Equations over groups and Cyclically presented groups

Julia Mary Paul, BSc.

Thesis submitted to The University of Nottingham
for the degree of Doctor of Philosophy

April 2010

Abstract

In Chapter 1, the concept of equations over groups is introduced and the two main conjectures and several theorems on the subject are discussed. The main theorem (Theorem 1.12) is stated, which is that when certain constraints are put on $r(t) \in G* < t| >$, where G is a group and t is distinct from G, then $r(t) = 1$ always has a solution over G. The corollary to the main theorem (Corollary 1.13) is proved, the method of proof is outlined and the key lemma is stated. In Chapters 2 and 3, the key lemma for the main theorem is proved and in Chapter 4, the proof of the main theorem is completed.

In Chapter 5, the concept of cyclically presented groups is introduced. The previous experiment which involved searching for trivial cyclically presented groups is discussed and the experiment undertaken here, which involves searching for finite cyclically presented groups, is briefly described. Results are stated, including the main theorem (Theorem 5.2.4), and the motivation for looking at the number of generators needed for finite groups is discussed.

In Chapter 6, the experiment for searching for finite cyclically presented groups is outlined in more detail. It is explained how a list of candidates for finite cyclically presented groups is found, and a table showing the numbers in the list is given. In Chapter 7, the methods used to check the list of candidates for finite groups is outlined.

In Chapter 8, a list is given of all finite groups found and their structures. The outstanding cases for which it is unknown whether or not the group is finite are mentioned. For those finite groups which appear to be a family, proofs are given. The results found for the number of generators for finite groups are discussed.

Acknowledgements

First, and most importantly, I would like to thank my supervisor Martin Edjvet for all the help and support he has given me over the past few years. I would also like to thank the University of Nottingham for providing my funding and for being a huge part of my life for the last eight years.

I would like to give a massive thank you to Dave Parkin who has helped me out more times than I can remember. Without his technical advice and assistance I would probably still be running my many computations now. I am also grateful to Alexander Hulpke for the assistance he has given me with the computer package GAP.

I would like to thank my boyfriend Martin Nelson. Firstly, for introducing me to Linux, teaching me Latex, and generally helping me out with computer related problems that have occurred all too frequently. Secondly, for all the support and understanding he has given me throughout the PhD process.

Finally, I thank my parents and family for all the support they have given me. In spite of knowing next to nothing about the subject of my work, they have always had confidence in me and I am truly grateful.

Contents

Chapter 1

Introduction – Equations over groups

1.1 Equations over groups

Definition 1.1. *Let G be a non-trivial group and let t be an element distinct from G. Let $r(t) = g_1 t^{l_1} \dots g_k t^{l_k},\ k \geq 1,\ g_i \in G \setminus \{1\},\ l_i \in \mathbb{Z} \setminus \{0\}$ be an element in the free product $G* < t >$.*
Then $r(t) = 1$ is said to be an equation over G *which has a* solution over G *if there is an embedding ϕ from G into a group H and an element $h \in H$ such that $\phi(g_1)h^{l_1} \dots \phi(g_k)h^{l_k} = 1$ in H.*
The length *of the equation is defined to be $|l_1| + \dots + |l_k|$ and the* exponent sum *is $l_1 + \dots + l_k$. An equation is called* singular *if its exponent sum is equal to zero, otherwise it is called* non-singular*.*

There are two main conjectures in the study of equations over groups.

Conjecture 1.2. *[21] Any equation over a torsion-free group G has a solution over G.*

Conjecture 1.3. *[22] Any non-singular equation over any group G has a solution over G.*

These conjectures remain unresolved, although some partial results have been proved for both.

For example, the following two results support Conjecture 1.2.

Theorem 1.4. *[18] Any equation of length at most 6 over a torsion-free group has a solution.*

Theorem 1.5. *[16] Any equation over a locally indicable group G has a solution over G.*

Locally indicable means each of the non-trivial finitely generated subgroups of the group admits an epimorphism onto the infinite cyclic group. Any locally indicable group is torsion-free.

The following theorem supports both Conjecture 1.2 and Conjecture 1.3.

Theorem 1.6. *[20] Any equation of exponent sum 1 over a torsion-free group has a solution.*

There have been two main approaches to Conjecture 1.3. One is to restrict the class of groups to which G belongs. The following theorem is an example of this approach.

Theorem 1.7. *[13] Any non-singular equation over a residually finite group G has a solution over G.*

Residually finite means that, for any non-trivial element g, there is a homomorphism θ to a finite group such that $\theta(g) \neq 1$. It is worth remarking that polycyclic groups are residually finite.

The other approach, which is the one adopted here, is to put constraints on $r(t)$, for example, on the length of $r(t)$. An example is given by the following.

Theorem 1.8. *[11] Any non-singular equation of length at most 5 has a solution.*

More recently, however, there has been the following theorem in which the free product length with respect to $G* < t >$ is unbounded.

Theorem 1.9. *[4, 7–9, 21, 22] Let* $r(t) = g_1 t^{l_1} \ldots g_k t^{l_k} \in G* < t >,\ k \geq 1,\ l_i \in \mathbb{Z} \setminus \{0\}$ *where:*

(i) $|g_i| > 2,\ (1 \leq i \leq k)$, *where* $|g_i|$ *refers to the order of* g_i *in* G.

(ii) $l_1 + \ldots + l_k \neq 0$ *(i.e. the equation* $r(t) = 1$ *is non-singular).*

(iii) $|l_i| \neq |l_j|$ *for* $i \neq j$.

Then $r(t) = 1$ has a solution over G.

Remark

The case $k = 1$ is a consequence of [21] in which it is shown that Conjecture 1.3 is true whenever the length of the equation equals its exponent sum. The case $k = 2$ is a particular result of the fact that if $r(t) = g_1 t^{l_1} g_2 t^{l_2}$ with $\{|g_1|, |g_2|\} \neq \{2, 3\}$ and $l_1 + l_2 \neq 0$, $r(t) = 1$ always has a solution, which was shown in [4]. Case $k = 3$ was proved in [7], case $k = 4$ was proved in [8], case $k = 5$ was proved in [9], and the result for $k \geq 6$ follows from a small cancellation argument [22]. In fact, if $k \geq 6$ and only condition *(iii)* holds then the theorem is still true.

We will consider $k \geq 5$ of Theorem 1.9 in a more general setting which we now describe.

1.2 Statement of results

Let G be a group and let

$$r(t) = w_1 t^{l_1} \dots w_k t^{l_k} \quad (k \geq 5)$$

where $w_i = g_{i,1} t^{m_{i,1}} g_{i,2} \dots t^{m_{i,k_i - 1}} g_{i,k_i}$ with $\;g_{i,j} \in G \setminus \{1\},\; k_i \geq 1,\; m_{i,j} \neq 0$.
We introduce the following condition:
(*) For $1 \leq i \leq k$, $|l_i|$ is distinct from $|l_j|$ for $j \neq i$ and is distinct from $|m_{u,v}|$ for any $u,\; v$.

The following can be proved using standard small cancellation arguments and its proof will be briefly discussed towards the end of the chapter.

Theorem 1.10. *If $k \geq 6$ in the above and condition (*) holds then $r(t) = 1$ has a solution over G.*

This statement generalises Theorem 1.9 for when $k \geq 6$. From now on therefore, it can be assumed that $k = 5$.

Lemma 1.11. *If $\forall i$, $\exists j \neq i$ such that $w_i = w_j^{\pm 1}$, then it can be assumed that $r(t)$ has one of the following forms (modulo cyclic permutation and inversion):*

A1	$wt^{l_1}wt^{l_2}wt^{l_3}wt^{l_4}wt^{l_5}$	*A3*	$wt^{l_1}wt^{l_2}wt^{l_3}w^{-1}t^{l_4}w^{-1}t^{l_5}$
A2	$wt^{l_1}wt^{l_2}wt^{l_3}wt^{l_4}w^{-1}t^{l_5}$	*A4*	$wt^{l_1}wt^{l_2}w^{-1}t^{l_3}wt^{l_4}w^{-1}t^{l_5}$

B1(a)	$wt^{l_1}wt^{l_2}wt^{l_3}vt^{l_4}vt^{l_5}$	*B4(a)*	$wt^{l_1}wt^{l_2}vt^{l_3}wt^{l_4}vt^{l_5}$
B1(b)	$wt^{l_1}wt^{l_2}wt^{l_3}vt^{l_4}v^{-1}t^{l_5}$	*B4(b)*	$wt^{l_1}wt^{l_2}vt^{l_3}wt^{l_4}v^{-1}t^{l_5}$
B2(a)	$wt^{l_1}wt^{l_2}w^{-1}t^{l_3}vt^{l_4}vt^{l_5}$	*B5(a)*	$wt^{l_1}wt^{l_2}vt^{l_3}w^{-1}t^{l_4}vt^{l_5}$
B2(b)	$wt^{l_1}wt^{l_2}w^{-1}t^{l_3}vt^{l_4}v^{-1}t^{l_5}$	*B5(b)*	$wt^{l_1}wt^{l_2}vt^{l_3}w^{-1}t^{l_4}v^{-1}t^{l_5}$
B3(a)	$wt^{l_1}w^{-1}t^{l_2}wt^{l_3}vt^{l_4}vt^{l_5}$	*B6(a)*	$wt^{l_1}w^{-1}t^{l_2}vt^{l_3}wt^{l_4}vt^{l_5}$
B3(b)	$wt^{l_1}w^{-1}t^{l_2}wt^{l_3}vt^{l_4}v^{-1}t^{l_5}$	*B6(b)*	$wt^{l_1}w^{-1}t^{l_2}vt^{l_3}wt^{l_4}v^{-1}t^{l_5}$

Proof. As we are assuming there is no single w_i distinct from all other w_j and their inverses, it must be the case that either the w_i are all equal to each other (or each other's inverses) or the w_i are split into a subset of three and a subset of two with the w_i in the same subset being equal to each other (or each other's inverses). With this in mind, $r(t)$ may always be rewritten to be in one of the above forms using cyclic permutation and inversion. For example, let $r(t) = vt^{l_1}wt^{l_2}w^{-1}t^{l_3}w^{-1}t^{l_4}v^{-1}t^{l_5}$. Inverting this gives us $t^{-l_5}vt^{-l_4}wt^{-l_3}wt^{-l_2}w^{-1}t^{-l_1}v^{-1}$. A cyclic permutation of this is $wt^{-l_3}wt^{-l_2}w^{-1}t^{-l_1}v^{-1}t^{-l_5}vt^{-l_4}$ which, after relabelling, is of the form B2(b). □

We define a *subword* of the word $w = g_1t^{m_1}g_2\dots g_{s-1}t^{m_{s-1}}g_s$ where $g_i \in G\setminus\{1\},\ m_i \in \mathbb{Z}\setminus\{0\},\ s \geq 1$ to be a word of the form $g_kt^{m_k}g_{k+1}\cdots g_{k+r-1}t^{m_{k+r-1}}g_{k+r}$ where $k \in \{1,\dots,s\}$ and $r \in \{0,\dots,s-k\}$. A subword is an *initial subword* if $k = 1$, an *end subword* if $r = s-k$ and a *proper subword* if $(k,r) \neq (1, s-k)$.

We are now ready to state our main theorem.

Theorem 1.12. *Let G be a group and let*

$$r(t) = w_1t^{l_1}w_2t^{l_2}w_3t^{l_3}w_4t^{l_4}w_5t^{l_5}$$

where $w_i = g_{i,1}t^{m_{i,1}}\dots t^{m_{i,k_i-1}}g_{i,k_i}$ with $g_{i,j} \in G\setminus\{1\},\ k_i \geq 1,\ m_{i,j} \neq 0$.
Assume that condition (*) holds and, in addition, that the following conditions hold.
(**) No w_i is a conjugate of an element of G of order 2.
(***) No w_i is a proper initial or end subword of any $w_j^{\pm1}$ for $j \neq i$.
Then the following statements are true.

(I) *Let $r(t)$ be given by one of the 16 forms listed in Lemma 1.11. Then $r(t) = 1$ has a solution over G if one of the following holds:*

(1) $r(t)$ *is of the form A1 or A4.*

(2) $r(t)$ *is of the form A2 and none of the following sets of relations hold:*

(i) $l_1 = l_2 + l_4$, $l_2 = l_3 + l_5$ *and* $l_3 = l_1 + l_5$,

(ii) $l_1 = l_3 + l_4$, $l_2 = l_1 + l_4$ *and* $l_3 = l_2 + l_5$,

(iii) $l_3 = l_2 + l_4 = l_1 + l_5$,

(iv) $l_1 = l_2 + l_5 = l_3 + l_4$.

(3) $r(t)$ *is of the form A3 and neither of the following sets of relations hold:*

(i) $l_2 = l_1 + l_3$ *and* $l_2 + l_4 + l_5 = 0$,

(ii) $l_1 = l_2 + l_5$ *and* $l_1 + l_3 + l_4 = 0$.

(4) $r(t)$ *is of the form B1-B6.*

(II) If $r(t)$ *is not one of the 16 forms listed in Lemma 1.11, i.e.* $\exists i$ *such that* $w_i \neq w_j^{\pm 1}$ *for all* $j \neq i$, *then* $r(t) = 1$ *has a solution over* G.

Remark

It is worth pointing out that $r(t)$ has a solution under the restrictions of 2(i) if and only if it has a solution under the restrictions of 2(ii). The same holds true for the pair 2(iii) and 2(iv) and the pair 3(i) and 3(ii). A full explanation for this is given in Chapter 2. The restrictions are required because our method of proof breaks down. We expect, however, that $r(t) = 1$ will have a solution in these cases.

Note also that we do not require $l_1 + \ldots + l_k \neq 0$ and this allows us to prove the following extension of Theorem 1.9 for when $k = 5$.

Corollary 1.13. *Let* $r(t) = g_1 t^{l_1} g_2 t^{l_2} g_3 t^{l_3} g_4 t^{l_4} g_5 t^{l_5} \in G* < t >,\ l_i \in \mathbb{Z} \setminus \{0\}$ *where:*

(i) $|g_i| > 2,\ (1 \leq i \leq 5)$.

(ii) $|l_i| \neq |l_j|$ *for* $i \neq j$.

Then $r(t) = 1$ *has a solution over* G.

Proof. The proof follows immediately from the theorem unless we have one of the exceptions in Case (2) or (3). If $r(t)$ is non-singular, $r(t)$ has a solution over G in the A cases by [13].

Let $r(t)$ be singular. Consider Case (3), so $r(t) = gt^{l_1}gt^{l_2}gt^{l_3}g^{-1}t^{l_4}g^{-1}t^{l_5}$. The exceptions are *(i)* $l_2 = l_1 + l_3$ and $l_2 + l_4 + l_5 = 0$, *(ii)* $l_1 = l_2 + l_5$ and $l_1 + l_3 + l_4 = 0$ and

we also have $l_1 + l_2 + l_3 + l_4 + l_5 = 0$. If *(i)* holds then the singularity condition implies $l_2 = 0$ while if *(ii)* holds we get $l_1 = 0$, either of which leads to a contradiction and the result holds in this case.
Now consider Case (2), so $r(t) = gt^{l_1}gt^{l_2}gt^{l_3}gt^{l_4}g^{-1}t^{l_5}$. Let condition *(iii)* hold so $l_3 = l_2 + l_4 = l_1 + l_5$. Let $h \in H$ be a solution in the overgroup H to the equation $gt^{l_3} = 1$. Then in H we have $r(h) = gh^{l_1}gh^{l_2}gh^{l_4}g^{-1}h^{l_5} = gh^{l_1}gh^{l_2}gh^{l_3}h^{-l_2}g^{-1}h^{l_5} = gh^{l_1}gh^{l_2}h^{-l_2}g^{-1}h^{l_5} = gh^{l_1}h^{l_5} = gh^{l_3} = 1$ so $r(t)$ has a solution over G. If *(iv)* holds, then the same result occurs by letting $h \in H$ be a solution to $gt^{l_1} = 1$. Cases *(i)* and *(ii)* require a different approach. Let *(i)* hold so $l_1 = l_2 + l_4$, $l_2 = l_3 + l_5$ and $l_3 = l_1 + l_5$ and we also have $l_1 + l_2 + l_3 + l_4 + l_5 = 0$. Using these relations, we can rewrite $r(t)$ as $gt^{-2l_3}gt^{4l_3}gt^{l_3}gt^{-6l_3}g^{-1}t^{3l_3}$. This equation has a solution if and only if the equation $r(t) = gt^{-2}gt^4gtgt^{-6}g^{-1}t^3$ has a solution (see remark below) so let us consider this equation instead. Since it can be assumed that $G =< g|g^n >$ (see remark below), we require $< g|g^n >$ to embed in the overgroup $H = \frac{<g|g^n>*<t|>}{<<r(t)>>}$. Let $K = \frac{<a|a^n>*<s|s^6>}{<<(as^4)^3>>}$. Define the following mapping $\theta : \{g, t\} \to K$ by $g \mapsto a,\ t \mapsto s$. Now substituting yields $as^{-2}as^4asas^{-6}a^{-1}s^3 = as^4as^4asaa^{-1}s^3 = (as^4)^3 = 1$ in K so θ extends to a group epimorphism. But since $|s^4| \neq 2$ in $< s|s^6 >$, it follows that $< a|a^n >$ embeds in K by [3], therefore g must have order n in H and so $G =< g|g^n >$ embeds in H and we are done. The same argument can be used for case *(ii)* by symmetry (See Subsection 2.3.2).
$\square$

Remarks

1. We can assume the greatest common divisor of the l_i's is equal to one. To see this, assume that $gcd(l_i : 1 \leq i \leq 5) = d > 1$. Then $l_i = d\alpha_i$ where $gcd(\alpha_i : 1 \leq i \leq 5) = 1$. If we know the natural map from G to $H_1 =< G, s|w_1s^{\alpha_1} \ldots w_5s^{\alpha_5} >$ is injective then G embeds in $H_1 *_{s=t^d} < t|t^m > =< G, t|r(t) >$, where $m = 0$ if s has infinite order and $m = dq$ if s has order $q < \infty$.

2. We assume, without loss of generality, that G is generated by the elements of G which appear in $r(t)$. For if $G_0 =< g_{i,j} >$ where $g_{i,j}$ are elements of G involved in $r(t)$ and $r(t)$ has a solution over G_0 in H, then $r(t) = 1$ has a solution over G with the amalgamated free product $H *_{G_0} G$ as the overgroup of G.

The only form of equation $r(t) = 1$ for which it is known that there is no solution is when $r(t) = u(G, t)gu(G, t)^{-1}\hat{g}$, where $u(G, t) \in G* < t >$ and $g, \hat{g} \in G$ have different orders. Note that this cannot happen under the conditions of Conjecture 1.2 and Conjecture 1.3. Observe that this situation is also ruled out by condition (*), even when we allow $k \geq 1$. This encourages us to make the following conjecture.

Conjecture 1.14. *Let G be a group and let $r(t) = g_1 t^{q_1} \dots g_l t^{q_l}$ where $l \geq 1$, $g_i \in G \setminus \{1\}$, $q_i \in \mathbb{Z} \setminus \{0\}$ and $\exists i$ such that $|q_i| \neq |q_j|$, $\forall j \neq i$. Then $r(t) = 1$ has a solution over G.*

1.3 Method of proof for Theorem 1.12

To show that $r(t) = 1$ has a solution over G, it is enough to show that the map $G \to H = < G, t \mid r(t) >$ given by $g \mapsto g$, $\forall g \in G$ is injective. Assume by way of contradiction that this map is not injective. Then there is a free product diagram K [22] whose boundary is a simple closed curve with an element $g_0 \in G \setminus \{1\}$ as its label.

We will now describe how such a diagram can be amended. The diagram will have two different types of regions. The first type is an $r(t)$-region whose boundary label is some cyclic permutation of $r(t)^{\pm 1}$. The second type is a G-region whose boundary label is a word in G which yields the identity.

The first amendment we make to the diagram is to contract each maximal t-segment to a point and label its corresponding corners with the l_i or $m_{u,v}$ as appropriate. The second amendment is to the G-regions. We place a new vertex in the interior of each G-region, including the infinite region external to K, and then make the following transformation. Create new edges between the newly added vertex and each vertex of the region and delete the old edges which form the boundary of the region. Label the corners around the new vertex with the element of G that corresponds to the label of the deleted edge. This transformation is shown in Figure 1.1.

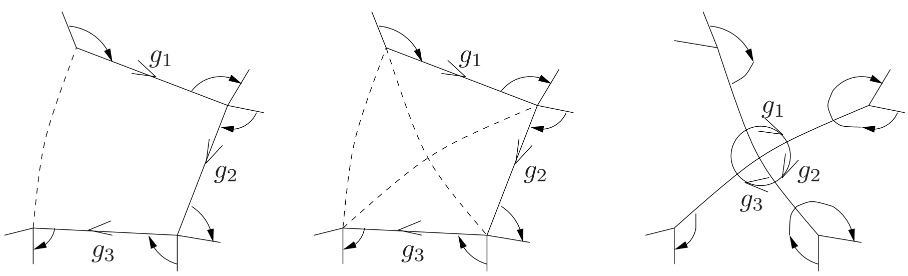


Figure 1.1: G-region amendment

What we have now obtained is a tessellation D of the 2-sphere, whose regions have corners labelled with some cyclic permutation of $r(t)^{\pm 1}$, reading around the region from any vertex.

Vertices which are labelled with powers of t will be referred to as *t-vertices*. Vertices labelled with elements of G are known as *g-vertices*. By convention we write i in place of l_i at the corners of the diagrams and we will use $\bar{i}$ to denote $-l_i$.

Given a region Δ, we refer to the five vertices with corner labels $l_1, \ldots, l_5$ within Δ as $v_1, \ldots, v_5$ respectively.

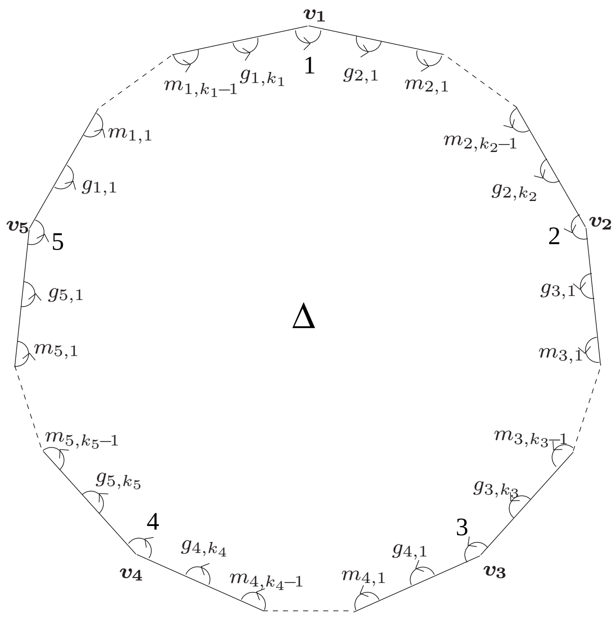


Figure 1.2: A typical region of D

The sum of the corner labels at a t-vertex must equal 0 (since $< t| >$ is one of the free factors) and the product of the corner labels at a g-vertex must give the identity in G, except for the vertex whose label is g_0.

Let v_0 be the vertex obtained from the amendment of the original G-region labelled with g_0. A region that has v_0 as one of its vertices is called a *boundary region*, otherwise it is called *interior*. The *degree* of a region is said to be the number of vertices of that region **with degree exceeding 2**. We denote the degree of a region Δ by $d(\Delta)$, the degree of a vertex v by $d(v)$ and the label of a vertex v by $l(v)$.

We may assume that D is minimal with respect to the number of regions and, subject to this, the number of interior vertices of D of degree 2 is maximal. These assumptions play a role of great importance and lead to the following Lemma which is analogous to one proved in [9].

Lemma 1.15. *(i) No vertex label of D can have as a sublabel $i\bar{i}$ or $\bar{i}i$,* $(1 \leq i \leq 5)$;

(ii) $d(v_i) > 2$ *for* $1 \leq i \leq 5$*;*

(iii) $d(\Delta) \geq 5$ *for any region* Δ *of* D*.*

Proof. (i) If we allowed such a sublabel then the diagram would not be reduced and it would be possible to cancel two regions and contradict our assumption of the minimality of D.
(ii) This comes from (i) and the fact that $|l_i| \neq |l_j|$ for $i \neq j$.
(iii) This fact follows of from (ii) and the fact that there are 5 of the v_i. □

We define the *curvature* of a region Δ to be $c(\Delta) = c(d_1, \ldots, d_m) = (2-m)\pi + 2\pi\Sigma_{i=1}^{m}\frac{1}{d_i}$, where $m = d(\Delta)$ and the d_i are the degrees of the vertices, $1 \leq i \leq m$. The total curvature of D, denoted by $c(D)$, is the sum of the curvatures of each region.

Lemma 1.16. $c(D) = 4\pi$.

Proof. Let V = number of vertices, E = number of edges and F = number of regions. Then:

$$\begin{aligned} c(D) = \Sigma_{\Delta\in D}c(\Delta) &= \Sigma_{\Delta\in D}[\pi(2-d(\Delta)) + 2\pi\left(\frac{1}{d_1} + \ldots + \frac{1}{d_k}\right)] \quad (k = d(\Delta)) \\ &= 2\pi\Sigma_{\Delta\in D}\left(\frac{1}{d_1} + \ldots + \frac{1}{d_k}\right) + 2\pi\Sigma_{\Delta\in D} - \pi\Sigma_{\Delta\in D}d(\Delta) \\ &= 2\pi V + 2\pi F - \pi 2E = 2\pi(V - E + F) = 4\pi. \ \square \end{aligned}$$

The contradiction required for our proof arises from being able to show that the total curvature of 4π is not obtainable, and thus the mapping $G \to H$, $g \mapsto g$, $\forall g \in G$ is injective, yielding Theorem 1.12.

From here, the next step is to locate interior regions Δ of D that have positive curvature and to show that for each such region, we can find a neighbouring region Δ' into which we can distribute this positive curvature. We do this by a numerical transfer of curvature between the regions of the diagram. Curvature is subtracted from positive regions and added to some negative regions that neighbour these positive regions. These movements of curvature are purely numerical reassignments and the diagram itself remains unchanged.

For the region Δ', let $c^*(\Delta')$ equal $c(\Delta')$ plus all positive curvature Δ' receives.

Our key lemma is the following, the proof of which will be given in later chapters.

Lemma 1.17. *If* Δ' *is an interior region of* D *such that* $c^*(\Delta') > c(\Delta')$ *then* $c^*(\Delta') \leq 0$.

We now state two more lemmas which will be useful later on.

Lemma 1.18. *For* $1 \leq i \leq 5$, $w_i^2 \neq 1$.

Proof. Let $w_i^2 = 1$ for some i. Let $w_i = g_1 t^{m_1} \dots t^{m_{k-1}} g_k$.
Then $w_i^2 = g_1 t^{m_1} \dots t^{m_{k-1}} g_k g_1 t^{m_1} \dots t^{m_{k-1}} g_k = 1$.
So $g_1 = g_k^{-1}$, $g_2 = g_{k-1}^{-1}, \dots$ and $m_1 = -m_{k-1}$, $m_2 = -m_{k-2}, \dots$.
If k even: $m_{\frac{k}{2}} = -m_{\frac{k}{2}} \Longrightarrow m_{\frac{k}{2}} = 0$: contradiction.
If k odd: $g_{\frac{k+1}{2}} = g_{\frac{k+1}{2}}^{-1} \Longrightarrow g_{\frac{k+1}{2}}^2 = 1$: contradiction by condition (**). □

Lemma 1.19. *Let* Δ *be an interior region of positive curvature in* D*. Then all* g*-vertices and* t*-vertices other than the* v_i *must have degree 2, four of the* v_i *must have degree 3, and the remaining* v_i *must have degree 3, 4 or 5,* $1 \leq i \leq 5$.

Proof. Since $c(3,3,3,3,3,3) = 0$, a region with positive curvature must have degree at most 5. By Lemma 1.15(iii), the region must have degree equal to 5. Therefore, as all the v_i have degree at least 3 by 1.15(ii), all g-vertices and t-vertices other than the v_i have degree 2. Since $c(3,3,3,4,4) = c(3,3,3,3,6) = 0$ there is at most one vertex v_i of degree exceeding 3 and its degree must not exceed 5. □

Remark
$c(3,3,3,3,3) = \frac{\pi}{3}$,
$c(3,3,3,3,4) = \frac{\pi}{6}$,
$c(3,3,3,3,5) = \frac{\pi}{15}$.
These are the only possible values for positive curvature of a region of degree 5.

When considering the distribution of curvature, we may not have complete information about regions which neighbour positive regions. We distribute curvature in steps i.e. we send curvature from a positive region Δ_1 into Δ' at step one, we send curvature from a positive region Δ_2 into Δ' at step two and so on.

We define *marking* to be an assignment of natural numbers ≥ 2 to all vertices of our diagram. The value assigned to a vertex is the *marked degree* of the vertex. At step 0 all vertices have marked degree 2. Then at each step we increase the marked degree of certain vertices, ensuring this value never exceeds the actual degree of the vertex. Let $d(v, n)$ be the marked degree of vertex v at step n.

If a region Δ has vertices $u_1, \dots, u_m$, we define the *marked curvature* of Δ at step n to be $c(\Delta, n) = c(d(u_1, n), \dots, d(u_m, n))$.

We perform the gradual transfer of curvature in steps from the positive regions into neighbouring negative regions, while simultaneously changing the marking of the diagram. At step n we check the curvature transferred into a region is compensated for by the difference $c(\Delta', n) - c(\Delta', n-1)$ arising from the remarking made at this step.

Remarks

1. We define $c^*(\Delta', n)$ to equal $c(\Delta', n)$ plus any curvature sent into Δ' at each step up to and including n.

2. The marked curvature of a region at any step is an upper bound on the actual curvature. In particular, if n is the final step, $c^*(\Delta', n) \leq 0$ implies $c^*(\Delta') \leq 0$ and so Lemma 1.17 holds for this region.

The following Lemma will be used later on in part of the proof of Lemma 1.17.

Lemma 1.20. *Suppose that at some step n, $c^*(\Delta', n) \leq 0$. Suppose $u_1, \ldots u_k$ are vertices of Δ' such that $d(u_i, n) = 2$, $d(u_i, n+1) > 2$ and suppose at step $n+1$, $x\pi$ of curvature is transferred into Δ'. Then $c^*(\Delta', n+1) \leq 0$ provided that $x - k + 2\Sigma_{i=1}^{k} \frac{1}{d(u_i, n+1)} \leq 0$.*

Proof. At step n, $c^*(\Delta', n) = c(d_1, \ldots, d_m) + p\pi = (2-m)\pi + 2\pi\Sigma_{i=1}^{m} \frac{1}{d_i} + p\pi \leq 0$, where $p\pi$ is the total curvature distributed to Δ' at steps $m \leq n$. Now at step $n+1$, let $a_i := d(u_i, n+1) > 2$ and distribute a further $x\pi$.
Then $c^*(\Delta', n+1)$ satisfies:

$$\begin{aligned}
c^*(\Delta', n+1) &\leq c(d_1, \ldots, d_m, a_1, \ldots, a_k) + p\pi + x\pi \\
&= (2 - (m+k))\pi + 2\pi\Sigma_{i=1}^{m} \frac{1}{d_i} + 2\pi\Sigma_{i=1}^{k} \frac{1}{a_i} + p\pi + x\pi \\
&= (2-m)\pi + 2\pi\Sigma_{i=1}^{m} \frac{1}{d_i} + p\pi - k\pi + 2\pi\Sigma_{i=1}^{k} \frac{1}{a_i} + x\pi \\
&= c(d_1, \ldots, d_m) + p\pi - k\pi + 2\pi\Sigma_{i=1}^{k} \frac{1}{a_i} + x\pi \\
&= c^*(\Delta', n) - k\pi + 2\pi\Sigma_{i=1}^{k} \frac{1}{a_i} + x\pi.
\end{aligned}$$

Therefore, $x - k + 2\Sigma_{i=1}^{k} \frac{1}{a_i} = x - k + 2\Sigma_{i=1}^{k} \frac{1}{d(u_i, n+1)} \leq 0 \implies c^*(\Delta', n+1) \leq 0$. $\square$

Remark

Given our equation $r(t)$, let D be the diagram whose construction is described above. Let Δ be an interior region of D with positive curvature. As all g-vertices and t-vertices other than the v_i in this region have degree 2, for reasons of presentation we may represent any line segment which gives us a w_i with one vertex labelled with the corresponding w_i. We call such a vertex a *w-vertex*. Note that a w-vertex is actually a subgraph containing a chain of vertices and when calculating curvature we convert the w-vertices back to g-vertices and t-vertices. We refer to the line segment between two v_i containing one of the w_i as an *edge*. The region Δ of Figure 1.2 is then represented by the following figure.

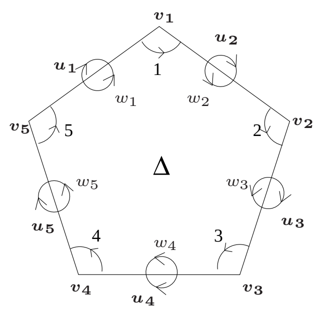


Figure 1.3: A typical region of D of positive curvature

$d(v_i) \geq 3,\ 1 \leq i \leq 5$ and $d(u_i) = 2,\ 1 \leq i \leq 5$.

Definition 1.21. *A vertex v is called a* split *if $d(v) > 2$ and v is not a v_i. Note that such a vertex must be either a g-vertex or a t-vertex. If such a vertex is found within an edge beginning with v_i and ending with v_{i+1}, we say* the edge $(i, i+1)$ splits.

Note that a split may not be found within a positive region or the degree of the region would exceed five, a contradiction by Lemma 1.15(iii).

Definition 1.22. *[4] Suppose some vertex v of D has label xy and we know that $x = 1$. Then we can change D by a* bridge move *as shown in Figure 1.4.*

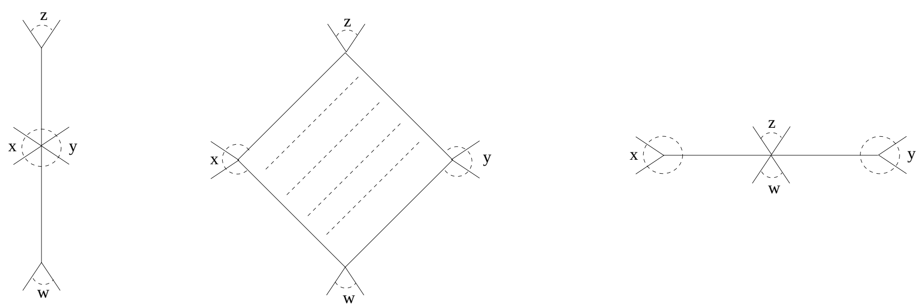


Figure 1.4: Bridge moves

Lemma 1.23. *If $w_i = w_j$, $i \neq j$, no vertex label of D can have as a proper sublabel $w_i w_j^{-1}$ or $w_i^{-1} w_j$. Also, if $w_i = w_j^{-1}$, $i \neq j$, no vertex label of D can have as a proper sublabel $w_i w_j$.*

Proof. If we were to allow such a label then we could perform bridge moves to increase the number of degree 2 vertices without changing the number of regions, which is a contradiction. See Figure 1.5 for an example of such a move. □

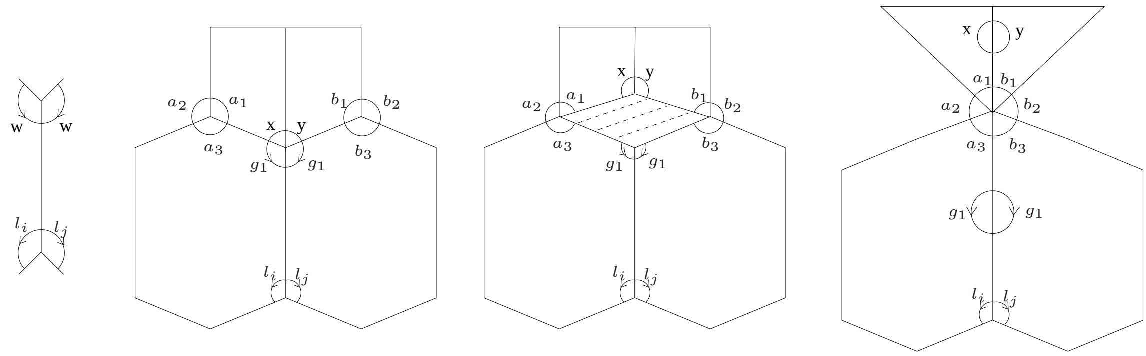


Figure 1.5: Performing bridge moves on a w-vertex

If $k \geq 6$, there are no regions of positive curvature and so Lemma 1.17 holds immediately. The completion of the proof of Theorem 1.10 follows the same argument as the completion of the proof of the main theorem, which will be dealt with in Chapter 4.

Chapters 2 and 3 prove Lemma 1.17 for Theorem 1.12 and Chapter 4 completes the proof of Theorem 1.12.

Chapter 2

Theorem 1.12 cases (I) 1-3

This chapter shall be concerned with the proof of Lemma 1.17 for the cases (I) (1)-(3) in Theorem 1.12, that is, the cases for which $r(t)$ is of the form (A1)-(A4).

Section 2.1 does preliminary work needed for the remainder of the chapter. Sections 2.2, 2.3 and 2.4 examine cases (I) (1), (2) and (3) respectively of Theorem 1.12, proving Lemma 1.17 in each case.

2.1 Positive regions

In this chapter we are assuming each w_i is equal to w_j or w_j^{-1} for $i \neq j$. Each vertex v_i has degree at least 3 and there are four different possible labellings for each corner of these vertices just outside of the region itself (each w within $r(t)$ matches up in the diagram with one of the other w's from $r(t)$ in order to avoid the situation of Lemma 1.15(i)). There are therefore a large number of potential combinations for labels, which have been worked out using a computer. The labels which give rise to a contradiction of the assumptions can be discarded. For example, reading around, the label 444 (i.e. the degree of the vertex is 3 and each corner is labelled in the same direction with 4 in place of l_4) would yield a contradiction as this would imply $l_4 + l_4 + l_4 = 3l_4 = 0 \Longrightarrow l_4 = 0$. Another example of a contradiction would be if we had the label 213 at vertex 1 and $13\bar{5}$ at vertex 3, as this would mean $l_1 + l_2 + l_3 = 0$ and $l_1 + l_3 - l_5 = 0$, which would imply that $l_2 = -l_5$. An example of a region whose labels do not give a contradiction is as follows.

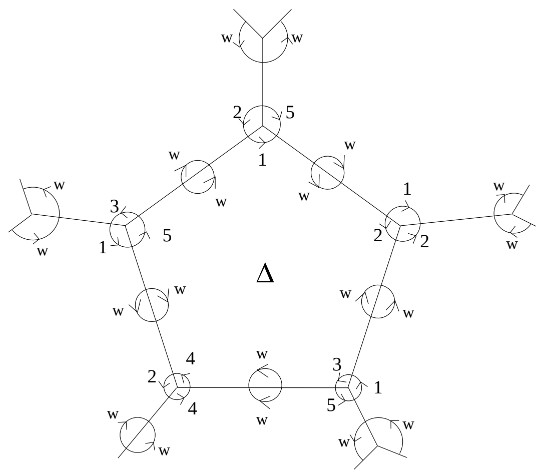


Figure 2.1: Example of a region of positive curvature

Using the notation introduced in Chapter 1, that $l(v_i)$ is the label of the vertex whose corner within the region is labelled i, then $l(v_1) = 21\bar{5}$, $l(v_2) = \bar{1}22$, $l(v_3) = 135$, $l(v_4) = 442$, $l(v_5) = 153$. This gives $l_1 + l_2 - l_5 = -l_1 + 2l_2 = l_1 + l_3 + l_5 = 2l_4 + l_2 = 0$, which does not yield any sort of contradiction.

Two different regions whose labels do not give a contradiction are known as *compatible regions*. Two regions which are not compatible may not both appear in the diagram.

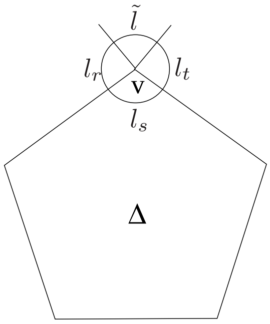
Possible labels

Lemma 2.1. *Let Δ be an interior region of positive curvature in diagram D. Then there is at most one v_i whose label involves t-powers other than the l_i (so involves at least one $m_{u,v}$) and, in this case, $d(v_i) = 4$ or 5. The two possibilities are shown in Figure 2.2.*

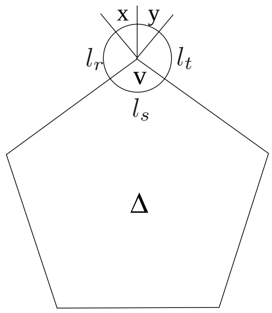
Proof. Let Δ be an interior region of positive curvature and let v be the vertex of Δ shown in Figure 2.2. Then, clearly $\{l_r, l_t\} \subseteq \{l_1^{\pm 1}, \dots, l_5^{\pm 1}\}$, for otherwise an edge would split and $d(\Delta) > 5$. $\square$

If such a vertex as mentioned in Lemma 2.1 exists, that is, a v_i containing an $m_{u,v}$ as a corner label, then we call this an *$\tilde{l}$-vertex* and let $\tilde{l}$ represent one of the $m_{u,v}$ which is a label of this vertex.

Let Δ be an interior region such that $c(\Delta) > 0$. Suppose that Δ contains an $\tilde{l}$-vertex v. Since $4 \leq d(v) \leq 5$, Δ is shown in Figure 2.2.



(a) $d(v) = 4$



(b) $d(v) = 5$

Figure 2.2: $\tilde{l}$-vertex in Δ (at least one of x, y is an $\tilde{l}$)

If the region Δ is given by Figure 2.2 (a) then add $c(\Delta) \leq c(3,3,3,3,4) = \frac{\pi}{6}$ to the region Δ' which contains the label $\tilde{l}$. If Δ is given by Figure 2.2 (b) then add $c(\Delta) \leq c(3,3,3,3,5) = \frac{\pi}{15}$ to the region Δ' which contains the label x if $x = \tilde{l}$ or y if $y = \tilde{l}$.

Lemma 2.2. *Assume at step n all $\tilde{l}$-vertices have marked degree 2 in Δ' and at step $n+1$ we mark all $\tilde{l}$-vertices of Δ' with their actual degree.*
If $c^*(\Delta', n) \leq 0$ then $c^*(\Delta', n+1) \leq 0$.

Proof. Let us assume we have such a region of positive curvature with a label involving an $\tilde{l}$ as in Figure 2.2 (a). Then the region Δ' which contains $\tilde{l}$ has degree at least 6 as it must also contain the five v_i, each of which have degree at least 3. If we send the $c(\Delta) \leq c(3,3,3,3,4) = \frac{\pi}{6}$ from Δ, into this region, then the curvature is successfully compensated for as $c(\Delta') \leq c(3,3,3,3,3,4) = -\frac{\pi}{6}$, provided the region receives this one lot of curvature only. If the degree of the vertex is 5 as in 2.2 (b) then it is possible to send up to two lots of curvature in across the same $\tilde{l}$-vertex. For example, if $x = \tilde{l}$ and $y \neq \tilde{l}$ then in principal, Δ'_x (the region containing the corner label x) may receive positive curvature through the $\tilde{l}$-vertex from both Δ and Δ_t (the region containing the corner label l_t). In this case however, the curvature is equal to $\frac{\pi}{15}$ each time and $\frac{2\pi}{15} < \frac{\pi}{6}$, so we can assume from now on that $\frac{\pi}{6}$ is being sent in and the degree of the vertex is 4. Let us assume now that $c^*(\Delta', n) \leq 0$, and at step $n+1$ we mark k $\tilde{l}$-vertices, each with their actual degree which must be at least 4. So at most $\frac{k\pi}{6}$ is distributed to Δ' at step $n+1$. Then by Lemma 1.20, because we have that $(\frac{1}{6} - 1 + \frac{2}{4})k < 0$, we know that $c^*(\Delta', n+1) \leq 0$. This completes the proof of the lemma. □

Lemma 2.2 tell us that if we are able to distribute curvature successfully, i.e. in order for Lemma 1.17 to be satisfied, for all regions while assuming no $\tilde{l}$-vertices, then we are able to distribute curvature successfully for all positive regions. Therefore, from now we will assume that there are no positive regions involving such an $\tilde{l}$-vertex. So for the rest of this chapter we need only consider positive regions whose vertices are either w-vertices or v_i whose labels involve only the l_i.

Computing regions

In order to compute the different labels for each region, it is necessary to check every possible combination of matching up the w_i's. We then compute every possible label assuming each of the following in turn (see Figure 2.3).

(i) $d(v_i) = 3,\ (1 \leq i \leq 5)$,

(ii) $d(v_1) \geq 4,\ d(v_i) = 3$ for $i \neq 1$,

(iii) $d(v_2) \geq 4,\ d(v_i) = 3$ for $i \neq 2$,

(iv) $d(v_3) \geq 4,\ d(v_i) = 3$ for $i \neq 3$,

(v) $d(v_4) \geq 4,\ d(v_i) = 3$ for $i \neq 4$,

(vi) $d(v_5) \geq 4,\ d(v_i) = 3$ for $i \neq 5$,

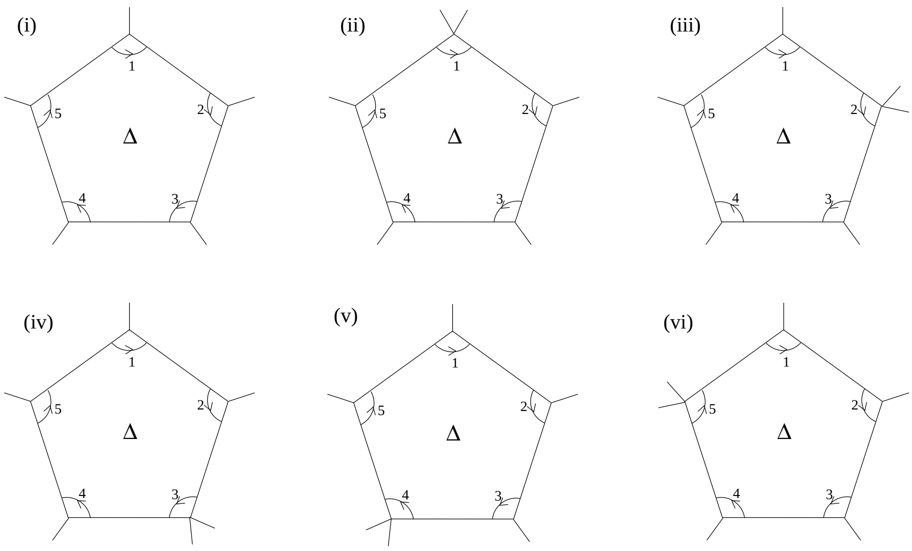


Figure 2.3: Possible vertex degrees

In the calculation we do not specify every combination of label for a vertex of degree exceeding 3 as there would be far too many. So, for example, if $d(v_1) > 3$ and we know part of the label for v_1 is $\bar{3}12$, we would write $l(v_1) = \bar{3}12\omega$ where ω is in place of either one or two numbers, depending of whether $d(v_1) = 4$ or 5 respectively. Using the methods mentioned above, we discard any labelling which gives a contradiction and list those which do not. We also use computational methods to list which region labellings are compatible and also to find out what so-called *type* each positive region is, which we define next.

Types of regions

Let Δ be a region of positive curvature. Then it must be one and only one of the following types:

(Type 1) Δ has a neighbour Δ' with two edges that split as shown in Figure 2.4.

(Type 2) The above does not hold but one of Δ's vertices has degree greater than 3, and one of the two neighbouring regions containing this vertex has a split as shown in Figure 2.4.

(Type 3) Δ is neither type 1 nor type 2.

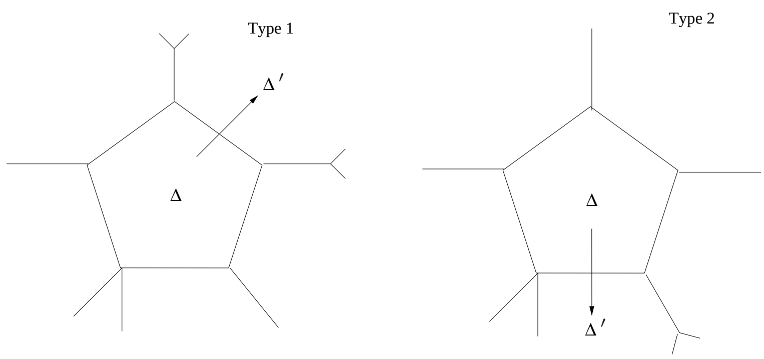


Figure 2.4: Type 1 and type 2 regions

For example, Figure 2.1 is of type 1 and has three neighbouring regions that fit the criteria of the definition.

From the definition, regions of type 3 are of one of the following forms, up to symmetry.

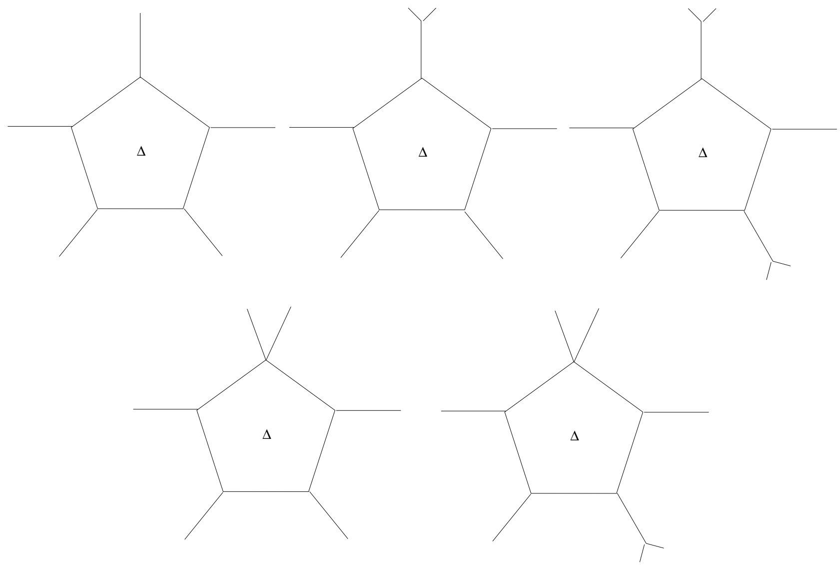


Figure 2.5: Type 3 regions

Receiving curvature from types 1 and 2 only

If a region Δ is of type 1, $c(\Delta) \leq \frac{\pi}{3}$ and this can be sent into Δ' as shown in Figure 2.4. If this is the only curvature the region receives, $c^*(\Delta') \leq c(3,3,3,3,3,3,3) + \frac{\pi}{3} = -\frac{\pi}{3} + \frac{\pi}{3} = 0$. If Δ is of type 2, $c(\Delta) \leq \frac{\pi}{6}$ and, again, send curvature into Δ' as shown in Figure 2.4. If this is the only curvature the region receives, $c^*(\Delta') \leq c(3,3,3,3,3,4) + \frac{\pi}{6} = -\frac{\pi}{6} + \frac{\pi}{6} = 0$. We need to see what happens now if a region Δ' receives curvature across more than one edge, from regions of type 1 and 2 only.

Assume a region Δ' receives curvature across two edges (see Figure 2.6):

1. Let both regions be of type 1. Then the remaining three edges all split: $c^*(\Delta') \leq c(3,3,3,3,3,3,3,3) + 2(\frac{\pi}{3}) = -\frac{2\pi}{3} + 2(\frac{\pi}{3}) = 0$.
2. Let one be of type 1 and the other of type 2, where the type 2 crossing shares one of the splits of the type 1 crossing: $c^*(\Delta') \leq c(3,3,3,3,3,3,4) + \frac{\pi}{3} + \frac{\pi}{6} = -\frac{\pi}{2} + \frac{\pi}{2} = 0$.
3. The same as 2, except the type 2 uses a third split along the remaining edge: $c^*(\Delta') \leq c(3,3,3,3,3,3,3,4) + \frac{\pi}{3} + \frac{\pi}{6} = -\frac{5\pi}{6} + \frac{\pi}{2} < 0$.
4. Both of type 2, sharing neither the vertex of degree > 3 nor the split: $c^*(\Delta') \leq c(3,3,3,3,3,4,4) + 2(\frac{\pi}{6}) = -\frac{2\pi}{3} + \frac{\pi}{3} < 0$.
5. Both of type 2, sharing the degree > 3 vertex: $c^*(\Delta') \leq c(3,3,3,3,3,3,4) + 2(\frac{\pi}{6}) = -\frac{\pi}{2} + \frac{\pi}{3} < 0$.

6. Both of type 2, sharing the split: $c^*(\Delta') \le c(3,3,3,3,4,4) + 2(\frac{\pi}{6}) = -\frac{\pi}{3} + \frac{\pi}{3} = 0$.

