho\sigma]} \chi^{IA}) - \partial_{[\mu} \text{tr} (V^{-1}\bar{\eta}_{iA} \gamma_{\nu\rho\sigma]} \eta^{iA})) \\ = & 4\frac{\kappa_5^2}{g^2} \partial_{[\mu} \left(\omega_{\nu\rho\sigma]}^Y - \frac{1}{4}V^{-1}\bar{\chi}_A^I \gamma_{\nu\rho\sigma]} \chi^{IA} - \frac{1}{4}V^{-1} \text{tr} (\bar{\eta}_{iA} \gamma_{\nu\rho\sigma]} \eta^{iA}) \right) \end{aligned} \quad (2.5.21)$$

where the I index runs over the adjoint of E_6 on $\partial\mathcal{M}_1$ and E_8 on $\partial\mathcal{M}_2$ and there is no chiral multiplet on $\partial\mathcal{M}_2$ (i.e. $\eta^{iA} = 0$ there). To dualize we replace the field strength $G_{\alpha\beta\gamma\delta}$ with a generic 4-form $g_{\alpha\beta\gamma\delta}$ which has the same supersymmetry transformation and we impose the Bianchi identity $\partial_{[\alpha} g_{\beta\gamma\delta\epsilon]} = 0$ and the boundary condition (2.5.20) by introducing σ as a Lagrange multiplier

$$\begin{aligned} S_{\text{Lagrange}} = & \frac{1}{2\kappa_5^2} \int_{\mathcal{M}} dv \left(-\frac{1}{24}\epsilon^{\alpha\beta\gamma\delta\epsilon} \sigma \partial_\alpha g_{\beta\gamma\delta\epsilon} \right) \\ & + \frac{1}{2\kappa_5^2} \int_{\partial\mathcal{M}_1} \left(\frac{1}{24}\epsilon^{\mu\nu\rho\sigma} (g_{\mu\nu\rho\sigma} + f_{\mu\nu\rho\sigma}) \sigma \right) \\ & + \frac{1}{2\kappa_5^2} \int_{\partial\mathcal{M}_2} \left(-\frac{1}{24}\epsilon^{\mu\nu\rho\sigma} (g_{\mu\nu\rho\sigma} - f_{\mu\nu\rho\sigma}) \sigma \right) \end{aligned} \quad (2.5.22)$$

The equation of motion for $g_{\alpha\beta\gamma\delta}$ is then

$$\begin{aligned}
 g_{\alpha\beta\gamma\delta} &= V^{-2}\epsilon_{\alpha\beta\gamma\delta}{}^\epsilon \left(\partial_\epsilon\sigma - i(\xi\partial_\epsilon\bar{\xi} - \bar{\xi}\partial_\epsilon\xi) - \alpha_i\mathcal{A}_\epsilon^i \right) \\
 &\quad - V^{-1}\epsilon_{\alpha\beta\gamma\delta}{}^\epsilon \left(\frac{\sqrt{2}}{2}\tau^A{}_B\bar{\zeta}_A\psi_\epsilon^B + \frac{3i}{4}\tau^A{}_B\bar{\zeta}_A\gamma_\epsilon\zeta^B + \frac{i}{4}G_{ij}^\perp\tau^A{}_B\bar{\lambda}_A^i\gamma_\epsilon\lambda^{jB} \right) \\
 &\quad - V^{-1}\epsilon_{\alpha\beta\gamma\delta}{}^\epsilon \left(\frac{i}{4}\tau^A{}_B\bar{\psi}_{A\zeta}\gamma_\epsilon\zeta^\eta\psi_\eta^B - \frac{\sqrt{2}}{2}\tau^A{}_B\bar{\zeta}_A\gamma_\epsilon\zeta\psi_\zeta^B \right) \quad (2.5.23)
 \end{aligned}$$

$$\begin{aligned}
 &\equiv V^{-2}\epsilon_{\alpha\beta\gamma\delta}{}^\epsilon \left(\mathcal{D}_\epsilon\sigma - \frac{i}{4}V\tau^A{}_B\bar{\psi}_{A\zeta}\gamma_\epsilon\zeta^\eta\psi_\eta^B - \frac{3i}{4}V\tau^A{}_B\bar{\zeta}_A\gamma_\epsilon\zeta^B \right. \\
 &\quad \left. - \frac{i}{4}VG_{ij}^\perp\tau^A{}_B\bar{\lambda}_A^i\gamma_\epsilon\lambda^{jB} - \frac{\sqrt{2}}{2}V\tau^A{}_B\bar{\zeta}_A\gamma_\epsilon\zeta\psi_\zeta^B \right) \quad (2.5.24)
 \end{aligned}$$

where we define

$$\mathcal{D}_\alpha\sigma = \partial_\alpha\sigma - i(\xi\partial_\alpha\bar{\xi} - \bar{\xi}\partial_\alpha\xi) - \alpha_i\mathcal{A}_\alpha^i \quad (2.5.25)$$

consistent with the quaternionic structure in Section (2.5.6).

The action for G before we introduced the Lagrange multiplier term was supersymmetric by virtue of being the reduction over a Calabi-Yau space of a supersymmetric eleven dimensional theory. However, the Lagrange multiplier clearly varies under supersymmetry itself, as in the bulk we have

$$\delta S_{Lagrange} = \frac{1}{2\kappa_5^2} \int_{\mathcal{M}} dv \left(\frac{1}{24}\epsilon^{\alpha\beta\gamma\delta\epsilon}\partial_\alpha\sigma\delta g_{\beta\gamma\delta\epsilon} - \frac{1}{24}\epsilon^{\alpha\beta\gamma\delta\epsilon}\delta\sigma\partial_\alpha g_{\beta\gamma\delta\epsilon} \right) \quad (2.5.26)$$

and we also have the terms which arise because $\partial_{[\alpha}g_{\beta\gamma\delta\epsilon]} \neq 0$ when we do not integrate out σ . Demanding that the action with both $g_{\alpha\beta\gamma\delta}$ and σ be supersymmetric fixes the variation of σ . Then eliminating $g_{\alpha\beta\gamma\delta}$ from this variation by its equation of motion we find that

$$\delta_s\sigma = -\frac{1}{\sqrt{2}}V\tau^A{}_B\bar{\zeta}_A\zeta^B + \frac{1}{\sqrt{2}}V^{\frac{1}{2}} \begin{pmatrix} 0 & \xi \\ \bar{\xi} & 0 \end{pmatrix}^A{}_B \bar{\zeta}_A\zeta^B \quad (2.5.27)$$

Substituting the equation of motion back into the action gives

$$\begin{aligned}
 S_\sigma &= \frac{1}{2\kappa_5^2} \int_{\mathcal{M}} dv \left(-\frac{1}{2}\mathcal{D}_\alpha\sigma\mathcal{D}^\alpha\sigma \right) \\
 &\quad + \frac{1}{2\kappa_5^2} \int_{\mathcal{M}} dv \left(\frac{i}{4}V\tau^A{}_B\bar{\psi}_{A\beta}\gamma_\alpha{}^{\beta\gamma}\psi_\gamma^B + \frac{3i}{4}V\tau^A{}_B\bar{\zeta}_A\gamma_\alpha\zeta^B + \frac{i}{4}VG_{ij}^\perp\tau^A{}_B\bar{\lambda}_A^i\gamma_\alpha\lambda^{jB} \right) \mathcal{D}^\alpha\sigma \\
 &\quad + \frac{1}{2\kappa_5^2} \int_{\mathcal{M}} dv \left(\frac{\sqrt{2}}{2}V\tau^A{}_B\bar{\zeta}_A\gamma^\beta\gamma_\alpha\psi_\beta^B \right) \mathcal{D}^\alpha\sigma \\
 &\quad - \frac{1}{2\kappa_5^2} \int_{\partial\mathcal{M}_1} dv \frac{1}{6}\epsilon^{\mu\nu\rho\sigma}\frac{\kappa_5^2}{g^2} \left(\omega_{\mu\nu\rho}^Y - \frac{1}{4}V^{-1}\bar{\chi}_A^I\gamma_{\mu\nu\rho}\chi^{IA} - \frac{1}{4}V^{-1}\text{tr}(\bar{\eta}_{iA}\gamma_{\mu\nu\rho}\eta^{iA}) \right) \partial_\sigma\sigma \\
 &\quad - \frac{1}{2\kappa_5^2} \int_{\partial\mathcal{M}_2} dv \frac{1}{6}\epsilon^{\mu\nu\rho\sigma}\frac{\kappa_5^2}{g^2} \left(\omega_{\mu\nu\rho}^Y - \frac{1}{4}V^{-1}\bar{\chi}_A^I\gamma_{\mu\nu\rho}\chi^{IA} \right) \partial_\sigma\sigma \\
 &\quad - \frac{1}{2\kappa_5^2} \int_{\partial\mathcal{M}_1} dv \frac{1}{6}\epsilon^{\mu\nu\rho\sigma}C_{\mu\nu\rho} \left(i(\xi\partial_\sigma\bar{\xi} - \bar{\xi}\partial_\sigma\xi) + \alpha_i\mathcal{A}_\sigma^i \right) \\
 &\quad + \frac{1}{2\kappa_5^2} \int_{\partial\mathcal{M}_2} dv \frac{1}{6}\epsilon^{\mu\nu\rho\sigma}C_{\mu\nu\rho} \left(i(\xi\partial_\sigma\bar{\xi} - \bar{\xi}\partial_\sigma\xi) + \alpha_i\mathcal{A}_\sigma^i \right) \quad (2.5.28)
 \end{aligned}$$

and finally by replacing $C_{\mu\nu\rho}$ by its boundary condition using (2.5.21) we find that the terms in the action involving σ are

$$\begin{aligned}
 S_\sigma &= \frac{1}{2\kappa_5^2} \int_{\mathcal{M}} dv \left(-\frac{1}{2} \mathcal{D}_\alpha \sigma \mathcal{D}^\alpha \sigma \right) \\
 &+ \frac{1}{2\kappa_5^2} \int_{\mathcal{M}} dv \left(\frac{i}{4} V \tau^A{}_B \bar{\psi}_{A\beta} \gamma_\alpha{}^{\beta\gamma} \psi_\gamma^B + \frac{3i}{4} V \tau^A{}_B \bar{\zeta}_A \gamma_\alpha \zeta^B + \frac{i}{4} V G_{ij}^\perp \tau^A{}_B \bar{\lambda}_A^i \gamma_\alpha \lambda^{jB} \right) \mathcal{D}^\alpha \sigma \\
 &+ \frac{1}{2\kappa_5^2} \int_{\mathcal{M}} dv \left(\frac{\sqrt{2}}{2} V \tau^A{}_B \bar{\zeta}_A \gamma^\beta \gamma_\alpha \psi_\beta^B \right) \mathcal{D}^\alpha \sigma \\
 &- \frac{1}{2\kappa_5^2} \int_{\partial\mathcal{M}_1} dv \frac{1}{6} \epsilon^{\mu\nu\rho\sigma} \frac{\kappa_5^2}{g^2} \left(\omega_{\mu\nu\rho}^Y - \frac{1}{4} V^{-1} \bar{\chi}_A^I \gamma_{\mu\nu\rho} \chi^{IA} - \frac{1}{4} V^{-1} \text{tr} (\bar{\eta}_{iA} \gamma_{\mu\nu\rho} \eta^{iA}) \right) \mathcal{D}_\sigma \sigma \\
 &- \frac{1}{2\kappa_5^2} \int_{\partial\mathcal{M}_2} dv \frac{1}{6} \epsilon^{\mu\nu\rho\sigma} \frac{\kappa_5^2}{g^2} \left(\omega_{\mu\nu\rho}^Y - \frac{1}{4} V^{-1} \bar{\chi}_A^I \gamma_{\mu\nu\rho} \chi^{IA} \right) \mathcal{D}_\sigma \sigma \tag{2.5.29}
 \end{aligned}$$

2.5.6 Hypermultiplet Quaternionic Structure

The hypermultiplet scalars can be regarded as the coordinates of a quaternionic manifold, as a particular case of the set-up described in Appendix D. The kinetic terms for the scalars in the hypermultiplet are

$$S_{sk} = \frac{1}{2\kappa_5^2} \int_{\mathcal{M}} \left(-\frac{1}{2} V^{-2} dV \wedge *dV - 2V^{-1} d\bar{\xi} \wedge d\xi - \frac{1}{2} V^{-2} \mathcal{D}\sigma \wedge *\mathcal{D}\sigma \right) \tag{2.5.30}$$

$$= \frac{1}{2\kappa_5^2} \int_{\mathcal{M}} (-f_x^A{}_B f_y{}^{\dagger B}{}_A Dq^x \wedge Dq^y) \tag{2.5.31}$$

with $f_x^A{}_B = \begin{pmatrix} \bar{v}_x & u_x \\ -\bar{u}_x & v_x \end{pmatrix}_B^A$ and

$$u = u_x dq^x = V^{-\frac{1}{2}} d\xi \tag{2.5.32}$$

$$v = v_x dq^x = \frac{V^{-1}}{2} (dV + id\sigma - \bar{\xi} d\xi + \xi d\bar{\xi}) \tag{2.5.33}$$

$$Dq^x = (dV, \mathcal{D}\sigma, d\xi, d\bar{\xi}) \tag{2.5.34}$$

so explicitly

$$f^A{}_B = \begin{pmatrix} \frac{1}{2} V^{-1} (dV - i\mathcal{D}\sigma) & V^{-\frac{1}{2}} d\xi \\ -V^{-\frac{1}{2}} d\bar{\xi} & \frac{1}{2} V^{-1} (dV + i\mathcal{D}\sigma) \end{pmatrix}_B^A \tag{2.5.35}$$

$\mathcal{D}\sigma$ being given by (2.5.25). The isometries here are shifts in σ , so the Killing vectors are $gk_i = -\alpha_i \frac{\partial}{\partial \sigma}$ which give the prepotential $g\mathcal{P}_i = -\frac{i\alpha_i}{4V} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and hence, from (D.0.6) and (D.0.7), we have gauged connections

$$\omega_{(L)}^A{}_B = \begin{pmatrix} \frac{i}{4} V^{-1} \mathcal{D}\sigma & -V^{-\frac{1}{2}} d\xi \\ V^{-\frac{1}{2}} d\bar{\xi} & -\frac{i}{4} V^{-1} \mathcal{D}\sigma \end{pmatrix}_B^A \tag{2.5.36}$$

$$\omega_{(R)}^A{}_B = \begin{pmatrix} \frac{3i}{4} V^{-1} \mathcal{D}\sigma & 0 \\ 0 & -\frac{3i}{4} V^{-1} \mathcal{D}\sigma \end{pmatrix}_B^A \tag{2.5.37}$$

We may use these connections to write covariant fermion derivatives

$$\mathcal{D}_\alpha \psi_\beta^A = D_\alpha \psi_\beta^A + \omega_{(L)}^A{}_{B} \psi_\beta^B \quad (2.5.38)$$

$$\mathcal{D}_\alpha \lambda^{iA} = D_\alpha \lambda^{iA} + \omega_{(L)}^A{}_{B} \lambda^{iB} \quad (2.5.39)$$

$$\mathcal{D}_\alpha \zeta^A = D_\alpha \zeta^A - \omega_{(R)}^A{}_{B} \zeta^B \quad (2.5.40)$$

so that we can package up the fermion terms in the bulk action from (2.5.8), (2.5.11) and (2.5.29) into

$$\begin{aligned} S_{C3} = \frac{1}{2\kappa_5^2} \int_{\mathcal{M}} dv \left(\bar{\psi}_{A\alpha} \gamma^{\alpha\beta\gamma} \mathcal{D}_\beta \psi_\gamma^A + \bar{\zeta}_A \gamma^\beta \mathcal{D}_\beta \zeta^A + G_{ij}^\perp \bar{\lambda}_A^i \gamma^\beta \mathcal{D}_\beta \lambda^{jA} \right. \\ - \sqrt{2} i \bar{\zeta}_A \gamma^\alpha \gamma^\beta \psi_\alpha^B f_{\beta B}^A + \frac{i}{2} G_{ij}^\perp \bar{\lambda}_A^i \gamma^\alpha \gamma^\beta \psi_\alpha^A \partial_\beta b^j \\ - \frac{\sqrt{2} i}{8} \left(\bar{\psi}_{A\gamma} \gamma^\alpha \gamma^\gamma \gamma^\delta \gamma^\beta \psi_\delta^A - \bar{\zeta}_A \gamma^{\alpha\beta} \zeta^A - \frac{1}{3} G_{jk}^\perp \bar{\lambda}_A^j \gamma^{\alpha\beta} \lambda^{Ak} \right) b_i \mathcal{F}_{\alpha\beta}^i \\ - \frac{\sqrt{2}}{4} G_{ij}^\perp \bar{\lambda}_A^i \gamma^\gamma \gamma^{\alpha\beta} \psi_\gamma^A \mathcal{F}_{\alpha\beta}^j + \frac{3\sqrt{2} i}{8} \mathcal{K}^{-1} \mathcal{K}_{ijk}^\perp \bar{\lambda}_A^i \gamma^{\alpha\beta} \lambda^{Aj} \mathcal{F}_{\alpha\beta}^k \\ - \frac{\sqrt{2}}{4} V^{-1} \alpha \tau^A{}_B \bar{\psi}_{A\alpha} \gamma^{\alpha\beta} \psi_\beta^B - \frac{\sqrt{2} i}{2} V^{-1} \alpha_i^\perp \tau^A{}_B \bar{\lambda}_A^i \gamma^\alpha \psi_\alpha^B \\ + i V^{-1} \alpha \tau^A{}_B \bar{\zeta}_A \gamma^\alpha \psi_\alpha^B \\ + \frac{3\sqrt{2}}{4} V^{-1} \left(\mathcal{K}^{-1} \mathcal{K}_{jk}^\perp + \frac{1}{9} b^i G_{jk}^\perp \right) \alpha_i \tau^A{}_B \bar{\lambda}_A^j \lambda^{kB} \\ \left. - 2V^{-1} \alpha_i^\perp \tau^A{}_B \bar{\zeta}_A \lambda^{Bi} + \frac{3\sqrt{2}}{4} V^{-1} \alpha \tau^A{}_B \bar{\zeta}_A \zeta^B \right) \quad (2.5.41) \end{aligned}$$

The quaternionic structure also appears in the supersymmetry transformations that we find by reducing (2.1.35) and (2.1.36). The individual components, (2.4.7-2.4.10) and (2.5.27), can be grouped together to find that

$$\delta q^x = -\frac{\sqrt{2} i}{2} f^{xA}{}_B \bar{s}_A \zeta^B \quad (2.5.42)$$

$$\delta \zeta^A = -\frac{\sqrt{2} i}{2} \gamma^\alpha s^B f_{xB}{}^A \mathcal{D}_\alpha q^x \quad (2.5.43)$$

while (2.4.2) becomes

$$\delta \psi_\alpha^A = \mathcal{D}_\alpha s^A - \frac{\sqrt{2} i}{12} (\gamma_\alpha^{\beta\gamma} - 4\delta_\alpha^\beta \gamma^\gamma) b_i \mathcal{F}_{\beta\gamma}^i s^A - \frac{2\sqrt{2} i}{3} g b^i \mathcal{P}_i{}^A{}_B \gamma_\alpha s^B \quad (2.5.44)$$

with

$$\mathcal{D}_\alpha s^A = D_\alpha s^A + \omega_{(L)}^A{}_{B} s^B \quad (2.5.45)$$

One might ask why we must gauge the isometry of the quaternionic manifold. From the point of view of the reduction of the eleven dimensional theory, it just turns out to be that way. From the perspective of five-dimensional supergravity however we see that the gauging is intimately bound up with the presence of the non-zero mode (which is forced upon us by the sources of G -flux on the

boundaries): if α_i were zero we would not have a gauged theory. The reason for the gauging is that the non-zero α_i gives a potential for V , which is $-\frac{1}{2}V^{-2}\alpha_i\alpha^i$ as we see in (2.5.9). A potential for hypermultiplet terms appears in gauged supergravity theories, such as the $N = 2$ four dimensional theories described in [65] which are rather similar to our five dimensional theory since they have the same amount of supersymmetry. We can see why such a potential term necessitates a gauged theory if we consider its supersymmetry transformation. Clearly the theory with $\alpha_i = 0$ is supersymmetric. So the only terms which might cancel the variation of the potential, $V\alpha^2\delta V = -\frac{i}{\sqrt{2}}V\alpha^2\tau^A{}_B\bar{s}_A\zeta^B$, are those which appear when we switch on α_i and these are just $\alpha_i\mathcal{A}_\alpha^i\mathcal{D}^\alpha\sigma$. The only terms in the supersymmetry variation of this which contain $\tau^A{}_B\bar{s}_A\zeta^B$ are $\alpha_i\mathcal{A}_\alpha^i\mathcal{D}^\alpha(\delta\sigma)$, since \mathcal{A}_α^i is not part of the hypermultiplet. This does not have the right form to cancel the variation of the potential directly. Therefore we see that the potential can only be present as part of a gauged theory in which the extra variations when $\alpha_i \neq 0$ can be identified as a gauge transformation of the hypermultiplet.

This completes the reduction of the theory in the bulk. We now turn to the description of the gauge theories on the boundaries and their couplings to the bulk fields.

2.6 The Boundaries

We find the boundary action and the boundary conditions coming from that of the eleven dimensional three-form C_{ABC} by substituting the ansatzes of Section 2.3 into the eleven-dimensional boundary action (2.1.30) and the boundary condition (2.1.31). It is easiest to find the remaining boundary conditions by varying the five dimensional action: these are given in Appendix E.2.

2.6.1 Boundary Action for the Gravity Multiplet

The ten dimensional metric determinant is $\sqrt{-g_{(10)}} = V^{-\frac{1}{3}}\sqrt{-g_{(4)}}\sqrt{g_{CY}}$ and so the volume element is $dv_{(10)} = V^{-\frac{1}{3}}dv_{(4)}dv_{CY}$. The unit vector normal to the boundaries (normalized by the five dimensional metric) is given in terms of the eleven dimensional normal vector by $n^\alpha = V^{-\frac{1}{3}}n_{(11)}^\alpha$. Thus we find that the Gibbons-Hawking term gives

$$-\frac{1}{2\kappa_{11}^2}\int_{\partial\mathcal{M}_{10,1}}dv(2K_{(10)}) = -\frac{1}{2\kappa_5^2}\int_{\partial\mathcal{M}_1}dv\left(2K - \frac{2}{3}V^{-1}\partial_z V\right) \quad (2.6.1)$$

and

$$\frac{1}{2\kappa_{11}^2}\int_{\partial\mathcal{M}_{10,2}}dv(2K_{(10)}) = \frac{1}{2\kappa_5^2}\int_{\partial\mathcal{M}_2}dv\left(2K - \frac{2}{3}V^{-1}\partial_z V\right) \quad (2.6.2)$$

As promised above the $V^{-1}\partial_z V$ contributions are appropriate to cancel those which come from the Einstein-Hilbert term (2.5.5).

The gravitino bilinear term in the boundary action gives

$$\begin{aligned} & \pm \frac{1}{2\kappa_{11}^2} \int_{\partial\mathcal{M}_{10}} dv \left(\frac{1}{2} \bar{\Psi}_A \Gamma^{AB} \Psi_B \right) \\ & = \pm \frac{1}{2\kappa_5^2} \int_{\partial\mathcal{M}} dv \left(\frac{1}{2} \tau^A{}_{\bar{B}} \bar{\psi}_{A\mu} \gamma^{\mu\nu} \psi_\nu^B + \frac{1}{2} \tau^A{}_{\bar{B}} \bar{\zeta}_A \zeta^B + \frac{1}{2} G_{ij}^\perp \tau^A{}_{\bar{B}} \bar{\lambda}^i \lambda^{Bj} \right) \end{aligned} \quad (2.6.3)$$

The term with the component of the gravitino normal to the boundary, $\mp \bar{\Psi}_A \Gamma^A \Psi_N$ is a fermionic torsion term which just cancels a total derivative of the torsion from the bulk Ricci scalar and never appears in the field equations. It is given explicitly by $\mathcal{T} = \mp \bar{\psi}_{A\mu} \gamma^\mu \left(\psi_z^A - \frac{\sqrt{2}i}{3} \gamma_z \zeta^A \right) \pm \frac{7\sqrt{2}i}{3} \bar{\zeta}_A \psi_z^A$.

2.6.2 Boundary Sources for G and the Embedding of the Spin Connection

As the only non-vanishing components of the curvature tensor on a Calabi-Yau manifold are $R_{\bar{a}\bar{b}\bar{c}\bar{d}}$, the part of $\text{tr}R^2$ in the tangent space of the Calabi-Yau manifold is a (2,2)-form and so we can expand it on the basis ν^i of (2,2) forms as

$$\text{tr}R^2 = -\frac{2\sqrt{2}g^2}{\kappa_5^2} \alpha_i \nu^i \quad (2.6.4)$$

with the coefficient being chosen to give (2.3.10) (which is in turn chosen to make α_i the charge associated with \mathcal{A}^i in the covariant derivative $\mathcal{D}_\alpha \sigma$). The integrals of $\text{tr}R^2$ over (2,2)-cycles of the Calabi-Yau, $\beta_i = -\frac{1}{8\pi} \int_{C_i} \text{tr}(R \wedge R)$, are integers characterizing the first Pontrjagin class of the particular Calabi-Yau. These fix α_i to be $\alpha_i = \frac{2\sqrt{2}\pi\kappa_5^2}{g^2} \beta_i$.

Due to the spin embedding, restricting to the Calabi-Yau components of the curvatures, $\text{tr}F^2 = \text{tr}R^2$ on $\partial\mathcal{M}_1$, as we see from (2.3.21), while $\text{tr}F^2 = 0$ on $\partial\mathcal{M}_2$, from (2.3.18). Therefore the boundary condition (2.1.31) gives $G_{\bar{a}\bar{b}\bar{c}\bar{d}} = -\sqrt{2}\alpha_i \nu_{\bar{a}\bar{b}\bar{c}\bar{d}}^i$ on both boundaries and we see that in the compactification the lowest energy mode is $G_{\bar{a}\bar{b}\bar{c}\bar{d}} = -\sqrt{2}\alpha_i \nu_{\bar{a}\bar{b}\bar{c}\bar{d}}^i$ everywhere; we cannot have $G_{\bar{a}\bar{b}\bar{c}\bar{d}} = 0$.

2.6.3 Reducing the Lorentz Multiplet

The eleven dimensional theory includes R^2 terms on the boundaries in a supersymmetric fashion by constructing a Lorentz multiplet from the spin connection and the gravitino. This transforms similarly under supersymmetry to a Yang-Mills multiplet and its action can thereby be constructed by analogy, though with some extra complications involving components of the gravitino normal to the boundary and G -flux fields. We have not, however, included this Lorentz multiplet in the reduction, except for the brane charges α_i described in the previous section. The reason for this is essentially that the analogy with the Yang-Mills multiplet, which in the eleven dimensional theory dictates the structure of the Lorentz multiplet, fails completely in the context of the reduction. One can find the contribution of the Lorentz multiplet to the ten dimensional boundary action and boundary conditions in a way which is in a sense independent of the bulk fields. In contrast once we have given the reduction ansatz for the metric, gravitino and three form in the bulk we have no freedom to choose an ansatz for the

Lorentz multiplet by analogy with the Yang-Mills ansatz. Of course this does not prohibit simply using the ansatz for the bulk fields to calculate the contribution of the Lorentz multiplet terms to the five dimensional theory and indeed we attempted this, for contributions at leading order in the derivative expansion of [31]. The problem with this approach is twofold: firstly the $SO(1, 9)$ traces in the ten dimensional terms give many different combinations of five dimensional fields arising from the permutations of spacetime and Calabi-Yau components; secondly we have lost any organizing principle to assemble the terms to exhibit some sort of multiplet structure in the reduced theory. The best candidate for such a structure would be a reprise in four dimensions of the ten dimensional Lorentz multiplet. However, such a four dimensional Lorentz multiplet could only include the five dimensional gravitino, ψ_α^A , and not the other fermions, ζ^A and $\lambda^{\perp Ai}$, which come from the eleven dimensional gravitino but are not part of the gravity multiplet in five dimensions. Unfortunately upon examining the results no alternative structure suggested itself to us either. The utility of such an unordered set of terms composed of bulk fermion fields in the boundary action and boundary conditions seems doubtful and so it seems better to us to adopt an approximation (of small spacetime curvature on the boundaries) which excludes them altogether.

2.6.4 Yang-Mills Action on the E_8 Boundary

On $\partial\mathcal{M}_2$ where the E_8 gauge group is unbroken the action is much simpler, so we will first examine that. We just have an E_8 Yang-Mills vector multiplet. The approximation made here is that $R_{\bar{a}\bar{b}\bar{c}\bar{d}}R^{\bar{a}\bar{b}\bar{c}\bar{d}}$ is the only non-negligible part of R^2 (i.e. the four dimensional spacetime curvature is small). So using (2.6.4), the bosonic part of the Yang-Mills action is just

$$\begin{aligned} & \frac{1}{\lambda^2} \int_{\partial\mathcal{M}_{10,2}} dv \left(-\frac{1}{4} \left(\text{tr} (F_{AB}F^{AB}) - \frac{1}{2} \text{tr} (R_{AB}R^{AB}) \right) \right) \\ &= \int_{\partial\mathcal{M}_2} dv \left(-\frac{1}{4g^2} V F_{\mu\nu}^I F^{I\mu\nu} + \frac{\sqrt{2}}{2\kappa_5^2} V^{-1} \alpha \right) \end{aligned} \quad (2.6.5)$$

where the four dimensional gauge coupling is $g^2 = \lambda^2/v$. The fermionic part is

$$\begin{aligned} & \frac{1}{\lambda^2} \int_{\partial\mathcal{M}_{10,2}} dv \left(\frac{1}{2} \text{tr} (\bar{\chi} \Gamma^A D_A \chi) - \frac{1}{4} \bar{\Psi}_C \Gamma^{AB} \Gamma^C \text{tr} (F_{AB} \chi) \right) \\ &= \frac{1}{g^2} \int_{\partial\mathcal{M}_2} dv \left(\frac{1}{2} \bar{\chi}_A^I \gamma^\mu D_\mu \chi^{IA} + \bar{\psi}_{A\mu} j^{A\mu} + \bar{\zeta}_A j^A \right) \end{aligned} \quad (2.6.6)$$

where the supercurrent is

$$j^{A\mu} = -\frac{1}{4} V^{\frac{1}{2}} F_{\nu\rho}^I \gamma^{\nu\rho} \gamma^\mu \chi^{IA} \quad (2.6.7)$$

and the current which couples to ζ^A is

$$j^A = -\frac{\sqrt{2}i}{2} V^{\frac{1}{2}} F_{\mu\nu}^I \gamma^{\mu\nu} \chi^{IA} \quad (2.6.8)$$

2.6.5 Chiral Multiplets

Given the field content in the reduction ansatz and the fact that reduction on a Calabi-Yau gives us a supersymmetric four dimensional theory on the boundary, we know that we must find an instance of a four dimensional super-Yang-Mills theory coupled to chiral multiplets, as described in the general case in, for example, [64]. The challenge in the reduction then is to arrange into this known structure the fields arising from substituting the ansatz into the eleven dimensional action, and in the process to identify the superpotential and Kähler potential which specify the particular four dimensional theory. This is indeed a challenge since the four dimensional supersymmetry structure is not manifest upon performing the reduction directly. For instance, the potential for the scalars arising from the reduction is

$$\begin{aligned}
 & V^{-1} (\omega_{ia}{}^b \omega_{jb}{}^c \omega_{kc}{}^d \omega_{ld}{}^a) \left(\frac{1}{4} C^{ip} \bar{C}^j{}_p C^{kr} \bar{C}^l{}_r - 5 d_{pst} d^{qrt} C^{ip} \bar{C}^j{}_q C^{ks} \bar{C}^l{}_r \right) \\
 & + V^{-1} (\omega_{ia}{}^b \omega_{lb}{}^a) (\omega_{jc}{}^d \omega_{kd}{}^c) \left(\frac{1}{4} C^{ip} \bar{C}^j{}_p C^{kr} \bar{C}^l{}_r + \frac{5}{2} d_{pst} d^{qrt} C^{ip} \bar{C}^j{}_q C^{ks} \bar{C}^l{}_r \right) \quad (2.6.9)
 \end{aligned}$$

which should be

$$-\frac{1}{2} V^{-1} G^{ij} \frac{\partial W}{\partial C^{ip}} \frac{\partial \bar{W}}{\partial \bar{C}^j{}_p} - 2V^{-1} D^I D^I \quad (2.6.10)$$

where W is the superpotential, which we must determine, and $D^I = G_{ij} \bar{C}^i{}_p \Lambda^I{}_q C^{jq}$ with Λ^I the generators of the fundamental of E_6 . The E_6 adjoint projector $\Lambda^I \Lambda^I$ is given in equation (C.0.18).

To reconcile these two forms we need some identities for contractions of strings of the (1,1)-form basis elements. Those identities we could find are summarized at the end of Appendix B. Unfortunately, for the contraction of four basis elements we could find only an identity for a symmetrized version and so in this one place we cannot complete the entire reduction directly. We do however have enough information to calculate the superpotential and Kähler potential and to cross-check.

The Kähler potential, $K(C, \bar{C})$ gives the coefficients of the kinetic terms as $-\frac{\partial^2 K}{\partial C^{ip} \partial \bar{C}^j{}_q} \mathcal{D}_\mu C^{ip} \mathcal{D}^\mu \bar{C}^j{}_q$ and since from the reduction this is $-2G_{ij} \mathcal{D}_\mu C^{ip} \mathcal{D}^\mu \bar{C}^j{}_p$ the Kähler potential is just $2G_{ij} C^{ip} \bar{C}^j{}_p$.

We can find the superpotential most easily from the mass terms for the η 's, which are given in terms of the superpotential by

$$\frac{1}{2} \frac{\partial^2 W}{\partial C^{ip} \partial C^{jq}} \bar{\eta}_R{}^{jq} \eta_L{}^{kr} + \frac{1}{2} \frac{\partial^2 \bar{W}}{\partial \bar{C}^i{}_p \partial \bar{C}^j{}_q} \bar{\eta}_L{}^j{}_q \eta_R{}^k{}_r \quad (2.6.11)$$

and from the reduction by

$$\frac{3\sqrt{10}}{2} V^{-\frac{1}{2}} \mathcal{K}^{-1} \mathcal{K}_{ijk} (d_{pqr} C^{ip} \bar{\eta}_R{}^{jq} \eta_L{}^{kr} + d^{pqr} \bar{C}^i{}_p \bar{\eta}_L{}^j{}_q \eta_R{}^k{}_r) \quad (2.6.12)$$

and so we find that the superpotential is

$$W = 3\sqrt{10} \mathcal{K}^{-1} \mathcal{K}_{ijk} d_{pqr} C^{ip} C^{jq} C^{kr} \quad (2.6.13)$$

The algebra of the gauge group generators T_{ap} and T^{ap} , the E_6 generators X^I and the $SU(3)$ generators S^i is vital to this chiral multiplet part of the reduction. To find (2.6.9) we need $\text{tr}([T_{ap}, T^{bq}][T_{cr}, T^{ds}])$ and to find (2.6.12) we need $\text{tr}([T_{ap}, T_{bq}]T_{cr})$ and so to find the correct coefficient of the superpotential requires that we know the algebra of the generators which is consistent with the normalizations we take for the traces. The algebra and normalizations we used are described in Appendix C.

The consistency check is that $D^I D^I$ does not contain any terms with coefficient $\mathcal{K}_{ijm}\mathcal{K}^m_{kl}d_{pst}d^{qrt}$ and so the only such term in (2.6.10) is $|\frac{\partial W}{\partial C}|^2$. On the other hand $d_{pst}d^{qrt}C^{ip}\bar{C}^j{}_q C^{ks}\bar{C}^l{}_r$ imposes enough symmetry on ω^4 to give the part we can calculate and so we know the corresponding term in (2.6.9). Indeed comparing these using the superpotential calculated from the fermion mass terms, (2.6.13), we find that the coefficients match correctly.

2.6.6 Yang-Mills Action on the E_6 Boundary

On $\partial\mathcal{M}_1$ we have an E_6 Yang-Mills multiplet with the same structure as the E_8 multiplet on $\partial\mathcal{M}_2$ but we also have a set of $h^{1,1}$ chiral multiplets from the (3, 27) and ($\bar{3}$, $\bar{27}$) parts of the eleven dimensional E_8 fields. We find, as described in the previous section, that the bosonic part of the action is

$$\begin{aligned} & \frac{1}{\lambda^2} \int_{\partial\mathcal{M}_{10,1}} dv \left(-\frac{1}{4} \left(\text{tr} (F_{AB}F^{AB}) - \frac{1}{2} \text{tr} (R_{AB}R^{AB}) \right) \right) \\ &= \frac{1}{g^2} \int_{\partial\mathcal{M}_1} dv \left(-\frac{1}{4} V F_{\mu\nu}^I F^{I\mu\nu} - 2G_{ij} \mathcal{D}_\mu C^{ip} \mathcal{D}^\mu \bar{C}^j{}_p - \frac{\sqrt{2}g^2}{2\kappa_5^2} V^{-1} \alpha \right. \\ & \quad \left. - \frac{1}{2} V^{-1} G^{ij} \frac{\partial W}{\partial C^{ip}} \frac{\partial \bar{W}}{\partial \bar{C}^j{}_p} - 2V^{-1} \bar{C}^i \Lambda^I C_i \bar{C}^j \Lambda^I C_j \right) \end{aligned} \quad (2.6.14)$$

with the superpotential $W = 3\sqrt{10}\mathcal{K}^{-1}\mathcal{K}_{ijk}d_{pqr}C^{ip}C^{jq}C^{kr}$, and the fermionic part is

$$\begin{aligned} & \frac{1}{\lambda^2} \int_{\partial\mathcal{M}_{10,1}} dv \left(\frac{1}{2} \text{tr} (\bar{\chi} \Gamma^A D_A \chi) - \frac{1}{4} \bar{\Psi}_C \Gamma^{AB} \Gamma^C \text{tr} (F_{AB} \chi) \right) \\ &= \frac{1}{g^2} \int_{\partial\mathcal{M}_1} dv \left(\frac{1}{2} \bar{\chi}_A^I \gamma^\mu \mathcal{D}_\mu \chi^{IA} + \frac{1}{2} G_{ij} \text{tr} (\bar{\eta}_A^i \gamma^\mu \mathcal{D}_\mu \eta^{Aj}) \right. \\ & \quad + \frac{3\sqrt{10}}{2} V^{-\frac{1}{2}} \mathcal{K}^{-1} \mathcal{K}_{ijk} (d_{pqr} C^{ip} \bar{\eta}_R^{jq} \eta_L^{kr} + d^{pqr} \bar{C}_p^i \bar{\eta}_L^j \eta_R^k{}_r) \\ & \quad + 2G_{ij} V^{-\frac{1}{2}} (\bar{\chi}_L^I \eta_R^i C^{jp} \Lambda^I{}_p - \bar{\chi}_R^I \eta_L^{ip} \bar{C}_q^j \Lambda^I{}_p) \\ & \quad \left. + \bar{\psi}_{A\mu} j^{A\mu} + \bar{\zeta}_{Aj}^A + G_{ij}^\perp \bar{\lambda}_{Aj}^i \right) \end{aligned} \quad (2.6.15)$$

The current coupling to the gravitino ψ_μ^A is

$$\begin{aligned}
 j^{A\mu} = & -\frac{1}{4}V^{\frac{1}{2}}\gamma^{\nu\rho}\gamma^\mu F_{\nu\rho}^I\chi^I \\
 & -\gamma^\nu\gamma^\mu P_R\eta^A{}_{ip}\mathcal{D}_\nu C^{ip} + \gamma^\nu\gamma^\mu P_L\eta^A{}_{ip}\mathcal{D}_\nu\bar{C}^i{}_p \\
 & + \frac{3\sqrt{10}}{2}V^{-\frac{1}{2}}\gamma^\mu\mathcal{K}^{-1}\mathcal{K}_{ijk}(d_{pqr}P_L\eta^{Aip}C^{jq}C^{kr} + d^{pqr}P_R\eta^{Ai}{}_p\bar{C}^j{}_q\bar{C}^k{}_r)
 \end{aligned} \tag{2.6.16}$$

which is, as one would expect, the supercurrent. The supercurrent is the part of the variation of the action due to the spacetime dependence of the supersymmetry parameter in a locally supersymmetric theory. By calculating directly the variation of the boundary Yang-Mills action we compute the supercurrent as the $D_\mu\bar{s}_A$ part of the variation and this indeed matches the current given above.

The current coupling to ζ^A is

$$\begin{aligned}
 j^A = & -\frac{\sqrt{2}i}{4}V^{\frac{1}{2}}\gamma^{\mu\nu}F_{\mu\nu}^I\chi^{IA} \\
 & + \frac{2\sqrt{2}i}{9}b_ib_j\gamma^\mu(P_R\eta^{Aj}{}_p\mathcal{D}_\mu C^{ip} - P_L\eta^{Aip}\mathcal{D}_\mu\bar{C}^j{}_p) \\
 & - \frac{\sqrt{2}i}{3}V^{-\frac{1}{2}}\left(G_{ij} + \frac{4}{3}b_ib_j\right)\tau^A{}_B\chi^{IB}\bar{C}^j\Lambda^I C^i \\
 & - 3\sqrt{5}iV^{-\frac{1}{2}}\mathcal{K}^{-1}\mathcal{K}_{ijk}(d_{pqr}C^{ip}C^{jq}P_L\eta^{Akr} + d^{pqr}\bar{C}^i{}_p\bar{C}^j{}_qP_R\eta^{Akr})
 \end{aligned} \tag{2.6.17}$$

and that coupling to λ^{Ai} is

$$\begin{aligned}
 j^{Ai} = & i\left(\Gamma^{\perp i}{}_{jk} - \frac{4}{3}\delta^{\perp i}{}_{(j}b_{k)}\right)\gamma^\mu(P_R\eta^{Ak}{}_p\mathcal{D}_\mu C^{jp} - P_L\eta^{Ajp}\mathcal{D}_\mu\bar{C}^k{}_p) \\
 & - 2iV^{-\frac{1}{2}}\left(\Gamma^{\perp i}{}_{jk} - \frac{4}{3}\delta^{\perp i}{}_{(j}b_{k)}\right)\tau^A{}_B\chi^{IB}\bar{C}^k\Lambda^I C^j
 \end{aligned} \tag{2.6.18}$$

2.6.7 Boundary Conditions

As in the eleven dimensional theory it is also necessary to specify some boundary conditions in addition to the action to complete the description of the theory, namely those arising from the eleven dimensional boundary condition on the three-form C . The $C_{\alpha\beta\gamma}$ parts have already been built in to the action in the process of dualizing $C_{\alpha\beta\gamma}$ to the scalar σ , the rest give the boundary conditions on ξ , $\bar{\xi}$ and \mathcal{A}_μ^i , which must be imposed separately. On $\partial\mathcal{M}_2$:

$$\xi = \frac{\kappa_5^2}{2g^2}V^{\frac{1}{2}}\bar{\chi}_2^I\chi^{I1} \tag{2.6.19}$$

$$\bar{\xi} = -\frac{\kappa_5^2}{2g^2}V^{\frac{1}{2}}\bar{\chi}_1^I\chi^{I2} \tag{2.6.20}$$

$$\mathcal{A}_\mu^i = -\frac{\sqrt{2}i\kappa_5^2}{4g^2}b^i\tau^A{}_B\bar{\chi}_A^I\gamma_\mu\chi^{IB} \tag{2.6.21}$$

and on $\partial\mathcal{M}_1$:

$$\xi = \frac{\kappa_5^2}{g^2} \left(\sqrt{10}\mathcal{K}^{-1}\mathcal{K}_{ijk}d_{pqr}C^{ip}C^{jq}C^{kr} + \frac{1}{2}V^{\frac{1}{2}}\bar{\chi}_1^I\chi^{I2} \right) \quad (2.6.22)$$

$$\bar{\xi} = \frac{\kappa_5^2}{g^2} \left(\sqrt{10}\mathcal{K}^{-1}\mathcal{K}_{ijk}d^{pqr}\bar{C}_p^i\bar{C}_q^j\bar{C}_r^k - \frac{1}{2}V^{\frac{1}{2}}\bar{\chi}_2^I\chi^{I1} \right) \quad (2.6.23)$$

$$\begin{aligned} \mathcal{A}_\mu^i = & -\frac{\sqrt{2}\kappa_5^2}{g^2} \left(i\Gamma^i{}_{jk} (\bar{C}_p^k \mathcal{D}_\mu C^{jp} - C^{jp} \mathcal{D}_\mu \bar{C}_p^k) - \frac{\sqrt{2}i}{4} b^i{}_{\tau A} \bar{\chi}_A^I \gamma_\mu \chi^{IB} \right. \\ & \left. - \frac{i}{4} (\Gamma^i{}_{jk} - b^i G_{jk}) (\bar{\eta}_L^{jp} \gamma_\mu \eta_R^k{}_p - \bar{\eta}_L^k{}_p \gamma_\mu \eta_L^{jp}) \right) \end{aligned} \quad (2.6.24)$$

The full set of boundary conditions, including those derived from the action, is given in the Appendix, in Section E.2.

2.7 Gaugino Condensation

Gaugino condensation (the outcome of the gauge theory becoming strongly coupled) provides a good example of the phenomenological impact of the improved version of Heterotic M-Theory, since the improved theory includes a contribution of the condensate to the boundary conditions which was missed by the original version. Gaugino condensation in the dimensional reduction of the improved Heterotic M-Theory and its contribution to the vacuum energy were considered, before the full reduction discussed here was completed, by Ahmed and Moss [36, 37]. We will discuss some of their results here, the main aim being to illustrate the effect of the improved theory.

If a gaugino condensate forms its contribution is

$$\bar{\chi}\Gamma_{abc}\chi = \Lambda\Omega_{abc} \quad (2.7.1)$$

or in terms of the four dimensional fields

$$\bar{\chi}_2\chi^1 = \Lambda \quad (2.7.2)$$

This contributes to the boundary conditions (2.1.31) or equivalently (2.6.22) and (2.6.23) giving a background flux of ξ and $\bar{\xi}$ in terms of the fields of the reduced theory described above. For the background described in [36], this gives a contribution to the superpotential of the four dimensional theory of $W_g \sim \Lambda_1 + \Lambda_2$, with Λ_1 and Λ_2 being the condensates on $\partial\mathcal{M}_1$ and $\partial\mathcal{M}_2$ respectively.

If we continue the calculation of the boundary condition on the gravitino which gives (2.1.18) to include all the fermion terms then we have

$$\Gamma^{AB} \left(P_- + \frac{\kappa_{11}^2}{\lambda^2} \Gamma P_+ \right) \Psi_A = \mp \frac{\kappa_{11}^2}{4\lambda^2} \Gamma^{BC} \Gamma^A \text{tr} (F_{BC}^* \chi) \quad (2.7.3)$$

where $F_{AB}^* = \frac{1}{2} (F_{AB} + \hat{F}_{AB})$ with \hat{F}_{AB} the supercovariant version of F_{AB} . The operator Γ is given by

$$\Gamma = \frac{1}{96} \text{tr} (\bar{\chi}\Gamma_{ABC}\chi) \Gamma^{ABC} \quad (2.7.4)$$

and represents a twist in the chirality condition which Ψ_A must obey on the boundaries. This contribution from Γ was missed by the original formulation of Heterotic M-Theory, so its effects are a novelty of the improved version. Clearly a non-zero gaugino condensate (2.7.1) gives us a source of such a twist.

Suppose we have background solutions to the five dimensional theory, such as those described in [36], which give us a four dimensional model with a potential which stabilizes the moduli fields in some minimum. The four dimensional moduli in that case were the values of the Calabi-Yau volume modulus on either boundary, V_1 and V_2 . The potential which fixes the moduli is negative at its minima, i.e. its vacua are anti-de Sitter. Therefore some contribution is needed to give the small positive vacuum energy that we observe in the real universe.

The twist in the chirality of the gravitino breaks the supersymmetry of the background. It also changes the quantum vacuum energy of the gravitino. Without the condensate the vacuum energy contributions of the bosonic and fermionic fields cancel each other exactly due to the supersymmetry. The condensate does not affect the vacuum energy of the bosons, but changes the vacuum energy of the gravitino, thus giving a change in the overall vacuum energy. The vacuum energy due to the twist given in (2.7.3) by the gaugino condensate (2.7.1) was calculated in [37] using the results of [66, 67]. It was found to be always positive and could, for some choices of Calabi-Yau in the reduction, be large enough to cancel the negative vacuum energy resulting from the moduli stabilization.

The conclusion of [37] is then that the improved version of Heterotic M-Theory unveils a new mechanism for the uplift of the vacuum from anti-de Sitter to de Sitter.

The fuller reduction described in this chapter, as compared to that used in [36, 37], sheds a little additional light on this mechanism, as was discussed in [34]. The only fermion considered in [36, 37] was the five dimensional gravitino ψ_α^A . Now that we have the ansatz for all the fermion fields coming from the eleven dimensional gravitino Ψ_I we can see that ψ_α is the only one which receives a contribution of the gaugino condensate to its boundary condition from (2.7.4). Recall from (2.3.15) that the fermions ζ^A and $\lambda^{\perp Ai}$, those which were omitted before, are given by the parts of Ψ_I whose Calabi-Yau spinor components are $\gamma^a u_A$ or $\gamma^{\bar{a}} u_A$. Now for the Calabi-Yau gamma matrices $\gamma^1 \gamma^1 = 0$, etc. and there are only three values a, b, c, d can take so $\gamma^{abc} \gamma^d u_A = 0$ while we also have that $\gamma^{abc} \gamma^{\bar{d}} u_A = \gamma^{abc} \gamma^{\bar{d}} u_1 = 2g^{cd} \gamma^{ab} u_1 = 0$ as $\gamma^a u_1 = 0$ and $\gamma^{\bar{a}} u_2 = 0$. Thus the contribution of the condensate (2.7.1) to the twist (2.7.4) annihilates the parts of Ψ_I which give ζ^A and $\lambda^{\perp Ai}$ and thus the condensate cannot affect their boundary conditions directly. We see then that the analysis of the Casimir energy of [37] should carry over to our more complete reduction with its conclusions intact.

2.8 Conclusions

We have described here the reduction to five dimensions of an improved version of Heterotic M-Theory over a Calabi-Yau manifold with (1,1) moduli only. Comparing to the reduction of Hořava and Witten's original formulation of Heterotic M-Theory, the bosonic sector is largely unaffected, and along with it most of the existing work on the phenomenology of Heterotic M-Theory.

Although phenomenology based upon the bosonic sector should not be much

affected by the improved version of Heterotic M-Theory, that previous phenomenological work is now on a firmer footing; we can now see that it can be built upon a theory which is consistent to all orders in the gravitational coupling and the fact that the analogous four dimensional expansion parameter $\epsilon = \kappa^{2/3} \rho V^{-2/3}$ seems to be of order one, when chosen to give reasonable values for Newton's constant and the GUT scale [24], is no longer a problem. On the other hand, work that has been done using explicitly terms higher order in the gravitational coupling, such as on the back reaction of anti-branes [25] and gaugino condensation [26] probably needs to be reconsidered in the light of the improved theory.

The inclusion of five-branes, and the further inclusion of anti-five-branes, has not yet been considered in the improved theory. Though such objects are vital to Heterotic M-Theory phenomenology, it is not possible with our current understanding of them to incorporate them in a fully consistent way which includes the backreaction of their matter and curvature, in the manner in which we have treated the boundary branes here. In the absence of such understanding, one might adopt a hybrid approach with boundary conditions at either end of the bulk, as we describe here, and junction conditions across the five-branes, as have been used in the Hořava-Witten theory.

Gaugino condensation has been considered already in the improved theory in the context of a much simpler reduction [36, 37]. The condensate appears in the boundary condition for the scalars ξ and $\bar{\xi}$ and so acts as a source of flux. It also appears in the three-fermion terms in the fermion boundary conditions, its effect being to introduce a 'twist' in the chirality of the fermions which contributes to the vacuum energy. By including just those higher fermion terms necessary for gaugino condensation into the full reduction given here, one can see that the gravitino contribution considered in [37] is the dominant one [34]. Therefore the calculation there of the vacuum energy and the conclusion that it is a candidate to uplift a negative cosmological constant, as left by moduli stabilization, to a positive one, in accordance with observation, still hold here.

A full treatment of moduli stabilization in the improved theory would require the inclusion of the (2,1) moduli for the Calabi-Yau space. There are no obvious obstacles to this other than complexity so we would expect it to be feasible if there is sufficient interest to warrant it.

Leaving aside the M-Theory context, the topic of supergravities with boundary matter is worthy of study of itself, particularly for its connection to braneworld models. Examples are hard to come by in dimensions greater than three, where there is an off-shell formulation to assist. Thus in providing a five dimensional example with four dimensional boundaries the theory described here may be of some interest.

Chapter 3

Big Crunch from Colliding Branes

Interest in braneworld scenarios began with attempts to confine matter to field theory domain walls [68, 69]. Suppose we have a scalar field, ϕ , coupled to a fermion, ψ , by a Yukawa coupling $\sim \phi\bar{\psi}\psi$ and we have a domain wall solution with $\phi = 0$ on some surface and $\phi = m$ far from the surface. Then the fermion is massless on the wall but massive away from it, so that it is energetically favourable for the fermion to be close to the wall. Indeed one may show [70] that in such a case there is a fermion zero mode which is both normalizable and confined to the wall.

Such scenarios attracted further interest more recently once it was realized that gravity could also be confined to the brane, as was shown by the construction of Randall and Sundrum [42]. They considered a brane separating two anti-de Sitter spacetimes so that the metric is warped in the z direction, perpendicular to the brane, as $ds^2 = e^{-\kappa|z|}\eta_{\mu\nu}dx^\mu dx^\nu + dz^2$. The warping suppresses the Kaluza-Klein modes of the gravitational field and means that the zero mode, i.e. the four dimensional graviton, dominates the gravitational interactions. Although the z direction is infinite in extent it actually has a finite volume so that the zero mode is normalizable, and in fact gives general relativity on the brane [71]. The possibility of confining gravity opened up the possibility of large extra dimensions and hence the possibility of building cosmological models using the dynamics of branes moving in them [72].

In a braneworld cosmological model it is natural to ask whether the big bang could have been caused by a collision of branes [18, 73–75] and in this context the scalar field domain wall returns as a convenient model of a braneworld whose detailed behaviour can be studied using numerical simulations. Particle production in a five dimensional model was studied in [46] and, with the inclusion of fermions on the branes, in [43–46] but in both cases without gravitational backreaction. When the backreaction was included in the scalar field theory model it was found [47, 48] that, as one might have expected, a singularity forms if the collision is energetic enough. The domain walls are assumed to be spatially homogeneous and isotropic, which reduces the 4+1 dimensional theory to a 1+1 dimensional problem, which is much more amenable to numerical solution. Given this symmetry the singularity formed was naturally assumed [48] to be the AdS black brane.