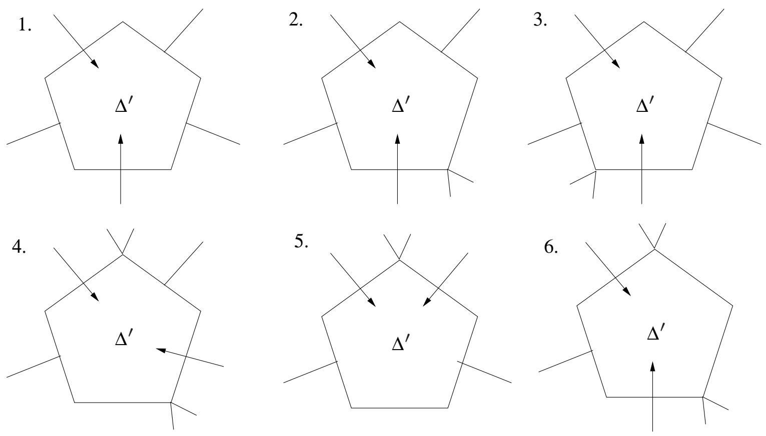


Figure 2.6: Receiving curvature across two edges

Assume a region Δ' receives curvature across three edges (see Figure 2.7):

1. All of type 2, in which case two must share a degree > 3 vertex and two must share a split: $c^*(\Delta') \le c(3,3,3,3,3,4,4) + 3(\frac{\pi}{6}) = -\frac{2\pi}{3} + \frac{\pi}{2} < 0$.

2. One of type 1, two of type 2, in which case each of the type 2 must share one of the type 1 splits and both must share the same degree > 3 vertex.

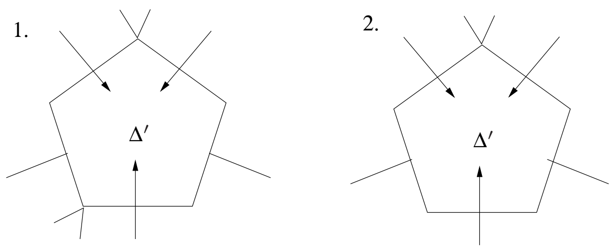


Figure 2.7: Receiving curvature across three edges

This last case is the only one which may cause problems when a region Δ' receives curvature from regions of types 1 and 2 only, as $c(\Delta') \le -\frac{\pi}{2}$ and as much as $\frac{2\pi}{3}$ could be sent in. This case depends upon the existence of a region with all its vertices of degree 3 being able to appear with regions that are of type 2, in which all the relevant labels match up. Note that the two regions of type 2 must be distinct in order for the labels to match up correctly. This is because any specific region of type 2 sends curvature across the same edge each time but the Δ' in question must receive curvature from type 2 regions across different edges. It is important to check if this situation can occur before moving on to proving Lemma 1.17 for a region receiving curvature from regions of type 3.

2.2 Case 1

2.2.1 $r(t) = wt^{l_1}wt^{l_2}wt^{l_3}wt^{l_4}wt^{l_5}$ (A1)

Let $r(t) = wt^{l_1}wt^{l_2}wt^{l_3}wt^{l_4}wt^{l_5}$ and let Δ be an interior region of the diagram D of positive curvature. As mentioned previously, the w-vertices all have degree 2 and the v_i must either all have degree 3 or have four vertices of degree 3 and one vertex of degree greater than 3.

Let us first assume that all vertices are degree 3. It can be observed from the following figure that every region sharing an edge with Δ has at least two splits. This comes from the fact that we cannot have $w^2 = 1$ by Lemma 1.18.

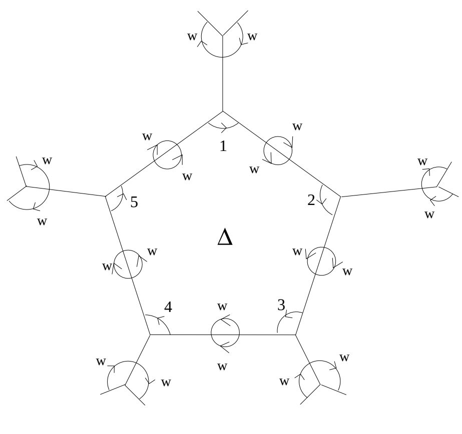


Figure 2.8: A region with all vertices of degree 3 in case A1

Now let us assume there are four vertices of degree 3 and one of degree greater than 3. If we observe the following figure, we see that the three neighbouring regions which do not contain the vertex of degree greater than 3 have at least two splits.

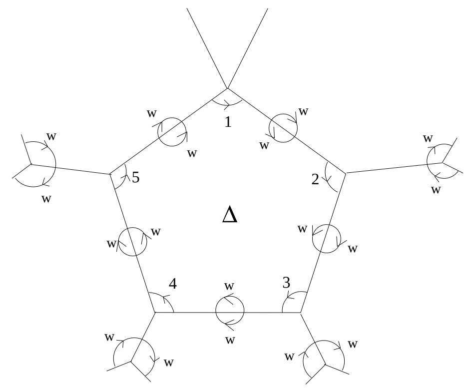


Figure 2.9: A region with one vertex of degree > 3 in case A1

This means that all regions of positive curvature in this case are of type 1 and so, by the argument at the end of Section 2.1, all curvature is successfully compensated for.

2.2.2 $r(t) = wt^{l_1}wt^{l_2}w^{-1}t^{l_3}wt^{l_4}w^{-1}t^{l_5}$ (A4)

Let $r(t) = wt^{l_1}wt^{l_2}w^{-1}t^{l_3}wt^{l_4}w^{-1}t^{l_5}$. This case does not have the same nice properties as the previous, and so the first thing to do is to work out all the different possible labellings of a region of positive curvature using computation methods as mentioned in Section 2.1, and decide which of the three types each of them is.

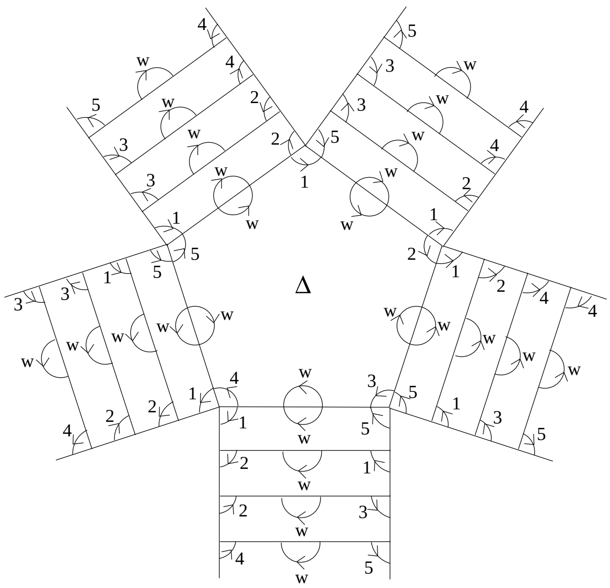


Figure 2.10: All possible labellings – A4

Figure 2.10 shows all possible ways in which the w's can match up.

We read around each vertex from left to right to obtain a possible label. For example, if $d(v_1) = 3$, $l(v_1) \in \{\bar{2}1\bar{5}, \bar{2}13, \bar{2}1\bar{3}, \bar{2}15, 21\bar{5}, 213, \ldots, 415\}$. If our choice for $l(v_1)$ ends in a 3 say, $l(v_2)$ must begin with a 2, and so on. We allow a computer to find all such combinations for each vertex and check if it is a valid labelling, that is, it does not give a contradiction. Figure 2.1 gives us a valid labelling in this case which does not give any contradiction.

The following results were obtained and we refer back to Figure 2.3 to consider each case.
In case (i) there are 8 possible labellings.
In case (ii) there are 48 possible labellings.
In case (iii) there are 87 possible labellings.
In case (iv) there are 94 possible labellings.
In case (v) there are 94 possible labellings.
In case (vi) there are 87 possible labellings.
A full list of possible labellings may be viewed in [24] (a hard copy of which is attached to this thesis for the convenience of the reader), in which the type of each region is indicated also.

We note that when we refer to 1_3, for example, we are referring to region number 1 in [24] for which all the v_i are of degree 3. Region numbers without a subscript refer to the regions which have one v_i of degree > 3.

A4 sendings

The way in which we send curvature for the type 3 regions is as follows.

Δ	Edge sent across in Δ	Δ	Edge sent across in Δ	Δ	Edge sent across in Δ	Δ	Edge sent across in Δ
19	$(3,4)$	177	$(5,1)$	225	$(5,1)$	288	$(5,1)$
34	$(3,4)$	178	$(1,2)$	229	$(1,2)$	290	$(5,1)$
35	$(3,4)$	181	$(1,2)$	251	$(5,1)$	295	$(5,1)$
44	$(3,4)$	185	$(1,2)$	254	$(5,1)$	298	$(1,2)$
173	$(5,1)$	203	$(1,2)$	286	$(5,1)$	310	$(1,2)$
174	$(1,2)$	206	$(5,1)$	287	$(5,1)$	314	$(1,2)$

The way in which we send curvature for certain type 1 and 2 regions is as follows.

Δ	Edge sent across in Δ	Δ	Edge sent across in Δ	Δ	Edge sent across in Δ	Δ	Edge sent across in Δ
52	$(2,3)$	124	$(1,2)$	297	$(3,4)$	367	$(5,1)$
64	$(1,2)$	125	$(1,2)$	323	$(3,4)$	379	$(5,1)$
68	$(1,2)$	126	$(1,2)$	327	$(5,1)$	381	$(5,1)$
84	$(1,2)$	155	$(3,4)$	345	$(5,1)$	398	$(5,1)$
88	$(1,2)$	191	$(3,4)$	357	$(5,1)$	400	$(5,1)$
90	$(1,2)$	213	$(3,4)$	365	$(5,1)$	401	$(5,1)$
104	$(2,3)$	296	$(3,4)$				

For the remaining type 2 regions, consider the vertex clockwise from the degree > 3 vertex. If this vertex gives a split, send positive curvature across the edge between this vertex and the degree > 3 vertex. Otherwise, the vertex anticlockwise from the degree > 3 vertex must give a split and so send positive curvature across the edge between this vertex and the degree > 3 vertex.

For the remaining type 1 regions, consider each pair of adjacent vertices, starting from v_1 and v_2, moving clockwise and ignoring any pair where one of the vertices has degree > 3. When the first pair of vertices is found where both give splits, send positive curvature between these two vertices.

We claim that, under the described sendings, Lemma 1.17 holds.

Proof of Lemma 1.17 for A4

It can also be viewed in [24] which regions of different labellings are compatible. We can therefore check if the situation of Figure 2.7 (2) can occur, which may cause a problem with regards to sending in three lots of positive curvature from regions of types 1 and 2 only. For this to happen we first require a region of type 1 in which all vertices have degree 3 such that it can appear with two different regions of type 2. Looking at the list shows us that there are only two region with all vertices of degree 3 appearing with a region of type 2, 8_3 which goes with 27 and 3_3 which goes with 42, but there are no type 1 regions appearing with two different regions of type 2. Therefore this situation cannot happen. So any regions receiving curvature from regions of types 1 and 2 only satisfy Lemma 1.17.

We now need to study regions of type 3 and for each such a region Δ, determine a neighbour whose curvature can compensate for the curvature of Δ, and then check this neighbour can compensate for any further curvature being sent into it.

For this case, there are 24 regions of type 3, which can be viewed in [24], and the curvature of each is at most $\frac{\pi}{6}$.

Recall that we are working with equations of the given form up to inversion and cyclic permutation so consider $r(t) = wt^{l_1}wt^{l_2}w^{-1}t^{l_3}wt^{l_4}w^{-1}t^{l_5}$.
Take the inverse of w to obtain the following: $t^{-l_5}wt^{-l_4}w^{-1}t^{-l_3}wt^{-l_2}w^{-1}t^{-l_1}w^{-1}$.
Cyclically permute this to obtain the following: $w^{-1}t^{-l_1}w^{-1}t^{-l_5}wt^{-l_4}w^{-1}t^{-l_3}wt^{-l_2}$.
But this is of the same form as $r(t)$ and so we obtain a symmetry from $(l_1, l_2, l_3, l_4, l_5)$ to $(l_1, l_5, l_4, l_3, l_2)$.

This means it is not necessary to find ways of allocating curvature for all the 24 regions of type 3 as the symmetry will cause some repetition. Once we have paired the regions so they are symmetrically equivalent, pick one of the symmetries, and we are only required to allocate curvature to both symmetries in a pair if the two symmetries are regions that may appear at the same time. This happens with 3 of our 12 pairings so we require allocation of curvature for 15 different regions.

The following table shows all regions of type 3 and how they pair up in symmetries. The 15 regions for which curvature needs to be allocated are highlighted.

	$l(v_1)$	$l(v_2)$	$l(v_3)$	$l(v_4)$	$l(v_5)$
34	$\boldsymbol{21\bar{3}\omega}$	$\boldsymbol{\bar{4}2\bar{4}}$	$\boldsymbol{\bar{5}3\bar{1}}$	$\boldsymbol{\bar{2}44}$	$\boldsymbol{353}$
44	$\bar{4}15\omega$	424	$33\bar{5}$	$\bar{1}4\bar{2}$	$\bar{3}5\bar{3}$
19	$\bar{4}13\omega$	224	$33\bar{5}$	$\bar{1}4\bar{2}$	$\bar{3}5\bar{3}$
35	$\boldsymbol{41\bar{3}\omega}$	$\boldsymbol{\bar{4}2\bar{4}}$	$\boldsymbol{\bar{5}3\bar{1}}$	$\boldsymbol{\bar{2}44}$	$\boldsymbol{355}$
177	$\boldsymbol{413}$	$\boldsymbol{22\bar{4}}$	$\boldsymbol{\bar{5}3\bar{1}\omega}$	$\boldsymbol{\bar{2}4\bar{2}}$	$\boldsymbol{\bar{3}55}$
295	$\boldsymbol{413}$	$\boldsymbol{22\bar{4}}$	$\boldsymbol{\bar{5}3\bar{5}}$	$\boldsymbol{\bar{1}4\bar{2}\omega}$	$\boldsymbol{\bar{3}55}$
178	$\boldsymbol{41\bar{3}}$	$\boldsymbol{\bar{4}22}$	$\boldsymbol{13\bar{1}\omega}$	$\boldsymbol{\bar{2}4\bar{2}}$	$\boldsymbol{\bar{3}55}$
251	$\bar{4}13$	$22\bar{4}$	$\bar{5}3\bar{5}$	$\bar{1}41\omega$	$55\bar{3}$
181	$\boldsymbol{21\bar{3}}$	$\boldsymbol{\bar{4}2\bar{4}}$	$\boldsymbol{\bar{5}3\bar{1}\omega}$	$\boldsymbol{\bar{2}44}$	$\boldsymbol{353}$
290	$\bar{4}15$	424	$33\bar{5}$	$\bar{1}4\bar{2}\omega$	$\bar{3}5\bar{3}$
185	$\boldsymbol{41\bar{3}}$	$\boldsymbol{\bar{4}2\bar{4}}$	$\boldsymbol{\bar{5}3\bar{1}\omega}$	$\boldsymbol{\bar{2}44}$	$\boldsymbol{355}$
286	$\bar{4}13$	224	$33\bar{5}$	$\bar{1}4\bar{2}\omega$	$\bar{3}5\bar{3}$
203	$\boldsymbol{215}$	$\boldsymbol{424}$	$\boldsymbol{333\omega}$	$\boldsymbol{244}$	$\boldsymbol{353}$
314	$\boldsymbol{215}$	$\boldsymbol{424}$	$\boldsymbol{335}$	$\boldsymbol{444\omega}$	$\boldsymbol{353}$

225	$\mathbf{\bar{4}1\bar{3}}$	$\mathbf{\bar{4}2\bar{4}}$	$\mathbf{\bar{5}35}\boldsymbol{\omega}$	$\mathbf{44\bar{2}}$	$\mathbf{\bar{3}5\bar{3}}$
288	$\mathbf{\bar{4}1\bar{3}}$	$\mathbf{\bar{4}2\bar{4}}$	$\mathbf{\bar{5}33}$	$\mathbf{24\bar{2}}\boldsymbol{\omega}$	$\mathbf{\bar{3}5\bar{3}}$
229	$\mathbf{41\bar{3}}$	$\mathbf{\bar{4}2\bar{4}}$	$\mathbf{\bar{5}35}\boldsymbol{\omega}$	$\mathbf{44\bar{2}}$	$\mathbf{\bar{3}55}$
287	$\bar{4}13$	$22\bar{4}$	$\bar{5}33$	$24\bar{2}\omega$	$\bar{3}5\bar{3}$
174	$\bar{4}1\bar{3}$	$\bar{4}22$	$13\bar{1}\omega$	$\bar{2}4\bar{2}$	$\bar{3}5\bar{3}$
254	$\mathbf{\bar{4}1\bar{3}}$	$\mathbf{\bar{4}2\bar{4}}$	$\mathbf{\bar{5}3\bar{5}}$	$\mathbf{\bar{1}41}\boldsymbol{\omega}$	$\mathbf{55\bar{3}}$
173	$\bar{4}13$	$22\bar{4}$	$\bar{5}3\bar{1}\omega$	$\bar{2}4\bar{2}$	$\bar{3}5\bar{3}$
298	$\mathbf{41\bar{3}}$	$\mathbf{\bar{4}2\bar{4}}$	$\mathbf{\bar{5}3\bar{5}}$	$\mathbf{\bar{1}4\bar{2}}\boldsymbol{\omega}$	$\mathbf{\bar{3}55}$
206	415	424	333ω	244	355
310	**213**	**224**	**335**	$\mathbf{444}\boldsymbol{\omega}$	**353**

For each of these 15 regions, there is a split in one of the edges off it and we are able to send curvature to one side of the split. For each region Δ, the same procedure follows:

1. If the region Δ' we are sending $c(\Delta) \leq \frac{\pi}{6}$ into contains a further split or a vertex of degree > 3, $c(\Delta') \leq c(3,3,3,3,3,4) = -\frac{\pi}{6}$.
2. We therefore assume there are no further splits in Δ' and all the vertices are degree 3, and in each of the 15 cases we obtain a contradiction.
3. Therefore, if Δ' receives curvature from one place only, the negative curvature is fully compensated for.

Following are the figures showing the way positive curvature can be sent in each of the 15 cases.

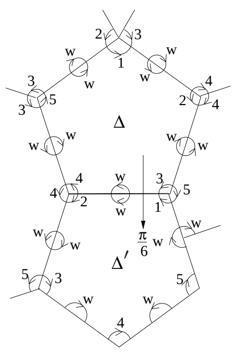


Figure 2.11: no. 34

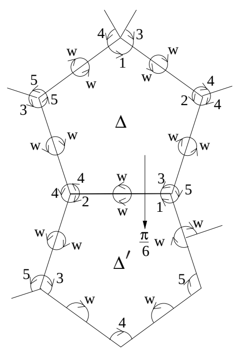


Figure 2.12: no. 35

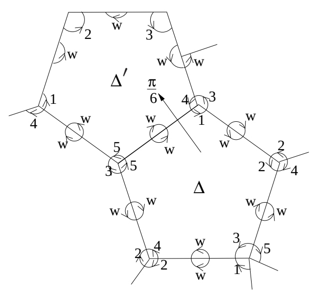


Figure 2.13: no. 177

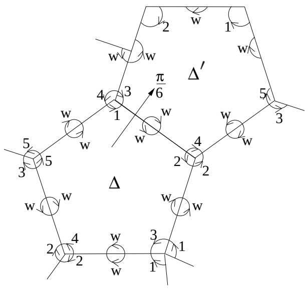


Figure 2.14: no. 178

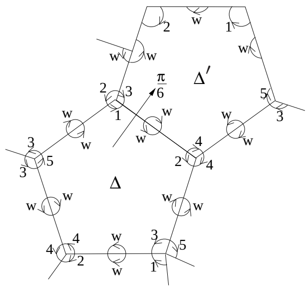


Figure 2.15: no. 181

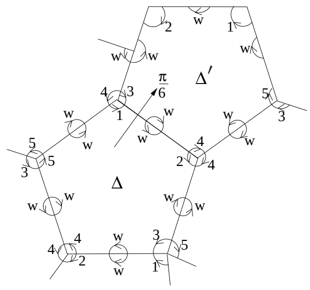


Figure 2.16: no. 185

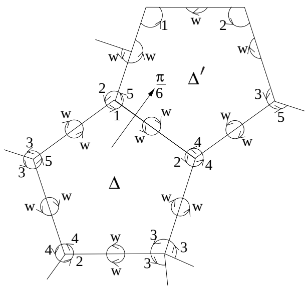


Figure 2.17: no. 203

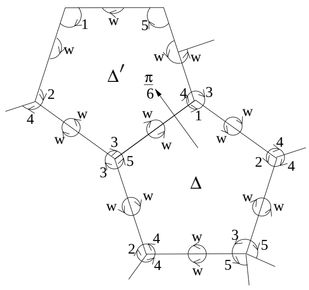


Figure 2.18: no. 225

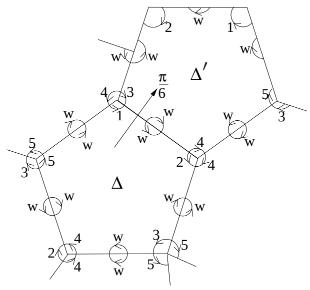


Figure 2.19: no. 229

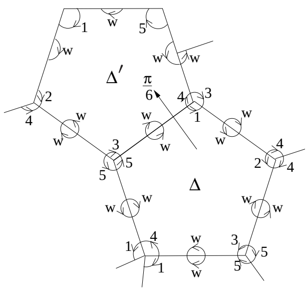


Figure 2.20: no. 254

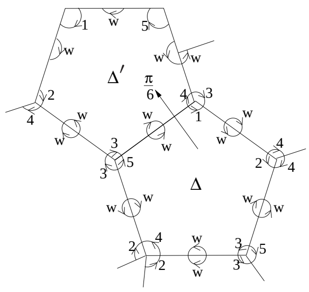


Figure 2.21: no. 288

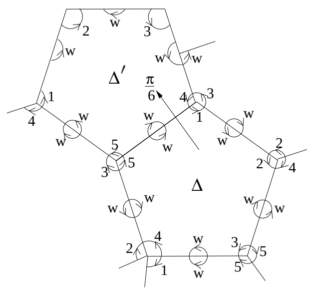


Figure 2.22: no. 295

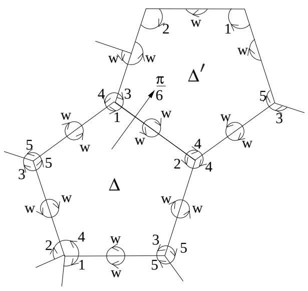


Figure 2.23: no. 298

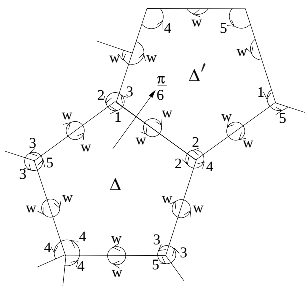


Figure 2.24: no. 310

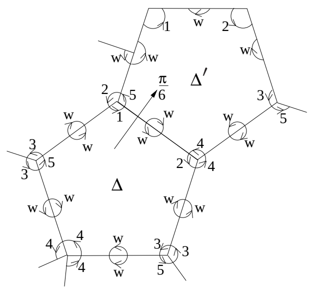


Figure 2.25: no. 314

We define u_i to be the vertex involving l_i in Δ'. In each case, we assume there are no further splits or vertices of degree > 3 in Δ'. Then the remaining labels of Δ' for the following cases must be:

34	$l(u_3) = \bar{5}3\bar{1}$	$l(u_4) = \bar{2}44$	$l(u_5) = 353$
35	$l(u_3) = \bar{5}3\bar{1}$	$l(u_4) = \bar{2}44$	$l(u_5) = 355$
178	$l(u_5) = \bar{3}55$	$l(u_1) = 41\bar{3}$	$l(u_2) = \bar{4}22$
181	$l(u_5) = 353$	$l(u_1) = 21\bar{3}$	$l(u_2) = \bar{2}44$
185	$l(u_5) = 355$	$l(u_1) = 41\bar{3}$	$l(u_2) = \bar{4}2\bar{4}$
203	$l(u_3) = 335$	$l(u_2) = 424$	(1, 2)-split
254	$l(u_2) = \bar{4}2\bar{4}$	$l(u_1) = \bar{4}1\bar{3}$	$l(u_5) = 55\bar{3}$
314	$l(u_3) = 335$	$l(u_2) = 424$	(1, 2)-split

But these all either give a new split or force the existing split to have proper sublabel ww^{-1} or $w^{-1}w$, which is a contradiction by Lemma 1.23.

For 177 and 295, $l(u_1) = \bar{1}45$ or $\bar{1}4\bar{5}$, but then we cannot complete u_2 with degree 3 without causing a split. For 310, $l(u_1) = 15\bar{2}$ or $15\bar{4}$, but then we cannot complete u_5 with degree 3 without causing a split. For 229 and 298 we cannot complete u_5 with degree 3 without causing a split and for 225 and 288 we cannot complete u_2 with degree 3 without causing a split. These all give a contradiction.

Therefore, there is sufficient negative curvature if Δ' only receives curvature across one edge. We now need to check what happens if Δ' receives curvature across more than one edge.

It is worth noting that if a type 1 or type 2 region has more than one possible neighbouring region to which we could send curvature, we may pick the one which is most useful to us. For example, if such a region is compatible with one of our type 3 regions and we are able choose it so that the two positive regions send curvature across the same edge, then we have made sure that no Δ' can receive curvature from both these two regions at the same time. An example of this situation is shown in Figure 2.26.

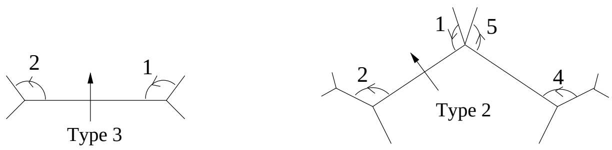


Figure 2.26: For the type 2 region we choose to send curvature to the left of the degree > 3 vertex

In 12 of the cases, the specified Δ' can only receive curvature from Δ as either there are no further regions compatible with Δ (See [24]), compatible regions send curvature across the same edge as Δ, compatible regions send curvature across the split edge in Δ', or the other region does not fit beside Δ due to having different labels. For example, the compatible region 126 for 35 distributes curvature across the $(2,3)$-edge and has label $\bar{5}3\bar{1}$ at vertex 3. However, Figure 2.12 shows that the region across the $(2,3)$-edge of Δ' would need to have the label 535 at vertex 3 and so this region cannot be 126.

Region no.	Compatible regions	Edges crossed by compatible region	Problem
34	124	(5, 1)	Crosses split edge (see Figure 2.11)
35	125	(5, 1)	Crosses split edge
	126	(2, 3)	Does not fit
177	52	(5, 1)	Does not fit
	104	(3, 4)	Crosses split edge
	295, 400, 401	(4, 5)	Crosses same edge
178	296	(2, 3)	Crosses split edge
	297	(4, 5)	Does not fit
181	None		
185	None		
225	88, 288, 381	(3, 4)	Crosses same edge
229	90, 298	(3, 4)	Crosses same edge
254	None		
288	88, 225, 381	(3, 4)	Crosses same edge
295	52	(5, 1)	Does not fit
	104	(3, 4)	Crosses split edge
	177, 400, 401	(4, 5)	Crosses same edge
298	90, 229	(3, 4)	Crosses same edge

Let us now look at the remaining 3 regions, 203, 310 and 314.

If we first consider the region 203, we can see it is compatible with region 314 and also with 64 and 367. However, 203, 314 and 64 all send curvature across (4, 5) so Δ' may not receive from more than one of these regions at a time. The only remaining possibility is for Δ' to receive from 367 as well as 203. The region 367 sends curvature across (2, 3) and forces $d(u_3) > 3$ and a $(2,1)$-split.

Next let us consider the region 314. This region is compatible with region 203 and also with 64 and 367. As above, only region 367 may send curvature to Δ' as well as 314, as all others cross the same edge. Again we have $d(u_3) > 3$ and a $(2, 1)$-split.

For both 203 and 314 at most $\frac{\pi}{3}$ is sent in and 367 introduces a new split and a new degree > 3 vertex so $c(\Delta') \leq c(3, 3, 3, 3, 3, 3, 4) = -\frac{\pi}{2}$. See Figure 2.27.

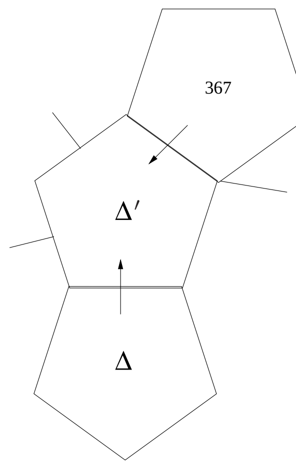


Figure 2.27: Receiving curvature from region $\Delta = 203$ or 314 along with region 367

Lastly, consider 310, which is compatible with 155 and 357. Both 310 and 357 send curvature across (2, 3). So Δ' may only receive curvature from 155 as well as 310. However, this would imply a $(2, 1)$-split with proper sublabel ww^{-1}, which is a contradiction by Lemma 1.23. Therefore, 310 is the only region from which Δ' can receive curvature.

We have checked for this case that all positive curvature is compensated for by negative curvature and so Lemma 1.17 holds for this case.

2.3 Case 2 (A2)

This section is concerned with the proof of Lemma 1.17 for Case (2) in Theorem 1.12. The method will be very similar to that of Subsection 2.2.2 and so this subsection may be referred to for further detail. Unlike Case (1), the theorem only holds in this case under further conditions, which come about due to some regions of positive curvature being unable to be successfully compensated for. We shall begin in the same way as Subsection 2.2.2 and take note of the regions which lead to the conditions later on.

Let $r(t) = wt^{l_1}wt^{l_2}wt^{l_3}wt^{l_4}w^{-1}t^{l_5}$. As in subsection 2.2.2, the first thing to do is to work out all the different possible labellings of a region of positive curvature using computation methods.

The following Figure 2.28 shows all possible ways in which the w's can match up.

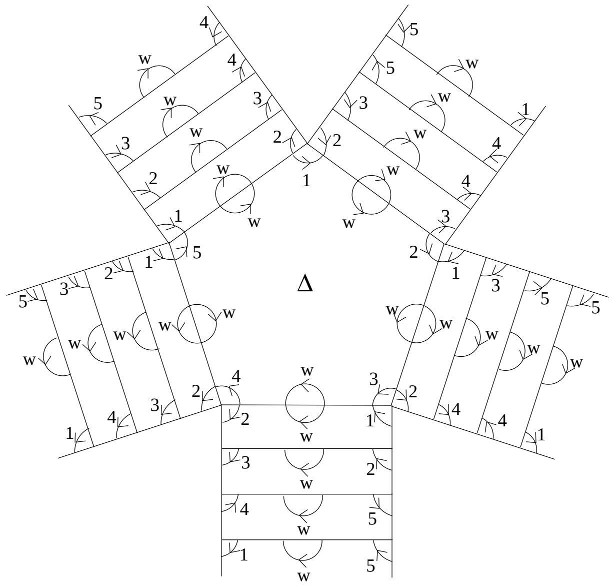


Figure 2.28: All possible labellings – A2

We again use a computer to find all possible labellings and their types, a full list of which can be found in [24].

Observe the cases in Figure 2.3.
In case (i) there are 6 possible labellings.
In case (ii) there are 82 possible labellings.
In case (iii) there are 66 possible labellings.
In case (iv) there are 82 possible labellings.
In case (v) there are 55 possible labellings.
In case (vi) there are 55 possible labellings.

2.3.1 A2 sendings

The way in which we send curvature for the type 3 regions is as follows.

Δ	Edge/vertex sent across in Δ	
1_3	N/A	
3_3	N/A	
4_3	$(5,1)$:	Send $\frac{1}{2}c(\Delta)$
	$(4,5)$:	Send $\frac{1}{2}c(\Delta)$
5_3	$(3,4)$:	Send $\frac{1}{2}c(\Delta)$
	$(4,5)$:	Send $\frac{1}{2}c(\Delta)$
12	$(4,5)$	
16	$(3,4)$	
19	$(3,4)$	
20	$(1,2)$:	When region across $(1,2)$ not positive.
	$(5,1)$:	Otherwise.
43	$(5,1)$:	When region across $(5,1)$ not positive.
	1-vertex:	Otherwise. $l(v_1) = \bar{5}\bar{1}3\bar{2}x$, $x \in \{4, \bar{1}, \bar{5}\}$. Send to region containing x.
44	N/A	
72	N/A	
89	$(3,4)$	
97	$(5,1)$	
99	$(2,3)$:	When this region has a split OR $d(v_2) = 5$ OR region across $(1,2)$ has no splits and $d(u_5) > 3$ in this region.
	$(1,2)$:	When region across $(2,3)$ positive OR $d(v_2) = 4$, region across $(2,3)$ has no splits and this region has a split.
	2-vertex:	Otherwise.
103	$(2,3)$	
106	$(1,2)$	
108	$(5,1)$	
125	$(4,5)$	
127	$(3,4)$	
128	$(5,1)$	
141	$(3,4)$	
146	$(1,2)$:	When this region has a split OR $d(v_2) = 5$ OR region across $(2,3)$ has no splits and $d(u_4) > 3$ in this region.
	$(2,3)$:	When region across $(1,2)$ positive OR $d(v_2) = 4$, region across $(1,2)$ has no splits and this region has a split.
	2-vertex:	Otherwise.

Δ	Edge/vertex sent across in Δ	
147	$(4,5)$	
149	$(3,4)$:	When region across $(3,4)$ not positive.
	3-vertex:	Otherwise. $l(v_3) = \bar{2}1\bar{3}\bar{4}x$, $x \in \{5, \bar{3}, \bar{4}\}$. Send to region containing x.
164	N/A	
166	$(4,5)$	
167	$(2,3)$:	When region across $(2,3)$ not positive.
	$(3,4)$:	Otherwise.
175	$(5,1)$	
212	N/A	
214	$(5,1)$	
232	$(4,5)$:	When this region has a split OR $d(v_4) = 5$ OR $l(v_4) = 442\bar{1}$ and $d(u_3) > 3$ in this region.
	$(3,4)$:	When region across $(4,5)$ does not split and $l(v_4) = 4425$ or $442\bar{5}$.
	4-vertex:	Otherwise.
239	N/A	
240	$(3,4)$	
255	$(4,5)$:	When region across $(3,4)$ is positive OR $l(v_4) = \bar{3}44\bar{5}$.
	$(3,4)$:	Otherwise.
256	$(1,2)$	
257	$(1,2)$	
258	N/A	
292	$(5,1)$	
295	$(2,3)$	
299	$(4,5)$:	When region across $(5,1)$ is positive OR $l(v_5) = 5\bar{1}\bar{4}5$.
	$(5,1)$:	Otherwise.
312	N/A	
327	N/A	
329	$(2,3)$	
331	$(4,5)$:	When this region has a split OR $d(v_5) = 5$ OR $l(v_5) = 55\bar{3}2$ and $d(u_1) > 3$ in this region.
	$(5,1)$:	When region across $(4,5)$ does not split and $l(v_5) = 5542$ or $55\bar{4}2$.
	5-vertex:	Otherwise.

The way in which we send curvature for certain type 1 and 2 regions is as follows.

Δ	Edge sent across in Δ	Δ	Edge sent across in Δ	Δ	Edge sent across in Δ	Δ	Edge sent across in Δ
5	$(4,5)$	82	$(4,5)$	189	$(3,4)$	324	$(3,4)$
11	$(5,1)$	105	$(1,2)$	197	$(4,5)$	330	$(4,5)$
26	$(3,4)$	122	$(3,4)$	209	$(2,3)$	334	$(5,1)$
37	$(1,2)$	129	$(5,1)$	213	$(2,3)$	179	$(5,1)$
42	$(5,1)$	130	$(5,1)$	219	$(3,4)$	235	$(4,5)$
45	$(5,1)$	135	$(3,4)$	220	$(3,4)$	243	$(3,4)$
50	$(1,2)$	138	$(2,3)$	223	$(4,5)$	245	$(3,4)$
51	$(5,1)$	153	$(3,4)$	266	$(4,5)$	260	$(4,5)$
62	$(1,2)$	154	$(2,3)$	307	$(5,1)$	273	$(5,1)$
74	$(5,1)$	171	$(3,4)$	316	$(4,5)$	278	$(4,5)$
75	$(5,1)$	189	$(3,4)$	317	$(4,5)$	298	$(4,5)$

The remaining type 1 and 2 regions are dealt with in the same way as in Subsection 2.2.2

We claim that, under the described sendings, Lemma 1.17 holds.

2.3.2 Proof of Lemma 1.17 for A2

We now have to check is the situation of Figure 2.7 (2). There are two type 1 regions, 2_3 and 6_3, but it can be observed in [24] that these regions are not compatible with any other region. Therefore this situation does not occur and any regions receiving curvature from regions of type 1 and 2 only satisfy Lemma 1.17.

We now move on to the regions of type 3. There are 44 regions of type 3, 40 with curvature at most $\frac{\pi}{6}$ (regions have one vertex of degree > 3) and 4 with curvature $\frac{\pi}{3}$ (regions have all vertices of degree 3).

As before, we can rewrite $r(t) = wt^{l_1}wt^{l_2}wt^{l_3}wt^{l_4}w^{-1}t^{l_5}$ to obtain a symmetry, which in this case is from $(l_1, l_2, l_3, l_4, l_5)$ to $(l_3, l_2, l_1, l_5, l_4)$. Following is the table showing all type 3 regions and their pairings with this symmetry. As before, those we will allocate curvature to are highlighted.

	$l(v_1)$	$l(v_2)$	$l(v_3)$	$l(v_4)$	$l(v_5)$
$\mathbf{1_3}$	$\boldsymbol{\bar{4}1\bar{2}}$	$\boldsymbol{\bar{3}2\bar{5}}$	$\boldsymbol{\bar{1}3\bar{5}}$	$\boldsymbol{\bar{1}42}$	$\boldsymbol{15\bar{3}}$
3_3	$\bar{4}1\bar{3}$	$\bar{4}2\bar{1}$	$\bar{2}3\bar{5}$	$\bar{1}43$	$25\bar{3}$
4_3	$41\bar{3}$	$\bar{4}2\bar{3}$	$\bar{4}3\bar{1}$	$\bar{2}43$	255
$\mathbf{5_3}$	$\boldsymbol{\bar{3}1\bar{5}}$	$\boldsymbol{\bar{1}2\bar{5}}$	$\boldsymbol{\bar{1}35}$	$\boldsymbol{442}$	$\boldsymbol{15\bar{2}}$
12	$\boldsymbol{\bar{3}1\bar{2}\omega}$	$\boldsymbol{\bar{3}2\bar{5}}$	$\boldsymbol{\bar{1}3\bar{5}}$	$\boldsymbol{\bar{1}44}$	$\boldsymbol{35\bar{2}}$
166	$\bar{4}1\bar{3}$	$\bar{4}2\bar{1}$	$\bar{2}3\bar{1}\omega$	$\bar{2}41$	$55\bar{3}$
16	$\boldsymbol{\bar{4}1\bar{3}\omega}$	$\boldsymbol{\bar{4}2\bar{1}}$	$\boldsymbol{\bar{2}35}$	$\boldsymbol{442}$	$\boldsymbol{15\bar{3}}$
214	$41\bar{2}$	$\bar{3}2\bar{5}$	$\bar{1}3\bar{5}\omega$	$\bar{1}43$	255
19	$\boldsymbol{\bar{2}1\bar{3}\omega}$	$\boldsymbol{\bar{4}2\bar{1}}$	$\boldsymbol{\bar{2}3\bar{5}}$	$\boldsymbol{\bar{1}44}$	$\boldsymbol{35\bar{1}}$
175	$\bar{4}1\bar{2}$	$\bar{3}2\bar{5}$	$\bar{1}3\bar{2}\omega$	$\bar{3}41$	$55\bar{3}$
43	$\boldsymbol{\bar{3}15\omega}$	$\boldsymbol{42\bar{3}}$	$\boldsymbol{\bar{4}3\bar{2}}$	$\boldsymbol{\bar{3}42}$	$\boldsymbol{15\bar{2}}$
149	$\bar{2}1\bar{5}$	$\bar{1}25$	$43\bar{1}\omega$	$\bar{2}43$	$25\bar{1}$
44	$\boldsymbol{\bar{4}15\omega}$	$\boldsymbol{42\bar{3}}$	$\boldsymbol{\bar{4}3\bar{2}}$	$\boldsymbol{\bar{3}42}$	$\boldsymbol{15\bar{3}}$
212	$\bar{2}1\bar{5}$	$\bar{1}25$	$43\bar{5}\omega$	$\bar{1}43$	$25\bar{1}$
103	$\bar{2}1\bar{5}$	$\bar{1}2\bar{3}\omega$	$\bar{4}3\bar{1}$	$\bar{2}43$	$25\bar{1}$
106	$\boldsymbol{\bar{3}1\bar{5}}$	$\boldsymbol{\bar{1}2\bar{3}\omega}$	$\boldsymbol{\bar{4}3\bar{2}}$	$\boldsymbol{\bar{3}42}$	$\boldsymbol{15\bar{2}}$
89	$\bar{2}1\bar{5}$	$\bar{1}2\bar{1}\omega$	$\bar{2}35$	443	$25\bar{1}$
108	$\boldsymbol{41\bar{2}}$	$\boldsymbol{\bar{3}2\bar{3}\omega}$	$\boldsymbol{\bar{4}3\bar{2}}$	$\boldsymbol{\bar{3}42}$	$\boldsymbol{155}$
127	$\boldsymbol{\bar{2}15}$	$\boldsymbol{425\omega}$	$\boldsymbol{43\bar{2}}$	$\boldsymbol{\bar{3}44}$	$\boldsymbol{35\bar{1}}$
128	$\bar{2}15$	425ω	$43\bar{2}$	$\bar{3}41$	$55\bar{1}$
97	$41\bar{3}$	$\bar{4}2\bar{1}\omega$	$\bar{2}3\bar{5}$	$\bar{1}42$	155
141	$\boldsymbol{\bar{4}1\bar{2}}$	$\boldsymbol{\bar{3}2\bar{5}\omega}$	$\boldsymbol{\bar{1}35}$	$\boldsymbol{443}$	$\boldsymbol{25\bar{3}}$
99	$\bar{4}1\bar{3}$	$\bar{4}2\bar{1}\omega$	$\bar{2}3\bar{5}$	$\bar{1}43$	$25\bar{3}$
146	$\boldsymbol{\bar{4}1\bar{2}}$	$\boldsymbol{\bar{3}2\bar{5}\omega}$	$\boldsymbol{\bar{1}3\bar{5}}$	$\boldsymbol{\bar{1}42}$	$\boldsymbol{15\bar{3}}$
125	$\bar{4}1\bar{3}$	$\bar{4}25\omega$	$43\bar{1}$	$\bar{2}41$	$55\bar{3}$
147	$\boldsymbol{\bar{3}15}$	$\boldsymbol{42\bar{5}\omega}$	$\boldsymbol{\bar{1}3\bar{5}}$	$\boldsymbol{\bar{1}44}$	$\boldsymbol{35\bar{2}}$
72	$\bar{3}1\bar{5}\omega$	$\bar{1}25$	$43\bar{1}$	$\bar{2}44$	$35\bar{2}$
164	$\boldsymbol{\bar{3}15}$	$\boldsymbol{42\bar{3}}$	$\boldsymbol{\bar{4}3\bar{1}\omega}$	$\boldsymbol{\bar{2}41}$	$\boldsymbol{55\bar{2}}$
20	$\bar{2}1\bar{3}\omega$	$\bar{4}2\bar{3}$	$\bar{4}3\bar{1}$	$\bar{2}43$	$25\bar{1}$
167	$\boldsymbol{\bar{3}1\bar{5}}$	$\boldsymbol{\bar{1}2\bar{5}}$	$\boldsymbol{\bar{1}3\bar{2}\omega}$	$\boldsymbol{\bar{3}42}$	$\boldsymbol{15\bar{2}}$
232	$\boldsymbol{\bar{3}1\bar{5}}$	$\boldsymbol{\bar{1}2\bar{5}}$	$\boldsymbol{\bar{1}35}$	$\boldsymbol{442\omega}$	$\boldsymbol{15\bar{2}}$
331	$41\bar{3}$	$\bar{4}2\bar{3}$	$\bar{4}3\bar{1}$	$\bar{2}43$	255ω
299	$\bar{2}1\bar{5}$	$\bar{1}25$	$43\bar{2}$	$\bar{3}41$	$55\bar{1}\omega$
255	$\boldsymbol{\bar{2}15}$	$\boldsymbol{42\bar{3}}$	$\boldsymbol{\bar{4}3\bar{2}}$	$\boldsymbol{\bar{3}44\omega}$	$\boldsymbol{35\bar{1}}$
256	$\boldsymbol{\bar{2}15}$	$\boldsymbol{425}$	$\boldsymbol{43\bar{2}}$	$\boldsymbol{\bar{3}44\omega}$	$\boldsymbol{35\bar{1}}$
295	$\bar{2}15$	425	$43\bar{2}$	$\bar{3}41$	$55\bar{1}\omega$

257	$\mathbf{\bar{3}1\bar{3}}$	$\mathbf{\bar{4}2\bar{1}}$	$\mathbf{\bar{2}35}$	$\mathbf{444\omega}$	$\mathbf{35\bar{2}}$
329	$41\bar{2}$	$\bar{3}2\bar{5}$	$\bar{1}3\bar{1}$	$\bar{2}41$	555ω
258	$\mathbf{\bar{3}15}$	$\mathbf{42\bar{1}}$	$\mathbf{\bar{2}35}$	$\mathbf{444\omega}$	$\mathbf{35\bar{2}}$
327	$41\bar{2}$	$\bar{3}25$	$43\bar{1}$	$\bar{2}41$	555ω
240	$\bar{2}1\bar{5}$	$\bar{1}25$	$43\bar{2}$	$\bar{3}43\omega$	$25\bar{1}$
292	$\mathbf{\bar{2}15}$	$\mathbf{42\bar{3}}$	$\mathbf{\bar{4}3\bar{2}}$	$\mathbf{\bar{3}42}$	$\mathbf{15\bar{1}\omega}$
239	$\bar{2}1\bar{5}$	$\bar{1}25$	$43\bar{1}$	$\bar{2}43\omega$	$25\bar{1}$
312	$\mathbf{\bar{3}15}$	$\mathbf{42\bar{3}}$	$\mathbf{\bar{4}3\bar{2}}$	$\mathbf{\bar{3}42}$	$\mathbf{15\bar{2}\omega}$

There are twenty-two pairs of symmetries and we are only required to distribute curvature to one of each of these pairs so there are twenty-two regions for which we need to know how to distribute symmetry. Nine of these regions can be dealt with in the same way as those in Subsection 2.2.2 so we shall deal with these first. The figures showing the way positive curvature can be sent in each of these cases are as follows.

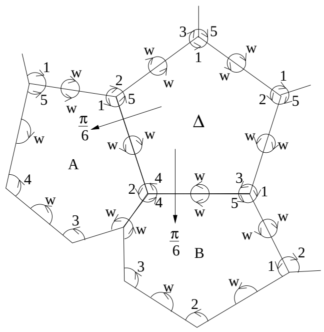


Figure 2.29: no. 5_3

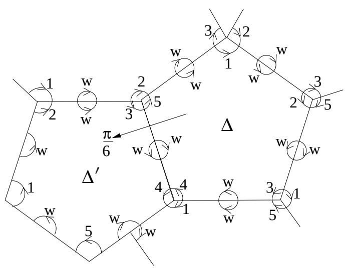


Figure 2.30: no. 12

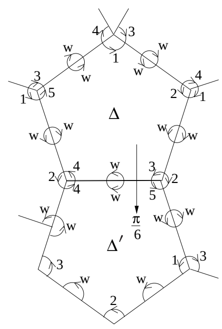


Figure 2.31: no. 16

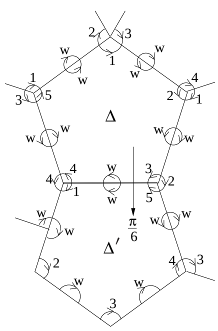


Figure 2.32: no. 19

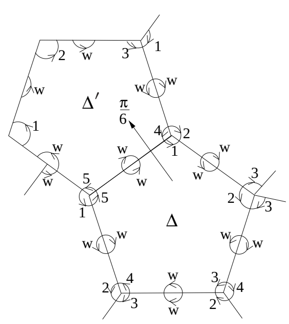


Figure 2.33: no. 108

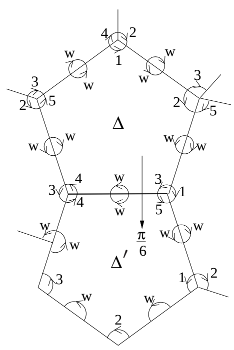


Figure 2.34: no. 141

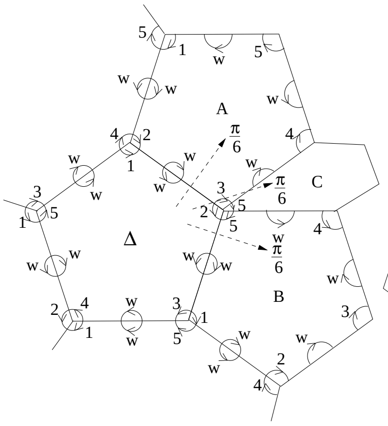


Figure 2.35: no. 146

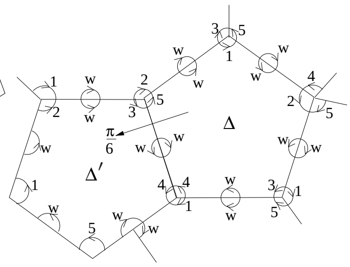


Figure 2.36: no. 147

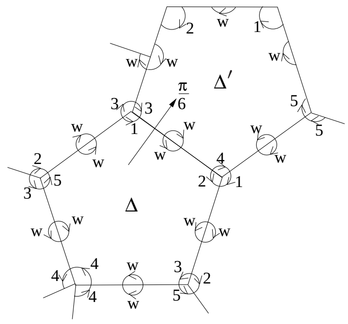


Figure 2.37: no. 257

For each region apart from 5_3 and 146, we assume by way of contradiction that there are no further splits and no vertices of degree > 3 in Δ'. We obtain the following:

12:	$l(u_2) = \bar{4}2\bar{1}$	$l(u_1) = 51\bar{3}$	(5, 1)-split, contradiction
16:	$l(u_1) = \bar{3}15$	$l(u_2) = 424$	(2, 3)-split, contradiction
19:	Cannot complete u_4		
108:	$l(u_3) = 53\bar{1}$	(2, 3)-split, contradiction	
141:	$l(u_1) = \bar{2}1\bar{4}$	(1, 2)-split, contradiction	
147:	$l(u_2) = \bar{4}2\bar{1}$	$l(u_1) = 51\bar{3}$	(5, 1)-split, contradiction
257:	Cannot complete u_5		

Therefore, Δ' has a further split or a further vertex of degree > 3, so $c(\Delta') \leq c(3,3,3,3,3,4) = -\frac{\pi}{6}$ and $c(\Delta) \leq \frac{\pi}{6}$.

Each specified Δ' for these seven regions can receive curvature from one region only as the following table shows:

Region no.	Compatible regions	Edges crossed	Problem with regions
12	147, 330	(3, 4)	Crosses same edge
16	None		
19	None		
108	45	(4, 5)	Crosses same edge
141	189, 334	(4, 5)	Crosses same edge
147	12, 330	(3, 4)	Crosses same edge
257	171, 307	(2, 3)	Crosses split edge

We now look at 5_3 and 146, which are more complicated than the previous regions.

5_3: Neither A nor B can be completed with no further splits or degree > 3 vertices, so $c(A), c(B) \leq c(3,3,3,3,3,4) = -\frac{\pi}{6}$ and we can send $\frac{1}{2}c(\Delta) \leq \frac{\pi}{6}$ to each of A and B.

146: Case I: A is positive. Then A must be 197 and so $l_A(u_3) = 5\bar{2}35\bar{3}$, which causes a $(4,5)$-split in B. So $c(B) \leq c(3,3,3,3,3,5) = -\frac{4\pi}{15}$ and send $c(\Delta) = \frac{\pi}{15}$ to B. Case II: A is not positive. So A must have a split or another vertex of degree > 3. If it has a split then $c(A) \leq c(3,3,3,3,3,4) = -\frac{\pi}{6}$ so send $c(\Delta) \leq \frac{\pi}{6}$ to A. Assume A has no splits and has at least one other vertex of degree > 3. If $d_A(u_3) = 5$ then send $c(\Delta) = \frac{\pi}{15}$ to A and observe that $c(A) \leq c(3,3,3,4,5) = -\frac{\pi}{10}$. Let $d_A(u_3) = 4$. If there is a split in B, $c(B) \leq c(3,3,3,3,3,4) = -\frac{\pi}{6}$ so send $c(\Delta) \leq \frac{\pi}{6}$ to B. Otherwise, if there is no split in B, $l_A(u_3) = 5\bar{2}35$. If $d_A(u_4) = 3$, $l_A(u_4) \in \{442, 443\}$, both of which split along $(3,4)$ in C so send $c(\Delta) \leq \frac{\pi}{6}$ to C, $c(C) \leq c(3,3,3,3,3,4) = -\frac{\pi}{6}$. If $d_A(u_4) > 3$, either $d_A(u_1) > 3$ or $l_A(u_1) = \bar{3}15$ and u_5 cannot be completed with degree 3, so $c(A) \leq c(3,3,4,4,4) = -\frac{\pi}{6}$ and send $c(\Delta) \leq \frac{\pi}{6}$ to A.