However, the asymptotic spacetimes of the domain wall solutions and the AdS black brane are not compatible, as we will show in Section 3.2. This prompted

us to further investigate the structure of the singularity, as reported in [57]. The results of our simulations support the conclusion that the singularity formed cannot be a black brane; they suggest instead a big crunch singularity which ends the spacetime.

3.1 Static Kinks

The model we use is a single real scalar canonically coupled to gravity in five dimensions

$$\mathcal{L} = \frac{m_p^3}{2}R - \frac{1}{2}\partial_a\phi\partial^a\phi - \mathcal{V}(\phi) \quad (3.1.1)$$

where m_p is the Planck mass.

A domain wall is a topological defect: we have one vacuum solution at $r \rightarrow -\infty$ and a different one at $r \rightarrow +\infty$. The transition region between the two vacua has a high cost in terms of the potential energy of the fields which are forced to leave the potential minima which they occupy in the vacuum. This cost is minimized by making the region small, so we have a localized structure: a domain wall. The domain wall is stable because of its topological character: neither vacuum could ever fill the whole spacetime on its own and so the transition between the two, the wall, must persist. In order to have a domain wall like this we must choose an appropriate potential which has two distinct minima, which give us the two vacua (with a potential barrier between them in field space) which we need to support a domain wall.

We examine domain walls which are spatially flat so for a static domain wall the metric and the scalar field only depend on the co-ordinate normal to the brane, r . We can thus describe the metric with an ansatz

$$ds^2 = e^{2U(r)}\eta_{\mu\nu}dx^\mu dx^\nu + dr^2 \quad (3.1.2)$$

while the scalar field is a function of r only

$$\phi = \phi(r) \quad (3.1.3)$$

With this ansatz the non-zero components of the Ricci tensor are

$$R_{tt} = e^{2U} (\partial_r^2 U + 4(\partial_r U)^2) \quad (3.1.4)$$

$$R_{xx} = -e^{2U} (\partial_r^2 U + 4(\partial_r U)^2) \quad (3.1.5)$$

$$R_{rr} = -4\partial_r^2 U - 4(\partial_r U)^2 \quad (3.1.6)$$

and the Ricci scalar is

$$R = -8\partial_r^2 U - 20(\partial_r U)^2 \quad (3.1.7)$$

and so the equations of motion, $R_{ab} = \kappa^2 (\partial_a\phi\partial_b\phi + \frac{2}{3}g_{ab}\mathcal{V})$ and $\nabla_a\nabla^a\phi - \frac{\partial\mathcal{V}}{\partial\phi} = 0$, are

$$(\partial_r U)^2 = \frac{1}{6m_p^3} \left(\frac{1}{2} (\partial_r\phi)^2 - \mathcal{V} \right) \quad (3.1.8)$$

$$\partial_r^2 U = -\frac{1}{3m_p^3} (\partial_r\phi)^2 \quad (3.1.9)$$

$$\partial_r^2\phi + 4\partial_r U\partial_r\phi = \frac{\partial\mathcal{V}}{\partial\phi} \quad (3.1.10)$$

In order to be able to find an exact solution, it is convenient to take a supergravity-inspired form for the potential [76]

$$\mathcal{V} = \frac{1}{2} \left(\left(\frac{\partial W}{\partial \phi} \right)^2 - \frac{4}{3m_p^3} W^2 \right) \quad (3.1.11)$$

with a superpotential $W(\phi)$ because if this is the case, then the equations of motion are solved if the BPS equations,

$$\begin{aligned} \frac{\partial \phi}{\partial r} &= \mp \frac{\partial W}{\partial \phi} \\ \frac{\partial U}{\partial r} &= \pm \frac{1}{3m_p^3} W \end{aligned} \quad (3.1.12)$$

are solved, where the upper and lower signs will give kinks and anti-kinks respectively. This is a great simplification as we need only solve the first order BPS equations rather than solving the second order equations of motion directly. Moreover, we do not need to know U to solve the BPS equation for ϕ and so we can solve for ϕ and U consecutively, rather than simultaneously as we would have to for the equations of motion.

Of course we must also pick an appropriate superpotential: we choose the sine-Gordon model

$$W = \mu^4 - \frac{4m}{\beta^2} \cos \left(\frac{\beta \phi}{2} \right) \quad (3.1.13)$$

which gives a potential

$$\mathcal{V} = \left(\frac{2m^2}{\beta^2} - \frac{2\mu^8}{3m_p^3} \right) + \frac{16m\mu^4}{3m_p^3\beta^2} \cos \left(\frac{\beta \phi}{2} \right) - \left(\frac{2m^2}{\beta^2} + \frac{32m^2}{3m_p^3\beta^4} \right) \cos^2 \left(\frac{\beta \phi}{2} \right) \quad (3.1.14)$$

whose parameters are: m which is the mass of the scalar field in the non-gravitating limit $m_p \rightarrow \infty$, and controls the curvature of the potential at the minima; β which sets the separation of the vacua in field space; and μ which controls the constant part of the potential and will allow us to make some of the minima be Minkowski vacua.

This potential has two sets of minima: one set at $\frac{\beta \phi}{2} = 2n\pi$ and another at $\frac{\beta \phi}{2} = (2n+1)\pi$. At the former the value of the potential is

$$\mathcal{V} \left(\frac{\beta \phi}{2} = 2n\pi \right) = -\frac{32}{3m_p^3\beta^4} \left(1 - \frac{\mu^4\beta^2}{4m} \right)^2 \quad (3.1.15)$$

so we can make these Minkowski vacua by setting $\mu^4 = \frac{4m}{\beta^2}$. With this choice its value at the latter is

$$\mathcal{V} \left(\frac{\beta \phi}{2} = (2n+1)\pi \right) = -6m_p^3 \left(\frac{8m}{3\beta^2 m_p^3} \right)^2 = m_p^3 \Lambda \quad (3.1.16)$$

giving us the value of the (negative) cosmological constant Λ in the AdS vacua.

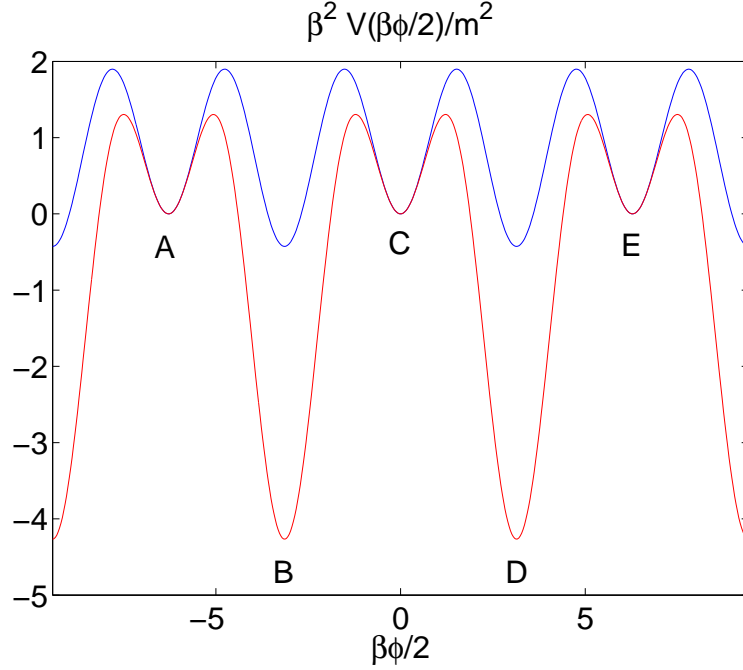


Figure 3.1: The potential with $\beta^2 m_p^3 = 10$ (red, lower curve) and $\beta^2 m_p^3 = 100$ (blue, upper curve).

The potential is plotted in Figure 3.1 where the Minkowski vacua are labelled A, C and E and the AdS vacua are B and D. Our simulations are performed with $\beta^2 m_p^3 = 100$, which is shown as the blue, upper curve.

The reason for the choice of this particular form of the sine-Gordon potential is this: we choose to have AdS vacua outside the domain walls so that gravity is confined, à la Randall-Sundrum, to the region around and between the domain walls. A priori we could choose the intermediate vacuum to be either AdS or Minkowski, noting from (3.1.15) that $\mathcal{V} \leq 0$ at a minimum so we cannot have dS vacua. However, in order to be able to construct a two-wall solution by superposition of the solutions for single walls we are forced to take the intermediate vacuum to be Minkowski. To superpose the solutions we must match their asymptotic spaces: the $r \rightarrow +\infty$ limit of the wall at smaller r and the $r \rightarrow -\infty$ limit of the wall at larger r . Since the solution corresponding to one of the walls enters the intermediate vacuum as r increases while that corresponding to the other leaves it, one wall must be a kink while the other must be an anti-kink. Since they share the intermediate vacuum, the value of ϕ there is the same, and constant asymptotically, and similarly for the superpotential $W(\phi)$. However, this being the case we can see from the BPS equations, (3.1.12), that if the intermediate vacuum is AdS, with $W < 0$, then for a kink U is monotonically decreasing with r while for the anti-kink U is monotonically increasing (or vice versa if $W > 0$). Thus there is no way that the solutions for $U(r)$ for the kink and anti-kink can match asymptotically in the intermediate region unless U is constant: we must have a Minkowski vacuum (with $W = 0$) in the intermediate region in order to superpose analytic single wall solutions to find a two wall solution.

Our initial set-up then will be a spacetime with two parallel domain walls which is asymptotically AdS⁵ and approaches Minkowski space between the do-

main walls. The scalar field evolves through space, in the direction normal to the domain walls, from $-\frac{4\pi}{\beta}$ (as $r \rightarrow -\infty$) through 0 (between the domain walls) to $\frac{4\pi}{\beta}$ (as $r \rightarrow +\infty$), that is from B through C to D in Figure 3.1. Since we have a Minkowski intermediate vacuum we can, as we have just seen, construct a two-wall solution by superposition of single-wall solutions. We will construct our initial conditions by adding the analytic solutions for a kink (with ϕ going from B to C) and an anti-kink (with ϕ going from C to D).

A BC kink is a solution to the BPS equations (3.1.12) with the upper sign, which is given by

$$\begin{aligned}\frac{\beta\phi}{2} &= 2 \tan^{-1} \left(\tanh \left(\frac{m(r-r_0)}{2} \right) \right) - \frac{\pi}{2} \\ U &= -\frac{4}{3\beta^2 m_p^3} \left(\ln \left(\cosh \left(m(r-r_0) \right) \right) - \frac{\beta^2 \mu^4}{4} (r-r_0) \right)\end{aligned}\quad (3.1.17)$$

and a CD anti-kink is a solution to the equations with the lower sign, given by

$$\begin{aligned}\frac{\beta\phi}{2} &= 2 \tan^{-1} \left(\tanh \left(\frac{m(r-r_0)}{2} \right) \right) + \frac{\pi}{2} \\ U &= -\frac{4}{3\beta^2 m_p^3} \left(\ln \left(\cosh \left(m(r-r_0) \right) \right) + \frac{\beta^2 \mu^4}{4} (r-r_0) \right)\end{aligned}\quad (3.1.18)$$

To construct the initial conditions which will give us colliding domain walls we will need to boost these static solutions, but first a comment on the asymptotic structure of systems such as these.

3.2 Asymptotic Structure

In the limit $r \rightarrow +\infty$ we are in the AdS⁵ region of the anti-kink solution so the metric is

$$\lim_{r \rightarrow +\infty} ds^2 = e^{-2\alpha r} \eta_{\mu\nu} dx^\mu dx^\nu + dr^2 \quad (3.2.1)$$

where $\alpha = \frac{8m}{3\beta^2 m_p^3}$ and making the co-ordinate transformation defined by $e^{\alpha r} = \alpha Z$ we see that the metric of the asymptotic region is

$$\lim_{Z \rightarrow +\infty} ds^2 = \frac{1}{\alpha^2 Z^2} (\eta_{\mu\nu} dx^\mu dx^\nu + dZ^2) \quad (3.2.2)$$

This is a portion of AdS⁵ that, in particular, does not include the AdS⁵ boundary, which is at $Z = 0$.

On the other hand, the metric of the AdS⁵ black brane is

$$ds_{bb}^2 = -f(R)dT^2 + \frac{1}{f(R)}dR^2 + R^2 \delta_{ij} dx^i dx^j \quad (3.2.3)$$

with

$$f(R) = -\frac{M}{R^2} - \frac{\Lambda R^2}{6} \quad (3.2.4)$$

and there is a version of Birkhoff's theorem [77, 78] which assures us that this is the only black brane solution which shares the symmetry of our domain walls, i.e. plane symmetry orthogonal to the r direction. For comparison with the domain wall solution (3.2.2) let us make the co-ordinate change

$$t = \sqrt{-\frac{\Lambda}{6}}T, \quad R = \frac{1}{\sqrt{-\frac{\Lambda}{6}}Z} \quad (3.2.5)$$

Then, remembering that the black brane is at $R = 0$ so the asymptotic region, far from the brane, is $Z \rightarrow 0$, the asymptotic metric of the black brane is

$$\lim_{Z \rightarrow 0} ds_{bb}^2 = -\frac{6}{\Lambda Z^2} (\eta_{\mu\nu} dx^\mu dx^\nu + dZ^2) \quad (3.2.6)$$

which is the same form as the asymptotic metric of the domain wall, but in a different region of AdS^5 : The asymptotic region of the domain wall is far from the AdS^5 boundary but the asymptotic region of the black brane is the region near the boundary. Thus it would be distinctly odd if a black brane were formed by the collision of domain walls.

3.3 Dynamical Solutions

If we define dimensionless variables by

$$x = \frac{\tilde{x}}{m}, \quad \phi = \frac{\tilde{\phi}}{\beta} \quad (3.3.1)$$

then we are left with just a single parameter, $\beta^2 m_p^3$. Henceforth we will use the dimensionless parameters, but drop the tildes. In other words we will choose to measure distances and the Planck mass in units of m , and ϕ in units of β .

We obtain solutions representing moving kinks and anti-kinks just by boosting the static ones given in (3.1.17) and (3.1.18). To do so it is convenient to change to co-ordinates which make manifest the $SO(1, 1)$ symmetry between the temporal direction and the direction perpendicular to the domain walls, so that the metric has the form

$$ds^2 = e^{2A(t,z)} (-dt^2 + dz^2) + e^{2B(t,z)} \delta_{ij} dx^i dx^j \quad (3.3.2)$$

This is accomplished for a static kink or anti-kink by

$$e^U dz = dr \quad (3.3.3)$$

since in the static case $A(t, z) = B(t, z) = U(r(z))$. In the new frame the equations of motion are

$$\partial_{\tilde{\mu}} \partial^{\tilde{\mu}} \phi + 3\partial_{\tilde{\mu}} B \partial^{\tilde{\mu}} \phi = e^{2A} \frac{\partial \mathcal{V}}{\partial \phi} \quad (3.3.4)$$

$$\partial_{\tilde{\mu}} \partial^{\tilde{\mu}} A - 3\partial_{\tilde{\mu}} B \partial^{\tilde{\mu}} B = \frac{1}{m_p^3} \left(-\frac{1}{2} \partial_{\tilde{\mu}} \phi \partial^{\tilde{\mu}} \phi + \frac{1}{3} e^{2A} \mathcal{V} \right) \quad (3.3.5)$$

$$\partial_{\tilde{\mu}} \partial^{\tilde{\mu}} B + 3\partial_{\tilde{\mu}} B \partial^{\tilde{\mu}} B = -\frac{2}{3m_p^3} e^{2A} \quad (3.3.6)$$

where $\tilde{\mu}, \tilde{\nu}, \dots = \{t, z\}$ and the constraint equations are

$$\begin{aligned} & \partial_{\tilde{\mu}}\partial_{\tilde{\nu}}B + \partial_{\tilde{\mu}}B\partial_{\tilde{\nu}}B + \eta_{\tilde{\mu}\tilde{\nu}}\partial_{\tilde{\rho}}B\partial^{\tilde{\rho}}B - \partial_{\tilde{\mu}}A\partial_{\tilde{\nu}}B - \partial_{\tilde{\mu}}B\partial_{\tilde{\nu}}A + \eta_{\tilde{\mu}\tilde{\nu}}\partial_{\tilde{\rho}}A\partial^{\tilde{\rho}}B \\ &= -\frac{1}{3m_p^3} \left(\partial_{\tilde{\mu}}\phi\partial_{\tilde{\nu}}\phi - \frac{1}{2}\eta_{\tilde{\mu}\tilde{\nu}}\partial_{\tilde{\rho}}\phi\partial^{\tilde{\rho}}\phi + \eta_{\tilde{\mu}\tilde{\nu}}e^{2A}\mathcal{V} \right) \end{aligned} \quad (3.3.7)$$

Clearly A , B and ϕ are scalars under $SO(1,1)$ boosts and so it is trivial to transform between frames: if \mathcal{O}' is moving at velocity v in the z direction with respect to \mathcal{O} , then

$$t' = \gamma(t - vz) \quad (3.3.8)$$

$$z' = \gamma(z - vt) \quad (3.3.9)$$

with the standard γ -factor $\gamma = \frac{1}{\sqrt{1-v^2}}$ and for a generic scalar Ψ

$$\Psi'(t', z') = \Psi(t, z) \quad (3.3.10)$$

These co-ordinates also have the advantage of making it easy to picture the causal structure, since null rays are just 45° lines in the $t - z$ plane.

In fact it is more convenient numerically to use double-null co-ordinates [48, 79]

$$u = \frac{1}{\sqrt{2}}(t - z), \quad v = \frac{1}{\sqrt{2}}(t + z) \quad (3.3.11)$$

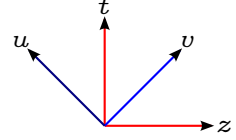


Figure 3.2: Double-null co-ordinates

because we want to investigate the behaviour when a singularity forms; such co-ordinates allow us to cut off the simulation when the singularity is reached, but continue the simulation in the regions outside of causal contact with the singularity. Thus we can try to investigate the asymptotic properties of the spacetime numerically despite the presence of the singularity. In double-null co-ordinates the equations of motion (3.3.4-3.3.6) are

$$\partial_u\partial_v\phi + \frac{3}{2}(\partial_uB\partial_v\phi + \partial_vB\partial_u\phi) = -\frac{1}{2}e^{2A}\frac{\partial\mathcal{V}}{\partial\phi} \quad (3.3.12)$$

$$\partial_u\partial_vA - 3\partial_uB\partial_vB = -\frac{1}{2m_p^3} \left(\partial_u\phi\partial_v\phi + \frac{1}{3}e^{2A}\mathcal{V} \right) \quad (3.3.13)$$

$$\partial_u\partial_vB + 3\partial_uB\partial_vB = \frac{1}{3m_p^3}e^{2A} \quad (3.3.14)$$

and the constraints (3.3.7) are

$$\partial_u\partial_uB + \partial_uB\partial_uB - \partial_uA\partial_uB - \partial_uB\partial_uA = -\frac{1}{3m_p^3}\partial_u\phi\partial_u\phi \quad (3.3.15)$$

$$\partial_v\partial_vB + \partial_vB\partial_vB - \partial_vA\partial_vB - \partial_vB\partial_vA = -\frac{1}{3m_p^3}\partial_v\phi\partial_v\phi \quad (3.3.16)$$

$$\partial_u\partial_vB - \partial_uB\partial_vB - 2(\partial_uA\partial_vB + \partial_uB\partial_vA) = \frac{1}{3m_p^3}e^{2A}\mathcal{V} \quad (3.3.17)$$

It is straightforward to solve the relation (3.3.3) between r and z numerically. This allows us to translate the analytic solutions (3.1.17) and (3.1.18) into the static kink and anti-kink solutions in the new frame. We then construct the initial conditions for colliding domain walls by adding boosted kink and anti-kink solutions equidistant from $z = 0$ and with equal and opposite velocities to allow us to exploit the reflection symmetry of the problem in the centre of mass frame.

3.4 Simulation Methods

Our first set of simulations were performed in $t - z$ co-ordinates using a fourth order Runge-Kutta method. These showed us the kink and anti-kink approaching and bouncing and the subsequent formation of a singularity at the origin. However, this set up has some limitations which prompted the later move to $u - v$ co-ordinates.

As the grid on which we run the simulation is finite in extent in the z direction, we are forced to apply some boundary condition at the edge. Since this boundary condition is not a physical one, it introduces errors into the numerical solution. In order to prevent these errors from affecting the region of physical interest, the grid has to be much larger than that region so that they do not have time during the length of the simulation to propagate to there from the boundary. This is not ideal from the point of view of computational efficiency, but is not a fundamental problem.

The more important issue is that since the method is fourth order, in order to calculate the fields at P (see Figure 3.3) we use the values at the points p_1 to p_4 which are up to two spatial steps away from P . This means that errors propagate through the simulation rather quickly. This is merely inconvenient for the errors introduced at the boundary but it is a fundamental problem for the errors introduced

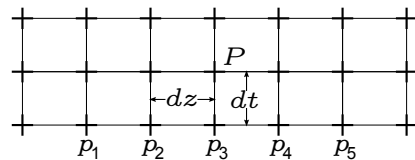


Figure 3.3: $t - z$ grid

by the formation of a singularity, near which the accuracy of the simulation obviously breaks down. Indeed the time interval of the grid, dt , must be small (with the speed of light taken to be unity, $c = 1$) compared to the spatial interval, dz . The timestep being small means that the effects of the singularity propagate superluminally and so prevents us from investigating with our simulations regions which are *not* in causal contact with the singularity and which should therefore be physically well behaved. This problem is resolved by our later simulation in double-null co-ordinates which, as we shall see, explicitly respects the causality structure of the spacetime. The reason the timestep must be small is in order to ensure that we satisfy the Courant condition (which states that we must have $dt < Cdz$ for a scheme dependent constant C of order unity): otherwise the Runge-Kutta method would be unstable. This condition arises from the fact that in one timestep this sort of numerical scheme can only transmit information a certain number of spatial points away (two in our case); if the timestep were too large then information would propagate faster physically than it could in the numerical scheme, which could lead to large errors.

Despite these issues, there is a window in which we have a numerical solution for the region between the kink and anti-kink and away from the singularity. Under the assumption that the singularity is a black brane we would expect that the formation of the singularity just replaces the initial Minkowski space between the walls with a black brane solution. In particular we expected to be able to measure the tension of the black brane. In fact the apparent ‘tension’ of the ‘black brane’ never settled down and so we were unable to measure it. This was our first hint that the singularity formed is not in fact a black brane, as

we confirmed by examination of the asymptotic structure, described above, and further simulations, described below.

In order to examine the structure of the singularity in more detail in our simulations, we switch to a scheme using double null co-ordinates [79] which alleviates the problems just described and also allows some useful optimizations. In order to describe this scheme let us denote the various fields by $h^i = \{\phi, A, B\}$, $i = 1, 2, 3$. To calculate the values of h^i at P (Figure 3.4) we now depend only on the values at p_1, p_2 and p_3 , which are all in the past light cone of P . To cover the space we calculate h^i along each ingoing ray at constant v , starting at $u = 0$ and continuing to $z = 0$ or until we get too close to the singularity, before moving on to the next ray at $v + dv$. In this way the only boundary condition we need is the initial data along $u = 0$, which we know exactly so we have no boundary errors as in the $t - z$ case above. More importantly, this scheme respects the causal structure: Therefore the singularity does not disrupt the simulation in regions that are outside of causal contact with it and so we can continue to simulate the whole external spacetime even after the singularity is formed.

The method we use to calculate h^i at P from the values at p_1, p_2 and p_3 (as shown in Figure 3.4) is second order [79]. We take

$$h^i(P) = -h^i(p_1) + h^i(p_2) + h^i(p_3) + F^i(p_4)dudv \quad (3.4.1)$$

with du and dv the lattice spacings in u and v and defining $F^i(p) = \partial_u \partial_v h^i$ as given by the equations of motion (3.3.12-3.3.14); F^i expresses the second derivative of h^i in terms of the fields and their first derivatives: $h^j, \partial_u h^k$ and $\partial_v h^l$. We need to know the values of these at p_4 to evaluate (3.4.1). We calculate these using the standard ‘‘predictor-corrector’’ method to iterate until the corrections are below the chosen threshold.

This numerical scheme only depends upon a rectangular set of grid points with one step in the u direction and one in the v direction. This makes it relatively straightforward to adapt the size of the grid, as compared to, for example, the Runge-Kutta method we used previously. To do so there we would have to take into account the changing step size between say $p_1 \rightarrow p_2$ and $p_2 \rightarrow p_3$ of Figure 3.3 within the single calculation of the point at P , whereas here we only have one step size in the calculation of a point, since we only have to consider one step. We make use of this property to implement an adaptive grid spacing, via ‘point splitting’ and ‘point reduction’. Near the domain walls and especially as we approach the singularity the fields are changing rapidly and so we need a fine lattice spacing to maintain the accuracy of our simulation. To maintain this fine spacing everywhere would be prohibitively expensive computationally, which creates the need for adaptive grid spacing.

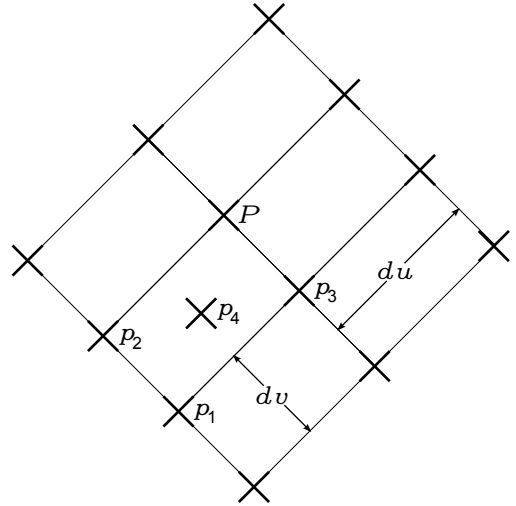


Figure 3.4: $u - v$ grid

When the difference between any h^i at adjacent points, (u_0, v_0) and $(u_0 + du, v_0)$, exceeds some threshold, we know that we need a finer grid. We obtain this by adding an extra point to our lattice at $(u_0 + \frac{1}{2}du, v_0)$ and calculating the values of h^i there by interpolation so that when we reach the same region on the next pass at $v_0 + dv$ we can use $h^i(u_0 + \frac{1}{2}du, v_0)$ to calculate $h^i(u_0 + \frac{1}{2}du, v_0 + dv)$ by our usual scheme described above. In this way regions where the fields are changing rapidly acquire more grid points so that we have a grid spacing fine enough to maintain the desired accuracy. This is what is called point splitting.

Points at a given value of u move away from the origin (and so from the singularity) at the speed of light as we increase v . Therefore if the influence of the singularity spreads subluminally the fields will be changing less rapidly at larger v than they were previously and the fine grid spacing required then becomes redundant. The domain wall, and hence the need for a finer mesh associated with it, also moves. This means that we can apply a method for removing points once they are no longer required, allowing us to avoid unnecessary computations. This is what is called point reduction. We have a second threshold, smaller than the threshold for point splitting, and if the difference in all the fields between some point and the previous one is lower than that threshold the the point is removed from the grid.

By the combination of these two procedures the spacing in u of our grid adapts itself to maintain accuracy in rapidly changing regions while not incurring superfluous computational cost in slowly changing regions.