The region 5_3 is not compatible with any other region and so the only possibility of some Δ' receiving curvature from 5_3 and from somewhere else also is if it receives from a second 5_3 region. This is possible in this case as curvature can be sent in from 5_3 across more than one edge. However, Δ' then has a $(2,3)$-split and a $(3,4)$-split and can therefore compensate for the total $\frac{\pi}{3}$ curvature being sent in, $\frac{\pi}{6}$ from each of the two 5_3 regions. As 5_3 sends curvature across two different edges only, Δ' can receive no more than two lots of curvature.

We now look at 146, using the notation $146A$ to mean Δ is 146 and curvature is sent into A. The region 146 is compatible with regions 37, 197 and 209 and we treat in case in Figure 2.35 in turn.

146*A*

Compatible regions which do not send curvature across the same edge are as follows: 37, $146B$ (a second region 146 sending curvature across the $(5,1)$-edge this time), $146C$, 197, 209. The regions 37, 197 and 209 do not fit so we are left with $146B$ and $146C$. Assume two lots are sent in from region $146B$ or $146C$ as well as from region $146A$. Then at most $\frac{\pi}{3}$ is sent in and we get a split and $d(u_5) > 3$, so $c(\Delta') \leq c(3,3,3,3,4,4) = -\frac{\pi}{3}$. We cannot have three lots sent in as $146A$, $146B$ and $146C$ do not all fit together.

146B

The region Δ' must contain a split. Compatible regions which do not cross the same edge (apart from 146A which has already been dealt with) are as follows: 37, 146C, 209. Region 146C does not fit. Assume two lots are sent in from region 37 or 209 as well as from region 146B, so at most $\frac{\pi}{3}$ is sent in. The region gives another vertex of degree > 3, so $c(\Delta') \leq c(3,3,3,3,4,4) = -\frac{\pi}{3}$. We cannot have three lots sent in as 37 and 209 cross the same edge.

146C

Compatible regions which do not cross the same edge (apart from 146A and 146B) are as follows: 37, 197, 209. However, none of them fit and this completes the case for region 146.

Lemma 1.17 therefore holds for these nine cases. The remaining fifteen cases are more complicated and need to be looked at in three groups, split depending on which other regions they are compatible with. The first of these groups contains eight type 3 regions and we are able to successfully distribute positive curvature for each of these regions.

Group I

This first group contains the regions 232, 43, 106, 127, 167, 255, 256 and 292. A table displaying the regions which may occur with each of the eight regions in this group is as follows. The type 3 regions have been highlighted.

232	26, **43**, 82, **106**, 122, **127**, 135, 153, 154, **167**, 219, 220, **255**, **256**, **292**, 316, 317
43	105, **106**, 153, **167**, 219, **232**, **255**, 266, **292**, 316, 317, 324
106	**43**, 153, **167**, 219, **232**, **255**, **292**, 316, 317
127	**232**
167	**43**, **106**, 153, 219, **232**, **255**, **292**, 316, 317
255	**43**, **106**, 153, **167**, 219, **232**, **292**, 316, 317
256	**232**, 317
292	**43**, 105, **106**, 153, **167**, 219, **232**, **255**, 266, 316, 317, 324

The following figures show the way in which curvature is distributed for each of the 8 regions.

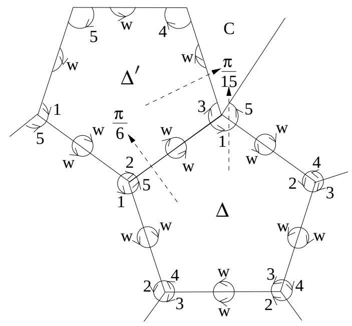


Figure 2.38: no. 43

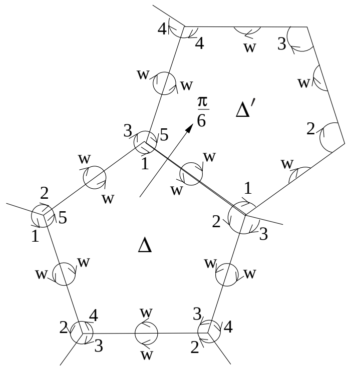


Figure 2.39: no. 106

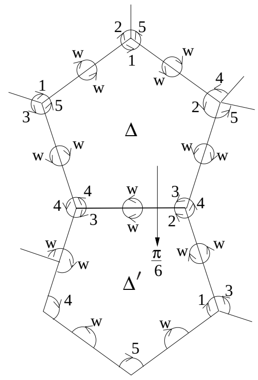


Figure 2.40: no. 127

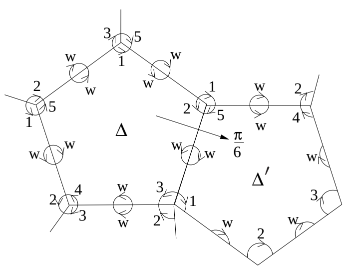


Figure 2.41: no. 167

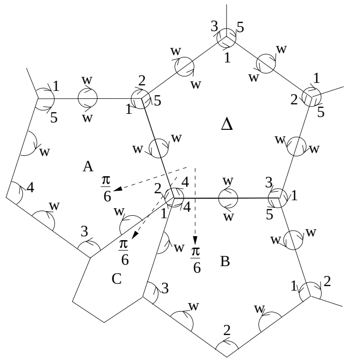


Figure 2.42: no. 232

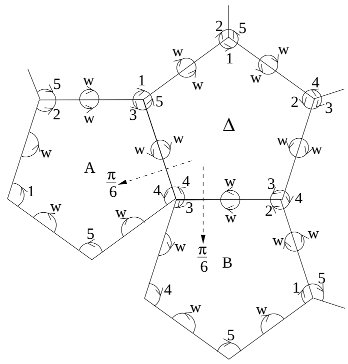


Figure 2.43: no. 255

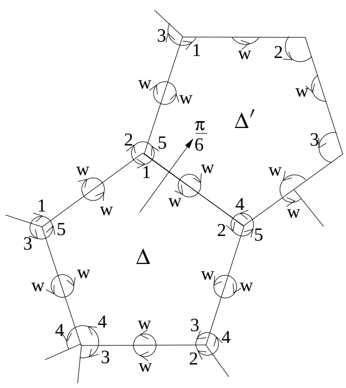


Figure 2.44: no. 256

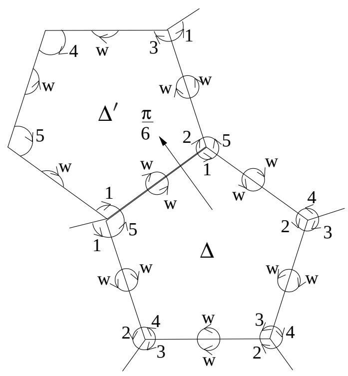


Figure 2.45: no. 292

Next we outline the exact manner in which curvature will be sent and give an explanation of how this curvature is compensated for, assuming only one lot is received.

43: Case I: Δ' is positive. Then it must be 167 as this is the only region that would fit. Then $l(v_1) = \bar{5}\bar{1}3\bar{2}x$ where $x \in \{4, \bar{1}, \bar{5}\}$. If $x = 4$ then we get a $(4,5)$-split in C. If $x = \bar{1}$ then we get a $(5,1)$ and a $(1,2)$-split in C. If $x = \bar{5}$ then we get a $(5,1)$-split in C. So send the $\frac{\pi}{15}$ from both 43 and 167 into C, so $c(C) \leq c(3,3,3,3,3,5) = -\frac{4\pi}{15}$ and $\frac{2\pi}{15}$ is sent in. Treat the curvature sending from 43 and 167 to the region C in Figure 2.38 as one sending of $\frac{2\pi}{15}$ from now on, as these sendings depend on one another. Refer to this sending as $43C$. Case II: Δ' is not positive. Assume $d(v_1) = 5$, so $c(\Delta) = \frac{\pi}{15}$. As Δ' is not positive there must be at least one split or further vertex of degree > 3 in Δ', which implies $c(\Delta') \leq c(3,3,3,4,5) = -\frac{\pi}{10}$, so we are done. Assume $d(v_1) = 4$, so $l(v_1) = \bar{3}154$, causing a $(3,4)$-split. Then $c(\Delta) = \frac{\pi}{6}$ and $c(\Delta') \leq c(3,3,3,3,3,4) = -\frac{\pi}{6}$.

106: Δ' is not positive so, as with region 43, if $d(v_2) = 5$ we are done. If $d(v_2) = 4$ then $l(v_2) = \bar{1}2\bar{3}\bar{3}$ or $\bar{1}2\bar{3}\bar{4}$, both of which cause a split, so $c(\Delta') \leq c(3,3,3,3,3,4) = -\frac{\pi}{6}$.

167: We assume Δ' is not positive, as the case when it is positive is dealt with in 43, so if $d(v_3) = 5$ we are done. If $d(v_3) = 4$ then $l(v_3) = \bar{1}3\bar{2}4$ or $\bar{1}3\bar{2}\bar{2}$, both of which cause a split, so we are done as $c(\Delta') \leq -\frac{\pi}{6}$ once again.

232: A is not positive, otherwise it would be one of the following regions: $122 \implies l(v_4) = 442\bar{3}\omega$, $127 \implies l(v_4) = 4425\omega$, $135 \implies l(v_4) = 4425\omega$, none of which complete with degree 4 or 5. If there is a split in A, send to A as $c(3,3,3,3,3,4) = -\frac{\pi}{6}$. Now assume there are no splits in A. The remaining vertices in A cannot complete with degree 3 (or v_4 would have degree > 5) so there is a further vertex of degree > 3 in A. If $d(v_4) = 5$, $c(A) \leq c(3,3,3,4,5) = -\frac{\pi}{10}$ so send to A. If $d(v_4) = 4$ then $l(v_4) = 4425, 442\bar{1}$ or $442\bar{5}$. If $l(v_4) = 4425$ or $442\bar{5}$, B splits along the $(3,4)$-edge so send to B. Assume $l(v_4) = 442\bar{1}$. Either $d_A(u_5) \geq 4$ or $l_A(u_5) = 35\bar{1}$ and $d_A(u_4) \geq 4$. If $d_A(u_3) \geq 4$, $c(A) \leq c(3,3,4,4,4) = -\frac{\pi}{6}$ so send to A. If $d_A(u_3) = 3$, $l_A(u_3) = \bar{2}3\bar{2}$, which splits $(2,3)$ in C, so send to C.

127: Another split or a vertex of degree > 3 in Δ' is enough as then $c(\Delta') \leq c(3,3,3,3,3,4) = -\frac{\pi}{6}$, so now assume otherwise. But then u_1 cannot be completed with degree 3 - contradiction.

255: Case I: B is positive, in which case B must be 219 (cannot be 153 or cannot complete v_4 with degree < 6) and $l(v_4) = \bar{3}4445$. A cannot be completed with $d_A(u_2) = d_A(u_1) = d_A(u_5) = 3$ so $c(A) \leq c(3,3,3,4,5) = -\frac{\pi}{10}$ and $c(\Delta) = \frac{\pi}{15}$, so send into A. Case II: B is not positive. Then if $d(v_4) = 5$, send to B. If $d(v_4) = 4$ then $l(v_4) = \bar{3}444, \bar{3}44\bar{1}$ or $\bar{3}44\bar{5}$. The first two cause a $(3,4)$-split in B which means $c(B) \leq c(3,3,3,3,3,4) = -\frac{\pi}{6}$ and so send to B. The last causes a $(4,5)$-split in A so send to A.

256: Another split or a vertex of degree > 3 in Δ' is enough so assume otherwise. But then the existing split has proper sublabel $w^{-1}w$ which is a contradiction.

292: Δ' is not positive so, as before, if $d(v_5) = 5$ we are done. If $d(v_5) = 4$ then $l(v_5) = 15\bar{1}\bar{5}$ which causes a split in Δ' and we are done.

Let us now check that curvature is still compensated for when more than one lot is sent in to the same region. We will look at a particular Δ' which curvature is being sent into and check if any further can be sent in.

232*A*

Compatible regions which do not cross the same edge are as follows: 26, 43, 43C, 82, 106, 122, 127, 135, 154, 167, 219, 220, 232B, 232C, 255A, 255B, 256, 316, 317.
Assume two lots are sent in:

Regions	Outcome
26, 43, 82, 106, 122, 127, 135, 154, 167, 219, 220, 232C, 255B, 316, 317	Does not fit.
43C	Cannot be sent across the 1 vertex as $d(u_1) = 3$ and cannot be sent across the 5 vertex as $l(u_5) = \bar{5}\bar{1}3\bar{2}\bar{5}$ would give a $(5,1)$-split with proper sublabel ww^{-1}, a contradiction by Lemma 1.23. Can only be across the 4 vertex, which has degree 5, giving a $(4,5)$-split. $\frac{\pi}{6} + \frac{2\pi}{15} = \frac{9\pi}{30}$ sent in. If $d(u_3) = 3$, splits (2, 3) so $c(\Delta') \leq c(3,3,3,3,4,5) = -\frac{13\pi}{30}$.
154, 256	Forces a $(5,1)$-split with proper sublabel ww^{-1} - contradiction.
232B	At most $\frac{\pi}{3}$ sent in. $d(u_4) \geq 4$, $(3,4)$-split, so $c(\Delta') \leq c(3,3,3,3,4,4) = -\frac{\pi}{3}$.
255A	Case I in 255A: $l_A(u_4) = \bar{3}4445$ and cannot complete $l_A(u_2) = l(v_4)$ with degree < 6 - contradiction. Case II in 255A: At most $\frac{\pi}{3}$ sent in. $d(u_4) \geq 4$, $(4,5)$-split: $c(\Delta') \leq c(3,3,3,3,4,4) = -\frac{\pi}{3}$.

We cannot have three lots sent in as potential regions do not fit together.

232*B*

Compatible regions which do not cross the same edge are as follows: 26, 43, 43C, 82, 106, 122, 127, 135, 153, 167, 219, 220, 232C, 255A, 255B, 292, 316, 317.

Assume two lots are sent in and note that $d(u_4) > 3$ and $(3, 4)$ splits:

Regions	Outcome
26, 43C, 82, 122, 135, 167, 219, 220, 255A, 317	Does not fit.
43, 255B	At most $\frac{\pi}{3}$ sent in. Region gives $d(u_3) \geq 4$ so $c(\Delta') \leq c(3,3,3,3,4,4) = -\frac{\pi}{3}$.
106	$d(u_1) \geq 4$. If $d(u_1) = 5$, at most $\frac{7\pi}{30}$ is sent in and $c(\Delta') \leq c(3,3,3,3,4,5) = -\frac{13\pi}{30}$. If $d(u_1) = 4$, at most $\frac{\pi}{3}$ sent in, $(1,2)$-split, so $c(\Delta') \leq c(3,3,3,3,3,4,4) = -\frac{2\pi}{3}$.
127	At most $\frac{\pi}{3}$ sent in. Either another split or $d(u_1) > 3$ so $c(\Delta') \leq c(3,3,3,3,4,4) = -\frac{\pi}{3}$.
153	At most $\frac{\pi}{3}$ sent in. Regions split $(2,3)$ and $d(u_1) \geq 4$ so $c(\Delta') \leq c(3,3,3,3,3,4,4) = -\frac{2\pi}{3}$.
292, 232C	At most $\frac{\pi}{3}$ sent in. Regions give $d(u_1) \geq 4$ so $c(\Delta') \leq c(3,3,3,3,4,4) = -\frac{\pi}{3}$.
316	At most $\frac{\pi}{3}$ sent in. Regions give $d(u_2) \geq 4$ so $c(\Delta') \leq c(3,3,3,3,4,4) = -\frac{\pi}{3}$.

Assume three lots are sent in:

Then it must be 232B with two from eight possible regions.

Crossing $(2, 3)$: 43, 255B, 127 (compatible with 232 only), 316.

Crossing $(5, 1)$: 106.

Crossing $(1, 2)$: 153 (gives a $(2, 3)$-split), 292.

Crossing vertex 1: 232C (gives a $(2, 3)$-split).

So there is no region sending across $(2, 3)$ compatible with 153 or 232C, and so possible pairs are: $\{43, 106\}$, $\{43, 292\}$, $\{106, 255B\}$, $\{255B, 292\}$, $\{106, 316\}$, $\{292, 316\}$, $\{106, 153\}$, $\{106, 292\}$, $\{106, 232C\}$, $\{153, 232C\}$, $\{232C, 292\}$.

Regions	Outcome
$\{106, 255B\}$, $\{255B, 292\}$, $\{292, 316\}$, $\{106, 153\}$, $\{106, 292\}$, $\{106, 232C\}$, $\{153, 232C\}$, $\{232C, 292\}$	Does not fit.

$\{43, 292\}$, $\{43, 106\}$	At most $\frac{\pi}{2}$ sent in. Regions give $d(u_1) \geq 4$ and $d(u_3) \geq 4$ so $c(\Delta') \leq c(3,3,3,4,4,4) = -\frac{\pi}{2}$.
$\{106, 316\}$	At most $\frac{\pi}{2}$ sent in. Regions give $d(u_1) \geq 4$ and $d(u_2) \geq 4$ so $c(\Delta') \leq c(3,3,3,4,4,4) = -\frac{\pi}{2}$.

If four lots were sent in it would be from $232B$ and three from 43, 106, 292, 316. But 43 and 316 cross the same edge and 106 and 292 do not fit together so four lots cannot be sent into Δ'.

$232C$

Compatible regions which do not cross the same edge are as follows: 26, 43, $43C$, 82, 106, 122, 127, 135, 153, 154, 167, 219, 220, $255A$, $255B$, 256, 292, 316, 317.
Assume two lots are sent in and note that $d(u_1) > 3$ and $(2,3)$-splits:

Regions	Outcome
26, 43, 82, 106, 122, 127, 135, 153, 154, 167, 219, 220, $255B$, 256, 292, 316, 317	Does not fit.
$43C$	Can only cross the 4 vertex which has degree 5. $232C$ gives a $(2,3)$-split and $43C$ gives a $(4,5)$-split. So $\frac{9\pi}{30}$ is sent in and $c(\Delta') \leq c(3,3,3,3,3,4,5) = -\frac{23\pi}{30}$.
$255A$	Forces a $(2,3)$-split with proper sublabel ww^{-1}, which is a contradiction.

Clearly we cannot have three lots sent in as $43C$ is the only region that can send into Δ' as well as $232C$.

127

We have completed 127 already as it is only compatible with 232. We do not have to consider 232 again in the remaining regions of this group or we will be repeating ourselves.

256

The only compatible region (apart from 232) is 317. This crosses the $(5,1)$-edge in Δ' and does not fit, so Δ' may not receive curvature from 317 also.

43

Compatible regions which do not cross the same edge are as follows: $43C$, 105, 106, 153, 167, 219, $255A$, 266, 292, 317, 324.

Regions	Outcome
105, 153, $255A$, 324	Does not fit.
$43C$	Cannot be across vertex 1 and cannot be across vertex 4 or cannot complete $l(v_1)$ with degree 5. Only possibility is across vertex 5, which splits $(5,1)$. If $d(v_1) = 4$, $l(v_1) = \bar{3}154$ which splits $(3,4)$. So $\frac{\pi}{6} + \frac{2\pi}{15} = \frac{9\pi}{30}$ is sent in and $c(\Delta') \leq c(3,3,3,3,3,4,5) = -\frac{23\pi}{30}$. If $d(v_1) = 5$, $\frac{\pi}{15} + \frac{2\pi}{15} = \frac{3\pi}{15}$ is sent in and $c(\Delta') \leq c(3,3,3,3,5,5) = -\frac{8\pi}{15}$.
106, 167, 292	If $d(u_1) = d(u_3) = 5$ then $\frac{2\pi}{15}$ is sent in and $c(\Delta') \leq c(3,3,3,5,5) = -\frac{\pi}{5}$. If at least one of these degrees is 4, at most $\frac{\pi}{3}$ is sent in and there is a split so $c(\Delta') \leq c(3,3,3,3,4,4) = -\frac{\pi}{3}$.
219, 317	Forces a $(1,2)$-split with proper sublabel ww^{-1}, which is a contradiction.
266	At most $\frac{\pi}{3}$ sent in. Region gives $d(u_1) \geq 4$ and a $(4,5)$-split so $c(3,3,3,3,4,4) = -\frac{\pi}{3}$.

Can only have three lots sent in if it is 292 and $43C$ as none of the others fit together. Each contribute a degree > 3 vertex and $43C$ gives a $(5,1)$-split. So $\frac{7\pi}{15}$ is sent in and $c(3,3,3,4,4,5) = -\frac{3\pi}{5}$.

43*C*

Compatible regions (with 167 also - see description of $43C$ sending above) which do not cross the same edge are as follows: 106, 153, 167, 219, $255A$, $255B$, 292, 316, 317.

Regions	Outcome
106, 167	Does not fit.
153	Through 1 vertex does not fit and through 5 vertex causes labels to give a contradiction, so through 4 vertex. $(4,5)$-split and $(2,3)$-split so $\frac{9\pi}{30}$ sent in and $c(3,3,3,3,3,4,5) = -\frac{23\pi}{30}$.
219	Only fits through 4 vertex but the labels give a contradiction.

$255A$	Through vertex 1 or 5. $\frac{9\pi}{30}$ sent in. At least one split so $c(\Delta') \leq c(3,3,3,3,4,5) = -\frac{13\pi}{30}$.
$255B$, 292	Through vertex 4 or 5. $\frac{9\pi}{30}$ sent in. At least one split so $c(\Delta') \leq c(3,3,3,3,4,5) = -\frac{13\pi}{30}$.
316, 317	All three crossings require a vertex whose label gives a contradiction.

The only possibility for three lots being sent in is $43C$ (through vertex 5) with $255A$ and 292 as the others do not fit. Each of these regions contribute a degree > 3 vertex and $43C$ gives a $(5,1)$-split. So $\frac{\pi}{6} + \frac{\pi}{6} + \frac{2\pi}{15} = \frac{7\pi}{15}$ is sent in and $c(3,3,3,4,4,5) = -\frac{3\pi}{5}$.

106 and **167**

Compatible regions which do not cross the same edge are as follows: 153, $255A$, $255B$, 292, 316.

Regions	Outcome
153, 292	Does not fit.
$255A$	Case I in 255A: Either $d(u_1) = 4$ so $\frac{7\pi}{30}$ is sent in, in which case there is a split and $c(3,3,3,3,4,5) = -\frac{13\pi}{30}$, or $d(u_1) = 5$ so $\frac{2\pi}{15}$ is sent in and $c(3,3,3,5,5) = -\frac{\pi}{5}$. Case II in 255A: Forces a $(4,5)$-split with proper sublabel $w^{-1}w$, which is a contradiction.
$255B$	If $d(u_1) = d(u_3) = 5$ then $\frac{2\pi}{15}$ is sent in and $c(\Delta') \leq c(3,3,3,5,5) = -\frac{\pi}{5}$. If at least one of these degrees is 4, at most $\frac{\pi}{3}$ is sent in and there is a split, so $c(\Delta') \leq c(3,3,3,3,4,4) = -\frac{\pi}{3}$.
316	At most $\frac{\pi}{3}$ sent in. Region gives $d(u_2) \geq 4$ and a $(3,4)$-split so $c(\Delta') \leq c(3,3,3,3,4,4) = -\frac{\pi}{3}$.

We cannot have three lots sent in as it would have to be two of 316, $255A$ and $255B$. But 316 and $255B$ cross the same edge, $255A$ crosses the $(3,4)$-edge and 316 splits the $(3,4)$-edge, and $255A$ and $255B$ do not fit together.

255A

Compatible regions which do not cross the same edge are as follows: 153, 219, 255B, 292, 316, 317.

Regions	Outcome
255B, 316	Does not fit.
153	Forces a $(2,3)$-split with proper sublabel ww^{-1}, which is a contradiction.
219, 317	At most $\frac{\pi}{3}$ sent in. Region gives $d(u_5) \geq 3$ and a $(1,2)$-split so $c(\Delta') \leq c(3,3,3,3,4,4) = -\frac{\pi}{3}$.
292	Region gives $d(u_1) > 3$. Assume case II in 255A: At most $\frac{\pi}{3}$ is sent in and a $(4,5)$-split, so $c(\Delta') \leq c(3,3,3,3,4,4) = -\frac{\pi}{3}$. Assume case I in 255A: If $d(u_1) = 4$, $\frac{7\pi}{30}$ is sent in and a $(5,1)$-split so $c(\Delta') \leq c(3,3,3,3,4,5) = -\frac{13\pi}{30}$. Otherwise, $d(u_1) = 5$, $\frac{2\pi}{15}$ is sent in and $c(\Delta') \leq c(3,3,3,5,5) = -\frac{\pi}{5}$

We cannot have three lots sent in as both 219 and 317 cross $(5,1)$ and split $(1,2)$, which is the edge crossed by 292.

255B

Compatible regions which do not cross the same edge are as follows: 153, 219, 292, 317.

Regions	Outcome
153, 292	Does not fit.
219, 317	Region gives a $(1,2)$-split with proper sublabel ww^{-1} - contradiction.

292

Compatible regions which do not cross the same edge are as follows: 105, 219, 266, 316, 317.

Regions	Outcome
219, 266, 316, 317	Does not fit.
105	Region gives a $(2,3)$-split with proper sublabel $w^{-1}w$ - contradiction.

Group II

44	38, 107, **164**, 188, 198, 241, 275, 301, **312**
164	**44**, 107, 275, **312**
312	38, **44**, 107, **164**, 188, 198, 241, 275, 301

This group brings about exception *(iii)* that the equalities $l_3 = l_2 + l_4 = l_1 + l_5$ do not hold, and by symmetry *(iv)*, in Case (2) of Theorem 1.12. The exception allows us to disregard regions 44 and 312, whose labellings are $l(v_1) = \bar{4}15\omega$, $l(v_2) = 42\bar{3}$, $l(v_3) = \bar{4}3\bar{2}$, $l(v_4) = \bar{3}42$, $l(v_5) = 15\bar{3}$ and $l(v_1) = \bar{3}15$, $l(v_2) = 42\bar{3}$, $l(v_3) = \bar{4}3\bar{2}$, $l(v_4) = \bar{3}42$, $l(v_5) = 15\bar{2}\omega$ respectively. The exception also rules out region 164, although this is not one of the regions which causes the problem, which we describe next.

If we were to allow the equalities $l_3 = l_2 + l_4 = l_1 + l_5$ and therefore the regions 44 and 312, it would be possible to end up with the following situation.

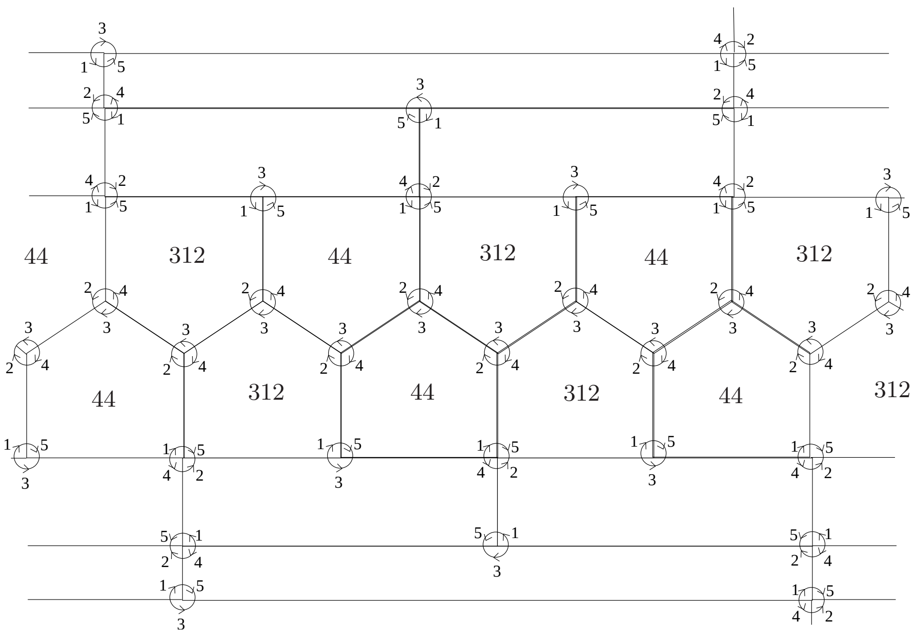


Figure 2.46: 44 and 312 together

All regions in this figure other than 44 and 312 have degree $-\frac{\pi}{6}$ and so far we have not been able to find a way to compensate for the positive regions. Therefore the restriction on the l_i's is required.

Group III

1_3	156, 173, **258**, 310, 311
258	$\mathbf{1_3}$, 156, 173, 310, 311

This group brings about exception *(i)* that the equalities $l_1 = l_2 + l_4$, $l_2 = l_3 + l_5$ and $l_3 = l_1 + l_5$ do not hold, and by symmetry *(ii)*, in Case (2) of Theorem 1.12. The exception allows us to disregard region 1_3, whose labelling is $l(v_1) = \bar{4}1\bar{2}$, $l(v_2) = \bar{3}2\bar{5}$, $l(v_3) = \bar{1}3\bar{5}$, $l(v_4) = \bar{1}42$, $l(v_5) = 15\bar{3}$. The exception also rules out region 258 and we now describe the problem that arises when allowing these two regions to occur.

If we allowed the mentioned equalities and therefore the regions 1_3 and 258, which are compatible, it is possible to end up with the following situation.

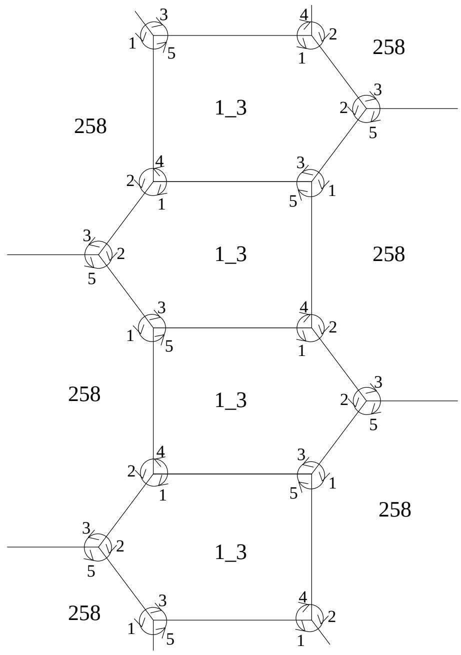


Figure 2.47: 1_3 and 258 together

Again we have been unable to compensative for the positive curvature so the exception on the l_i's is required.

This completes the proof of Lemma 1.17 in Case (2), with the mentioned exceptions.

2.4 Case 3 (A3)

This section is concerned with the proof of Lemma 1.17 for case (3) in Theorem 1.12.

Let $r(t) = wt^{l_1}wt^{l_2}wt^{l_3}w^{-1}t^{l_4}w^{-1}t^{l_5}$.

The following figure shows all possible ways in which the w's can match up.

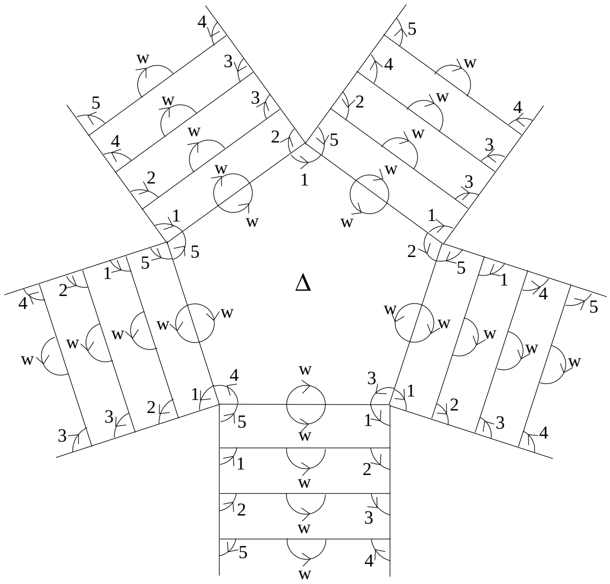


Figure 2.48: All possible labellings – A3

A full list of all possible labellings and their types can be found in [24].

Observe the cases in Figure 2.3.
In case (i) there are 0 possible labellings.
In case (ii) there are 64 possible labellings.
In case (iii) there are 65 possible labellings.
In case (iv) there are 51 possible labellings.
In case (v) there are 48 possible labellings.
In case (vi) there are 51 possible labellings.

2.4.1 A3 sendings

The way in which we send curvature for the type 3 regions is as follows.

Δ	Edge/vertex sent across in Δ	
4	$(3,4)$	
6	$(5,1)$	
16	N/A	
34	$(4,5)$	
35	$(4,5)$	
63	$(5,1)$:	When region across $(1,2)$ is positive.
	$(1,2)$:	Otherwise.
66	$(2,3)$:	When region across $(1,2)$ is positive.
	$(1,2)$:	Otherwise.
78	$(3,4)$	
79	N/A	
91	$(4,5)$	
92	$(3,4)$	
123	$(2,3)$	
130	$(4,5)$	
132	$(5,1)$	
150	N/A	
152	$(2,3)$	
162	$(4,5)$	
165	$(1,2)$	
169	$(4,5)$:	When region across $(3,4)$ is positive
	$(3,4)$:	When this region is not positive and either $d(v_3)=5$ or $d(v_3)=4$, $l(v_3)\neq 3343$.
	$(2,3)$:	Otherwise.
173	$(4,5)$	
175	$(2,3)$:	When this region has a split OR region across $(3,4)$ does not split and either $d(v_3)=5$ or $d(v_3)=4$ and $d(u_3)=3$ and one of $d(u_4), d(u_5)>3$ in this region.
	$(3,4)$:	When region across $(2,3)$ does not split and this region has a split OR region across $(2,3)$ is positive.
	3-vertex:	When none of the above hold and this region splits.
	Send $\frac{1}{2}c(\Delta)$ across $(2,3)$ and 3-vertex otherwise.	

Δ	Edge/vertex sent across in Δ	
200	$(4,5)$:	When region across $(3,4)$ is positive
	$(3,4)$:	Otherwise.
205	N/A	
206	$(3,4)$:	When region across $(4,5)$ is positive
	$(4,5)$:	Otherwise.
209	$(4,5)$	
210	N/A	
220	$(3,4)$	
231	$(5,1)$:	When this region has a split OR region across $(4,5)$ does not split and either $d(v_5)=5$ or $d(v_5)=4$ and $d(u_5)=3$ and one of $d(u_4), d(u_3)>3$ in this region.
	$(4,5)$:	When region across $(5,1)$ does not split and this region has a split OR region across $(5,1)$ is positive.
	5-vertex:	When none of the above hold and this region splits.
	Send $\frac{1}{2}c(\Delta)$ across $(5,1)$ and 5-vertex otherwise.	
233	$(3,4)$	
241	$(5,1)$	
248	$(2,3)$	
258	$(3,4)$	
259	N/A	
270	$(3,4)$	
271	$(3,4)$:	When region across $(4,5)$ is positive
	$(4,5)$:	When this region is not positive and either $d(v_5)=5$ or $d(v_5)=4$, $l(v_5)\neq 5545$.
	$(5,1)$:	Otherwise.
272	$(1,2)$	

The way in which we send curvature for certain type 1 and 2 regions is as follows.

Δ	Edge sent across in Δ	Δ	Edge sent across in Δ	Δ	Edge sent across in Δ	Δ	Edge sent across in Δ
7	$(1,2)$	50	$(5,1)$	98	$(2,3)$	176	$(3,4)$
18	$(4,5)$	52	$(3,4)$	99	$(2,3)$	182	$(2,3)$
19	$(5,1)$	58	$(5,1)$	109	$(2,3)$	185	$(3,4)$
23	$(1,2)$	64	$(5,1)$	114	$(2,3)$	193	$(3,4)$
24	$(5,1)$	71	$(3,4)$	121	$(4,5)$	198	$(5,1)$
26	$(2,3)$	72	$(2,3)$	122	$(1,2)$	207	$(4,5)$
28	$(1,2)$	74	$(2,3)$	124	$(4,5)$	212	$(4,5)$
30	$(5,1)$	75	$(2,3)$	128	$(1,2)$	230	$(4,5)$
31	$(5,1)$	80	$(5,1)$	136	$(4,5)$	249	$(3,4)$
42	$(3,4)$	82	$(1,2)$	142	$(4,5)$	261	$(4,5)$
47	$(5,1)$	90	$(2,3)$	159	$(3,4)$	279	$(3,4)$

The remaining type 1 and 2 regions are dealt with in the same way as in Subsection 2.2.2

We claim that, under the described sendings, Lemma 1.17 holds.

2.4.2 Proof of Lemma 1.17 for A3

Since there are no positive regions whose vertices are all of degree 3, the situation of Figure 2.7 (2) does not occur.

There are 36 regions of type 3, each with curvature $\frac{\pi}{6}$.

We can rewrite $r(t) = wt^{l_1}wt^{l_2}wt^{l_3}w^{-1}t^{l_4}w^{-1}t^{l_5}$ to obtain a symmetry from $(l_1, l_2, l_3, l_4, l_5)$ to $(l_2, l_1, l_5, l_4, l_3)$. Following is the table showing all type 3 regions and their pairings with this symmetry. As before, those we will allocate curvature to are highlighted.

	$l(v_1)$	$l(v_2)$	$l(v_3)$	$l(v_4)$	$l(v_5)$
4	$\mathbf{\bar{2}1\bar{5}}\boldsymbol{\omega}$	$\mathbf{\bar{1}2\bar{5}}$	$\mathbf{\bar{1}33}$	$\mathbf{24\bar{3}}$	$\mathbf{\bar{4}5\bar{1}}$
91	$\bar{3}1\bar{2}$	$\bar{3}2\bar{1}\omega$	$\bar{2}3\bar{4}$	$\bar{5}41$	$55\bar{2}$
6	$\mathbf{31\bar{5}}\boldsymbol{\omega}$	$\mathbf{\bar{1}2\bar{5}}$	$\mathbf{\bar{1}3\bar{4}}$	$\mathbf{\bar{5}43}$	$\mathbf{254}$
123	$\bar{3}1\bar{2}$	$\bar{3}25\omega$	431	$54\bar{3}$	$\bar{4}5\bar{2}$
34	$\mathbf{\bar{2}1\bar{2}}\boldsymbol{\omega}$	$\mathbf{\bar{3}2\bar{1}}$	$\mathbf{\bar{2}31}$	$\mathbf{54\bar{3}}$	$\mathbf{\bar{4}5\bar{1}}$
92	$\bar{2}1\bar{5}$	$\bar{1}2\bar{1}\omega$	$\bar{2}3\bar{4}$	$\bar{5}43$	$25\bar{1}$
35	$\mathbf{\bar{3}1\bar{2}}\boldsymbol{\omega}$	$\mathbf{\bar{3}2\bar{1}}$	$\mathbf{\bar{2}31}$	$\mathbf{54\bar{3}}$	$\mathbf{\bar{4}5\bar{2}}$
78	$\bar{2}1\bar{5}$	$\bar{1}2\bar{5}\omega$	$\bar{1}3\bar{4}$	$\bar{5}43$	$25\bar{1}$

63	$\bar{3}15\omega$	425	431	542	$15\bar{2}$
66	$\boldsymbol{314}$	$\boldsymbol{32\bar{5}\omega}$	$\boldsymbol{\bar{1}32}$	$\boldsymbol{143}$	$\boldsymbol{254}$
16	$\bar{3}1\bar{5}\omega$	$\bar{1}25$	431	$54\bar{3}$	$\bar{4}5\bar{2}$
79	$\boldsymbol{31\bar{2}}$	$\boldsymbol{\bar{3}2\bar{5}\omega}$	$\boldsymbol{\bar{1}3\bar{4}}$	$\boldsymbol{\bar{5}43}$	$\boldsymbol{254}$
130	$\boldsymbol{\bar{3}1\bar{2}}$	$\boldsymbol{\bar{3}24}$	$\boldsymbol{331\omega}$	$\boldsymbol{541}$	$\boldsymbol{55\bar{2}}$
270	$41\bar{5}$	$\bar{1}2\bar{5}$	$\bar{1}33$	243	255ω
152	$\boldsymbol{314}$	$\boldsymbol{32\bar{1}}$	$\boldsymbol{\bar{2}32\omega}$	$\boldsymbol{143}$	$\boldsymbol{254}$
241	$\bar{2}15$	425	431	542	$15\bar{1}\omega$
165	$\boldsymbol{\bar{2}1\bar{2}}$	$\boldsymbol{\bar{3}24}$	$\boldsymbol{333\omega}$	$\boldsymbol{24\bar{3}}$	$\boldsymbol{\bar{4}5\bar{1}}$
272	$41\bar{5}$	$\bar{1}2\bar{1}$	$\bar{2}3\bar{4}$	$\bar{5}41$	555ω
173	$\boldsymbol{31\bar{2}}$	$\boldsymbol{\bar{3}2\bar{1}}$	$\boldsymbol{\bar{2}3\bar{4}\omega}$	$\boldsymbol{\bar{5}41}$	$\boldsymbol{554}$
233	$\bar{2}1\bar{5}$	$\bar{1}25$	433	$24\bar{3}$	$\bar{4}5\bar{1}\omega$
175	$\boldsymbol{31\bar{2}}$	$\boldsymbol{\bar{3}2\bar{1}}$	$\boldsymbol{\bar{2}3\bar{4}\omega}$	$\boldsymbol{\bar{5}42}$	$\boldsymbol{154}$
231	$\bar{2}1\bar{5}$	$\bar{1}25$	432	$14\bar{3}$	$\bar{4}5\bar{1}\omega$
200	$\boldsymbol{31\bar{2}}$	$\boldsymbol{\bar{3}2\bar{1}}$	$\boldsymbol{\bar{2}31}$	$\boldsymbol{542\omega}$	$\boldsymbol{154}$
206	$\bar{2}1\bar{5}$	$\bar{1}25$	432	143ω	$25\bar{1}$
209	$\boldsymbol{31\bar{5}}$	$\boldsymbol{\bar{1}2\bar{5}}$	$\boldsymbol{\bar{1}3\bar{4}}$	$\boldsymbol{\bar{5}43\omega}$	$\boldsymbol{254}$
220	$\bar{3}1\bar{2}$	$\bar{3}25$	431	$54\bar{3}\omega$	$\bar{4}5\bar{2}$
205	$\bar{2}1\bar{5}$	$\bar{1}25$	431	543ω	$25\bar{1}$
210	$\boldsymbol{31\bar{2}}$	$\boldsymbol{\bar{3}2\bar{1}}$	$\boldsymbol{\bar{2}31}$	$\boldsymbol{543\omega}$	$\boldsymbol{254}$
132	$\bar{3}1\bar{5}$	$\bar{1}2\bar{5}$	$\bar{1}31\omega$	542	$15\bar{2}$
248	$\boldsymbol{\bar{3}1\bar{2}}$	$\boldsymbol{\bar{3}2\bar{5}}$	$\boldsymbol{\bar{1}32}$	$\boldsymbol{143}$	$\boldsymbol{25\bar{2}\omega}$
162	$\bar{2}1\bar{5}$	$\bar{1}25$	433ω	243	$25\bar{1}$
258	$\boldsymbol{31\bar{2}}$	$\boldsymbol{\bar{3}2\bar{1}}$	$\boldsymbol{\bar{2}31}$	$\boldsymbol{541}$	$\boldsymbol{554\omega}$
150	$\bar{2}1\bar{5}$	$\bar{1}25$	432ω	143	$25\bar{1}$
259	$\boldsymbol{31\bar{2}}$	$\boldsymbol{\bar{3}2\bar{1}}$	$\boldsymbol{\bar{2}31}$	$\boldsymbol{542}$	$\boldsymbol{154\omega}$
169	$\bar{3}1\bar{2}$	$\bar{3}24$	$33\bar{4}\omega$	$\bar{5}41$	$55\bar{2}$
271	$\boldsymbol{41\bar{5}}$	$\boldsymbol{\bar{1}2\bar{5}}$	$\boldsymbol{\bar{1}33}$	$\boldsymbol{24\bar{3}}$	$\boldsymbol{\bar{4}55\omega}$

In this case we do not get a pair for which the two regions are compatible with each other. Therefore, as there are eighteen pairs of symmetries and we only require to distribute curvature from one of each of these pairs, there are eighteen regions for which we need to know how to distribute curvature. Nine of these regions can be dealt with individually so we shall deal with these first. The figures showing the way positive curvature can be sent in each of these cases is as follows:

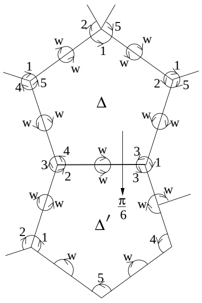


Figure 2.49: no. 4

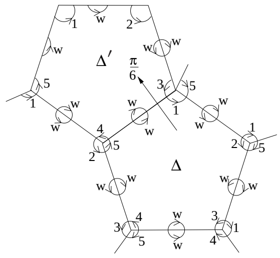


Figure 2.50: no. 6

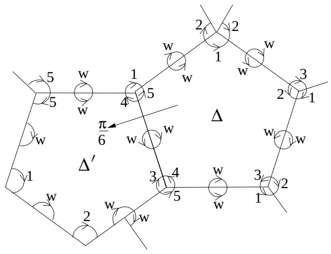


Figure 2.51: no. 34

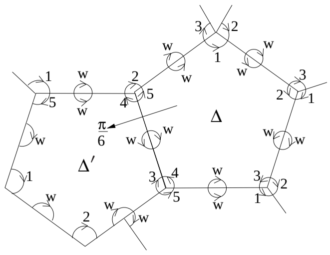


Figure 2.52: no. 35

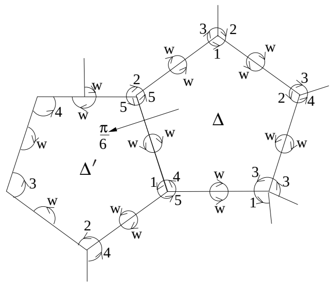


Figure 2.53: no. 130

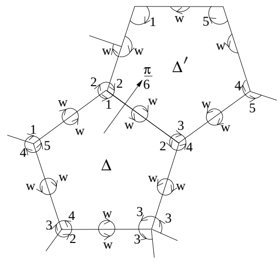


Figure 2.54: no. 165

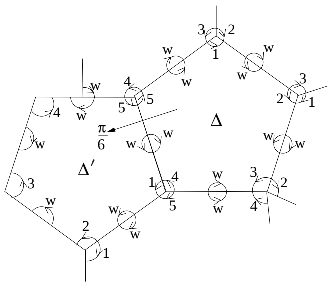


Figure 2.55: no. 173

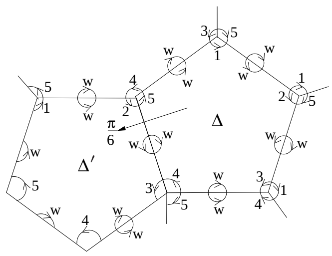


Figure 2.56: no. 209

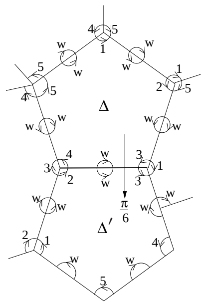


Figure 2.57: no. 271

For the regions where we already know Δ' has a split, we assume there are no further splits and no vertices of degree > 3 in Δ' and obtain a contradiction for each.

4:	Cannot complete u_1.
34:	$l(u_5) = 55\bar{2}$. Cannot complete u_1.
35:	Cannot complete u_5.
130:	Cannot complete u_2.
165:	$l(u_4) = \bar{5}41$, $l(u_5) = 554$. Cannot complete u_1.
173:	Cannot complete u_2.
271:	Cannot complete u_1.

So $c(\Delta) \leq \frac{\pi}{6}$ and $c(\Delta') \leq c(3,3,3,3,3,4) = -\frac{\pi}{6}$.

Let us now look at the remaining two regions. In these cases, we can use the fact that the degree > 3 vertex must, in fact, be degree 5.

6: $d(v_1) = 5$ so $c(\Delta) = \frac{\pi}{15}$. Cannot have no splits and all remaining vertices of degree 3 in Δ', or would have a compatible positive region with vertex 3 of degree 5. This is not the case or Δ' would have to be 6, which is clearly not true, or 99, which has vertex 3 of degree 3. $c(\Delta') \leq c(3,3,3,4,5) = -\frac{\pi}{10}$.

209: $d(v_4) = 5$ so $c(\Delta) = \frac{\pi}{15}$. Cannot have all remaining degrees 3 in Δ' or would have a compatible positive region with vertex 3 of degree 5, which is not the case. $c(\Delta') \leq c(3,3,3,4,5) = -\frac{\pi}{10}$.

The Δ' for the following regions cannot receive curvature from elsewhere:

Region no.	Compatible regions	Edges crossed	Problem with regions
6	99	(3, 4)	Crosses same edge
34	261	(3, 4)	Crosses same edge
130	None		
165	128	(2, 3)	Crosses same edge
173	193	(5, 1)	Crosses same edge
	198	(4, 5)	Crosses split edge
209	None		

Each of the remaining three cases, 4, 35, and 271, can only appear with one other region that does not send curvature across the same edge: 75, 82, and 75 respectively. However, 75 forces a $(1,2)$-split and 82 forces a $(4,5)$-split, both with proper sublabel ww^{-1} which is a contradiction by Lemma 1.23. Therefore, no more than one lot of curvature can be sent into Δ'.

The remaining cases are split into three groups as in the previous section.

Group I

A table displaying the regions in this group along with their compatible regions is as follows. As before, the compatible type 3 regions have been highlighted.

66	23, 64, **152**, **248**
152	23, 64, **66**, **248**
248	23, 24, 64, **66**, **152**

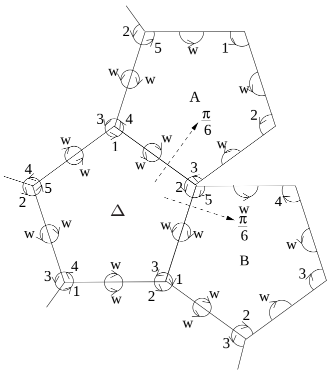


Figure 2.58: no. 66

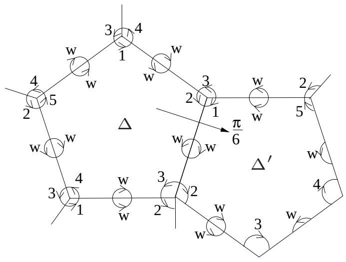


Figure 2.59: no. 152

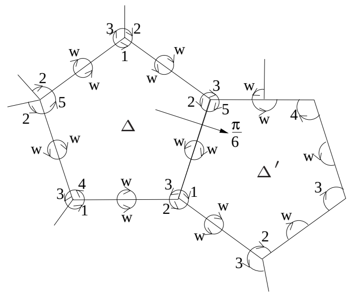


Figure 2.60: no. 248

66: Case I: A is positive. A must be 152 and $l(v_2) = 5\bar{2}\bar{3}2\bar{3}$, which causes a $(4,5)$-split in B. So send to B as $c(B) \leq c(3,3,3,3,3,5) = -\frac{4\pi}{15}$ and $\frac{\pi}{15}$ is sent in. Case II: B is positive. B must be 248 and $l(v_2) = 32\bar{5}\bar{2}3$. If there is a split or another degree > 3 vertex then $c(A) \leq c(3,3,3,4,5) = -\frac{\pi}{10}$ and $\frac{\pi}{15}$ sent in, so send to A. Now assume otherwise. But then $l_A(u_2) = 245$, which splits the $(1,2)$-edge - contradiction. Case III: A and B are not positive. If $d(v_2) = 5$ then there must be at least one split or degree > 3 vertex in A, otherwise A would be positive. So $c(A) \leq c(3,3,3,4,5) = -\frac{\pi}{10}$ and send the $\frac{\pi}{15}$ to A. If $d(v_2) = 4$ then $l(v_2) \in \{32\bar{5}\bar{5}, 32\bar{5}2\}$, both of which split A so $c(A) \leq c(3,3,3,3,3,4) = -\frac{\pi}{6}$ and send the $\frac{\pi}{6}$ to A.

152: Any splits and we are done so assume no splits. If $d(u_5) = d(u_4) = 3$ then $l(u_5) = 5\bar{2}5$. But then u_4 cannot be completed with degree 3 and no splits. Let $d(u_5) > 3$ or $d(u_4) > 3$. If $d(v_3) = 5$ then we are done as $c(\Delta') \leq -\frac{\pi}{10}$ and $\frac{\pi}{15}$ is sent in. If $d(v_3) = 4$, $l(v_3) = \bar{2}32\bar{3}$, which gives a split - contradiction.

248: If a further split or degree > 3 vertex then we are done so assume otherwise. But then the labels force the split to have sublabel ww^{-1}, which is a contradiction.