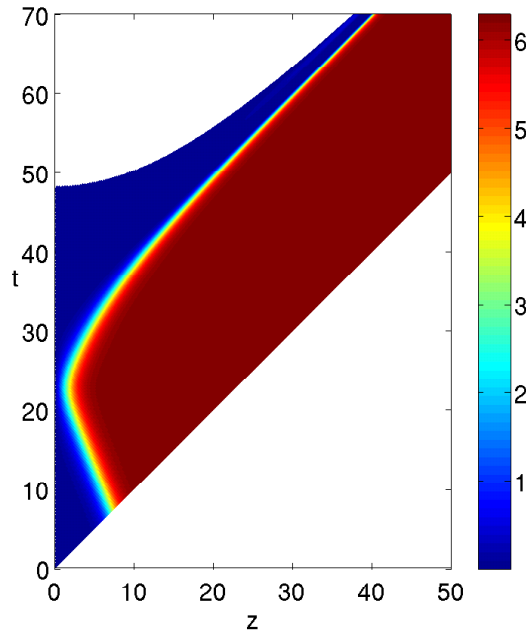


Figure 3.5: The scalar field during a collision in which a singularity forms. The value of ϕ is shown by the colours from 0 (blue) in the Minkowski vacuum to 2π (red) in the AdS^5 vacuum. The position of the brane is the transition region between the two, i.e. the green-yellow-orange band. We see the brane bouncing away from its partner in the $z < 0$ region just after $t = 20$ before the singularity forms at around $t = 50$.

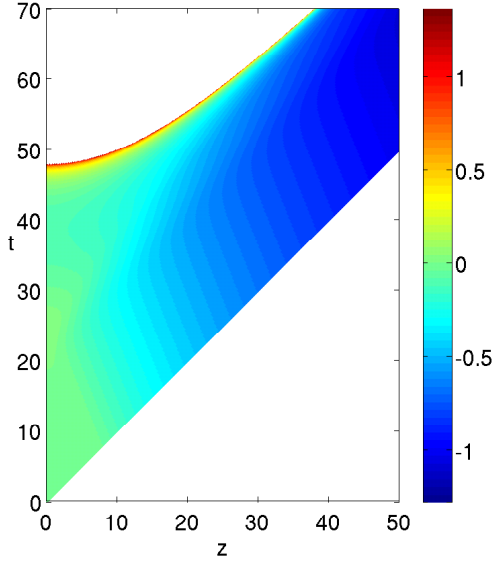


Figure 3.6: The value of the metric function A , as defined in Equation (3.3.2), over the $t-z$ plane. A is not dramatically affected by the bounce just after $t = 20$. It diverges later, driven by B .

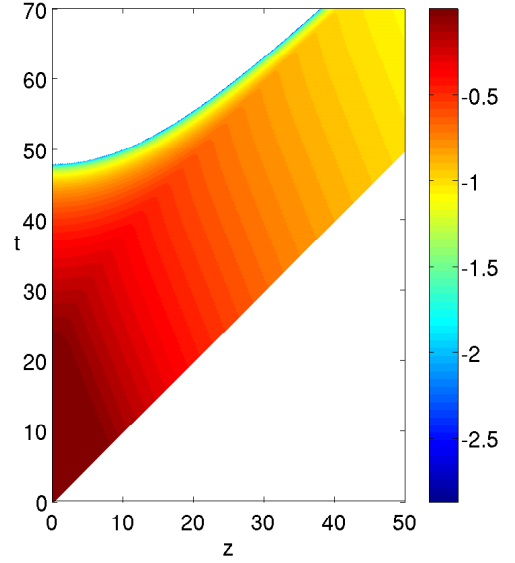


Figure 3.7: The value of the metric function B , as defined in Equation (3.3.2), over the $t-z$ plane. We can see that after the branes bounce just after $t = 20$ B begins to decrease, continuing until the singularity forms around $t = 50$. Hence the singularity formation is driven by the collapse in the size of the transverse slices, that is the x^i directions.

3.5 Simulation Results

The outcome of a simulation, using the numerical method in double-null coordinates, is shown in Figure 3.5. In this simulation a singularity forms. Only half of the spacetime is shown, the other follows from reflection symmetry about $z = 0$. The initial conditions are imposed along the $u = 0$ line which borders the blank triangle in the bottom right of the plot. We see the domain wall between the AdS^5 region with $\phi \rightarrow 2\pi$ at large z and the Minkowski region with $\phi \rightarrow 0$ around $z = 0$. The other domain wall is its mirror image in the $z < 0$ region. The domain walls approach each other initially before bouncing apart. After the bounce a singularity forms in the centre. The simulation is cut off when the curvature becomes too large near the singularity: this is the blank area in the top left of the plot. The formation of the singularity is driven by the metric function B , as defined in Equation (3.3.2), which controls the x^i components of the metric. As we see in Figure 3.7, B begins to decrease after the bounce, and diverges as the singularity is approached. This causes the other metric function, A , also to diverge (Figure 3.6).

3.6 Horizon Structure

Having seen that there is a singularity, we would like to check whether there is a horizon, and examine its structure if there is one. Indeed we may expect a horizon, since we see from Figure 3.5 that there is a region within which all timelike geodesics will end on the singularity. To find a horizon we study the structure of the null geodesics. However, given the dynamical nature of the spacetime it is hard to study global properties of the geometry, like the event horizon; it is more convenient to find an alternative with a local definition. This we do in the definition by Hayward [80] of trapping horizons, which are given in terms of the expansion of incoming and outgoing null geodesics.

Since we are using double-null co-ordinates, we have null vectors to hand already,

$$N_+ = \partial_u, \quad N_- = \partial_v \quad (3.6.1)$$

where for $z > 0$ N_+ is ingoing and N_- is outgoing. The dual one-forms to these are

$$n_+ = -e^{2A}dv, \quad n_- = -e^{2A}du \quad (3.6.2)$$

and we can define unit normalized null vectors

$$u_+ = e^{-2A}N_-, \quad u_- = e^{-2A}N_+ \quad (3.6.3)$$

so that $n_{\pm}(u_{\pm}) = -1$. Then the induced metric on three-dimensional surfaces normal to both u_+ and u_- is

$$h = g + e^{-2A}n_+ \otimes n_- + e^{-2A}n_- \otimes n_+ \quad (3.6.4)$$

where g is the full metric. We can use the induced metric to define the expansions of null geodesics by

$$\Theta_{\pm} = \frac{1}{2}h^{ij}\mathcal{L}_{\pm}h_{ij} \quad (3.6.5)$$

with \mathcal{L}_{\pm} being the Lie derivatives along u_{\pm} , $\mathcal{L}_{\pm} \equiv \mathcal{L}_{u_{\pm}}$. A marginal surface is a three-surface on which one of the expansions vanishes, i.e. it is a point in the $t-z$ plane. Finally we can define a trapping horizon as the locus of marginal surfaces, which is a line in the $t-z$ plane. We may classify trapping horizons firstly as past or future and secondly as inner or outer. Consider a trapping horizon defined by $\Theta_- = 0$. It is future if $\Theta_+ < 0$ and past if $\Theta_+ > 0$. It is outer if $\mathcal{L}_+\Theta_- < 0$ and inner if $\mathcal{L}_+\Theta_- > 0$.

The expansions in our example are given in Figures 3.8 and 3.9 where the values of the expansions are shown by the colours over the $t-z$ plane. We can see from Figure 3.8 that there is a region where Θ_- is negative (near the singularity) and another where it is positive (far from the singularity) so there is a line where Θ_- changes sign, which is a trapping horizon. Figure 3.9 shows that Θ_+ is negative along the trapping horizon, so it is a future horizon. Evaluating $\mathcal{L}_+\Theta_-$ on the horizon shows that it is positive (the horizon is slightly steeper than a null ray) and so it is an inner horizon. Now a black hole has a future-outer trapping horizon, suggesting again that our singularity is not a black brane.

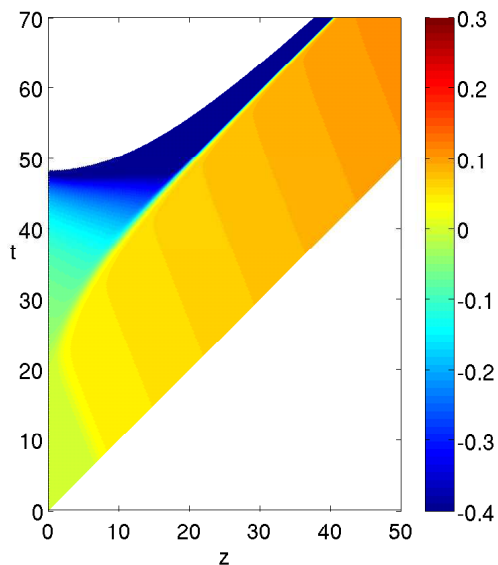


Figure 3.8: Θ_- over the $t - z$ plane. The trapping horizon is the line where $\Theta_- = 0$. Θ_- is negative inside the horizon and positive outside, which shows that we have an inner horizon, since u_+ crosses the horizon from inside to outside.

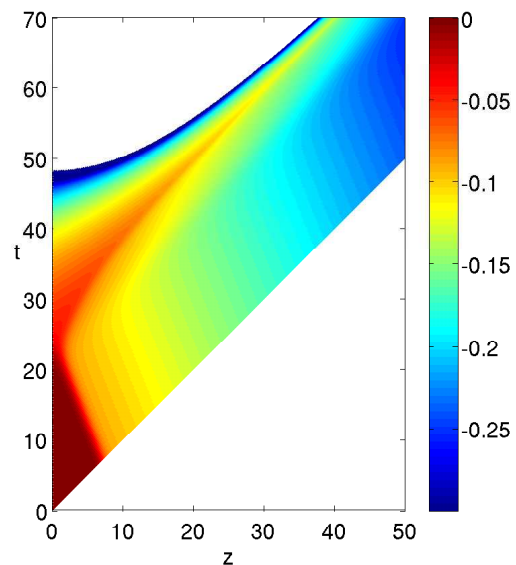


Figure 3.9: Θ_+ over the $t - z$ plane. Θ_+ is negative everywhere (apart from the initial Minkowski region between the branes, where it vanishes) and in particular on the horizon, showing that the horizon is future.

However, it might be a big crunch singularity which does have a future-inner trapping horizon.

By way of confirmation of this conclusion, we can measure the area of the horizon, which for a future-inner trapping horizon should be non-increasing. Figure 3.10, which gives the value of B along the horizon, shows that this is indeed the case.

3.7 Closing Off the Spacetime

If our singularity is a big crunch then it should close off the entire spacetime: there should be no region where an observer could survive forever without encountering the singularity. It is challenging to show this robustly numerically as it would require us to simulate the spacetime out to very large values of v . As something of a substitute we can look at the level surfaces of the Ricci scalar. If they were all to reach $u = 0$ eventually, then any observer would ultimately have to cross all of them and thus reach the singularity. The level set for $R = -5$, plotted in Figure 3.11, shows u decreasing rather slowly as v increases but it is at least consistent with crossing $u = 0$ eventually and thus consistent with a big crunch singularity.

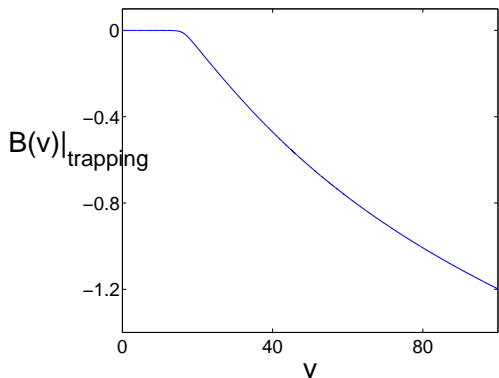


Figure 3.10: B evaluated on the trapping horizon, showing that the area of the horizon decreases monotonically.

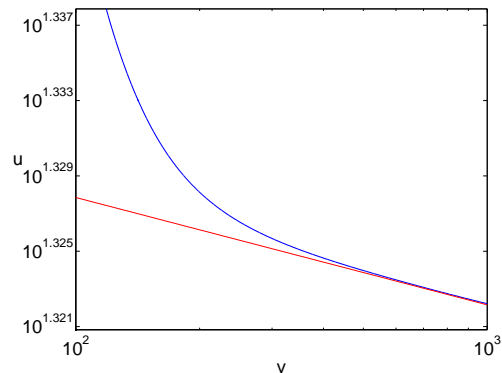


Figure 3.11: The level set for $R = -5$, in blue, and the line $u = \frac{1}{v^{5.7 \times 10^{-3}}}$, in red.

3.8 Conclusions

We have reconsidered the formation of curvature singularities in domain wall collisions [48] with an eye on their global structure. Our initial observation was that the asymptotic AdS⁵ structures of the domain walls and black branes are mutually incompatible and so the singularities formed by domain walls cannot be black branes. This is corroborated by looking at the dynamical geometry using numerical simulations. These show us that the trapping horizons formed around the singularity after the domain wall collision are of the type associated with a big crunch rather than a black brane. Further confirmation comes from the non-increasing area of the horizon and the trend of the Ricci level sets across $u = 0$ at large v , though the evidence for the latter is tentative because of the numerical difficulties presented by the very slow rate of decrease. As the type of singularity seems to be dictated by the asymptotic structure, one might expect similar qualitative behaviour as we see for the scalar domain wall to apply also to a generic braneworld collision if they share the same asymptotic structure.

In [76] it was predicted that the AdS⁵ Cauchy horizon would generically be replaced by a pp singularity if the AdS region were perturbed. A pp singularity is a singularity where all the scalar curvature invariants are finite, but the components of the Riemann curvature diverge so that an observer in an orthonormal frame parallelly propagated along a timelike geodesic (i.e. a freely falling observer) would observe the divergence, which would manifest in the tidal forces experienced by that observer. Here we find that when a curvature singularity is formed it closes off the geometry in a big crunch and no pp singularity is formed. However, when the collision is not sufficiently violent to form a curvature singularity there is no reason not to expect the prediction of a pp singularity to hold.

Now that it is established that the singularity formed in the collision of domain walls is a big crunch, further simulations would be useful to map out the parameter space for where singularities do and do not form. If a singularity does form then the universe on the domain wall must end in finite time, since it will encounter the singularity. Therefore in order to construct models of an eternally expanding universe one would have to set the initial conditions in order to avoid

singularity formation. One could also now return to the investigation of particle dynamics on the domain walls, as in [43–46], but including the gravitational dynamics.

Chapter 4

Braneworlds and Galileons

For modifications of gravity to be phenomenologically viable (as solutions to the dark energy problem) they must give undetectably small corrections to general relativity on solar system scales, but large corrections approaching horizon scales. This forces such modifications to be non-linear, but in non-linear theories instabilities, such as ghosts, often become problematic. By generalizing the four dimensional effective field theory arising from the DGP model, Nicolis et al. found a promising class of models [50]. These have a scalar field, π , coupled to gravity whose Lagrangian is symmetric under the ‘Galilean’ transformation $\pi \rightarrow \pi + b_\mu x^\mu + c$, leading them to christen π the ‘galileon’. Ghosts can be avoided by demanding that the equations of motion contain at most second derivatives and this demand together with Galilean symmetry restricts the models to a family with only five free parameters (in four dimensions): in dimension d the terms in the Lagrangian take the form $\eta^{\mu_1}_{[\nu_1} \dots \eta^{\mu_m}_{\nu_m]} \pi \partial_{\mu_1} \partial^{\nu_1} \pi \dots \partial_{\mu_m} \partial^{\nu_m} \pi$ for $m \leq d$.

As the galileon theory was inspired initially by the DGP braneworld model, it is not surprising that it can itself be embedded in a braneworld model as the effective theory of a codimension one probe brane, as was found by de Rham and Tolley for flat spacetime [51] and generalized by Goon et al. [53]. This is achieved by using more general gravity theories than the DGP model which has just the Einstein-Hilbert term in both brane and bulk actions. The requirement of second order equations of motion means that those gravity theories must be what one might call Lovelock-Myers theories, with Lovelock terms [52] both in the bulk and on the brane supplemented by Myers-type surface terms [23] on the brane. On a co-dimension one brane (which is equivalent to a boundary) there are a series of Lovelock terms for the intrinsic curvature (cosmological constant Λ , Einstein-Hilbert R , Gauss-Bonnet $R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}, \dots$) and Myers terms for the extrinsic curvature, which are the surface terms corresponding to the bulk Lovelock terms (Gibbons-Hawking K , $K_{GB} = -\frac{1}{3}K^3 + K_{\mu\nu}K^{\mu\nu}K - \frac{2}{3}K_\mu^\nu K_\nu^\rho K_\rho^\mu - 2(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu})K^{\mu\nu}, \dots$). In a given dimension these series terminate, with all the higher terms only contributing total derivatives to the action. The galileon theory is constructed by introducing a four dimensional probe brane into a maximally symmetric five dimensional bulk, neglecting the backreaction of the brane modes, i.e. of the galileon. The galileon is the displacement of the brane in the fifth dimension with the Galilean symmetry thus following from the Poincaré symmetry of the bulk: an infinitesimal transformation is given by

$x^a \rightarrow x^a + \epsilon^a_b x^b + v^a$ so if we shift the brane from $y = \pi(x)$ to $y = \pi(x) + \epsilon^y_\mu x^\mu + v^y$ there can be no physical change, and so we can see that the theory is symmetric under Galilean transformations of π . Four of the five parameters of the galileon field theory then come from the coefficients of the first two Lovelock terms and the first two Myers terms. The fifth comes from the coefficient of the tadpole term $\int^\pi \sqrt{g} dy$ which also turns out to be allowed.

On the other hand, one can also construct four dimensional field theories with more than one galileon [54, 81]. These share many of the attractive properties of the single galileon theories, although as the number of galileons increases the number of possible terms in the action, which is [55] $\sum_{n=1}^5 \frac{(m+n-1)!}{n!(m-1)!}$ for m galileons, increases rather fast giving 55 free parameters for three galileons and 125 for four galileons. Just as a single galileon is the displacement of a codimension one brane in the transverse direction, so should m galileons be the transverse displacements of a codimension m brane.

Such a construction has been performed for a Minkowski probe brane in a Minkowski bulk by Hinterbichler et al. [56] but with an extremely restricted choice of terms in the action, leaving only one free parameter. This can only be a braneworld embedding of a rather small class of the possible field theory galileon models. This choice was made on the strength of the assertion, based on the results of [82, 83], that the only possible terms in the action for a four dimensional brane of even codimension N are: a cosmological constant and a term $\sqrt{-g} (R(g) - (K^i)^2 + K^i_{\mu\nu} K_i^{\mu\nu})$ for $N = 2$; and a cosmological constant and the Einstein-Hilbert term of the induced metric for $N > 2$. However, the actions found in [82, 83] are those which allow consistent matching conditions for a distributional (i.e. zero ‘thickness’) brane with a tension which backreacts on the bulk geometry. They are not relevant to the case here where we are interested in probe branes which do not backreact. For example, black hole theorems restricting the possible solutions and thus the allowed sources cannot be a restriction here because there is no backreaction and thus no source for the bulk curvature. We are allowed any curvature terms we can write down which result in second order equations of motion. This certainly includes all the Lovelock terms of the induced curvature (which in four dimensions are just the cosmological constant and the Einstein-Hilbert term). Presumably some terms in the extrinsic curvature are also allowed but it is not clear how to construct the analogues in codimension greater than one of the Myers terms now that we have an extra (transverse) index on the extrinsic curvature.

Here we describe some work done in collaboration with Ian Moss, Antonio Padilla and Paul Saffin. We describe a slightly more general class of models, allowing for maximally symmetric spaces with arbitrary curvature but with the same restriction as [56] on the terms in the action. This restriction arises from the $SO(m)$ symmetry of the extra dimensions which follows if both brane and bulk are maximally symmetric (and so restricts the case in [56] as well). It seems that the only way to find the complete set of the models which can be constructed in the field theory approach is to construct less than maximally symmetric models, because of the strong restriction placed by $SO(m)$ symmetry on the possible terms in the action. This remains yet to be done. It appears to be a rather challenging task since the analogues in higher codimension of the Myers surface terms induced by bulk Lovelock invariants in codimension one are unclear: though in the case

we consider here they are incompatible with the $SO(m)$ symmetry up to the order to which we work. Also unclear is the role of the tadpole term in higher codimension since in this case there is ambiguity over how one integrates up to the brane, in contrast to codimension one. However, the formalism we have developed for constructing the braneworld-galileon actions may be of some interest; we hope that it might be useful for constructing more general galileon actions than just those considered here if a scheme for breaking the $SO(m)$ symmetry can be specified.

4.1 Galileons in Codimension One

The codimension one case has two properties which make it much more straightforward than the higher codimension case. The first is that as mentioned above the possible terms in the action are clear in arbitrary dimension. The second is that one can find exact expressions for all these in terms of the galileon π , the displacement of the brane in the transverse direction. The reason for this is that one can analytically invert the induced metric on the brane in this case.

Consider a codimension one probe brane. Taking a Gaussian normal foliation, the full metric of the bulk spacetime is

$$ds^2 = d\rho^2 + f(\rho)^2 g_{\mu\nu}(x) dx^\mu dx^\nu \quad (4.1.1)$$

with the co-ordinates x^μ propagated through the leaves of the foliation at constant ρ by the trajectories normal to them. Using the co-ordinates x^μ on the brane also and defining the galileon, π , as the ρ -value of each point x^μ on the brane,

$$\pi(x) = \rho|_{\text{brane}(x)} \quad (4.1.2)$$

then the induced metric on the brane is

$$\tilde{g}_{\mu\nu} = f(\pi)^2 g_{\mu\nu} + \partial_\mu \pi \partial_\nu \pi \quad (4.1.3)$$

whose inverse is

$$\tilde{g}^{\mu\nu} = \frac{1}{f^2} \left(g^{\mu\nu} - \frac{\partial^\mu \pi \partial^\nu \pi}{f^2 \left(1 + \frac{1}{f^2} \partial_\xi \pi \partial^\xi \pi \right)} \right) \quad (4.1.4)$$

With these two expressions it is possible to compute exactly the determinant and intrinsic curvature of the induced metric, and also the normal vector and hence the extrinsic curvature.

Note however that we only have an analytic expression of the inverse, (4.1.4), because the order of the π 's does not matter: $(\partial_\mu \pi \partial_\nu \pi) (\partial^\xi \pi \partial^\nu \pi) = \partial_\mu \pi \partial_\nu \pi (\partial_\xi \pi \partial^\xi \pi)$. In the higher codimension case which is our focus here, this trick fails. We now have a galileon for each transverse direction so if the codimension is m we have π^I with $I = 1, \dots, m$. The induced metric is similar

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} + g_{IJ} \nabla_\mu \pi^I \nabla_\nu \pi^J \quad (4.1.5)$$

except that now π^I fields come in pairs with contracted transverse indices. Thus $(\nabla_\mu \pi^I \nabla_\xi \pi^I) (\nabla^\xi \pi^J \nabla^\nu \pi^J) \neq \nabla_\mu \pi^I \nabla_\nu \pi^I (\nabla_\xi \pi^J \nabla^\xi \pi^J)$ and so we cannot write an

analytic expression for the inverse metric like (4.1.4). Since we therefore cannot use a fully explicit, analytic formalism in the higher codimension case we instead work perturbatively from the outset. We write the full metric of the bulk space as a Taylor expansion in the distance from the ‘zero position’ of the brane (its position when all the galileon fields π^I vanish) and truncate the rest of our expressions to fourth order in the π^I . This approach allows us to use the existing machinery for calculating, for example, the Ricci scalar in terms of metric perturbations. We now turn to the elaboration of this perturbative formalism.

4.2 Perturbative Formalism

Consider a codimension m probe 3-brane in an $(m+4)$ -dimensional bulk. We have then m galileons π^I ($I = 1, \dots, m$) which are the displacements of the brane in the transverse directions. We will work perturbatively, assuming from the outset that the galileons are small. We set the co-ordinates on the bulk spacetime by choosing a foliation adapted to the brane. Let Σ_0 denote a four dimensional slice of the spacetime (the position of the unperturbed brane with $\pi^I = 0$) with co-ordinates x^μ ($\mu = 0, 1, 2, 3$). Then we extend the co-ordinates to the neighbourhood of Σ_0 along the m directions normal to the surface, i.e. $x^a = \{x^\mu, x^I\}$ ($a = 0, \dots, 3+m$) so that the co-ordinate basis vectors $\mathbf{e}_\mu \equiv \frac{\partial}{\partial x^\mu}$ and $\mathbf{e}_I \equiv \frac{\partial}{\partial x^I}$ are orthogonal. The galileons are just the displacements of the brane, Σ , away from Σ_0 : the co-ordinates of points on the brane are $x^a|_{brane} = \{x^\mu, \pi^I(x^\mu)\}$.

Let $\boldsymbol{\xi} = \xi^I \mathbf{e}_I$ denote the normalized ($\boldsymbol{\xi} \cdot \boldsymbol{\xi} = 1$) tangent vector to some geodesic normal to Σ_0 and σ denote the proper distance along this geodesic from Σ_0 . Since we are working perturbatively, we take σ to be small and we can write quantities on the brane as a truncated Taylor series in σ about their values on Σ_0 . The (full) metric at the point a distance σ away from Σ_0 along the geodesic whose tangent is $\boldsymbol{\xi}$ is

$$g_{ab} = \sum_{n=0}^{\infty} \frac{1}{n!} \sigma^n [\nabla_{\boldsymbol{\xi}}^n (\mathbf{e}_a \cdot \mathbf{e}_b)]_{\Sigma_0} \quad (4.2.1)$$

where of course $\mathbf{e}_a \cdot \mathbf{e}_b$ is just g_{ab} but this form is convenient since we can determine how $\nabla_{\boldsymbol{\xi}}$ acts on the basis vectors \mathbf{e}_a :

$$\nabla_{\boldsymbol{\xi}} \mathbf{e}_\mu|_{\Sigma_0} = \xi^I k_{I\mu}{}^\nu \mathbf{e}_\nu + \xi^I a_{I\mu}{}^J \mathbf{e}_J \quad (4.2.2)$$

$$\nabla_{\boldsymbol{\xi}}^n \mathbf{e}_\mu|_{\Sigma_0} = \nabla_{\boldsymbol{\xi}}^{n-2} \mathbf{R}(\boldsymbol{\xi}, \mathbf{e}_\mu) \boldsymbol{\xi} \quad (4.2.3)$$

$$\nabla_{\boldsymbol{\xi}} \mathbf{e}_I|_{\Sigma_0} = 0 \quad (4.2.4)$$

$$\nabla_{\boldsymbol{\xi}}^n \mathbf{e}_I|_{\Sigma_0} = \frac{(n-1)}{(n+1)} \nabla_{\boldsymbol{\xi}}^{n-2} \mathbf{R}(\boldsymbol{\xi}, \mathbf{e}_I) \boldsymbol{\xi} \quad (4.2.5)$$

where $k_{I\mu}{}^\nu$ and $a_{IJ}{}^\mu$ are the Weingarten coefficients of the surface Σ_0 . For proofs of these statements see Appendix F. In our normal co-ordinates they are defined quite straightforwardly: the extrinsic curvature¹ is

$$k_{I\mu}{}^\nu = -\Gamma^I{}_{\mu\nu}|_{\Sigma_0} \quad (4.2.6)$$

¹The relation of the extrinsic curvature defined like this to the usual definition for a codimension one hypersurface is discussed in Appendix F

and the ‘twist connection’ is

$$a_{IJ}{}^\mu = -\Gamma^\mu{}_{IJ}|_{\Sigma_0} \quad (4.2.7)$$

Up to this point we are entirely agnostic as to the structure of either the bulk spacetime or the unperturbed brane. To make further progress, we assume that both are maximally symmetric, as this greatly simplifies the expressions for the bulk curvature, and the extrinsic curvature and twist connection of the unperturbed brane. This assumption would need to be relaxed somehow to construct more general galileon theories, but this being achieved one could presumably continue in a similar spirit to the method described below for the maximally symmetric case.