We now check that curvature is still compensated for when more than one lot of curvature is sent in.

248

Compatible regions which do not cross the same edge are as follows: 23, 24, 64, 66A, 152.

Assume two lots are sent in:

Regions	Outcome
24, 64	Does not fit.
23	At most $\frac{\pi}{3}$ sent in. Region gives a $(3,4)$-split and $d(u_2) > 3$ so $c(\Delta') \leq c(3,3,3,3,3,3,4) = -\frac{\pi}{2}$.
66A	Forces the split to have sublabel ww^{-1}, which is a contradiction.
152	Region gives $d(u_2) > 3$. If $d(u_2) = 4$, at most $\frac{\pi}{3}$ is sent in and $l(u_2) = \bar{2}32\bar{3}$, which gives a $(2,3)$-split, so $c(\Delta') \leq c(3,3,3,3,3,3,4) = -\frac{\pi}{2}$. If $d(u_2) = 5$, at most $\frac{7\pi}{30}$ is sent in and $c(\Delta') \leq c(3,3,3,3,3,5) = -\frac{8\pi}{30}$.

If three lots are sent in it must be with 23 and 152. But 23 and 152 do not fit together unless $d(u_2) > 5$, so three lots cannot be sent in.

152

Compatible regions which do not cross the same edge are as follows: 23, 64, 66A, 66B.

Assume two lots are sent in:

Regions	Outcome
23	Does not fit.
64	Region gives a $(5,1)$-split with proper sublabel ww^{-1} - contradiction.
66A	We have $d(u_2) > 3$ and the region gives $d(u_3) > 3$ and also $d(u_5) > 3$ from the labels. If $d(u_2) = 4$, at most $\frac{\pi}{3}$ is sent in and $l(u_2) = \bar{2}32\bar{3}$, which causes a split, so $c(\Delta') \leq c(3,3,3,4,4,4) = -\frac{\pi}{2}$. If $d(u_2) = 5$, at most $\frac{7\pi}{30}$ is sent in and $c(\Delta') \leq c(3,3,4,4,5) = -\frac{8\pi}{30}$.

$66B$	At most $\frac{7\pi}{30}$ sent in. Region gives a $(4,5)$-split and $d(u_5) = 5$, $l(u_5) = 5\bar{2}\bar{3}2\bar{3}$, so $c(3,3,3,3,4,5) = -\frac{13\pi}{30}$.

If three lots are sent in it must be with $66A$ and $66B$, but the three do not fit without forcing a $(4,5)$-split with proper sublabel ww^{-1}, a contradiction.

66*A*

Compatible regions which do not cross the same edge are as follows: 23, 64, $66B$.
Assume two lots are sent in:

Regions	Outcome
23, 64	Does not fit.
$66B$	Region forces a $(4,5)$-split with proper sublabel ww^{-1} - contradiction.

66*B*

Compatible regions which do not cross the same edge are as follows: 23, 64.
Assume two lots are sent in:

Regions	Outcome
64	Does not fit.
23	At most $\frac{\pi}{3}$ sent in. Region gives a $(3,4)$-split and $d(u_2) > 3$ so $c(\Delta') \leq c(3,3,3,3,4,4) = -\frac{\pi}{3}$.

Clearly we cannot have more than two lots being sent in.

Group II

175	30, 114, **200**, **258**
200	30, 31, 42, 71, 74, 80, 114, 121, 159, **175**, **258**, 279
258	30, 31, 42, 71, 74, 80, 114, 121, 159, **175**, **200**, 279

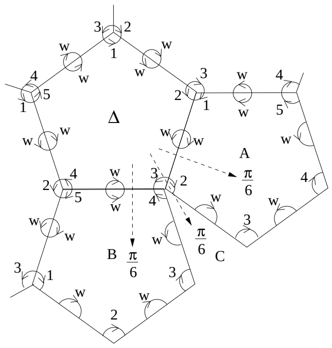


Figure 2.61: no. 175

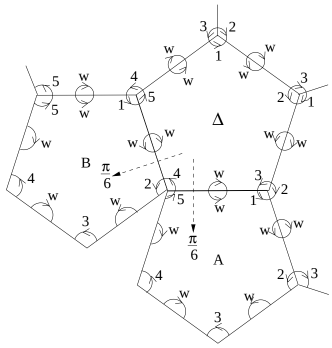


Figure 2.62: no. 200

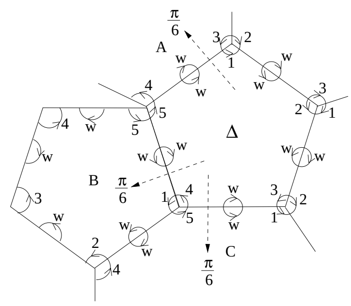


Figure 2.63: no. 258

175: If there are any splits in A or B then send there. Now assume there are no splits in A or B. Case I: A is positive. Then A is 114 and $d(v_3) = 5$, $l(v_3) = 4\bar{3}242$. If B has another vertex of degree > 3 then $c(B) \leq -\frac{\pi}{10}$ and send the $\frac{\pi}{15}$ there, so now assume otherwise. But then u_1 cannot be completed with degree 3 - contradiction. Case II: A is not positive. If $d(v_3) = 5$, there must be a split or another vertex of degree > 3 in A or A would be positive, so send to A. Let $d(v_3) = 4$, so $l(v_3) = \bar{2}3\bar{4}\bar{5}$. If C splits, send there and now assume otherwise. Cannot complete $l_A(u_3)$ with degree 3 without splitting A or C so $d_A(u_3) > 3$. If $d_A(u_4) > 3$ or $d_A(u_5) > 3$, $c(A) \leq c(3,3,4,4,4) = -\frac{\pi}{6}$ so send to A. Now assume $d_A(u_4) = d_A(u_5) = 3$, so $l_A(u_5) = 541$ and $l_A(u_4) = 42\bar{5}$. If $d_A(u_3) = 4$, $l_A(u_3) = 43\bar{4}\bar{3}$, which splits C - contradiction. So $d_A(u_3) = 5$, $c(A), c(C) \leq c(3,3,3,4,5) = -\frac{\pi}{10}$ and so send $\frac{\pi}{12}$ each to A and C.

200: Case I: A is positive. Then A is 258 and $d(v_4) = 5$. If we have a split or further vertex of degree > 3 in B then send to B, so now assume otherwise. Now $l(v_4) = 55425$ or $5542\bar{1}$, both of which give a contradiction when completing the labels of B with degree 3 with no splits. Case II: A is not positive and $d(v_4) = 4$. Then $l(v_4) = 542\bar{3}$ which splits A so send to A. Case III: A is not positive and $d(v_4) = 5$. Must be a split or another vertex of degree > 3 or A would be positive we are so done.

258: Case I: B is positive. Then B is 279. This splits the $(3,4)$-edge of C so if there is another split or a vertex of degree > 3 in C then send to C. Now assume otherwise, which gives a contradiction when trying to complete the labels with degree 3 with no splits. Case II: B is not positive and $d(v_5) = 5$. Must be a split or another vertex of degree > 3 in B or would be positive so send to B. Case III: B is not positive and $d(v_5) = 4$. If $l(v_5) = 5545$, A splits along $(4,5)$ so send to A. The other potential labels split B along $(4,5)$ so then send to B.

175A

Compatible regions which do not cross the same edge are as follows: 114, 175B, 175C, 200A, 258A, 258B, 258C.

Assume two lots are sent in:

Regions	Outcome
175B, 258B, 258C	Does not fit (recall that 258C requires 279, which is not compatible with 175).
114, 258A	At most $\frac{\pi}{3}$ sent in. Region gives $d(u_4) > 3$ and a split, so $c(\Delta') \leq c(3,3,3,3,4,4) = -\frac{\pi}{3}$.
175C	$d(u_2) \geq 4$ and $d(u_5) \geq 4$ and if there is a split we are done. If $d(u_2) = 5$, $c(\Delta') \leq c(3,3,4,4,5) = -\frac{4\pi}{15}$ and $\frac{4\pi}{15}$ send in so assume $d(u_2) = 4$, $d(u_3) > 3$. If $d(u_3) = 4$, region gives a split so we are done. Otherwise, $d(u_3) = 5$ and at most $\frac{\pi}{6} + \frac{\pi}{12} = \frac{\pi}{4}$ is sent in, so $c(\Delta') \leq c(3,3,4,4,5) = -\frac{4\pi}{15}$.
200A	$d(u_2) \geq 4$ and $d(u_5) \geq 4$. Either $d(u_5) = 4$ and region causes a split so we are done or $d(u_5) = 5$. If $d(u_2) = 5$ also, $\frac{2\pi}{15}$ is sent in and $c(3,3,3,5,5) = -\frac{\pi}{5}$. If $d(u_2) = 4$ then either a split or $d(u_3) > 3$, so $\frac{7\pi}{30}$ sent in and $c(3,3,4,4,5) = -\frac{4\pi}{15}$.

Assume three lots are sent in:

Regions	Outcome
$\{175C, 200A\}$	Does not fit.
$\{114, 175C\}$, $\{114, 200A\}$, $\{175C, 258A\}$, $\{200A, 258A\}$	At most $\frac{\pi}{2}$ sent in. One region gives a split and $d(u_4) > 3$ and the other gives $d(u_5) > 3$, so $c(\Delta') \leq c(3,3,3,4,4,4) = -\frac{\pi}{2}$.

We cannot have four lots sent in as would need to be three of 114, 175C, 200A and 258A, but 114 and 258A cross the same edge and 175C and 200A do not fit together.

175B

Compatible regions which do not cross the same edge are as follows: 30, 114, 175C, 200A, 200B, 258A, 258B, 258C.

Assume two lots are sent in:

Regions	Outcome
114, $175C$, $200A$, $258A$, $258B$, $258C$	Does not fit.
30, $200B$	Region gives a $(5,1)$-split with proper sublabel $w^{-1}w$ - contradiction.

Clearly we cannot have three lots sent in.

175*C*

Compatible regions which do not cross the same edge are as follows: 30, 114, $200A$, $200B$, $258A$, $258B$, $258C$.

Assume two lots are sent in:

Regions	Outcome
$200A$, $258B$, $258C$	Does not fit.
30, $200B$	Region gives a $(5,1)$-split with proper sublabel $w^{-1}w$ - contradiction.
114	At most $\frac{\pi}{3}$ sent in. Region gives $d(u_4) > 3$ and a $(2,3)$-split, so $c(\Delta') \leq c(3,3,3,3,4,4) = -\frac{\pi}{3}$.
$258A$	Region gives a $(4,5)$-split with proper sublabel $w^{-1}w$ - contradiction.

Clearly we cannot have three lots sent in.

200*B*

Compatible regions which do not cross the same edge are as follows: 31, 42, 71, 74, 80, 114, 121, 159, $200A$, $258A$, $258B$, $258C$, 279.

Assume two lots are sent in and note that $d(u_2) = 5$:

Regions	Outcome
31, 42, 71, 74, 80, 121, 159, $200A$, $258C$	Does not fit.
114, $258A$	At most $\frac{7\pi}{30}$ sent in. Region gives $d(u_4) > 3$ and a split, so $c(\Delta') \leq c(3,3,3,3,4,5) = -\frac{13\pi}{30}$.
$258B$	At most $\frac{7\pi}{30}$ sent in. Region gives $d(u_5) > 3$. If $d(u_5) = 5$ then $c(3,3,3,5,5) = -\frac{\pi}{5}$ and $\frac{2\pi}{15}$ is sent in. If $d(u_5) = 4$, there is a $(4,5)$-split and $c(3,3,3,3,4,5) = -\frac{13\pi}{30}$.

279	Region gives a $(5,1)$-split with proper sublabel ww^{-1} - contradiction.

If three lots are sent in must be with $258B$ and either 114 or $258A$ as 114 and $258A$ cross the same edge. Then $d(u_2)=5,\ d(u_5)\geq 4,\ d(u_4)\geq 4$ and there is a $(2,3)$-split or a $(4,5)$-split from 114 or $258A$ respectively, so $c(3,3,3,4,4,5)=-\frac{3\pi}{5}$ and $\frac{2\pi}{5}$ is sent in.

200*A*

Compatible regions which do not cross the same edge are as follows: 30, 31, 42, 71, 80, 114, 121, 159, $258A$, 279.
Assume two lots are sent in:

Regions	Outcome
30, 279	Does not fit.
31, 42, 71, 80, 121, 159	Region gives a $(1,2)$-split with proper sublabel $w^{-1}w$ - contradiction.
114, $258A$	At most $\frac{\pi}{3}$ sent in. Region gives $d(u_4)>3$ and a split, so $c(\Delta')\leq c(3,3,3,3,4,4)=-\frac{\pi}{3}$.

Three lots cannot be sent in as 144 and 258A cross the same edge.

258*A*

Compatible regions which do not cross the same edge are as follows: 30, 31, 42, 71, 74, 80, 121, 159, $258B$, $258C$, 279.
Assume two lots are sent in and note that $d(u_4)=4$ and $(4,5)$ splits:

Regions	Outcome
31, 42, 71, 80, 121, 159, $258C$, 279	Does not fit.
30	At most $\frac{\pi}{3}$ sent in. Region gives a $(5,1)$-split and $d(u_2)>3$ so $c(\Delta')\leq c(3,3,3,3,3,4,4)=-\frac{2\pi}{3}$
74	At most $\frac{\pi}{3}$ sent in. Region gives a $(1,2)$-split and $d(u_5)>3$ so $c(\Delta')\leq c(3,3,3,3,3,4,4)=-\frac{2\pi}{3}$.
$258B$	At most $\frac{\pi}{3}$ sent in. Region gives $d(u_5)>3$ so $c(\Delta')\leq c(3,3,3,3,4,4)=-\frac{\pi}{3}$.

Three lots cannot be sent in as 74 and $258B$ cross $(5,1)$ and 30 splits $(5,1)$.

258*B* and **258*C***

Compatible regions which do not cross the same edge are as follows: 30, 31, 42, 71, 80, 114, 121, 159, 279.

Assume two lots are sent in:

Regions	Outcome
30, 279	Does not fit.
31, 42, 71, 80, 159	Region gives a $(1,2)$-split with proper sublabel $w^{-1}w$ - contradiction.
114	At most $\frac{\pi}{3}$ sent in. Can only be $258B$ as $258C$ requires 279 and 114 and 279 are not compatible. Region gives $d(u_4) > 3$ and a $(2,3)$-split, so $c(3,3,3,3,4,4) = -\frac{\pi}{3}$.

Clearly we cannot have three lots sent in.

Group III

79	32, 177, **210**, **259**
210	32, **79**, 112, 117, 148, 172, 177, 189, **259**
259	32, **79**, 112, 117, 148, 172, 177, 189, **210**

This group brings about exception *(i)* that the equalities $l_2 = l_1 + l_3$ and $l_2 + l_4 + l_5 = 0$ do not hold, and by symmetry *(ii)*, in Case (3) of Theorem 1.12. The exception given allows us to disregard regions 210 and 259, whose labels are $l(v_1) = 31\bar{2}$, $l(v_2) = \bar{3}2\bar{1}$, $l(v_3) = \bar{2}31$, $l(v_4) = 543\omega$, $l(v_5) = 254$ and $l(v_1) = 31\bar{2}$, $l(v_2) = \bar{3}2\bar{1}$, $l(v_3) = \bar{2}31$, $l(v_4) = 542$, $l(v_5) = 154\omega$ respectively. The exception also rules out region 79, although this is not one of the regions which causes the problem, which we describe next.

If we were to allow the mentioned equalities and therefore the regions 210 and 259, it is possible to end up with the following situation.

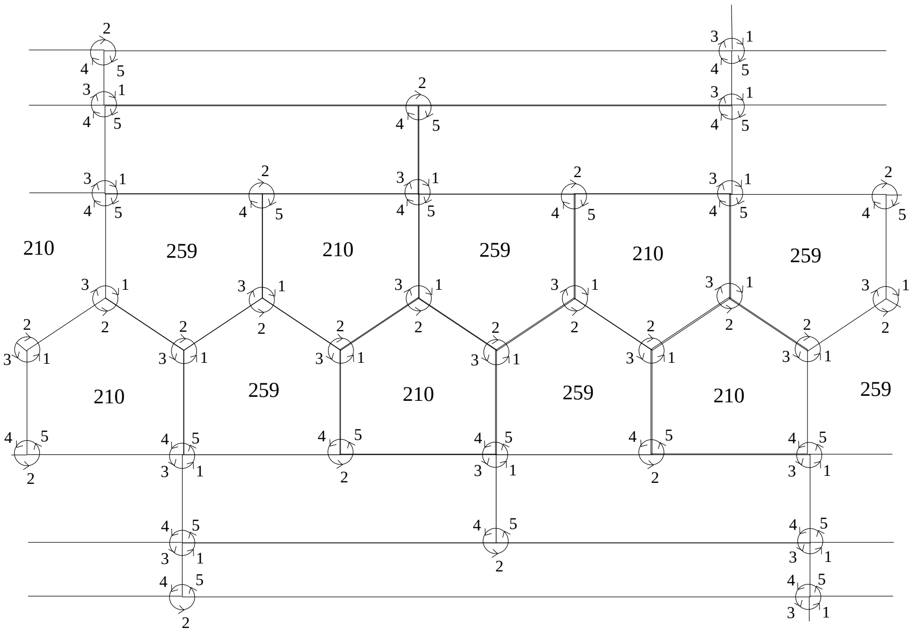


Figure 2.64: 210 and 259 together

All regions in this figure other than 210 and 259 have degree $-\frac{\pi}{6}$ and, again, we have not been able to compensate for the positive regions. Therefore the restriction on the l_i's is required.

This completes the proof of Lemma 1.17 in this case, with the mentioned exceptions.

Chapter 3

Theorem 1.12 cases (I) 4 and (II)

This chapter shall be concerned with the proof of Lemma 1.17 for the cases (I)(4) and (II) in Theorem 1.12. We will concentrate on (I)(4) first of all, that is, the cases for which $r(t)$ is of the form (B1)-(B6).

As in the previous chapter, we need to locate regions of positive curvature and distribute this curvature to one or more nearby regions which are able to compensate for this curvature. We will see that this is done in a similar way to the previous chapter, apart from the cases B4(b) and B5(b), which require a slightly different approach.

We have the added complication for the B cases that we have the potential for v and w to be subwords of each other. We assume that they are not equal to each other or to the other's inverse or we are back to the A cases. However, added complications arise when v or w is a proper subword of the other, or a proper subword of the other's inverse. For now, we assume this is not the case and we will see in Section 3.3 why this situation causes further complication and in which particular situations we get a problem.

The same result as in Lemma 2.2 from the previous chapter, for regions of the type shown in Figure 2.2, follows to this chapter. This, along with the assumption that no v or w is a proper subword of the other, means that for now we need only consider positive regions whose vertices are either w-vertices (which here can have label w or v) or v_i.

3.1 Case 4

For each case, we will display the possible labellings of Δ, an interior region of positive curvature, and check if there are any which do not give a contradiction and therefore give us a region of positive curvature. As before, we allow the degree of at most one of the v_i to exceed 3, and this degree must then be 4 or 5.

Unlike the A cases, there are much fewer possible labellings and so the work may be done by hand instead of using a computer. Much of the work in this section is the same as in the original case of Theorem 1.9, which can be viewed in [9].

3.1.1 $r(t) = wt^{l_1}wt^{l_2}wt^{l_3}vt^{l_4}vt^{l_5}$ (B1(a))

Let $r(t)$ be of the form B1(a) and so $r(t) = wt^{l_1}wt^{l_2}wt^{l_3}vt^{l_4}vt^{l_5}$.

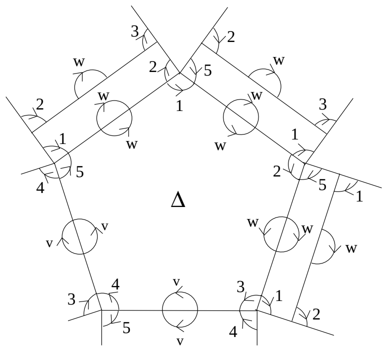


Figure 3.1: All possible labellings for B1(a)

If $d(v_4) = 3$, $l(v_4) = \bar{5}4\bar{3}$.
If $d(v_3) = 3$, $l(v_3) \in \{\bar{1}3\bar{4}, \bar{2}3\bar{4}\}$.
If $d(v_5) = 3$, $l(v_5) \in \{\bar{4}5\bar{1}, \bar{4}5\bar{2}\}$.
So $d(v_4) = d(v_3) = 3$ gives a contradiction and $d(v_4) = d(v_5) = 3$ gives a contradiction and so, as there can only be one vertex of degree > 3, $d(v_4) > 3$ and $d(v_i) = 3$, $\forall i \neq 4$.
Let $l(v_5) = \bar{4}5\bar{1}$. Then $l(v_3) = \bar{2}3\bar{4}$, $l(v_2) = \bar{1}2\bar{1}$, which leaves us with $l(v_1) = \bar{2}1\bar{5}$ which gives a contradiction.
Let $l(v_5) = \bar{4}5\bar{2}$. Then $l(v_3) = \bar{1}3\bar{4}$ and $l(v_2) \in \{\bar{1}2\bar{5}, \bar{3}2\bar{5}\}$, both of which give a contradiction.

There are no regions of positive curvature in this case.

3.1.2 $r(t) = wt^{l_1}wt^{l_2}wt^{l_3}vt^{l_4}v^{-1}t^{l_5}$ (B1(b))

Let $r(t)$ be of the form B1(b) and so $r(t) = wt^{l_1}wt^{l_2}wt^{l_3}vt^{l_4}v^{-1}t^{l_5}$.

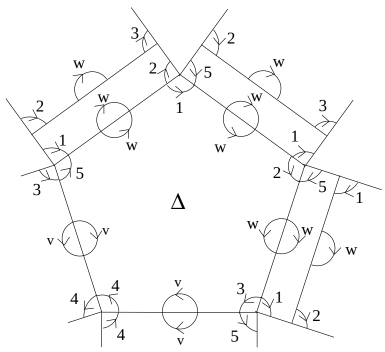


Figure 3.2: All possible labellings for B1(b)

In this case $d(v_4) > 3$, $d(v_i) = 3$, $\forall i \neq 4$.
Let $l(v_5) = 35\bar{1}$. Then $l(v_3) = \bar{1}35$, $l(v_2) = \bar{1}2\bar{5}$, leaving $l(v_1) = \bar{2}1\bar{5}$ which is a contradiction.
Let $l(v_5) = 35\bar{2}$. The $l(v_3) = \bar{2}35$, $l(v_2) = \bar{1}2\bar{1}$, leaving $l(v_1) = \bar{3}1\bar{5}$ which is a contradiction.

There are no regions of positive curvature in this case.

3.1.3 $r(t) = wt^{l_1}wt^{l_2}w^{-1}t^{l_3}vt^{l_4}vt^{l_5}$ (B2(a))

Let $r(t)$ be of the form B2(a) and so $r(t) = wt^{l_1}wt^{l_2}w^{-1}t^{l_3}vt^{l_4}vt^{l_5}$.

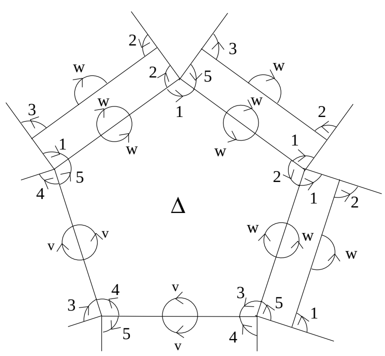


Figure 3.3: All possible labellings for B2(a)

Let $d(v_i) = 3, \ \forall i$.
Then $l(v_3) = l(v_4) = l(v_5) = 53\bar{4}$, $l(v_2) = 221$, forcing $l(v_1) = 213$, contradiction.

Let $d(v_1) > 3$, $d(v_i) = 3, \ \forall i \neq 1$.
Then $l(v_3) = l(v_4) = l(v_5) = 53\bar{4}$, $l(v_2) = 221$. Send curvature as shown in (a) of Diagram 3.4 below.

Let $d(v_2) > 3$, $d(v_i) = 3, \ \forall i \neq 2$.
Then $l(v_3) = l(v_4) = l(v_5) = 53\bar{4}$, $l(v_1) \in \{213, 21\bar{5}\}$.
If $l(v_1) = 213$ then send curvature as shown in (b) of Diagram 3.4 below.
If $l(v_1) = 21\bar{5}$ then send curvature as shown in (c) of Diagram 3.4 below.

Let $d(v_3) > 3$, $d(v_i) = 3, \ \forall i \neq 3$.
Then $l(v_4) = l(v_5) = 53\bar{4}$, $l(v_1) = 21\bar{5}$, $l(v_2) = \bar{1}22$. Send curvature as shown in (d) of Diagram 3.4 below.

Let $d(v_4) > 3$, $d(v_i) = 3, \ \forall i \neq 4$.
Then $l(v_5) \in \{\bar{4}5\bar{1}, \bar{4}53\}$.
If $l(v_5) = \bar{4}5\bar{1}$ then $l(v_1) = \bar{2}13$, $l(v_2) = 221$, forcing $l(v_3) = 53\bar{4}$, contradiction.
If $l(v_5) = \bar{4}53$ then $l(v_3) = 53\bar{4}$, $l(v_2) = 221$, $l(v_1) = 213$, contradiction.

Let $d(v_5) > 3$, $d(v_i) = 3, \ \forall i \neq 5$.
Then $l(v_3) = l(v_4) = 53\bar{4}$, $l(v_2) = 122$, $l(v_1) = 13\bar{2}$: send curvature as shown in (e) of Diagram 3.4 below.

In the diagram we have the following:

(a) Curvature sent across the $(2,3)$-edge, $d(u_3) > 3$, $(1,2)$-split.

(b) Curvature sent across the $(2,3)$-edge, $d(u_2) > 3$, $(3,4)$-split.

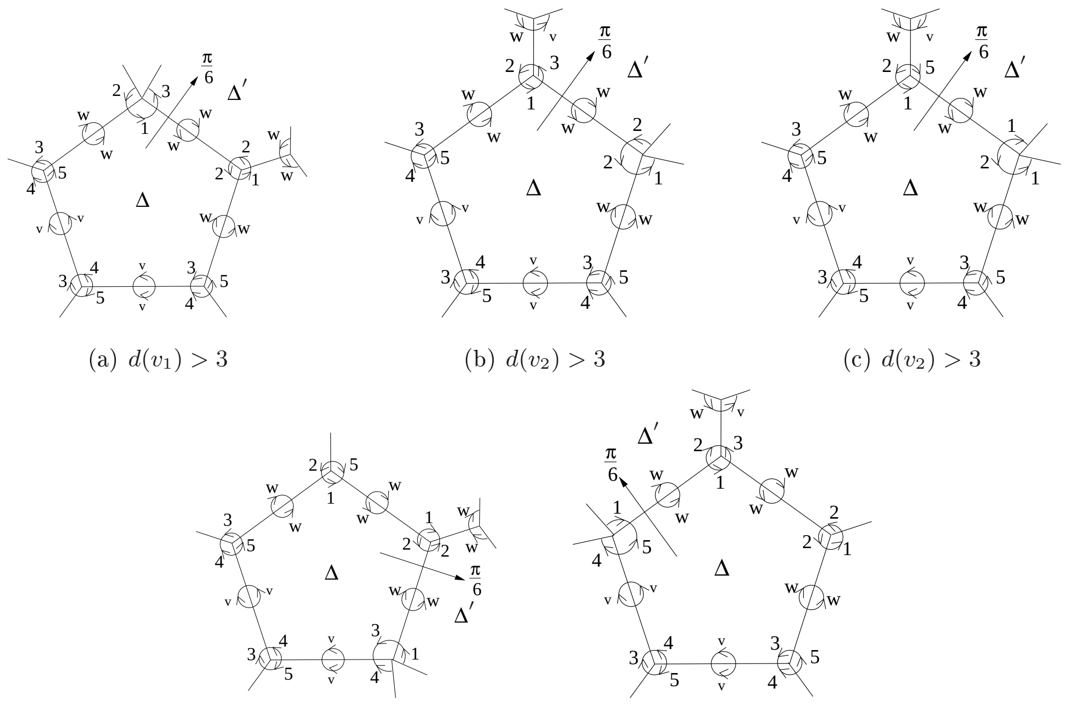
(c) Curvature sent across the $(5,1)$-edge, $d(u_1) > 3$, $(4,5)$-split.

(d) Curvature sent across the $(1,2)$-edge, $d(u_1) > 3$, $(2,3)$-split.

(e) Curvature sent across the $(1,2)$-edge, $d(u_1) > 3$, $(2,3)$-split.

In each case, if there is only one sending of curvature to any of the Δ', then $\frac{\pi}{6}$ is sent in and $c(\Delta') \leq c(3,3,3,3,3,4) = -\frac{\pi}{6}$.

If two lots of curvature is sent in to any of the Δ', it must be from case (c) plus one of the others as (d) and (e) both split $(2,3)$. In case (c), $l(v_1) = 21\bar{5}$, which contradict the labels in (a), (b) and (e), so the only possibility is (c) with (d). At most $\frac{\pi}{3}$ is being sent in and we get a $(2,3)$-split, a $(4,5)$-split and $d(u_1) > 3$ so $c(\Delta') \leq c(3,3,3,3,3,3,4) = -\frac{\pi}{2}$, which is more than enough to compensate.



(a) $d(v_1) > 3$ (b) $d(v_2) > 3$ (c) $d(v_2) > 3$

(d) $d(v_3) > 3$ (e) $d(v_5) > 3$

Figure 3.4: Curvature distribution in B2(a)

3.1.4 $r(t) = wt^{l_1}wt^{l_2}w^{-1}t^{l_3}vt^{l_4}v^{-1}t^{l_5}$ (B2(b))

Let $r(t)$ be of the form B2(b) and so $r(t) = wt^{l_1}wt^{l_2}w^{-1}t^{l_3}vt^{l_4}v^{-1}t^{l_5}$.

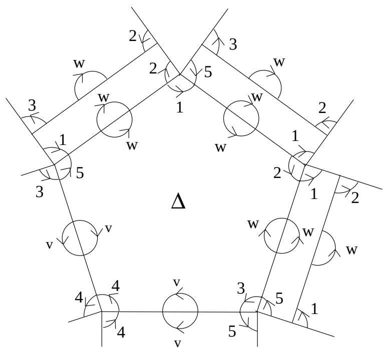


Figure 3.5: All possible labellings for B2(b)

In this case $d(v_4) > 3$, $d(v_i) = 3$, $\forall i \neq 4$.
$l(v_5) \in \{35\bar{1}, 353\}$, $l(v_3) \in \{535, 135\}$. Any choice for $l(v_5)$ and $l(v_3)$ gives a contradiction.

There are no regions of positive curvature in this case.

3.1.5 $r(t) = wt^{l_1}w^{-1}t^{l_2}wt^{l_3}vt^{l_4}vt^{l_5}$ (B3(a))

Let $r(t)$ be of the form B3(a) and so $r(t) = wt^{l_1}w^{-1}t^{l_2}wt^{l_3}vt^{l_4}vt^{l_5}$.

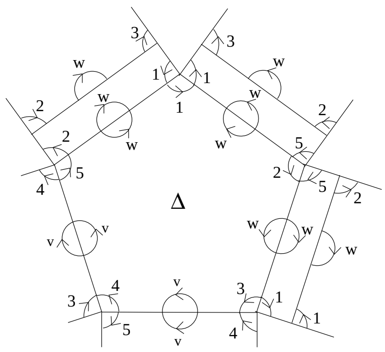


Figure 3.6: All possible labellings for B3(a)

If $d(v_4) = 3$, $l(v_4) = \bar{5}4\bar{3}$.
If $d(v_3) = 3$, $l(v_3) \in \{\bar{1}3\bar{4}, 13\bar{4}\}$.
If $d(v_5) = 3$, $l(v_5) \in \{\bar{4}52, \bar{4}5\bar{2}\}$.
So $d(v_4) = d(v_3) = 3$ gives a contradiction and $d(v_4) = d(v_5) = 3$ gives a contradiction and so, as there can only be one vertex of degree > 3, we must have $d(v_4) > 3$. Therefore, $d(v_i) = 3$, $\forall i \neq 4$.
Let $l(v_5) = \bar{4}52$. Then $l(v_1) = 113$, $l(v_2) = 22\bar{5}$, $l(v_3) = \bar{1}3\bar{4}$. $l(v_1)$ and $l(v_3) \implies -l_4 = 3l_1$. $l(v_2)$ and $l(v_5) \implies l_4 = 3l_2 \implies l_1 = -l_2$, which is a contradiction.
Let $l(v_5) = \bar{4}5\bar{2}$. Then $l(v_1) = \bar{3}11$, $l(v_2) = 522$, $l(v_3) = 13\bar{4}$. $l(v_1)$ and $l(v_3) \implies l_4 = 3l_1$. $l(v_2)$ and $l(v_5) \implies l_4 = -3l_2 \implies l_1 = -l_2$, which is a contradiction.

There are no regions of positive curvature in this case.

3.1.6 $r(t) = wt^{l_1}w^{-1}t^{l_2}wt^{l_3}vt^{l_4}v^{-1}t^{l_5}$ (B3(b))

Let $r(t)$ be of the form B3(b) and so $r(t) = wt^{l_1}w^{-1}t^{l_2}wt^{l_3}vt^{l_4}v^{-1}t^{l_5}$.

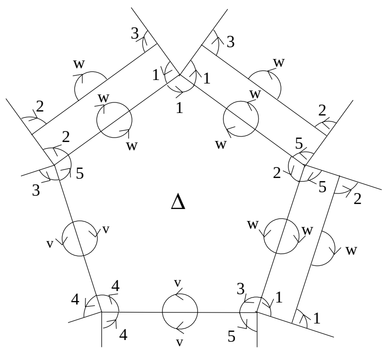


Figure 3.7: All possible labellings for B3(b)

In this case $d(v_4) > 3$, $d(v_i) = 3$, $\forall i \neq 4$.
$l(v_5) \in \{352, 35\bar{2}\}$, $l(v_3) \in \{\bar{1}35, 135\}$. Any choice for $l(v_5)$ and $l(v_3)$ gives a contradiction.

There are no regions of positive curvature in this case.

3.1.7 $r(t) = wt^{l_1}wt^{l_2}vt^{l_3}wt^{l_4}vt^{l_5}$ (B4(a))

Let $r(t)$ be of the form B4(a) and so $r(t) = wt^{l_1}wt^{l_2}vt^{l_3}wt^{l_4}vt^{l_5}$.

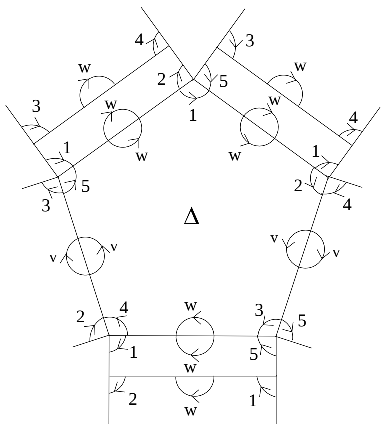


Figure 3.8: All possible labellings for B4(a)

If $d(v_3) = d(v_5) = 3$, $l(v_3) \in \{\bar{3}55, \bar{3}51\}$, $l(v_5) \in \{\bar{3}5\bar{1}, \bar{3}5\bar{3}\}$, which gives a contradiction, so one of the degrees must be > 3.
If $d(v_2) = d(v_4) = 3$, $l(v_2) \in \{\bar{1}2\bar{4}, \bar{4}2\bar{4}\}$, $l(v_4) \in \{\bar{1}4\bar{2}, \bar{2}4\bar{2}\}$, which gives a contradiction, so one of the degrees must be > 3.
But there can be at most one vertex of degree > 3 so this gives a contradiction.

There are no regions of positive curvature in this case.

3.1.8 $r(t) = wt^{l_1}wt^{l_2}vt^{l_3}w^{-1}t^{l_4}vt^{l_5}$ (B5(a))

Let $r(t)$ be of the form B5(a) and so $r(t) = wt^{l_1}wt^{l_2}vt^{l_3}w^{-1}t^{l_4}vt^{l_5}$.

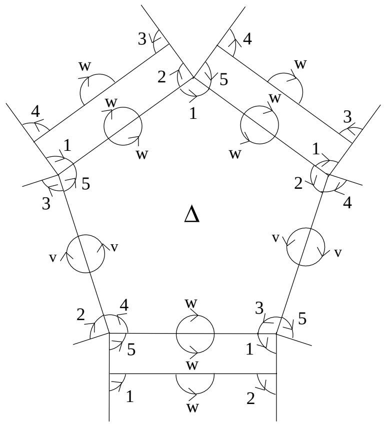


Figure 3.9: All possible labellings for B5(a)

Let $d(v_3) = d(v_5) = 3$. Then $l(v_3) = l(v_5) = \bar{3}5\bar{1}$. If $d(v_4) = 3$, $l(v_4) = 54\bar{2}$, if $d(v_1) = 3$, $l(v_1) \in \{\bar{2}1\bar{5}, \bar{2}14\}$ and if $d(v_2) = 3$, $l(v_2) \in \{\bar{1}2\bar{4}, 32\bar{4}\}$. So $d(v_4) = d(v_1) = 3$ and $d(v_4) = d(v_2) = 3$ both give a contradiction so either $d(v_4) > 3$, $d(v_3) > 3$ or $d(v_5) > 3$.
Let $d(v_4) > 3$, $d(v_i) = 3$, $\forall i \neq 4$.
Then $l(v_1) = \bar{2}14$ or would contradict $l(v_3) = l(v_5) = \bar{3}5\bar{1}$, so $l(v_2) = 32\bar{4}$, contradiction.
Let $d(v_3) > 3$, $d(v_i) = 3$, $\forall i \neq 3$.
Then $l(v_2) = l(v_4) = 2\bar{4}\bar{1}$, $l(v_1) \in \{\bar{2}1\bar{5}, 31\bar{5}\}$ so it must be $31\bar{5}$ or we get a contradiction.
But then $l(v_5) = \bar{3}54$, contradiction.
Let $d(v_5) > 3$, $d(v_i) = 3$, $\forall i \neq 5$.
$l(v_2) = l(v_4) = 2\bar{4}\bar{1}$, $l(v_3) = \bar{5}32$ and $l(v_1) \in \{\bar{2}1\bar{5}, 31\bar{5}\}$, both of which give a contradiction.

There are no regions of positive curvature in this case.

3.1.9 $r(t) = wt^{l_1}w^{-1}t^{l_2}vt^{l_3}wt^{l_4}vt^{l_5}$ (B6(a))

Let $r(t)$ be of the form B6(a) and so $r(t) = wt^{l_1}w^{-1}t^{l_2}vt^{l_3}wt^{l_4}vt^{l_5}$.

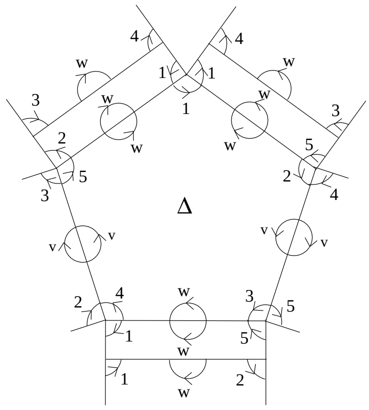


Figure 3.10: All possible labellings for B6(a)

If $d(v_3) = d(v_5) = 3$, $l(v_3) \in \{\bar{3}55, \bar{3}5\bar{2}\}$, $l(v_5) \in \{\bar{3}52, \bar{3}5\bar{3}\}$, which gives a contradiction, so one of the degrees must be > 3.
If $d(v_2) = d(v_4) = 3$, $l(v_2) \in \{52\bar{4}, 32\bar{4}\}$, $l(v_4) \in \{\bar{1}4\bar{2}, 14\bar{2}\}$, which gives a contradiction, so one of the degrees must be > 3.
But there can be at most one vertex of degree > 3 so this gives a contradiction.

There are no regions of positive curvature in this case.

3.1.10 $r(t) = wt^{l_1}w^{-1}t^{l_2}vt^{l_3}wt^{l_4}v^{-1}t^{l_5}$ (B6(b))

Let $r(t)$ be of the form B6(b) and so $r(t) = wt^{l_1}w^{-1}t^{l_2}vt^{l_3}wt^{l_4}v^{-1}t^{l_5}$.

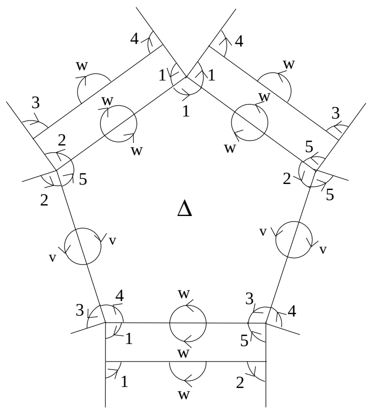


Figure 3.11: All possible labellings for B6(b)

If $d(v_3) = d(v_4) = 3$, $l(v_3) \in \{43\bar{5}, 432\}$, $l(v_4) \in \{\bar{1}43, 143\}$, which gives a contradiction, so one of the degrees must be > 3.
If $d(v_2) = d(v_5) = 3$, $l(v_2) \in \{525, 325\}$, $l(v_5) \in \{252, 25\bar{3}\}$, which gives a contradiction, so one of the degrees must be > 3.
But there can be at most one vertex of degree > 3 so this gives a contradiction.

There are no regions of positive curvature in this case.

We now look at cases B4(b) and B5(b), which were missed out previously due to them being more difficult cases.

3.1.11 $r(t) = wt^{l_1}wt^{l_2}vt^{l_3}wt^{l_4}v^{-1}t^{l_5}$ (B4(b))

For this case, $r(t)$ is of the form B4(b) and so $r(t) = wt^{l_1}wt^{l_2}vt^{l_3}wt^{l_4}v^{-1}t^{l_5}$. We shall examine the possible labellings of a region with positive curvature and see that the situation becomes more difficult when $l_1 = l_2 + l_5 = l_3 + l_4$.

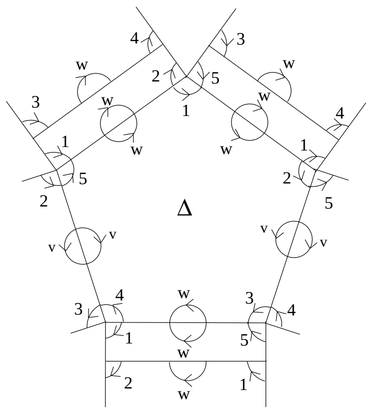


Figure 3.12: All possible labellings for B4(b)

If $d(v_3) = 3$, $l(v_3) \in \{43\bar{5}, 43\bar{1}\}$.
If $d(v_4) = 3$, $l(v_4) \in \{\bar{1}43, \bar{2}43\}$.
We cannot have $l(v_3) = l(v_4) = 43\bar{1}$ so we must have $d(v_3) > 3$ or $d(v_4) > 3$.
Then $l(v_5) \in \{25\bar{1}, 25\bar{3}\}$, $l(v_2) \in \{\bar{1}25, \bar{4}25\}$ so $l(v_2) = l(v_5) = 25\bar{1} = l(v_1)$.

There are 4 possibilities for regions of positive curvature.
1: $l(v_1) = \bar{2}1\bar{5}$, $l(v_2) = \bar{1}25$, $l(v_3) = 43\bar{1}\omega$, $l(v_4) = \bar{2}43$, $l(v_5) = 25\bar{1}$.
2: $l(v_1) = \bar{2}1\bar{5}$, $l(v_2) = \bar{1}25$, $l(v_3) = 43\bar{5}$, $l(v_4) = \bar{1}43\omega$, $l(v_5) = 25\bar{1}$.
3: $l(v_1) = \bar{2}1\bar{5}$, $l(v_2) = \bar{1}25$, $l(v_3) = 43\bar{5}\omega$, $l(v_4) = \bar{1}43$, $l(v_5) = 25\bar{1}$.
4: $l(v_1) = \bar{2}1\bar{5}$, $l(v_2) = \bar{1}25$, $l(v_3) = 43\bar{1}$, $l(v_4) = \bar{2}43\omega$, $l(v_5) = 25\bar{1}$.

The way in which curvature can be distributed for 1 and 2 is as follows.

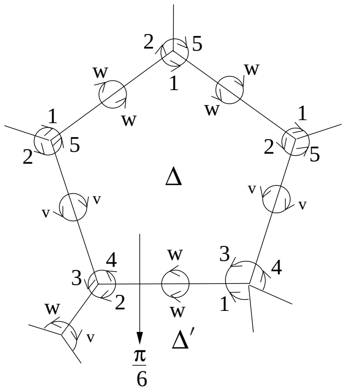


Figure 3.13: no. 1

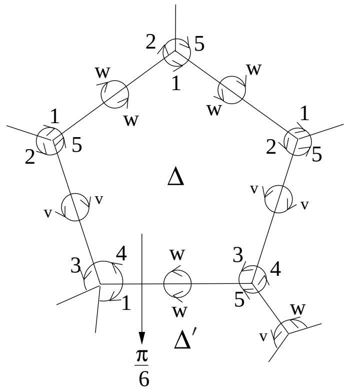


Figure 3.14: no. 2

The labels for these two regions cannot appear together and so Δ' cannot receive curvature from both 1 and 2 at the same time. Regions 1 and 2 can also not appear with regions 3 and 4. Therefore Lemma 1.17 holds for these cases.

Now we consider regions 3 and 4 and we may obtain the following diagram which, like in the previous section, shows that the positive curvature cannot be compensated for in the usual way.

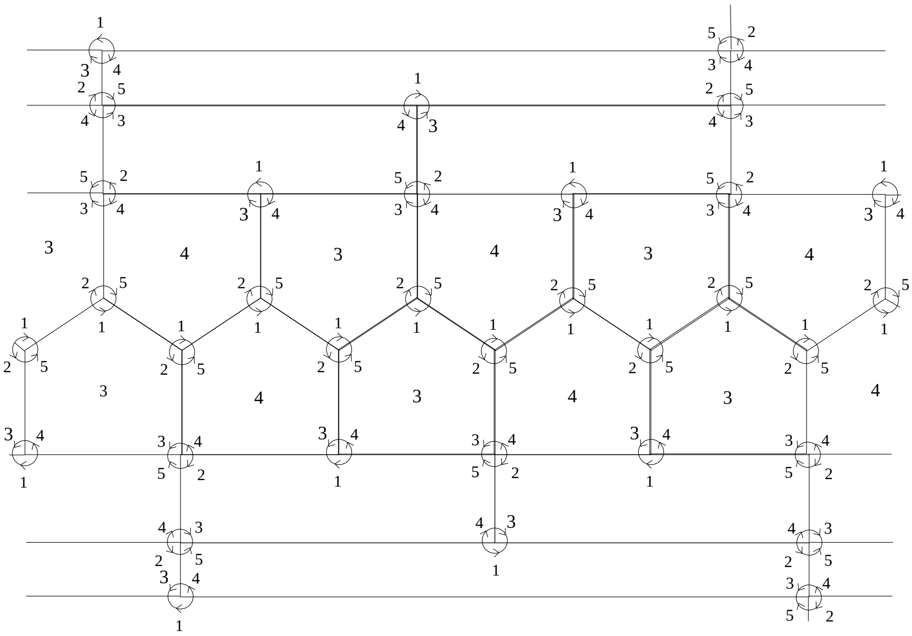


Figure 3.15: 3 and 4 together

All regions in this diagram other than 3 and 4 have degree $-\frac{\pi}{6}$ and so there are not enough sufficiently negative regions to compensate for the positive regions. Therefore, restrictions on the l_i seem to be required as in the previous section. However, for this particular situation we are able to use a different method so that restrictions on the l_i are not required. We deal with this situation in Section 3.2.

3.1.12 $r(t) = wt^{l_1}wt^{l_2}vt^{l_3}w^{-1}t^{l_4}v^{-1}t^{l_5}$ (B5(b))

For this case, $r(t)$ is of the form B5(b) and so $r(t) = wt^{l_1}wt^{l_2}vt^{l_3}w^{-1}t^{l_4}v^{-1}t^{l_5}$. We shall examine the possible labellings of a region with positive curvature and see that, similar to case B4(b), the situation becomes more difficult when $l_1 = l_2 + l_5$ and $l_1 + l_3 + l_4 = 0$.

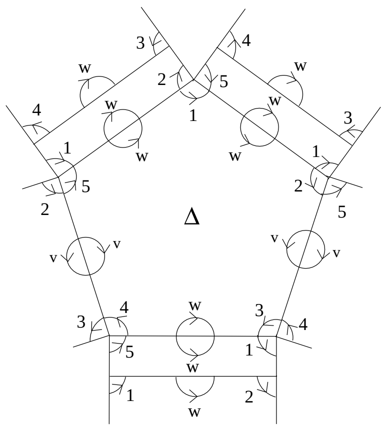


Figure 3.16: All possible labellings for B5(b)

If $d(v_3) = 3$, $l(v_3) \in \{431, 432\}$.
If $d(v_4) = 3$, $l(v_4) \in \{543, 143\}$.
We cannot have $l(v_3) = l(v_4) = 431$ so we must have $d(v_3) > 3$ or $d(v_4) > 3$.
Then $l(v_5) \in \{25\bar{1}, 254\}$, $l(v_2) \in \{\bar{1}25, 325\}$ so $l(v_5) = l(v_2) = 25\bar{1} = l(v_1)$.

There are 4 possibilities for regions of positive curvature.
1: $l(v_1) = \bar{2}1\bar{5}$, $l(v_2) = \bar{1}25$, $l(v_3) = 431\omega$, $l(v_4) = 543$, $l(v_5) = 25\bar{1}$.
2: $l(v_1) = \bar{2}1\bar{5}$, $l(v_2) = \bar{1}25$, $l(v_3) = 432$, $l(v_4) = 143\omega$, $l(v_5) = 25\bar{1}$.
3: $l(v_1) = \bar{2}1\bar{5}$, $l(v_2) = \bar{1}25$, $l(v_3) = 432\omega$, $l(v_4) = 143$, $l(v_5) = 25\bar{1}$.
4: $l(v_1) = \bar{2}1\bar{5}$, $l(v_2) = \bar{1}25$, $l(v_3) = 431$, $l(v_4) = 543\omega$, $l(v_5) = 25\bar{1}$.

The way in which curvature can be distributed for 1 and 2 is as follows.

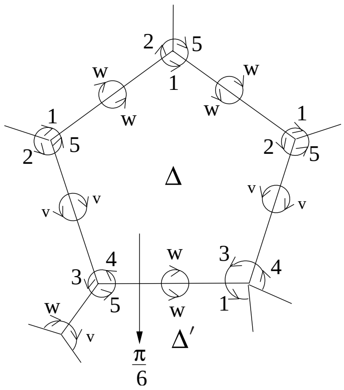


Figure 3.17: no. 1

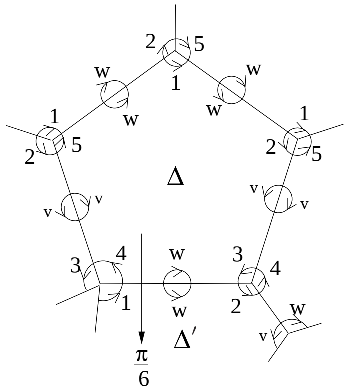


Figure 3.18: no. 2

The labels for these two regions cannot appear together and so Δ' cannot receive curvature from both 1 and 2 at the same time. Regions 1 and 2 can also not appear with regions 3 and 4. Therefore Lemma 1.17 holds for these cases.

Now we consider regions 3 and 4 and we may obtain the following picture which, like in the previous section, shows that the positive curvature cannot be compensated for in the usual way.

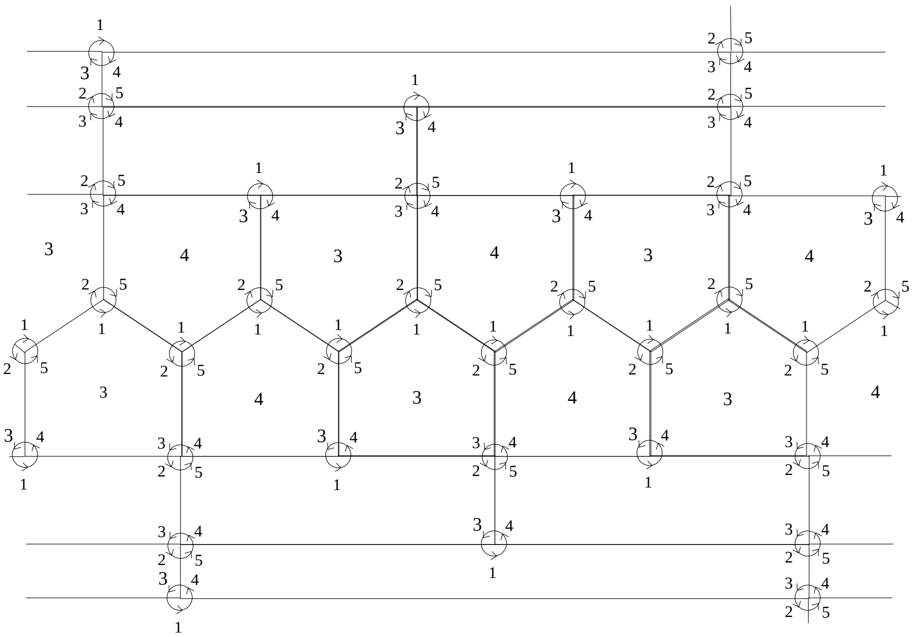


Figure 3.19: 3 and 4 together

Again, for this particular situation we are able to use a different method to prevent the restrictions being required and we deal with this in the next section.