In the maximally symmetric case we have

$$k_{I\mu\nu} = c_I g_{\mu\nu} \quad (4.2.8)$$

$$a_{IJ\mu} = 0 \quad (4.2.9)$$

$$R_{abcd} = \kappa (g_{ac}g_{bd} - g_{ad}g_{bc}) \quad (4.2.10)$$

where c_I and κ are constants. We will work at up to fourth order in the galileon fields. To do so we need to evaluate the terms in the expansion of the metric, (4.2.1), at the brane, Σ . There $\sigma\xi = \pi^I e_I$ and so, writing $\pi = c_I \pi^I$ and $\boldsymbol{\pi} \cdot \boldsymbol{\pi} = g_{IJ} \pi^I \pi^J$,

$$\sigma \nabla_\xi e_\mu = \pi e_\mu \quad (4.2.11)$$

$$\sigma^2 \nabla_\xi^2 e_\mu = \sigma^2 \mathbf{R}(\boldsymbol{\xi}, e_\mu) \boldsymbol{\xi} = \pi^I \pi^J R^\nu{}_{JI\mu} e_\nu = -\kappa \boldsymbol{\pi} \cdot \boldsymbol{\pi} e_\mu \quad (4.2.12)$$

$$\sigma^3 \nabla_\xi^3 e_\mu = \sigma^3 \nabla_\xi \mathbf{R}(\boldsymbol{\xi}, e_\mu) \boldsymbol{\xi} = \sigma^3 \mathbf{R}(\boldsymbol{\xi}, \nabla_\xi e_\mu) \boldsymbol{\xi} = \pi^I \pi^J R^\nu{}_{JI\mu} e_\nu = -\kappa \boldsymbol{\pi} \cdot \boldsymbol{\pi} e_\mu \quad (4.2.13)$$

$$\sigma^4 \nabla_\xi^4 e_\mu = \sigma^4 \mathbf{R}(\boldsymbol{\xi}, \nabla_\xi^2 e_\mu) \boldsymbol{\xi} = -\kappa \pi^I \pi^2 \pi^J R^\nu{}_{JI\mu} e_\nu = \kappa^2 (\boldsymbol{\pi} \cdot \boldsymbol{\pi})^2 e_\mu \quad (4.2.14)$$

$$\sigma^2 \nabla_\xi^2 e_I = \sigma^2 \frac{1}{3} \mathbf{R}(\boldsymbol{\xi}, e_I) \boldsymbol{\xi} = \frac{1}{3} \pi^J \pi^K R^L{}_{KJI} e_L = \frac{1}{3} \kappa (\pi_I \pi^J e_J - \boldsymbol{\pi} \cdot \boldsymbol{\pi} e_I) \quad (4.2.15)$$

Then from 4.2.1, and calling the metric at Σ_0 \bar{g}_{ab} , the metric at Σ is

$$\begin{aligned} g_{\mu\nu} &= \bar{g}_{\mu\nu} + 2\pi \bar{g}_{\mu\nu} - \kappa \boldsymbol{\pi} \cdot \boldsymbol{\pi} \bar{g}_{\mu\nu} + \pi^2 \bar{g}_{\mu\nu} - \frac{4}{3} \kappa \boldsymbol{\pi} \cdot \boldsymbol{\pi} \bar{g}_{\mu\nu} \\ &\quad + \frac{1}{3} \kappa^2 (\boldsymbol{\pi} \cdot \boldsymbol{\pi})^2 \bar{g}_{\mu\nu} - \frac{1}{3} \kappa \pi^2 \boldsymbol{\pi} \cdot \boldsymbol{\pi} \bar{g}_{\mu\nu} + \mathcal{O}(\pi^5) \end{aligned} \quad (4.2.16)$$

$$g_{\mu I} = 0 \quad (4.2.17)$$

$$g_{IJ} = \bar{g}_{IJ} + \frac{1}{3} \kappa (\pi_I \pi_J - \boldsymbol{\pi} \cdot \boldsymbol{\pi} \bar{g}_{IJ}) + \mathcal{O}(\pi^4) \quad (4.2.18)$$

We use the co-ordinates x^μ on Σ , with $\pi^I = \pi^I(x^\mu)$ so the induced metric on Σ is

$$\begin{aligned} \tilde{g}_{\mu\nu} &= g_{\mu\nu} + g_{IJ} \nabla_\mu \pi^I \nabla_\nu \pi^J \\ &= \Omega^2 (\bar{g}_{\mu\nu} + \delta g_{\mu\nu}) \end{aligned} \quad (4.2.19)$$

where we have

$$\Omega^2 = 1 + 2\pi + \pi^2 - \kappa \boldsymbol{\pi} \cdot \boldsymbol{\pi} - \frac{4}{3} \kappa \pi \boldsymbol{\pi} \cdot \boldsymbol{\pi} - \frac{1}{3} \kappa \pi^2 \boldsymbol{\pi} \cdot \boldsymbol{\pi} + \frac{1}{3} \kappa^2 (\boldsymbol{\pi} \cdot \boldsymbol{\pi})^2 + \mathcal{O}(\pi^5) \quad (4.2.20)$$

$$\delta g_{\mu\nu} = \Omega^{-2} \left(\pi_\mu \cdot \pi_\nu + \frac{1}{3} \kappa (\pi \cdot \pi_\mu \boldsymbol{\pi} \cdot \pi_\nu - \boldsymbol{\pi} \cdot \boldsymbol{\pi} \pi_\mu \cdot \pi_\nu) + \mathcal{O}(\pi^6) \right) \quad (4.2.21)$$

We define the scalar $\pi = c_I \pi^I$ and the vector $\boldsymbol{\pi} = \pi^I \mathbf{e}_I$. Spacetime indices on these denote derivatives: $\pi_\mu = \nabla_\mu \pi$, $\boldsymbol{\pi}_\mu = \nabla_\mu \boldsymbol{\pi}$, $\boldsymbol{\pi}_{\mu\nu} = \nabla_\mu \nabla_\nu \pi$.

With the full metric g_{ab} and the induced metric $\tilde{g}_{\mu\nu}$ we now have the ingredients we need to evaluate the possible terms in the action as expansions in π^I 's.

4.3 Possible Actions

What terms can be present in the brane action? In the codimension one case one has the Lovelock terms in the intrinsic curvature ($\Lambda, R, R_{GB} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}, \dots$); the Myers surface terms involving the extrinsic curvature, which correspond to the Lovelock terms in the bulk, ($K, K_{GB} = -\frac{1}{3}K^3 + K_{\mu\nu}K^{\mu\nu}K - \frac{2}{3}K_\mu^\nu K_\nu^\rho K_\rho^\mu - 2(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu})K^{\mu\nu}, \dots$); and a tadpole term $\int^\pi \sqrt{\tilde{g}} dy$. For a 3-brane in a five dimensional bulk the non-trivial terms (i.e. those which are not just total derivatives) are, labelling them as in [53],

$$\mathcal{L}_{(1)} = \int^\pi \sqrt{\tilde{g}} dy \quad (4.3.1)$$

$$\mathcal{L}_{(2)} = \sqrt{-\tilde{g}} \quad (4.3.2)$$

$$\mathcal{L}_{(3)} = \sqrt{-\tilde{g}} K \quad (4.3.3)$$

$$\mathcal{L}_{(4)} = \sqrt{-\tilde{g}} R(\tilde{g}) \quad (4.3.4)$$

$$\mathcal{L}_{(5)} = \sqrt{-\tilde{g}} K_{GB} \quad (4.3.5)$$

The question is what the analogues are in the higher codimension case. The brane is still four dimensional so as before we have the Lovelock terms $\mathcal{L}_{(2)}$ and $\mathcal{L}_{(4)}$ and the higher Lovelock terms (R_{GB}, \dots) are total derivatives. However, since a brane with codimension greater than one is not a boundary the analogues of $\mathcal{L}_{(3)}$ and $\mathcal{L}_{(5)}$ are unclear as the extrinsic curvature now has an extra, transverse index. However for a Minkowski brane in a Minkowski bulk, the extrinsic curvature is at least $\mathcal{O}(\pi^3)$ (see Appendix F). With $SO(m)$ symmetry the only way to contract the transverse index on the extrinsic curvature is with another extrinsic curvature. Therefore any such terms compatible with $SO(m)$ symmetry are in this case $\mathcal{O}(\pi^6)$ and beyond the order we consider here. The definition of the tadpole term $\mathcal{L}_{(1)}$ also becomes ambiguous since one now has freedom in choosing the integration contour. On the other hand, presumably one could dimensionally reduce the codimension m theory to codimension one with $(m-1)$ of the galileons becoming very massive. In that case one would have to recover all the terms $\mathcal{L}_{(1)}$ to $\mathcal{L}_{(5)}$: analogues of all these terms should therefore exist. However, such a dimensional reduction would obviously have to break the $SO(m)$ symmetry in picking out a single large dimension and so it is perhaps not too surprising that when working within this restriction it is not clear how $\mathcal{L}_{(1)}$, $\mathcal{L}_{(3)}$ and $\mathcal{L}_{(5)}$ could appear. The general Lagrangian respecting $SO(m)$ symmetry is, up to this order, the sum with arbitrary coefficients of $\mathcal{L}_{(2)}$ and $\mathcal{L}_{(4)}$ which are worked out in detail for the codimension greater than one case below.

4.3.1 Volume Term $\mathcal{L}_{(2)}$

The volume term is

$$\mathcal{L}_{(2)} = \sqrt{-\tilde{g}} \quad (4.3.6)$$

Now if $\hat{g}_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}$ then

$$\sqrt{-\hat{g}} = \sqrt{-\bar{g}} \left(1 + \frac{1}{2} \bar{g}^{\mu\nu} \delta g_{\mu\nu} - \frac{1}{8} (\bar{g}^{\alpha\gamma} \bar{g}^{\beta\delta} + \bar{g}^{\alpha\delta} \bar{g}^{\beta\gamma} - \bar{g}^{\alpha\beta} \bar{g}^{\gamma\delta}) \delta g_{\alpha\beta} \delta g_{\gamma\delta} + \mathcal{O}(\delta g^3) \right) \quad (4.3.7)$$

so using (4.2.21) and (F.0.13) we can expand $\mathcal{L}_{(2)}$ in terms of the π 's as

$$\begin{aligned} \mathcal{L}_{(2)} &= \Omega^4 \sqrt{-\hat{g}} \\ &= \sqrt{-\bar{g}} \left(1 + 4\pi + 6\pi^2 + \frac{1}{2} \boldsymbol{\pi} \cdot \boldsymbol{\pi}^\mu - 2\kappa \boldsymbol{\pi} \cdot \boldsymbol{\pi} + 4\pi^3 + \pi \boldsymbol{\pi} \cdot \boldsymbol{\pi}^\mu - \frac{20}{3} \kappa \boldsymbol{\pi} \cdot \boldsymbol{\pi} \right. \\ &\quad + \pi^4 + \frac{1}{2} \pi^2 \boldsymbol{\pi} \cdot \boldsymbol{\pi}^\mu - \frac{1}{4} \boldsymbol{\pi} \cdot \boldsymbol{\pi}^\nu \boldsymbol{\pi}^\mu \cdot \boldsymbol{\pi}^\nu + \frac{1}{8} (\boldsymbol{\pi} \cdot \boldsymbol{\pi}^\mu)^2 - 8\kappa \pi^2 \boldsymbol{\pi} \cdot \boldsymbol{\pi} + \\ &\quad \left. \frac{1}{6} \kappa \boldsymbol{\pi} \cdot \boldsymbol{\pi} \boldsymbol{\pi} \cdot \boldsymbol{\pi}^\mu - \frac{2}{3} \kappa \boldsymbol{\pi} \cdot \boldsymbol{\pi} \boldsymbol{\pi} \cdot \boldsymbol{\pi}^\mu + \frac{5}{3} \kappa^2 (\boldsymbol{\pi} \cdot \boldsymbol{\pi})^2 + \mathcal{O}(\pi^5) \right) \quad (4.3.8) \end{aligned}$$

4.3.2 Einstein-Hilbert Term $\mathcal{L}_{(4)}$

The Einstein-Hilbert term is

$$\mathcal{L}_{(4)} = \sqrt{-\tilde{g}} R(\tilde{g}_{\mu\nu}) \quad (4.3.9)$$

The Ricci scalar for a conformally transformed metric is

$$R(\Omega^2 \hat{g}_{\mu\nu}) = \Omega^{-2} R(\hat{g}_{\mu\nu}) - 6\Omega^{-3} \hat{\nabla}^2 \Omega \quad (4.3.10)$$

in four dimensions, where $\hat{\nabla}$ is the derivative covariant with respect to $\hat{g}_{\mu\nu}$.

For small perturbations $\delta g_{\mu\nu}$ about a background metric $\bar{g}_{\mu\nu}$, dropping total derivatives and up to second order in the perturbation,

$$\sqrt{-\hat{g}} R(\hat{g}) = \sqrt{\bar{g}} \left(R(\bar{g}) - G(\bar{g})^{\mu\nu} \delta g_{\mu\nu} - \frac{1}{2} \delta g_{\mu\nu} \Delta_L^{(\mu\nu)(\rho\sigma)} \delta g_{\rho\sigma} + \bar{g}^{\mu\nu} \mathfrak{F}_\mu \mathfrak{F}_\nu \right) \quad (4.3.11)$$

where $G(\bar{g})_{\mu\nu} = R(\bar{g})_{\mu\nu} - \frac{1}{2} \bar{g}_{\mu\nu} R(\bar{g})$ is the Einstein tensor and \mathfrak{F} is given by

$$\mathfrak{F}_\mu = \bar{g}^{\rho\sigma} \left(\nabla_\sigma \delta g_{\mu\rho} - \frac{1}{2} \nabla_\mu \delta g_{\rho\sigma} \right) \quad (4.3.12)$$

and the Lichnerowicz operator Δ_L is

$$\Delta_L^{(\mu\nu)(\rho\sigma)} = -g^{(\mu\nu)(\rho\sigma)} \nabla^2 + g^{(\mu\nu)(\tau\nu)} (R(\bar{g})_{\tau v}^\rho{}^\sigma + G(\bar{g})_{\tau}^\rho \delta_v^\sigma) \quad (4.3.13)$$

with the derivatives being covariant with respect to $\bar{g}_{\mu\nu}$ and defining $g^{(\mu\nu)(\rho\sigma)} = \frac{1}{2} (\bar{g}^{\mu\rho} \bar{g}^{\nu\sigma} + \bar{g}^{\mu\sigma} \bar{g}^{\nu\rho} - \bar{g}^{\mu\nu} \bar{g}^{\rho\sigma})$.

As Σ_0 is maximally symmetric we have that $R(\bar{g})_{\mu\nu\rho\sigma} = \bar{\kappa}(\bar{g}_{\mu\rho}\bar{g}_{\nu\sigma} - \bar{g}_{\mu\sigma}\bar{g}_{\nu\rho})$. (Using the relations (4.2.8) and (4.2.10) it is easy to see that $\bar{\kappa}$ is related to κ and c_I by $\bar{\kappa} = \kappa + c_I c^I$.) This implies that also $G(\bar{g})_{\mu\nu} = -3\bar{\kappa}\bar{g}_{\mu\nu}$ and $R(\bar{g}) = 12\bar{\kappa}$. These relations along with (4.2.21) allow us to expand the terms in (4.3.11), (given in equations (F.0.16-F.0.22)) and find

$$\begin{aligned} \mathcal{L}_{(4)} = \sqrt{-\bar{g}} & \left(12\bar{\kappa}\Omega^2 + 3\bar{\kappa}\boldsymbol{\pi}_\mu \cdot \boldsymbol{\pi}^\mu + 6\pi_\mu\pi^\mu - 12\kappa\pi_\mu\boldsymbol{\pi} \cdot \boldsymbol{\pi}^\mu \right. \\ & + \kappa\bar{\kappa}(\boldsymbol{\pi} \cdot \boldsymbol{\pi}_\mu\boldsymbol{\pi} \cdot \boldsymbol{\pi}^\mu - \boldsymbol{\pi} \cdot \boldsymbol{\pi}\boldsymbol{\pi}_\mu \cdot \boldsymbol{\pi}^\mu) + 12\kappa^2\boldsymbol{\pi} \cdot \boldsymbol{\pi}_\mu\boldsymbol{\pi} \cdot \boldsymbol{\pi}^\mu \\ & + 4\bar{\kappa}\pi_\mu \cdot \pi_\nu\pi^\mu \cdot \pi^\nu - \bar{\kappa}(\pi_\mu \cdot \pi^\mu)^2 - 6\pi_\mu\pi_\nu\pi^\mu \cdot \pi^\nu \\ & + 3\pi_\mu\pi^\mu\pi_\nu \cdot \pi^\nu - \pi_\mu \cdot \pi_{\nu\rho}\pi^\mu \cdot \pi^{\nu\rho} + \pi_\mu \cdot \pi_\nu{}^\nu\pi^\mu\pi_\rho{}^\rho \\ & \left. - \pi_\mu \cdot \pi_{\nu\rho}\pi^\nu \cdot \pi^{\mu\rho} + \pi^\mu \cdot \pi_{\mu\rho}\pi^\nu \cdot \pi_\nu{}^\rho + \mathcal{O}(\pi^5) \right) \end{aligned} \quad (4.3.14)$$

4.3.3 Tadpole Term $\mathcal{L}_{(1)}$

Unbroken $SO(m)$ in the transverse directions restricts the candidates for the tadpole term to just the volume of the surface, S , given by the locus of the geodesics connecting Σ_0 and Σ , whose co-ordinates in the transverse dimensions are $x^I = s\pi^I$, $s \in [0, 1]$. Clearly we can obtain the induced metric on the slice at each value of s just by scaling (4.3.8) by a factor of s for each π while the component of the metric along the s direction is given by $g_{ss}ds^2 = g_{IJ}dx^I dx^J = ds^2\boldsymbol{\pi} \cdot \boldsymbol{\pi}$. So the volume element on S is $\sqrt{\tilde{g}(s) \cdot g_{ss}} = \sqrt{\boldsymbol{\pi} \cdot \boldsymbol{\pi}}\sqrt{\tilde{g}(s)}$ and

$$\begin{aligned} \mathcal{L}_{(1)} &= \int_0^1 ds \sqrt{-\tilde{g}(s) \cdot g_{ss}} \\ &= \sqrt{\bar{g}}\sqrt{\boldsymbol{\pi} \cdot \boldsymbol{\pi}} \left(1 + 2\pi + 2\pi^2 + \frac{1}{6}\boldsymbol{\pi}_\mu \cdot \boldsymbol{\pi}^\mu - \frac{2}{3}\kappa\boldsymbol{\pi} \cdot \boldsymbol{\pi} \right. \\ & \quad \left. + \pi^3 + \frac{1}{4}\pi\boldsymbol{\pi}_\mu \cdot \boldsymbol{\pi}^\mu - \frac{5}{3}\kappa\pi\boldsymbol{\pi} \cdot \boldsymbol{\pi} \right) + \mathcal{O}(\pi^5) \end{aligned} \quad (4.3.15)$$

It seems likely that this is not in fact the correct prescription for the tadpole term, which is probably simply absent in the $SO(m)$ symmetric case, as seems to be suggested by the field theory [55]. If (4.3.15) were to be the correct form, the interpretation of $\sqrt{\boldsymbol{\pi} \cdot \boldsymbol{\pi}}$ would present a puzzle, or perhaps suggest new possibilities for the field theory galileons.

If instead the $SO(m)$ symmetry were broken so that a constant vector field, V^I , were available somehow, one could integrate along it from ∞ to the brane and have something like $\mathcal{L}_{(1)} = \sqrt{\bar{g}}V_I\pi^I(1 + \dots)$ which would be rather more like the field theory tadpole term $\alpha_I\pi^I$.

4.4 Conclusions

The results presented in this chapter, (4.3.8) and (4.3.14), are only slightly more general than those which have been given before. We obtained these results by working perturbatively from the outset so that the geometrical quantities on the brane could be constructed straightforwardly as Taylor expansions. This

contrasts to the approach in [56] which was rather closer to the method used in the codimension one case. There, with the assumption that both brane and bulk are Minkowski, the induced curvature of the brane was calculated directly from the induced metric and its inverse. In this simpler case the only thing without an exact expression (and therefore requiring a perturbative expansion) is the inverse induced metric of the brane, which makes the direct approach feasible. However as we can see in equations (4.2.16-4.2.19) the metric has a trivial exact expression only for this very restricted case with no bulk curvature and no extrinsic curvature (i.e. $\kappa = 0$ and $c_I = 0$) which is lost even for the rather modest relaxation which we allow here.

We hope that techniques such as those we have introduced here may prove to be of further use in constructing models in which the $SO(m)$ symmetry of the bulk is broken, which might thereby describe all of the models allowed in the field theory approach.

Chapter 5

Conclusions

In this thesis we have considered several topics concerning different aspects of braneworld models, in which the structure at boundaries of gravitational theories plays a crucial role.

We have constructed a five dimensional reduction of Ian Moss's improved version of Heterotic M-Theory [28–31], whose action, boundary conditions and supersymmetry transformations are summarized in Appendix E. Since this formulation of the eleven dimensional theory is consistent to all orders in the gravitational coupling, it imposes fewer restrictions on model building than the original version due to Hořava and Witten [7, 8]. This five dimensional reduction provides a starting point for the development of the phenomenology of the improved theory. Most of the previous phenomenological work concerning the bosonic sector of the theory, which called only for terms of first order in $\kappa^{2/3}$, carries over unchanged to the improved theory. However, it is now possible to consider the effects of higher order terms consistently and so this reduction opens up a new avenue of research in the phenomenology of Heterotic M-Theory. This may include qualitatively new features of the improved theory, such as the appearance of the gaugino condensate in the boundary condition of the gravitino discovered by Ahmed and Moss [36, 37] which produces a positive Casimir energy.

We have examined singularity formation in brane collisions. The formation of a singularity when two scalar field domain walls collide had been observed before [48] but it had been assumed that the singularity was a black brane. We observe that in fact the asymptotic structure is not consistent with that of a black brane, and further investigation through numerical simulations showed that the trapping horizon formed was a future-inner horizon, not the future-outer horizon of a black brane. This horizon structure suggests rather a big crunch singularity; tentative confirmation of this conclusion comes from the level sets of the Ricci scalar which, up to the limits imposed by the finite size of our simulations, seem to be consistent with the hypothesis that they cross all the outgoing null rays so that the spacetime is indeed cut off by the singularity (which would be the level set $R = \infty$). The consequence of this is that the universes on the branes must also end in a finite time; they cannot persist outside the horizon of the singularity forever, as they could perhaps if it were a black brane. To further examine the implications of this big crunch for cosmological models it would be useful to continue the simulations to map out the parameter space (in the approach velocity of the domain walls) where the singularity forms. Having established the nature

of the singularity one could also pursue further the investigation of topics such as particle exchange and reheating in gravitating brane collisions, presumably while taking care in this case to avoid singularity formation.

We have developed a formalism for the calculation of the geometrical quantities associated with a probe brane of arbitrary codimension. This allows us to recover previous results for the action of multi-galileon theories which arise as the perturbations of a flat probe brane in a flat bulk but more importantly it provides a tool-kit for the construction of more general multi-galileon theories. However, another ingredient is also needed: that is, how to break the symmetry of the bulk spacetime so that we can have preferred directions and thereby break the $SO(m)$ symmetry between the several galileon fields, which is a necessary consequence of having a maximally symmetric bulk and brane. The specification of a breaking which would allow the recovery of the most general galileon theories, as found in the four dimensional effective field theory approach, remains to be discovered.

The project to find extra dimensional models relevant to the description of the real world is an ongoing one which still presents many challenges. We have described here some results in such models which move them forward a step or two towards the ultimate goal of representing the universe in which we live and which, as discussed above, also suggest directions for further inquiry.

Appendix A

Heterotic M-Theory Conventions

Eleven Dimensions

- \mathcal{M}_{11} denotes the bulk spacetime with boundaries $\partial\mathcal{M}_{10,1}$ (with inward-pointing normal) and $\partial\mathcal{M}_{10,2}$ (with outward pointing normal).
- Indices:
 - $I, J, K, \dots = 0, \dots, 10$ are the bulk spacetime indices.
 - $A, B, C, \dots = 0, \dots, 9$ are the boundary spacetime indices.
 - $\hat{I}, \hat{J}, \hat{K}, \dots$ and $\hat{A}, \hat{B}, \hat{C}, \dots$ are the corresponding tangent space indices.
 - N is the direction normal to the boundaries (i.e. $V^N = g_{IJ}^{(11)} V^I n_{(11)}^J$ where $n_{(11)}^J$ is the unit normal vector).
- The metric is mostly plus.
- The volume element is $dv = \sqrt{-g^{(11)}} d^{11}x$.
- The Riemann tensor is $R(\Gamma)^I_{JKL} = \partial_K \Gamma^I_{LJ} - \partial_L \Gamma^I_{KJ} + \Gamma^I_{KM} \Gamma^M_{LJ} - \Gamma^I_{LM} \Gamma^M_{KJ}$ or $R(\omega)_{\hat{I}\hat{J}\hat{K}\hat{L}} = \partial_{\hat{K}} \omega_{\hat{L}\hat{I}\hat{J}} - \partial_{\hat{L}} \omega_{\hat{K}\hat{I}\hat{J}} + \omega_{\hat{K}\hat{I}}^{\hat{M}} \omega_{\hat{L}\hat{M}\hat{J}} - \omega_{\hat{L}\hat{I}}^{\hat{M}} \omega_{\hat{K}\hat{M}\hat{J}}$, the Ricci tensor is $R_{IJ} = R^K_{IKJ}$ and the Ricci scalar is $g^{(11)IJ} R_{IJ}$.
- The trace over $SO(1,9)$ indices of the curvature two-form is given by a standard matrix trace so for example $\text{tr}(R_{AB}R_{CD}) = R_{ABE}{}^F R_{CDF}{}^E$.
- The Γ_A are imaginary, $\Gamma_A^* = -\Gamma_A$, while Γ_N is real, $\Gamma_N^* = \Gamma_N$, and satisfy $\{\Gamma_I, \Gamma_J\} = 2g_{IJ}^{(11)}$ and $\Gamma_{IJ} = \Gamma_{[I}\Gamma_{J]}$, etc.
- The Dirac conjugate is $\bar{\Psi} = -i\Psi^\dagger\Gamma_0$.
- Fermionic fields swap places under complex conjugation, i.e. $(\psi\chi)^* = \chi^*\psi^*$.
- \pm and \mp refer to the signs of boundary terms, with the upper sign referring to $\partial\mathcal{M}_{10,1}$ and the lower to $\partial\mathcal{M}_{10,2}$.

Five Dimensions

- \mathcal{M} denotes the bulk spacetime with boundaries $\partial\mathcal{M}_1$ (with inward-pointing normal) and $\partial\mathcal{M}_2$ (with outward pointing normal).
- Indices:
 - $\alpha, \beta, \gamma, \dots = 0, \dots, 4$ are the bulk spacetime indices.
 - $\mu, \nu, \rho, \dots = 0, \dots, 3$ are the boundary spacetime indices.
 - $\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \dots$ and $\hat{\mu}, \hat{\nu}, \hat{\rho}, \dots$ are the corresponding tangent space indices.
 - z is the direction normal to the boundaries (i.e. $V^z = g_{\alpha\beta} V^\alpha n^\beta$).
 - I, J, K, \dots are E_6 gauge indices on $\partial\mathcal{M}_1$ and E_8 gauge indices on $\partial\mathcal{M}_2$.
 - $A, B, C, \dots = 1, 2$ are $SU(2)$ indices.
- The metric is mostly plus.
- The volume element is $dv = \sqrt{-g} d^5x$.
- The Riemann tensor is $R(\Gamma)^\alpha{}_{\beta\gamma\delta} = \partial_\gamma \Gamma^\alpha{}_{\delta\beta} - \partial_\delta \Gamma^\alpha{}_{\gamma\beta} + \Gamma^\alpha{}_{\gamma\epsilon} \Gamma^\epsilon{}_{\delta\beta} - \Gamma^\alpha{}_{\delta\epsilon} \Gamma^\epsilon{}_{\gamma\beta}$ or $R(\omega)_{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}} = \partial_{\hat{\gamma}} \omega_{\delta\hat{\alpha}\hat{\beta}} - \partial_{\hat{\delta}} \omega_{\gamma\hat{\alpha}\hat{\beta}} + \omega_{\gamma\hat{\alpha}}{}^{\hat{\epsilon}} \omega_{\delta\hat{\epsilon}\hat{\beta}} - \omega_{\delta\hat{\alpha}}{}^{\hat{\epsilon}} \omega_{\gamma\hat{\epsilon}\hat{\beta}}$, the Ricci tensor is $R_{\alpha\beta} = R^\gamma{}_{\alpha\gamma\beta}$ and the Ricci scalar is $g^{\alpha\beta} R_{\alpha\beta}$.
- The alternating tensor $\epsilon_{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}}$ is normalized so that $\epsilon_{0\dots 4} = 1, \epsilon^{0\dots 4} = -1$ and $\epsilon_{\mu\nu\rho\sigma} = \epsilon_{\mu\nu\rho\sigma 5}$.
- The gamma matrices satisfy $\{\gamma_\alpha, \gamma_\beta\} = 2g_{\alpha\beta}$ and $\gamma_{\alpha\beta} = \gamma_{[\alpha}\gamma_{\beta]}$, etc. The γ_μ are real, $\gamma_\mu^* = \gamma_\mu$ and $\gamma_5 \equiv \gamma_z$ is imaginary $\gamma_5^* = -\gamma_5$.
- The Dirac conjugate is $\bar{\psi} = -i\psi^\dagger \gamma_0$.
- Fermionic fields swap places under complex conjugation, i.e. $(\psi\chi)^* = \chi^* \psi^*$.
- \pm and \mp refer to the signs of boundary terms, with the upper sign referring to $\partial\mathcal{M}_1$ and the lower to $\partial\mathcal{M}_2$.

Calabi-Yau Manifold

- X denotes the Calabi-Yau space over which the reduction is performed, so $\mathcal{M}_{11} = \mathcal{M}_5 \times X$, $\partial\mathcal{M}_{10,1} = \partial\mathcal{M}_1 \times X$ and $\partial\mathcal{M}_{10,2} = \partial\mathcal{M}_2 \times X$.
- Indices:
 - $\underline{a}, \underline{b}, \underline{c}, \dots = \underline{1}, \dots, \underline{6}$ are real indices.
 - $a, b, c, \dots = 1, 2, 3$ are holomorphic indices, related to the real indices by $V^1 = V_{\underline{1}} + iV_{\underline{2}}, V_{\underline{1}} = \frac{1}{2}(V_1 - iV_2)$, etc.
 - $\bar{a}, \bar{b}, \bar{c}, \dots = \bar{1}, \bar{2}, \bar{3}$ are anti-holomorphic indices on the Calabi-Yau space, related to the real indices by $V^{\bar{1}} = V_{\underline{1}} - iV_{\underline{2}}, V_{\bar{1}} = \frac{1}{2}(V_{\underline{1}} + iV_{\underline{2}})$, etc.

- The metric is Euclidean.
- The volume element is $dv = \sqrt{g^{CY}} d^6x$.
- The reference volume of X after scaling out the geometric moduli is $v = \int_X dv$.
- The alternating tensor ϵ_{abc} is normalized so that $\epsilon_{123} = 1, \epsilon_{abc}\epsilon^{abc} = 48$.
- The gamma matrices satisfy $\{\gamma_a, \gamma_{\bar{b}}\} = 2g_{a\bar{b}}$ and $\gamma_{a\bar{b}} = \gamma_{[a}\gamma_{\bar{b}]}$, etc. The γ_a are imaginary $(\gamma_a)^* = -\gamma_{\bar{a}}$ and $\gamma_7^* = -\gamma_7$.
- The Hermitian conjugate of the covariantly constant spinor u^A is defined to be $(u^A)^\dagger = \bar{u}_A$ and then the Dirac conjugate is given by $\overline{(u^A)} = \bar{u}_A \gamma_7$

Appendix B

Calabi-Yau Geometry

Useful identities for spinors and gamma matrices on the Calabi-Yau follow from the fact that there is a unique constant spinor, u_A ; there is no constant vector, so $\bar{u}^A \gamma_a u_A = 0$; there is a unique holomorphic three-form, so $\bar{u}^A \gamma_{abc} u_A \propto \Omega_{abc} = i\epsilon_{abc}$; and the Dirac algebra $\{\gamma_a, \gamma_{\bar{b}}\} = 2g_{a\bar{b}}$. We choose the spinor basis so that $\gamma^a u_1 = 0$, and as $u_1^* = u_2$ it follows that

$$\gamma^a u_1 = 0 \tag{B.0.1}$$

$$\gamma^{\bar{a}} u_2 = 0 \tag{B.0.2}$$

$$\gamma^{a\bar{b}} u_A = g^{a\bar{b}} \tau^B{}_{A} u_B \tag{B.0.3}$$

$$\gamma_{abc} u_1 = i\epsilon_{abc} u_2 \tag{B.0.4}$$

$$\gamma_{\bar{a}\bar{b}\bar{c}} u_2 = i\epsilon_{\bar{a}\bar{b}\bar{c}} u_1 \tag{B.0.5}$$

where we have the matrix $\tau^A{}_B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^A{}_B$. Tensors formed from arbitrary products of gamma matrices using the constant spinor, such as those that appear in all the two fermion terms, are given by some combination of $g_{a\bar{b}}$, ϵ_{abc} and $\epsilon_{\bar{a}\bar{b}\bar{c}}$ by using these identities and the Dirac algebra.

The following results concerning the geometry of the (1,1) moduli of the Calabi-Yau space are based on [84].

The components of the complex structure ω and the metric g are related by

$$\omega_{a\bar{b}} = i g_{a\bar{b}} \tag{B.0.6}$$

and the reference volume of the Calabi-Yau (the unit in which V is measured) is

$$v = \frac{1}{6} \int \omega \wedge \omega \wedge \omega \tag{B.0.7}$$

Given a basis $\omega_{i a\bar{b}}$ of the cohomology group $H^{1,1}$ the complex structure is given by a set of real moduli fields b^i

$$\omega_{a\bar{b}} = b^i \omega_{i a\bar{b}} \tag{B.0.8}$$

The metric on the moduli space is

$$G_{ij} = \frac{1}{2v} \int \omega_i \wedge * \omega_j \tag{B.0.9}$$

The dual basis of $H^{2,2}$, ν^i , is defined by

$$\int \omega_i \wedge \nu^j = v \delta_i^j \quad (\text{B.0.10})$$

and we can see by comparing this to (B.0.9) that $\nu_{\bar{a}\bar{b}\bar{c}\bar{d}}^i = \frac{1}{2} G^{ij} (*\omega)_{\bar{a}\bar{b}\bar{c}\bar{d}}$

The intersection numbers are

$$\mathcal{K}_{ijk} = \int \omega_i \wedge \omega_j \wedge \omega_k \quad (\text{B.0.11})$$

and it is useful to define

$$\mathcal{K} = \mathcal{K}_{ijk} b^i b^j b^k = 6v \quad (\text{B.0.12})$$

From (B.0.7) and (B.0.9) we may find that

$$*\omega_i = b_i \omega \wedge \omega - \omega \wedge \omega_i \quad (\text{B.0.13})$$

$$G_{ij} = \frac{1}{2} \omega_{i\bar{a}\bar{b}} \omega_j^{a\bar{b}} \quad (\text{B.0.14})$$

and using (B.0.9-B.0.13)

$$G_{ij} = -\frac{1}{2} \frac{\partial \ln \mathcal{K}}{\partial b^i \partial b^j} \quad (\text{B.0.15})$$

The contractions of \mathcal{K}_{ijk} are

$$\mathcal{K}^{-1} \mathcal{K}_{ijk} b^k = \frac{2}{3} b_i b_j - \frac{1}{3} G_{ij} \quad (\text{B.0.16})$$

$$\mathcal{K}^{-1} \mathcal{K}_{ijk} b^j b^k = \frac{2}{3} b_i \quad (\text{B.0.17})$$

and

$$b_i b^i = \frac{3}{2} \quad (\text{B.0.18})$$

From (B.0.8) and (B.0.14) we may see that the connection coefficients

$$\Gamma_{ijk} = -\frac{i}{2} \omega_{ia}{}^b \omega_{jb}{}^c \omega_{kc}{}^a \quad (\text{B.0.19})$$

can be used to define the covariant derivative on fields with i, j, k, \dots indices

$$\mathcal{D}_\alpha C^i = \nabla_\alpha C^i + \partial_\alpha b^k \Gamma_{jk}^i C^j \quad (\text{B.0.20})$$

which is compatible with the metric G_{ij} , $\mathcal{D}_\alpha G_{ij} = 0$.

The Levi-Civita components follow from (B.0.15)

$$\Gamma_{i(jk)} = -\frac{3}{2} \mathcal{K}^{-1} \mathcal{K}_{ijk} - 3b_{(i} G_{jk)} + 2b_i b_j b_k \quad (\text{B.0.21})$$

with a corresponding curvature tensor

$$R_{ijkl} = \frac{9}{4} \mathcal{K}^{-2} \mathcal{K}_{il}{}^m \mathcal{K}_{kjm} - \frac{9}{4} \mathcal{K}^{-2} \mathcal{K}_{ik}{}^m \mathcal{K}_{ljm} + \frac{1}{2} G_{il} G_{kj} - \frac{1}{2} G_{ik} G_{lj} \quad (\text{B.0.22})$$

It is useful to decompose the tangent space to the moduli into a direction parallel to b^i and the perpendicular directions, which we may do using the projection tensor

$$\delta^{\perp i}_j = \delta^i_j - \frac{2}{3}b^i b_j \quad (\text{B.0.23})$$

and, extending the notation to G and \mathcal{K} ,

$$G^{\perp}_{ij} = G_{ij} - \frac{2}{3}b_i b_j \quad (\text{B.0.24})$$

$$G^{\perp ij} = G^{ij} - \frac{2}{3}b^i b^j \quad (\text{B.0.25})$$

$$\mathcal{K}^{-1}\mathcal{K}^{\perp}_{ijk} = \mathcal{K}^{-1}\mathcal{K}_{ijk} + \frac{2}{3}b_{(i}G_{jk)} - \frac{20}{27}b_i b_j b_k \quad (\text{B.0.26})$$

Cyclic contractions of the cohomology generators appear frequently in the calculations of the reduction. A collection of useful identities for these are

$$\omega_{ia}{}^a = 2ib_i \quad (\text{B.0.27})$$

$$\omega_{ia}{}^b \omega_{jb}{}^a = -2G_{ij} \quad (\text{B.0.28})$$

$$\omega_{ia}{}^b \omega_{jb}{}^c \omega_{kc}{}^a = 2i\Gamma_{ijk} \quad (\text{B.0.29})$$

$$\omega_{ia}{}^{[d} \omega_{jb}{}^e \omega_{kc}{}^f] = -\frac{i}{48}\epsilon_{abc}\epsilon^{def}\mathcal{K}^{-1}\mathcal{K}_{ijk} \quad (\text{B.0.30})$$

$$\frac{1}{2}(\omega_{ia}{}^b \omega_{jb}{}^c \omega_{kc}{}^a + \omega_{ja}{}^b \omega_{ib}{}^c \omega_{kc}{}^a) = -3i\mathcal{K}^{-1}\mathcal{K}_{ijk} - 6ib_{(i}G_{jk)} + 4ib_i b_j b_k \quad (\text{B.0.31})$$

$$\frac{1}{2}(\omega_{ia}{}^b \omega_{jb}{}^c \omega_{kc}{}^d \omega_{ld}{}^a + \omega_{ka}{}^b \omega_{jb}{}^c \omega_{ic}{}^d \omega_{ld}{}^a) = -\frac{9}{2}\mathcal{K}^{-2}\mathcal{K}_{ikm}\mathcal{K}_{jl}{}^m + G_{il}G_{jk} + G_{ij}G_{kl} \quad (\text{B.0.32})$$

Appendix C

E_8 Gauge Theory

Our reference for the gauge theory was [85]. We include here the particulars needed above for ease of reference and to make clear the normalizations used.

We need to split the (imaginary) generators of E_8 , Y^A , into: S^i which generate $SU(3)$; X^I which generate E_6 ; $T_{\hat{a}p}$ and $T^{\hat{a}p} = (T_{\hat{a}p})^\dagger$ which generate the cosets. This corresponds to $\mathbf{248} = (\mathbf{8}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{78}) \oplus (\mathbf{3}, \mathbf{27}) \oplus (\bar{\mathbf{3}}, \bar{\mathbf{27}})$. So the S^i and X^I form adjoint representations of $SU(3)$ and E_6 respectively, i.e.

$$[S^i, S^j] = if^{ij}_k S^k; (S^i)^j_k = -if^{ij}_k \quad (\text{C.0.1})$$

$$[X^I, X^J] = iF^{IJ}_K X^K; (X^I)^J_K = -iF^{IJ}_K \quad (\text{C.0.2})$$

where f^{ij}_k and F^{IJ}_K are the (real) structure constants of $SU(3)$ and E_6 . The $T_{\hat{a}p}$ transform as the fundamental both of $SU(3)$ and of E_6

$$[S^i, T_{\hat{a}p}] = -\lambda^{i\hat{a}}_{\hat{b}} T_{\hat{b}p}; [S^i, T^{\hat{a}p}] = \lambda^{i\hat{a}}_{\hat{b}} T^{\hat{b}p} \quad (\text{C.0.3})$$

$$[X^I, T_{\hat{a}p}] = -\Lambda^I_{p\hat{q}} T_{\hat{a}q}; [X^I, T^{\hat{a}p}] = \Lambda^I_{p\hat{q}} T^{\hat{a}q} \quad (\text{C.0.4})$$

where $\lambda^{i\hat{a}}_{\hat{b}}$ and $\Lambda^I_{p\hat{q}}$ are the fundamental generators of $SU(3)$ and E_6 respectively, and of course the $SU(3)$ and E_6 subgroups commute

$$[S^i, X^I] = 0 \quad (\text{C.0.5})$$

We normalize the antisymmetric invariant tensor of $SU(3)$ by

$$\hat{\epsilon}_{123} = 1, \text{ or } \hat{\epsilon}_{\hat{a}\hat{b}\hat{c}} \hat{\epsilon}^{\hat{a}\hat{b}\hat{c}} = 6 \quad (\text{C.0.6})$$

the symmetric invariant tensor of E_6 by

$$d_{prs} d^{qrs} = \delta_p^q \quad (\text{C.0.7})$$

the generators by

$$\text{Tr}(S^i S^j) = 30\delta^{ij} \quad (\text{C.0.8})$$

$$\text{Tr}(X^I X^J) = 30\delta^{IJ} \quad (\text{C.0.9})$$

$$\text{Tr}(T^{\hat{a}p} T_{\hat{b}q}) = 30\delta_{\hat{b}}^{\hat{a}} \delta_q^p \quad (\text{C.0.10})$$

and the fundamental generators by

$$(\lambda^i)_{\hat{b}}^{\hat{a}} (\lambda^j)^{\hat{b}}_{\hat{a}} = \frac{1}{2} \delta^{ij} \quad (\text{C.0.11})$$

$$(\Lambda^I)^p_q (\Lambda^J)^q_p = 3\delta^{IJ} \quad (\text{C.0.12})$$

where Tr is the trace in the adjoint of E_8 . Then by considering the Jacobi identities the algebra of the T^{ap} and T_{ap} is fixed to be

$$[T_{\hat{a}p}, T_{\hat{b}q}] = \sqrt{5} \hat{\epsilon}_{\hat{a}\hat{b}\hat{c}} d_{pqr} T^{\hat{c}r} \quad (\text{C.0.13})$$

$$[T^{\hat{a}p}, T^{\hat{b}q}] = -\sqrt{5} \hat{\epsilon}^{\hat{a}\hat{b}\hat{c}} d^{pqr} T_{\hat{c}r} \quad (\text{C.0.14})$$

$$[T_{\hat{a}p}, T^{\hat{b}q}] = -\delta_{\hat{a}}^{\hat{b}} \Lambda^I{}^q{}_p X^I - \delta_p^q \lambda_{\hat{a}}^i{}^{\hat{b}} S^i \quad (\text{C.0.15})$$

Whence we can also find that

$$\text{Tr} \left(T_{\hat{a}p} T_{\hat{b}q} T_{\hat{c}r} \right) = 15\sqrt{5} \hat{\epsilon}_{\hat{a}\hat{b}\hat{c}} d_{pqr} \quad (\text{C.0.16})$$

By considering the singlet and adjoint projectors, we can find the contractions of the fundamental generators,

$$(\lambda^i)_{\hat{c}}^{\hat{a}} (\lambda^j)^{\hat{b}}_{\hat{d}} = \frac{1}{2} \left(\delta_{\hat{d}}^{\hat{a}} \delta_{\hat{c}}^{\hat{b}} - \frac{1}{3} \delta_{\hat{c}}^{\hat{a}} \delta_{\hat{d}}^{\hat{b}} \right) \quad (\text{C.0.17})$$

$$(\Lambda^I)^p_q (\Lambda^J)^r_s = \frac{1}{6} \left(\delta_q^p \delta_s^r + 3\delta_s^p \delta_q^r - 30d_{qst} d^{prt} \right) \quad (\text{C.0.18})$$

In using these results it is important to note that in the reduction we have everywhere tr rather than Tr , which is defined for E_8 (since there is no fundamental with lower dimension than the adjoint for E_8) as $\text{tr} = \frac{1}{30} \text{Tr}$ so

$$\text{tr} (S^i S^j) = \delta^{ij} \quad (\text{C.0.19})$$

$$\text{tr} (X^I X^J) = \delta^{IJ} \quad (\text{C.0.20})$$

$$\text{tr} \left(T^{\hat{a}p} T_{\hat{b}q} \right) = \delta_{\hat{b}}^{\hat{a}} \delta_q^p \quad (\text{C.0.21})$$

$$\text{tr} \left(T_{\hat{a}p} T_{\hat{b}q} T_{\hat{c}r} \right) = \frac{\sqrt{5}}{2} \hat{\epsilon}_{\hat{a}\hat{b}\hat{c}} d_{pqr} \quad (\text{C.0.22})$$

and that the $SU(3)$ indices used here, $\hat{a}, \hat{b}, \hat{c}, \dots$ are raised and lowered by complex conjugation, so they are not the same as the (anti-)holomorphic indices of the Calabi-Yau, a, b, c, \dots and we must use a dreibein to convert between them. This is particularly relevant with respect to the ϵ tensors since $\hat{\epsilon}_{\hat{a}\hat{b}\hat{c}} \hat{\epsilon}^{\hat{a}\hat{b}\hat{c}} = 6$ while $\epsilon_{abc} \epsilon^{abc} = 48$ and so it is necessary to keep track of which is which: we have

$$\hat{\epsilon}_{abc} = \hat{\epsilon}_{\hat{a}\hat{b}\hat{c}} e_a^{\hat{a}} e_b^{\hat{b}} e_c^{\hat{c}} = \frac{1}{2\sqrt{2}} \epsilon_{abc} \quad (\text{C.0.23})$$

Appendix D

Gauging a Quaternionic Isometry

The material in this appendix is not in any way new, but is given for ease of reference and clarity of conventions.

Suppose we have a quaternionic manifold with coordinates q^x . The holonomy group is $SU(2) \otimes SU(2)$ so we have a vierbein which converts vector indices $x, y, z, \dots = 1, \dots, 4$ into pairs of $SU(2)$ indices $A, B, C, \dots = 1, 2$ which we can write as

$$f_x^A{}_B = \begin{pmatrix} \bar{v}_x & u_x \\ -\bar{u}_x & v_x \end{pmatrix}_B^A \quad (\text{D.0.1})$$

$SU(2)$ indices are raised and lowered with ϵ^{AB} and ϵ_{AB} ($X_A = X^B \epsilon_{BA}$, $X^A = \epsilon^{AB} X_B$) and the x, y, z metric is $g_{xy} = \epsilon_{AC} \epsilon^{BD} f_x^A{}_B f_y^C{}_D$.

Metric compatibility $\nabla_x f_y^A{}_B = 0 \Rightarrow df^A{}_B + \tilde{\omega}_{(L)}^A{}_C \wedge f^C{}_B + f^A{}_C \wedge \tilde{\omega}_{(R)}^C{}_B = 0$ tells us that the connections are

$$\tilde{\omega}_{(L)}^A{}_B = \begin{pmatrix} \frac{1}{4}(v - \bar{v}) & -u \\ \bar{u} & -\frac{1}{4}(v - \bar{v}) \end{pmatrix}_B^A \quad (\text{D.0.2})$$

$$\tilde{\omega}_{(R)}^A{}_B = \begin{pmatrix} \frac{3}{4}(v - \bar{v}) & 0 \\ 0 & -\frac{3}{4}(v - \bar{v}) \end{pmatrix}_B^A \quad (\text{D.0.3})$$

Since the manifold is quaternionic it has three complex structures $(J^u)^x{}_y$ (where $u, v, w = 1, 2, 3$) satisfying the quaternion algebra $J^u J^v = -\delta^{uv} + \epsilon^{uvw} J^w$. There are correspondingly three Kähler forms satisfying $(K^u)_x{}^z (K^v)_{zy} = -\delta^{uv} g_{xy} + \epsilon^{uvw} (K^w)_{xy}$. A suitable choice of K^u is given by

$$\begin{aligned} K^1 &= -i(u \wedge \bar{v} - \bar{u} \wedge v) \\ K^2 &= -(u \wedge \bar{v} + \bar{u} \wedge v) \\ K^3 &= -i(u \wedge \bar{u} - v \wedge \bar{v}) \end{aligned} \quad (\text{D.0.4})$$

which we may write as $K_A{}^B = \frac{i}{2} (\sigma^u)_A{}^B K^u$, where $(\sigma^u)_A{}^B$ are the standard Pauli matrices, so that

$$K_A{}^B = \begin{pmatrix} \frac{1}{2}(u \wedge \bar{u} - v \wedge \bar{v}) & -\bar{u} \wedge v \\ u \wedge \bar{v} & -\frac{1}{2}(u \wedge \bar{u} - v \wedge \bar{v}) \end{pmatrix}_A^B \quad (\text{D.0.5})$$

Now if the manifold has some isometries, labelled by indices i, j, k, \dots , generated by Killing vectors k_i^x then we may gauge them by adding a prepotential

term to the connections

$$\omega_{(L)}{}^A{}_B = \tilde{\omega}_{(L)}{}^A{}_B - \frac{1}{2}g\mathcal{A}^i\nabla_x k_i^y f_y^A f_B^x{}^C \quad (\text{D.0.6})$$

$$\omega_{(R)}{}^A{}_B = \tilde{\omega}_{(R)}{}^A{}_B - \frac{1}{2}g\mathcal{A}^i\nabla_x k_i^y f_C^x{}^A f_y^C{}_B \quad (\text{D.0.7})$$

where $-\frac{1}{2}\nabla_x k_i^y f_y^A f_B^x{}^C = \mathcal{P}_i^A{}_B$ is the prepotential defined by $i_{k_i}K^A{}_B = d\mathcal{P}_i^A{}_B + [\omega_{(L)}, \mathcal{P}_i^A{}_B]$ (i_{k_i} stands for the inner product $i_{k_i}K^A{}_B = (k_i)^x (K^A{}_B)_{xy} dq^y$).