3.2 Difficult cases for B4(b) and B5(b)

The two diagrams obtained in the previous section have the same structure as those that bring about the second two restrictions in Case (2) and the restrictions in Case (3). Therefore, it seems likely that we would require the same restrictions for these cases. However, using a new technique we are able to remove these restrictions and prove Lemma 1.17 for these situations also.

3.2.1 Case B4(b)

Let $r(t) = wt^{l_1}wt^{l_2}vt^{l_3}wt^{l_4}v^{-1}t^{l_5}$ and assume that $l_1 = l_2 + l_5 = l_3 + l_4$.
The problems are brought about by the situation occurring in Figure 3.15. If we return to examine this diagram, we see that if we remove all edges in the diagram with corner labels 43 at one end and 25 at the other, then we obtain strips.

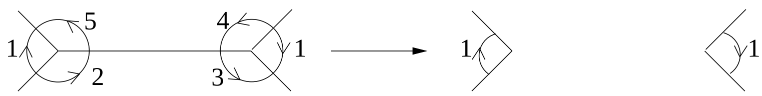


Figure 3.20: Removing edges

Lemma 3.1. *The boundaries of the strips obtained in the diagram are connected, i.e. the strips do not form an annulus.*

Proof. Assume by way of contradiction that these strips form an annulus. If we then deleted the two strips alternating regions 3 and 4, then we could rejoin the diagram as follows. Shift the bottom layer to the right so that it joins up with the top layer to create vertices with labels $\bar{1}(25)$. But this creates a diagram with fewer regions than before. However, we assumed that our original diagram was minimal with respect to the number of regions, and so we obtain a contradiction. □

Note that this lemma is required as the resulting diagram must be connected for Lemma 1.16 to hold.

It is the important to know what labellings may occur at each end of the strips to see whether or not our Lemma 1.17 now holds. We explain more precisely the method we are now using.

We begin with the amended diagram as described in Section 1.3. We now remove any vertex sublabel 43 or 25 and replace them with new labels (43) and (25). We also remove the corresponding edge, so that the new labels (43) and (25) each add one to the degree of the vertex. So an old vertex $43\bar{1}$ of degree 3 would now become $(43)\bar{1}$, which is a vertex of degree 2.

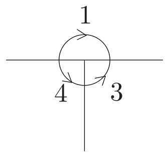


Figure 3.21: Amending vertices

We may assume from now on that we do not have the sublabels 43, $\bar{3}\bar{4}$, 25 or $\bar{5}\bar{2}$ anywhere in the diagram. We look for potential positive regions under all these conditions.

Assume there are no splits in Δ and observe Figure 3.12, which shows the same possible corner labels either side of the two vs, as the new labels begin and end with a w. Therefore, because only a v may match up with another v, v_2 has sublabel 25, v_3 has sublabel 43, v_4 has sublabel 43 and v_5 has sublabel 25. However, none of these sublabels are allowed and so we must have at least two splits, along the $(2, 3)$-edge and the $(4, 5)$-edge of Δ. The only original v_i whose degree could potentially be 2 is v_1 and so Δ has degree at least 6. Therefore, Δ is not positive and we have no regions of positive curvature in this case.

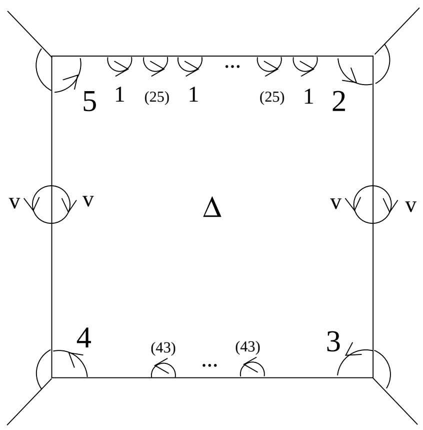


Figure 3.22: Typical region for B4(b)

3.2.2 Case B5(b)

Let $r(t) = wt^{l_1}wt^{l_2}vt^{l_3}w^{-1}t^{l_4}v^{-1}t^{l_5}$ and assume that $l_1 = l_2 + l_5$ and $l_1 + l_3 + l_4 = 0$. As in Subsection 3.2.1, we allow the new labels $(43), (25), (\overline{43}), (\overline{25})$ and we no longer allow the sublabels $43, 25, \bar{3}\bar{4}, \bar{5}\bar{2}$.

By the same argument as in Subsection 3.2.1, there are no regions of positive curvature in this case.

3.3 Subword problems

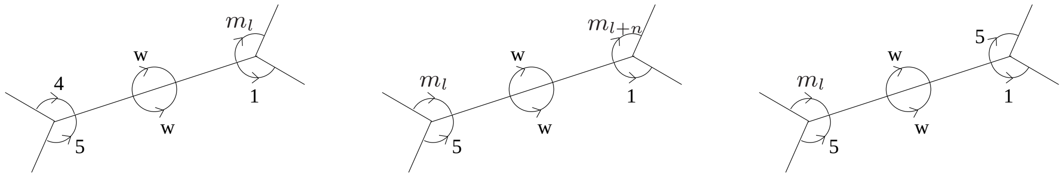
The results obtained so far are with us assuming that v and w are not subwords of each other. Before we explain the problems that may arise when dealing with the case when they are subwords, we will recall exactly what is meant by a subword.

We define a *subword* of the word $w = g_1 t^{m_1} g_2 \dots g_{s-1} t^{m_{s-1}} g_s$ where $g_i \in G \setminus \{1\},\ m_i \in \mathbb{Z} \setminus \{0\},\ s \geq 1$ to be a word of the form $g_k t^{m_k} g_{k+1} \dots g_{k+r-1} t^{m_{k+r-1}} g_{k+r}$ where $k \in \{1, \dots, s\}$ and $r \in \{0, \dots, s-k\}$. A subword is an *initial subword* if $k = 1$, an *end subword* if $r = s - k$ and a *proper subword* if $(k, r) \neq (1, s-k)$.

So for example, if in the above word we have $g_1 = g'g'',\ g', g'' \in G$ then g_1 is a subword but g' and g'' are not subwords. So a subword must begin either at the start of the original word or right after one of the t's and must end either at the end of the word or just before one of the t's.

Note that when we talk about v or w being a subword of the other, we are also talking about the situation where v^{-1} is a subword of w or w^{-1} is a subword of v.

The reason why v and w being subwords of each other causes difficulties is as follows. In the diagrams, we match up the v's and w's to look at all potential labellings. So for example, if w runs along the $(5,1)$-edge and the $(1,2)$-edge then $5\bar{1}$ is a potential sublabel for v_5 and $\bar{2}1$ is a potential sublabel for v_1 and we have so far assumed that this is the only type of label possible. However, if w is a sublabel of v then we may also obtain the following situations.



(a) $v = wu$ (b) $v = u_1wu_2$ (c) $v = uw$

Figure 3.23: A possible example for when w is a subword of v

In this diagram we let $v = g_1t^{m_1}g_2 \dots g_{s-1}t^{m_{s-1}}g_s$ and let w be a subword of the following form.

(a) $v = wt^{m_l}g_{l+1} \cdots g_{s-1}t^{m_{s-1}}g_s$,

(b) $v = g_1t^{m_1}g_2 \dots g_lt^{m_l}wt^{m_{l+n}}g_{l+n+1} \cdots g_{s-1}t^{m_{s-1}}g_s$,

(c) $v = g_1t^{m_1}g_2 \dots g_lt^{m_l}w$.

This demonstrates that when w is a subword of v, and similarly when v is a subword of w, there are many more possible labellings.

Theorem 1.12 states that we will not allow v or w to be a proper initial or end subwords of each other, which are the situations displayed in Diagram 3.23(a) and Diagram 3.23(c) respectively. Before explaining why these two situations cause such a problem, we will first show that allowing the situation in Diagram 3.23(b) does not cause us any further problems and therefore that Lemma 3.2 holds.

Lemma 3.2. *If we allow subwords of the form (b) above only, then Lemma 1.17 still holds.*

Proof. In this case, the region on the other side of the positive region, call it Δ' as usual (i.e. the region containing the labels m_l and m_{l+n}) has degree at least 7 as we must also have vertices labelled with the l_1 up to l_5. These m_i vertices give us a new kind of split, which we refer to as a *t-split*.
Now, every time we have such an edge, let us send curvature across this edge. We now have every way of sending curvature in all cases of the theorem and we need to check that curvature is still compensated for.
Assume that at step n all t-splits have marked degree 2 and that $c^*(\Delta', n) \leq 0$. Then one sending in the way just described allows us to mark two t-splits with marked degree

3 and any further allows us to mark at least one more t-split with marked degree 3. This is equivalent to each sending allowing us to mark at least one vertex with marked degree at least 3, and distributing at most $\frac{\pi}{3}$ of curvature to Δ'. Therefore, as $\frac{k}{3} - k + \frac{2k}{3} = 0$, after k sendings of curvature in this new way at step $n+1$, $c^*(\Delta', n+1) \leq 0$ by Lemma 1.20. Therefore, curvature is still able to be compensated for when allowing v or w to be a middle subword of the other. □

The same result as in Lemma 3.2 is not true for when v or w is an initial or end subword of the other. In these cases, if we look back at Diagram 3.23(a) and Diagram 3.23(c), we see that the degree of the region containing m_l could be 6 and therefore we could get the case that $\frac{\pi}{3}$ is being sent in and $c(\Delta') \leq c(3,3,3,3,3,4) = -\frac{\pi}{6}$ or $\frac{\pi}{6}$ is being sent in and $c(\Delta') \leq c(3,3,3,3,3,3) = 0$. In these cases, curvature is not obviously compensated for by Δ' and it would need to be checked for every possible case. There are so many more cases if this occurs that it is not done here and we therefore assume that we cannot have v or w as an initial or end subword of the other.

3.4 Case (II)

This case states that if at least one of the w_i's within $r(t) = w_1 t^{l_1} w_2 t^{l_2} w_3 t^{l_3} w_4 t^{l_4} w_5 t^{l_5}$ is not equal to plus or minus any other w_j and the mentioned conditions hold, then $r(t)$ has a solution over G.

Let w_i be such a word for a particular i.

Let us first assume that w_i is not a subword of any other w_j. Given the fact that the w-vertices must have degree 2 in order for a region to have positive curvature, such a region must have a w-vertex corresponding to w_i with the same w_i on the other side as this is the only possible matching. But this contradicts Lemma 1.15(i) so no such region of positive curvature exists.

Let w_i be a middle subword of w_j as described in Section 3.3 for some j. Then the region containing the w_j is of degree at least 7 and so we can send any positive curvature into this region by the same argument as that stated in Lemma 3.2.

If w_i were an initial or end subword then the corresponding region could be degree 6 and the same problems as mentioned in Subsection 3.3 could occur. By (***), we do not allow this situation to occur. Therefore, we have proved Lemma 1.17 for this case also.

Chapter 4

Proof of Theorem 1.12

Let D be the tessellation of the 2-sphere as described in Chapter 1. Then $c(D) = 4\pi$ by Lemma 1.16. After making all the described curvature distributions from interior regions Δ with $c(\Delta) > 0$, $c(D) \leq \Sigma c^*(\Delta')$ where the sum is taken over all interior regions of D such that $c^*(\Delta') > c(\Delta')$, and over all boundary regions of D. Now, Lemma 1.17 implies that $c(D) \leq \Sigma c^*(\Delta')$ where the sum is taken over all boundary regions Δ' of D only.

It is now necessary to prove that the equality $c(D) = 4\pi$ is not obtainable, which yields a contradiction.

For now we look at the case $k = 5$ only, where k is the free-product length of the equation as mentioned in Theorem 1.12.

4.1 Maximum curvature sent to a boundary region

We need to consider how much curvature may be sent into a boundary region. If we consider the crossings mentioned in Section 2.1 and in Section 3.3 involving t-powers other than the l_i, we see that the total amount is unbounded.

Let us recall what happens to the curvature of a region when such crossings take place. Recall that $c^*(\Delta') = (2-m)\pi + 2\pi\Sigma_{i=1}^{m}\frac{1}{d_i} + p$, where p is any curvature being sent in. The first type of crossing, mentioned in 2.1, sends $\frac{\pi}{6}$ and the second type of crossing, mentioned in 3.3, sends $\frac{\pi}{3}$ so at most $\frac{\pi}{3}$ is sent across each time and each such crossing increases the degree of the region by at least 1. As $\frac{1}{3} - 1 + \frac{2}{3} = 0$, these sorts of crossings either have no effect or decrease the total curvature of the region they are being sent to by Lemma 1.20. Therefore we may disregard these types of crossings.

Recall that v_0 is the distinguished vertex and let $d(v_0) = k_0$. A boundary region must contain this vertex plus the five t-vertices $v_1, \dots v_5$.

Curvature can be sent across each of the four edges not adjacent to v_0, as it must be sent from an interior region, and curvature may be also sent across at most two of the five v_i.

Note that we may in fact assume that curvature is sent across at most one of the v_i. This is because, if there were two sending across the v_i, they would be the sendings for A2 regions 43 and 232 as described in Section 2.3, Group I. These two crossings send in $\frac{2\pi}{15} + \frac{\pi}{6}$ but cause two splits and these splits each prohibit crossings of $\frac{\pi}{6}$. Therefore the curvature sendings would not be maximal if we assumed two sending across the v_i.

Whenever $\frac{\pi}{3}$ is sent in we get two splits and so we may assume that $\frac{\pi}{6}$ is sent in from each of the four edges in order to maximize potential curvature. Whenever $\frac{\pi}{6}$ is sent across a vertex we get a split so, again to maximise curvature sent in, we assume at most $\frac{\pi}{12}$ is sent in across one of the vertices, which happens in Section 2.4 without causing a split.

Therefore, without any loss we can assume that the most is being sent in to a boundary region is $\frac{4\pi}{6} + \frac{\pi}{12} = \frac{3\pi}{4}$.

4.2 Checking total curvature

Suppose $k_0 = 1$: a single boundary region, denoted $\hat{\Delta}$.
Since any region contains at least five t-vertices of degree at least 3, it follows that $c(\hat{\Delta}) \leq c(k_0, 3, 3, 3, 3, 3) = c(1, 3, 3, 3, 3, 3) = \frac{4\pi}{3}$.
At most $\frac{3\pi}{4}$ can be sent in to any one boundary region so $c^*(\hat{\Delta}) \leq \frac{4\pi}{3} + \frac{3\pi}{4} = \frac{25\pi}{12} < 4\pi$, and so $c(D) < 4\pi$.

Suppose $k_0 = 2$: at most two boundary regions.
$c(\hat{\Delta}) \leq c(2, 3, 3, 3, 3, 3) = \frac{\pi}{3}$.
$c(D) \leq 2(\frac{\pi}{3} + \frac{3\pi}{4}) = \frac{13\pi}{6} < 4\pi$.

Suppose $k_0 \geq 3$:
Let $\hat{\Delta}$ be a boundary region of degree $n \geq 6$. Suppose n_1 of the vertices coincide with

v_0.

$$\begin{aligned}
c^*(\Delta') &\leq c(k_0,\ldots,k_0,3,\ldots,3) + \frac{3\pi}{4} \\
&= \pi\left[(2-n) + \frac{2n_1}{k_0} + \frac{2(n-n_1)}{3} + \frac{3}{4}\right] \\
&= \pi\left[\frac{4}{k_0} + \frac{2(n_1-2)}{k_0} + \frac{11}{4} - \frac{n+2n_1}{3}\right] \\
&\leq \pi\left[\frac{4}{k_0} + \frac{2(n_1-2)}{3} + \frac{11}{4} - \frac{n+2n_1}{3}\right] \\
&= \pi\left[\frac{4}{k_0} + \frac{17}{12} - \frac{n}{3}\right] < \frac{4\pi}{k_0}.
\end{aligned}$$

So $c(D) < k_0\left(\frac{4\pi}{k_0}\right) = 4\pi$.

Therefore $c(D) < 4\pi$ for all values of k_0, which gives a contradiction and so completes the proof of Theorem 1.12.

Remark

If $k \geq 6$, there are no regions of positive curvature and so Lemma 1.17 is seen to hold straight away. An argument similar to the above tells us that the total curvature of 4π cannot be obtained, which proves Theorem 1.10.

Chapter 5

Introduction – Cyclically presented groups

5.1 Irreducible cyclic presentations

Let $F_n =< x_0, \dots, x_{n-1} >$ be the free group on n elements and let $\theta : F_n \to F_n$ be the automorphism for which $x_i\theta = x_{i+1}$, where subscripts are taken mod n.

Let $\omega \in F_n$ be a cyclically reduced word. Define $G_n(\omega) =< x_0, \dots, x_{n-1} | \omega, \omega\theta, \dots, \omega\theta^{n-1} >$.

Definition 5.1.1. *A group G is said to have a* cyclic presentation *(or to be* cyclically presented*) if $G \cong G_n(\omega)$ for some n and some ω.*

Note that the group $G_{3m}(x_0^{-1}x_m x_0 x_m^{-2})$ is a trivial cyclically presented group for $m \geq 1$. This follows from the fact that each group is isomorphic to a free product of m copies of the known trivial group $G_3(x_0^{-1}x_1x_0x_1^{-2})$. To avoid this situation where a cyclically presented group may be decomposed into a free product of cyclically presented groups on fewer generators, we introduce the notion of irreducibility.

Definition 5.1.2. *$G_n(\omega)$ is defined to be* irreducible *if $n = 1$ or $n > 1$ and the following two conditions are satisfied:*

(1) w involves at least two of the x_i.

(2) If ω involves only $x_{i_1} \dots x_{i_k}$ where $i_j < i_{j+1}$, $1 \leq j \leq k-1$, and where $k \geq 2$, then $gcd(i_2 - i_1, \dots, i_k - i_{k-1}, n) = 1$.

From now on we assume that any cyclic presentation we refer to is irreducible.

The above automorphism θ induces an automorphism of $G_n(\omega)$ and we obtain the following split extension of $G_n(\omega)$ by the cyclic group of order n.

$$H_n(\omega) =< x, t \mid t^n, w(x,t) >,$$

where $w(x,t)$ is in the normal closure of x and t^n in the free group on x and t [19]. It can also be verified that any group with such a presentation is a split extension of some cyclically presented group $G_n(\omega)$.

Note that $w(x,t)$ is obtained from ω by the rewrite $x_i \mapsto t^{-i}xt^i$. So, for example, if $\omega = x_0x_3x_4^{-1}x_3$ then $w(x,t) = x(t^{-3}xt^3)(t^{-4}x^{-1}t^4)(t^{-3}xt^3) = xt^{-3}xt^{-1}x^{-1}txt^3$.

We refer to $G_n(\omega)$ as the cyclically presented group *associated with* $w(x,t)$ where $w(x,t)$ is as above.

We denote by $l(w(x,t))$ the *length* of $w(x,t)$ regarded as a word in the free group on x and t.

5.2 Motivation and results

The following theorem was proved in [5].

Theorem 5.2.1. *Let $w(x,t)$ be a cyclically reduced element in the normal closure of x and t^n in the free group on x and t. If $6 \leq n \leq 10$, $l(w(x,t)) \leq 15$ and the cyclically presented group $G_n(\omega)$ associated with $w(x,t)$ is irreducible then $G_n(\omega)$ is non-trivial.*

This theorem was then extended in [2] up to $n = 100$.

The aim is to extend the experiment which looks for examples of trivial cyclically presented groups under certain parameters by looking at when cyclically presented groups are finite.

The experiment looks at cyclically presented groups associated with $w(x,t)$ when $l(w(x,t)) \leq 15$ and $n \geq 4$.

Definition 5.2.2. *By a* family, *we mean $G_n(\omega)$ where ω is fixed and n takes infinitely many values.*

Definition 5.2.3. *The word $w_1(x,t)$ is* equivalent *to $w_2(x,t)$ if and only if $w_1(x,t)$ can be obtained from $w_2(x,t)$ by a sequence of the following moves.*

(1) Cyclic permutation,

(2) Inversion,

(3) $x \rightarrow x^{-1}$,

(4) $t \rightarrow t^{-1}$.

*For a particular n, $w_1(x,t)$ is n-*equivalent *to $w_2(x,t)$ if and only if $w_1(x,t)$ can be obtained from $w_2(x,t)$ by a sequence of any of the above or the following moves.*

(5) Replace t^{k_1} by t^{k_2} where $k_1 \equiv k_2$ mod n,

(5) Multiply the powers of t by m where $(m,n) = 1$.

We obtain the following result, which is motivated by Theorem 5.2.1.

Theorem 5.2.4. *Let $w(x,t)$ be a cyclically reduced element in the normal closure of x and t^n in the free group on x and t. Let $6 \leq n \leq 50$, $l(w(x,t)) \leq 10$ and assume the cyclically presented group $G_n(\omega)$ associated with each $w(x,t)$ is irreducible and that ω involves at least three of the x_is. Assume $w(x,t)$ is not n-equivalent to one of the following:*

(i) $x^{-1}t^{-1}x^{-1}t^{-1}xtx^2t$ for $n = 11, 13, 17, 19, 21, 23, 25, 29, 31, 33, 37, 41, 43, 47$ or 49

(ii) $x^{-1}t^{-1}x^{-1}t^{-1}x^{-2}tx^2t$ for $n = 7$,

(iii) $x^{-1}t^{-3}xtx^{-1}t^2$ for $n = 9$,

(iv) $x^{-1}t^{-3}xtxt^2$ for $n = 9$.

Then $G_n(\omega)$ is finite if and only if, up to n-equivalence, the associated $w(x,t)$ is one of the following:

(i) $x^{-1}t^{-2}x^{-1}tx^{-1}t$ for $n \not\equiv 0$ mod 3,

(ii) $x^{-1}t^{-2}x^{-1}tx^{-2}t$ for n odd,

(iii) $x^{-1}t^{-1}x^{-1}t^{-1}x^{-1}tx^{-1}t$ for n odd,

(iv) $x^{-1}t^{-3}x^{-1}tx^{-1}tx^{-1}t$ *for* n *odd,*

(v) $x^{-1}t^{-2}xtxt^{-1}xt^2$ *for* n *odd,*

(vi) $x^{-1}t^{-2}xtx^{-1}t$ *for* $n = 7$,

(vii) $x^{-1}t^{-2}x^{-2}tx^{-1}t$ *for* $n = 6$,

(viii) $x^{-1}t^{-3}x^{-1}tx^{-1}t^2$ *for* $n = 6$ *and 9,*

(ix) $x^{-1}t^{-3}x^{-1}txt^2$ *for* $n = 6$ *and 8,*

(x) $x^{-1}t^{-3}xtx^{-1}t^2$ *for* $n = 6$,

(xi) $x^{-1}t^{-3}xtxt^2$ *for* $n = 6$,

(xii) $x^{-1}t^{-2}x^{-2}tx^{-2}t$ *for* $n = 6$,

(xiii) $x^{-1}t^{-1}x^{-1}t^{-1}xtx^2t$ *for* $n = 7$,

(xiv) $x^{-1}t^{-3}xtx^{-1}tx^{-1}t$ *for* $n = 6$.

Remarks

1. Although we are running the experiment for $n \geq 4$, Theorem 5.2.4 is for $n \geq 6$. There are many words for which we have not been able to determine if the corresponding groups are finite or infinite for $n = 4$ and $n = 5$. There are precisely 20 such words when $l \leq 10$.

2. We used the restriction that ω involves at least three of the x_is as we already know otherwise that, if the group is finite, it must be cyclic [25], and we are interested in finding non-cyclic finite groups.

3. The words *(iii)* and *(iv)* in the first list in Theorem 5.2.4 are mentioned in [27] as groups for which it is open to determine whether or not they are finite.

Proposition 5.2.5. *For each fixed* $k \geq 3$*, the groups* $G_n(x_0 \dots x_{k-1})$ *are families of finite cyclically presented groups. In fact,* $G_n(x_0 \dots x_{k-1})$ *is finite if and only if* $gcd(n, k) = 1$.

The proof of this proposition will appear in Section 8.4.

Corollary 5.2.6. *There is no upper bound on the length of* ω *for which a family of finite cyclically presented groups* $G_n(\omega)$ *exists.*

A further motivation for performing the described experiment is to look for groups which are referred to as *interesting groups* in [19]. A finite group is interesting if it has a balanced presentation. No examples have yet been found of an interesting group needing more than three generators. So, while performing this experiment, we have searched for finite groups of the form $< x_1, x_2, x_3, x_4 \mid r_1, r_2, r_3, r_4 >$, where all four generators are required. We have, however, been unable to find any such groups and so this supports the belief that no such groups exist.

Chapter 6

Obtaining lists of possible words

6.1 Experiment

We describe the experiment in which we locate all possible irreducible presentations under certain parameters, and attempt to identify which of the groups are finite.

Note that $H_n(\omega) =< x, t \mid t^n, w(x,t) >$ is finite if and only if the associated cyclically presented group $G_n(\omega)$ is finite. In fact, in this case $|G_n(\omega)| = |H_n(\omega)|/n$ which means we may study the group $H_n(\omega)$ in order to find out if $G_n(\omega)$ is finite or infinite.

We make the assumptions that ω involves at least three of the x_is and that $l(w(x,t)) \leq 15$ and $n \geq 4$. Initially, we also require $n \leq 15$ in order for us to obtain a finite number of potential groups.

Under these assumptions there are, in theory, $2(3^{15} - 1)$ reduced $w(x,t)$ which we are required to consider. We may make the following restrictions however, in order to allow our words to be contenders for giving us a finite associated cyclically presented group.

(1) The word $w(x,t)$ must be cyclically reduced.

(2) The exponent sum of t in $w(x,t)$ must be equal to 0 mod n.

(3) We work modulo equivalence.

(4) The exponent sum of x in $w(x,t)$ must not equal 0.

(5) No cyclic permutation of $w(x,t)$ may contain the subwords t^{-k}, t^{k+1} (when $n = 2k$) or $t^{-(k+1)}, t^{k+1}$ (when $n = 2k+1$).

(6) The resulting presentation must be irreducible.

(7) The determinant of the relation matrix of the resulting presentation must not equal 0.

(8) The rewritten word ω must involve at least 3 of the x_i.

Recall that we use the restriction that ω involves at least three of the x_is as we already know otherwise that, if the group is finite, it must be cyclic [25]. Restrictions (1), (3) and (5) are in place as otherwise, equivalent words will appear more than once. Restrictions (4) and (7) are in place as otherwise, if they do not hold, the group is known to be infinite. Restriction (2) must hold as this is true for any $w(x,t)$ obtained from rewriting ω as described in Section 5.1.

A computer program has been produced which lists all possible words under the above restrictions. To optimise the speed of this program it was further assumed that each word begins with x^{-1}. This assumption may be made as any word is equivalent to a word beginning with x^{-1}.

The following table gives the total number of words modulo (1)-(8) above for $l = l(w(x,t)) \leq 15$ and $4 \leq n \leq 15$.

$l \backslash n$	4	5	6	7	8	9	10	11	12	13	14	15
≤ 7	3	3	1	3	3	2	3	3	1	3	3	2
8	5	11	5	8	4	8	5	8	4	8	5	8
9	32	34	28	30	30	29	30	30	28	30	30	29
10	45	87	49	72	48	64	50	66	44	66	50	64
11	171	237	239	234	220	215	217	220	209	220	217	215
12	273	585	414	584	357	483	367	484	343	484	367	477
13	1148	1648	1710	1787	1712	1608	1575	1604	1520	1604	1578	1571
14	1870	4208	3074	4352	2918	3750	2804	3534	2628	3521	2789	3484
15	7191	11698	12807	13340	13106	12258	11973	11807	11266	11741	11652	11537

A full list of these words may be viewed in [24].

Definition 6.1.1. *Let $J(l,n)$ refer to the set of words $w(x,t)$ in the (l,n)-entry of the table above.*

Note that $|J(l,n)| = 0$ for $l < 7$.

Lemma 6.1.2. *For $l \leq 15$, $J(l,n) \subseteq J(l,13)$ for $n \geq 13$. Moreover, $J(l,p) = J(l,13)$ when $p \geq 13$ is prime.*

Proof. To prove the first part, we need to show that there is no $w(x,t) \in J(l(w(x,t)),n)$, $n > 13$ such that $w \notin J(l(w(x,t)),13)$. This is true if we cannot find a word that fails the test for $n = 13$ but passes for some $n > 13$.

Let us look at the above restrictions. Restrictions (1), (3) and (4) do not depend on n and so the test cannot fail for $n = 13$ and pass for $n > 13$ from any of these. When $n \geq 13$ and $l \leq 15$, the exponent sum of t must be equal to 0 for restriction (2) to hold and so this restriction does not depend on n at these values. When $n \geq 13$ and $l \leq 15$, $t^{\pm 7}$ cannot occur and so restriction (5) always holds for $n \geq 13$.

Restriction (8) holding for $n = 13$ implies it holds for $n > 13$ also as the same three or more x_i involved when $n > 13$ will be involved when $n = 13$. This is because, if ω involves only two of the x_i when $n = 13$, x_0 and x_k say, this is the rewrite of a word $w(t,x)$ involving $t^{\pm k}$ as the only powers of t. To appear in the list when $n = 13$, k must be at most 6 and so this word $w(t,x)$ rewrites to the same word ω involving only the two generators x_0 and x_k when $n > 13$.

If $l \leq 15$ and w involves $x_{i_1}, \ldots, x_{i_k}$ then 13 does not divide $i_2 - i_1, \ldots, i_k - i_{k-1}$ so restriction (6) always holds at $n = 13$ and therefore it cannot occur that this restriction fails for $n = 13$ and passes for some $n > 13$.

The only restriction left to look at is restriction (7), which states that the determinant of the resulting presentation must be non-zero. If we rerun the test without this restriction then we see $J(l,13)$ remains unchanged. This shows that the determinant test never fails at $n = 13$ and so restriction (7) cannot fail for a word at $n = 13$ and pass for some $n > 13$. So there is no word that fails any test for some $n = 13$ but passes for $n > 13$ and so the first part of the lemma holds.

To prove that $J(l,p) = J(l,13)$ when $p \geq 13$ is prime, we need to show that any word in $J(l,13)$ is also in $J(l,p)$. The only way this could potentially not be the case is if a word passes the determinant test for $n = 13$ and fails for $n = p$.

Any word ω fails the determinant test at $p \iff det(M) = 0$, where M is the relation matrix $\iff \exists a$ such that $f(a) = 0$, f associated polynomial, and $a^p = 1$ [2].

Assume $a \neq 1$. The minimal polynomial for the primitive p^{th} root of unity (i.e. the polynomial with the smallest degree such that a p^{th} root of unity is a root of the polynomial) has degree $p - 1$ [26]. Therefore, because the maximum degree of f is 6, a p^{th} root of unity may not be the root of f when $p > 7$ and so, in particular, when $p \geq 13$. Therefore the determinant test never fails when $n = p \geq 13$ is prime. □

Remark

For each individual l, there is a particular smallest prime p_l such that $J(l,n) \subseteq J(l,p_l)$ for $n \geq p_l$. For $l = 15$, we can see from the tables that this prime is 13 and so this prime is the value we use in the lemma.

Lemma 6.1.2 tells us that, with the exception of sporadics, which occur at smaller values of n and shall be discussed in a moment, the words occurring for when n is prime ($n \geq 13$) are all the possible words. Therefore, apart from sporadics, there is a finite set of potential words for each value l and out of these, the potentials for each n is a subset of this set.

We can therefore, for each length l, examine each of the words in this set individually and see for which values of n, if any, this word produces a finite group. Examining the words in this way is useful for being systematic as, if we have proved the group associated with this word is infinite for some n, then the group associated with this word is infinite for kn where k is any positive integer.

Definition 6.1.3. *A* sporadic *is a word that appears in only finitely many $J(l, n)$. If $l \leq 15$, Lemma 6.1.2 implies w is sporadic iff $w \notin J(l, 13)$.*

Remarks

1. A *sporadic* will appear for small n only.
2. Sporadics exist due to restriction number (2) requiring only that the exponent sum of t is equal to 0 mod n and they are precisely the words for which n divides the t exponent sum but the exponent sum is non-zero.
3. Sporadics cannot form families but may give us finite groups for specific n.

Sporadics will be discussed in Section 8.5. For now we will put sporadics to one side and concentrate on the list of words for each length. We see from the above table that the number of words for each l is as follows.

Length	7	8	9	10	11	12	13	14	15
Total number of words	3	8	30	66	220	484	1604	3521	11741

Chapter 7

Checking for finiteness

7.1 Special cases

Before we begin the computational methods for working out which groups are finite, which we describe in the next section, we can reduce our number of words in the lists once again by using existing results on words that are of a certain form.

We describe the different forms we deal with in the remainder of this section and then give the totals left after these words have been dealt with. It is worth noting that, although the removal of these special cases before continuing with computational methods is now seen to be a good way, for lower values of l we found some of the results for such words using computational methods instead.

We give an example for each of the special cases and explain how the relevant results are used. However, there is usually more than one example in each case and details of these can be found in [24].

7.1.1 Fibonacci groups

Let F_n be defined as a group of the following form.

$$F_n =< x_0, \ldots, x_{n-1} \mid x_i x_{i+1} = x_{i+2} \ (0 \leq i \leq n-1) >, \text{ subscripts taken mod } n.$$

This family of groups is known as the *Fibonacci groups* and it is known that the only finite Fibonacci groups are F_1, F_2, F_3, F_4, F_5 and F_7.

The length 7 word $x^{-1}t^{-2}xtx^{-1}t$ is equivalent to a word that rewrites to $x_0x_1x_2^{-1}$. But the group $G_n(x_0x_1x_2^{-1}) = F_n$ and so we know that this word is finite when $n = 4, 5$ and 7 and infinite otherwise. See Section 8.1 for a full list of all finite groups in which this group appears.

7.1.2 Generalized Fibonacci groups

Let $G_n(m, k)$ be defined by the following presentation.

$$G_n(m,k) =< x_0, \ldots, x_{n-1} \mid x_i x_{i+m} = x_{i+k} \ (0 \le i \le n-1) >, \text{ subscripts taken mod } n.$$

Assume that the presentation is *irreducible*, i.e. that $0 < m < k < n$ and $(n, m, k) = 1$.

The following results are taken from [27].

Lemma 7.1.1. *1. If $(n, k) = 1$ then $G_n(m, k) \cong H(n, z)$ where $zk \equiv m$ mod n.*

2. If $(n, k - m) = 1$ then $G_n(m, k) \cong H(n, z)$ where $z(k - m) \equiv n - m$ mod n.

Here, $H(n, z)$ refers to the Gilbert-Howie groups which are studied in [14].

Theorem 7.1.2. *Suppose $(n, z) \neq (8, 3), (9, 3), (9, 4), (9, 6), (9, 7)$. Then $H(n, z)$ is finite if and only if one of the following holds:*

1. $z = 0, 1$;
2. $(n, z) = (2s, s + 1)$ *where* $s \geq 1$;
3. $(n, z) = (3, 2), (4, 2), (4, 3), (5, 2), (5, 3), (5, 4), (6, 3), (6, 4), (7, 4), (7, 6), (8, 5)$.

Remark
The groups $H(9, 3)$ and $H(9, 6)$ are known to be infinite.
It remains unknown as to whether $H(9, 4)$ and $H(9, 7)$ are finite or infinite.
The group $H(8, 3)$ is finite of order 295,245.

If $(n, k) > 1$ and $(n, k - m) > 1$ then the presentation is called *strongly irreducible* and Lemma 7.1.1 and therefore Theorem 7.1.2 cannot be used. Instead we have the following corollary from [27].

Corollary 7.1.3. *Let $G = G_n(m,k)$ be strongly irreducible and assume $G \neq 1$. Then G is finite if and only if $(m,k) = 1$ and $n = 2k$ or $n = 2(k-m)$, in which case $G \cong \mathbb{Z}_s$ where $s = 2^{\frac{n}{2}} - (-1)^{\frac{m+n}{2}}$.*

Now let us look at an example of how the above results can be applied.

The length nine word $x^{-1}t^{-3}x^{-1}txt^2$ rewrites to $x_0^{-1}x_3^{-1}x_1$ which is equivalent to $x_0x_{n-3}x_{n-1}^{-1}$. Therefore, this word gives us the presentation $G_n(n-3,n-1)$, i.e. $m = n-3$, $k = n-1$. As $(n,k) = (n,n-1) = 1$, we can use part (i) of Lemma 7.1.1 and we find that $z = 3$. This means $G_n(m,k) \cong H(n,3)$ and the lemma tells us that the group is finite when $n = 4, 5, 6$ and 8 and infinite otherwise.

7.1.3 Positive words of length 3

Let $\Gamma_n(k,l)$ be defined by the following presentation.

$$\Gamma_n(k,l) =< x_1, \ldots, x_n \mid x_i x_{i+k} x_{i+l} = 1 \ (1 \leq i \leq n) > .$$

The following two conditions will be used in the results:
(A) $k + l \equiv 0 \bmod n$ or $2l - k \equiv 0 \bmod n$ or $2k - l \equiv 0 \bmod n$;
(B) $n \equiv 0 \bmod 3$ and $k + l \equiv 0 \bmod 3$.

The following theorem is deduced from [10].

Theorem 7.1.4. *Let $(n,k,l) = 1$. Then $\Gamma_n(k,l)$ is finite if and only if one of the following holds:*

(i) $k \equiv l \bmod n$, in which case $\Gamma_n(k,l) \cong \mathbb{Z}_s$ where $s = 2^n - (-1)^n$,

(ii) (A) holds and (B) does not, in which case $\Gamma_n(k,l) \cong \mathbb{Z}_3$,

(iii) (A) does not hold, $n \equiv 0 \bmod 3$, $k + l \not\equiv 0 \bmod 3$ (so (B) does not hold) and $n|3k$ or $n|3l$ or $n|3(l-k)$, in which case $\Gamma_n(k,l) \cong \mathbb{Z}_s$ where $s = 2^n - (-1)^n$.

There is one length nine word of this form which is $x^{-1}t^{-3}x^{-1}tx^{-1}t^2$. This word is equivalent to a word that rewrites to $x_0x_2x_3$ and so $k = 2$ and $l = 3$. Case (i) of the theorem does not hold, case (ii) gives us that the group is finite for $n = 4$ and $n = 5$, and case (iii) gives us that the group is finite for $n = 6$ and $n = 9$. The group is infinite for all other values of n.

7.1.4 Positive words of length 4

Consider the following presentation.

$$G_n(j,k,l) =< x_1,\ldots,x_n \mid x_i x_{i+j} x_{i+k} x_{i+l} = 1\ (1 \leq i \leq n) >.$$

Assume that $G_n(j,k,l)$ is non-trivial.

The following theorem is deduced from [1].

Theorem 7.1.5. *If $-j$, $j-k$, $k-l$ and l are all distinct mod n then $G_n(j,k,l)$ is infinite. Therefore, $G_n(j,k,l)$ is finite only if one of the following conditions holds:*

(i) $n|(2j-k)$;

(ii) $n|(j+k-l)$;

(iii) $n|(j+l)$;

(iv) $n|(j-2k+l)$;

(v) $n|(j-k-l)$; *or*

(vi) $n|(k-2l)$.

The length ten word $x^{-1}t^{-2}x^{-1}tx^{-1}t^{-1}x^{-1}t^2$ is of this form with $j=2$, $k=1$ and $l=2$. We know that $G_n(2,1,2)$ is finite only if n divides 3, 1, 4, 2, -1, -3. Therefore, we know $G_n(2,1,2)$ is infinite whenever $n>4$.

7.1.5 Positive words of length 5

Consider the following presentation.

$$G_n(j,k,l,m) =< x_1,\ldots,x_n \mid x_i x_{i+j} x_{i+k} x_{i+l} x_{i+m} = 1\ (1 \leq i \leq n) >.$$

Assume that we do not have any of the following:

(i) $n|(j+k-l)$ and $n|(k+l)$,

(ii) $n|(j-k-l+m)$ and $n|(2j-l-m)$,

(iii) $n|(k-l-m)$ and $n|(2k-m)$,

(iv) $n|(j+l-m)$ and $n|(j-2l)$,

(v) $n|(j-k-m)$ and $n|(j+k-2m)$.

The following theorem is deduced from [17].

Theorem 7.1.6. *$G_n(j,k,l,m)$ is finite only if two of the following hold:*

(1) $n|(2j-k)$

(2) $n|(j+k-l)$

(3) $n|(j+l-m)$

(4) $n|(j+m)$

(5) $n|(j-2k+l)$

(6) $n|(j-k-l+m)$

(7) $n|(j-k-m)$

(8) $n|(k-2l+m)$

(9) $n|(k-l-m)$

(10) $n|(l-2m)$

The length eleven word $x^{-1}t^{-2}x^{-1}tx^{-2}t^{-1}x^{-1}t^2$ is of this form with $j=2$, $k=1$, $l=1$ and $m=2$. As we are looking at $n \geq 4$, it is true that we do not have n dividing any of the following pairs of numbers: $\{2,2\},\{2,1\},\{-2,0\},\{1,0\},\{-1,-1\}$. Then $G_n(2,1,1,2)$ is finite only if n divides two of the following: 3, 2, 1, 4, 1, 2, -1, 1, -2, -3. This is never true for $n \geq 4$ and so $G_n(2,1,1,2)$ is infinite for all $n \geq 4$.

7.1.6 Exceptional intersections

The following theorem was proved in [6].

Let us assume that the word $\omega(x_0,\ldots,x_k)$ and any cyclic permutation of this word is not able to be written in either of the following forms:

(i) $\omega_1^{\alpha_1}\omega_2^{\beta_1}\omega_1^{\alpha_2}\omega_2^{\beta_2}\ldots\omega_1^{\alpha_l}\omega_2^{\beta_l}$, $\omega_1 \in < x_0,\ldots,x_{k-1} >$, $\omega_2 \in < x_1,\ldots,x_k >$;

(ii) $\omega_3^{\alpha_1}(\nu_1\nu_2)^{\beta_1}\omega_3^{\alpha_2}(\nu_1\nu_2)^{\beta_2}\ldots\omega_3^{\alpha_l}(\nu_1\nu_2)^{\beta_l}$, $\omega_3 \in < x_1,\ldots,x_{k-1} >$, $\nu_1 \in < x_0,\ldots,x_{k-1} >$, $\nu_2 \in < x_1,\ldots,x_k >$;

where $\alpha_i, \beta_i \in \mathbb{Z}$.

Then $G_n(\omega)$ is infinite for $n \geq 4k$.

Furthermore, if $\omega(x_0,\ldots,x_k)$ involves every x_i, $0 \leq i \leq k$ then $G_n(\omega)$ is infinite for $n \geq 2(k+1)$.

The length thirteen word $x^{-1}t^{-2}x^{-1}tx^{-1}tx^{-1}t^{-2}xt^2$ rewrites to $x_0^{-1}x_2^{-1}x_1^{-1}x_0^{-1}x_2$. This cannot be written in the first form as the existence of two separated x_0s and x_2s means that we would have $l = 2$ in the formula, but x_1 could not appear in either ω_1 or ω_2 as it only occurs once. If it were in the second form then we would have $\omega_3^{\alpha_1} = x_1$. But then either $(\nu_1\nu_2)^{\beta_1} = x_0^{-1}x_2x_0^{-1}x_2^{-1}$ or $x_0^{-1}x_2^{-1}$, neither of which are valid possibilities. Therefore, this word is not in either of the above forms. Since $k = 2$ in this case and x_0, x_1 and x_2 are all involved in the word, we get that $G_n(\omega)$ is infinite for $n \geq 6$.

7.1.7 Special cases results

Although initially several of the special cases were dealt with by hand, a computer program was created subsequently to check which words were special cases, record the results, and remove the some of the special case words from the list. Note that only some of the special case words were removed, depending upon how complete a list of results we are able to obtain using this computational method. The results of the program for $l \geq 13$, which are the l values for which we consider the list without the special words, can be viewed in [24]. After this process, the number of words remaining is shown in the following table.

Length	7	8	9	10	11	12	13	14	15
Total number of words before	3	8	30	66	220	484	1604	3521	11741
Total after special case words removed	0	8	26	60	205	481	1572	3512	11671

7.2 Method for testing finiteness

After removing the words which fall under the category of special cases, the next step is to look at each word separately and try to determine for each $l \leq 15$ which ones are finite and which are infinite using computational methods. There are two different computer packages that are used for the majority of cases. The first is called KBMAG [15] and works by trying to find an automatic structure for the group $H_n(\omega)$ and, if successful, it is able to tell us the size of the group. This method frequently fails for when n is small, in which case we often obtain success by using KBMAG with the group $G_n(\omega)$ instead. If both these methods fail then we may try to determine the size of the group $H_n(\omega)$ using the computer package GAP [12]. This package allows various commands and, if the group is finite, it is often more successful than KBMAG in discovering this by means of coset enumeration. Various tests can be put in place for checking if a group is infinite, which we state below.

1. Low index subgroups are found for the group and checked to see if there is a zero in the abelian invariants for one of these subgroups.

2. Subgroups in the derived series are checked for a zero in the abelian invariants.

3. The derived subgroups of low index subgroups are checked for a zero in the abelian invariants.

4. The cores of low index subgroups are checked for a zero in the abelian invariants.

5. A mapping onto $PSL(m, q)$ is found, where q is a power of a prime and subgroups of $H_n(\omega)$ containing the kernel of this mapping are checked for a zero in the abelian invariants.

6. A mapping as mentioned above is found and the derived subgroups of the mentioned subgroups are checked for a zero in the abelian invariants.

7. A mapping as mentioned above but from a low index subgroup instead of $H_n(\omega)$ itself, mentioned subgroups are checked for a zero in the abelian invariants.

8. A factor p-group is found and checked for a zero in the abelian invariants.

9. If a zero has failed to be found in the abelian invariants for any of the above tests, then we many perform the Newman Infinity Criterion (see below) to see if this can prove the group to be infinite.

We note that we have not been able to use these methods, either with KBMAG or GAP, in the same way for when $l = 13$, $l = 14$ and $l = 15$, due to the large number of words involved for each. The results we were able to obtain are discussed in Section 8.3.

7.2.1 Newman Infinity Criterion

Let G be a group. Let p be a prime and let G_1 be the subgroup of G generated by all commutators and pth powers, so $G_1 = [G, G]G^p$. Let $G_2 = [G_1, G]G^p$. Define $d_p(G)$ to be the rank of G/G_1 and $e_p(G)$ to be the rank of G/G_2.

The following result was given in [23].

Theorem 7.2.1. *Let G be a group with a finite presentation on b generators and r relations. For some prime p, let $d = d_p(G)$ and $e = e_p(G)$. If*

$$r - b + d < \tfrac{d^2}{2} - \tfrac{d}{2} - e$$

or

$$r - b + d \leq \tfrac{d^2}{2} - \tfrac{d}{2} - e + \tfrac{d}{2}(e + \tfrac{d}{2} - \tfrac{d^2}{4}),$$

then G has arbitrarily large quotients of p-power order.

In particular, if the criteria in the above theorem hold, then the group G is infinite. This result has been implemented into GAP as a method and returns *true* if the group is found to be infinite using this method, or *fail* if the method cannot tell us whether or not the group is infinite. Candidates for the prime p are prime divisors of the order of G/G'. If a prime p appears several times in the abelian invariants of a group then it is an indication that the Newman Infinity Criterion using this value p may give us the return value *true*, and it is this observation that has given us our success with this method for finding groups to be infinite.

The length twelve word $x^{-1}t^{-1}x^{-2}t^{-1}x^2tx^3t$ was found to be infinite when $n = 5$ using the Newman Infinity Criterion in GAP. A subgroup of index 10 was found and the Newman test on the derived subgroup of this subgroup returned true.

Chapter 8

Results for finitely presented groups

In this chapter we list all finite cyclically presented groups which have been found using special cases, KBMAG or GAP, starting with $l \leq 10$. Where we do not know the exact structure of the group, we instead give its derived series. All of the finite groups in this chapter can be generated by three generators.

Note that, in order to avoid repetition, we work up to n-equivalence of each word $w(x, t)$ and therefore each word ω. Words which are equivalent for all n are discarded by restriction (3) in Section 6.1. However, it is possible for repetition to occur when n is fixed if two words are n-equivalence but not equivalent e.g. $x_0^{-1}x_3x_2^2$ is equivalent to $x_0^{-1}x_1x_2^2$ when $n = 4$ ($i \to n-i$) but not when $n = 5$. Also, when n is odd, the subscripts of $x_0x_1x_2x_1$ may be multiplied by 2 by Definition 5.2.3 (6) so this word is n-equivalent to $x_0x_2x_4x_2$ in this instance.

When such repetitions have been found in our results they have been omitted from the lists in this chapter. Therefore, there may be finite groups appearing in the results pages in [24] which are equivalent to a word that is already in the list, and is therefore not itself in the list.

8.1 Finite groups for $l \leq 12$

In this section we list all finite cyclically presented groups we have found for $l \leq 12$, apart from the sporadics which will be dealt with in Section 8.5.

8.1.1 $l \leq 10$

The following are families of finite groups and therefore each word produces an infinite number of finite groups. Note that the full proofs that these and all other families in this chapter are indeed families are given in Section 8.4, as well as an explanation as to why the groups in the family are infinite for all values of n other than those given in the conditions.

Group G	$\lvert G \rvert$	Structure of G
$l = 7$		
$< x_0, \ldots, x_{n-1} \mid x_i x_{i+2} x_{i+1} \ (0 \leq i \leq n-1) >, \ n \not\equiv 0 \bmod 3$	3	$\mathbb{Z}_3$
$l = 8$		
$< x_0, \ldots, x_{n-1} \mid x_i x_{i+2} x_{i+1}^2 \ (0 \leq i \leq n-1) >, \ n$ odd	4	$\mathbb{Z}_4$
$< x_0, \ldots, x_{n-1} \mid x_i x_{i+1} x_{i+2} x_{i+1} \ (0 \leq i \leq n-1) >, \ n$ odd	$4n$	$\mathbb{Z}_n \rtimes \mathbb{Z}_4$
$l = 10$		
$< x_0, \ldots, x_{n-1} \mid x_i x_{i+3} x_{i+2} x_{i+1} \ (0 \leq i \leq n-1) >, \ n$ odd	4	$\mathbb{Z}_4$
$< x_0, \ldots, x_{n-1} \mid x_i x_{i+2}^{-1} x_{i+1}^{-1} x_{i+2}^{-1} \ (0 \leq i \leq n-1) >, \ n$ odd	$\frac{2}{3}(4^n - 1)$	$1 \trianglelefteq \mathbb{Z}_{\frac{1}{3}(2^n+1)} \trianglelefteq G$

The following are the rest of the finite groups, which occur for specific and generally small n.

Group G	$\lvert G \rvert$	Structure of G
$l = 7$		
$< x_0, \ldots, x_3 \mid x_i x_{i+2} x_{i+1}^{-1} \ (0 \leq i \leq 3) >$	24	$SL(2,3)$
$< x_0, \ldots, x_4 \mid x_i x_{i+2} x_{i+1}^{-1} \ (0 \leq i \leq 4) >$	120	$SL(2,5)$
$< x_0, \ldots, x_3 \mid x_i x_{i+2}^{-1} x_{i+1} \ (0 \leq i \leq 3) >$	5	$\mathbb{Z}_5$
$< x_0, \ldots, x_4 \mid x_i x_{i+2}^{-1} x_{i+1} \ (0 \leq i \leq 4) >$	11	$\mathbb{Z}_{11}$
$< x_0, \ldots, x_6 \mid x_i x_{i+2}^{-1} x_{i+1} \ (0 \leq i \leq 6) >$	29	$\mathbb{Z}_{29}$
$l = 8$		
$< x_0, \ldots, x_3 \mid x_i x_{i+2}^2 x_{i+1} \ (0 \leq i \leq 3) >$	80	$\mathbb{Z}_7 \rtimes \mathbb{Z}_{16}$
$< x_0, \ldots, x_4 \mid x_i x_{i+2}^2 x_{i+1} \ (0 \leq i \leq 4) >$	220	$\mathbb{Z}_{11} \times (\mathbb{Z}_5 \rtimes \mathbb{Z}_4)$
$< x_0, \ldots, x_5 \mid x_i x_{i+2}^2 x_{i+1} \ (0 \leq i \leq 5) >$	4088448	$G_2 \trianglelefteq G_1 \trianglelefteq G$, G_2 perfect
$l = 9$		
$< x_0, \ldots, x_5 \mid x_i x_{i+3} x_{i+2} \ (0 \leq i \leq 5) >$	63	$\mathbb{Z}_7 \rtimes \mathbb{Z}_9$
$< x_0, \ldots, x_8 \mid x_i x_{i+3} x_{i+2} \ (0 \leq i \leq 8) >$	513	$\mathbb{Z}_{19} \rtimes \mathbb{Z}_{27}$
$< x_0, \ldots, x_5 \mid x_i x_{i+3} x_{i+2}^{-1} \ (0 \leq i \leq 5) >$	56	$(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_7$
$< x_0, \ldots, x_7 \mid x_i x_{i+3} x_{i+2}^{-1} \ (0 \leq i \leq 7) >$	295245	$1 \trianglelefteq G_2 \trianglelefteq G_1 \trianglelefteq G$
$< x_0, \ldots, x_5 \mid x_i x_{i+3}^{-1} x_{i+2} \ (0 \leq i \leq 5) >$	9	$\mathbb{Z}_9$
$< x_0, \ldots, x_5 \mid x_i x_{i+3}^{-1} x_{i+2}^{-1} \ (0 \leq i \leq 5) >$	7	$\mathbb{Z}_7$

Group G	$\|G\|$	Structure of G
$< x_0,\ldots,x_3 \mid x_i x_{i+2}^2 x_{i+1}^2 \ (0 \le i \le 3) >$	125	$\mathbb{Z}_{25} \rtimes \mathbb{Z}_5$
$< x_0,\ldots,x_4 \mid x_i x_{i+2}^2 x_{i+1}^2 \ (0 \le i \le 4) >$	275	$\mathbb{Z}_{11} \rtimes \mathbb{Z}_{25}$
$< x_0,\ldots,x_5 \mid x_i x_{i+2}^2 x_{i+1}^2 \ (0 \le i \le 5) >$	2015	$1 \trianglelefteq \mathbb{Z}_{31} \trianglelefteq G$
$< x_0,\ldots,x_3 \mid x_i x_{i+2}^{-3} x_{i+1} \ (0 \le i \le 3) >$	51	$\mathbb{Z}_{51}$
$< x_0,\ldots,x_3 \mid x_i x_{i+2}^{-2} x_{i+1}^{-2} \ (0 \le i \le 3) >$	39	$\mathbb{Z}_{39}$
$< x_0,\ldots,x_3 \mid x_i x_{i+1}^2 x_{i+3}^2 \ (0 \le i \le 3) >$	120	$\mathbb{Z}_5 \times SL(2,3)$
$< x_0,\ldots,x_3 \mid x_i x_{i+1} x_{i+2}^{-1} x_{i+1}^{-2} \ (0 \le i \le 3) >$	5	$\mathbb{Z}_5$
$< x_0,\ldots,x_4 \mid x_i x_{i+1} x_{i+2}^{-1} x_{i+1}^{-2} \ (0 \le i \le 4) >$	11	$\mathbb{Z}_{11}$
$< x_0,\ldots,x_6 \mid x_i x_{i+1} x_{i+2}^{-1} x_{i+1}^{-2} \ (0 \le i \le 6) >$	29	$\mathbb{Z}_{29}$
$< x_0,\ldots,x_3 \mid x_i x_{i+3}^2 x_{i+1}^{-2} \ (0 \le i \le 3) >$	17	$\mathbb{Z}_{17}$
$< x_0,\ldots,x_3 \mid x_i x_{i+2}^{-2} x_{i+1}^2 \ (0 \le i \le 3) >$	39000	$1 \trianglelefteq \mathbb{Z}_5 \trianglelefteq (\mathbb{Z}_5^2) \rtimes \mathbb{Z}_5 \trianglelefteq$
		$G_2 \trianglelefteq G_1 \trianglelefteq G$
$l = 10$		
$< x_0,\ldots,x_3 \mid x_i x_{i+2} x_{i+3} x_{i+2} \ (0 \le i \le 3) >$	80	$\mathbb{Z}_5 \rtimes \mathbb{Z}_{16}$
$< x_0,\ldots,x_3 \mid x_i x_{i+2} x_{i+3}^{-1} x_{i+2} \ (0 \le i \le 3) >$	80	$\mathbb{Z}_5 \rtimes \mathbb{Z}_{16}$
$< x_0,\ldots,x_5 \mid x_i x_{i+3}^{-1} x_{i+2} x_{i+1} \ (0 \le i \le 5) >$	1512	$\mathbb{Z}_5 \rtimes \mathbb{Z}_{16}$

8.1.2 $l = 11, 12$

The following are families of finite groups for $l = 11, 12$.

Group G	$\|G\|$	Structure of G
$l = 12$		
$< x_0,\ldots,x_{n-1} \mid x_i x_{i+4} x_{i+3} x_{i+1} \ (0 \le i \le n-1) >,\ n$ odd, $n \not\equiv 0 \bmod 3$	4	$\mathbb{Z}_4$
$< x_0,\ldots,x_{n-1} \mid x_i x_{i+3} x_{i+2}^2 x_{i+1}^2 \ (0 \le i \le n-1) >,\ n$ odd, $n \not\equiv 0 \bmod 3$	6	$\mathbb{Z}_6$
$< x_0,\ldots,x_{n-1} \mid x_i x_{i+3} x_{i+1} x_{i+2} \ (0 \le i \le n-1) >,\ n$ odd	4	$\mathbb{Z}_4$
$< x_0,\ldots,x_{n-1} \mid x_i x_{i+1} x_{i+2}^{-1} x_{i+1} x_{i+2} x_{i+1}^{-1} \ (0 \le i \le n-1) >,\ n$ odd	$2^{n+1} - 2$	$D_{2^{n+1}-2}$

The following are the rest of the finite groups for $l = 11, 12$.

Group G	$\|G\|$	Structure of G
$l = 11$		
$< x_0,\ldots,x_{11} \mid x_i x_{i+4} x_{i+3} \ (0 \le i \le 11) >$	4095	$1 \trianglelefteq \mathbb{Z}_{91} \trianglelefteq G$
$< x_0,\ldots,x_7 \mid x_i x_{i+4}^{-1} x_{i+3} \ (0 \le i \le 7) >$	17	$\mathbb{Z}_{17}$
$< x_0,\ldots,x_3 \mid x_i x_{i+2} x_{i+3}^2 x_{i+1} \ (0 \le i \le 3) >$	205	$\mathbb{Z}_{41} \rtimes \mathbb{Z}_5$
$< x_0,\ldots,x_3 \mid x_i x_{i+2} x_{i+3} x_{i+2} x_{i+1} \ (0 \le i \le 3) >$	205	$\mathbb{Z}_{41} \rtimes \mathbb{Z}_5$

Group G	$\|G\|$	Structure of G
$< x_0, \ldots, x_4 \mid x_i x_{i+2} x_{i+3} x_{i+2} x_{i+1}\ (0 \leq i \leq 4) >$	1025	$\mathbb{Z}_{41} \rtimes \mathbb{Z}_{25}$
$< x_0, \ldots, x_5 \mid x_i x_{i+2} x_{i+3} x_{i+2} x_{i+1}\ (0 \leq i \leq 5) >$	6335	$\mathbb{Z}_{181} \rtimes (\mathbb{Z}_5 \times \mathbb{Z}_7)$
$< x_0, \ldots, x_3 \mid x_i x_{i+2} x_{i+3}^{-2} x_{i+1}^{-1}\ (0 \leq i \leq 3) >$	295245	$1 \trianglelefteq \mathbb{Z}_3^6 \trianglelefteq G_1 \trianglelefteq G$
$< x_0, \ldots, x_3 \mid x_i x_{i+2} x_{i+3}^{-1} x_{i+2} x_{i+1}\ (0 \leq i \leq 3) >$	1755	$1 \trianglelefteq \mathbb{Z}_{39} \trianglelefteq G$
$< x_0, \ldots, x_3 \mid x_i x_{i+2} x_{i+3}^{-1} x_{i+2} x_{i+1}^{-1}\ (0 \leq i \leq 3) >$	295245	$1 \trianglelefteq \mathbb{Z}_3^6 \trianglelefteq G_1 \trianglelefteq G$
$< x_0, \ldots, x_5 \mid x_i^2 x_{i+1}^{-1} x_{i+3}^{-2}\ (0 \leq i \leq 5) >$	65	$\mathbb{Z}_{65}$
$< x_0, \ldots, x_3 \mid x_i x_{i+2}^{-1} x_{i+3}^{2} x_{i+1}^{-1}\ (0 \leq i \leq 3) >$	13	$\mathbb{Z}_{13}$
$< x_0, \ldots, x_3 \mid x_i x_{i+2}^{-1} x_{i+3}^{-1} x_{i+2} x_{i+1}^{-1}\ (0 \leq i \leq 3) >$	9375	$1 \trianglelefteq \mathbb{Z}_5^3 \trianglelefteq G_1 \trianglelefteq G$
$< x_0, \ldots, x_3 \mid x_i x_{i+2}^{-1} x_{i+3}^{-1} x_{i+2}^{-1} x_{i+1}\ (0 \leq i \leq 3) >$	13	$\mathbb{Z}_{13}$
$< x_0, \ldots, x_3 \mid x_i x_{i+2}^{-1} x_{i+3}^{-1} x_{i+2}^{-1} x_{i+1}^{-1}\ (0 \leq i \leq 3) >$	6561	$1 \trianglelefteq G_1 \trianglelefteq G$
$< x_0, \ldots, x_3 \mid x_i x_{i+2}^{-1} x_{i+3}^{-1} x_{i+1}^{2}\ (0 \leq i \leq 3) >$	13	$\mathbb{Z}_{13}$
$< x_0, \ldots, x_3 \mid x_i x_{i+2}^{-1} x_{i+1} x_{i+2} x_{i+1}\ (0 \leq i \leq 3) >$	195	$\mathbb{Z}_{13} \rtimes (\mathbb{Z}_3 \times \mathbb{Z}_5)$
$< x_0, \ldots, x_3 \mid x_i x_{i+2}^{-1} x_{i+1} x_{i+2}^{-1} x_{i+1}\ (0 \leq i \leq 3) >$	39000	$1 \trianglelefteq \mathbb{Z}_5 \trianglelefteq (\mathbb{Z}_5^2) \rtimes \mathbb{Z}_5 \trianglelefteq$
		$G_2 \trianglelefteq G_1 \trianglelefteq G$
$< x_0, \ldots, x_3 \mid x_i x_{i+2}^{-1} x_{i+1}^{-1} x_{i+2} x_{i+1}\ (0 \leq i \leq 3) >$	1	Trivial
$< x_0, \ldots, x_3 \mid x_i x_{i+1}^{3} x_{i+2}^{-2} x_{i+1}^{-1}\ (0 \leq i \leq 3) >$	39000	$1 \trianglelefteq \mathbb{Z}_5 \trianglelefteq (\mathbb{Z}_5^2) \rtimes \mathbb{Z}_5 \trianglelefteq$
		$G_2 \trianglelefteq G_1 \trianglelefteq G$
$< x_0, \ldots, x_3 \mid x_i x_{i+2}^{-1} x_{i+3} x_{i+2} x_{i+1}^{-1}\ (0 \leq i \leq 3) >$	5	$\mathbb{Z}_5$
$< x_0, \ldots, x_5 \mid x_i^2 x_{i+1} x_{i+3}^{-2}\ (0 \leq i \leq 5) >$	63	$\mathbb{Z}_{63}$
$< x_0, \ldots, x_3 \mid x_i x_{i+2}^{-1} x_{i+1} x_i x_{i+1}\ (0 \leq i \leq 3) >$	39	$\mathbb{Z}_{39}$
$< x_0, \ldots, x_3 \mid x_i x_{i+2} x_{i+3} x_{i+2}^{-1} x_{i+1}^{-1}\ (0 \leq i \leq 3) >$	5	$\mathbb{Z}_5$
$< x_0, \ldots, x_4 \mid x_i x_{i+2} x_{i+3} x_{i+2}^{-1} x_{i+1}^{-1}\ (0 \leq i \leq 4) >$	11	$\mathbb{Z}_{11}$
$< x_0, \ldots, x_5 \mid x_i x_{i+2} x_{i+3} x_{i+2}^{-1} x_{i+1}^{-1}\ (0 \leq i \leq 5) >$	56	$1 \trianglelefteq \mathbb{Z}_2^3 \trianglelefteq G$
$< x_0, \ldots, x_3 \mid x_i x_{i+2} x_{i+1} x_{i+2} x_{i+1}\ (0 \leq i \leq 3) >$	125	$\mathbb{Z}_{25} \rtimes \mathbb{Z}_5$
$< x_0, \ldots, x_4 \mid x_i x_{i+2} x_{i+1} x_{i+2} x_{i+1}\ (0 \leq i \leq 4) >$	275	$\mathbb{Z}_{11} \rtimes \mathbb{Z}_{25}$
$< x_0, \ldots, x_3 \mid x_i x_{i+2}^{-1} x_{i+3} x_{i+2} x_{i+1}\ (0 \leq i \leq 3) >$	39	$\mathbb{Z}_{13} \rtimes \mathbb{Z}_3$
$< x_0, \ldots, x_3 \mid x_i x_{i+2}^{-1} x_{i+3} x_{i+2}^{-1} x_{i+1}^{-1}\ (0 \leq i \leq 3) >$	13	$\mathbb{Z}_{13}$
$< x_0, \ldots, x_3 \mid x_i x_{i+2}^{-1} x_{i+3}^{-1} x_{i+2} x_{i+1}\ (0 \leq i \leq 3) >$	5	$\mathbb{Z}_5$
$< x_0, \ldots, x_4 \mid x_i x_{i+2}^{-1} x_{i+3}^{-1} x_{i+2} x_{i+1}\ (0 \leq i \leq 4) >$	1	Trivial
$< x_0, \ldots, x_4 \mid x_i x_{i+2}^{-1} x_{i+1}^{-1} x_{i+2}^{-1} x_{i+1}\ (0 \leq i \leq 4) >$	31	$\mathbb{Z}_{31}$
$< x_0, \ldots, x_6 \mid x_i x_{i+2}^{-1} x_{i+1}^{-1} x_{i+2}^{-1} x_{i+1}\ (0 \leq i \leq 6) >$	127	$\mathbb{Z}_{127}$
$< x_0, \ldots, x_8 \mid x_i x_{i+2}^{-1} x_{i+1}^{-1} x_{i+2}^{-1} x_{i+1}\ (0 \leq i \leq 8) >$	511	$\mathbb{Z}_{511}$
$< x_0, \ldots, x_3 \mid x_i x_{i+1}^{3} x_{i+3}^{-3}\ (0 \leq i \leq 3) >$	37	$\mathbb{Z}_{37}$
$< x_0, \ldots, x_3 \mid x_i x_{i+2}^{3} x_{i+1}^{3}\ (0 \leq i \leq 4) >$	2639	$\mathbb{Z}_{29} \rtimes (\mathbb{Z}_7 \times \mathbb{Z}_{13})$

Group G	$\|G\|$	Structure of G
$l = 12$		
$< x_0, \ldots, x_7 \mid x_i x_{i+3} x_{i+4} x_{i+2}\ (0 \leq i \leq 7) >$	6560	$1 \trianglelefteq \mathbb{Z}_{205} \trianglelefteq G$
$< x_0, \ldots, x_5 \mid x_i x_{i+3} x_{i+4}^{-1} x_{i+3}\ (0 \leq i \leq 5) >$	728	$1 \trianglelefteq \mathbb{Z}_{14} \trianglelefteq G$

Group G	$\|G\|$	Structure of G
$< x_0, \ldots, x_5 \mid x_i x_{i+3}^{-1} x_{i+4} x_{i+2}\ (0 \leq i \leq 5) >$	1512	$1 \trianglelefteq \mathbb{Z}_3 \trianglelefteq G_1 \trianglelefteq G$
$< x_0, \ldots, x_5 \mid x_i x_{i+3}^{-1} x_{i+4}^{-1} x_{i+3}^{-1}\ (0 \leq i \leq 5) >$	728	$\mathbb{Z}_{13} \rtimes \mathbb{Z}_{56}$
$< x_0, \ldots, x_4 \mid x_i x_{i+1} x_{i+2} x_{i+1} x_{i+2} x_{i+1}\ (0 \leq i \leq 4) >$	2046	$1 \trianglelefteq \mathbb{Z}_{31} \trianglelefteq G$
$< x_0, \ldots, x_6 \mid x_i x_{i+1} x_{i+2} x_{i+1} x_{i+2} x_{i+1}\ (0 \leq i \leq 6) >$	32766	$1 \trianglelefteq \mathbb{Z}_{127} \trianglelefteq G$

8.1.3 Words left over

After using KBMAG and GAP to find whether a group is finite or infinite we are left with the following number of words which are undecided for at least one $4 \leq n \leq 50$:

Length	7	8	9	10	11	12
Total number of words before	3	8	30	66	220	484
Total after special case words removed	0	8	26	60	205	481
Total after tests	0	0	10	8	77	46

8.2 Remaining groups

It is often difficult to ascertain whether a group is finite or infinite for small n. Often it has been necessary to put aside a word which has results for when $n \geq 6$. The following table shows the number of words that remain:

Length	7	8	9	10	11	12
Total before	3	8	30	66	220	484
Total after tests	0	0	10	8	77	46
Total after tests $n \geq 6$	0	0	3	1	27	28

The partial results for the words in the bottom two rows can be viewed in [24]. They leave us with the following groups for $6 \leq n \leq 50$, $l \leq 10$, for which we do not know whether the group is finite or infinite.

- $G_7(x_0^{-1}x_1^{-1}x_2^{-2}x_1^2)$;
- $G_9(x_0^{-1}x_3x_2^{-1})$;
- $G_9(x_0^{-1}x_3x_2)$;
- $G_n(x_0^{-1}x_1^{-1}x_2x_1^2)$, $n \in \{11, 13, 17, 19, 21, 23, 25, 29, 31, 33, 37, 41, 43, 47, 49\}$.

The words $w(x,t)$ corresponding to the above groups $G_n(\omega)$ are those mentioned in Theorem 5.2.4.

We list the remaining groups for $4 \leq n \leq 5$, $l \leq 10$.

$l = 9$

- $G_4(x_0^{-1}x_2^{-3}x_1^{-1})$;
- $G_5(x_0^{-1}x_2^{-2}x_1^2)$;
- $G_4(x_0^{-1}x_2^3x_1)$;
- $G_5(x_0^{-1}x_2^2x_1^2)$;
- $G_5(x_0^{-1}x_1^{-1}x_2^{-1}x_1^2)$;
- $G_4(x_0^{-1}x_1^{-1}x_2^2x_1)$;
- $G_5(x_0^{-1}x_1x_2^{-2}x_1)$.

$l = 10$

- $G_5(x_0^{-1}x_2^{-1}x_3^{-1}x_1)$;
- $G_5(x_0^{-1}x_2^{-1}x_3x_1^{-1})$;
- $G_4(x_0^{-1}x_2^{-4}x_1^{-1})$;
- $G_4(x_0^{-1}x_2^4x_1^{-1})$;
- $G_5(x_0^{-1}x_1^{-1}x_2^{-2}x_1^{-2})$;
- $G_4(x_0^{-1}x_1^{-1}x_2^3x_1)$;
- $G_5(x_0^{-1}x_1^{-1}x_2x_1^3)$;
- $G_4(x_0^{-1}x_1^{-1}x_2^{-2}x_1^2)$;

- $G_5(x_0^{-1}x_1^{-1}x_2^{-2}x_1^2)$;

Note that the words $x_0^{-1}x_2^{-1}x_3^{-1}x_1$ and $x_0^{-1}x_2^{-1}x_3x_1^{-1}$ appearing in the above list are n-equivalent to each other.

8.3 13s, 14s and 15s

Due to the large number of words for when $l = 13$, $l = 14$ and $l = 15$, it has not been possible in the time given to perform the tests on the individual words in the same way as we did for the lower values of l. Instead, we ran all words in GAP and simply asked if it could tell us which ones it knew to be finite. Doing things this way means that we cannot be certain if any of the groups which GAP did not find to be finite are definitely infinite. However, it does give us a great number of groups which it can be sure are finite.

8.3.1 $l = 13$

Below is a table showing the finite family found and its structure.

Group G	$\|G\|$	Structure
$< x_0, \ldots, x_{n-1} \mid x_i x_{i+4} x_{i+3} x_{i+2} x_{i+1}\ (0 \leq i \leq n-1) >,\ \ n \not\equiv 0 \bmod 5$	5	$\mathbb{Z}_5$

The following table shows the remaining groups we have found after removing those which are cyclic as the number of cyclic groups found is very large.

Group G	$\|G\|$	Structure of G
$< x_0, \ldots, x_{14} \mid x_i x_{i+5} x_{i+3}\ (0 \leq i \leq 14) >$	32769	$1 \trianglelefteq \mathbb{Z}_{331} \trianglelefteq G$
$< x_0, \ldots, x_5 \mid x_i x_{i+3}^{-1} x_{i+4} x_{i+2} x_{i+1}^{-1}\ (0 \leq i \leq 5) >$	320	$1 \trianglelefteq \mathbb{Z}_2^2 \trianglelefteq G_1 \trianglelefteq G$
$< x_0, \ldots, x_5 \mid x_i x_{i+3}^{-1} x_{i+4} x_{i+2}^{-1} x_{i+1}^{-1}\ (0 \leq i \leq 5) >$	19683	$1 \trianglelefteq \mathbb{Z}_3 \trianglelefteq G_1 \trianglelefteq G$
$< x_0, \ldots, x_5 \mid x_i x_{i+3}^{-1} x_{i+4}^{-1} x_{i+2}^{-1} x_{i+1}\ (0 \leq i \leq 5) >$	6552	$G_1 \trianglelefteq G$ (G_1 perfect)
$< x_0, \ldots, x_3 \mid x_i x_{i+2} x_i x_{i+3} x_{i+2}\ (0 \leq i \leq 3) >$	120	$\mathbb{Z}_5 \times SL(2,3)$
$< x_0, \ldots, x_3 \mid x_i x_{i+2} x_i^{-1} x_{i+3} x_{i+1}\ (0 \leq i \leq 3) >$	39	$\mathbb{Z}_{13} \rtimes \mathbb{Z}_3$
$< x_0, \ldots, x_3 \mid x_i x_{i+2} x_i^{-1} x_{i+3}^{-1} x_{i+1}^{-1}\ (0 \leq i \leq 3) >$	9375	$1 \trianglelefteq \mathbb{Z}_5^3 \trianglelefteq G_1 \trianglelefteq G$
$< x_0, \ldots, x_3 \mid x_i x_{i+2} x_i^{-1} x_{i+2} x_{i+1}\ (0 \leq i \leq 3) >$	35520	$1 \trianglelefteq G_2 \trianglelefteq G_1 \trianglelefteq G$

Group G	$\lvert G\rvert$	Structure of G
$< x_0,\ldots,x_4 \mid x_ix_{i+2}x_{i+4}^{-1}x_{i+2}x_{i+1}\ (0\le i\le 4) >$	14043	$\mathbb{Z}_{151} \rtimes (\mathbb{Z}_3\times\mathbb{Z}_{31})$
$< x_0,\ldots,x_4 \mid x_ix_{i+2}x_{i+4}^{-1}x_{i+2}^{-1}x_{i+1}^{-1}\ (0\le i\le 4) >$	120	$SL(2,5)$
$< x_0,\ldots,x_5 \mid x_ix_{i+2}x_{i+4}^{-1}x_{i+2}^{-1}x_{i+1}^{-1}\ (0\le i\le 5) >$	56	$1 \trianglelefteq \mathbb{Z}_2^3 \trianglelefteq G$
$< x_0,\ldots,x_3 \mid x_ix_{i+2}x_{i+3}x_{i+2}^2x_{i+1}^2\ (0\le i\le 3) >$	1015	$\mathbb{Z}_{29} \rtimes (\mathbb{Z}_7\times\mathbb{Z}_5)$
$< x_0,\ldots,x_4 \mid x_ix_{i+2}x_{i+3}x_{i+1}x_{i+2}\ (0\le i\le 4) >$	275	$\mathbb{Z}_{11} \rtimes \mathbb{Z}_{25}$
$< x_0,\ldots,x_3 \mid x_ix_{i+2}^{-1}x_{i+3}x_{i+2}^2x_{i+1}^2\ (0\le i\le 3) >$	975	$\mathbb{Z}_{13} \rtimes \mathbb{Z}_{75}$
$< x_0,\ldots,x_3 \mid x_ix_{i+2}^{-1}x_{i+3}^{-1}x_{i+2}^2x_{i+1}^2\ (0\le i\le 3) >$	6561	$1 \trianglelefteq \mathbb{Z}_9^2\times\mathbb{Z}_3 \trianglelefteq G$
$< x_0,\ldots,x_3 \mid x_ix_{i+2}^{-1}x_{i+3}^{-1}x_{i+2}x_{i+1}^3\ (0\le i\le 3) >$	663	$\mathbb{Z}_{13} \rtimes (\mathbb{Z}_3\times\mathbb{Z}_{17})$
$< x_0,\ldots,x_3 \mid x_ix_{i+1}x_{i+2}x_{i+3}^2x_{i+1}^{-2}\ (0\le i\le 3) >$	6561	$1 \trianglelefteq \mathbb{Z}_9^2\times\mathbb{Z}_3 \trianglelefteq G$
$< x_0,\ldots,x_3 \mid x_ix_{i+1}x_{i+2}x_{i+3}^{-1}x_{i+2}^{-1}x_{i+1}^{-2}\ (0\le i\le 3) >$	24	$SL(2,3)$

8.3.2 $l = 14$

Finite families:

Group G	$\lvert G\rvert$	Structure of G
$< x_0,\ldots,x_{n-1} \mid x_ix_{i+5}x_{i+4}x_{i+1}\ (0\le i\le n-1) >,\ n$ odd	4	$\mathbb{Z}_4$
$< x_0,\ldots,x_{n-1} \mid x_ix_{i+5}x_{i+3}x_{i+2}\ (0\le i\le n-1) >,\ n$ odd, $n\not\equiv 0 \bmod 3$	4	$\mathbb{Z}_4$
$< x_0,\ldots,x_{n-1} \mid x_ix_{i+4}x_{i+3}x_{i+2}^2x_{i+1}\ (0\le i\le n-1) >,\ n$ odd, $n\not\equiv 0 \bmod 3$	6	$\mathbb{Z}_6$
$< x_0,\ldots,x_{n-1} \mid x_ix_{i+3}x_{i+2}x_{i+1}x_{i+2}x_{i+1}\ (0\le i\le n-1) >,\ n$ odd, $n\not\equiv 0 \bmod 3$	6	$\mathbb{Z}_6$
$< x_0,\ldots,x_{n-1} \mid x_ix_{i+2}x_{i+1}x_{i+3}x_{i+2}x_{i+1}\ (0\le i\le n-1) >,\ n$ odd, $n\not\equiv 0 \bmod 3$	6	$\mathbb{Z}_6$
$< x_0,\ldots,x_{n-1} \mid x_ix_{i+2}^{-1}x_{i+3}x_{i+2}x_i^{-1}x_{i+1}\ (0\le i\le n-1) >,\ n$ odd	$2^{n+1}-2$	$D_{2^{n+1}-2}$

The following table shows the remaining groups we have found after removing those which are cyclic.

Group G	$\lvert G\rvert$	Structure of G
$< x_0,\ldots,x_7 \mid x_ix_{i+4}x_{i+5}x_{i+4}\ (0\le i\le 7) >$	6560	$1 \trianglelefteq \mathbb{Z}_{41} \trianglelefteq G$
$< x_0,\ldots,x_7 \mid x_ix_{i+4}x_{i+3}^{-1}x_{i+4}\ (0\le i\le 8) >$	6560	$1 \trianglelefteq \mathbb{Z}_{41} \trianglelefteq G$
$< x_0,\ldots,x_5 \mid x_ix_{i+3}x_{i+5}x_{i+3}\ (0\le i\le 5) >$	728	$\mathbb{Z}_{13} \rtimes \mathbb{Z}_{56}$
$< x_0,\ldots,x_6 \mid x_ix_{i+3}^{-1}x_{i+5}^{-1}x_{i+3}^{-1}\ (0\le i\le 6) >$	10922	$1 \trianglelefteq \mathbb{Z}_{43} \trianglelefteq G$
$< x_0,\ldots,x_4 \mid x_ix_{i+2}x_{i+3}x_{i+2}x_ix_{i+1}\ (0\le i\le 4) >$	2046	$1 \trianglelefteq \mathbb{Z}_{31} \trianglelefteq G$
$< x_0,\ldots,x_4 \mid x_ix_{i+2}^{-1}x_{i+1}^{-1}x_{i+2}^{-1}x_{i+1}^{-1}x_{i+2}^{-1}\ (0\le i\le 4) >$	29524	$1 \trianglelefteq \mathbb{Z}_{61} \trianglelefteq G$
$< x_0,\ldots,x_3 \mid x_ix_{i+1}^2x_{i+2}^2x_{i+3}^3\ (0\le i\le 3) >$	6560	$1 \trianglelefteq \mathbb{Z}_{205} \trianglelefteq G$

8.3.3 $l = 15$

There are no finite families and the following table shows the finite groups we have found after removing those which are cyclic.

Group G	$\lvert G\rvert$	Structure of G
$< x_0, \ldots, x_5 \mid x_i x_{i+3}^{-1} x_{i+5} x_{i+4}^{-1} x_{i+2}^{-1} \ (0 \le i \le 5) >$	6552	$G_1 \trianglelefteq G$ (G_1 perfect)
$< x_0, \ldots, x_5 \mid x_i x_{i+3}^{-1} x_{i+5} x_{i+2}^{-1} x_{i+1} \ (0 \le i \le 5) >$	6552	$G_1 \trianglelefteq G$ (G_1 perfect)
$< x_0, \ldots, x_5 \mid x_i x_{i+3}^{-1} x_{i+5}^{-1} x_{i+4} x_{i+2} \ (0 \le i \le 5) >$	320	$1 \trianglelefteq \mathbb{Z}_2^2 \trianglelefteq G_1 \trianglelefteq G$
$< x_0, \ldots, x_5 \mid x_i x_{i+3}^{-1} x_{i+4} x_{i+1}^{-1} x_{i+2} \ (0 \le i \le 5) >$	320	$1 \trianglelefteq \mathbb{Z}_2^2 \trianglelefteq G_1 \trianglelefteq G$
$< x_0, \ldots, x_3 \mid x_i x_{i+2} x_i^2 x_{i+3} x_{i+1}^2 \ (0 \le i \le 3) >$	1015	$\mathbb{Z}_{29} \rtimes \mathbb{Z}_{35}$
$< x_0, \ldots, x_3 \mid x_i x_{i+2} x_i^{-1} x_{i+3}^3 x_{i+1}^{-1} \ (0 \le i \le 3) >$	663	$\mathbb{Z}_{13} \rtimes (\mathbb{Z}_3 \times \mathbb{Z}_{17})$
$< x_0, \ldots, x_4 \mid x_i x_{i+2}^2 x_{i+4}^2 x_{i+1}^2 \ (0 \le i \le 4) >$	5467	$\mathbb{Z}_{71} \rtimes (\mathbb{Z}_7 \times \mathbb{Z}_{11})$
$< x_0, \ldots, x_3 \mid x_i x_{i+2}^2 x_i x_{i+3}^2 x_{i+1} \ (0 \le i \le 3) >$	791	$\mathbb{Z}_{113} \rtimes \mathbb{Z}_7$
$< x_0, \ldots, x_3 \mid x_i x_{i+2}^2 x_i^{-1} x_{i+3}^2 x_{i+1}^{-1} \ (0 \le i \le 3) >$	507	$\mathbb{Z}_{13} \rtimes (\mathbb{Z}_3 \times \mathbb{Z}_{13})$
$< x_0, \ldots, x_3 \mid x_i x_{i+2}^2 x_{i+1}^2 x_i^2 x_{i+3}^2 \ (0 \le i \le 3) >$	4329	$1 \trianglelefteq \mathbb{Z}_{481} \trianglelefteq G$
$< x_0, \ldots, x_4 \mid x_i x_{i+2}^2 x_{i+3}^2 x_{i+1}^2 \ (0 \le i \le 4) >$	5467	$\mathbb{Z}_{71} \rtimes (\mathbb{Z}_7 \times \mathbb{Z}_{11})$
$< x_0, \ldots, x_3 \mid x_i x_{i+2}^{-1} x_{i+3} x_{i+4}^{-1} x_{i+3} x_{i+2} x_{i+1}^{-1} \ (0 \le i \le 3) >$	243	$1 \trianglelefteq \mathbb{Z}_3^3 \trianglelefteq G$
$< x_0, \ldots, x_3 \mid x_i x_{i+2}^{-1} x_{i+3} x_i^{-1} x_{i+2}^2 x_{i+1}^{-1} \ (0 \le i \le 3) >$	320	$1 \trianglelefteq \mathbb{Z}_2^2 \trianglelefteq G_1 \trianglelefteq G$
$< x_0, \ldots, x_3 \mid x_i x_{i+2}^{-1} x_{i+3}^2 x_i^2 x_{i+1}^{-1} \ (0 \le i \le 3) >$	975	$\mathbb{Z}_{13} \rtimes \mathbb{Z}_{75}$
$< x_0, \ldots, x_3 \mid x_i x_{i+2}^{-1} x_{i+3} x_{i+2}^2 x_i^{-1} x_{i+1}^{-1} \ (0 \le i \le 3) >$	320	$1 \trianglelefteq \mathbb{Z}_2^2 \trianglelefteq G_1 \trianglelefteq G$
$< x_0, \ldots, x_3 \mid x_i x_{i+2}^{-1} x_{i+3} x_{i+2} x_{i+1}^{-1} x_i^{-1} x_{i+1}^{-1} \ (0 \le i \le 3) >$	243	$1 \trianglelefteq \mathbb{Z}_3^3 \trianglelefteq G$
$< x_0, \ldots, x_3 \mid x_i x_{i+2}^{-1} x_{i+3} x_{i+2} x_i^{-2} x_{i+1}^{-1} \ (0 \le i \le 3) >$	320	$1 \trianglelefteq \mathbb{Z}_2^2 \trianglelefteq G_1 \trianglelefteq G$
$< x_0, \ldots, x_3 \mid x_i x_{i+2}^{-1} x_{i+3}^{-1} x_i^{-2} x_{i+2} x_{i+1} \ (0 \le i \le 3) >$	320	$1 \trianglelefteq \mathbb{Z}_2^2 \trianglelefteq G_1 \trianglelefteq G$
$< x_0, \ldots, x_3 \mid x_i x_{i+2}^{-1} x_{i+3}^{-1} x_i^{-1} x_{i+3}^{-1} x_{i+2} x_{i+1} \ (0 \le i \le 3) >$	243	$1 \trianglelefteq \mathbb{Z}_3^3 \trianglelefteq G$
$< x_0, \ldots, x_3 \mid x_i x_{i+2}^{-1} x_{i+3}^{-1} x_{i+2}^2 x_i^{-1} x_{i+1} \ (0 \le i \le 3) >$	320	$1 \trianglelefteq \mathbb{Z}_2^2 \trianglelefteq G_1 \trianglelefteq G$
$< x_0, \ldots, x_3 \mid x_i x_{i+2}^{-1} x_{i+3}^{-1} x_{i+2} x_{i+1} x_i x_{i+1} \ (0 \le i \le 3) >$	507	$\mathbb{Z}_{13} \rtimes (\mathbb{Z}_3 \times \mathbb{Z}_{13})$
$< x_0, \ldots, x_3 \mid x_i x_{i+2}^{-1} x_{i+3}^{-1} x_{i+2} x_{i+1} x_i^{-1} x_{i+1}^{-1} \ (0 \le i \le 3) >$	243	$1 \trianglelefteq \mathbb{Z}_3^3 \trianglelefteq G$
$< x_0, \ldots, x_3 \mid x_i x_{i+2}^{-1} x_{i+3}^{-1} x_{i+2}^{-1} x_{i+1}^{-1} x_{i+2}^{-1} x_{i+1} \ (0 \le i \le 3) >$	663	$\mathbb{Z}_{13} \rtimes (\mathbb{Z}_3 \times \mathbb{Z}_{17})$
$< x_0, \ldots, x_3 \mid x_i x_{i+2}^{-2} x_{i+1} x_{i+3}^{-1} x_{i+2}^{-1} x_{i+1}^{-1} \ (0 \le i \le 3) >$	663	$\mathbb{Z}_{13} \rtimes (\mathbb{Z}_3 \times \mathbb{Z}_{17})$
$< x_0, \ldots, x_3 \mid x_i x_{i+2}^2 x_{i+1}^2 x_i^2 x_{i+3}^2 \ (0 \le i \le 3) >$	791	$\mathbb{Z}_{113} \rtimes \mathbb{Z}_7$
$< x_0, \ldots, x_4 \mid x_i x_{i+2}^{-1} x_{i+1}^{-1} x_{i+2}^{-1} x_{i+3}^{-1} x_{i+2} x_{i+1} \ (0 \le i \le 4) >$	120	$SL(2,5)$
$< x_0, \ldots, x_4 \mid x_i x_{i+1} x_{i+2} x_{i+3}^2 x_{i+4}^2 \ (0 \le i \le 4) >$	840	$SL(2,5) \trianglelefteq G$
		($SL(2,5)$ perfect)
$< x_0, \ldots, x_3 \mid x_i x_{i+1}^2 x_{i+3}^{-2} x_{i+2}^{-2} x_{i+1}^{-2} \ (0 \le i \le 3) >$	507	$\mathbb{Z}_{13} \rtimes (\mathbb{Z}_3 \times \mathbb{Z}_{13})$
$< x_0, \ldots, x_3 \mid x_i x_{i+1}^2 x_{i+2}^2 x_i^{-2} x_{i+3}^{-2} \ (0 \le i \le 3) >$	1600	$1 \trianglelefteq \mathbb{Z}_2^2 \trianglelefteq G_1 \trianglelefteq G$
$< x_0, \ldots, x_3 \mid x_i x_{i+1}^2 x_i^{-2} x_{i+2}^2 x_{i+3}^{-2} \ (0 \le i \le 3) >$	1600	$1 \trianglelefteq \mathbb{Z}_2^2 \trianglelefteq G_1 \trianglelefteq G$

8.4 Proofs for finite families

The sixteen finite families mentioned in this chapter have been discovered for certain n using computation methods and conjectured to be families by observation. Here, we prove that the groups are in fact families, i.e. that there is no bound on n for which a finite group exists with the given word.

We begin by proving Proposition 5.2.5, which will cover the proofs for several of the listed finite families.

8.4.1 Proof of Proposition 5.2.5

To prove the groups $G_n(x_0 \ldots x_{k-1})$ for $k \geq 3$ are families of finite cyclically presented groups, we first examine the families of this form which we have already found.

The group $G = G_n(x_0x_2x_1) =< x_0, \ldots, x_{n-1} \mid x_ix_{i+2}x_{i+1}\ (0 \leq i \leq n-1) >$ is equivalent to the group $G_n(x_0x_1x_2)$ and this group is isomorphic to the group $\mathbb{Z}_3$ when $n \not\equiv 0$ mod 3 and infinite otherwise, by observing the results in Subsection 8.1.1.

Let us now consider how we might prove this.

Let $n = 3$ so $G =< x_0, x_1, x_2 \mid x_0x_1x_2, x_1x_2x_0, x_2x_0x_1 >$.
If we let $x_2 = x_1^{-1}x_0^{-1}$ from the last relator and remove the generator x_2, then we get $G =< x_0, x_1 \mid\ >$.
So G is infinite, as expected.

Let $n = 4$ so $G =< x_0, x_1, x_2, x_3 \mid x_0x_1x_2, x_1x_2x_3, x_2x_3x_0, x_3x_0x_1 >$.
Now let $x_3 = x_1^{-1}x_0^{-1}$ and use Tietze transformations to obtain the following:

$$\begin{aligned} G &= < x_0, x_1, x_2 \mid x_0x_1x_2, x_1x_2x_1^{-1}x_0^{-1}, x_2x_1^{-1} > \\ &= < x_0, x_1 \mid x_0x_1^2, x_1x_0^{-1} > = < x_0 \mid x_0^3 > \cong\ \mathbb{Z}_3. \end{aligned}$$

Let $n = 5$ so $G =< x_0, x_1, x_2, x_3, x_4 \mid x_0x_1x_2, x_1x_2x_3, x_2x_3x_4, x_3x_4x_0, x_4x_0x_1 >$.
Let $x_4 = x_1^{-1}x_0^{-1}$:

$$\begin{aligned} G &= < x_0, x_1, x_2, x_3 \mid x_0x_1x_2, x_1x_2x_3, x_2x_3x_1^{-1}x_0^{-1}, x_3x_1^{-1} > \\ &= < x_0, x_1, x_2 \mid x_0x_1x_2, x_1x_2x_1, x_2x_0^{-1} > \\ &= < x_0, x_1 \mid x_0x_1x_0, x_1x_0x_1 > = < x_0 \mid x_0^3 > \cong\ \mathbb{Z}_3. \end{aligned}$$

Now assume $n > 5$.

So $G =< x_0,\ \ldots,\ x_{n-1} \mid x_0x_1x_2,\ \ldots,\ x_{n-3}x_{n-2}x_{n-1},\ x_{n-2}x_{n-1}x_0,\ x_{n-1}x_0x_1 >$

Let $x_{n-1} = x_1^{-1}x_0^{-1}$:

$G =< x_0,\ \ldots,\ x_{n-2} \mid x_0x_1x_2,\ \ldots,\ x_{n-4}x_{n-3}x_{n-2},\ x_{n-3}x_{n-2}x_1^{-1}x_0^{-1},\ x_{n-2}x_1^{-1} >$

Let $x_{n-2} = x_1$:

$G =< x_0,\ \ldots,\ x_{n-3} \mid x_0x_1x_2,\ \ldots,\ x_{n-5}x_{n-4}x_{n-3},\ x_{n-4}x_{n-3}x_1,\ x_{n-3}x_0^{-1} >$

Let $x_{n-3} = x_0$:

$G =< x_0,\ \ldots,\ x_{n-4} \mid x_0x_1x_2,\ \ldots,\ x_{n-6}x_{n-5}x_{n-4},\ x_{n-5}x_{n-4}x_0,\ \ x_{n-4}x_0x_1 >$.

But then $G_n(\omega) \cong G_{n-3}(\omega)$, where $\omega = x_0x_1x_2$, and so by induction, $n \equiv m \bmod 3 \implies G_n(\omega) \cong G_m(\omega)$.

So, as $G_4(\omega) = G_5(\omega) = \mathbb{Z}_3$ and $G_3(\omega)$ is infinite, $n \equiv 0 \bmod 3 \implies G_n(\omega)$ is infinite, and $n \not\equiv 0 \bmod 3 \implies G_n(\omega) = \mathbb{Z}_3$. So $G_3(\omega)$ is infinite $\iff n \equiv 0 \bmod 3 \iff gcd(n, 3) = 1$.

Also in our list of finite families is the group $G = G_n(x_0x_3x_2x_1) = G_n(x_0x_1x_2x_3)$, isomorphic to $\mathbb{Z}_4$ when n is odd and infinite otherwise.

$$G =< x_0,\ \ldots,\ x_{n-1} \mid \begin{array}{l} x_0x_1x_2x_3,\ \ldots,\ x_{n-4}x_{n-3}x_{n-2}x_{n-1},\ x_{n-3}x_{n-2}x_{n-1}x_0, \\ x_{n-2}x_{n-1}x_0x_1,\ x_{n-1}x_0x_1x_2 >. \end{array}$$

Let $x_{n-1} = x_2^{-1}x_1^{-1}x_0^{-1}$:

$$G =< x_0,\ \ldots,\ x_{n-2} \mid \begin{array}{l} x_0x_1x_2x_3,\ \ldots,\ x_{n-5}x_{n-4}x_{n-3}x_{n-2},\ x_{n-4}x_{n-3}x_{n-2}x_2^{-1}x_1^{-1}x_0^{-1}, \\ x_{n-3}x_{n-2}x_2^{-1}x_1^{-1}, x_{n-2}x_2^{-1} >. \end{array}$$

$$G =< x_0,\ \ldots,\ x_{n-3} \mid \begin{array}{l} x_0x_1x_2x_3,\ \ldots,\ x_{n-6}x_{n-5}x_{n-4}x_{n-3},\ x_{n-5}x_{n-4}x_{n-3}x_2, \\ x_{n-4}x_{n-3}x_1^{-1}x_0^{-1},\ x_{n-3}x_1^{-1} >. \end{array}$$

$$G =< x_0,\ \ldots,\ x_{n-4} \mid \begin{array}{l} x_0x_1x_2x_3,\ \ldots,\ x_{n-7}x_{n-6}x_{n-5}x_{n-4},\ x_{n-6}x_{n-5}x_{n-4}x_1, \\ x_{n-5}x_{n-4}x_1x_2,\ x_{n-4}x_0^{-1} >. \end{array}$$

$$G =< x_0,\ \ldots,\ x_{n-5} \mid \begin{array}{l} x_0x_1x_2x_3,\ \ldots,\ x_{n-8}x_{n-7}x_{n-6}x_{n-5},\ x_{n-7}x_{n-6}x_{n-5}x_0, \\ x_{n-6}x_{n-5}x_0x_1,\ x_{n-5}x_0x_1x_2 >. \end{array}$$

So this time $G_n(\omega) \cong G_{n-4}(\omega)$ and so $n \equiv m \bmod 4 \implies G_n(\omega) \cong G_m(\omega)$ where $\omega = x_0x_1x_2x_3$.

More generally, if $\omega = x_0x_1\dots x_{k-1}$ then $n \equiv m \bmod k \implies G_n(\omega) \cong G_m(\omega)$.

To show this, let $G_n(\omega) =< x_0,\ \dots,\ x_{n-1} \mid r_0,\dots,r_{n-k},\ s_0,\dots,s_{k-2} >$, where $n > k$, and where the relators are defined as follows:

$$\begin{aligned} r_0 &= x_0\dots x_{k-1}, & s_0 &= x_{n-(k-1)}\dots x_{n-1}x_0,\\ r_1 &= x_1\dots x_k, & s_1 &= x_{n-(k-2)}\dots x_{n-1}x_0x_1,\\ &\vdots & &\vdots\\ r_{n-k} &= x_{n-k}\dots x_{n-1} & s_{k-2} &= x_{n-1}x_0\dots x_{k-2}. \end{aligned}$$

There are $n-k+1$ relators of the type r_i and $k-1$ relators of the type s_i, which makes n relators in total, as expected.

$$\begin{aligned} s_{k-2} = x_{n-1}x_0\dots x_{k-2} &\implies x_{n-1} = x_{k-2}^{-1}x_{k-3}^{-1}\dots x_1^{-1}x_0^{-1}\\ s_{k-3} = x_{n-2}x_{n-1}x_0\dots x_{k-3} &\implies x_{n-2} = x_{k-2}\\ s_{k-4} = x_{n-3}x_{n-2}x_{n-1}x_0\dots x_{k-4} &\implies x_{n-3} = x_{k-3}\\ &\vdots\\ s_0 = x_{n-(k-1)}\dots x_{n-1}x_0 &\implies x_{n-(k-1)} = x_1 \end{aligned}$$

So $x_i = x_{i+n-k}$, for $1 \le i \le k-2$. This allows us to remove all s_i and all x_i for $n-(k-1) \le i \le n-1$.

$$\begin{aligned} \text{Then } r_{n-k} &= x_{n-k}x_{n-(k-1)}x_{n-(k-2)}\dots x_{n-3}x_{n-2}x_{n-1}\\ &= x_{n-k}x_1x_2\dots x_{k-3}x_{k-2}x_{k-2}^{-1}x_{k-3}^{-1}\dots x_2^{-1}x_1^{-1}x_0^{-1}\\ &= x_{n-k}x_0^{-1}. \end{aligned}$$

So $r_{n-k} \implies x_{n-k} = x_0$ and therefore remove r_{n-k} and x_{n-k}.

We are left with the relators $r_0,\dots,r_{n-k-1}$, which are those that originally only involved x_i for $i < n-1$.
As we used the fact that $x_i = x_{i+n-k}$ for $0 \le i \le k-2$ where necessary in these relators, we are now working mod $n-k$ with the indices and we have obtained $G_{n-k}(\omega)$.
Therefore, $G_n(\omega) = G_{n-k}(\omega)$, as expected and we have shown that $n \equiv m \bmod k \implies G_n(\omega) \cong G_m(\omega)$.

Now let us assume that $n = k$. In this case, $G_n(\omega) =< x_0, \ \ldots, \ x_{n-1} \mid x_0 \ldots x_{n-1} >$ as all the n relators will be cyclic permutations of each other. The relator implies that $x_{n-1} = x_{n-2}^{-1} \ldots x_0^{-1}$ and we so can remove x_{n-1} and the relator, leaving us with $< x_0, \ \ldots, \ x_{n-2} \mid \ > = \ F_{n-1}$, which is infinite.

Finally, let us assume that $1 \leq n < k$.
Then $G = G_n(\omega) =< x_0, \ \ldots, \ x_{n-1} \mid r_0, \ldots, r_{n-1} >$, where $r_i = x_i \ldots x_{n-1} x_0 \ldots x_{n-1} x_0 \ldots$, where the pattern continues so that r_i has length k for each i. We claim that G is finite cyclic of order k if and only if $gcd(n, k) = 1$.

$$\begin{aligned} r_0 &= (x_0 \ldots x_{n-1})^s x_0 \ldots x_{r-1}, \\ r_1 &= (x_1 \ldots x_{n-1} x_0)^s x_1 \ldots x_r, \\ &\vdots \\ r_i &= (x_i \ldots x_{n-1} x_0 \ldots x_{i-1})^s x_i \ldots x_{r-1+i}, \\ r_{i+1} &= (x_{i+1} \ldots x_{n-1} x_0 \ldots x_i)^s x_{i+1} \ldots x_{r+i}, \end{aligned}$$

where $r \equiv k \bmod n, \ s = \frac{k-r}{n}$, i.e. $k = ns + r$.
Then $r_0 \implies (x_1 \ldots x_{n-1} x_0)^s x_1 \ldots x_{r-1} x_0 = 1$ so $r_0, r_1 \implies x_0 = x_r$.
$r_i \implies (x_{i+1} \ldots x_{n-1} x_0 \ldots x_i)^s x_{i+1} \ldots x_{r-1+i} x_i = 1$ so $r_i, r_{i+1} \implies x_i = x_{i+r}$.
So $i = j + ar \bmod n$ for some $a \in \mathbb{Z} \implies x_i = x_j$ (*).

Let $gcd(n, k) = 1$. Then $gcd(n, r) = 1 \implies ar \bmod n$ generates $\mathbb{Z}_n, \ a \in \mathbb{Z} \implies \forall j, \exists a_j \in \mathbb{Z}$ such that $j \equiv a_j r \bmod n \implies x_0 = x_j, \ \forall j$ by (*).
We end up with one generator for G, x_0 say, and one relator, x_0^k, so $G \equiv \mathbb{Z}_k$ as predicted.

Let $gcd(n, k) = d > 1$. Then $gcd(n, r) = d \implies x_i = x_{i+d}$. We are left with the generators $x_0, \ldots, x_{d-1}$ and our relators are $r_0 = (x_0 \ldots x_{d-1})^m, \ldots, r_i = (x_i \ldots x_{d-1} x_0 \ldots x_{i-1})^m$, $m = \frac{k}{d}$. These relators are all equal, so we are left with $G =< x_0, \ldots, x_{d-1} \mid (x_0 \ldots x_{d-1})^m >$, which is infinite, as expected.

So far we have proved:

(A) $n \equiv m \bmod k \implies G_n(\omega) \cong G_m(\omega)$.

(B) If $1 \leq n \leq k$, $G_n(\omega)$ is finite $\iff gcd(n, k) = 1$.

Assume $gcd(n, k) = 1$. Then $gcd(m, k) = 1$ for $m = n \bmod k$, $1 \leq m \leq k$, which implies $G_m(\omega)$ is finite (from (B)), and $G_n(\omega) = G_m(\omega)$ (from (A)). So $G_n(\omega)$ is finite.

Assume $gcd(n, k) = d > 1$. Then $gcd(m, k) = 1$ for $m = n \bmod k$, $1 \leq m \leq k$, which implies $G_m(\omega)$ is infinite (from (B)), and $G_n(\omega) = G_m(\omega)$ (from (A)). So $G_n(\omega)$ is infinite.

In conclusion, $G_n(x_0 \dots x_{k-1})$ is finite $\iff$ $gcd(n,k) = 1$. Also, for any k there is always an n for which $gcd(n,k) = 1$ and so $G_{n+ak}(\omega)$ is finite for all $a \in \mathbb{Z}$. This proves Proposition 5.2.5.

8.4.2 $\omega = x_0x_2x_1$

$G =< x_0, \dots, x_{n-1} \mid x_ix_{i+2}x_{i+1}\ (0 \leq i \leq n-1) >, \quad n \not\equiv 0 \bmod 3. \quad G = \mathbb{Z}_3,\ l = 7.$

See Subsection 8.4.1.

8.4.3 $\omega = x_0x_2x_1^2$

$G =< x_0, \dots, x_{n-1} \mid x_ix_{i+2}x_{i+1}^2\ (0 \leq i \leq n-1) >, \quad n$ odd. $\quad G = \mathbb{Z}_4,\ l = 8.$

Assume n odd.