Appendix E

Summary of the Reduced Heterotic M-Theory

E.1 Action

The bulk action is

$$\begin{aligned}
S(\mathcal{M}) = \frac{1}{2\kappa_5^2} \int_{\mathcal{M}} dv \left(R - \frac{1}{2} V^{-2} \partial_\alpha V \partial^\alpha V - \frac{1}{2} \mathcal{D}_\alpha \sigma \mathcal{D}^\alpha \sigma - 2V^{-1} \partial_\alpha \bar{\xi} \partial^\alpha \xi \right. \\
- G_{ij}^\perp \partial_\alpha b^i \partial^\alpha b^j - \frac{1}{4} \mathcal{F}_{i\alpha\beta} \mathcal{F}^{i\alpha\beta} - \frac{1}{2} V^{-2} \alpha_i \alpha^i \\
+ \bar{\psi}_{A\alpha} \gamma^{\alpha\beta\gamma} \mathcal{D}_\beta \psi_\gamma^A + \bar{\zeta}_A \gamma^\beta \mathcal{D}_\beta \zeta^A + G_{ij}^\perp \bar{\lambda}_A^i \gamma^\beta \mathcal{D}_\beta \lambda^{jA} \\
- \sqrt{2} i \bar{\zeta}_A \gamma^\alpha \gamma^\beta \psi_\alpha^B f_{\beta B}^A + \frac{i}{2} G_{ij} \bar{\lambda}_A^i \gamma^\alpha \gamma^\beta \psi_\alpha^A \partial_\beta b^j \\
- \frac{\sqrt{2} i}{8} \left(\bar{\psi}_{A\gamma} \gamma^\alpha \gamma^\gamma \gamma^\delta \gamma^\beta \psi_\delta^A - \bar{\zeta}_A \gamma^{\alpha\beta} \zeta^A - \frac{1}{3} G_{jk}^\perp \bar{\lambda}_A^j \gamma^{\alpha\beta} \lambda^{Ak} \right) b_i \mathcal{F}_{\alpha\beta}^i \\
- \frac{\sqrt{2}}{4} G_{ij}^\perp \bar{\lambda}_A^i \gamma^\gamma \gamma^{\alpha\beta} \psi_\gamma^A \mathcal{F}_{\alpha\beta}^j + \frac{3\sqrt{2} i}{8} \mathcal{K}^{-1} \mathcal{K}_{ijk}^\perp \bar{\lambda}_A^i \gamma^{\alpha\beta} \lambda^{Aj} \mathcal{F}_{\alpha\beta}^k \\
- \frac{\sqrt{2}}{4} V^{-1} \alpha \tau^A_B \bar{\psi}_{A\alpha} \gamma^{\alpha\beta} \psi_\beta^B + i V^{-1} \alpha \tau^A_B \bar{\zeta}_A \gamma^\alpha \psi_\alpha^B \\
- \frac{\sqrt{2} i}{2} V^{-1} \alpha_i^\perp \tau^A_B \bar{\lambda}_A^i \gamma^\alpha \psi_\alpha^B - 2V^{-1} \alpha_i^\perp \tau^A_B \bar{\zeta}_A \lambda^{Bi} \\
+ \frac{3\sqrt{2}}{4} V^{-1} \alpha \tau^A_B \bar{\zeta}_A \zeta^B \\
+ \left. \frac{3\sqrt{2}}{4} V^{-1} \left(\mathcal{K}^{-1} \mathcal{K}_{jk}^\perp + \frac{1}{9} b^i G_{jk}^\perp \right) \alpha_i \tau^A_B \bar{\lambda}_A^j \lambda^{kB} \right) \quad (\text{E.1.1})
\end{aligned}$$

The action on $\partial\mathcal{M}_1$ is

$$\begin{aligned}
 S(\partial\mathcal{M}_1) = & \frac{1}{2\kappa_5^2} \int_{\partial\mathcal{M}_1} dv \left(-2K - \sqrt{2}V^{-1}\alpha \right. \\
 & \left. + \frac{1}{2}\tau^A{}_B \bar{\psi}_{A\mu} \gamma^{\mu\nu} \psi_\nu^B + \frac{1}{2}\tau^A{}_B \bar{\zeta}_A \zeta^B + \frac{1}{2}G_{ij}^\perp \tau^A{}_B \bar{\lambda}_A^i \lambda^{Bj} + \mathcal{T} \right) \\
 & + \frac{1}{g^2} \int_{\partial\mathcal{M}_1} dv \left(-\frac{1}{4}V F_{\mu\nu}^I F^{I\mu\nu} - 2G_{ij} \mathcal{D}_\mu C^{ip} \mathcal{D}^\mu \bar{C}^j{}_p \right. \\
 & - \frac{1}{2}V^{-1} G_{ij} \frac{\partial W}{\partial C^{ip}} \frac{\partial \bar{W}}{\partial \bar{C}^j{}_p} - 2V^{-1} \bar{C}^i \Lambda^I C_i \bar{C}^j \Lambda^I C_j \\
 & + \frac{1}{2} \bar{\chi}_A^I \gamma^\mu \mathcal{D}_\mu \chi^{IA} + \frac{1}{2} G_{ij} \text{tr} (\bar{\eta}_A^i \gamma^\mu \mathcal{D}_\mu \eta^{Aj}) \\
 & + \frac{3\sqrt{10}}{2} V^{-\frac{1}{2}} \mathcal{K}^{-1} \mathcal{K}_{ijk} (d_{pqr} C^{ip} \bar{\eta}_R^{jq} \eta_L^{kr} + d^{pqr} \bar{C}^i{}_p \bar{\eta}_L^j{}_q \eta_R^k{}_r) \\
 & + 2G_{ij} V^{-\frac{1}{2}} (\bar{\chi}_L^I \eta_{Rq}^i C^{jp} \Lambda^I{}_p - \bar{\chi}_R^I \eta_{Lq}^i \bar{C}^j{}_p \Lambda^I{}_q) \\
 & \left. + \bar{\psi}_{A\mu} j^{A\mu} + \bar{\zeta}_A j^A + G_{ij}^\perp \bar{\lambda}_A^i j^{Aj} + \Theta^\mu \mathcal{D}_\mu \sigma \right) \quad (\text{E.1.2})
 \end{aligned}$$

Where $\mathcal{T} = -\bar{\psi}_{A\mu} \gamma^\mu (\psi_z^A - \frac{\sqrt{2}i}{3} \gamma_z \zeta^A) + \frac{7\sqrt{2}i}{3} \bar{\zeta}_A \psi_z^A$, the superpotential is

$$W = 3\sqrt{10} \mathcal{K}^{-1} \mathcal{K}_{ijk} d_{pqr} C^{ip} C^{jq} C^{kr} \quad (\text{E.1.3})$$

and the currents which couple to bulk fields are

$$\begin{aligned}
 j^{A\mu} = & -\frac{1}{4} V^{\frac{1}{2}} \gamma^{\nu\rho} \gamma^\mu F_{\nu\rho}^I \chi^{IA} \\
 & - \gamma^\nu \gamma^\mu P_R \eta^A{}_{ip} \mathcal{D}_\nu C^{ip} + \gamma^\nu \gamma^\mu P_L \eta^A{}_{ip} \mathcal{D}_\nu \bar{C}^i{}_p \\
 & + \frac{3\sqrt{10}}{2} V^{-\frac{1}{2}} \gamma^\mu \mathcal{K}^{-1} \mathcal{K}_{ijk} (d_{pqr} P_L \eta^{Aip} C^{jq} C^{kr} + d^{pqr} P_R \eta^{Aip} \bar{C}^j{}_q \bar{C}^k{}_r) \\
 j^A = & -\frac{\sqrt{2}i}{4} V^{\frac{1}{2}} \gamma^{\mu\nu} F_{\mu\nu}^I \chi^{IA} \\
 & + \frac{2\sqrt{2}i}{9} b_i b_j \gamma^\mu (P_R \eta^{Aj}{}_p \mathcal{D}_\mu C^{ip} - P_L \eta^{Aip} \mathcal{D}_\mu \bar{C}^j{}_p) \\
 & - \frac{\sqrt{2}i}{3} V^{-\frac{1}{2}} \left(G_{ij} + \frac{4}{3} b_i b_j \right) \tau^A{}_B \chi^{IB} \bar{C}^j \Lambda^I C^i \\
 & - 3\sqrt{5}i V^{-\frac{1}{2}} \mathcal{K}^{-1} \mathcal{K}_{ijk} (d_{pqr} C^{ip} C^{jq} P_L \eta^{Akr} + d^{pqr} \bar{C}^i{}_p \bar{C}^j{}_q P_R \eta^{Akr}) \\
 j^{Ai} = & i \left(\Gamma^{\perp i}{}_{jk} - \frac{4}{3} \delta^{\perp i}{}_{(j} b_{k)} \right) \gamma^\mu (P_R \eta^{Ak}{}_p \mathcal{D}_\mu C^{jp} - P_L \eta^{Ajp} \mathcal{D}_\mu \bar{C}^k{}_p) \\
 & - 2i V^{-\frac{1}{2}} \left(\Gamma^{\perp i}{}_{jk} - \frac{4}{3} \delta^{\perp i}{}_{(j} b_{k)} \right) \tau^A{}_B \chi^{IB} \bar{C}^k \Lambda^I C^j \\
 \Theta^\mu = & \frac{1}{12} \epsilon^{\mu\nu\rho\sigma} \left(\omega_{\nu\rho\sigma}^Y - \frac{1}{4} V^{-1} \bar{\chi}_A^I \gamma_{\nu\rho\sigma} \chi^{IA} - \frac{1}{4} V^{-1} \text{tr} (\bar{\eta}_{iA} \gamma_{\nu\rho\sigma} \eta^{iA}) \right) \quad (\text{E.1.4})
 \end{aligned}$$

The action on $\partial\mathcal{M}_2$ is

$$\begin{aligned}
 S(\partial\mathcal{M}_2) = & \frac{1}{2\kappa_5^2} \int_{\partial\mathcal{M}_2} dv \left(2K + \sqrt{2}V^{-1}\alpha \right. \\
 & \left. - \frac{1}{2}\tau^A{}_B \bar{\psi}_{A\mu} \gamma^{\mu\nu} \psi_\nu^B - \frac{1}{2}\tau^A{}_B \bar{\zeta}_A \zeta^B - \frac{1}{2}G_{ij}^\perp \tau^A{}_B \bar{\lambda}^i \lambda^{Bj} + \mathcal{T} \right) \\
 & + \frac{1}{g^2} \int_{\partial\mathcal{M}_2} dv \left(-\frac{1}{4}V F_{\mu\nu}^I F^{I\mu\nu} + \frac{1}{2}\bar{\chi}_A^I \gamma^\mu D_\mu \chi^{IA} \right. \\
 & \left. + \bar{\psi}_{A\mu} j^{A\mu} + \bar{\zeta}_A j^A + \Theta^\mu \mathcal{D}_\mu \sigma \right)
 \end{aligned}$$

where $\mathcal{T} = \bar{\psi}_{A\mu} \gamma^\mu \left(\psi_z^A - \frac{\sqrt{2}i}{3} \gamma_z \zeta^A \right) - \frac{7\sqrt{2}i}{3} \bar{\zeta}_A \psi_z^A$ and the currents on $\partial\mathcal{M}_2$ are:

$$\begin{aligned}
 j^{A\mu} &= -\frac{1}{4}V^{\frac{1}{2}} F_{\nu\rho}^I \gamma^{\nu\rho} \gamma^\mu \chi^{IA} \\
 j^A &= -\frac{\sqrt{2}i}{2}V^{\frac{1}{2}} F_{\mu\nu}^I \gamma^{\mu\nu} \chi^{IA} \\
 \Theta^\mu &= \frac{1}{12} \epsilon^{\mu\nu\rho\sigma} \frac{\kappa_5^2}{g^2} \left(\omega_{\nu\rho\sigma}^Y - \frac{1}{4}V^{-1} \bar{\chi}_A^I \gamma_{\nu\rho\sigma} \chi^{IA} \right)
 \end{aligned} \tag{E.1.5}$$

E.2 Boundary Conditions

On $\partial\mathcal{M}_1$:

$$\xi = \frac{\kappa_5^2}{g^2} \left(\sqrt{10}\mathcal{K}^{-1}\mathcal{K}_{ijk}d_{pqr}C^{ip}C^{jq}C^{kr} + \frac{1}{2}V^{\frac{1}{2}}\bar{\chi}_1^I\chi^{I2} \right) \quad (\text{E.2.1})$$

$$\bar{\xi} = \frac{\kappa_5^2}{g^2} \left(\sqrt{10}\mathcal{K}^{-1}\mathcal{K}_{ijk}d^{pqr}\bar{C}^i{}_p\bar{C}^j{}_q\bar{C}^k{}_r - \frac{1}{2}V^{\frac{1}{2}}\bar{\chi}_2^I\chi^{I1} \right) \quad (\text{E.2.2})$$

$$\begin{aligned} \mathcal{A}_\mu^i = & -\frac{\sqrt{2}\kappa_5^2}{g^2} \left(i\Gamma^i{}_{jk} (\bar{C}^k{}_p\mathcal{D}_\mu C^{jp} - C^{jp}\mathcal{D}_\mu\bar{C}^k{}_p) - \frac{i}{4}b^i\tau^A{}_B\bar{\chi}_A^I\gamma_\mu\chi^{IB} \right. \\ & \left. - \frac{i}{4}(\Gamma^i{}_{jk} - b^i G_{jk}) (\bar{\eta}_L{}^{jp}\gamma_\mu\eta_R{}^k{}_p - \bar{\eta}_L{}^k{}_p\gamma_\mu\eta_L{}^{jp}) \right) \quad (\text{E.2.3}) \end{aligned}$$

$$\begin{aligned} \mathcal{D}_5\sigma = & -\frac{1}{12g^2}V^2\partial_\mu \left(\omega_{\mu\nu\rho}^Y - \frac{1}{4}V^{-1}\bar{\chi}_A^I\gamma_{\mu\nu\rho}\chi^{IA} - \frac{1}{4}V^{-1}\text{tr}(\bar{\eta}_{iA}\gamma_{\mu\nu\rho}\eta^{iA}) \right) \\ & - \frac{i}{4}V\tau^A{}_B\bar{\psi}_{A\alpha}\gamma^{\alpha\beta}\gamma_5\psi_\beta^B - \frac{3i}{4}V\tau^A{}_B\bar{\zeta}_A\gamma_5\zeta^B + \frac{i}{4}V\tau^A{}_B G_{ij}^\perp\bar{\lambda}_A^i\gamma_5\lambda^{Bj} \\ & + \frac{\sqrt{2}}{2}V\tau^A{}_B\bar{\zeta}_A\gamma^\alpha\gamma_5\psi_\alpha^B \quad (\text{E.2.4}) \end{aligned}$$

$$\begin{aligned} K^{\mu\nu} - Kg^{\mu\nu} = & -\kappa_5^2 T_{YM}^{\mu\nu} - \sqrt{2}\alpha V^{-1}g^{\mu\nu} + \frac{1}{2}\tau^A{}_B\bar{\psi}_A^{(\mu}\gamma^{\nu)\rho}\psi_\rho^B \\ & + \frac{1}{4}\tau^A{}_B (\bar{\psi}_{A\rho}\gamma^{\rho\sigma}\psi_\sigma^B + \bar{\zeta}_A\zeta^B + G_{ij}^\perp\bar{\lambda}_A^i\lambda^{Bj}) g^{\mu\nu} \quad (\text{E.2.5}) \end{aligned}$$

$$\partial_5 V = -\sqrt{2}\alpha - \frac{\kappa_5^2}{g^2}V^2\frac{\partial\mathcal{L}_{YM}}{\partial V} - \frac{\sqrt{2}i}{2}V\bar{\zeta}_A\gamma^\alpha\gamma_5\psi_\alpha^A \quad (\text{E.2.6})$$

$$P_-{}^A{}_B\psi_\mu^B = -\frac{1}{3}(\gamma_{\mu\nu} - 3g_{\mu\nu})\tau^A{}_B j^{B\nu} \quad (\text{E.2.7})$$

$$P_+{}^A{}_B\zeta^B = -\tau^A{}_B j^B \quad (\text{E.2.8})$$

$$P_+{}^A{}_B\lambda^{\perp iB} = -\tau^A{}_B j^{iB} \quad (\text{E.2.9})$$

where

$$T_{YM}^{\mu\nu} = 2\frac{\delta\mathcal{L}_{YM}}{\delta g_{\mu\nu}} - g^{\mu\nu}\mathcal{L}_{YM} \quad (\text{E.2.10})$$

and the currents $j^{B\nu}$, j^B and j^{iB} are given by (E.1.4).

On $\partial\mathcal{M}_2$:

$$\xi = \frac{\kappa_5^2}{2g^2} V^{\frac{1}{2}} \bar{\chi}_2^I \chi^{I1} \quad (\text{E.2.11})$$

$$\bar{\xi} = -\frac{\kappa_5^2}{2g^2} V^{\frac{1}{2}} \bar{\chi}_1^I \chi^{I2} \quad (\text{E.2.12})$$

$$\mathcal{A}_\mu^i = -\frac{\sqrt{2}i\kappa_5^2}{4g^2} b^i \tau^A{}_{B\bar{\chi}_A^I} \gamma_\mu \chi^{IB} \quad (\text{E.2.13})$$

$$\begin{aligned} \mathcal{D}_5\sigma &= \frac{1}{12g^2} \partial_\mu \left(\omega_{\mu\nu\rho}^Y - \frac{1}{4} V^{-1} \bar{\chi}_A^I \gamma_{\mu\nu\rho} \chi^{IA} \right) \\ &\quad - \frac{i}{4} V \tau^A{}_{B\bar{\psi}_{A\alpha}} \gamma^{\alpha\beta} \gamma_5 \psi_\beta^B - \frac{3i}{4} V \tau^A{}_{B\bar{\zeta}_A} \gamma_5 \zeta^B + \frac{i}{4} V \tau^A{}_{B\bar{G}_{ij}^\perp} \bar{\lambda}_A^i \gamma_5 \lambda^{Bj} \\ &\quad + \frac{\sqrt{2}}{2} V \tau^A{}_{B\bar{\zeta}_A} \gamma^\alpha \gamma_5 \psi_\alpha^B \end{aligned} \quad (\text{E.2.14})$$

$$\begin{aligned} K^{\mu\nu} - K g^{\mu\nu} &= \kappa_5^2 T_{YM}^{\mu\nu} - \sqrt{2}\alpha V^{-1} g^{\mu\nu} + \frac{1}{2} \tau^A{}_{B\bar{\psi}_A} (\gamma^{\mu\nu})^\rho \psi_\rho^B \\ &\quad + \frac{1}{4} \tau^A{}_{B\bar{\psi}_{A\rho}} \gamma^{\rho\sigma} \psi_\sigma^B + \bar{\zeta}_A \zeta^B + G_{ij}^\perp \bar{\lambda}_A^i \lambda^{Bj} \quad g^{\mu\nu} \end{aligned} \quad (\text{E.2.15})$$

$$\partial_5 V = -\sqrt{2}\alpha + \frac{\kappa_5^2}{g^2} V^2 \frac{\partial \mathcal{L}_{YM}}{\partial V} - \frac{\sqrt{2}i}{2} V \bar{\zeta}_A \gamma^\alpha \gamma_5 \psi_\alpha^A \quad (\text{E.2.16})$$

$$P_-{}^A{}_B \psi_\mu^B = \frac{1}{3} (\gamma_{\mu\nu} - 3g_{\mu\nu}) \tau^A{}_{Bj}{}^{B\nu} \quad (\text{E.2.17})$$

$$P_+{}^A{}_B \zeta^B = \tau^A{}_{Bj}{}^{iB} \quad (\text{E.2.18})$$

$$P_+{}^A{}_B \lambda^{\perp iB} = \tau^A{}_{Bj}{}^{iB} \quad (\text{E.2.19})$$

with the currents $j^{B\nu}$, j^B and j^{iB} given by (E.1.5).

E.3 Supersymmetry

The supersymmetry transformations of the bulk fields are:

$$\delta e^{\hat{\alpha}}_{\alpha} = -\frac{1}{2}\bar{s}_A\gamma^{\hat{\alpha}}\psi_{\alpha}^A \quad (\text{E.3.1})$$

$$\delta\psi_{\alpha}^A = \mathcal{D}_{\alpha}S^A - \frac{\sqrt{2}i}{12}(\gamma_{\alpha}^{\beta\gamma} - 4\delta_{\alpha}^{\beta}\gamma^{\gamma})b_i\mathcal{F}_{\beta\gamma}^i S^A - \frac{\sqrt{2}}{6}V^{-1}\alpha\tau^A{}_B\gamma_{\alpha}S^B \quad (\text{E.3.2})$$

$$\delta\zeta^A = -\frac{\sqrt{2}i}{2}\gamma^{\alpha}S^B f_{xB}{}^A D_{\alpha}q^x \quad (\text{E.3.3})$$

$$\delta V = -\frac{i}{\sqrt{2}}\tau^A{}_B\bar{s}_A\zeta^B \quad (\text{E.3.4})$$

$$\delta\sigma = \frac{\sqrt{2}}{2}\begin{pmatrix} V & -V^{\frac{1}{2}}\xi \\ -V^{\frac{1}{2}}\bar{\xi} & -V \end{pmatrix}^A{}_B \bar{s}_A\zeta^B \quad (\text{E.3.5})$$

$$\delta\xi = -\frac{\sqrt{2}i}{2}V^{\frac{1}{2}}\bar{s}_2\zeta^1 \quad (\text{E.3.6})$$

$$\delta\bar{\xi} = \frac{\sqrt{2}i}{2}V^{\frac{1}{2}}\bar{s}_1\zeta^2 \quad (\text{E.3.7})$$

$$\left(\text{i.e. } \delta q^x = -\frac{\sqrt{2}i}{2}f^{xA}{}_B\bar{s}_A\zeta^B\right) \quad (\text{E.3.8})$$

$$\delta\mathcal{A}_{\alpha}^i = -\frac{\sqrt{2}i}{2}b^i\tau^A{}_B\bar{s}_A\psi_{\alpha}^B + \frac{\sqrt{2}}{2}\tau^A{}_B\bar{s}_A\gamma_{\alpha}\lambda^{\perp Bi} \quad (\text{E.3.9})$$

$$\delta\lambda^{\perp iA} = -\frac{i}{2}\partial_{\alpha}b^i\gamma^{\alpha}S^A + \frac{\sqrt{2}}{4}\mathcal{F}_{\alpha\beta}^{i\perp}\gamma^{\alpha\beta}S^A - \frac{\sqrt{2}i}{4}V^{-1}G^{\perp ij}\alpha_j\tau^A{}_B S^B \quad (\text{E.3.10})$$

$$\delta b^i = -\frac{i}{2}\tau^A{}_B\bar{s}_A\lambda^{\perp iB} \quad (\text{E.3.11})$$

Those of the E_6 gauge fields on $\partial\mathcal{M}_1$ are

$$\delta A_{\mu}^I = \frac{1}{2}V^{-\frac{1}{2}}\bar{s}_A\gamma_{\mu}\chi^A \quad (\text{E.3.12})$$

$$\delta\chi^{IA} = \frac{1}{4}V^{\frac{1}{2}}\gamma^{\mu\nu}F_{\mu\nu}^I S^A + V^{-\frac{1}{2}}G_{ij}\bar{C}^i\Lambda^I C^j\tau^A{}_B S^B \quad (\text{E.3.13})$$

$$\delta\eta_L{}^{ip} = -\mathcal{D}_{\mu}C^{ip}\gamma^{\mu}S^2 - \frac{3\sqrt{10}}{2}V^{-\frac{1}{2}}\mathcal{K}^{-1}\mathcal{K}^i{}_{jk}d^{pqr}\bar{C}^j{}_q\bar{C}^k{}_r S^1 \quad (\text{E.3.14})$$

$$\delta\eta_R{}^i{}_p = \mathcal{D}_{\mu}\bar{C}^i{}_p\gamma^{\mu}S^1 - \frac{3\sqrt{10}}{2}V^{-\frac{1}{2}}\mathcal{K}^{-1}\mathcal{K}^i{}_{jk}d_{pqr}C^{jq}C^{kr} \quad (\text{E.3.15})$$

$$\delta C^{ip} = -\frac{1}{2}\bar{s}_2\eta_L{}^{ip} - \frac{i}{4}\Gamma^i{}_{jk}C^{jp}\bar{s}_A\lambda^{\perp Ak} \quad (\text{E.3.16})$$

$$\delta\bar{C}^i{}_p = -\frac{1}{2}\bar{s}_1\eta_R{}^i{}_p - \frac{i}{4}\Gamma^i{}_{kj}\bar{C}^j{}_p\bar{s}_A\lambda^{\perp Ak} \quad (\text{E.3.17})$$

and of the E_8 fields on $\partial\mathcal{M}_2$ are

$$\delta A_{\mu}^I = \frac{1}{2}V^{-\frac{1}{2}}\bar{s}_A\gamma_{\mu}\chi^A \quad (\text{E.3.18})$$

$$\delta\chi^{IA} = \frac{1}{4}V^{\frac{1}{2}}\gamma^{\mu\nu}F_{\mu\nu}^I S^A \quad (\text{E.3.19})$$

Appendix F

Galileons: Further Details

Derivatives of the co-ordinate vectors

Tangential

$\nabla_{\xi} e_{\mu} = \xi^I \nabla_I e_{\mu} = \xi^I \Gamma_{I\mu}^b e_b$ and due to the normal co-ordinates $\Gamma_{I\mu\nu} = -\Gamma_{\mu I\nu}$ and $\Gamma_{\mu IJ} = -\Gamma_{I\mu J}$ so using the definitions (4.2.6) and (4.2.7) we can see that

$$\nabla_{\xi} e_{\mu}|_{\Sigma_0} = \xi^I k_{I\mu}^{\nu} e_{\nu} + \xi^I a_I^J e_J \quad (\text{F.0.1})$$

ξ and e_{μ} are orthogonal, $[\xi, e_{\mu}] = 0$, so $\nabla_{\xi} e_{\mu} = \nabla_{\mu} \xi$ and $\nabla_{\xi} \nabla_{\xi} e_{\mu} = \nabla_{\xi} \nabla_{\mu} \xi$. Then since ξ is the tangent of a geodesic we have $\nabla_{\xi} \xi = 0$ and so $\nabla_{\xi} \nabla_{\mu} \xi = [\nabla_{\xi}, \nabla_{\mu}] \xi = \mathbf{R}(\xi, e_{\mu}) \xi$ (recall that $\mathbf{R}(\mathbf{A}, \mathbf{B})\mathbf{C} = R^a{}_{bcd} A^c B^d C^b e_a$). Applying ∇_{ξ}^{n-2} to this expression we find

$$\nabla_{\xi}^n e_{\mu}|_{\Sigma_0} = \nabla_{\xi}^{n-2} \mathbf{R}(\xi, e_{\mu}) \xi \quad (\text{F.0.2})$$

Transverse

σ is the proper distance along the geodesic whose tangent is ξ from Σ_0 , so $\nabla_{\xi} \sigma = 1$, $\partial_I \sigma = \xi_I$ and points along the geodesic have co-ordinates $x^I = \sigma \xi^I$. Thus we can evaluate the Lie bracket to be $[\xi, \sigma e_I] = \partial_{\xi} \sigma e_I - \sigma \partial_I (\frac{1}{\sigma} x^J) e_J = \xi_I \xi$ and so $\nabla_{\xi} (\sigma e_I) = \sigma \nabla_I \xi + \xi_I \xi$. Combining this with the geodesic equation $\nabla_{\xi} \xi = 0$ and the fact that ξ is normalized, $\nabla_{\xi} \xi^I = 0$, we find that

$$\nabla_{\xi}^2 (\sigma e_I) = \nabla_I \xi + \sigma \nabla_{\xi} \nabla_I \xi \quad (\text{F.0.3})$$

On the other hand, factoring out σ , we have $\sigma [\xi, e_I] = (\xi_I \xi - e_I)$, so using $\nabla_{\xi} \xi = 0$ again

$$\mathbf{R}(\xi, e_I) \xi \equiv \nabla_{\xi} \nabla_I \xi - \nabla_I \nabla_{\xi} \xi - \nabla_{[\xi, e_I]} \xi = \nabla_{\xi} \nabla_I \xi + \frac{1}{\sigma} \nabla_I \xi \quad (\text{F.0.4})$$

Then comparing (F.0.3) and (F.0.4) we can see that

$$\nabla_{\xi}^2 (\sigma e_I) = \sigma \mathbf{R}(\xi, e_I) \xi \quad (\text{F.0.5})$$

Using this result, since $\nabla_{\xi}^2 (\sigma e_I) = 2 \nabla_{\xi} e_I + \sigma \nabla_{\xi}^2 e_I$ then if we evaluate on Σ_0 , where $\sigma = 0$, then we find

$$\nabla_{\xi} e_I|_{\Sigma_0} = 0 \quad (\text{F.0.6})$$

while applying ∇_{ξ}^n gives

$$\nabla_{\xi}^n (\nabla_{\xi}^2 (\sigma e_I)) = \nabla_{\xi}^n (2\nabla_{\xi} e_I + \sigma \nabla_{\xi}^2 e_I) = (2+n) \nabla_{\xi}^{n+1} e_I + \sigma \nabla_{\xi}^{n+2} e_I \quad (\text{F.0.7})$$

and

$$\nabla_{\xi}^n (\sigma \mathbf{R}(\xi, e_I) \xi) = n \nabla_{\xi}^{n-1} \mathbf{R}(\xi, e_I) \xi + \sigma \nabla_{\xi}^n \mathbf{R}(\xi, e_I) \xi \quad (\text{F.0.8})$$

so evaluating on Σ_0 once again

$$\nabla_{\xi}^n e_I|_{\Sigma_0} = \frac{(n-1)}{(n+1)} \nabla_{\xi}^{n-2} \mathbf{R}(\xi, e_I) \xi \quad (\text{F.0.9})$$

Relation of $k^I_{\mu\nu}$ to codimension one

Here we define the extrinsic curvature as

$$k^I_{\mu\nu} = -\Gamma^I_{\mu\nu}|_{\Sigma_0} \quad (\text{F.0.10})$$

The definition of extrinsic curvature which does not depend on choosing normal co-ordinates is

$$k^I_{\mu\nu} = -h_{\mu}^a h_{\nu}^b \sigma^I_c \nabla_a h_b^c \quad (\text{F.0.11})$$

where $h_{\mu\nu} = e_{\mu} \cdot e_{\nu}$ and $\sigma_{IJ} = e_I \cdot e_J$ are the parts of the metric transverse and normal respectively to the surface whose extrinsic curvature we are calculating. These definitions are equivalent since $\nabla_a g_b^c = 0 \Rightarrow \nabla_a h_b^c = -\nabla_a \sigma_b^c$ so $k^I_{\mu\nu} = h_{\mu}^a h_{\nu}^b \sigma^I_c \nabla_a \sigma_b^c$ while $h_{\nu}^b \sigma_b^d = 0$ and $h_{\nu}^b \sigma^I_c \partial_a \sigma_b^c = -h_{\nu}^b \sigma_b^c \partial_a \sigma^I_c = 0$ which leaves just the one connection term on the right hand side, corresponding to [F.0.10](#). In co-dimension one, $h_{ab} = g_{ab} - n_a n_b$, with n_a being the normal to the surface, and $\sigma^I_c = \delta_c^I$ while I only takes one value so $\sigma^I_c \nabla_a h_b^c = -\nabla_a (n_b n^I)$ but $n^I = n^5 = 1$ and so we recover

$$k_{\mu\nu} = h_{\mu}^a h_{\nu}^b \nabla_a n_b \quad (\text{F.0.12})$$

which is the familiar definition.