Relators are:
$x_0x_2x_1x_1$
$x_1x_3x_2x_2$
$x_2x_4x_3x_3$
$\dots$
$x_{n-2}x_0x_{n-1}x_{n-1}$
$x_{n-1}x_1x_0x_0$

Let $y_i = x_i^{-1}$.

$x_0 = y_1y_1y_2$
$x_1 = y_2y_2y_3$
$x_2 = y_3y_3y_4$
$\dots$
$x_{n-4} = y_{n-3}y_{n-3}y_{n-2}$
$x_{n-3} = y_{n-2}y_{n-2}y_{n-1}$

So we can remove $x_0, x_1, \dots, x_{n-3}$ and the first $n-2$ relators and write the last two relators in terms of x_{n-2} and x_{n-1}.

Consider the relator $x_{n-1}x_{n-1}x_{n-2}x_0$:

$$\begin{aligned}
x_0 &= y_1y_1y_2\\
&= (x_3x_2x_2)(x_3x_2x_2)y_2 = x_3x_2x_2x_3x_2 = (x_3x_2)x_2(x_3x_2)\\
&= y_3y_4y_3y_3y_4y_3y_4 = (y_3y_4)y_3(y_3y_4)^2\\
&= x_5x_4x_5x_4x_4x_5x_4x_5x_4 = (x_5x_4)^2x_4(x_5x_4)^2\\
&= y_5y_6y_5y_6y_5y_5y_6y_5y_6y_5 = (y_5y_6)^2y_5(y_5y_6)^3\\
&\dots\\
&= [(x_{i+1}x_i)^{\frac{i}{2}}]x_i[(x_{i+1}x_i)^{\frac{i}{2}}] \text{ when removing } x_{i-1},\ i \text{ is even}\\
&= [(y_iy_{i+1})^{\frac{i-1}{2}}]y_i[(y_iy_{i+1})^{\frac{i+1}{2}}] \text{ when removing } x_{i-1},\ i \text{ is odd}\\
&\dots\\
&= [(y_{n-2}y_{n-1})^{\frac{n-3}{2}}]y_{n-2}[(y_{n-2}y_{n-1})^{\frac{n-1}{2}}]
\end{aligned}$$

$$\begin{aligned}
x_{n-1}x_{n-1}x_{n-2}x_0 = 1 \ &\Leftrightarrow\ x_{n-1}x_{n-1}x_{n-2}[(y_{n-2}y_{n-1})^{\frac{n-3}{2}}]y_{n-2}[(y_{n-2}y_{n-1})^{\frac{n-1}{2}}] = 1\\
&\Leftrightarrow\ x_{n-1}x_{n-1}x_{n-2}[(y_{n-2}y_{n-1})^{\frac{n-3}{2}}]y_{n-2}[(y_{n-2}y_{n-1})^{\frac{n-3}{2}}]y_{n-2}y_{n-1} = 1\\
&\Leftrightarrow\ x_{n-1}x_{n-2}[(y_{n-2}y_{n-1})^{\frac{n-3}{2}}]y_{n-2}[(y_{n-2}y_{n-1})^{\frac{n-3}{2}}]y_{n-2} = 1\\
&\Leftrightarrow\ x_{n-1}x_{n-2}y_{n-2}y_{n-1}[(y_{n-2}y_{n-1})^{\frac{n-5}{2}}]y_{n-2}[(y_{n-2}y_{n-1})^{\frac{n-3}{2}}]y_{n-2} = 1\\
&\Leftrightarrow\ x_{n-1}y_{n-1}[(y_{n-2}y_{n-1})^{\frac{n-5}{2}}]y_{n-2}[(y_{n-2}y_{n-1})^{\frac{n-3}{2}}]y_{n-2} = 1\\
&\Leftrightarrow\ [(y_{n-2}y_{n-1})^{\frac{n-5}{2}}]y_{n-2}[(y_{n-2}y_{n-1})^{\frac{n-3}{2}}]y_{n-2} = 1\\
&\Leftrightarrow\ y_{n-2}[(y_{n-2}y_{n-1})^{\frac{n-5}{2}}]y_{n-2}[(y_{n-2}y_{n-1})^{\frac{n-3}{2}}] = 1.
\end{aligned}$$

Consider the relator $x_{n-1}x_1x_0x_0$:

$$\begin{aligned}
x_1x_0x_0 &= x_1(y_1y_1y_2)(y_1y_1y_2) = y_1y_2y_1y_1y_2\\
&= x_3x_2x_3x_2x_2x_3x_2 = (x_3x_2)^2x_2(x_3x_2)\\
&= y_3y_4y_3y_4y_3y_3y_4y_3y_4 = (y_3y_4)^2y_3(y_3y_4)^2\\
&= x_5x_4x_5x_4x_5x_4x_4x_5x_4x_5x_4 = (x_5x_4)^3x_4(x_5x_4)^2\\
&= y_5y_6y_5y_6y_5y_6y_5y_5y_6y_5y_6y_5y_6 = (y_5y_6)^3y_5(y_5y_6)^3\\
&\dots\\
&= [(x_{i+1}x_i)^{\frac{i}{2}+1}]x_i[(x_{i+1}x_i)^{\frac{i}{2}}] \text{ when removing } x_{i-1},\ i \text{ is even}\\
&= [(y_iy_{i+1})^{\frac{i+1}{2}}]y_i[(y_iy_{i+1})^{\frac{i+1}{2}}] \text{ when removing } x_{i-1},\ i \text{ is odd}\\
&\dots\\
&= [(y_{n-2}y_{n-1})^{\frac{n-1}{2}}]y_{n-2}[(y_{n-2}y_{n-1})^{\frac{n-1}{2}}]
\end{aligned}$$

$$\begin{aligned}
x_{n-1}x_1x_0x_0 = 1 \ &\Leftrightarrow\ x_{n-1}[(y_{n-2}y_{n-1})^{\frac{n-1}{2}}]y_{n-2}[(y_{n-2}y_{n-1})^{\frac{n-1}{2}}] = 1\\
&\Leftrightarrow\ x_{n-1}[(y_{n-2}y_{n-1})^{\frac{n-1}{2}}]y_{n-2}[(y_{n-2}y_{n-1})^{\frac{n-3}{2}}]y_{n-2}y_{n-1} = 1
\end{aligned}$$

$$\begin{aligned}&\Leftrightarrow\ [(y_{n-2}y_{n-1})^{\frac{n-1}{2}}]y_{n-2}[(y_{n-2}y_{n-1})^{\frac{n-3}{2}}]y_{n-2}=1\\&\Leftrightarrow\ y_{n-2}[(y_{n-2}y_{n-1})^{\frac{n-1}{2}}]y_{n-2}[(y_{n-2}y_{n-1})^{\frac{n-3}{2}}]=1\end{aligned}$$

Consider:
$y_{n-2}[(y_{n-2}y_{n-1})^{\frac{n-1}{2}}]y_{n-2}[(y_{n-2}y_{n-1})^{\frac{n-3}{2}}]$
$y_{n-2}[(y_{n-2}y_{n-1})^{\frac{n-5}{2}}]y_{n-2}[(y_{n-2}y_{n-1})^{\frac{n-3}{2}}]$

So $(y_{n-2}y_{n-1})^{\frac{n-1}{2}}=(y_{n-2}y_{n-1})^{\frac{n-5}{2}}$
$[(y_{n-2}y_{n-1})^{\frac{n-5}{2}}](y_{n-2}y_{n-1})(y_{n-2}y_{n-1})=(y_{n-2}y_{n-1})^{\frac{n-5}{2}}$

So $(y_{n-2}y_{n-1})(y_{n-2}y_{n-1})=1$.

Into $y_{n-2}[(y_{n-2}y_{n-1})^{\frac{n-1}{2}}]y_{n-2}[(y_{n-2}y_{n-1})^{\frac{n-3}{2}}]$:

Let $\frac{n-1}{2}$ even, so $\frac{n-3}{2}$ odd:
$y_{n-2}[(y_{n-2}y_{n-1})^{\frac{n-1}{2}}]y_{n-2}[(y_{n-2}y_{n-1})^{\frac{n-3}{2}}]=y_{n-2}y_{n-2}y_{n-2}y_{n-1}$:
$y_{n-1}=x_{n-2}^3$

Let $\frac{n-1}{2}$ odd, so $\frac{n-3}{2}$ even:
$y_{n-2}[(y_{n-2}y_{n-1})^{\frac{n-1}{2}}]y_{n-2}[(y_{n-2}y_{n-1})^{\frac{n-3}{2}}]=y_{n-2}y_{n-2}y_{n-1}y_{n-2}$:
$y_{n-1}=x_{n-2}^3$.

So we can remove x_{n-1}.

$$\begin{aligned}y_{n-2}[(y_{n-2}y_{n-1})^{\frac{n-1}{2}}]y_{n-2}[(y_{n-2}y_{n-1})^{\frac{n-3}{2}}]&=y_{n-2}[(y_{n-2}x_{n-2}^3)^{\frac{n-1}{2}}]y_{n-2}[(y_{n-2}x_{n-2}^3)^{\frac{n-3}{2}}]\\&=y_{n-2}[(x_{n-2}^2)^{\frac{n-1}{2}}]y_{n-2}[(x_{n-2}^2)^{\frac{n-3}{2}}]\\&=y_{n-2}[(x_{n-2})^{n-1}]y_{n-2}[(x_{n-2})^{n-3}]\\&=y_{n-2}x_{n-2}[(x_{n-2})^{n-2}]y_{n-2}x_{n-2}[(x_{n-2})^{n-4}]\\&=[(x_{n-2})^{n-2}][(x_{n-2})^{n-4}]=x_{n-2}^{2n-6}\end{aligned}$$

So $x_{n-2}^{2n-6}=1$

$$\begin{aligned}y_{n-2}[(y_{n-2}y_{n-1})^{\frac{n-5}{2}}]y_{n-2}[(y_{n-2}y_{n-1})^{\frac{n-3}{2}}]&=y_{n-2}[(y_{n-2}x_{n-2}^3)^{\frac{n-5}{2}}]y_{n-2}[(y_{n-2}x_{n-2}^3)^{\frac{n-3}{2}}]\\&=y_{n-2}[(x_{n-2}^2)^{\frac{n-5}{2}}]y_{n-2}[(x_{n-2}^2)^{\frac{n-3}{2}}]\\&=y_{n-2}[(x_{n-2})^{n-5}]y_{n-2}[(x_{n-2})^{n-3}]\\&=y_{n-2}x_{n-2}[(x_{n-2})^{n-6}]y_{n-2}x_{n-2}[(x_{n-2})^{n-4}]\\&=[(x_{n-2})^{n-6}][(x_{n-2})^{n-4}]\ =\ x_{n-2}^{2n-10}\end{aligned}$$

So $x_{n-2}^{2n-10}=1$

$x_{n-2}^{2n-6} = x_{n-2}^{2n-10} x_{n-2}^4 = x_{n-2}^{2n-10}$ so $x_{n-2}^4 = 1$
So $G =< x_{n-2} \mid x_{n-2}^4 >$

8.4.4 $\omega = x_0 x_1 x_2 x_1$

$G =< x_0, \ldots, x_{n-1} \mid x_i x_{i+1} x_{i+2} x_{i+1}\ (0 \leq i \leq n-1) >, \quad n$ odd. $\quad G = \mathbb{Z}_n \rtimes \mathbb{Z}_4$, $l = 8$.

Relators:

$x_0 x_1 x_2 x_1$

$x_1 x_2 x_3 x_2$

$x_2 x_3 x_4 x_3$

$\ldots$

$x_{n-3} x_{n-2} x_{n-1} x_{n-2}$

$x_{n-2} x_{n-1} x_0 x_{n-1}$

$x_{n-1} x_0 x_1 x_0$

Let $y_i = x_i^{-1}$

$x_0 = y_1 y_2 y_1$

$x_1 = y_2 y_3 y_2$

$x_2 = y_2 y_3 y_2$

$\ldots$

$x_{n-3} = y_{n-2} y_{n-1} y_{n-2}$

Get the generators $x_0, \ldots, x_{n-3}$ in terms of x_{n-2} and x_{n-1} in order to remove them.

$x_{n-3} = y_{n-2} y_{n-1} y_{n-2}$

$$
\begin{aligned}
x_{n-4} &= y_{n-3} y_{n-2} y_{n-3} \\
&= x_{n-2} x_{n-1} x_{n-2} y_{n-2} x_{n-2} x_{n-1} x_{n-2} \\
&= x_{n-2} x_{n-1} x_{n-2} x_{n-1} x_{n-2} \\
x_{n-5} &= y_{n-4} y_{n-3} y_{n-4} \\
&= x_{n-3} x_{n-2} x_{n-3} x_{n-2} x_{n-3} \\
&= y_{n-2} y_{n-1} y_{n-2} y_{n-1} y_{n-2} y_{n-1} y_{n-2} \\
&\ldots \\
x_i &= y_{n-2} (y_{n-1} y_{n-2})^{n-i-2} && i \text{ even} \\
x_i &= x_{n-2} (x_{n-1} x_{n-2})^{n-i-2} && i \text{ odd} \\
&\ldots
\end{aligned}
$$

$$\begin{aligned}
x_1 &= x_{n-2}(x_{n-1}x_{n-2})^{n-3} \\
x_0 &= y_{n-2}(y_{n-1}y_{n-2})^{n-2}
\end{aligned}$$

Relators left:

$x_{n-1}x_{n-2}x_{n-1}x_0$

$x_{n-1}x_0x_1x_0$

$$\begin{aligned}
x_{n-1}x_{n-2}x_{n-1}x_0 &= x_{n-1}x_{n-2}x_{n-1}y_{n-2}(y_{n-1}y_{n-2})^{n-2} \\
x_{n-1}x_0x_1x_0 \qquad &= x_{n-1}y_{n-2}(y_{n-1}y_{n-2})^{n-2}x_{n-2}(x_{n-1}x_{n-2})^{n-3}y_{n-2}(y_{n-1}y_{n-2})^{(n-2)} \\
&= x_{n-1}y_{n-2}(y_{n-1}y_{n-2})^{n-2}x_{n-2}(x_{n-1}x_{n-2})^{n-3}(y_{n-2}y_{n-1})^{n-2}y_{n-2} \\
&= x_{n-1}y_{n-2}(y_{n-1}y_{n-2})^{n-2}x_{n-2}(x_{n-1}x_{n-2})^{n-3}(y_{n-2}y_{n-1})^{n-3}y_{n-2}y_{n-1}y_{n-2} \\
&= x_{n-1}y_{n-2}(y_{n-1}y_{n-2})^{n-2}y_{n-1}y_{n-2} \\
&= x_{n-1}y_{n-2}(y_{n-1}y_{n-2})^{n-1}
\end{aligned}$$

Compare:

$x_{n-1}x_{n-2}x_{n-1}y_{n-2}(y_{n-1}y_{n-2})^{n-2}$

$x_{n-1}y_{n-2}y_{n-1}y_{n-2}(y_{n-1}y_{n-2})^{n-2}$

Then $x_{n-2}x_{n-1} = y_{n-2}y_{n-1}$

The relator $x_{n-1}x_{n-2}x_{n-1}y_{n-2}(y_{n-1}y_{n-2})^{n-2}$ can now be removed so relators are: $x_{n-2}x_{n-2}x_{n-1}x_{n-1}$, $(x_{n-1}x_{n-2})^{n-1}x_{n-1}y_{n-2}$.

Let $x = x_{n-1}$, $y = x_{n-2}$

$G =< x, y \mid x^2y^2, (xy)^{n-1}xy^{-1} >$

So $yx^2y = yx^2y^{-1}x^{-2} = 1$
and $xy^2x = y^{-2}x^{-1}y^2x = 1$

So $[x^2, y] = [y^2, x] = 1$

$y = (xy)^{n-1}x$ so $y^2 = ((xy)^{n-1}x)(x(yx)^{n-1}) = x(yx(yx...(yx(yxxy)xy)...xy)xy)x = x^2$
So $y^2 = x^2$ and $y^4 = x^4 = 1$

$$\begin{aligned}
\text{Then } (xy)^{n-1}xy^{-1} = (xy^{-1}y^2)^{n-1}xy^{-1} &= [(xy^{-1})^n][y^{2(n-1)}] && \text{as } y^2 \text{ commutes with } x \text{ and } y \\
&= (xy^{-1})^n && \text{as } n \text{ odd so } 4|2(n-1)
\end{aligned}$$

So $G = < x, y \mid y^4, x^2y^2, (xy^{-1})^n >$

$$= < x, y, z \mid zyx^{-1}, y^4, x^2y^2, z^n >$$
$$= < y, z \mid y^4, z^n, zyzy^3 >$$
$$= < y, z \mid y^4, z^n, y^{-1}zy = z^{n-1} >$$

8.4.5 $\omega = x_0x_3x_2x_1$

$G =< x_0, \ldots, x_{n-1} \mid x_ix_{i+3}x_{i+2}x_{i+1}\ (0 \le i \le n-1) >$, n odd. $G = \mathbb{Z}_4$, $l = 10$.

See Subsection 8.4.1.

8.4.6 $\omega = x_0x_2^{-1}x_1^{-1}x_2^{-1}$

$G =< x_0, \ldots, x_{n-1} \mid x_ix_{i+2}^{-1}x_{i+1}^{-1}x_{i+2}^{-1}\ (0 \le i \le n-1) >$, n odd. $1 \trianglelefteq \mathbb{Z}_{\frac{1}{3}(2^n+1)} \trianglelefteq G$, $|G| = \frac{2}{3}(4^n - 1)$, $l = 10$.

This particular family of groups is infinite when n is even (since $H_2(\omega)$ is the infinite dihedral group) and finite of order $\frac{2}{3}(4^n - 1)$ when n is odd, the latter of which we now prove.

Let us assume $n > 4$ is odd.

The group $< x_0, \ldots, x_{n-1} \mid x_ix_{i+2}^{-1}x_{i+1}^{-1}x_{i+2}^{-1}\ (0 \le i \le n-1) >$ is the associated cyclically presented group of $H = H_n(w) =< x, t \mid t^n, xtxt^{-1}xt^2x^{-1}t^{-2} >$.

To show the group G is finite and has order $\frac{2}{3}(4^n - 1)$, we first find the order of H. Let us note that $H_{ab} = < x, t \mid t^n, x^2, [x, t] > = \mathbb{Z}_2 \times \mathbb{Z}_n$ and therefore $|H : H'| = 2n$, where H' denotes the derived subgroup of H. Next, we use the use the Schreier method to find a presentation for H'.

We can see that $U = \{e, t, t^2, \ldots, t^{n-1}, x, xt, xt^2, \ldots, xt^{n-1}\}$ is a Schreier Transversal for H' in H. We note that we should find $(g-1)i + 1 = 2n + 1$ generators for H', where $g =$ no. generators of H and $i = |H : H'|$.

The next step is to find the generators for H' in terms of x and t by finding all words of the form $uy(\overline{uy})^{-1}$, where $u \in U$ and $y \in \{x, t\}$:

$u \in U$	$y \in \{x,t\}$	$\overline{uy}$	$uy(\overline{uy})^{-1}$
e	x	x	e
t	x	xt	$txt^{-1}x^{-1}$
t^2	x	xt^2	$t^2xt^{-2}x^{-1}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
t^{n-2}	x	xt^{n-2}	$t^{n-2}xt^{-(n-2)}x^{-1}$
t^{n-1}	x	xt^{n-1}	$t^{n-1}xt^{-(n-1)}x^{-1}$
x	x	e	x^2
xt	x	t	$xtxt^{-1}$
xt^2	x	t^2	xt^2xt^{-2}
$\vdots$	$\vdots$	$\vdots$	$\vdots$
xt^{n-2}	x	t^{n-2}	$xt^{n-2}xt^{-(n-2)}$
xt^{n-1}	x	t^{n-1}	$xt^{n-1}xt^{-(n-1)}$

$u \in U$	$y \in \{x,t\}$	$\overline{uy}$	$uy(\overline{uy})^{-1}$
e	t	t	e
t	t	t^2	e
t^2	t	t^3	e
$\vdots$	$\vdots$	$\vdots$	$\vdots$
t^{n-2}	t	t^{n-1}	e
t^{n-1}	t	e	t^n
x	t	xt	e
xt	t	xt^2	e
xt^2	t	xt^3	e
$\vdots$	$\vdots$	$\vdots$	$\vdots$
xt^{n-2}	t	xt^{n-1}	e
xt^{n-1}	t	x	xt^nx^{-1}

So generators for H' are:

$a_i = t^ixt^{-i}x^{-1}$ $\qquad$ $c_1 = x^2$

$b_i = xt^ixt^{-i}$ $\qquad$ $c_2 = t^n$

$(1 \leq i \leq n-1)$ $\qquad$ $c_3 = xt^nx^{-1}$

As predicted, there are $2n+1$ generators for H'. We now need to find the defining relators of H' in terms of the a_i, b_i and c_i. These relators are all words of the form uru^{-1} where $u \in U$ and r is a relator for H. To aid us we first calculate all conjugates of the generators of H' by the generators of H.

g	a_1	a_2	$\dots$	a_{n-2}	a_{n-1}	b_1	b_2	$\dots$	b_{n-2}	b_{n-1}	c_1	c_2	c_3
tgt^{-1}	$a_2a_1^{-1}$	$a_3a_1^{-1}$	$\dots$	$a_{n-1}a_1^{-1}$	$c_2c_3^{-1}a_1^{-1}$	a_1b_2	a_1b_3	$\dots$	a_1b_{n-1}	$a_1c_3c_1c_2^{-1}$	a_1b_1	c_2	$a_1c_3a_1^{-1}$

g	a_1	a_2	$\dots$	a_{n-2}	a_{n-1}	b_1	b_2	$\dots$	b_{n-2}	b_{n-1}	c_1	c_2	c_3
xgx^{-1}	$b_1c_1^{-1}$	$b_2c_1^{-1}$	$\dots$	$b_{n-2}c_1^{-1}$	$b_{n-1}c_1^{-1}$	c_1a_1	c_1a_2	$\dots$	c_1a_{n-2}	c_1a_{n-1}	c_1	c_3	$c_1c_2c_1^{-1}$

Our relators for H' in term of x and t are the following:

uru^{-1}:

$u \backslash r$	t^n	$xtxt^{-1}xt^2x^{-1}t^{-2}$
t^i	t^n	$t^ixtxt^{-1}xt^2x^{-1}t^{-2-i}$, $(0 \leq i \leq n-1)$
xt^i	xt^nx^{-1}	$xt^ixtxt^{-1}xt^2x^{-1}t^{-2-i}x^{-1}$, $(0 \leq i \leq n-1)$

The final step in finding a presentation for H' is to rewrite all relators in terms of the a_i, b_i and c_i. For this we can use our conjugacy tables as the relators corresponding to t^i and xt^i, $0 < i \leq n-1$, are simply the relators corresponding to t^{i-1} conjugated by t and t^i conjugated by x respectively. We use the fact that $a^{-1}bbb...ba = (a^{-1}ba)(a^{-1}ba)...(a^{-1}ba)$. So, for example, if $r = xtxt^{-1}xt^2x^{-1}t^{-2}$ then the relator $t^0rt^{-0} = r = b_1a_2^{-1}$ by inspection. The relator t^1rt^{-1} is simply r conjugated by t, so $t^1rt^{-1} = tb_1a_2^{-1}t^{-1} = (tb_1t^{-1})(ta_2^{-1}t^{-1}) = (a_1b_2)(a_1a_3^{-1}) = a_1b_2a_1a_3^{-1}$, from the conjugacy tables. Continuing in this way, we obtain all of the relators for H':

$p_1 = c_2$

$p_2 = c_3$

$q_1 = b_1a_2^{-1}$

$q_2 = a_1b_2a_1a_3^{-1}$

$q_3 = a_2b_3a_2a_4^{-1}$

$\vdots$

$q_i = a_{i-1}b_ia_{i-1}a_{i+1}^{-1}$

$\vdots$

$q_{n-2} = a_{n-3}b_{n-2}a_{n-3}a_{n-1}^{-1}$

$q_{n-1} = a_{n-2}b_{n-1}a_{n-2}c_3c_2^{-1}$

$q_n = a_{n-1}c_3c_1c_2^{-1}a_{n-1}c_3a_1^{-1}c_2^{-1}$

$r_1 = c_1a_1c_1b_2^{-1}$

$r_2 = b_1a_2b_1b_3^{-1}$

$r_3 = b_2a_3b_2b_4^{-1}$

$\vdots$

$r_i = b_{i-1}a_ib_{i-1}b_{i+1}^{-1}$

$\vdots$

$r_{n-2} = b_{n-3}a_{n-2}b_{n-3}b_{n-1}^{-1}$

$r_{n-1} = b_{n-2}a_{n-1}b_{n-2}c_2c_1^{-1}c_3^{-1}$

$r_n = b_{n-1}c_2c_3^{-1}b_{n-1}c_2b_1^{-1}c_3^{-1}$

The relator r_i is obtained from q_i using the second conjugacy table.

We have now found a presentation for H' which is as follows:
$H' = < a_1, \ldots, a_{n-1}, b_1, \ldots, b_{n-1}, c_1, c_2, c_3 \mid p_1, p_2, q_1, \ldots, q_n, r_1, \ldots, r_n >$

We can simplify the presentation in the following way:

$c_2 = c_3 = 1$.
$b_1 = a_2$. Remove b_1: (removes q_1)

$$H' = < a_1,\ \ldots,\ a_{n-1},\ b_2,\ \ldots,\ b_{n-1},\ c_1,\ \mid\ a_{i-1}b_ia_{i-1}a_{i+1}^{-1}\ (2 \leq i \leq n-2),\ a_{n-2}b_{n-1}a_{n-2},\ a_{n-1}c_1a_{n-1}a_1^{-1},\ c_1a_1c_1b_2^{-1},\ a_2^3b_3^{-1},\ b_{i-1}a_ib_{i-1}b_{i+1}^{-1}\ (3 \leq i \leq n-2),\ b_{n-2}a_{n-1}b_{n-2}c_1^{-1},\ b_{n-1}^2a_2^{-1} >$$

$a_2 = b_{n-1}^2$. Remove a_2: (removes r_n)

$$H' =< a_1, a_3, a_4, \ldots, a_{n-1}, b_2, \ldots, b_{n-1}, c_1 \mid \ a_1b_2a_1a_3^{-1},\ b_{n-1}^2b_3b_{n-1}^2a_4^{-1}, \\ a_{i-1}b_ia_{i-1}a_{i+1}^{-1}\ (4 \le i \le n-2),\ a_{n-2}b_{n-1}a_{n-2}, \\ a_{n-1}c_1a_{n-1}a_1^{-1},\ c_1a_1c_1b_2^{-1}, b_{n-1}^6b_3^{-1}, \\ b_{i-1}a_ib_{i-1}b_{i+1}^{-1}\ (3 \le i \le n-2), b_{n-2}a_{n-1}b_{n-2}c_1^{-1} >$$

$b_3 = b_{n-1}^6$. Remove b_3: (removes r_2)

$$H' =< a_1, a_3, a_4, \ldots, a_{n-1}, b_2, b_4, b_5 \ldots, b_{n-1}, c_1, \mid \ a_1b_2a_1a_3^{-1},\ b_{n-1}^{10}a_4^{-1}, \\ a_{i-1}b_ia_{i-1}a_{i+1}^{-1}\ (4 \le i \le n-2), \\ a_{n-2}b_{n-1}a_{n-2},\ a_{n-1}c_1a_{n-1}a_1^{-1}, \\ c_1a_1c_1b_2^{-1},\ b_2a_3b_2b_4^{-1},\ b_{n-1}^6a_4b_{n-1}^6b_5^{-1}, \\ b_{i-1}a_ib_{i-1}b_{i+1}^{-1}\ (5 \le i \le n-2), \\ b_{n-2}a_{n-1}b_{n-2}c_1^{-1} >$$

$a_4 = b_{n-1}^{10}$. Remove a_4: (removes q_3)

$$H' =< a_1, a_3, a_5, a_6, \ldots, a_{n-1}, b_2, b_4, b_5 \ldots, b_{n-1}, c_1, \mid a_1b_2a_1a_3^{-1},\ a_3b_4a_3a_5^{-1}, \\ b_{n-1}^{10}b_5b_{n-1}^{10}a_6^{-1},\ a_{i-1}b_ia_{i-1}a_{i+1}^{-1}\ (6 \le i \le n-2),\ a_{n-2}b_{n-1}a_{n-2},\ a_{n-1}c_1a_{n-1}a_1^{-1}, \\ c_1a_1c_1b_2^{-1},\ b_2a_3b_2b_4^{-1},\ b_{n-1}^{22}b_5^{-1},\ b_{i-1}a_ib_{i-1}b_{i+1}^{-1}\ (5 \le i \le n-2),\ b_{n-2}a_{n-1}b_{n-2}c_1^{-1} >$$

Continue in this way, removing the generators $b_5, a_6, b_7, a_8 \ldots$ and so the relators $r_4, q_5, r_6, q_7, \ldots$ respectively.

Removing b_k, k odd where $b_k = b_{n-1}^d$ for some d yields the following presentation:

$$H' =< a_1, a_3, \ldots, a_k, a_{k+1}, \ldots a_{n-1}, b_2, b_4, \ldots, b_{k+1}, b_{k+2}, \ldots, b_{n-1}, c_1, \mid \\ a_{i-1}b_ia_{i-1}a_{i+1}^{-1}\ (2 \le i \le k-1,\ i \text{ even, and } k+1 \le i \le n-2),\ b_{n-1}^{d'}a_{k+1}^{-1},\ a_{n-2}b_{n-1}a_{n-2}, \\ a_{n-1}c_1a_{n-1}a_1^{-1},\ c_1a_1c_1b_2^{-1},\ b_{i-1}a_ib_{i-1}b_{i+1}^{-1}\ (3 \le i \le k,\ i \text{ odd, and } k+2 \le i \le n-2), \\ b_{n-1}^da_{k+1}b_{n-1}^db_{k+2}^{-1},\ b_{n-2}a_{n-1}b_{n-2}c_1^{-1} >$$

Removing a_k, k even where $a_k = b_{n-1}^d$ for some d yields the following presentation:

$$H' =< a_1, a_3, \ldots, a_{k+1}, a_{k+2}, \ldots a_{n-1}, b_2, b_4, \ldots, b_k, b_{k+1}, \ldots, b_{n-1}, c_1, \mid \\ a_{i-1}b_ia_{i-1}a_{i+1}^{-1}\ (2 \le i \le k,\ i \text{ even, and } k+2 \le i \le n-2),\ b_{n-1}^db_{k+1}b_{n-1}^da_{k+2}^{-1}, \\ a_{n-2}b_{n-1}a_{n-2},\ a_{n-1}c_1a_{n-1}a_1^{-1},\ c_1a_1c_1b_2^{-1},\ b_{i-1}a_ib_{i-1}b_{i+1}^{-1} \\ (3 \le i \le k-1,\ i \text{ odd, and } k+1 \le i \le n-2),\ b_{n-1}^{d'}b_{k+1}^{-1},\ b_{n-2}a_{n-1}b_{n-2}c_1^{-1} >$$

Once we have removed $a_{n-1} = b_{n-1}^{d}$ (and so q_{n-2}) we have the following presentation:

$$\begin{gathered} H' =< a_1, a_3, \ldots, a_{n-4}, a_{n-2}, b_2, b_4, \ldots, b_{n-3}, b_{n-1}, c_1 \mid a_{i-1}b_ia_{i-1}a_{i+1}^{-1} \\ (2 \leq i \leq n-3,\ i \text{ even}),\ a_{n-2}b_{n-1}a_{n-2},\ b_{n-1}^{d}c_1b_{n-1}^{d}a_1^{-1},\ c_1a_1c_1b_2^{-1}, \\ b_{i-1}a_ib_{i-1}b_{i+1}^{-1}\ (3 \leq i \leq n-2,\ i \text{ odd}),\ b_{n-1}^{d'}c_1^{-1} > \end{gathered}$$

We remove generators that equal powers of b_{n-1} and these powers are the following: $2, 6, 10, 22, \ldots$. These come from the relator of the form $b_{n-1}^{d''}b_mb_{n-1}^{d''}a_{m+1}^{-1}$: $10 = 2(2) + 6$, $22 = 2(6) + 10$ etc.

Let x_i denote the relevant power and let $x_0 = 2, x_1 = 6$. Then $x_i = 2x_{i-2} + x_{i-1}$.
Let $x_i = \lambda^i$ so $\lambda^i = 2\lambda^{i-2} + \lambda^{i-1} \implies \lambda^2 = 2 + \lambda \implies \lambda = 2$ or $\lambda = -1$.
$x_i = A(2)^i + B(-1)^i$. $i = 0:\ 2 = A + B,\ i = 1:\ 6 = 2A - B \implies A = \frac{8}{3}, B = \frac{-2}{3}$.
So $x_i = \frac{1}{3}(8(2)^i - 2(-1)^i)$.

Once we have removed a_{n-1} and obtained the above presentation, $d = x_{n-3}, d' = x_{n-2}$ and as n is odd: $d = \frac{1}{3}(8(2)^{n-3} - 2)$, $d' = \frac{1}{3}(8(2)^{n-2} + 2)$.

Next we remove c_1: $c_1 = b_{n-1}^{d'}$ (removes r_{n-1}).

$$\begin{gathered} H' =< a_1, a_3, \ldots, a_{n-4}, a_{n-2}, b_2, b_4, \ldots, b_{n-3}, b_{n-1} \mid a_{i-1}b_ia_{i-1}a_{i+1}^{-1} \\ (2 \leq i \leq n-3,\ i \text{ even}),\ a_{n-2}b_{n-1}a_{n-2},\ b_{n-1}^{d''}a_1^{-1},\ b_{n-1}^{d'}a_1b_{n-1}^{d'}b_2^{-1}, \\ b_{i-1}a_ib_{i-1}b_{i+1}^{-1}\ (3 \leq i \leq n-2,\ i \text{ odd}) > \end{gathered}$$

Now we have $a_1 = b_{n-1}^{d''}$ where $d'' = x_{n-1} = \frac{1}{3}(8(2)^{n-1} - 2)$.
So we remove $a_1, b_2, a_3, b_4, \ldots$ as before and each time remove $q_n, r_1, q_2, r_3, \ldots$.

Once we have removed b_{n-3} we are left only with a_{n-2} and b_{n-1} as generators and we have removed the relators $q_1, q_2, \ldots, q_{n-2}, q_n, r_1, r_2, \ldots, r_{n-4}, r_n$ so we have $q_{n-1}, r_{n-3}, r_{n-2}$ left as relators.

$$H' =< a_{n-2}, b_{n-1} \mid b_{n-1}^{m'}a_{n-2}^{-1}, a_{n-2}b_{n-1}a_{n-2}, b_{n-1}^{m}a_{n-2}b_{n-1}^{m}b_{n-1}^{-1} >$$

$a_{n-2} = b_{n-1}^{m'}$. Remove a_{n-2}:

$$H' =< b_{n-1} \mid b_{n-1}^{m'}b_{n-1}b_{n-1}^{m'}, b_{n-1}^{m}b_{n-1}^{m'}b_{n-1}^{m}b_{n-1}^{-1} >$$

As we had $d' = x_{n-2}$, we have $m' = x_{2(n-2)} = x_{2n-4}$ and $m = x_{2n-5}$.
$m = \frac{1}{3}(8(2)^{2n-5} + 2)$ and $m' = \frac{1}{3}(8(2)^{2n-4} - 2)$.

First relator: $m'+m'+1 = 2\frac{1}{3}(8(2)^{2n-4}-2)+1 = \frac{1}{3}(8(2^{2n-3})-1) = \frac{1}{3}(2^{2n}-1) = \frac{1}{3}(4^n-1)$.
Second relator: $m+m'+m-1 = 2\frac{1}{3}(8(2)^{2n-5}+2)+\frac{1}{3}(8(2)^{2n-4}-2)-1 = \frac{2}{3}(8(2)^{2n-4})-\frac{1}{3} = \frac{1}{3}(2^{2n} - 1) = \frac{1}{3}(4^n - 1)$. Therefore:

$$H' =< b_{n-1} \mid b_{n-1}^s >= \mathbb{Z}_s,$$

where $s = \frac{1}{3}(4^n - 1)$.

So $|H'| = \frac{1}{3}(4^n - 1)$ and $|H : H'| = 2n$. As $|G| = |H|/n$ and $|H| = |H : H'||H'|$ then $|G| = 2|H'| = \frac{2}{3}(4^n - 1)$, which is what we were trying to prove.

8.4.7 $\omega = x_0x_4x_3x_1$

$G =< x_0, \dots, x_{n-1} \mid x_ix_{i+4}x_{i+3}x_{i+1}\ (0 \le i \le n-1) >$, n odd, $n \not\equiv 0 \bmod 3$. $G = \mathbb{Z}_4$, $l = 12$.

Relators:
$x_ix_{i+4}x_{i+3}x_{i+1},\quad 0 \le i \le n - 1$

For all i, $x_ix_{i+4}x_{i+3}x_{i+1}$ and $x_{i+3}x_{i+7}x_{i+6}x_{i+4}$ are relators and can be rearranged to give the following:
$x_{i+4}x_{i+3}x_{i+1}x_i$
$x_{i+4}x_{i+3}x_{i+7}x_{i+6}$

So $x_{i+1}x_i = x_{i+7}x_{i+6}$ for all i, subscripts taken mod n.
So $x_1x_0 = x_{7k}x_{6k}$ for all k.
As n is odd and $n \not\equiv 0 \bmod 3$, $gcd(n, 6) = 1$.

Therefore, $x_1x_0 = x_{i+1}x_i$ for all i.

Let $z = x_{i+1}x_i$
Each relator is of the form $x_{i+1}x_ix_{i+4}x_{i+3} = z^2$

$G =< x_0, x_1, \dots, x_{n-2}, x_{n-1}, z \mid z^{-1}x_{n-1}x_{n-2}, ..., z^{-1}x_2x_1, z^{-1}x_1x_0, z^{-1}x_0x_{n-1}, z^2 >$

Remove $x_0 = x_1^{-1}z$:
$G =< x_1, \dots, x_{n-2}, x_{n-1}, z \mid z^{-1}x_{n-1}x_{n-2}, ..., z^{-1}x_2x_1, z^{-1}x_1^{-1}zx_{n-1}, z^2 >$

Remove $x_1 = x_2^{-1}z$:
$G =< x_2, ..., x_{n-2}, x_{n-1}, z \mid z^{-1}x_{n-1}x_{n-2}, ..., z^{-1}x_3x_2, z^{-2}x_2zx_{n-1}, z^2 >$

Remove $x_2 = x_3^{-1}z$:
$G =< x_3, ..., x_{n-2}, x_{n-1}, z \mid z^{-1}x_{n-1}x_{n-2}, ..., z^{-1}x_4x_3, z^{-2}x_3^{-1}z^2x_{n-1}, z^2 >$
...
Remove $x_{n-3} = x_{n-2}^{-1}z$:
$G =< x_{n-2}, x_{n-1}, z \mid z^{-1}x_{n-1}x_{n-2}, z^{-\frac{n-1}{2}}x_{n-2}^{-1}z^{\frac{n-1}{2}}x_{n-1}, z^2 >$

Remove $x_{n-2} = x_{n-1}^{-1}z$:
$G =< x_{n-1}, z \mid z^{-\frac{n+1}{2}}x_{n-1}z^{\frac{n-1}{2}}x_{n-1}, z^2 >$

$z^2 = 1$ so reduces to:
$G =< x_{n-1}, z \mid zx_{n-1}^2, z^2 >$ When $\frac{n+1}{2}$ even.
$G =< x_{n-1}, z \mid z^{-1}x_{n-1}^2, z^2 >$ When $\frac{n+1}{2}$ odd.

Remove $z = x_{n-1}^{\pm 2}$:
$G =< x_{n-1} \mid x_{n-1}^4 >$

8.4.8 $\omega = x_0x_3x_2^2x_1^2$

$G =< x_0, \ldots, x_{n-1} \mid x_ix_{i+3}x_{i+2}^2x_{i+1}^2 \ (0 \leq i \leq n-1) >$, n odd, $n \not\equiv 0 \bmod 3$. $G = \mathbb{Z}_6$, $l = 12$.

Relators:
$x_ix_{i+3}x_{i+2}x_{i+2}x_{i+1}x_{i+1},\ 0 \leq i \leq n-1$

For all i, $x_ix_{i+3}x_{i+2}x_{i+2}x_{i+1}x_{i+1}$ and $x_{i+1}x_{i+4}x_{i+3}x_{i+3}x_{i+2}x_{i+2}$ are relators and can be rearranged to give the following:
$x_{i+3}x_{i+2}x_{i+2}x_{i+1}x_{i+1}x_i$
$x_{i+3}x_{i+2}x_{i+2}x_{i+1}x_{i+4}x_{i+3}$

So $x_{i+1}x_i = x_{i+4}x_{i+3}$ for all i, subscripts taken mod n

So $x_1x_0 = x_{4k}x_{3k}$ for all k.
As n is odd and $n \not\equiv 0$ mod 3, $gcd(n, 3) = 1$.
Therefore, $x_1x_0 = x_{i+1}x_i$ for all i.

Let $z = x_{i+1}x_i$

Each relator is of the form $x_{i+1}x_ix_{i+3}x_{i+2}x_{i+2}x_{i+1} = z^3$

$G =< x_0, x_1, \ldots, x_{n-2}, x_{n-1}, z \mid z^{-1}x_{n-1}x_{n-2}, \ldots, z^{-1}x_2x_1, z^{-1}x_1x_0, z^{-1}x_0x_{n-1}, z^3 >$

Remove $x_0 = x_1^{-1}z$:
$G =< x_1, \ldots, x_{n-2}, x_{n-1}, z \mid z^{-1}x_{n-1}x_{n-2}, \ldots, z^{-1}x_2x_1, z^{-1}x_1^{-1}zx_{n-1}, z^3 >$

Remove $x_1 = x_2^{-1}z$:
$G =< x_2, \ldots, x_{n-2}, x_{n-1}, z \mid z^{-1}x_{n-1}x_{n-2}, \ldots, z^{-1}x_3x_2, z^{-2}x_2zx_{n-1}, z^3 >$

Remove $x_2 = x_3^{-1}z$:
$G =< x_3, \ldots, x_{n-2}, x_{n-1}, z \mid z^{-1}x_{n-1}x_{n-2}, \ldots, z^{-1}x_4x_3, z^{-2}x_3^{-1}z^2x_{n-1}, z^3 >$
...
Remove $x_{n-3} = x_{n-2}^{-1}z$:
$G =< x_{n-2}, x_{n-1}, z \mid z^{-1}x_{n-1}x_{n-2}, z^{-\frac{n-1}{2}}x_{n-2}^{-1}z^{\frac{n-1}{2}}x_{n-1}, z^3 >$

Remove $x_{n-2} = x_{n-1}^{-1}z$:
$G =< x_{n-1}, z \mid z^{-\frac{n+1}{2}}x_{n-1}z^{\frac{n-1}{2}}x_{n-1}, z^3 >$

$z^3 = 1$ so we get one of the following:

(a) $\frac{n+1}{2} \equiv 0 \bmod 3$, $\frac{n-1}{2} \equiv 2 \bmod 3$
$G =< x_{n-1}, z \mid zx_{n-1}^2, z^3 >$

(b) $\frac{n+1}{2} \equiv 1 \bmod 3$, $\frac{n-1}{2} \equiv 0 \bmod 3$
$G =< x_{n-1}, z \mid zx_{n-1}^2, z^3 >$

(c) $\frac{n+1}{2} \equiv 2 \bmod 3$, $\frac{n-1}{2} \equiv 1 \bmod 3$ - not possible as $n \not\equiv 0 \bmod 3$

Remove $z = x_{n-1}^{-2}$:
$G =< x_{n-1} \mid x_{n-1}^6 >$

8.4.9 $\omega = x_0x_3x_1x_2$

Relators:
$x_ix_{i+3}x_{i+1}x_{i+2},\ 0 \leq i \leq n-1$

For all i, $x_ix_{i+3}x_{i+1}x_{i+2}$ and $x_{i+1}x_{i+4}x_{i+2}x_{i+3}$ are relators and can be rearranged to give the following:

$x_{i+3}x_{i+1}x_{i+2}x_i$

$x_{i+3}x_{i+1}x_{i+4}x_{i+2}$

So $x_{i+2}x_i = x_{i+4}x_{i+2}$ for all i, subscripts taken mod n
So $x_2x_0 = x_{4k}x_{2k}$ for all k.

As n is odd, $gcd(n, 2) = 1$.
Therefore, $x_2x_0 = x_{i+2}x_i$ for all i.
Let $z = x_{i+2}x_i$

Each relator is of the form $x_{i+3}x_{i+1}x_{i+2}x_i = z^2$.

$$G =< x_0, x_1, \ldots, x_{n-2}, x_{n-1}, z \mid \ z^{-1}x_{n-1}x_{n-3}, \ldots, z^{-1}x_3x_1, z^{-1}x_2x_0,\ z^{-1}x_1x_{n-1}, z^{-1}x_0x_{n-2},\ z^2 >$$

Remove $x_0 = x_2^{-1}z$:
$G =< x_1, \ldots, x_{n-2}, x_{n-1}, z \mid z^{-1}x_{n-1}x_{n-3}, \ldots, z^{-1}x_3x_1, z^{-1}x_1x_{n-1}, z^{-1}x_2^{-1}zx_{n-2}, z^2 >$

Remove $x_1 = x_3^{-1}z$:
$G =< x_2, \ldots, x_{n-2}, x_{n-1}, z \mid z^{-1}x_{n-1}x_{n-3}, \ldots, z^{-1}x_4x_2, z^{-1}x_3^{-1}zx_{n-1}, z^{-1}x_2^{-1}zx_{n-2}, z^2 >$

Remove $x_2 = x_4^{-1}z$:
$G =< x_3, \ldots, x_{n-2}, x_{n-1}, z \mid z^{-1}x_{n-1}x_{n-3}, \ldots, z^{-1}x_5x_3, z^{-1}x_3^{-1}zx_{n-1}, z^{-2}x_4zx_{n-2}, z^2 >$

Remove $x_3 = x_5^{-1}z$:
$G =< x_4, \ldots, x_{n-2}, x_{n-1}, z \mid z^{-1}x_{n-1}x_{n-3}, \ldots, z^{-1}x_6x_4, z^{-2}x_5zx_{n-1}, z^{-2}x_4zx_{n-2}, z^2 >$
$\ldots$

Removing relator $x_i = x_{i+2}^{-1}z$ leaves us with relators $z^{-1}x_{j+2}x_j$, z^2 and
$z^{-A_1}x_{A_2}^a z^{A_3}x_{n-1},\ \ z^{-B_1}x_{B_2}^b z^{B_3}x_{n-2}$ where:

$A_1 = \frac{1}{4}(i + 1 - (i + 1) \bmod 4) + 1$

$B_1 = \frac{1}{4}(i + 2 - (i + 2) \bmod 4) + 1$

$$A_2 = \begin{cases} i + 1, & \text{if } i \text{ even} \\ i + 2, & \text{if } i \text{ odd} \end{cases}$$

$$B_2 = \begin{cases} i + 2, & \text{if } i \text{ even} \\ i + 1, & \text{if } i \text{ odd} \end{cases}$$

$A_3 = \frac{1}{4}(i - 1 - (i - 1) \bmod 4) + 1$

$B_3 = \frac{1}{4}(i - i \bmod 4) + 1$

$$a = \begin{cases} 1, & \text{if } i \equiv 0, 3 \bmod 4 \\ -1, & \text{if } i \equiv 1, 2 \bmod 4 \end{cases}$$

$$b = \begin{cases} 1, & \text{if } i \equiv 2, 3 \bmod 4 \\ -1, & \text{if } i \equiv 0, 1 \bmod 4 \end{cases}$$

As n is odd, $n-4$ is odd.

Remove $x_{n-4} = x_{n-2}^{-1}z$:

Assume $n-4 \equiv 1 \bmod 4$:
$G =< x_{n-3}, x_{n-2}, x_{n-1}, z \mid z^{-1}x_{n-1}x_{n-3}, z^{-\frac{n-1}{4}}x_{n-2}^{-1}z^{\frac{n-1}{4}}x_{n-1}, z^{-\frac{n-1}{4}}x_{n-3}^{-1}z^{\frac{n-1}{4}}x_{n-2}, z^2 >$
Remove $x_{n-3} = x_{n-1}^{-1}z$:
$G =< x_{n-2}, x_{n-1}, z \mid z^{-\frac{n-1}{4}}x_{n-2}^{-1}z^{\frac{n-1}{4}}x_{n-1}, z^{-\frac{n+3}{4}}x_{n-1}z^{\frac{n-1}{4}}x_{n-2}, z^2 >$
If $\frac{n-1}{4}$ even:
$G =< x_{n-2}, x_{n-1}, z \mid x_{n-2}^{-1}x_{n-1}, zx_{n-1}x_{n-2}, z^2 > = < x_{n-1}, z \mid zx_{n-1}^2, z^2 >$
If $\frac{n-1}{4}$ odd:
$G =< x_{n-2}, x_{n-1}, z \mid zx_{n-2}^{-1}zx_{n-1}, x_{n-1}zx_{n-2}, z^2 > = < x_{n-1}, z \mid zx_{n-1}^2, z^2 >$

Assume $n-4 \equiv 3 \bmod 4$:
$G =< x_{n-3}, x_{n-2}, x_{n-1}, z \mid z^{-1}x_{n-1}x_{n-3}, z^{-\frac{n+1}{4}}x_{n-2}z^{\frac{n-3}{4}}x_{n-1}, z^{-\frac{n+1}{4}}x_{n-3}z^{\frac{n-3}{4}}x_{n-2}, z^2 >$
Remove $x_{n-3} = x_{n-1}^{-1}z$:
$G =< x_{n-2}, x_{n-1}, z \mid z^{-\frac{n+1}{4}}x_{n-2}z^{\frac{n-3}{4}}x_{n-1}, z^{-\frac{n+1}{4}}x_{n-1}^{-1}z^{\frac{n+1}{4}}x_{n-2}, z^2 >$
If $\frac{n+1}{4}$ even:
$G =< x_{n-2}, x_{n-1}, z \mid x_{n-2}zx_{n-1}, x_{n-1}^{-1}x_{n-2}, z^2 > = < x_{n-1}, z \mid zx_{n-1}^2, z^2 >$
If $\frac{n+1}{4}$ odd:
$G =< x_{n-2}, x_{n-1}, z \mid zx_{n-2}x_{n-1}, zx_{n-1}^{-1}zx_{n-2}, z^2 > = < x_{n-1}, z \mid zx_{n-1}^2, z^2 >$

So $G \ = < x_{n-1}, z \mid zx_{n-1}^2, z^2 > = < x_{n-1} \mid x_{n-1}^4 >$, as required.

8.4.10 $\omega = x_0x_4x_3x_2x_1$

$G =< x_0, \ldots, x_{n-1} \mid x_ix_{i+4}x_{i+3}x_{i+2}x_{i+1}\ (0 \leq i \leq n-1) >, \quad n \not\equiv 0 \bmod 5. \quad G = \mathbb{Z}_5$, $l = 13$.

See Subsection 8.4.1.

8.4.11 $\omega = x_0x_5x_4x_1$

$G =< x_0, \ldots, x_{n-1} \mid x_ix_{i+5}x_{i+4}x_{i+1}\ (0 \leq i \leq n-1) >, \quad n$ odd. $\quad G = \mathbb{Z}_4$, $l = 14$.

Relators:
$x_ix_{i+5}x_{i+4}x_{i+1},\ 0 \leq i \leq n-1$

For all i, $x_ix_{i+5}x_{i+4}x_{i+1}$ and $x_{i+4}x_{i+9}x_{i+8}x_{i+5}$ are relators and can be rearranged to give the following:

$x_{i+5}x_{i+4}x_{i+1}x_i$

$x_{i+5}x_{i+4}x_{i+9}x_{i+8}$

So $x_{i+1}x_i = x_{i+9}x_{i+8}$ for all i, subscripts taken mod n.
So $x_1x_0 = x_{9k}x_{8k}$ for all k.

As n is odd, $gcd(n, 8) = 1$.

Therefore, $x_1x_0 = x_{i+1}x_i$ for all i.

Let $z = x_{i+1}x_i$
Each relator is of the form $x_{i+1}x_ix_{i+5}x_{i+4} = z^2$

$G =< x_0, x_1, \ldots, x_{n-2}, x_{n-1}, z \mid z^{-1}x_{n-1}x_{n-2}, ..., z^{-1}x_2x_1, z^{-1}x_1x_0, z^{-1}x_0x_{n-1}, z^2 >$

The remainder of the proof is the same as in Subsection 8.4.7.

8.4.12 $\omega = x_0x_5x_3x_2$

$G =< x_0, \ldots, x_{n-1} \mid x_ix_{i+5}x_{i+3}x_{i+2}\ (0 \le i \le n-1) >$, n odd, $n \not\equiv 0 \bmod 3$. $G = \mathbb{Z}_4$, $l = 14$.

Relators:
$x_ix_{i+5}x_{i+3}x_{i+2},\ 0 \le i \le n-1$

For all i, $x_ix_{i+5}x_{i+3}x_{i+2}$ and $x_{i+3}x_{i+8}x_{i+6}x_{i+5}$ are relators and can be rearranged to give the following:

$x_{i+5}x_{i+3}x_{i+2}x_i$

$x_{i+5}x_{i+3}x_{i+8}x_{i+6}$

So $x_{i+2}x_i = x_{i+8}x_{i+6}$ for all i, subscripts taken mod n
So $x_2x_0 = x_{8k}x_{6k}$ for all k.

As n is odd and $n \not\equiv 0 \bmod 3$, $gcd(n, 6) = 1$.
Therefore, $x_2x_0 = x_{i+2}x_i$ for all i.
Let $z = x_{i+2}x_i$

Each relator is of the form $x_{i+2}x_ix_{i+5}x_{i+3} = z^2$.

$$G =< x_0, x_1, \ldots, x_{n-2}, x_{n-1}, z \mid \; z^{-1}x_{n-1}x_{n-3}, \ldots, z^{-1}x_3x_1, z^{-1}x_2x_0, \; z^{-1}x_1x_{n-1}, \; z^{-1}x_0x_{n-2}, \; z^2 >$$

The remainder of the proof is the same as in Subsection 8.4.9.

8.4.13 $\omega = x_0x_4x_3x_2^2x_1$

$G =< x_0, \ldots, x_{n-1} \mid x_ix_{i+4}x_{i+3}x_{i+2}^2x_{i+1} \; (0 \leq i \leq n-1) >$, $\quad n$ odd, $n \not\equiv 0 \bmod 3$.
$G = \mathbb{Z}_6$, $l = 14$.

Relators:
$x_ix_{i+4}x_{i+3}x_{i+2}x_{i+2}x_{i+1}, \; 0 \leq i \leq n-1$

For all i, $x_ix_{i+4}x_{i+3}x_{i+2}x_{i+2}x_{i+1}$ and $x_{i+2}x_{i+6}x_{i+5}x_{i+4}x_{i+4}x_{i+3}$ are relators and can be rearranged to give the following:
$x_{i+4}x_{i+3}x_{i+2}x_{i+2}x_{i+1}x_i$
$x_{i+4}x_{i+3}x_{i+2}x_{i+6}x_{i+5}x_{i+4}$

So $x_{i+2}x_{i+1}x_i = x_{i+6}x_{i+5}x_{i+4}$ for all i, subscripts taken mod n

So $x_2x_1x_0 = x_{6k}x_{5k}x_{4k}$ for all k.
As n is odd, $gcd(n, 4) = 1$.
Therefore, $x_2x_1x_0 = x_{i+2}x_{i+1}x_i$ for all i.

Let $z = x_{i+2}x_{i+1}x_i$

Each relator is of the form $x_{i+2}x_{i+1}x_ix_{i+4}x_{i+3}x_{i+2} = z^2$

$$G \; = < x_0, \; x_1, \; \ldots, \; x_{n-2}, \; x_{n-1}, \; z \mid \; z^{-1}x_{n-1}x_{n-2}x_{n-3}, \; \ldots, \; z^{-1}x_3x_2x_1, \; z^{-1}x_2x_1x_0, \; z^{-1}x_1x_0x_{n-1}, \; z^{-1}x_0x_{n-1}x_{n-2}, \; z^2 >$$

Remove $x_0 = x_1^{-1}x_2^{-1}z$:

$$G \; = < x_1, \; \ldots, \; x_{n-2}, \; x_{n-1}, \; z \mid \; z^{-1}x_{n-1}x_{n-2}x_{n-3}, \; \ldots, \; z^{-1}x_3x_2x_1, \; z^{-1}x_2^{-1}zx_{n-1}, \; z^{-1}x_1^{-1}x_2^{-1}zx_{n-1}x_{n-2}, \; z^2 >$$

Remove $x_1 = x_2^{-1}x_3^{-1}z$:

$$G = < x_2,\ \ldots,\ x_{n-2},\ x_{n-1},\ z \mid \ z^{-1}x_{n-1}x_{n-2}x_{n-3},\ \ldots,\ z^{-1}x_4x_3x_2,\ z^{-1}x_2^{-1}zx_{n-1},\ z^{-2}x_3zx_{n-1}x_{n-2},\ z^2\ >$$

Remove $x_2 = x_3^{-1}x_4^{-1}z$:

$$G = < x_3,\ \ldots,\ x_{n-2},\ x_{n-1},\ z \mid \ z^{-1}x_{n-1}x_{n-2}x_{n-3},\ \ldots,\ z^{-1}x_5x_4x_3,\ z^{-2}x_4x_3zx_{n-1},\ z^{-2}x_3zx_{n-1}x_{n-2},\ z^2 >$$

Remove $x_3 = x_4^{-1}x_5^{-1}z$:

$$G = < x_4,\ \ldots,\ x_{n-2},\ x_{n-1},\ z \mid \ z^{-1}x_{n-1}x_{n-2}x_{n-3},\ \ldots,\ z^{-1}x_6x_5x_4,\ z^{-2}x_5^{-1}z^2x_{n-1},\ z^{-2}x_4^{-1}x_5^{-1}z^2x_{n-1}x_{n-2},\ z^2 >$$

Remove $x_4 = x_5^{-1}x_6^{-1}z$:

$$G = < x_5,\ \ldots,\ x_{n-2},\ x_{n-1},\ z \mid \ z^{-1}x_{n-1}x_{n-2}x_{n-3},\ \ldots,\ z^{-1}x_7x_6x_5,\ z^{-2}x_5^{-1}z^2x_{n-1},\ z^{-3}x_6z^2x_{n-1}x_{n-2},\ z^2 >$$

...

Remove $x_{n-4} = x_{n-3}^{-1}x_{n-2}^{-1}z$

$$G = < x_{n-3},\ x_{n-2},\ x_{n-1},\ z \mid z^{-1}x_{n-1}x_{n-2}x_{n-3},\ z^{-a_1}A_1z^bx_{n-1},\ z^{-a_2}A_2z^bx_{n-1}x_{n-2},\ z^2 >$$

We have the following cases:

(a) $n-4 = 0 \bmod 3$
$A_1 = x_{n-2}^{-1}$
$A_2 = x_{n-3}^{-1}x_{n-2}^{-1}$
$a_1 = \frac{n-4}{3} + 1$
$a_2 = \frac{n-4}{3} + 1$
$b = \frac{n-4}{3} + 1$

(b) $n-4 = 1 \bmod 3$
$A_1 = x_{n-3}^{-1}$
$A_2 = x_{n-2}$
$a_1 = \frac{n-5}{3} + 1$
$a_2 = \frac{n-5}{3} + 2$
$b = \frac{n-5}{3} + 1$

(c) $n-4=2 \bmod 3$ - cannot happen as $n \not\equiv 0 \bmod 3$

(a) $n-4=0 \bmod 3$

If $\frac{n-4}{3}+1$ odd $\implies$ $\frac{n-4}{3}$ even $\implies$ $n-4$ even $\implies$ n even - contradiction so $\frac{n-4}{3}+1$ even, $\frac{n-4}{3}+1=0 \bmod 2$.

$A_1=x_{n-2}^{-1},\ A_2=x_{n-3}^{-1}x_{n-2}^{-1},\ a_1=a_2=b=0.$

$G=<x_{n-3},x_{n-2},x_{n-1},z \mid z^{-1}x_{n-1}x_{n-2}x_{n-3},x_{n-2}^{-1}x_{n-1},x_{n-3}^{-1}x_{n-2}^{-1}x_{n-1}x_{n-2},z^2>$

Remove $x_{n-1}=x_{n-2}$:

$G=<x_{n-3},x_{n-2},z \mid z^{-1}x_{n-2}^2x_{n-3},x_{n-3}^{-1}x_{n-2},z^2>$

Remove $x_{n-3}=x_{n-2}$:

$G=<x_{n-2},z \mid z^{-1}x_{n-2}^3,z^2>$

Remove $z=x_{n-2}^3$:

$G=<x_{n-2} \mid x_{n-2}^6>$

(b) $n-4=1 \bmod 3$

If $\frac{n-5}{3}+1$ even $\implies$ $\frac{n-5}{3}$ odd $\implies$ $n-5$ odd $\implies$ n even - contradiction so $\frac{n-5}{3}+1$ odd, $\frac{n-5}{3}+1=1 \bmod 2$.

$A_1=x_{n-3}^{-1},\ A_2=x_{n-2},\ a_1=1,\ a_2=0,\ b=1.$

$G=<x_{n-3},x_{n-2},x_{n-1},z \mid z^{-1}x_{n-1}x_{n-2}x_{n-3},zx_{n-3}^{-1}zx_{n-1},x_{n-2}zx_{n-1}x_{n-2},z^2>$

Remove $x_{n-3}=x_{n-2}^{-1}x_{n-1}^{-1}z$

$G=<x_{n-2},x_{n-1},z \mid x_{n-1}^2x_{n-2}z,zx_{n-1}x_{n-2}^2,z^2>$

Remove $x_{n-2}=x_{n-1}^{-2}z^{-1}$

$G=<x_{n-1},z \mid x_{n-1}^{-3}z^{-1},z^2>$

Remove $z=x_{n-1}^{-3}$

$G=<x_{n-1} \mid x_{n-1}^{-6}>$

8.4.14 $\omega=x_0x_3x_2x_1x_2x_1$

$G=<x_0,\dots,x_{n-1} \mid x_ix_{i+3}x_{i+2}x_{i+1}x_{i+2}x_{i+1}\ (0\le i\le n-1)>$, $\quad n$ odd, $n \not\equiv 0 \bmod 3$.
$G=\mathbb{Z}_6$, $l=14$.

Relators are:

$x_0x_3x_2x_1x_2x_1$

$x_1x_4x_3x_2x_3x_2$

$x_2x_5x_4x_3x_4x_3$

$x_3x_6x_5x_4x_5x_4$

$\ldots$

$x_{n-4}x_{n-1}x_{n-2}x_{n-3}x_{n-2}x_{n-3}$

$x_{n-3}x_0x_{n-1}x_{n-2}x_{n-1}x_{n-2}$

$x_{n-2}x_1x_0x_{n-1}x_0x_{n-1}$

$x_{n-1}x_2x_1x_0x_1x_0$

Let $y_i = x_i^{-1}$

$$\begin{aligned} x_0 &= y_1y_2y_1y_2y_3 \\ &= x_4x_3x_2x_3x_2y_2x_4x_3x_2x_3x_2y_2y_3 \\ &= x_4x_3x_2x_3x_4x_3x_2 \\ &= x_4x_3y_3y_4y_3y_4y_5x_3x_4x_3y_3y_4y_3y_4y_5 \\ &= y_3y_4y_5y_4y_5 \\ &= x_6x_5x_4x_5x_4y_4y_5y_4y_5 \\ &= x_6 \end{aligned}$$

So $x_0 = x_6$
By the symmetry of the relators we get $x_0 = x_6 = x_{12}...$, so $x_0 = x_{6i}$ where subscripts are taken mod n.
As n is odd and $n \not\equiv 0 \bmod 3$, $gcd(n,6) = 1$ and so x_{6i} will run though $x_0, ..., x_{n-1}$.
So $x_0 = x_1 = ... = x_{n-1}$ and, by the relators, $x_0^6 = 1$.

8.4.15 $\omega = x_0x_2x_1x_3x_2x_1$

$G =< x_0, \ldots, x_{n-1} \mid x_ix_{i+2}x_{i+1}x_{i+3}x_{i+2}x_{i+1}\ (0 \leq i \leq n-1) >$, n odd, $n \not\equiv 0 \bmod 3$
$G = \mathbb{Z}_6$, $l = 14$.

Relators are:

$x_0x_2x_1x_3x_2x_1$

$x_1x_3x_2x_4x_3x_2$

$x_2x_4x_3x_5x_4x_3$

$x_3x_5x_4x_6x_5x_4$

$\ldots$

$x_{n-4}x_{n-2}x_{n-3}x_{n-1}x_{n-2}x_{n-3}$

$x_{n-3}x_{n-1}x_{n-2}x_0x_{n-1}x_{n-2}$

$x_{n-2}x_0x_{n-1}x_1x_0x_{n-1}$

$x_{n-1}x_1x_0x_2x_1x_0$

Let $y_i = x_i^{-1}$

$$\begin{aligned} x_0 &= y_1y_2y_3y_1y_2 \\ &= x_3x_2x_4x_3x_2y_2y_3x_3x_2x_4x_3x_2y_2 \\ &= x_3x_2x_4x_3x_2x_4x_3 \\ &= x_3y_3y_4y_5y_3y_4x_4x_3y_3y_4y_5y_3y_4x_4x_3 \\ &= y_4y_5y_3y_4y_5 \\ &= y_4y_5x_5x_4x_6x_5x_4y_4y_5 \\ &= x_6 \end{aligned}$$

So $x_0 = x_6$

By the symmetry of the relators we get $x_0 = x_6 = x_{12}...$, so $x_0 = x_{6i}$ where subscripts are taken mod n.

As n is odd and $n \not\equiv 0$ mod 3, $gcd(n, 6i) = 1$ and so x_{6i} will run though $x_0, ..., x_{n-1}$.

So $x_0 = x_1 = ... = x_{n-1}$ and, by the relators, $x_0^6 = 1$.

8.4.16 $\omega = x_0x_1x_2^{-1}x_1x_2x_1^{-1}$

$< x_0, \dots, x_{n-1} \mid x_ix_{i+1}x_{i+2}^{-1}x_{i+1}x_{i+2}x_{i+1}^{-1}\ (0 \le i \le n-1) >$, n odd. $G = D_{2^{n+1}-2}$

Relators are:

$x_ix_{i+1}x_{i+2}^{-1}x_{i+1}x_{i+2}x_{i+1}^{-1},\ 0 \le i \le n-1$

For all i, $x_ix_{i+1}x_{i+2}^{-1}x_{i+1}x_{i+2}x_{i+1}^{-1}$ and $x_{i+1}x_{i+2}x_{i+3}^{-1}x_{i+2}x_{i+3}x_{i+2}^{-1}$ are relators and can be rearranged to give the following:

$x_{i+2}^{-1}x_{i+1}x_{i+2}x_{i+1}^{-1}x_ix_{i+1}$

$x_{i+2}^{-1}x_{i+1}x_{i+2}x_{i+3}^{-1}x_{i+2}x_{i+3}$

So $x_{i+1}^{-1}x_ix_{i+1} = x_{i+3}^{-1}x_{i+2}x_{i+3}$ for all i, subscripts taken mod n

So $x_1^{-1}x_0x_1 = x_{3k}^{-1}x_{2k}x_{3k}$ for all k.

As n is odd, $gcd(n, 2) = 1$

Therefore, $x_1^{-1}x_0x_1 = x_{i+1}^{-1}x_ix_{i+1}$ for all i.

Let $z = x_{i+1}^{-1}x_ix_{i+1}$

Each relator is of the form $x_{i+2}^{-1}x_{i+1}x_{i+2}x_{i+1}^{-1}x_ix_{i+1} = z^2$

$G =< x_0, x_1, \ldots, x_{n-2}, x_{n-1}, z \mid zx_{n-1}^{-1}x_{n-2}x_{n-1}, \ldots, zx_2^{-1}x_1x_2, zx_1^{-1}x_0x_1, x_{n-1}x_0zx_0^{-1}, z^2 >$

Remove $x_0 = x_1zx_1^{-1}$:
$G =< x_1, \ldots, x_{n-2}, x_{n-1}, z \mid zx_{n-1}^{-1}x_{n-2}x_{n-1}, \ldots, zx_2^{-1}x_1x_2, x_{n-1}x_1zx_1^{-1}(zx_1zx_1^{-1}), z^2 >$

Remove $x_1 = x_2zx_2^{-1}$:
$G =< x_2, \ldots, x_{n-2}, x_{n-1}, z \mid zx_{n-1}^{-1}x_{n-2}x_{n-1}, \ldots, zx_3^{-1}x_2x_3, x_{n-1}x_2zx_2^{-1}(zx_2zx_2^{-1})^3, z^2 >$

Remove $x_2 = x_3zx_3^{-1}$:
$G =< x_3, \ldots, x_{n-2}, x_{n-1}, z \mid zx_{n-1}^{-1}x_{n-2}x_{n-1}, \ldots, zx_4^{-1}x_3x_4, x_{n-1}x_3zx_3^{-1}(zx_3zx_3^{-1})^7, z^2 >$
...
Remove $x_{n-3} = x_{n-2}zx_{n-2}^{-1}$:
$G =< x_{n-2}, x_{n-1}, z \mid zx_{n-1}^{-1}x_{n-2}x_{n-1}, x_{n-1}x_{n-2}zx_{n-2}^{-1}(zx_{n-2}zx_{n-2}^{-1})^{2^{n-2}-1}, z^2 >$

Remove $x_{n-2} = x_{n-1}zx_{n-1}^{-1}$

$G =< x_{n-1}, z \mid x_{n-1}x_{n-1}zx_{n-1}^{-1}(zx_{n-1}zx_{n-1}^{-1})^{2^{n-1}-1}, z^2 >$
$G =< x_{n-1}, z \mid x_{n-1}x_{n-1}z(x_{n-1}^{-1}zx_{n-1}z)^{2^{n-1}-1}x_{n-1}^{-1}, z^2 >$
$G =< x_{n-1}, z \mid x_{n-1}z(x_{n-1}^{-1}zx_{n-1}z)^{2^{n-1}-1}, z^2 >$

$$\begin{aligned}
\text{So } x_{n-1}^{-1} &= z(x_{n-1}^{-1}zx_{n-1}z)^{2^{n-1}-1} \\
&= z(x_{n-1}^{-1}zx_{n-1}z)(x_{n-1}^{-1}zx_{n-1}z)^{2^{n-2}-1}(x_{n-1}^{-1}zx_{n-1}z)^{2^{n-2}-1} \\
&= zx_{n-1}^{-1}(zx_{n-1}zx_{n-1}^{-1})^{2^{n-2}-1}zx_{n-1}z(x_{n-1}^{-1}zx_{n-1}z)^{2^{n-2}-1} \\
&= zx_{n-1}^{-1}(zx_{n-1}zx_{n-1}^{-1})^{2^{n-2}-1}z(x_{n-1}zx_{n-1}^{-1}z)^{2^{n-2}-1}x_{n-1}z \\
&= wzw^{-1},
\end{aligned}$$
where $w = zx_{n-1}^{-1}(zx_{n-1}zx_{n-1}^{-1})^{2^{n-2}-1}$

So x_{n-1}^{-1} is a conjugate of z and so $x_{n-1}^2 = 1$

$G =< x_{n-1}, z \mid x_{n-1}^2, x_{n-1}z(x_{n-1}zx_{n-1}z)^{2^{n-1}-1}, z^2 >$
$G =< x_{n-1}, z \mid x_{n-1}^2, (x_{n-1}z)^{2^n-1}, z^2 >= D_{2^{n+1}-2}$

8.4.17 $\omega = x_0x_2^{-1}x_3x_2x_0^{-1}x_1$

$< x_0, \ldots, x_{n-1} \mid x_ix_{i+2}^{-1}x_{i+3}x_{i+2}x_i^{-1}x_{i+1}\ (0 \le i \le n-1) >$, n odd. $G = D_{2^{n+1}-2}$

Relators are:
$x_ix_{i+2}^{-1}x_{i+3}x_{i+2}x_i^{-1}x_{i+1},\ 0 \le i \le n-1$

For all i, $x_i x_{i+2}^{-1} x_{i+3} x_{i+2} x_i^{-1} x_{i+1}$ and $x_{i+2} x_{i+4}^{-1} x_{i+5} x_{i+4} x_{i+2}^{-1} x_{i+3}$ are relators and can be rearranged to give the following:
$x_{i+2}^{-1} x_{i+3} x_{i+2} x_i^{-1} x_{i+1} x_i$
$x_{i+2}^{-1} x_{i+3} x_{i+2} x_{i+4}^{-1} x_{i+5} x_{i+4}$

So $x_i^{-1} x_{i+1} x_i = x_{i+4}^{-1} x_{i+5} x_{i+4}$ for all i, subscripts taken mod n

So $x_0^{-1} x_1 x_0 = x_{4k}^{-1} x_{5k} x_{4k}$ for all k.
As n is odd, $gcd(n, 4) = 1$
Therefore, $x_0^{-1} x_1 x_0 = x_i^{-1} x_{i+1} x_i$ for all i.

Let $z = x_i^{-1} x_{i+1} x_i$

Each relator is of the form $x_{i+2}^{-1} x_{i+3} x_{i+2} x_i^{-1} x_{i+1} x_i = z^2$

$G =< x_0, x_1, \ldots, x_{n-2}, x_{n-1}, z \mid z x_{n-2}^{-1} x_{n-1} x_{n-2}, \ldots, z x_1^{-1} x_2 x_1, z x_0^{-1} x_1 x_0, x_0 x_{n-1} z x_{n-1}^{-1}, z^2 >$

Remove $x_{n-1} = x_{n-2} z x_{n-2}^{-1}$:
$G =< x_0, x_1, \ldots, x_{n-2}, z \mid z x_{n-3}^{-1} x_{n-2} x_{n-3}, \ldots, z x_0^{-1} x_1 x_0, x_0 x_{n-2} z x_{n-2}^{-1} (z x_{n-2} z x_{n-2}^{-1}), z^2 >$

Remove $x_{n-2} = x_{n-3} z x_{n-3}^{-1}$:
$G =< x_0, x_1, \ldots, x_{n-3}, z \mid z x_{n-4}^{-1} x_{n-3} x_{n-4}, \ldots, z x_0^{-1} x_1 x_0, x_0 x_{n-3} z x_{n-3}^{-1} (z x_{n-3} z x_{n-3}^{-1})^3, z^2 >$

Remove $x_{n-3} = x_{n-4} z x_{n-4}^{-1}$:
$G =< x_0, x_1, \ldots, x_{n-4}, z \mid z x_{n-5}^{-1} x_{n-4} x_{n-5}, \ldots, z x_0^{-1} x_1 x_0, x_0 x_{n-4} z x_{n-4}^{-1} (z x_{n-4} z x_{n-4}^{-1})^7, z^2 >$
$\ldots$
Remove $x_2 = x_1 z x_1^{-1}$:
$G =< x_0, x_1, z \mid z x_0^{-1} x_1 x_0, x_0 x_1 z x_1^{-1} (z x_1 z x_1^{-1})^{2^{n-2}-1}, z^2 >$

Remove $x_1 = x_0 z x_0^{-1}$:

$G =< x_0, z \mid x_0 x_0 z x_0^{-1} (z x_0 z x_0^{-1})^{2^{n-1}-1}, z^2 >$
$G =< x_0, z \mid x_0 x_0 z (z x_0 z x_0^{-1})^{2^{n-1}-1} x_0^{-1}, z^2 >$
$G =< x_0, z \mid x_0 z (z x_0 z x_0^{-1})^{2^{n-1}-1}, z^2 >$

The remainder of the proof is the same as in Subsection 8.4.16.

8.4.18 The remaining groups in each family

In this section we have proved each group in the family is finite for the specified n. What we have not yet proved, apart from the case where $\omega = x_0 \ldots x_{k-1}$ which is dealt with in Subsection 8.4.1, is that the groups in the family are infinite for all other values of n.

Apart from those of the form $\omega = x_0 \ldots x_{k-1}$, the other families are either finite when n is odd or when n is odd and $n \not\equiv 0 \bmod 3$. For each of these families we look at the case when $n = 2$ and for those which also require $n \not\equiv 0 \bmod 3$, we also look at the case when $n = 3$.

Finite when n is odd:
$G_2(x_0x_2x_1^2) \; = < x_0, x_1 \mid x_0^2x_1^2 >$
$G_2(x_0x_1x_2x_1) \; = < x_0, x_1 \mid (x_0x_1)^2 >$
$G_2(x_0x_2^{-1}x_1^{-1}x_2^{-1}) \; = < x_0, x_1 \mid x_0x_1 >$
$G_2(x_0x_3x_1x_2) \; = < x_0, x_1 \mid x_0^2x_1^2 >$
$G_2(x_0x_5x_4x_1) \; = < x_0, x_1 \mid (x_0x_1)^2 >$
$G_2(x_0x_1x_2^{-1}x_1x_2x_1^{-1}) \; = < x_0, x_1 \mid x_0x_1x_0^{-1}x_1x_0x_1^{-1} >$
$G_2(x_0x_2^{-1}x_3x_2x_0^{-1}x_1) \; = < x_0, x_1 \mid x_1^2 >$

Finite when n is odd and $n \not\equiv 0 \bmod 3$:
$G_2(x_0x_4x_3x_1) \; = < x_0, x_1 \mid x_0^2x_1^2 >$
$G_3(x_0x_4x_3x_1) \; = < x_0, x_1, x_2 \mid (x_0x_1)^2, (x_1x_2)^2, (x_2x_0)^2 >$
$G_2(x_0x_3x_2^2x_1^2) \; = < x_0, x_1 \mid x_0x_1x_0^2x_1^2 >$
$G_3(x_0x_3x_2^2x_1^2) \; = < x_0, x_1, x_2 \mid x_0^2x_2^2x_1^2 >$
$G_2(x_0x_5x_3x_2) \; = < x_0, x_1 \mid x_0^2x_1^2 >$
$G_3(x_0x_5x_3x_2) \; = < x_0, x_1, x_2 \mid (x_0x_1)^2, (x_1x_2)^2, (x_2x_0)^2 >$
$G_2(x_0x_4x_3x_2^2x_1) \; = < x_0, x_1 \mid (x_0^2x_1)^2, (x_1^2x_0)^2 >$
$G_3(x_0x_4x_3x_2^2x_1) \; = < x_0, x_1, x_2 \mid x_0x_1x_0x_2^2x_1, x_1x_2x_1x_0^2x_2, x_2x_0x_2x_1^2x_0 >$
$G_2(x_0x_3x_2x_1x_2x_1) \; = < x_0, x_1 \mid (x_0x_1)^3 >$
$G_3(x_0x_3x_2x_1x_2x_1) \; = < x_0, x_1, x_2 \mid x_0x_1x_0x_2^2x_1, x_1x_2x_1x_0^2x_2, x_2x_0x_2x_1^2x_0 >$
$G_2(x_0x_2x_1x_3x_2x_1) \; = < x_0, x_1 \mid x_0x_1x_0^2x_1^2 >$
$G_3(x_0x_2x_1x_3x_2x_1) \; = < x_0, x_1, x_2 \mid (x_0x_2x_1)^2 >$

The groups $< x_0, x_1 \mid x_0^2x_1^2 >$, $< x_0, x_1 \mid (x_0x_1)^2 >$, $< x_0, x_1 \mid x_0x_1 >$, $< x_0, x_1 \mid x_0x_1x_0^{-1}x_1x_0x_1^{-1} >$, $< x_0, x_1 \mid x_1^2 >$, $< x_0, x_1 \mid x_0x_1x_0^2x_1^2 >$, $< x_0, x_1, x_2 \mid x_0^2x_2^2x_1^2 >$, $< x_0, x_1 \mid (x_0x_1)^3 >$ and $< x_0, x_1, x_2 \mid (x_0x_2x_1)^2 >$ all have more generators than relators so they are infinite.

The remaining three groups, $< x_0, x_1, x_2 \mid (x_0x_1)^2, (x_1x_2)^2, (x_2x_0)^2 >$,
$< x_0, x_1 \mid (x_0^2x_1)^2, (x_1^2x_0)^2 >$ and $< x_0, x_1, x_2 \mid x_0x_1x_0x_2^2x_1, x_1x_2x_1x_0^2x_2, x_2x_0x_2x_1^2x_0 >$, can all be found to be infinite by KBMAG.

Therefore, each group in the family is finite if and only if n meets the stated conditions.

8.5 Sporadics

In Chapter 6 we defined a *sporadic* to mean a word which appears for small n but does not occur in the set of words for each l which we have used throughout. They are precisely the words for which n divides the t exponent sum but the exponent sum is non-zero. For example, when $l = 8$, the word $x^{-1}t^{-1}x^{-1}t^{-1}x^{-1}t^{-1}xt^{-1}$ has t-exponent 4 and is a valid word for an irreducible presentation only when $n = 4$. There are 3 words valid for when $n = 5$ and for no other n when $l = 8$, and these 4 words are the only sporadics for $l = 8$. A full list of sporadics can be view in [24]. It is worth noting that there are no sporadics when $l = 7$.

As with the previous words for which the t-exponent is zero, our list of sporadic words increases as l increases. We therefore handle the words in the same way as before, looking at each one for $8 \leq l \leq 12$ and simply trying to compute the finite groups for $13 \leq l \leq 15$. Note that, whereas before we looked at each word for $4 \leq n \leq 50$, we now require only to look at the one relevant n. It is possible a particular sporadic word may be valid for more than one n, for example, if the t-exponent is 8 then it is valid for $n = 8$ and $n = 4$. However, we treat such a sporadic word separately for the different values of n so that we work using a list of words for each different l and each different n.

The following table shows how many sporadics there are for each l and n.

$l\backslash n$	4	5	6	7	8	9	10	11	12	Total
8	1	3	0	0	0	0	0	0	0	4
9	6	8	0	0	0	0	0	0	0	14
10	11	33	3	6	0	0	0	0	0	53
11	56	64	34	21	3	0	0	0	0	178
12	85	210	71	122	9	6	0	0	0	503
13	363	489	328	287	142	41	3	0	0	1653
14	636	1766	606	1074	285	300	19	13	0	4699
15	2788	3960	2886	2589	1693	894	346	77	5	15238

8.5.1 $l \leq 12$

As sporadics only exist for small n, there are no families of groups. Following are the finite groups found for the sporadics when $l \leq 12$.

Group G	$\|G\|$	Structure of G
$l = 9$		
$< x_0, \ldots, x_3 \mid x_i x_{i+1} x_{i+2} x_{i+3}^2 \ (0 \leq i \leq 3) >$	5	$\mathbb{Z}_5$
$< x_0, \ldots, x_4 \mid x_i x_{i+2}^{-1} x_{i+3} x_{i+4} \ (0 \leq i \leq 4) >$	22	$\mathbb{Z}_{22}$
$l = 10$		
$< x_0, \ldots, x_4 \mid x_i x_{i+2} x_{i+3} x_{i+4}^2 \ (0 \leq i \leq 4) >$	275	$\mathbb{Z}_{11} \rtimes \mathbb{Z}_{25}$
$< x_0, \ldots, x_4 \mid x_i x_{i+2}^2 x_{i+3} x_{i+4} \ (0 \leq i \leq 4) >$	1025	$\mathbb{Z}_{41} \rtimes \mathbb{Z}_{25}$
$< x_0, \ldots, x_5 \mid x_i x_{i+2} x_{i+4} x_{i+5} \ (0 \leq i \leq 5) >$	24	$\mathbb{Z}_3 \rtimes \mathbb{Z}_8$
$l = 11$		
$< x_0, \ldots, x_3 \mid x_i x_{i+1}^2 x_{i+2} x_{i+3}^3 \ (0 \leq i \leq 3) >$	168	$\mathbb{Z}_7 \times SL(2,3)$
$< x_0, \ldots, x_3 \mid x_i x_{i+1}^2 x_{i+2}^2 x_{i+3}^2 \ (0 \leq i \leq 3) >$	7	$\mathbb{Z}_7$
$< x_0, \ldots, x_3 \mid x_i x_{i+1}^{-2} x_i x_{i+2}^{-1} \ (0 \leq i \leq 3) >$	13	$\mathbb{Z}_{13}$
$l = 12$		
$< x_0, \ldots, x_3 \mid x_i x_{i+1} x_{i+2} x_{i+3} x_i x_{i+1}^{-1} \ (0 \leq i \leq 3) >$	624	$\mathbb{Z}_{39} \rtimes \mathbb{Z}_{16}$
$< x_0, \ldots, x_3 \mid x_i x_{i+1}^2 x_{i+2}^2 x_{i+3}^3 \ (0 \leq i \leq 3) >$	6260	$1 \trianglelefteq \mathbb{Z}_{205} \trianglelefteq G$
$< x_0, \ldots, x_4 \mid x_i x_{i+2} x_{i+4} x_{i+1}^2 \ (0 \leq i \leq 4) >$	1025	$\mathbb{Z}_{41} \rtimes \mathbb{Z}_{25}$
$< x_0, \ldots, x_4 \mid x_i x_{i+2} x_{i+4}^2 x_{i+1} \ (0 \leq i \leq 4) >$	275	$\mathbb{Z}_{11} \rtimes \mathbb{Z}_{25}$
$< x_0, \ldots, x_4 \mid x_i x_{i+2} x_{i+4}^{-1} x_i^{-1} x_{i+4} \ (0 \leq i \leq 4) >$	120	$SL(2,5)$
$< x_0, \ldots, x_4 \mid x_i x_{i+2}^{-1} x_{i+3} x_i^{-1} x_{i+1} \ (0 \leq i \leq 4) >$	1	$\mathbb{Z}_1$
$< x_0, \ldots, x_4 \mid x_i x_{i+1} x_{i+2}^2 x_{i+3} x_{i+4}^2 \ (0 \leq i \leq 4) >$	7	$\mathbb{Z}_7$
$< x_0, \ldots, x_5 \mid x_i x_{i+2}^2 x_{i+3} x_{i+4} x_{i+5} \ (0 \leq i \leq 5) >$	15624	$1 \trianglelefteq \mathbb{Z}_{434} \trianglelefteq G$
$< x_0, \ldots, x_6 \mid x_i x_{i+2} x_{i+3} x_{i+5} x_{i+6} \ (0 \leq i \leq 6) >$	5	$\mathbb{Z}_5$

The following table lists the number of unknown cases for $l \leq 12$. There are no unknowns for $l \leq 10$, $n \geq 6$ and so these unknowns do not affect Theorem 5.2.4.

Length	8	9	10	11	12
Total remaining	0	0	4	20	30
Total remaining $n \geq 6$	0	0	0	6	10

For $l \leq 10$ the remaining groups are as follows.

- $G_5(x_0^{-1}x_2x_3^{-1}x_4^{-2})$;
- $G_5(x_0^{-1}x_2x_3^{-1}x_4^{2})$;
- $G_5(x_0^{-1}x_2^2x_4^{-2})$;
- $G_5(x_0^{-1}x_2^2x_4^2)$.

Note that the words $x_0^{-1}x_2^2x_4^{-2}$ and $x_0^{-1}x_2^2x_4^2$ appearing in the above list are n-equivalent to $x_0^{-1}x_2^{-2}x_1^2$ and $x_0^{-1}x_2^2x_1^2$ respectively, which appear in the list of remaining groups in Section 8.2.

8.5.2 $13 \leq l \leq 15$

Following are the list of finite sporadics found for $l = 13$, 14 and 15 by asking GAP to tell us which of the sporadic words bring about finite groups.

$l = 13$

Group G	$\lvert G\rvert$	Structure of G
$< x_0, \ldots, x_3 \mid x_ix_{i+1}x_{i+2}x_{i+3}x_ix_{i+1}^2 \ (0 \leq i \leq 3) >$	1015	$\mathbb{Z}_{29} \rtimes (\mathbb{Z}_7 \times \mathbb{Z}_5)$
$< x_0, \ldots, x_3 \mid x_ix_{i+1}x_{i+2}x_{i+3}x_i^2x_{i+1} \ (0 \leq i \leq 3) >$	1015	$\mathbb{Z}_{29} \rtimes (\mathbb{Z}_7 \times \mathbb{Z}_5)$
$< x_0, \ldots, x_3 \mid x_ix_{i+1}x_{i+2}x_{i+3}x_i^2x_{i+1}^{-1} \ (0 \leq i \leq 3) >$	169125	$1 \trianglelefteq \mathbb{Z}_{2255} \trianglelefteq G$
$< x_0, \ldots, x_3 \mid x_ix_{i+1}x_{i+2}x_{i+3}^2x_ix_{i+1} \ (0 \leq i \leq 3) >$	791	$\mathbb{Z}_{113} \rtimes \mathbb{Z}_7$
$< x_0, \ldots, x_3 \mid x_ix_{i+1}x_{i+2}^{-1}x_ix_{i+2}^{-1} \ (0 \leq i \leq 3) >$	17	$\mathbb{Z}_{17}$
$< x_0, \ldots, x_3 \mid x_ix_{i+1}x_{i+2}^{-1}x_i^{-1}x_{i+2} \ (0 \leq i \leq 3) >$	1	Trivial
$< x_0, \ldots, x_3 \mid x_ix_{i+1}^3x_{i+3}^{-2}x_{i+2}^{-1} \ (0 \leq i \leq 3) >$	29	$\mathbb{Z}_{29}$
$< x_0, \ldots, x_3 \mid x_ix_{i+1}^2x_{i+3}^{-2}x_{i+2}^{-2} \ (0 \leq i \leq 3) >$	25	$\mathbb{Z}_{25}$
$< x_0, \ldots, x_3 \mid x_ix_{i+1}x_{i+3}x_ix_{i+2} \ (0 \leq i \leq 3) >$	5	$\mathbb{Z}_5$
$< x_0, \ldots, x_3 \mid x_ix_{i+1}x_{i+3}^{-1}x_ix_{i+2}^{-1} \ (0 \leq i \leq 3) >$	13	$\mathbb{Z}_{13}$
$< x_0, \ldots, x_3 \mid x_ix_{i+1}^{-3}x_{i+3}^2x_{i+2}^{-1} \ (0 \leq i \leq 3) >$	29	$\mathbb{Z}_{29}$
$< x_0, \ldots, x_3 \mid x_ix_{i+1}^{-2}x_{i+3}^2x_{i+2}^{-2} \ (0 \leq i \leq 3) >$	25	$\mathbb{Z}_{25}$
$< x_0, \ldots, x_3 \mid x_ix_{i+1}^{-1}x_{i+3}x_i^{-1}x_{i+2} \ (0 \leq i \leq 3) >$	5	$\mathbb{Z}_5$
$< x_0, \ldots, x_3 \mid x_i^2x_{i+1}^2x_{i+2}^2x_{i+3}^3 \ (0 \leq i \leq 3) >$	9	$\mathbb{Z}_9$
$< x_0, \ldots, x_4 \mid x_ix_{i+1}x_{i+2}^2x_{i+3}^2x_{i+4}^2 \ (0 \leq i \leq 4) >$	8	$\mathbb{Z}_8$
$< x_0, \ldots, x_4 \mid x_ix_{i+1}^2x_{i+2}x_{i+3}^2x_{i+4}^2 \ (0 \leq i \leq 4) >$	8	$\mathbb{Z}_8$
$< x_0, \ldots, x_5 \mid x_ix_{i+2}x_{i+4}x_0x_{i+1}^{-1} \ (0 \leq i \leq 5) >$	4095	$1 \trianglelefteq \mathbb{Z}_{91} \trianglelefteq G$
$< x_0, \ldots, x_5 \mid x_ix_{i+1}x_{i+2}x_{i+3}x_{i+4}x_{i+5}^2 \ (0 \leq i \leq 5) >$	7	$\mathbb{Z}_7$
$< x_0, \ldots, x_5 \mid x_ix_{i+1}^2x_{i+4}^{-1}x_{i+3}^{-1} \ (0 \leq i \leq 5) >$	19	$\mathbb{Z}_{19}$

Group G	$\lvert G\rvert$	Structure of G
$< x_0, \dots, x_5 \mid x_i x_{i+1}^{-2} x_{i+4} x_{i+3}^{-1} \ (0 \le i \le 5) >$	35	$\mathbb{Z}_{35}$
$< x_0, \dots, x_6 \mid x_i x_{i+2} x_{i+3} x_{i+5} x_{i+6}^{2} \ (0 \le i \le 6) >$	6	$\mathbb{Z}_{6}$
$< x_0, \dots, x_6 \mid x_i x_{i+2} x_{i+3} x_{i+4} x_{i+5} x_{i+6} \ (0 \le i \le 6) >$	6	$\mathbb{Z}_{6}$
$< x_0, \dots, x_7 \mid x_i x_{i+2} x_{i+4} x_{i+5} x_{i+7} \ (0 \le i \le 7) >$	5	$\mathbb{Z}_{5}$

$l = 14$

Group G	$\lvert G\rvert$	Structure of G
$< x_0, \dots, x_3 \mid x_i x_{i+1} x_{i+2}^{2} x_{i+3} x_i^{2} x_{i+1}^{-1} \ (0 \le i \le 3) >$	122640	$1 \trianglelefteq \mathbb{Z}_{2255} \trianglelefteq G$
$< x_0, \dots, x_4 \mid x_i x_{i+1} x_{i+2} x_{i+3} x_{i+4} x_i x_{i+1} \ (0 \le i \le 4) >$	7	$\mathbb{Z}_{7}$
$< x_0, \dots, x_4 \mid x_i x_{i+1} x_{i+2} x_{i+3} x_{i+4} x_i x_{i+1}^{-1} \ (0 \le i \le 4) >$	7775	$1 \trianglelefteq \mathbb{Z}_{311} \trianglelefteq G$
$< x_0, \dots, x_4 \mid x_i x_{i+1}^{2} x_{i+2}^{2} x_{i+3} x_{i+4}^{3} \ (0 \le i \le 4) >$	9	$\mathbb{Z}_{9}$
$< x_0, \dots, x_4 \mid x_i x_{i+1}^{2} x_{i+2}^{2} x_{i+3}^{2} x_{i+4}^{2} \ (0 \le i \le 4) >$	9	$\mathbb{Z}_{9}$
$< x_0, \dots, x_5 \mid x_i x_{i+2} x_i x_{i+3}^{-1} \ (0 \le i \le 4) >$	728	$\mathbb{Z}_{13} \rtimes \mathbb{Z}_{56}$
$< x_0, \dots, x_5 \mid x_i x_{i+1} x_{i+2} x_{i+3}^{2} x_{i+4} x_{i+5}^{2} \ (0 \le i \le 4) >$	728	$\mathbb{Z}_{13} \rtimes \mathbb{Z}_{56}$
$< x_0, \dots, x_8 \mid x_i x_{i+3} x_{i+4} x_{i+7} x_{i+8} \ (0 \le i \le 8) >$	5	$\mathbb{Z}_{5}$
$< x_0, \dots, x_9 \mid x_i x_{i+4} x_{i+8} x_{i+9} \ (0 \le i \le 9) >$	40	$\mathbb{Z}_{5} \rtimes \mathbb{Z}_{8}$

$l = 15$

Group G	$\lvert G\rvert$	Structure of G
$< x_0, \dots, x_3 \mid x_i x_{i+1} x_{i+2} x_{i+3} x_i x_{i+1} x_{i+2} \ (0 \le i \le 3) >$	7	$\mathbb{Z}_{7}$
$< x_0, \dots, x_3 \mid x_i x_{i+1} x_{i+2} x_{i+3} x_i x_{i+1} x_{i+2}^{-1} \ (0 \le i \le 3) >$	125	$\mathbb{Z}_{25} \rtimes \mathbb{Z}_{5}$
$< x_0, \dots, x_3 \mid x_i x_{i+1} x_{i+2}^{-2} x_i x_{i+3}^{-1} x_{i+2}^{-1} \ (0 \le i \le 3) >$	29	$\mathbb{Z}_{29}$
$< x_0, \dots, x_3 \mid x_i x_{i+1} x_{i+2}^{-1} x_i^{2} x_{i+2}^{-2} \ (0 \le i \le 3) >$	37	$\mathbb{Z}_{37}$
$< x_0, \dots, x_3 \mid x_i x_{i+1} x_{i+2}^{-1} x_i x_{i+3}^{-1} x_{i+2}^{-2} \ (0 \le i \le 3) >$	29	$\mathbb{Z}_{29}$
$< x_0, \dots, x_3 \mid x_i x_{i+1} x_{i+2}^{-1} x_i^{-2} x_{i+2}^{2} \ (0 \le i \le 3) >$	5	$\mathbb{Z}_{5}$
$< x_0, \dots, x_3 \mid x_i x_{i+1} x_{i+2}^{-1} x_i^{-1} x_{i+3}^{-2} x_{i+2}^{-1} \ (0 \le i \le 3) >$	39	$\mathbb{Z}_{39}$
$< x_0, \dots, x_3 \mid x_i x_{i+1}^{2} x_{i+2}^{-1} x_{i+1}^{-1} x_{i+3}^{-1} x_{i+2} \ (0 \le i \le 3) >$	5	$\mathbb{Z}_{5}$
$< x_0, \dots, x_3 \mid x_i x_{i+1}^{2} x_{i+2}^{-1} x_{i+1}^{-1} x_{i+3}^{-1} x_{i+2}^{-1} \ (0 \le i \le 3) >$	13	$\mathbb{Z}_{13}$
$< x_0, \dots, x_3 \mid x_i x_{i+1}^{4} x_{i+3}^{-3} x_{i+2}^{-1} \ (0 \le i \le 3) >$	53	$\mathbb{Z}_{53}$
$< x_0, \dots, x_3 \mid x_i x_{i+1}^{2} x_{i+3} x_i x_{i+2}^{2} \ (0 \le i \le 3) >$	791	$\mathbb{Z}_{113} \rtimes \mathbb{Z}_{7}$
$< x_0, \dots, x_3 \mid x_i x_{i+1}^{2} x_{i+3} x_{i+1}^{-1} x_{i+3}^{-2} \ (0 \le i \le 3) >$	5	$\mathbb{Z}_{5}$
$< x_0, \dots, x_3 \mid x_i x_{i+1}^{2} x_{i+3}^{-2} x_{i+2}^{-2} x_{i+1}^{-2} \ (0 \le i \le 3) >$	507	$\mathbb{Z}_{13} \rtimes (\mathbb{Z}_{3} \times \mathbb{Z}_{13})$
$< x_0, \dots, x_3 \mid x_i x_{i+1}^{2} x_{i+3}^{-1} x_{i+2}^{-1} x_{i+3}^{-1} x_{i+1} \ (0 \le i \le 3) >$	29	$\mathbb{Z}_{29}$

Group G	$\lvert G \rvert$	Structure of G
$< x_0, \ldots, x_3 \mid x_i x_{i+1}^2 x_{i+3}^{-1} x_{i+1} x_{i+3}^{-2} \ (0 \le i \le 3) >$	37	$\mathbb{Z}_{37}$
$< x_0, \ldots, x_3 \mid x_i x_{i+1}^{-1} x_{i+2} x_{i+1}^{-1} x_{i+3}^2 x_{i+2}^{-1} \ (0 \le i \le 3) >$	17	$\mathbb{Z}_{17}$
$< x_0, \ldots, x_3 \mid x_i x_{i+1}^{-4} x_{i+3}^3 x_{i+2}^{-1} \ (0 \le i \le 3) >$	53	$\mathbb{Z}_{53}$
$< x_0, \ldots, x_3 \mid x_i x_{i+1}^{-2} x_i^{-2} x_{i+3}^{-2} x_{i+2}^{-2} \ (0 \le i \le 3) >$	791	$\mathbb{Z}_{113} \rtimes \mathbb{Z}_7$
$< x_0, \ldots, x_3 \mid x_i x_{i+1}^{-2} x_{i+3} x_i^{-2} x_{i+2} \ (0 \le i \le 3) >$	13	$\mathbb{Z}_{13}$
$< x_0, \ldots, x_3 \mid x_i x_{i+1}^{-2} x_{i+3} x_i^{-1} x_{i+2}^{-2} \ (0 \le i \le 3) >$	507	$\mathbb{Z}_{13} \rtimes (\mathbb{Z}_3 \times \mathbb{Z}_{13})$
$< x_0, \ldots, x_3 \mid x_i x_{i+1}^{-2} x_{i+3} x_{i+2}^{-1} x_{i+3} x_{i+1}^{-1} \ (0 \le i \le 3) >$	29	$\mathbb{Z}_{29}$
$< x_0, \ldots, x_3 \mid x_i^2 x_{i+1}^3 x_{i+2}^2 x_{i+3}^4 \ (0 \le i \le 3) >$	264	$1 \trianglelefteq \mathbb{Z}_2 \trianglelefteq \mathbb{Q}_8 \trianglelefteq G$
$< x_0, \ldots, x_3 \mid x_i^2 x_{i+1}^3 x_{i+2}^3 x_{i+3}^3 \ (0 \le i \le 3) >$	11	$\mathbb{Z}_{11}$
$< x_0, \ldots, x_3 \mid x_i^2 x_{i+1}^3 x_{i+3}^{-2} x_{i+2}^{-2} \ (0 \le i \le 3) >$	41	$\mathbb{Z}_{41}$
$< x_0, \ldots, x_3 \mid x_i^2 x_{i+1}^{-3} x_{i+3}^2 x_{i+2}^{-2} \ (0 \le i \le 3) >$	41	$\mathbb{Z}_{41}$
$< x_0, \ldots, x_4 \mid x_i x_{i+2} x_{i+3} x_i x_{i+1} x_{i+4} \ (0 \le i \le 4) >$	6	$\mathbb{Z}_6$
$< x_0, \ldots, x_4 \mid x_i x_{i+2} x_{i+3}^2 x_{i+4} x_{i+3} x_{i+4}^2 \ (0 \le i \le 4) >$	8	$\mathbb{Z}_8$
$< x_0, \ldots, x_4 \mid x_i x_{i+1} x_{i+2} x_{i+3}^2 x_{i+4} x_i x_{i+1} \ (0 \le i \le 4) >$	8	$\mathbb{Z}_8$
$< x_0, \ldots, x_5 \mid x_i x_{i+2}^2 x_{i+5}^{-1} x_{i+3}^{-1} \ (0 \le i \le 5) >$	35	$\mathbb{Z}_{35}$
$< x_0, \ldots, x_5 \mid x_i x_{i+2}^{-2} x_{i+5} x_{i+3}^{-1} \ (0 \le i \le 5) >$	19	$\mathbb{Z}_{19}$
$< x_0, \ldots, x_5 \mid x_i x_{i+1} x_{i+2}^2 x_{i+3} x_{i+4}^2 x_{i+5}^2 \ (0 \le i \le 5) >$	262143	$1 \trianglelefteq \mathbb{Z}_{9709} \trianglelefteq G$
$< x_0, \ldots, x_5 \mid x_i x_{i+3}^3 x_{i+4}^{-2} x_{i+3}^{-1} \ (0 \le i \le 5) >$	117	$\mathbb{Z}_{117}$
$< x_0, \ldots, x_5 \mid x_i x_{i+1}^2 x_{i+4}^{-2} x_{i+3}^{-2} \ (0 \le i \le 5) >$	37	$\mathbb{Z}_{37}$
$< x_0, \ldots, x_5 \mid x_i x_{i+1}^{-3} x_{i+4}^2 x_{i+3}^{-1} \ (0 \le i \le 5) >$	133	$\mathbb{Z}_{133}$
$< x_0, \ldots, x_5 \mid x_i x_{i+1}^{-2} x_{i+4}^2 x_{i+3}^{-2} \ (0 \le i \le 5) >$	91	$\mathbb{Z}_{91}$
$< x_0, \ldots, x_6 \mid x_i x_{i+3} x_{i+6} x_{i+2} \ (0 \le i \le 6) >$	7	$\mathbb{Z}_7$
$< x_0, \ldots, x_6 \mid x_i x_{i+3} x_{i+4}^2 x_{i+5}^2 x_{i+6}^2 \ (0 \le i \le 6) >$	8	$\mathbb{Z}_8$
$< x_0, \ldots, x_6 \mid x_i x_{i+2} x_{i+3} x_{i+5} x_{i+6} x_{i+1} \ (0 \le i \le 6) >$	6	$\mathbb{Z}_6$
$< x_0, \ldots, x_6 \mid x_i x_{i+2} x_{i+3}^2 x_{i+4} x_{i+5} x_{i+6}^2 \ (0 \le i \le 6) >$	8	$\mathbb{Z}_8$
$< x_0, \ldots, x_6 \mid x_i x_{i+1} x_{i+2} x_{i+3} x_{i+4} x_{i+5} x_{i+6}^2 \ (0 \le i \le 6) >$	8	$\mathbb{Z}_8$
$< x_0, \ldots, x_7 \mid x_i x_{i+1}^2 x_{i+5}^{-1} x_{i+4}^{-1} \ (0 \le i \le 7) >$	97	$\mathbb{Z}_{97}$
$< x_0, \ldots, x_7 \mid x_i x_{i+1}^{-2} x_{i+5} x_{i+4}^{-1} \ (0 \le i \le 7) >$	97	$\mathbb{Z}_{97}$

8.6 Number of generators required for finite groups

All of the finite groups in this chapter can be generated by up to three generators. We know a group can be generated by up to two generators if the group is cyclic or metacyclic. Otherwise GAP has been used to simplify the presentations.

While we have not found any interesting groups needing four generators, we have found groups which may require three. We list those groups for which GAP has not found a presentation using fewer than three generators, each of which can be presented on three generators.

- $G =< x_0, \ldots, x_3 \mid x_i x_{i+2} x_{i+3} x_{i+2}^2 x_{i+1}^2 \ (0 \leq i \leq 3) >$, $l = 13$, $|G| = 1015$,
- $G =< x_0, \ldots, x_5 \mid x_i x_{i+3}^{-1} x_{i+5} x_{i+2}^{-1} x_{i+1} \ (0 \leq i \leq 5) >$, $l = 15$, $|G| = 6552$.

All the remaining groups mentioned in this chapter require at most two generators.

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