Ω^{2n}

Since expansions for various powers of Ω are employed, it is useful to note that for any n

$$\begin{aligned} \Omega^{2n} &= 1 + 2n\pi + n(2n-1)\pi^2 - n\kappa\boldsymbol{\pi} \cdot \boldsymbol{\pi} + \frac{2}{3}n(n-1)(2n-1)\pi^3 \\ &\quad - \frac{2}{3}n(3n-1)\kappa\boldsymbol{\pi} \cdot \boldsymbol{\pi} + \frac{1}{6}n(n-1)(4n^2 - 14n + 15)\pi^4 \\ &\quad - \frac{1}{3}n(3n^2 + 2n - 4)\kappa\pi^2 \boldsymbol{\pi} \cdot \boldsymbol{\pi} + \frac{1}{6}n(3n-1)\kappa^2 (\boldsymbol{\pi} \cdot \boldsymbol{\pi})^2 + \mathcal{O}(\pi^5) \end{aligned} \quad (\text{F.0.13})$$

Definition of π , $\boldsymbol{\pi}$, ...

As defined at the end of Section [\(4.2\)](#) the symbols π and $\boldsymbol{\pi}$ with various indices are

$$\pi = c_I \pi^I; \quad \pi_{\mu} = \nabla_{\mu} (c_I \pi^I); \quad \text{etc.}$$

$$\boldsymbol{\pi} = \pi^I \mathbf{e}_I; \quad \boldsymbol{\pi}_\mu = \nabla_\mu (\pi^I \mathbf{e}_I); \quad \boldsymbol{\pi}_{\mu\nu} = \nabla_\mu \nabla_\nu (\pi^I \mathbf{e}_I); \quad \text{etc.}$$

and the $\boldsymbol{\pi}$'s always appear in scalar products, for example $\boldsymbol{\pi} \cdot \boldsymbol{\pi} = \pi^I \pi^J \mathbf{e}_I \cdot \mathbf{e}_J = \bar{g}_{IJ} \pi^I \pi^J$.

Extrinsic Curvature is $\mathcal{O}(\pi^3)$

The extrinsic curvature is

$$k^I{}_{\mu\nu} = -h_\mu{}^a h_\nu{}^b \sigma^I{}_c \nabla_a h_b{}^c \quad (\text{F.0.14})$$

for the brane $h_{ab} = g_{ab} + \nabla_a \pi_I \nabla_b \pi^I$ and $\sigma_{ab} = g_{ab} - \nabla_\mu \pi_a \nabla^\mu \pi_b$. The leading term in the extrinsic curvature is therefore

$$k^I{}_{\mu\nu} = -\nabla_\mu (\nabla_\nu \pi_J \nabla^I \pi^J) + \dots \quad (\text{F.0.15})$$

but since $\partial_I \pi^J = 0$ and for a Minkowski brane in a Minkowski bulk $\partial_I \bar{g}_{ab} = 0$, $\nabla_I \pi^J = \mathcal{O}(\pi^2)$ and $k^I{}_{\mu\nu} = \mathcal{O}(\pi^3)$, as claimed.

Expansion of terms in $\mathcal{L}_{(4)}$

Using (4.2.21), and remembering that we will keep only the terms necessary to find $\sqrt{-\bar{g}}R(\bar{g})$ up to fourth order in π , we can write

$$\mathfrak{F}_\mu = \nabla_\mu \pi_I \nabla^2 \pi^I \equiv \boldsymbol{\pi}_\mu \cdot \boldsymbol{\pi}_\nu{}^\nu + \mathcal{O}(\pi^3) \quad (\text{F.0.16})$$

Dropping the total derivative allows us to write

$$\begin{aligned} -\frac{1}{2} g^{(\mu\nu)(\rho\sigma)} \delta g_{\mu\nu} \nabla^2 \delta g_{\rho\sigma} &= -\frac{1}{2} \nabla_\tau (g^{(\mu\nu)(\rho\sigma)} \delta g_{\mu\nu} \nabla^\tau \delta g_{\rho\sigma}) + \frac{1}{2} g^{(\mu\nu)(\rho\sigma)} \nabla_\tau \delta g_{\mu\nu} \nabla^\tau \delta g_{\rho\sigma} \\ &= \frac{1}{2} g^{(\mu\nu)(\rho\sigma)} \nabla_\tau (\boldsymbol{\pi}_\mu \cdot \boldsymbol{\pi}_\nu) \nabla^\tau (\boldsymbol{\pi}_\rho \cdot \boldsymbol{\pi}_\sigma) + \mathcal{O}(\pi^5) \\ &= (\boldsymbol{\pi}_\nu \cdot \boldsymbol{\pi}_{\mu\rho}) (\boldsymbol{\pi}^\nu \cdot \boldsymbol{\pi}^{\mu\rho}) + (\boldsymbol{\pi}_\nu \cdot \boldsymbol{\pi}_{\mu\rho}) (\boldsymbol{\pi}^\mu \cdot \boldsymbol{\pi}^{\nu\rho}) \\ &\quad - (\boldsymbol{\pi}^\mu \cdot \boldsymbol{\pi}_{\mu\rho}) (\boldsymbol{\pi}^\nu \cdot \boldsymbol{\pi}_\nu{}^\rho) + \mathcal{O}(\pi^5) \end{aligned} \quad (\text{F.0.17})$$

Since $R(\bar{g})_{\mu\nu\rho\sigma} = \bar{\kappa} (\bar{g}_{\mu\rho} \bar{g}_{\nu\sigma} - \bar{g}_{\mu\sigma} \bar{g}_{\nu\rho})$ and $G(\bar{g})_{\mu\nu} = -3\bar{\kappa} \bar{g}_{\mu\nu}$ for the maximally symmetric slice Σ_0

$$\begin{aligned} G(\bar{g})^{\mu\nu} \delta g_{\mu\nu} &= -3\bar{\kappa} \left(\boldsymbol{\pi}_\mu \cdot \boldsymbol{\pi}^\mu + 2\pi \boldsymbol{\pi}_\mu \cdot \boldsymbol{\pi}^\mu - \pi^2 \boldsymbol{\pi}_\mu \cdot \boldsymbol{\pi}^\mu \right. \\ &\quad \left. + \frac{2}{3} \kappa \boldsymbol{\pi} \cdot \boldsymbol{\pi} \boldsymbol{\pi}_\mu \cdot \boldsymbol{\pi}^\mu + \frac{1}{3} \kappa \boldsymbol{\pi} \cdot \boldsymbol{\pi}_\mu \boldsymbol{\pi} \cdot \boldsymbol{\pi}^\mu \right) + \mathcal{O}(\pi^5) \end{aligned} \quad (\text{F.0.18})$$

and

$$-\frac{1}{2} \delta g_{\mu\nu} g^{(\mu\nu)(\tau\nu)} (R(\bar{g})_{\tau\nu}{}^{\rho\sigma} + G(\bar{g})_{\tau\nu}{}^{\rho\sigma} \delta_v^\sigma) \delta g_{\rho\sigma} \quad (\text{F.0.19})$$

$$= 4\bar{\kappa} (\boldsymbol{\pi}_\mu \cdot \boldsymbol{\pi}_\nu) (\boldsymbol{\pi}^\mu \cdot \boldsymbol{\pi}^\nu) - \bar{\kappa} (\boldsymbol{\pi}_\mu \cdot \boldsymbol{\pi}^\mu)^2 + \mathcal{O}(\pi^5) \quad (\text{F.0.20})$$

Finally since

$$\begin{aligned}
 \sqrt{-\hat{g}}\hat{g}^{\mu\nu} &= \sqrt{-\bar{g}} \left(\bar{g}^{\mu\nu} - \delta g^{\mu\nu} + \frac{1}{2}\bar{g}^{\mu\nu}\delta g_\rho{}^\rho + \mathcal{O}(\delta g^2) \right) \\
 &= \sqrt{-\bar{g}} \left(\bar{g}^{\mu\nu} - \boldsymbol{\pi}^\mu \cdot \boldsymbol{\pi}^\nu + \frac{1}{2}\bar{g}^{\mu\nu}\boldsymbol{\pi}_\rho \cdot \boldsymbol{\pi}^\rho + \mathcal{O}(\pi^4) \right)
 \end{aligned} \tag{F.0.21}$$

we find that dropping the total derivative

$$\begin{aligned}
 -6\Omega^4 \sqrt{\hat{g}}\Omega^{-3}\hat{\nabla}^2\Omega &= -6\sqrt{-\hat{g}}\hat{\nabla}_\mu \left(\Omega\hat{\nabla}^\mu\Omega \right) + 6\sqrt{\hat{g}}\hat{g}^{\mu\nu}\partial_\mu\Omega\partial_\nu\Omega \\
 &= 6\sqrt{\hat{g}}\hat{g}^{\mu\nu}\partial_\mu\Omega\partial_\nu\Omega \\
 &= 6\sqrt{-\bar{g}} \left(\pi_\mu\pi^\mu - 2\kappa\pi_\mu\boldsymbol{\pi} \cdot \boldsymbol{\pi}^\mu + 2\kappa^2\boldsymbol{\pi} \cdot \pi_\mu\boldsymbol{\pi} \cdot \boldsymbol{\pi}^\mu \right. \\
 &\quad \left. - \pi_\mu\pi_\nu\boldsymbol{\pi}^\mu \cdot \boldsymbol{\pi}^\nu + \frac{1}{2}\pi_\mu\pi^\mu\pi_\nu \cdot \boldsymbol{\pi}^\nu + \mathcal{O}(\pi^5) \right)
 \end{aligned} \tag{F.0.22}$$

Bibliography

- [1] S. R. Coleman and J. Mandula, “All Possible Symmetries of the S Matrix,” *Phys. Rev.* **159** (1967) 1251–1256.
- [2] E. Cremmer, B. Julia, and J. Scherk, “Supergravity theory in 11 dimensions,” *Phys. Lett.* **B76** (1978) 409–412.
- [3] L. Alvarez-Gaume and P. H. Ginsparg, “The Structure of Gauge and Gravitational Anomalies,” *Ann. Phys.* **161** (1985) 423.
- [4] A. Bilal and S. Metzger, “Anomaly cancellation in M-theory: A critical review,” *Nucl. Phys.* **B675** (2003) 416–446, [arXiv:hep-th/0307152](#).
- [5] P. K. Townsend, “The eleven-dimensional supermembrane revisited,” *Phys. Lett.* **B350** (1995) 184–187, [arXiv:hep-th/9501068](#).
- [6] E. Witten, “String theory dynamics in various dimensions,” *Nucl. Phys.* **B443** (1995) 85–126, [arXiv:hep-th/9503124](#).
- [7] P. Horava and E. Witten, “Heterotic and type I string dynamics from eleven dimensions,” *Nucl. Phys.* **B460** (1996) 506–524, [arXiv:hep-th/9510209](#).
- [8] P. Horava and E. Witten, “Eleven-Dimensional Supergravity on a Manifold with Boundary,” *Nucl. Phys.* **B475** (1996) 94–114, [arXiv:hep-th/9603142](#).
- [9] R. Donagi, B. A. Ovrut, T. Pantev, and D. Waldram, “Standard models from heterotic M-theory,” *Adv. Theor. Math. Phys.* **5** (2002) 93–137, [arXiv:hep-th/9912208](#).
- [10] L. B. Anderson, J. Gray, Y.-H. He, and A. Lukas, “Exploring Positive Monad Bundles And A New Heterotic Standard Model,” *JHEP* **02** (2010) 054, [arXiv:0911.1569 \[hep-th\]](#).
- [11] Y.-H. He, M. Kreuzer, S.-J. Lee, and A. Lukas, “Heterotic Bundles on Calabi-Yau Manifolds with Small Picard Number,” [arXiv:1108.1031 \[hep-th\]](#).
- [12] L. B. Anderson, J. Gray, A. Lukas, and E. Palti, “Two Hundred Heterotic Standard Models on Smooth Calabi-Yau Threefolds,” [arXiv:1106.4804 \[hep-th\]](#).
- [13] L. B. Anderson, J. Gray, A. Lukas, and B. Ovrut, “Stabilizing All Geometric Moduli in Heterotic Calabi-Yau Vacua,” *Phys. Rev.* **D83** (2011) 106011, [arXiv:1102.0011 \[hep-th\]](#).

- [14] L. B. Anderson, J. Gray, A. Lukas, and B. Ovrut, “The Atiyah Class and Complex Structure Stabilization in Heterotic Calabi-Yau Compactifications,” [arXiv:1107.5076 \[hep-th\]](#).
- [15] A. Lukas, B. A. Ovrut, and D. Waldram, “Cosmological solutions of Horava-Witten theory,” *Phys. Rev.* **D60** (1999) 086001, [arXiv:hep-th/9806022](#).
- [16] K. Becker, M. Becker, and A. Krause, “M-Theory Inflation from Multi M5-Brane Dynamics,” *Nucl. Phys.* **B715** (2005) 349–371, [arXiv:hep-th/0501130](#).
- [17] P. V. Moniz, S. Panda, and J. Ward, “Higher order corrections to Heterotic M-theory inflation,” *Class. Quant. Grav.* **26** (2009) 245003, [arXiv:0907.0711 \[astro-ph.CO\]](#).
- [18] J. Khoury, B. A. Ovrut, P. J. Steinhardt, and N. Turok, “The ekpyrotic universe: Colliding branes and the origin of the hot big bang,” *Phys. Rev.* **D64** (2001) 123522, [arXiv:hep-th/0103239](#).
- [19] J.-L. Lehners, P. McFadden, and N. Turok, “Colliding Branes in Heterotic M-theory,” *Phys. Rev.* **D75** (2007) 103510, [arXiv:hep-th/0611259](#).
- [20] J.-L. Lehners and P. J. Steinhardt, “Dynamical Selection of the Primordial Density Fluctuation Amplitude,” *Phys. Rev. Lett.* **106** (2011) 081301, [arXiv:1008.4567 \[hep-th\]](#).
- [21] G. W. Gibbons and S. W. Hawking, “Action Integrals and Partition Functions in Quantum Gravity,” *Phys. Rev.* **D15** (1977) 2752–2756.
- [22] H. Luckock and I. Moss, “The quantum geometry of random surfaces and spinning membranes,” *Class. Quant. Grav.* **6** (1989) 1993.
- [23] R. C. Myers, “Higher Derivative Gravity, Surface Terms and String Theory,” *Phys. Rev.* **D36** (1987) 392.
- [24] T. Banks and M. Dine, “Couplings and Scales in Strongly Coupled Heterotic String Theory,” *Nucl. Phys.* **B479** (1996) 173–196, [arXiv:hep-th/9605136](#).
- [25] J. Gray, A. Lukas, and B. Ovrut, “Perturbative anti-brane potentials in heterotic M- theory,” *Phys. Rev.* **D76** (2007) 066007, [arXiv:hep-th/0701025](#).
- [26] J. Gray, A. Lukas, and B. Ovrut, “Flux, Gaugino Condensation and Anti-Branes in Heterotic M- theory,” *Phys. Rev.* **D76** (2007) 126012, [arXiv:0709.2914 \[hep-th\]](#).
- [27] A. Lukas, B. A. Ovrut, and D. Waldram, “On the four-dimensional effective action of strongly coupled heterotic string theory,” *Nucl. Phys.* **B532** (1998) 43–82, [arXiv:hep-th/9710208](#).

- [28] I. G. Moss, “Boundary terms for eleven-dimensional supergravity and M-theory,” *Phys. Lett.* **B577** (2003) 71–75, [arXiv:hep-th/0308159](#).
- [29] I. G. Moss, “Boundary terms for supergravity and heterotic M-theory,” *Nucl. Phys.* **B729** (2005) 179–202, [arXiv:hep-th/0403106](#).
- [30] I. G. Moss, “A new look at anomaly cancellation in heterotic M-theory,” *Phys. Lett.* **B637** (2006) 93–96, [arXiv:hep-th/0508227](#).
- [31] I. G. Moss, “Higher order terms in an improved heterotic M theory,” *JHEP* **11** (2008) 067, [arXiv:0810.1662 \[hep-th\]](#).
- [32] W. Israel, “Singular hypersurfaces and thin shells in general relativity,” *Nuovo Cim.* **B44S10** (1966) 1.
- [33] A. Lukas, B. A. Ovrut, K. S. Stelle, and D. Waldram, “Heterotic M-theory in five dimensions,” *Nucl. Phys.* **B552** (1999) 246–290, [arXiv:hep-th/9806051](#).
- [34] I. G. Moss, J. T. Omotani, and P. M. Saffin, “Reducing heterotic M-theory to five dimensional supergravity on a manifold with boundary,” *JHEP* **10** (2011) 088, [arXiv:1108.5456 \[hep-th\]](#).
- [35] A. Lukas, B. A. Ovrut, and D. Waldram, “Five-branes and supersymmetry breaking in M-theory,” *JHEP* **04** (1999) 009, [arXiv:hep-th/9901017](#).
- [36] N. Ahmed and I. G. Moss, “Gaugino condensation in an improved heterotic M-theory,” *JHEP* **12** (2008) 108, [arXiv:0809.2244 \[hep-th\]](#).
- [37] N. Ahmed and I. G. Moss, “Balancing the vacuum energy in heterotic M-theory,” *Nucl. Phys.* **B833** (2010) 133–152, [arXiv:0907.1602 \[hep-th\]](#).
- [38] D. V. Belyaev and P. van Nieuwenhuizen, “Tensor calculus for supergravity on a manifold with boundary,” *JHEP* **02** (2008) 047, [arXiv:0711.2272 \[hep-th\]](#).
- [39] D. V. Belyaev and T. G. Pugh, “The supermultiplet of boundary conditions in supergravity,” *JHEP* **10** (2010) 031, [arXiv:1008.1574 \[hep-th\]](#).
- [40] D. V. Belyaev, “Boundary conditions in supergravity on a manifold with boundary,” *JHEP* **01** (2006) 047, [arXiv:hep-th/0509172](#).
- [41] T. G. Pugh, E. Sezgin, and K. S. Stelle, “D=7 / D=6 Heterotic Supergravity with Gauged R-Symmetry,” *JHEP* **02** (2011) 115, [arXiv:1008.0726 \[hep-th\]](#).
- [42] L. Randall and R. Sundrum, “An alternative to compactification,” *Phys. Rev. Lett.* **83** (1999) 4690–4693, [arXiv:hep-th/9906064](#).
- [43] G. Gibbons, K.-i. Maeda, and Y.-i. Takamizu, “Fermions on colliding branes,” *Phys. Lett.* **B647** (2007) 1–7, [arXiv:hep-th/0610286](#).

- [44] P. M. Saffin and A. Tranberg, “Particle transfer in braneworld collisions,” *JHEP* **08** (2007) 072, [arXiv:0705.3606 \[hep-th\]](#).
- [45] P. M. Saffin and A. Tranberg, “The fermion spectrum in braneworld collisions,” *JHEP* **12** (2007) 053, [arXiv:0710.3272 \[hep-th\]](#).
- [46] Y.-i. Takamizu and K.-i. Maeda, “Collision of domain walls and reheating of the brane universe,” *Phys. Rev.* **D70** (2004) 123514, [arXiv:hep-th/0406235](#).
- [47] Y.-i. Takamizu and K.-i. Maeda, “Collision of domain walls in asymptotically anti de Sitter spacetime,” *Phys. Rev.* **D73** (2006) 103508, [arXiv:hep-th/0603076](#).
- [48] Y.-i. Takamizu, H. Kudoh, and K.-i. Maeda, “Dynamics of colliding branes and black brane production,” *Phys. Rev.* **D75** (2007) 061304, [arXiv:gr-qc/0702138](#).
- [49] G. R. Dvali, G. Gabadadze, and M. Porrati, “4D gravity on a brane in 5D Minkowski space,” *Phys. Lett.* **B485** (2000) 208–214, [arXiv:hep-th/0005016](#).
- [50] A. Nicolis, R. Rattazzi, and E. Trincherini, “The galileon as a local modification of gravity,” *Phys. Rev.* **D79** (2009) 064036, [arXiv:0811.2197 \[hep-th\]](#).
- [51] C. de Rham and A. J. Tolley, “DBI and the Galileon reunited,” *JCAP* **1005** (2010) 015, [arXiv:1003.5917 \[hep-th\]](#).
- [52] D. Lovelock, “The Einstein tensor and its generalizations,” *J. Math. Phys.* **12** (1971) 498–501.
- [53] G. Goon, K. Hinterbichler, and M. Trodden, “Symmetries for Galileons and DBI scalars on curved space,” *JCAP* **1107** (2011) 017, [arXiv:1103.5745 \[hep-th\]](#).
- [54] A. Padilla, P. M. Saffin, and S.-Y. Zhou, “Bi-galileon theory I: motivation and formulation,” *JHEP* **12** (2010) 031, [arXiv:1007.5424 \[hep-th\]](#).
- [55] A. Padilla, P. M. Saffin, and S.-Y. Zhou, “Multi-galileons, solitons and Derrick’s theorem,” *Phys. Rev.* **D83** (2011) 045009, [arXiv:1008.0745 \[hep-th\]](#).
- [56] K. Hinterbichler, M. Trodden, and D. Wesley, “Multi-field galileons and higher co-dimension branes,” *Phys. Rev.* **D82** (2010) 124018, [arXiv:1008.1305 \[hep-th\]](#).
- [57] J. Omotani, P. M. Saffin, and J. Louko, “Colliding branes and big crunches,” *Phys. Rev.* **D84** (2011) 063526, [arXiv:1107.3938 \[hep-th\]](#).
- [58] S. Kachru, R. Kallosh, A. D. Linde, and S. P. Trivedi, “De Sitter vacua in string theory,” *Phys. Rev.* **D68** (2003) 046005, [arXiv:hep-th/0301240](#).

- [59] A. Lukas, B. A. Ovrut, and D. Waldram, “Non-standard embedding and five-branes in heterotic M- theory,” *Phys. Rev.* **D59** (1999) 106005, [arXiv:hep-th/9808101](#).
- [60] J. W. York, Jr., “Role of conformal three geometry in the dynamics of gravitation,” *Phys. Rev. Lett.* **28** (1972) 1082–1085.
- [61] S. Metzger, “M-theory compactifications, G(2)-manifolds and anomalies,” [arXiv:hep-th/0308085](#).
- [62] E. Bergshoeff and M. de Roo, “Supersymmetric Chern-Simons Terms In Ten-Dimensions,” *Phys. Lett.* **B218** (1989) 210.
- [63] E. A. Bergshoeff and M. de Roo, “The Quartic Effective Action Of The Heterotic String And Supersymmetry,” *Nucl. Phys.* **B328** (1989) 439.
- [64] S. Weinberg, “The quantum theory of fields. Vol. 3: Supersymmetry,” Cambridge, UK: Univ. Pr. (2000) 419 p.
- [65] L. Andrianopoli *et al.*, “General Matter Coupled N=2 Supergravity,” *Nucl. Phys.* **B476** (1996) 397–417, [arXiv:hep-th/9603004](#).
- [66] A. Flachi, I. G. Moss, and D. J. Toms, “Fermion vacuum energies in brane world models,” *Phys. Lett.* **B518** (2001) 153–156, [arXiv:hep-th/0103138](#).
- [67] A. Flachi, I. G. Moss, and D. J. Toms, “Quantized bulk fermions in the Randall-Sundrum brane model,” *Phys. Rev.* **D64** (2001) 105029, [arXiv:hep-th/0106076](#).
- [68] K. Akama, “An Early Proposal of ‘Brane World’,” *Lect. Notes Phys.* **176** (1982) 267–271, [arXiv:hep-th/0001113](#).
- [69] V. A. Rubakov and M. E. Shaposhnikov, “Do We Live Inside a Domain Wall?,” *Phys. Lett.* **B125** (1983) 136–138.
- [70] R. Jackiw and C. Rebbi, “Solitons with Fermion Number 1/2,” *Phys. Rev.* **D13** (1976) 3398–3409.
- [71] J. Garriga and T. Tanaka, “Gravity in the brane-world,” *Phys. Rev. Lett.* **84** (2000) 2778–2781, [arXiv:hep-th/9911055](#).
- [72] D. Langlois, “Brane cosmology: An introduction,” *Prog. Theor. Phys. Suppl.* **148** (2003) 181–212, [arXiv:hep-th/0209261](#).
- [73] M. Bucher, “A braneworld universe from colliding bubbles,” [arXiv:hep-th/0107148](#).
- [74] U. Gen, A. Ishibashi, and T. Tanaka, “Brane big-bang brought by bulk bubble,” *Phys. Rev.* **D66** (2002) 023519, [arXiv:hep-th/0110286](#).
- [75] D. Langlois, K.-i. Maeda, and D. Wands, “Conservation laws for collisions of branes (or shells) in general relativity,” *Phys. Rev. Lett.* **88** (2002) 181301, [arXiv:gr-qc/0111013](#).

- [76] A. Chamblin and G. W. Gibbons, “Nonlinear supergravity on a brane without compactification,” *Phys. Rev. Lett.* **84** (2000) 1090–1093, [arXiv:hep-th/9909130](#).
- [77] C. Charmousis and J.-F. Dufaux, “General Gauss-Bonnet brane cosmology,” *Class. Quant. Grav.* **19** (2002) 4671–4682, [arXiv:hep-th/0202107](#).
- [78] R. Zegers, “Birkhoff’s theorem in Lovelock gravity,” *J. Math. Phys.* **46** (2005) 072502, [arXiv:gr-qc/0505016](#).
- [79] L. M. Burko and A. Ori, “Late-time evolution of nonlinear gravitational collapse,” *Phys. Rev.* **D56** (1997) 7820–7832, [arXiv:gr-qc/9703067](#).
- [80] S. A. Hayward, “General laws of black hole dynamics,” *Phys. Rev.* **D49** (1994) 6467–6474.
- [81] C. Deffayet, S. Deser, and G. Esposito-Farese, “Arbitrary p-form Galileons,” *Phys. Rev.* **D82** (2010) 061501, [arXiv:1007.5278 \[gr-qc\]](#).
- [82] C. Charmousis and R. Zegers, “Matching conditions for a brane of arbitrary codimension,” *JHEP* **08** (2005) 075, [arXiv:hep-th/0502170](#).
- [83] C. Charmousis and R. Zegers, “Einstein gravity on an even codimension brane,” *Phys. Rev.* **D72** (2005) 064005, [arXiv:hep-th/0502171](#).
- [84] P. Candelas and X. de la Ossa, “Moduli Space of Calabi-Yau Manifolds,” *Nucl. Phys.* **B355** (1991) 455–481.
- [85] P. Cvitanovic, “Group theory: Birdtracks, Lie’s and exceptional groups,” Princeton, USA: Univ. Pr. (2008) 273 p.